CIRCLE

Basic Geometry with Circles

1. Equal chords subtends equal angles at the centre and vice-versa.

2. Equal chords of a circle are equidistant from the centre and vice – versa.

3. Angle subtended by an arc at the centre is double the angle subtended at any point on the remaining part of the circle.

4. Angles in the same segment of a circle are equal.

5. The sum of the opposite angles of a cyclic quadrilateral is 180° and vice-versa.

6. If two chords of a circle intersect either inside or outside the circle, the rectangle contained by the parts one chord is equal in area to the rectangle contained by the parts of the other.

7. The greater of the two chords in a circle is nearer to the centre than lesser.

8. A chord drawn across the circular region divides it into parts each of which is called a segment of the circle.

9. The tangents at the extremities of a chord of a circle are equal.

Definition of Circle

Circle is defined as a locus of a point ‘P’ which moves in x - y plane in such a way such that its distance from the fixed point in the same plane is always constant.

Standard Form

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$$(x - a)^2 + (y - b)^2 = r^2$$

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$a, b \equiv$ centre

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$a, b \equiv$ centre

$r \equiv$ radius

General Equation of the Circle

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$$x^2 + y^2 + 2gx + 2fy + c = 0.$$

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$$\text{Centre} \equiv (-g, -f) \text{ i.e. } \left(-\frac{1}{2}\text{coefficient of x}; -\frac{1}{2}\text{coefficient of y}\right)$$

$$\text{Radius} \equiv \sqrt{g^2 + f^2 - c}$$

Examples

Q.1 Find equation of circle whose radius is 3 and centre is (-1, 2)

Q.2 Find equation of circle whose radius is 10 and centre is (-5, -6)

Necessary and Sufficient Condition for General Equation of 2° to Represent a Circle

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(a) Coefficient of x^2 = coefficient of y^2 (not necessarily unity) and

(b) Coefficient of $xy = 0$

Note

The general equation of circle
$x^2 + y^2 + 2gx + 2fy + c = 0$ contains 3 independent arbitrary constants g, f and c which means that a unique circle passes through 3 non-collinear points. Hence 3 points on a circle must be given to determine the unique equation of the circle.

Nature of Circle

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

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$$x^2 + y^2 + 2gx + 2fy + c = 0$$

(i) If $g^2 + f^2 - c > 0$

$\Rightarrow$ Real circle with finite radius

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(i) If $g^2 + f^2 - c > 0$

$\Rightarrow$ *Real circle with finite radius*

(ii) If $g^2 + f^2 - c = 0$

$\Rightarrow$ *Point circle*

Nature of Circle

$x^2 + y^2 + 2gx + 2fy + c = 0$

(i) If $g^2 + f^2 - c > 0$

$\Rightarrow$ Real circle with finite radius

(ii) If $g^2 + f^2 - c = 0$

$\Rightarrow$ Point circle

(iii) If $g^2 + f^2 - c < 0$

$\Rightarrow$ imaginary circle

Examples

Q.1 Find the equation of the circle passing through the points (3, 4), (-3, -4), (0, 5)

Q.2 Find the equation of the circle having lines $2x - 3y = 5$ and $3x - 4y = 7$ as its diameter / Normal / longest chord and whose area is 154 sq. units.

Q.3 Find the equation of the Circumcircle of Δ formed by the lines

$xy + 2x + 2y + 4 = 0 \; ; \; x + y + 2 = 0$

Q.4 Find centre and radius of the circle

$$2x^2 + 2y^2 - 6x + 8y - 5 = 0$$

Q.5 Find equation of circle concentric with $3x^2 + 3y^2 - 5x - 6y - 14 = 0$ and perimeter of its semicircle is 36.

Q.6 Find equation of the circle which passes through (2, 3) and centre on the x-axis, radius being 5.

Q.7 Find the equation of the circle for which centre is on the line $y=2x$ and passing through $(-1, 2)$ and $(3, -2)$.

Q.8 Find the equation of circle whose centre is $(4, 3)$ and touches the line $5x - 12y - 10 = 0$.

S.L. Loney
Assignment - 1

Find the equation to the circle :

Q.1 Whose radius is 3 and whose centre is (-1, 2)

Q.2 Whose radius is 10 and whose centre is (-5, -6)

Q.3 Whose radius is a + b and whose centre is (a, -b)

Q.4 Whose radius is $\sqrt{a^2 - b^2}$ and whose centre is (-a, -b).

Find the coordinates of the centres and the radii of the circle whose equations are :

Q.5 $x^2 + y^2 - 4x - 8y = 41$

Q.6 $3x^2 + 3y^2 - 5x - 6y + 4 = 0$

Q.7 $x^2 + y^2 = k(x + k)$

Q.8 $x^2 + y^2 = 2gx - 2fy$

Q.9 $\sqrt{1+m^2}\,(x^2 + y^2) - 2cx - 2mcy = 0$

Find the equations to the circles which pass through the points :

Q.10 $(0, 0)$, $(a, 0)$ and $(0, b)$

Q.11 $(1, 2)$, $(3, -4)$ and $(5, -6)$

Q.12 $(1, 1)$ $(2, -1)$ and $(3, 2)$

Q.13 $(5, 7)$, $(8, 1)$ and $(1, 3)$

Q.14 (a, b), $(a, -b)$ and $(a + b, a - b)$

Diametrical Form of Circle

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$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$$

Diametrical Form of Circle

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$$

Where (x_1, y_1) & (x_2, y_2) are diametrical opposite ends

Example

Q.1 Find the equation of the circle of least radius passing through the points (2,3), (3,1).

Q.2 Find equation of tangent to circle parallel to tangent $x + y = 5$, center is $(1, 2)$.

Q.3 The abscissa of 2 points ‘A’ and ‘B’ are the roots of the equation $x^2+2x-3=0$ and the ordinate are the roots of the equation $y^2-4y+1=0$.

Find the equation of circle with AB as diameter.

Q.4 Find the equation of the circle which touches the lines $x = 0$, $y = 0$ and $x = 4$?

Q.5 Line $y = mx + c$ cuts the curve $y^2 = 4ax$ at A and B. Find the equation of circle with AB as diameter.

Q.6 If line $y = x + c$ and $y^2 = 8x$ intersect in A & B. Circle with AB as diameter passes through (0,0) find c ?

Q.7 Find locus of point of intersection of

$x + 2y + \lambda (x - 2y) = 0$ and

$(x + y - 2) + \lambda (x - 2) = 0$

if these lines are always perpendicular to each other.

INTERCEPT

Length of chord

Angle between Line & Circle

$$\cos\theta = \frac{p}{r}$$

X-Intercept

X-Intercept

$$\left|x_1 - x_2\right| = 2\left|\sqrt{g^2 - c}\right|$$

X-Intercept

(i) If $g^2 - c > 0$

$\Rightarrow$ circle cuts the x-axis at 2 distinct points.

X-Intercept

(ii) If $g^2 - c = 0$

$\Rightarrow$ circle touches the x-axis

X-Intercept

(iii) If $g^2 < c$

$\Rightarrow$ circle lies completely above or below the x-axis

Y-Intercept

Y-Intercept

$$\left|y_1 - y_2\right| = 2\left|\sqrt{f^2 - c}\right|$$

Y-Intercept

(i) If $f^2 - c > 0$

$\Rightarrow$ circle cuts the y-axis at 2 distinct points

Y-Intercept

(ii) If $f^2 = c$

$\Rightarrow$ circle touches the y - axis

Y-Intercept

(iii) If $f^2 < c$

$\Rightarrow$ circle lies completely either on right or on left of y – axis.

Examples

Q.1 Find the equation of the circle which touches the +ve axis of y at a distance of 4 units from origin and cuts off an intercept of 6 unit from the positive axis.

Q.2 Find the equation of the circle which touches the coordinate axes and whose radius = 5

Q.3 Find the equation of a circle passing through origin cutting off intercept equals to unity on the lines $y^2 - x^2 = 0$.

Q.4 Find the equation of the locus of the centre of a circle which touches the positive y-axis and having intercept on x-axis equals to $2l$.

Q.5 Find the equation of incircle and circumcircle of the quadrilateral formed by the lines $x = 0$ & $y = 0$, $x = 16$, $y = 16$

Q.6 Find λ if length of intercept by line $3x - 4y + \lambda = 0$ on the circle $x^2 + y^2 = 25$ is of 8 unit.

Q.7 Find the equation of circle whose centre is (5, 0) and touches the circle $x^2 + y^2 = 4$

Q.8 Two rods whose lengths are 2a & 2b move along the rectangular axes (one on X-axis and other on Y-axis) in such a way that their extremities are always concyclic. Find the equation of the locus of the centre of the circle.

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Assignment - 2

Q.1 Find the equation to the circle which passes through the points (1, -2) and (4, -3) and which has its centre on the straight lines $3x + 4y = 7$

Q.2 Find the equation to the circle passing through the point (0, a) and (b, h), and having its centre on the axis of x.

Q.3 ABCD is a square whose side is a; taking AB and AD as axes, prove that the equation to the circle circumscribing the square is,

$$x^2 + y^2 = a(x + y).$$

Q.4 Find the equation to the circle which passes through the origin and cuts off intercepts equal to 3 and 4 from the axes.

Q.5 Find the equation to the circle passing through the origin and the points (a, b) and (b, a). Find the lengths of the chords that it cuts off from the axes.

Q.6 Fine the equation to the circle which goes through the origin and cuts off intercepts equal to h and k from the positive parts of the axes.

Q.7 Find the equation to the circle, of radius a, which passes through the two points on the axis of x which are at a distance b from the origin.

Find the equation to the circle which :

Q.8 Touches each axis at a distance 5 from the origin.

Q.9 Touches each axis and is of radius a.

Q.10 Touches both axes and passes through the point (-2, -3).

Q.11 Touches the axis of x and passes through the two points (1, -2) and (3, -4)

Q.12 Touches the axis of y at the origin and passes through the point (b,c)

Q.13 Touches the axis of x at a distance 3 from the origin and intercepts a distance 6 on the axis of y.

Q.14 Points (1, 0) wand (2, 0) are taken on the axis of x, the axes being rectangular. On the line joining these points an equilateral triangle is described, its vertex being in the positive quadrant. Find the equation to the circle described on its sides as diameters.

Q.15 If $y = mx$ be the equations of a chord of a circle whose radius is a, the origin of coordinates being one extremity of the chord and the axis of x being a diameter of the circle, prove that the equation of a circle of which this chord is the diameter is,

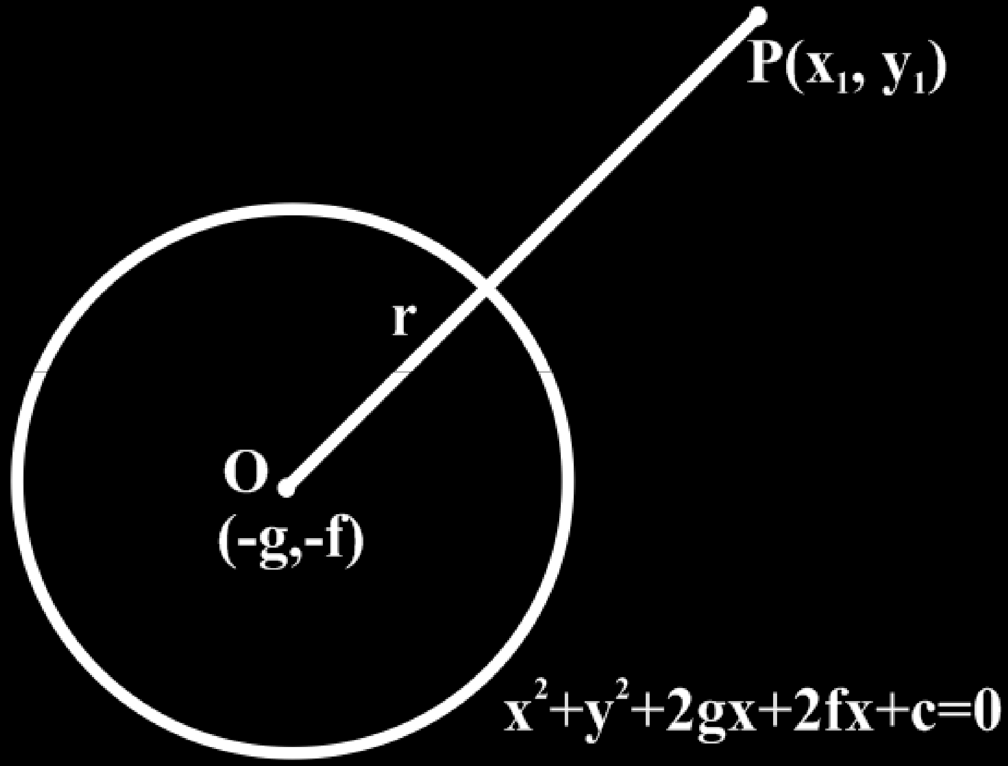
$$(1 + m^2)(x^2 + y^2) - 2a(x + my) = 0$$

Q.16 Prove that the equation to the circle of which the points (x_1, y_1) and (x_2, y_2) are the ends of a chord of a segment containing an angle θ is,

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2)$$
$$\pm \cot\theta \, [(x - x_1)(y - y_2) - (x - x_2)(y - y_1)] = 0$$

Q.17 Find the equations to the circles in which the line joining the points (a, b) and (b, -a) is a chord subtending an angle of 45° at any point on its circumference.

Position of A Point w.r.t. A Circle



$OP \geq r$

Position of A Point w.r.t. A Circle

$S_1 \equiv$ Power of point

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$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c > 0$$

Position of A Point w.r.t. A Circle

$S_1 \equiv$ Power of point

$$\mathbf{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c > 0}$$

If $S_1 > 0 \Rightarrow$ Point exterior of a circle

Position of A Point w.r.t. A Circle

$S_1 \equiv$ Power of point

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c > 0$$

If $S_1 > 0 \Rightarrow$ Point exterior of a circle

If $S_1 = 0 \Rightarrow$ Point on circle

Position of A Point w.r.t. A Circle

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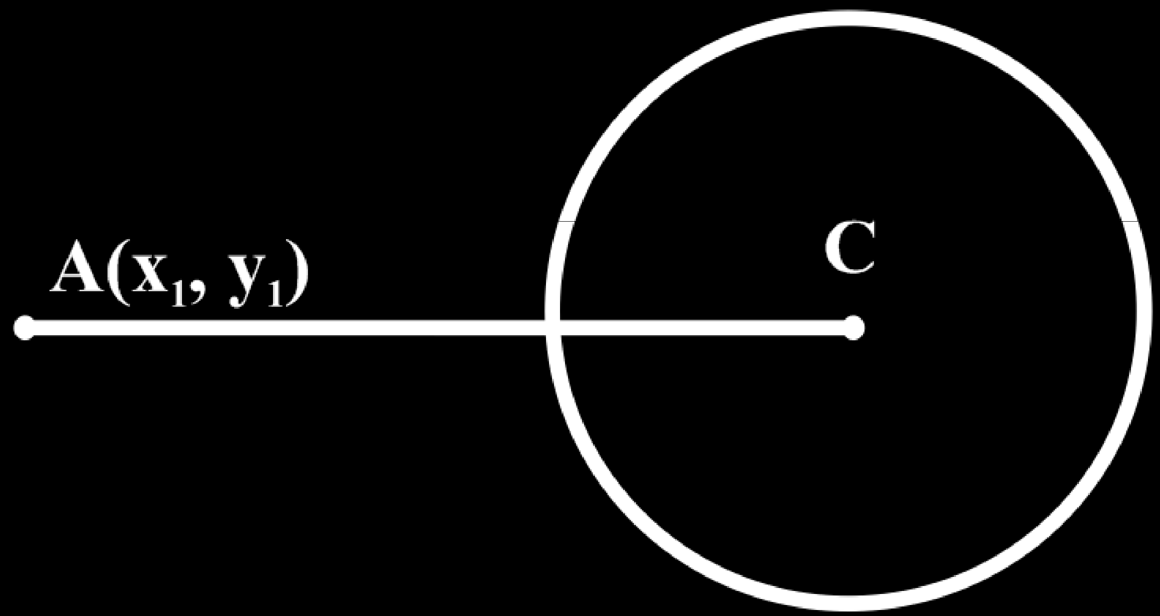
$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c > 0$$

If $S_1 > 0 \Rightarrow$ Point exterior of a circle

If $S_1 = 0 \Rightarrow$ Point on circle

If $S_1 < 0 \Rightarrow$ Point interior of circle

Greatest and least distance of a point A (x_1, y_1)



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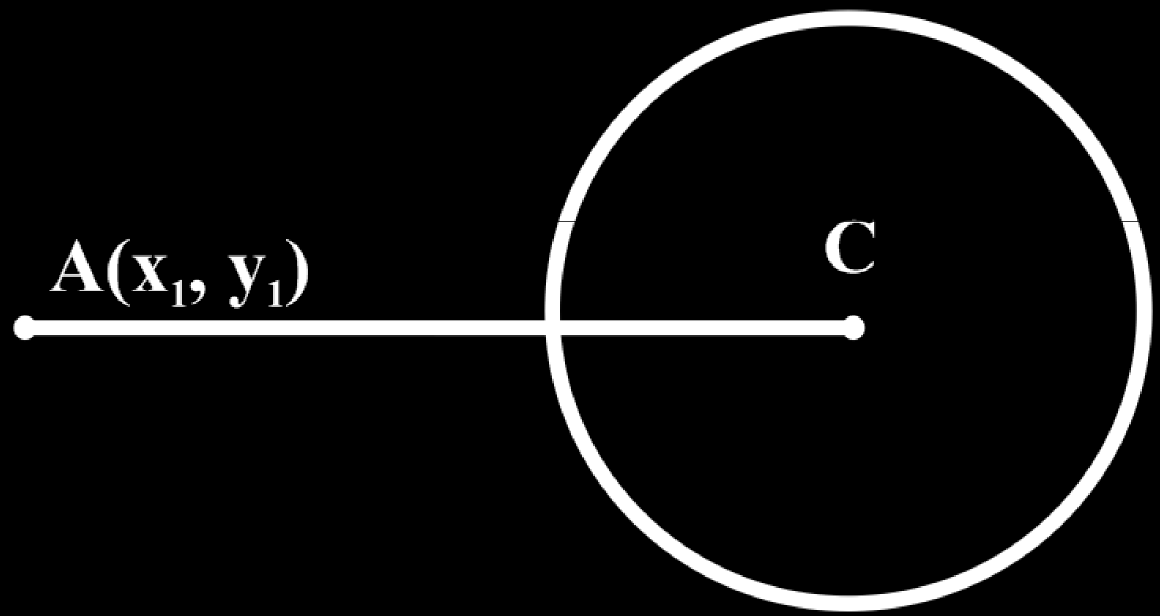

Maximum distance = | AC + r | and

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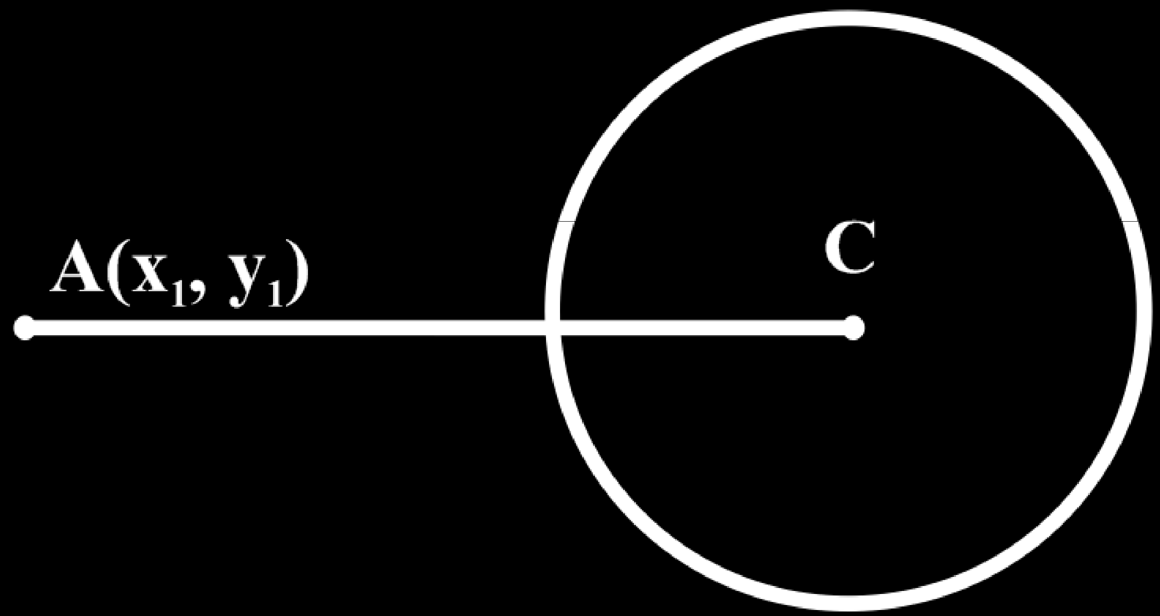

Maximum distance = | AC + r | and
Minimum distance = | AC – r |

Examples

Q.1 If the join of (x_1, y_1) & (x_2, y_2) makes on obtuse angle at (x_3, y_3) then prove that
$(x_3 - x_1)(x_3 - x_2) + (y_3 - y_1)(y_3 - y_2) < 0$

Q.2 $S_1 = x^2 + y^2 - 4x + 6y - 3 = 0$

$S_2 = x^2 + y^2 + 4x - 6y - 3 = 0$

point (1,2) lies

(A) inside $S_1 = 0$ an inside $S_2 = 0$

(B) outside $S_1 = 0$ an outside $S_2 = 0$

(C) inside $S_1 = 0$ an outside $S_2 = 0$

(D) outside $S_1 = 0$ an inside $S_2 = 0$

Q.3 Find the minimum and maximum distance between two points one lying on the circle $x^2 + y^2 = 144$ and other lying on $(x - 15)^2 + (y - 20)^2 = 1$

Q.4 Find minimum and maximum distance between any point on circle $x^2 + y^2 = 25$ & point (6, 8)

Line and A Circle

Let L = 0 be a line and S = 0 be a circle. If ‘r’ is the radius of the circle and ‘p’ is the length of perpendicular from the centre on the line, then

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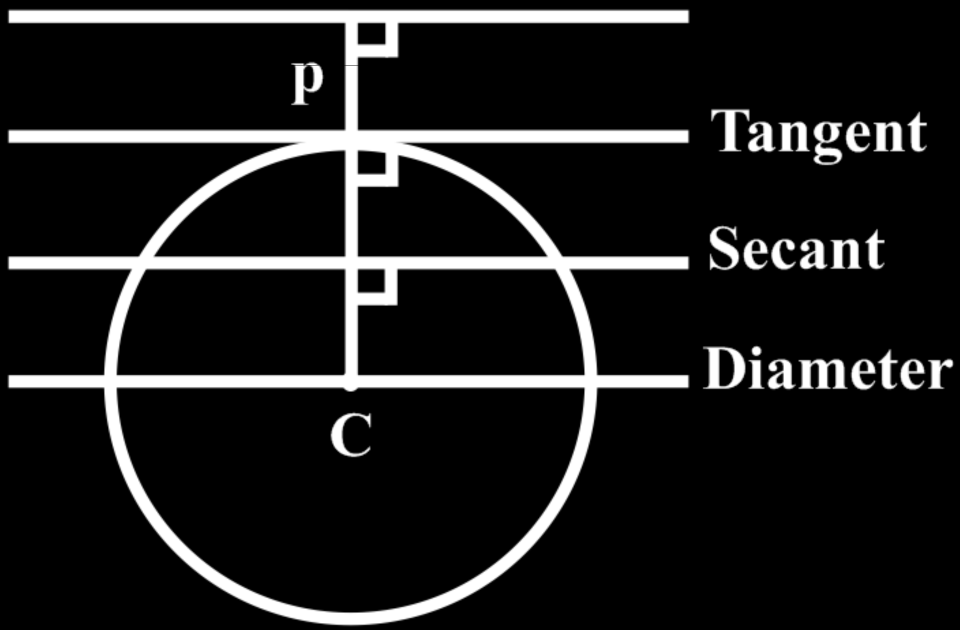

Line and A Circle

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$\Rightarrow$ line is neither secant nor tangent ; passes outside the circle

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(iii) If $p < r \Rightarrow$ line is a secant.

(iv) If $p = 0 \Rightarrow$ line is a diameter.

II Method

Solve the line with the circle and if

(i) $D > 0$ $\Rightarrow$ line is a Secant

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Solve the line with the circle and if

(i) $D > 0$ $\Rightarrow$ line is a Secant

(ii) $D = 0$ $\Rightarrow$ line is a Tangent

(iii) $D < 0$ $\Rightarrow$ line passes outside the circle.

Examples

Q.1 For what value of 'm' the line $3x - my + 6 = 0$ is tangent to the circle $x^2 + y^2 - 4x + 6y - 3 = 0$

Q.2 Find k if line $3x + 4y = k$ touches the circle $x^2 + y^2 - 10x = 0$

Q.3 Find the equation of $T = 0$ to circle $x^2 + y^2 = 4$ and parallel to line $x + 2y + 3 = 0$

Q.4 Find radius of circle whose tangents are

$6x + 8y + 26 = 0$

$3x + 4y - 17 = 0$

Note

(i) Number of tangents from external point to circle is 2

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(ii) Number of tangents from interior point to circle is 0

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(ii) Number of tangents from interior point to circle is 0

(iii) If point is on periphery the number of tangent is 1

Parametric Equation of A Circle

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$$x = x_1 + r\cos\theta \text{ and } y = y_1 + r\sin\theta$$

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$$x = x_1 + r\cos\theta \text{ and } y = y_1 + r\sin\theta$$

$x_1, y_1 \rightarrow$ fixed centre

$r \rightarrow$ fixed radius and $\theta \in [0, 2\pi)$ is a parameter.

Note

If θ is eliminated we get Cartesian form of a circle i.e. $(x - x_1)^2 + (y - y_1)^2 = r^2$.

Example

Q.1 $x^2 + y^2 - 6x + 4y - 3 = 0$. convert into parametric form

Q.2 If $x^2 + y^2 - 2x - 4y - 4 = 0$ find max/min value of $3x + 4y$

Q.3 Find circumcentre of ΔABC, where co-ordinates of

$$A \equiv \left(2+\cos\frac{\pi}{3}, 3+\sin\frac{\pi}{3}\right)$$

$$B \equiv (2+\cos\pi, 3+\sin\pi)$$

$$C \equiv \left(2+\cos\frac{4\pi}{3}, 3+\sin\frac{4\pi}{3}\right)$$

Q.4 If A $(\cos\theta_1, \sin\theta_1)$; B $(\cos\theta_2, \sin\theta_2)$; C $(\cos\theta_3, \sin\theta_3)$ are the vertices of the ΔABC then. Find

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(i) centroid of ΔABC

Q.4 If A $(\cos\theta_1, \sin\theta_1)$; B $(\cos\theta_2, \sin\theta_2)$; C $(\cos\theta_3, \sin\theta_3)$ are the vertices of the ΔABC then. Find

(ii) circumcentre of ΔABC

Q.4 If A $(\cos\theta_1, \sin\theta_1)$; B $(\cos\theta_2, \sin\theta_2)$; C $(\cos\theta_3, \sin\theta_3)$ are the vertices of the ΔABC then. Find

(iii) orthocentre of ΔABC

Tangent

Tangent

Tangent is the limiting case of the secant as the point B $\rightarrow$ A

Normal

Normal

Normal is a line perpendicular to the tangent passing through the point of tangency. In case of circle normal always passes through centre.

Equation of the tangent drawn to the circle in the various forms

Cartesian Form

Tangent drawn to the circle

$x^2 + y^2 + 2gx + 2fy + c = 0$ at its point (x_1, y_1) is

$xx_1 + yy_1 + g(x+x_1) + f(y+y_1) + c = 0$

Cartesian Form

If circle is $x^2 + y^2 = a^2$ then equation of tangent is

$$xx_1 + yy_1 = a^2$$

Example

Q. If equation of circle is $x^2 + y^2 = 25$

Find equation of tangent at (3, 4)

Parametric form

Parametric form

$$\left.\begin{aligned} x_1 &= r\cos\theta \\ y_1 &= r\sin\theta \end{aligned}\right\}$$

Parametric form

$$\left.\begin{aligned} x_1 &= r\cos\theta \\ y_1 &= r\sin\theta \end{aligned}\right\}$$

Equation of tangent is $x\cos\theta + y\sin\theta = r$

Slope form

Slope form

$$x^2 + y^2 = a^2$$

Slope form

$$x^2 + y^2 = a^2$$

Equation of the tangent is $y = mx \pm a\sqrt{1+m^2}$

Note

For a unique value of m there will be 2 tangent which are parallel to each other

Point of Tangency

Method - I

Step 1 : Write equation of normal {$\perp$ to T = 0 & passing through (-g, -f)}

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Step 1 : Write equation of normal {$\perp$ to T = 0 & passing through (-g, -f)}

Step 2 : Intersection of N = 0, T = 0 is coordinate of that point.

Method - II

Example

Q.1 Find point of tangency if equation of tangent $3x + 4y = 50$ to circle $x^2 + y^2 - 6x - 8y = 0$

Q.2 Find the equations of the tangents to the circle $x^2 + y^2 - 2x - 4y - 4 = 0$ which are

(i) perpendicular to the line $3x - 4y + 7 = 0$

Q.2 Find the equations of the tangents to the circle $x^2 + y^2 - 2x - 4y - 4 = 0$ which are

(ii) parallel to the line $3x - 4y + 7 = 0$

Q.3 Find the equation of the tangent to the circle $x^2 + y^2 = 4$ drawn from the point (2, 3).

Q.4 Find the equation of the tangent drawn to the circle $x^2 + y^2 - 6x + 4y - 3 = 0$ from the point (7, 4) lying outside the circle. Also find the point of contact.

Q.5 Find shortest distance between line

$3x + 4y = 25$ and circle $x^2 + y^2 - 6x + 8y = 0$

Q.6 If equation of tangent line on circle $x^2 + y^2 = 1^2$ is $y = x + \sqrt{2}$ then Find point of contact.

Q.7 Tangent is drawn from the point P (4, 0) to the circle $x^2 + y^2 = 8$ touches it at the point A in the 1st quadrant. Find the coordinates of another point B on the circle such that AB = 4.

Note

Point of intersection of the tangent drawn to the circle $x^2 + y^2 = a^2$ at the point P (α) and Q (β) is

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Point of intersection of the tangent drawn to the circle $x^2 + y^2 = a^2$ at the point P (α) and Q (β) is

$P(\alpha) \equiv (a \cos\alpha, a \sin\alpha)$

$Q(\beta) \equiv (a \cos\beta, a \sin\beta)$

Note

Point of intersection of the tangent drawn to the circle $x^2 + y^2 = a^2$ at the point P (α) and Q (β) is

$P(\alpha) \equiv (a \cos\alpha, a \sin\alpha)$

$Q(\beta) \equiv (a \cos\beta, a \sin\beta)$

$$h = \frac{a\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}\ ;\ k = \frac{a\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}$$

Example

Q. Find the locus of the point of intersection of the pair of tangents drawn to a circle

$x^2+y^2 = a^2$ at P (α) and Q (β), where $|\alpha-\beta|=120°$.

Note

Equation of a chord line joining two points α and β on the $x^2 + y^2 = a^2$ is

Note

Equation of a chord line joining two points α and β on the $x^2 + y^2 = a^2$ is

$$x \cos\frac{\alpha+\beta}{2} + y \sin\frac{\alpha+\beta}{2} = a \cos\frac{\alpha-\beta}{2}$$

Example

Q.1 In a ΔABC the equation of line $BC \equiv x - y = 0$, $O \equiv (2,3)$, $H(5,8)$. Find equation of circumcircle

Length of Tangent & Power of a point.

Length of Tangent & Power of a point.

"Length of the tangent from an external point (x_1, y_1) to a given circle"

$$S = x^2 + y^2 + 2gx + 2fy + c = 0$$

Length of Tangent & Power of a point.

"Length of the tangent from an external point (x_1, y_1) to a given circle"

$S = x^2 + y^2 + 2gx + 2fy + c = 0$

$$L_1 = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c} = \sqrt{S_1}$$

Example

Q. Find length of tangent from (6,8) to circle $x^2 + y^2 = 25$

5 Important Deduction

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(i) Area of Quad PAOB = rL

5 Important Deduction

(ii) Area of $\Delta PAB = \dfrac{rL^3}{r^2 + L^2}$

5 Important Deduction

(iii) Length of chord of contact

$$AB = \frac{2rL}{\sqrt{r^2 + L^2}}$$

5 Important Deduction

(iv) Angle 2θ between the pair of Tangents

$$2\theta = \tan^{-1}\left(\frac{2rL}{L^2 - r^2}\right)$$

5 Important Deduction

(v) Equation of the circle circumscribing the ΔPAB. $(x - x_1)(x + g) + (y - y_1)(y + f) = 0$

Example

Q.1 Tangents PA & PB are drawn from P(4,3) to circle $x^2+y^2 = a^2$ Find

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(a) PA

Example

Q.1 Tangents PA & PB are drawn from P(4,3) to circle $x^2+y^2 = a^2$ Find

(b) area of quadrilateral PAOB

Example

Q.1 Tangents PA & PB are drawn from P(4,3) to circle $x^2+y^2 = a^2$ Find

(c) AB

Example

Q.1 Tangents PA & PB are drawn from P(4,3) to circle $x^2+y^2 = a^2$ Find

(d) area ΔPAB

Example

Q.1 Tangents PA & PB are drawn from P(4,3) to circle $x^2+y^2 = a^2$ Find

(e) $\angle APB$

Example

Q.1 Tangents PA & PB are drawn from P(4,3) to circle $x^2+y^2 = a^2$ Find

(f) equation of circumcircle of ΔPAB

Q.2 Find the length of the Tangent from any point on the circle $x^2+y^2 = 25$ to the circle $x^2+y^2 = 16$

Q.3 Find the range of ‘p’ for which the power of a point P(2,5) is negative w.r.t. a circle $x^2 + y^2 – 8x – 12y + p = 0$ and the circle neither touches nor intersects the coordinates axis.

Q.4 Find the locus of a point the tangents from which to the circles $4x^2 + 4y^2 - 9 = 0$ and $9x^2 + 9y^2 - 16 = 0$ are in the ratio 3 : 4.

Director Circle

Director Circle

Locus of intersection of two mutually perpendicular tangents

Director Circle

$$(x - \alpha)^2 + (y - \beta)^2 = 2r^2$$

Examples

Q. Find the range of 'a' such that the angle 'θ' between the pair of tangents drawn from the point (a, 0) to the circle $x^2 + y^2 = 4$ satisfies

$$\frac{\pi}{2} < \theta < \pi$$

Chord in Terms of Mid Point

Chord in Terms of Mid Point

$$T = S_1$$

Examples

Q.1 Find mid point of the chord $2x - 5y + 18 = 0$ of the circle $x^2 + y^2 - 6x + 2y - 54 = 0$

Q.2 Locus of the middle point of the chords of the circles $x^2 + y^2 + 2gx + 2fy + c = 0$ which passes through a fixed point (a, b) lying outside the circle.

Q.3 Find the equation to the locus of the middle point of the chord of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ which subtends right angle at a given point (a, b)

Q.4 Tangents are drawn to a unit circle with centre at origin from every point on the line $2x + y = 4$, prove that

Q.4 Tangents are drawn to a unit circle with centre at origin from every point on the line $2x + y = 4$, prove that

(i) chord of contact passes through a fixed point

Q.4 Tangents are drawn to a unit circle with centre at origin from every point on the line $2x + y = 4$, prove that

(ii) equation to the locus of the middle point of chord of contact.

Q.5 Chord of contact of the tangent drawn from a point on the circle $x^2 + y^2 = a^2$ to the circle $x^2 + y^2 = b^2$ touches the circle $x^2 + y^2 = c^2$. Prove that a, b, c are in G.P.

Q.6 If the chord of contact of tangents drawn from P to the circle $x^2 + y^2 = a^2$ subtends a right angle at the centre, find the locus of P.

Pair of Tangents

Pair of Tangents

$$SS_1 = T^2$$

Pair of Tangents

$$SS_1 = T^2$$

where $S \equiv x^2 + y^2 - a^2$; $S_1 \equiv x_1^2 + y_1^2 - a^2$.

Examples

Q.1 Show that the equation to the pair of tangents drawn from the origin to the circle

$x^2 + y^2 + 2gx + 2fy + c = 0$ is

$(gx + fy)^2 = c(x^2 + y^2)$

Q.2 Tangents are drawn to the circle $x^2 + y^2 = a^2$ from two points on the axis of x, equidistant from the point $(\alpha, 0)$. Show that the locus of their intersection is $\alpha y^2 = a^2(\alpha - x)$

Family of Circles

Type - 1

Equation of the family of circles which passes through the points of intersection of two circles $S_1 = 0$ and $S_2 = 0$ is

$$S_1 + \lambda S_2 = 0 \qquad ; \qquad \lambda \neq -1$$

Examples

Q.1 Find the equation of a circle which passes through the point of intersection of

$S_1 = 0$ and $S_2 = 0$

$S_1 \equiv x^2 + y^2 - 4x + 6y - 3 = 0$

$S_2 \equiv x^2 + y^2 + 4x - 6y - 12 = 0$

(i) Which passes through (0,0)

Examples

Q.1 Find the equation of a circle which passes through the point of intersection of

$S_1 = 0$ and $S_2 = 0$

$S_1 \equiv x^2 + y^2 - 4x + 6y - 3 = 0$

$S_2 \equiv x^2 + y^2 + 4x - 6y - 12 = 0$

(ii) Centre lies on x-axis

or centre lies on y-axis

Type - 2

Equation of the family of circles passes through the point of intersection of a circles $S = 0$ and a line $L = 0$ is given by $S + \lambda L = 0$

Modifying Type - 1 Using Type - 2

$$S_1 + \lambda\,(S_2 - S_1) = 0$$

Examples

Q.1 Find the equation of a circle drawn on the chord $x \cos \alpha + y \sin \alpha = p$ of the circle $x^2 + y^2 = a^2$ as its diameter.

Q.2 Show that the equation $x^2 + y^2 - 2x + \lambda y - 8 = 0$ represents for different values of λ, a system of circles passing through two fixed points A and B on the X-axis, and also find the equation of that circle of the system the tangent to which at A and B meet on the line $x + 2y + 5 = 0$

Q.3 Find the equation of a circle which passes through the point of contact of the tangents drawn from the origin to the circle

$x^2 + y^2 - 11x + 13y + 17 = 0$

Type - 3

Equation of the family of circles passes through two given points $A(x_1, y_1)$ & $B(x_2, y_2)$

Equation of Circle Passing Through $(x_1, y_1), (x_2, y_2)$ in Diametrical Form

$$S + \lambda L = 0$$

Equation of Circle Passing Through $(x_1, y_1), (x_2, y_2)$ in Diametrical Form

$$S + \lambda L = 0$$

$$S \equiv (x-x_1)(x-x_2) + (y-y_1)(y-y_2) = 0$$

Equation of Circle Passing Through $(x_1, y_1), (x_2, y_2)$ in Diametrical Form

$$S + \lambda L = 0$$

$$S \equiv (x-x_1)(x-x_2) + (y-y_1)(y-y_2) = 0$$

$l \equiv$ line in 2 point form

Example

Q. Find equation of circumcircle of Δ whose vertices are (1,0), (2,0), (3,1)

Type – 4 (Point Circle)

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Equation of family of circles touching a line at its fixed point (x_1, y_1) is

Type – 4 (Point Circle)

Equation of family of circles touching a line at its fixed point (x_1, y_1) is

$$(x-x_1)^2 + (y-y_1)^2 + \lambda L = 0$$

Examples

Q.1 Find the equation of a circle which touches the line $2x - y = 4$ at the point $(1, -2)$ and passes through $(3,4)$

Q.2 Find the equation of the circle which passes through the point (-1, 2) & touches the circle $x^2 + y^2 - 8x + 6y = 0$ at origin.

Q.3 Find equation of circle to when line $4x + 3y = 10$ is a common tangent at (1,2) and radius of each circle is 5.

Type - 5

Type - 5

Equation of a circle passing through points of intersection of lines l_1, l_2, $l_3 = 0$ / equation of circumcircle of ΔABC where equation of sides are given

Type - 5

Equation of a circle passing through points of intersection of lines l_1, l_2, $l_3 = 0$ / equation of circumcircle of ΔABC where equation of sides are given

$$l_1 l_2 + \lambda\, l_2 l_3 + \mu\, l_3 l_1 = 0$$

Note

To find λ & μ coefficient of x^2 = coefficient of y^2 & coefficient of $xy = 0$

Type - 6

Type - 6

Equation of a circle circumscribing a quadrilateral whose sides in order are represented by the line $l_1 = 0$; $l_2, = 0$; $l_3, = 0$; $l_4, = 0$ is given by

Type - 6

Equation of a circle circumscribing a quadrilateral whose sides in order are represented by the line $l_1 = 0$; $l_2, = 0$; $l_3, = 0$; $l_4, = 0$ is given by

$$l_1 l_3 + \lambda l_2 l_4 = 0$$

Note

To find λ coefficient of x^2 = coefficient of y^2 & coefficient of $xy = 0$

Common Tangents

Common Tangents

(1) Direct Common Tangent (DCT)
(External Common Tangent)

Common Tangents

(1) Direct Common Tangent (DCT)
(External Common Tangent)

(2) Transverse Common Tangent (TCT)
(Internal Common Tangent)

Direct Common Tangent

The centres of both the circles lie on the same side of the tangent line.

Transverse Common Tangent

The centres of both the circles lie on the opposite side of the tangent line.

Length of DCT/TCT

Length of DCT/TCT

$$L_{ext} = \sqrt{d^2 - \left(r_1 - r_2\right)^2}$$

Length of DCT/TCT

$$L_{ext} = \sqrt{d^2 - (r_1 - r_2)^2}$$

$$L_{int} = \sqrt{d^2 - (r_1 + r_2)^2}$$

Equation of DCT/TCT

Position of Circles
(And Number of Common Tangents)

(1) If 2 circles are separated, then $d > r_1 + r_2$

(1) If 2 circles are separated, then $d > r_1 + r_2$

4 common tangent
- 2 D.C.T.
- 2 T.C.T.

(2) If 2 circles touch externally then $d = r_1 + r_2$

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3 common tangents — 2 D.C.T.

3 common tangents — 1 T.C.T.

(3) If 2 circles touches internally then $d = | r_1 - r_2 |$

(3) If 2 circles touches internally then $d = |r_1 - r_2|$

one common tangent (1 D.C.T.)

(4) If 2 circles intersect each other then

$| r_2 – r_1 | < d < r_1 - r_2$

(4) If 2 circles intersect each other then

$| r_2 - r_1 | < d < r_1 - r_2$

2 common tangent (2 D.C.T.)

(5) If $d < | r_1 - r_2 |$ no tangent

Example

Q.1 Find the range of 'r' so that the circles :

$(x - 1)^2 + (y - 3)^2 = r^2$ and

$(x - 4)^2 + (y - 1)^2 = 9$

intersects at 2 distinct points

Q.2 Find common tangent to the circles
$x^2 + y^2 = 1$ and $(x - 1)^2 + (y - 3)^2 = 4$

Q.3 Find the equation of the circles to which the line $4x + 3y = 10$ is a common tangent at (1,2) and radius of each of the circle is 5.

Radical Axis

Radical Axis

Radical axis of 2 circles is the locus of a point whose powers w.r.t. the two circles are equal.

Radical Axis

The equation of radical axis of two circles $S_1 = 0$ and $S_2 = 0$ is given by

Radical Axis

The equation of radical axis of two circles $S_1 = 0$ and $S_2 = 0$ is given by

$$S_1 - S_2 = 0$$

Note that

(a) If two circles intersect, then the radical axis is the common chord of the two circles.

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(b) If two circles touch each other then the radical axis is the common tangent of the two circles at the common point of contact.

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(a) If two circles intersect, then the radical axis is the common chord of the two circles.

(b) If two circles touch each other then the radical axis is the common tangent of the two circles at the common point of contact.

(c) Radical axis is always perpendicular to the line joining the centres of the two circles.

Note that

(d) Radical axis need not always pass through the mid point of the line joining the centres of the two circles.

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(e) Radical axis bisects a common tangent between the two circles.

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(e) Radical axis bisects a common tangent between the two circles.

(f) If one circle is contained in another circle when radical axis passes outside to both the circles.

Examples

Q.1 Show that the equation of a straight line meeting the circle $x^2 + y^2 = a^2$ in 2 points at equal distance 'd' from the point (x_1, y_1) on its circumference is

$$xx_1 + yy_1 - a^2 + \frac{d^2}{2} = 0$$

Q.2 Prove that the circle $x^2+y^2+2gx+2fy+c = 0$ will bisect the circumference of the circle $x^2 + y^2 + 2g'x + 2f'y + c' = 0$ if $2g' (g - g') + 2f' (f - f') = c - c'$.

Q.3 Tangent are drawn to the circle $x^2 + y^2 = 12$ at the points where it is met by the circle $x^2 + y^2 - 5x + 3y - 2 = 0$.

Find the point of intersection of the tangents.

Q.4 Find the equation of a circle which bisects the circumferences of the circles

$x^2 + y^2 = 1$, $x^2 + y^2 + 2x = 3$ and

$x^2 + y^2 + 2y = 3$.

Q.5 Find the locus of the centre of circles which bisect the circumference of the circles $x^2 + y^2 = 4$ and $x^2 + y^2 - 2x + 6y + 1 = 0$.

Q.6 Find the equation of the circle which bisects the circumference of the circle $x^2 + y^2 + 2y - 3 = 0$ and touches the line $x - y = 0$ at origin.

Radical Centre

Point of intersection of the radical axis of 3 circles taken 2 at a time is called the ***Radical Centre***

Note

Radical axis taken 2 at a time will be concurred at a point.

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Radical axis taken 2 at a time will be concurred at a point.

Radical centre of three circles described on sides of a Δ as diameter is orthocenter of the Δ

Coaxial System of Circles

Definition : A system of circles, every 2 of which have the same radical axis, is called Coaxial system of circles.

Example

Q.1 Find the equation of the circle passes through (1,1) belonging to the system of coaxial circles which touches $x^2 + y^2 = 8$ at (2, 2)

Q.2 From a point P tangents drawn to the circles $x^2 + y^2 + x - 3 = 0$; $3x^2 + 3y^2 - 5x + 3y = 0$ and $4x^2 + 4y^2 + 8x + 7y + 9 = 0$ are of equal length. Find the equation of the circle passes through P and which touches the line $x+y = 5$ at $(6, -1)$

Orthogonality of Two Circles

Orthogonality of Two Circles

Two curves are said to be orthogonal if angle between them is 90° at point of intersection

Angle Between Two Curve

Note

Line is tangent to itself

Condition for Orthogonality of 2 Circles

$$2\, g_1\, g_2 + 2\, f_1\, f_2 = c_1 + c_2$$

Examples

Q.1 The circle $x^2 + y^2 + 2x + 4y + 1 = 0$ & line $x = 0$, $y = 0$ Orthogonally. Find the equation of circle

Q.2 Prove that locus of the centre of a variable circle $x^2 + y^2 + 2gx + 2fy + c = 0$ which cuts the 2 given circles $x^2 + y^2 + 2g_1x + 2f_1y = 0$ and $x^2 + y^2 + 2g_2x + 2f_2y + c = 0$ orthogonally is the radical axis of 2 given circles.

Q.3 If the circles, $S_1 : x^2 + y^2 + 2x + 2ky + 6 = 0$ and $S_2 : x^2 + y^2 + 2kx + k = 0$ intersects orthogonally then find k.

S.L. Loney
Assignment – 3

Write down the equation of the tangent to the circle.

Q.1 $x^2 + y^2 - 3x + 10y = 15$ at the point $(4, 11)$

Q.2 $4x^2 + 4y^2 - 10x + 24y = 117$ at the point $\left(-4, -\frac{11}{2}\right)$

Find the equations to the tangents to the circle

Q.3 $x^2 + y^2 = 4$ which are parallel to the line

$x + 2y + 3 = 0$

Q.4 $x^2 + y^2 + 2gx + 2fy + c = 0$ which are parallel to

the lines $x + 2y - 6 = 0$

Q.5 Prove that the straight line $y = x + \sqrt{2}$ touches the circle $x^2 + y^2 = c^2$ and find its point of contact.

Q.6 Find the condition that the straight line $cx - by + b^2 = 0$ may touch the circle $x^2 + y^2 = ax + by$ and find the point of correct.

Q.7 Find whether the straight line $x + y = 2 + \sqrt{2}$ touches the circle $x^2 + y^2 - 2x - 2y + 1 = 0$

Q.8 Find the condition that the straight line

$3x + 4y = k$ may touch the circle $x^2 + y^2 = 10x$.

Q.9 Find the value of p so that the straight line,

$x \cos \alpha + y \sin \alpha - p = 0$

may touch the circle

$x^2 + y^2 - 2ax \cos \alpha - 2by \sin \alpha - a^2 \sin^2 \alpha = 0$

Q.10 Find the condition that the straight line

$Ax + By + C = 0$ may touch the circle

$(x - a)^2 + (y - b)^2 = c^2$

Q.11 Find the equation to the tangent to the circle

$x^2 + y^2 = a^2$ which,

(i) is parallel to the straight line $y = mx + c$,

(ii) is perpendicular to the straight line

$y = mx + c$

(iii) passes through the point $(b, 0)$,

and (iv) makes with the axes a triangle whose area is a^2.

Q.12 Find the length of the chord joining the points in which the straight line,

$$\frac{x}{a} + \frac{y}{b} = 1$$

Meets the circle, $x^2 + y^2 = r^2$

Q.13 Find the equation to the circles which pass through the origin and cut off equal chords a from the straight lines $y = x$ and $y = -x$.

Q.14 Find the equation of the straight lines joining the origin to the points in which the straight line $y = mx + c$ cuts the circle,

$x^2 + y^2 = 2ax + 2by$

Hence, find the condition that these points may be subtend a right angle at the origin.

Find also the condition that the straight line may touch the circle.

Find the equation to the circle which :

Q.15 Has its centre at the point (3, 4) and touches the straight line, $5x + 12y = 1$

Q.16 Touches the axes of coordinates and also the line $\frac{x}{a} + \frac{y}{b} = 1$ the centre being in the positive quadrant.

Q.17 Has its centre at the point (1, -3) and touches the straight line, $2x - y - 4 = 0$

Q.18 Find the general equation of a circle referred to two perpendicular tangents as axes.

Q.19 Find the equation to a circle of radius r which touches the axis of y at a point distant h from the origin, the centre of the circle being in the positive quadrant.

Prove also that the equation to the other tangent which passes through the origin is,

$$(r^2 - h^2)x + 2rhy = 0$$

Q.20 Find the equation to the circle whose centre is at the point (α, β) and which passes through the origin, and prove that the equation of the tangent at the origin is $a\alpha + \beta y = 0$

Q.21 A circle passes through the points $(-1, 1)$, $(0, 6)$ and $(5, 5)$. Find the points on this circle the tangents at which are parallel to the straight line joining the origin to its centre.