Complex Numbers

General Introduction :

Complete development of the number system can be summarised as

$N \subset W \subset I \subset Q \subset R \subset Z$

Every complex number z can be written as $z = x + i\,y$ where $x, y \in R$ and $i = \sqrt{-1}$. x is called the real part of z and y is the imaginary part of complex.

Note :

$\sqrt{a}\sqrt{b} = \sqrt{ab}$ only if atleast one of either a or b is non negative.

(If a and b are positive reals then $\sqrt{-a}\sqrt{-b} = -\sqrt{ab}$]

Complex Plane

$$z = a + ib$$

Purely real	Purely imaginary	imaginary
If $b = 0$	if $a = 0$	if $b \neq 0$

Hence 0 + 0i is both a purely real as well as purely imaginary but not imaginary.

Note :

(i) The symbol i combines itself and with real number as per the rule of algebra together with $i^2 = -1$; $i^3 = -i$; $i^4 = 1$, $i^{2005} = i$, ; $i^{2006} = -1$. Infact $i^{4n} = 1$, $n \in I$ note that $(1 + i + i^2 + \dots + i^{2006} = i)$]

(ii) Every real number can also be treated as complex with its imaginary part zero. Hence there is one-one imapping between the set of complex numbers and the set of points are the complex plane.

Algebra of Complex

Addition, substraction and multiplication of complex numbers are carried out like in ordinary algebra using $i^2 = -1$, $i^3 = -i$ etc. treating i as a polynomial.

Differences between algebra of complex and algebra of real number are.

(i) Inequality in complex numbers are never talked. If $a + i b > c + id$ has to be meaningful $\Rightarrow b = d = 0$. Equalities however in complex numbers are meaningful. Two complex numbers z_1 and z_2 are said to be equal if

$\text{Re } z_1 = \text{Re } z_2$ and $\text{Im } (z_1) = \text{Im } (z_2)$

(i.e. they occupy the same position on complex plane)

(ii) In real number system if $a^2 + b^2 = 0 \Rightarrow a = 0 = b$ but if z_1 and z_2 are complex numbers then $z_1^2 + z_2^2 = 0$ does not imply $z_1 = z_2 = 0$ e.g. $z_1 = 1 + i$ and $z_2 = 1 - i$.

However if the product of two complex numbers is zero then at least one of them must be zero, same as in case of real numbers.

(iii) In case x is real then

$$|x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$$ but in case of complex.

$|z|$ altogether has a different meaning.

Conjugate

If $z = a + ib$ then its conjugate complex is obtained by changing the sign of its imaginary part and denoted by $\bar{z}$ i.e. $\bar{z} = a - ib$.

Note that :

(i) $z + \bar{z} = 2\text{Re } z$

(ii) $z - \bar{z} = 2i \text{ Im } z$

(iii) $z\bar{z} = a^2 + b^2$

(iv) If z lies in I^{st} quadrant then $\bar{z}$ lies in 4^{th} quadrant and $-z$ in the 2^{nd} quad.

If $z, z_1, z_2 \in C$ then ;

(a) $\overline{(\bar{z})} = z$;

$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$; $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$; $\overline{z_1 z_2} = \bar{z}_1 . \bar{z}_2$

$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}; z_2 \neq 0$$

Modulus :

If $z = x + iy$ then $|z| = \sqrt{x^2 + y^2}$

Note that $|z| \geq 0$.

Note :

All complex number having the same modulus lie on a circle with centre as origin and $r = | z |$.

(b) $|z| \ge 0$; $|z| \ge \text{Re}(z)$; $|z| \ge \text{Im}(z)$; $|z| = |\bar{z}| = |-z|$; $z\bar{z} = |z|^2$; if $|\bar{z}| = 1$ then $\bar{z} = \frac{1}{z}$

$|z_1 z_2| = |z_1| \cdot |z_2|$; $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, $z_2 \ne 0$,

$|z^n| = |z|^n$;

Argument :

If OP makes an angle θ with real axis then θ is called one of the argument of z.

Note :

By specifying the modulus and argument, a complex number is completely defined. However for the complex number 0 + 0i the argument is not defined and this is the only complex number which is completely defined by talking in terms of its modulus.

Amplitude
(Principal Value of Argument)

The unique value of θ such that $-\pi < \theta \leq \pi$ is called principal value of argument. Unless otherwise stated, amp z refers to the principal value of argument.

Q. Among the complex number z which satisfy $| z - 25 i | = 15$, find the one having the least +ve argument.

Q. $i^n + i^{n+1} + i^{n+2} + i^{n+3} = ?\ n \in N$

Q. If $\frac{(x-2)+(y-3)i}{1+i} = 1 - 3i$, Find (x, y).

Q. If $z = (x, y) \in C$ then find z, satisfying $z^2 (1, 1) = (-1, 7)$.

Q. If $z^2 + 2(1 + 2i)\, z - (11 + 2i) = 0$. find z in the form of $a + ib$.

Q. If $f(x) = x^4 - 4x^3 + 4x^2 + 8x + 44$, find $f(3 + 2i)$.

Q. If $\operatorname{Arg} z = \frac{\pi}{4}$ and $|z + 3 - i| = 4$, find z.

Q. If $| z - i | = 1$ and $\text{Arg } z = \frac{\pi}{2}$, find z **?**

Q. If $z = \dfrac{\sqrt{9+40i}+\sqrt{9-40i}}{\sqrt{9+40i}-\sqrt{9-40i}}$, find $|z|$ and amp z.

Q. Compute (a) $\sqrt{-4}\sqrt{-\frac{9}{4}}$

Q. $\sqrt{-25}+3\sqrt{-4}+2\sqrt{-9}$

Q. Find the least positive $n \in N$ if $\left(\frac{1+i}{1-i}\right)^n = 1$

Representation of a complex in different forms

(i) Cartesian form / Algebric form :

$z = x + iy$; Here $|z| = \sqrt{x^2 + y^2}$; $\bar{z} = x - iy$ $\theta = \tan^{-1}\frac{y}{x}$

Note :

Generally this form is used in locus problems or while solving equations.

Find locus of point in complex plane

Q. $\mathrm{Re}\left(\frac{1}{z}\right) = \frac{1}{2}$

Q. Find the set of points on the complex plane for which $z^2 + z + 1$ is real and positive.

Q. Show that the locus of the point $P(\omega)$ denoting the complex number $z + \frac{1}{z}$ on the complex plane is a standard ellipse where $|z| = 2$.

Polar form

(ii) Trigonometric form / Polar form :

$z = x + iy = r(\cos\theta + i\sin\theta) = r\,\text{CiS}\,\theta$ where

$|z| = r$; $\text{amp}\, z = \theta$

Note :

$$(\text{CiS}\ \alpha)\ (\text{CiS}\ \beta) = \text{CiS}\ (\alpha + \beta)$$

$$(\text{CiS}\ \alpha)\ (\text{CiS}\ (-\beta) = \text{CiS}\ (\alpha - \beta)$$

$$\frac{1}{(\text{CiS})\alpha} = (\text{CiS}\ \alpha)^{-1} = \text{CiS}\ (-\alpha)$$

Q. If $z = 1 + \cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}$ find r and amp z.

Q. Find $|z|$ & amp (z) if $z = \dfrac{1+i\sqrt{3}}{2i\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)}$;

Exponential form :

$e^{ix} = \cos x + i \sin x$

$z = re^{i\theta}$ is the exponential representation.

Note :

(a) $\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$ and $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$

are known as Eulers identities.

(b) $\cos ix = \dfrac{e^{ix} + e^{-ix}}{2} = \cos hx$ is always positive

real $\forall\ x \in R$ and is ≥ 1. note that $f(x) = \cos ix$.

Q. $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\,[|z_1|^2 + |z_2|^2]$ Give proof and its geometrical interpretation.

Triangle Inequality

$$||z_1| - |z_2|| \le |z_1 + z_2| \le |z_1| + |z_2|$$

Note :

(1) amp $(z_1 . z_2)$ = amp z_1 + amp z_2 + $2k\pi$, $k \in I$

(2) amp $\left(\frac{z_1}{z_2}\right)$ = amp z_1 – amp z_2 + $2k\pi$, $k \in I$

(3) amp (z^n) = n amp (z) + $2k\pi$.
where proper value of k must be chosen so that RHS lies in $(-\pi, \pi]$.

Q. Show that amp z + amp of $(-\bar{z}) = \pi$

Q. $z=\frac{(2\sqrt{3}+2i)^8}{(1-i)^6}+\frac{(1+i)^6}{(2\sqrt{3}-2i)^8}$ Then amp (z) is :

(A) $-\pi/3$

(B) $5\pi/6$

(C) $-2\pi/3$

(D) $5\pi/12$

Q. Let z be a complex number $\frac{1+z+z^2}{1-z+z^2} \in R$ then prove that $|z| = 1$.

Q. Let $z_1, z_2, z_3, \dots z_n$ are the complex numbers such that $|z_1| = |z_2| = \dots = |z_n| = 1$.

If $z = \left(\sum_{k=1}^{n} z_k\right)\left(\sum_{k=1}^{n} \frac{1}{z_k}\right)$ then prove that

(i) z is a real number

(ii) $|z| \le n^2$

Q. Find the greatest and least values of $|z|$ if z satisfies $\left|z-\frac{4}{z}\right|=2$.

Q. Find z satisfying simultaneously $\left|\frac{z-12}{z-8i}\right| = \frac{5}{3}$ and $\left|\frac{z-4}{z-8}\right| = 1$

n^{th} Roots of Unity :

Complex cube roots of unity

Vectorial Representation of A Complex

Every complex number can be considered as if it is the position vector of that point. If the point P represents the complex number z then,

$$\vec{op} = z \;\&\; |\vec{op}| = |z|$$

Geometrical meaning of $e^{i\theta}$

What does $z_1 z_2$ "means" (Rotation)

Q. Section formula, centroid, incentre, orthocentre and circumcentre for a triangle whose vertices are z_1, z_2, z_3.

Examples on Vectorial Representation & Rotation Of A Vector

Q. If z_1, z_2, z_3 are the vertices of an isosceles traingle right angled at z_2 then prove that

$$z_1^2 + 2z_2^2 + z_3^2 = 2z_2(z_1 + z_3)$$

Q. If z_1, z_2, z_3 are the vertices of an equilateral triangle then prove that

$$z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1$$

and if z_0 is its circumcentre then $3z_0^2 = z_1^2 + z_2^2 + z_3^2$

Q. If z_r(r = 1, 2, ...6) are the vertices of a regular hexagon then $\sum_{r=1}^{6} z_r^2 = 6z_0^2$ where z_0 is the circumcentre.

Q. Prove that the triangle whose vertices are the points z_1, z_2, z_3 on the Argand plane is an equilateral triangle if and only if

$$\frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} + \frac{1}{z_1 - z_2} = 0$$

Q. Let z_1 and z_2 be roots of the equation $z^2 + pz + q = 0$, where the coefficients p and q may be complex numbers. Let A and B represent z_1 and z_2 in the complex plane. If $\angle AOB = \alpha \neq 0$ and $OA = OB$, where O is the origin, prove that $p^2 = 4q \cos^2\alpha/2$.

Q. On the Argand plane z_1, z_2 and z_3 are respectively the vertices of an isosceles triangle ABC with AC = BC and equal angles are θ. If z_4 is the incentre of the triangle then prove that $(z_2 - z_1)(z_3 - z_1) = (1 + \sec\theta)(z_4 - z_1)^2$

Q. Interpret locus of z

$$|z - (1 + 2i)| = 3$$

Q. $|z-1| = |z-i|$

Q. $|z - 4i| + |z + 4i| = 10$

Q. $|z-1|+|z+1|=1$

Q. $1 \le |z - 1| < 3$

Q. $0 \le \text{Arg } Z \le \frac{\pi}{4}$

Q. $\mathrm{Re}\,(z^2) = 0$

Q. True or False :

If z_1, z_2, z_3, z_4 in order are the vertices of the square taken in order then

(i) $\dfrac{z_1 - z_3}{z_2 - z_4}$ is purely imaginary

(ii) $\dfrac{z_1 - z_2}{z_3 - z_4}$ is purely real

(iii) $\dfrac{z_4 - z_3}{z_2 - z_3}$ is purely imaginary

(iv) $\dfrac{z_2 + z_4}{z_1 + z_3} = 1$

Q. Let z_1, z_2, z_3 are the vertices of a triangle with origin as the circumcentre. If z is the orthocentre then $z = z_1 + z_2 + z_3$. (T/F)

Q. If z_1, z_2, z_3 are the vertices of a triangle such that $|z_1 - 1| = |z_2 - 1| = |z_3 - 1|$ and $z_1 + z_2 + z_3 = 3$ then the triangle is an equilateral triangle. (T/F)

Q. If the area of the triangle formed by z, iz and $z + iz$ is 8 sq. units then find $|z|$.

Q. If z_1, z_2, z_3 are the vertices of an equilateral triangle with circumcentre at $(1 - 2i)$. Find z_2 and z_3 if $z_1 = 1 + i$

Demoivre's Theorem : (D M T)

Statement :

cos nθ + i sin nθ is the value or one of the values of $(\cos\theta + i\sin\theta)^n$ $\forall\ n \in Q$. One Value if n is an integer, one of the values if n is rational which is not an integer. The theorem is very useful in determining the roots of any complex quantity.

Basic steps to determine the roots of a complex number

(a) Write the complex number whose roots are to be determined in polar form.

(b) Add $2m\pi$ to the argument

(c) Apply D M T

(d) Put m = 0, 1, 2, 3, …. (n – 1) to get all the n roots. You can also put m = 1, 2, 3, …. N

Application of DMT

to determine n^{th} roots of unity

Q. Find z if $z = \left(1 + i\sqrt{3}\right)^{1/4}$

Q. $2\sqrt{2}z^4 = \left(\sqrt{3}-1\right)+i\left(\sqrt{3}+1\right)$

Q. Find the roots of the equation

$$z^5 + z^4 + z^3 + z^2 + z + 1 = 0$$

Q. $z^4 - z^3 + z^2 - z + 1 = 0$

Q. Find the number of roots of the equation $z^{10} - z^5 - 992 = 0$ with real part –ve.

Q. Prove that $\tan^{-1}\dfrac{1}{5} \approx \dfrac{\pi}{16}$

[Hint : Let $z = 5 + i$]

Q. The following factorisation should be remembered :

(i) $x^2 + x + 1 = (x - \omega)(x - \omega^2)$

Q. If the area of the triangle in the Argand diagram, formed by Z, ωZ and $Z + \omega Z$ where ω is the usual complex cube root of unity is $16\sqrt{3}$ square units, then $|Z|$ is

(A) 16 (B) 4

(C) 8 (D) 3

Q. If $(a + w)^{-1} + (b + w)^{-1} + (c + w)^{-1} + (d + w)^{-1} = 2\,w^{-1}$ and $(a + w^2)^{-1} + (b + w^2)^{-1} + (c + w^2)^{-1} + (d + w^2)^{-1} = 2\,w^{-2}$ where w is the complex cube root of unity then show that :

(i) $\Sigma\, abc = 2$ & (ii) $\Sigma\, a = 2\, \Pi\, a$

Hence show that $(a + 1)^{-1} + (b + 1)^{-1} + (c + 1)^{-1} + (d + 1)^{-1} = 2$, $a, b, c, d, \in R$.

n^{th} Roots of Unity :

If 1, α_1, α_2, α_3, ……. α_{n-1}, are the n, n^{th} root of unity then :

They are in G.P. with common ratio

$$e^{i(2\pi/n)} = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$$

Q. Show that $1^P + \alpha_1^P + \alpha_2^P + \ldots + \alpha_{n-1}^P = 0$ if p is not an integral multiple of n

Q. Show that $1^P + (\alpha_1)^P + (\alpha_2)^P + \ldots + (\alpha_{n-1})^P = n$ if p is an integral multiple of n

Q. $(1 - \alpha_1)(1 - \alpha_2) \ldots (1 - \alpha_{n-1}) = n$

Q. $(1 + \alpha_1)(1 + \alpha_2) \ldots (1 + \alpha_{n-1}) = 0$ if n is even and 1 if n is odd.

Q. 1. $\alpha_1 . \alpha_2 . \alpha_3 \ldots\ldots\ldots\alpha_{n-1} = 1$ or -1 according as n is odd or even.

Q. $(w - \alpha_1)(w - \alpha_2) \ldots\ldots (w - \alpha_{n-1})$

$$= \begin{cases} 0 & \text{if } n = 3k \\ 1 & \text{if } n = 3k+1 \\ 1+w & \text{if } n = 3k+2 \end{cases}$$

Q. Sum of all the n, n^{th} roots always vanishes.

Q. $\sum_{\lambda=1}^{12}\left(\sin\frac{2\lambda\pi}{13}-i\cos\frac{2\lambda\pi}{13}\right)=$

(A) 1 (B) –1

(C) i (D) –i

Q. If $\cos(\alpha - \beta) + \cos(\beta - \gamma) + \cos(\gamma - \alpha) = -3/2$ then prove that :

(a) $\Sigma \cos 2\alpha = 0 = \Sigma \sin 2\alpha$

(b) $\Sigma \sin(\alpha + \beta) = 0 = \Sigma \cos(\alpha + \beta)$

(c) $\Sigma \sin 2\alpha = \Sigma \cos 2\alpha = 3/2$

(d) $\Sigma \sin 3\alpha = 3 \sin(\alpha + \beta + \gamma)$

(e) $\Sigma \cos 3\alpha = 3 \cos(\alpha + \beta + \gamma)$

(f) $\cos^3(\theta + \alpha) + \cos^3(\theta + \beta) + \cos^3(\theta + \gamma)$
$= 3 \cos(\theta + \alpha) \cdot \cos(\theta + \beta) \cdot \cos(\theta + \gamma)$
where $\theta \in R$.

Q. Prove that all roots of the equation $\left(\frac{z+1}{z}\right)^n = 1$ are collinear on the complex plane & lie on $x = -1/2$.

Q. If z_r, $r = 1, 2, 3, \ldots\ldots\ldots 2m$, $m \in N$ are the roots of the equation

$Z^{2m} + Z^{2m-1} + Z^{2m-2} + \ldots\ldots + Z + 1 = 0$ then

prove that $\displaystyle\sum_{r=1}^{2m} \frac{1}{z_r - 1} = -m$

Complex numbers and binomial coefficients

(i) $C_0 + C_5 + C_8 + \ldots..$

(ii) $C_1 + C_5 + C_9 + \ldots..$

(iii) $C_2 + C_6 + C_{10} + \ldots..$

(iv) $C_3 + C_7 + C_{11} + \ldots..$

(v) $C_0 + C_3 + C_6 + C_9 + \ldots..$

Straight lines & Circles on Complex Plane

(i) Equation of a line passing through z_1 & z_2 on argand plane.

$z = z_1 + \lambda(z_2 - z_1)$ (see vector equation of line)

(ii) Circle $| z - z_0 | = r$

Q. Find the area bounded by the curves Arg $z = \frac{\pi}{3}$. Arg $z = \frac{2\pi}{3}$ & Arg $(z - 2 - 2\sqrt{3}i) = \pi$ on the complex plane.

Q. Find all the points in the complex plane which satisfy the equations

$\log_5 (|z| + 3) - \log_{\sqrt{5}} ||z| - 1| = 1$ and

$\arg(z - 1 - i) = \dfrac{\pi}{4}$

Parametric Equation Of A Line

$\vec{OP} = \vec{OA} + \vec{AP}$

$z = z_1 + \lambda (z_2 - z_1)$

where $\lambda \in$ R which is the same as equation number 3 i.e. $\dfrac{z - z_1}{z_2 - z_1}$ is purely real.

Reflection Points For A Line (Image of a point in a line)

Use concept of straight line. Write $z = x + iy$

Equation of a circle described on the line joining z_1 & z_2 as diameter

Note that the equation

$$\text{Arg}\,(z + i) - \text{Arg}\,(z - i) = \frac{\pi}{2}$$

Does not represent a complete circle but only a semi circle described on the line segment joining (0, 1) & (0, –1) as diameter. (in I^{st} and 4^{th} quadrant)

General locii on complex plane

(a) $| z - z_1 | + | z - z_2 | =$ constant (constant $> | z_1 - z_2 |$) is an ellipse with its two foci at z_1 and z_2

(b) $| z - z_1 | - | z - z_2 | =$ constant (constant $< | z_1 - z_2 |$) is a hyperbola with its foci as z_1 and z_2.

(c) $| z - z_1 |^2 + | z - z_2 |^2 = | z_1 - z_2 |^2$ represent locus of a circle with z_1 and z_2 as its diameter

(d) $(z - \bar{z})^2 + 8a(z + \bar{z}) = 0$ represent a standard equation of parabola

(e) $| z - z_1 | + | z - z_2 | = | z_1 - z_2 |$ represent a line segment.