University of Cape Town
Department of Physics

PHY2014F
Vibrations and Waves

Part 3
Travelling waves
Boundary conditions
Sound
Interference and diffraction

… covering (more or less)
French Chapters 7 & 8

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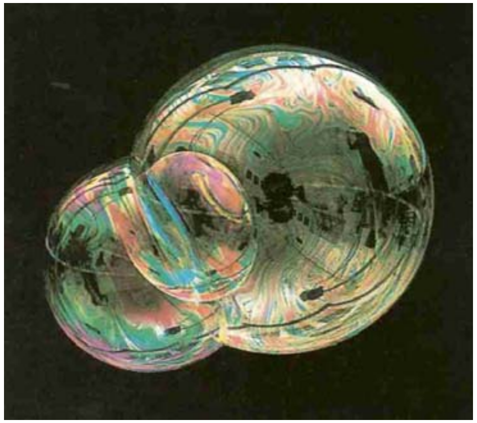
Problem-solving and homework



Each week you will be given a take-home problem set to complete and hand in for marks ...

In addition to this, you need to work through the following problems in *French*, in you own time, at home. You will not be asked to hand these in for marks. Get help from you friends, the course tutor, lecturer, ... Do not take shortcuts. Mastering these problems is a fundamental aspect of this course.

The problems associated with Part 3 are:

7-1, 7-2, 7-3, 7-4, 7-5, 7-6, 7-7, 7-8, 7-9, 7-10, 7-11, 7-15, 7-17, 7-18, 7-19, 7-21

You might find these tougher:

Travelling waves

For a string clamped at both ends the displacement of the n^{th} normal mode is

$$y(x,t) = A\sin\left(\frac{n\pi x}{L}\right)\cos\omega_n t \qquad \omega_n = \frac{n\pi v}{L}$$

Since $L = n\dfrac{\lambda}{2}$ then $\dfrac{n\pi vt}{L} = \dfrac{2\pi vt}{\lambda}$

Then write: $\psi(x,t) = A\sin\left(\dfrac{n\pi x}{L}\right)\cos\left(\dfrac{2\pi vt}{\lambda}\right)$

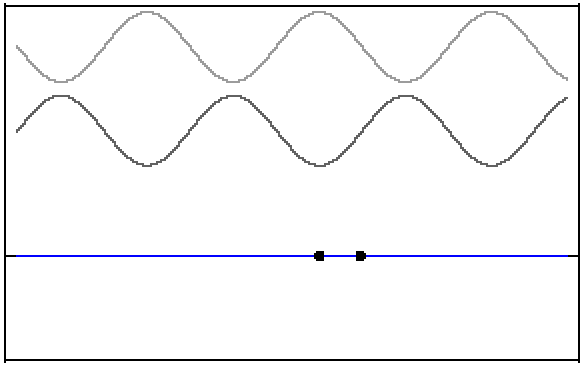
And using $\sin\alpha\sin\beta = \frac{1}{2}\left\{\sin(\alpha+\beta)+\sin(\alpha-\beta)\right\}$

$$\psi(x,t) = \frac{A}{2}\sin\left(\frac{2\pi x}{\lambda}+\frac{2\pi vt}{\lambda}\right)+\frac{A}{2}\sin\left(\frac{2\pi x}{\lambda}-\frac{2\pi vt}{\lambda}\right)$$

$$= \frac{A}{2}\sin\left\{\frac{2\pi}{\lambda}(x+vt)\right\}+\frac{A}{2}\sin\left\{\frac{2\pi}{\lambda}(x-vt)\right\}$$

Travelling waves ...2

$$\psi(x,t) = \frac{A}{2}\underbrace{\sin\left\{\frac{2\pi}{\lambda}(x - vt)\right\}}_{\substack{\text{A wave travelling in}\\ \text{the +ve } x\text{-direction}\\ \text{with speed } v}} + \frac{A}{2}\underbrace{\sin\left\{\frac{2\pi}{\lambda}(x + vt)\right\}}_{\substack{\text{A wave travelling in}\\ \text{the }-\text{ve } x\text{-direction}\\ \text{with speed } v}}$$



The speed v is called the **phase velocity** and is the speed of a crest (or any other point of specified phase)

$$v = \frac{\Delta x}{\Delta t}$$

Travelling waves ...3

We have shown that a standing wave

$$\psi(x,t) = A\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{2\pi vt}{\lambda}\right)$$

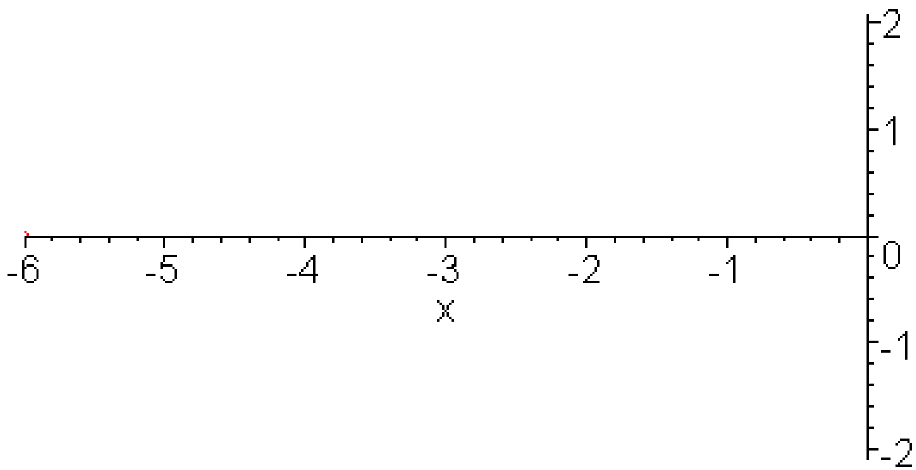
can be regarded as the superposition of two travelling waves:

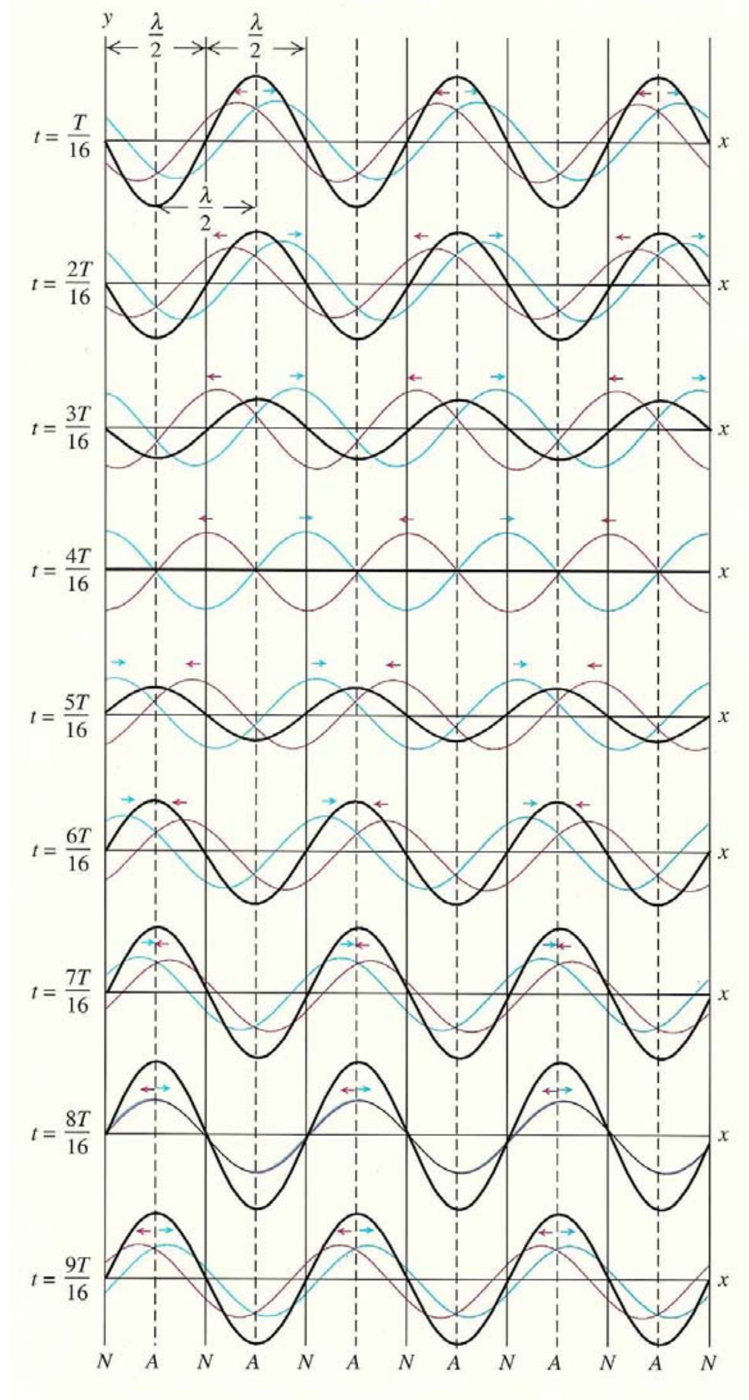
$$\psi_1(x,t) = \frac{A}{2}\sin\left\{\frac{2\pi}{\lambda}\left(x+vt\right)\right\} \quad \text{and} \quad \psi_2(x,t) = \frac{A}{2}\sin\left\{\frac{2\pi}{\lambda}\left(x-vt\right)\right\}$$

We can think of the standing wave being made up of a travelling wave being reflected back and forth from the two ends. The quantity v introduced in the standing wave treatment acquires a definite physical meaning: **phase velocity**

Formation of standing waves

…two waves having the same amplitude and frequency, travelling in opposite directions, interfere with each other.

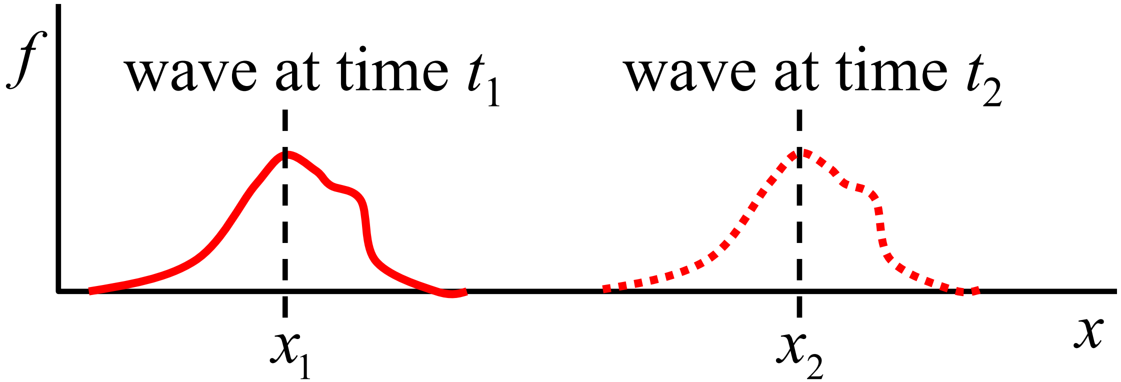




Travelling waves ...4

The travelling waves we have considered are sinusoidal in shape. Other shapes (wave pulses etc) are possible … and can be represented as the superposition of sinusoidal waves.

Any functions $f(x-vt)$ or $g(x+vt)$ are **solutions of the wave equation** and represent travelling waves.



$$f(x_1 - vt_1) = f(x_2 - vt_2)$$

$$\therefore x_1 - vt_1 = x_2 - vt_2$$

$$\therefore v = \frac{x_1 - x_2}{t_1 - t_2}$$

Travelling waves ...5

Travelling waves can be represented by sine or cosine functions:

$$\psi(x,t) = A\sin\left\{\frac{2\pi}{\lambda}(x-vt)\right\}$$

or $$\psi(x,t) = A\cos\left\{\frac{2\pi}{\lambda}(x-vt)\right\}$$

$$= A\cos\left\{2\pi\left(\frac{x}{\lambda}-\frac{v}{\lambda}t\right)\right\}$$

$$= A\cos\left\{2\pi\left(\frac{x}{\lambda}-ft\right)\right\}$$

$$= A\cos\left\{2\pi\left(\frac{x}{\lambda}-\frac{t}{T}\right)\right\}$$

$$= A\cos\{kx-\omega t\}$$

wave number

$$k = \frac{2\pi}{\lambda}$$

(French: $k = 1/\lambda$!)

Travelling waves ...6

Sinusoidal travelling waves:

$$\psi(x,t) = A\sin\{kx - \omega t + \phi\}$$

or $$\psi(x,t) = A\cos\{kx - \omega t + \phi'\}$$

waves in +ve x-direction

$$\psi(x,t) = A\sin\{kx + \omega t + \phi\}$$

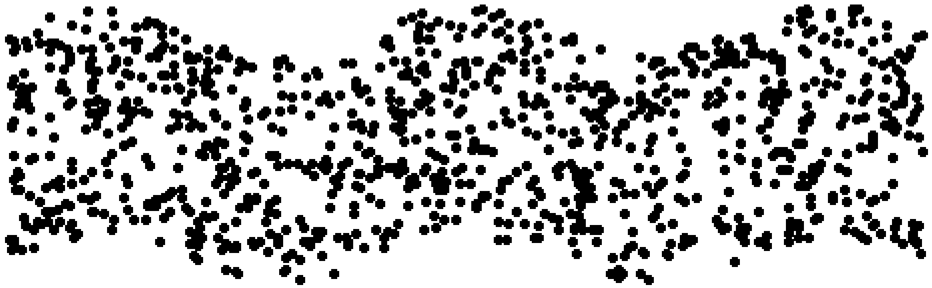
or $$\psi(x,t) = A\cos\{kx + \omega t + \phi'\}$$

waves in –ve x-direction

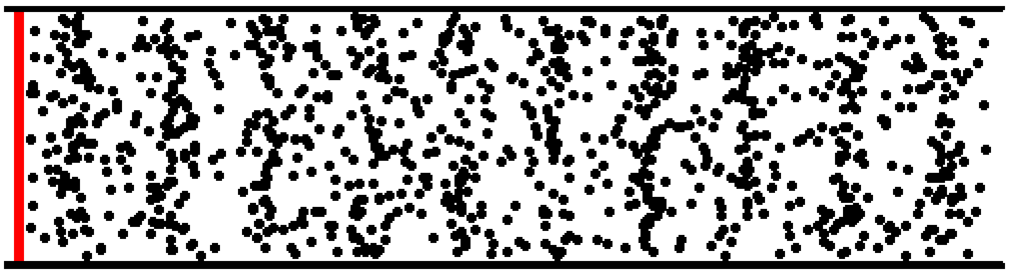
Phase velocity $v_{\text{phase}} = f\lambda = 2\pi f \dfrac{\lambda}{2\pi} = \dfrac{\omega}{k}$

Different types of travelling waves

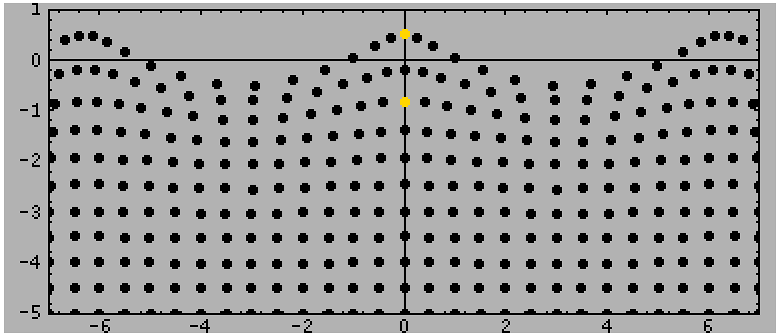
“transverse”



“longitudinal”



... and water waves?



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Wave speeds in specific media

Transverse waves on a stretched string: $v = \sqrt{\dfrac{T}{\mu}}$

Longitudinal waves in a thin rod: $v = \sqrt{\dfrac{Y}{\rho}}$

Liquids and gases: mainly longitudinal waves

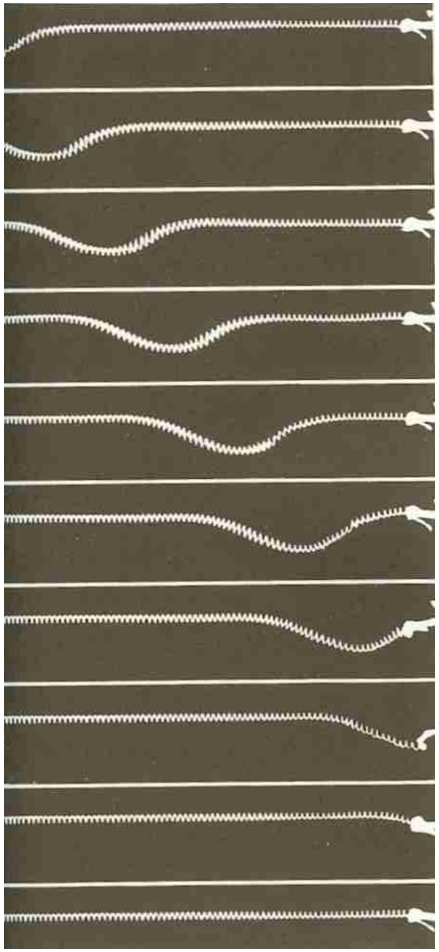
Liquids: $v = \sqrt{\dfrac{B}{\rho}}$ B: bulk modulus

Gases: $v = \sqrt{\dfrac{\gamma p}{\rho}} = \sqrt{\dfrac{\gamma RT}{M}}$

γ : ratio of specific heats

p : pressure

Wave pulses



... not covered in detail.

Read French 216 - 230

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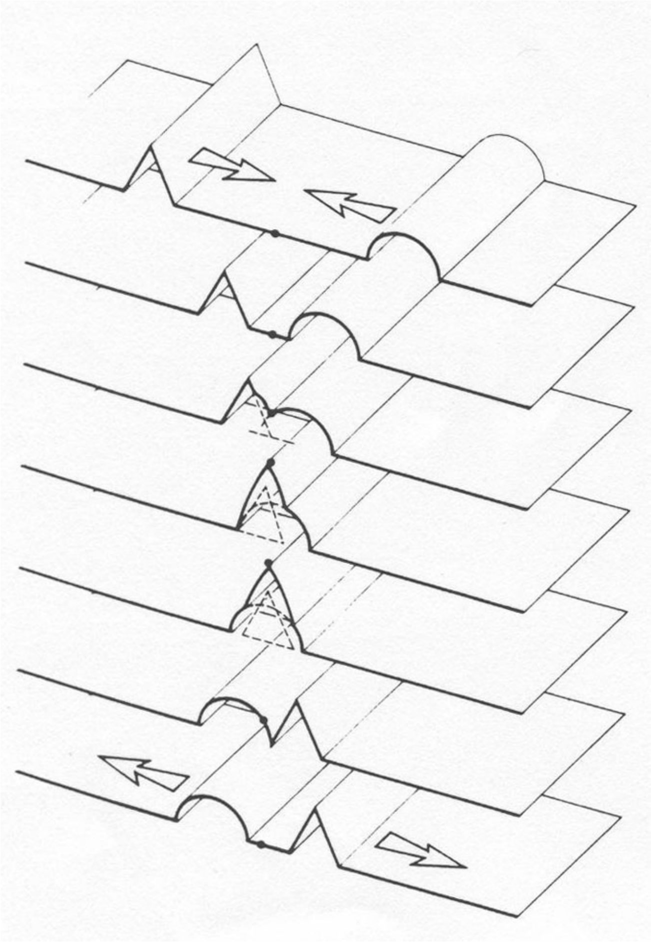
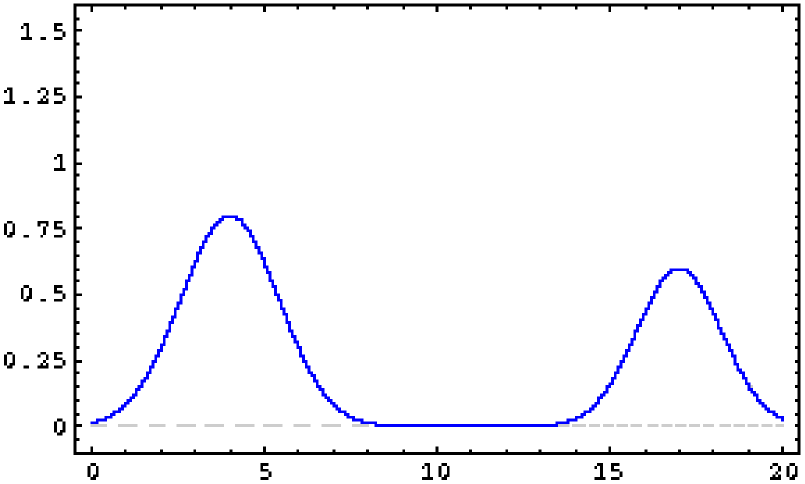
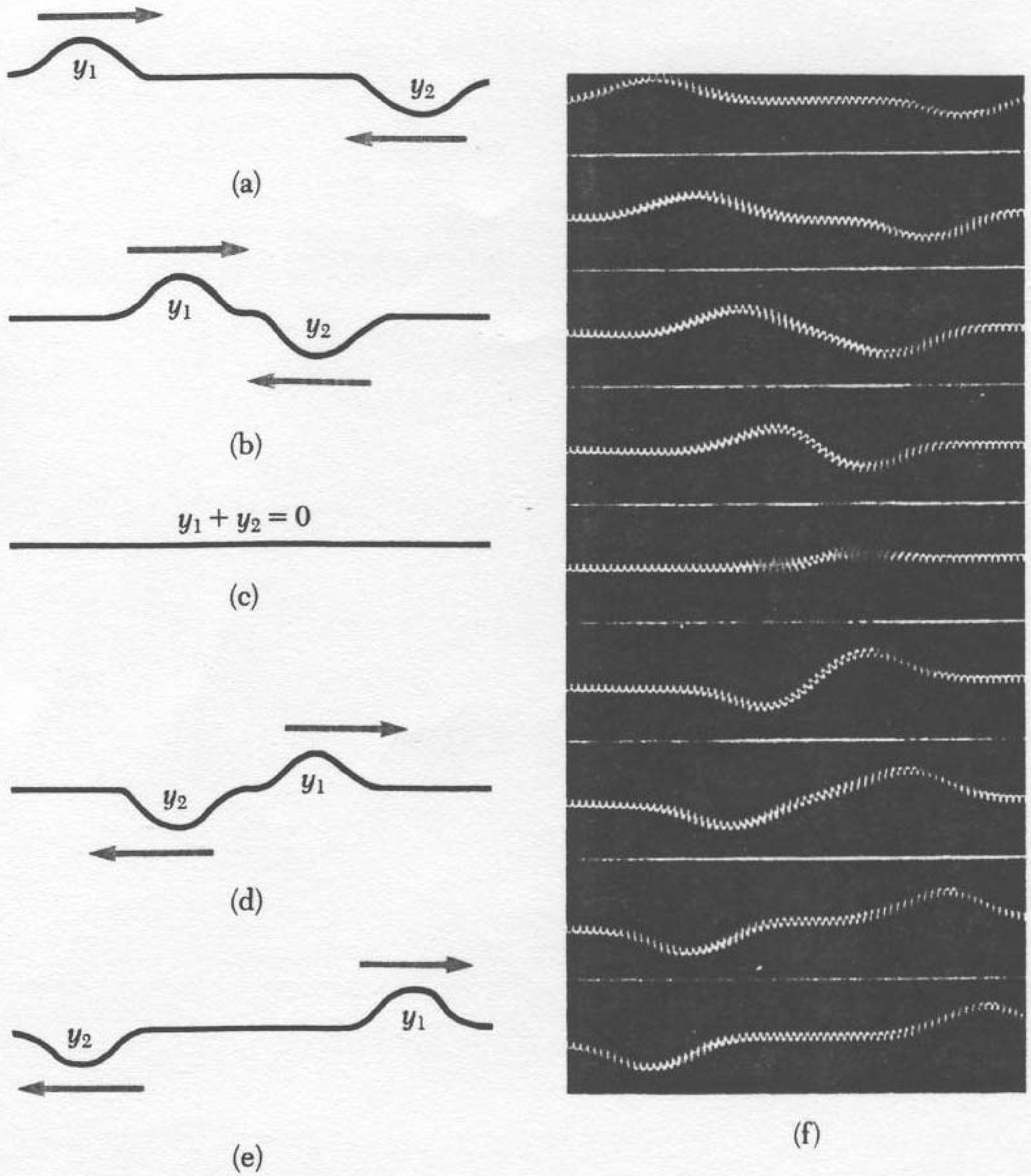
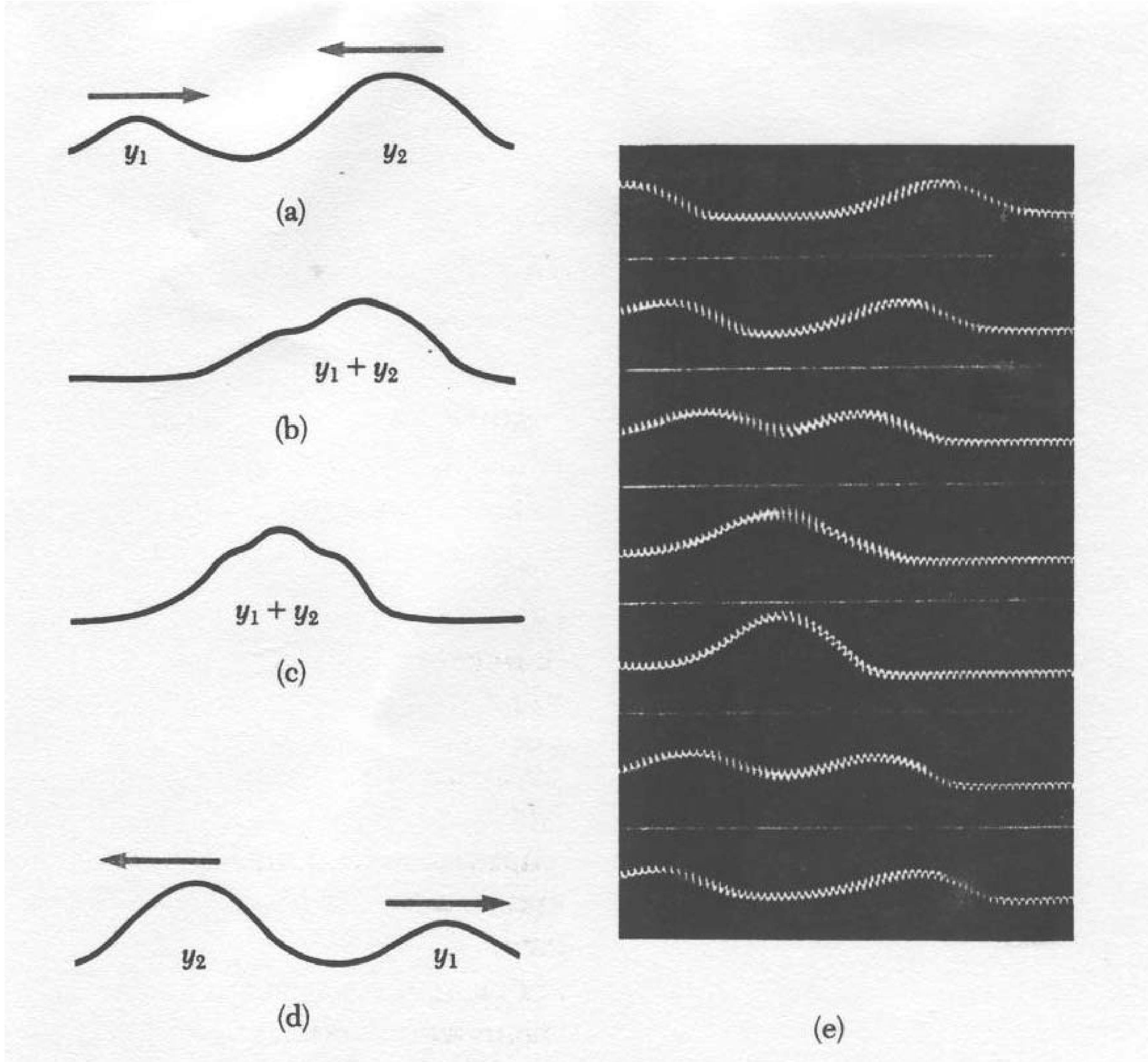
Superposition of wave pulses







y_1
y_2
(a)
y_1
y_2
(b)
$y_1 + y_2 = 0$
(c)
y_2
y_1
(d)
y_2
y_1
(e)
(f)



y_1
y_2
(a)
$y_1 + y_2$
(b)
$y_1 + y_2$
(c)
y_2
y_1
(d)
(e)

Reflection of wave pulses

heavy spring light spring very heavy spring

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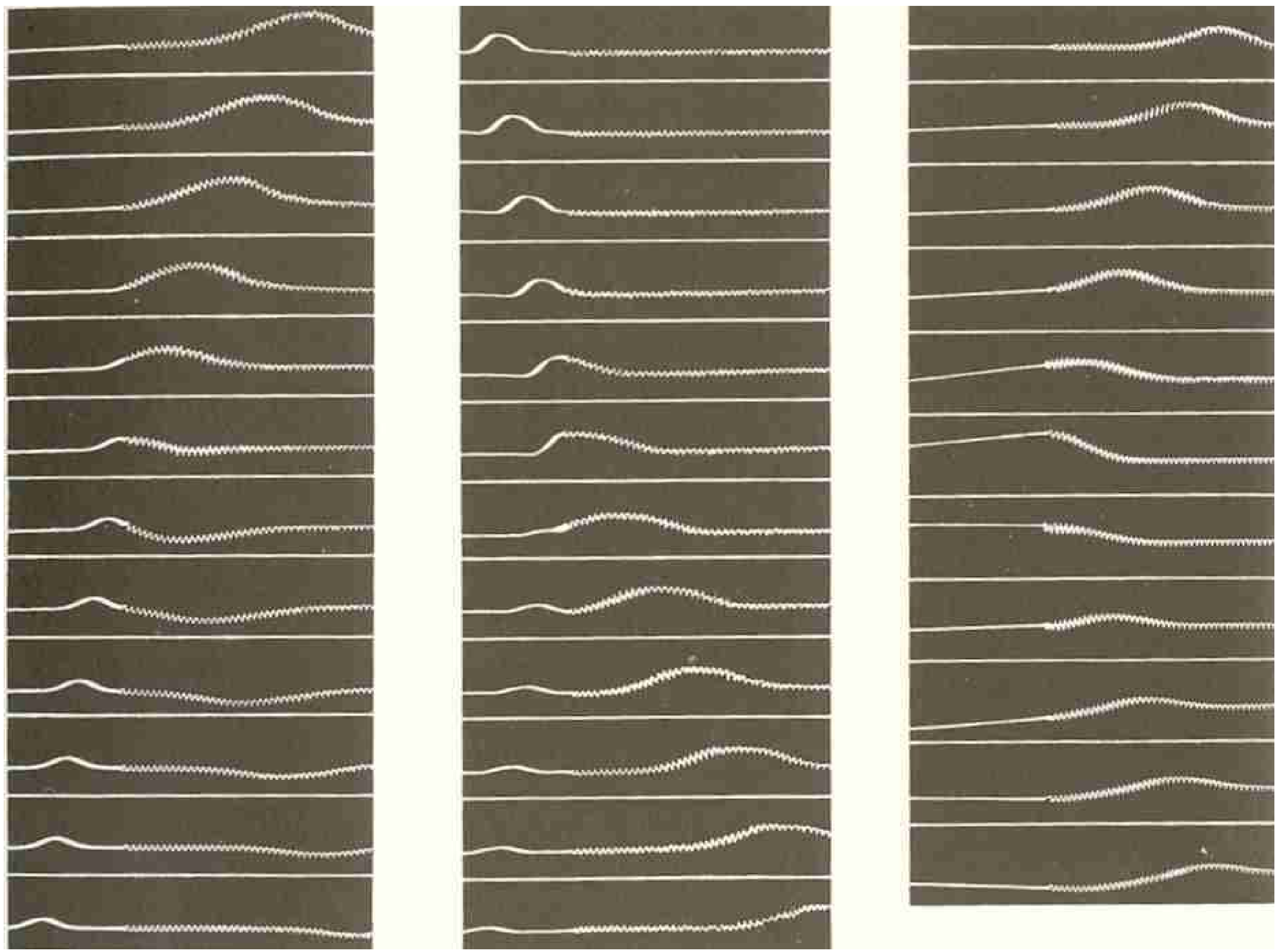


Reflection of wave pulses

heavy string light string

Reflection of wave pulses

Phase change of π
(i.e. inversion) on
reflection from fixed end.

No inversion on
reflection from free end.

Dispersion

The waves we are most familiar with (e.g. sound waves in air) are **non-dispersive** … i.e. waves of different frequency travel at the same speed.

Travelling wave on a string: $y(x,t) = A\sin\left\{\frac{2\pi}{\lambda}(x - vt)\right\}$

For a continuous string: $v = \sqrt{\frac{T}{\mu}}$ $\qquad f_n = nf_1$

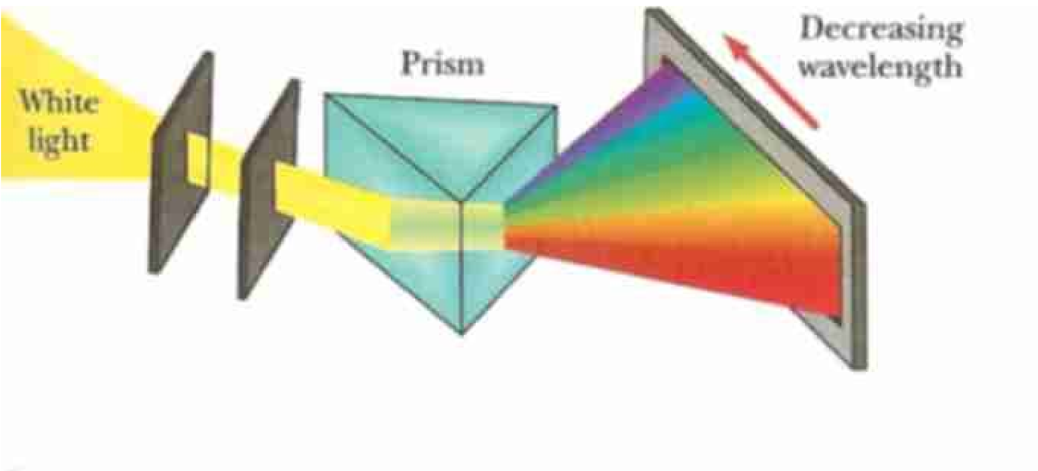
For a lumpy, beaded string: $f_n = 2f_0\sin\left(\frac{n\pi}{2(N+1)}\right)$

… high frequency waves may have a different velocity than low frequency.

Dispersion …2

No dispersion of light in a vacuum.

But light is dispersed into colours by a prism:



Snell's law: $$\frac{\sin i}{\sin r} = n(\lambda) = \frac{c}{v(\lambda)}$$

refractive index

… important for optic fibre communications …

French
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Dispersion …3

Take two sinusoidal travelling waves of slightly different frequencies:

$$\psi_1(x,t) = A\cos\{k_1x - \omega_1 t\}$$
$$\psi_2(x,t) = A\cos\{k_2x - \omega_2 t\}$$

Then $\psi(x,t) = \psi_1(x,t) + \psi_2(x,t)$

$$= A\cos\{k_1x - \omega_1 t\} + A\cos\{k_2x - \omega_2 t\}$$

$$= 2A\cos\left\{\frac{(k_1x - \omega_1 t) + (k_2x - \omega_2 t)}{2}\right\}\cos\left\{\frac{(k_1x - \omega_1 t) - (k_2x - \omega_2 t)}{2}\right\}$$

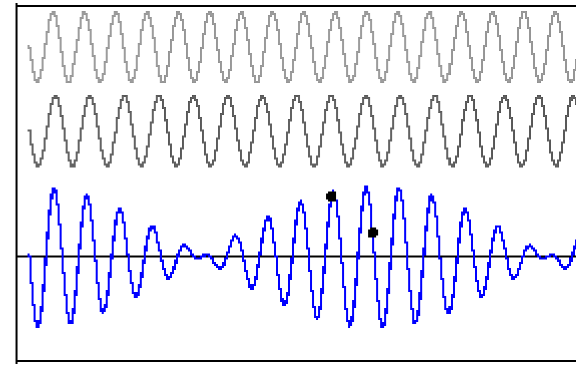
$$= 2A\cos\left\{\frac{k_1 + k_2}{2}x - \frac{\omega_1 + \omega_2}{2}t\right\}\cos\left\{\frac{k_1 - k_2}{2}x - \frac{\omega_1 - \omega_2}{2}t\right\}$$

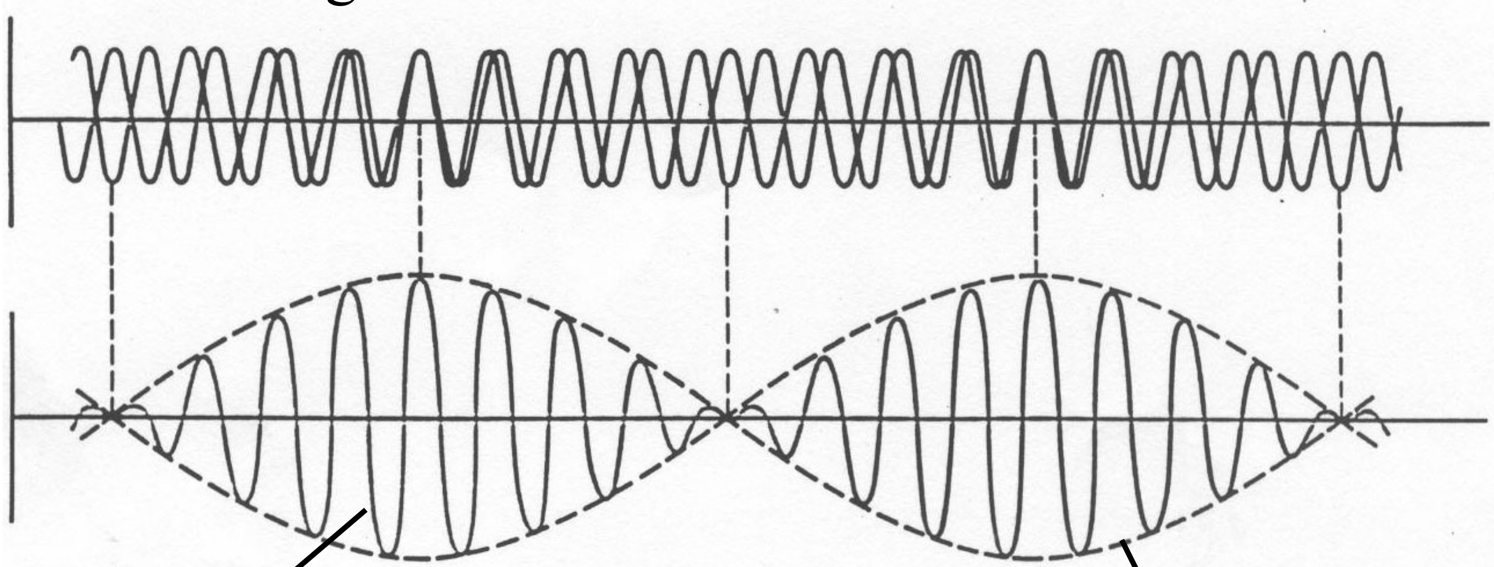
$$= 2A\cos\{kx - \omega t\}\cos\{\tfrac{1}{2}\Delta kx - \tfrac{1}{2}\Delta\omega t\}$$

where $k = \dfrac{k_1 + k_2}{2}$ $\quad \omega = \dfrac{\omega_1 + \omega_2}{2}$ $\quad \Delta k = k_1 - k_2$ $\quad \Delta\omega = \omega_1 - \omega_2$

Dispersion …4

$$\psi(x,t) = 2A\underbrace{\cos\{kx-\omega t\}}_{\text{Average travelling wave}}\underbrace{\cos\{\tfrac{1}{2}\Delta kx-\tfrac{1}{2}\Delta\omega t\}}_{\text{Envelope}}$$





$\cos\{kx-\omega t\}$

$\cos\{\tfrac{1}{2}\Delta kx-\tfrac{1}{2}\Delta\omega t\}$

$$v_{\text{phase}} = \frac{\omega}{k}$$

Group velocity $= v_{\text{group}} = \dfrac{\Delta\omega/2}{\Delta k/2} = \dfrac{\Delta\omega}{\Delta k}$

Dispersion …5

Phase velocity: $v_{\text{phase}} = \dfrac{\omega}{k}$

Group velocity: $v_{\text{group}} = \dfrac{\Delta\omega}{\Delta k} = \dfrac{d\omega}{dk}$

… the wave packet moves at v_g … so does transport of energy

When analyzing an arbitrary pulse into pure sinusoids …
… if these sinusoids have different characteristic speeds, then the shape of the disturbance must change over time …
… a pulse that is highly localized at $t = 0$ will become more and more spread out as it moves along.

Dispersion …6

Consider surface waves on liquids …

For long wavelength waves ($\lambda \sim$ m) on deep water (gravity waves):

$$\omega = \sqrt{gk} \qquad \text{then} \qquad \left.\begin{aligned} v_\phi &= \frac{\omega}{k} = \sqrt{\frac{g}{k}} \\ v_g &= \frac{d\omega}{dk} = \frac{1}{2}\sqrt{\frac{g}{k}} \end{aligned}\right\} \quad \therefore v_\phi = \frac{1}{2} v_g$$

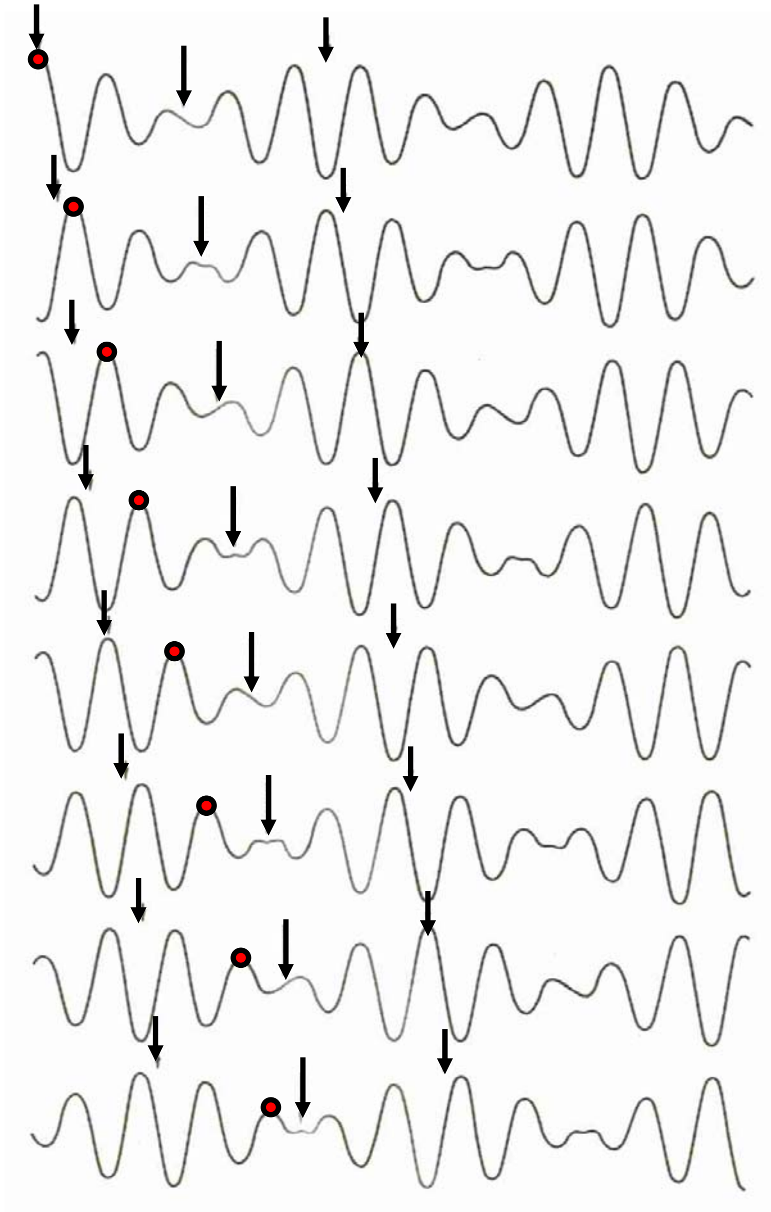
For short wavelength waves ($\lambda \sim$ mm) on deep water (capillary waves or ripples):

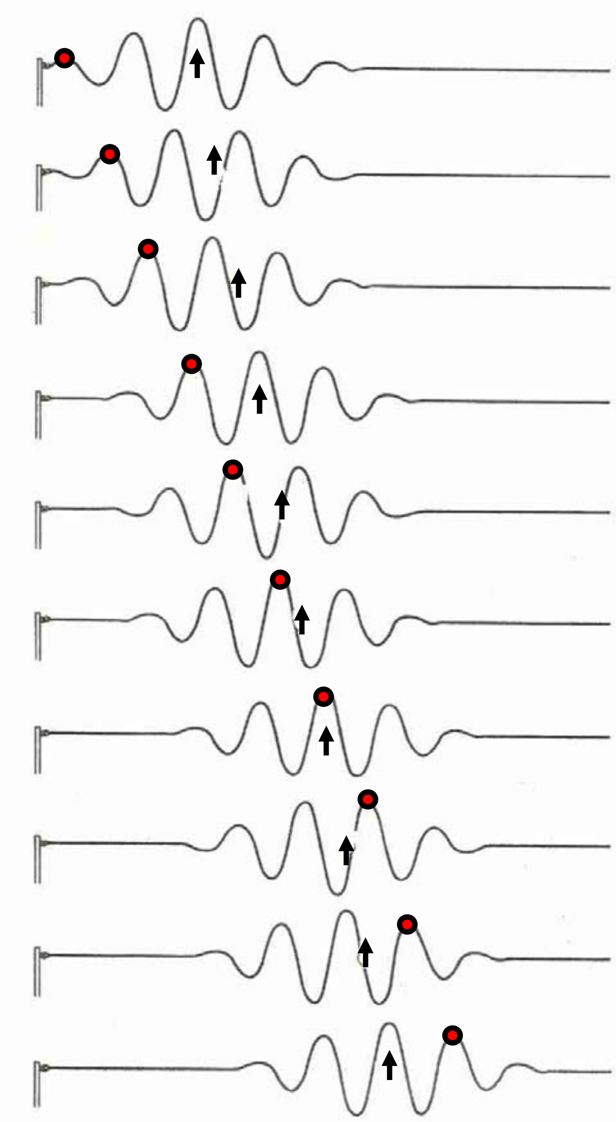
$$\omega = \sqrt{\frac{S}{\rho}k^3} \qquad \text{then} \qquad \left.\begin{aligned} v_\phi &= \frac{\omega}{k} = \sqrt{\frac{Sk}{\rho}} \\ v_g &= \frac{d\omega}{dk} = \frac{3}{2}\sqrt{\frac{Sk}{\rho}} \end{aligned}\right\} \quad \therefore v_\phi = \frac{3}{2} v_g$$

S = surface tension

Phase and group velocities of a wave and a pulse

with $v_\phi = 2v_g$





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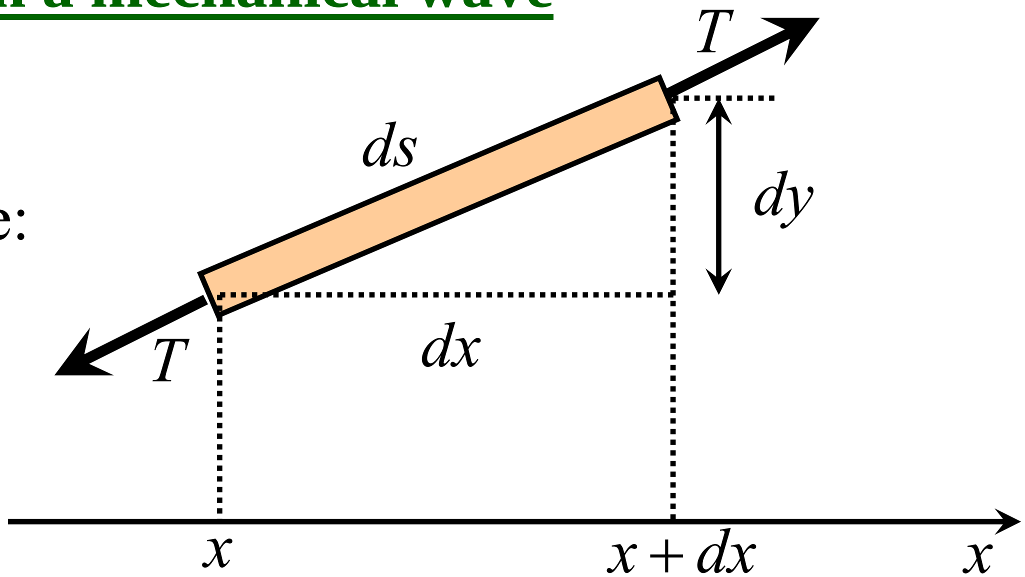
The energy in a mechanical wave



Consider a small segment of a string carrying a wave:

Mass of segment = μdx

If $u_y = \dfrac{\partial y}{\partial t}$

then kinetic energy per unit length = $\dfrac{dK}{dx} = \dfrac{1}{2}\mu\left(\dfrac{\partial y}{\partial t}\right)^2$

Potential energy = $T(ds - dx)$

where $ds = \sqrt{dx^2 + dy^2} = dx\sqrt{1+\left(\dfrac{\partial y}{\partial x}\right)^2} = dx\left(1+\dfrac{1}{2}\left(\dfrac{\partial y}{\partial x}\right)^2 + \ldots\right)$

Then potential energy per unit length = $\dfrac{dU}{dx} = \dfrac{1}{2}T\left(\dfrac{\partial y}{\partial x}\right)^2$

The energy in a mechanical wave …2

For a travelling wave on a string: $y(x,t) = A\sin\left\{2\pi f\left(t-\frac{x}{v}\right)\right\}$

Then $u_y(x,t) = \frac{\partial y}{\partial t} = 2\pi fA\cos\left\{2\pi f\left(t-\frac{x}{v}\right)\right\}$

$$= u_0\cos\left\{2\pi f\left(t-\frac{x}{v}\right)\right\}$$

At $t = 0$: $u_y(x) = \frac{\partial y}{\partial t} = u_0\cos\left\{\frac{2\pi fx}{v}\right\} = u_0\cos\left\{\frac{2\pi x}{\lambda}\right\}$

then $\frac{dK}{dx} = \frac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2 = \frac{1}{2}\mu u_0^2\cos^2\left\{\frac{2\pi x}{\lambda}\right\}$

and $K = \int_0^\lambda \frac{1}{2}\mu u_0^2\cos^2\left\{\frac{2\pi x}{\lambda}\right\}dx = \frac{1}{4}\lambda\mu u_0^2$ in one wavelength

The energy in a mechanical wave …3

Similarly for potential energy:

At $t = 0$: $$\frac{\partial y}{\partial x} = -\frac{2\pi A}{\lambda}\cos\left\{\frac{2\pi x}{\lambda}\right\}$$

then $$\frac{dU}{dx} = \frac{2\pi^2 A^2 T}{\lambda^2}\cos^2\left\{\frac{2\pi x}{\lambda}\right\}$$

and $$U = \int_0^\lambda \frac{2\pi^2 A^2 T}{\lambda^2}\cos^2\left\{\frac{2\pi x}{\lambda}\right\}dx = \frac{2\pi^2 A^2 T}{\lambda^2}\frac{\lambda}{2}$$ in one wavelength

$$\therefore U = \pi^2 A^2 \mu f^2 \lambda = \frac{1}{4}\lambda\mu u_0^2 \qquad u_0 = 2\pi f A$$

Over one wavelength, total energy: $$E = K + U = \frac{1}{2}\lambda\mu u_0^2$$

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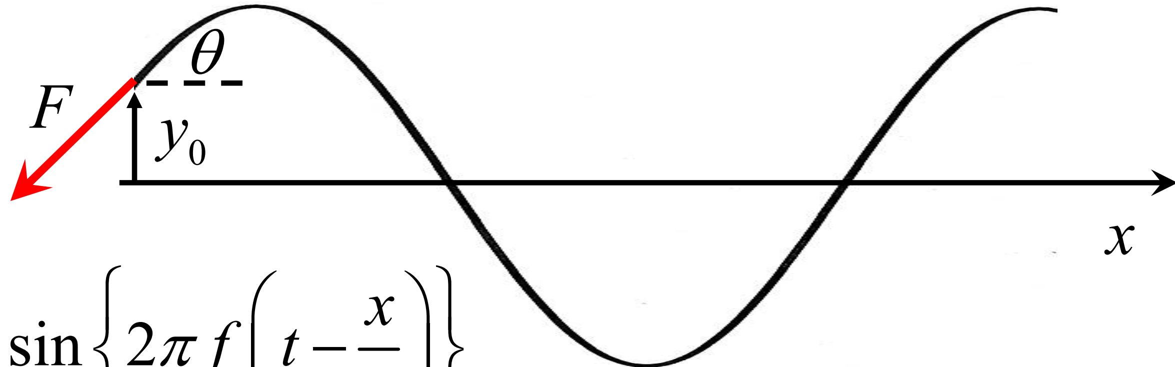
Transport of energy by a wave



$$y(x,t) = A\sin\left\{2\pi f\left(t - \frac{x}{v}\right)\right\}$$

A long string has one end at $x = 0$ and is driven at this point by a driving force F equal in magnitude to the tension T and applied in a direction tangent to the string.

At $x = 0$: $y(0,t) = A\sin 2\pi ft$

and $F_y = -T\sin\theta \approx -T\left(\frac{\partial y}{\partial x}\right)_{x=0} = -T\left(-\frac{2\pi fA}{v}\cos 2\pi ft\right)$

Then work done:

$$W = \int F_y dy_0 = \frac{2\pi fAT}{v}\int\left(\cos 2\pi ft\right) d\left(A\sin 2\pi ft\right)$$

$$= \frac{\left(2\pi fA\right)^2 T}{v}\int\left(\cos^2 2\pi ft\right)dt$$

Transport of energy by a wave …2

For one complete cycle: from $t = 0$ to $t = 1/f$:

$$W_{\text{cycle}} = \frac{u_0^2 T}{2v} \int_0^{1/f} (1 + \cos 4\pi f t)\; dt = \frac{u_0^2 T}{2vf} = \frac{1}{2} \lambda \mu u_0^2$$

Then mean power input $P = W_{\text{cycle}} \Big/ \tfrac{1}{f} = \dfrac{u_0^2 T}{2v} = \dfrac{1}{2} \mu u_0^2 v$

… thus P = energy per unit length × velocity

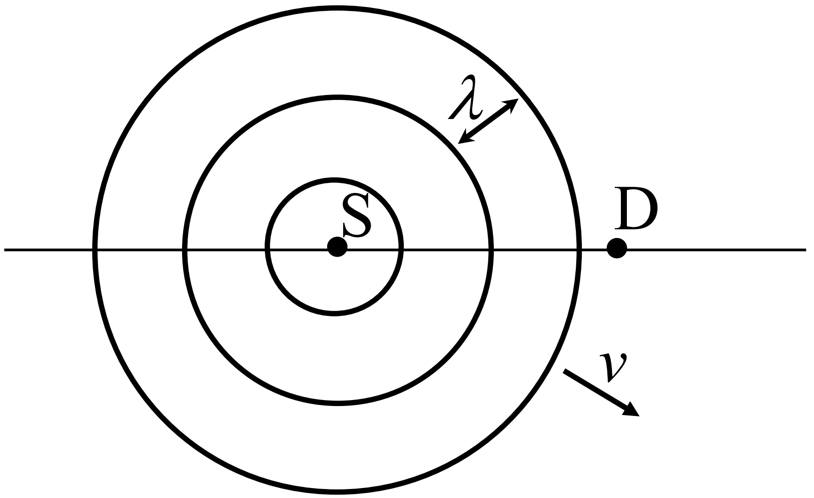
… energy flows along medium at velocity v ...

… and at v_g if the medium is dispersive

Doppler effect

... frequency changes of traveling waves due to motion of source and/or detector ...

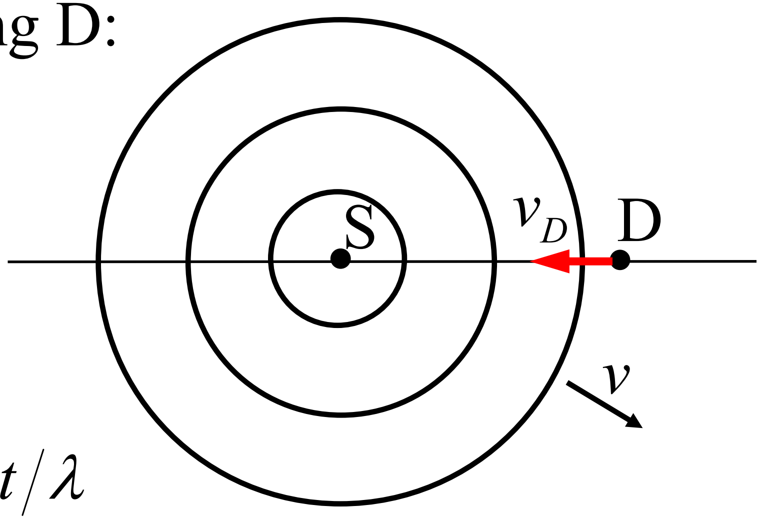
S: source of waves (sound)
D: detector
v_S : speed of source
v_D: speed of detector
v : speed of travelling wave = $f\lambda$



When both v_S and v_D are zero,then the number of wavelengths passing D in time t is $\dfrac{vt}{\lambda}$

Doppler effect ...2

For the case of stationary S and moving D:



For moving D we must add (or subtract) $v_D t/\lambda$ wavelengths.

Observed frequency: $$f' = \frac{vt/\lambda \pm v_D t/\lambda}{t}$$

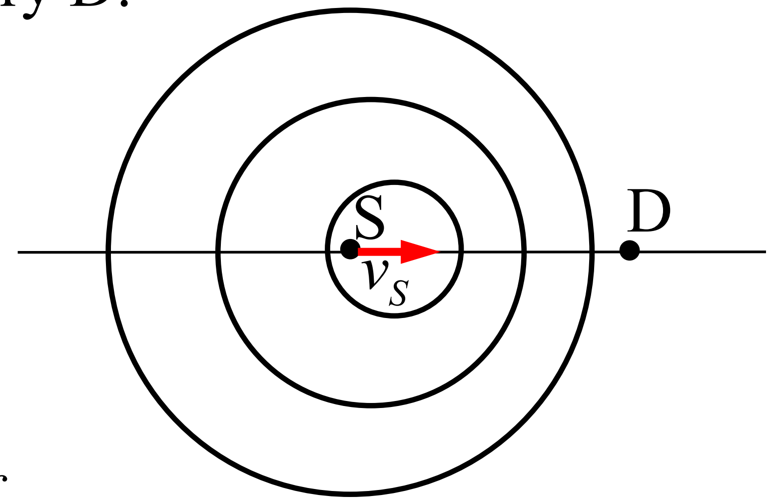
$$= \frac{v \pm v_D}{\lambda}$$

Source frequency: $f = v/\lambda$

$$\boxed{\therefore f' = f\,\frac{v \pm v_D}{v}}$$

Doppler effect ...3

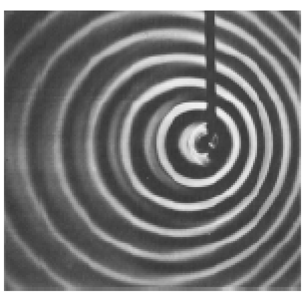
For the case of moving S and stationary D:



In time period T, S moves a distance $v_S T$.

Wavelength at D is therefore shortened (or lengthened) by $v_S T$.

Observed wavelength: $\lambda' = \lambda \mp v_S / f$



$$\therefore \frac{v}{f'} = \frac{v}{f} \mp \frac{v_S}{f} = \frac{v \mp v_S}{f}$$

$$\boxed{\therefore f' = f \frac{v}{v \mp v_S}}$$

For both source and detector moving:

$$f' = f \frac{v \pm v_D}{v \mp v_S}$$

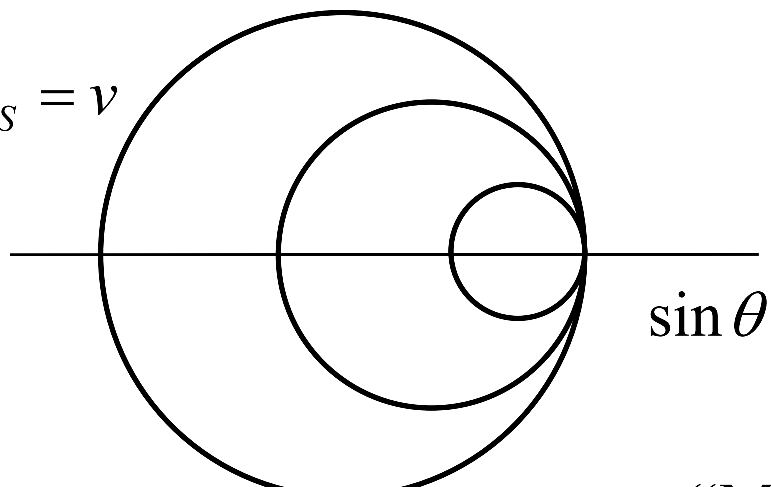
Doppler effect ...4

At supersonic speeds $v_S > v$, this relationship no longer applies:

All wavefronts are buched along a V-shaped envelope in 3D ... called a Mach cone ... a shock wave exists alongs the surface of this cone since the bunching of the wavefronts causes an abrupt rise and fall of air pressure as the surface passes any point ... causes a "sonic boom"

$v_S = v$

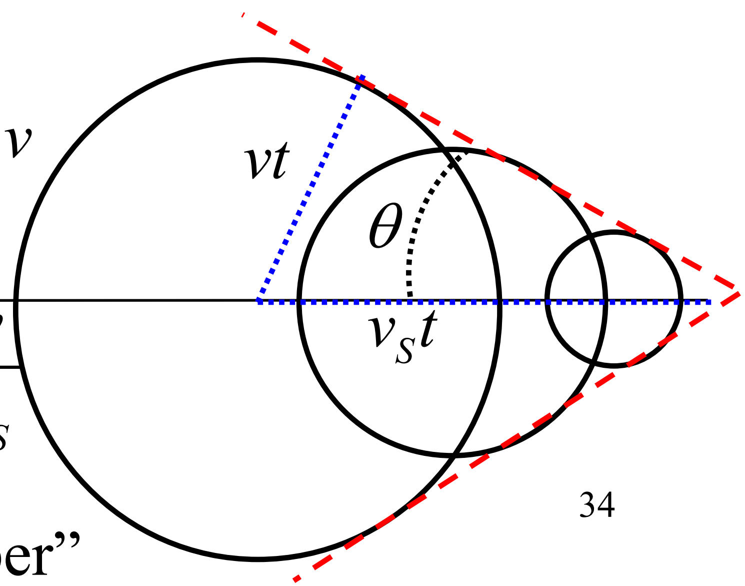
$v_S = v$



$$\sin\theta = \frac{vt}{v_S t} = \frac{v}{v_S}$$

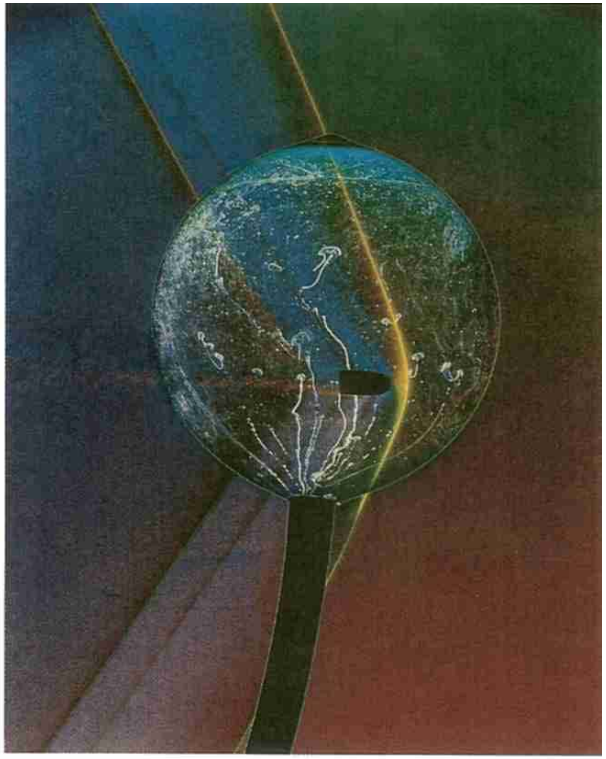
"Mach number"

$v_S > v$



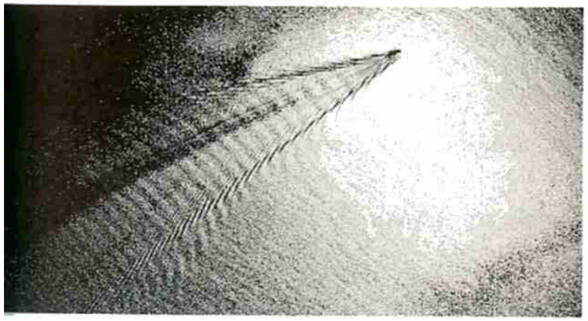
Doppler effect ...5

supersonic aircraft



travelling bullet

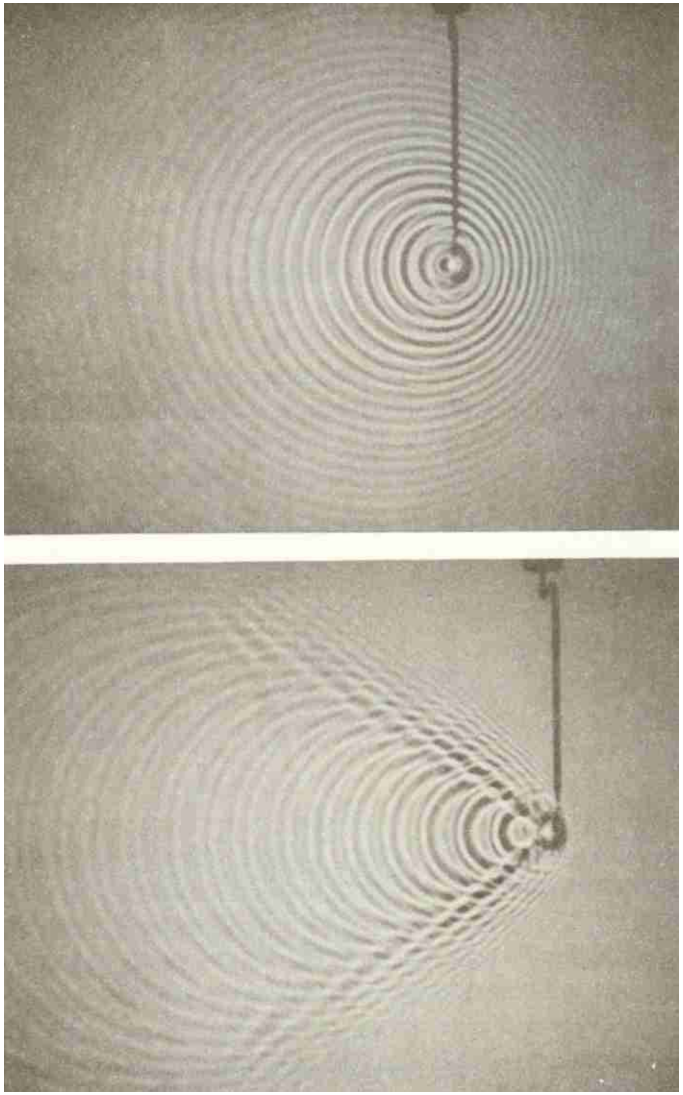
bow wave of a boat

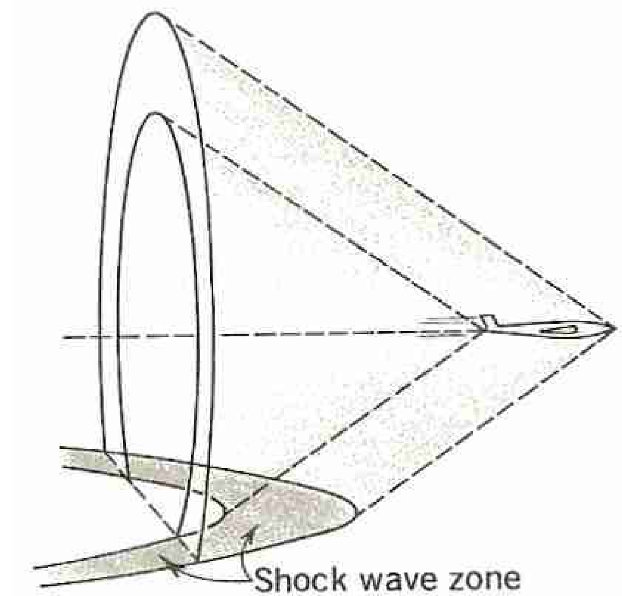


... get a "sonic boom" whenever the speed of the source of the waves is greater than the speed of waves <u>in that medium</u>.

Cherenkov radiation

Doppler effect ...6





Physical optics

Electromagnetic radiation can be modelled as a wave or a beam of particles ... all observed phenomena can be described by either model, although the treatment my be easier with one, or the other.

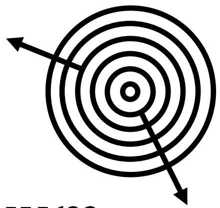
EM radiation propagates in vacuo, and may be thought of as E and B fields in phase with each other, and propagating at right angles to each other and to the direction of propagation.

Velocity of EM radiation in vacuum = constant, c

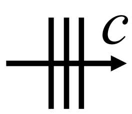
In a medium of refractive index n, $\quad v = c/n$

EM wave from a point source:

E is in phase around each circle ...



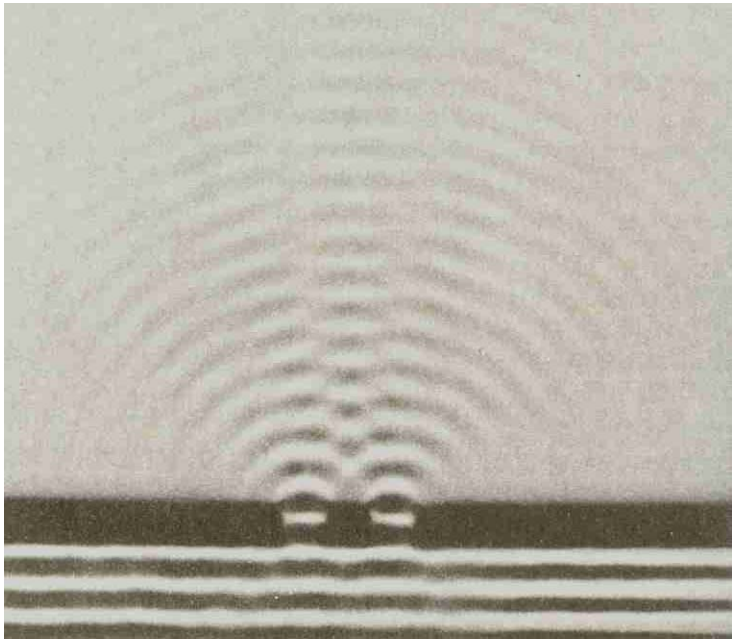
vacuum



... get a coherent plane wave at a large distance from point source

Interference

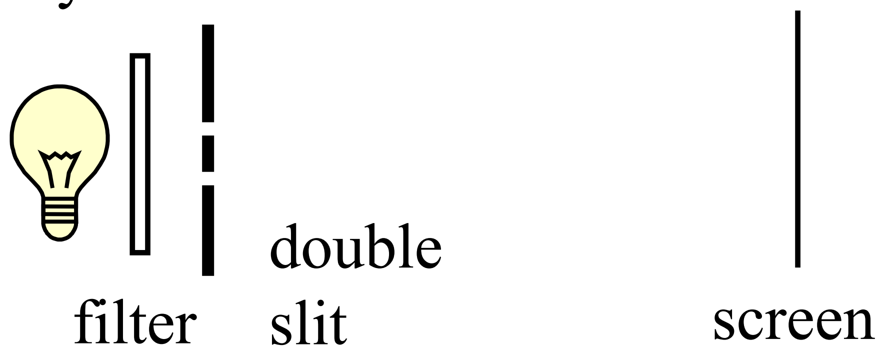
If we use two in-phase sources to generate waves on the surface of a liquid it is easy to observe interference effects. At certain points the waves are in phase and add constructively, and at other points they are out of phase, interfere destructively, giving zero amplitude.



... this is not an everyday observation with light sources ... the wavelength of light is small (400-800 nm) ... ordinary light sources are enormously larger than this ... and are not monochromatic.

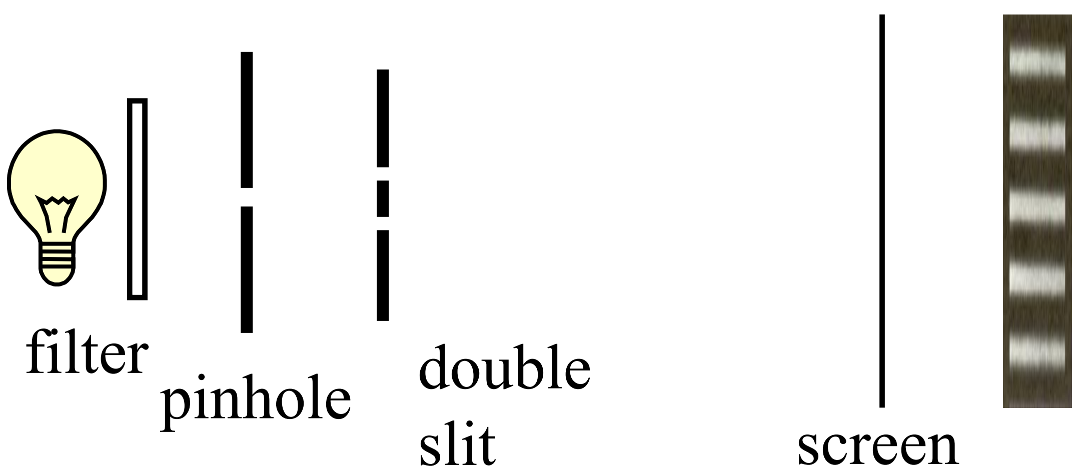
Interference ...2

Try this ...?



... no interference fringes observed ... light from different portions of source is incoherent ... no fixed phase relationship

Thomas Young performed a classic interference experiment in 1801:

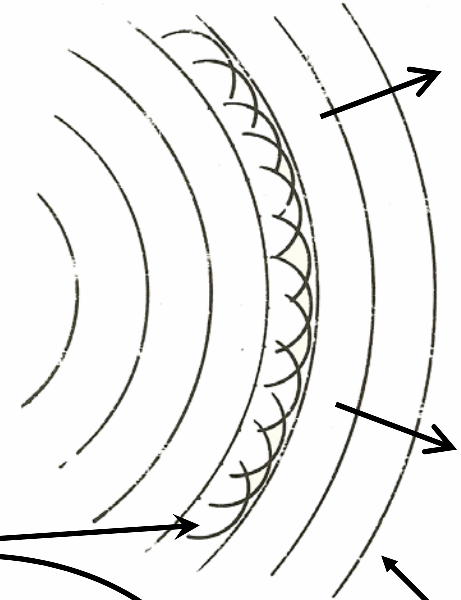


... interference fringes observed on screen ...

Interference ...3

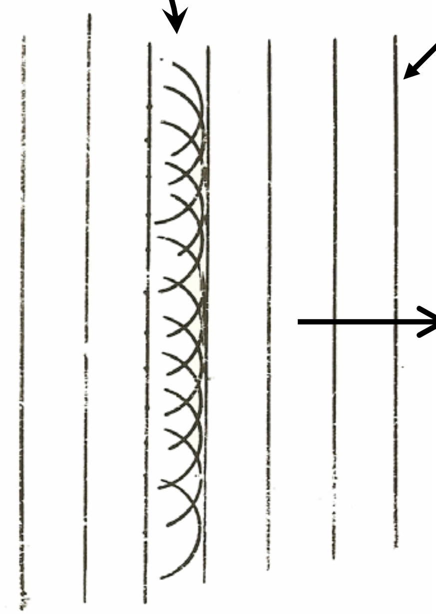
... but why do we see fringes at all ... doesn’t light travel in straight lines?

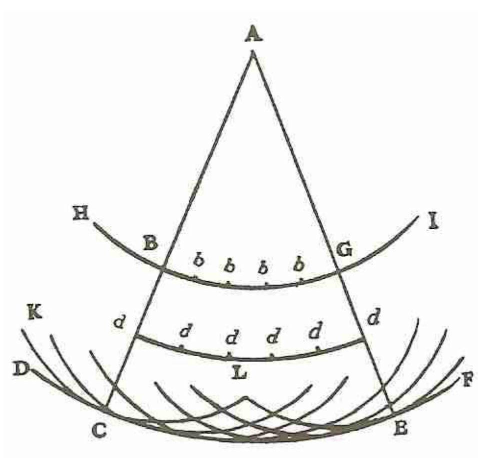
Huyghens showed that one could regard each point on a wavefront as being a source of “secondary wavelets”



... the envelope of these secondary wavelets form a subsequent wavefront ...

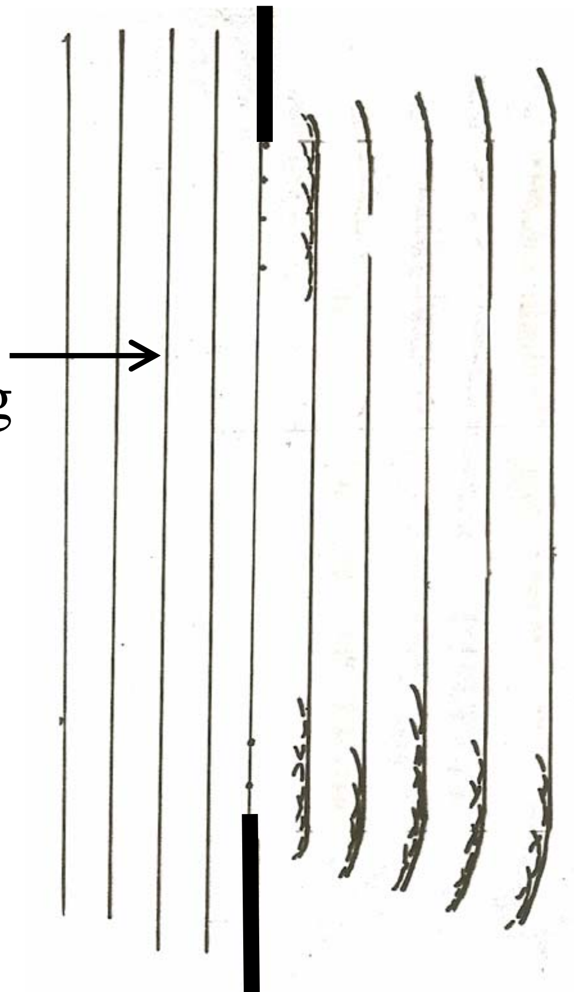
wavefronts



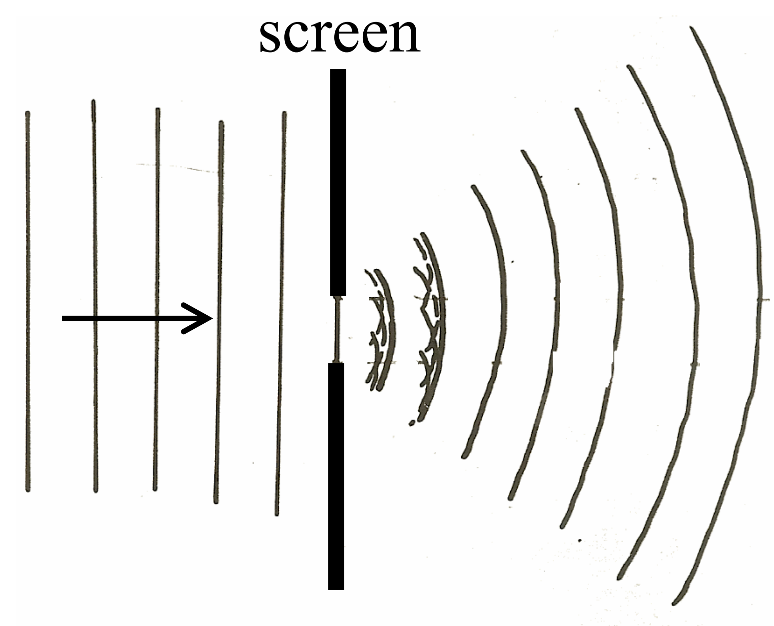


Huyghens' Principle

plane
wave
travelling
in this
direction



Aperture large compared with
a wavelength
... some spreading of wave

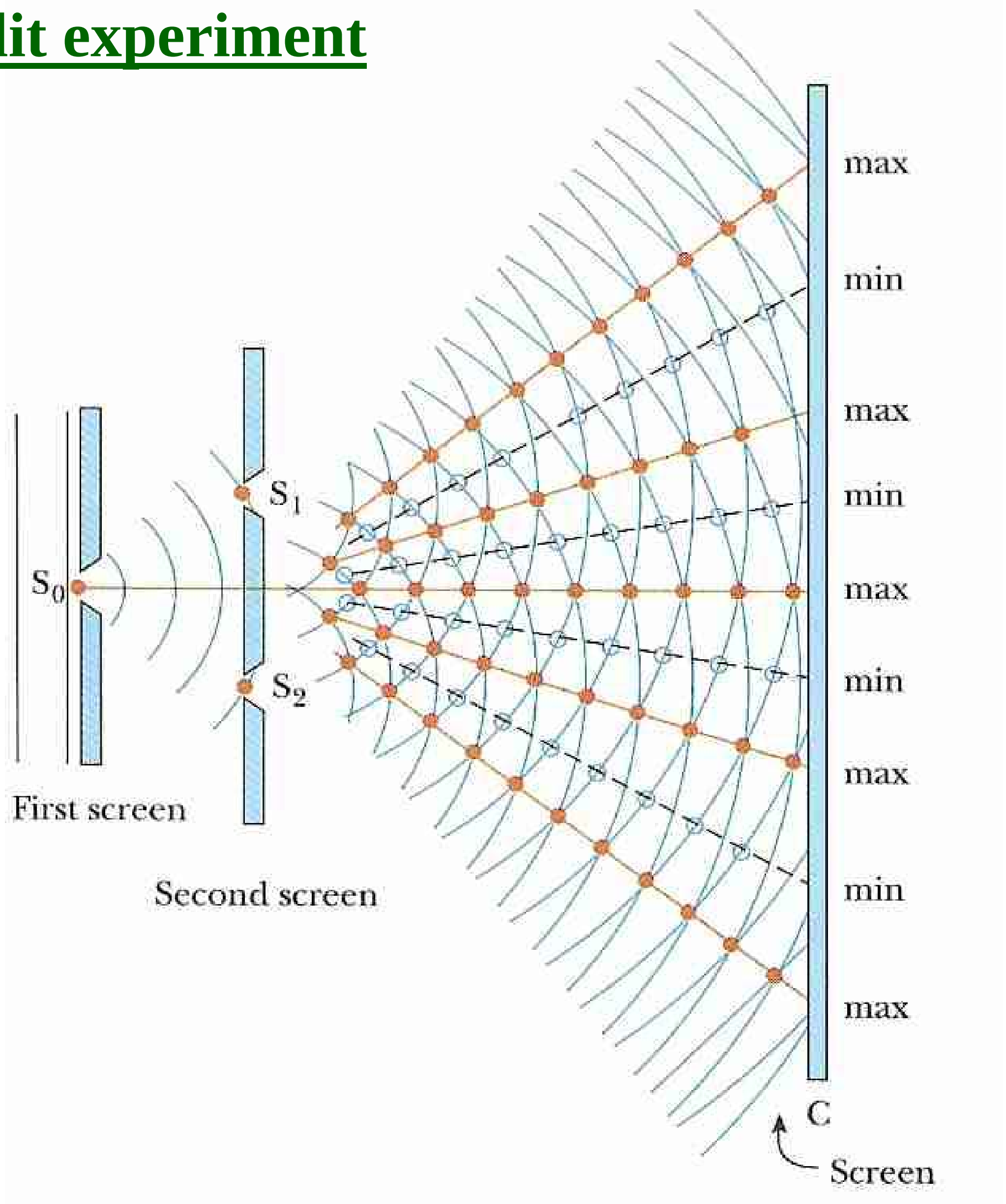


Aperture similar size
to a wavelength ...
large angular spread
from small aperture

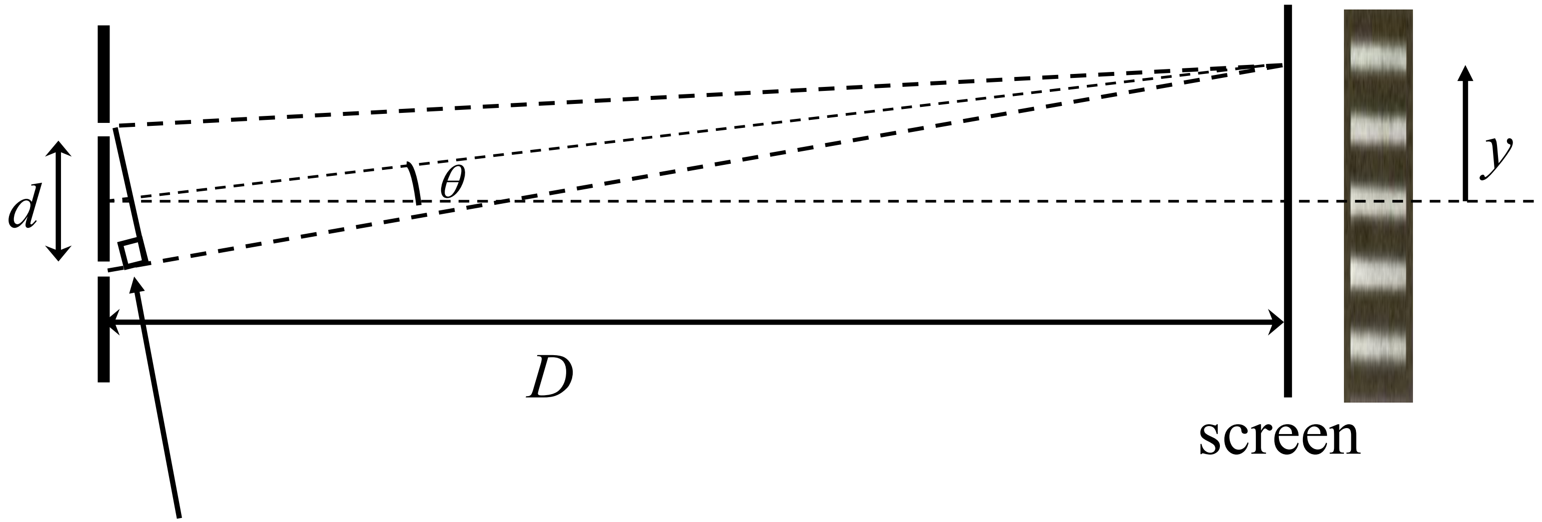
Young's double slit experiment

back to Young's experiment ...

... the two slits in the second screen act as coherent sources



Young's double slit experiment ...2



Path difference = $d\sin\theta$

Maxima on screen when $d\sin\theta = m\lambda$

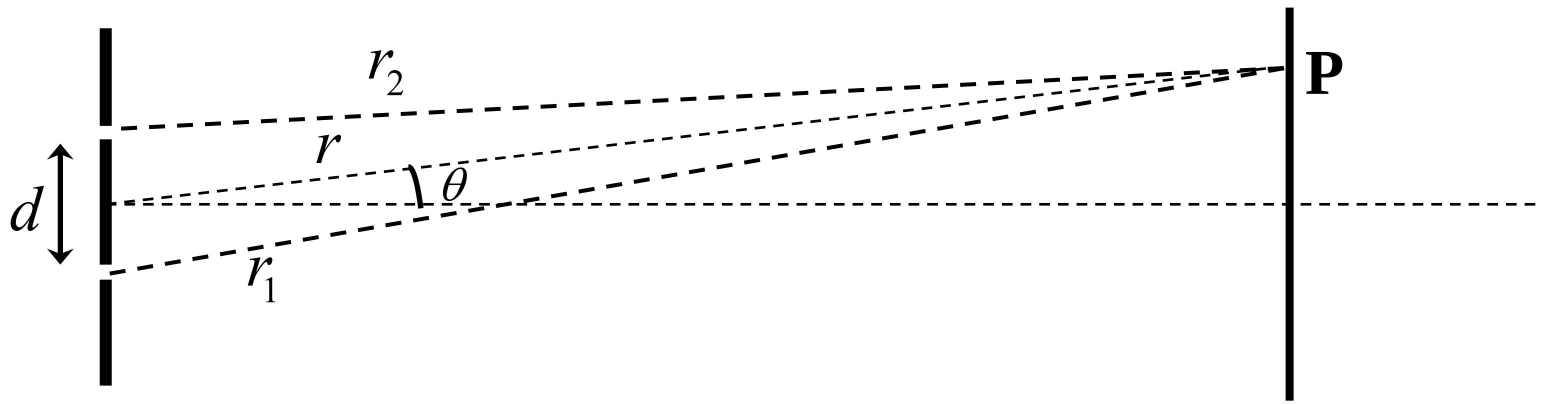
$m = 0, \pm 1, \pm 2, \ldots$

Minima on screen when $d\sin\theta = \left(m + \tfrac{1}{2}\right)\lambda$

$y = D\sin\theta$ for small θ

$\therefore y = D\dfrac{m\lambda}{d}$ for maxima

Young's double slit experiment ...3 ... a more detailed treatment



Suppose a plane wave illuminates the slits ... the waves from the two slits are in phase at the slits.
Since the distances r_1 and r_2 differ, the waves will have a phase difference at **P**.

Electric fields of the two waves:

$$E_1 = E_0 \cos(kr_1 - \omega t)$$
$$E_2 = E_0 \cos(kr_2 - \omega t)$$

... both waves have the same amplitude at **P** ... not quite right since r_1 and r_2 are different distances ... but ok.

Young's double slit experiment ...4

Total field: $E = E_1 + E_2 = E_0 \cos(kr_1 - \omega t) + E_0 \cos(kr_2 - \omega t)$

$$= 2E_0 \cos\left\{\frac{k(r_1 + r_2)}{2} - \omega t\right\} \cos\left\{\frac{k(r_1 - r_2)}{2}\right\}$$

$$= 2E_0 \cos\{kr - \omega t\} \cos\left\{\frac{k}{2} d \sin\theta\right\}$$

$$= 2E_0 \cos\{kr - \omega t\} \cos\left\{\frac{\pi d}{\lambda} \sin\theta\right\}$$

We now take $I = \overline{E^2}$ (time average of E^2)

Thus for one slit only: $I_1 = E_0^2 \overline{\cos^2(kr_1 - \omega t)} = \dfrac{E_0^2}{2}$

similarly: $I_2 = \dfrac{E_0^2}{2}$

Young's double slit experiment ...5

Write $I_0 = I_1 = I_2 = \frac{1}{2} E_0^2$

For both slits: $I = 4E_0^2 \overline{\cos^2\{kr - \omega t\}} \cos^2\left\{\frac{\pi d}{\lambda}\sin\theta\right\}$

$$\therefore I = 4I_0 \cos^2\left\{\frac{\pi d}{\lambda}\sin\theta\right\} = 4I_0 \cos^2\frac{\phi}{2}$$

So I will have maxima when $\cos^2\left\{\frac{\pi d}{\lambda}\sin\theta\right\} = 1$

or when $\frac{\pi d}{\lambda}\sin\theta = m\pi \qquad m = 0, \pm 1, \pm 2, \ldots$

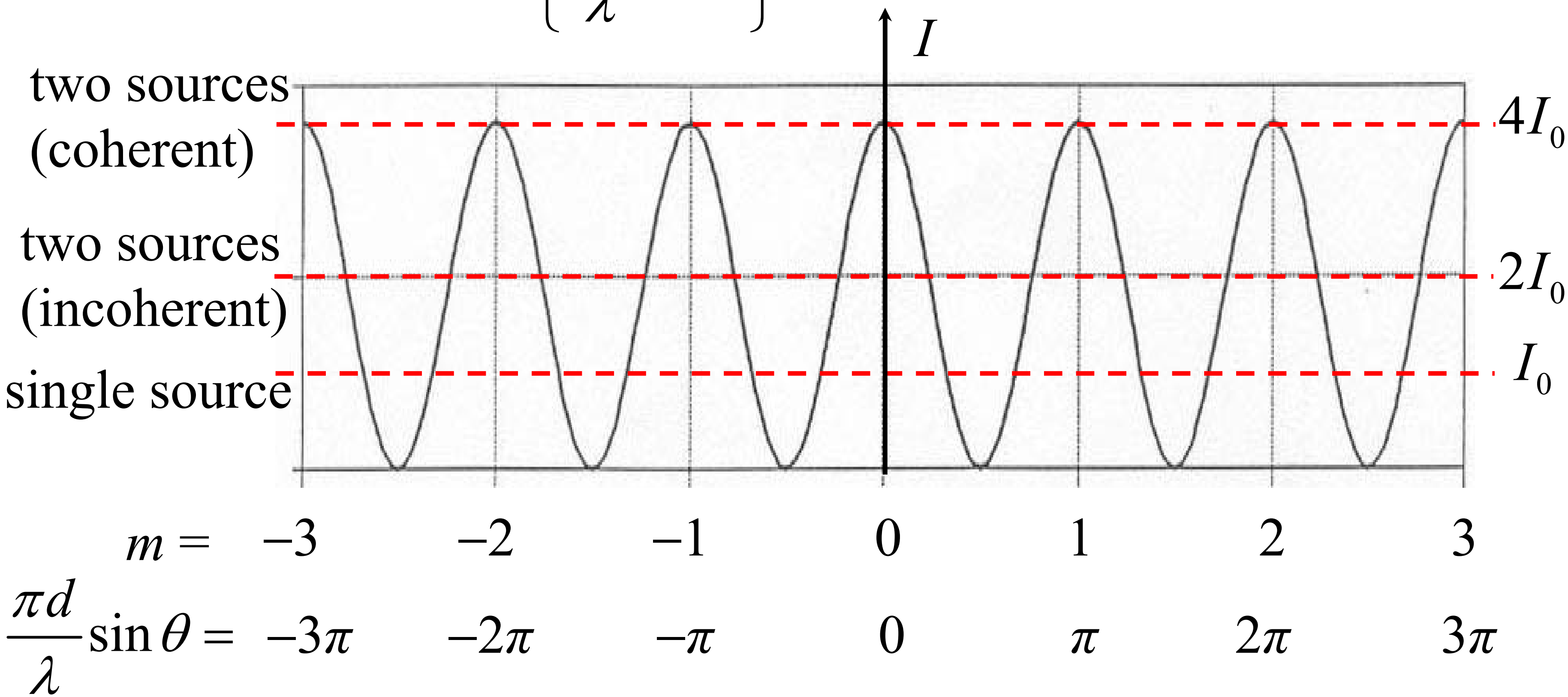
or $d\sin\theta = m\lambda$

This approach gives us additional information ...

Young's double slit experiment ...6

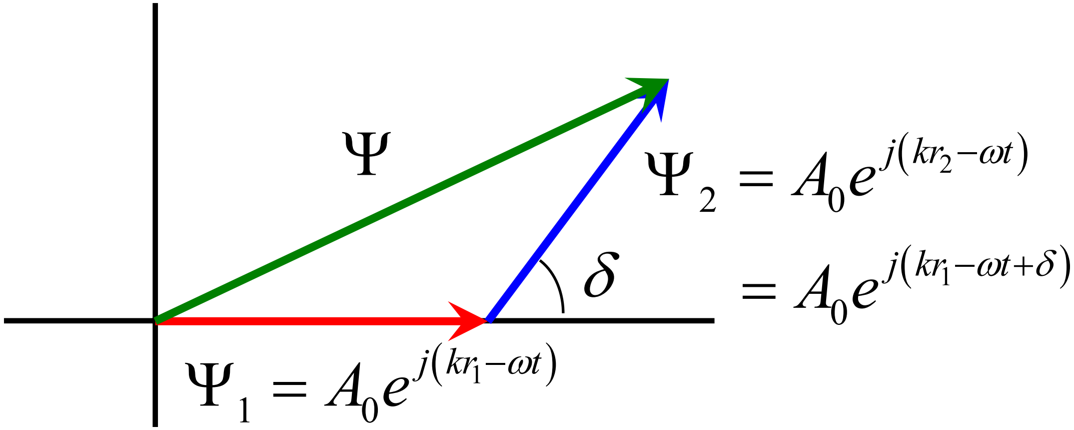
Double slit interference pattern

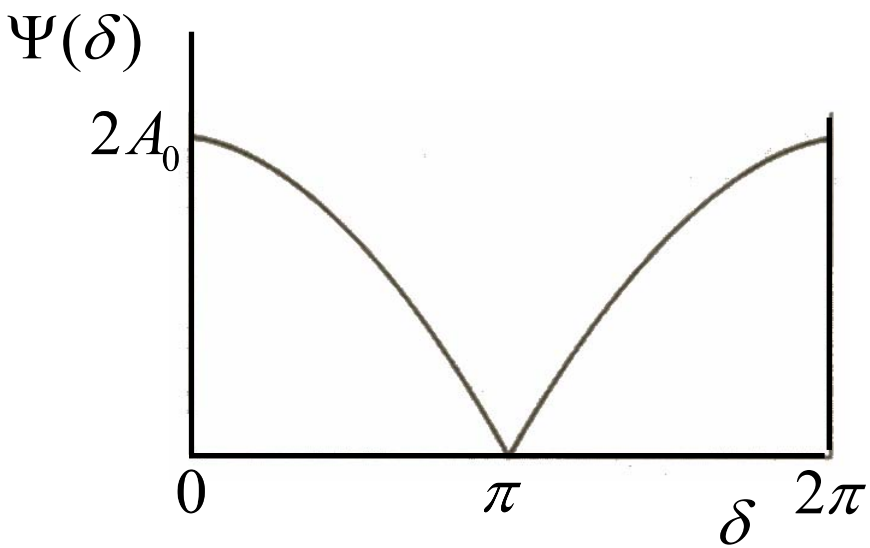
$$I(\theta) = 4I_0 \cos^2\left\{\frac{\pi d}{\lambda}\sin\theta\right\}$$

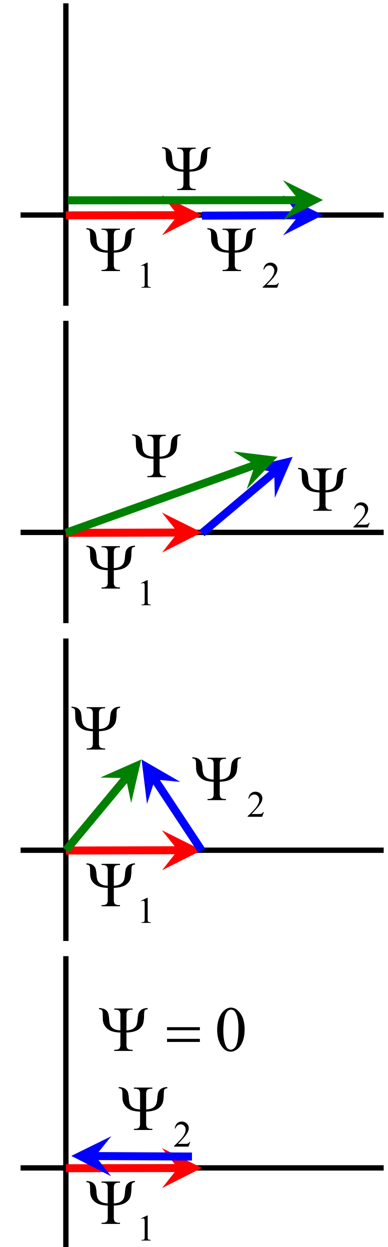


Young's double slit experiment ...7

... can also use phasors ...

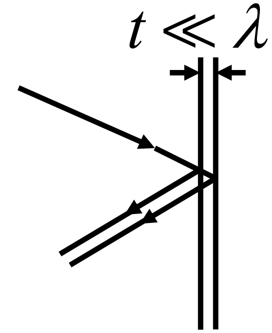




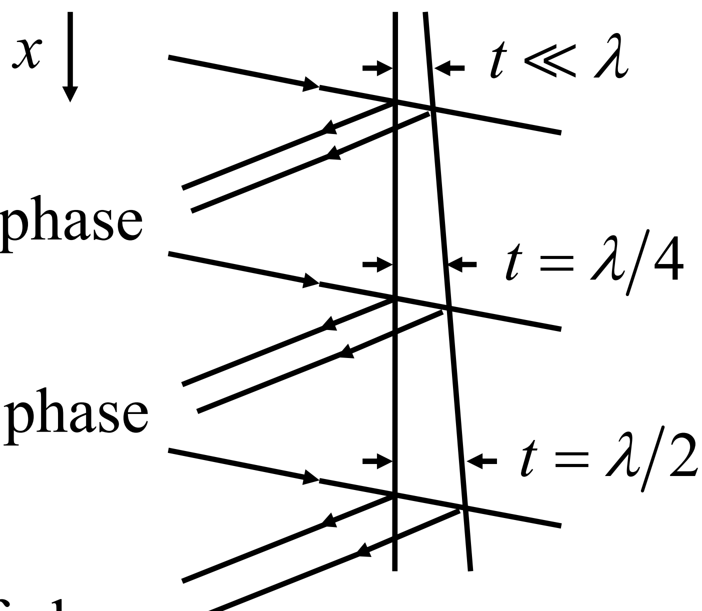


Interference patterns from thin films

"Black" (i.e. destructive intereference) from very thin film indicates that there is a phase change of π at one of the reflections.



For wedges having small angle α, bright fringes correspond to an increase of $\lambda/2$ in thickness t :



dark
π out of phase

bright
2π out of phase

dark
3π out of phase

For nearly normal incidence, path difference = $2t$

Minima occur when the path difference = $n\lambda$

For minima, $2t = n\lambda$

or $2x_n\alpha = n\lambda$

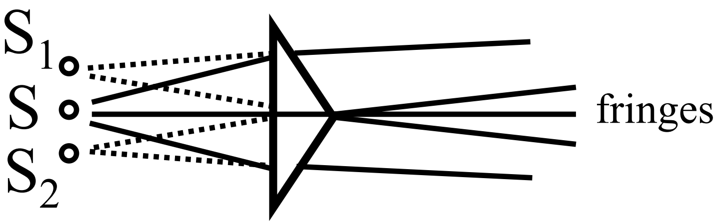
Distance between fringes:

$$x_{n+1} - x_n = \lambda/2\alpha$$

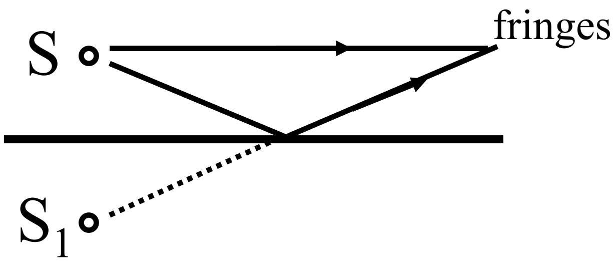
To obtain fringes on a visible scale a very small angle is necessary ... such fringes may be seen on viewing the light reflected from a **soap film** held vertically as it “drains” ...

For other interference effects, look up for yourself:

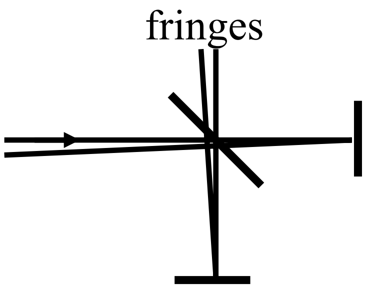
Fresnel bi-prism



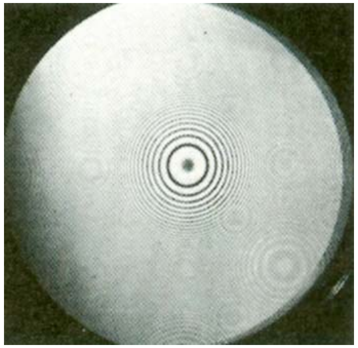
Lloyd’s mirror

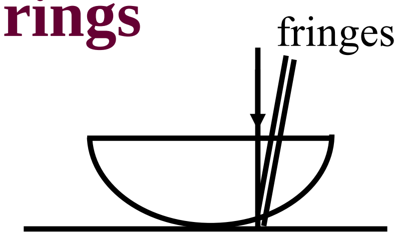


Michelson interferometer



Newton’s rings



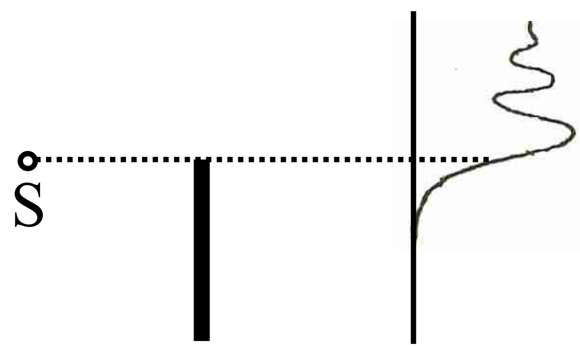


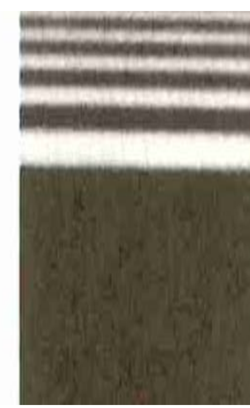
Fresnel diffraction

Around 1818 Augustin Fresnel applied Huyghen’s priciple to the problem of diffraction of light by apertures and obstacles ... took into account the realtive phases of the secondary wavelets consequent upon their having to travel diferent distances to the point of observation ... analytical treatment is complicated
... look at the results of a few special cases ...

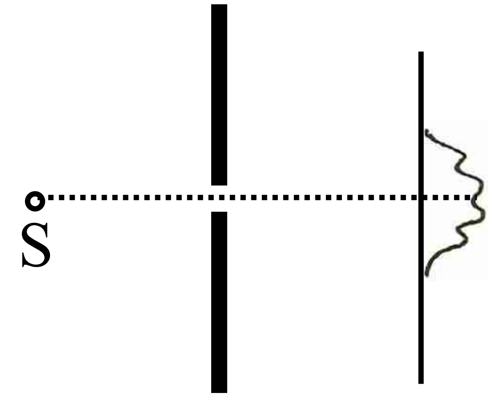
Shadow of a straight edge

(cast by a point source S)
... no sharp edge to the shadow
... illumination diminishes smoothly into the shadow
... with fringes outside the shadow.

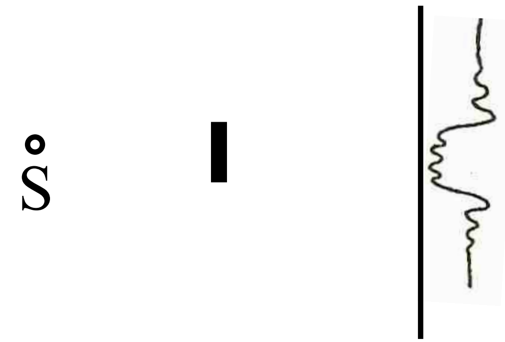




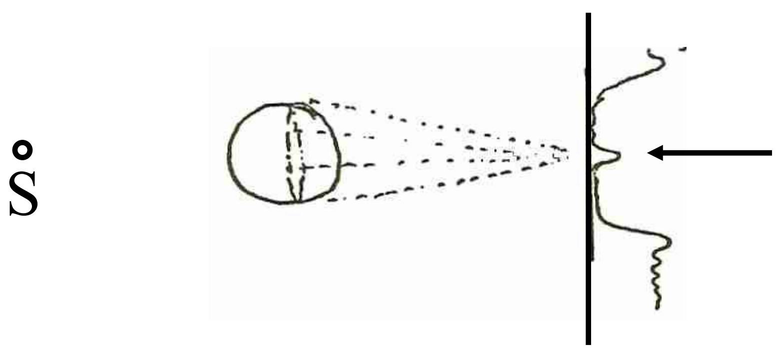
Shadow of a narrow slit: centre of pattern at P may either be a relative maximum or minimum, depending on ratio of slit width to wavelength.



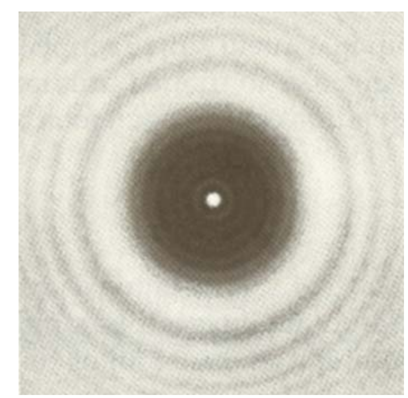
Shadow of a narrow parallel-sided obstacle: centre always a relative maximum ... but very faint unless obstacle is very narrow.



Shadow of a circular obstacle: wavelets from all round circumference all reinforce at centre of shadow.

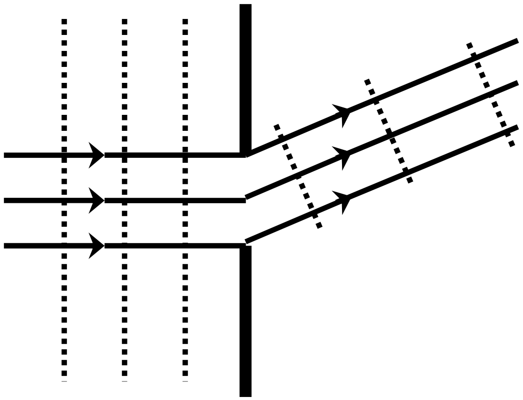


"Poisson" bright spot (found by Arago)

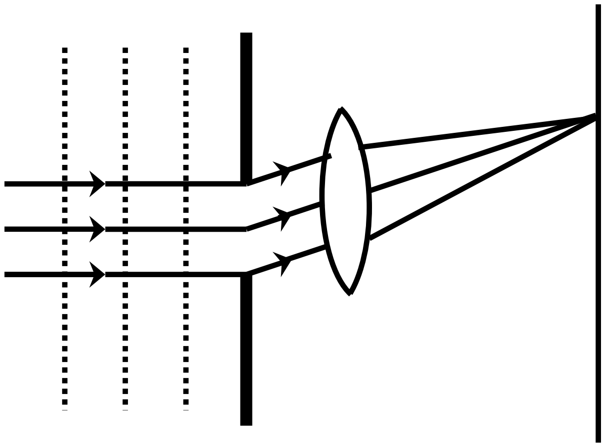


Fraunhofer diffraction

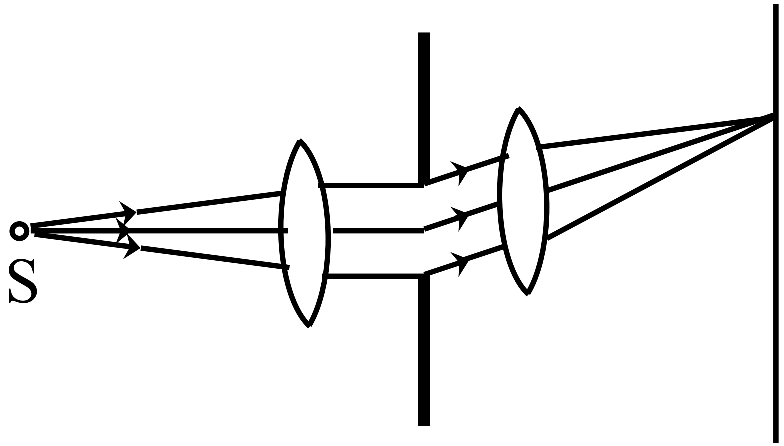
... special (limiting) cases of Fresnel diffraction ... when both the distances to source and screen both tend to **infinity** ... treatment is easier since the paths of all the wavelets to any point of interest are parallel.

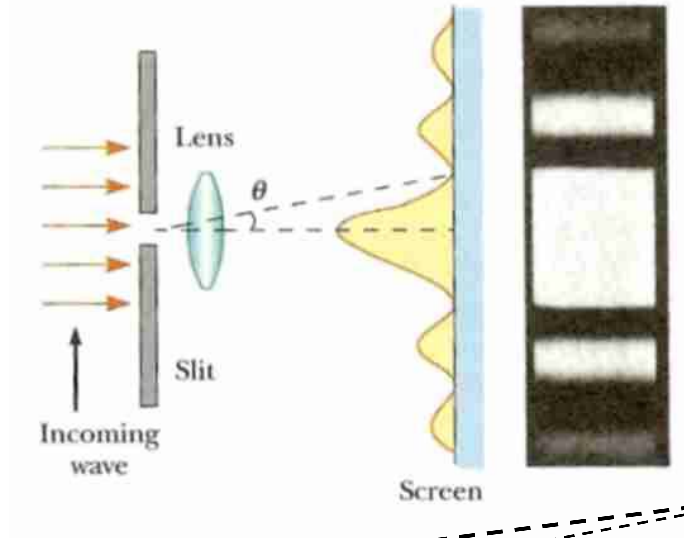


It is practical sometimes to use a lens ... what would be seen at infinity is then found in its focal plane.



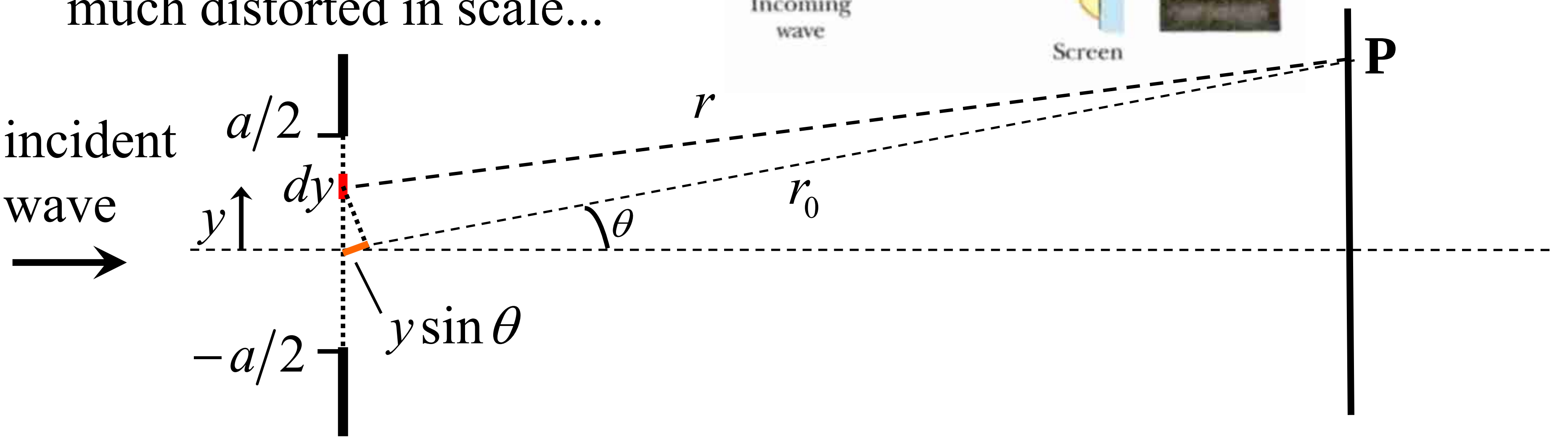
or





Fraunhofer diffraction by a single slit

Let slit width be *a*
... picture below very much distorted in scale...



To find the intensity of the wave at **P** we "add" the contributions at **P** of waves originating from all points of the aperture.

$$\Psi(\mathbf{P}) = \int_{-a/2}^{a/2} Ae^{j(kr-\omega t)}dy = A\int_{-a/2}^{a/2} e^{j(kr_0-\omega t)-jky\sin\theta}dy \;= Ae^{j(kr_0-\omega t)}\int_{-a/2}^{a/2} e^{-jky\sin\theta}dy$$

Fraunhofer diffraction by a single slit ...2

$$\Psi(\mathbf{P}) = A_0 e^{j(kr_0-\omega t)} \int_{-a/2}^{a/2} e^{-jky\sin\theta}\,dy$$

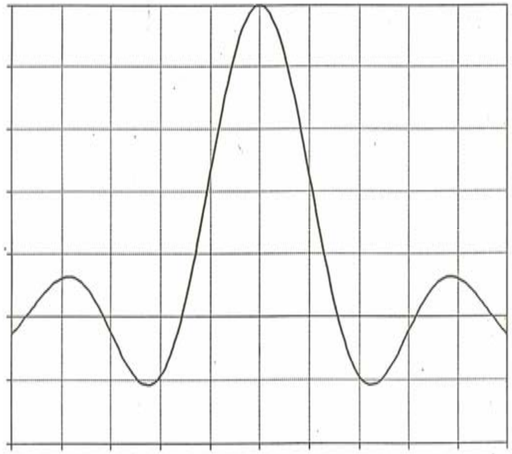
$$= A_0 e^{j(kr_0-\omega t)} \left[\frac{1}{-jk\sin\theta} e^{-jky\sin\theta} \right]_{-a/2}^{a/2}$$

$$= A_0 e^{j(kr_0-\omega t)} \frac{e^{jk\frac{a}{2}\sin\theta} - e^{-jk\frac{a}{2}\sin\theta}}{jk\sin\theta}$$

$$\left[\text{use} \quad \sin\phi = \frac{e^{j\phi} - e^{-j\phi}}{2j} \right]$$

$$\therefore \Psi(\mathbf{P}) = A_0 e^{j(kr_0-\omega t)} a \frac{\sin\left(\frac{1}{2}ka\sin\theta\right)}{\frac{1}{2}ka\sin\theta}$$

$$\left(\frac{\sin\alpha}{\alpha}\right)$$



$$\alpha$$

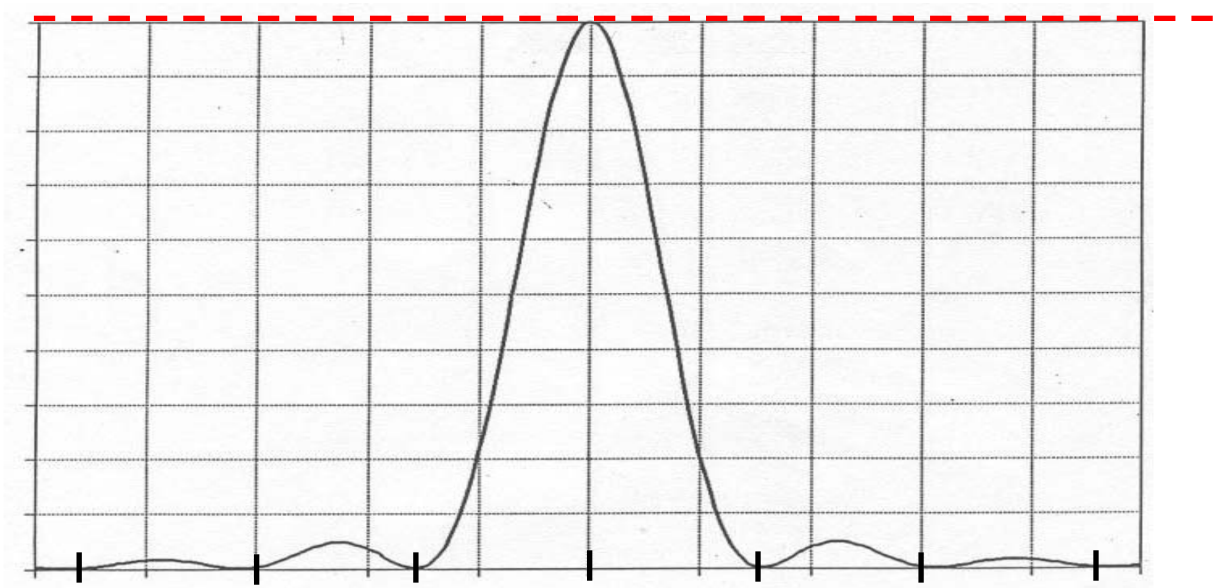
Then $\quad I = \overline{\Psi^2(\mathbf{P})}$

$$\therefore I(\theta) = I_0 \left(\frac{\sin\alpha}{\alpha}\right)^2 \qquad \text{where} \quad \alpha = \tfrac{1}{2}ka\sin\theta$$

Fraunhofer diffraction by a single slit ...3

$$\therefore I(\theta) = I_0 \left(\frac{\sin\alpha}{\alpha} \right)^2 \qquad \text{where} \quad \alpha = \tfrac{1}{2} k a \sin\theta$$

$I(\theta)$



I_0

$$\alpha = \; -3\pi \quad -2\pi \quad -\pi \quad 0 \quad \pi \quad 2\pi \quad 3\pi$$

$$\sin\theta = -3\frac{\lambda}{a} \quad -2\frac{\lambda}{a} \quad -\frac{\lambda}{a} \qquad \frac{\lambda}{a} \quad 2\frac{\lambda}{a} \quad 3\frac{\lambda}{a}$$

Minima at $\alpha = m\pi$ where $m = \pm 1, \pm 2, \ldots$

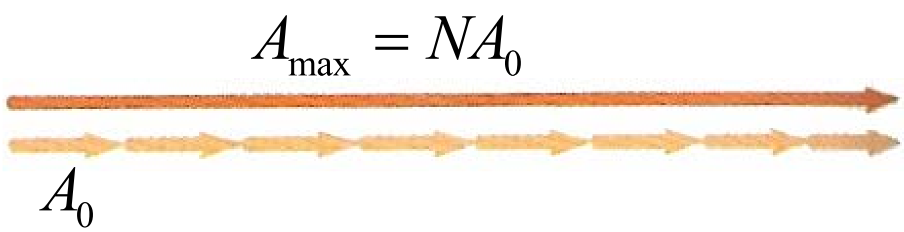
or $\tfrac{1}{2} k a \sin\theta = m\pi$ or $\sin\theta = \dfrac{m}{a}\dfrac{2\pi}{k} = m\dfrac{\lambda}{a}$

Fraunhofer diffraction by a single slit ...4

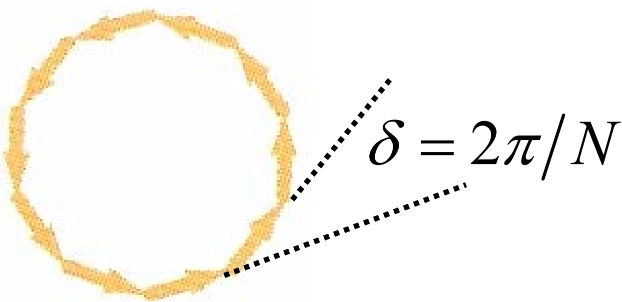
Using **phasors** ... consider a single slit as N sources, each of amplitude A_0 ... in the example below, $N = 10$.

The path difference between any two adjacent sources is $d\sin\theta$ where $d = a/N$ and the phase difference is $\delta = dk\sin\theta$.

At the central maximum point at $\theta = 0$, the waves from the N sources add in phase, giving the resultant amplitude $A_{max} = NA_0$.

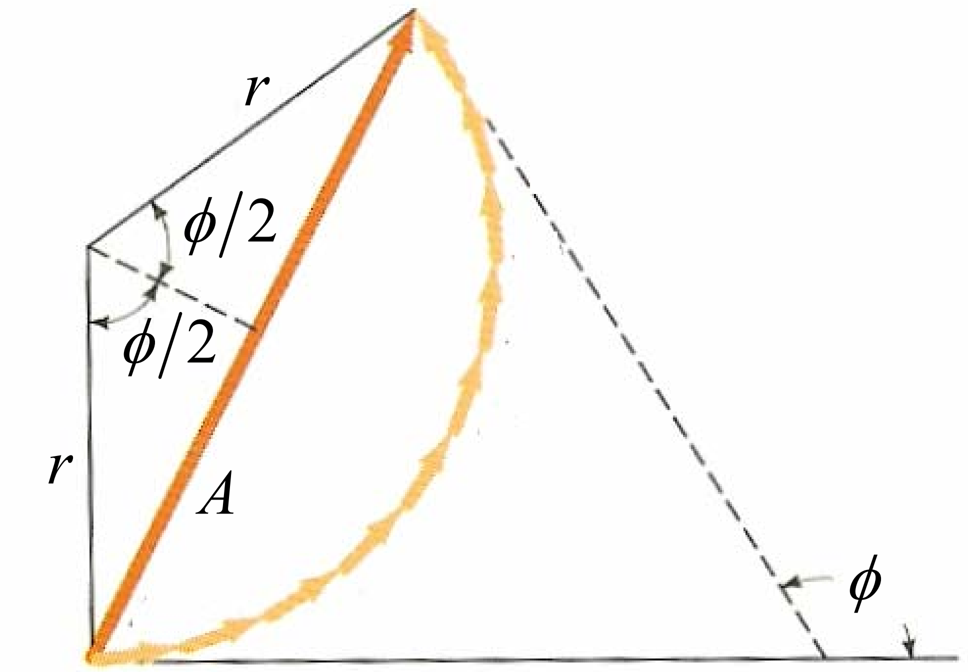


At the first minimum, the N phasors form a closed polygon ... the phase difference between adjacent sources is then $\delta = 2\pi/N$
When N is large, the waves from the first and last sources are approximately in phase



Fraunhofer diffraction by a single slit ...5

At a general point
... where the waves from two adjacent sources differ in phase by δ ...
The phase difference between the first and last wave is ϕ .



As N increases, the phasor diagram approximates the arc of a circle ... the resultant amplitude A is the length of the chord of this arc.

From the figure: $\sin(\phi/2) = A/2r$ or $A = 2r\sin\frac{1}{2}\phi$

The length of the arc is $A_{\max}\left(= NA_0\right)$

Therefore $\phi = \dfrac{A_{\max}}{r}$ or $r = \dfrac{A_{\max}}{\phi}$

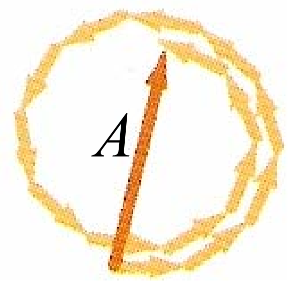
Fraunhofer diffraction by a single slit ...6

Combining ... $A = 2r\sin\tfrac{1}{2}\phi = 2\dfrac{A_{\max}}{\phi}\sin\tfrac{1}{2}\phi = A_{\max}\dfrac{\sin\tfrac{1}{2}\phi}{\tfrac{1}{2}\phi}$

Then $\dfrac{I}{I_0} = \dfrac{A^2}{A_{\max}^2} = \left(\dfrac{\sin\tfrac{1}{2}\phi}{\tfrac{1}{2}\phi}\right)^2$ or $I = I_0\left(\dfrac{\sin\tfrac{1}{2}\phi}{\tfrac{1}{2}\phi}\right)^2$

...where ϕ is the phase difference between the first and last waves $= (2\pi/\lambda)a\sin\theta = ka\sin\theta$

The second maximum occurs when the N phasors complete $1\tfrac{1}{2}$ circles:



... and the resultant amplitude is the diameter of the circle ...

The diameter $A = \dfrac{C}{\pi} = \dfrac{\tfrac{2}{3}A_{\max}}{\pi} = \dfrac{2A_{\max}}{3\pi}$ then $A^2 = \dfrac{4A_{\max}^2}{9\pi^2}$

and $I = I_0\dfrac{4}{9\pi^2} = I_0\dfrac{1}{22.2}$

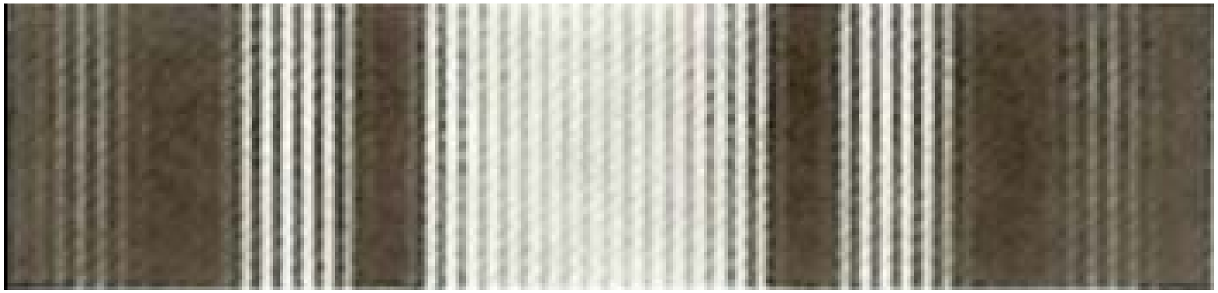
Interference and diffraction from a double slit

When there are two or more slits, the intensity pattern on a screen far away is a combination of the single slit diffraction pattern and the multiple slit interference pattern.

$$I(\theta) = 4I_0\left(\frac{\sin\alpha}{\alpha}\right)^2 \cos^2\beta$$

$$\alpha = \frac{\pi a}{\lambda}\sin\theta$$

$$\beta = \frac{\pi d}{\lambda}\sin\theta$$

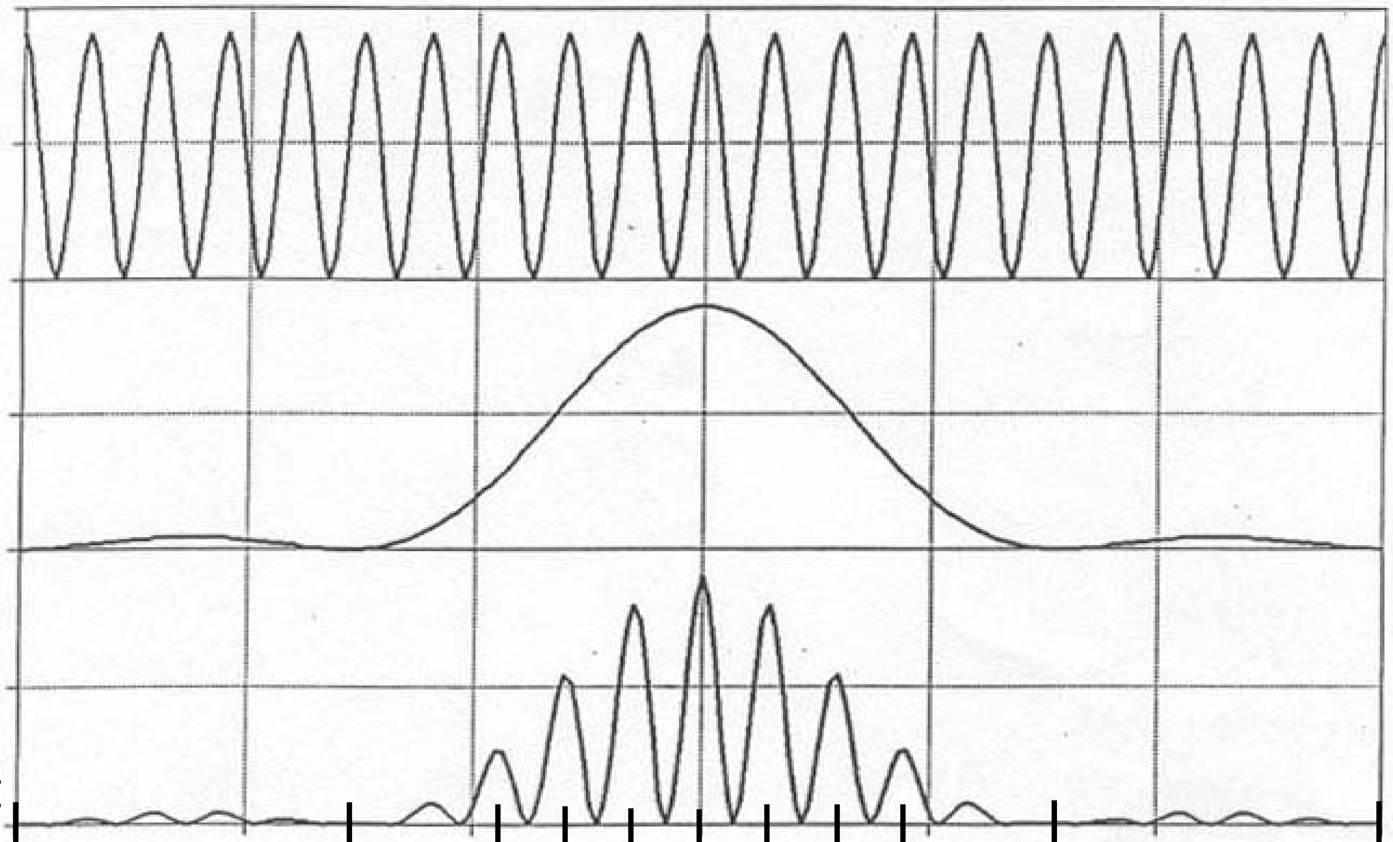


Interference and diffraction from a double slit ...2

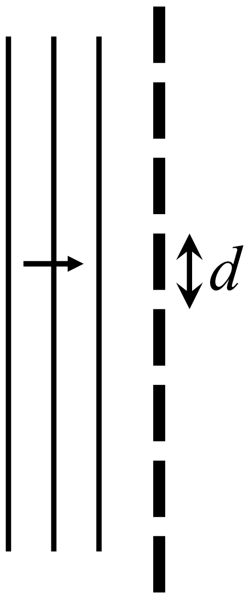
intensity pattern for two slits if $a \rightarrow 0$:

intensity pattern for one slit of width a :

intensity pattern for two slits each of width a :



Interference patterns from multiple slits (diffraction gratings)



... many slits, with inter-slit spacing d, illuminated by a plane wave ...

As with the double slit, each slit acts as a source of waves.

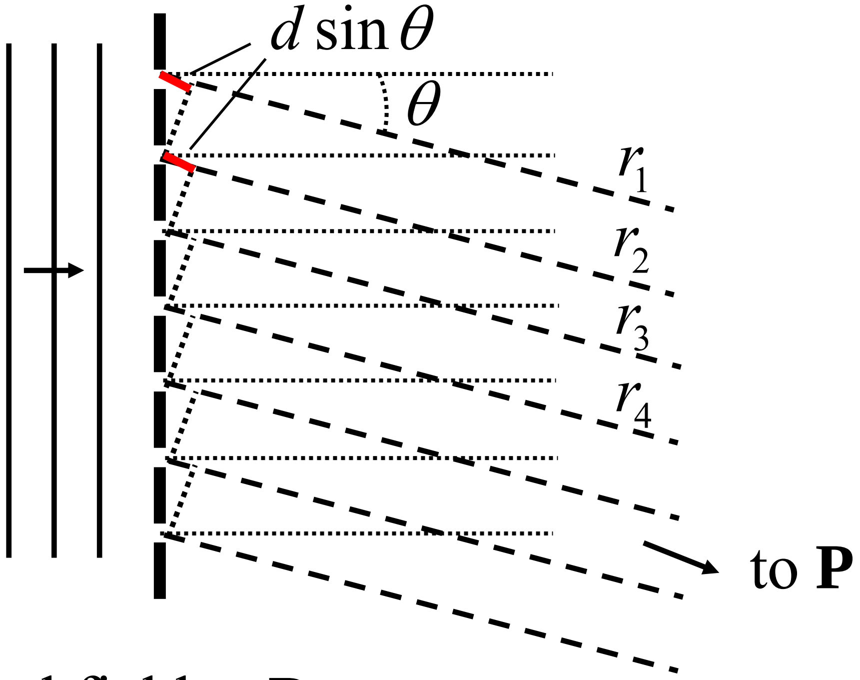
Consider each pair of slits as a Young's double slit. Slits are coherent, in phase sources for light at an angle θ_n to the n^{th} bright fringe.

For constructive interference $d \sin \theta_n = n\lambda$ $\quad n = 0, \pm 1, \pm 2, \ldots$

If m is the number of slits, then amplitude at θ_n is $A = ma$ where a is the amplitude form a single slit

Then $I \sim A^2 \sim m^2 a^2$ (...many slits give intense fringes)

Interference patterns from multiple slits ...2



$$r_2 = r_1 + d\sin\theta$$

$$r_3 = r_2 + 2d\sin\theta$$

$$r_4 = r_1 + 3d\sin\theta$$

etc.

Total field at **P**:

$$E = E_1 + E_2 + E_3 + \ldots$$
$$= E_0\cos(kr_1 - \omega t) + E_0\cos(kr_2 - \omega t) + E_0\cos(kr_3 - \omega t) + \ldots$$

Use complex numbers:

$$E = E_0 e^{j(kr_1 - \omega t)} + E_0 e^{j(kr_2 - \omega t)} + E_0 e^{j(kr_3 - \omega t)} + \ldots$$

Interference patterns from multiple slits ...3

$$E = E_0 e^{j(kr_1-\omega t)} + E_0 e^{j(kr_2-\omega t)} + E_0 e^{j(kr_3-\omega t)} + \ldots$$

$$= E_0 e^{j(kr_1-\omega t)} + E_0 e^{j(kr_1-\omega t+kd\sin\theta)} + E_0 e^{j(kr_1-\omega t+2kd\sin\theta)} + \ldots$$

$$= E_0 e^{j(kr_1-\omega t)} \underbrace{\left\{1 + e^{jkd\sin\theta} + e^{2jkd\sin\theta} + \ldots + e^{(N-1)jkd\sin\theta}\right\}}_{\text{geometric progression}} \quad \ldots \text{ for } N \text{ slits}$$

$$\therefore E = E_0 e^{j(kr_1-\omega t)} \frac{1-\left(e^{jkd\sin\theta}\right)^N}{1-e^{jkd\sin\theta}}$$

$$= E_0 e^{j(kr_1-\omega t)} \frac{e^{\frac{N}{2}jkd\sin\theta}\left(e^{-\frac{N}{2}jkd\sin\theta} - e^{\frac{N}{2}jkd\sin\theta}\right)}{e^{\frac{1}{2}jkd\sin\theta}\left(e^{-\frac{1}{2}jkd\sin\theta} - e^{\frac{1}{2}jkd\sin\theta}\right)}$$

and using $\sin\phi = \dfrac{e^{j\phi} - e^{-j\phi}}{2j}$ ...

Interference patterns from multiple slits ...4

$$\therefore E = E_0 e^{j(kr_1 - \omega t)} E_0 e^{\frac{N-1}{2} jkd\sin\theta} \frac{\sin\left(\frac{1}{2} Nkd\sin\theta\right)}{\sin\left(\frac{1}{2} kd\sin\theta\right)}$$

$$= E_0 e^{j\left\{k\left(r_1 + \frac{N-1}{2} jkd\sin\theta\right) - \omega t\right\}} \frac{\sin N\beta}{\sin\beta}$$

where $\beta = \frac{1}{2} kd\sin\theta = \dfrac{\pi d}{\lambda}\sin\theta$

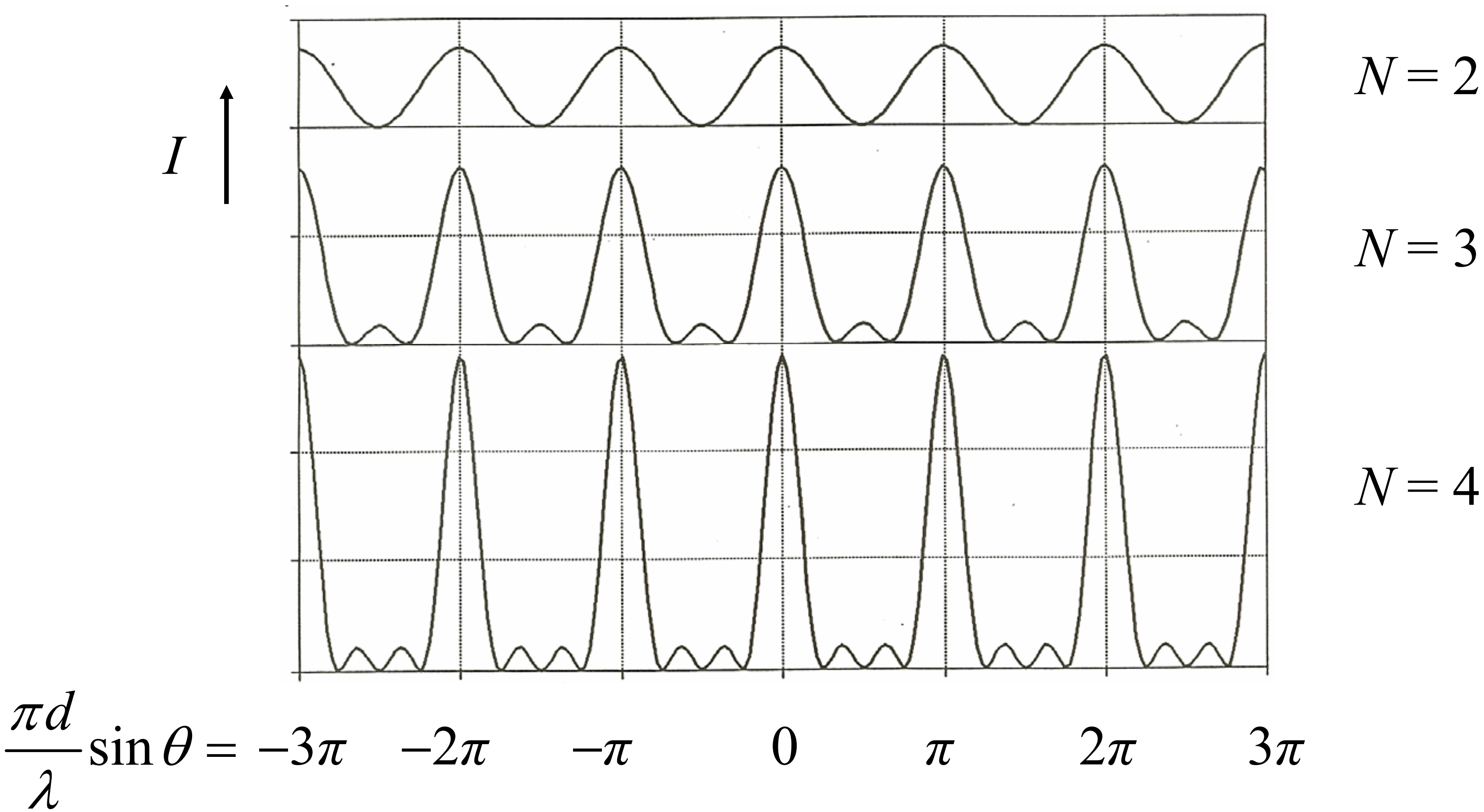
Write

$$E = E_0 e^{j\{kr - \omega t\}} \frac{\sin N\beta}{\sin\beta}$$

Then, as before, $I(\theta) = I_0 \left(\dfrac{\sin N\beta}{\sin\beta}\right)^2$

Interference patterns from multiple slits ...5

$$I(\theta) = I_0 \left(\frac{\sin N\beta}{\sin \beta} \right)^2 \qquad \beta = \frac{\pi d}{\lambda} \sin\theta$$

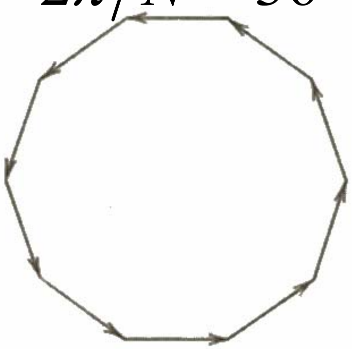


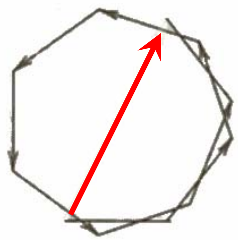
Interference patterns from multiple slits ...6

Phasors for a 10 slit grating ($N = 10$) ...

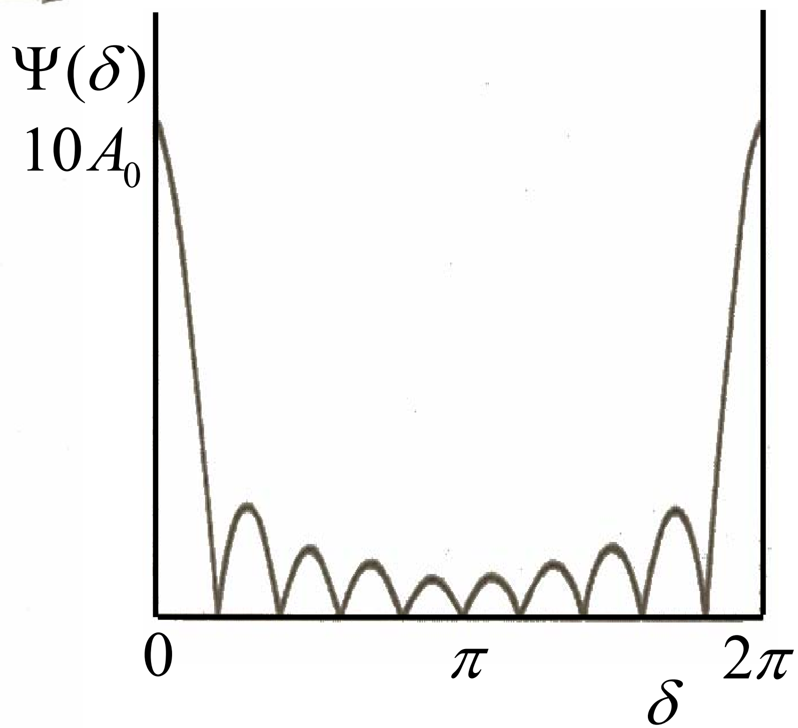
$\delta = 0, 2\pi$

$\delta = 2\pi/N = 36°$

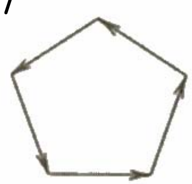




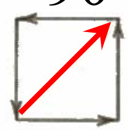
$\delta = 3\pi/N = 54°$



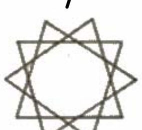
$\delta = 4\pi/N = 72°$



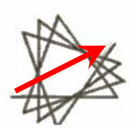
$\delta = 5\pi/N = 90°$



$\delta = 6\pi/N = 108°$



$\delta = 7\pi/N = 126°$



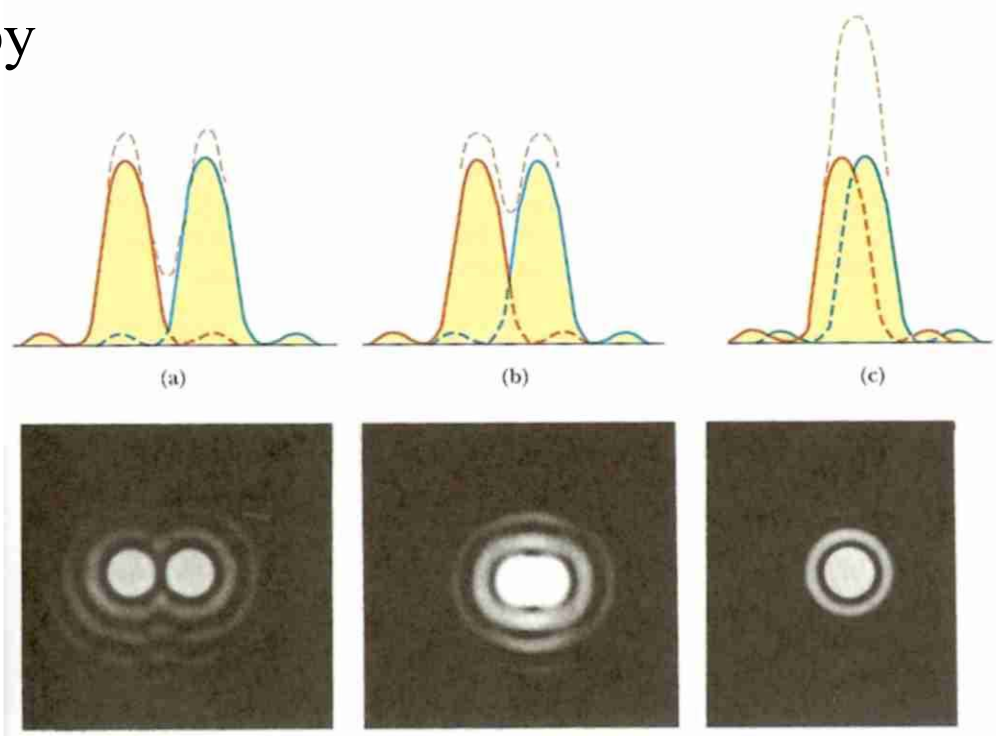
Circular apertures

Different shaped apertures will give different diffraction patterns.
A commonly encountered aperture in optics is the circular aperture.

... diffraction pattern described by a Bessel function, with the first minimum at

$$\sin\theta = 1.22\frac{\lambda}{d}$$

... where d is the diameter



To resolve two objects, $\theta > \theta_R$

where $\theta_R = \sin^{-1} 1.22\frac{\lambda}{d}$

$\theta > \theta_R$ $\theta = \theta_R$ $\theta < \theta_R$

Rayleigh criterion

Interference and diffraction with water waves

See *French* 280 - 294