APPLICATION OF DERIVATIVE

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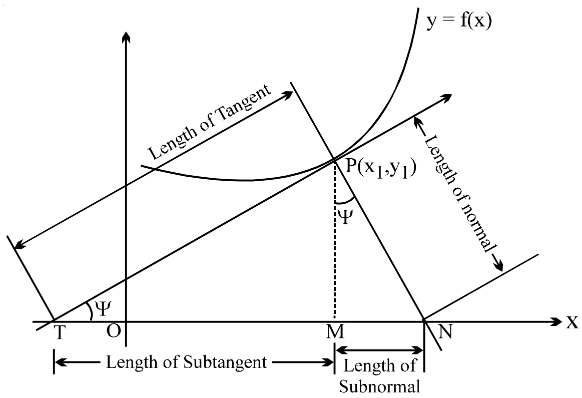
TANGENT & NORMAL

THINGS TO REMEMBER :



I The value of the derivative at P (x_1, y_1) gives the slope of the tangent to the curve at P. Symbolically

$$f'(x_1) = \left.\frac{dy}{dx}\right]_{x_1 y_1} = \text{Slope of tangent at}$$

$P(x_1 y_1) = m$ (say).

II Equation of tangent at (x_1, y_1) is ;

$$y - y_1 = \left.\frac{dy}{dx}\right]_{x_1 y_1} (x - x_1).$$

III Equation of normal at (x_1, y_1) is ;

$$y - y_1 = -\frac{1}{\left.\frac{dy}{dx}\right]_{x_1 y_1}} (x - x_1).$$

NOTE :

1. The point P (x_1, y_1) will satisfy the equation of the curve & the equation of tangent & normal line.

2. If the tangent at any point P on the curve is parallel to the axis of x then $dy/dx = 0$ at the point P.

3. If the tangent at any point on the curve is parallel to the axis of y, then $dy/dx = \infty$ or $dx/dy = 0$.

4. If the tangent at any point on the curve is equally inclined to both the axes then $dy/dx = \pm 1$.

5. If the tangent at any point makes equal intercept on the coordinate axes then $dy/dx = -1$.

6. Tangent to a curve at the point P (x_1, y_1) can be drawn even through dy/dx at P does not exist.
e.g. $x = 0$ is a tangent to $y = x^{2/3}$ at (0, 0).

7. If a curve passing through the origin be given by a rational integral algebraic equation, the equation of the tangent (or tangents) at the origin is obtained by equating to zero the terms of the lowest degree in the equation.
e.g. If the equation of a curve be $x^2 - y^2 + x^3 + 3x^2y - y^3 = 0$, the tangents at the origin are given by $x^2 - y^2 = 0$ i.e. $x + y = 0$ and $x - y = 0$.

IV Angle of intersection between two curves is defined as the angle between the 2 tangents drawn to the 2 curves at their point of intersection. If the angle between two curves is 90° every where then they are called **ORTHOGONAL** curves.

V **(a)** Length of the tangent (PT) = $\dfrac{y_1\sqrt{1 + [f'(x_1)]^2}}{f'(x_1)}$ **(b)** Length of Subtangent (MT) = $\dfrac{y_1}{f'(x_1)}$

(c) Length of Normal (PN) = $y_1\sqrt{1 + [f'(x_1)]^2}$ **(d)** Length of Subnormal (MN) = $y_1 f'(x_1)$

VI **DIFFERENTIALS :**
The differential of a function is equal to its derivative multiplied by the differential of the independent variable. Thus if, $y = \tan x$ then $dy = \sec^2 x\, dx$.
In general $dy = f'(x)\, dx$.
Note that : $d(c) = 0$ where 'c' is a constant.
$d(u + v - w) = du + dv - dw$ $d(uv) = u\,dv + v\,du$

Note :

1. For the independent variable 'x', increment Δx and differential dx are equal but this is not the case with the dependent variable 'y' i.e. $\Delta y \neq dy$.

2. The relation $dy = f'(x)\, dx$ can be written as $\dfrac{dy}{dx} = f'(x)$; thus the quotient of the differentials of 'y' and 'x' is equal to the derivative of 'y' w.r.t. 'x'.

EXERCISE–I

Q.1 Find the equation of the normal to the curve $y = (1+x)^y + \sin^{-1}(\sin^2 x)$ at $x = 0$.

Q.2 Find the equations of the tangents drawn to the curve $y^2 - 2x^3 - 4y + 8 = 0$ from the point (1, 2).

Q.3 Find the point of intersection of the tangents drawn to the curve $x^2y = 1 - y$ at the points where it is intersected by the curve $xy = 1 - y$.

Q.4 Find all the lines that pass through the point (1, 1) and are tangent to the curve represented parametrically as $x = 2t - t^2$ and $y = t + t^2$.

Q.5 The tangent to $y = ax^2 + bx + \frac{7}{2}$ at (1, 2) is parallel to the normal at the point (–2, 2) on the curve $y = x^2 + 6x + 10$. Find the value of a and b.

Q.6 A straight line is drawn through the origin and parallel to the tangent to a curve $\frac{x+\sqrt{a^2-y^2}}{a} = ln\left(\frac{a+\sqrt{a^2-y^2}}{y}\right)$ at an arbitary point M. Show that the locus of the point P of intersection of the straight line through the origin & the straight line parallel to the x-axis & passing through the point M is $x^2 + y^2 = a^2$.

Q.7 A line is tangent to the curve $f(x) = \frac{41x^3}{3}$ at the point P in the first quadrant, and has a slope of 2009. This line intersects the y-axis at (0, b). Find the value of 'b'.

Q.8 A function is defined parametrically by the equations

$$f(t) = x = \begin{cases} 2t + t^2 \sin\frac{1}{t} & \text{if } t \neq 0 \\ 0 & \text{if } t = 0 \end{cases} \quad \text{and } g(t) = y = \begin{cases} \frac{1}{t}\sin t^2 & \text{if } t \neq 0 \\ o & \text{if } t = 0 \end{cases}$$

Find the equation of the tangent and normal at the point for t = 0 if exist.

Q.9 Find all the tangents to the curve $y = \cos(x+y), -2\pi \le x \le 2\pi$, that are parallel to the line x + 2y = 0.

Q.10 Prove that the segment of the normal to the curve $x = 2a\sin t + a\sin t\cos^2 t$; $y = -a\cos^3 t$ contained between the co-ordinate axes is equal to 2a.

Q.11 Show that the normals to the curve $x = a(\cos t + t\sin t)$; $y = a(\sin t - t\cos t)$ are tangent lines to the circle $x^2 + y^2 = a^2$.

Q.12 The chord of the parabola $y = -a^2x^2 + 5ax - 4$ touches the curve $y = \frac{1}{1-x}$ at the point x = 2 and is bisected by that point. Find 'a'.

Q.13 If the tangent at the point (x_1, y_1) to the curve $x^3 + y^3 = a^3$ $(a \neq 0)$ meets the curve again in (x_2, y_2) then show that $\frac{x_2}{x_1} + \frac{y_2}{y_1} = -1$.

Q.14 Determine a differentiable function y = f (x) which satisfies $f'(x) = [f(x)]^2$ and $f(0) = -\frac{1}{2}$. Find also the equation of the tangent at the point where the curve crosses the y-axis.

Q.15 Tangent at a point P_1 [other than (0 , 0)] on the curve $y = x^3$ meets the curve again at P_2. The tangent at P_2 meets the curve at P_3 & so on. Show that the abscissae of P_1, P_2, P_3, ... P_n, form a GP. Also find the ratio $\frac{\text{area}\ (P_1P_2P_3)}{\text{area}\ (P_2P_3P_4)}$.

Q.16 The curve $y = ax^3 + bx^2 + cx + 5$, touches the x - axis at P (–2 , 0) & cuts the y-axis at a point Q where its gradient is 3. Find a , b , c.

Q.17 The tangent at a variable point P of the curve $y = x^2 - x^3$ meets it again at Q. Show that the locus of the middle point of PQ is $y = 1 - 9x + 28x^2 - 28x^3$.

Q.18 Show that the distance from the origin of the normal at any point of the curve $x = a\,e^{\theta}\left(\sin\frac{\theta}{2} + 2\cos\frac{\theta}{2}\right)$ & $y = a\,e^{\theta}\left(\cos\frac{\theta}{2} - 2\sin\frac{\theta}{2}\right)$ is twice the distance of the tangent at the point from the origin.

Q.19 Show that the condition that the curves $x^{2/3} + y^{2/3} = c^{2/3}$ & $\mathbf{(x^2/a^2) + (y^2/b^2) = 1}$ may touch if c = a + b.

Q.20 The graph of a certain function f contains the point (0, 2) and has the property that for each number 'p' the line tangent to $y = f(x)$ at $(p, f(p))$ intersect the x-axis at p + 2. Find $f(x)$.

Q.21 A curve is given by the equations $x = at^2$ & $y = at^3$. A variable pair of perpendicular lines through the origin 'O' meet the curve at P & Q. Show that the locus of the point of intersection of the tangents at P & Q is $4y^2 = 3ax - a^2$.

Q.22 A and B are points of the parabola $y = x^2$. The tangents at A and B meet at C. The median of the triangle ABC from C has length 'm' units. Find the area of the triangle in terms of 'm'.

Q.23 (a) Find the value of n so that the subnormal at any point on the curve $xy^n = a^{n+1}$ may be constant.
(b) Show that in the curve $y = a.\ ln\,(x^2 - a^2)$, sum of the length of tangent & subtangent varies as the product of the coordinates of the point of contact.

Q.24(a) Show that the curves $\frac{x^2}{a^2+K_1} + \frac{y^2}{b^2+K_1} = 1$ & $\frac{x^2}{a^2+K_2} + \frac{y^2}{b^2+K_2} = 1$ intersect orthogonally.
(b) If the two curves $C_1 : x = y^2$ and $C_2 : xy = k$ cut at right angles find the value of k.

Q.25 Show that the angle between the tangent at any point 'A' of the curve $ln\,(x^2 + y^2) = C\tan^{-1}\frac{y}{x}$ and the line joining A to the origin is independent of the position of A on the curve.

EXERCISE–II

RATE MEASURE AND APPROXIMATIONS

Q.1 Water is being poured on to a cylindrical vessel at the rate of 1 m^3/min. If the vessel has a circular base of radius 3 m, find the rate at which the level of water is rising in the vessel.

Q.2 A man 1.5 m tall walks away from a lamp post 4.5 m high at the rate of 4 km/hr.
(i) how fast is the farther end of the shadow moving on the pavement ?
(ii) how fast is his shadow lengthening ?

Q.3 A particle moves along the curve $6y = x^3 + 2$. Find the points on the curve at which the y coordinate is changing 8 times as fast as the x coordinate.

Q.4 An inverted cone has a depth of 10 cm & a base of radius 5 cm. Water is poured into it at the rate of 1.5 cm^3/min. Find the rate at which level of water in the cone is rising, when the depth of water is 4 cm.

Q.5 A water tank has the shape of a right circular cone with its vertex down. Its altitude is 10 cm and the radius of the base is 15 cm. Water leaks out of the bottom at a constant rate of 1cu. cm/sec. Water is poured into the tank at a constant rate of C cu. cm/sec. Compute C so that the water level will be rising at the rate of 4 cm/sec at the instant when the water is 2 cm deep.

Q.6 Sand is pouring from a pipe at the rate of 12 cc/sec. The falling sand forms a cone on the ground in such a way that the height of the cone is always 1/6th of the radius of the base. How fast is the height of the sand cone increasing when the height is 4 cm.

Q.7 An open Can of oil is accidently dropped into a lake; assume the oil spreads over the surface as a circular disc of uniform thickness whose radius increases steadily at the rate of 10 cm/sec. At the moment when the radius is 1 meter, the thickness of the oil slick is decreasing at the rate of 4 mm/sec, how fast is it decreasing when the radius is 2 meters.

Q.8 Water is dripping out from a conical funnel of semi vertical angle $\pi/4$, at the uniform rate of 2 cm^3/sec through a tiny hole at the vertex at the bottom. When the slant height of the water is 4 cm, find the rate of decrease of the slant height of the water.

Q.9 An air force plane is ascending vertically at the rate of 100 km/h. If the radius of the earth is R Km, how fast the area of the earth, visible from the plane increasing at 3min after it started ascending. Take visible area $A = \dfrac{2\pi R^2 h}{R + h}$ Where h is the height of the plane in kms above the earth.

Q.10 A variable ΔABC in the xy plane has its orthocentre at vertex 'B' , a fixed vertex 'A' at the origin and the third vertex 'C' restricted to lie on the parabola $y = 1 + \dfrac{7x^2}{36}$. The point B starts at the point (0, 1) at time t = 0 and moves upward along the y axis at a constant velocity of 2 cm/sec. How fast is the area of the triangle increasing when $t = \dfrac{7}{2}$ sec.

Q.11 A circular ink blot grows at the rate of 2 cm^2 per second. Find the rate at which the radius is increasing after $2\dfrac{6}{11}$ seconds. Use $\pi = \dfrac{22}{7}$.

Q.12 Water is flowing out at the rate of 6 m^3/min from a reservoir shaped like a hemispherical bowl of radius R = 13 m. The volume of water in the hemispherical bowl is given by $V = \frac{\pi}{3} \cdot y^2 (3R - y)$ when the water is y meter deep. Find

(a) At what rate is the water level changing when the water is 8 m deep.

(b) At what rate is the radius of the water surface changing when the water is 8 m deep.

Q.13 If in a triangle ABC, the side 'c' and the angle 'C' remain constant, while the remaining elements are changed slightly, show that $\frac{da}{\cos A} + \frac{db}{\cos B} = 0$.

Q.14 At time $t > 0$, the volume of a sphere is increasing at a rate proportional to the reciprocal of its radius. At $t = 0$, the radius of the sphere is 1 unit and at $t = 15$ the radius is 2 units.

(a) Find the radius of the sphere as a function of time t.

(b) At what time t will the volume of the sphere be 27 times its volume at $t = 0$.

Q.15(i) Use differentials to a approximate the values of; (a) $\sqrt{36.6}$ and (b) $\sqrt[3]{26}$.

(ii) If the radius of a sphere is measured as 9 cm with an error of 0.03 cm, then find the approximate error in calculating its volume.

EXERCISE–III

Q.1 Find the equation of the straight line which is tangent at one point and normal at another point of the curve, $x = 3t^2$, $y = 2t^3$. [REE 2000 (Mains) 5 out of 100]

Q.2 If the normal to the curve, $y = f(x)$ at the point (3, 4) makes an angle $\frac{3\pi}{4}$ with the positive x–axis. Then $f'(3) =$

(A) – 1 (B) $-\frac{3}{4}$ (C) $\frac{4}{3}$ (D) 1

[JEE 2000 (Scr.) 1 out of 35]

Q.3 The point(s) on the curve $y^3 + 3x^2 = 12y$ where the tangent is vertical, is(are)

(A) $\left(\pm\frac{4}{\sqrt{3}}, -2\right)$ (B) $\left(\pm\sqrt{\frac{11}{3}}, 1\right)$ (C) (0, 0) (D) $\left(\pm\frac{4}{\sqrt{3}}, 2\right)$

[JEE 2002 (Scr.), 3]

Q.4 Tangent to the curve $y = x^2 + 6$ at a point P (1, 7) touches the circle $x^2 + y^2 + 16x + 12y + c = 0$ at a point Q. Then the coordinates of Q are

(A) (– 6, –11) (B) (–9, –13) (C) (– 10, – 15) (D) (–6, –7)

[JEE 2005 (Scr.), 3]

Q.5 The tangent to the curve $y = e^x$ drawn at the point (c, e^c) intersects the line joining the points $(c - 1, e^{c-1})$ and $(c + 1, e^{c+1})$

(A) on the left of x = c (B) on the right of x = c

(C) at no point (D) at all points [JEE 2007, 3]

MONOTONOCITY

(Significance of the sign of the first order derivative)

DEFINITIONS :

1. A function f(x) is called an Increasing Function at a point x = a if in a sufficiently small neighbourhood around x = a we have $\left.\begin{array}{l} f(a+h) > f(a) \text{ and} \\ f(a-h) < f(a) \end{array}\right\}$ increasing;

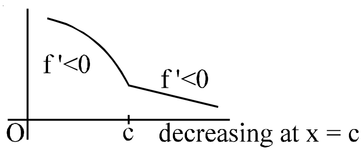
Similarly decreasing if $\left.\begin{array}{l} f(a+h) < f(a) \text{ and} \\ f(a-h) > f(a) \end{array}\right\}$ decreasing.

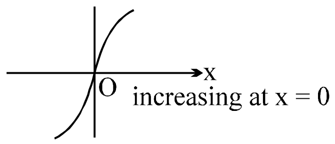
disregards whether f is non derivable or even discontinuous at x = a

2. A differentiable function is called increasing in an interval (a, b) if it is increasing at every point within the interval (but not necessarily at the end points).A function decreasing in an interval (a, b) is similarly defined.

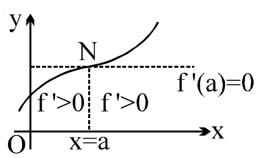
3. A function which in a given interval is increasing or decreasing is called **"Monotonic"** in that interval.

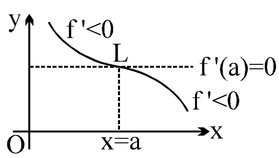
4. **Tests for increasing and decreasing of a function at a point :**
If the derivative $f'(x)$ is positive at a point x = a, then the function f (x) at this point is increasing. If it is negative, then the function is decreasing. Even if f'(a) is not defined, f can still be increasing or decreasing.





Note : If $f'(a) = 0$, then for x = a the function may be still increasing or it may be decreasing as shown. It has to be identified by a seperate rule. e.g. $f(x) = x^3$ is increasing at every point.
Note that, **dy/dx = 3 x²**.





5. **Tests for Increasing & Decreasing of a function in an interval :**
SUFFICIENCY TEST : If the derivative function $f'(x)$ in an interval (a , b) is every where positive, then the function f(x) in this interval is Increasing ;
If $f'(x)$ is every where negative, then f(x) is Decreasing.

General Note :

(1) If a continuous function is invertible it has to be either increasing or decreasing.

(2) If a function is continuous the intervals in which it rises and falls may be separated by points at which its derivative fails to exist.

(3) If f is increasing in [a, b] and is continuous then f (b) is the greatest and f (c) is the least value of f in [a, b]. Similarly if f is decreasing in [a, b] then f (a) is the greatest value and f (b) is the least value.

6.

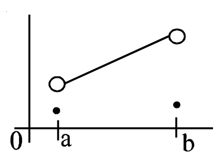
(a) **ROLLE'S THEOREM :**
Let f(x) be a function of x subject to the following conditions :

(i) f(x) is a continuous function of x in the closed interval of $a \le x \le b$.

(ii) $f'(x)$ exists for every point in the open interval $a < x < b$.

(iii) f (a) = f (b).

Then there exists at least one point x = c such that a < c < b where $f'(c) = 0$.
Note that if f is not continuous in closed [a, b] then it may lead to the adjacent graph where all the 3 conditions of Rolles will be valid but the assertion will not be true in (a, b).



(b) LMVT THEOREM :

Let f(x) be a function of x subject to the following conditions :

(i) f(x) is a continuous function of x in the closed interval of $a \le x \le b$.

(ii) $f'(x)$ exists for every point in the open interval $a < x < b$.

(iii) $f(a) \ne f(b)$.

Then there exists at least one point x = c such that a < c < b where $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

Geometrically, the slope of the secant line joining the curve at x = a & x = b is equal to the slope of the tangent line drawn to the curve at x = c. Note the following :

✍ Rolles theorem is a special case of LMVT since

$f(a) = f(b) \Rightarrow f'(c) = \dfrac{f(b) - f(a)}{b - a} = 0.$

Note : Now [f (b) – f (a)] is the change in the function f as x changes from a to b so that [f (b) – f (a)] / (b – a) is the *average rate of change* of the function over the interval [a, b]. Also f '(c) is the actual rate of change of the function for x = c. Thus, the theorem states that the average rate of change of a function over an interval is also the actual rate of change of the function at some point of the interval. In particular, for instance, the average velocity of a particle over an interval of time is equal to the velocity at some instant belonging to the interval.

This interpretation of the theorem justifies the name "Mean Value" for the theorem.

(c) APPLICATION OF ROLLES THEOREM FOR ISOLATING THE REAL ROOTS OF AN EQUATION f (x)=0

Suppose a & b are two real numbers such that ;

(i) f(x) & its first derivative $f'(x)$ are continuous for $a \le x \le b$.

(ii) f(a) & f(b) have opposite signs.

(iii) $f'(x)$ is different from zero for all values of x between a & b.

Then there is one & only one real root of the equation f(x) = 0 between a & b.

EXERCISE–I

Q.1 Find the intervals of monotonocity for the following functions & represent your solution set on the number line.

(a) $f(x) = 2.\ e^{x^2-4x}$ (b) $f(x) = e^x/x$ (c) $f(x) = x^2 e^{-x}$ (d) $f(x) = 2x^2 - \ln|x|$

Also plot the graphs in each case & state their range.

Q.2 Let $f(x) = 1 - x - x^3$. Find all real values of x satisfying the inequality, $1 - f(x) - f^3(x) > f(1 - 5x)$

Q.3 Find the intervals of monotonocity of the functions in [0, 2π]

(a) $f(x) = \sin x - \cos x$ in $x \in [0, 2\pi]$ **(b)** $g(x) = 2\sin x + \cos 2x$ in $(0 \le x \le 2\pi)$.

(c) $f(x) = \dfrac{4\sin x - 2x - x\cos x}{2 + \cos x}$

Q.4 Let $f(x)$ be a increasing function defined on $(0, \infty)$. If $f(2a^2 + a + 1) > f(3a^2 - 4a + 1)$. Find the range of a.

Q.5 Let $f(x) = x^3 - x^2 + x + 1$ and $g(x) = \begin{cases} \max\{f(t) : 0 \le t \le x\} & , 0 \le x \le 1 \\ 3 - x & , 1 < x \le 2 \end{cases}$

Discuss the conti. & differentiability of g(x) in the interval (0,2).

Q.6 Find the set of all values of the parameter 'a' for which the function,

$f(x) = \sin 2x - 8(a+1)\sin x + (4a^2 + 8a - 14)x$ increases for all $x \in R$ and has no critical points for all $x \in R$.

Q.7 Find the greatest & the least values of the following functions in the given interval if they exist.

(a) $f(x) = \sin^{-1}\dfrac{x}{\sqrt{x^2+1}} - \ln x$ in $\left[\dfrac{1}{\sqrt{3}}, \sqrt{3}\right]$ **(b)** $f(x) = 12x^{4/3} - 6x^{1/3}$, $x \in [-1, 1]$

(c) $y = x^5 - 5x^4 + 5x^3 + 1$ in $[-1, 2]$

Q.8 Find the values of 'a' for which the function $f(x) = \sin x - a\sin 2x - \frac{1}{3}\sin 3x + 2ax$ increases throughout the number line.

Q.9 If $f(x) = \left(\frac{a^2-1}{3}\right)x^3 + (a-1)x^2 + 2x + 1$ is monotonic increasing for every $x \in R$ then find the range of values of 'a'.

Q.10 Find the set of values of 'a' for which the function,

$$f(x) = \left(1 - \frac{\sqrt{21-4a-a^2}}{a+1}\right)x^3 + 5x + \sqrt{7}$$ is increasing at every point of its domain.

Q.11 Let $a + b = 4$, where $a < 2$ and let $g(x)$ be a differentiable function. If $\frac{dg}{dx} > 0$ for all x, prove that $\int_0^a g(x)\,dx + \int_0^b g(x)\,dx$ increases as $(b - a)$ increases.

Q.12 Let $f(x) = \left(1+\frac{1}{x}\right)^x$ and $g(x) = \left(1+\frac{1}{x}\right)^{x+1}$, both f and g being defined for $x > 0$, then prove that $f(x)$ is increasing and $g(x)$ is decreasing.

Q.13 Find the value of $x > 1$ for which the function

$$F(x) = \int_x^{x^2} \frac{1}{t} \ln\left(\frac{t-1}{32}\right) dt$$ is increasing and decreasing.

Q.14 Find all the values of the parameter 'a' for which the function ;
$f(x) = 8ax - a\sin 6x - 7x - \sin 5x$ increases & has no critical points for all $x \in R$.

Q.15 If $f(x) = 2e^x - ae^{-x} + (2a+1)x - 3$ monotonically increases for every $x \in R$ then find the range of values of 'a'.

Q.16 Prove that, $x^2 - 1 > 2x \ln x > 4(x-1) - 2\ln x$ for $x > 1$.

Q.17 Prove that $\tan^2 x + 6\ln \sec x + 2\cos x + 4 > 6\sec x$ for $x \in \left(\frac{3\pi}{2}, 2\pi\right)$.

Q.18 Find the set of values of x for which the inequality $\ln(1+x) > x/(1+x)$ is valid.

Q.19 If $b > a$, find the minimum value of $|(x-a)^3| + |(x-b)^3|$, $x \in R$.

Q.20 Suppose that the function $f(x) = \log_c \frac{x-2}{x+2}$ is defined for all x in the interval [a, b], is monotonic decreasing. Find the value of 'c' for which there exists 'a' and 'b' (b>a>2) such that the range of the function is $[\log_c c(b-1), \log_c c(a-1)]$.

EXERCISE–II

Q.1 Verify Rolles throrem for $f(x) = (x-a)^m (x-b)^n$ on [a, b] ; m, n being positive integer.

Q.2 Let $f(x) = 4x^3 - 3x^2 - 2x + 1$, use Rolle's theorem to prove that there exist c, $0 < c < 1$ such that $f(c) = 0$.

Q.3 Using LMVT prove that : **(a)** $\tan x > x$ in $\left(0, \frac{\pi}{2}\right)$, **(b)** $\sin x < x$ for $x > 0$

Q.4 Let f be continuous on [a, b] and assume the second derivative f'' exists on (a, b). Suppose that the graph of f and the line segment joining the point $(a, f(a))$ and $(b, f(b))$ intersect at a point $(x_0, f(x_0))$ where $a < x_0 < b$. Show that there exists a point $c \in (a, b)$ such that $f''(c) = 0$.

Q.5 Prove that if f is differentiable on [a, b] and if $f(a) = f(b) = 0$ then for any real α there is an $x \in (a, b)$ such that $\alpha f(x) + f'(x) = 0$.

Q.6 For what value of a, m and b does the function $f(x) = \begin{cases} 3 & x = 0 \\ -x^2 + 3x + a & 0 < x < 1 \\ mx + b & 1 \le x \le 2 \end{cases}$

satisfy the hypothesis of the mean value theorem for the interval [0, 2].

Q.7 Assume that f is continuous on [a, b], a > 0 and differentiable on an open interval (a, b). Show that if $\frac{f(a)}{a} = \frac{f(b)}{b}$, then there exist $x_0 \in (a, b)$ such that $x_0 f'(x_0) = f(x_0)$.

Q.8 Let f, g be differentiable on R and suppose that $f(0) = g(0)$ and $f'(x) \le g'(x)$ for all $x \ge 0$. Show that $f(x) \le g(x)$ for all $x \ge 0$.

Q.9 Let f be continuous on [a, b] and differentiable on (a, b). If $f(a) = a$ and $f(b) = b$, show that there exist distinct c_1, c_2 in (a, b) such that $f'(c_1) + f'(c_2) = 2$.

Q.10 Let f defined on [0, 1] be a twice differentiable function such that, $|f''(x)| \le 1$ for all $x \in [0, 1]$
If $f(0) = f(1)$, then show that, $|f'(x)| < 1$ for all $x \in [0, 1]$

Q.11 f (x) and g (x) are differentiable functions for $0 \le x \le 2$ such that $f(0) = 5$, $g(0) = 0$, $f(2) = 8$, $g(2) = 1$. Show that there exists a number c satisfying $0 < c < 2$ and $f'(c) = 3g'(c)$.

Q.12 If f, ϕ, ψ are continuous in [a, b] and derivable in]a, b[then show that there is a value of c lying between a & b such that,

$$\begin{vmatrix} f(a) & f(b) & f'(c) \\ \phi(a) & \phi(b) & \phi'(c) \\ \Psi(a) & \Psi(b) & \Psi'(c) \end{vmatrix} = 0$$

Q.13 Show that exactly two real values of x satisfy the equation $x^2 = x \sin x + \cos x$.

Q.14 Let $a > 0$ and f be continuous in [–a, a]. Suppose that $f'(x)$ exists and $f'(x) \le 1$ for all $x \in (-a, a)$. If $f(a) = a$ and $f(-a) = -a$, show that $f(0) = 0$.

Q.15 Prove the inequality $e^x > (1 + x)$ using LMVT for all $x \in R_0$ and use it to determine which of the two numbers e^π and π^e is greater.

EXERCISE–III

Q.1(a) For all $x \in (0, 1)$:

(A) $e^x < 1 + x$ (B) $\log_e(1 + x) < x$ (C) $\sin x > x$ (D) $\log_e x > x$

(b) Consider the following statements S and R :

S : Both sin x & cos x are decreasing functions in the interval $(\pi/2, \pi)$.

R : If a differentiable function decreases in an interval (a, b), then its derivative also decreases in (a, b).

Which of the following is true ?

(A) both S and R are wrong

(B) both S and R are correct, but R is not the correct explanation for S

(C) S is correct and R is the correct explanation for S

(D) S is correct and R is wrong.

(c) Let $f(x) = \int e^x (x-1)(x-2)\, dx$ then f decreases in the interval :
(A) $(-\infty, 2)$ (B) $(-2, -1)$ (C) $(1, 2)$ (D) $(2, +\infty)$

[JEE 2000 (Scr.) 1+1+1 out of 35]

Q.2(a) If $f(x) = xe^{x(1-x)}$, then f(x) is

(A) increasing on $\left(-\frac{1}{2}, 1\right)$ (B) decreasing on $\left[-\frac{1}{2}, 1\right]$

(C) increasing on R (C) decreasing on R

(b) Let $-1 \le p \le 1$. Show that the equation $4x^3 - 3x - p = 0$ has a unique root in the interval $\left[\frac{1}{2}, 1\right]$ and identify it. [JEE 2001, 1 + 5]

Q.3 The length of a longest interval in which the function $f(x) = 3 \sin x - 4 \sin^3 x$ is increasing, is
(A) $\pi/3$ (B) $\pi/2$ (C) $3\pi/2$ (D) π

[JEE 2002 (Screening), 3]

Q.4(a) Using the relation $2(1 - \cos x) < x^2$, $x \neq 0$ or otherwise, prove that $\sin(\tan x) \ge x$, $\forall\ x \in \left[0, \frac{\pi}{4}\right]$.

(b) Let $f : [0, 4] \to R$ be a differentiable function.
(i) Show that there exist a, b $\in [0, 4]$, $(f(4))^2 - (f(0))^2 = 8 f'(a) f(b)$
(ii) Show that there exist α, β with $0 < \alpha < \beta < 2$ such that

$$\int_0^4 f(t)\, dt = 2(\alpha f(\alpha^2) + \beta f(\beta^2))$$ [JEE 2003 (Mains), 4 + 4 out of 60]

Q.5(a) Let $f(x) = \begin{cases} x^{\alpha} \ln x, & x > 0 \\ 0, & x = 0 \end{cases}$. Rolle's theorem is applicable to f for $x \in [0, 1]$, if $\alpha =$

(A) –2 (B) –1 (C) 0 (D) 1/2

(b) If f is a strictly increasing function, then $\lim\limits_{x \to 0} \frac{f(x^2) - f(x)}{f(x) - f(0)}$ is equal to

(A) 0 (B) 1 (C) –1 (D) 2

[JEE 2004 (Scr)]

Q.6 If $p(x) = 51x^{101} - 2323x^{100} - 45x + 1035$, using Rolle's theorem, prove that at least one root of p(x) lies between $(45^{1/100}, 46)$. [JEE 2004, 2 out of 60]

Q.7 If $f(x)$ is a twice differentiable function and given that $f(1) = 1$, $f(2) = 4$, $f(3) = 9$, then
(A) $f''(x) = 2$, for $\forall\ x \in (1, 3)$ (B) $f''(x) = f'(x) = 2$, for some $x \in (2, 3)$
(C) $f''(x) = 3$, for $\forall\ x \in (2, 3)$ (D) $f''(x) = 2$, for some $x \in (1, 3)$

[JEE 2005 (Scr), 3]

Q.8(a) Let $f(x) = 2 + \cos x$ for all real x.
Statement-1: For each real t, there exists a point 'c' in $[t, t + \pi]$ such that $f'(c) = 0$.
because
Statement-2: $f(t) = f(t + 2\pi)$ for each real t.
(A) Statement-1 is true, statement-2 is true; statement-2 is correct explanation for statement-1.
(B) Statement-1 is true, statement-2 is true; statement-2 is NOT a correct explanation for statement-1.
(C) Statement-1 is true, statement-2 is false.
(D) Statement-1 is false, statement-2 is true. [JEE 2007, 3]
Paragraph [JEE 2007, 4+4+4]

Q.8(b) If a continuous function f defined on the real line **R**, assumes positive and negative values in **R** then the equation $f(x) = 0$ has a root in **R**. For example, if it is known that a continuous function f on **R** is positive at some point and its minimum value is negative then the equation $f(x) = 0$ has a root in **R**.

Consider $f(x) = ke^x - x$ for all real x where k is a real constant.

(i) The line $y = x$ meets $y = ke^x$ for $k \le 0$ at
(A) no point (B) one point (C) two points (D) more than two points

(ii) The positive value of k for which $ke^x - x = 0$ has only one root is
(A) $1/e$ (B) 1 (C) e (D) $\log_e 2$

(iii) For $k > 0$, the set of all values of k for which $ke^x - x = 0$ has two distinct roots is
(A) $(0, 1/e)$ (B) $(1/e, 1)$ (C) $(1/e, \infty)$ (D) $(0, 1)$

Match the column. [JEE 2007, 6]

Q.8(c) In the following [x] denotes the greatest integer less than or equal to x.
Match the functions in **Column I** with the properties in **Column II**.

	Column I		**Column II**
(A)	$x\,|x|$	(P)	continuous in $(-1, 1)$
(B)	$\sqrt{|x|}$	(Q)	differentiable in $(-1, 1)$
(C)	$x + [x]$	(R)	strictly increasing in $(-1, 1)$
(D)	$|x - 1| + |x + 1|$	(S)	non differentiable at least at one point in $(-1, 1)$

Q.9(a) Let the function $g : (-\infty, \infty) \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ be given by $g(u) = 2\tan^{-1}(e^u) - \frac{\pi}{2}$. Then, g is
(A) even and is strictly increasing in $(0, \infty)$ (B) odd and is strictly decreasing in $(-\infty, \infty)$
(C) odd and is strictly increasing in $(-\infty, \infty)$
(D) neither even nor odd, but is strictly increasing in $(-\infty, \infty)$

Q.9(b) Let $f(x)$ be a non-constant twice differentiable function defined on $(-\infty, \infty)$ such that $f(x) = f(1 - x)$ and $f'(1/4) = 0$. Then
(A) $f''(x)$ vanishes at least twice on $[0, 1]$ (B) $f'(1/2) = 0$
(C) $\int_{-1/2}^{1/2} f\left(x + \frac{1}{2}\right)\sin x\, dx = 0$ (D) $\int_{0}^{1/2} f(t)\, e^{\sin \pi t}\, dt = \int_{1/2}^{1} f(1-t)\, e^{\sin \pi t}\, dt$

[JEE 2008, 3 + 4]

Q.10 For the function $f(x) = x\cos\frac{1}{x},\ x \ge 1$,
(A) for at least one x in the interval $[1, \infty)$, $f(x + 2) - f(x) < 2$
(B) $\lim_{x\to\infty} f'(x) = 1$
(C) for all x in the interval $[1, \infty)$, $f(x + 2) - f(x) > 2$
(D) $f'(x)$ is strictly decreasing in the interval $[1, \infty)$ [JEE 2009, 4]

MAXIMA - MINIMA

FUNCTIONS OF A SINGLE VARIABLE

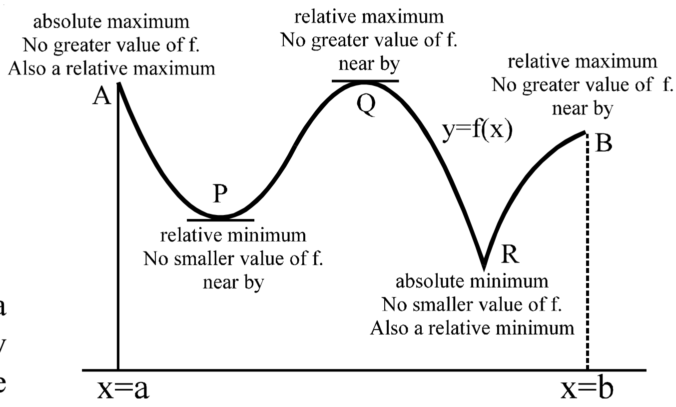
HOW MAXIMA & MINIMA ARE CLASSIFIED

1. A function f(x) is said to have a maximum at x = a if f(a) is greater than every other value assumed by f(x) in the immediate neighbourhood of x = a. Symbolically

$$\left.\begin{array}{l} f(a) > f(a+h) \\ f(a) > f(a-h) \end{array}\right] \Rightarrow x = a \text{ gives maxima for}$$

a sufficiently small positive h.

Similarly, a function f(x) is said to have a minimum value at x = b if f(b) is least than every other value assumed by f(x) in the immediate neighbourhood at x = b. Symbolically if



$$\left.\begin{array}{l} f(b) < f(b+h) \\ f(b) < f(b-h) \end{array}\right] \Rightarrow x = b \text{ gives minima for a sufficiently small positive h.}$$

Note that :

(i) the maximum & minimum values of a function are also known as local/relative maxima or local/relative minima as these are the greatest & least values of the function relative to some neighbourhood of the point in question.

(ii) the term 'extremum' or (extremal) or 'turning value' is used both for maximum or a minimum value.

(iii) a maximum (minimum) value of a function may not be the greatest (least) value in a finite interval.

(iv) a function can have several maximum & minimum values & a minimum value may even be greater than a maximum value.

(v) maximum & minimum values of a continuous function occur alternately & between two consecutive maximum values there is a minimum value & vice versa.

2. A NECESSARY CONDITION FOR MAXIMUM & MINIMUM :

If f(x) is a maximum or minimum at x = c & if $f'(c)$ exists then $f'(c) = 0$.

Note :

(i) The set of values of x for which $f'(x) = 0$ are often called as stationary points or critical points. The rate of change of function is zero at a stationary point.

(ii) In case $f'(c)$ does not exist f(c) may be a maximum or a minimum & in this case left hand and right hand derivatives are of opposite signs.

(iii) The greatest (global maxima) and the least (global minima) values of a function f in an interval [a, b] are f(a) or f(b) or are given by the values of x for which $f'(x) = 0$.

(iv) Critical points are those where $\frac{dy}{dx} = 0$, if it exists , or it fails to exist either by virtue of a vertical tangent or by virtue of a geometrical sharp corner but not because of discontinuity of function.

3. SUFFICIENT CONDITION FOR EXTREME VALUES :

$$\left.\begin{array}{l} f'(c-h) > 0 \\ f'(c+h) < 0 \end{array}\right] \Rightarrow x = c \text{ is a point of local maxima, where } f'(c) = 0.$$

Similarly $\left.\begin{array}{l} f'(c-h) < 0 \\ f'(c+h) > 0 \end{array}\right] \Rightarrow x = c$ is a point of local minima, where $f'(c) = 0$.

] *h is a sufficiently small positive quantity*

Note : If $f'(x)$ does not change sign i.e. has the same sign in a certain complete neighbourhood of c, then f(x) is either strictly increasing or decreasing throughout this neighbourhood implying that f(c) is not an extreme value of f.

4. USE OF SECOND ORDER DERIVATIVE IN ASCERTAINING THE MAXIMA OR MINIMA:

(a) f(c) is a minimum value of the function f, if $f'(c) = 0$ & $f''(c) > 0$.

(b) f(c) is a maximum value of the function f, $f'(c) = 0$ & $f''(c) < 0$.

Note : if $f''(c) = 0$ then the test fails. Revert back to the first order derivative check for ascertaning the maxima or minima.

5. SUMMARY–WORKING RULE :

FIRST :

When possible , draw a figure to illustrate the problem & label those parts that are important in the problem. Constants & variables should be clearly distinguished.

SECOND :

Write an equation for the quantity that is to be maximised or minimised. If this quantity is denoted by 'y', it must be expressed in terms of a single independent variable x. his may require some algebraic manipulations.

THIRD :

If $y = f(x)$ is a quantity to be maximum or minimum, find those values of x for which $dy/dx = f'(x) = 0$.

FOURTH :

Test each values of x for which $f'(x) = 0$ to determine whether it provides a maximum or minimum or neither. The usual tests are :

(a) If d^2y/dx^2 is positive when $dy/dx = 0 \Rightarrow$ y is minimum.
If d^2y/dx^2 is negative when $dy/dx = 0 \Rightarrow$ y is maximum.
If $d^2y/dx^2 = 0$ when $dy/dx = 0$, the test fails.

(b) $$\text{If } \frac{dy}{dx} \text{ is } \left.\begin{matrix} \text{positive} & \text{for} & x < x_0 \\ \text{zero} & \text{for} & x = x_0 \\ \text{negative} & \text{for} & x > x_0 \end{matrix}\right] \Rightarrow \text{a maximum occurs at } x = x_0.$$

But if dy/dx changes sign from negative to zero to positive as x advances through x_0 there is a minimum. If dy/dx does not change sign, neither a maximum nor a minimum. Such points are called **INFLECTION POINTS**.

FIFTH :

If the function $y = f(x)$ is defined for only a limited range of values $a \le x \le b$ then examine $x = a$ & $x = b$ for possible extreme values.

SIXTH :

If the derivative fails to exist at some point, examine this point as possible maximum or minimum.

Important Note :

– Given a fixed point $A(x_1, y_1)$ and a moving point $P(x, f(x))$ on the curve $y = f(x)$. Then AP will be maximum or minimum if it is normal to the curve at P.

– If the sum of two positive numbers x and y is constant than their product is maximum if they are equal, i.e. $x + y = c$, $x > 0$, $y > 0$, then
$$xy = \frac{1}{4}\,[\,(x + y)^2 - (x - y)^2\,]$$

– If the product of two positive numbers is constant then their sum is least if they are equal. i.e. $(x + y)^2 = (x - y)^2 + 4xy$

6. USEFUL FORMULAE OF MENSURATION TO REMEMBER :

☞ Volume of a cuboid = lbh.

☞ Surface area of a cuboid = $2(lb + bh + hl)$.

☞ Volume of a prism = area of the base x height.

☞ Lateral surface of a prism = perimeter of the base x height.

☞ Total surface of a prism = lateral surface + 2 area of the base
(Note that lateral surfaces of a prism are all rectangles).

☞ Volume of a pyramid $= \frac{1}{3}$ area of the base x height.

☞ Curved surface of a pyramid $= \frac{1}{2}$ (perimeter of the base) x slant height.
(Note that slant surfaces of a pyramid are triangles).

☞ Volume of a cone $= \frac{1}{3}\pi r^2 h$.

☞ Curved surface of a cylinder $= 2\pi rh$.

☞ Total surface of a cylinder $= 2\pi rh + 2\pi r^2$.

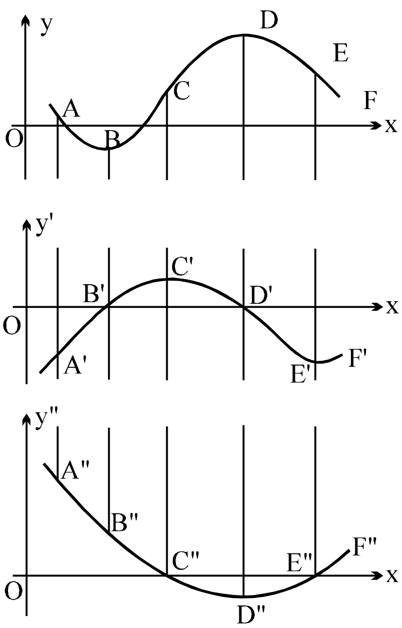
☞ Volume of a sphere $= \frac{4}{3}\pi r^3$.

☞ Surface area of a sphere $= 4\pi r^2$.

☞ Area of a circular sector $= \frac{1}{2} r^2\theta$, when θ is in radians.

7. SIGNIFICANCE OF THE SIGN OF 2ND ORDER DERIVATIVE AND POINTS OF INFLECTION :

The sign of the 2nd order derivative determines the concavity of the curve. Such points such as C & E on the graph where the concavity of the curve changes are called the points of inflection. From the graph we find that if:



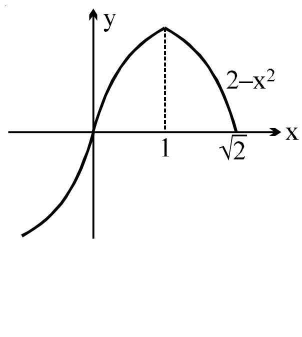
(i) $\frac{d^2y}{dx^2} > 0 \Rightarrow$ concave upwards

(ii) $\frac{d^2y}{dx^2} < 0 \Rightarrow$ concave downwards.

At the point of inflection we find that $\frac{d^2y}{dx^2} = 0$ &

$\frac{d^2y}{dx^2}$ changes sign.

Inflection points can also occur if $\frac{d^2y}{dx^2}$ fails to exist. For example, consider the graph of the function defined as,



$$f(x) = \begin{cases} x^{3/5} & \text{for } x \in (-\infty, 1) \\ 2 - x^2 & \text{for } x \in (1, \infty) \end{cases}$$

Note that the graph exhibits two critical points one is a point of local maximum & the other a point of inflection.

EXERCISE–I

Q.1 A cubic f(x) vanishes at $x = -2$ & has relative minimum/maximum at $x = -1$ and $x = 1/3$. If $\int_{-1}^{1} f(x)\,dx = \frac{14}{3}$, find the cubic $f(x)$.

Q.2 Investigate for maxima & minima for the function, $f(x) = \int_{1}^{x} [2(t-1)(t-2)^3 + 3(t-1)^2(t-2)^2]\,dt$

Q.3 Find the greatest & least value for the function ;

(a) $y = x + \sin 2x,\ 0 \le x \le 2\pi$ (b) $y = 2\cos 2x - \cos 4x,\ 0 \le x \le \pi$

Q.4 Suppose f(x) is a function satisfying the following conditions :

(i) $f(0) = 2,\ f(1) = 1$ (ii) f has a minimum value at $x = \frac{5}{2}$ and

(iii) for all x, $f'(x) = \begin{vmatrix} 2ax & 2ax-1 & 2ax+b+1 \\ b & b+1 & -1 \\ 2(ax+b) & 2ax+2b+1 & 2ax+b \end{vmatrix}$

Where a, b are some constants. Determine the constants a, b & the function f(x).

Q.5 Suppose f(x) is real valued polynomial function of degree 6 satisfying the following conditions ;

(a) f has minimum value at $x = 0$ and 2

(b) f has maximum value at $x = 1$

(c) for all x, $\operatorname*{Limit}_{x\to 0} \frac{1}{x}\, ln \begin{vmatrix} \frac{f(x)}{x} & 1 & 0 \\ 0 & \frac{1}{x} & 1 \\ 1 & 0 & \frac{1}{x} \end{vmatrix} = 2.$

Determine f(x).

Q.6 Find the maximum perimeter of a triangle on a given base ‘a’ and having the given vertical angle α.

Q.7 The length of three sides of a trapezium are equal, each being 10 cms. Find the maximum area of such a trapezium.

Q.8 The plan view of a swimming pool consists of a semicircle of radius r attached to a rectangle of length '2r' and width 's'. If the surface area A of the pool is fixed, for what value of 'r' and 's' the perimeter 'P' of the pool is minimum.

Q.9 For a given curved surface of a right circular cone when the volume is maximum, prove that the semi vertical angle is $\sin^{-1}\frac{1}{\sqrt{3}}$.

Q.10 Of all the lines tangent to the graph of the curve $y = \frac{6}{x^2+3}$, find the equations of the tangent lines of minimum and maximum slope.

Q.11 A statue 4 metres high sits on a column 5.6 metres high. How far from the column must a man, whose eye level is 1.6 metres from the ground, stand in order to have the most favourable view of statue.

Q.12 By the post office regulations, the combined length & girth of a parcel must not exceed 3 metre. Find the volume of the biggest cylindrical (right circular) packet that can be sent by the parcel post.

Q.13 A running track of 440 ft. is to be laid out enclosing a football field, the shape of which is a rectangle with semi circle at each end. If the area of the rectangular portion is to be maximum, find the length of its sides. Use : $\pi \approx 22/7$.

Q.14 A window of fixed perimeter (including the base of the arch) is in the form of a rectangle surmounted by a semicircle. The semicircular portion is fitted with coloured glass while the rectangular part is fitted with clean glass. The clear glass transmits three times as much light per square meter as the coloured glass does. What is the ratio of the sides of the rectangle so that the window transmits the maximum light?

Q.15 A closed rectangular box with a square base is to be made to contain 1000 cubic feet. The cost of the material per square foot for the bottom is 15 paise, for the top 25 paise and for the sides 20 paise. The labour charges for making the box are Rs. 3/-. Find the dimensions of the box when the cost is minimum.

Q.16 Find the area of the largest rectangle with lower base on the x-axis & upper vertices on the curve $y = 12 - x^2$.

Q.17 A trapezium ABCD is inscribed into a semicircle of radius l so that the base AD of the trapezium is a diameter and the vertices B & C lie on the circumference. Find the base angle θ of the trapezium ABCD which has the greatest perimeter.

Q.18 If $y = \dfrac{ax+b}{(x-1)(x-4)}$ has a turning value at $(2, -1)$ find a & b and show that the turning value is a maximum.

Q.19 If r is a real number then find the smallest possible distance from the origin (0, 0) to the vertex of the parabola whose equation is $y = x^2 + rx + 1$.

Q.20 A sheet of poster has its area 18 m². The margin at the top & bottom are 75 cms and at the sides 50 cms. What are the dimensions of the poster if the area of the printed space is maximum?

Q.21 A perpendicular is drawn from the centre to a tangent to an ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. Find the greatest value of the intercept between the point of contact and the foot of the perpendicular.

Q.22 A beam of rectangular cross section must be sawn from a round log of diameter d. What should the width x and height y of the cross section be for the beam to offer the greatest resistance (a) to compression; (b) to bending. Assume that the compressive strength of a beam is proportional to the area of the cross section and the bending strength is proportional to the product of the width of section by the square of its height.

Q.23 What are the dimensions of the rectangular plot of the greatest area which can be laid out within a triangle of base 36 ft. & altitude 12 ft ? Assume that one side of the rectangle lies on the base of the triangle.

Q.24 The flower bed is to be in the shape of a circular sector of radius r & central angle θ. If the area is fixed & perimeter is minimum, find r and θ.

Q.25 The circle $x^2 + y^2 = 1$ cuts the x-axis at P & Q. Another circle with centre at Q and varable radius intersects the first circle at R above the x-axis & the line segment PQ at S. Find the maximum area of the triangle QSR.

EXERCISE–II

Q.1 The mass of a cell culture at time t is given by, $M(t) = \dfrac{3}{1+4e^{-t}}$

(a) Find $\lim\limits_{t\to-\infty} M(t)$ and $\lim\limits_{t\to\infty} M(t)$

(b) Show that $\dfrac{dM}{dt} = \dfrac{1}{3}M(3-M)$

(c) Find the maximum rate of growth of M and also the vlaue of t at which occurs.

Q.2 Find the cosine of the angle at the vertex of an isosceles triangle having the greatest area for the given constant length l of the median drawn to its lateral side.

Q.3 From a fixed point A on the circumference of a circle of radius 'a', let the perpendicular AY fall on the tangent at a point P on the circle, prove that the greatest area which the ΔAPY can have is $3\sqrt{3}\,\dfrac{a^2}{8}$ sq. units.

Q.4 Given two points A $(-2, 0)$ & B $(0, 4)$ and a line $y = x$. Find the co-ordinates of a point M on this line so that the perimeter of the Δ AMB is least.

Q.5 A given quantity of metal is to be casted into a half cylinder i.e. with a rectangular base and semicircular ends. Show that in order that total surface area may be minimum, the ratio of the height of the cylinder to the diameter of the semi circular ends is $\pi/(\pi+2)$.

Q.6 Let α, β be real numbers with $0 \le \alpha \le \beta$ and $f(x) = x^2 - (\alpha+\beta)x + \alpha\beta$ such that $\int_{-1}^{1} f(x)\,dx = 1$. Find the maximum value of $\int_{0}^{\alpha} f(x)\,dx$.

Q.7 Show that for each $a > 0$ the function $e^{-ax}.\,x^{a^2}$ has a maximum value say $F(a)$, and that $F(x)$ has a minimum value, $e^{-e/2}$.

Q.8 For $a > 0$, find the minimum value of the integral $\int_{0}^{1/a} (a^3 + 4x - a^5x^2)e^{ax}\,dx$.

Q.9 Consider the function $f(x) = \begin{cases} \sqrt{x}\,\ln x & \text{when } x > 0 \\ 0 & \text{for } x = 0 \end{cases}$

(a) Find whether f is continuous at $x = 0$ or not.

(b) Find the minima and maxima if they exist.

(c) Does $f'(0)$? Find $\lim\limits_{x\to0} f'(x)$.

(d) Find the inflection points of the graph of $y = f(x)$..

Q.10 Consider the function $y = f(x) = \ln(1 + \sin x)$ with $-2\pi \le x \le 2\pi$. Find

(a) the zeroes of $f(x)$

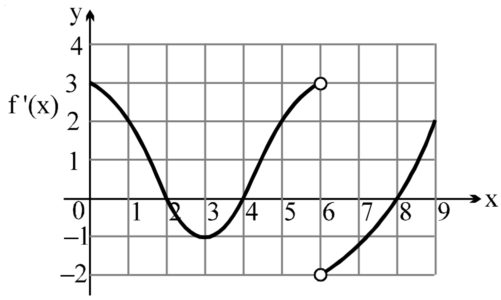
(b) inflection points if any on the graph

(c) local maxima and minima of $f(x)$

(d) asymptotes of the graph

(e) sketch the graph of $f(x)$ and compute the value of the definite integral $\int_{-\pi/2}^{\pi/2} f(x)\,dx$.

Q.11 The graph of the derivative f' of a continuous function f is shown with $f(0) = 0$. If



(i) f is monotomic increasing in the interval $[a, b) \cup (c, d) \cup (e, f]$ and decreasing in $(p, q) \cup (r, s)$.

(ii) f has a local minima at $x = x_1$ and $x = x_2$.

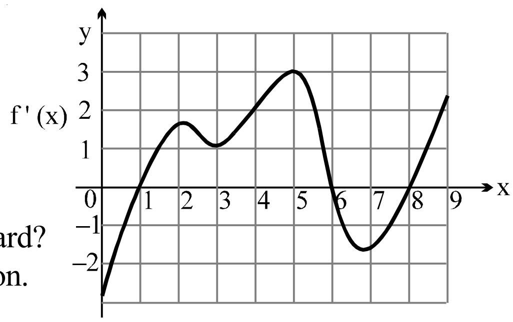
(iii) f is concave up in $(l, m) \cup (n, t]$

(iv) f has inflection point at $x = k$

(v) number of critical points of $y = f(x)$ is 'w'.

Find the value of $(a + b + c + d + e) + (p + q + r + s) + (l + m + n) + (x_1 + x_2) + (k + w)$.

Q.12 The graph of the derivative f' of a continuous function f is shown with $f(0) = 0$



(i) On what intervals is f increasing or decreasing?

(ii) At what values of x does f have a local maximum or minimum?

(iii) On what intervals is f concave upward or downward?

(iv) State the x-coordinate(s) of the point(s) of inflection.

(v) Assuming that $f(0) = 0$, sketch a graph of f.

Q.13 Find the set of value of m for the cubic $x^3 - \frac{3}{2}x^2 + \frac{5}{2} = \log_{1/4}(m)$ has 3 distinct solutions.

Q.14 Find the positive value of k for the value of the definite integral $\int_0^{\pi/2} |\cos x - kx|\, dx$ is minimised.

Q.15 A cylinder is obtained by revolving a rectangle about the x – axis, the base of the rectangle lying on the x – axis and the entire rectangle lying in the region between the curve

$y = \frac{x}{x^2+1}$ & the x – axis. Find the maximum possible volume of the cylinder.

Q.16 The value of 'a' for which $f(x) = x^3 + 3(a - 7)x^2 + 3(a^2 - 9)x - 1$ have a positive point of maximum lies in the interval $(a_1, a_2) \cup (a_3, a_4)$. Find the value of $a_2 + 11a_3 + 70a_4$.

Q.17 What is the radius of the smallest circular disk large enough to cover every acute isosceles triangle of a given perimeter L?

Q.18 Find the magnitude of the vertex angle 'α' of an isosceles triangle of the given area 'A' such that the radius 'r' of the circle inscribed into the triangle is the maximum.

Q.19 The function $f(x)$ defined for all real numbers x has the following properties

(i) $f(0) = 0$, $f(2) = 2$ and $f'(x) = k(2x - x^2)e^{-x}$ for some constant $k > 0$. Find

(a) the intervals on which f is increasing and decreasing and any local maximum or minimum values.

(b) the intervals on which the graph f is concave down and concave up.

(c) the function $f(x)$ and plot its graph.

Q.20 Use calculus to prove the inequality, $\sin x \ge 2x/\pi$ in $0 \le x \le \pi/2$.

Use this inequality to prove that, $\cos x \le 1 - x^2/\pi$ in $0 \le x \le \pi/2$.

EXERCISE–III

Q.1 Let $f(x) = \begin{cases} |x| & \text{for } 0 < |x| \le 2 \\ 1 & \text{for } x = 0 \end{cases}$. Then at x = 0, 'f' has :

(A) a local maximum (B) no local maximum

(C) a local minimum (D) no extremum.

[JEE 2000 Screening, 1 out of 35]

Q.2 Find the area of the right angled triangle of least area that can be drawn so as to circumscribe a rectangle of sides 'a' and 'b', the right angle of the triangle coinciding with one of the angles of the rectangle.

[REE 2001 Mains, 5 out of 100]

Q.3(a) Let $f(x) = (1 + b^2)x^2 + 2bx + 1$ and let m(b) be the minimum value of f(x). As b varies, the range of m (b) is

(A) [0, 1] (B) $\left(0, \frac{1}{2}\right]$ (C) $\left[\frac{1}{2}, 1\right]$ (D) (0, 1]

(b) The maximum value of $(\cos \alpha_1) \cdot (\cos \alpha_2) \ldots\ldots\ldots (\cos \alpha_n)$, under the restrictions

$0 \le \alpha_1, \alpha_2, \ldots\ldots\ldots, \alpha_n \le \frac{\pi}{2}$ and $\cot \alpha_1 \cdot \cot \alpha_2 \ldots\ldots\ldots \cot \alpha_n = 1$ is

(A) $\frac{1}{2^{n/2}}$ (B) $\frac{1}{2^n}$ (C) $\frac{1}{2n}$ (D) 1

[JEE 2001 Screening, 1 + 1 out of 35]

Q.4 If $a_1, a_2, \ldots\ldots, a_n$ are positive real numbers whose product is a fixed number e, the minimum value of $a_1 + a_2 + a_3 + \ldots\ldots + a_{n-1} + 2a_n$ is

(A) $n(2e)^{1/n}$ (B) $(n+1)e^{1/n}$ (C) $2ne^{1/n}$ (D) $(n+1)(2e)^{1/n}$

[JEE 2002 Screening]

Q.5(a) Find a point on the curve $x^2 + 2y^2 = 6$ whose distance from the line $x + y = 7$, is minimum.

(b) For a circle $x^2 + y^2 = r^2$, find the value of 'r' for which the area enclosed by the tangents drawn from the point P(6, 8) to the circle and the chord of contact is maximum. [JEE 2003, Mains, 2 + 2 out of 60]

Q.6(a) Let $f(x) = x^3 + bx^2 + cx + d$, $0 < b^2 < c$. Then f

(A) is bounded (B) has a local maxima

(C) has a local minima (D) is strictly increasing [JEE 2004 (Scr.)]

(b) Prove that $\sin x + 2x \ge \frac{3x \cdot (x+1)}{\pi}$ $\forall x \in \left[0, \frac{\pi}{2}\right]$. (Justify the inequality, if any used).

[JEE 2004, 4 out of 60]

Q.7 If P(x) be a polynomial of degree 3 satisfying P(–1) = 10, P(1) = – 6 and P(x) has maximum at x = – 1 and P'(x) has minima at x = 1. Find the distance between the local maximum and local minimum of the curve. [JEE 2005 (Mains), 4 out of 60]

Q.8(a) If $f(x)$ is cubic polynomial which has local maximum at x = – 1. If $f(2) = 18$, $f(1) = -1$ and $f'(x)$ has local maxima at x = 0, then

(A) the distance between (–1, 2) and $(a, f(a))$, where x = a is the point of local minima is $2\sqrt{5}$.

(B) $f(x)$ is increasing for $x \in (1, 2\sqrt{5}]$

(C) $f(x)$ has local minima at x = 1

(D) the value of $f(0) = 5$

(b) $f(x) = \begin{cases} e^x & 0 \le x \le 1 \\ 2 - e^{x-1} & 1 < x \le 2 \\ x - e & 2 < x \le 3 \end{cases}$ and $g(x) = \int_0^x f(t)dt$, $x \in [1, 3]$ then g(x) has

(A) local maxima at $x = 1 + ln\,2$ and local minima at $x = e$
(B) local maxima at x = 1 and local minima at x = 2
(C) no local maxima
(D) no local minima [JEE 2006, 5 marks each]

(c) If $f(x)$ is twice differentiable function such that $f(a) = 0, f(b) = 2, f(c) = -1, f(d) = 2, f(e) = 0$, where a < b < c < d < e, then find the minimum number of zeros of $g(x) = (f'(x))^2 + f(x).f''(x)$ in the interval [a, e]. [JEE 2006, 6]

Q.9**(a)** The total number of local maxima and local minima of the function $f(x) = \begin{cases} (2+x)^3, & -3 < x \le -1 \\ x^{2/3}, & -1 < x < 2 \end{cases}$ is

(A) 0 (B) 1 (C) 2 (D) 3

(b) Comprehension:
Consider the function $f : (-\infty, \infty) \to (-\infty, \infty)$ defined by

$$f(x) = \frac{x^2 - ax + 1}{x^2 + ax + 1}, \ 0 < a < 2$$

(i) Which of the following is true?
(A) $(2 + a)^2 f''(1) + (2 - a)^2 f''(-1) = 0$ (B) $(2 - a)^2 f''(1) - (2 + a)^2 f''(-1) = 0$
(C) $f'(1) f'(-1) = (2 - a)^2$ (D) $f'(1) f'(-1) = -(2 + a)^2$

(ii) Which of the following is true?
(A) f (x) is decreasing on (–1, 1) and has a local minimum at x = 1
(B) f (x) is increasing on (–1, 1) and has a local maximum at x = 1
(C) f (x) is increasing on (–1, 1) but has neither a local maximum and nor a local minimum at x=1.
(D) f (x) is decreasing on (–1, 1) but has neither a local maximum and nor a local minimum at x = 1.

(iii) Let $g(x) = \int_0^{e^x} \frac{f'(t)}{1+t^2} dt$

Which of the following is true?
(A) g'(x) is positive on $(-\infty, 0)$ and negative on $(0, \infty)$
(B) g'(x) is negative on $(-\infty, 0)$ and positive on $(0, \infty)$
(C) g'(x) changes sign on both $(-\infty, 0)$ and $(0, \infty)$
(D) g'(x) does not change sign on $(-\infty, 0)$ and $(0, \infty)$ [JEE 2008, 3 + 4 + 4 + 4]

Q.10**(a)** Let p(x) be a polynomial of degree 4 having extremum at x = 1, 2 and $\lim_{x \to 0} \left(1 + \frac{p(x)}{x^2}\right) = 2$. Then the value of p(2) is

(b) The maximum value of the function $f(x) = 2x^3 - 15x^2 + 36x - 48$ on the set $A = \{x \mid x^2 + 20 \le 9x\}$ is
[JEE 2008, 4 + 4]

ANSWER KEY
TANGENT & NORMAL

EXERCISE–I

Q.1 $x + y - 1 = 0$ **Q.2** $2\sqrt{3}\ x - y = 2\left(\sqrt{3} - 1\right)$ or $2\sqrt{3}\ x + y = 2\left(\sqrt{3} + 1\right)$

Q.3 (0, 1) **Q.4** $x = 1$ when $t = 1$, $m \to \infty$; $5x - 4y = 1$ if $t \neq 1$, $t = 1/3$

Q.5 $a = 1, b = \frac{-5}{2}$ **Q.7** $-\frac{82 \cdot 7^3}{3}$

Q.8 $T : x - 2y = 0$; $N : 2x + y = 0$ **Q.9** $x + 2y = \pi/2$ & $x + 2y = -3\pi/2$

Q.12 a = 1 **Q.14** $-\frac{1}{x+2}$; $x - 4y = 2$ **Q.15** 1/16 **Q.16** $a = -1/2$; $b = -3/4$; $c = 3$

Q.20 $2e^{-x/2}$ **Q.22** $\frac{m\sqrt{m}}{\sqrt{2}}$ **Q.23** **(a)** $n = -2$

Q.24 **(b)** $\pm\frac{1}{2\sqrt{2}}$ **Q.25** $\theta = \tan^{-1}\frac{2}{C}$

EXERCISE–II

Q.1 $1/9\pi$ m/min **Q.2** **(i)** 6 km/h ; **(ii)** 2 km/hr **Q.3** (4 , 11) & (– 4, – 31/3)

Q.4 $3/8\pi$ cm/min **Q.5** $1 + 36\pi$ cu. cm/sec **Q.6** $1/48\pi$ cm/s

Q.7 0.05 cm/sec **Q.8** $\frac{\sqrt{2}}{4\pi}$ cm/s **Q.9** $200\pi r^3 / (r+5)^2$ km²/h

Q.10 $\frac{66}{7}$ **Q.11** $\frac{1}{4}$ cm/sec. **Q.12** (a) $-\frac{1}{24\pi}$ m/min., (b) $-\frac{5}{288\pi}$ m/min.

Q.14 (a) $r = (1 + t)^{1/4}$, (b) t = 80 **Q.15** **(i)** (a) 6.05, (b) $\frac{80}{27}$; (ii) 9.72π cm³

EXERCISE–III

Q.1 $\sqrt{2}\ x + y - 2\sqrt{2} = 0$ or $\sqrt{2}\ x - y - 2\sqrt{2} = 0$ **Q.2** D

Q.3 D **Q.4** D **Q.5** A

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MONOTONOCITY

EXERCISE–I

Q.1 **(a)** I in $(2, \infty)$ & D in $(-\infty, 2)$ **(b)** I in $(1, \infty)$ & D in $(-\infty, 0) \cup (0, 1)$

(c) I in $(0, 2)$ & D in $(-\infty, 0) \cup (2, \infty)$

(d) I for $x > \frac{1}{2}$ or $-\frac{1}{2} < x < 0$ & D for $x < -\frac{1}{2}$ or $0 < x < \frac{1}{2}$

Q.2 $(-2, 0) \cup (2, \infty)$

Q.3 **(a)** I in $[0, 3\pi/4) \cup (7\pi/4, 2\pi]$ & D in $(3\pi/4, 7\pi/4)$

(b) I in $[0, \pi/6) \cup (\pi/2, 5\pi/6) \cup (3\pi/2, 2\pi]$ & D in $(\pi/6, \pi/2) \cup (5\pi/6, 3\pi/2)]$

(c) I in $[0, \pi/2) \cup (3\pi/2, 2\pi]$ and D in $(\pi/2, 3\pi/2)$

Q.4 $(0, 1/3) \cup (1, 5)$ **Q.5** continuous but not diff. at $x = 1$ **Q.6** $a < -\left(2+\sqrt{5}\right)$ or $a > \sqrt{5}$

Q.7 **(a)** $(\pi/6)+(1/2)\ln 3, (\pi/3) - (1/2)\ln 3$
(b) Maximum at $x = 1$ and $f(-1) = 18$; Minimum at $x = 1/8$ and $f(1/8) = -9/4$
(c) $2 \& -10$

Q.8 $[1, \infty)$ **Q.9** $a \in (-\infty, -3] \cup [1, \infty)$ **Q.10** $[-7, -1) \cup [2, 3]$
Q.13 $\uparrow$ in $(3, \infty)$ and $\downarrow$ in $(1, 3)$ **Q.14** $(6, \infty)$ **Q.15** $a \geq 0$

Q.18 $(-1, 0) \cup (0, \infty)$ **Q.19** $(b - a)^3/4$ **Q.20** $0 < c < \frac{1}{9}$

EXERCISE–II

Q.1 $c = \frac{mb + na}{m + n}$ which lies between a & b **Q.6** $a = 3, b = 4$ and $m = 1$

EXERCISE–III

Q.1 **(a)** B ; **(b)** D ; **(c)** C **Q.2** **(a)** A, **(b)** $\cos\left(\frac{1}{3}\cos^{-1} p\right)$ **Q.3** A **Q.5** **(a)** D; **(b)** C

Q.7 D **Q.8** **(a)** B; **(b)** (i) B, (ii) A, (iii) A; **(c)** (A) P,Q,R; (B) P,S; (C) R,S; (D) P, Q
Q.9 (a) C, (b) A, B, C, D **Q.10** B, C, D

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MAXIMA - MINIMA

EXERCISE–I

Q.1 $f(x) = x^3 + x^2 - x + 2$ **Q.2** max. at $x = 1$; $f(1) = 0$, min. at $x = 7/5$; $f(7/5) = -108/3125$

Q.3 **(a)** Max at $x = 2\pi$, Max value $= 2\pi$, Min. at $x = 0$, Min value $= 0$
(b) Max at $x = \pi/6$ & also at $x = 5\pi/6$ and
Max value $= 3/2$, Min at $x = \pi/2$, Min value $= -3$

Q.4 $a = \frac{1}{4}$; $b = -\frac{5}{4}$; $f(x) = \frac{1}{4}(x^2 - 5x + 8)$

Q.5 $f(x) = \frac{2}{3}x^6 - \frac{12}{5}x^5 + 2x^4$ **Q.6** $P_{max} = a\left(1 + \operatorname{cosec}\frac{\alpha}{2}\right)$ **Q.7** $75\sqrt{3}$ sq. units

Q.9 $r = \sqrt{\frac{2A}{\pi+4}}$, $s = \sqrt{\frac{2A}{\pi+4}}$ **Q.10** $3x + 4y - 9 = 0$; $3x - 4y + 9 = 0$

Q.11 $4\sqrt{2}$ m **Q.12** $1/\pi$ cu m
Q.13 110' , 70' **Q.14** $6/(6 + \pi)$ **Q.15** side 10', height 10'
Q.16 32 sq. units **Q.17** $\theta = 60^0$ **Q.18** $a = 1, b = 0$

Q.19 $d_{min} = \frac{\sqrt{3}}{2}$ when $r = \sqrt{2}$ or $-\sqrt{2}$

Q.20 width $2\sqrt{3}$ m, length $3\sqrt{3}$ m **Q.21** $|a - b|$

Q.22 **(a)** $x = y = \frac{d}{\sqrt{2}}$, **(b)** $x = \frac{d}{\sqrt{3}}$, $y = \sqrt{\frac{2}{3}}\,d$ **Q.23** $6' \times 18'$

Q.24 $r = \sqrt{A}$, $\theta = 2$ radians **Q.25** $\frac{4}{3\sqrt{3}}$

EXERCISE–II

Q.1 **(a)** 0, 3, **(c)** $\frac{3}{4}$, t = $\ln 4$ **Q.2** $\cos A = 0.8$ **Q.4** (0 , 0)

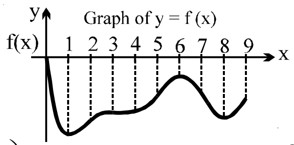
Q.6 $\frac{\sqrt{6}}{108}$ **Q.8** 4 when $a = \sqrt{2}$

Q.9 **(a)** f is continuous at $x = 0$; **(b)** – 2/e ; **(c)** does not exist, does not exist ; **(d)** pt. of inflection $x = 1$

Q.10 (a) $x = -2\pi, -\pi, 0, \pi, 2\pi$, (b) no inflection point, (c) maxima at $x = \pi/2$ and $-3\pi/2$ and no minima, (d) $x = 3\pi/2$ and $x = -\pi/2$, (e) $-\pi \ln 2$

Q.11 74

Q.12 **(i)** I in $(1, 6) \cup (8, 9)$ and D in $(0, 1) \cup (6, 8)$; **(ii)** L.Min. at $x = 1$ and $x = 8$; L.Max. $x = 6$
(iii) CU in $(0, 2) \cup (3, 5) \cup (7, 9)$ and CD in $(2, 3) \cup (5, 7)$; **(iv)** $x = 2, 3, 5, 7$

(v) Graph is 

Q.13 $m \in \left(\frac{1}{32}, \frac{1}{16}\right)$ **Q.14** $k = \frac{2\sqrt{2}}{\pi}\cos\left(\frac{\pi}{2\sqrt{2}}\right)$ **Q.15** $\pi/4$ **Q.16** 320 **Q.17** L/4

Q.18 $\pi/3$

Q.19 **(a)** increasing in (0, 2) and decreasing in $(-\infty, 0) \cup (2, \infty)$, local min. value = 0 and local max. value = 2
(b) concave up for $(-\infty, 2-\sqrt{2}) \cup (2+\sqrt{2}, \infty)$ and concave down in $(2-\sqrt{2}), (2+\sqrt{2})$
(c) $f(x) = \frac{1}{2}e^{2\cdot x}\cdot x^2$

EXERCISE–III

Q.1 A **Q.2** 2ab **Q.3** (a) D ; **(b)** A **Q.4** A **Q.5** **(a)** (2, 1) ; **(b)** 5

Q.6 **(a)** D **Q.7** $4\sqrt{65}$ **Q.8** (a) B, C; (b) A (c) 6 solutions

Q.9 **(a)** C; **(b)** **(i)** A, **(ii)** A, **(iii)** B **Q.10** **(a)** 0; **(b)** 7

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