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# Complex Numbers

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# *KEY CONCEPTS*

**1. DEFINITION :**

Complex numbers are definited as expressions of the form a + ib where a, b $\in$ R & i = $\sqrt{-1}$ . It is denoted by z i.e. z = a + ib. 'a' is called as real part of z (Re z) and 'b' is called as imaginary part of z (Im z).

**EVERY COMPLEX NUMBER CAN BE REGARDED AS**

| Purely real | Purely imaginary | Imaginary |
|---|---|---|
| if b = 0 | if a = 0 | if b $\neq$ 0 |

**Note :**

**(a)** The set R of real numbers is a proper subset of the Complex Numbers. Hence the Complete Number system is $N \subset W \subset I \subset Q \subset R \subset C$.

**(b)** Zero is both purely real as well as purely imaginary but not imaginary.

**(c)** i = $\sqrt{-1}$ is called the imaginary unit. Also $i^2 = -1$ ; $i^3 = -i$ ; $i^4 = 1$ etc.

**(d)** $\sqrt{a}\,\sqrt{b} = \sqrt{a\,b}$ only if atleast one of either a or b is non-negative.

**2. CONJUGATE COMPLEX :**

If z = a + ib then its conjugate complex is obtained by changing the sign of its imaginary part & is denoted by $\bar{z}$ . i.e. $\bar{z}$ = a − ib.

**Note that :**

**(i)** $z + \bar{z} = 2\,\text{Re}(z)$ **(ii)** $z - \bar{z} = 2i\,\text{Im}(z)$ **(iii)** $z\bar{z} = a^2 + b^2$ which is real

**(iv)** If z lies in the 1st quadrant then $\bar{z}$ lies in the 4th quadrant and $-\bar{z}$ lies in the 2nd quadrant.

**3. ALGEBRAIC OPERATIONS :**

The algebraic operations on complex numbers are similiar to those on real numbers treating *i* as a polynomial. Inequalities in complex numbers are not defined. There is no validity if we say that complex number is positive or negative.

e.g. z > 0, 4 + 2i < 2 + 4 i are meaningless .

However in real numbers if $a^2 + b^2 = 0$ then a = 0 = b but in complex numbers,

$z_1^2 + z_2^2 = 0$ does not imply $z_1 = z_2 = 0$.

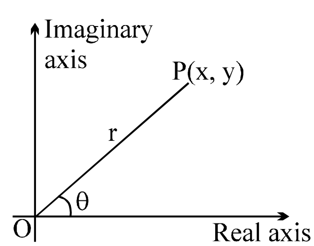
**4. EQUALITY IN COMPLEX NUMBER :**

Two complex numbers $z_1 = a_1 + ib_1$ & $z_2 = a_2 + ib_2$ are equal if and only if their real & imaginary parts coincide.

**5. REPRESENTATION OF A COMPLEX NUMBER IN VARIOUS FORMS :**

**(a) Cartesian Form (Geometric Representation) :**

Every complex number z = x + i y can be represented by a point on the cartesian plane known as complex plane (Argand diagram) by the ordered pair (x, y).




length OP is called modulus of the complex number denoted by $|z|$ & $\theta$ is called the argument or amplitude .

eg. $|z| = \sqrt{x^2 + y^2}$ &

$\theta = \tan^{-1}\dfrac{y}{x}$ (angle made by OP with positive x−axis)

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**NOTE :**

**(i)** $|z|$ is always non negative . Unlike real numbers $|z| = \begin{bmatrix} z & \text{if } z > 0 \\ -z & \text{if } z < 0 \end{bmatrix}$ is **not correct**

**(ii)** Argument of a complex number is a many valued function . If $\theta$ is the argument of a complex number then $2n\pi + \theta$ ; $n \in I$ will also be the argument of that complex number. Any two arguments of a complex number differ by $2n\pi$.

**(iii)** The unique value of $\theta$ such that $-\pi < \theta \le \pi$ is called the principal value of the argument.

**(iv)** Unless otherwise stated, amp z implies principal value of the argument.

**(v)** By specifying the modulus & argument a complex number is defined completely. For the complex number $0 + 0i$ the argument is not defined and this is the only complex number which is given by its modulus.

**(vi)** There exists a one-one correspondence between the points of the plane and the members of the set of complex numbers.

**(b) Trignometric / Polar Representation :**

$z = r(\cos\theta + i\sin\theta)$ where $|z| = r$ ; $\arg z = \theta$ ; $\bar{z} = r(\cos\theta - i\sin\theta)$

**Note:** $\cos\theta + i\sin\theta$ is also written as $\mathrm{CiS}\,\theta$.

Also $\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$ & $\sin x = \dfrac{e^{ix} - e^{-ix}}{2}$ are known as Euler's identities.

**(c) Exponential Representation :**

$z = re^{i\theta}$ ; $|z| = r$ ; $\arg z = \theta$ ; $\bar{z} = re^{-i\theta}$

**6. IMPORTANT PROPERTIES OF CONJUGATE / MODULI / AMPLITUDE :**

If $z, z_1, z_2 \in C$ then ;

**(a)** $z + \bar{z} = 2\,\mathrm{Re}(z)$ ; $z - \bar{z} = 2i\,\mathrm{Im}(z)$ ; $\overline{(\bar{z})} = z$ ; $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$ ;

$\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$ ; $\overline{z_1 z_2} = \bar{z}_1 \cdot \bar{z}_2$ $\quad \overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}$ ; $z_2 \ne 0$

**(b)** $|z| \ge 0$ ; $|z| \ge \mathrm{Re}(z)$ ; $|z| \ge \mathrm{Im}(z)$ ; $|z| = |\bar{z}| = |-z|$ ; $z\bar{z} = |z|^2$ ;

$|z_1 z_2| = |z_1| \cdot |z_2|$ ; $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$, $z_2 \ne 0$, $|z^n| = |z|^n$ ;

$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\left[|z_1|^2 + |z_2|^2\right]$

$\big||z_1| - |z_2|\big| \le |z_1 + z_2| \le |z_1| + |z_2|$ **[ TRIANGLE INEQUALITY ]**

**(c)** **(i)** $\mathrm{amp}(z_1 \cdot z_2) = \mathrm{amp}\, z_1 + \mathrm{amp}\, z_2 + 2k\pi$. $k \in I$

**(ii)** $\mathrm{amp}\left(\dfrac{z_1}{z_2}\right) = \mathrm{amp}\, z_1 - \mathrm{amp}\, z_2 + 2k\pi$ ; $k \in I$

**(iii)** $\mathrm{amp}(z^n) = n\,\mathrm{amp}(z) + 2k\pi$ .
where proper value of k must be chosen so that RHS lies in $(-\pi, \pi]$.

**(7) VECTORIAL REPRESENTATION OF A COMPLEX :**

Every complex number can be considered as if it is the position vector of that point. If the point P represents the complex number z then, $\vec{OP} = z$ & $|\vec{OP}| = |z|$.

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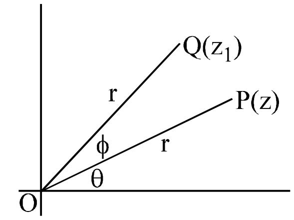
**NOTE :**




**(i)** If $\vec{OP} = z = r e^{i\theta}$ then $\vec{OQ} = z_1 = r e^{i(\theta+\phi)} = z \,.\, e^{i\phi}$. If $\vec{OP}$ and $\vec{OQ}$ are of unequal magnitude then $\hat{OQ} = \hat{OP}\, e^{i\phi}$

**(ii)** If A, B, C & D are four points representing the complex numbers $z_1, z_2, z_3$ & $z_4$ then

AB || CD if $\dfrac{z_4 - z_3}{z_2 - z_1}$ is purely real ;

AB ⊥ CD if $\dfrac{z_4 - z_3}{z_2 - z_1}$ is purely imaginary ]

**(iii)** If $z_1, z_2, z_3$ are the vertices of an equilateral triangle where $z_0$ is its circumcentre then

**(a)** $z_1^2 + z_2^2 + z_3^2 - z_1 z_2 - z_2 z_3 - z_3 z_1 = 0$ **(b)** $z_1^2 + z_2^2 + z_3^2 = 3 z_0^2$

**8. DEMOIVRE'S THEOREM :**

**Statement :** $\cos n\theta + i \sin n\theta$ is the value or one of the values of $(\cos\theta + i \sin\theta)^n \;\forall\; n \in Q$. The theorem is very useful in determining the roots of any complex quantity

**Note :** Continued product of the roots of a complex quantity should be determined using theory of equations.

**9. CUBE ROOT OF UNITY :**

**(i)** The cube roots of unity are $1, \dfrac{-1+i\sqrt{3}}{2}, \dfrac{-1-i\sqrt{3}}{2}$.

**(ii)** If w is one of the imaginary cube roots of unity then $1 + w + w^2 = 0$. In general $1 + w^r + w^{2r} = 0$ ; where $r \in I$ but is not the multiple of 3.

**(iii)** In polar form the cube roots of unity are :

$\cos 0 + i \sin 0$ ; $\cos\dfrac{2\pi}{3} + i\sin\dfrac{2\pi}{3}$, $\cos\dfrac{4\pi}{3} + i\sin\dfrac{4\pi}{3}$

**(iv)** The three cube roots of unity when plotted on the argand plane constitute the verties of an equilateral triangle.

**(v)** The following factorisation should be remembered :

(a, b, c $\in$ R & $\omega$ is the cube root of unity)

$a^3 - b^3 = (a - b)(a - \omega b)(a - \omega^2 b)$ ; $x^2 + x + 1 = (x - \omega)(x - \omega^2)$ ;

$a^3 + b^3 = (a + b)(a + \omega b)(a + \omega^2 b)$ ;

$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a + \omega b + \omega^2 c)(a + \omega^2 b + \omega c)$

**10. $n^{th}$ ROOTS OF UNITY :**

If $1, \alpha_1, \alpha_2, \alpha_3 \,.....\, \alpha_{n-1}$ are the n, $n^{th}$ root of unity then :

**(i)** They are in G.P. with common ratio $e^{i(2\pi/n)}$ **&**

**(ii)** $1^p + \alpha_1^p + \alpha_2^p + .... + \alpha_{n-1}^p = 0$ if p is not an integral multiple of n
$= n$ if p is an integral multiple of n

**(iii)** $(1 - \alpha_1)(1 - \alpha_2) \,......\, (1 - \alpha_{n-1}) = n$ **&**
$(1 + \alpha_1)(1 + \alpha_2) \,.......\, (1 + \alpha_{n-1}) = 0$ if n is even and 1 if n is odd.

**(iv)** $1 \,.\, \alpha_1 \,.\, \alpha_2 \,.\, \alpha_3 \,.........\, \alpha_{n-1} = 1$ or $-1$ according as n is odd or even.

**11. THE SUM OF THE FOLLOWING SERIES SHOULD BE REMEMBERED :**

**(i)** $\cos\theta + \cos 2\theta + \cos 3\theta + ..... + \cos n\theta = \dfrac{\sin(n\theta/2)}{\sin(\theta/2)} \cos\left(\dfrac{n+1}{2}\right)\theta.$

**(ii)** $\sin\theta + \sin 2\theta + \sin 3\theta + ..... + \sin n\theta = \dfrac{\sin(n\theta/2)}{\sin(\theta/2)} \sin\left(\dfrac{n+1}{2}\right)\theta.$

**Note :** If $\theta = (2\pi/n)$ then the sum of the above series vanishes.



MC Sir

## 12. STRAIGHT LINES & CIRCLES IN TERMS OF COMPLEX NUMBERS :

**(A)** If $z_1$ & $z_2$ are two complex numbers then the complex number $z=\dfrac{nz_1+mz_2}{m+n}$ divides the joins of $z_1$ & $z_2$ in the ratio m : n.

**Note:**

**(i)** If a , b , c are three real numbers such that $az_1+bz_2+cz_3=0$ ; where $a+b+c=0$ and a,b,c are not all simultaneously zero, then the complex numbers $z_1$ , $z_2$ & $z_3$ are collinear.

**(ii)** If the vertices A, B, C of a $\Delta$ represent the complex nos. $z_1, z_2, z_3$ respectively, then :

**(a)** Centroid of the $\Delta ABC = \dfrac{z_1+z_2+z_3}{3}$ :

**(b)** Orthocentre of the $\Delta ABC =$

$$\frac{(a\sec A)z_1+(b\sec B)z_2+(c\sec C)z_3}{a\sec A+b\sec B+c\sec C} \quad \textbf{OR} \quad \frac{z_1\tan A+z_2\tan B+z_3\tan C}{\tan A+\tan B+\tan C}$$

**(c)** Incentre of the $\Delta ABC = (az_1+bz_2+cz_3)\div(a+b+c)$ .

**(d)** Circumcentre of the $\Delta ABC =$ :
$(Z_1\sin 2A+Z_2\sin 2B+Z_3\sin 2C)\div(\sin 2A+\sin 2B+\sin 2C)$ .

**(B)** $\text{amp}(z)=\theta$ is a ray emanating from the origin inclined at an angle $\theta$ to the x− axis.

**(C)** $|z-a| = |z-b|$ is the perpendicular bisector of the line joining a to b.

**(D)** The equation of a line joining $z_1$ & $z_2$ is given by ;
$z=z_1+t\,(z_1-z_2)$ where t is a perameter.

**(E)** $z=z_1\,(1+it)$ where t is a real parameter is a line through the point $z_1$ & perpendicular to $oz_1$.

**(F)** The equation of a line passing through $z_1$ & $z_2$ can be expressed in the determinant form as

$$\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix}=0.$$ This is also the condition for three complex numbers to be collinear.

**(G)** Complex equation of a straight line through two given points $z_1$ & $z_2$ can be written as $z(\bar{z}_1-\bar{z}_2)-\bar{z}(z_1-z_2)+(z_1\bar{z}_2-\bar{z}_1z_2)=0$, which on manipulating takes the form as $\bar{\alpha}\,z+\alpha\,\bar{z}+r=0$ where r is real and $\alpha$ is a non zero complex constant.

**(H)** The equation of circle having centre $z_0$ & radius $\rho$ is :
$|z-z_0|=\rho$ or $z\bar{z}-z_0\bar{z}-\bar{z}_0z+\bar{z}_0z_0-\rho^2=0$ which is of the form

$z\bar{z}+\bar{\alpha}z+\alpha\bar{z}+r=0$ , r is real centre $-\alpha$ & radius $\sqrt{\alpha\bar{\alpha}-r}$ .

Circle will be real if $\alpha\,\bar{\alpha}-r\geq 0$ .

**(I)** The equation of the circle described on the line segment joining $z_1$ & $z_2$ as diameter is :

**(i)** $\arg\dfrac{z-z_2}{z-z_1}=\pm\dfrac{\pi}{2}$ or $(z-z_1)(\bar{z}-\bar{z}_2)+(z-z_2)(\bar{z}-\bar{z}_1)=0$

**(J)** Condition for four given points $z_1$ , $z_2$ , $z_3$ & $z_4$ to be concyclic is, the number $\dfrac{z_3-z_1}{z_3-z_2}\cdot\dfrac{z_4-z_2}{z_4-z_1}$ is real. Hence the equation of a circle through 3 non collinear points $z_1$, $z_2$ & $z_3$ can be taken as $\dfrac{(z-z_2)(z_3-z_1)}{(z-z_1)(z_3-z_2)}$ is real $\Rightarrow \dfrac{(z-z_2)(z_3-z_1)}{(z-z_1)(z_3-z_2)}=\dfrac{(\bar{z}-\bar{z}_2)(\bar{z}_3-\bar{z}_1)}{(\bar{z}-\bar{z}_1)(\bar{z}_3-\bar{z}_2)}$

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**13.(a) Reflection points for a straight line :**

Two given points P & Q are the reflection points for a given straight line if the given line is the right bisector of the segment PQ. Note that the two points denoted by the complex numbers $z_1$ & $z_2$ will be the reflection points for the straight line $\bar{\alpha} z + \alpha \bar{z} + r = 0$ if and only if; $\bar{\alpha} z_1 + \alpha \bar{z}_2 + r = 0$, where r is real and $\alpha$ is non zero complex constant.

**(b) Inverse points w.r.t. a circle :**

Two points P & Q are said to be inverse w.r.t. a circle with centre 'O' and radius $\rho$, if :

**(i)** the point O, P, Q are collinear and on the same side of O. **(ii)** OP . OQ = $\rho^2$.

Note that the two points $z_1$ & $z_2$ will be the inverse points w.r.t. the circle $z\bar{z}+\bar{\alpha}z+\alpha\bar{z}+r=0$ if and only if $z_1\bar{z}_2+\bar{\alpha}z_1+\alpha\bar{z}_2+r=0$.

**14. PTOLEMY'S THEOREM :**

It states that the product of the lengths of the diagonals of a convex quadrilateral inscribed in a circle is equal to the sum of the lengths of the two pairs of its opposite sides.

i.e. $|z_1 - z_3|\ |z_2 - z_4| = |z_1 - z_2|\ |z_3 - z_4| + |z_1 - z_4|\ |z_2 - z_3|$.

**15. LOGARITHM OF A COMPLEX QUANTITY :**

**(i)** $\text{Log}_e(\alpha + i\,\beta) = \dfrac{1}{2}\text{Log}_e(\alpha^2 + \beta^2) + i\left(2n\pi + \tan^{-1}\dfrac{\beta}{\alpha}\right)$ where $n \in I$.

**(ii)** $i^i$ represents a set of positive real numbers given by $e^{-\left(2n\pi+\frac{\pi}{2}\right)}$, $n \in I$.

## VERY ELEMENTARY EXERCISE

Q.1 Simplify and express the result in the form of $a + bi$

(a) $\left(\dfrac{1+2i}{2+i}\right)^2$ (b) $-i\,(9 + 6\,i)\,(2 - i)^{-1}$ (c) $\left(\dfrac{4i^3 - i}{2i+1}\right)^2$ (d) $\dfrac{3+2i}{2-5i} + \dfrac{3-2i}{2+5i}$ (e) $\dfrac{(2+i)^2}{2-i} - \dfrac{(2-i)^2}{2+i}$

(f) A square $P_1P_2P_3P_4$ is drawn in the complex plane with $P_1$ at (1, 0) and $P_3$ at (3, 0). Let $P_n$ denotes the point $(x_n, y_n)$ n = 1, 2, 3, 4. Find the numerical value of the product of complex numbers $(x_1 + i\,y_1)(x_2 + i\,y_2)(x_3 + i\,y_3)(x_4 + i\,y_4)$.

Q.2 Given that $x, y \in R$, solve : (a) $(x + 2y) + i\,(2x - 3y) = 5 - 4i$ (b) $(x + iy) + (7 - 5i) = 9 + 4i$
(c) $x^2 - y^2 - i\,(2x + y) = 2i$ (d) $(2 + 3i)\,x^2 - (3 - 2i)\,y = 2x - 3y + 5i$

Q.3 Find the square root of : (a) $9 + 40\,i$ (b) $-11 - 60\,i$ (c) $50\,i$

Q.4 (a) If $f(x) = x^4 + 9x^3 + 35x^2 - x + 4$, find $f(-5 + 4i)$
(b) If $g(x) = x^4 - x^3 + x^2 + 3x - 5$, find $g(2 + 3i)$

Q.5 Among the complex numbers z satisfying the condition $\left|z + 3 - \sqrt{3}\,i\right| = \sqrt{3}$, find the number having the least positive argument.

Q.6 Solve the following equations over C and express the result in the form $a + ib$, $a, b \in R$.
(a) $ix^2 - 3x - 2i = 0$ (b) $2\,(1 + i)\,x^2 - 4\,(2 - i)\,x - 5 - 3\,i = 0$

Q.7 Locate the points representing the complex number z on the Argand plane:

**(a)** $|z + 1 - 2i| = \sqrt{7}$ ; **(b)** $|z - 1|^2 + |z + 1|^2 = 4$ ; **(c)** $\left|\dfrac{z-3}{z+3}\right| = 3$ ; **(d)** $|z - 3| = |z - 6|$

Q.8 If a & b are real numbers between 0 & 1 such that the points $z_1 = a + i$, $z_2 = 1 + bi$ & $z_3 = 0$ form an equilateral triangle, then find the values of 'a' and 'b'.

Q.9 Let $z_1 = 1 + i$ and $z_2 = -1 - i$. Find $z_3 \in C$ such that triangle $z_1, z_2, z_3$ is equilaterial.

Q.10 For what real values of x & y are the numbers $-3 + ix^2y$ & $x^2 + y + 4i$ conjugate complex?

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Q.11 Find the modulus, argument and the principal argument of the complex numbers.

(i) $6(\cos 310° - i \sin 310°)$ (ii) $-2(\cos 30° + i \sin 30°)$ (iii) $\frac{2+i}{4i+(1+i)^2}$

Q.12 If $(x+iy)^{1/3} = a + bi$ ; prove that $4(a^2 - b^2) = \frac{x}{a} + \frac{y}{b}$.

Q.13 Let z be a complex number such that $z \in c\backslash R$ and $\frac{1+z+z^2}{1-z+z^2} \in R$, then prove that $|z| = 1$.

Q.14 Prove the identity, $|1 - z_1\bar{z}_2|^2 - |z_1 - z_2|^2 = \left(1 - |z_1|^2\right)\left(1 - |z_2|^2\right)$

Q.15 Prove the identity, $|1 + z_1\bar{z}_2|^2 + |z_1 - z_2|^2 = \left(1 + |z_1|^2\right)\left(1 + |z_2|^2\right)$

Q.16 For any two complex numbers, prove that $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\left[|z_1|^2 + |z_2|^2\right]$. Also give the geometrical interpretation of this identity.

Q.17 (a) Find all non–zero complex numbers Z satisfying $\bar{Z} = iZ^2$.

(b) If the complex numbers $z_1, z_2, ................z_n$ lie on the unit circle $|z| = 1$ then show that

$$|z_1 + z_2 + .............+z_n| = |z_1^{-1} + z_2^{-1} + ................+z_n^{-1}|.$$

Q.18 Find the Cartesian equation of the locus of 'z' in the complex plane satisfying, $|z - 4| + |z + 4| = 16$.

Q.19 Let $z = (0, 1) \in C$. Express $\sum_{k=0}^{n} z^k$ in terms of the positive integer n.

**Paragraph for question nos. 20 to 22**

Consider a complex number $w = \frac{z-i}{2z+1}$ where $z = x + iy$, where $x, y \in R$.

Q.20 If the complex number w is purely imaginary then locus of z is

(A) a straight line

(B) a circle with centre $\left(-\frac{1}{4}, \frac{1}{2}\right)$ and radius $\frac{\sqrt{5}}{4}$.

(C) a circle with centre $\left(\frac{1}{4}, -\frac{1}{2}\right)$ and passing through origin..

(D) neither a circle nor a straight line.

Q.21 If the complex number w is purely real then locus of z is

(A) a straight line passing through origin

(B) a straight line with gradient 3 and y intercept (–1)

(C) a straight line with gradient 2 and y intercept 1.

(D) none

Q.22 If $|w| = 1$ then the locus of P is

(A) a point circle (B) an imaginary circle

(C) a real circle (D) not a circle.

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# EXERCISE–I

Q.1 Simplify and express the result in the form of $a+bi$ :

(a) $-i\,(9+6\,i)\,(2-i)^{-1}$ (b) $\left(\dfrac{4i^3-i}{2i+1}\right)^2$ (c) $\dfrac{3+2i}{2-5i}+\dfrac{3-2i}{2+5i}$

(d) $\dfrac{(2+i)^2}{2-i}-\dfrac{(2-i)^2}{2+i}$ (e) $\sqrt{i}+\sqrt{-i}$

Q.2 Find the modulus, argument and the principal argument of the complex numbers.

(i) $z=1+\cos\left(\dfrac{10\pi}{9}\right)+i\sin\left(\dfrac{10\pi}{9}\right)$ (ii) $(\tan 1-i)^2$

(iii) $z=\dfrac{\sqrt{5+12i}+\sqrt{5-12i}}{\sqrt{5+12i}-\sqrt{5-12i}}$ (iv) $\dfrac{i-1}{i\left(1-\cos\dfrac{2\pi}{5}\right)+\sin\dfrac{2\pi}{5}}$

Q.3 Given that $x, y \in R$, solve :

(a) $(x+2y)+i\,(2x-3y)=5-4i$ (b) $\dfrac{x}{1+2i}+\dfrac{y}{3+2i}=\dfrac{5+6i}{8i-1}$

(c) $x^2-y^2-i\,(2x+y)=2i$ (d) $(2+3i)\,x^2-(3-2i)\,y=2x-3y+5i$

(e) $4x^2+3xy+(2xy-3x^2)i=4y^2-(x^2/2)+(3xy-2y^2)i$

Q.4(a) Let Z is complex satisfying the equation, $z^2-(3+i)z+m+2i=0$, where $m\in R$. Suppose the equation has a real root, then find the value of *m*.

(b) a, b, c are real numbers in the polynomial, $P(Z)=2Z^4+aZ^3+bZ^2+cZ+3$. If two roots of the equation $P(Z)=0$ are 2 and i, then find the value of 'a'.

Q.5(a) Find the real values of $x$ & $y$ for which $z_1=9y^2-4-10\,i\,x$ and $z_2=8y^2-20\,i$ are conjugate complex of each other.

(b) Find the value of $x^4-x^3+x^2+3x-5$ if $x=2+3i$

Q.6 Solve the following for z:

$z^2-(3-2\,i)z=(5i-5)$

Q.7(a) If $i\,Z^3+Z^2-Z+i=0$, then show that $|Z|=1$.

(b) Let $z_1$ and $z_2$ be two complex numbers such that $\left|\dfrac{z_1-2z_2}{2-z_1\bar{z}_2}\right|=1$ and $|z_2|\neq 1$, find $|z_1|$.

(c) Let $z_1=10+6i$ & $z_2=4+6i$. If $z$ is any complex number such that the argument of, $\dfrac{z-z_1}{z-z_2}$ is $\dfrac{\pi}{4}$, then prove that $|z-7-9i|=3\sqrt{2}$.

Q.8 Show that the product,

$$\left[1+\left(\frac{1+i}{2}\right)\right]\left[1+\left(\frac{1+i}{2}\right)^2\right]\left[1+\left(\frac{1+i}{2}\right)^{2^2}\right]\ldots\ldots\left[1+\left(\frac{1+i}{2}\right)^{2^n}\right]$$ is equal to $\left(1-\dfrac{1}{2^{2^n}}\right)(1+i)$ where $n\geq 2$.

Q.9 Let $z_1, z_2$ be complex numbers with $|z_1|=|z_2|=1$, prove that $|z_1+1|+|z_2+1|+|z_1z_2+1|\geq 2$.

MC Sir

Q.10 Interpret the following locii in $z \in C$.

(a) $1 < |z - 2i| < 3$ (b) $\operatorname{Re}\left(\dfrac{z+2i}{iz+2}\right) \le 4 \quad (z \ne 2i)$

(c) $\operatorname{Arg}(z+i) - \operatorname{Arg}(z-i) = \pi/2$ (d) $\operatorname{Arg}(z-a) = \pi/3$ where $a = 3 + 4i$.

Q.11 Let $A = \{a \in R \mid$ the equation $(1+2i)x^3 - 2(3+i)x^2 + (5-4i)x + 2a^2 = 0\}$ has at least one real root. Find the value of $\sum\limits_{a \in A} a^2$.

Q.12 P is a point on the Aragand diagram. On the circle with OP as diameter two points Q & R are taken such that $\angle POQ = \angle QOR = \theta$. If 'O' is the origin & P, Q & R are represented by the complex numbers $Z_1$, $Z_2$ & $Z_3$ respectively, show that : $Z_2^2 . \cos 2\theta = Z_1 . Z_3 \cos^2\theta$.

Q.13 Let $z_1, z_2, z_3$ are three pair wise distinct complex numbers and $t_1, t_2, t_3$ are non-negative real numbers such that $t_1 + t_2 + t_3 = 1$. Prove that the complex number $z = t_1z_1 + t_2z_2 + t_3z_3$ lies inside a triangle with vertices $z_1, z_2, z_3$ or on its boundry.

Q.14 Let $A \equiv z_1$ ; $B \equiv z_2$; $C \equiv z_3$ are three complex numbers denoting the vertices of an acute angled triangle. If the origin 'O' is the orthocentre of the triangle, then prove that

$$z_1\bar{z}_2 + \bar{z}_1 z_2 = z_2\bar{z}_3 + \bar{z}_2 z_3 = z_3\bar{z}_1 + \bar{z}_3 z_1$$

hence show that the $\Delta ABC$ is a right angled triangle $\Leftrightarrow z_1\bar{z}_2 + \bar{z}_1 z_2 = z_2\bar{z}_3 + \bar{z}_2 z_3 = z_3\bar{z}_1 + \bar{z}_3 z_1 = 0$

Q.15 Let $\alpha + i\beta$; $\alpha, \beta \in R$, be a root of the equation $x^3 + qx + r = 0$; $q, r \in R$. Find a real cubic equation, independent of $\alpha$ & $\beta$, whose one root is $2\alpha$.

Q.16 Find the sum of the series $1(2-\omega)(2-\omega^2) + 2(3-\omega)(3-\omega^2)$ ....... $(n-1)(n-\omega)(n-\omega^2)$ where $\omega$ is one of the imaginary cube root of unity.

Q.17 If A, B and C are the angles of a triangle

$$D = \begin{vmatrix} e^{-2iA} & e^{iC} & e^{iB} \\ e^{iC} & e^{-2iB} & e^{iA} \\ e^{iB} & e^{iA} & e^{-2iC} \end{vmatrix} \quad \text{where } i = \sqrt{-1}$$

then find the value of D.

Q.18 If w is an imaginary cube root of unity then prove that :

(a) $(1 - w + w^2)(1 - w^2 + w^4)(1 - w^4 + w^8)$ ..... to 2n factors $= 2^{2n}$.

(b) If w is a complex cube root of unity, find the value of $(1 + w)(1 + w^2)(1 + w^4)(1 + w^8)$ ..... to n factors .

Q.19 Prove that $\left(\dfrac{1+\sin\theta + i\cos\theta}{1+\sin\theta - i\cos\theta}\right)^n = \cos\left(\dfrac{n\pi}{2} - n\theta\right) + i\sin\left(\dfrac{n\pi}{2} - n\theta\right)$. Hence deduce that

$$\left(1+\sin\frac{\pi}{5} + i\cos\frac{\pi}{5}\right)^5 + i\left(1+\sin\frac{\pi}{5} - i\cos\frac{\pi}{5}\right)^5 = 0$$

Q.20 If $\cos(\alpha - \beta) + \cos(\beta - \gamma) + \cos(\gamma - \alpha) = -3/2$ then prove that:

(a) $\Sigma \cos 2\alpha = 0 = \Sigma \sin 2\alpha$ (b) $\Sigma \sin(\alpha + \beta) = 0 = \Sigma \cos(\alpha + \beta)$

(c) $\Sigma \sin^2\alpha = \Sigma \cos^2\alpha = 3/2$ (d) $\Sigma \sin 3\alpha = 3\sin(\alpha + \beta + \gamma)$

(e) $\Sigma \cos 3\alpha = 3\cos(\alpha + \beta + \gamma)$

(f) $\cos^3(\theta + \alpha) + \cos^3(\theta + \beta) + \cos^3(\theta + \gamma) = 3\cos(\theta + \alpha) . \cos(\theta + \beta) . \cos(\theta + \gamma)$ where $\theta \in R$.

MC sir

Q.21 Resolve $Z^5 + 1$ into linear & quadratic factors with real coefficients. Deduce that : $4 \cdot \sin\frac{\pi}{10} \cdot \cos\frac{\pi}{5} = 1$.

Q.22 If $x = 1 + i\sqrt{3}$ ; $y = 1 - i\sqrt{3}$ & $z = 2$, then prove that $x^p + y^p = z^p$ for every prime $p > 3$.

Q.23 Dividing f(z) by $z - i$, we get the remainder i and dividing it by $z + i$, we get the remainder $1 + i$. Find the remainder upon the division of f(z) by $z^2 + 1$.

Q.24(a) Let $z = x + iy$ be a complex number, where *x* and *y* are real numbers. Let A and B be the sets defined by $A = \{z \mid |z| \le 2\}$ and $B = \{z \mid (1 - i)z + (1 + i)\bar{z} \ge 4\}$. Find the area of the region $A \cap B$.

(b) For all real numbers x, let the mapping $f(x) = \frac{1}{x - i}$, where $i = \sqrt{-1}$. If there exist real number *a*, *b*, *c* and *d* for which *f*(a), *f*(b), *f*(c) and *f*(d) form a square on the complex plane. Find the area of the square.

Q.25

| | **Column-I** | | **Column-II** |
|---|---|---|---|
| (A) | Let w be a non real cube root of unity then the number of distinct elements in the set $\{(1 + w + w^2 + ....... + w^n)^m \mid m, n \in N\}$ is | (P) | 4 |
| (B) | Let 1, w, $w^2$ be the cube root of unity. The least possible degree of a polynomial with real coefficients having roots 2w, $(2 + 3w)$, $(2 + 3w^2)$, $(2 - w - w^2)$, is | (Q) | 5 |
| (C) | $\alpha = 6 + 4i$ and $\beta = (2 + 4i)$ are two complex numbers on the complex plane. A complex number z satisfying $\text{amp}\left(\frac{z-\alpha}{z-\beta}\right) = \frac{\pi}{6}$ moves on the major segment of a circle whose radius is | (R) | 6 |
| | | (S) | 8 |

## ***EXERCISE–II***

Q.1 If $\begin{vmatrix} p & q & r \\ q & r & p \\ r & p & q \end{vmatrix} = 0$ ; where p , q , r are the moduli of non–zero complex numbers u, v, w respectively, prove that, $\arg\frac{w}{v} = \arg\left(\frac{w-u}{v-u}\right)^2$.

Q.2 Let $Z = 18 + 26i$ where $Z_0 = x_0 + iy_0$ $(x_0, y_0 \in R)$ is the cube root of Z having least positive argument. Find the value of $x_0y_0(x_0 + y_0)$.

Q.3 Show that the locus formed by z in the equation $z^3 + iz = 1$ never crosses the co-ordinate axes in the Argand's plane. Further show that $|z| = \sqrt{\frac{-\text{Im}(z)}{2\text{Re}(z)\text{Im}(z) + 1}}$

Q.4 If ω is the fifth root of 2 and $x = \omega + \omega^2$, prove that $x^5 = 10x^2 + 10x + 6$.

Q.5 Prove that, with regard to the quadratic equation $z^2 + (p + ip')z + q + iq' = 0$ where p , p′, q , q′ are all real.

(i) if the equation has one real root then $q'^2 - pp'q' + qp'^2 = 0$.

(ii) if the equation has two equal roots then $p^2 - p'^2 = 4q$ & $pp' = 2q'$. State whether these equal roots are real or complex.

Q.6 If the equation $(z + 1)^7 + z^7 = 0$ has roots $z_1, z_2, .... z_7$, find the value of

(a) $\sum_{r=1}^{7}\text{Re}(Z_r)$ and (b) $\sum_{r=1}^{7}\text{Im}(Z_r)$

MC Sir

Q.7 Find the roots of the equation $Z^n = (Z+1)^n$ and show that the points which represent them are collinear on the complex plane. Hence show that these roots are also the roots of the equation

$$\left(2\sin\frac{m\pi}{n}\right)^2 \overline{Z}^2 + \left(2\sin\frac{m\pi}{n}\right)^2 \overline{Z} + 1 = 0.$$

Q.8 If the expression $z^5 - 32$ can be factorised into linear and quadratic factors over real coefficients as $(z^5 - 32) = (z-2)(z^2 - pz + 4)(z^2 - qz + 4)$ then find the value of $(p^2 + 2p)$.

Q.9 Let $z_1$ & $z_2$ be any two arbitrary complex numbers then prove that :

$$|z_1 + z_2| \geq \frac{1}{2}(|z_1| + |z_2|)\left|\frac{z_1}{|z_1|} + \frac{z_2}{|z_2|}\right|.$$

Q.10 If $Z_r$, r = 1, 2, 3, ......... 2m, m ε N are the roots of the equation

$Z^{2m} + Z^{2m-1} + Z^{2m-2} + \ldots\ldots\ldots\ldots + Z + 1 = 0$ then prove that $\sum_{r=1}^{2m} \frac{1}{Z_r - 1} = -m$

Q.11**(i)** Let Cr's denotes the combinatorial coefficients in the expansion of $(1+x)^n$, $n \in N$. If the integers

$a_n = C_0 + C_3 + C_6 + C_9 + \ldots\ldots$

$b_n = C_1 + C_4 + C_7 + C_{10} + \ldots\ldots$

and $c_n = C_2 + C_5 + C_8 + C_{11} + \ldots\ldots$, then

prove that **(a)** $a_n^3 + b_n^3 + c_n^3 - 3a_nb_nc_n = 2^n$, **(b)** $(a_n - b_n)^2 + (b_n - c_n)^2 + (c_n - a_n)^2 = 2$.

**(ii)** Prove the identity: $(C_0 - C_2 + C_4 - C_6 + \ldots..)^2 + (C_1 - C_3 + C_5 - C_7 + \ldots\ldots.)^2 = 2^n$

Q.12 Let $z_1, z_2, z_3, z_4$ be the vertices A, B, C, D respectively of a square on the Argand diagram taken in anticlockwise direction then prove that :

(i) $2z_2 = (1+i)z_1 + (1-i)z_3$ & (ii) $2z_4 = (1-i)z_1 + (1+i)z_3$

Q.13 Show that all the roots of the equation $\left(\frac{1+ix}{1-ix}\right)^n = \frac{1+ia}{1-ia}$ $a \in R$ are real and distinct.

Q.14 Prove that:

(a) $\cos x + {}^nC_1 \cos 2x + {}^nC_2 \cos 3x + \ldots.. + {}^nC_n \cos(n+1)x = 2^n \cdot \cos^n \frac{x}{2} \cdot \cos\left(\frac{n+2}{2}\right)x$

(b) $\sin x + {}^nC_1 \sin 2x + {}^nC_2 \sin 3x + \ldots.. + {}^nC_n \sin(n+1)x = 2^n \cdot \cos^n \frac{x}{2} \cdot \sin\left(\frac{n+2}{2}\right)x$

(c) $\cos\left(\frac{2\pi}{2n+1}\right) + \cos\left(\frac{4\pi}{2n+1}\right) + \cos\left(\frac{6\pi}{2n+1}\right) + \ldots.. + \cos\left(\frac{2n\pi}{2n+1}\right) = -\frac{1}{2}$ When $n \in N$.

Q.15 Show that all roots of the equation $a_0z^n + a_1z^{n-1} + \ldots\ldots + a_{n-1}z + a_n = n$,

where $|a_i| \leq 1$, i = 0, 1, 2, .... , n lie outside the circle with centre at the origin and radius $\frac{n-1}{n}$.

Q.16 The points A, B, C depict the complex numbers $z_1, z_2, z_3$ respectively on a complex plane & the angle B & C of the triangle ABC are each equal to $\frac{1}{2}(\pi - \alpha)$. Show that

$$(z_2 - z_3)^2 = 4(z_3 - z_1)(z_1 - z_2)\sin^2\frac{\alpha}{2}.$$

Q.17 Evaluate: $\sum_{p=1}^{32} (3p+2)\left(\sum_{q=1}^{10}\left(\sin\frac{2q\pi}{11} - i\cos\frac{2q\pi}{11}\right)\right)^p$.

Q.18 Let a, b, c be distinct complex numbers such that $\frac{a}{1-b} = \frac{b}{1-c} = \frac{c}{1-a} = k$. Find the value of k.

Q.19 Let $\alpha$, $\beta$ be fixed complex numbers and z is a variable complex number such that,

$$|z-\alpha|^2 + |z-\beta|^2 = k.$$

Find out the limits for 'k' such that the locus of $z$ is a circle. Find also the centre and radius of the circle.

Q.20 C is the complex number. $f: C \to R$ is defined by $f(z) = |z^3 - z + 2|$. Find the maximum value of $f(z)$ if $|z| = 1$.

Q.21 Let $f(x) = \log_{\cos 3x}(\cos 2ix)$ if $x \neq 0$ and $f(0) = K$ (where $i = \sqrt{-1}$) is continuous at $x = 0$ then find the value of K.

Q.22 If $\alpha = e^{\frac{2\pi i}{7}}$ and $f(x) = A_0 + \sum_{k=1}^{20} A_k x^k$, then find the value of, $f(x) + f(\alpha x) + ...... + f(\alpha^6 x)$ independent of $\alpha$.

Q.23 Find the set of points on the argand plane for which the real part of the complex number $(1+i)z^2$ is positive where $z = x + iy$, $x, y \in R$ and $i = \sqrt{-1}$.

Q.24 If a and b are positive integer such that $N = (a+ib)^3 - 107i$ is a positive integer. Find N.

Q.25 If the biquadratic $x^4 + ax^3 + bx^2 + cx + d = 0$ (a, b, c, d $\in$ R) has 4 non real roots, two with sum $3+4i$ and the other two with product $13+i$. Find the value of 'b'.

## ***EXERCISE–III***

Q.1(a) If $z_1, z_2, z_3$ are complex numbers such that $|z_1| = |z_2| = |z_3| = \left|\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right| = 1$, then $|z_1 + z_2 + z_3|$ is :

(A) equal to 1 (B) less than 1 (C) greater than 3 (D) equal to 3

(b) If $\arg(z) < 0$, then $\arg(-z) - \arg(z) =$

(A) $\pi$ (B) $-\pi$ (C) $-\frac{\pi}{2}$ (D) $\frac{\pi}{2}$

[ JEE 2000 (Screening) 1 + 1 out of 35 ]

Q.2 Given, $z = \cos\frac{2\pi}{2n+1} + i\sin\frac{2\pi}{2n+1}$, 'n' a positive integer, find the equation whose roots are, $\alpha = z + z^3 + ...... + z^{2n-1}$ & $\beta = z^2 + z^4 + ...... + z^{2n}$.

[ REE 2000 (Mains) 3 out of 100 ]

Q.3 Find all those roots of the equation $z^{12} - 56z^6 - 512 = 0$ whose imaginary part is positive.

[ REE 2000, 3 out of 100 ]

MCSir

Q.4(a) The complex numbers $z_1, z_2$ and $z_3$ satisfying $\dfrac{z_1 - z_3}{z_2 - z_3} = \dfrac{1 - i\sqrt{3}}{2}$ are the vertices of a triangle which is

(A) of area zero (B) right-angled isosceles
(C) equilateral (D) obtuse – angled isosceles

(b) Let $z_1$ and $z_2$ be nth roots of unity which subtend a right angle at the origin. Then n must be of the form
(A) 4k + 1 (B) 4k + 2 (C) 4k + 3 (D) 4k
[ JEE 2001 (Scr) 1 + 1 out of 35 ]

Q.5(a) Let $\omega = -\dfrac{1}{2} + i\dfrac{\sqrt{3}}{2}$. Then the value of the determinant $\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1-\omega^2 & \omega^2 \\ 1 & \omega^2 & \omega^4 \end{vmatrix}$ is

(A) $3\omega$ (B) $3\omega(\omega - 1)$ (C) $3\omega^2$ (D) $3\omega(1 - \omega)$

(b) For all complex numbers $z_1$, $z_2$ satisfying $|z_1| = 12$ and $|z_2 - 3 - 4i| = 5$, the minimum value of $|z_1 - z_2|$ is
(A) 0 (B) 2 (C) 7 (D) 17 [JEE 2002 (Scr) 3+3]

(c) Let a complex number $\alpha$ , $\alpha \neq 1$, be a root of the equation
$z^{p+q} - z^p - z^q + 1 = 0$ where p, q are distinct primes.
Show that either $1 + \alpha + \alpha^2 + ...... + \alpha^{p-1} = 0$ or $1 + \alpha + \alpha^2 + ...... + \alpha^{q-1} = 0$ , but not both together.
[JEE 2002, (5) ]

Q.6(a) If $z_1$ and $z_2$ are two complex numbers such that $|z_1| < 1 < |z_2|$ then prove that $\left|\dfrac{1 - z_1\bar{z}_2}{z_1 - z_2}\right| < 1$.

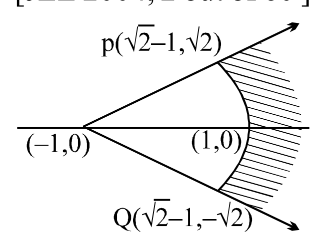
(b) Prove that there exists no complex number z such that $|z| < \dfrac{1}{3}$ and $\sum_{r=1}^{n} a_r z^r = 1$ where $|a_r| < 2$.

[JEE-03, 2 + 2 out of 60]

Q.7(a) $\omega$ is an imaginary cube root of unity. If $(1 + \omega^2)^m = (1 + \omega^4)^m$ , then least positive integral value of m is
(A) 6 (B) 5 (C) 4 (D) 3
[JEE 2004 (Scr)]

(b) Find centre and radius of the circle determined by all complex numbers $z = x + iy$ satisfying $\left|\dfrac{(z - \alpha)}{(z - \beta)}\right| = k$ , where $\alpha = \alpha_1 + i\alpha_2$, $\beta = \beta_1 + i\beta_2$ are fixed complex and $k \neq 1$. [JEE 2004, 2 out of 60 ]

Q.8(a) The locus of z which lies in shaded region is best represented by




(A) z : $|z + 1| > 2$, $|\arg(z + 1)| < \pi/4$
(B) z : $|z - 1| > 2$, $|\arg(z - 1)| < \pi/4$
(C) z : $|z + 1| < 2$, $|\arg(z + 1)| < \pi/2$
(D) z : $|z - 1| < 2$, $|\arg(z - 1)| < \pi/2$

(b) If a, b, c are integers not all equal and w is a cube root of unity ($w \neq 1$), then the minimum value of $|a + bw + cw^2|$ is

(A) 0 (B) 1 (C) $\dfrac{\sqrt{3}}{2}$ (D) $\dfrac{1}{2}$

[JEE 2005 (Scr), 3 + 3]

(c) If one of the vertices of the square circumscribing the circle $|z - 1| = \sqrt{2}$ is $2 + \sqrt{3}\,i$. Find the other vertices of square. [JEE 2005 (Mains), 4]

MC Sir

Q.9 If $w = \alpha + i\beta$ where $\beta \neq 0$ and $z \neq 1$, satisfies the condition that $\dfrac{w - \overline{w}z}{1-z}$ is purely real, then the set of values of z is

(A) $\{z : |z| = 1\}$ (B) $\{z : z = \overline{z})$ (C) $\{z : z \neq 1\}$ (D) $\{z : |z| = 1, z \neq 1\}$

[JEE 2006, 3]

Q.10**(a)** A man walks a distance of 3 units from the origin towards the North-East (N 45° E) direction. From there, he walks a distance of 4 units towards the North-West (N 45° W) direction to reach a point P. Then the position of P in the Argand plane is

(A) $3e^{i\pi/4} + 4i$ (B) $(3-4i)e^{i\pi/4}$ (C) $(4+3i)e^{i\pi/4}$ (D) $(3+4i)e^{i\pi/4}$

**(b)** If $|z| = 1$ and $z \neq \pm 1$, then all the values of $\dfrac{z}{1-z^2}$ lie on

(A) a line not passing through the origin (B) $|z| = \sqrt{2}$

(C) the x-axis (D) the y-axis [JEE 2007, 3+3]

Q.11**(a)** A particle P starts from the point $z_0 = 1 + 2i$, where $i = \sqrt{-1}$. It moves first horizontally away from origin by 5 units and then vertically away from origin by 3 units to reach a point $z_1$. From $z_1$ the particle moves $\sqrt{2}$ units in the direction of the vector $\hat{i} + \hat{j}$ and then it moves through an angle $\dfrac{\pi}{2}$ in anticlockwise direction on a circle with centre at origin, to reach a point $z_2$. The point $z_2$ is given by

(A) $6 + 7i$ (B) $-7 + 6i$ (C) $7 + 6i$ (D) $-6 + 7i$

**(b) Comprehension (3 questions together)**

Let A, B, C be three sets of complex numbers as defined below

$$A = \{z : \operatorname{Im} z \geq 1\}$$

$$B = \{z : |z - 2 - i| = 3\}$$

$$C = \{z : \operatorname{Re}((1-i)z) = \sqrt{2}\}$$

(i) The number of elements in the set $A \cap B \cap C$ is

(A) 0 (B) 1 (C) 2 (D) $\infty$

(ii) Let $z$ be any point in $A \cap B \cap C$. Then, $|z + 1 - i|^2 + |z - 5 - i|^2$ lies between

(A) 25 and 29 (B) 30 and 34 (C) 35 and 39 (D) 40 and 44

(iii) Let $z$ be any point in $A \cap B \cap C$ and let $w$ be any point satisfying $|w - 2 - i| < 3$. Then, $|z| - |w| + 3$ lies between

(A) –6 and 3 (B) –3 and 6 (C) –6 and 6 (D) –3 and 9

[JEE 2008, 3 + 4 + 4 + 4]

Q.12(a) Let $z = \cos\theta + i\sin\theta$. Then the value of $\sum_{m=1}^{15} \operatorname{Im}(z^{2m-1})$ at $\theta = 2°$ is

(A) $\dfrac{1}{\sin 2°}$ (B) $\dfrac{1}{3\sin 2°}$ (C) $\dfrac{1}{2\sin 2°}$ (D) $\dfrac{1}{4\sin 2°}$

(b) Let $z = x + iy$ be a complex number where x and y are integers. Then the area of the rectangle whose vertices are the roots of the equation $z\overline{z}^3 + \overline{z}z^3 = 350$ is

(A) 48 (B) 32 (C) 40 (D) 80

[JEE 2009, 3 + 3]

# ANSWER KEY

## VERY ELEMENTARY EXERCISE

**Q.1** **(a)** $\frac{7}{25}+\frac{24}{25}$ i; **(b)** $\frac{21}{5}-\frac{12}{5}$ i; **(c)** $3+4i$; **(d)** $-\frac{8}{29}+0i$; **(e)** $\frac{22}{5}$ i; **(f)** 15

**Q.2** **(a)** $x=1, y=2$; **(b)** $(2, 9)$; **(c)** $(-2, 2)$ or $\left(-\frac{2}{3}, -\frac{2}{3}\right)$; **(d)** $(1, 1)$ $\left(0, \frac{5}{2}\right)$

**Q.3** **(a)** $\pm(5+4i)$; **(b)** $\pm(5-6i)$ **(c)** $\pm 5(1+i)$

**Q.4** **(a)** $-160$ ; **(b)** $-(77+108\,i)$

**Q.5** $-\frac{3}{2}+\frac{3\sqrt{3}}{2}i$

**Q.6** **(a)** $-i, -2i$ **(b)** $\frac{3-5i}{2}$ or $-\frac{1+i}{2}$

**Q.7** **(a)** on a circle of radius $\sqrt{7}$ with centre $(-1, 2)$ **;** **(b)** on a unit circle with centre at origin
**(c)** on a circle with centre $(-15/4, 0)$ & radius $9/4$ **;** **(d)** a straight line

**Q.8** $a=b=2-\sqrt{3}$ ;

**Q.9** $z_3=\sqrt{3}(1-i)$ and $z'_3=\sqrt{3}(-1-i)$

**Q.10** $x=1,\ y=-4$ or $x=-1,\ y=-4$

**Q.11** **(i)** Modulus = 6 , Arg $= 2k\pi+\frac{5\pi}{18}$ $(K \in I)$ , Principal Arg $= \frac{5\pi}{18}$ $(K \in I)$

**(ii)** Modulus = 2 , Arg $= 2k\pi+\frac{7\pi}{6}$ , Principal Arg $= -\frac{5\pi}{6}$

**(iii)** Modulus $= \frac{\sqrt{5}}{6}$ , Arg $= 2k\pi-\tan^{-1}2$ $(K \in I)$ , Principal Arg $= -\tan^{-1}2$

**Q.17** **(a)** $\frac{\sqrt{3}}{2}-\frac{i}{2}, -\frac{\sqrt{3}}{2}-\frac{i}{2}, i$

**Q.18** $\frac{x^2}{64}+\frac{y^2}{48}=1$

**Q.19** $\begin{cases}(1,0) & \text{for } n=4k\\ (1,1) & \text{for } n=4k+1\\ (0,1) & \text{for } n=4k+2\\ (0,0) & \text{for } n=4k+3\end{cases}$

Q.20 B  Q.21 C  Q.22 C

## *EXERCISE–I*

**Q.1 (a)** $\frac{21}{5}-\frac{12}{5}i$ **(b)** $3+4i$ **(c)** $-\frac{8}{29}+0i$ **(d)** $\frac{22}{5}i$ **(e)** $\pm\sqrt{2}+0i$ or $0\pm\sqrt{2}i$

**Q.2** **(i)** Principal Arg $z=-\frac{4\pi}{9}$ ; $|z|=2\cos\frac{4\pi}{9}$ ; Arg $z = 2k\pi-\frac{4\pi}{9}$ $k \in I$

**(ii)** Modulus $= \sec^2 1$ , Arg $= 2n\pi+(2-\pi)$ , Principal Arg $= (2-\pi)$

**(iii)** Principal value of Agr $z=-\frac{\pi}{2}$ & $|z|=\frac{3}{2}$ ; Principal value of Arg $z=\frac{\pi}{2}$ & $|z|=\frac{2}{3}$

**(iv)** Modulus $= \frac{1}{\sqrt{2}}\operatorname{cosec}\frac{\pi}{5}$ , Arg $z = 2n\pi+\frac{11\pi}{20}$ , Principal Arg $= \frac{11\pi}{20}$

**Q.3(a)** $x=1, y=2$; **(b)** $x=1$ & $y=2$ ; **(c)** $(-2, 2)$ or $\left(-\frac{2}{3}, -\frac{2}{3}\right)$; **(d)** $(1, 1)$ $\left(0, \frac{5}{2}\right)$; **(e)** $x=K, y=\frac{3K}{2}$ $K \in R$

**Q.4** **(a)** 2, **(b)** $-11/2$

**Q.5** **(a)** $[(-2, 2) ; (-2, -2)]$ **(b)** $-(77+108\,i)$

**Q.6** $z=(2+i)$ or $(1-3i)$

**Q.7** **(b)** 2



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**Q.10** **(a)** The region between the co encentric circles with centre at $(0, 2)$ & radii 1 & 3 units

**(b)** region outside or on the circle with centre $\frac{1}{2} + 2i$ and radius $\frac{1}{2}$.

**(c)** semi circle (in the 1st & 4th quadrant) $x^2 + y^2 = 1$ **(d)** a ray emanating from the point $(3 + 4i)$ directed away from the origin & having equation $\sqrt{3}x - y + 4 - 3\sqrt{3} = 0$

**Q.11** 18 **Q.15** $x^3 + qx - r = 0$ **Q.16** $\left[\frac{n(n+1)}{n}\right]^2 - n$

**Q.17** $-4$ **Q.18** **(b)** one if n is even ; $-w^2$ if n is odd

**Q.21** $(Z + 1)(Z^2 - 2Z\cos 36° + 1)(Z^2 - 2Z\cos 108° + 1)$ **Q.23** $\frac{iz}{2} + \frac{1}{2} + i$

**Q.24** **(a)** $\pi - 2$ ; **(b)** 1/2 **Q.25** (A) R; (B) Q; (C) P

## *EXERCISE–II*

**Q.2** 12 **Q.6** (a) $-\frac{7}{2}$, (b) zero **Q.24** 4 **Q.17** 48(1 - i)

**Q.18** $-\omega$ or $-\omega^2$ **Q.19** $k > \frac{1}{2}|\alpha - \beta|^2$

**Q.20** $|f(z)|$ is maximum when $z = \omega$, where $\omega$ is the cube root unity and $|f(z)| = \sqrt{13}$

**Q.21** $K = -\frac{4}{9}$ **Q.22** $7A_0 + 7A_7x^7 + 7A_{14}x^{14}$

**Q.23** required set is constituted by the angles without their boundaries, whose sides are the straight lines $y = (\sqrt{2} - 1)x$ and $y + (\sqrt{2} + 1)x = 0$ containing the $x$ – axis

**Q.24** 198 **Q.25** 51

## *EXERCISE–III*

**Q.1** **(a)** A **(b)** A **Q.2** $z^2 + z + \frac{\sin^2 n\theta}{\sin^2\theta} = 0$, where $\theta = \frac{2\pi}{2n+1}$

**Q.3** $\pm 1 + i\sqrt{3}, \frac{(\pm\sqrt{3} + i)}{\sqrt{2}}, \sqrt{2}i$ **Q.4** **(a)** C, **(b)** D **Q.5** **(a)** B ; **(b)** B

**Q.7** **(a)** D; **(b)** Centre $\equiv \frac{k^2\beta - \alpha}{k^2 - 1}$, Radius $= \frac{1}{(k^2 - 1)}\sqrt{|\alpha - k^2\beta|^2 - (k^2.|\beta|^2 - |\alpha|^2)(k^2 - 1)}$

**Q.8** **(a)** A, **(b)** B, **(c)** $z_2 = -\sqrt{3}i$ ; $z_3 = (1 - \sqrt{3}) + i$ ; $z_4 = (1 + \sqrt{3}) - i$ **Q.9** D

**Q.10** **(a)** D ; **(b)** D

**Q.11** **(a)** D ; **(b)** (i) B; (ii) C; (iii) D **Q.12** (a) D; (b) A

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