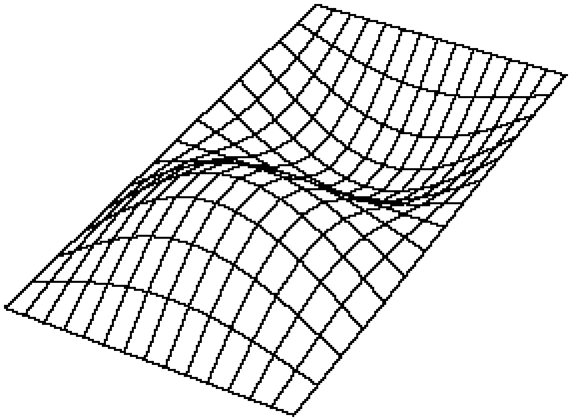
University of Cape Town
Department of Physics

PHY2014F

Vibrations and Waves



Part 2

Coupled oscillators
Normal modes of continuous systems
The wave equation
Fourier analysis

… covering (more or less)
French Chapters 2, 5 & 6

Andy Buffler
Department of Physics
University of Cape Town
andy.buffler@uct.ac.za

Problem-solving and homework

Each week you will be given a take-home problem set to complete and hand in for marks ...

In addition to this, you need to work through the following problems in *French*, in you own time, at home. You will not be asked to hand these in for marks. Get help from you friends, the course tutor, lecturer, ... Do not take shortcuts. Mastering these problems is a fundamental aspect of this course.

The problems associated with Part 2 are:

2-2, 2-3, 2-4, 2-5, 2-6, 5-2, 5-8, 5-9, 6-1, 6-2, 6-6, 6-7, 6-10, 6-11, 6-14

You might find these tougher: 5-4, 5-5, 5-6, 5-7

French
page 20

The superposition of periodic motions

Two superimposed vibrations of equal frequency

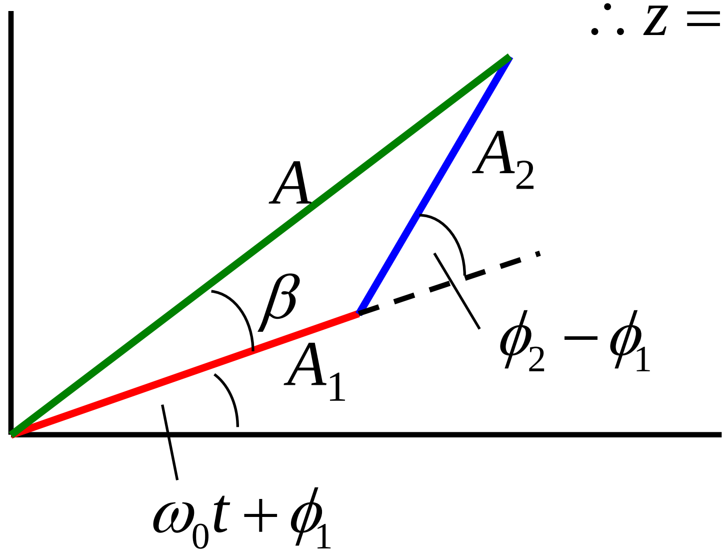
$$\left.\begin{aligned} x_1 &= A_1\cos(\omega_0 t+\phi_1) \\ x_2 &= A_2\cos(\omega_0 t+\phi_2) \end{aligned}\right\}$$

combination can be written as

$$x = A\cos(\omega_0 t+\phi)$$

Using complex numbers: $\left.\begin{aligned} z_1 &= A_1 e^{j(\omega_0 t+\phi_1)} \\ z_2 &= A_2 e^{j(\omega_0 t+\phi_2)} \end{aligned}\right\} z = z_1 + z_2$

$$\therefore z = e^{j(\omega_0 t+\phi_1)}\left\{A_1 + A_2 e^{j(\phi_2-\phi_1)}\right\}$$



Phase difference $\phi = \phi_2 - \phi_1$

Then

$$A^2 = A_1^2 + A_2^2 + 2A_1A_2\cos(\phi_2-\phi_1)$$

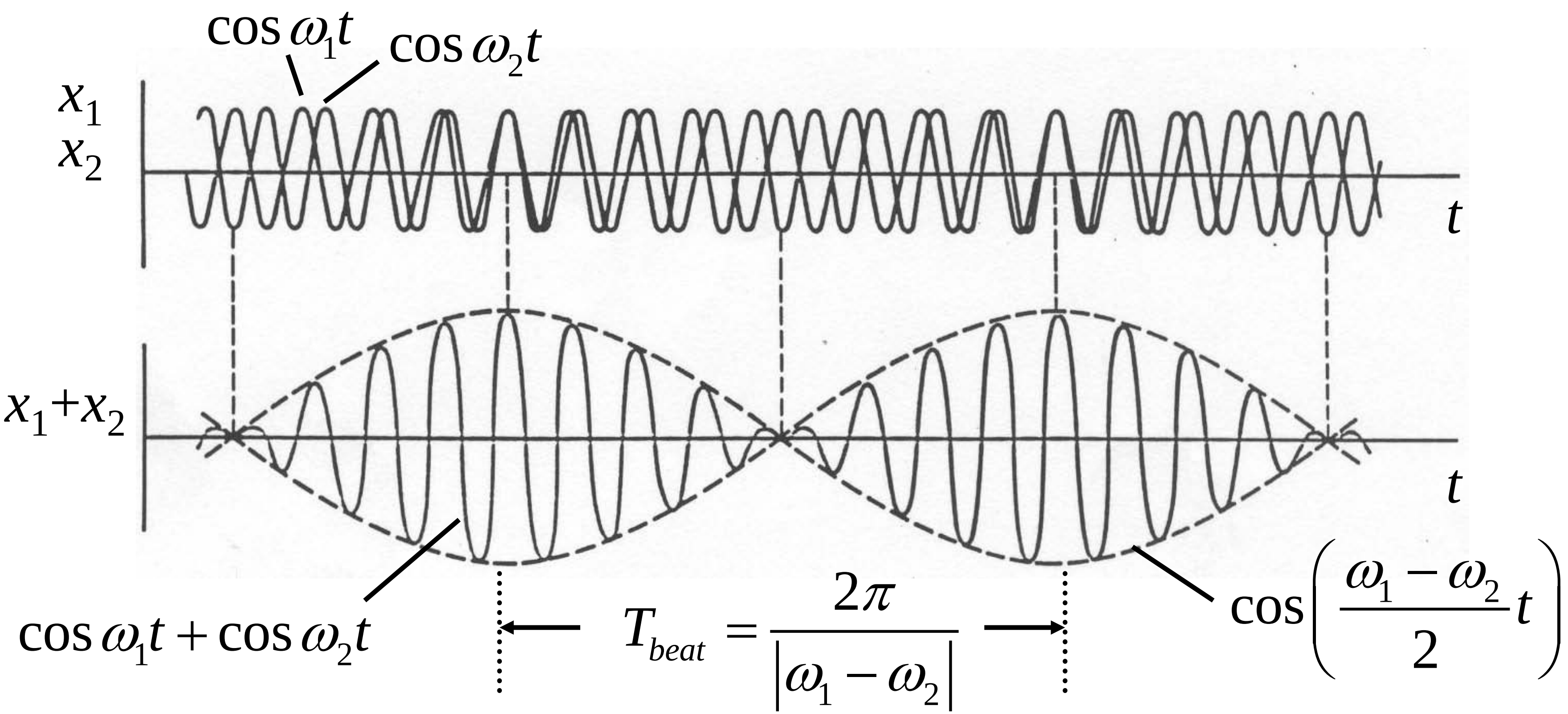
and $A\sin\beta = A_2\sin\left(\phi_2-\phi_1\right)$

Superposed vibrations of slightly different frequency: Beats

If we add two sinusoids of slightly different frequency ω_1 and ω_2
… we observe "beats"…

$$\cos\omega_1 t + \cos\omega_2 t = 2\cos\left(\frac{\omega_1-\omega_2}{2}t\right)\cos\left(\frac{\omega_1+\omega_2}{2}t\right)$$

French
page 22



Combination of two vibrations at right angles

French page 29

$$\left.\begin{aligned} x &= A_1\cos(\omega_1 t+\phi_1) \\ y &= A_2\cos(\omega_2 t+\phi_2) \end{aligned}\right\} \quad ???$$

Consider case where frequencies are equal and let initial phase difference be ϕ

Write $x = A_1\cos(\omega_0 t)$ and $y = A_2\cos(\omega_0 t+\phi)$

Case 1 : $\phi = 0$ $\left.\begin{aligned} x &= A_1\cos(\omega_0 t) \\ y &= A_2\cos(\omega_0 t) \end{aligned}\right\}$ $y = \dfrac{A_2}{A_1}x$ Rectilinear motion

Case 2 : $\phi = \pi/2$ $x = A_1\cos(\omega_0 t)$

$$y = A_2\cos(\omega_0 t+\pi/2) = -A_2\sin(\omega_0 t)$$

$$\therefore \frac{x^2}{A_1^2}+\frac{y^2}{A_2^2} = 1$$

Elliptical path in clockwise direction

Combination of two vibrations at right angles …2

Case 3 : $\phi = \pi$

$$\left.\begin{aligned} x &= A_1 \cos(\omega_0 t) \\ y &= A_2 \cos(\omega_0 t + \pi) = -A_2 \cos(\omega_0 t) \end{aligned}\right\} \quad y = -\frac{A_2}{A_1} x$$

Case 4 : $\phi = 3\pi/2$

$$x = A_1 \cos(\omega_0 t)$$

$$y = A_2 \cos(\omega_0 t + 3\pi/2) = +A_2 \sin(\omega_0 t)$$

$$\therefore \frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} = -1$$

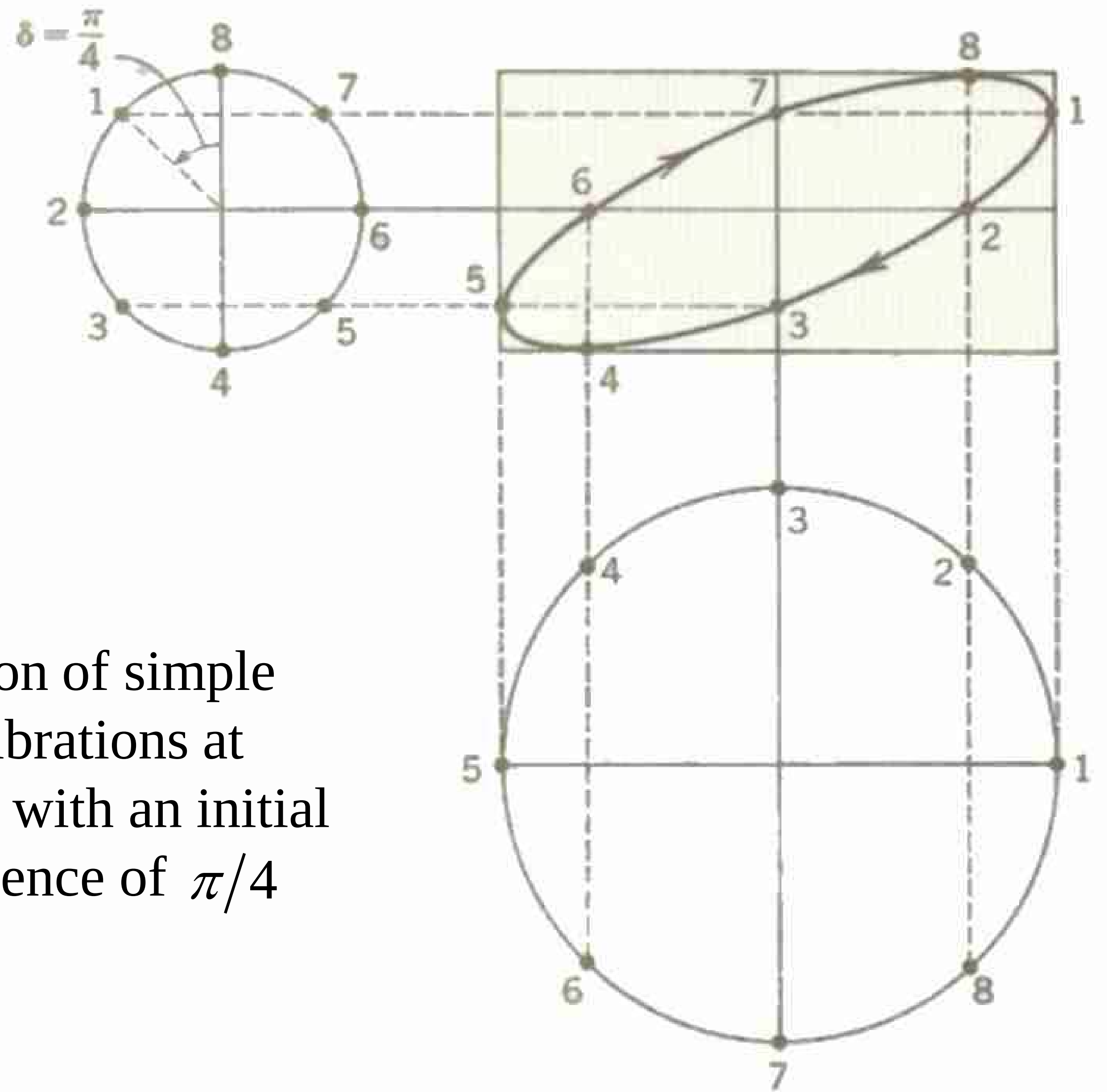
Elliptical path in anticlockwise direction

Case 5 : $\phi = \pi/4$

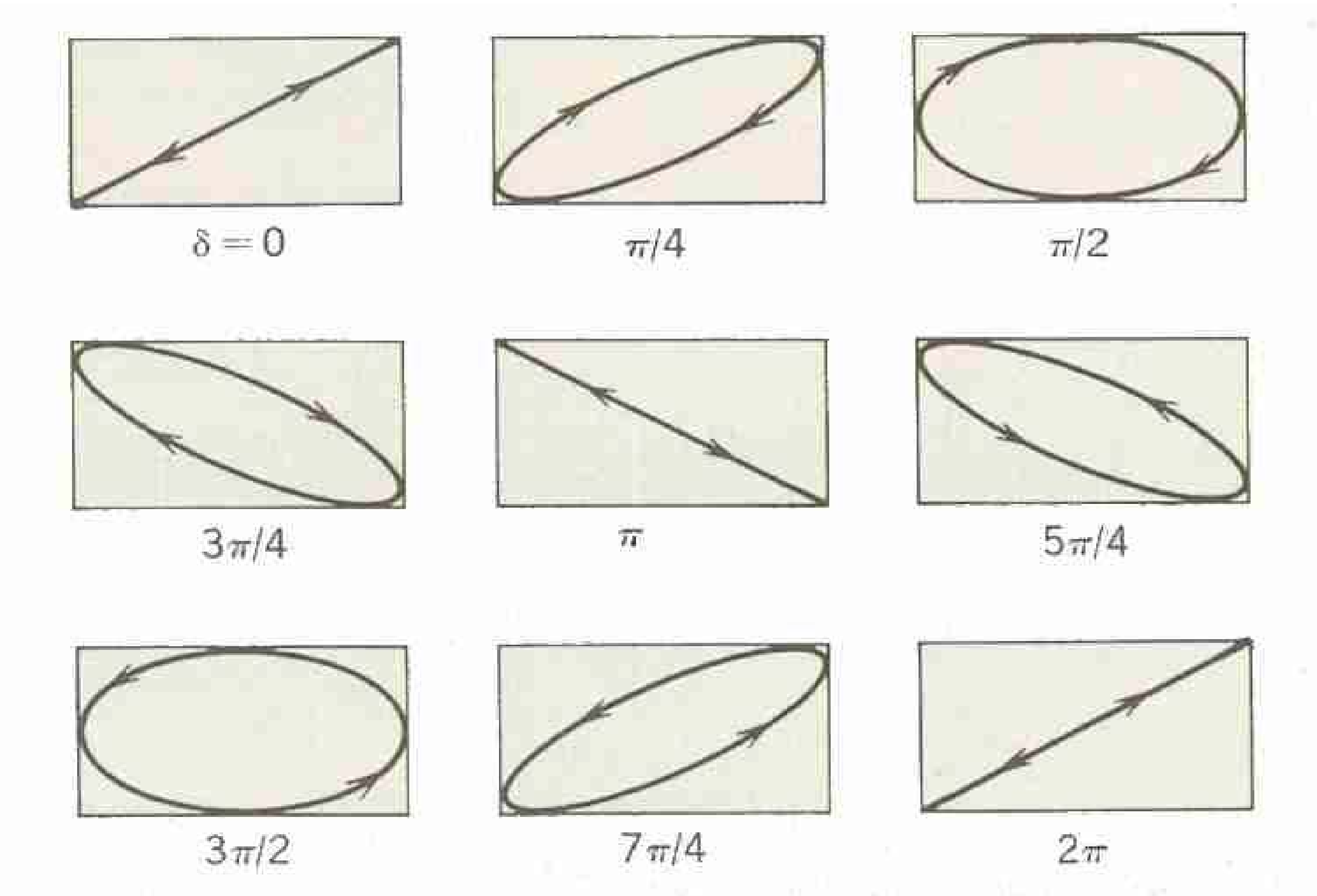
$$x = A_1 \cos(\omega_0 t)$$

$$y = A_2 \cos(\omega_0 t + \pi/4)$$

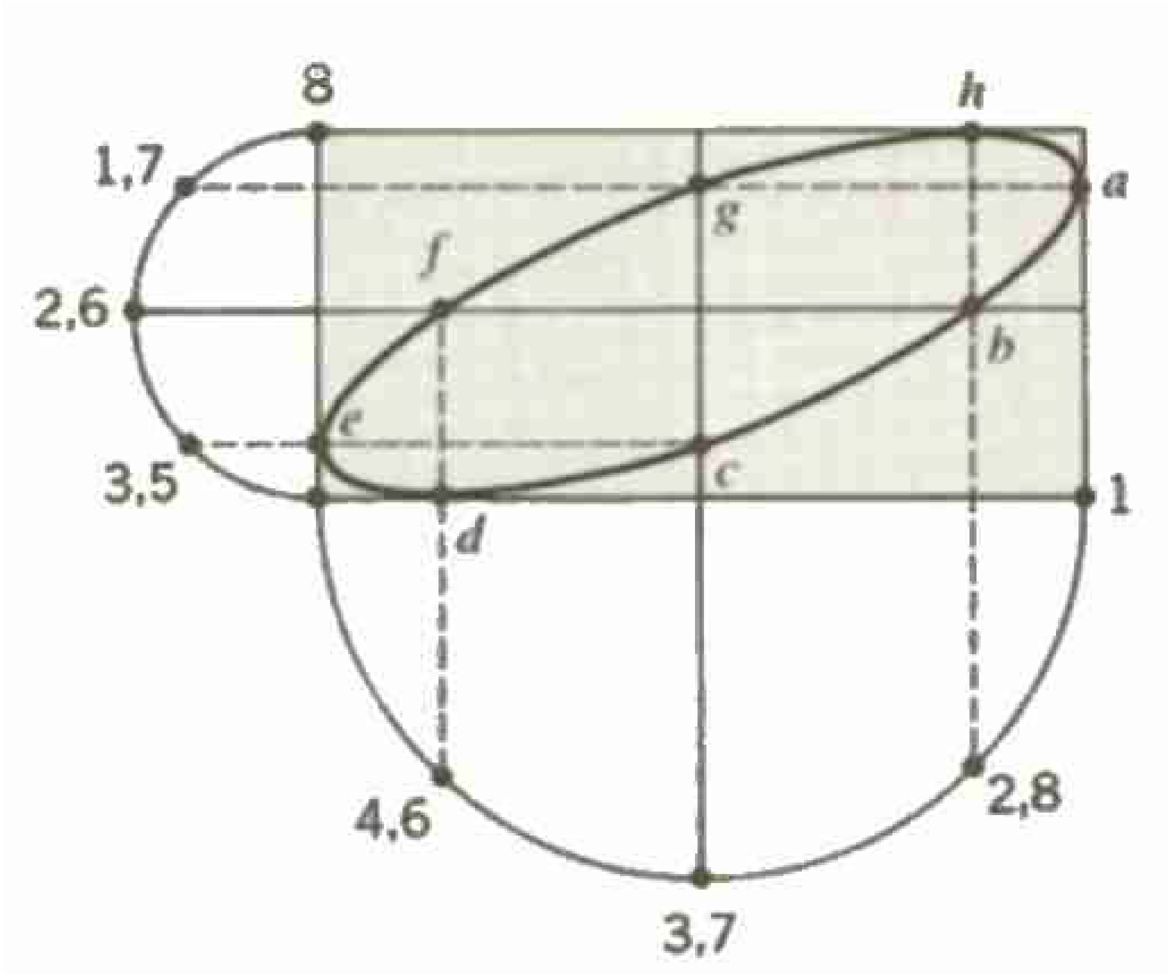
Harder to see … use a graphical approach …



Superposition of simple harmonic vibrations at right angles with an initial phase difference of $\pi/4$

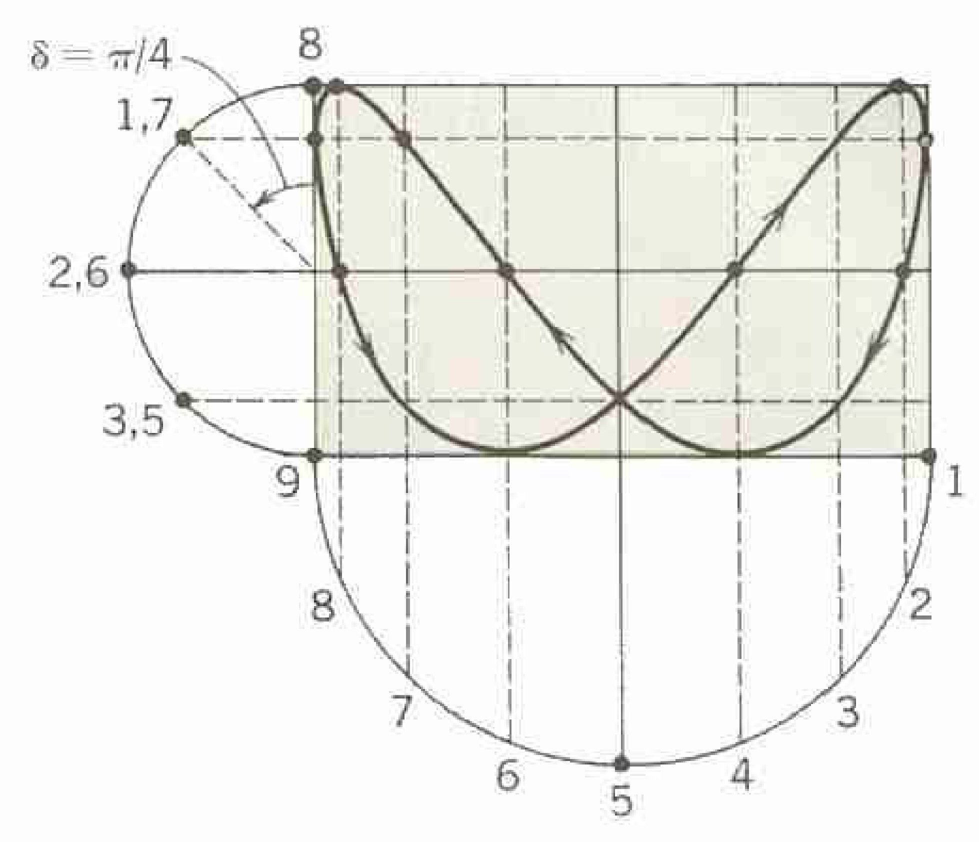


Superposition of two perpendicular simple harmonic motions of the same frequency for various initial phase differences.



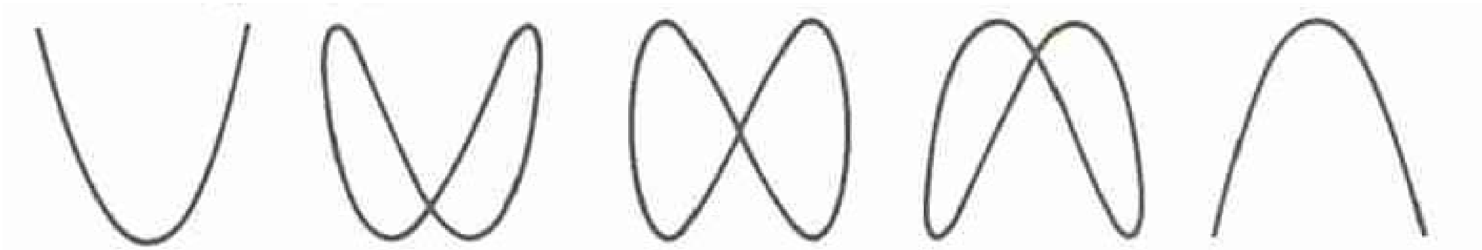
Abbreviated construction for the superposition of vibrations at right angles … see French page 34.

Perpendicular motions with different frequencies: Lissajous figures



See French page 35.

Lissajous figures for $\omega_2 = 2\omega_1$ with various initial phase differences.



$\phi =$ 0 $\pi/4$ $\pi/2$ $3\pi/4$ π

Lissajous figures

$\omega_2 : \omega_1$

1:1

1:2

1:3

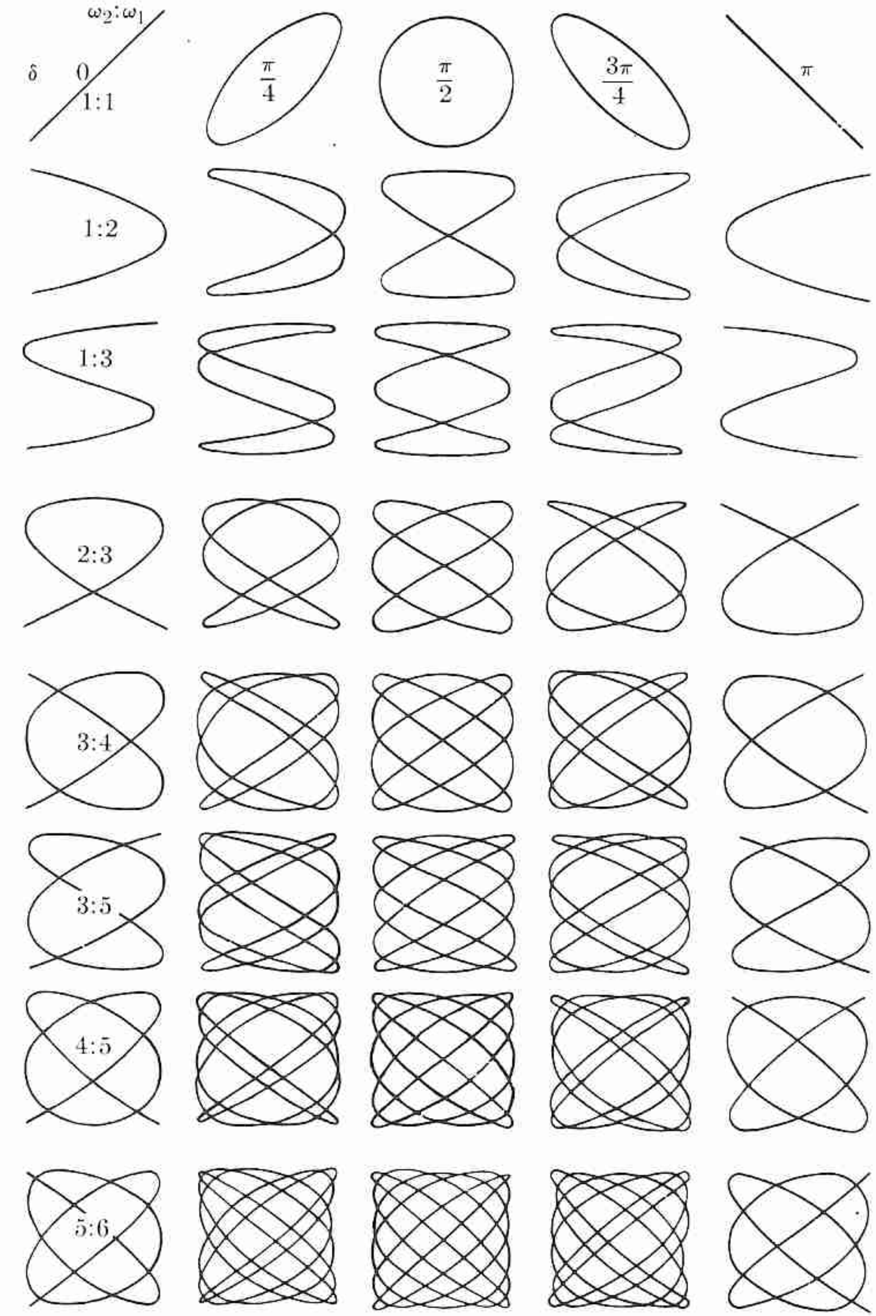
2:3

3:4

3:5

4:5

5:6

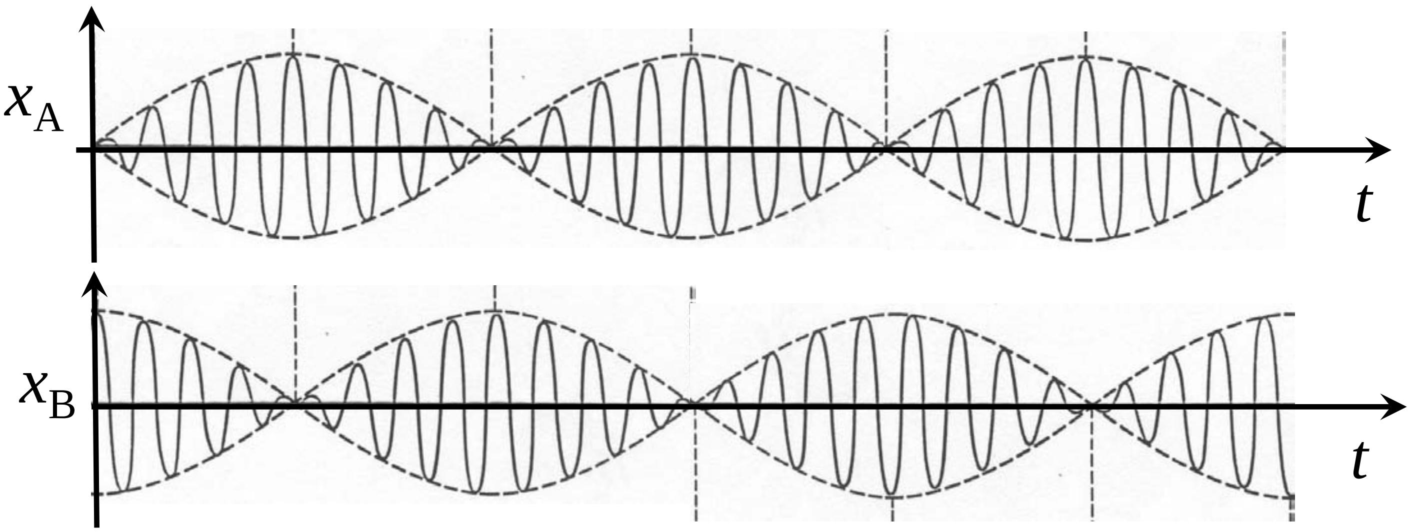


$\phi = 0 \quad \pi/4 \quad \pi/2 \quad 3\pi/4 \quad \pi$

French
page 121

Coupled oscillators

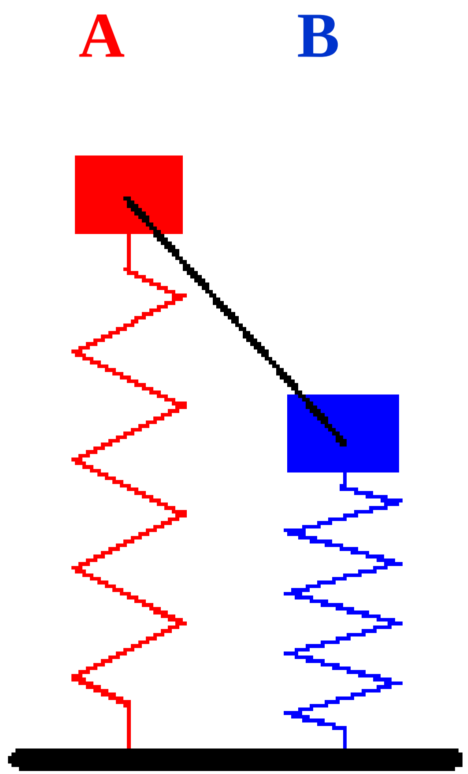
When we observe two weakly coupled identical oscillators A and B, we see:

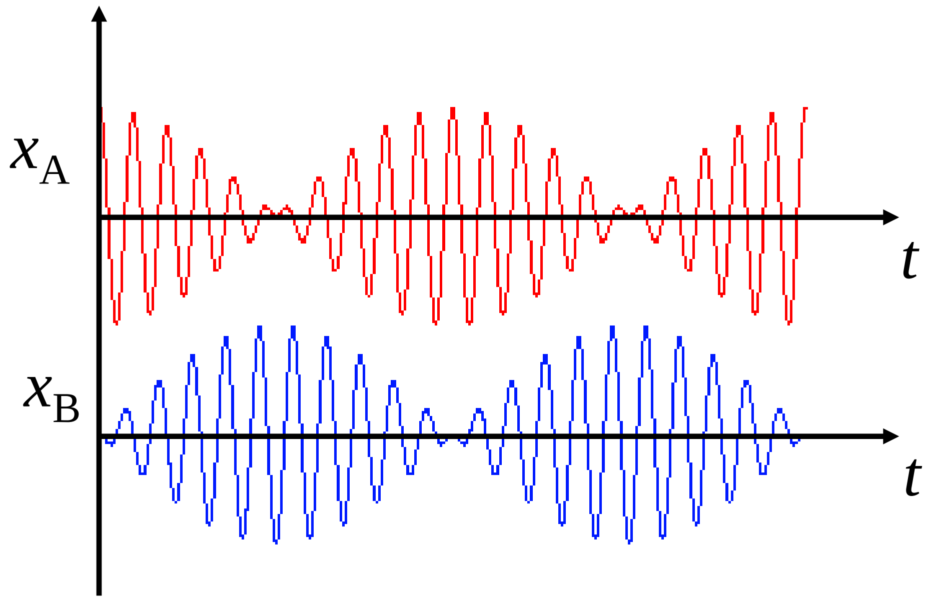


… these functions arise mathematically from the addition of two SHMs of similar frequencies … so what are these two SHMs?

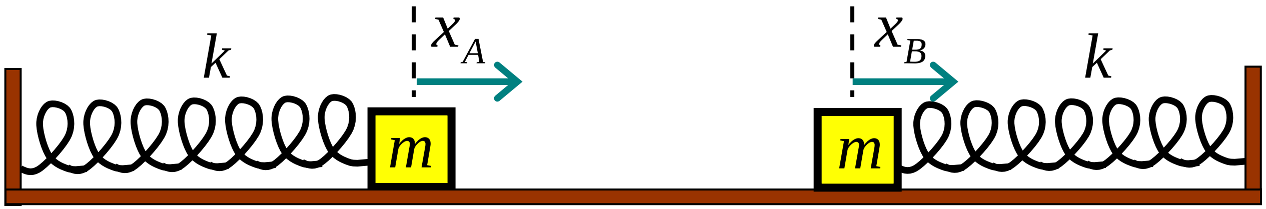
These two modes are known as **normal modes** … which are states of the system in which all parts of the system oscillate with SHM either in phase or in antiphase.

Coupled oscillators





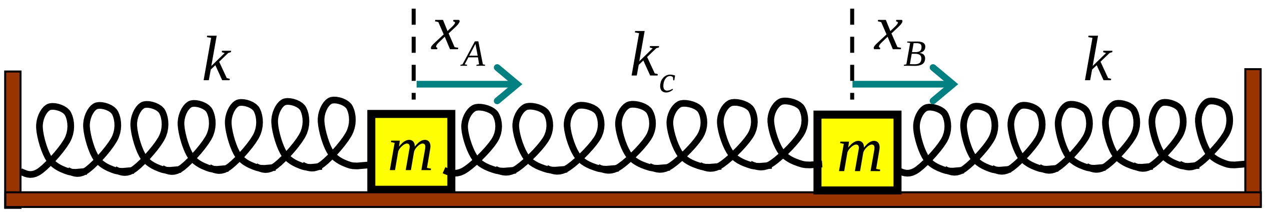
The double mass-spring oscillator



Individually we know that $m\ddot{x}_A = -kx_A$ and $m\ddot{x}_B = -kx_B$

For both oscillators: $\omega_0 = \sqrt{\dfrac{k}{m}}$

Now add a weak coupling force:



For mass A: $m\ddot{x}_A = -kx_A + k_c(x_B - x_A)$

or $\ddot{x}_A = -\omega_0^2 x_A + \Omega^2 (x_B - x_A)$ where $\omega_0^2 = \dfrac{k}{m}$, $\Omega^2 = \dfrac{k_c}{m}$

The double mass-spring oscillator …2

For mass A: $\ddot{x}_A = -\omega_0^2 x_A + \Omega^2 (x_B - x_A)$

For mass B: $\ddot{x}_B = -\omega_0^2 x_B - \Omega^2 (x_B - x_A)$

… two coupled differential equations … how to solve ?

Adding them: $$\frac{d^2}{dt^2}(x_A + x_B) = -\omega_0^2 (x_A + x_B)$$

Subtract B from A: $$\frac{d^2}{dt^2}(x_A - x_B) = -\omega_0^2 (x_A - x_B) - 2\Omega^2 (x_A - x_B)$$

Define two new variables: $$\begin{cases} q_1 = x_A + x_B \\ q_2 = x_A - x_B \end{cases}$$ … called "normal coordinates"

Then $$\frac{d^2 q_1}{dt^2} = -\omega_0^2 q_1$$ and $$\frac{d^2 q_2}{dt^2} = -(\omega_0^2 + 2\Omega^2) q_2$$

The double mass-spring oscillator …3

The two equations are now decoupled …

Write $\begin{cases} \dfrac{d^2 q_1}{dt^2} = -\omega_s^2 q_1 \\ \dfrac{d^2 q_2}{dt^2} = -\omega_f^2 q_2 \end{cases}$

$\omega_s^2 = \omega_0^2$ $\qquad s =$ “slow”

$\omega_f^2 = \omega_0^2 + 2\Omega^2$ $\qquad f =$ “fast”

… which have the solutions: $\begin{cases} q_1 = C\cos(\omega_s t + \phi_1) \\ q_2 = D\cos(\omega_f t + \phi_2) \end{cases}$

Since $q_1 = x_A + x_B$ and $q_2 = x_A - x_B$

We can write $x_A = \frac{1}{2}\left(q_1 + q_2\right)$ and $x_B = \frac{1}{2}\left(q_1 - q_2\right)$

The double mass-spring oscillator …4

Then
$$\begin{cases} x_A = \dfrac{C}{2}\cos(\omega_s t+\phi_1)+\dfrac{D}{2}\cos(\omega_f t+\phi_2) \\ x_B = \dfrac{C}{2}\cos(\omega_s t+\phi_1)-\dfrac{D}{2}\cos(\omega_f t+\phi_2) \end{cases}$$

So x_A and x_B have been expressed as the sum and difference of two SHMs as expected from observation.

… C, D, ϕ_1 and ϕ_2 may be determined from the initial conditions.

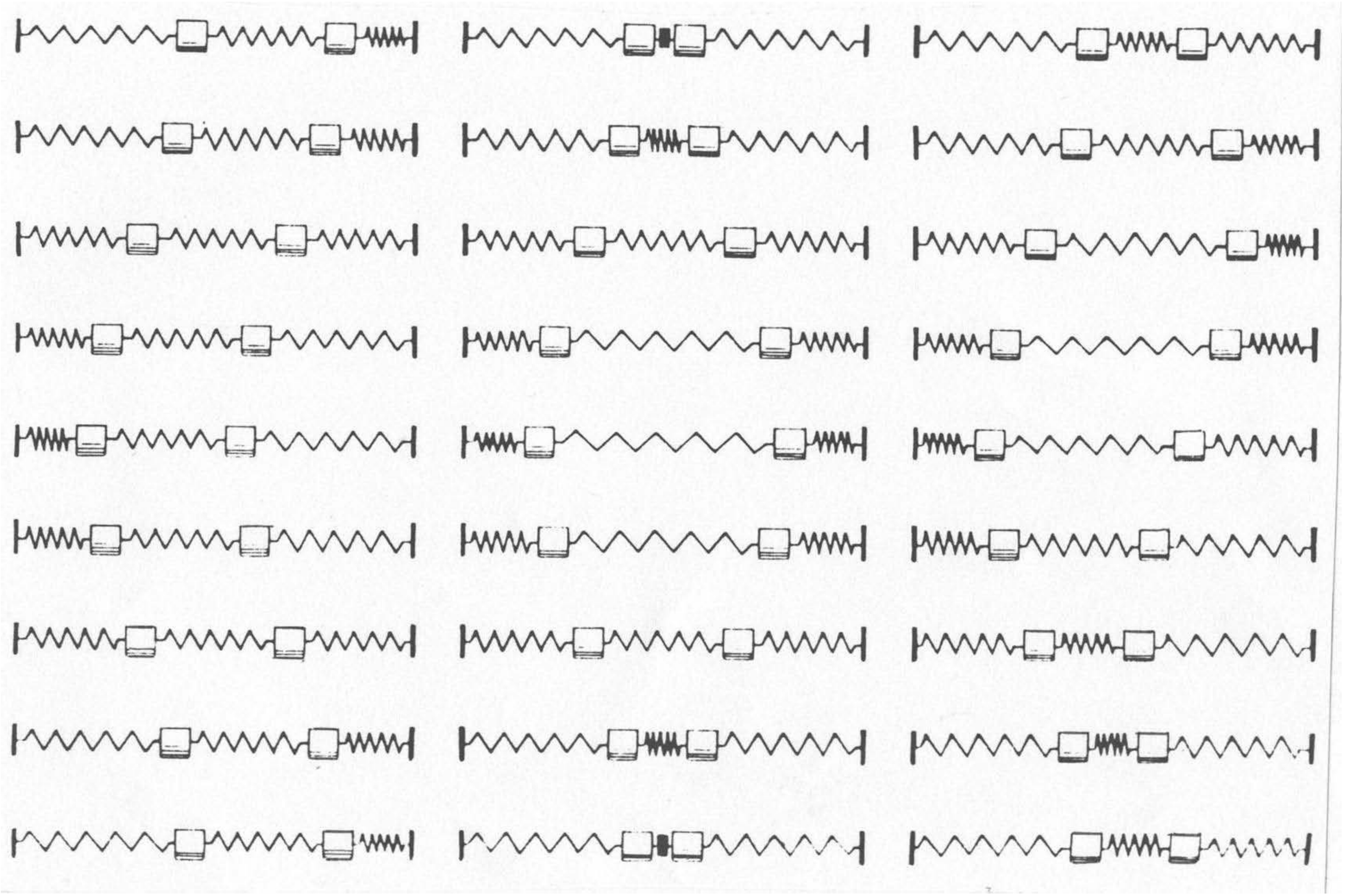
… when $x_A = x_B$,then $q_2 = 0$ … there is no contribution from the fast mode and the two masses move in phase … the coupling spring does not change length and has no effect on the motion … $\omega_s = \omega_0$

… when $x_A = -x_B$,then $q_1 = 0$ … there is no contribution from the slow mode … the coupling spring gives an extra force … each mass experiences a force $-(k+2k_c)x$ giving

$$\omega_f^2 = \frac{k+2k_c}{m}$$
$$= \omega_0^2 + 2\Omega^2$$

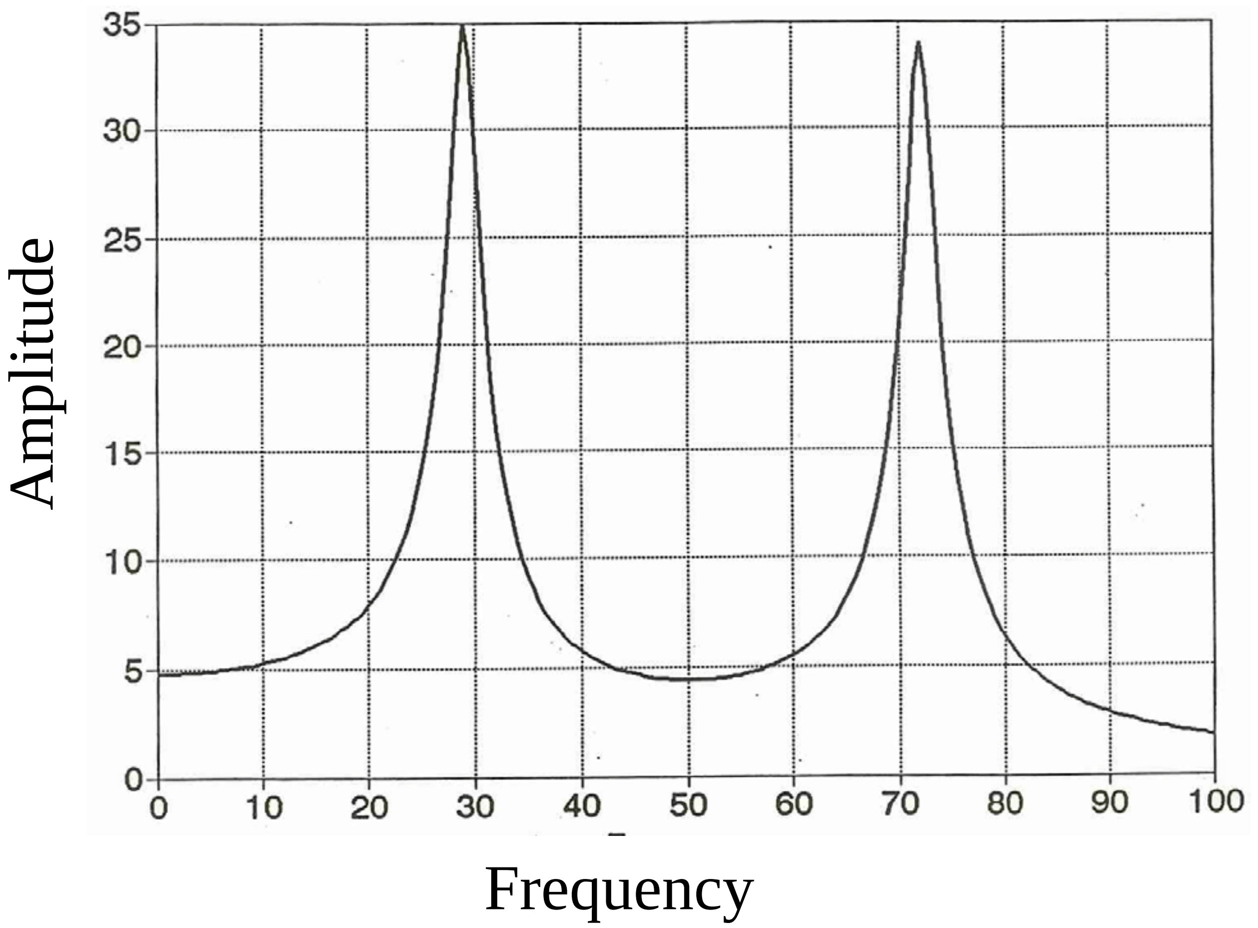
The double mass-spring oscillator …5

symmetric mode antisymmetric mode mixed mode



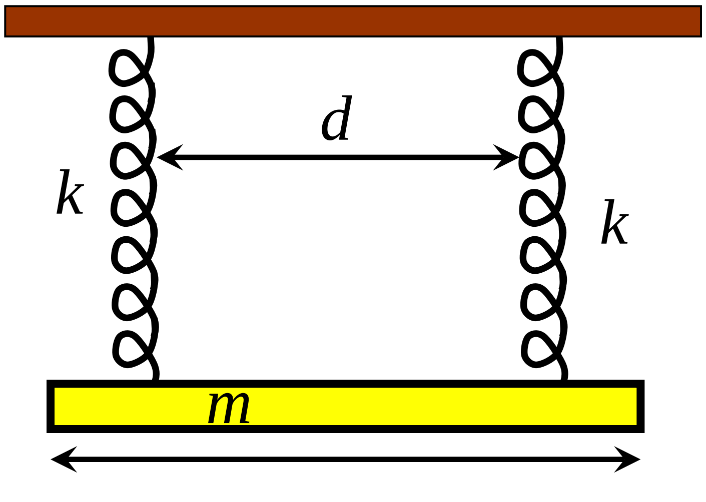
The double mass-spring oscillator …6

We now have a system with two natural frequencies, and experimentally find two resonances.



Pitch and bounce oscillator

French
page 127



Two normal modes (by inspection):

Bouncing

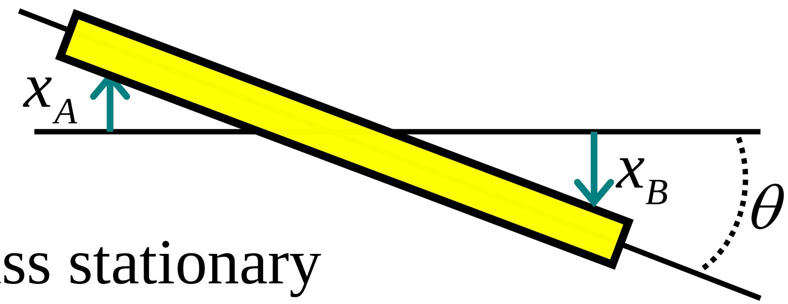
$x_A = x_B$

Restoring force = $-2kx$

$$\therefore m\frac{d^2x}{dt^2} = -2kx \qquad \omega^2_{\text{bounce}} = \frac{2k}{m}$$

Pitching

$x_A = -x_B$

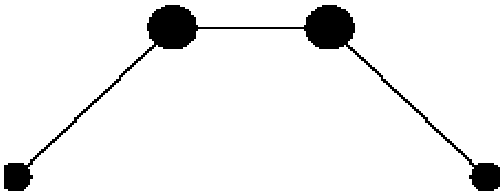


Centre of mass stationary

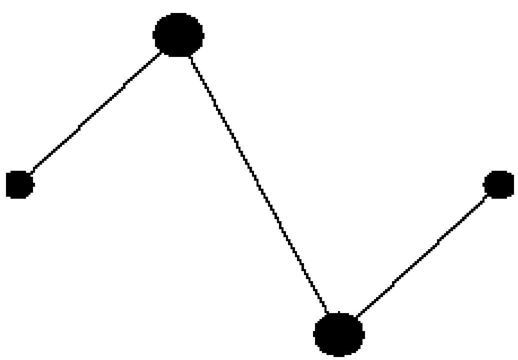
$$\tau = -I\ddot{\theta}$$

$$kd\left(\tfrac{1}{2}\theta d\right) = -\tfrac{1}{12}mL^2\ddot{\theta} \qquad \rightarrow \ddot{\theta} = \frac{6kd^2}{mL^2}\theta \qquad \omega^2_{\text{pitch}} = \frac{6k}{m}\frac{d^2}{L^2}$$

$N = 2$

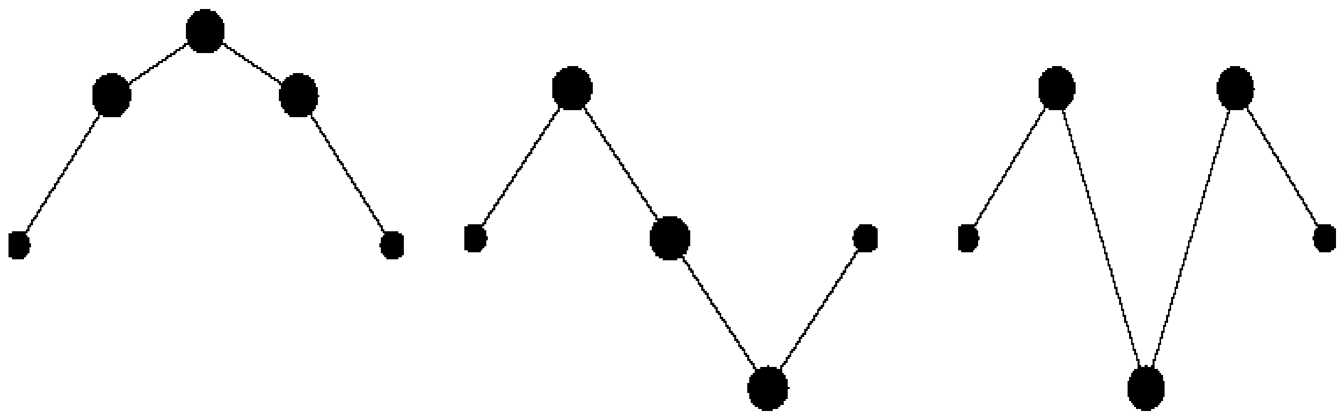


$$\omega_1 = 2\omega_0 \sin\frac{1\pi}{2(2+1)} = \omega_0$$

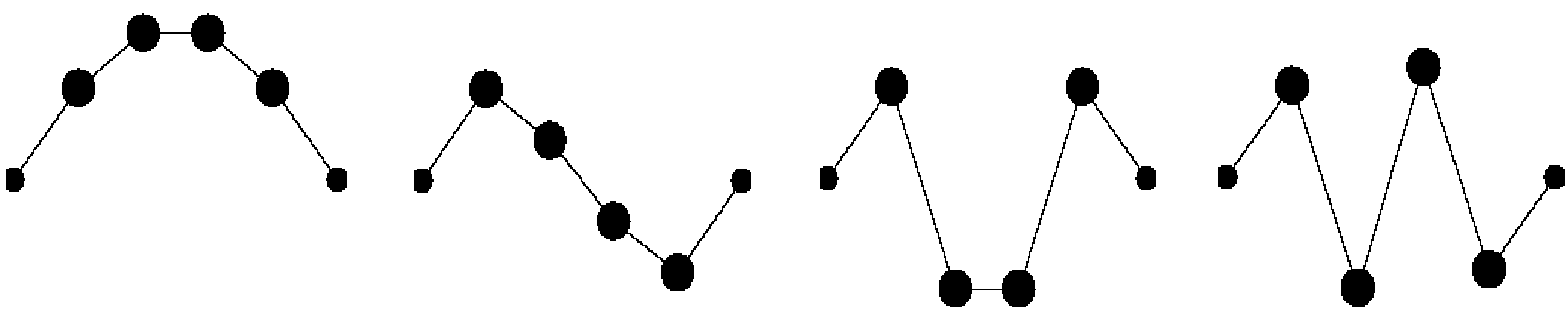


$$\omega_2 = 2\omega_0 \sin\frac{2\pi}{2(2+1)} = \sqrt{3}\omega_0$$

$N = 3$

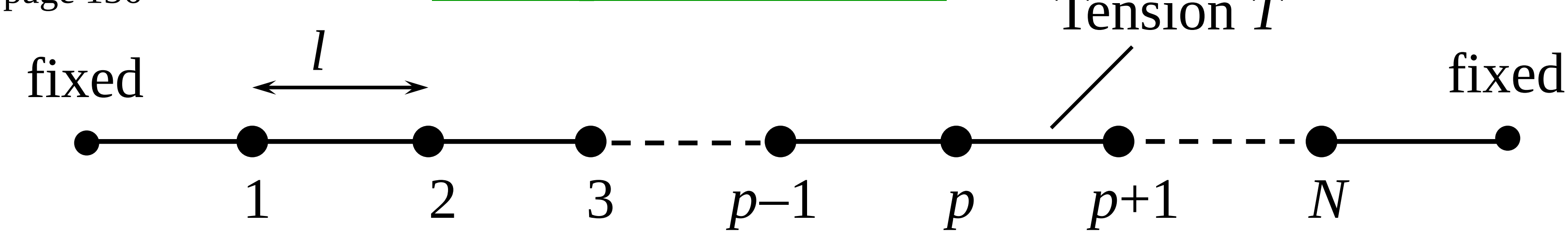


$N = 4$



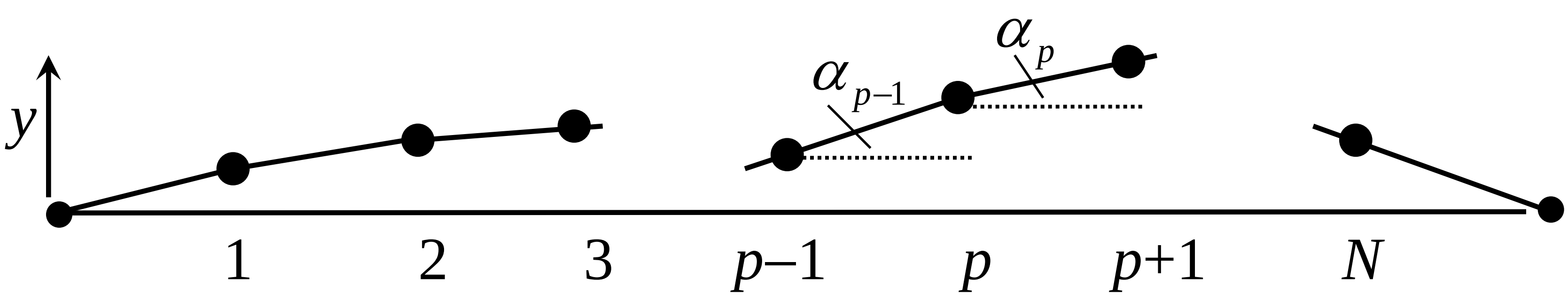
French
page 136

N-coupled oscillators



Each bead has mass m

… consider transverse displacements that are small.



Transverse force on p^{th} particle: $F_p = -T\sin\alpha_{p-1} + T\sin\alpha_p$

$$= -T\frac{y_p - y_{p-1}}{l} + T\frac{y_{p+1} - y_p}{l}$$

for small α

N-coupled oscillators …2

$$F_p = m\frac{d^2 y_p}{dt^2} = -T\,\frac{y_p - y_{p-1}}{l} + T\,\frac{y_{p+1} - y_p}{l}$$

$$\therefore \frac{d^2 y_p}{dt^2} + 2\omega_0^2 y_p - \omega_0^2\left(y_{p+1} - y_{p-1}\right) = 0$$

where $\omega_0^2 = \dfrac{T}{ml}$, $p = 1, 2 \ldots \mathrm{N}$

… a set of N coupled differential equations.

Normal mode solutions: $y_p = A_p \sin \omega t$

Substitute to obtain N simultaneous equations

$$\left(-\omega^2 + 2\omega_0^2\right)A_p - \omega_0^2\left(A_{p+1} + A_{p-1}\right) = 0$$

or $$\frac{A_{p+1} + A_{p-1}}{A_p} = \frac{-\omega^2 + 2\omega_0^2}{\omega_0^2}$$

N-coupled oscillators …3

From observation of physical systems we expect sinusoidal "shape functions" of the form $A_p = C \sin p\theta$

Substitute into $\dfrac{A_{p+1} + A_{p-1}}{A_p} = \dfrac{-\omega^2 + 2\omega_0^2}{\omega_0^2}$

And apply boundary conditions $A_0 = 0$ and $A_{N+1} = 0$

… find that $\theta = \dfrac{n\pi}{N+1}$ $n = 1, 2, 3, \ldots N$ (modes)

There are N modes:

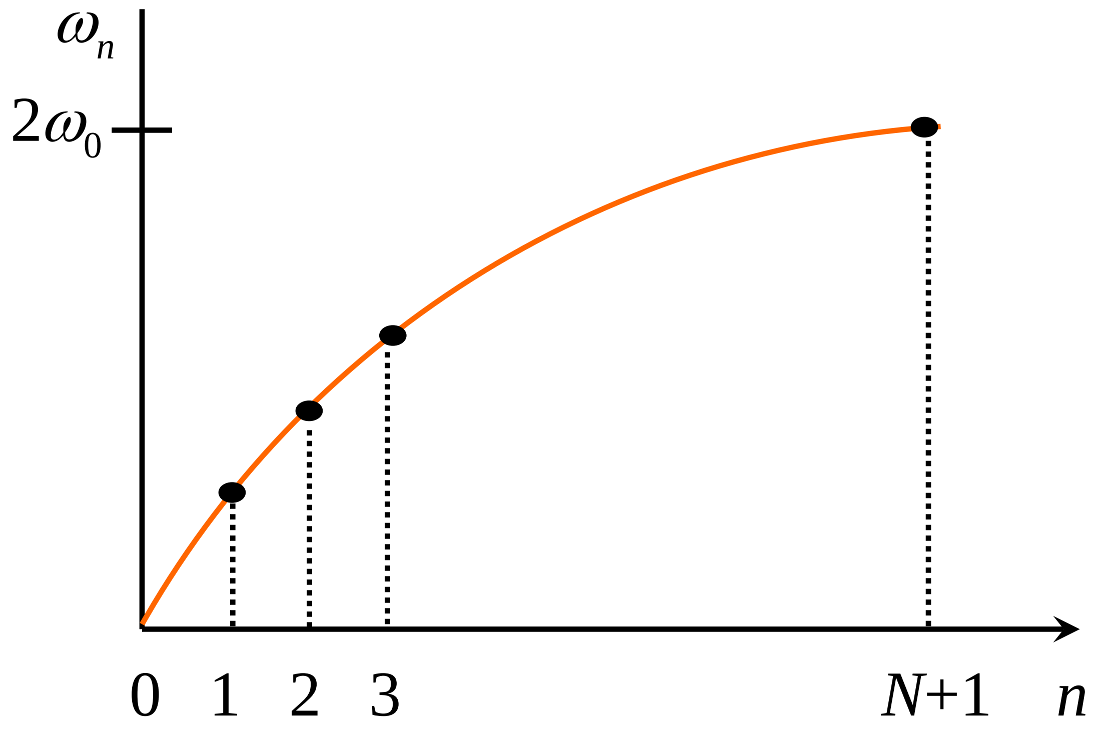
$$y_{pn} = A_{pn} \sin \omega_n t = C_n \sin \frac{pn\pi}{N+1} \sin \omega_n t$$

and $$\omega_n = 2\omega_0 \sin \frac{n\pi}{2(N+1)}$$

N-coupled oscillators …4

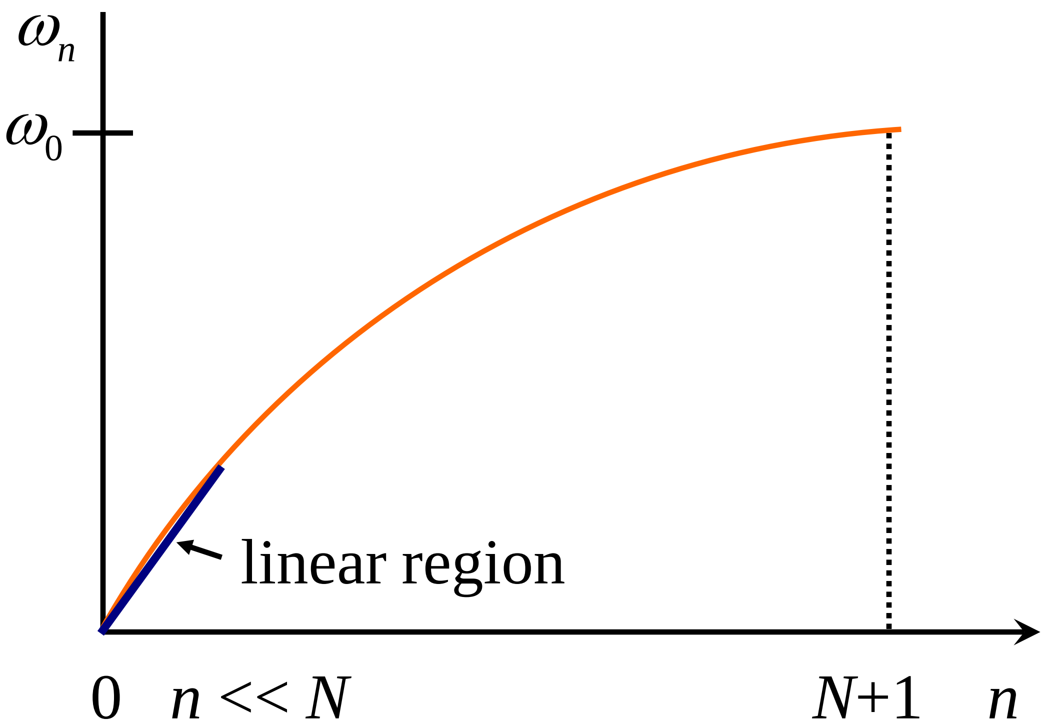
$$\omega_n = 2\omega_0 \sin\frac{n\pi}{2\left(N+1\right)}$$

For small N:



N-coupled oscillators …5

In many systems of interest N is very large… and we are only interested in the lowest frequency modes.



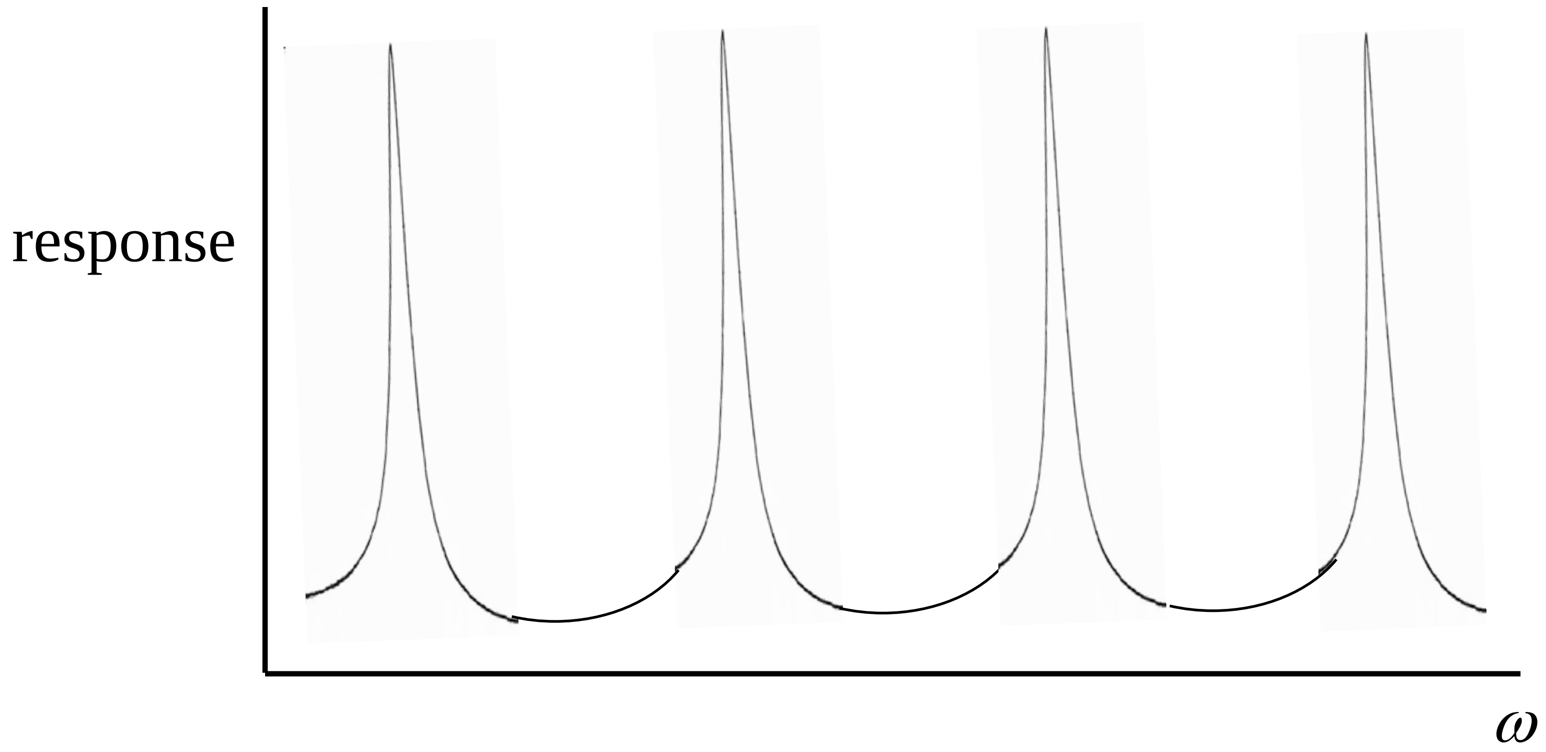
$$\text{For } n << N: \ \sin\frac{n\pi}{2(N+1)} = \frac{n\pi}{2(N+1)}$$

$$\text{then} \quad \omega_n = 2\omega_0 \frac{n\pi}{2(N+1)} = \left(\frac{\pi\omega_0}{N+1}\right) n$$

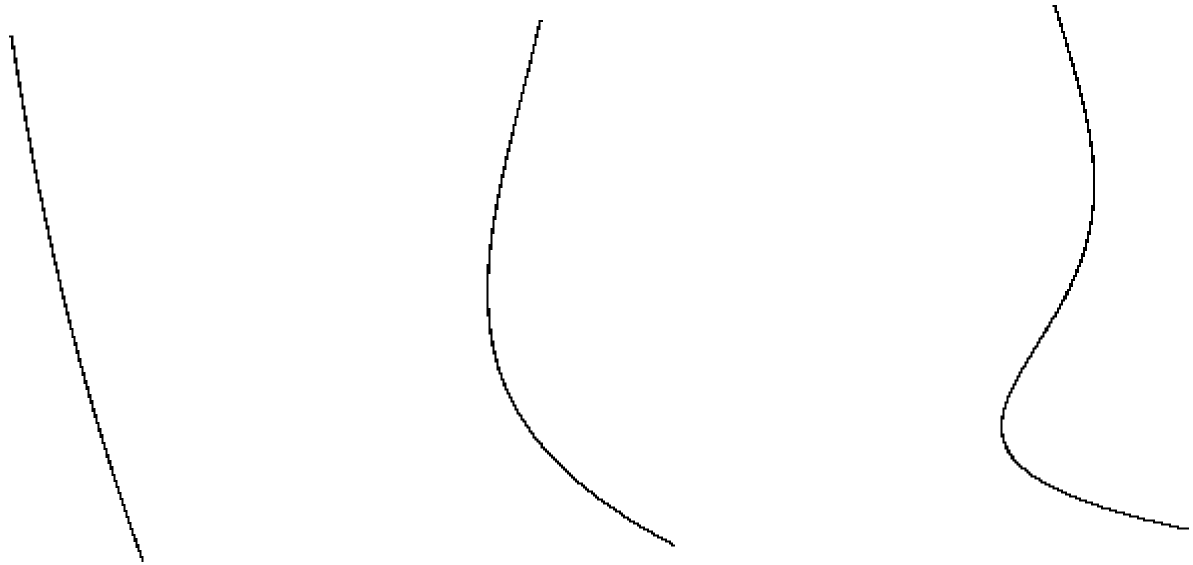
i.e. $\omega_n \propto n$ for $n << N$

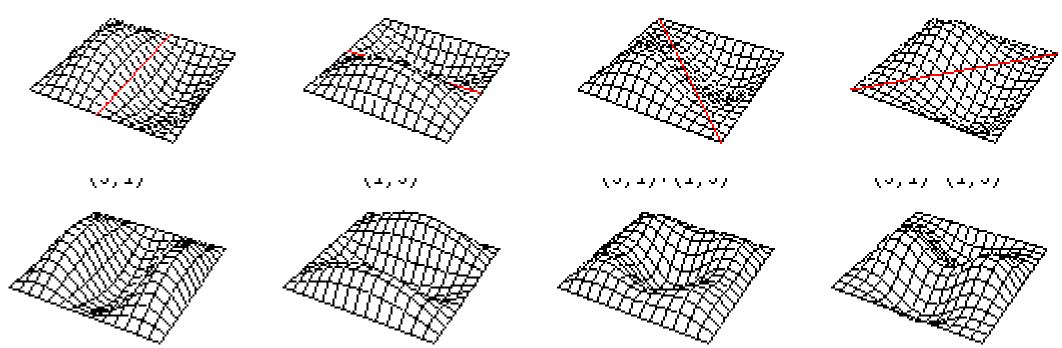
N-coupled oscillators …6

N coupled oscillators have *N* normal modes and hence *N* resonances



Continuous systems



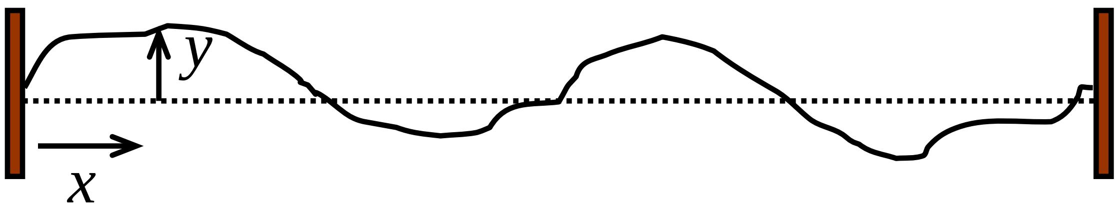


Continuous systems

Consider a string stretched between two rigid supports …

String has mass m and mass per unit length $\mu = m/L$

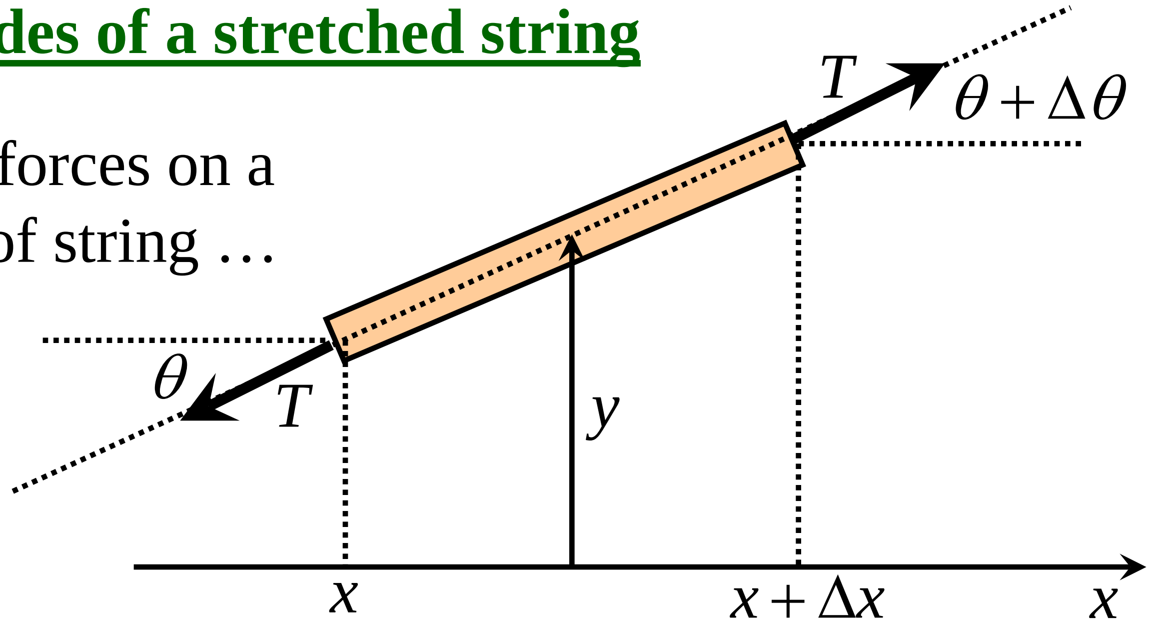
Suppose that the string is disturbed in some way:



The displacement y is a function of x and t : $y(x,t)$

Normal modes of a stretched string

Consider the forces on a small length of string …



French page 162

Restrict to small amplitude disturbances … then θ is small and

$$\cos\theta = 1 \qquad \sin\theta = \tan\theta = \theta = \frac{\partial y}{\partial x}$$

The tension T is uniform throughout the string.

Net horizontal force is zero: $T\cos(\theta+\Delta\theta) - T\cos\theta = 0$

Vertical force: $F = T\sin(\theta+\Delta\theta) - T\sin\theta$

Then $F = T\left.\frac{\partial y}{\partial x}\right|_{x+\Delta x} - T\left.\frac{\partial y}{\partial x}\right|_{x}$

Normal modes of a stretched string …2

$$F = T\frac{\partial y}{\partial x}\bigg|_{x+\Delta x} - T\frac{\partial y}{\partial x}\bigg|_{x}$$

Use $\quad \dfrac{dg}{dx} = \dfrac{g(x+\Delta x) - g(x)}{\Delta x}$

Then $\quad F = T\dfrac{\partial^2 y}{\partial x^2}\Delta x$

or $\quad (\mu\Delta x)\dfrac{\partial^2 y}{\partial t^2} = T\dfrac{\partial^2 y}{\partial x^2}\Delta x$

giving $\quad \dfrac{\partial^2 y}{\partial x^2} = \left(\dfrac{\mu}{T}\right)\dfrac{\partial^2 y}{\partial t^2}$

Write $\quad \dfrac{\partial^2 y}{\partial x^2} = \dfrac{1}{v^2}\dfrac{\partial^2 y}{\partial t^2}$

One dimensional wave equation

μ: mass per unit length

Check: μ/T has the dimensions $1/v^2$

Then $v = \sqrt{T/\mu}$ is the speed at which a wave propagates along the string … see later

Normal modes of a stretched string …3

Look the standing wave (normal mode) solutions …

Normal mode: all parts of the system move in SHM at the same frequency …

Write: $y(x,t) = f(x)\cos\omega t$

$f(x)$ is the "shape" function … substitute into wave equation

$$\left.\begin{aligned} \frac{\partial y^2(x,t)}{\partial x^2} &= \frac{d^2 f(x)}{dx^2}\cos\omega t \\ \frac{\partial y^2(x,t)}{\partial t^2} &= f(x)\left(-\omega^2\cos\omega t\right) \end{aligned}\right\} \frac{d^2 f(x)}{dx^2}\cos\omega t = -\frac{1}{v^2}\omega^2 f(x)\cos\omega t$$

which must be true for all t

then $\dfrac{d^2 f}{dx^2} = -\dfrac{\omega^2}{v^2} f(x)$

Normal modes of a stretched string …4

$$\frac{d^2 f}{dx^2} = -\frac{\omega^2}{v^2} f(x)$$

… which has the same form as the eq. of SHM: $\dfrac{d^2 x}{dt^2} = -\omega_0^2 x$

… has general solution: $x = A\sin(\omega_0 t + \phi)$

Thus we must have: $f(x) = A\sin\left(\dfrac{\omega}{v}x + \Phi\right)$

Apply boundary conditions: $y = 0$ at $x = 0$ and $x = L$

$$\therefore\ f(0) = 0 \quad \text{and} \quad f(L) = 0$$

$x = 0, f = 0$: $0 = A\sin\left(\dfrac{\omega}{v}0 + \Phi\right)$ i.e. $\Phi = 0$

$x = L, f = 0$: $0 = A\sin\left(\dfrac{\omega}{v}L\right)$ i.e. $\dfrac{\omega}{v}L = n\pi$ $n = 1,2,3,\ldots$

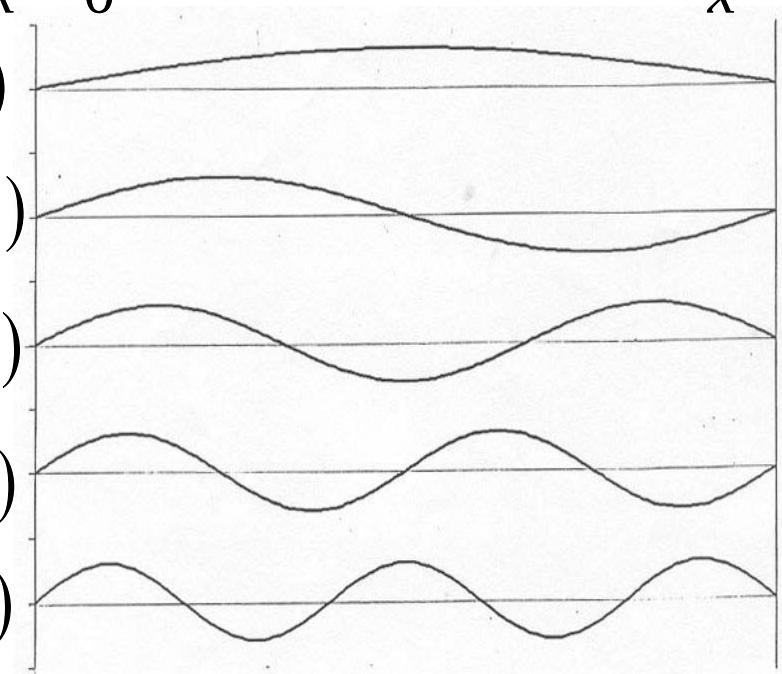
Normal modes of a stretched string …5

Write $\omega_n = \dfrac{n\pi v}{L}$ $n = 1,2,3,\ldots$

Therefore $f(x) = A_n \sin\left(\dfrac{x}{v}\dfrac{n\pi v}{L}\right) = A_n \sin\left(\dfrac{n\pi x}{L}\right)$

shape function, or "eigenfunction"

$x = 0$ $x = L$



$f(x) = A_1 \sin(\pi x/L)$ $n = 1$ $\omega_1 = \pi v/L$

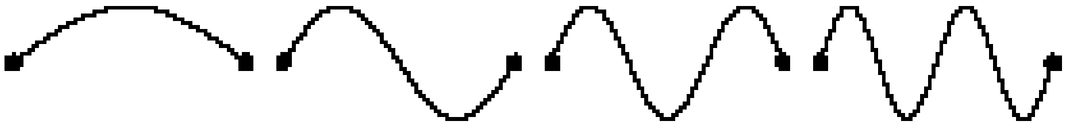
$f(x) = A_2 \sin(2\pi x/L)$ $n = 2$ $\omega_2 = 2\pi v/L$

$f(x) = A_3 \sin(3\pi x/L)$ $n = 3$ $\omega_3 = 3\pi v/L$

$f(x) = A_4 \sin(4\pi x/L)$ $n = 4$ $\omega_4 = 4\pi v/L$

$f(x) = A_5 \sin(5\pi x/L)$ $n = 5$ $\omega_5 = 5\pi v/L$

Normal modes of a stretched string

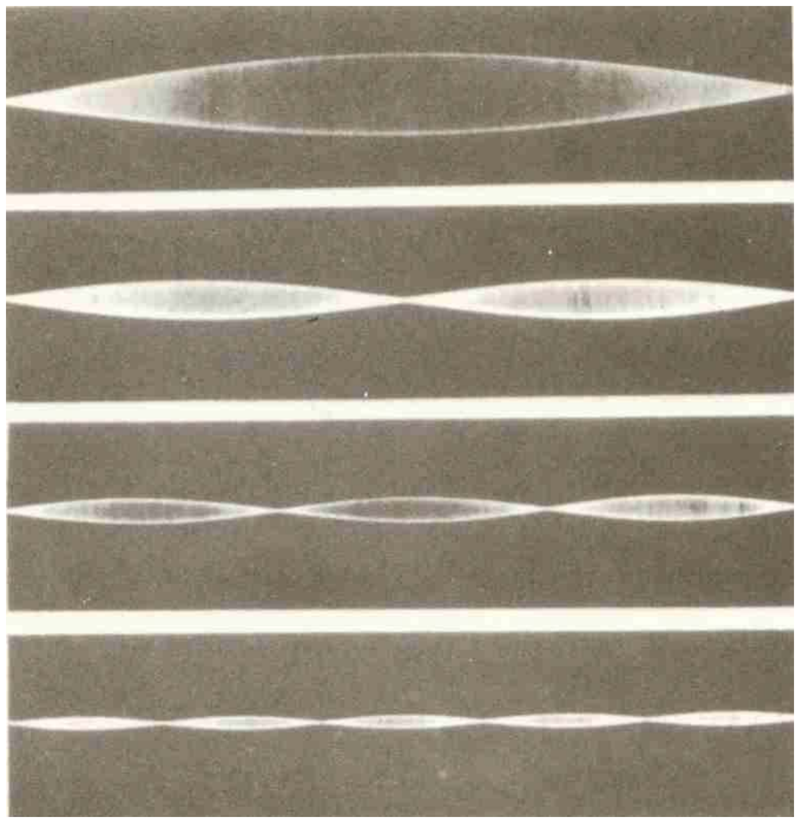


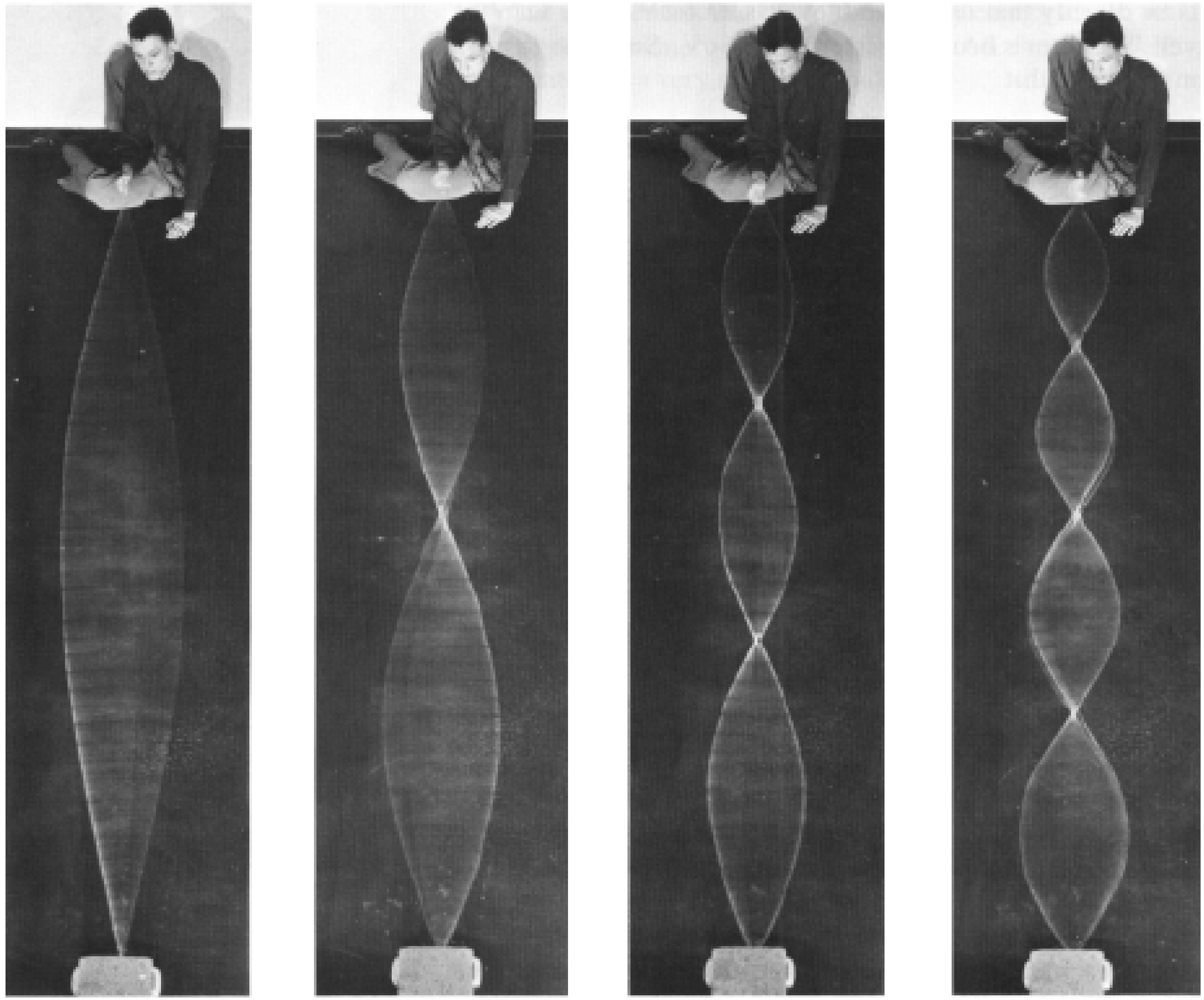
$n = 1$

$n = 2$

$n = 3$

$n = 4$





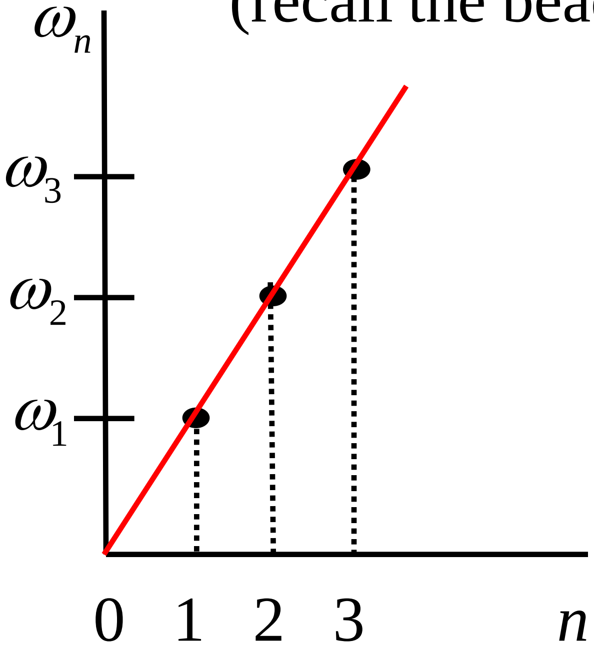
Normal modes of a stretched string …6

Full solution for our standing waves:

$$y(x,t) = A_n \sin\left(\frac{n\pi x}{L}\right)\cos\omega_n t \qquad \omega_n = \frac{n\pi v}{L}$$

The mode frequencies are evenly spaced: $\omega_n = n\omega_1$

(recall the beaded string)



This continuum approach breaks down as the wavelength approaches atomic dimensions… also if there is any "stiffness" in the spring which adds an additional restoring force which is more pronounced in the high frequency modes.

Normal modes of a stretched string …7

All motions of the system can be made up from the superposition of normal modes

$$y(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)\cos(\omega_n t + \phi_n)$$

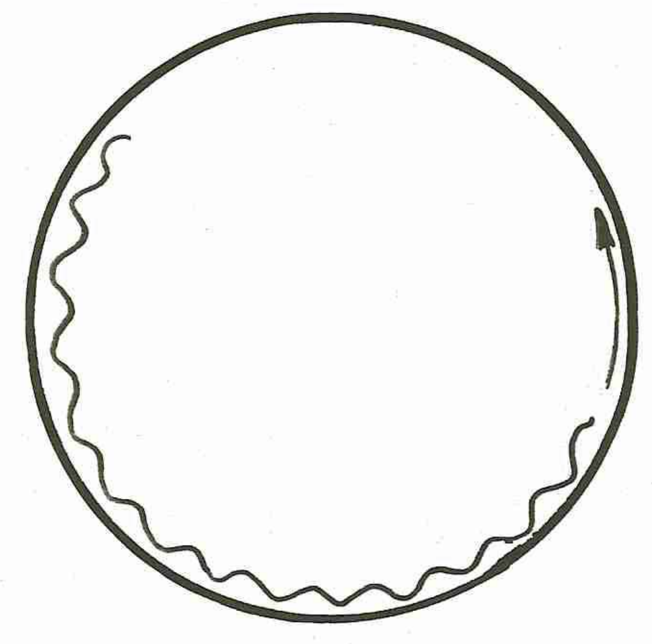
with $\omega_n = \dfrac{n\pi v}{L}$

Note that the phase angle is back since the modes may not be in phase with each other.

Whispering galleries

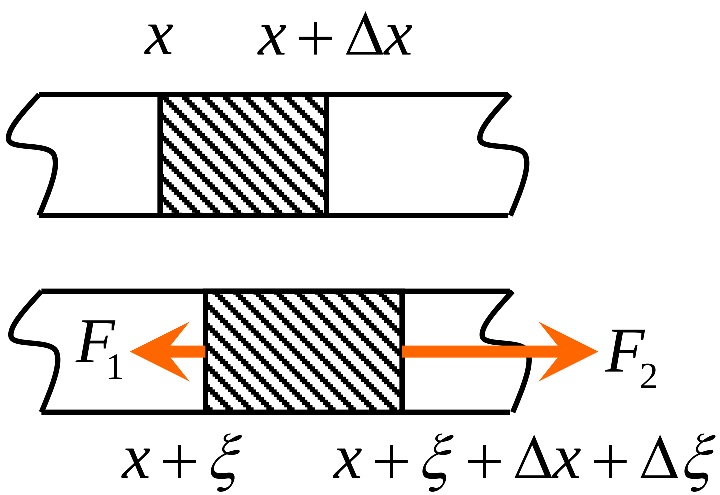
… best example is the inside dome of St. Paul’s cathedral.

If you whisper just inside the dome, then an observer close to you can hear the whisper coming from the opposite direction … it has travelled right round the inside of the dome.



Longitudinal vibrations of a rod

French
page 170



section of massive rod

section is displaced and stretched by an unbalanced force

Average strain = $\dfrac{\Delta\xi}{\Delta x}$

Average stress = $Y\dfrac{\Delta\xi}{\Delta x}$

Y : Young's modulus

stress at $x+\Delta x$ = (stress at x) + $\dfrac{\partial(\text{stress})}{\partial x}\Delta x$

Longitudinal vibrations of a rod …2

If the cross sectional area of the rod is α

$$F_1 = \alpha Y \frac{\partial \xi}{\partial x} \quad \text{and} \quad F_2 = \alpha Y \frac{\partial \xi}{\partial x} + \alpha Y \frac{\partial^2 \xi}{\partial x^2}\Delta x$$

$$F_1 - F_2 = \alpha Y \frac{\partial^2 \xi}{\partial x^2}\Delta x = ma$$

$$\therefore \alpha Y \frac{\partial^2 \xi}{\partial x^2}\Delta x = \rho \alpha \Delta x \frac{\partial^2 \xi}{\partial t^2}$$

or
$$\frac{\partial^2 \xi}{\partial x^2} = \frac{\rho}{Y}\frac{\partial^2 \xi}{\partial t^2}$$

$$\therefore \frac{\partial^2 \xi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \xi}{\partial t^2} \qquad v = \sqrt{\frac{Y}{\rho}}$$

Longitudinal vibrations of a rod …3

Look for solutions of the type: $\xi(x,t) = f(x)\cos\omega t$

where $f(x) = A\sin\left(\dfrac{\omega}{v}x + \Phi\right)$

Apply boundary conditions: one end fixed and the other free

$x = 0 :\quad \xi(0,t) = 0 \qquad$ i.e. $\Phi = 0$

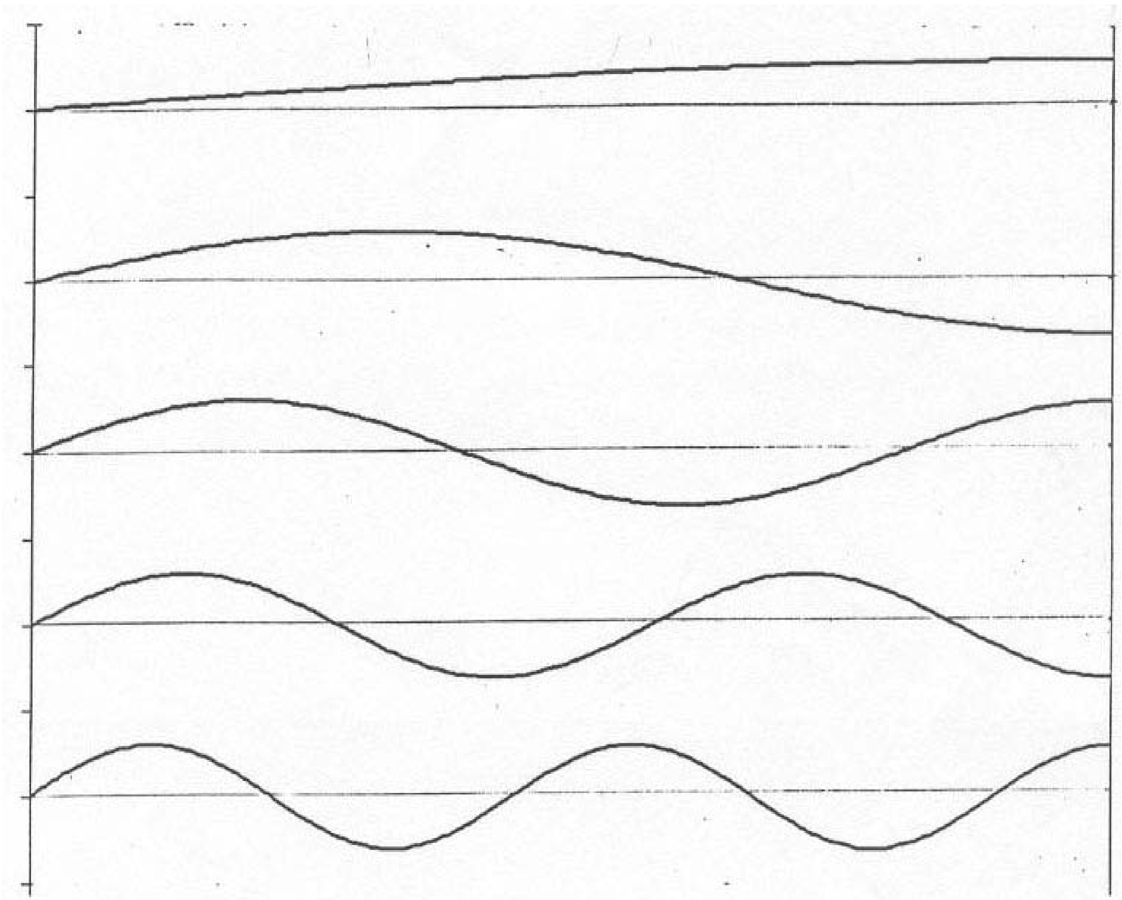
$$x = L :\ F = \alpha Y \frac{\partial \xi}{\partial x} = 0$$

… then $\cos\left(\dfrac{\omega L}{v}\right) = 0$

or $\dfrac{\omega}{v}L = \left(n - \tfrac{1}{2}\right)\pi \qquad n = 1,2,3,\ldots$

The natural angular frequencies $\omega_n = \dfrac{\left(n-\frac{1}{2}\right)\pi v}{L} = \dfrac{\left(n-\frac{1}{2}\right)\pi}{L}\sqrt{\dfrac{Y}{\rho}}$

$x = 0$ $x = L$



$n = 1$ $$\omega_1 = \frac{\pi}{2L}\sqrt{\frac{Y}{\rho}}$$

$n = 2$ $$\omega_2 = \frac{3\pi}{2L}\sqrt{\frac{Y}{\rho}}$$

$n = 3$ $$\omega_3 = \frac{5\pi}{2L}\sqrt{\frac{Y}{\rho}}$$

$n = 4$

$n = 5$

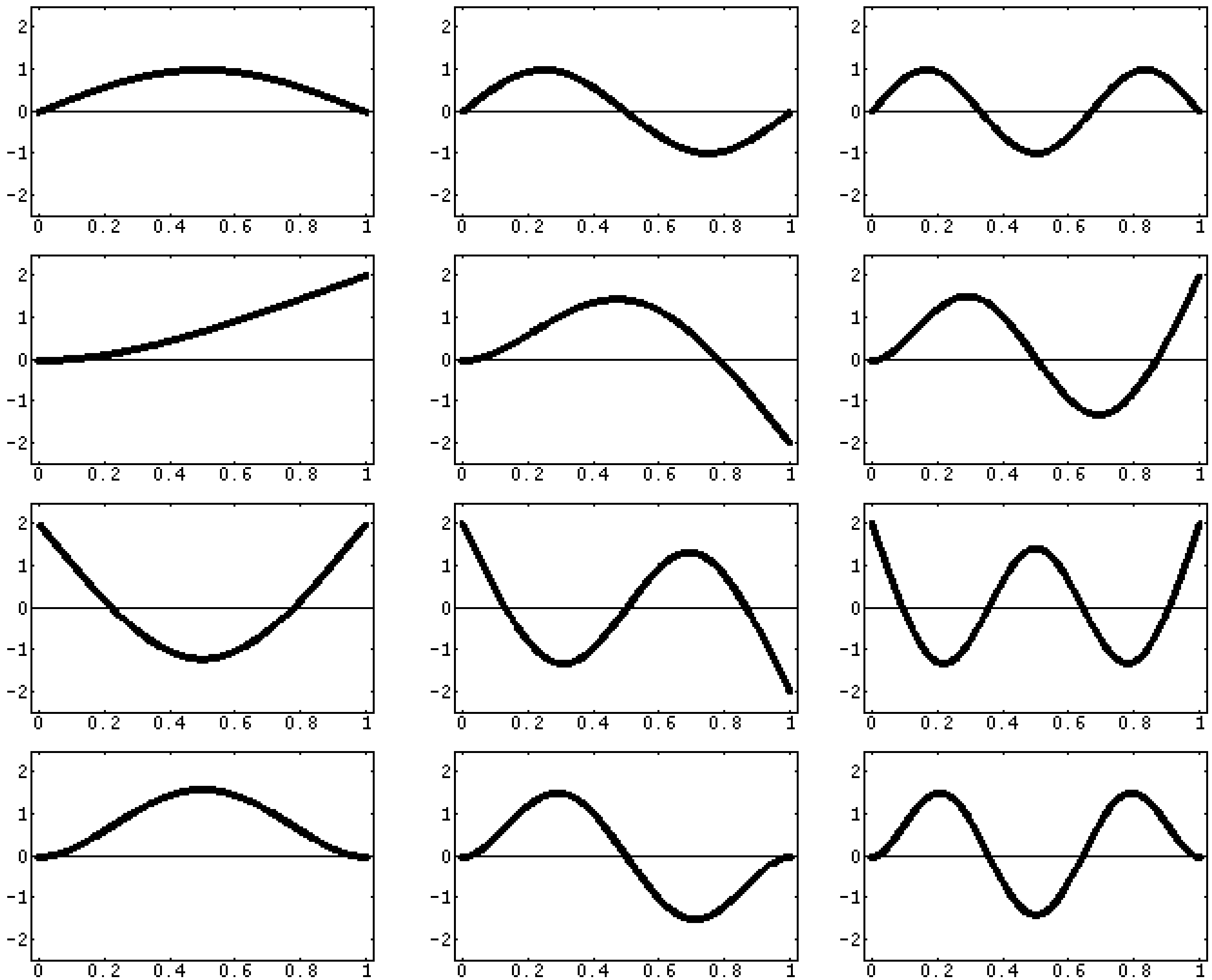
Normal modes for different boundary conditions

Simply supported

Clamped one end

Free both ends

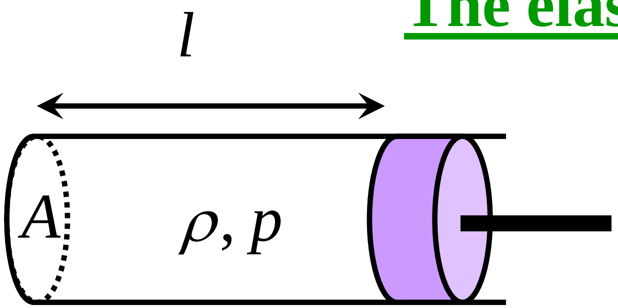
Clamped both ends



$n = 1$ $n = 2$ $n = 3$

The elasticity of a gas

French
page 176



Bulk modulus: $K = -V\dfrac{dp}{dV}$

Kinetic theory of gases: Pressure $p = \frac{1}{3}\rho v_{\text{rms}}^2 = \dfrac{m}{3Al}v_{\text{rms}}^2$

If $E_k = \frac{1}{2}mv_{\text{rms}}^2$ then $p = \dfrac{2}{3A}\dfrac{E_k}{l}$

Now move piston so as to compress the gas

… work done <u>on</u> gas: $\Delta W = -pA\Delta l = \Delta E_k$

Then $\Delta p = \dfrac{2}{3A}\dfrac{\Delta E_k}{l} - \dfrac{2}{3A}\dfrac{\Delta l}{l^2}\Delta E_k = \dfrac{2}{3A}\left(-pA\Delta l\right) - \dfrac{\Delta l}{l}(p) = -\dfrac{5}{3}p\dfrac{\Delta l}{l}$

giving $K_{\text{adiabatic}} = -V\dfrac{\Delta p}{\Delta V} = \dfrac{5}{3}p$

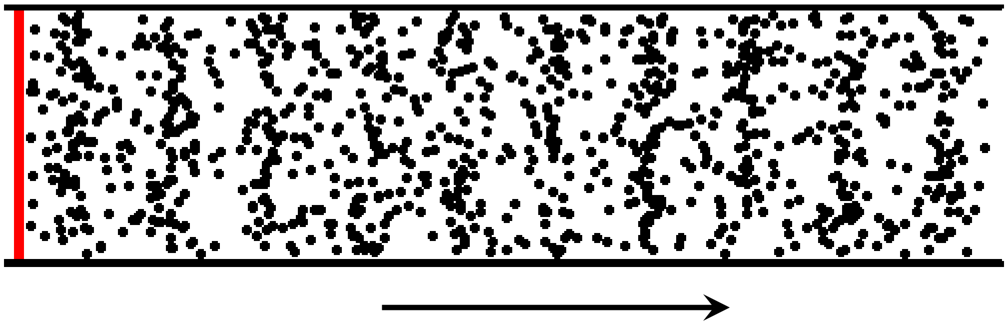
and $v = \sqrt{\dfrac{K}{\rho}} = \sqrt{\dfrac{1.667\,p}{\rho}}$

French
page 174

Sound waves in pipes

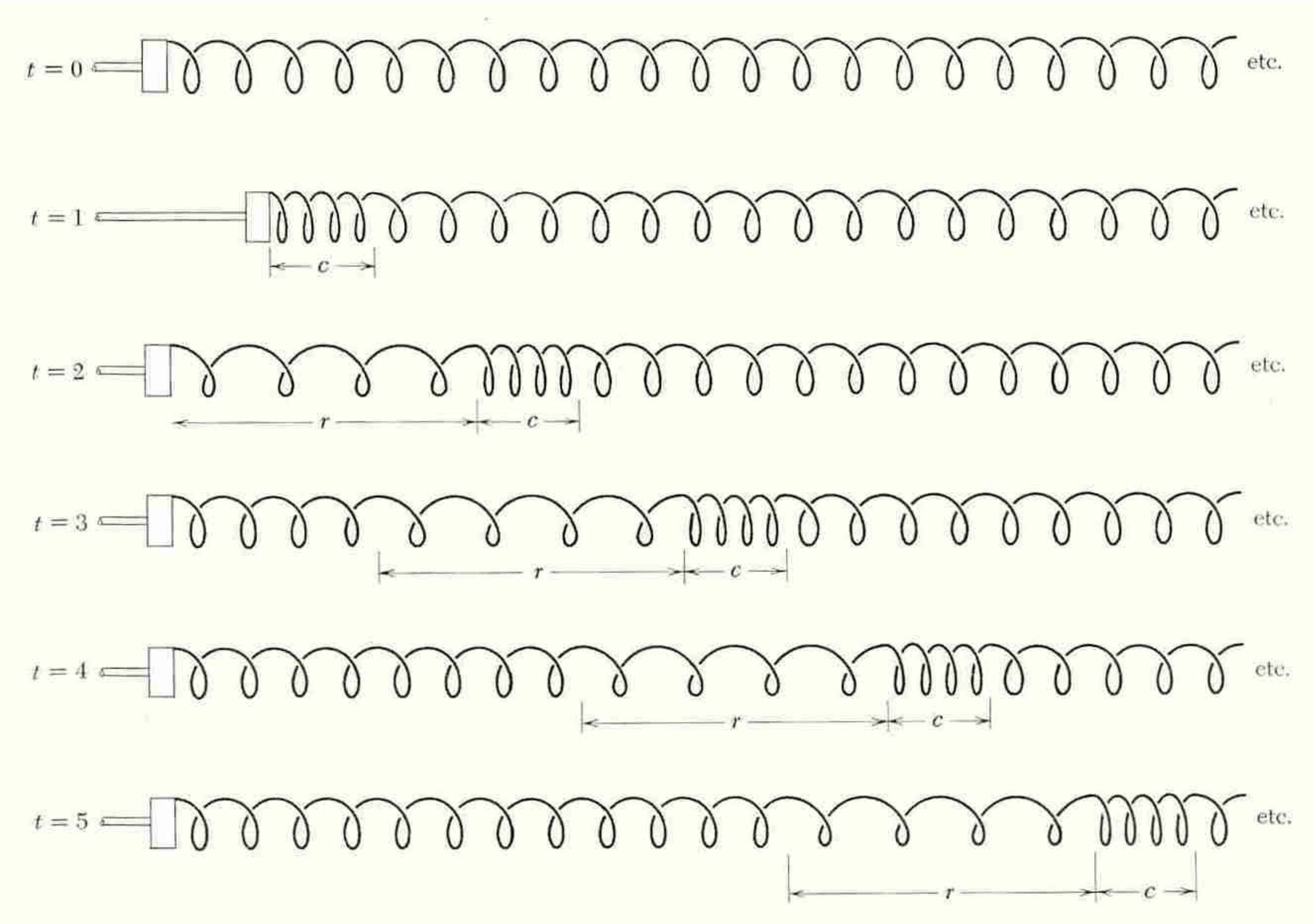
A sound wave consists of a series of compressions and rarefactions of the supporting medium (gas, liquid, solid)

In this wave individual molecules move longitudinally with SHM. Thus a pressure maximum represents regions in which the molecules have approached from both sides, receding from the pressure minima.

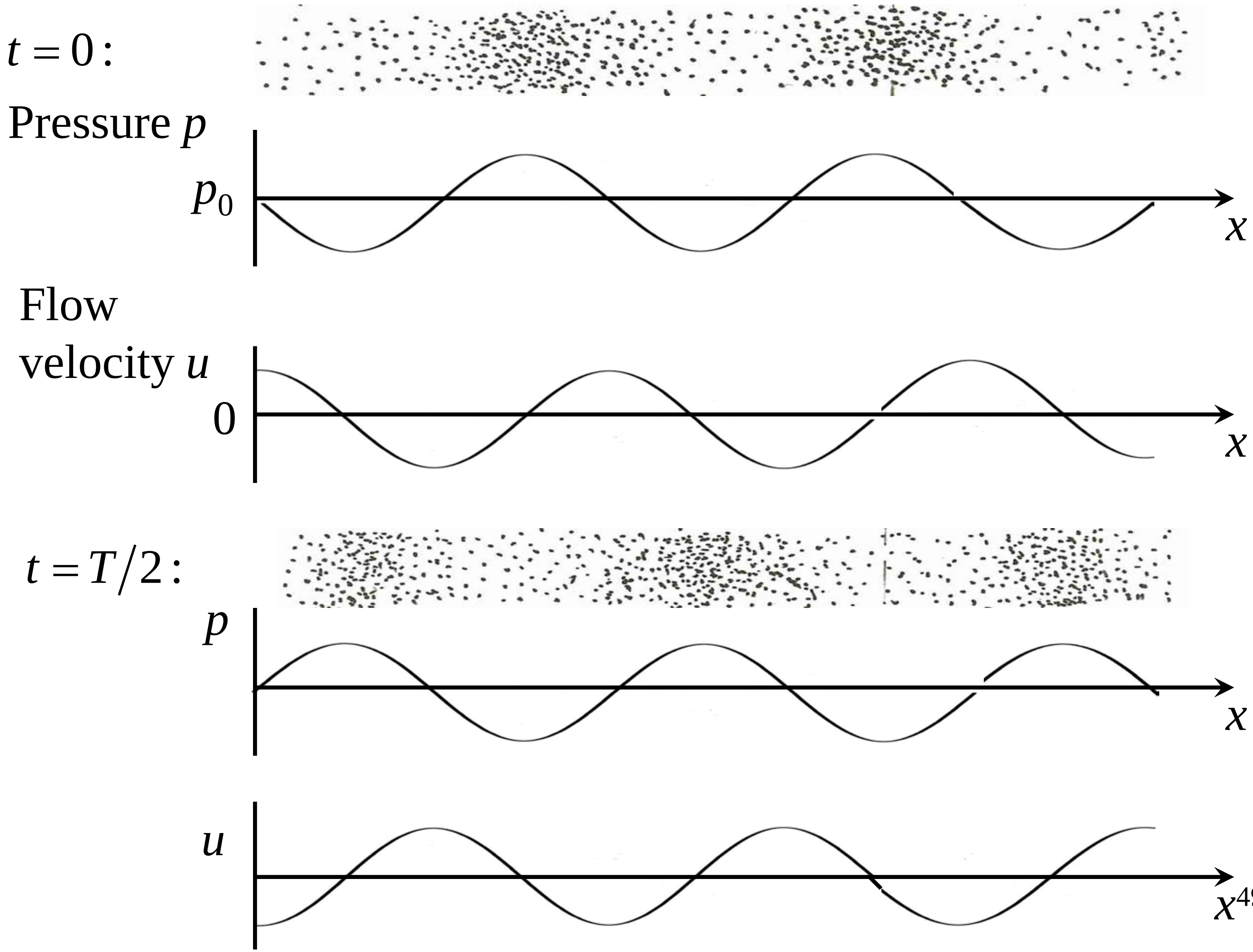


wave propagation

Longitudinal wave on a spring

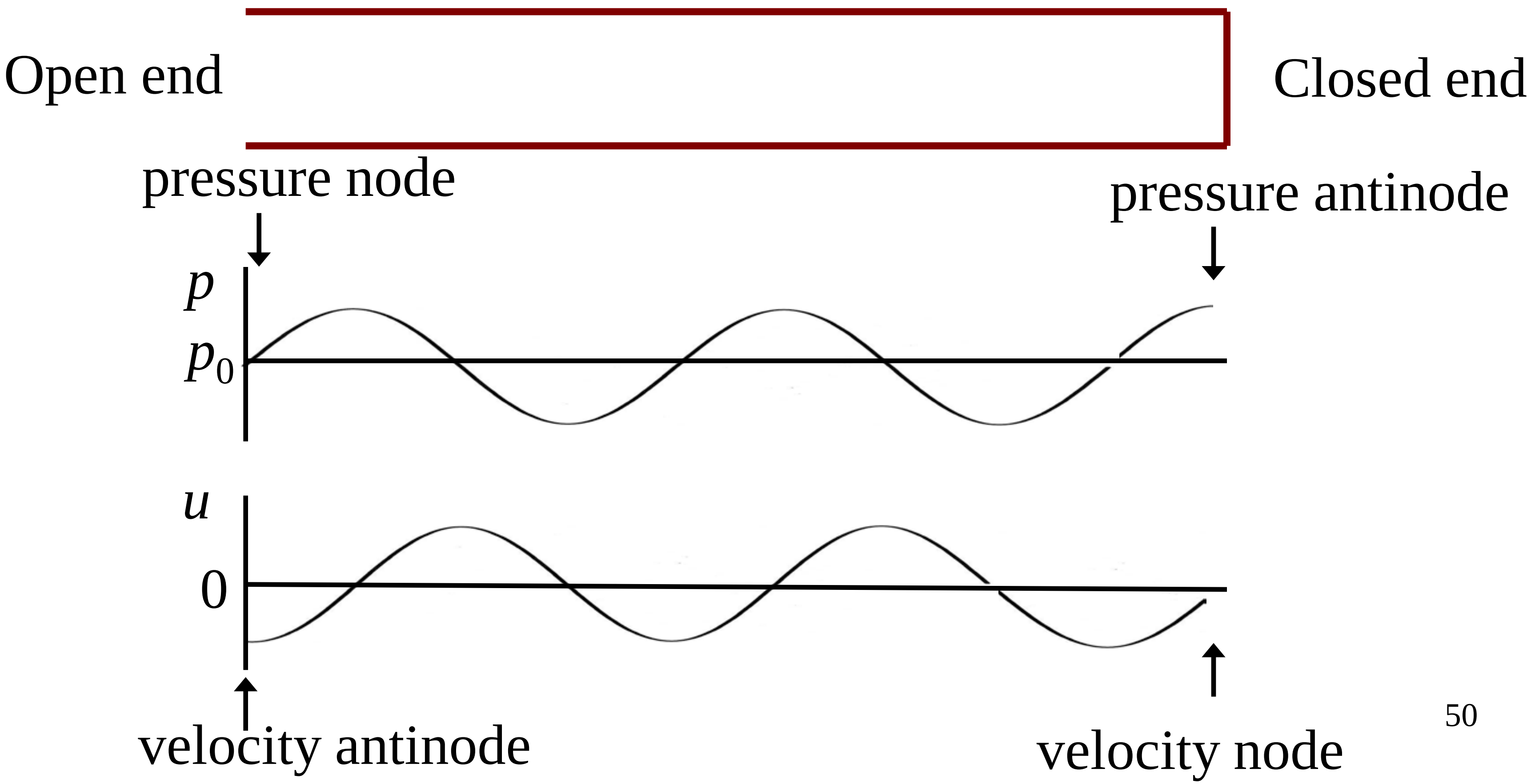


Standing sound waves in pipes



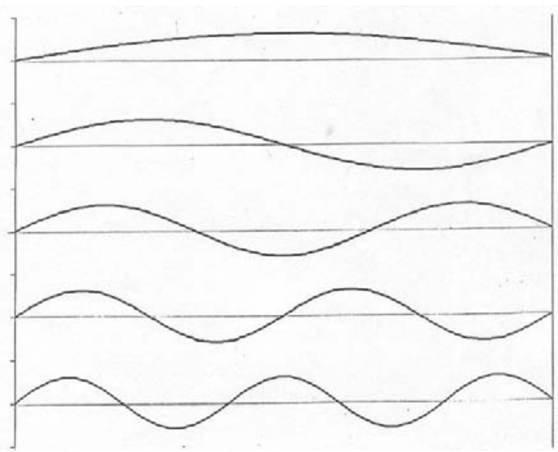
Standing sound waves in pipes …2

Consider a sound wave in a pipe. At the closed end the flow velocity is zero (velocity node, pressure antinode).
At the open end the gas is in contact with the atmosphere, i.e. $p = p_0$ (pressure node and velocity antinode).

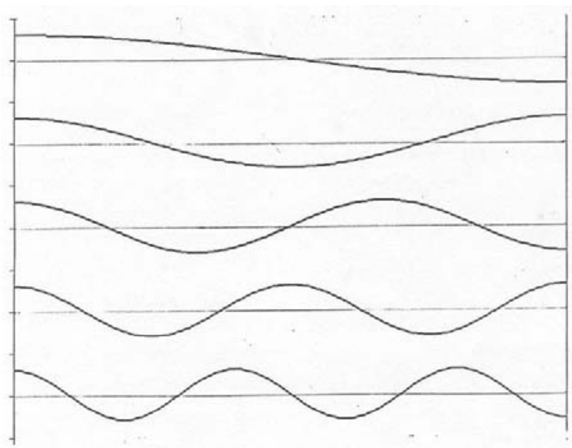


Standing sound waves in pipes …3

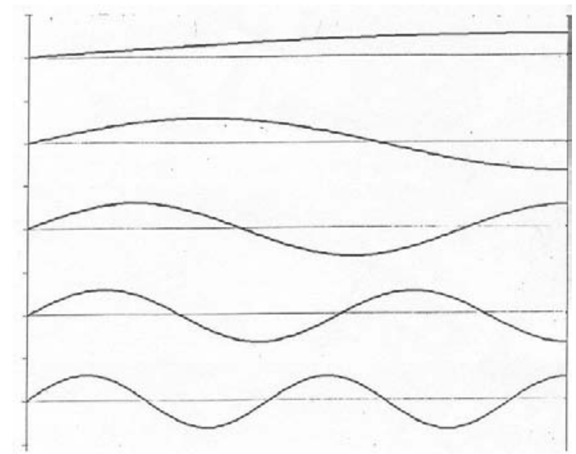
Pipe closed at both ends



Pipe open at both ends



Pipe open at one end



$$L = \frac{n\lambda}{2} = \frac{nv}{2f}$$

$$f = \frac{nv}{2L}$$

$$\omega_n = \frac{n\pi v}{L}$$

$$L = \frac{(2n-1)\lambda}{4} = \frac{(2n-1)}{4}\frac{v}{f}$$

$$f = \frac{(2n-1)v}{4L}$$

$$\omega_n = \frac{(2n-1)\pi v}{2L}$$

Sound

Audible sound is usually a longitudinal compression wave in air to which the eardrum responds.

Velocity of sound (at NTP) ~ 330 m s^{-1}

By considering the transport of energy by a compression wave, can show that $P = 2\pi^2 f^2 \rho A v s_m^2$

… where A is cross sectional area of air column and s_m is maximum displacement of air particle in longitudinal wave

Then intensity = $\dfrac{P}{A} = 2\pi^2 f^2 \rho v s_m^2$ unit: W m^{-2}

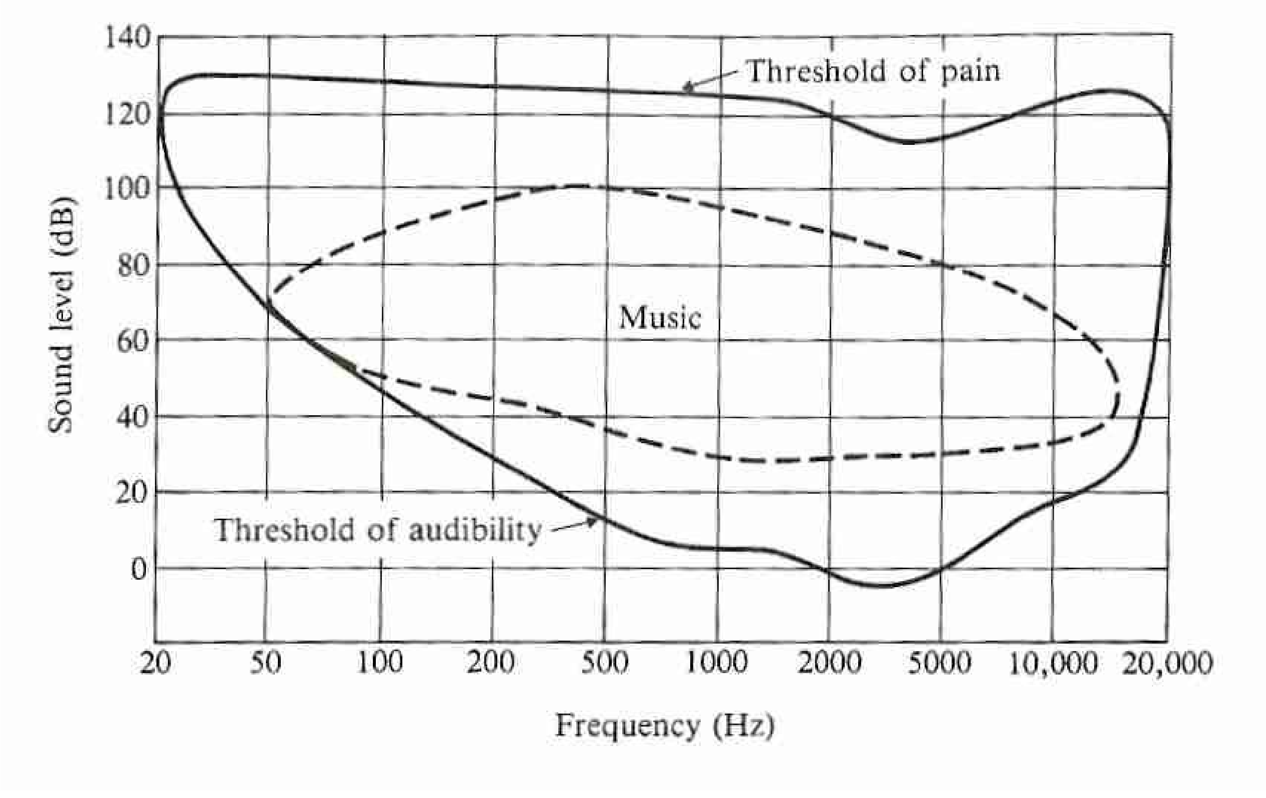
Sound …2

The human ear detects sound from ~10^{-12} W m^{-2} to ~1 W m^{-2}

… use a logarithmic scale for I :

Intensity **level** or "loudness": $\beta = 10\log_{10}\left(\frac{I}{I_0}\right)$ decibels

where I_0 = "reference intensity" = 10^{-12} W m^{-2}



Musical sounds

Waveforms from real musical instruments are complex, and may contain multiple harmonics, different phases, vibrato, ...

Pitch is the characteristic of a sound which allows sounds to be ordered on a scale from high to low (!?). For a pure tone, pitch is determined mainly by the frequency, although sound level may also change the pitch. Pitch is a subjective sensation and is a subject in “psychoacoustics”.

The basic unit in most musical scales is the **octave**. Notes judged an octave apart have frequencies nearly (not exactly) in the ratio 2:1. Western music normally divides the octave into 12 intervals called **semitones** ... which are given note names (A through G with sharps and flats) and designated on musical scales.

Musical sounds ...2

Timbre is used to denote “tone quality” or “tone colour” of a sound and may be understood as that attribute of auditory sensation whereby a listener can judge that two sounds are dissimilar using any criteria other than pitch, loudness or duration. Timbre depends primarily on the spectrum of the stimulus, but also on the waveform, sound pressure and temporal characteristics of the stimulus.

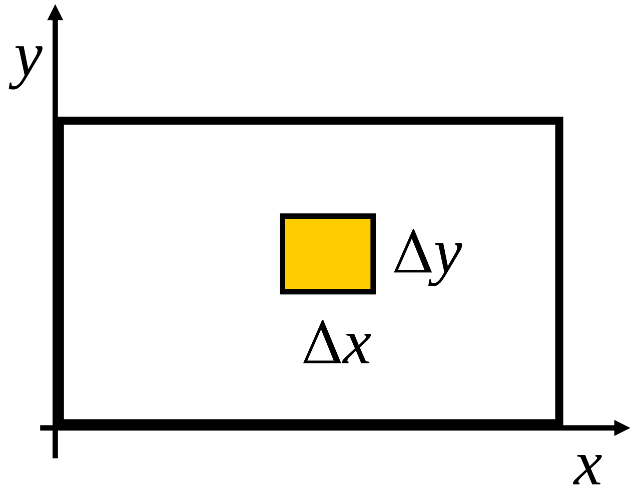
One subjective rating scale for timbre (von Bismarck, 1974)

dull	———————	sharp
compact	———————	scattered
full	———————	empty
colourful	———————	colourless

Two dimensional systems

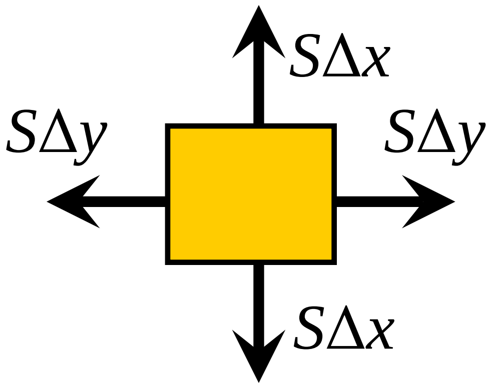
French
page 181

Consider an elastic membrane clamped at its edges …



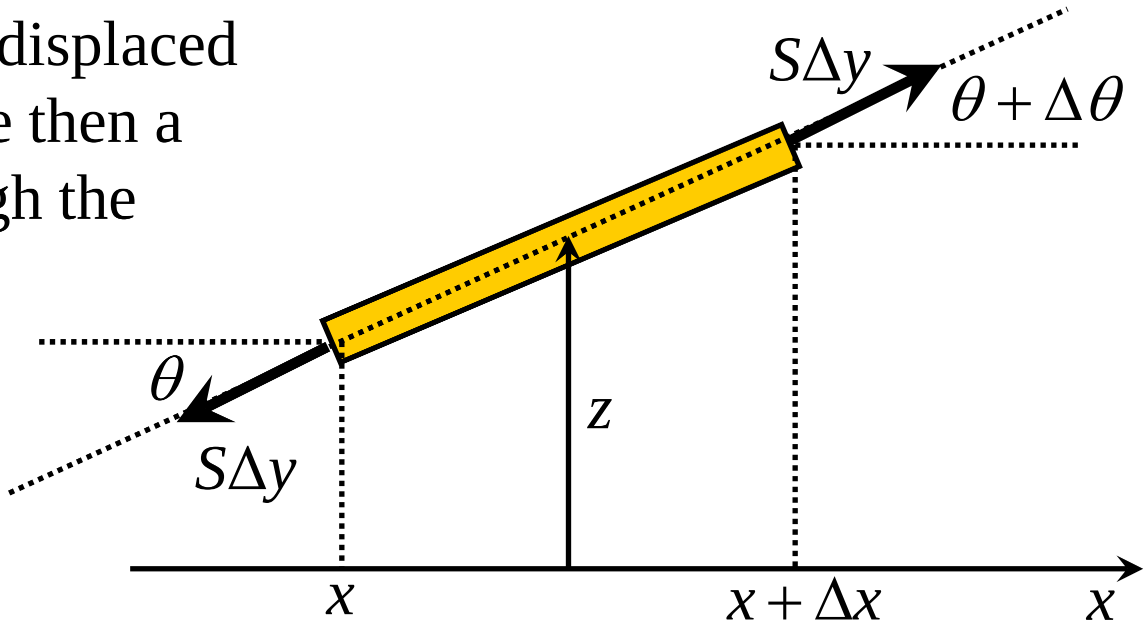
… the membrane has mass per unit area σ , and a surface tension S which gives a force $S\Delta l$ perpendicular to a length Δl in the surface …

The forces on the shaded portion are …



Two dimensional systems …2

If the membrane is displaced from the $z = 0$ plane then a cross section through the shaded area shows:



… looks exactly like the case of the stretched string.

The transverse force on the element will be $S\Delta y \dfrac{\partial^2 z}{\partial x^2}\Delta x$

And if we looked at a cross section perpendicular to this … the transverse force will be $S\Delta x \dfrac{\partial^2 z}{\partial y^2}\Delta y$

Two dimensional systems …3

The mass of the element is $\sigma\Delta x\Delta y$.

Thus $$S\Delta y\frac{\partial^2 z}{\partial x^2}\Delta x + S\Delta x\frac{\partial^2 z}{\partial y^2}\Delta y = \sigma\Delta x\Delta y\frac{\partial^2 z}{\partial t^2}$$

or $$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \left(\frac{\sigma}{S}\right)\frac{\partial^2 z}{\partial t^2}$$

… a two dimensional wave equation

… with the wave velocity $v = \sqrt{\dfrac{S}{\sigma}}$

Two dimensional systems …4

Look for normal mode solutions of the form:

$$z(x,y,t) = f(x)g(y)\cos\omega t$$

$$\left.\begin{aligned}
\frac{\partial z^2}{\partial x^2} &= \frac{d^2 f}{dx^2} g(y)\cos\omega t \\
\frac{\partial z^2}{\partial y^2} &= \frac{d^2 g}{dy^2} f(x)\cos\omega t \\
\frac{\partial z^2}{\partial t^2} &= f(x)g(y)\left(-\omega^2\cos\omega t\right)
\end{aligned}\right\}
\quad
\begin{aligned}
&\frac{d^2 f}{dx^2} g(y)\cos\omega t + \frac{d^2 g}{dy^2} f(x)\cos\omega t = \\
&\qquad -\frac{\omega^2}{v^2} f(x)g(y)\cos\omega t
\end{aligned}$$

i.e. $\quad \dfrac{1}{f}\dfrac{d^2 f}{dx^2} + \dfrac{1}{g}\dfrac{d^2 g}{dy^2} = -\dfrac{\omega^2}{v^2}$

In a similar fashion to the 1D case, find …

$$f(x) = A_{n_1}\sin\left(\frac{n_1 \pi x}{L_x}\right) \qquad \text{and} \qquad g(y) = B_{n_2}\sin\left(\frac{n_2 \pi y}{L_y}\right)$$

Two dimensional systems …5

…then $$z(x,y,t) = C_{n_1 n_2} \sin\left(\frac{n_1 \pi x}{L_x}\right) \sin\left(\frac{n_2 \pi y}{L_y}\right) \cos \omega_{n_1,n_2} t$$

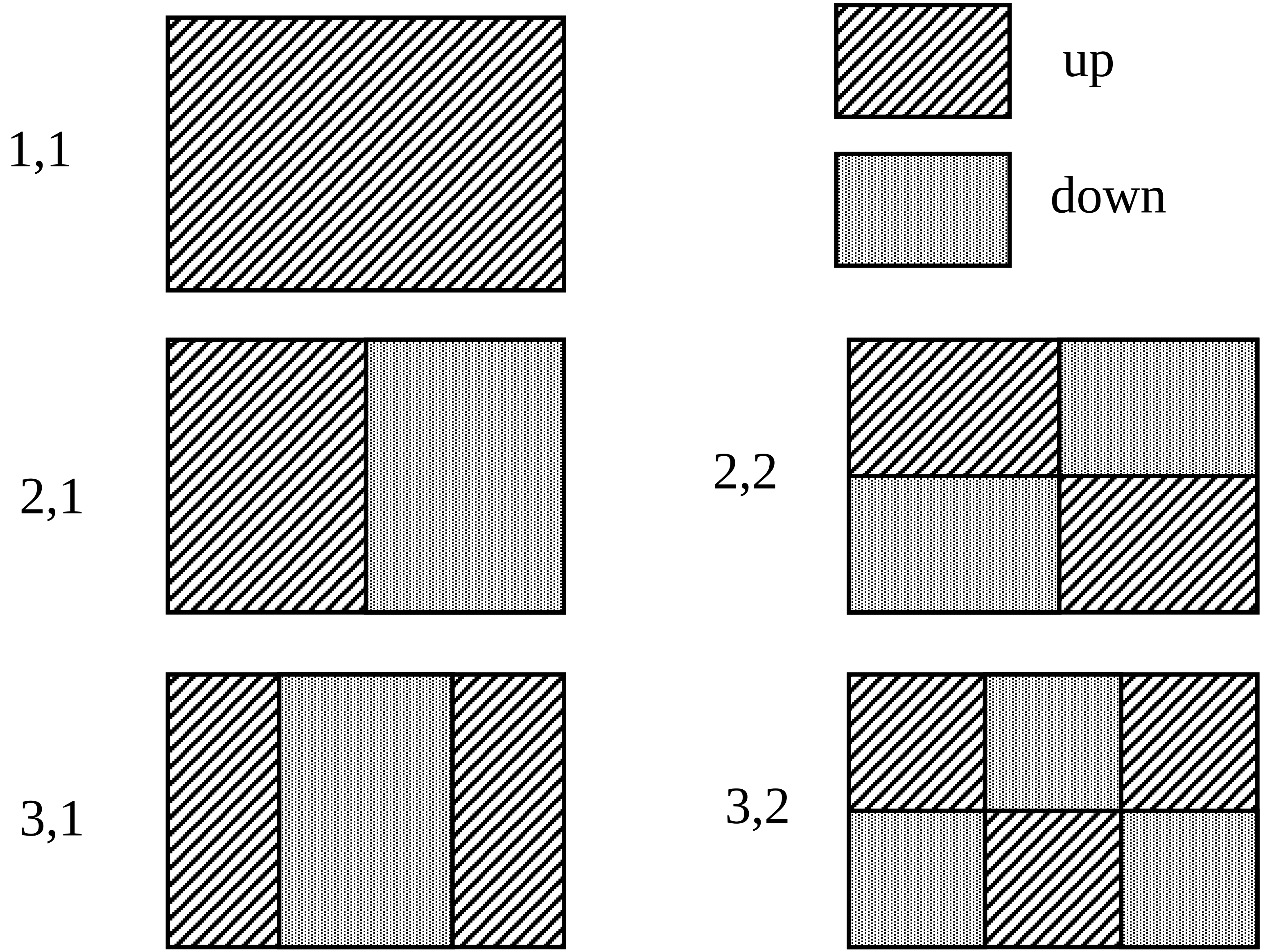
where the normal mode frequencies are

$$\omega_{n_1,n_2} = \sqrt{\left(\frac{n_1 \pi v}{L_x}\right)^2 + \left(\frac{n_2 \pi v}{L_y}\right)^2}$$

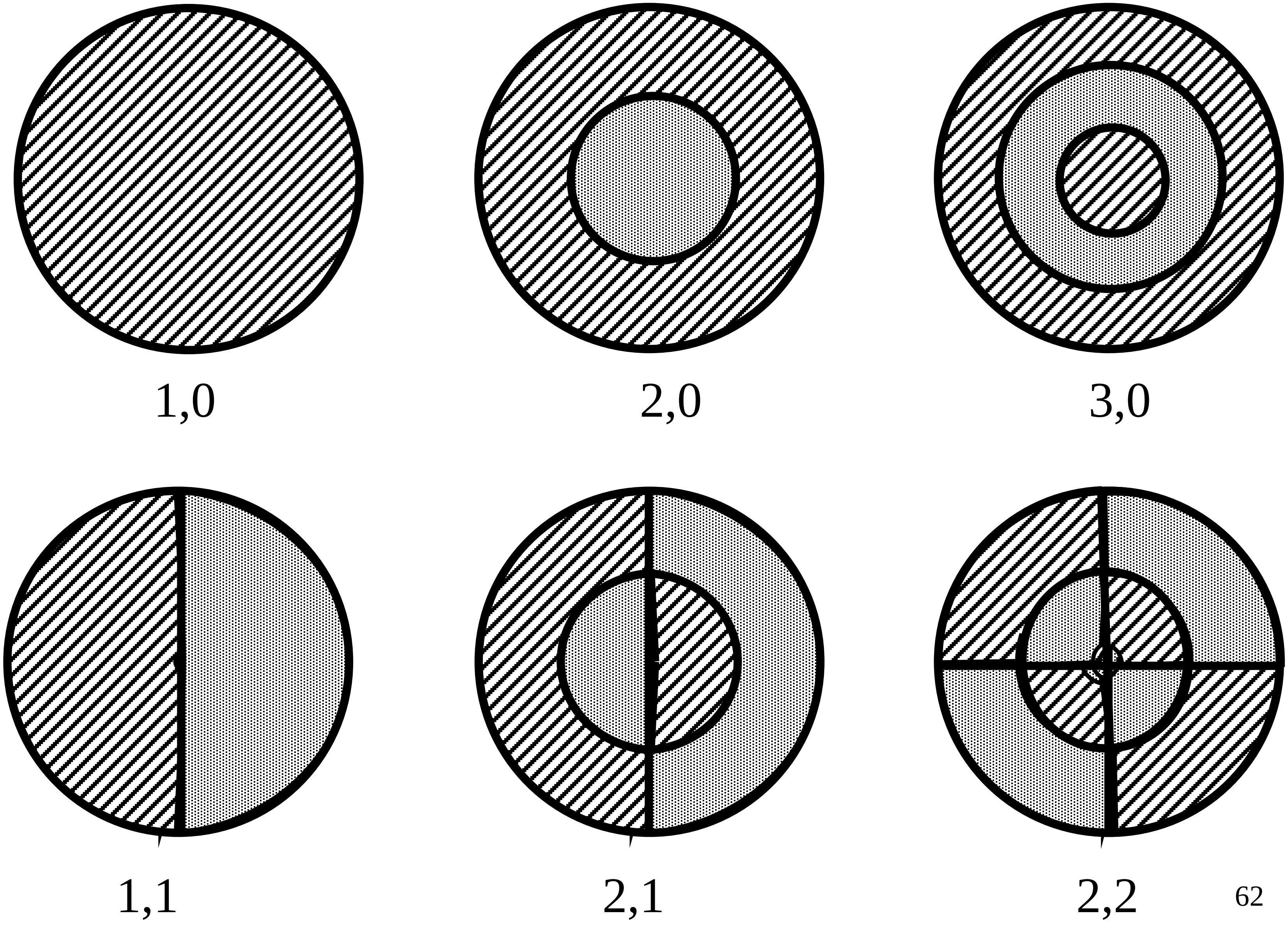
e.g. for a membrane having sides 1.05L and 0.95L

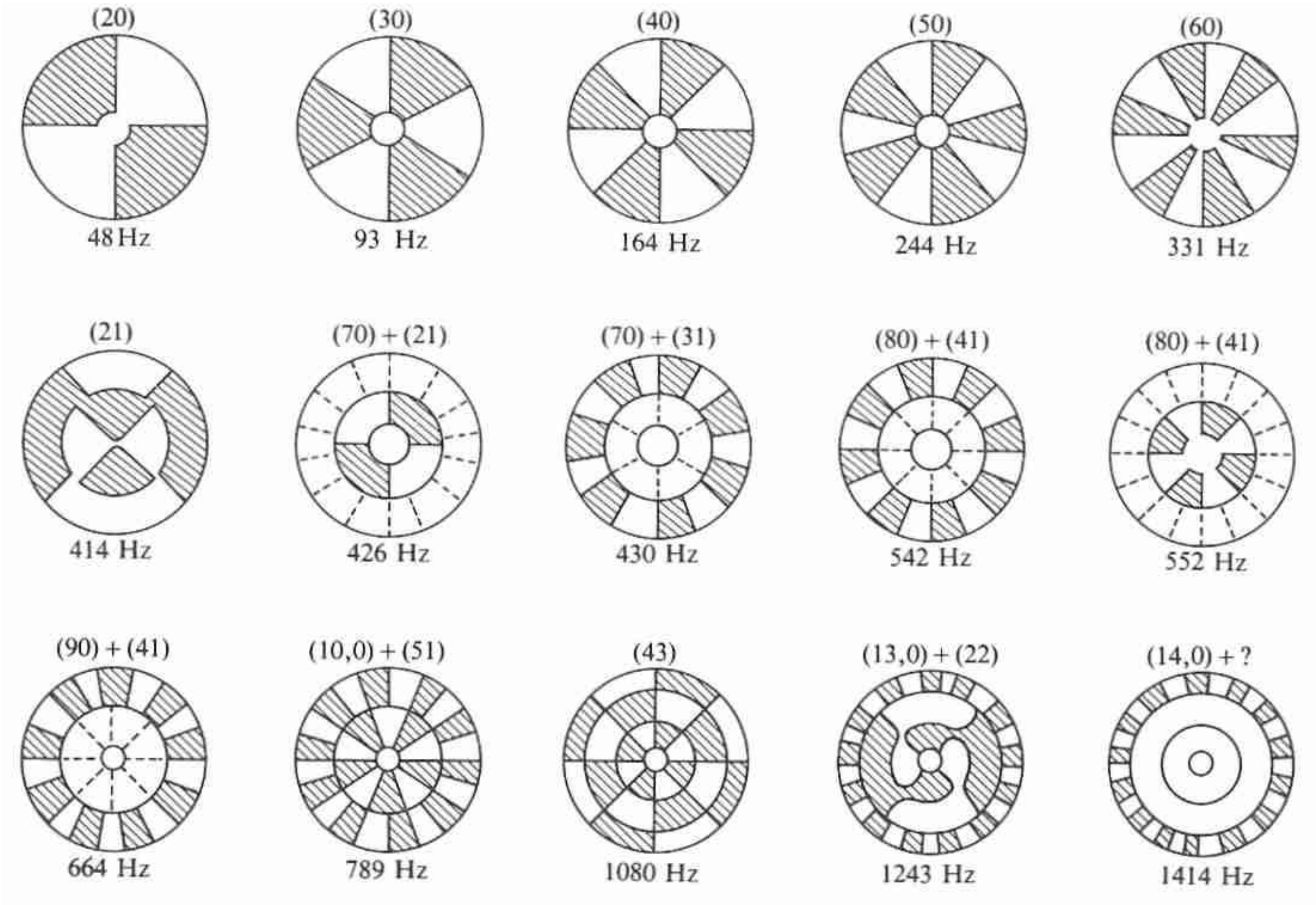
then $$\omega_{n_1,n_2} = \frac{\pi v}{L} \sqrt{\left(\frac{n_1}{1.05}\right)^2 + \left(\frac{n_2}{0.95}\right)^2}$$

Normal modes of a rectangular membrane



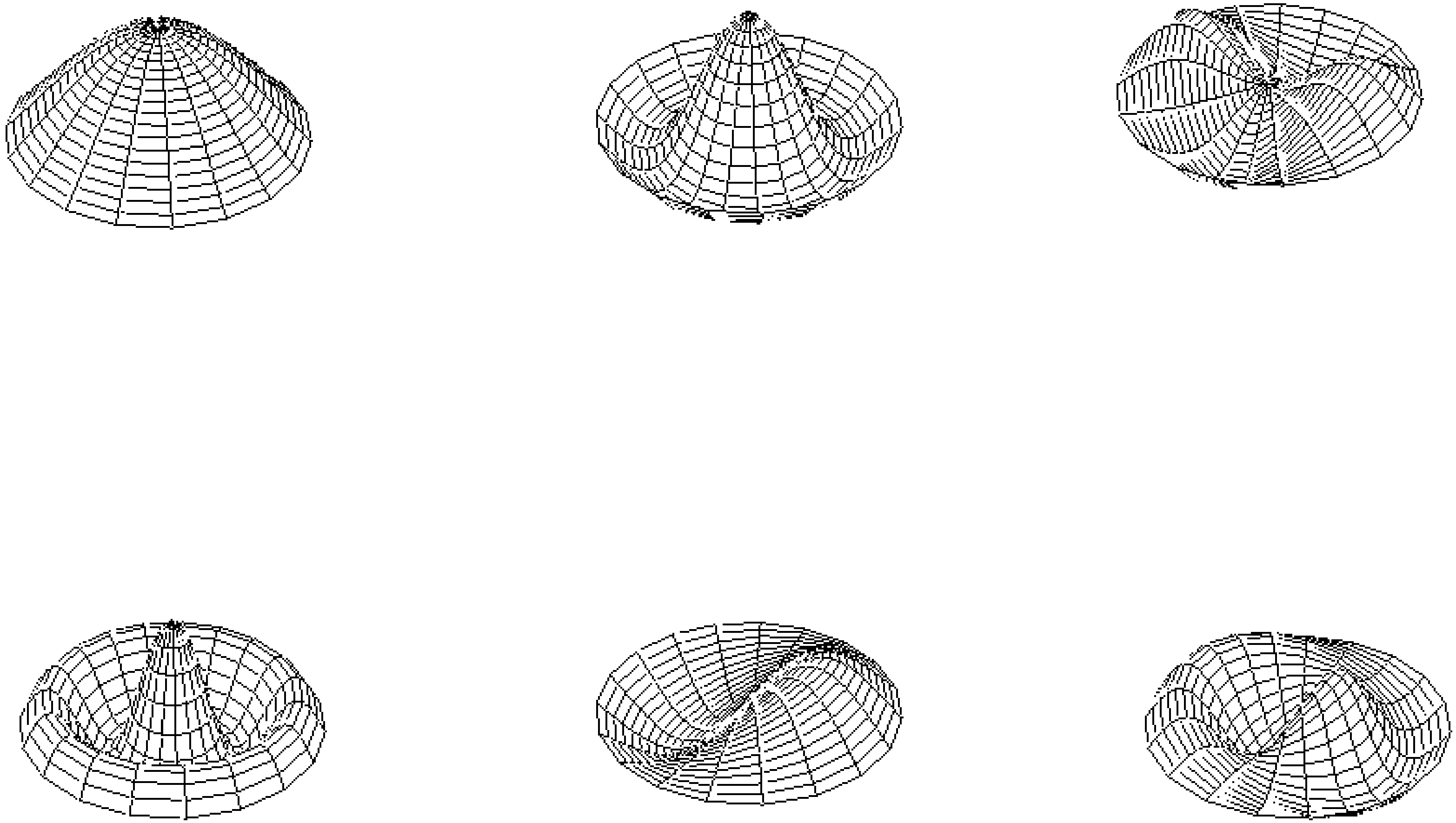
Normal modes of a circular membrane



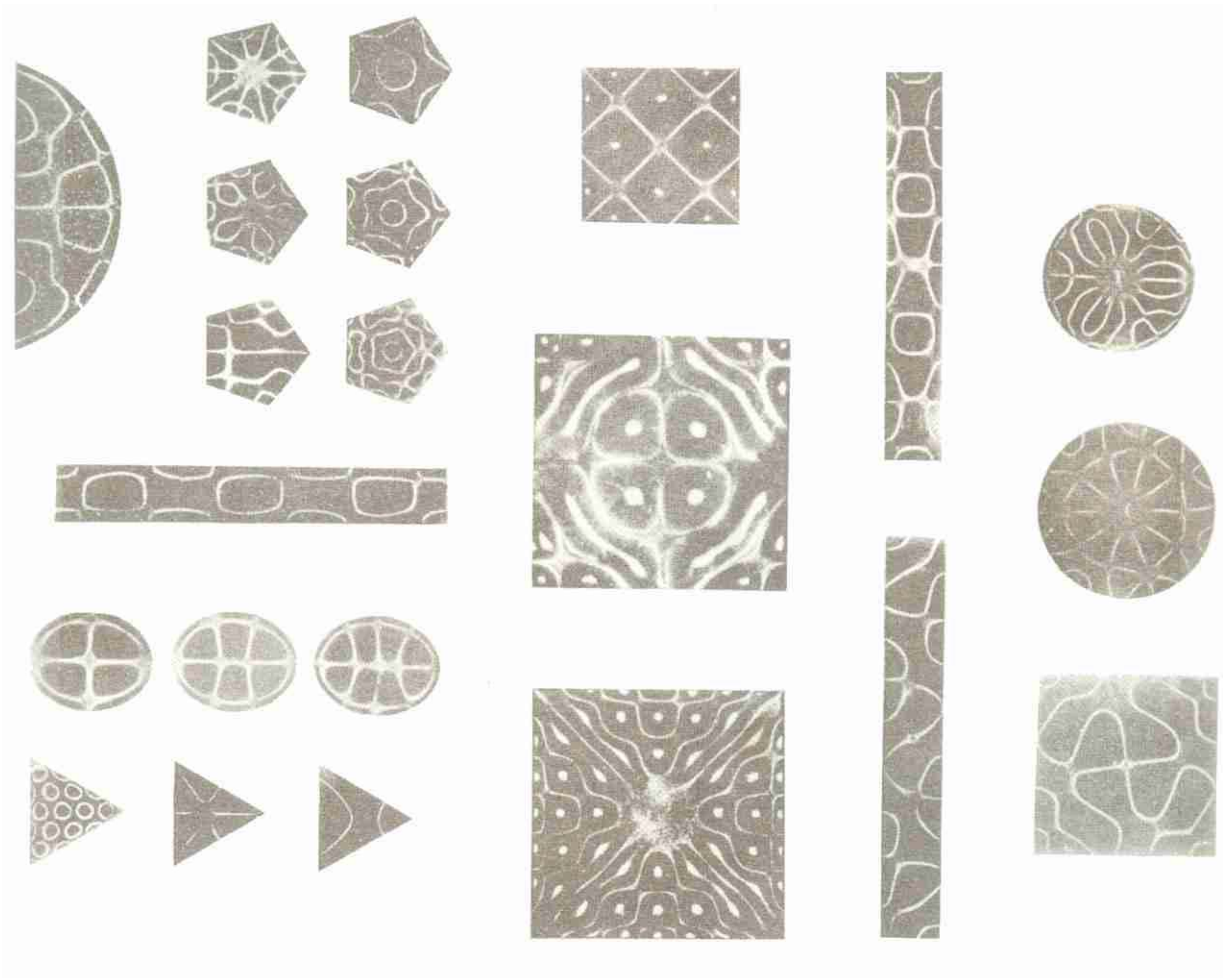


Modes of vibration of a 38 cm cymbal. The first 6 modes resemble those of a flat plate ... but after that the resonances tend to be combinations of two or more modes.

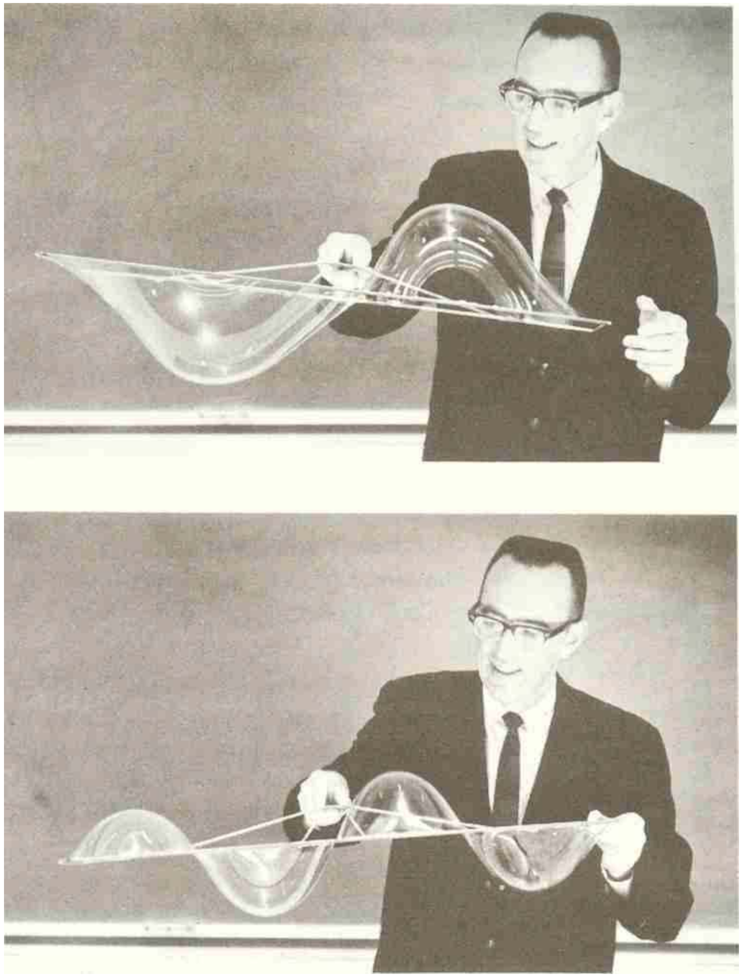
Normal modes of a circular drum

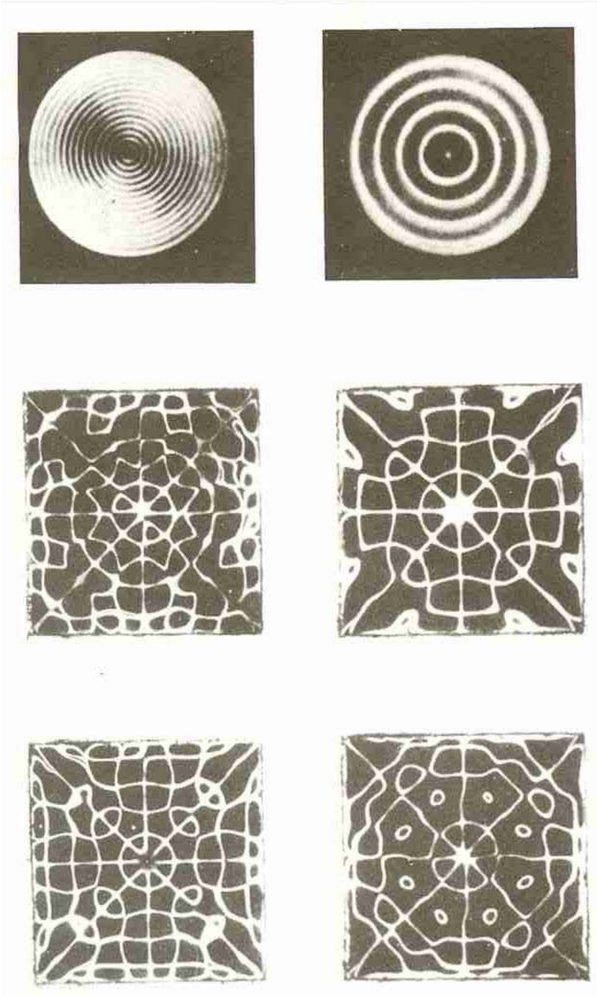


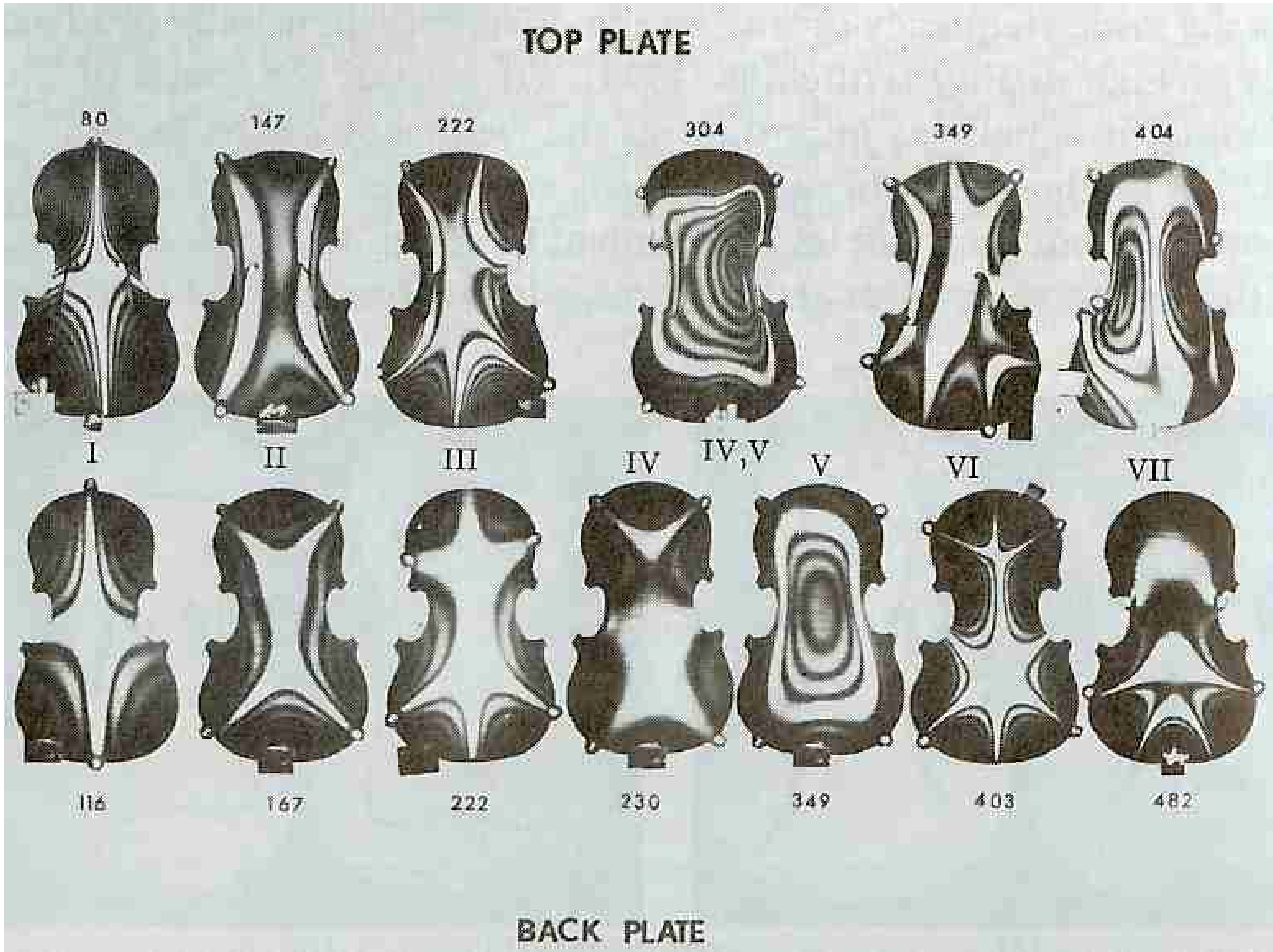
Chladni plates



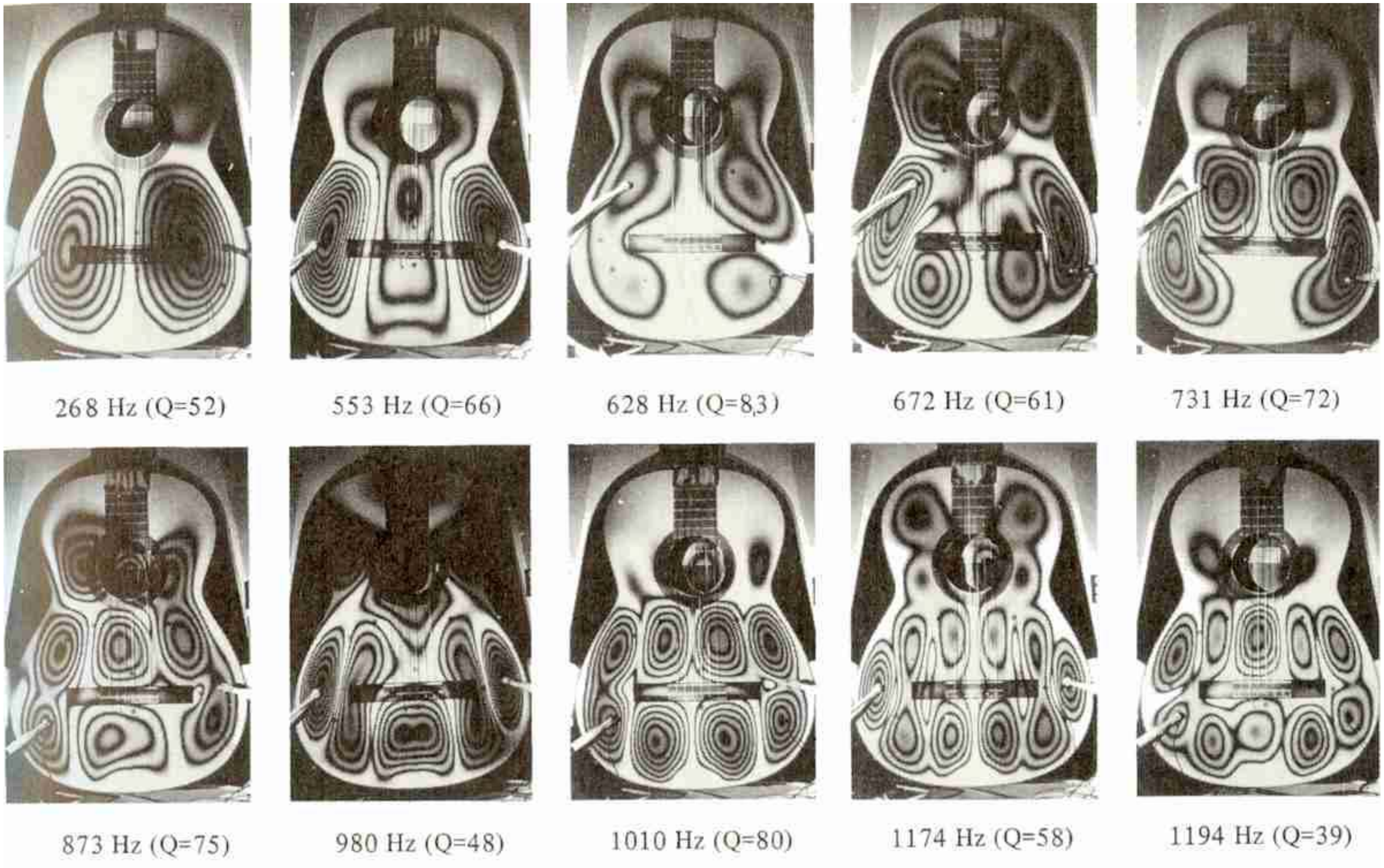
Soap films



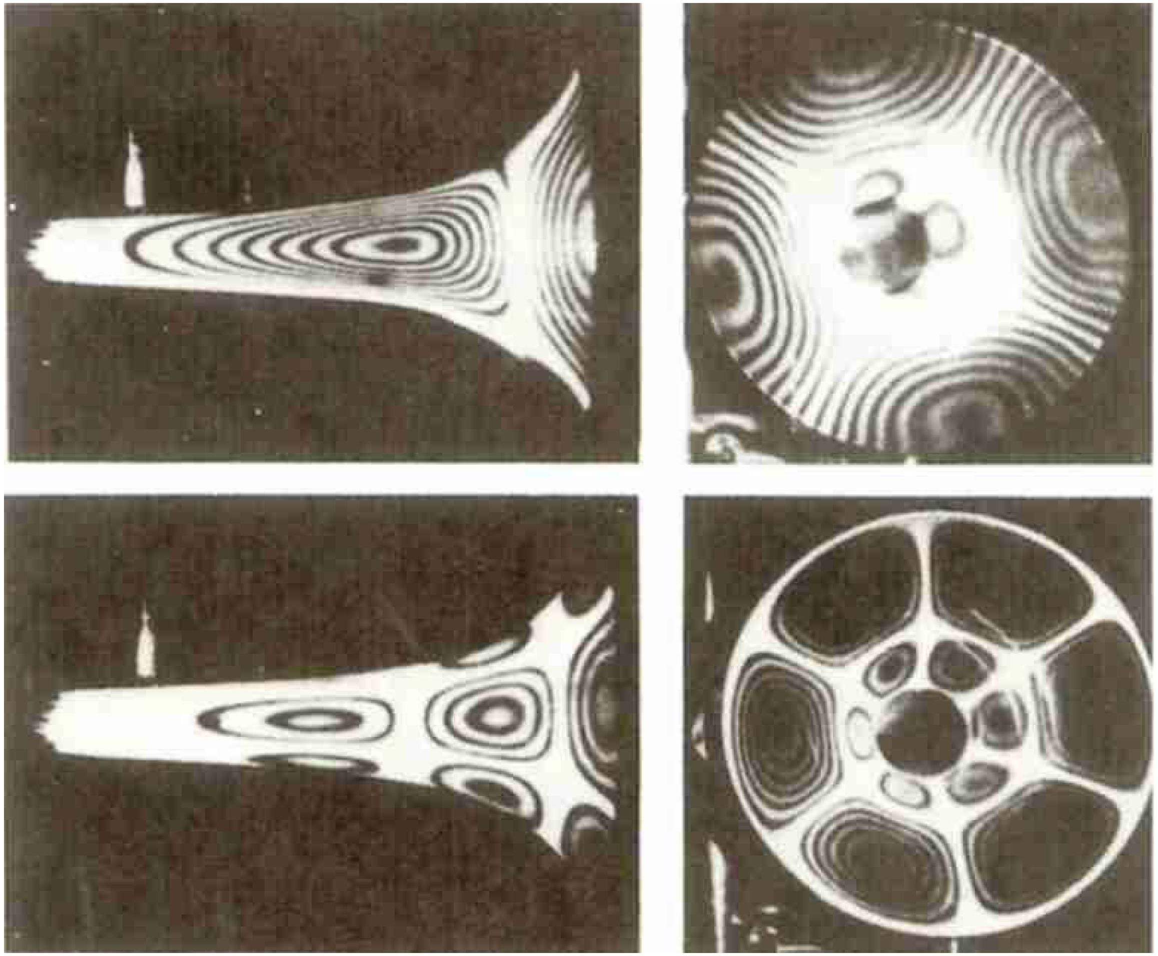




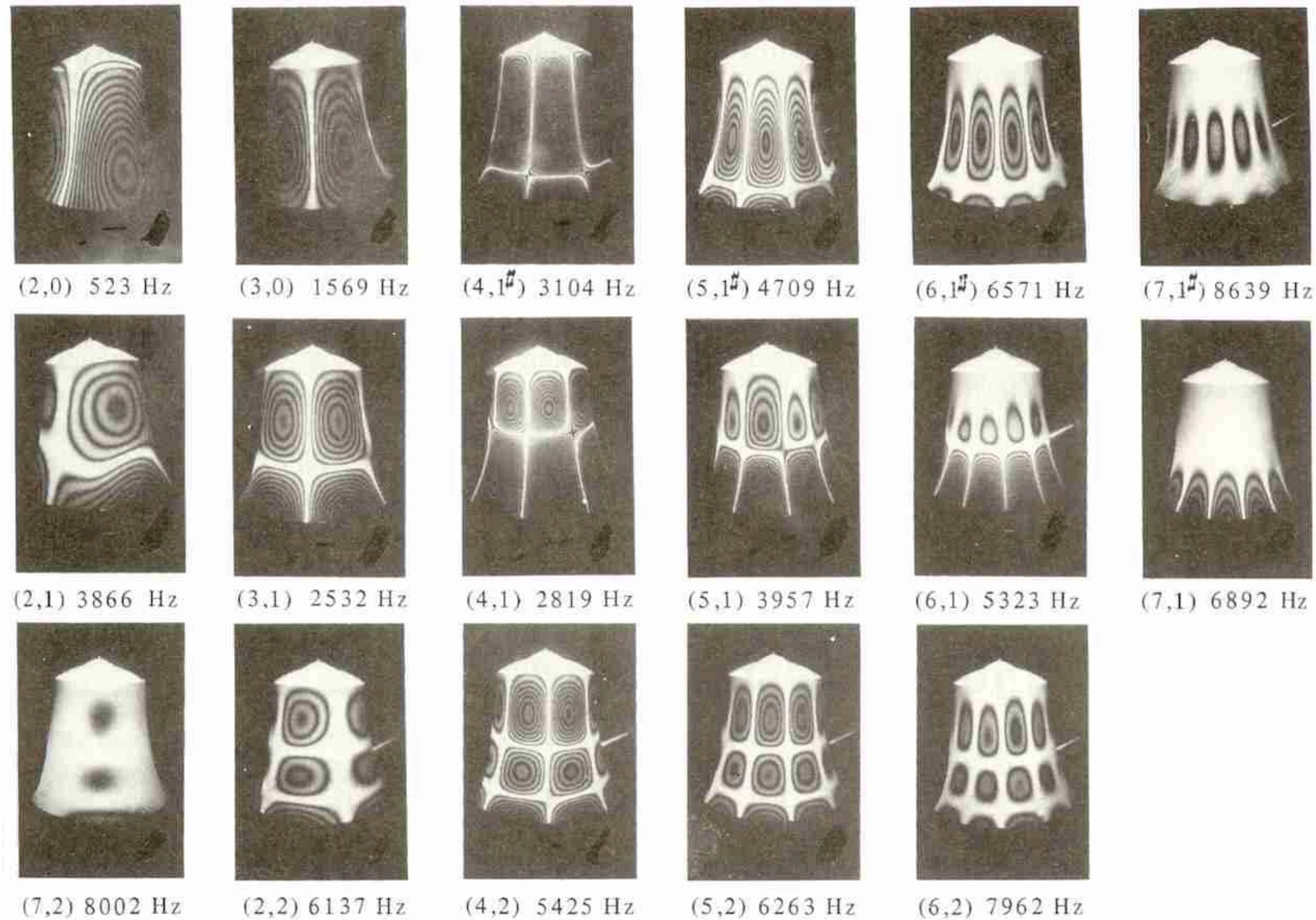
Holographic interferograms of the top and bottom plates of a violin at several resonances.



Holographic interferograms of a classical guitar top plate at several resonances.

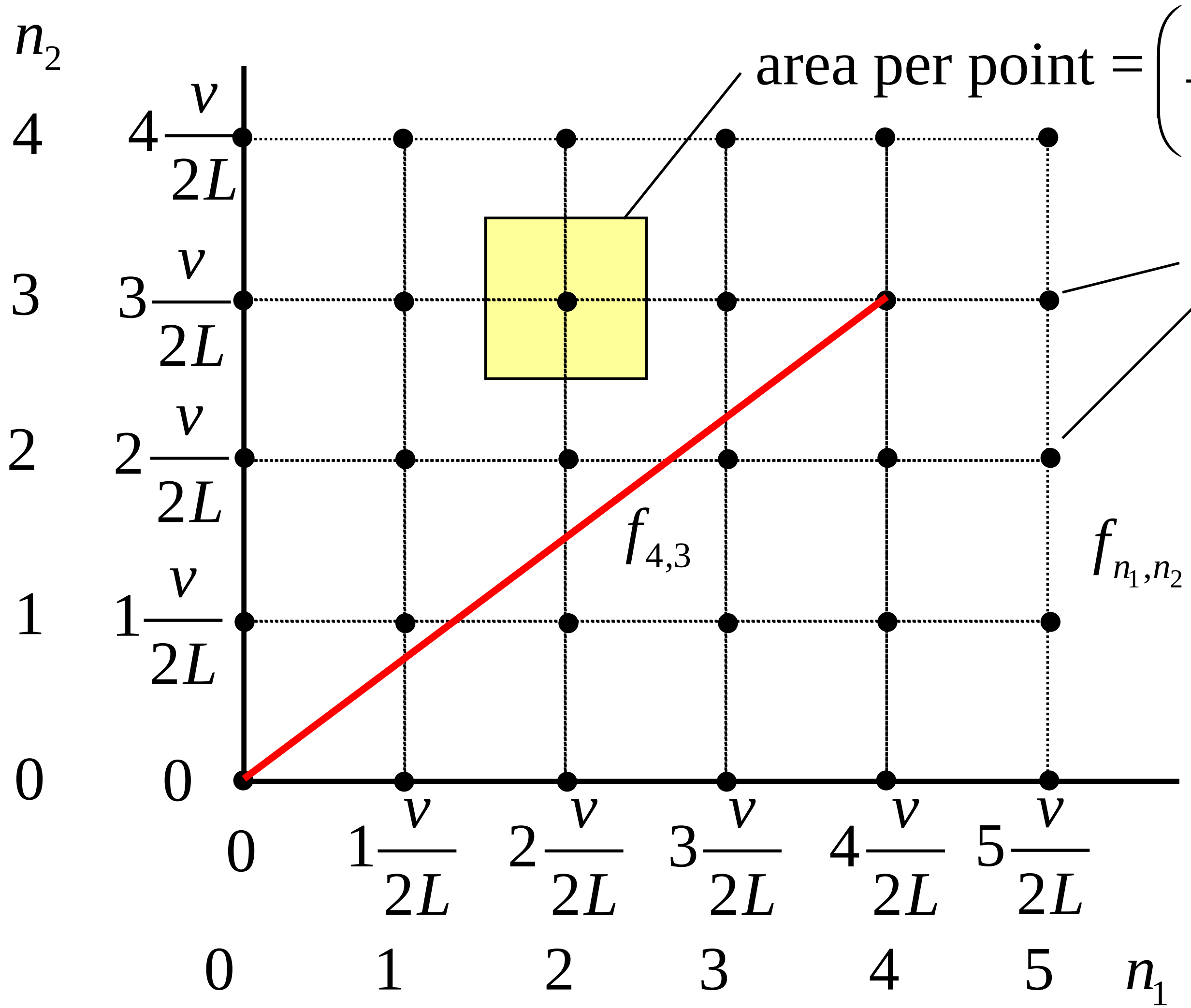


Holographic interferograms showing the vibrations of a 0.3 mm thick trombone driven acoustically at 240 and 630 Hz.



Time-average hologram interferograms of inextensional modes in a C_5 handbell

Normal modes of a square membrane



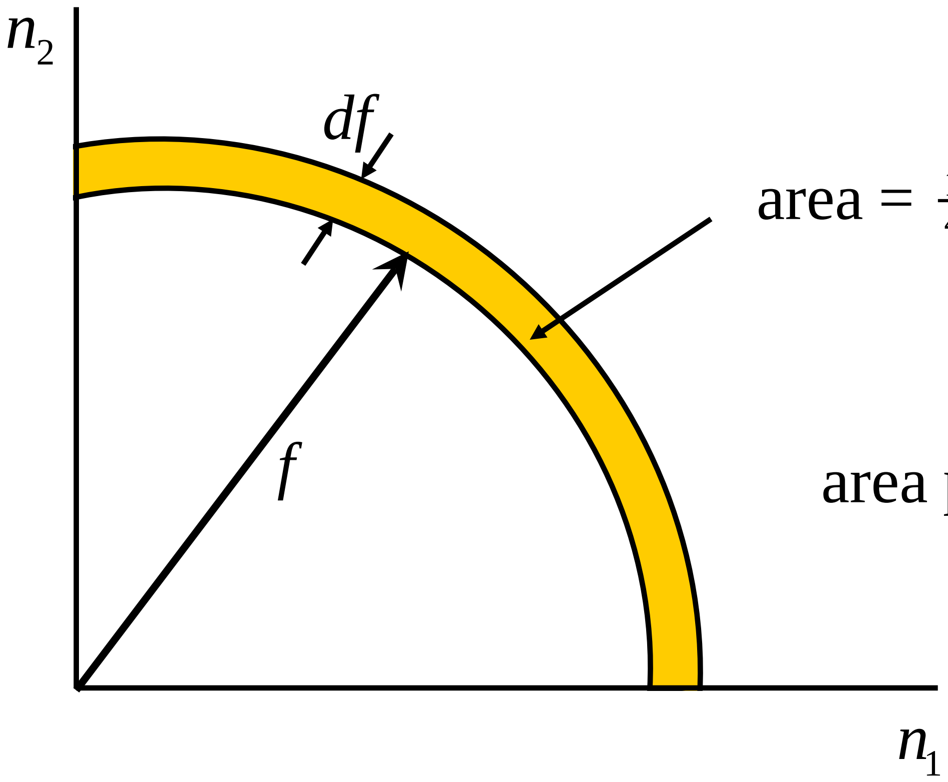
area per point $= \left(\frac{v}{2L}\right)^2$

one point per normal mode

$$f_{n_1,n_2} = \sqrt{\left(\frac{n_1 v}{2L}\right)^2 + \left(\frac{n_2 v}{2L}\right)^2}$$

Normal modes having the same frequency are said to be **degenerate**

Normal modes of a square membrane … for large n_1 and n_2



area = $\frac{1}{4}(2\pi f)df$

area per mode = $\left(\frac{v}{2L}\right)^2$

Number of modes with
frequencies between f and $(f + df) = \frac{1}{4}(2\pi f)df\left(\frac{2L}{v}\right)^2$

$$= \frac{2\pi L^2 f \; df}{v^2}$$

French
page 188

Three dimensional systems

Consider some quantity Ψ which depends on x , y , z and t , e.g. the density of air in a room.

In three dimensions: $$\frac{\partial^2\Psi}{\partial x^2}+\frac{\partial^2\Psi}{\partial y^2}+\frac{\partial^2\Psi}{\partial z^2}=\frac{1}{v^2}\frac{\partial^2\Psi}{\partial t^2}$$

which can be written: $$\nabla^2\Psi=\frac{1}{v^2}\frac{\partial^2\Psi}{\partial t^2}$$

The solutions for a rectangular enclosure:

$$\omega_{n_1,n_2,n_3}=\sqrt{\left(\frac{n_1\pi v}{L_x}\right)^2+\left(\frac{n_2\pi v}{L_y}\right)^2+\left(\frac{n_3\pi v}{L_z}\right)^2}$$

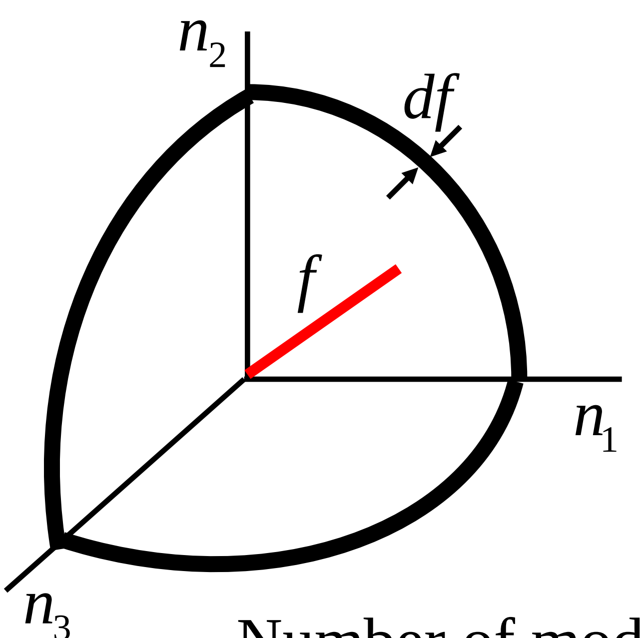
…and for a cube: $$\omega_{n_1,n_2,n_3}=\frac{\pi v}{L}\sqrt{n_1^2+n_2^2+n_3^2}$$

Three dimensional systems …2

How many modes are there with frequencies in the range f and $(f + df)$ … ?

Set up an imaginary cubic lattice with spacing $v/2L$

… and consider positive frequencies only.



$$\text{Volume of shell} = \tfrac{1}{8}(4\pi f^2)df$$

$$\text{Volume per mode} = \left(\frac{v}{2L}\right)^3$$

Number of modes with frequencies between f and $(f + df)$ $= \tfrac{1}{8}(4\pi f^2)df\left(\dfrac{2L}{v}\right)^3$

$$= \frac{4\pi L^3 f^2\ df}{v^3}$$

Three dimensional systems …3

Number of modes with frequencies between f and $(f + df)$ $= \dfrac{4\pi V\ f^2\ df}{v^3}$

… holds for any volume V … provided its dimensions are much greater than the wavelengths involved.

… need to multiply by a factor of 2 when dealing with electromagnetic radiation (2 polarization states) …

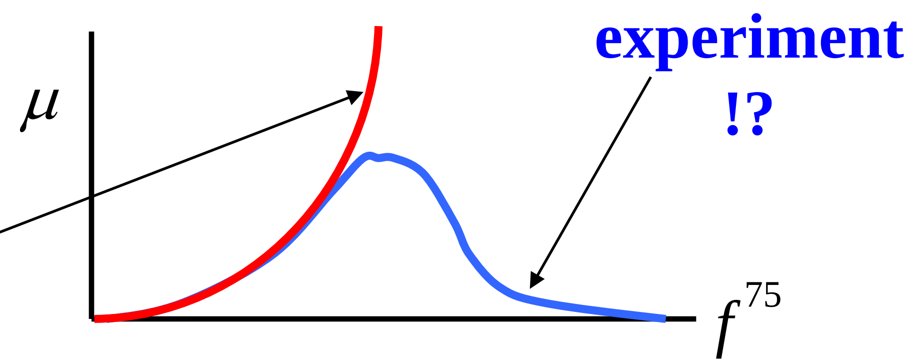
"Ultraviolet catastrophe" for blackbody radiation …

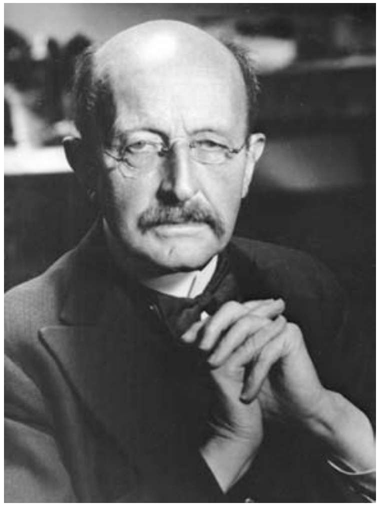
Equipartition theorem: in thermal equilibrium each mode has an average energy $\tfrac{1}{2}k_B T$ in each of its two energy stores

Hence, energy density of radiation in a cavity:

$$\mu df = 2\frac{4\pi V\ f^2\ df}{c^3}2\left(\tfrac{1}{2}kT\right)$$

or $$\mu = \frac{8\pi\ f^2}{c^3}kT$$





Planck was able to show, effectively by assuming that energy was emitted an absorbed in quanta of energy hf , that the average energy of a cavity mode was not kT but

$$\frac{hf}{e^{hf/kT}-1}$$

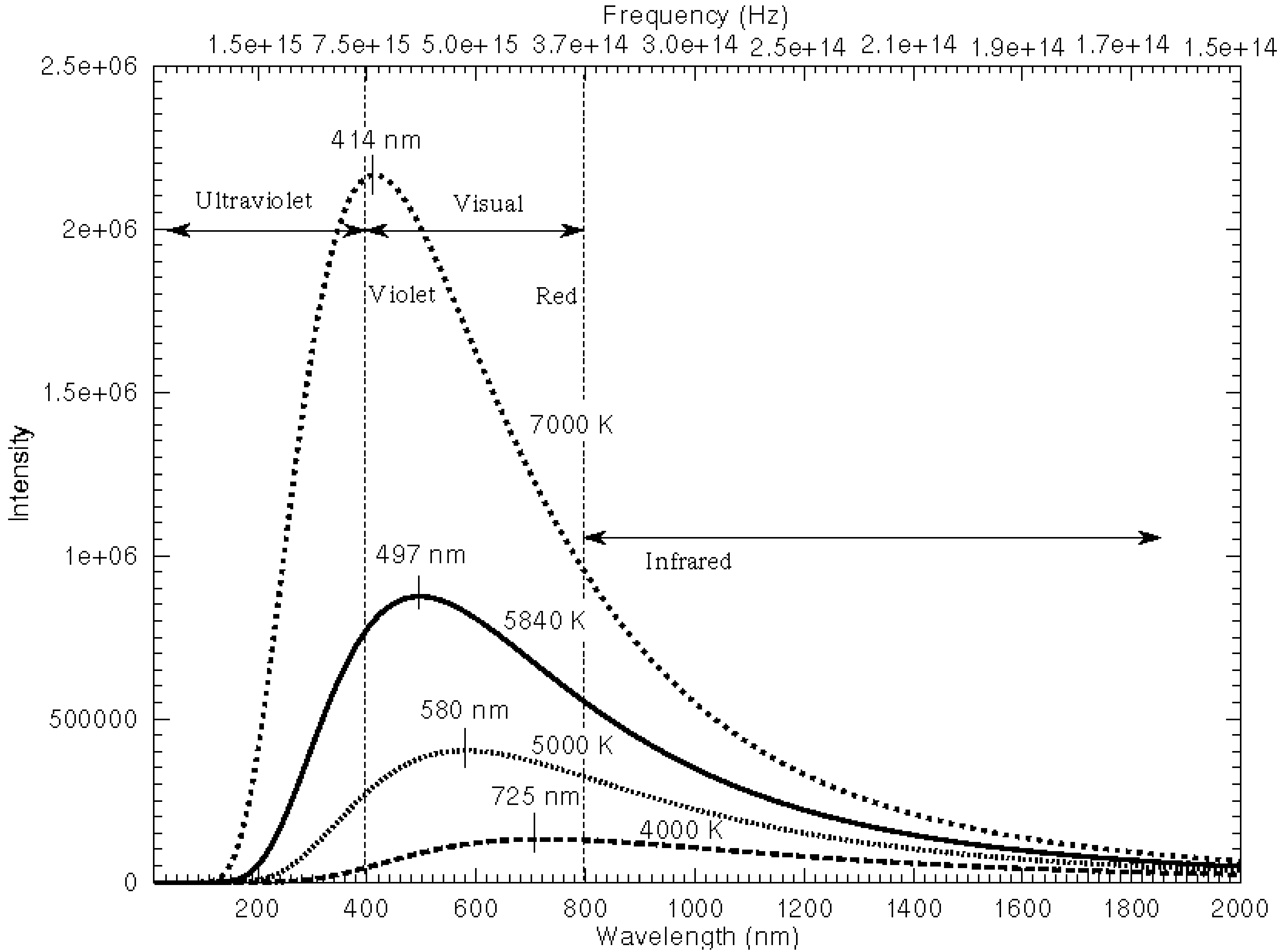
where Planck's constant $h = 6.67 \quad 10^{-34}$ J K^{-1}

Then

$$\underbrace{\mu\, df}_{\text{energy density}} = \underbrace{\frac{8\pi f^2 df}{c^3}}_{\text{no. of modes in range } f \text{ to } f+df} \underbrace{\frac{hf}{e^{hf/kT}-1}}_{\text{average energy per mode}}$$

Planck's law

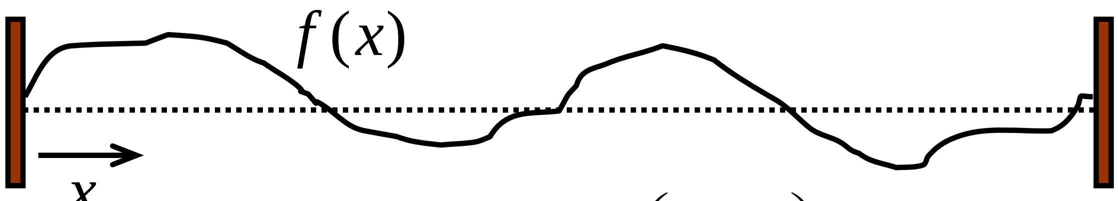
… which agrees extremely well with experiment.



Frequency (Hz)
1.5e+15 7.5e+15 5.0e+15 3.7e+14 3.0e+14 2.5e+14 2.1e+14 1.9e+14 1.7e+14 1.5e+14
2.5e+06
2e+06
1.5e+06
1e+06
500000
0
Intensity
414 nm
Ultraviolet
Visual
Violet
Red
7000 K
Infrared
497 nm
5840 K
580 nm
5000 K
725 nm
4000 K
200 400 600 800 1000 1200 1400 1600 1800 2000
Wavelength (nm)

Introduction to Fourier methods

We return to our claim that any physically observed shape function of a stretched string can be made up from normal mode shape functions.



i.e. $f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$

… a surprising claim … ? … first find B_n

… multiply both sides by $\sin\left(\frac{n_1 \pi x}{L}\right)$

and integrate over the range $x = 0$ to $x = L$

$$\int_0^L f(x)\sin\left(\frac{n_1\pi x}{L}\right)dx = \int_0^L \sin\left(\frac{n_1\pi x}{L}\right)\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)dx$$

Fourier methods …2

$$\int_0^L f(x)\sin\left(\frac{n_1\pi x}{L}\right)dx = \int_0^L \sin\left(\frac{n_1\pi x}{L}\right)\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)dx$$

If the functions are well behaved, then we can re-order things:

$$\int_0^L f(x)\sin\left(\frac{n_1\pi x}{L}\right)dx = \sum_{n=1}^{\infty} B_n \int_0^L \sin\left(\frac{n_1\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx$$

[n_1 is a particular integer and n can have any value between 1 and ∞ .]

Integral on rhs:

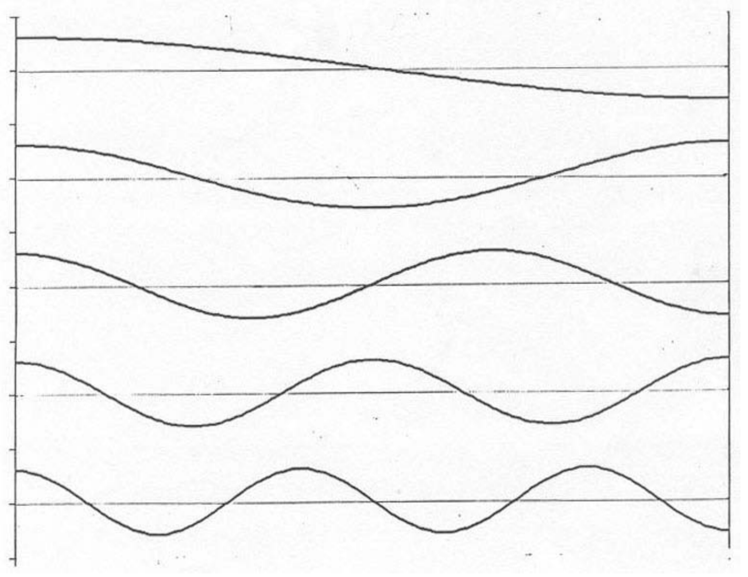
$$\int_0^L \sin\left(\frac{n_1\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = \int_0^L \frac{1}{2}\left\{\cos\left(\frac{(n_1-n)\pi x}{L}\right) - \cos\frac{(n_1+n)\pi x}{L}\right\}dx$$

Fourier methods …3

Both $(n_1 + n)$ and $(n_1 - n)$ are integers, so the functions

$$\cos\left(\frac{(n_1 \pm n)\pi x}{L}\right)$$

on the interval $x = 0$ to L must look like $\longrightarrow$



… from which it is evident that

$$\int_0^L \cos\left(\frac{(n_1 \pm n)\pi x}{L}\right) dx = 0$$

Except for the special case when n_1 and n are equal … then …

$$\cos\left(\frac{(n_1 - n)\pi x}{L}\right) = 1 \quad \text{and} \quad \int_0^L \cos\left(\frac{(n_1 - n)\pi x}{L}\right) dx = L$$

Fourier methods …4

Thus all the terms in the summation are zero, except for the single case when $n_1 = n$ i.e.

$$\int_0^L f(x)\sin\left(\frac{n_1\pi x}{L}\right)dx = B_{n_1}\int_0^L \frac{1}{2}\left\{\cos\left(\frac{(n_1-n)\pi x}{L}\right)-\cos\frac{(n_1+n)\pi x}{L}\right\}dx$$

$$= B_{n_1}\frac{L}{2}$$

i.e. $$B_{n_1} = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n_1\pi x}{L}\right)dx$$

We have found the value of the coefficient for some particular value of n_1 … the same recipe must work for any value, so we can write:

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Fourier methods …5

The important property we have used is that the functions

$$\sin\left(\frac{n_1 \pi x}{L}\right) \text{ and } \sin\left(\frac{n \pi x}{L}\right)$$

are "**orthogonal** over the interval $x = 0$ to $x = L$."

i.e. $$\int_0^L \sin\left(\frac{n_1 \pi x}{L}\right)\sin\left(\frac{n \pi x}{L}\right)dx = \begin{cases} 0 \text{ if } n_1 \neq n \\ \dfrac{L}{2} \text{ if } n_1 = n \end{cases}$$

Read French pages 195-6

Fourier methods …6

The most general case (where there can be nodal or antinodal boundary conditions at $x = 0$ and $x = L$) is

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left\{ A_n \cos\left(\frac{n\pi x}{L}\right) + B_n \sin\left(\frac{n\pi x}{L}\right) \right\}$$

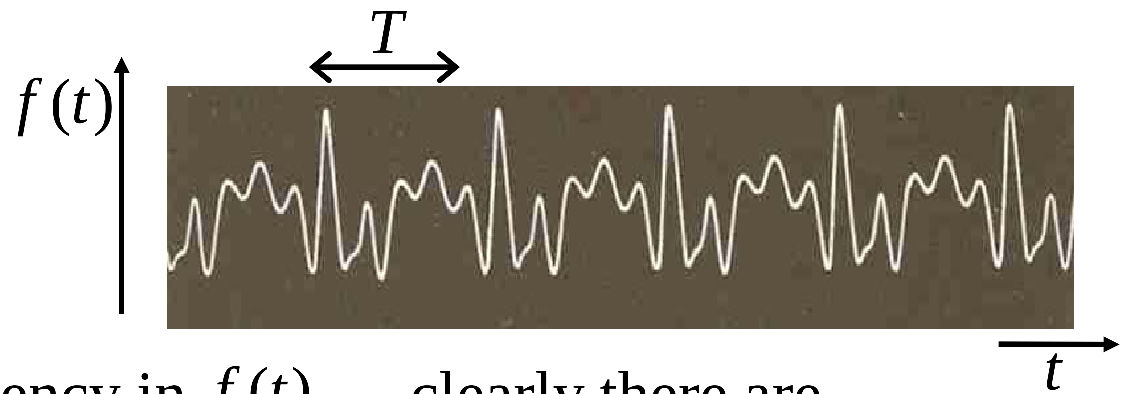
where

$$A_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

$$B_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

Fourier methods …7

One of the most commonly encountered uses of Fourier methods is the representation of periodic functions of time in terms of sine and cosine functions …



Put $\Omega = \dfrac{2\pi}{T}$

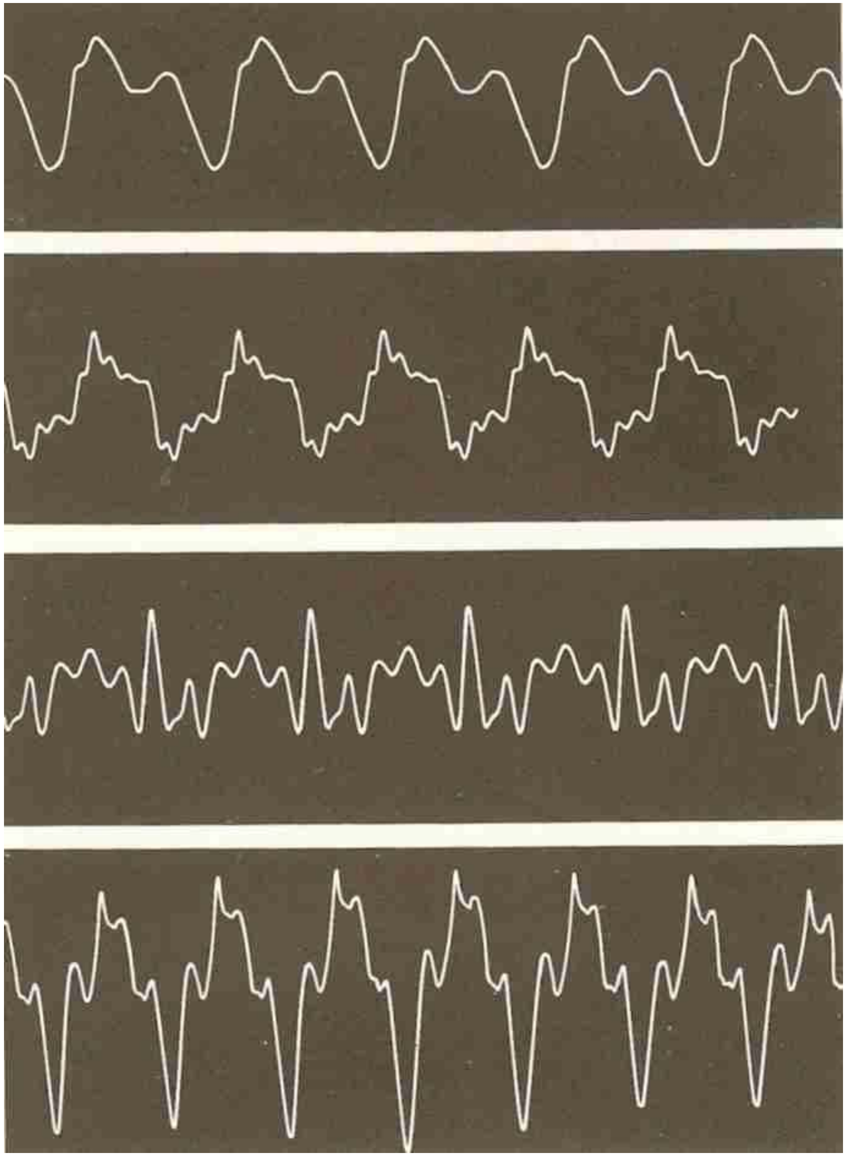
This is the lowest frequency in $f(t)$ … clearly there are higher frequencies … by the same method as before, write …

$$f(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{2\pi nt}{T}\right) + B_n \sin\left(\frac{2\pi nt}{T}\right)$$

$$= \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos n\Omega t + B_n \sin n\Omega t$$

where $A_n = \dfrac{2}{T}\displaystyle\int_0^T f(t)\cos(n\Omega t)\,dt$ and $B_n = \dfrac{2}{T}\displaystyle\int_0^T f(t)\sin(n\Omega t)\,dt$

Waveforms of ...



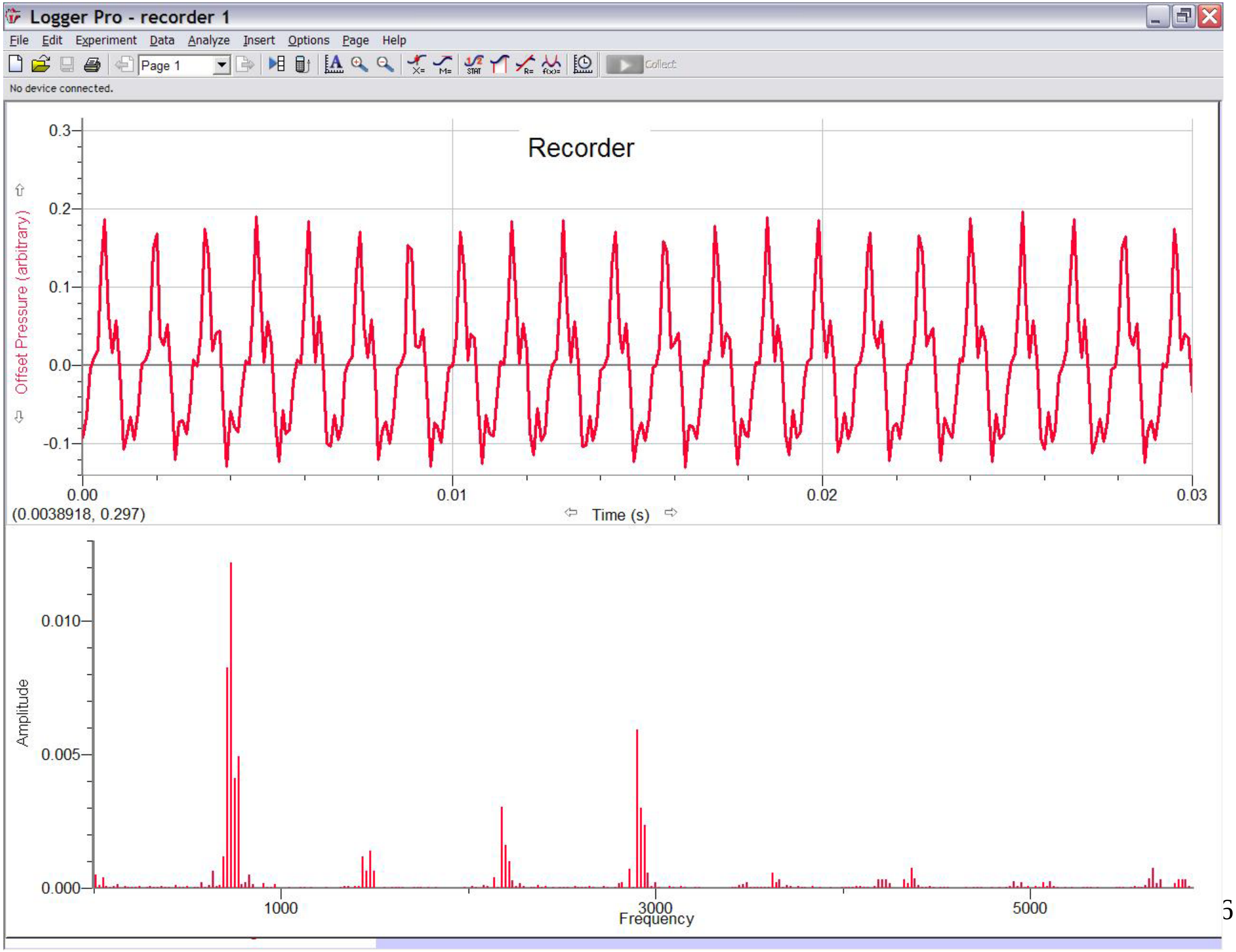
a flute

a clarinet

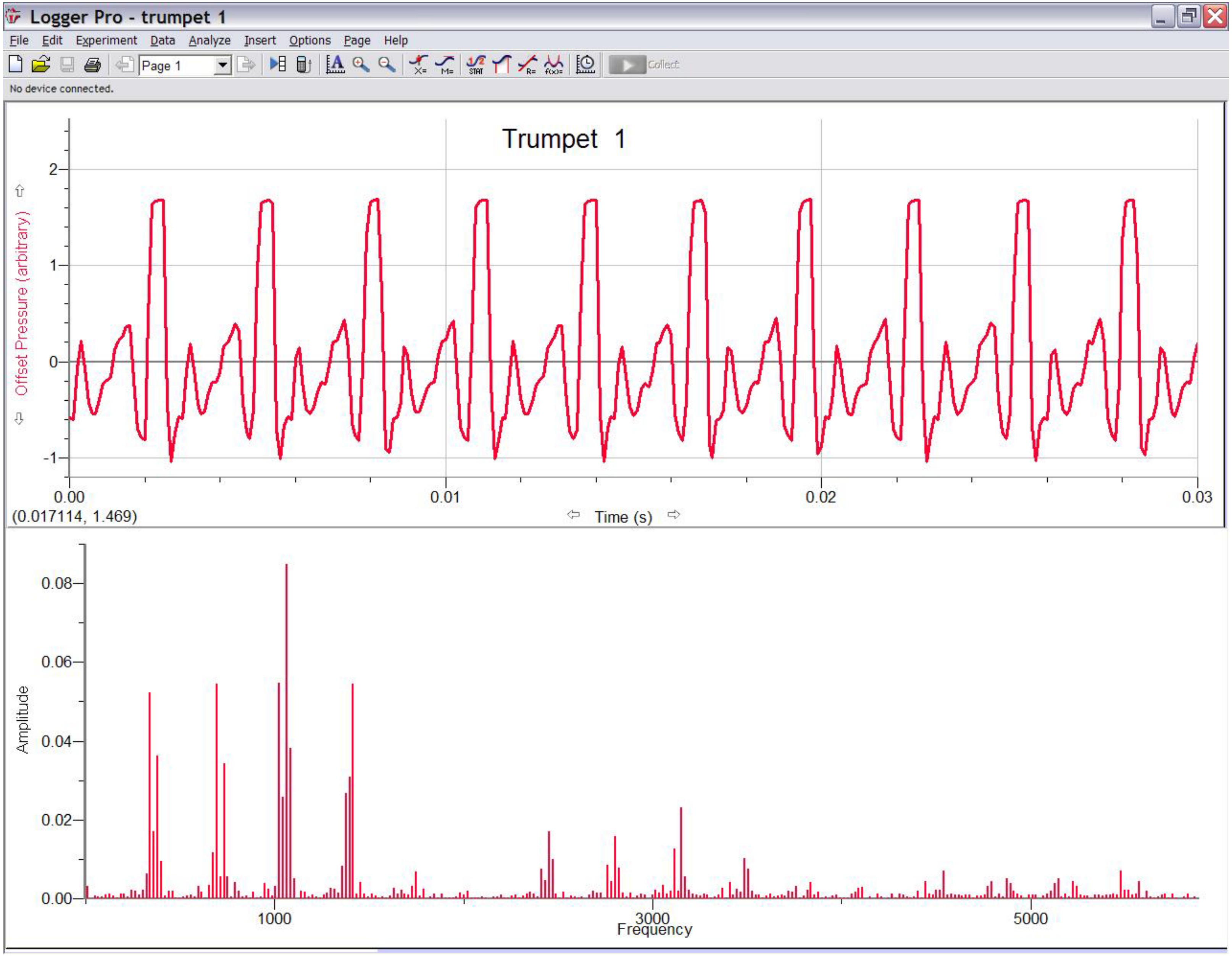
an oboe

a saxophone

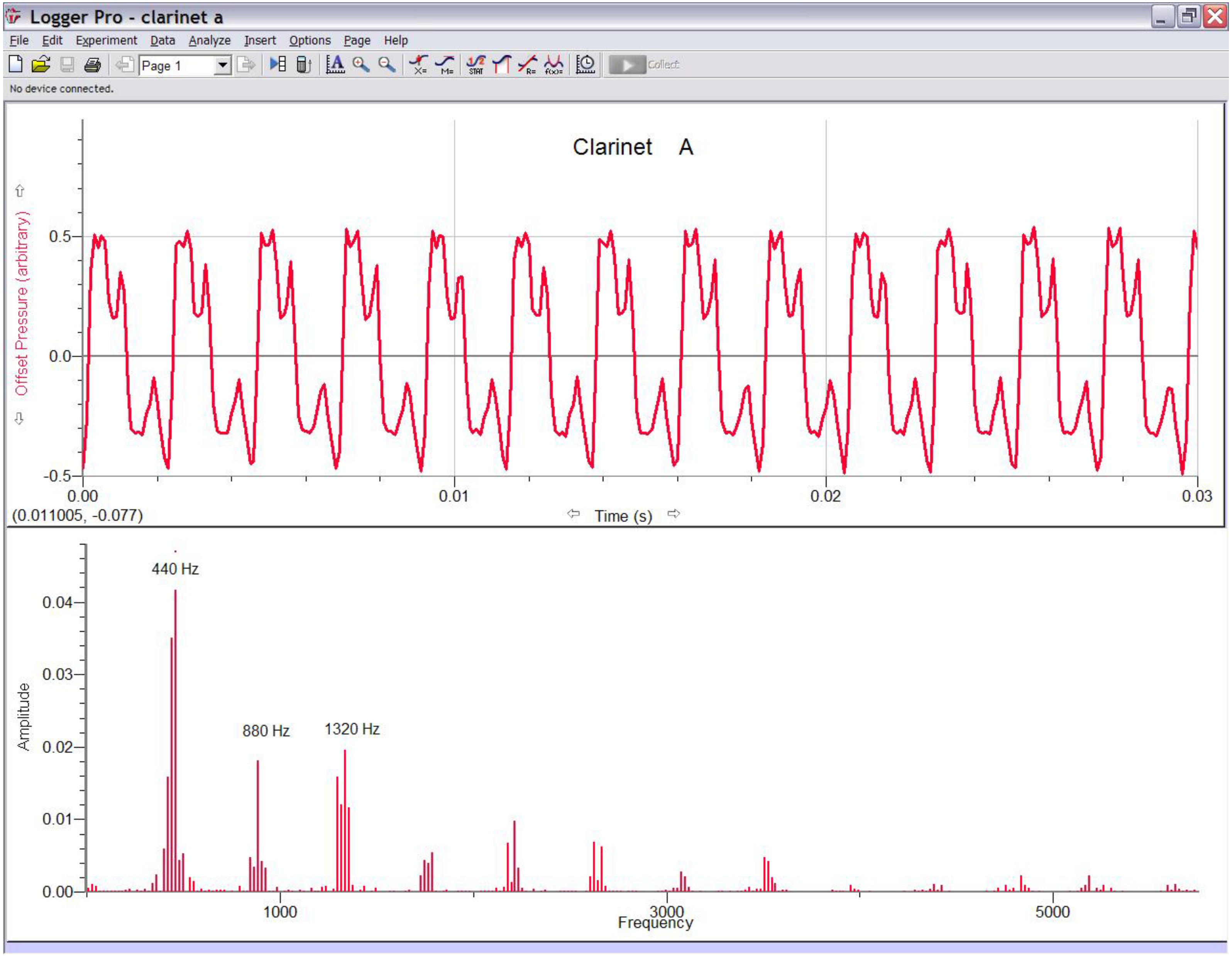
Fast Fourier transform experiments, 10 March 2008



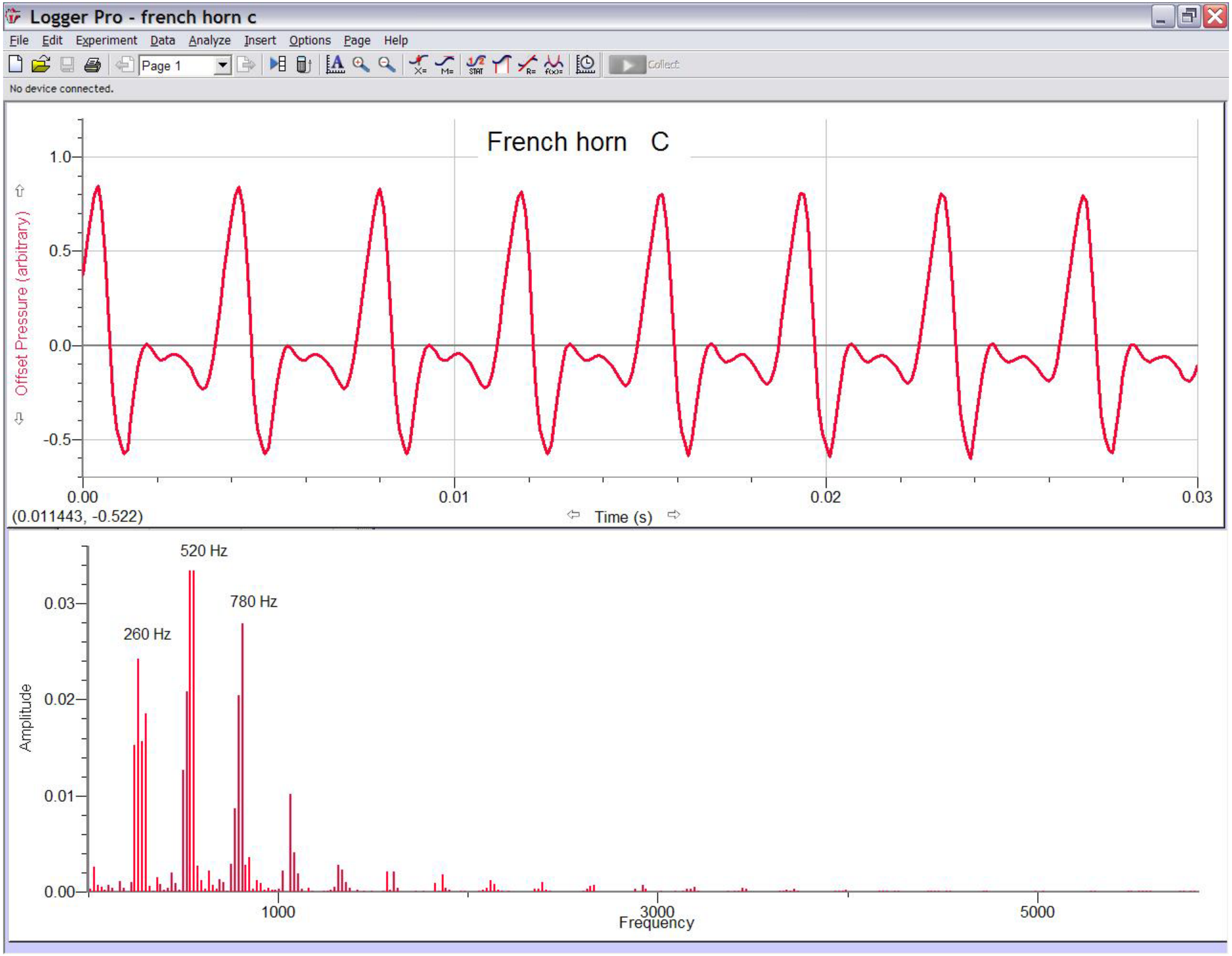
Fast Fourier transform experiments, 10 March 2008



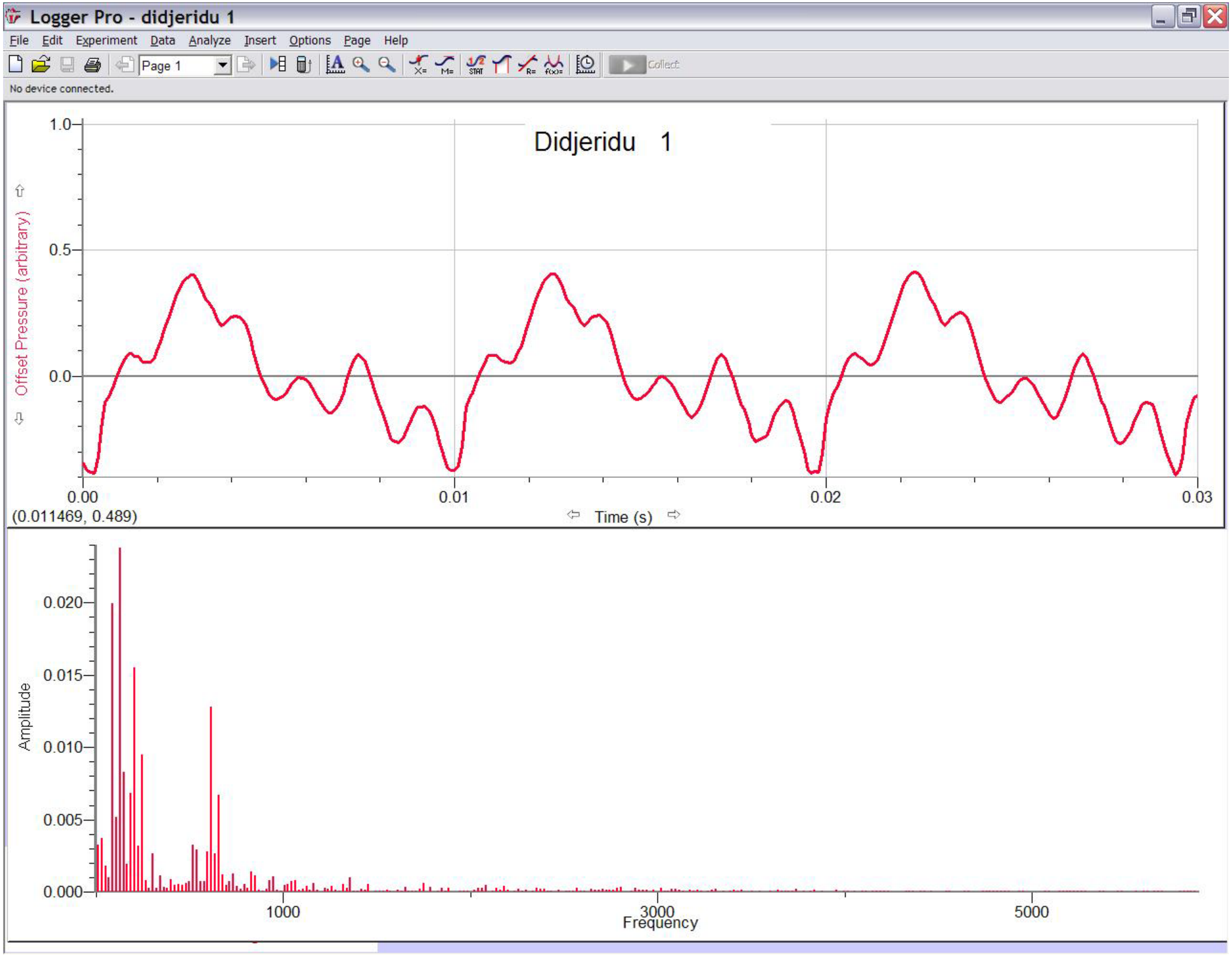
Fast Fourier transform experiments, 10 March 2008



Fast Fourier transform experiments, 10 March 2008



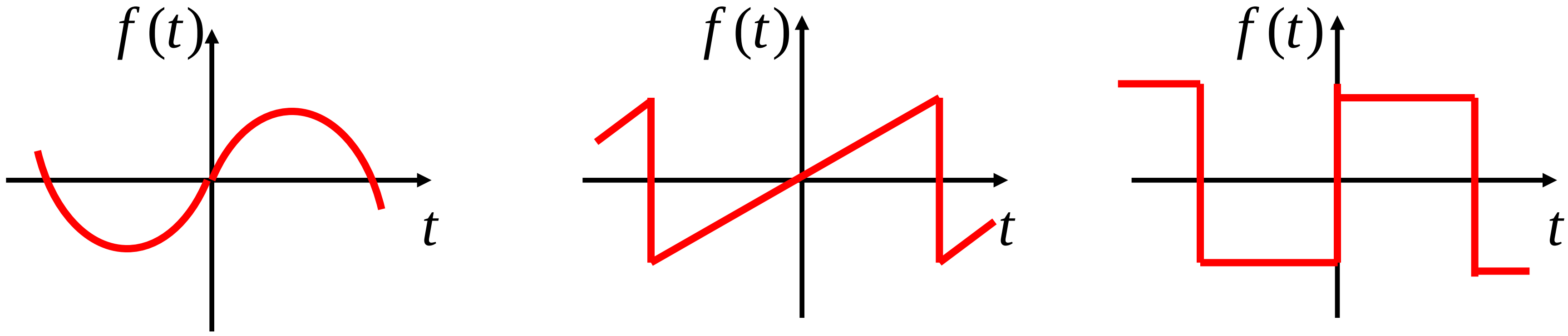
Fast Fourier transform experiments, 10 March 2008



Odd functions

An odd periodic function $f(-t) = -f(t)$

where $-T/2 < t < T/2$



…can be expressed as a sum of sine functions

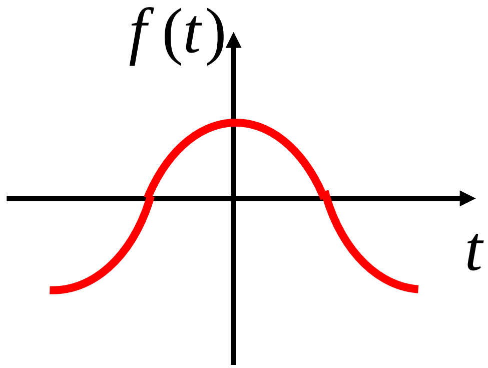
$$f(t) = \sum_{n=0}^{\infty} B_n \sin n\Omega t \qquad \Omega = \frac{2\pi}{T}$$

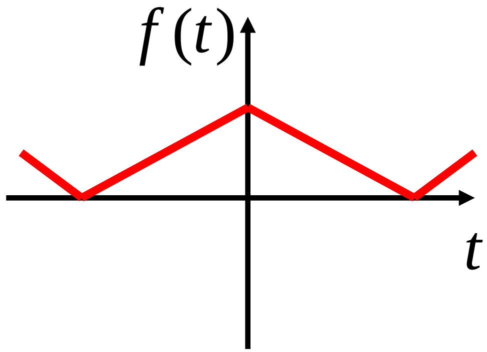
$$B_n = \frac{2}{T}\int_0^T f(t)\sin\left(n\Omega t\right)dt$$

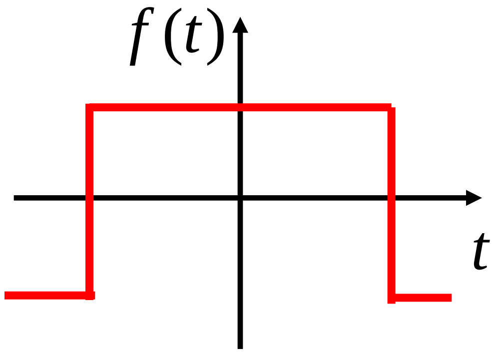
Even functions

An even periodic function $f(-t) = +f(t)$

where $-T/2 < t < T/2$







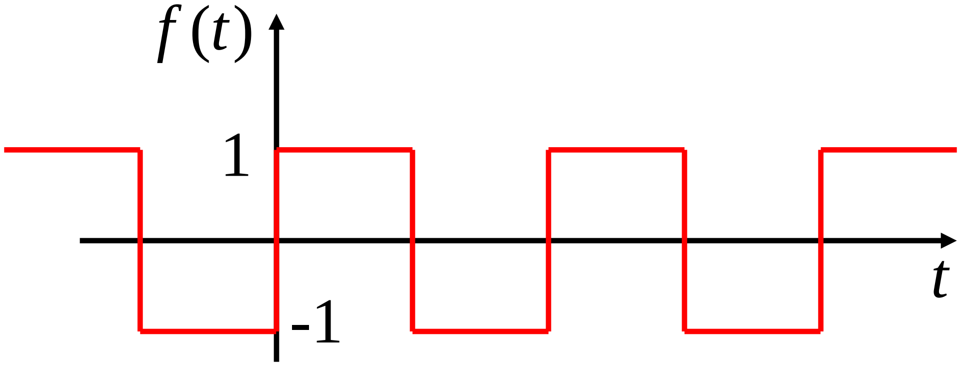
…can be expressed as a sum of cosine functions

$$f(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos n\Omega t \qquad \Omega = \frac{2\pi}{T}$$

$$A_n = \frac{2}{T}\int_0^T f(t)\cos(n\Omega t)\,dt$$

Fourier methods … Example

Find Fourier coefficients for the case:



This is an odd function: $f(-t) = -f(t)$ $\qquad \therefore\ f(t) = \sum_{n=1}^{\infty} B_n \sin n\Omega t$

$$B_n = \frac{2}{T}\int_0^T f(t)\sin\left(n\Omega t\right)dt$$

$$= \frac{2}{T}\int_0^{T/2} (1)\sin\left(n\Omega t\right)dt + \frac{2}{T}\int_{T/2}^{T} (-1)\sin\left(n\Omega t\right)dt$$

$$= \frac{2}{T}(1)\left(-\frac{1}{n\Omega}\right)\left[\cos\left(n\Omega t\right)\right]_0^{T/2} + \frac{2}{T}(-1)\left(-\frac{1}{n\Omega}\right)\left[\cos\left(n\Omega t\right)\right]_{T/2}^{T}$$

Fourier methods … Example cont.

$$\Omega = \frac{2\pi}{T}$$

$$\therefore B_n = -\frac{1}{n\pi}\left[\cos n\pi - \cos 0\right] + \frac{1}{n\pi}\left[\cos 2n\pi - \cos n\pi\right]$$

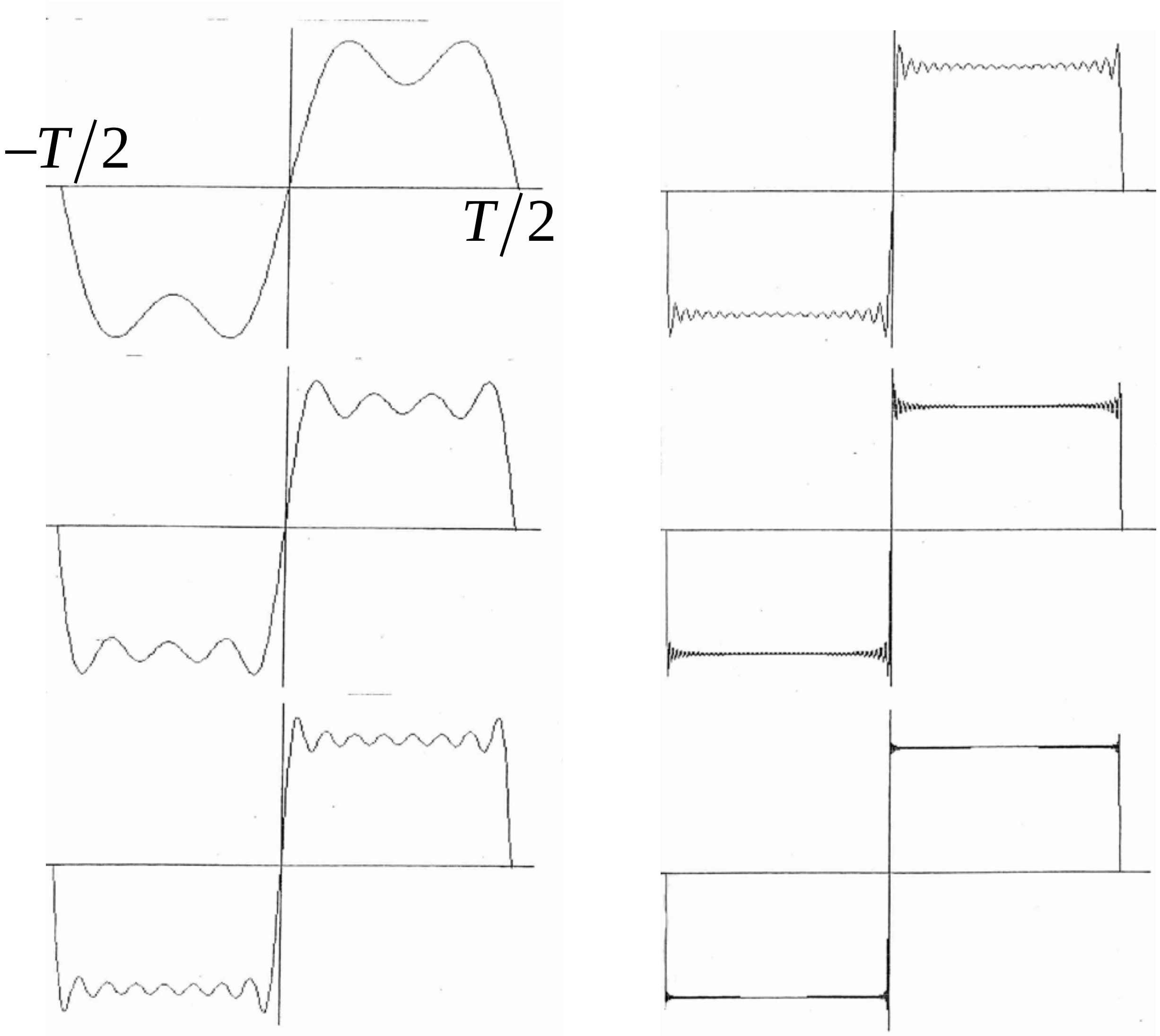
For even n: $\cos 2n\pi = \cos n\pi = 1$ $\quad\therefore B_{\text{even } n} = 0$

For odd n: $\cos n\pi = -1$ and $\cos 2n\pi = +1$

$$\therefore B_{\text{odd } n} = -\frac{1}{n\pi}(-1-1) + \frac{1}{n\pi}(1+1) = \frac{4}{n\pi}$$

$$\therefore f(t) = \frac{4}{\pi}\left\{\sin \Omega t + \frac{1}{3}\sin 3\Omega t + \frac{1}{5}\sin 5\Omega t + \ldots\right\}$$

Fourier sums … Example 3



2 terms

20 terms

4 terms

50 terms

8 terms

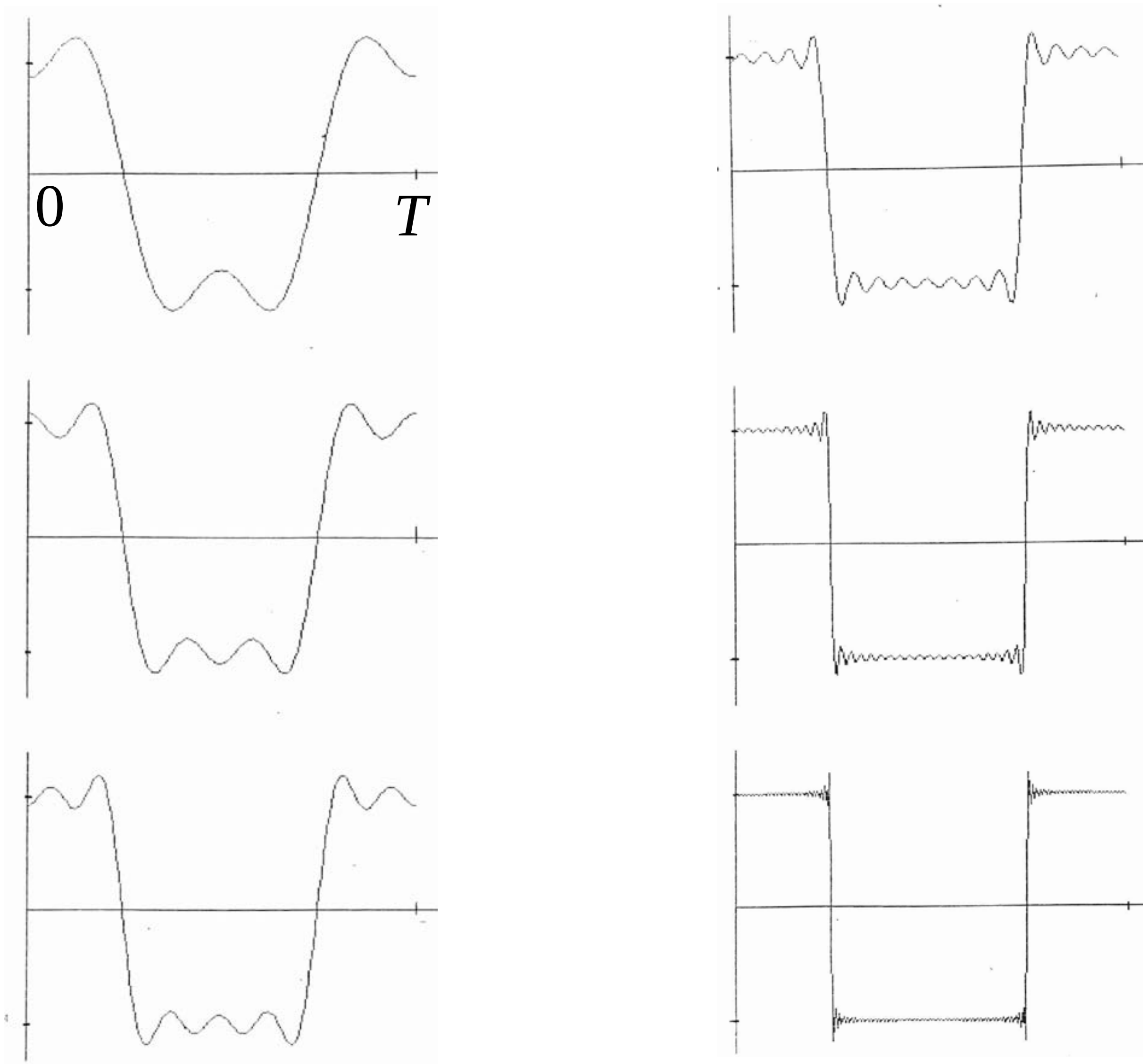
200 terms

Fourier sums … Example 1

2 terms

3 terms

4 terms



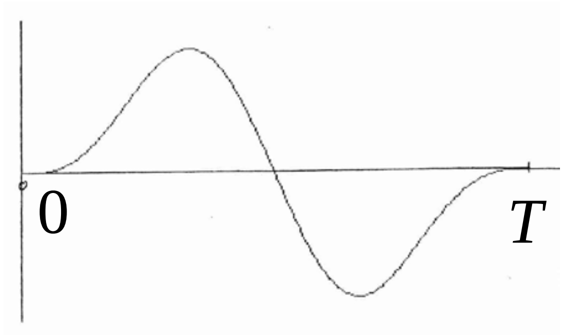
8 terms

20 terms

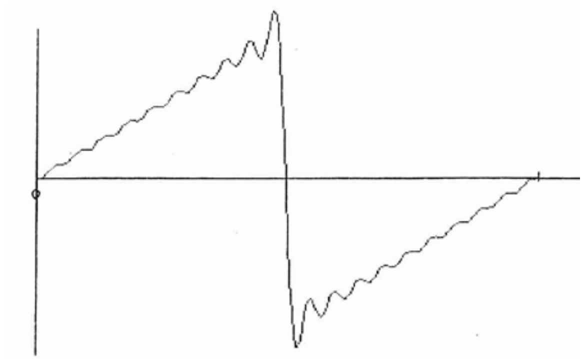
50 terms

Fourier sums … Example 2

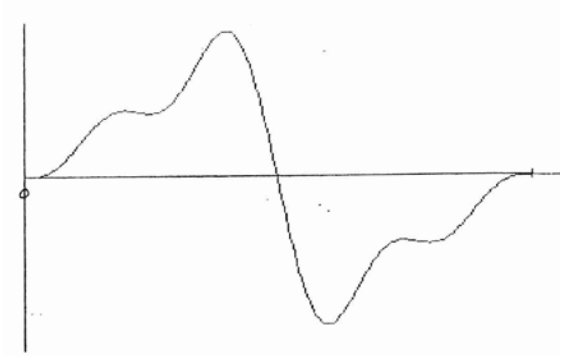
2 terms



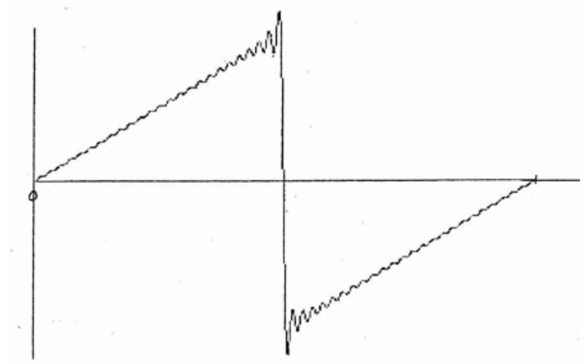
20 terms



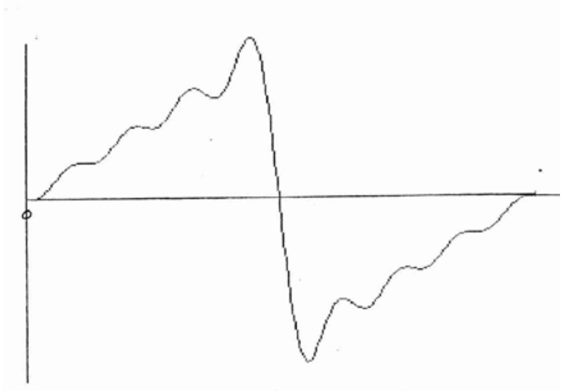
4 terms



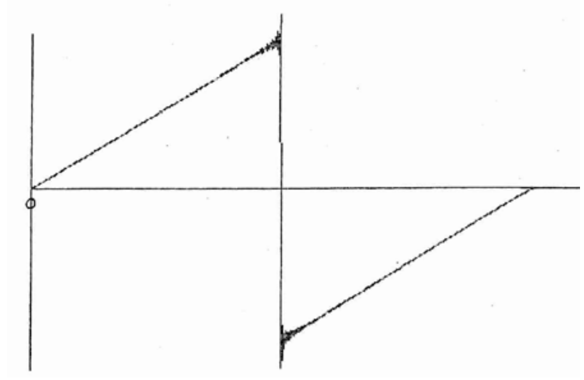
50 terms



8 terms

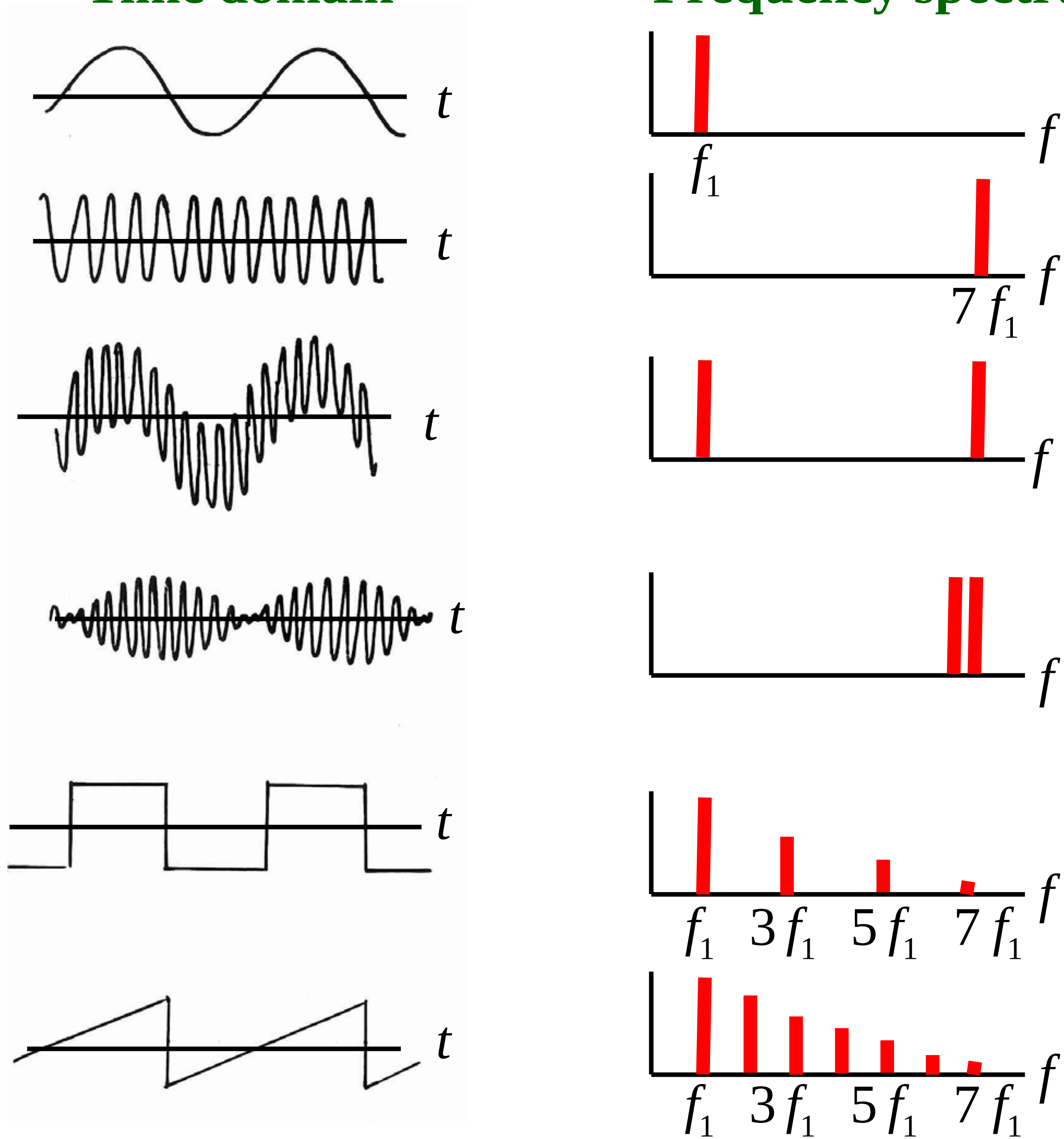


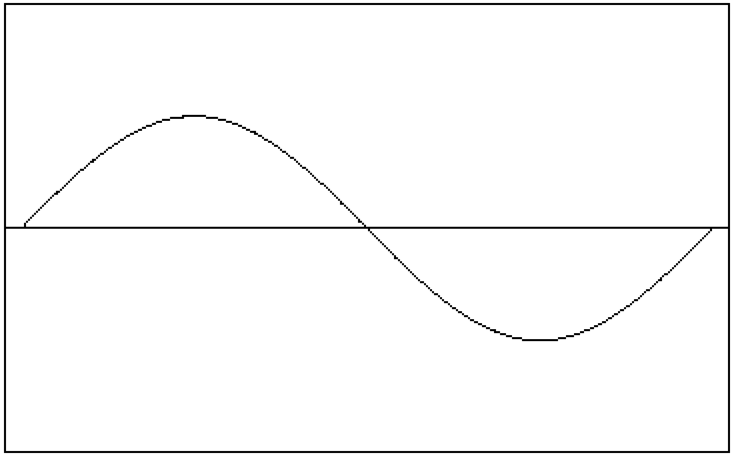
200 terms



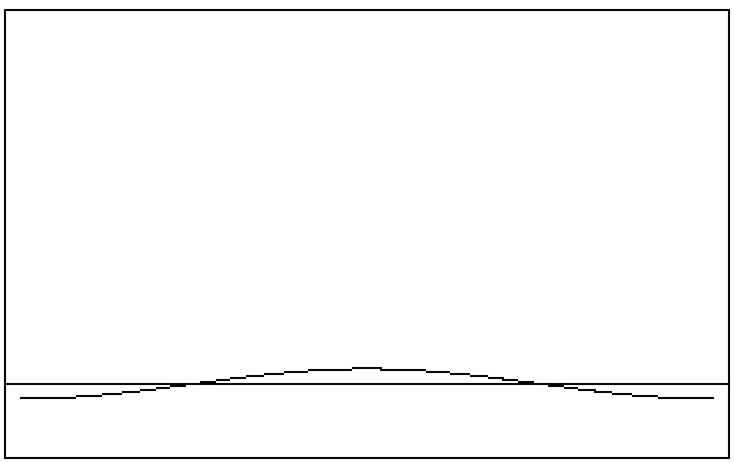
Fourier transforms

Time domain | Frequency spectrum

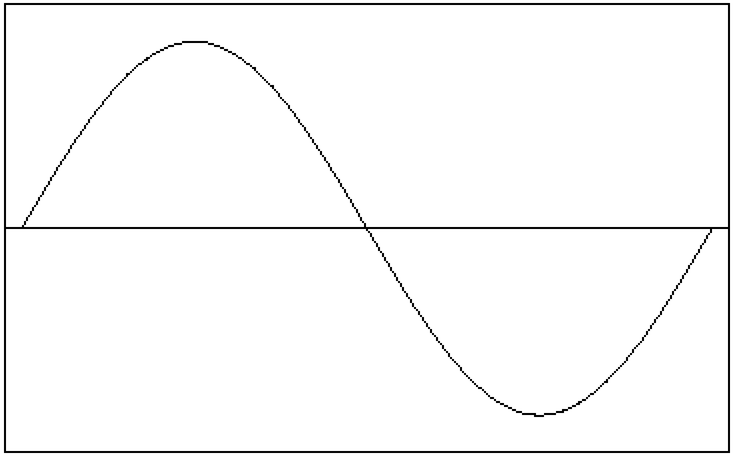




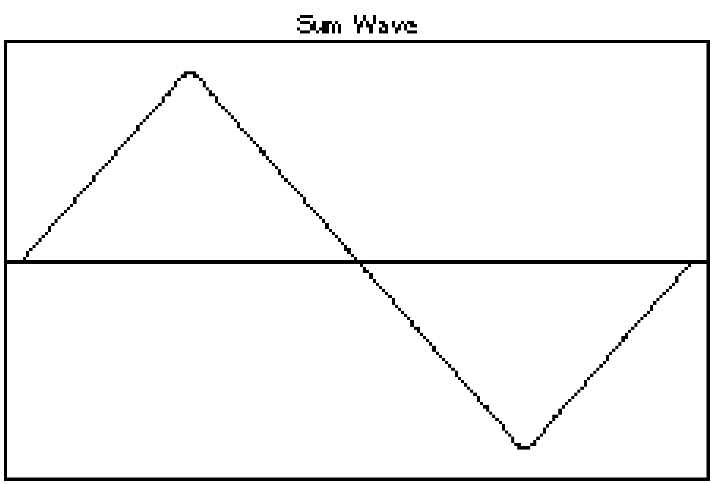

1
0.8
0.6
0.4
0.2
1

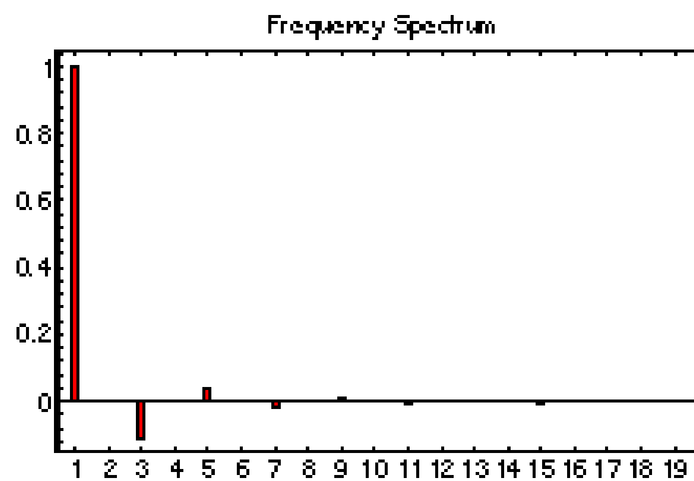



1
0.8
0.6
0.4
0.2
1




1
0.8
0.6
0.4
0.2
1


Sum Wave


Frequency Spectrum
1
0.8
0.6
0.4
0.2
0
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19