AREA UNDER CURVE

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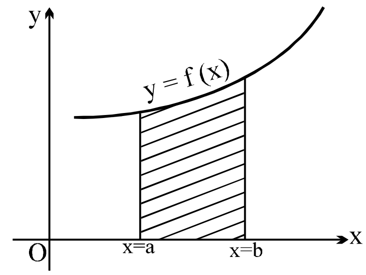
KEY CONCEPTS (AREA UNDER THE CURVE)

THINGS TO REMEMBER :

1. The area bounded by the curve y = f(x), the x-axis and the ordinates at x = a & x = b is given by,

$$A = \int_a^b f(x)\,dx = \int_a^b y\,dx.$$

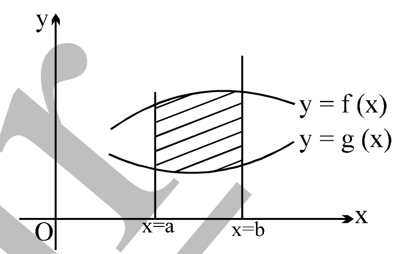


2. If the area is below the x–axis then A is negative. The convention is to consider the magnitude only i.e.

$$A = \left|\int_a^b y\;dx\right| \text{ in this case.}$$

3. Area between the curves y = f (x) & y = g (x) between the ordinates at x = a & x = b is given by,

$$A = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b [\,f(x) - g(x)\,]\,dx.$$



4. Average value of a function y = f (x) w.r.t. *x* over an interval $a \le x \le b$ is defined as :

$$y(av) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

5. The area function A_a^x satisfies the differential equation $\frac{dA_a^x}{dx} = f(x)$ with initial condition $A_a^a = 0$.

Note : If F (x) is any integral of f (x) then ,

$A_a^x = \int f(x)\,dx = F(x) + c$ $\qquad A_a^a = 0 = F(a) + c \Rightarrow c = -F(a)$

hence $A_a^x = F(x) - F(a)$. Finally by taking x = b we get , $A_a^b = F(b) - F(a)$.

6. CURVE TRACING :

The following outline procedure is to be applied in Sketching the graph of a function y = f(x) which in turn will be extremely useful to quickly and correctly evaluate the area under the curves.

(a) Symmetry : The symmetry of the curve is judged as follows :

(i) If all the powers of *y* in the equation are even then the curve is symmetrical about the axis of *x*.

(ii) If all the powers of *x* are even , the curve is symmetrical about the axis of *y*.

(iii) If powers of *x* & *y* both are even, the curve is symmetrical about the axis of *x* as well as *y*.

(iv) If the equation of the curve remains unchanged on interchanging *x* and *y*, then the curve is symmetrical about y = x.

(v) If on interchanging the signs of *x* & *y* both the equation of the curve is unaltered then there is symmetry in opposite quadrants.

(b) Find dy/dx & equate it to zero to find the points on the curve where you have horizontal tangents.

(c) Find the points where the curve crosses the x–axis & also the y–axis.

(d) Examine if possible the intervals when f (x) is increasing or decreasing. Examine what happens to 'y' when $x \to \infty$ or $-\infty$.

7. USEFUL RESULTS :

(i) Whole area of the ellipse, $x^2/a^2 + y^2/b^2 = 1$ is πab.

(ii) Area enclosed between the parabolas $y^2 = 4ax$ & $x^2 = 4by$ is 16ab/3.

(iii) Area included between the parabola $y^2 = 4ax$ & the line y = mx is $8a^2/3m^3$.

EXERCISE–I

Q.1 Find the area bounded on the right by the line $x + y = 2$, on the left by the parabola $y = x^2$ and below by the x-axis.

Q.2 Find the area of the region $\{(x, y) : 0 \le y \le x^2 + 1,\ 0 \le y \le x + 1,\ 0 \le x \le 2\}$.

Q.3 Find the value of c for which the area of the figure bounded by the curves $y = \sin 2x$, the straight lines $x = \pi/6$, $x = c$ & the abscissa axis is equal to 1/2.

Q.4 Compute the area of the region bounded by the curves $y = e.\ x.\ \ln x$ & $y = \ln x/(e.x)$ where $\ln e = 1$.

Q.5 A figure is bounded by the curves $y = \left|\sqrt{2}\sin\frac{\pi x}{4}\right|$, $y = 0$, $x = 2$ & $x = 4$. At what angles to the positive x–axis straight lines must be drawn through (4 , 0) so that these lines partition the figure into three parts of the same size.

Q.6 Find the area bounded by the curves $y = \sqrt{1 - x^2}$ and $y = x^3 - x$. Also find the ratio in which the y-axis divided this area.

Q.7 If the area enclosed by the parabolas $y = a - x^2$ and $y = x^2$ is $18\sqrt{2}$ sq. units. Find the value of 'a'.

Q.8 The line $3x + 2y = 13$ divides the area enclosed by the curve, $9x^2 + 4y^2 - 18x - 16y - 11 = 0$ into two parts. Find the ratio of the larger area to the smaller area.

Q.9 Find the values of m (m > 0) for which the area bounded by the line $y = mx + 2$ and $x = 2y - y^2$ is, (i) 9/2 square units & (ii) minimum. Also find the minimum area.

Q.10 Consider two curves $C_1 : y = \frac{1}{x}$ and $C_2 : y = \ln x$ on the xy plane. Let D_1 denotes the region surrounded by C_1, C_2 and the line $x = 1$ and D_2 denotes the region surrounded by C_1, C_2 and the line $x = a$. If $D_1 = D_2$. Find the value of 'a'.

Q.11 Find the area enclosed between the curves : $y = \log_e(x + e)$, $x = \log_e(1/y)$ & the x–axis.

Q.12 Find the value (s) of the parameter 'a' (a > 0) for each of which the area of the figure bounded by the straight line, $y = \frac{a^2 - ax}{1 + a^4}$ & the parabola $y = \frac{x^2 + 2ax + 3a^2}{1 + a^4}$ is the greatest.

Q.13 For what value of 'a' is the area bounded by the curve $y = a^2x^2 + ax + 1$ and the straight line $y = 0$, $x = 0$ & $x = 1$ the least ?

Q.14 Find the positive value of 'a' for which the parabola $y = x^2 + 1$ bisects the area of the rectangle with vertices (0, 0), (a, 0), $(0, a^2 + 1)$ and $(a, a^2 + 1)$.

Q.15 Compute the area of the curvilinear triangle bounded by the y-axis & the curve, $y = \tan x$ & $y = (2/3)\cos x$.

Q.16 Let $f(x) = \text{Maximum}\ \{x^2, (1 - x)^2, 2x(1 - x)\}$, where $0 \le x \le 1$. Determine the area of the region bounded by the curves $y = f(x)$, x– axis, $x = 0$ & $x = 1$.

Q.17 Find the area bounded by the curve $y = x\,e^{-x}$; $xy = 0$ and $x = c$ where c is the x-coordinate of the curve's inflection point.

Q.18 Find the value of 'c' for which the area of the figure bounded by the curve, $y = 8x^2 - x^5$, the straight lines $x = 1$ & $x = c$ & the abscissa axis is equal to 16/3.

Q.19 Find the area bounded by the curve $y = x\,e^{-x^2}$, the x-axis, and the line x = c where y(c) is maximum.

Q.20 Find the area bounded by the polynomial $y = x^2 - \left|x^2 - 1\right| + 2\left||x| - 1\right| + 2|x| - 7$ and the x-axis.

Q.21 Consider a circle $x^2 + (y-1)^2 = 1$ and the parabola $y = -\frac{x^2}{4}$. The common tangents to the two curves constitute a triangle ABC, the point A and B being on the x-axis and C on the y-axis. CA produced touches the parabola at P and CB produced touches the parabola at Q.

(a) Find the equation of the common tangent BC.

(b) Find the area of the portion between the upper arc of the circle and the common tangents QC and PC.

(c) Find the area enclosed by the parabola $y = -\frac{x^2}{4}$, the x-axis and the lines AP and BQ.

Q.22 Consider one side AB of a square ABCD, (read in order) on the line $y = 2x - 17$, and the other two vertices C, D on the parabola $y = x^2$.

(a) Find the minimum intercept of the line CD on y-axis.

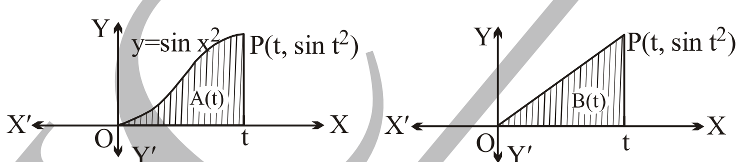
(b) Find the maximum possible area of the square ABCD.

(c) Find the area enclosed by the line CD with minimum y-intercept and the parabola $y = x^2$.

EXERCISE–II

Q.1 A polynomial function $f(x)$ satisfies the condition $f(x+1) = f(x) + 2x + 1$. Find $f(x)$ if $f(0) = 1$. Find also the equations of the pair of tangents from the origin on the curve $y = f(x)$ and compute the area enclosed by the curve and the pair of tangents.

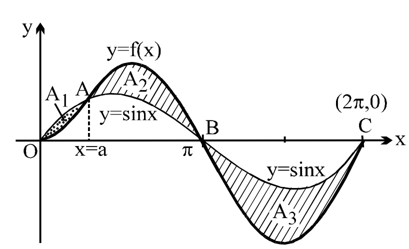
Q.2 The figure shows two regions in the first quadrant.



A(t) is the area under the curve $y = \sin x^2$ from 0 to t and B(t) is the area of the triangle with vertices O, P and M(t, 0). Find $\lim_{t \to 0} \frac{A(t)}{B(t)}$.

Q.3 Consider the curve $y = x^n$ where $n > 1$ in the 1st quadrant. If the area bounded by the curve, the x-axis and the tangent line to the graph of $y = x^n$ at the point (1, 1) is maximum then find the value of n.

Q.4 In the adjacent figure, graphs of two functions y = f(x) and y = sinx are given. y = sinx intersects, y = f(x) at A (a, f(a)); B(π, 0) and C(2π, 0). A_i (i = 1, 2, 3,) is the area bounded by the curves y = f (x) and y = sinx between x=0 and x= a; i = 1, between x = a and x = π; i = 2, between x = π and x = 2π; i = 3. If $A_1 = 1 - \sin a + (a-1)\cos a$, determine the function f(x). Hence determine 'a' and A_1. Also calculate A_2 and A_3.



Q.5 Consider the two curves $y = 1/x^2$ & $y = 1/[4(x-1)]$.

(i) At what value of 'a' (a > 2) is the reciprocal of the area of the fig. bounded by the curves, the lines $x = 2$ & $x = a$ equal to 'a' itself?

(ii) At what value of 'b' (1 < b < 2) the area of the figure bounded by these curves, the lines $x = b$ & $x = 2$ equal to 1 – 1/b.

Q.6 Show that the area bounded by the curve $y = \dfrac{\ln x - c}{x}$, the x-axis and the vertical line through the maximum point of the curve is independent of the constant c.

Q.7 For what value of 'a' is the area of the figure bounded by the lines, $y = \dfrac{1}{x}$, $y = \dfrac{1}{2x-1}$, $x = 2$ & $x = a$ equal to $\ln\dfrac{4}{\sqrt{5}}$?

Q.8 Compute the area of the loop of the curve $y^2 = x^2\,[(1+x)/(1-x)]$.

Q.9 For the curve $f(x) = \dfrac{1}{1+x^2}$, let two points on it are $A(\alpha, f(\alpha))$, $B\left(-\dfrac{1}{\alpha}, f\left(-\dfrac{1}{\alpha}\right)\right)$ $(\alpha > 0)$. Find the minimum area bounded by the line segments OA, OB and $f(x)$, where 'O' is the origin.

Q.10 Let 'c' be the constant number such that $c > 1$. If the least area of the figure given by the line passing through the point $(1, c)$ with gradient 'm' and the parabola $y = x^2$ is 36 sq. units find the value of $(c^2 + m^2)$.

Q.11 Let A_n be the area bounded by the curve $y = (\tan x)^n$ & the lines $x = 0$, $y = 0$ & $x = \pi/4$. Prove that for $n > 2$, $A_n + A_{n-2} = 1/(n-1)$ & deduce that $1/(2n+2) < A_n < 1/(2n-2)$.

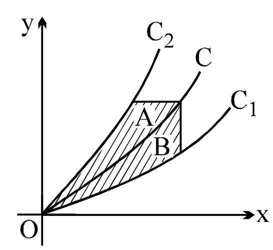
Q.12 If $f(x)$ is monotonic in (a, b) then prove that the area bounded by the ordinates at $x = a$; $x = b$; $y = f(x)$ and $y = f(c)$, $c \in (a, b)$ is minimum when $c = \dfrac{a+b}{2}$.

Hence if the area bounded by the graph of $f(x) = \dfrac{x^3}{3} - x^2 + a$, the straight lines $x = 0$, $x = 2$ and the x-axis is minimum then find the value of 'a'.

Q.13 Consider the two curves $C_1 : y = 1 + \cos x$ & $C_2 : y = 1 + \cos(x - \alpha)$ for $\alpha \in (0, \pi/2)$; $x \in [0, \pi]$. Find the value of α, for which the area of the figure bounded by the curves C_1, C_2 & $x = 0$ is same as that of the figure bounded by C_2, $y = 1$ & $x = \pi$. For this value of α, find the ratio in which the line $y = 1$ divides the area of the figure by the curves C_1, C_2 & $x = \pi$.

Q.14 For what values of $a \in [0, 1]$ does the area of the figure bounded by the graph of the function $y = f(x)$ and the straight lines $x = 0$, $x = 1$ & $y = f(a)$ is at a minimum & for what values it is at a maximum if $f(x) = \sqrt{1 - x^2}$. Find also the maximum & the minimum areas.

Q.15 Let C_1 & C_2 be two curves passing through the origin as shown in the figure. A curve C is said to "bisect the area" the region between C_1 & C_2, if for each point P of C, the two shaded regions A & B shown in the figure have equal areas. Determine the upper curve C_2, given that the bisecting curve C has the equation $y = x^2$ & that the lower curve C_1 has the equation $y = x^2/2$.



Q.16**(a)** Given $f(x) = \int_0^x e^t(\ln \sec t - \sec^2 t)\,dt$; $g(x) = -2e^x \tan x$. Find the area bounded by the curves $y = f(x)$ and $y = g(x)$ between the ordinates $x = 0$ and $x = \dfrac{\pi}{3}$.

(b) Let $f : [0, \infty) \to R$ be a continuous and strictly increasing function such that $f^3(x) = \int_0^x t\,f^2(t)\,dt$, $\forall\, x > 0$. Find the area enclosed by $y = f(x)$, the x-axis and the ordinate at $x = 3$.

EXERCISE–III

Q.1 For which of the following values of m, is the area of the region bounded by the curve $y = x - x^2$ and the line $y = mx$ equals 9/2 ?

(A) – 4 (B) – 2 (C) 2 (D) 4

[JEE '99, 3 (out of 200)]

Q.2 Find the area of the region lying inside $x^2 + (y-1)^2 = 1$ and outside $c^2x^2 + y^2 = c^2$ where $c = \sqrt{2} - 1$.

[REE '99, 6]

Q.3 Find the area enclosed by the parabola $(y-2)^2 = x - 1$, the tangent to the parabola at (2, 3) and the x-axis. [REE 2000,3]

Q.4 The area bounded by the curves $y = |x| - 1$ and $y = -|x| + 1$ is

(A) 1 (B) 2 (C) $2\sqrt{2}$ (D) 4 [JEE'2002, (Scr)]

Q.5 Find the area of the region bounded by the curves $y = x^2$, $y = |2 - x^2|$ and $y = 2$, which lies to the right of the line $x = 1$. [JEE '2002, (Mains)]

Q.6 If the area bounded by $y = ax^2$ and $x = ay^2$, $a > 0$, is 1, then a =

(A) 1 (B) $\frac{1}{\sqrt{3}}$ (C) $\frac{1}{3}$ (D) $-\frac{1}{\sqrt{3}}$ [JEE '2004, (Scr)]

Q.7(a) The area bounded by the parabolas $y = (x+1)^2$ and $y = (x-1)^2$ and the line $y = 1/4$ is

(A) 4 sq. units (B) 1/6 sq. units (C) 4/3 sq. units (D) 1/3 sq. units

[JEE '2005 (Screening)]

(b) Find the area bounded by the curves $x^2 = y$, $x^2 = -y$ and $y^2 = 4x - 3$.

(c) Let f(x) be a quadratic polynomial and a, b, c be distinct real numbers such that

$$\begin{bmatrix} 4a^2 & 4a & 1 \\ 4b^2 & 4b & 1 \\ 4c^2 & 4c & 1 \end{bmatrix} \begin{bmatrix} f(-1) \\ f(1) \\ f(2) \end{bmatrix} = \begin{bmatrix} 3a^2 + 3a \\ 3b^2 + 3b \\ 3c^2 + 3c \end{bmatrix}.$$

Let V be the point of maximum of the curve $y = f(x)$. If A and B are the points on this curve such that the curve meets the positive x-axis at A and the chord AB subtends a right angle at V, then find the area enclosed by the curve and the chord AB. [JEE '2005 (Mains), 4 + 6]

Q.8 **Match the following**

Column-I	**Column-II**
(A) The cosine of the angle between the curves $y = 3^{x-1}\,\ln x$ and $y = x^x - 1$ at their point of intersection on the line $y = 0$, is	(P) 0
(B) The area bounded by the curves $x = -4y^2$ and $(x-1) = -5y^2$ is	(Q) 1
(C) The value of the integral $\int_0^{\pi/2} (\sin x)^{\cos x}(\cos x \cot x - \ln(\sin x)^{\sin x})\,dx$, is	(R) $\frac{4}{3}$
(D) A continuous function $f : [1, 6] \to [0, \infty)$ is such that $f'(x) = \frac{2}{x + f(x)}$ and $f(1) = 0$, then the maximum value of f cannot exceed	(S) $2\ln 6$

[JEE 2006, 6]

Q.9(a) The area of the region between the curves $y = \sqrt{\frac{1+\sin x}{\cos x}}$ and $y = \sqrt{\frac{1-\sin x}{\cos x}}$ bounded by the lines $x = 0$ and $x = \frac{\pi}{4}$ is

(A) $\int_0^{\sqrt{2}-1} \frac{t}{(1+t^2)\sqrt{1-t^2}}\,dt$ (B) $\int_0^{\sqrt{2}-1} \frac{4t}{(1+t^2)\sqrt{1-t^2}}\,dt$

(C) $\int_0^{\sqrt{2}+1} \frac{4t}{(1+t^2)\sqrt{1-t^2}}\,dt$ (D) $\int_0^{\sqrt{2}+1} \frac{t}{(1+t^2)\sqrt{1-t^2}}\,dt$

(b) **Comprehension (3 questions together):**
Consider the functions defined implicitly by the equation $y^3 - 3y + x = 0$ on various intervals in the real line.
If $x \in (-\infty, -2) \cup (2, \infty)$, the equation implicitly defines a unique real valued differentiable function $y = f(x)$.
If $x \in (-2, 2)$, the equation implicitly defines a unique real valued differentiable function $y = g(x)$ satisfying $g(0) = 0$.

(i) If $f(-10\sqrt{2}) = 2\sqrt{2}$, then $f''(-10\sqrt{2}) =$

(A) $\frac{4\sqrt{2}}{7^3 3^2}$ (B) $-\frac{4\sqrt{2}}{7^3 3^2}$ (C) $\frac{4\sqrt{2}}{7^3 3}$ (D) $-\frac{4\sqrt{2}}{7^3 3}$

(ii) The area of the region bounded by the curve $y = f(x)$, the x-axis, and the lines $x = a$ and $x = b$, where $-\infty < a < b < -2$, is

(A) $\int_a^b \frac{x}{3\left((f(x))^2 - 1\right)}dx + bf(b) - af(a)$ (B) $-\int_a^b \frac{x}{3\left((f(x))^2 - 1\right)}dx + bf(b) - af(a)$

(C) $\int_a^b \frac{x}{3\left((f(x))^2 - 1\right)}dx - bf(b) + af(a)$ (D) $-\int_a^b \frac{x}{3\left((f(x))^2 - 1\right)}dx - bf(b) + af(a)$

(iii) $\int_{-1}^{1} g'(x)\,dx =$

(A) 2g(–1) (B) 0 (C) – 2 g(1) (D) 2 g(1)

[JEE 2008, 3 + 4 + 4 + 4]

Q.10 Area of the region bounded by the curve $y = e^x$ and lines $x = 0$ and $y = e$ is

(A) $e - 1$ (B) $\int_1^e \ln(e+1-y)\,dy$ (C) $e - \int_0^1 e^x\,dx$ (D) $\int_1^e \ln y\,dy$

[JEE 2009, 4]

AREA UNDER THE CURVE

EXERCISE-I

Q.1 5/6 sq. units **Q.2** 23/6 sq. units **Q.3** $c = -\pi/6$ or $\pi/3$

Q.4 $(e^2 - 5)/4e$ sq. units **Q.5** $\pi - \tan^{-1}\dfrac{2\sqrt{2}}{3\pi}$; $\pi - \tan^{-1}\dfrac{4\sqrt{2}}{3\pi}$

Q.6 $\dfrac{\pi}{2}$; $\dfrac{\pi-1}{\pi+1}$ **Q.7** a = 9 **Q.8** $\dfrac{3\pi+2}{\pi-2}$

Q.9 **(i)** m = 1, **(ii)** $m = \infty$; $A_{min} = 4/3$ **Q.10** e **Q.11** 2 sq. units **Q.12** $a = 3^{1/4}$

Q.13 $a = -3/4$ **Q.14** $\sqrt{3}$ **Q.15** $\dfrac{1}{3} + \ell n\left(\dfrac{\sqrt{3}}{2}\right)$ sq. units **Q.16** 17/27 **Q.17** $1 - 3e^{-2}$

Q.18 $C = -1$ or $\left(8 - \sqrt{17}\right)^{1/3}$ **Q.19** $\dfrac{1}{2}(1 - e^{-1/2})$ **Q.20** $\dfrac{44}{3}$

Q.21 (a) $\sqrt{3}x - y + 3 = 0$; (b) $\left(\sqrt{3} - \dfrac{\pi}{3}\right)$; (c) $\sqrt{3}$ **Q.22** (a) 3; (b) 1280;

EXERCISE-II

Q.1 $f(x) = x^2 + 1$; $y = \pm 2x$; $A = \dfrac{2}{3}$ sq. units **Q.2** 2/3

Q.3 $\sqrt{2} + 1$

Q.4 $f(x) = x \sin x$, a = 1; $A_1 = 1 - \sin 1$; $A_2 = \pi - 1 - \sin 1$; $A_3 = (3\pi - 2)$ sq. units

Q.5 $a = 1 + e^2$, $b = 1 + e^{-2}$ **Q.6** 1/2 **Q.7** a = 8 or $\dfrac{2}{5}\left(6 - \sqrt{21}\right)$ **Q.8** $2 - (\pi/2)$ sq. units

Q.9 $\dfrac{(\pi - 1)}{2}$ **Q.10** 104 **Q.12** $a = \dfrac{2}{3}$ **Q.13** $\alpha = \pi/3$, ratio = $2 : \sqrt{3}$

Q.14 a = 1/2 gives minima, $A\left(\dfrac{1}{2}\right) = \dfrac{3\sqrt{3} - \pi}{12}$; a = 0 gives local maxima $A(0) = 1 - \dfrac{\pi}{4}$;
a = 1 gives maximum value, $A(1) = \pi/4$

Q.15 $(16/9)x^2$ **Q.16** **(a)** $e^{\pi/3}\log 2$ sq. units, **(b)** 3/2

EXERCISE-III

Q.1 B, D **Q.2** $\left(\pi - \dfrac{\pi - 2}{2\sqrt{2}}\right)$ sq. units **Q.3** 9 sq. units

Q.4 B **Q.5** $\left(\dfrac{20}{3} - 4\sqrt{2}\right)$ sq. units **Q.6** B **Q.7** (a) D ; (b) $\dfrac{1}{3}$ sq. units ; (c) $\dfrac{125}{3}$ sq. units

Q.8 (A) Q, (B) R, (C) Q, (D) S **Q.9** **(a)** B, **(b)** (i) B, (ii) A, (iii) D **Q.10** **B, C, D**