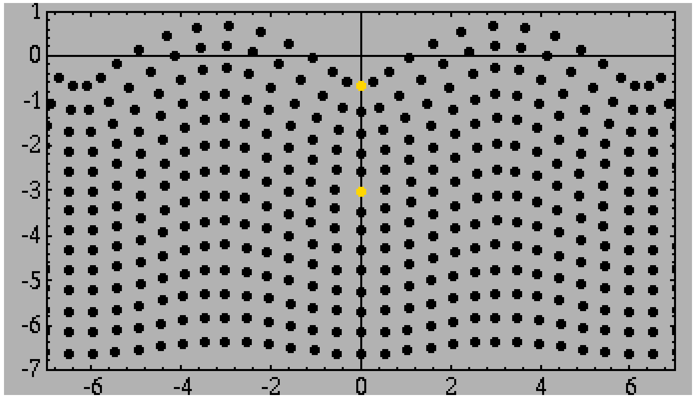
University of Cape Town
Department of Physics

PHY2014F

Vibrations and Waves

<u>Part 1</u>
Simple harmonic oscillators
Damped oscillators
Driven oscillators
Resonance



… covering (more or less)
French Chapters 1, 3 & 4

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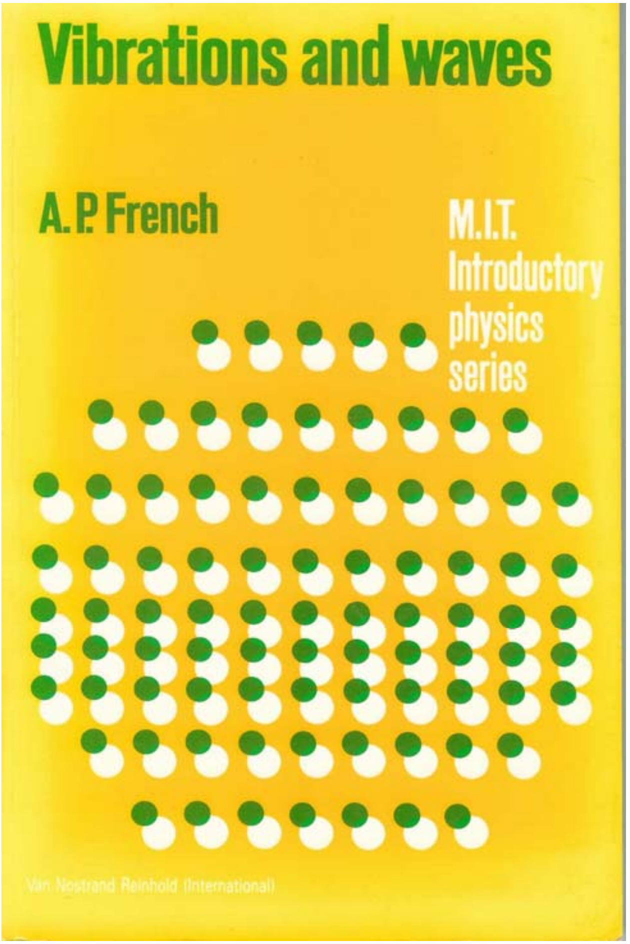
PHY2014F

Prescribed textbook:



… with significant acknowledgment to the notes of Steve Driver, who was brilliant for decades …

… the animated gif’s are from Dr. Daniel A. Russell, Kettering University (http://www.kettering.edu/~drussell/)

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What to do in this course:

1. Read the relevant sections in the textbook … the course notes will guide you.

2. Do all homework and problem sets.

3. Get help early … from AB, the course tutor Gary Tupper, …

4. As in PHY1004W, there are no shortcuts … put effort in to understand things …

Problem-solving and homework

Each week you will be given a take-home problem set to complete and hand in for marks ...

In addition to this, you need to work through the following problems in *French*, in you own time, at home. You will not be asked to hand these in for marks. Get help from you friends, the course tutor, lecturer, ... Do not take shortcuts. Mastering these problems is a fundamental aspect of this course.

The problems associated with Part 1 are:

1-8, 1-11, 1-12, 2-1, 3-1, 3-2, 3-3, 3-4, 3-5, 3-6, 3-8, 3-9, 3-11, 3-14, 3-15, 4-3, 4-4, 4-5, 4-7, 4-10, 4-11, 4-12, 4-13, 4-14, 4-15, 4-16, 4-17

You might find these tougher: 3-7, 3-10, 3-16, 3-17, 3-18, 4-8, 4-9

Some useful (?) trigonometric identities

$$\sin^2 A + \cos^2 A = 1$$

$$\sin\left(\pi/2 - A\right) = \cos A$$

$$\sin\left(-A\right) = -\sin A$$
$$\cos\left(-A\right) = \cos A$$
$$\tan\left(-A\right) = -\tan A$$

$$\sin\left(A \pm B\right) = \sin A \cos B \pm \cos A \sin B$$
$$\cos\left(A \pm B\right) = \cos A \cos B \mp \sin A \sin B$$
$$\tan\left(A \pm B\right) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin\left(2A\right) = 2\sin A \cos A$$
$$\cos\left(2A\right) = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A$$
$$\tan\left(2A\right) = \frac{2\tan A}{1 - \tan^2 A}$$

More useful (?) trigonometric identities

$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

$$\sin A \sin B = \tfrac{1}{2}\left[\cos\left(A-B\right) - \cos\left(A+B\right)\right]$$

$$\cos A \cos B = \tfrac{1}{2}\left[\cos\left(A-B\right) + \cos\left(A+B\right)\right]$$

$$\sin A \cos B = \tfrac{1}{2}\left[\sin\left(A+B\right) + \sin\left(A-B\right)\right]$$

$$\cos A \sin B = \tfrac{1}{2}\left[\sin\left(A+B\right) - \sin\left(A-B\right)\right]$$

Oscillatory Phenomena

… observed in many physical systems …
from the very small…(e.g. dipole resonance in nuclei)
… to the very large (earthquake waves, stars,…)

Mechanical systems …

… to electrical systems …

… from violin strings …

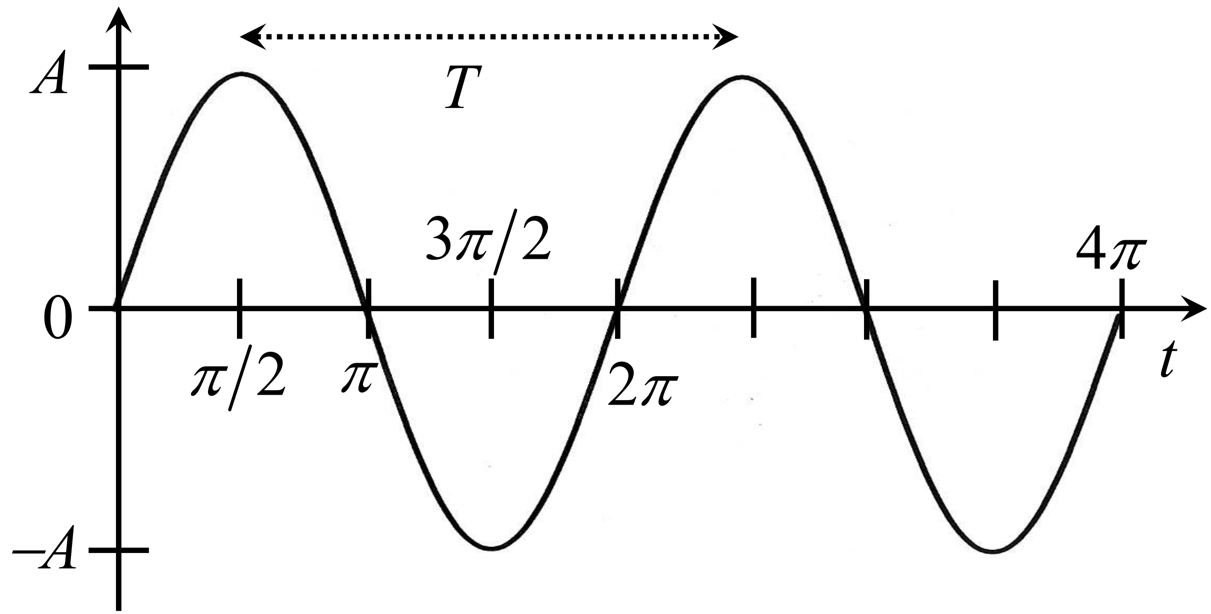
… to lasers …

All periodic phenomena can be represented as the sum of sine and cosine functions:

$$x(t) = \sum (a_n \cos n\omega t + b_n \sin n\omega t)$$

… useful to study the **simple harmonic oscillator** …

Consider the function $x(t) = A\cos(\omega t + \phi)$



$\omega T = 2\pi$ where T : period (s)

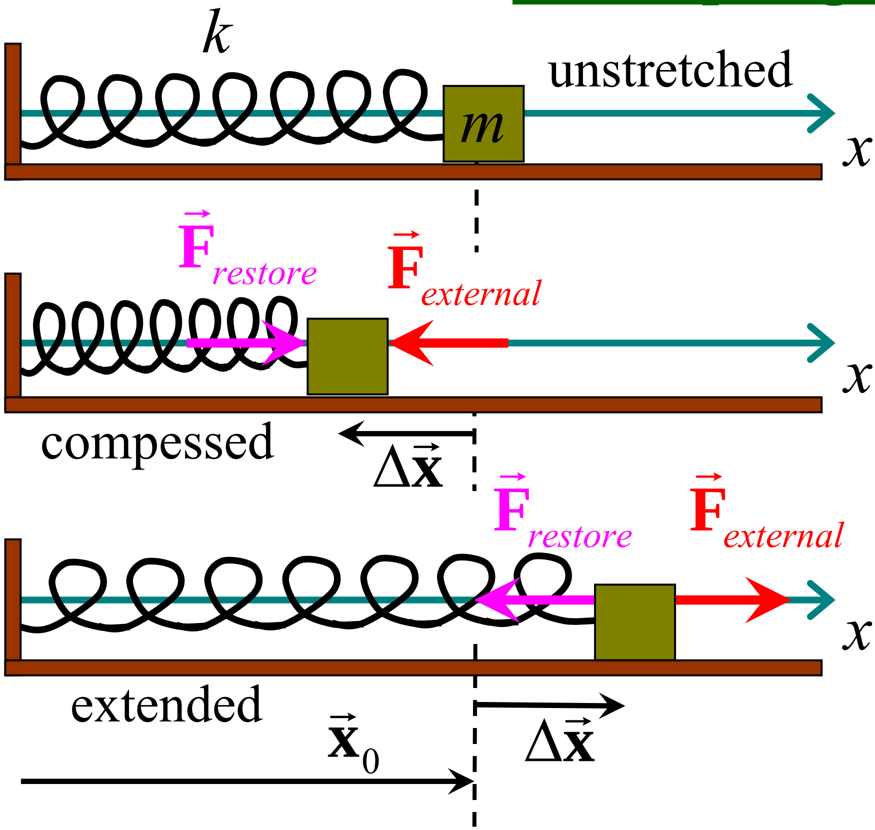
ω : angular frequency (rad s^{-1})

some books use υ $f = \frac{1}{T}$ where f : frequency (Hz)

A : Amplitude

ϕ : phase angle, initial phase or phase constant

Mass-spring oscillator



Hooke's Law:

Restoring force,

$$\vec{\mathbf{F}}_{restore} = -k\Delta\vec{\mathbf{x}}$$

where $\Delta\vec{\mathbf{x}} = \vec{\mathbf{x}} - \vec{\mathbf{x}}_0$

and k is the "spring constant" [N m^{-1}]

Start with the momentum principle: $\dfrac{d\vec{\mathbf{p}}}{dt} = \vec{\mathbf{F}}_{\text{net}}$

For horizontal forces on the mass: $\dfrac{dp_x}{dt} = -kx$

$$\therefore \frac{d(mv_x)}{dt} = -kx \quad \text{or} \quad \frac{d}{dt}\left(m\frac{dx}{dt}\right) = -kx$$

$$\therefore \frac{d^2x}{dt^2} = -\frac{k}{m}x$$

Mass-spring oscillator ...2

$$\frac{d^2x(t)}{dt^2} = \frac{-k}{m}x(t)$$

... a second order differential equation

... we know that if we displace a mass-spring system from its rest position and then release it, it will perform SHM ...

Guess a trial solution: $x(t) = A\cos(\omega t + \phi)$

then $\dfrac{d^2x}{dt^2} = -A\omega^2\cos(\omega t + \phi)$

and substitute into our DE: $-A\omega^2\cos(\omega t + \phi) = -A\dfrac{k}{m}\cos(\omega t + \phi)$

... which is true provided $\omega^2 = \dfrac{k}{m}$

Therefore our solution is $x(t) = A\cos(\omega t + \phi)$ where $\omega = \sqrt{\dfrac{k}{m}}$

Mass-spring oscillator ...3

We will write a particular value of ω as ω_0 , as the **natural angular frequency** of the oscillator – the frequency that it "wants" to oscillate at.

Mass-spring system: $\omega_0 = \sqrt{\dfrac{k}{m}}$ and $T = 2\pi\sqrt{\dfrac{m}{k}}$

System parameters: $m, k \longrightarrow \omega_0$

Initial conditions: $\longrightarrow A, \phi$

(Note that ω_0 is independent of A)

Simple harmonic oscillator

French
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$$x(t) = A\cos(\omega_0 t + \phi)$$

$$v(t) = \frac{dx(t)}{dt} = -A\omega_0 \sin(\omega_0 t + \phi)$$

$$a(t) = \frac{d^2x(t)}{dt^2} = \frac{dv(t)}{dt} = -A\omega_0{}^2 \cos(\omega_0 t + \phi)$$

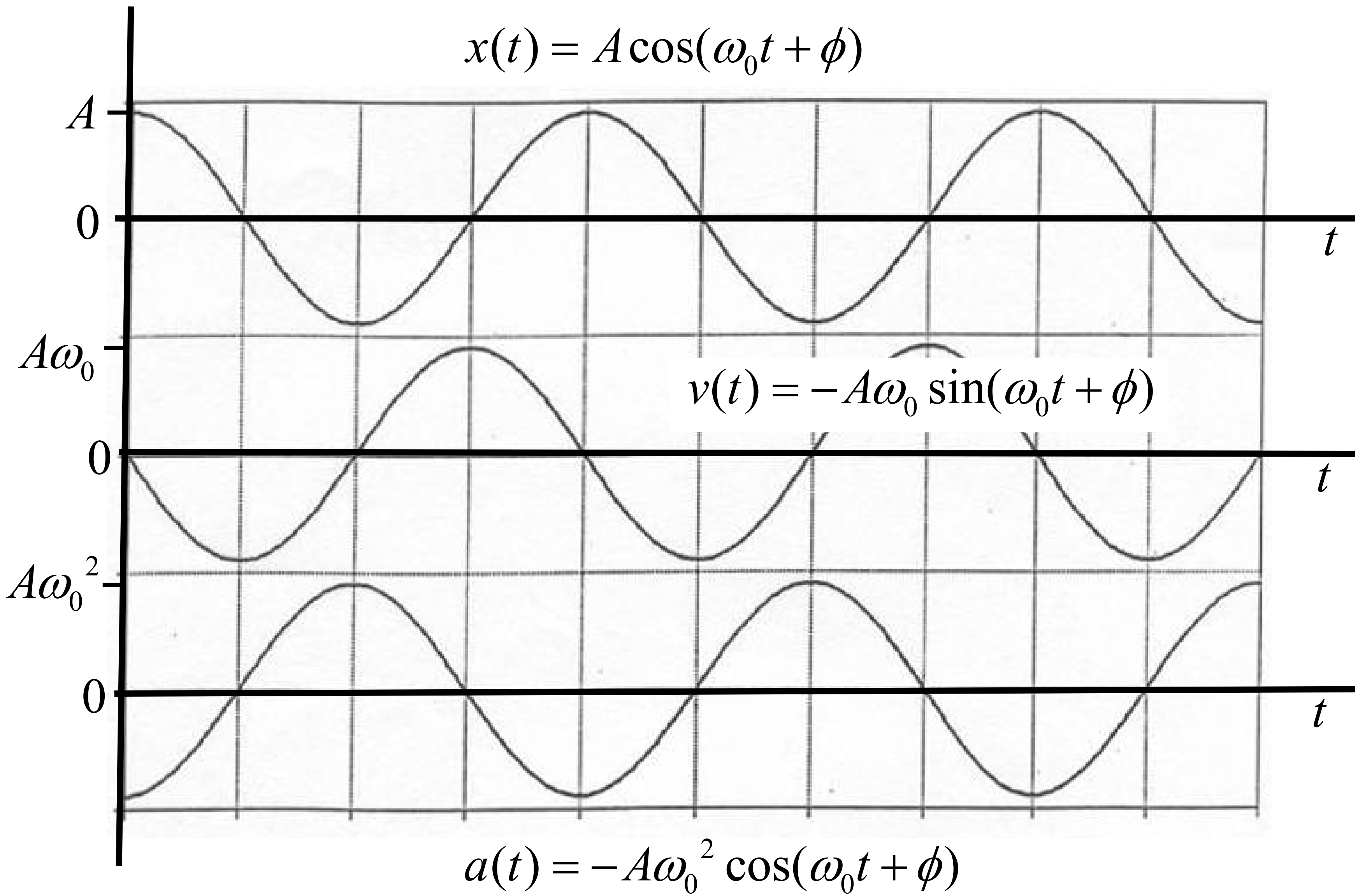
… acceleration = – (constant) . (displacement)

$$= -A\omega_0^2 \cos(\omega_0 t + \phi)$$

$$= A\omega_0^2 \cos(\omega_0 t + \phi + \pi)$$

Phase difference between acceleration and displacement is π

Phase difference between v and x (and v & a) is $\frac{\pi}{2}$


$x(t) = A\cos(\omega_0 t + \phi)$
A
0
t
$A\omega_0$
$v(t) = -A\omega_0 \sin(\omega_0 t + \phi)$
0
t
$A\omega_0^{\ 2}$
0
t
$a(t) = -A\omega_0^{\ 2} \cos(\omega_0 t + \phi)$

Simple harmonic oscillator

At $t = 0$, write $x = x_0$ and $v = v_0$.

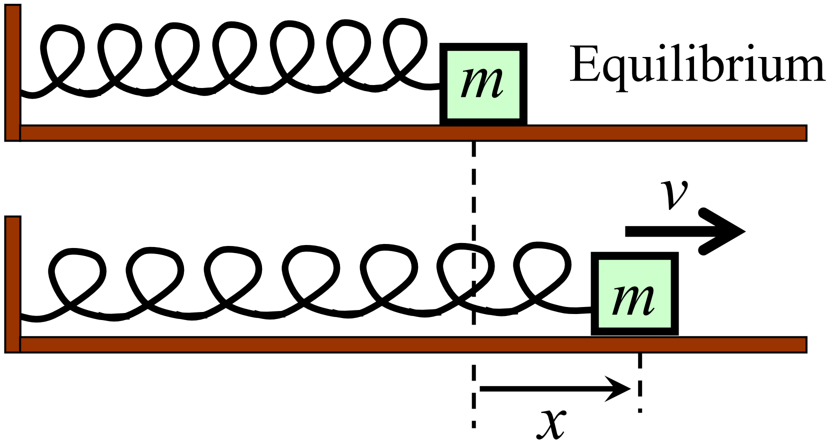
Then at $t = 0$:

$$\left.\begin{aligned} x_0 &= A\cos(\phi) \\ v_0 &= -\omega_0 A\sin(\phi) \end{aligned}\right\} \quad \tan\phi = -\frac{v_0}{\omega_0 x_0}$$

… and $\quad x_0{}^2 + \left(\frac{v_0}{\omega_0}\right)^2 = A^2\cos^2(\phi) + A^2\sin^2(\phi) = A^2$

$$\therefore A = \sqrt{x_0{}^2 + \left(\frac{v_0}{\omega_0}\right)^2}$$

Mass-spring oscillator: an energy approach



Suppose that the mass has a speed v when it has displacement x

Kinetic energy of mass = $\frac{1}{2}mv^2$

Potential energy of spring = $\int_0^x F\,dx' = \int_0^x kx'\,dx' = \frac{1}{2}kx^2$

There are no dissipative mechanisms in our model (no friction).
… the total energy of the mass-spring system is conserved.

$$\tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2 = \text{constant}$$

Mass-spring oscillator: an energy approach …2

For our mass-spring system: $\tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2 = \text{constant}$

$$\therefore \frac{d}{dt}\left(\tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2\right) = 0$$

$$\therefore mv\frac{dv}{dt} + kx\frac{dx}{dt} = 0$$

$$\therefore mv\frac{dv}{dt} + kxv = 0$$

$$\therefore m\frac{dv}{dt} + kx = 0$$

$$\therefore \frac{d^2x}{dt^2} = -\frac{k}{m}x$$

… as before

Mass-spring oscillator: an energy approach …3

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For the mass-spring system: $x = A\cos(\omega_0 t + \phi)$

Potential energy = $\frac{1}{2}kx^2 = \frac{1}{2}kA^2\cos^2(\omega_0 t + \phi)$

k.e. = $\frac{1}{2}mv^2 = \frac{1}{2}m[-A\omega_0\sin(\omega_0 t + \phi)]^2 = \frac{1}{2}mA^2\omega_0^2\sin^2(\omega_0 t + \phi)$

Total energy = p.e. + k.e

$$= \frac{1}{2}kA^2\cos^2(\omega_0 t + \phi) + \frac{1}{2}mA^2\omega_0^2\sin^2(\omega_0 t + \phi)$$

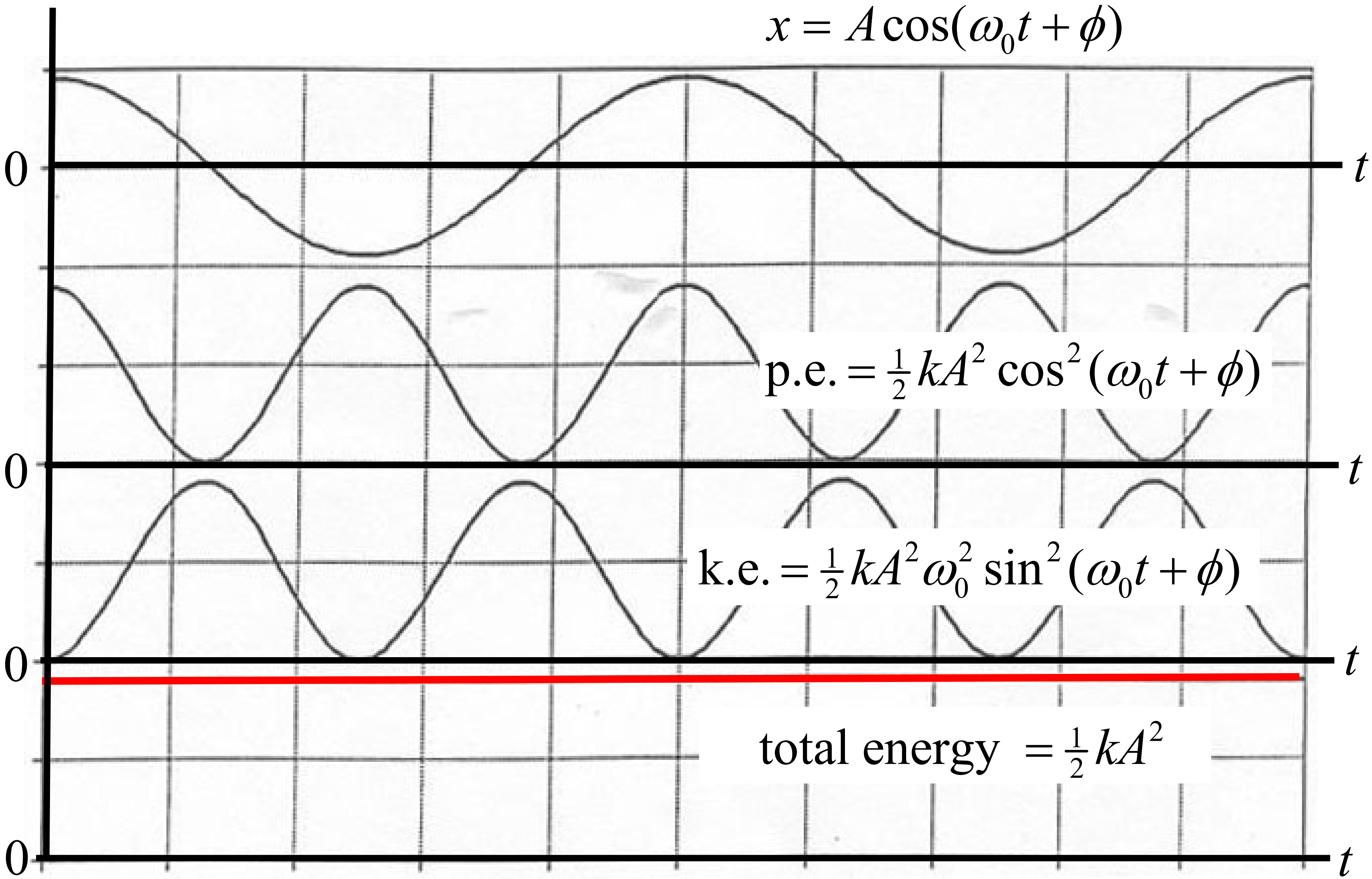
$$= \frac{1}{2}kA^2 \quad (= \frac{1}{2}m\omega_0{}^2A^2) \qquad (\therefore E \propto A^2)$$

We can now write: $\frac{1}{2}kx^2 + \frac{1}{2}mv^2 = \frac{1}{2}kA^2$

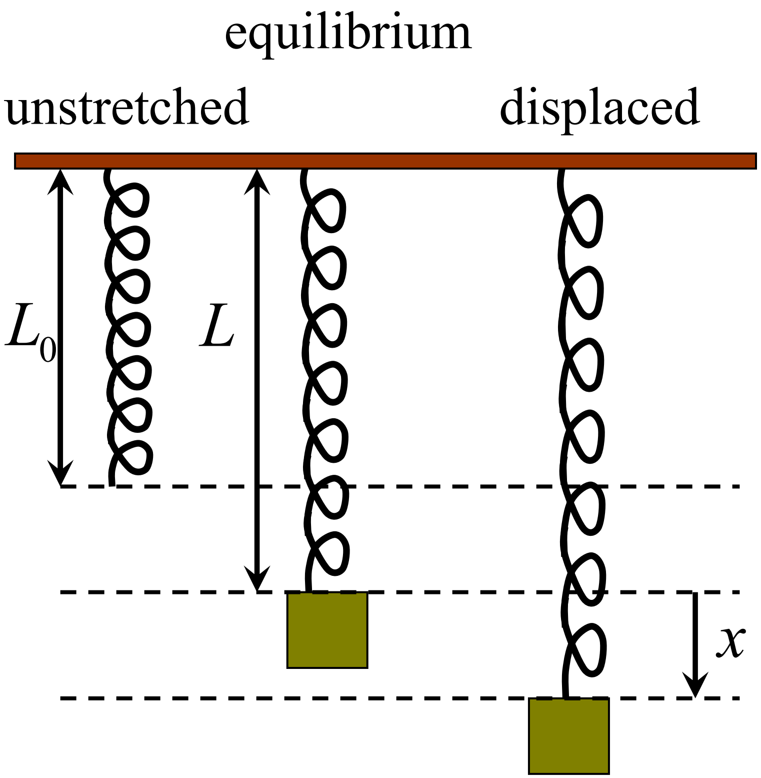
$$\therefore v = \pm\sqrt{\frac{k}{m}(A^2 - x^2)} \quad \text{or} \quad v(x) = \pm\omega_0\sqrt{A^2 - x^2}$$

(useful)

Energy of the mass-spring simple harmonic oscillator



Mass suspended from a light spring



Equilibrium: $k(L - L_0) = mg$

Displaced:

Force on mass due to spring:

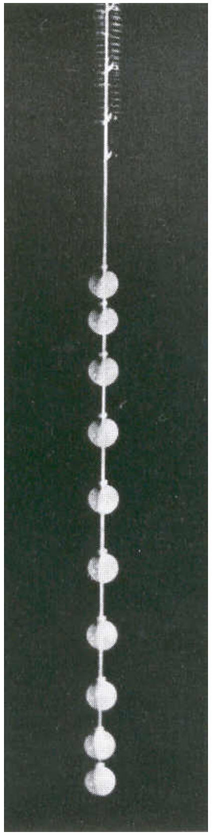
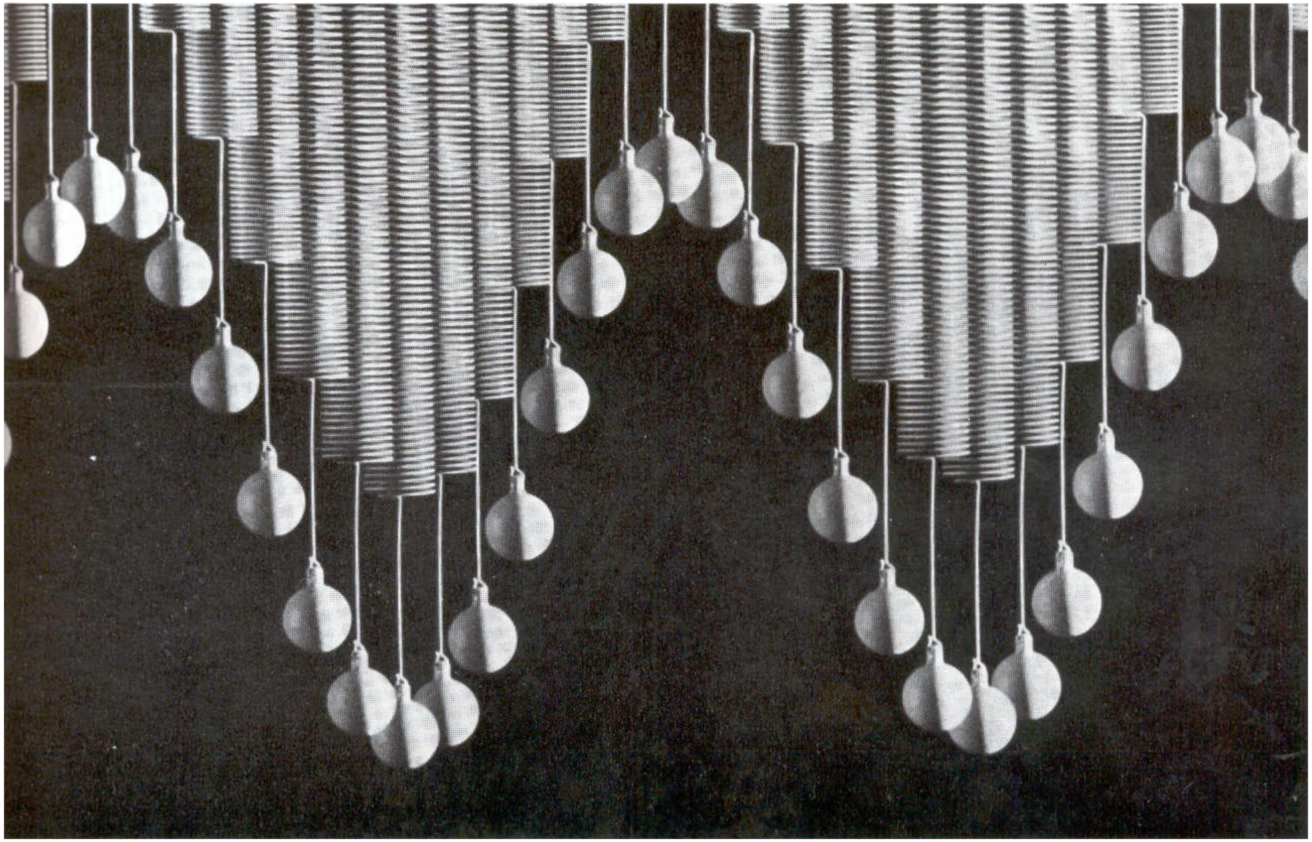
$$= k(L - L_0) + kx \quad = mg + kx$$

(upwards)

Net force on mass: $mg - (mg + kx) = -kx$ (downwards)

$$\therefore m\frac{d^2x}{dt^2} = -kx$$

(Same equation as for horizontal case)

The pendulum: general case

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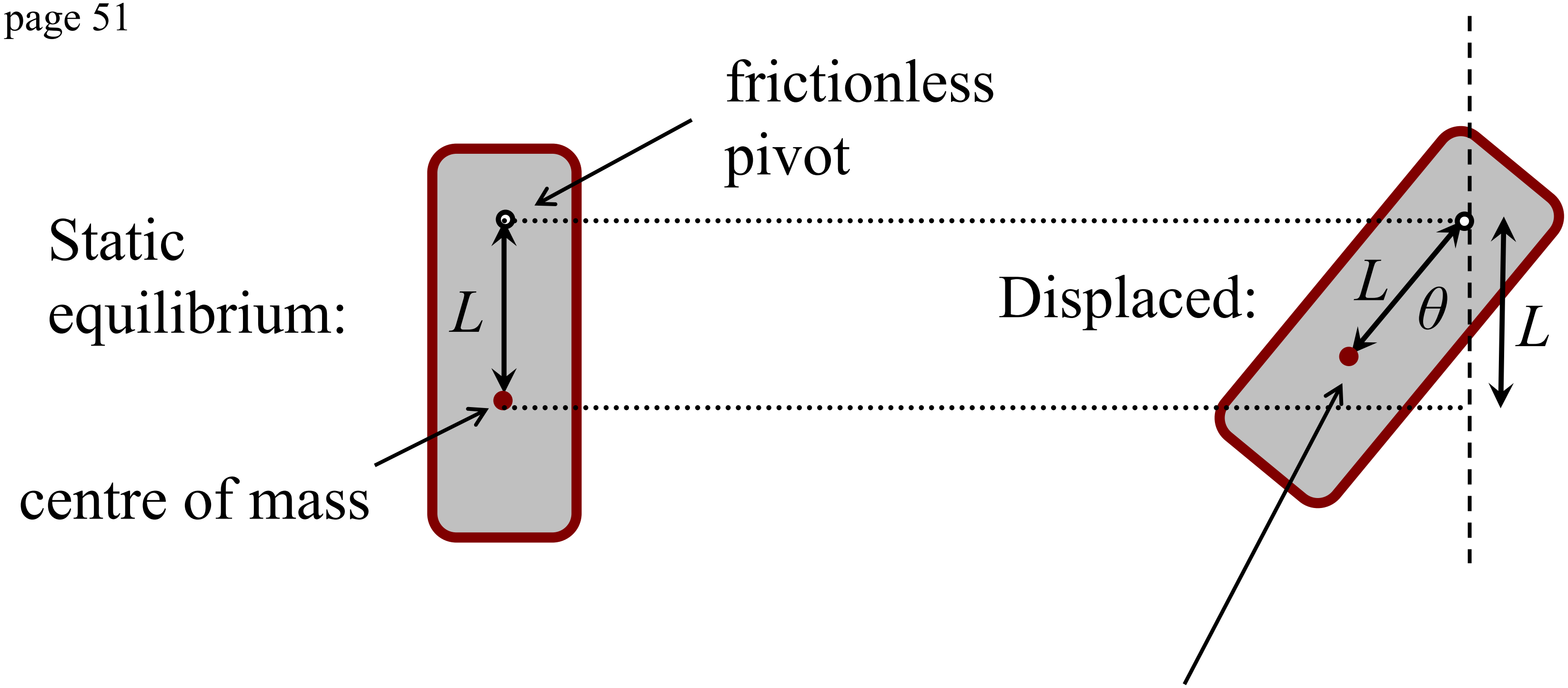


In displaced position, centre of mass is $L - L\cos\theta$ above the equilibrium position.

Recall $\cos\theta = 1 - \dfrac{\theta^2}{2!} + \dfrac{\theta^4}{4!} - \ldots$ For small angles, $\cos\theta \approx 1 - \dfrac{\theta^2}{2}$

Gravitational potential energy = $mgL(1-\cos\theta) = mgL\dfrac{\theta^2}{2}$

The pendulum: general case …2

Gravitational potential energy = $\frac{1}{2} mgL\theta^2$

Kinetic energy = $\frac{1}{2} I\left(\frac{d\theta}{dt}\right)^2$

Total energy = $\frac{1}{2} I\left(\frac{d\theta}{dt}\right)^2 + \frac{1}{2} mgL\theta^2$ = constant

$$\therefore I\frac{d\theta}{dt}\frac{d^2\theta}{dt^2} + mgL\theta\frac{d\theta}{dt} = 0$$

… true for all $\frac{d\theta}{dt}$

$$\therefore \frac{d^2\theta}{dt^2} = -\frac{mgL}{I}\theta = -\omega_0^2\theta$$

where $\omega_0 = \sqrt{\frac{mgL}{I}}$

Equation of SHM

The pendulum: general case …3

Sometimes we need the moment of inertia about an axis parallel to the axis through the centre of mass (which might be easier to calculate).

Then by the **parallel axis theorem**: $I = I_{CM} + mL^2$

... where I is the moment of inertia a distance L from the centre of mass, under the condition that the two axes of rotation are parallel.

Then for the pendulum: $\omega_0 = \sqrt{\dfrac{mgL}{I_{CM} + mL^2}}$

write: $I_{CM} = mk^2$ ... where k is the "radius of gyration"

Then $\omega_0 = \sqrt{\dfrac{gL}{k^2 + L^2}}$ and $T = 2\pi\sqrt{\dfrac{k^2 + L^2}{gL}}$

Simple harmonic systems

In general, to "show SHM" …
…. get an equation of motion of the form:

$$\boxed{\frac{d^2\vartheta}{dt^2} = -\omega_0^2\vartheta}$$

Also useful to consider conservation of energy:

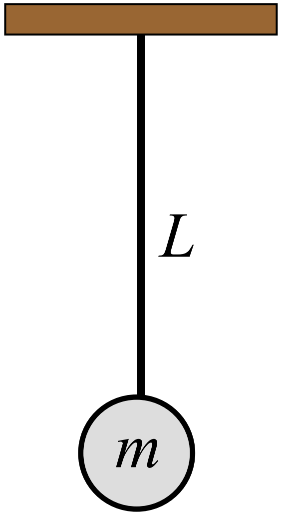
$$E = \tfrac{1}{2}(*)\frac{d^2x}{dt^2} + (\#)x^2 \qquad (E = \tfrac{1}{2}mv^2 + \underbrace{mgh}_{\frac{1}{2}kx^2})$$

then: $\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{2\#}{*}}$

Look at French pages 45-59 very carefully:

- Elasticity and Young's modulus
- Floating objects
- Pendulums
- Water in a U-tube
- Torsional oscillations
- The spring of air

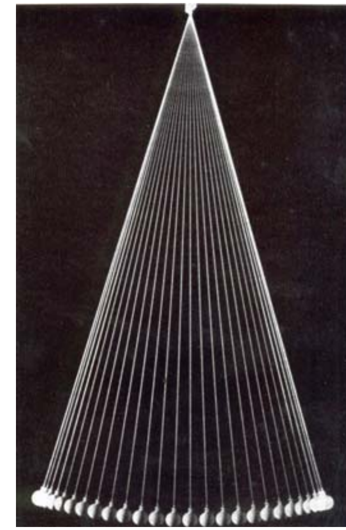
The simple pendulum



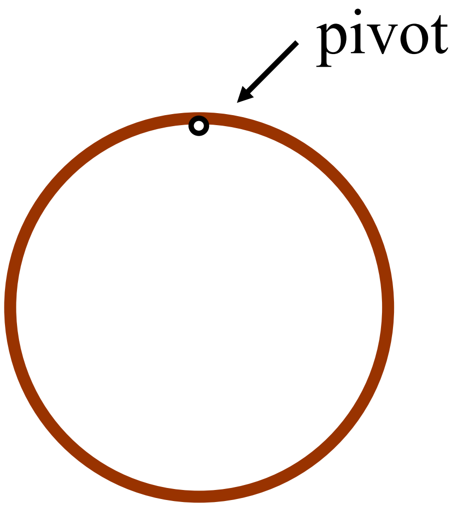
$$I = mL^2$$

$$\omega_0 = \sqrt{\frac{mgL}{mL^2}} = \sqrt{\frac{g}{L}}$$

$$T = 2\pi\sqrt{\frac{L}{g}}$$



The oscillating hoop

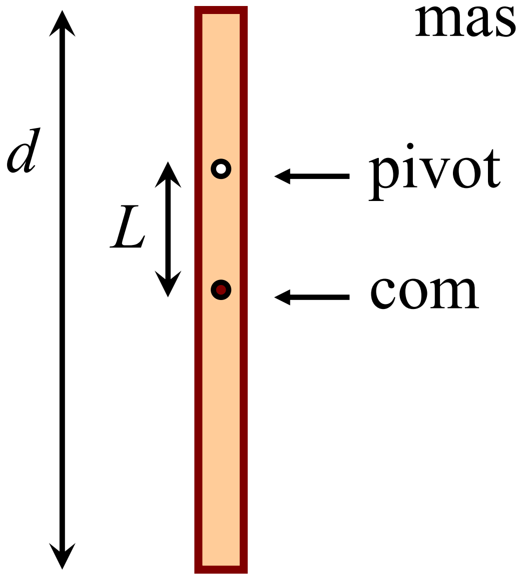


radius a, mass m

$$\omega_0 = \sqrt{\frac{g}{2a}}$$

The oscillating rod

mass m

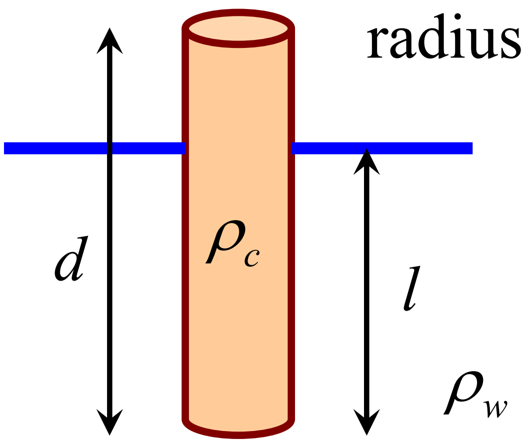


$$I = \frac{md^2}{12} + mL^2$$

$$\omega_0 = \sqrt{\frac{mgL}{\frac{1}{12}md^2 + mL^2}} = \sqrt{\frac{g}{d^2/12L + L}}$$

The wooden cylinder bouncing (vertically) in water

radius a, mass m



… can show that …

$$ma = -\left(\pi a^2 \rho_w g\right)x$$

… get … $\omega_0 = \sqrt{\frac{g}{l}}$

Elasticity

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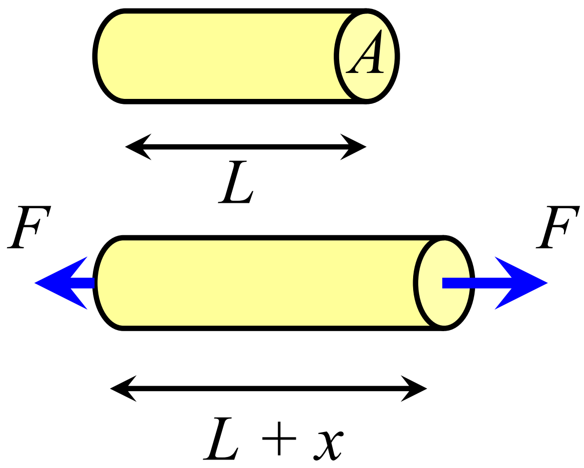
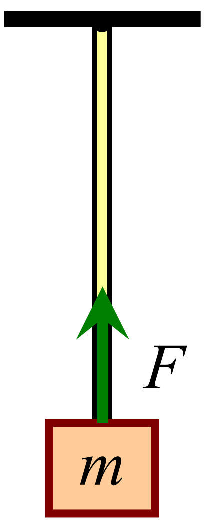


$$\frac{\text{stress}}{\text{strain}} = \text{constant for small stretching}$$

Write $$\frac{F/A}{x/L} = Y$$

Y: Young's modulus
= constant for a particular material

Elastic oscillations

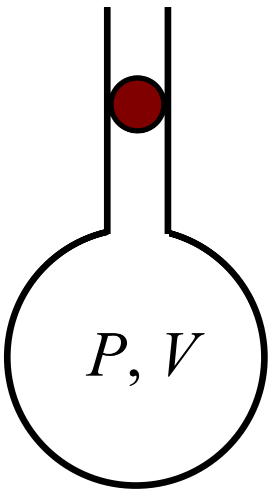


Here F is restoring force in wire

Hence $$ma = -\frac{AYx}{L}$$

and $$\omega_0 = \sqrt{\frac{AY}{mL}}$$

Ball bearing bouncing (vertically) in a cylinder of air



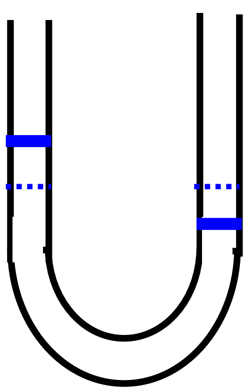
ball radius a, mass m, density ρ_b

If pressure change in flask is adiabatic:

$$PV^{\gamma} = \text{constant} \qquad \text{where} \quad \gamma = \frac{C_P}{C_V}$$

$$\omega_0 = \sqrt{\frac{3\pi\gamma Pa}{4\rho_b V}}$$

Water sloshing in a U-tube



U-tube radius a

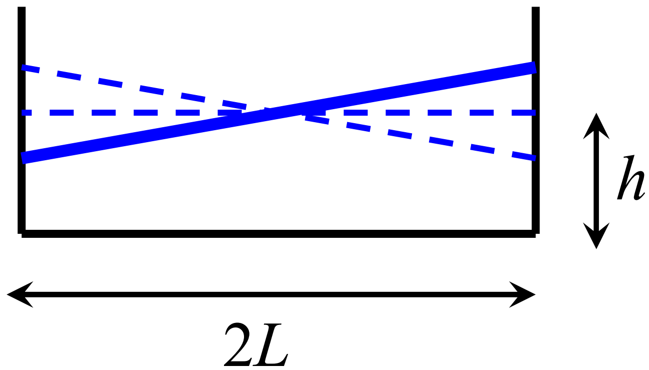
Water volume V

… use conservation of energy …

$$\omega_0 = \sqrt{\frac{2g\pi a^2}{V}}$$

Water sloshing in a rectangular pan

… tough ?



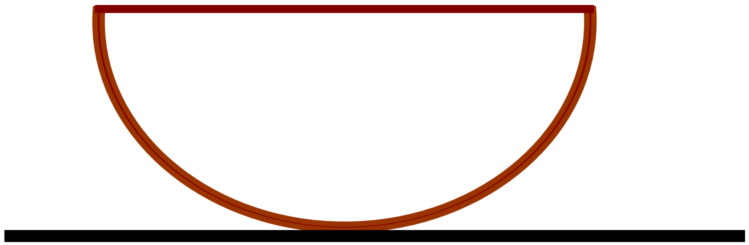
… use conservation of energy …

$$\omega_0 = \sqrt{\frac{3gh}{L^2}}$$

Rocking solid hemisphere

radius a, mass m

… tough ?



… use conservation of energy …

$$\omega_0 = \sqrt{\frac{15g}{26a}}$$

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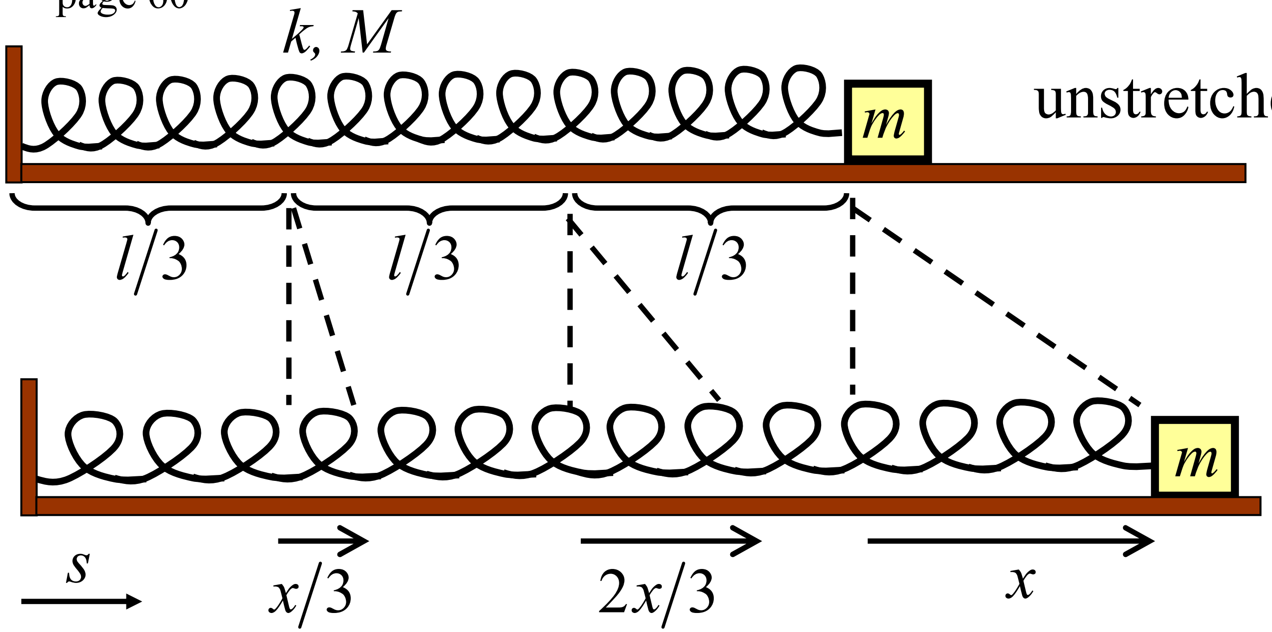
Oscillations involving massive springs



stretched
(x depends on distance s from fixed end)

$$\text{k.e. of element of spring lying between } s \text{ and } ds = \frac{1}{2}\left(\frac{M}{l}ds\right)\left(\frac{s}{l}\frac{dx}{dt}\right)^2$$

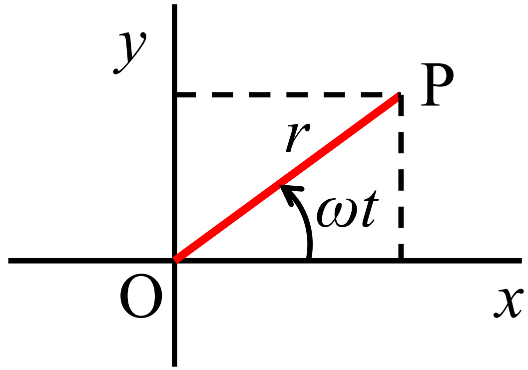
$$\text{Total kinetic energy of spring} = \frac{M}{2l^3}\left(\frac{dx}{dt}\right)^2\int_0^l s^2 ds = \frac{M}{6}\left(\frac{dx}{dt}\right)^2$$

$$\text{Total energy of mass-spring system} = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}kx^2 + \frac{M}{6}\left(\frac{dx}{dt}\right)^2$$

… giving $\omega_0 = \sqrt{\dfrac{k}{m+M/3}}$

Complex numbers

Consider a vector $\overrightarrow{\mathrm{OP}}$ of length r which rotates with angular velocity ω



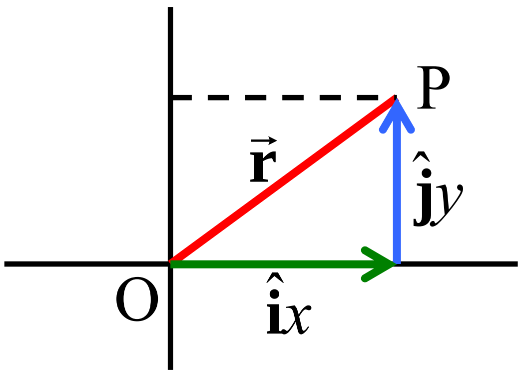
The point P has coordinates

$$x = r\cos\omega t \qquad\qquad y = r\sin\omega t$$

We see that the x coordinate of P, or the projection of $\overrightarrow{\mathrm{OP}}$ onto the x-axis, executes SHM

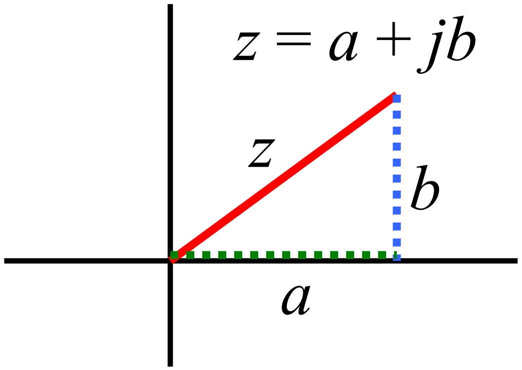
Can also introduce the unit vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$

and write $\vec{\mathbf{r}} = \hat{\mathbf{i}}x + \hat{\mathbf{j}}y$



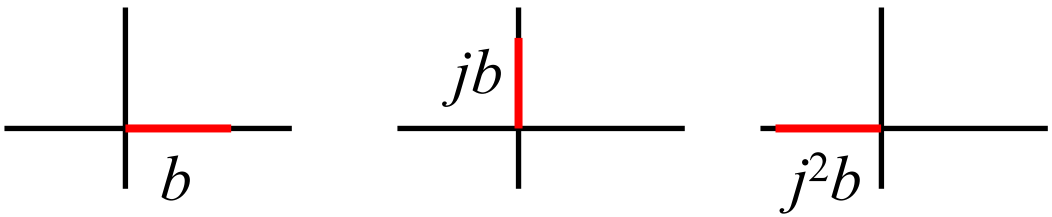
Complex numbers …2

Modify our notation to $z = x + jy$



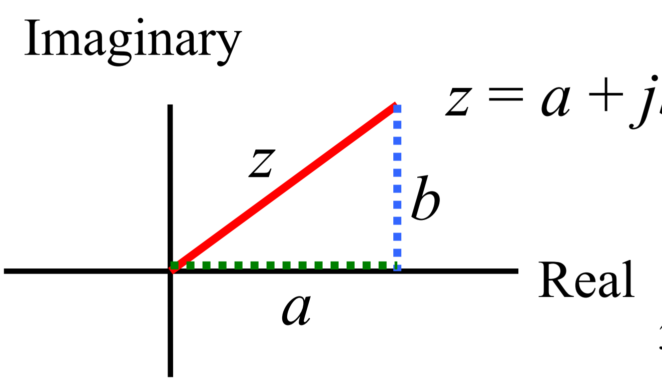
... where x means a displacement in the x-direction and jy means a displacement in the y-direction

We can also think of j as a rotation through $\pi/2$ anticlockwise



Hence $j^2 = -1$

... really talking about vectors in the complex number plane:



Complex numbers …3

From Taylor's theorem: $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$

therefore $e^{j\theta} = 1 + j\theta - \frac{\theta^2}{2!} - \frac{j\theta^3}{3!} + \frac{\theta^4}{4!} + \ldots$

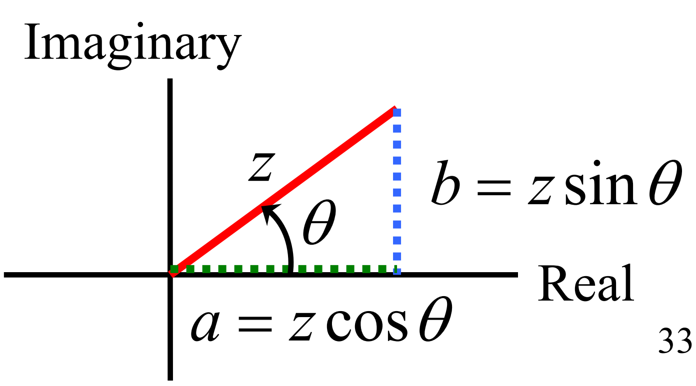
and $\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \ldots$ and $j\sin\theta = j\theta - \frac{j\theta^3}{3!} + \ldots$

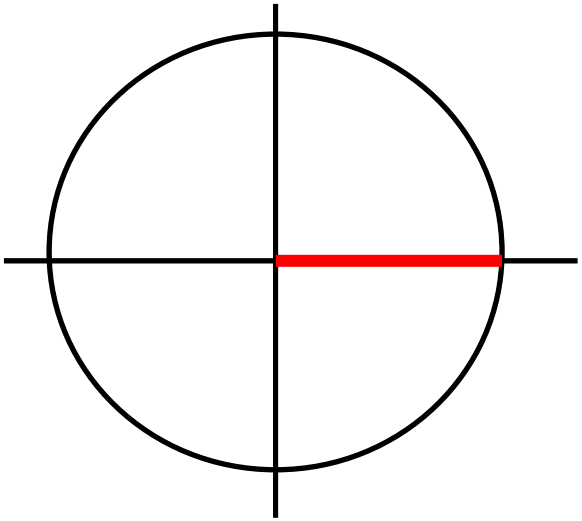
Hence $\boxed{e^{j\theta} = \cos\theta + j\sin\theta}$ **Euler relation**

Then $z = a + jb = |z| e^{j\theta}$

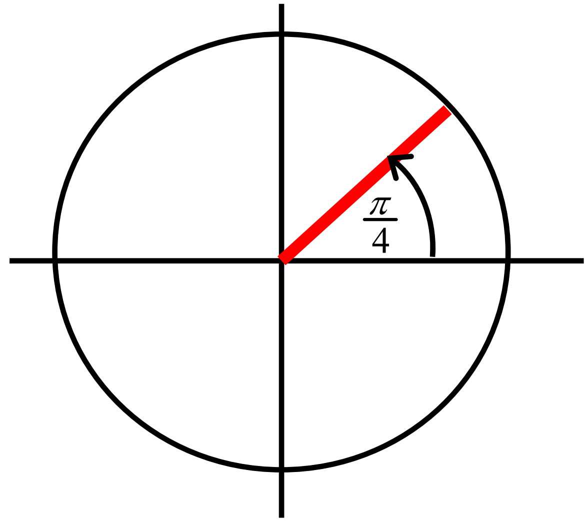
where $|z| = \sqrt{a^2 + b^2}$

$\tan\theta = b/a$

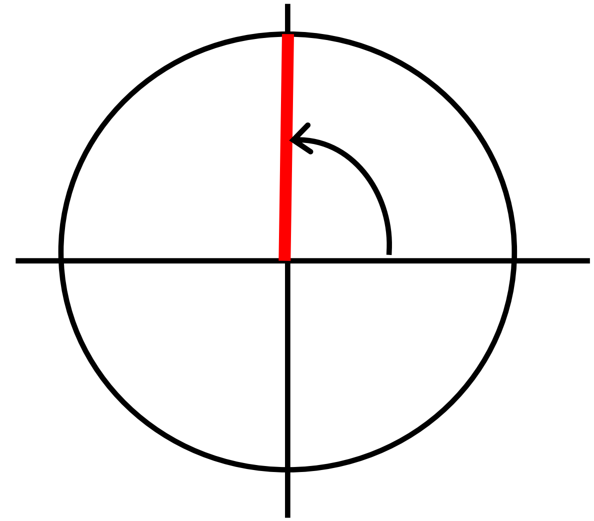




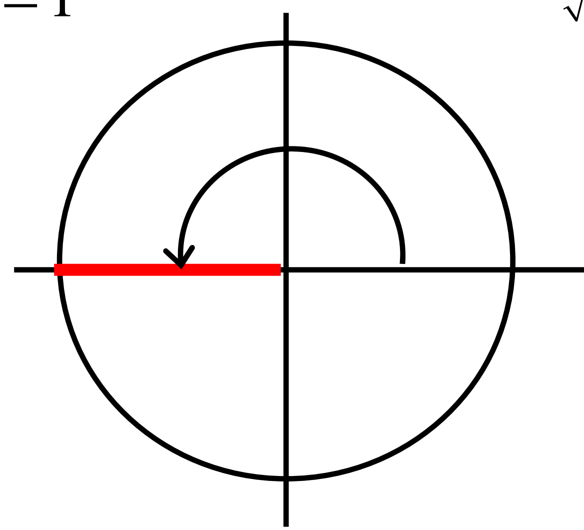
$$e^{j0} = 1$$



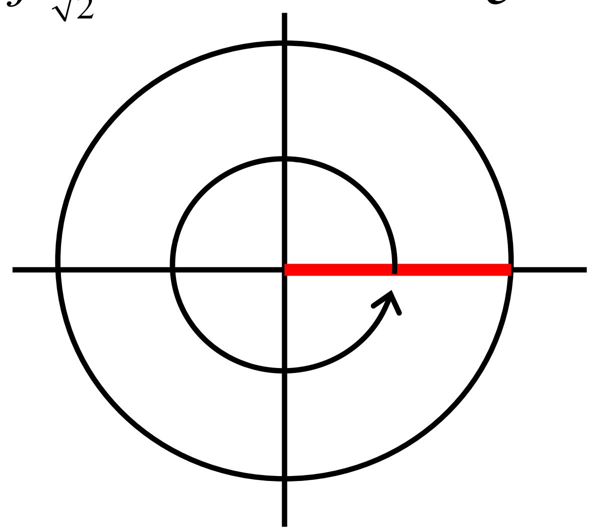
$$e^{j\frac{\pi}{4}} = \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}$$



$$e^{j\frac{\pi}{2}} = j$$



$$e^{j\pi} = -1$$

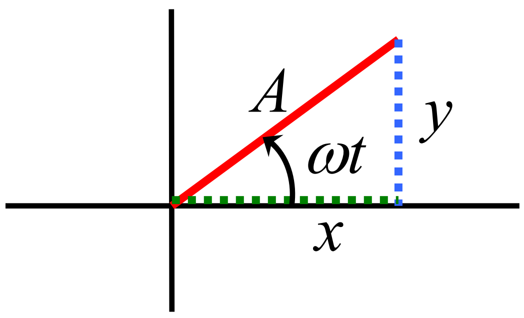


$$e^{j2\pi} = 1$$

Complex numbers …4

For our rotating vectors:

$$z = x + jy$$
$$= A\cos\omega t + jA\sin\omega t$$
$$= A\left(\cos\omega t + j\sin\omega t\right)$$
$$= Ae^{j\omega t}$$



Now write: $Ae^{j(\omega t+\phi)} = A\cos(\omega t+\phi) + jA\sin(\omega t+\phi)$

... and remember that the physical quantity x (e.g. a displacement) is the real part of z :

i.e. $x = \mathrm{Re}[z]$

French
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SHM using complex numbers

$$\frac{d^2x}{dt^2} + \omega_0^2 x = 0$$

Using $z = x + jy$ becomes $\frac{d^2z}{dt^2} + \omega_0^2 z = 0$

Try $z = Ae^{j(\omega t + \phi)}$

$$\therefore A(j\omega)^2 e^{j(\omega t+\phi)} + \omega^2 A e^{j(\omega t+\phi)} = 0$$

Therefore $z = Ae^{j(\omega t+\phi)}$ is the most general solution

A and ϕ are determined from the initial conditions.

Take real part of z:

$$x = \mathrm{Re}[z] = A\cos(\omega_0 t + \phi)$$

SHM using complex numbers

$$x = A\cos(\omega_0 t + \phi)$$

$$z = Ae^{j(\omega t+\phi)}$$

$$x = \mathrm{Re}[z]$$

$$\frac{dx}{dt} = -A\omega_0 \sin(\omega_0 t + \phi)$$

$$\frac{dz}{dt} = j\omega Ae^{j(\omega t+\phi)} = j\omega z$$

$$\frac{dx}{dt} = \mathrm{Re}\left[\frac{dz}{dt}\right]$$

$$\frac{d^2x}{dt^2} = -A\omega_0{}^2 \cos(\omega_0 t + \phi)$$

$$\frac{d^2z}{dt^2} = \left(j\omega\right)^2 Ae^{j(\omega t+\phi)} = -\omega^2 z$$

$$\frac{d^2x}{dt^2} = \mathrm{Re}\left[\frac{d^2z}{dt^2}\right]$$

Solving $\dfrac{d^2x(t)}{dt^2}+\omega_0^2 x(t)=0$

Let $x = Be^{pt}$

Then $\dfrac{dx}{dt} = Bpe^{pt}$ and $\dfrac{d^2x}{dt^2} = Bp^2e^{pt}$

Substituting into DE: $Bp^2e^{pt} + B\omega_0^2 e^{pt} = 0$

This holds true for all t if and only if $p^2 = -\omega_0^2$ or $p = \pm j\omega_0$

$$\therefore x = B_1 e^{j\omega_0 t} + B_2 e^{-j\omega_0 t}$$

How to get B_1 and B_2 ? … need to know the initial conditions

… but consider $v = \dfrac{dx}{dt} = j\omega_0 B_1 e^{j\omega_0 t} - j\omega_0 B_2 e^{-j\omega_0 t}$

Solving $\dfrac{d^2x(t)}{dt^2}+\omega_0^2 x(t)=0$ continued

$$v=\frac{dx}{dt}=j\omega_0 B_1 e^{j\omega_0 t}-j\omega_0 B_2 e^{-j\omega_0 t}$$

At $t=0$, $v(0)=j\omega B_1-j\omega_0 B_2$

Choose $v(0)=0$

Since v must be real, then $B_1=B_2=B$

i.e. $x=Be^{j\omega_0 t}+Be^{-j\omega_0 t}$

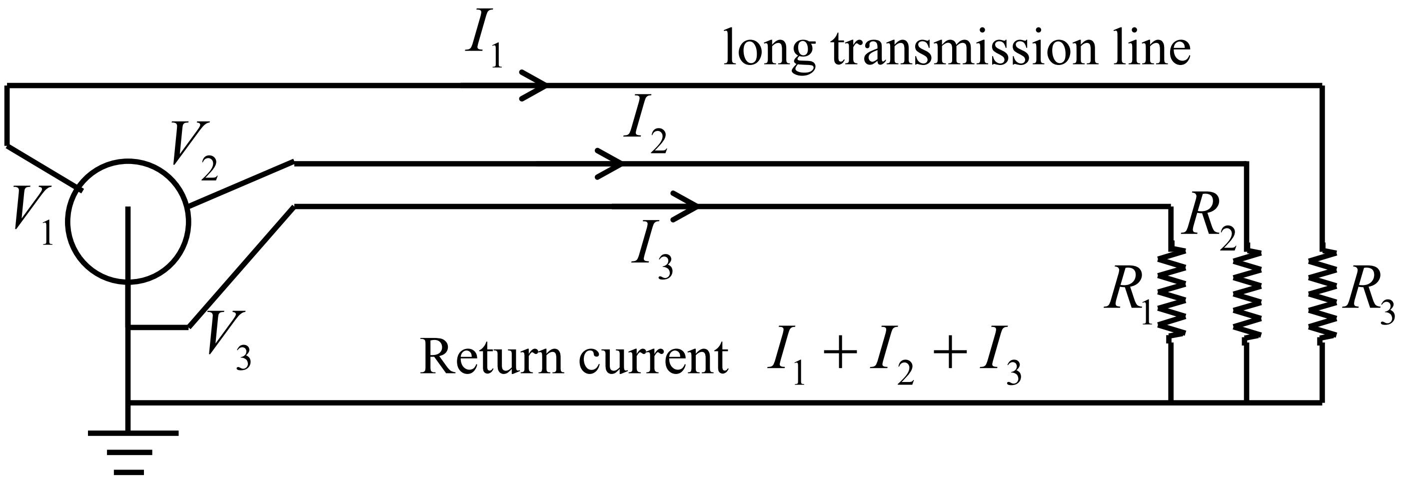
$$=2B\cos\omega_0 t$$

$$\therefore x=A\cos\omega_0 t$$

[… with a little more effort we could have got the more general solution $x=A\cos\left(\omega_0 t+\phi\right)$]

Example: 3-phase current

Large scale electrical power transmission makes use of "3 phases"



The three voltages are 120° out of phase with each other

$$V_1 = V_0 \cos(\omega t)$$

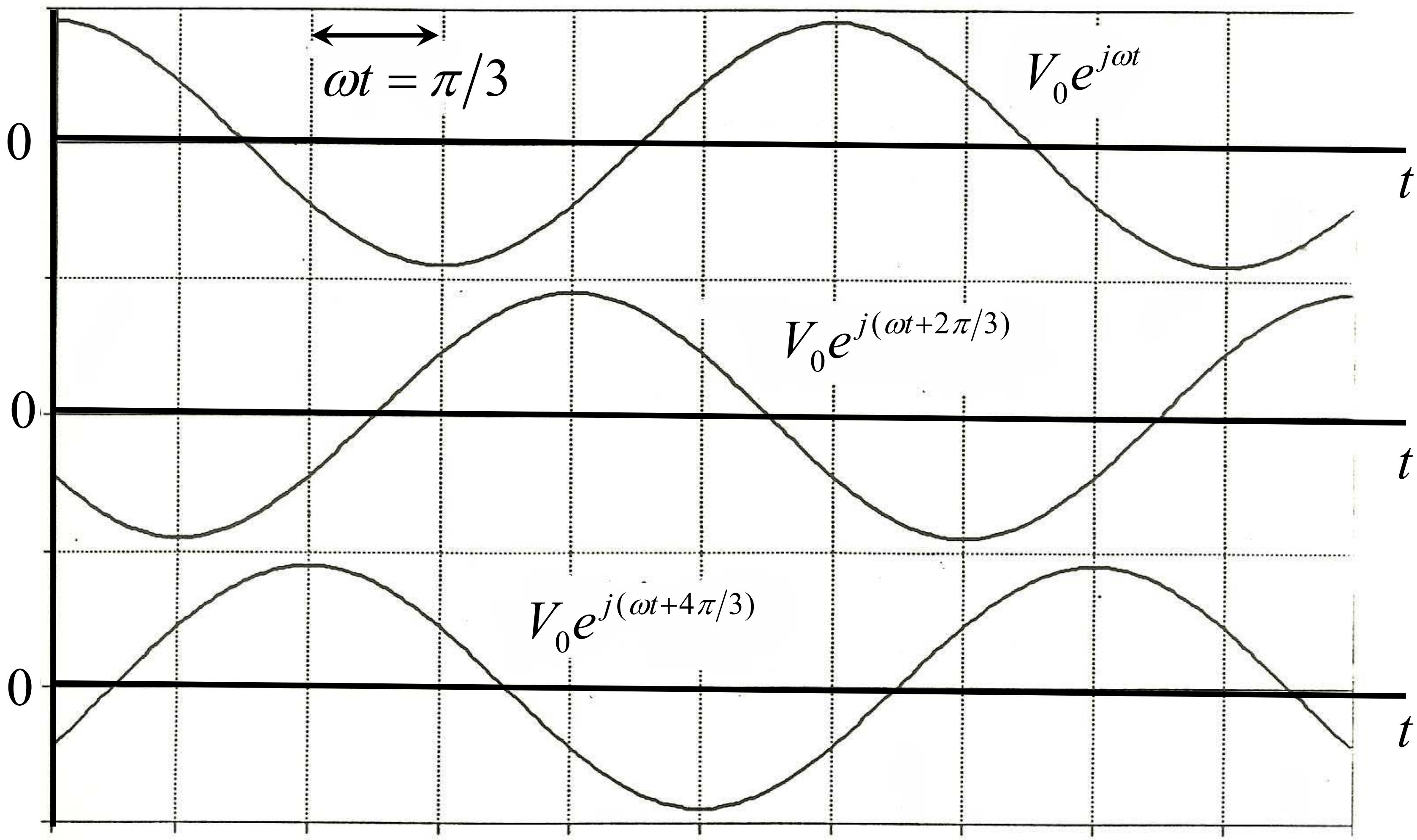
$$V_2 = V_0 \cos(\omega t + 2\pi/3)$$

$$V_3 = V_0 \cos(\omega t + 4\pi/3)$$

3-phase current …2

Use complex numbers:

$$V_1 = V_0 e^{j\omega t}$$
$$V_2 = V_0 e^{j(\omega t + 2\pi/3)}$$
$$V_3 = V_0 e^{j(\omega t + 4\pi/3)}$$



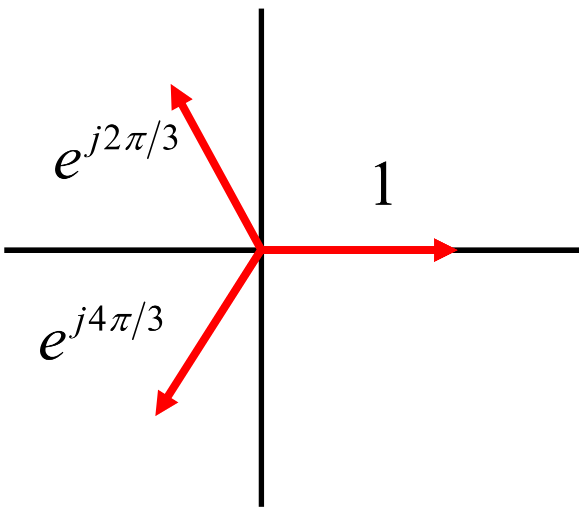
3-phase current …3

The currents are given by: $I_1 = V_1/R_1$ $\quad I_2 = V_2/R_2$ $\quad I_3 = V_3/R_3$

Consider important case where $R_1 = R_2 = R_3 = R$

Then $\quad I_1 = \dfrac{V_0}{R} e^{j\omega t}$ $\quad I_2 = \dfrac{V_0}{R} e^{j(\omega t + 2\pi/3)}$ $\quad I_3 = \dfrac{V_0}{R} e^{j(\omega t + 4\pi/3)}$

Return current $I = I_1 + I_2 + I_3 \quad = \dfrac{V_0}{R} e^{j\omega t} \underbrace{\left\{1 + e^{j2\pi/3} + e^{j4\pi/3}\right\}}_{=0}$



Why is this useful ? ... and cost-saving?

A short diversion … A.C.

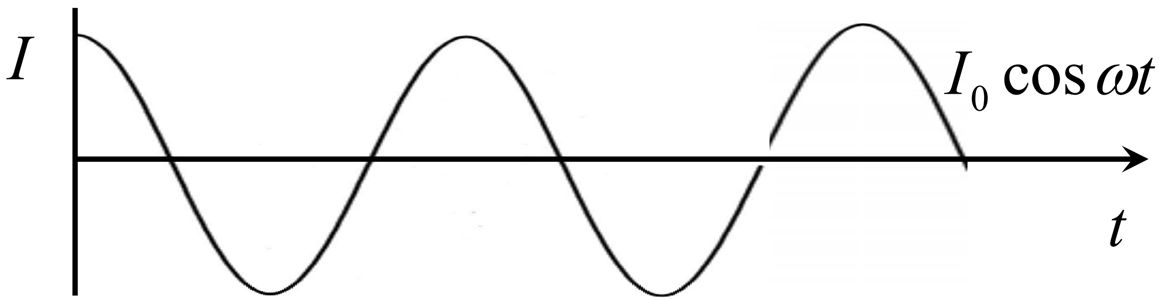
Consider the heating effect of alternating current in a resistor.

At any instant $P = VI = I^2 R$

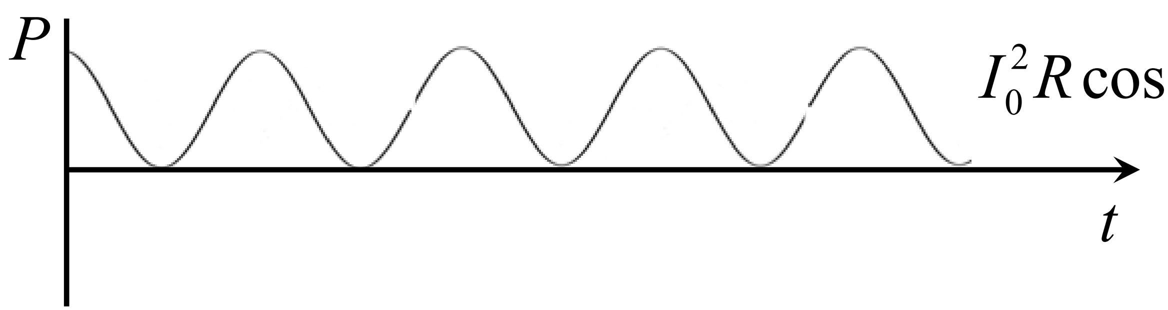
For $I = I_0 \cos \omega t$, $P = I_0^2 R \cos^2 \omega t$

Then the average power heating: $\overline{P} = I_0^2 R \overline{\cos^2 \omega t}$

$$\overline{\cos \omega t} = 0$$



$$\overline{\cos^2 \omega t} = \tfrac{1}{2}$$



A.C. …2

Then the **average power heating**: $\overline{P} = I_0^2 R \overline{\cos^2 \omega t}$

$$= \frac{1}{2} I_0^2 R = \left(\frac{I_0}{\sqrt{2}} \right)^2 R$$

… a direct current of magnitude $I_0 / \sqrt{2}$ would have the same heating power as an alternating current of amplitude I_0 .
… $I_0 / \sqrt{2}$ is known as the "effective" value of the current, or the **root mean square value.**

With respect to the 50 Hz a.c. mains in South Africa:

… the r.m.s. voltage is quoted as 220 volts

Hence $V_0 / \sqrt{2} = 220$

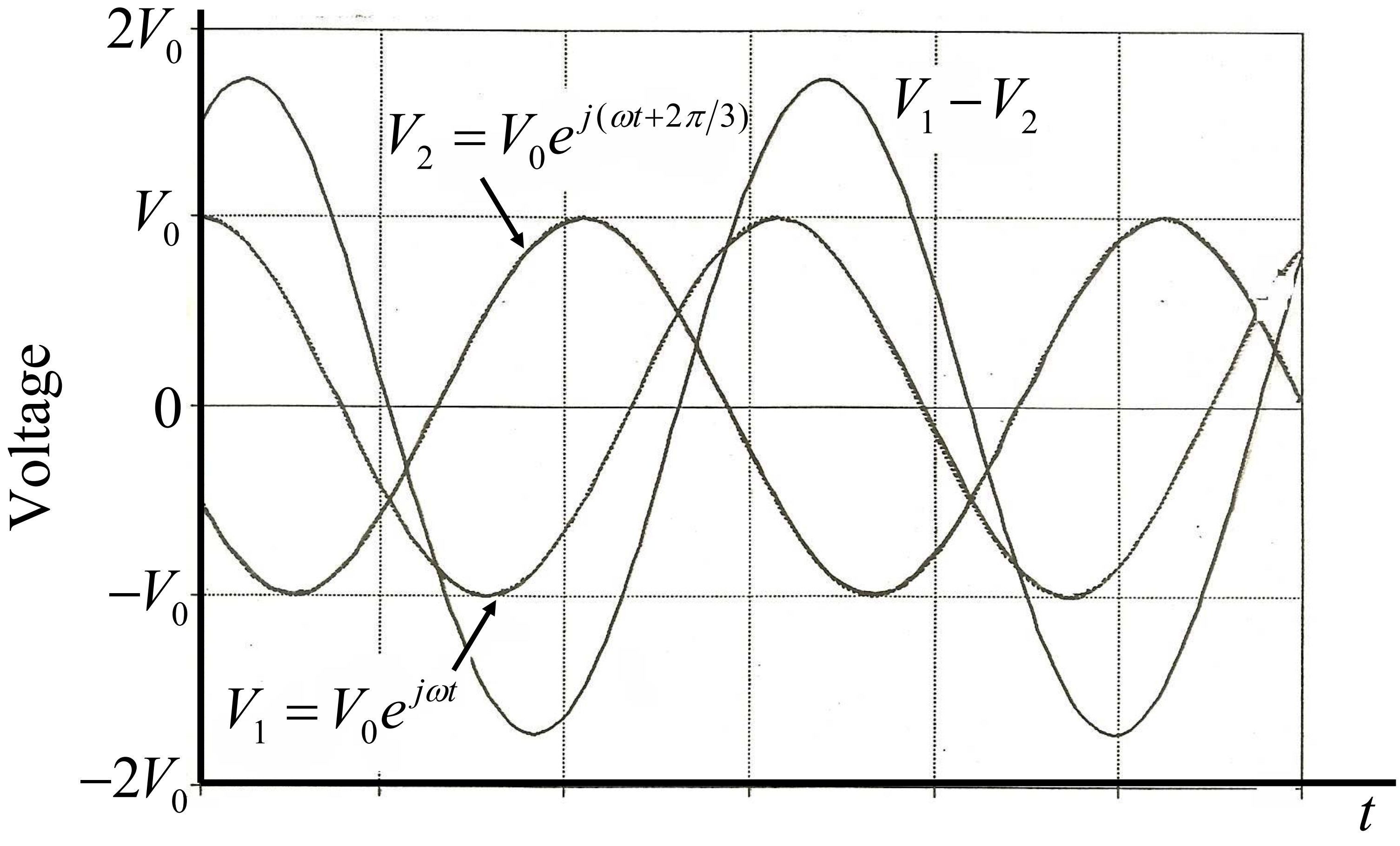
$$V_0 = 220\sqrt{2} \;\; = 311 \text{ volts}$$

… back to three phase power …

What is the maximum voltage between two of the lines?

… e.g. what is $\left(V_1 - V_2\right)_{\text{maximum}}$?

$$V_1 = V_0 e^{j\omega t}$$
$$V_2 = V_0 e^{j(\omega t + 2\pi/3)}$$
$$V_3 = V_0 e^{j(\omega t + 4\pi/3)}$$



A.C. …4

$\left(V_1 - V_2\right)_{\text{maximum}}$ in three phase power ?

$$V_1 = V_0 e^{j\omega t}$$
$$V_2 = V_0 e^{j(\omega t + 2\pi/3)}$$

$$V_1 - V_2 = V_0 e^{j\omega t} - V_0 e^{j(\omega t + 2\pi/3)}$$

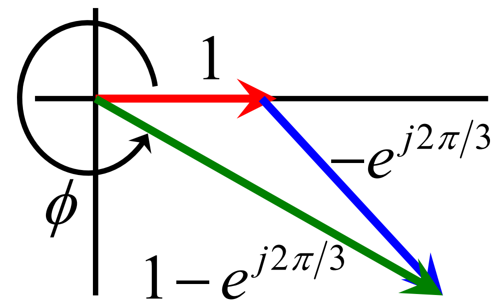
$$= V_0 e^{j\omega t}\left\{1 - e^{j2\pi/3}\right\}$$

$$= V_0 e^{j\omega t}\left\{1 - \cos\frac{2\pi}{3} - j\sin\frac{2\pi}{3}\right\}$$

$$= V_0 e^{j\omega t}\left\{1 + \frac{1}{2} - j\frac{\sqrt{3}}{2}\right\}$$

$$= V_0 e^{j\omega t}\left\{1 + \frac{1}{2} - j\frac{\sqrt{3}}{2}\right\}$$

$$= V_0 e^{j\omega t}\sqrt{3}e^{j\phi}$$



$$\left|1 - e^{j2\pi/3}\right| = \sqrt{\left(\tfrac{3}{2}\right)^2 + \left(\tfrac{\sqrt{3}}{2}\right)^2}$$

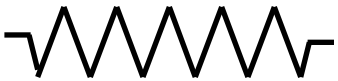
$$= \sqrt{3}$$

$$1 - e^{j2\pi/3} = \sqrt{3}e^{j\phi}$$

$$\therefore \left(V_1 - V_2\right)_{\text{maximum}} = \left(220\sqrt{2}\right)\sqrt{3} \qquad = 539 \text{ volts}$$

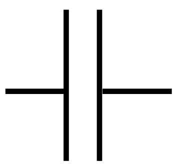
Oscillations in electric circuits

Resistor R



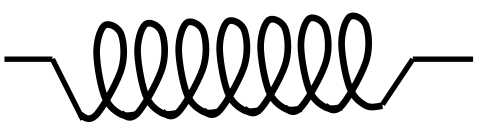
$$|\Delta V| = IR \qquad P = I^2 R$$

Capacitor C



$$|\Delta V| = \frac{Q}{C} \qquad W = \tfrac{1}{2}\frac{Q^2}{C}$$

Inductor L

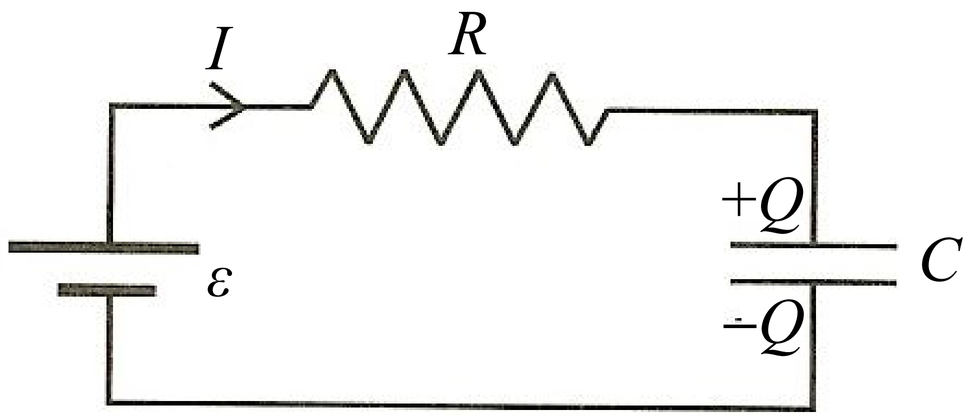


$$|\Delta V| = L\frac{dI}{dt} \qquad W = \tfrac{1}{2}LI^2$$

where $I = \dfrac{dQ}{dt}$

Charging a capacitor through a resistor

Suppose the capacitor is uncharged at $t = 0$.
At some time $t > 0$, a current I is established and a charge Q builds up on the capacitor.



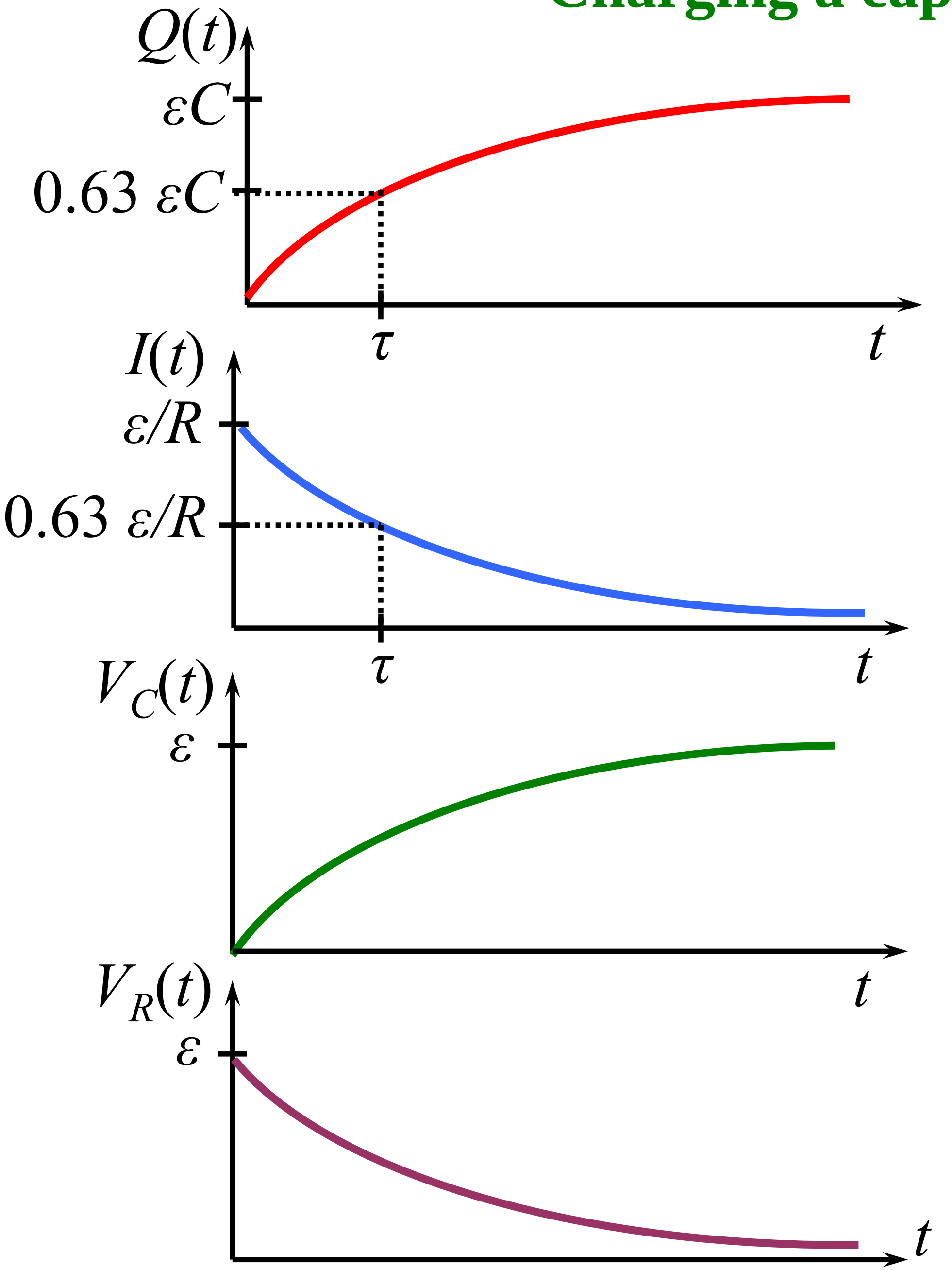
Energy equation for this circuit: $$\Delta V_{\text{round trip}} = \varepsilon - RI - \frac{Q}{C} = 0$$

or $$\varepsilon = RI + \frac{Q}{C} = R\frac{dQ}{dt} + \frac{Q}{C}$$

Differential equation

Solution: $$Q(t) = C\varepsilon\left(1 - e^{-\frac{t}{RC}}\right)$$

Charging a capacitor through resistor …2



$$Q(t) = C\varepsilon\left(1 - e^{-\frac{t}{RC}}\right)$$

$RC =$ **time constant**

$$I(t) = \frac{dQ(t)}{dt}$$

$$= \frac{\varepsilon}{R} e^{-\frac{t}{RC}} = \frac{\varepsilon}{R} e^{-\frac{t}{\tau}}$$

$$V_C = \left(\varepsilon - V_R\right) = \left(\varepsilon - IR\right)$$

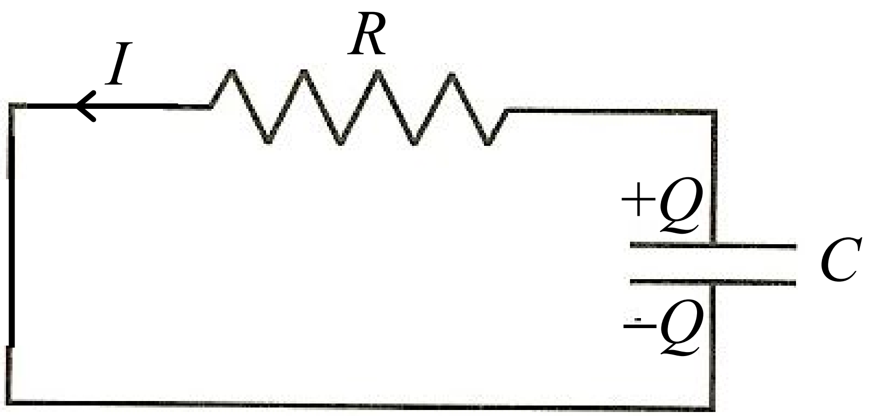
$$= \varepsilon(1 - e^{-\frac{t}{\tau}})$$

$$V_R = \left(\varepsilon - V_C\right)$$

$$= \varepsilon e^{-\frac{t}{\tau}}$$

Discharging a capacitor through resistor

Suppose the capacitor has charge Q_0 at $t = 0$.
At some time $t > 0$, a current is established as the charge drains off the capacitor.



Energy equation for this circuit: $$\Delta V_{\text{round trip}} = RI + \frac{Q}{C} = 0$$

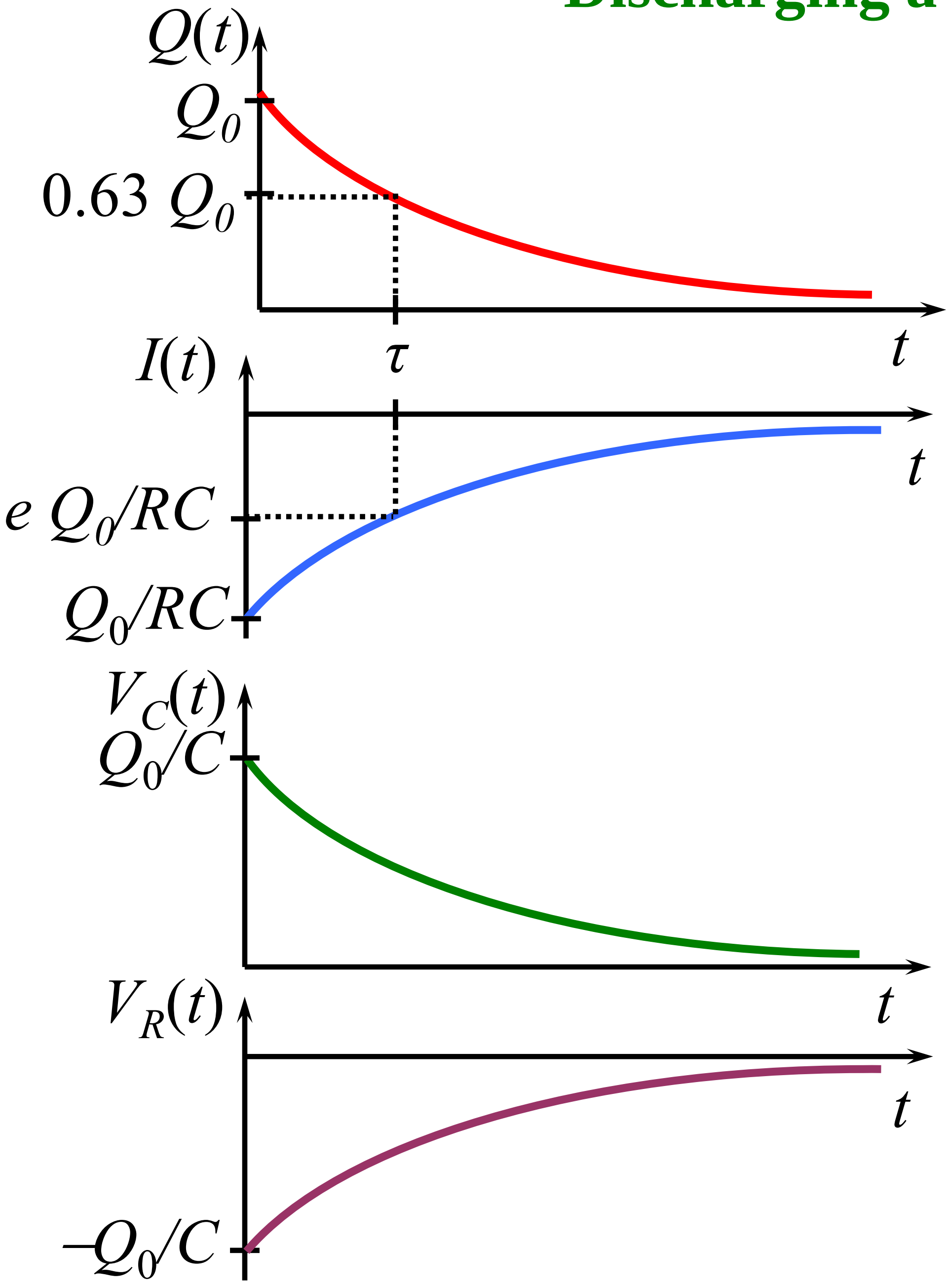
or $$0 = RI + \frac{Q}{C} = R\frac{dQ}{dt} + \frac{Q}{C}$$

Write $$\int \frac{dQ}{Q} = -\frac{1}{RC}\int dt$$ giving $$\ln Q = -\frac{1}{RC}t + K$$

$$K = \ln Q_0$$

or $$Q(t) = Q_0 e^{-\frac{t}{RC}}$$

Discharging a capacitor through resistor …2



$$Q(t) = Q_0 e^{-\frac{t}{\tau}}$$

RC = **time constant**

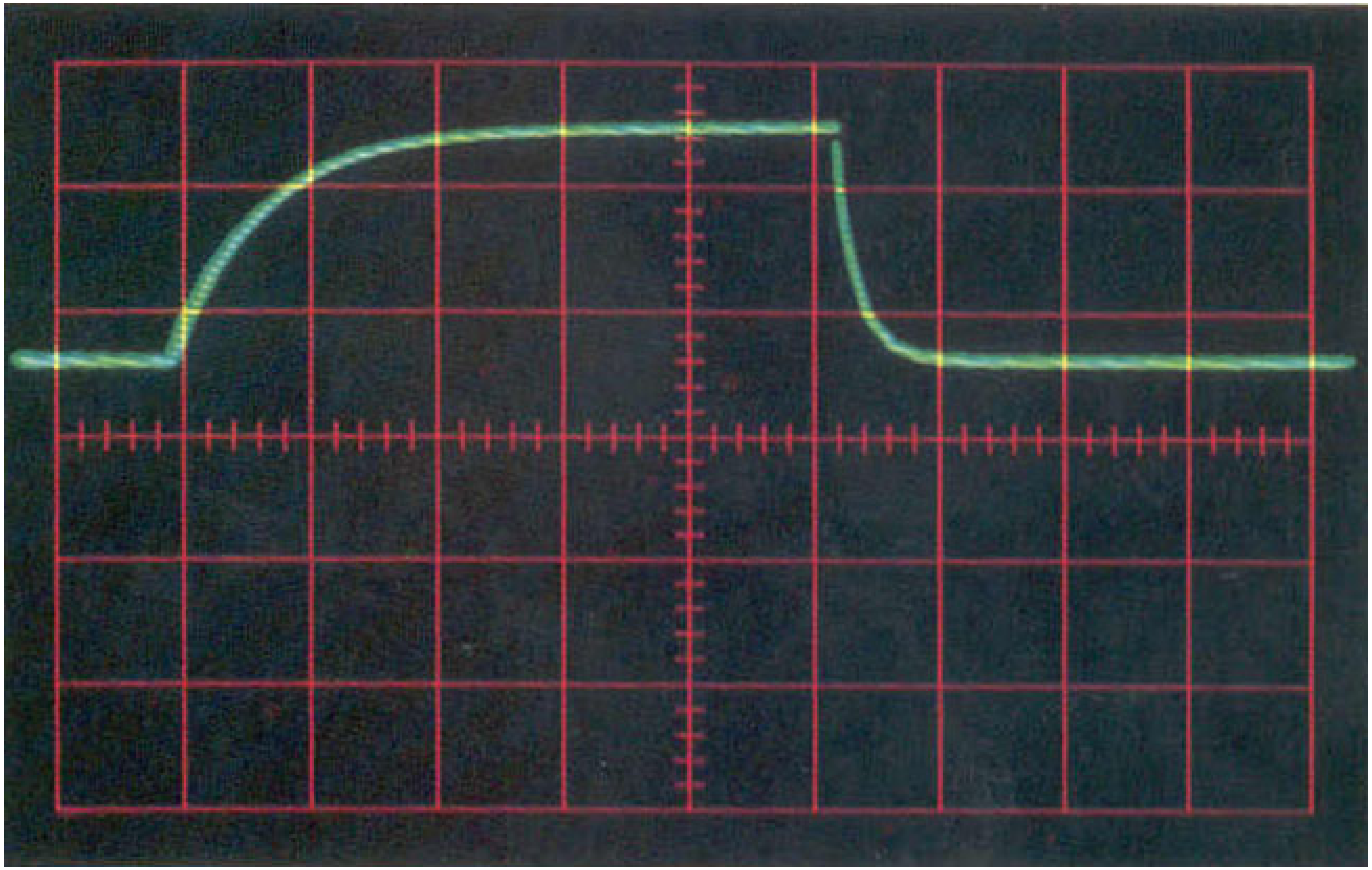
$$I(t) = \frac{dQ(t)}{dt}$$

$$= -\frac{Q_0}{RC} e^{-\frac{t}{RC}} = -\frac{Q_0}{RC} e^{-\frac{t}{\tau}}$$

$$V_C = \frac{Q}{C} = \frac{Q_0}{C} e^{-\frac{t}{\tau}}$$

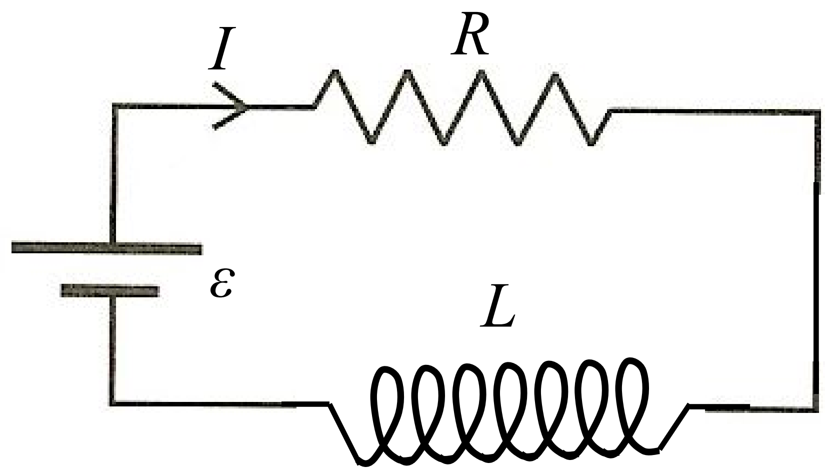
$$V_R = -IR = -\frac{RQ_0}{RC} e^{-\frac{t}{\tau}}$$

$$= -\frac{Q_0}{C} e^{-\frac{t}{\tau}}$$



Establishing a current in an inductor

Suppose the current in the circuit is zero at $t = 0$. At some time $t > 0$, a current I is established.



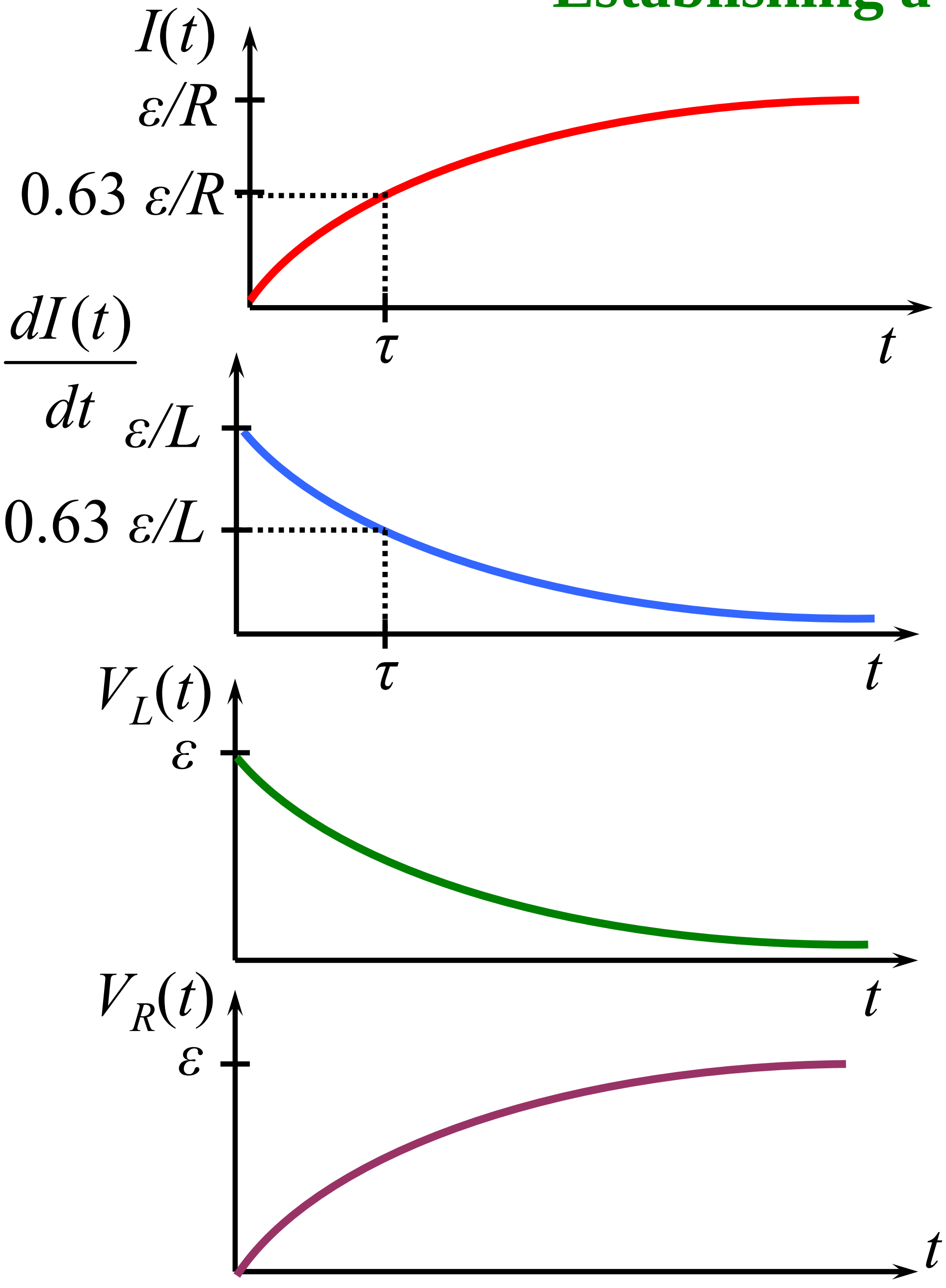
Energy equation for this circuit: $$\Delta V_{\text{round trip}} = \varepsilon - RI - L\frac{dI}{dt} = 0$$

or $$\frac{dI}{dt} + \left(\frac{R}{L}\right)I = \frac{\varepsilon}{L}$$

Differential equation

Solution: $$I(t) = \frac{\varepsilon}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$

Establishing a current in an inductor …2



$$I(t) = \frac{\varepsilon}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$

R/L = **time constant**

$$\frac{dI(t)}{dt} = \frac{\varepsilon}{L} e^{-\frac{R}{L}t}$$

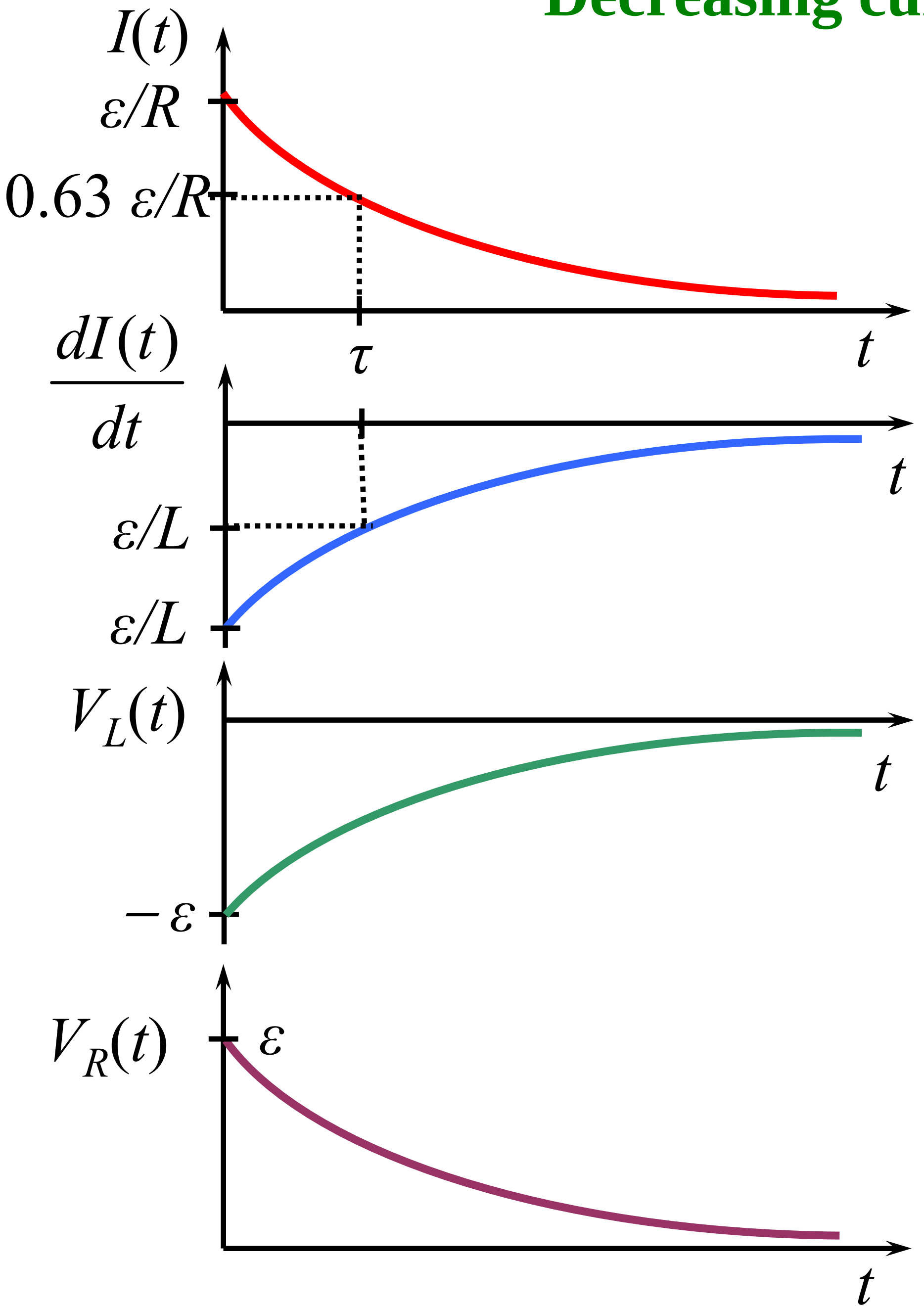
$$V_L = (\varepsilon - V_R) = (\varepsilon - IR)$$

$$= \varepsilon e^{-\frac{t}{\tau}}$$

$$V_R = (\varepsilon - V_L)$$

$$= \varepsilon(1 - e^{-\frac{t}{\tau}})$$

Decreasing current in an inductor



$$I(t) = C\varepsilon\left(1 - e^{-\frac{R}{L}t}\right)$$

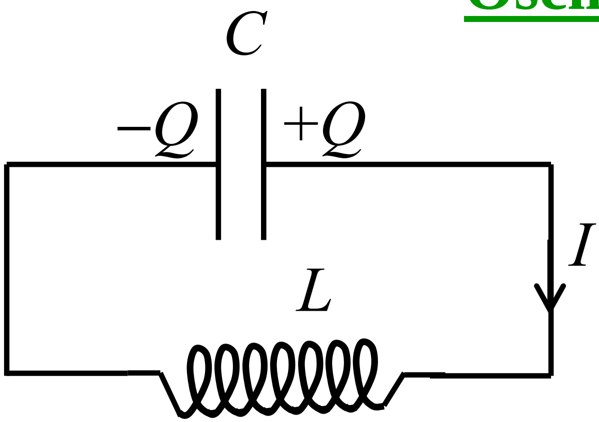
R/L = **time constant**

$$\frac{dI(t)}{dt} = -\frac{\varepsilon}{L}e^{-\frac{R}{L}t}$$

$$V_L = L\frac{dI(t)}{dt} = -\varepsilon e^{-\frac{R}{L}t}$$

$$V_R = IR = \varepsilon e^{-\frac{t}{\tau}}$$

Oscillations in an LC circuit



Capacitor fully charged at $t = 0$. At $t > 0$, charge on capacitor decreases, giving current in the direction shown.

$$I = -\frac{dQ}{dt}$$

$$\left|\Delta V\right|_C = \left|\Delta V\right|_L$$

$$\therefore \frac{Q}{C} = L\frac{dI}{dt} = L\frac{d}{dt}\left(-\frac{dQ}{dt}\right)$$

Hence: $\dfrac{d^2Q}{dt^2} = -\dfrac{1}{LC}Q$ … compare with $\dfrac{d^2x}{dt^2} = -\dfrac{k}{m}x$

Solution of the form $Q(t) = Q_0\cos(\omega_0 t + \phi)$

where $\omega_0 = \sqrt{\dfrac{1}{LC}}$

Oscillations in an *LC* circuit: an energy approach

Energy stored in capacitor: $E = \tfrac{1}{2}\dfrac{Q^2}{C}$

Energy stored in inductor: $E = \tfrac{1}{2}LI^2$

No dissipation (circuit has zero resistance)

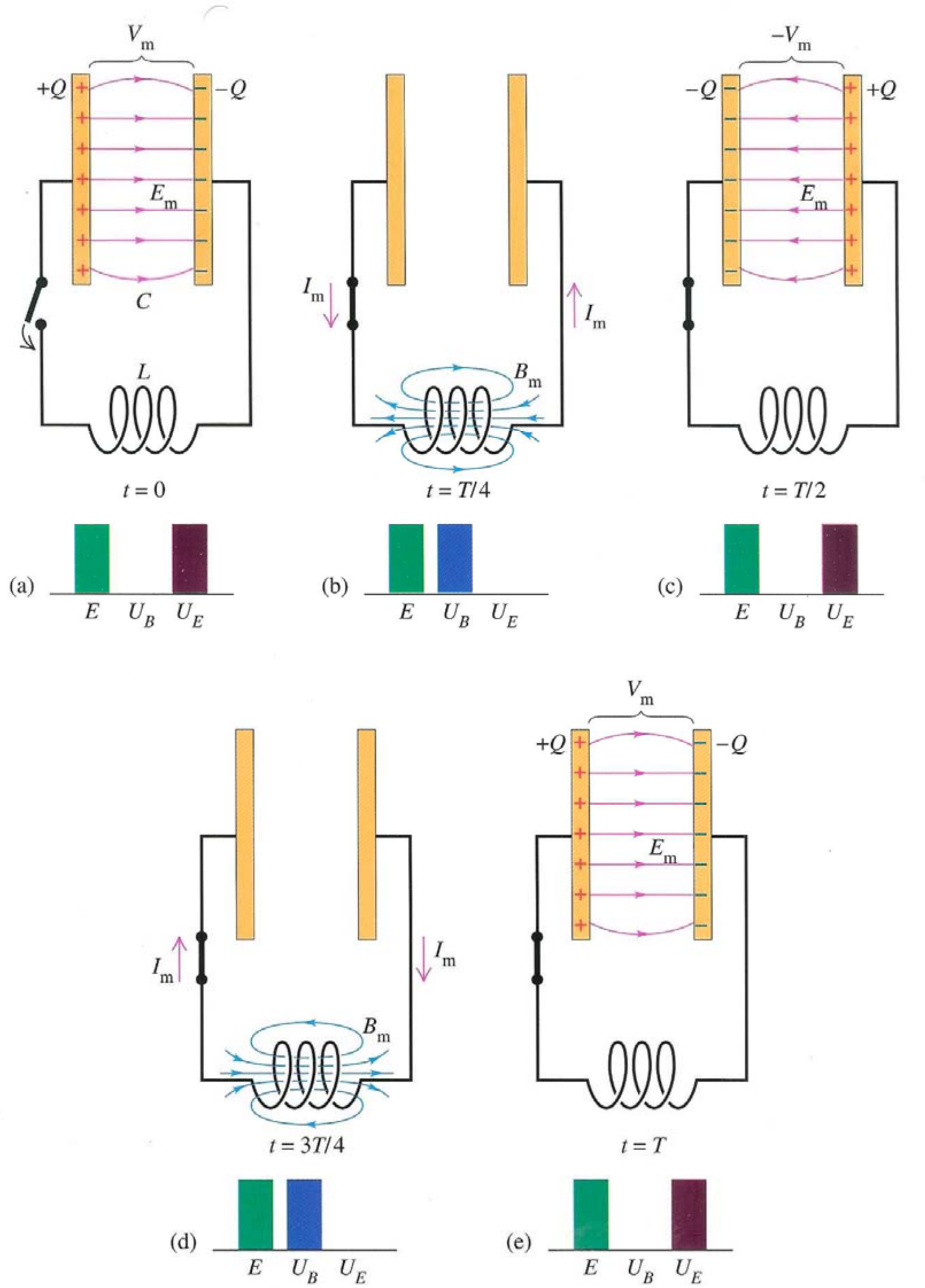
$$\therefore \tfrac{1}{2}\frac{Q^2}{C} + \tfrac{1}{2}LI^2 = \text{constant}$$

$$\therefore \frac{d}{dt}\left(\tfrac{1}{2}\frac{Q^2}{C} + \tfrac{1}{2}LI^2\right) = 0$$

$$\therefore \frac{Q}{C}\frac{dQ}{dt} + LI\frac{dI}{dt} = 0 \qquad\qquad I = -\frac{dQ}{dt}$$

$$\therefore \frac{Q}{C} - L\frac{d}{dt}\left(\frac{dQ}{dt}\right) = 0$$

or $\dfrac{d^2Q}{dt^2} = -\dfrac{1}{LC}Q$ …as before



V_m
$+Q$
$-Q$
E_m
C
L
$t = 0$
(a)
E U_B U_E
I_m
B_m
$t = T/4$
(b)
E U_B U_E
$-V_m$
$-Q$
$+Q$
E_m
$t = T/2$
(c)
E U_B U_E
I_m
B_m
$t = 3T/4$
(d)
E U_B U_E
V_m
$+Q$
$-Q$
E_m
$t = T$
(e)
E U_B U_E

Mechanical	**Electrical**
displacement x	charge Q
velocity v	current I
mass m	inductance L
spring constant k	$\frac{1}{\text{capacitance}}$ $\frac{1}{C}$
$\omega_0 = \sqrt{\frac{k}{m}}$	$\omega_0 = \sqrt{\frac{1}{LC}}$
potential energy: $\frac{1}{2}kx^2$	Electric energy stored in capacitor: $\frac{1}{2}\frac{Q^2}{C}$
kinetic energy: $\frac{1}{2}mv^2$	Magnetic energy stored in inductor: $\frac{1}{2}LI^2$

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Damped oscillations

We have thus far neglected all dissipative mechanisms

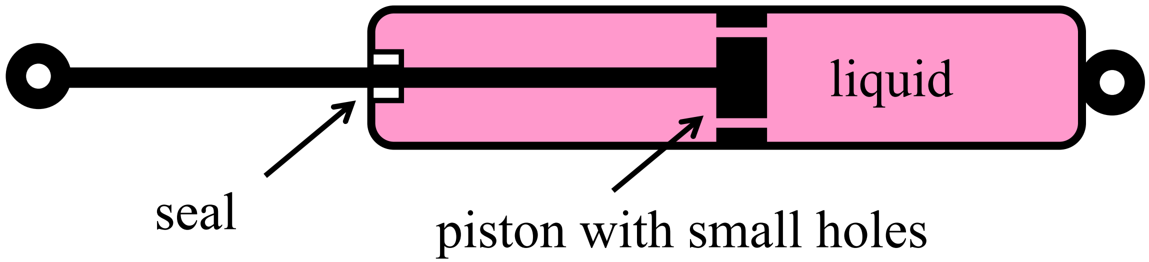
… our oscillations can continue oscillating with the same amplitude forever …

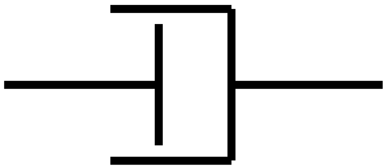
Various physical damping mechanisms will contribute towards the damping…

- friction between mass and table
- "air resistance"
- internal friction in spring
- …….

… model these by introducing a damping force which is proportional to the <u>velocity</u> of the oscillator …

In mechanical systems, we draw in a "dashpot" … a device similar to a shock absorber in a car …





dashpot

b: damping coefficient

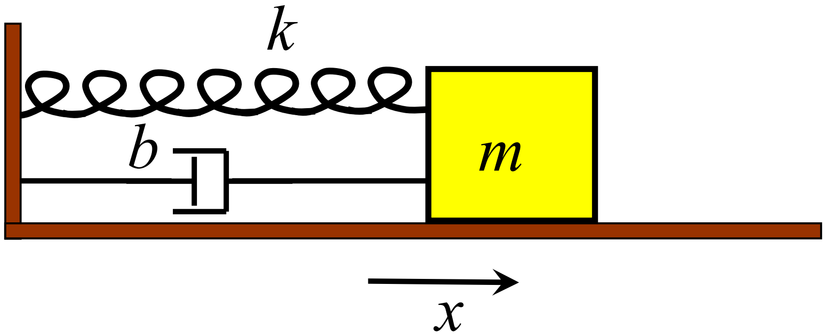
In these systems, the damping force $\vec{\mathbf{R}} = -b\vec{\mathbf{v}}$

… deals with <u>many</u> damping mechanism – but not all …

…talk about …

"viscous damping" → mechanical systems
"resistive damping" → electrical systems

Damped mass-spring system



For horizontal forces on the mass: $ma = -kx - bv$

or $m\dfrac{d^2x}{dt^2} = -kx - b\dfrac{dx}{dt}$

or $\dfrac{d^2x}{dt^2} + \gamma\dfrac{dx}{dt} + \omega_0{}^2 x = 0$ where $\begin{cases} \omega_0 = \sqrt{\dfrac{k}{m}} \\ \gamma = \dfrac{b}{m} \end{cases}$

γ : ”damping constant” unit: s^{-1}

• “life time” $= \dfrac{1}{\gamma}$

Damped oscillator equation

$$\frac{d^2x}{dt^2}+\gamma\frac{dx}{dt}+\omega_0{}^2x=0$$

Let $x=Be^{pt}$

Then $\dfrac{dx}{dt}=Bpe^{pt}$ and $\dfrac{d^2x}{dt^2}=Bp^2e^{pt}$

Substituting into DE: $Bp^2e^{pt}+\gamma Bpe^{pt}+\omega_0{}^2Be^{pt}=0$

Thus $p^2+\gamma p+\omega_0{}^2=0$

$$\therefore p=\tfrac{1}{2}\left\{-\gamma\pm\sqrt{\gamma^2-4\omega_0{}^2}\right\}$$

or $p=-\dfrac{\gamma}{2}\pm\sqrt{\dfrac{\gamma^2}{4}-\omega_0{}^2}$

Damped oscillator equation …2

$$p = -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \omega_0{}^2}$$

We can distinguish three cases:

(i) $\omega_0{}^2 > \dfrac{\gamma^2}{4}$ **Oscillatory behaviour**

(ii) $\omega_0{}^2 = \dfrac{\gamma^2}{4}$ **Critical damping**

(iii) $\omega_0{}^2 < \dfrac{\gamma^2}{4}$ **Overdamping**

$$\boxed{\text{Case (i):} \qquad \omega_0^{\,2} > \frac{\gamma^2}{4}}$$

$$\therefore \sqrt{\gamma^2/4 - \omega_0^{\,2}} = \sqrt{-\left(\omega_0^{\,2} - \gamma^2/4\right)}$$

Put $\quad \omega_1^{\,2} = \omega_0^{\,2} - \gamma^2/4$

$$\therefore p = -\frac{\gamma}{2} \pm \sqrt{-\omega_1^{\,2}} = -\frac{\gamma}{2} \pm j\omega_1$$

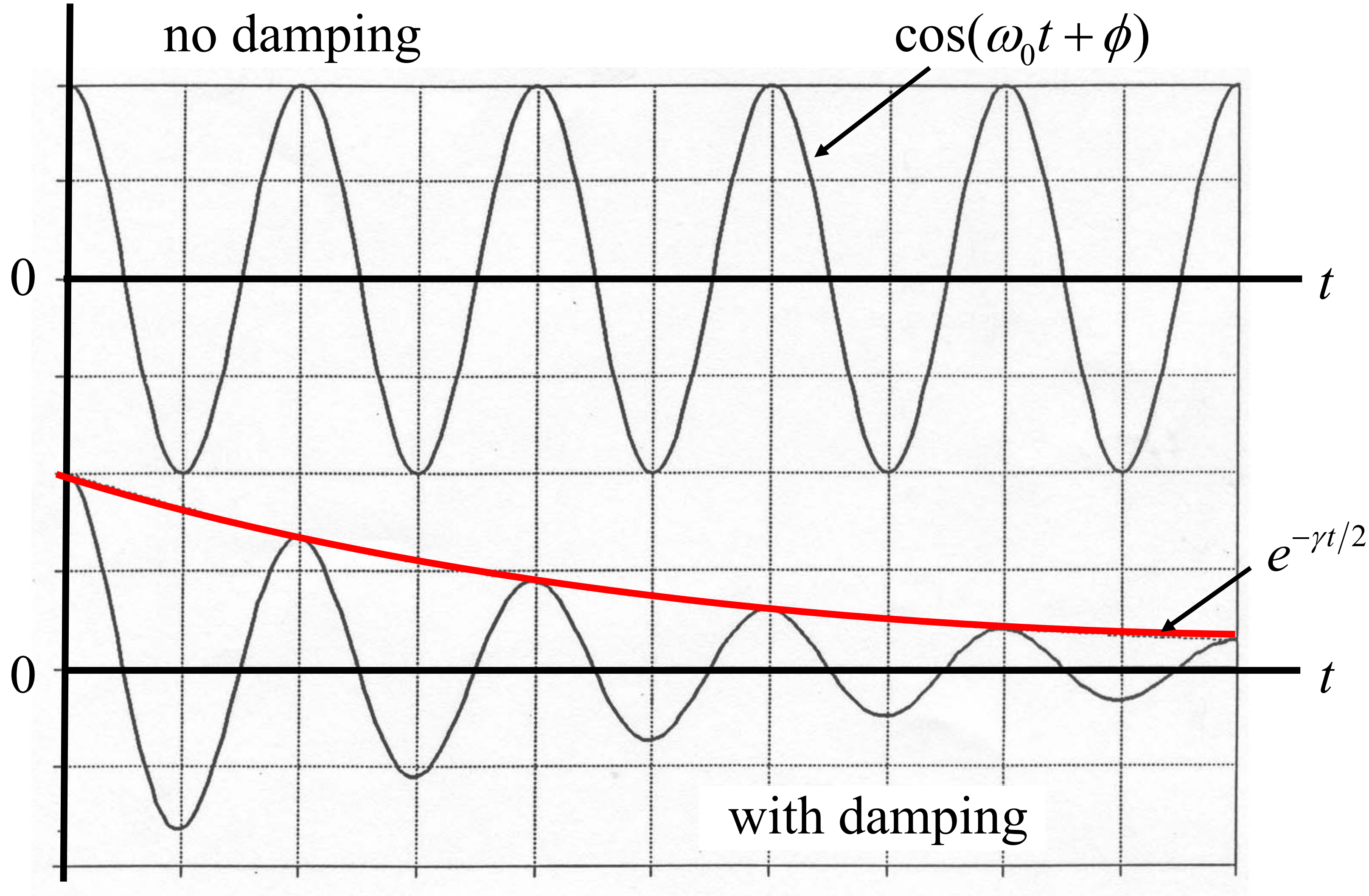
The solution will be:

$$x = B_1 e^{\left(-\frac{\gamma}{2} + j\omega_1\right)t} + B_2 e^{\left(-\frac{\gamma}{2} - j\omega_1\right)t} = e^{-\frac{\gamma}{2}t}\left\{B_1 e^{j\omega_1 t} + B_2 e^{-j\omega_1 t}\right\}$$

… leading to $\quad x(t) = Ae^{-\frac{\gamma t}{2}}\cos(\omega_1 t + \phi)$

This is an **oscillatory solution** $\; A\cos(\omega_1 t + \phi) \;$ multiplied by a damping factor $\; e^{-\gamma t/2}$.

As $\; \gamma \to 0 \;$ we approach our undamped oscillator.


no damping
$\cos(\omega_0 t + \phi)$
0
t
$e^{-\gamma t/2}$
0
t
with damping

$$\boxed{\text{Case (ii):} \quad \omega_0^{\ 2} = \frac{\gamma^2}{4}}$$

The two roots coincide: $p = -\dfrac{\gamma}{2}$

The solution will be $x(t) = (A + Bt)e^{-\frac{\gamma}{2}t}$

The condition $\omega_0^{\ 2} = \gamma^2/4$ is referred to as the "**critical damping**" condition.

If $\omega_0^{\ 2} < \gamma^2/4$ a system released from rest will oscillate.

As γ is increased the oscillations decay more rapidly, until at $\omega_0^{\ 2} = \gamma^2/4$ oscillation no longer occurs.

[… many practical applications …]

Case (iii): $\omega_0^{\,2} < \dfrac{\gamma^2}{4}$

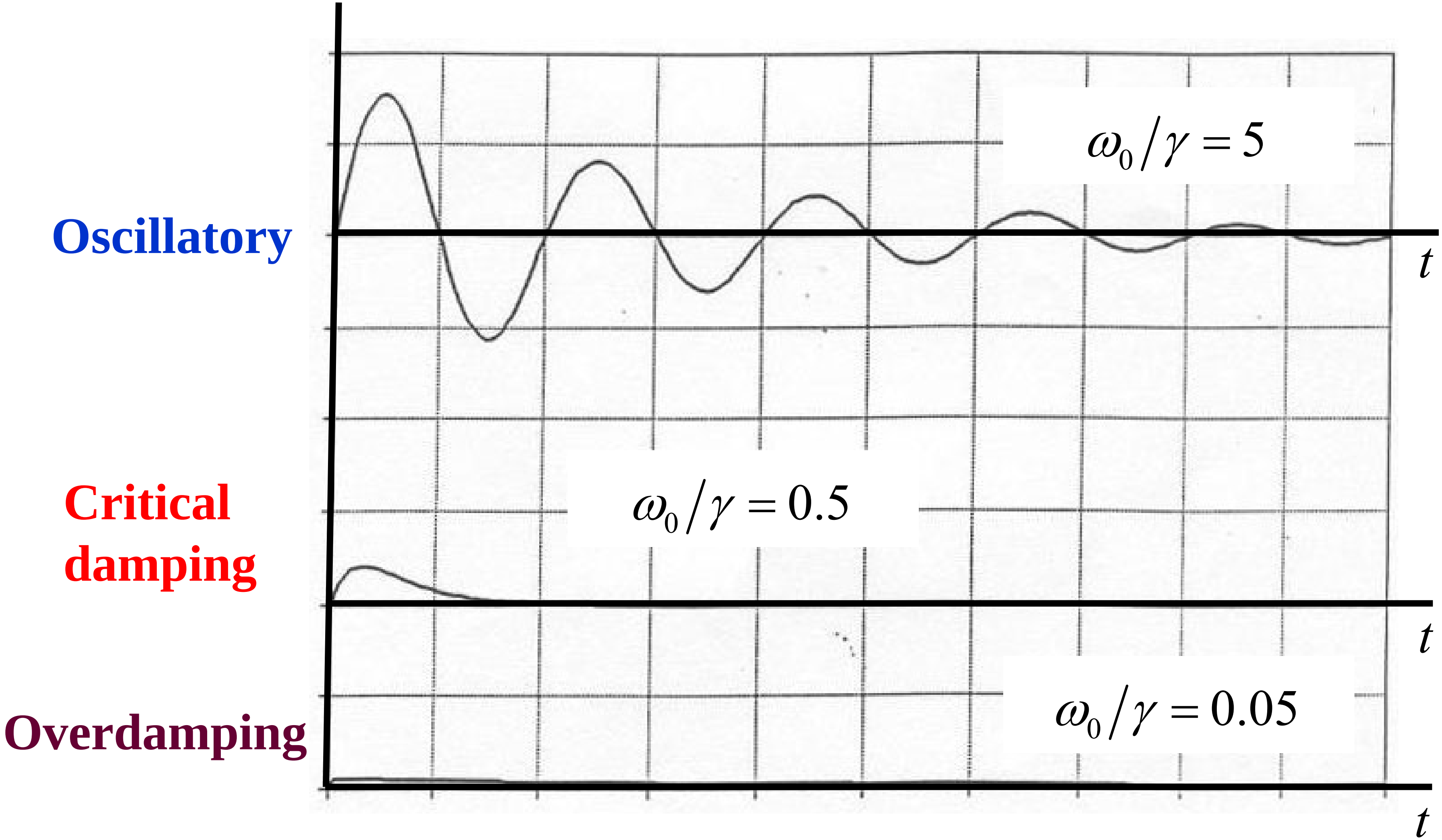
$$p = -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \omega_0^{\,2}}$$

$$= -\frac{\gamma}{2} \pm \lambda \qquad \text{say}$$

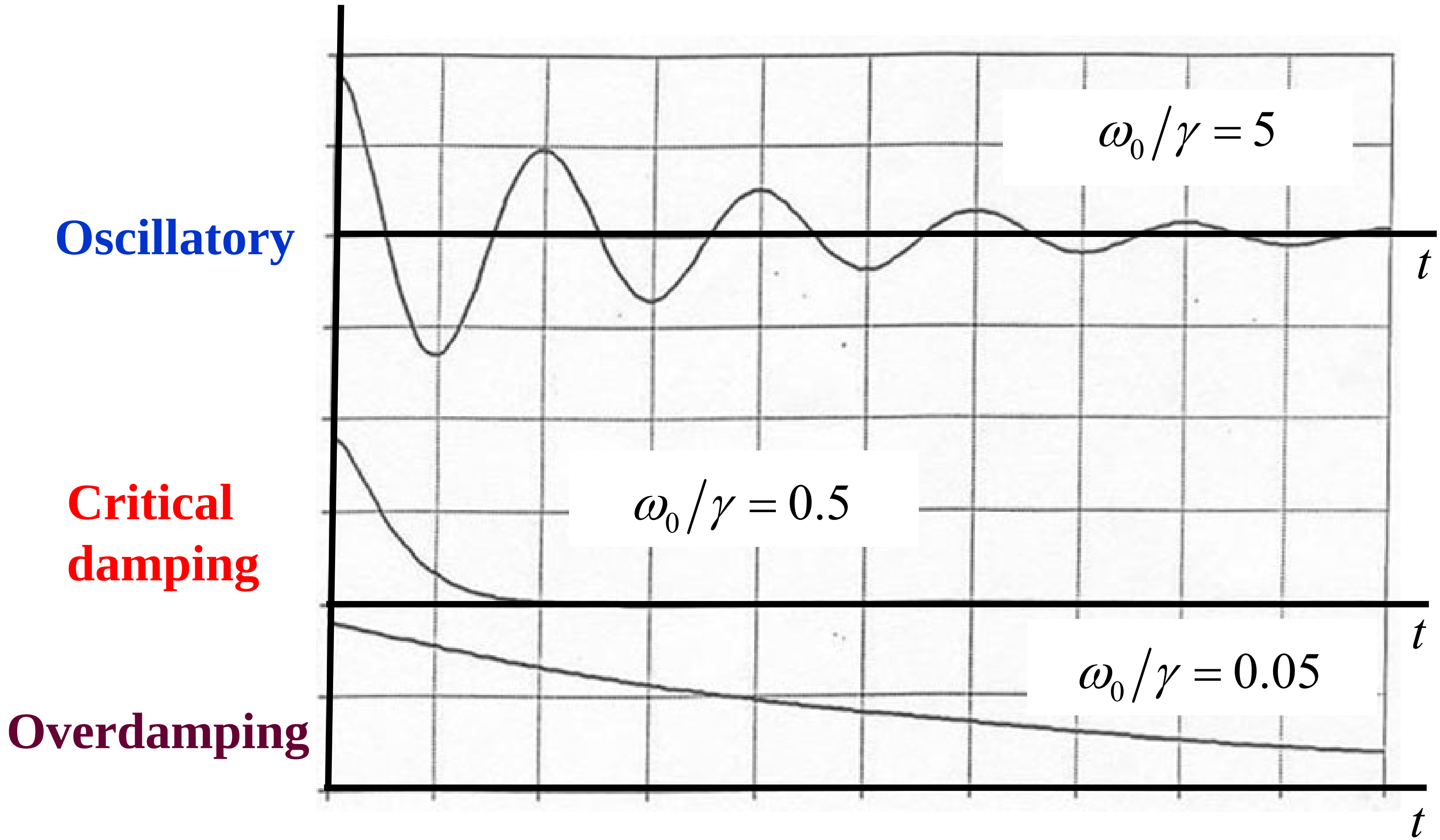
The solution will be $x(t) = B_1 e^{\left(-\frac{\gamma}{2}+\lambda\right)t} + B_2 e^{\left(-\frac{\gamma}{2}-\lambda\right)t}$

The condition $\omega_0^{\,2} < \dfrac{\gamma^2}{4}$ is referred to as **overdamping**

… a slower approach to the rest position is observed.



Oscillatory
$\omega_0/\gamma = 5$
t
Critical damping
$\omega_0/\gamma = 0.5$
t
Overdamping
$\omega_0/\gamma = 0.05$
t



Oscillatory
$\omega_0/\gamma = 5$
t
Critical damping
$\omega_0/\gamma = 0.5$
t
Overdamping
$\omega_0/\gamma = 0.05$
t

Convenient to define a parameter $Q = \frac{\omega_0}{\gamma}$

Q: quality factor or the "**Q of the oscillator**"

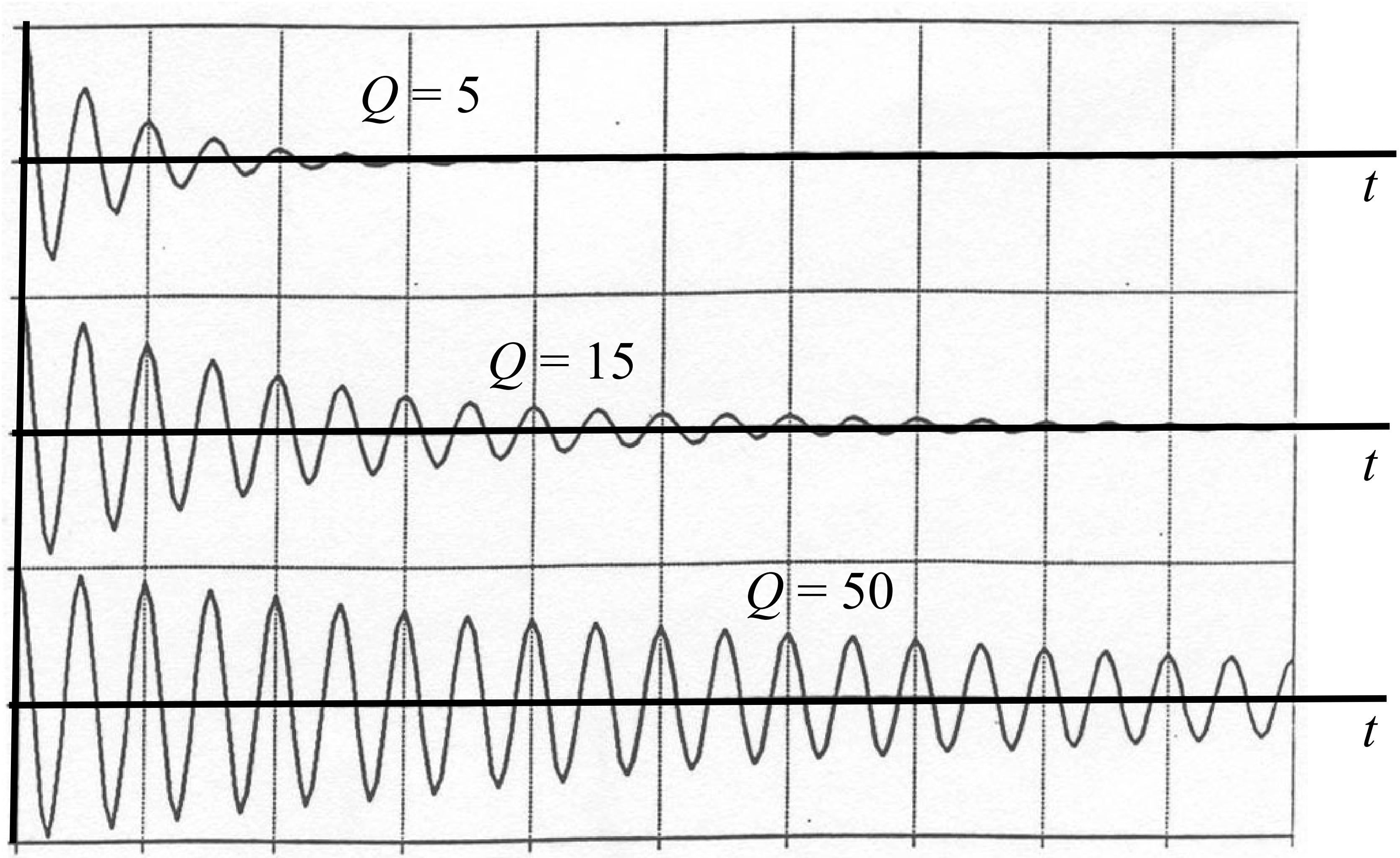
… gives a measure of $\frac{\text{damping time}}{\text{period}}$

Notice that … $\omega^2 = \omega_0{}^2 - \frac{\gamma^2}{4} = \omega_0{}^2\left(1 - \frac{\gamma^2}{4\omega_0{}^2}\right) = \omega_0{}^2\left(1 - \frac{1}{4Q^2}\right)$

So for $Q \gg 1$ (light damping) : $\omega_1 \approx \omega_0$

e.g. for $Q = 10$ $\omega_1 = \omega_0\sqrt{1 - \frac{1}{400}} \simeq \omega_0\left(1 - \frac{1}{800}\right)$

Note also $\omega_1 < \omega_0$ → Damping increases the period of the oscillator, as one may intuitively expect.



$Q = 5$
t
$Q = 15$
t
$Q = 50$
t

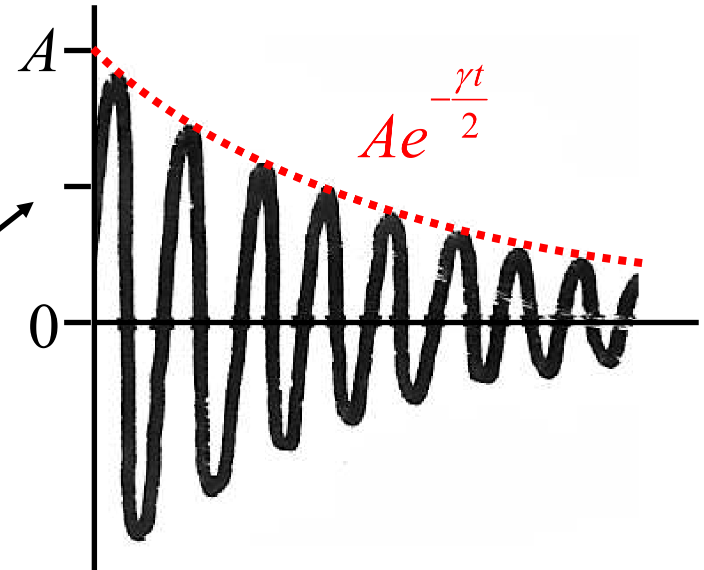
Damped oscillator: summary

The differential equation $$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0{}^2 x = 0$$

has the general solution $x(t) = Ae^{-\frac{\gamma t}{2}}\cos(\omega_1 t + \phi)$

where $\omega_1 = \sqrt{\omega_0{}^2 - \gamma^2/4}$

… the two constants A and ϕ must be determined by some initial conditions



A subtle point:

$A \geq$ the initial amplitude

… A is not the amplitude … it is a constant related to the initial amplitude by (initial amplitude) $= A\cos\phi$

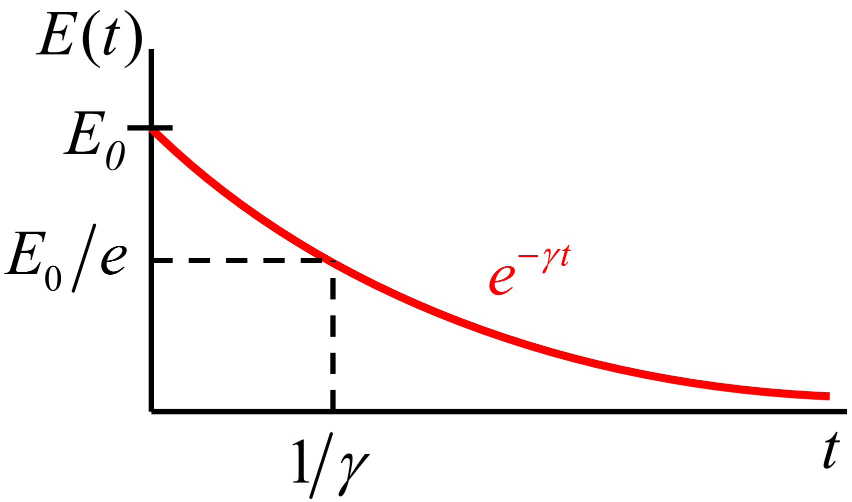
Energy of a damped oscillator

For a lightly damped oscillator, the amplitude is slowly varying …

Total energy of the undamped oscillator: $E = \frac{1}{2}kA^2$

For the lightly damped oscillator: $E = \frac{1}{2}k\left(Ae^{-\frac{\gamma t}{2}}\right)^2$

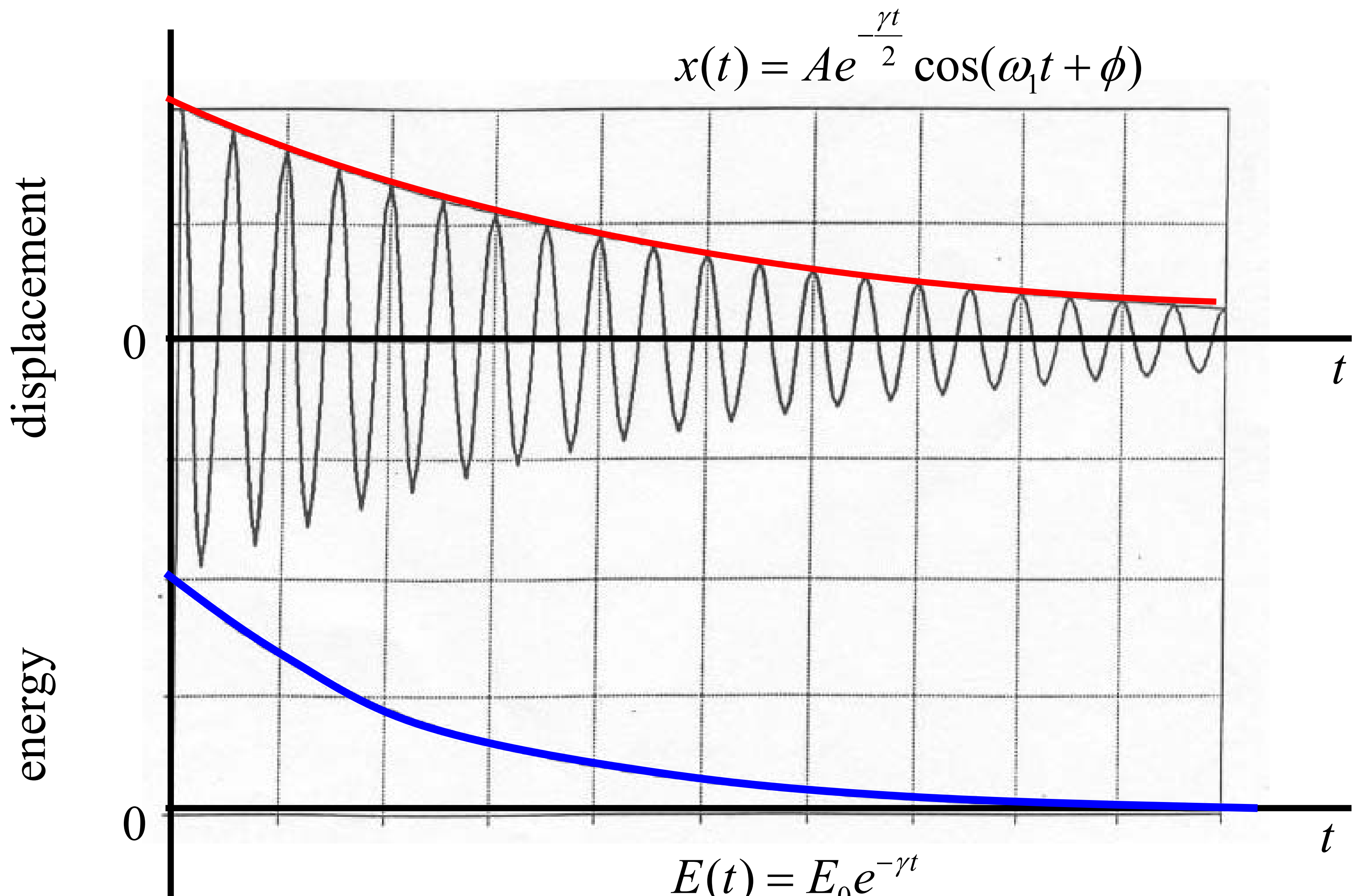
$$= \frac{1}{2}kA^2e^{-\gamma t}$$



$$\therefore E(t) = E_0e^{-\gamma t}$$

Then:

$$P = -\frac{dE}{dt} = \gamma E_0 e^{-\gamma t} = \gamma E$$



$x(t) = Ae^{-\frac{\gamma t}{2}}\cos(\omega_1 t + \phi)$
displacement
0
t
energy
0
t
$E(t) = E_0 e^{-\gamma t}$

Energy of a damped oscillator …2

Now write:

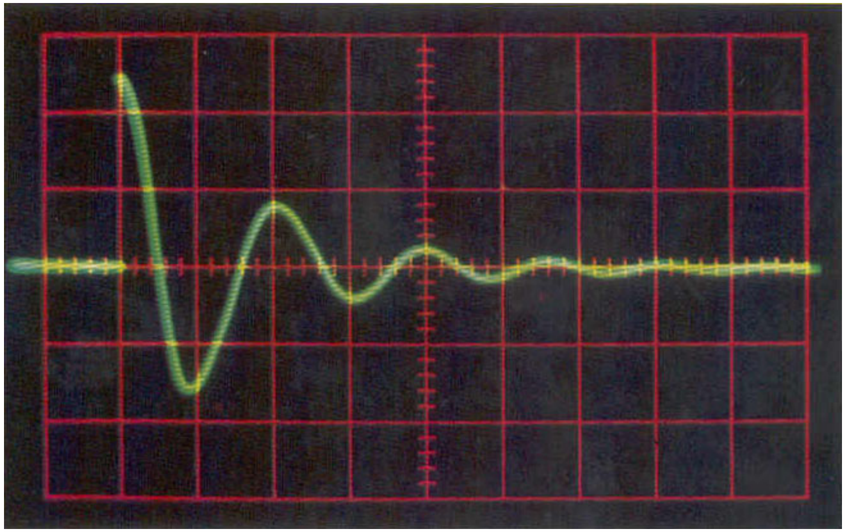
$$Q = \frac{\omega_0}{\gamma} \quad = \quad \omega_0 \frac{E}{P} \quad = \quad \omega_0 \frac{\text{Oscillator energy}}{\text{Power dissipation}}$$

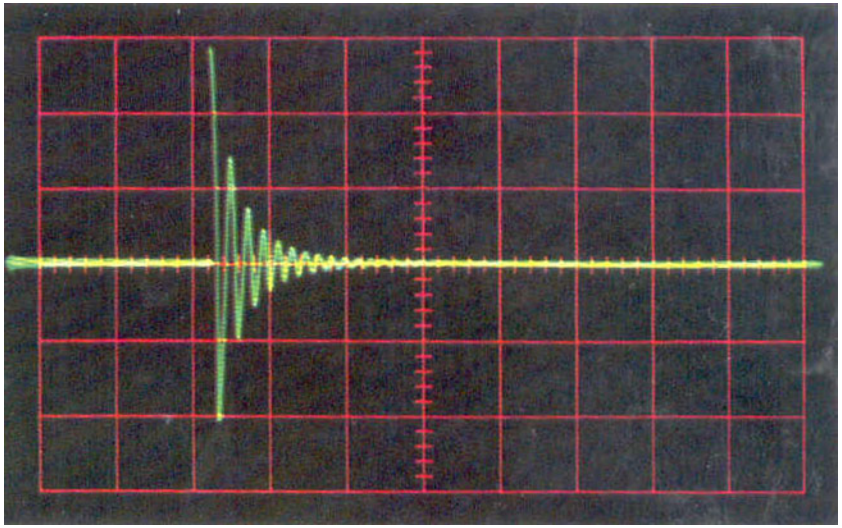
$$= \frac{2\pi}{T} \; \frac{\text{Oscillator energy}}{\text{Power dissipation}}$$

$$\therefore Q = 2\pi \; \frac{\text{Oscillator energy}}{\text{Energy dissipation per cycle}}$$

Number of oscillator cycles in one lifetime = $\left(\frac{1}{\gamma}\right)\left(\frac{1}{T}\right) \approx \frac{1}{\gamma}\frac{\omega_0}{2\pi} = \frac{Q}{2\pi}$

… e.g. a oscillator having $Q = 10$ performs 1.6 cycles in one lifetime
… which may be regarded as quite significant damping …





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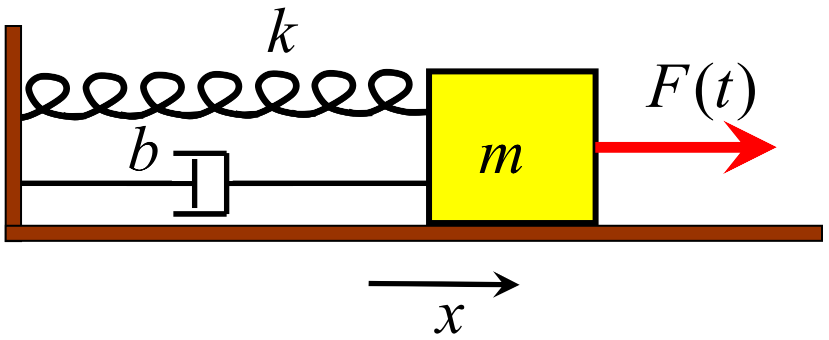
Forced oscillations and resonance

We now look at oscillators which are subject to an externally applied force $F(t) = F_0 \cos \omega t$



… $F(t)$ is a periodic force with driving frequency ω

$$\sum F_x = ma_x$$

$$\therefore F_0 \cos \omega t - kx - bv = ma$$

$$\frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + \omega_0{}^2 x = \frac{F_0}{m} \cos \omega t \qquad \begin{cases} \omega_0{}^2 = k/m \\ \gamma = b/m \end{cases}$$

Forced oscillations in a mass-spring system

Equation of motion: $$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0{}^2 x = \frac{F_0}{m}\cos\omega t$$

Two parts observed to the motion of the system:
Transient response: while the system continues to oscillate at its natural frequency ω_0 … which dies out with the characteristic $1/\gamma$ decay time.
Steady state response: at times >> $1/\gamma$ the response has the same frequency ω as the applied force

Observe also that there might be a phase difference (usually a lag) between the response and the driving force

For $F(t) = F_0 \cos\omega t$

write $x(t) = A\cos(\omega t - \delta)$

Forced oscillations in a mass-spring system…2

Using complex numbers: $F(t) = F_0 e^{j\omega t}$ and $x(t) = Ae^{j(\omega t-\delta)}$

Then $\dfrac{dx}{dt} = j\omega A e^{j(\omega t-\delta)}$ and $\dfrac{d^2x}{dt^2} = -\omega^2 A e^{j(\omega t-\delta)}$

Substituting into our differential equation of motion:

$$-\omega^2 A e^{j(\omega t-\delta)} + \gamma(j\omega) A e^{j(\omega t-\delta)} + \omega_0^2 A e^{j(\omega t-\delta)} = \frac{F_0}{m} e^{j\omega t}$$

Divide by $e^{j\omega t}$ … since this holds for all *t* :

$$(-\omega^2 + j\gamma\omega + \omega_0^2) A e^{-j\delta} = \frac{F_0}{m}$$

or $$\left\{(\omega_0^2 - \omega^2) + j\gamma\omega\right\} A = \frac{F_0}{m} e^{j\delta}$$

$$= \frac{F_0}{m}\left(\cos\delta + j\sin\delta\right)$$

Forced oscillations in a mass-spring system…3

$$\left\{(\omega_0^2-\omega^2)+j\gamma\omega\right\}A=\frac{F_0}{m}\left(\cos\delta+j\sin\delta\right)$$

Equating the real and imaginary parts: $\begin{cases}(\omega_0^2-\omega^2)A=\dfrac{F_0}{m}\cos\delta \\ \omega\gamma A=\dfrac{F_0}{m}\sin\delta\end{cases}$

Then $\tan\delta=\dfrac{\omega\gamma}{(\omega_0^2-\omega^2)}$

and $\left(\dfrac{F_0}{m}\right)^2\cos^2\delta+\left(\dfrac{F_0}{m}\right)^2\sin^2\delta=A^2(\omega_0^2-\omega^2)^2+A^2\omega^2\gamma^2$

$\therefore A^2\left\{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2\right\}=\left(\dfrac{F_0}{m}\right)^2$ or $A(\omega)=\dfrac{F_0/m}{\sqrt{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}}$

Forced oscillations in a mass-spring system…summary

Steady state solution of $$\ddot{x} + \gamma \dot{x} + {\omega_0}^2 x = \frac{F_0}{m} \cos \omega t$$

is $$x(t) = A(\omega) \cos(\omega t - \delta)$$

where $$A(\omega) = \frac{F_0 / m}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2 \gamma^2}}$$

and $$\tan \delta(\omega) = \frac{\omega \gamma}{(\omega_0^2 - \omega^2)}$$

Resonance

Note (for now) that $A(\omega)$ has a maximum at a frequency ω_m slightly less than ω_0 … see later …

At low frequencies, $\omega \ll \omega_0$

$$A(\omega) = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}} \approx \frac{F_0/m}{\sqrt{\omega_0^4 + \omega^2\gamma^2}} \approx \frac{F_0/m}{\omega_0^2} = \frac{F_0}{k}$$

The amplitude is approximately F_0/k which is the amplitude for a steady force … the oscillator behaviour is determined essentially by the spring (or the capacitor) … i.e. the oscillator is "spring limited"

And $$\tan\delta(\omega) = \frac{\omega\gamma}{(\omega_0^2 - \omega^2)} \approx \frac{\omega\gamma}{\omega_0^2} = \frac{1}{Q}\frac{\omega}{\omega_0}$$

i.e. $\tan\delta \ll 1$ … δ is small

… the response is almost in phase with the driving force.

Resonance …2

At high frequencies, $\omega \gg \omega_0$

$$A(\omega) = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}} \approx \frac{F_0/m}{\sqrt{\omega^4 + \omega^2\gamma^2}} \approx \frac{F_0/m}{\sqrt{\omega^4 + \omega^2\,\omega_0^2/Q^2}}$$

$$= \frac{F_0}{m\omega^2}$$

… the oscillator is "inertia limited"

And $\quad \tan\delta(\omega) = \dfrac{\omega\gamma}{(\omega_0^2 - \omega^2)} \approx -\dfrac{\omega\gamma}{\omega^2} = -\dfrac{\gamma}{\omega} = -\dfrac{\omega_0}{Q}\dfrac{1}{\omega} = -\dfrac{1}{Q}\dfrac{\omega_0}{\omega}$

i.e. $\quad \tan\delta$ is small and negative … $\delta \to \pi$

… the response is π out of phase with the driving force.

Resonance …3

In the resonance region , $\omega \approx \omega_0$

$$A(\omega) = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}} \approx \frac{F_0/m}{\sqrt{\omega_0^2\gamma^2}} \approx \frac{F_0}{\omega_0 m\gamma} = \frac{F_0}{\omega_0 b} = Q\frac{F_0}{k}$$

… the oscillator amplitude is strongly dependent on the damping constant b … it is "resistance limited"

And $\tan\delta(\omega) = \dfrac{\omega\gamma}{(\omega_0^2 - \omega^2)} \to \infty$ as $\omega \to \omega_0$

i.e. $\delta \to \dfrac{\pi}{2}$

… the response lags the driving force by $\dfrac{\pi}{2}$.

$$A(\omega_0) = \frac{F_0/m}{\sqrt{0+\omega^2\gamma^2}}$$

Resonance …4

Close to resonance, it is the $(\omega_0^2 - \omega^2)^2$ term which gives the rapid variation with frequency … as we move away from ω_0 we see that we will get

$$A(\omega) = \frac{1}{\sqrt{2}} A(\omega_0)$$

when: $$\frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega_0{}^2\gamma^2}} = \frac{1}{\sqrt{2}} \frac{F_0/m}{\sqrt{\omega_0{}^2\gamma^2}}$$

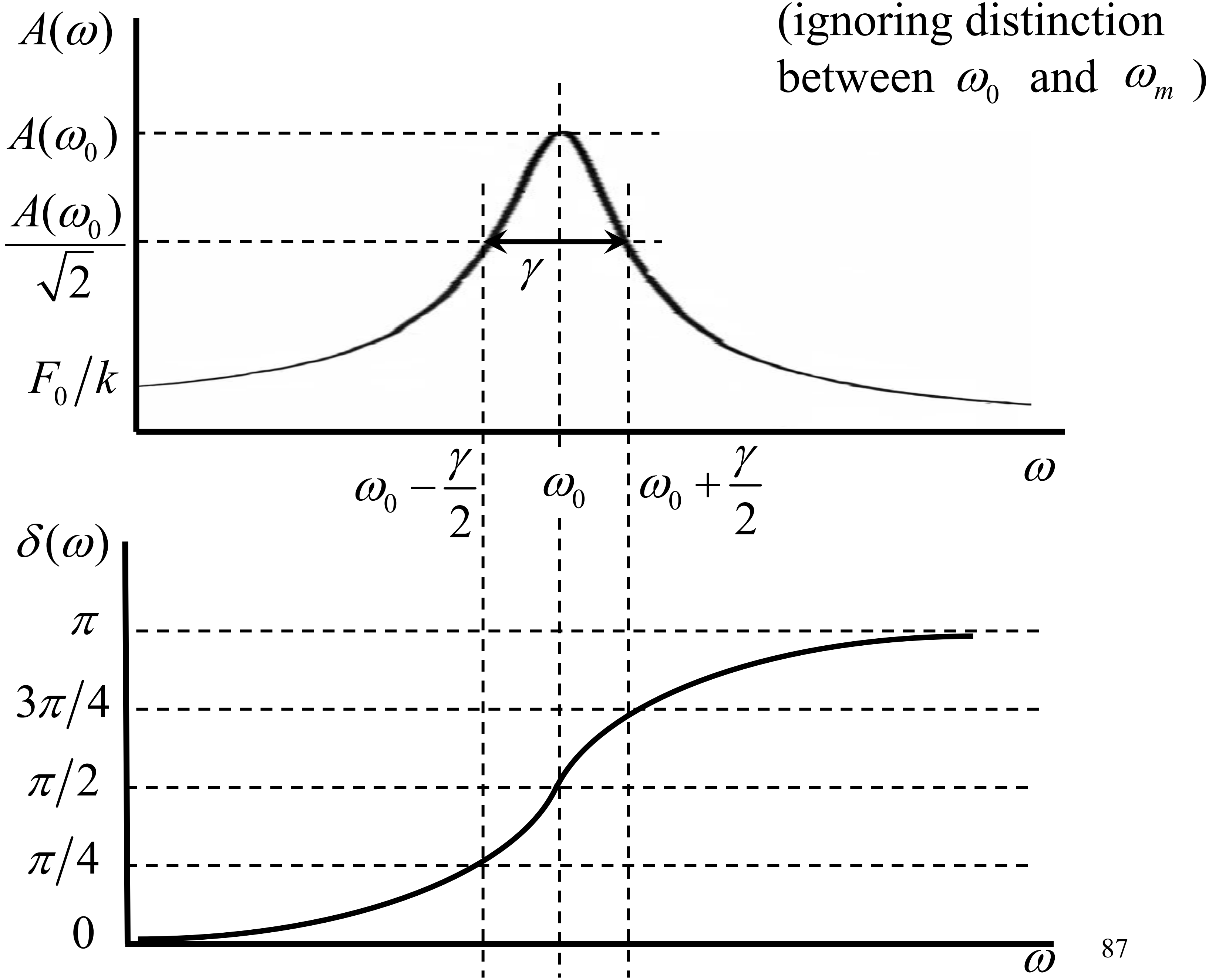
$$\therefore (\omega_0^2 - \omega^2) = \pm\omega_0\gamma$$

$$\therefore (\omega_0 + \omega)(\omega_0 - \omega) = \pm\omega_0\gamma$$

$$\therefore 2\omega_0(\omega_0 - \omega) = \pm\omega_0\gamma$$

$$\therefore (\omega_0 - \omega) = \pm\frac{\gamma}{2}$$

When $A(\omega) = \frac{1}{\sqrt{2}} A(\omega_0)$ then $\omega = \omega_0 \pm \frac{\gamma}{2}$



(ignoring distinction between ω_0 and ω_m)
$A(\omega)$
$A(\omega_0)$
$\frac{A(\omega_0)}{\sqrt{2}}$
γ
F_0/k
$\omega_0 - \frac{\gamma}{2}$
ω_0
$\omega_0 + \frac{\gamma}{2}$
ω
$\delta(\omega)$
π
$3\pi/4$
$\pi/2$
$\pi/4$
0
ω

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Resonance …5

At $\omega = \omega_0 + \dfrac{\gamma}{2}$ and $\omega = \omega_0 - \dfrac{\gamma}{2}$, the oscillator amplitude is decreased by $1/\sqrt{2}$ of its maximum (resonance) value.

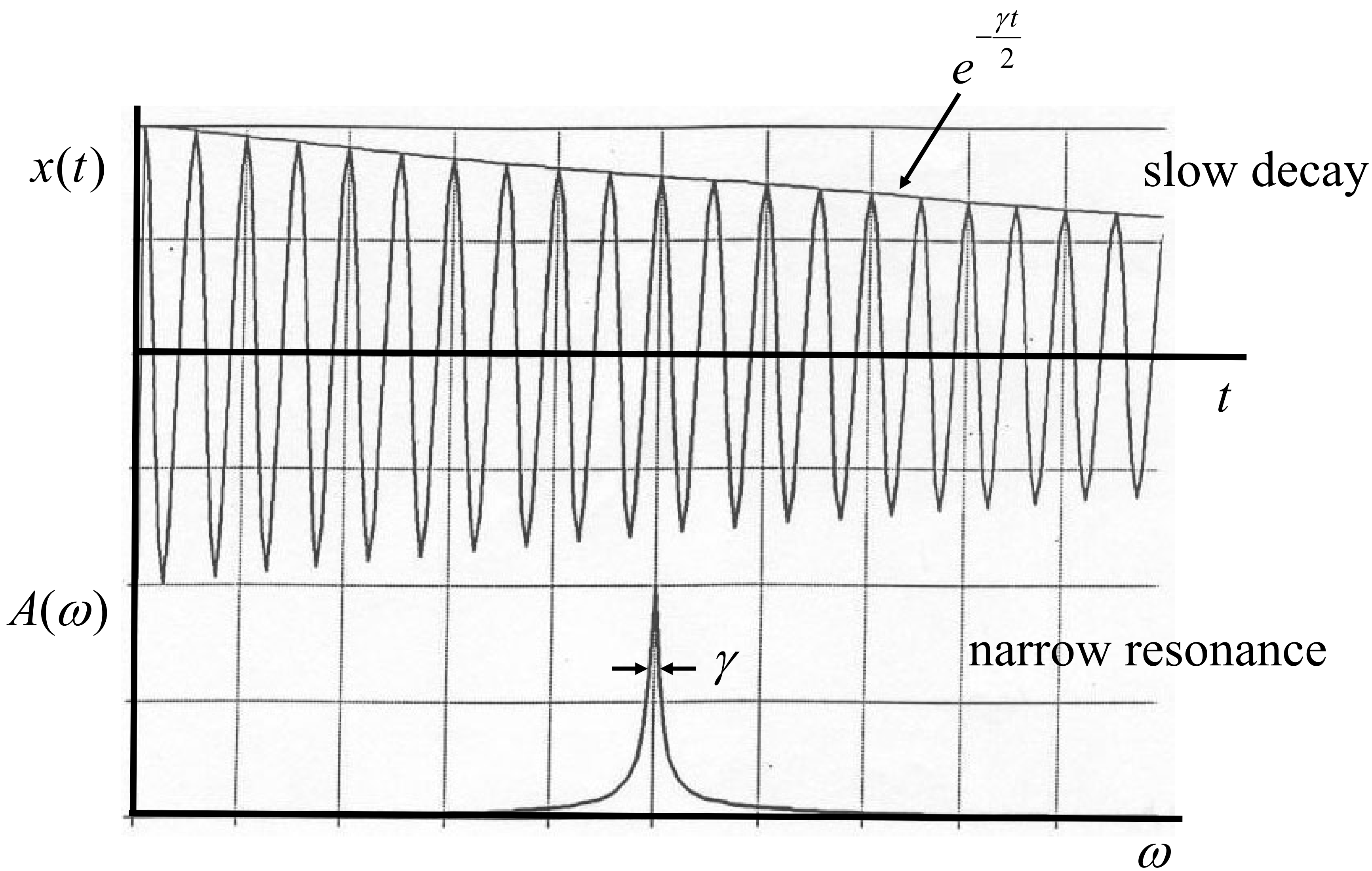
These two frequencies are known as the "half power frequencies"

… and γ is known as the "resonance width"

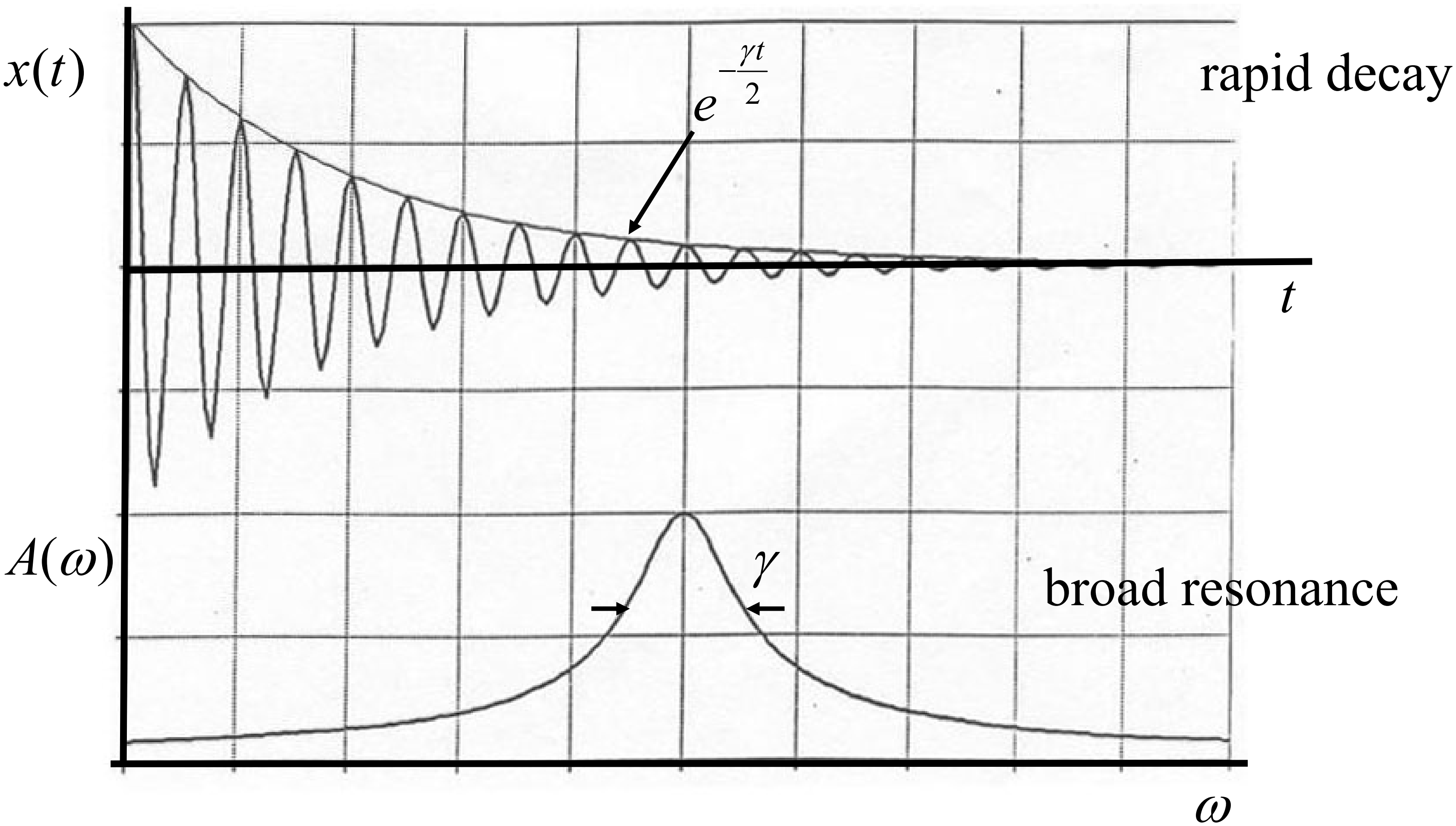
$Q = \dfrac{\omega_0}{\gamma}$ can be understood to be

$$\frac{\text{resonance frequency}}{\text{width of resonance curve at half power points}}$$

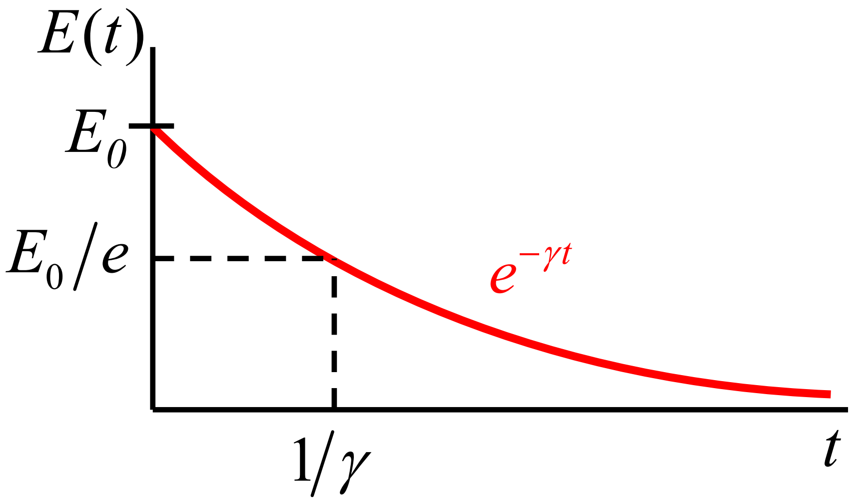
… as the damping term decreases, i.e. as Q increases, the resonance curve gets "sharper"



$e^{-\frac{\gamma t}{2}}$
$x(t)$
slow decay
t
$A(\omega)$
narrow resonance
γ
ω



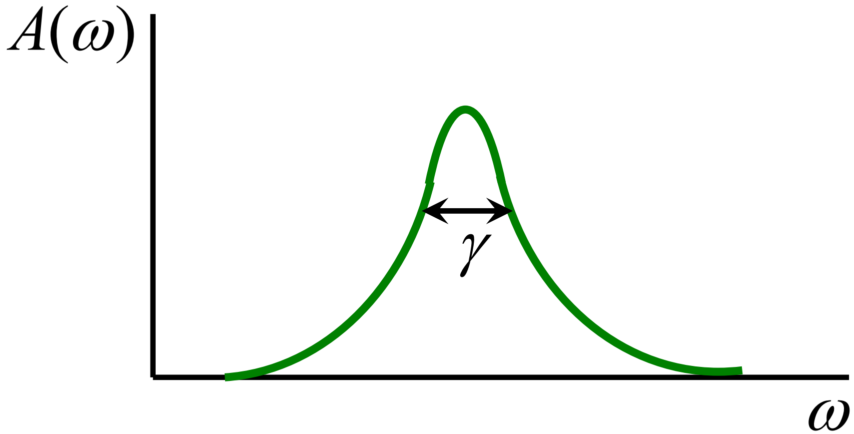
$x(t)$
$e^{-\frac{\gamma t}{2}}$
rapid decay
t
$A(\omega)$
γ
broad resonance
ω



Free vibration

$$E(t) = E_0 e^{-\gamma t}$$

$$\textbf{Lifetime} = \frac{1}{\gamma}$$



Forced vibration

$$\textbf{Resonance width} = \gamma$$

$$\boxed{\textbf{lifetime} \times \textbf{resonance width} = 1}$$

Distinction between ω_0 and ω_m

$$A(\omega) = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}}$$

$$\frac{dA}{d\omega} = \frac{F_0}{m}\left(-\tfrac{1}{2}\right)\left((\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2\right)^{-\frac{3}{2}}\left\{2(\omega_0^2 - \omega^2)\left(-2\omega\right) + 2\omega\gamma^2\right\}$$

Thus $\dfrac{dA}{d\omega} = 0$ for $\left(\omega_0{}^2 - \omega^2\right)2\omega = \omega\gamma^2$

or $\omega^2 = \omega_0{}^2 - \dfrac{\gamma^2}{2} = \omega_0{}^2\left(1 - \dfrac{\gamma^2}{2\omega_0{}^2}\right) = \omega_0{}^2\left(1 - \dfrac{1}{2Q^2}\right)$

i.e. $\omega_m = \omega_0\sqrt{1 - \dfrac{1}{2Q^2}}$

where ω_m is the frequency at which $A(\omega)$ is a maximum.

e.g. for $Q = 8$, $\omega_m = 0.996\omega_0$

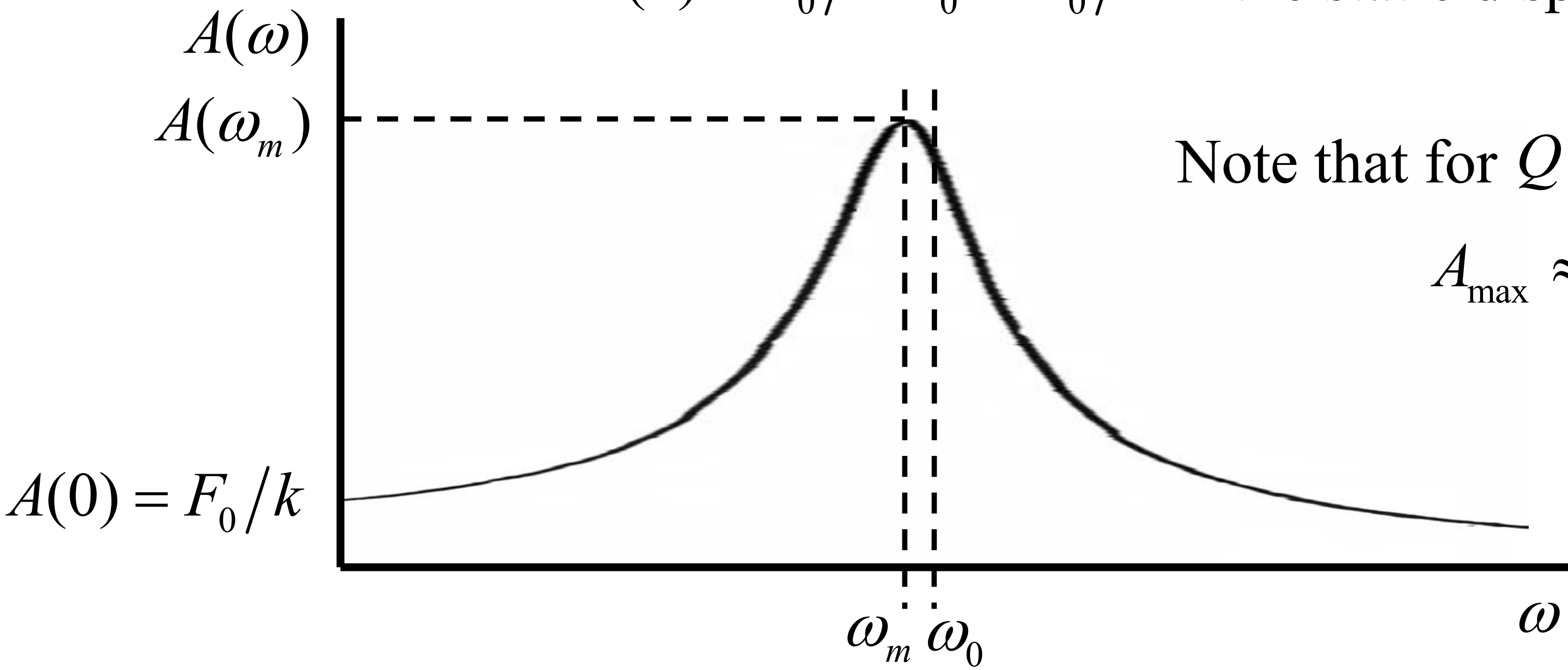
Distinction between ω_0 and ω_m … 2

What is the maximum value of $A(\omega)$?

Substitute $\omega_m = \omega_0\sqrt{1-\dfrac{1}{2Q^2}}$ into $A(\omega) = \dfrac{F_0/m}{\sqrt{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}}$

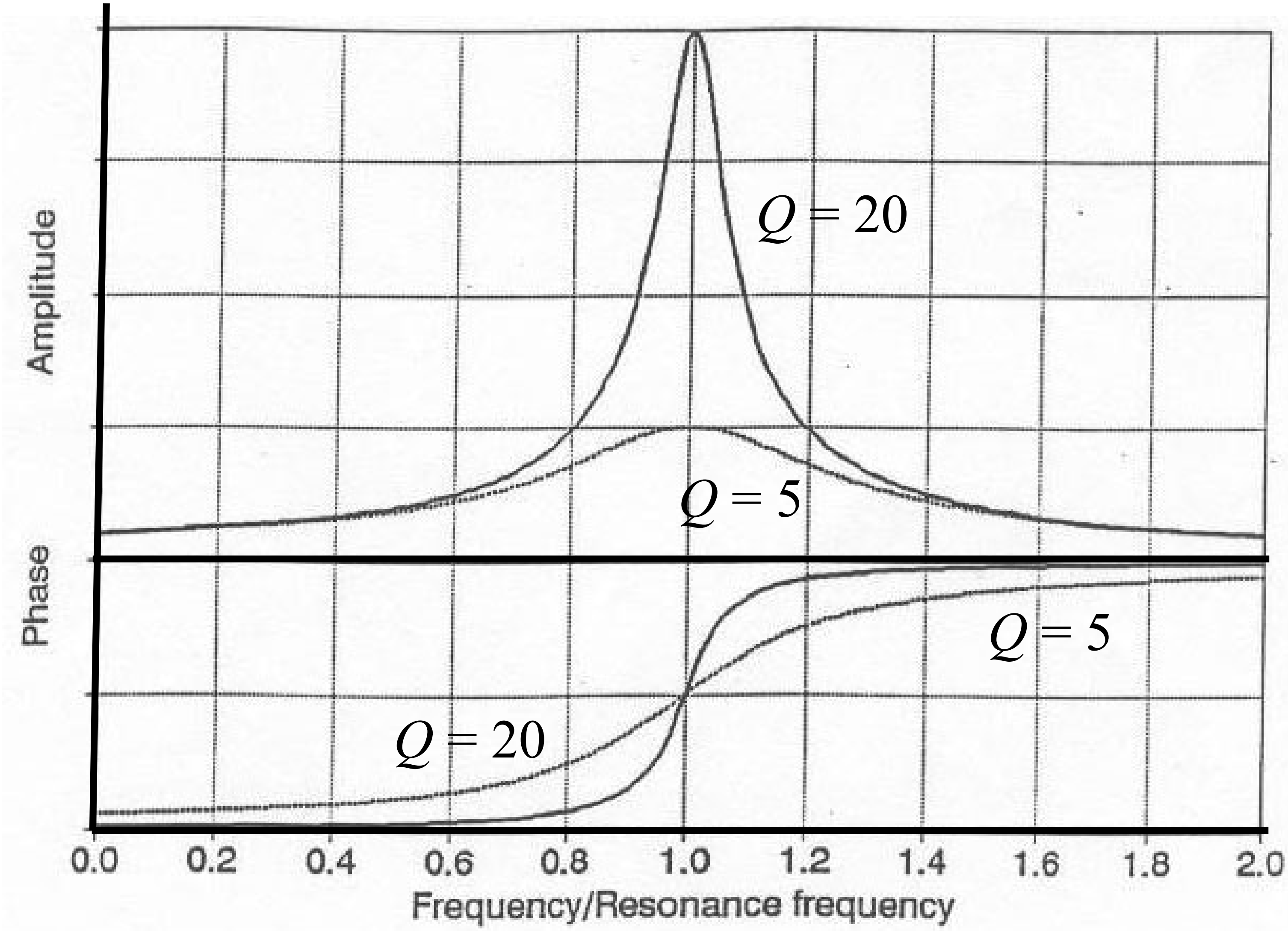
and find $A_{\max} = A(\omega_m) = \dfrac{QA(0)}{\sqrt{1-1/4Q^2}}$

where $A(0) = F_0/m\omega_0^2 = F_0/k$ = the static displacement



Note that for $Q \gtrsim 5$,

$A_{\max} \approx QA(0)$


Q = 20
Q = 5
Amplitude
Phase
Q = 5
Q = 20
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
2.0
Frequency/Resonance frequency

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Transients

Consider a mass-spring system which is at rest at $t = 0$ when a driving force is turned on.

Equation of motion: $$m\frac{d^2x}{dt^2} + kx = F_0 \cos \omega t$$ (no damping)

or $$\frac{d^2x}{dt^2} + \omega_0{}^2 x = \frac{F_0}{m}\cos \omega t$$

Then $$x = \frac{F_0/m}{(\omega_0^2 - \omega^2)}\cos \omega t$$

But what are the adjustable constants of integration?

… when the force is switched on, then x jumps from 0 to $\dfrac{F_0/m}{(\omega_0^2 - \omega^2)}$!

Transients …2

Say that $\dfrac{d^2x}{dt^2} + \omega_0{}^2 x = \dfrac{F_0}{m}\cos\omega t$ has solution x_1

and $\dfrac{d^2x}{dt^2} + \omega_0{}^2 x = 0$ has solution x_2

Then $x_1 + x_2$ is also a solution to $\dfrac{d^2x}{dt^2} + \omega_0{}^2 x = \dfrac{F_0}{m}\cos\omega t$

The complete solution of the forced-motion equation is

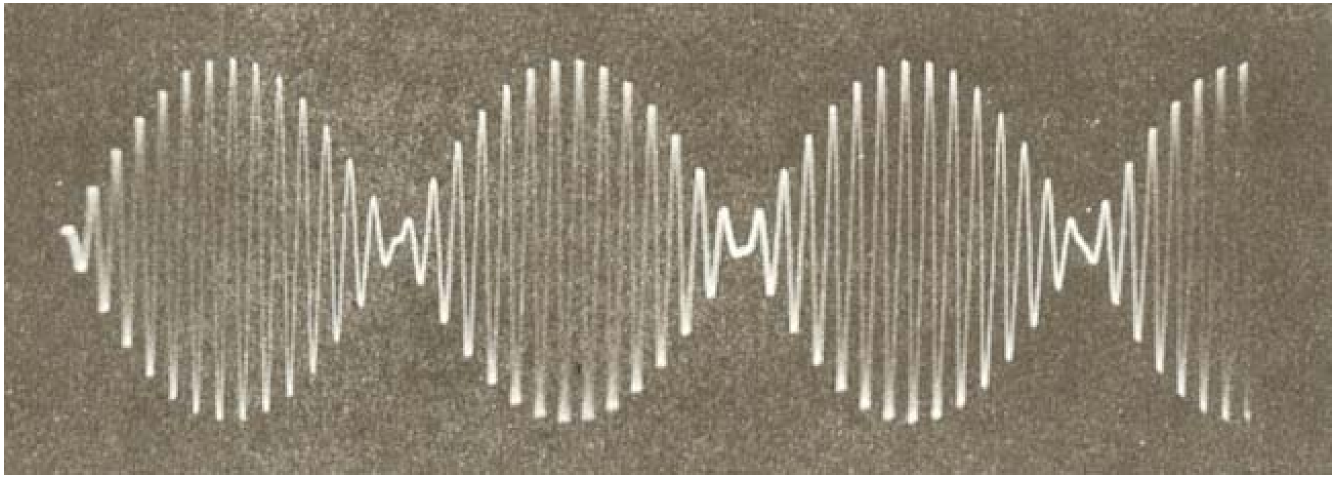
$$x(t) = B\cos(\omega_0 t + \beta) + C\cos\omega t \quad \text{where} \quad C = \frac{F_0/m}{(\omega_0^2 - \omega^2)}$$

Initial conditions:

$$\left.\begin{array}{ll} x = 0 \ \text{ at } t = 0: & 0 = B\cos\beta + C \\ \dfrac{dx}{dt} = 0 \ \text{ at } t = 0: & 0 = -\omega_0 B\sin\beta \end{array}\right\} \quad \begin{array}{c} \beta = 0 \ \text{ or } \ \pi \\ B = -C \end{array}$$

Transients …3

Then $x(t) = C\left(\cos\omega t - \cos\omega_0 t\right)$ Beats!



Since $\cos\omega t \approx 1 - \dfrac{\omega^2 t^2}{2}$

then $x \approx \dfrac{F_0/m}{(\omega_0^2 - \omega^2)} \dfrac{(\omega_0^2 - \omega^2)t^2}{2} = \dfrac{1}{2}\dfrac{F_0}{m}t^2$ as expected

Transients …4

If damping is present then:

$$x(t) = \underbrace{Be^{-\frac{\gamma t}{2}}\cos(\omega_1 t + \beta)}_{\text{transient term}} + \underbrace{A\cos(\omega t - \delta)}_{\text{steady state term}}$$

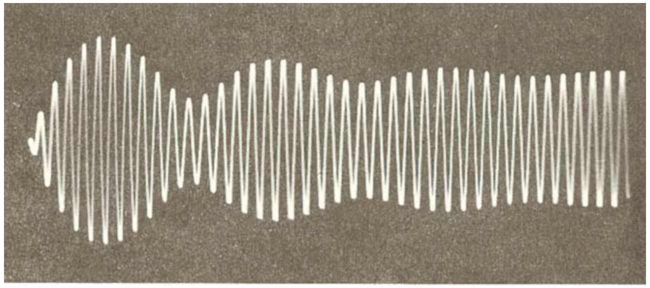
where:

$$\omega_1 = \sqrt{\omega_0{}^2 - \frac{\gamma^2}{4}}$$

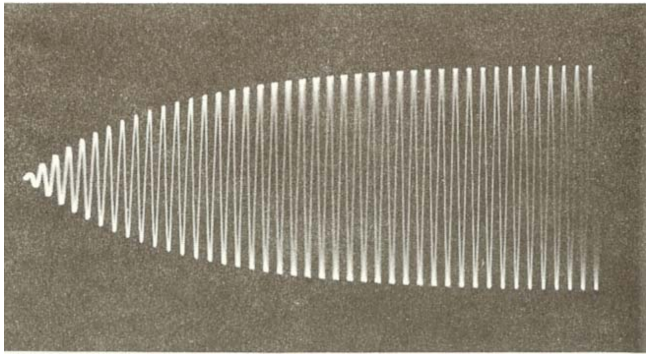
$$A(\omega) = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}}$$

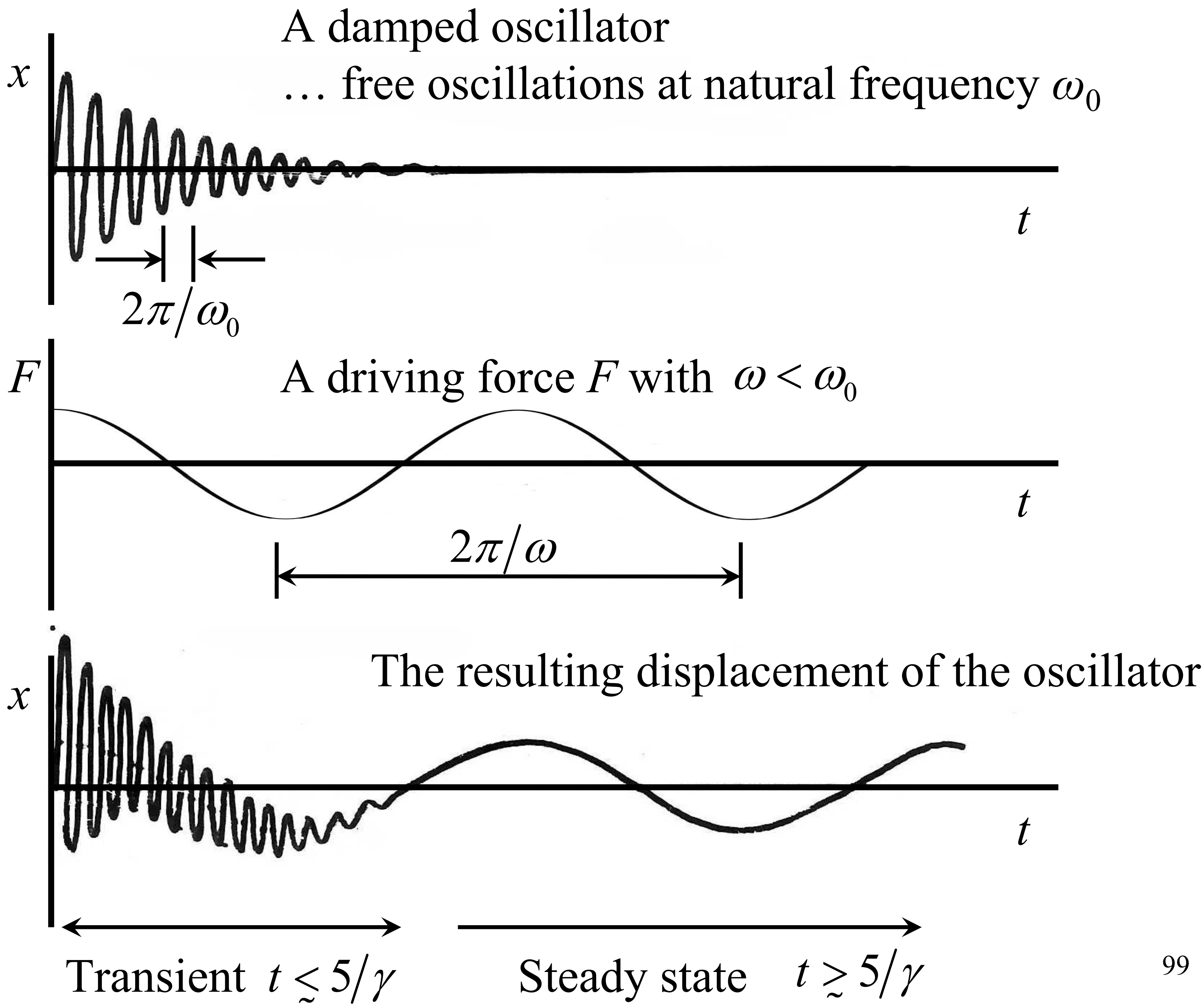
$$\tan\delta(\omega) = \frac{\omega\gamma}{(\omega_0^2 - \omega^2)}$$

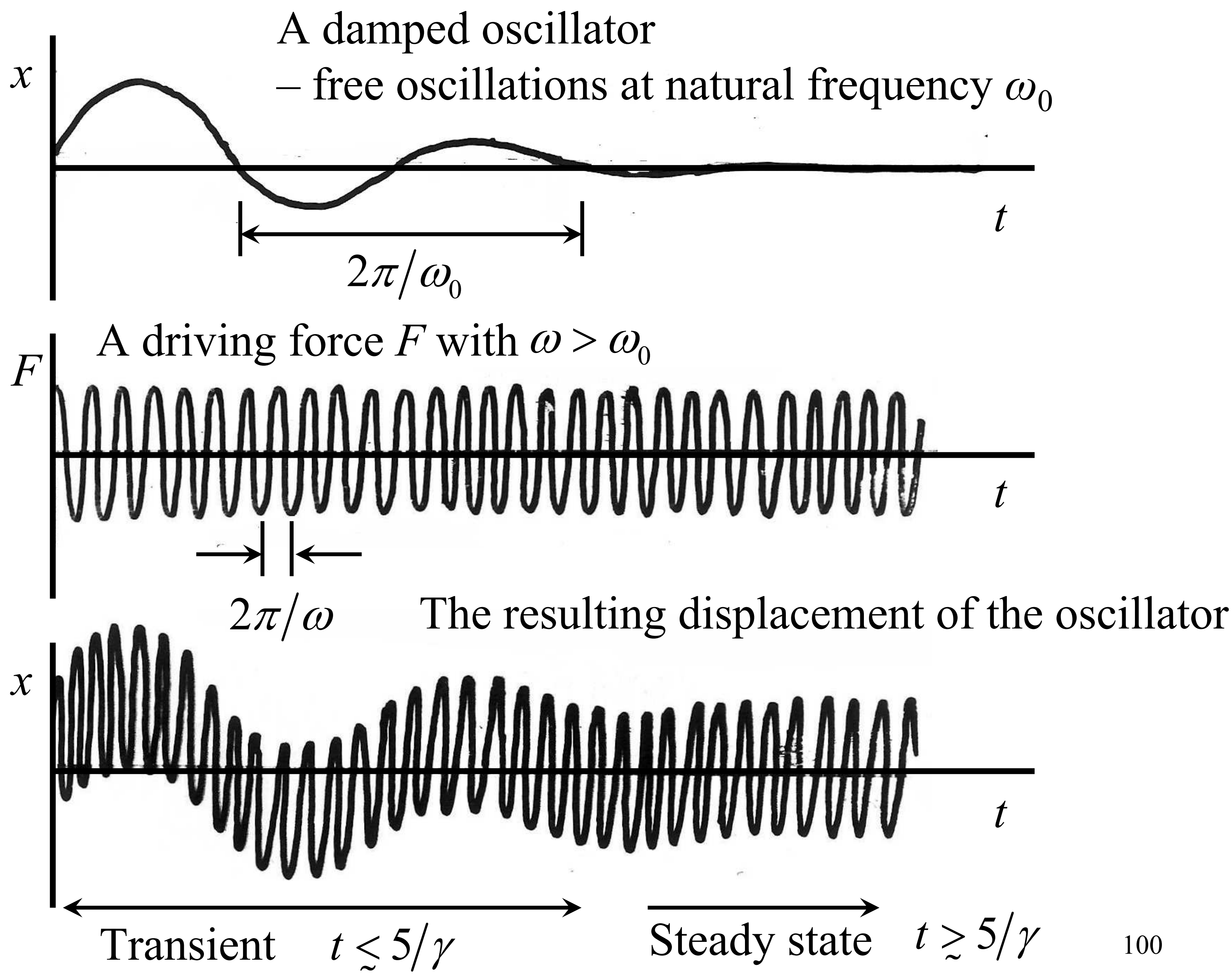
off resonance:

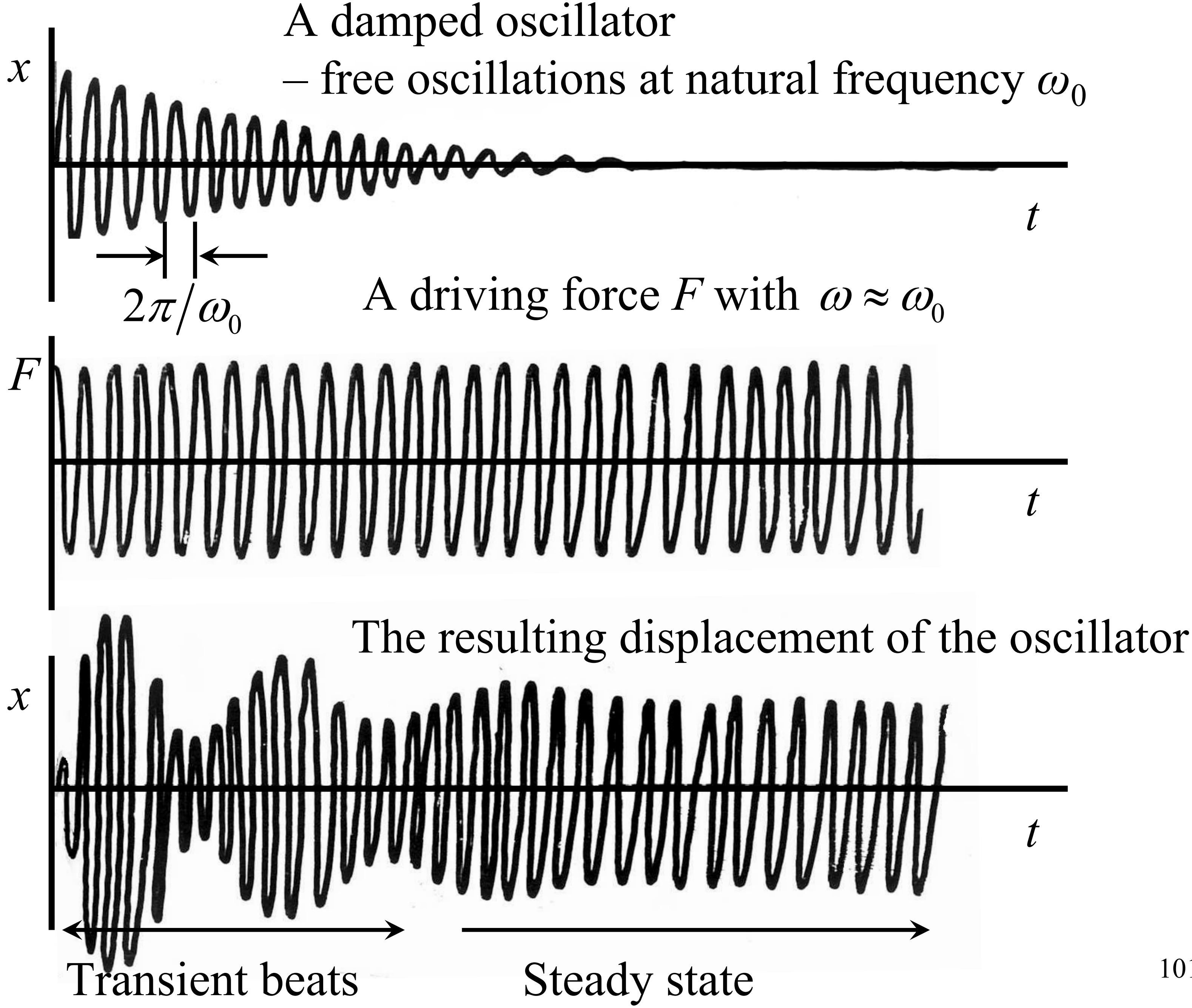


at resonance:



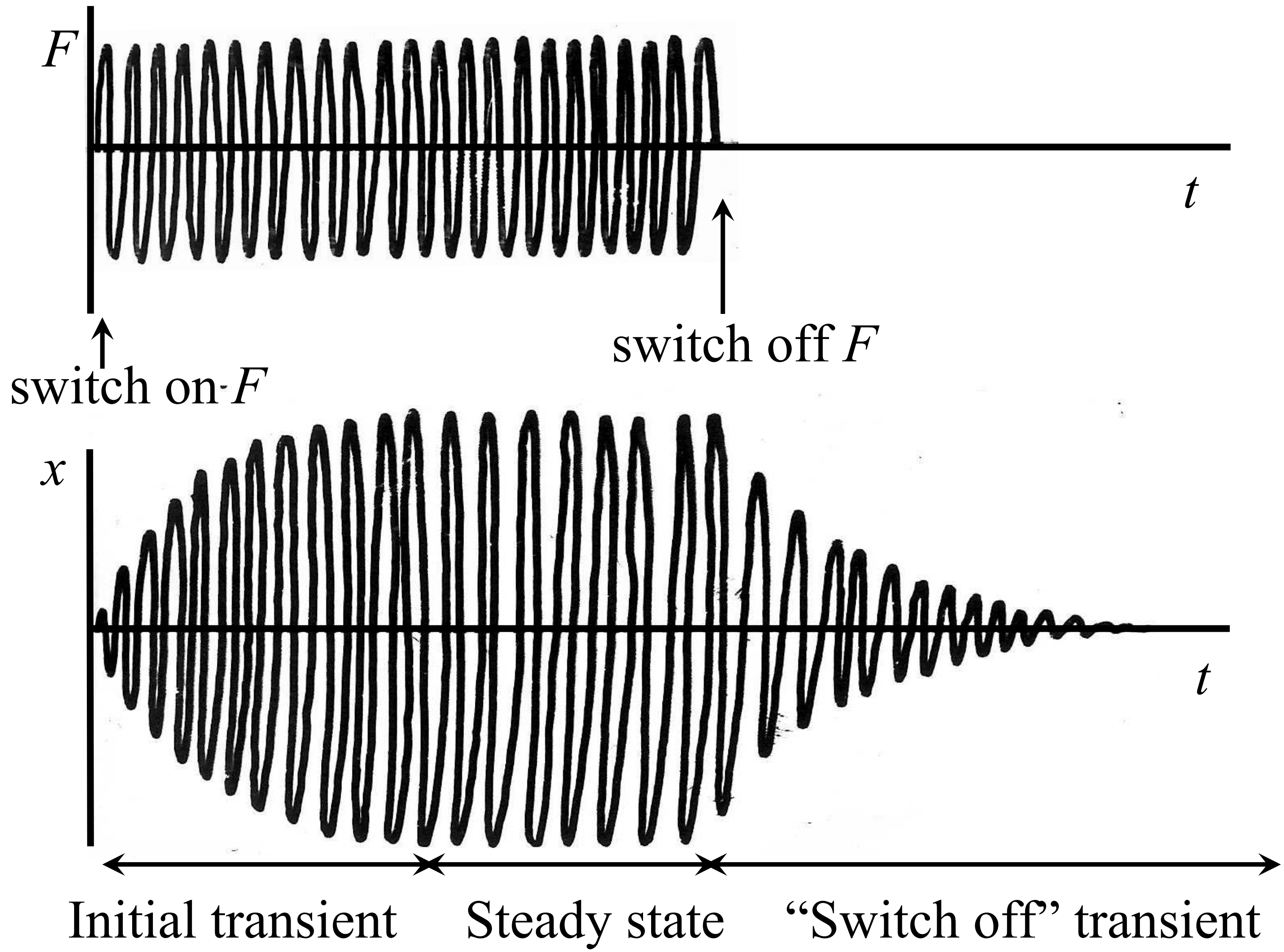

A damped oscillator
… free oscillations at natural frequency ω_0
x
t
$2\pi/\omega_0$
F
A driving force F with $\omega < \omega_0$
t
$2\pi/\omega$
The resulting displacement of the oscillator
x
t
Transient $t \lesssim 5/\gamma$
Steady state $t \gtrsim 5/\gamma$


A damped oscillator
– free oscillations at natural frequency ω_0
x
t
$2\pi/\omega_0$
A driving force F with $\omega > \omega_0$
F
t
$2\pi/\omega$
The resulting displacement of the oscillator
x
t
Transient $t \lesssim 5/\gamma$
Steady state $t \gtrsim 5/\gamma$


A damped oscillator
– free oscillations at natural frequency ω_0
x
t
$2\pi/\omega_0$
A driving force F with $\omega \approx \omega_0$
F
t
The resulting displacement of the oscillator
x
t
Transient beats
Steady state

An oscillator driven at resonance

… driving frequency ω = natural frequency ω_0



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The power absorbed by a driven oscillator

We now consider the power required to keep an oscillator vibrating with constant amplitude.

$$F(t) = F_0 \cos \omega t \qquad \text{and} \qquad x(t) = A(\omega)\cos(\omega t - \delta)$$

$$\therefore v(t) = -\omega A(\omega)\sin(\omega t - \delta)$$

Power $P = Fv = -\omega A(\omega) F_0 \cos \omega t \sin(\omega t - \delta)$

$$= -\omega A(\omega) F_0 \cos \omega t \left\{\sin \omega t \cos \delta - \cos \omega t \sin \delta\right\}$$

$$= F_0 \omega A(\omega) \cos^2 \omega t \sin \delta - F_0 \omega A(\omega) \cos \omega t \sin \omega t \cos \delta$$

Now $\overline{\cos^2 \omega t} = \dfrac{1}{2}$ and $\overline{\cos \omega t \sin \omega t} = 0$

Therefore $\overline{P} = \tfrac{1}{2} F_0 \omega A(\omega) \sin \delta$

The power absorbed by a driven oscillator …2

$$\overline{P} = \tfrac{1}{2} F_0 \omega A(\omega) \sin\delta \quad \begin{cases} A(\omega) = \dfrac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}} \\ \tan\delta = \dfrac{\omega\gamma}{(\omega_0^2 - \omega^2)} \\ \sin\delta = \dfrac{\omega\gamma}{\sqrt{(\omega_0^2 - \omega^2) + \omega^2\gamma^2}} \end{cases}$$

$$\therefore \overline{P} = \tfrac{1}{2} F_0 \omega \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}} \frac{\omega\gamma}{\sqrt{(\omega_0^2 - \omega^2) + \omega^2\gamma^2}}$$

or $$\overline{P}(\omega) = \frac{F_0^2 \omega^2 \gamma}{2m} \frac{1}{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}$$

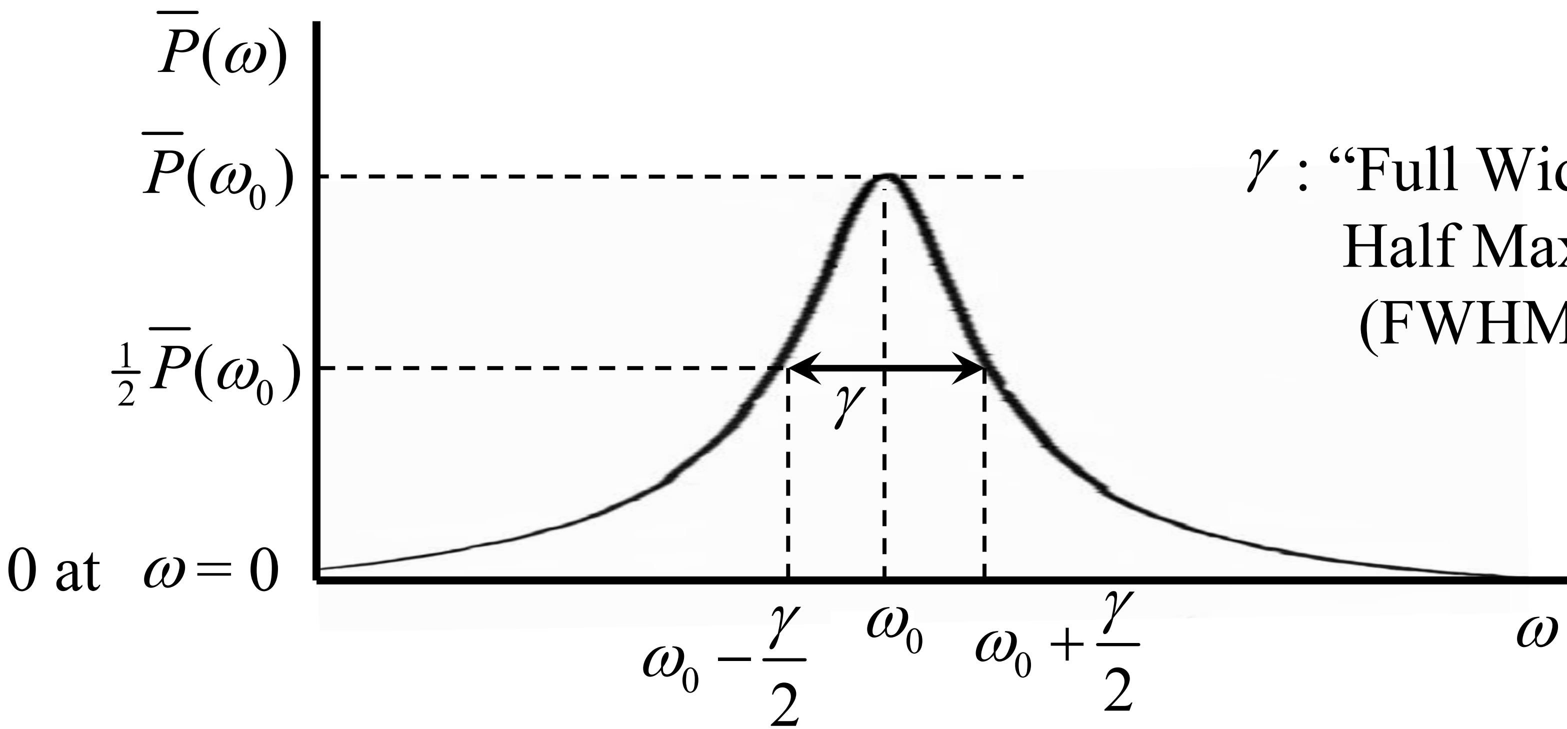
Can show that $$\overline{P}(\omega) = \frac{F_0^2 \omega_0}{2kQ} \left\{ \left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0} \right)^2 + \frac{1}{Q^2} \right\}^{-1}$$

The power absorbed by a driven oscillator …3

$$A(\omega) = \frac{F_0/m}{\sqrt{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}} \quad \text{and} \quad \overline{P}(\omega) = \frac{F_0^2\omega^2\gamma}{2m}\frac{1}{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}$$

$A(\omega)$ drops to $1/\sqrt{2}$ of its maximum value at $\omega = \omega_0 \pm \gamma/2$

… so $\overline{P}(\omega)$ will drop to $1/2$ its maximum value at these frequencies



γ : "Full Width at Half Maximum" (FWHM)

An alternative form for $A(\omega)$ for high Q oscillators close to resonance …

$$A(\omega) = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}}$$

$$(\omega_0^2 - \omega^2)^2 = (\omega_0 + \omega)^2(\omega_0 - \omega)^2$$

$$\approx (2\omega_0)^2(\omega_0 - \omega)^2 \quad \text{near resonance}$$

Near resonance, $\omega \approx \omega_0$ $\qquad \therefore A(\omega) = \dfrac{F_0/m}{\sqrt{4\omega_0^2(\omega_0 - \omega)^2 + \omega_0^2\gamma^2}}$

$$\therefore A(\omega) = \frac{F_0/m}{2\omega_0\sqrt{(\omega_0 - \omega)^2 + \gamma^2/4}}$$

Near resonance $A(\omega) = \dfrac{F_0/m}{2\omega_0\sqrt{(\omega_0-\omega)^2+\gamma^2/4}}$

The energy $E = \frac{1}{2}mv_{\max}^2 = \frac{1}{2}m\omega^2 A^2(\omega)$

$$= \frac{1}{2}m\omega^2 \frac{\left(F_0/m\right)^2}{4\omega_0^2\left\{(\omega_0-\omega)^2+\gamma^2/4\right\}}$$

$$= m\frac{F_0^2\omega^2}{8m\omega_0^2}\frac{1}{(\omega_0-\omega)^2+\gamma^2/4}$$

…this form is encountered in various branches of physics
… e.g. the Breit-Wigner formula in nuclear physics

$$\text{lineshape} \propto \frac{1}{\left(E_0-E\right)^2+\Gamma^2/4} \qquad \left(E=\hbar\omega\right)$$

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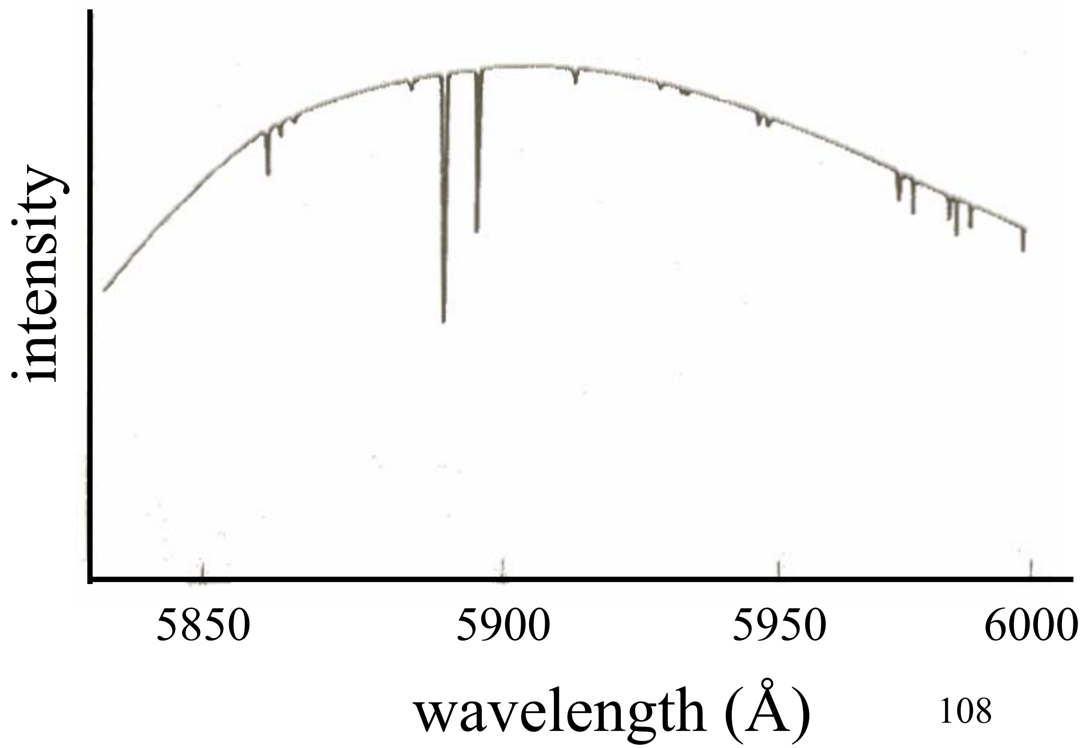
Optical resonance

Atoms behave like sharply tuned oscillators in the processes of emitting and absorbing light.

Example: Fraunhofer lines in the absorption spectrum of the sun … which are the result of resonant absorption processes

… Doppler broadening of the spectral lines is about 100 times greater than any effect due to the true lifetime of the radiating atom!



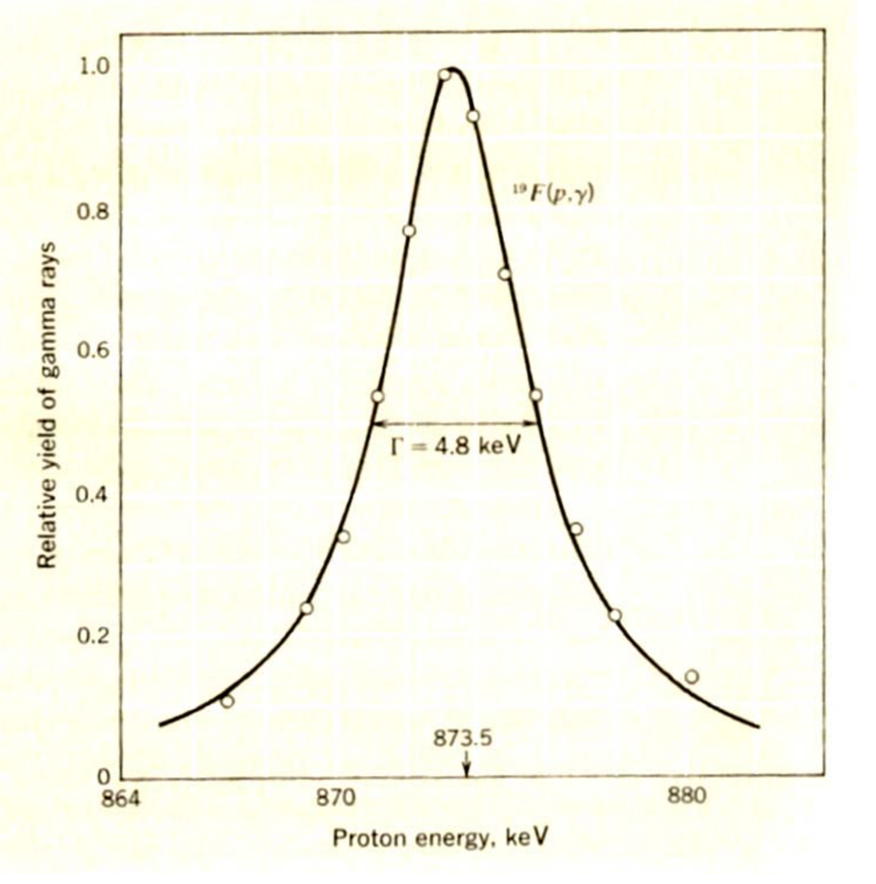
Nuclear resonance

Nuclear cross sections $\sigma(E)$ display resonances at certain energies, which depend on the particular reaction and incident energy E_0…

Many of these resonances can be well-described by the equation

$$\sigma(E) = \frac{\sigma(E_0)}{\frac{4(E_0 - E)^2}{\Gamma^2} + 1}$$

Γ : FWHM



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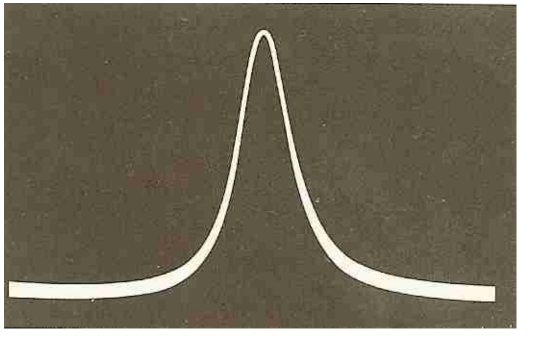
Nuclear magnetic resonance

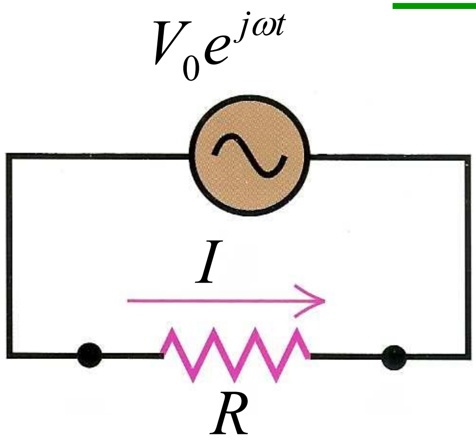
Atomic nuclei are limited to only a few discrete orientations when placed in a magnetic field.
… protons have only two orientations and can be caused to flip from one state to the other, in a magnetic field, by injecting electromagnetic radiation of just the right frequency
… for protons in a magnetic field of 5000 G, the resonance frequency is about 21 MHz … can observe a voltage which varies (resonantly) as a function of frequency and can be described by

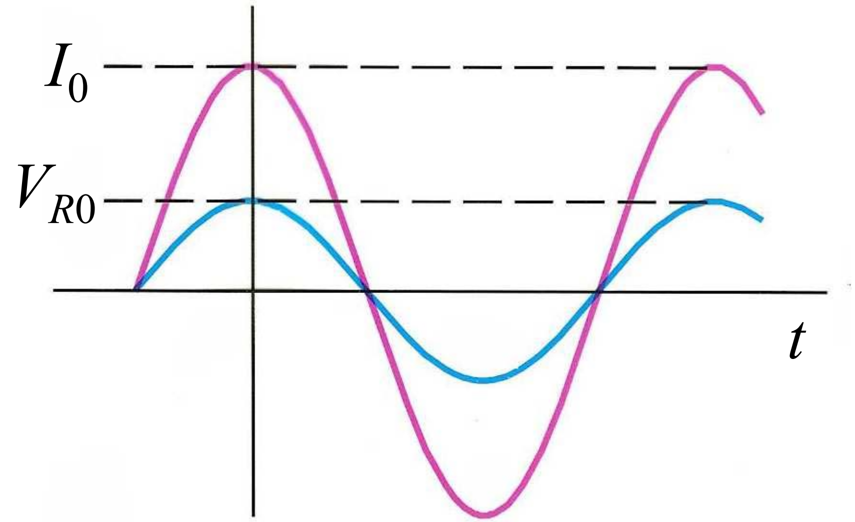
$$V(B) = \frac{V_0}{\frac{4(B_0 - B)^2}{(\Delta B)^2} + 1}$$



… “nuclear magnetic resonance”
Nobel Prize in physics (1952): F. Bloch and E.M. Purcell

A.C.: A resistive circuit



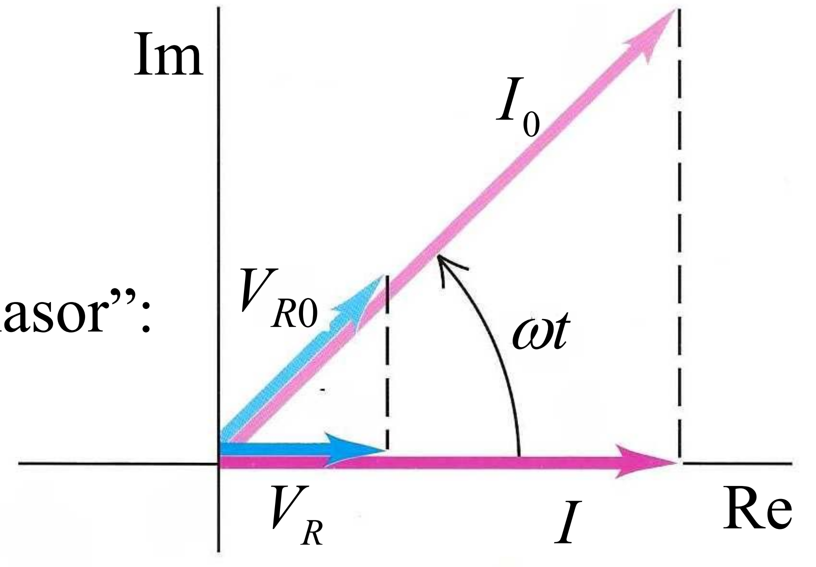


$$V_0 e^{j\omega t} = V_R$$

$$IR = V_{R0} e^{j\omega t}$$

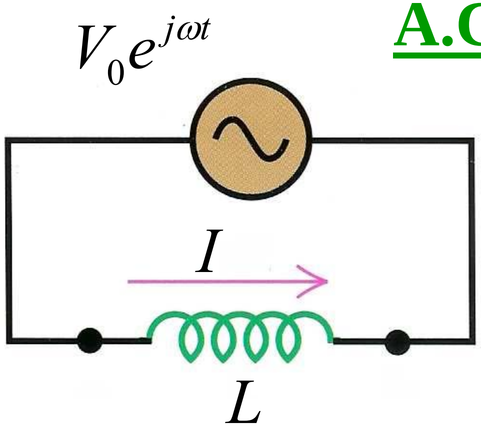
$$\therefore I = \frac{V_{R0}}{R} e^{j\omega t} = I_0 e^{j\omega t}$$

"Phasor":



Current <u>in phase</u> with voltage

A.C.: An inductive circuit



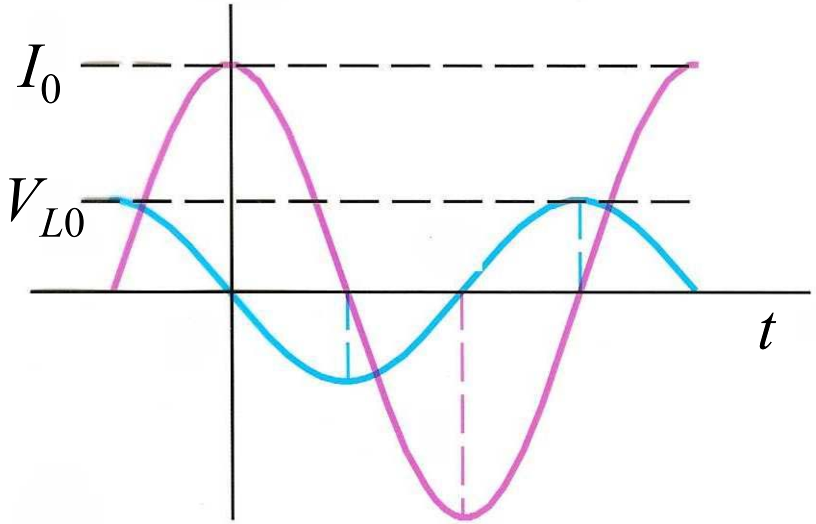
$$V_0 e^{j\omega t} = V_L$$

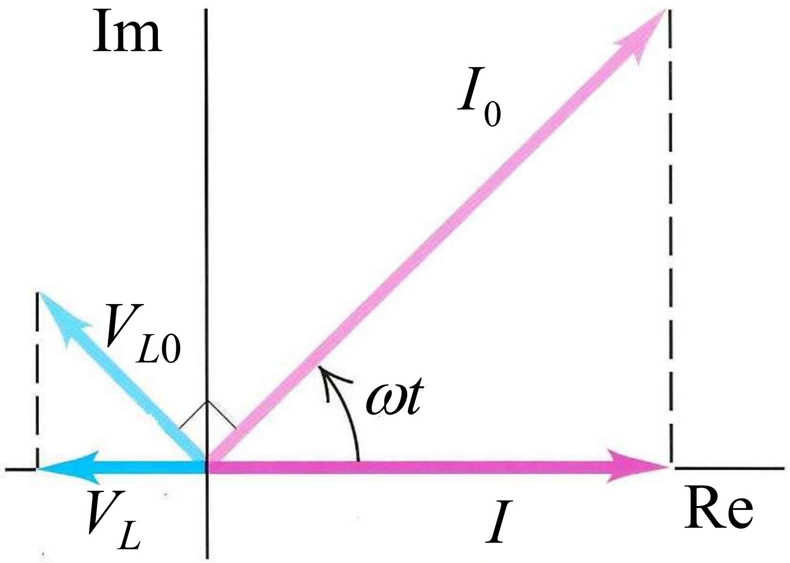
$$\therefore V_L = L\frac{dI}{dt} = L\frac{d}{dt}\left(I_0 e^{j\omega t}\right) = j\omega L I$$

$$\therefore I = \frac{V_L}{j\omega L} = \frac{V_0}{j\omega L}e^{j\omega t} = \frac{V_0}{\omega L}(-j)e^{j\omega t}$$

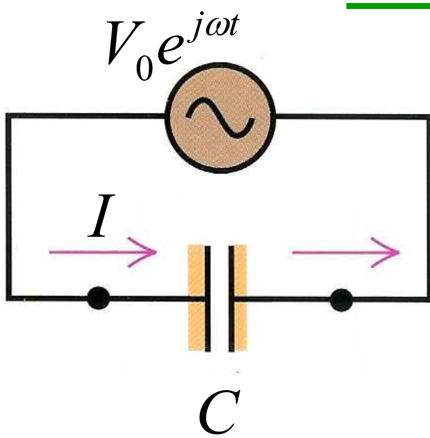
$$\therefore I = \frac{V_0}{\omega L}e^{j(\omega t - \pi/2)} \qquad -j = e^{-j\pi/2}$$

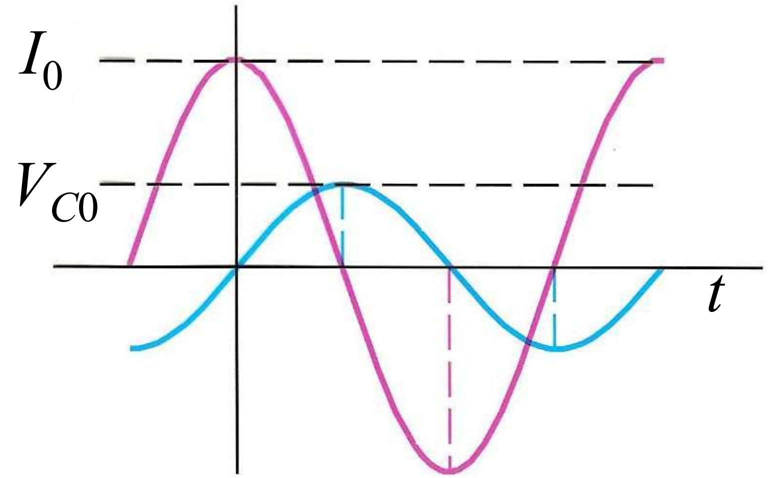
Current lags voltage by $\pi/2$





A.C.: A capacitive circuit





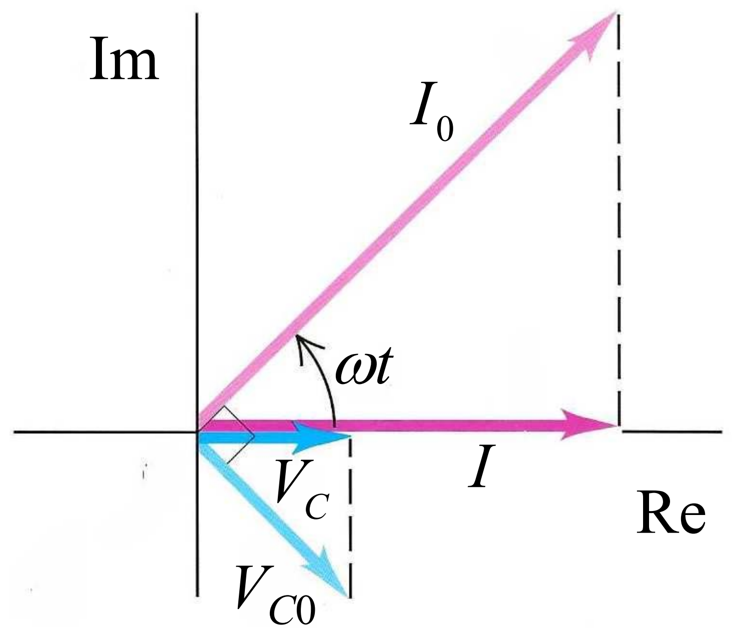
$$V_0 e^{j\omega t} = V_C$$

$$Q = CV_C = CV_0 e^{j\omega t}$$

$$I = \frac{dQ}{dt} = j\omega C V_0 e^{j\omega t}$$

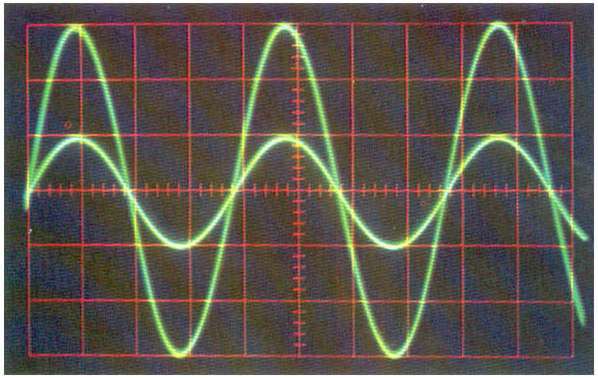
$$j = e^{j\pi/2}$$

$$I = \frac{dQ}{dt} = \omega C V_0 e^{j(\omega t + \pi/2)}$$

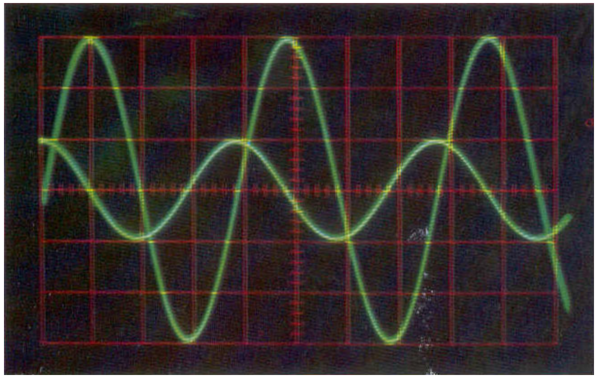


Current <u>leads</u> voltage by $\pi/2$

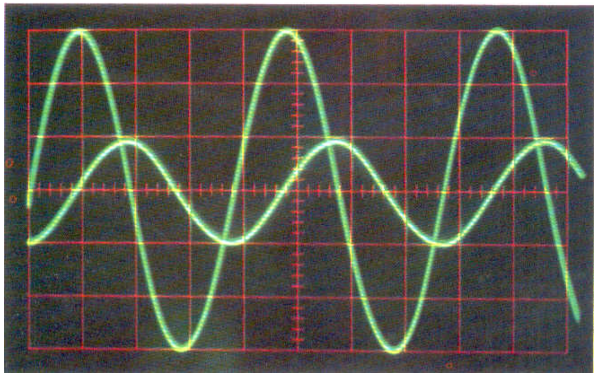
A resistive circuit



An inductive circuit



A capacitive circuit



A.C.: LRC circuit

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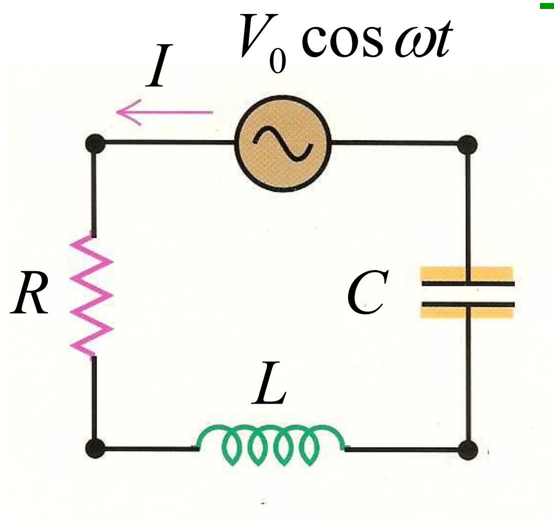


$$V_0 \cos \omega t = V_L + V_R + V_C$$

$$\therefore L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = V_0 \cos \omega t$$

$$I = \frac{dQ}{dt}$$

$$\therefore L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = \frac{dV}{dt}$$

... I will be of the form $I_0 e^{j\omega t}$

Therefore write $(j\omega)^2 LI + j\omega RI + \frac{1}{C}I = j\omega V$

Rearrange: $\left(R + j\omega L + \frac{1}{j\omega C}\right)I = V$

LRC circuit …2

$$\left(R + j\omega L + \frac{1}{j\omega C}\right) I = V$$

… call $R + j\omega L + \dfrac{1}{j\omega C}$ the **impedance** Z

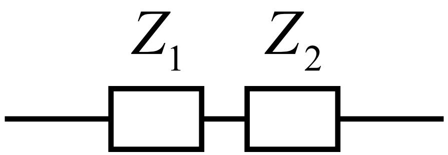
… which is the a.c. analogue of resistance.

Write $\quad Z = R + j\omega L - \dfrac{j}{\omega C}$

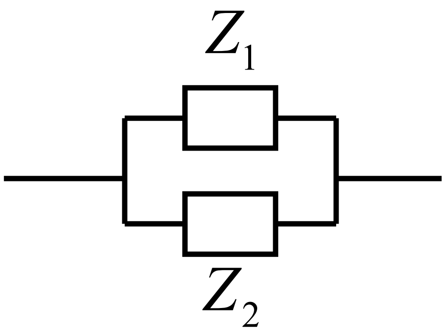
Circuit element	**Impendence**
resistance	R
inductance	$j\omega L$
capacitance	$-\dfrac{j}{\omega C}$

LRC circuit …3

Write $V = IZ$



$$Z = Z_1 + Z_2$$



$$\frac{1}{Z} = \frac{1}{Z_1} + \frac{1}{Z_2}$$

Impedance is **complex**:

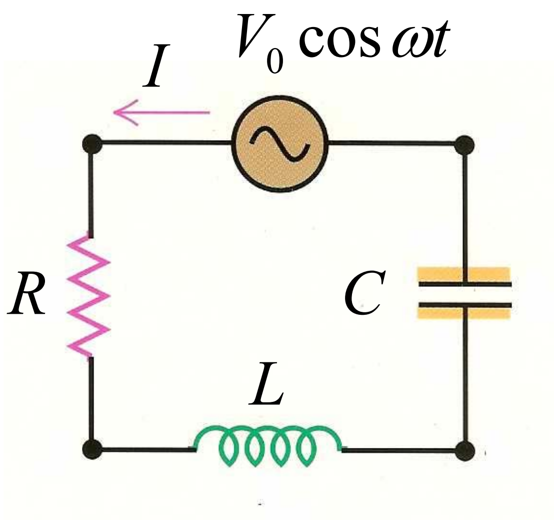
$$Z = R + j\left(\omega L - \frac{1}{\omega C}\right)$$

or $Z = |Z| e^{j\delta}$

… can use
Kirchhoff's Rules,
Thenevin's theorem, etc.

where $|Z| = \sqrt{R^2 + (\omega L - 1/\omega C)^2}$ and $\tan\delta = \dfrac{(\omega L - 1/\omega C)}{R}$

LRC circuit …4



$$I = \frac{V}{Z} = \frac{V_0 e^{j\omega t}}{R + j\left(\omega L - 1/\omega C\right)}$$

$$= \frac{V_0 e^{j\omega t}}{\sqrt{R^2 + \left(\omega L - 1/\omega C\right)^2}\; e^{j\delta}}$$

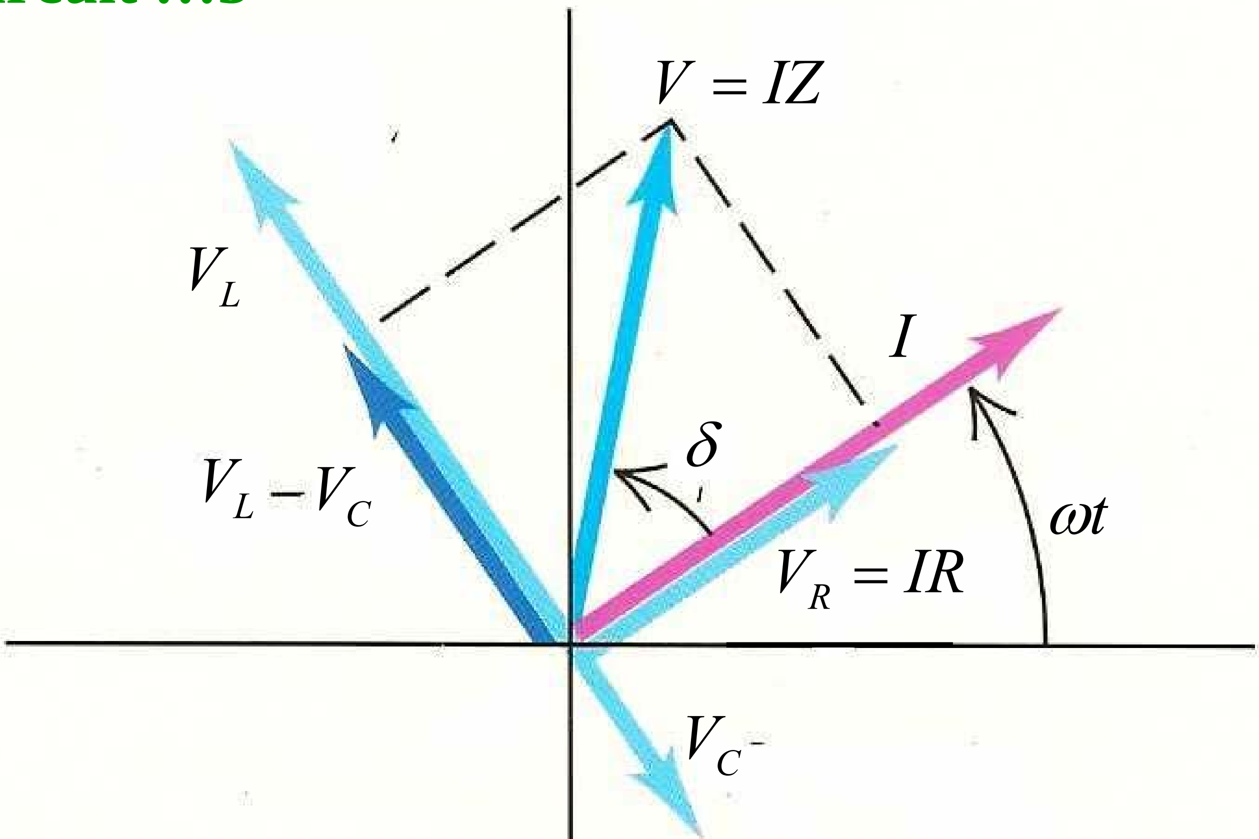
$$= \frac{V_0}{\sqrt{R^2 + \left(\omega L - 1/\omega C\right)^2}} e^{j(\omega t - \delta)}$$

…and the physical current is $\dfrac{V_0}{\sqrt{R^2 + \left(\omega L - 1/\omega C\right)^2}} \cos\left(\omega t - \delta\right)$

The current has amplitude $\dfrac{V_0}{\sqrt{R^2 + \left(\omega L - 1/\omega C\right)^2}}$

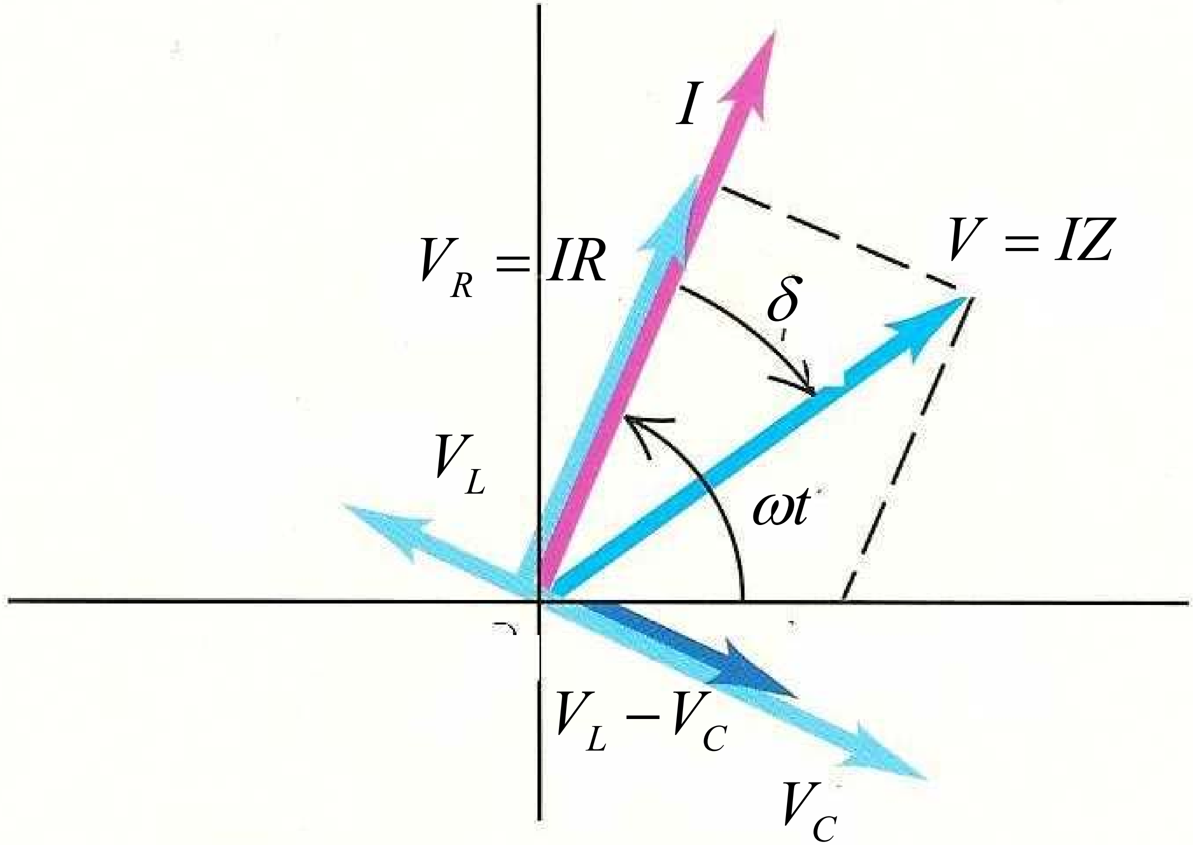
and lags the applied voltage by δ .

LRC circuit …5



At <u>high</u> frequencies the impedance of the inductance is dominant, the phase lag is positive,
i.e. the current <u>lags</u> the voltage.

LRC circuit …6



At <u>low</u> frequencies the impedance of the capacitance is dominant, the phase lag is negative, i.e. the current <u>leads</u> the voltage.

LRC circuit …7

The amplitude and phase lag depend on frequency.

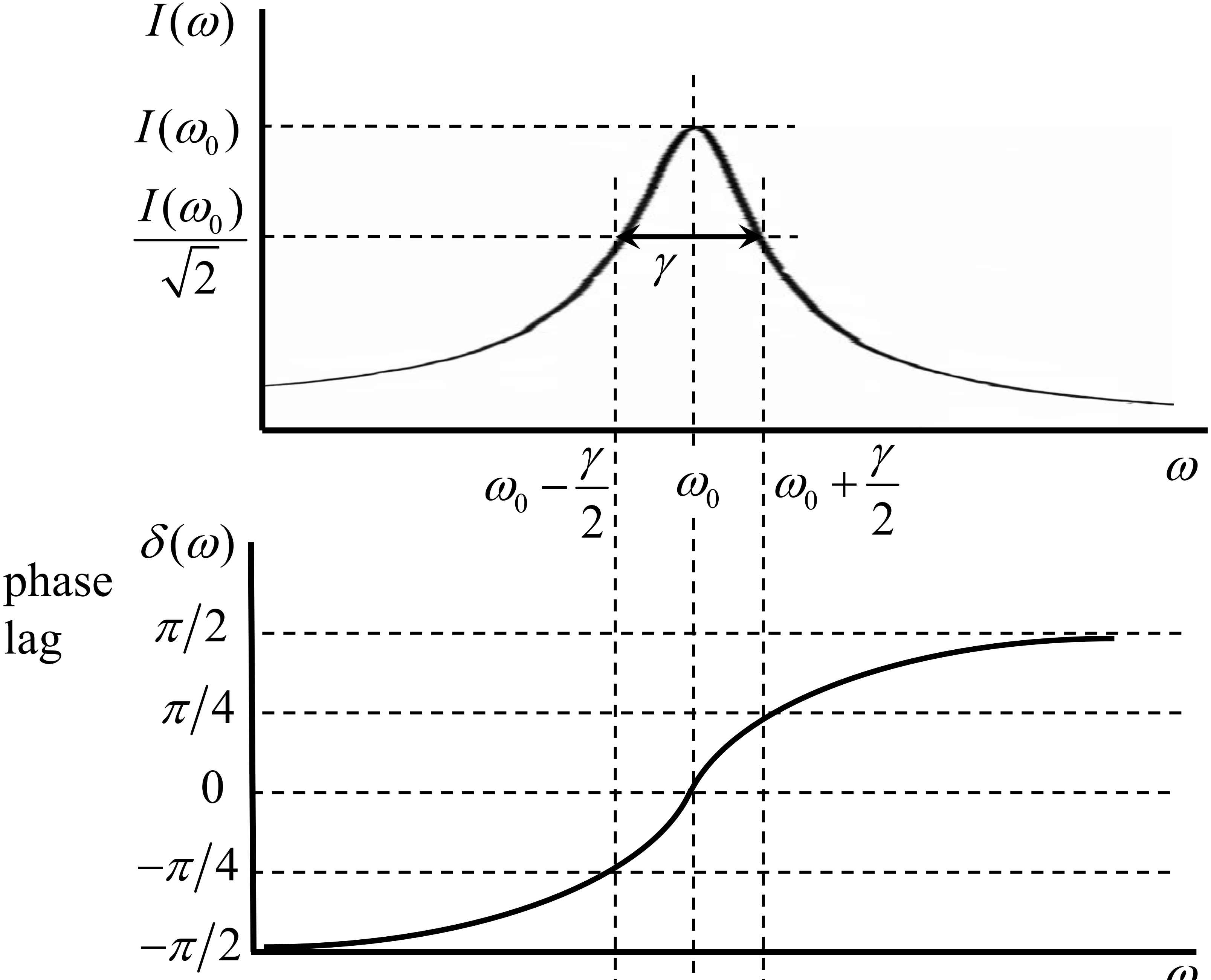
… as $\left(\omega L - 1/\omega C\right) \to 0$ the current amplitude is a maximum.

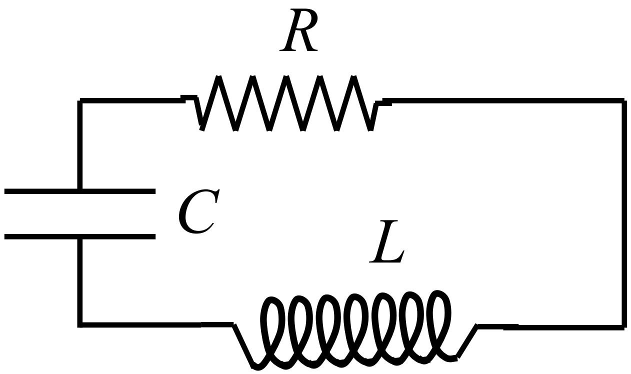
For $\omega L = \dfrac{1}{\omega C}$, $\omega^2 = \dfrac{1}{LC}$

… i.e. the amplitude is a maximum when the applied frequency ω is equal to the natural frequency of the (undamped) oscillator.

At resonance $\omega L = \dfrac{1}{\omega C}$ the circuit behaves like a pure resistance.

$$I = \frac{V_0}{R}\cos\omega t \qquad \delta = 0$$


$I(\omega)$
$I(\omega_0)$
$\frac{I(\omega_0)}{\sqrt{2}}$
γ
$\omega_0 - \frac{\gamma}{2}$
ω_0
$\omega_0 + \frac{\gamma}{2}$
ω
$\delta(\omega)$
phase
lag
$\pi/2$
$\pi/4$
0
$-\pi/4$
$-\pi/2$
ω



Oscillations in an *LRC* circuit … an energy approach

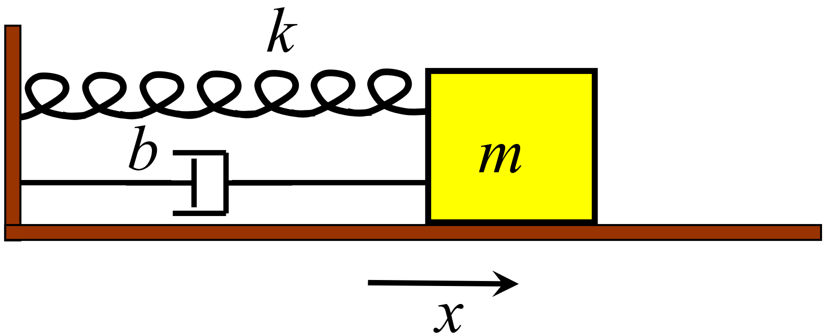
Power dissipated in resistor = rate of decrease of stored energy

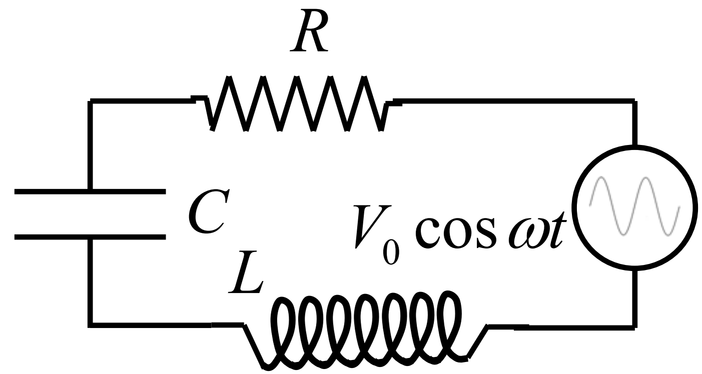
$$I^2 R = -\frac{d}{dt}\left\{\tfrac{1}{2}\frac{Q^2}{C} + \tfrac{1}{2} L I^2\right\}$$

$$\therefore \left(\frac{dQ}{dt}\right)^2 R + \frac{d}{dt}\left\{\tfrac{1}{2}\frac{Q^2}{C} + \tfrac{1}{2} L \left(\frac{dQ}{dt}\right)^2\right\} = 0$$

$$\therefore \left(\frac{dQ}{dt}\right)^2 R + \frac{Q}{C}\frac{dQ}{dt} + L\frac{dQ}{dt}\frac{d^2Q}{dt^2} = 0 \qquad \div \frac{dQ}{dt}$$

$$\therefore \frac{d^2Q}{dt^2} + \gamma\frac{dQ}{dt} + \omega_0^2 Q = 0 \quad \text{with} \quad \omega_0 = \sqrt{\frac{1}{LC}} \quad \text{and} \quad \gamma = \frac{R}{L}$$





$$\frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + \omega_0^2 x = \frac{F_0}{m} \cos \omega t$$

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = V_0 \cos \omega t$$

$$\gamma = \frac{b}{m} \qquad \omega_0^2 = \frac{k}{m}$$

$$\gamma = \frac{R}{L} \qquad \omega_0^2 = \frac{1}{LC}$$

Hence

$$x(t) = A(\omega)\cos(\omega t - \delta)$$

$$x(t) = \mathfrak{A}(\omega)\cos(\omega t - \delta)$$

where

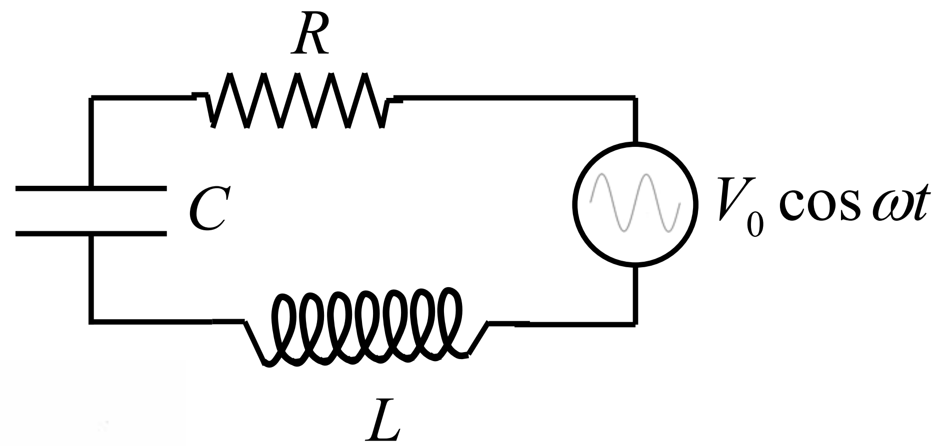
$$A(\omega) = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}}$$

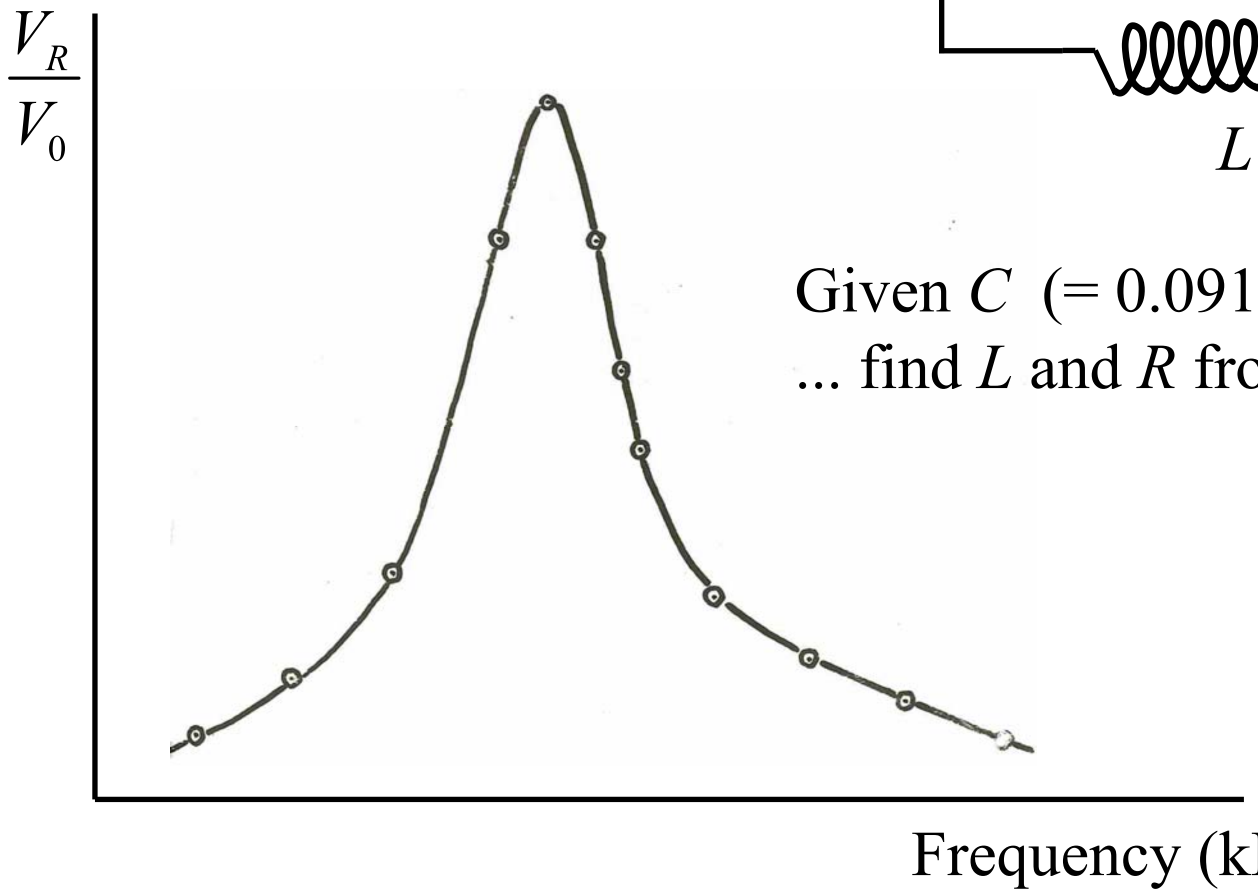
$$\mathfrak{A}(\omega) = \frac{V_0/L}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}}$$

$$\tan\delta(\omega) = \frac{\omega\gamma}{(\omega_0^2 - \omega^2)}$$

$$\tan\delta(\omega) = \frac{\omega\gamma}{(\omega_0^2 - \omega^2)}$$

LRC resonance … CI laboratory experiment





Given C (= 0.0914 μF)
... find L and R from resonance curve