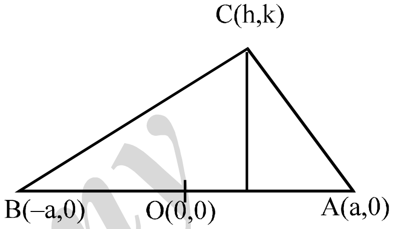
MC SIR

CLASS : XIII` ***DPP. NO.- 1***

Q.1 If on a given base, a triangle be described such that the sum of the tangents of the base angles is a constant, then the locus of the vertex is :

(A) a circle **(B*)** a parabola **(C)** an ellipse **(D)** a hyperbola

[Sol. $\tan A = \dfrac{k}{a-h}$; $\tan B = \dfrac{k}{a+h}$



$\tan A + \tan B = \text{'C'}$ (a constant)

$$\frac{k}{a-h} + \frac{k}{a+h} = C \quad \Rightarrow \quad \frac{k(a+h)+k(a-h)}{a^2-h^2} = C$$

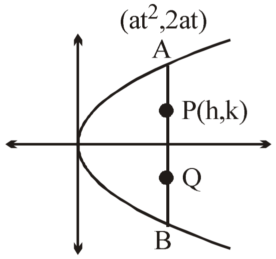
$$\therefore \quad a^2 - h^2 = \frac{2a}{C}\cdot k \quad \Rightarrow \quad \text{locus of } (h, k) \text{ will be}$$

$$x^2 = a^2 - \frac{2a}{C}\cdot y \quad \Rightarrow \quad \text{A parabola } \textbf{Ans.}]$$

Q.2 The locus of the point of trisection of all the double ordinates of the parabola $y^2 = lx$ is a parabola whose latus rectum is

(A*) $\dfrac{l}{9}$ (B) $\dfrac{2l}{9}$ (C) $\dfrac{4l}{9}$ (D) $\dfrac{l}{36}$

[Sol. Let $y^2 = 4ax$; $4a = l$



$A(at^2, 2at)$

hence $h = at^2$

$3k = 2at$

$9k^2 = 4a^2 \cdot \dfrac{h}{a}$

$y^2 = \dfrac{4a}{9}x \quad \Rightarrow \quad y^2 = \dfrac{l}{9}x \Rightarrow$ (A)]

Q.3 Let a variable circle is drawn so that it always touches a fixed line and also a given circle, the line not passing through the centre of the circle. The locus of the centre of the variable circle, is

(A*) a parabola (B) a circle

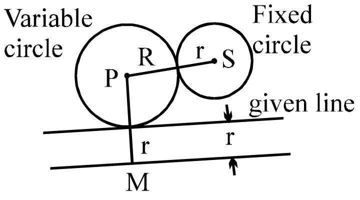
(C) an ellipse (D) a hyperbola

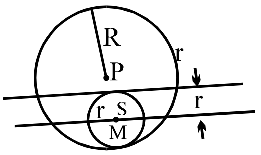
[Sol. Case-I: $PS = R + r_1 = PM$

$\therefore$ locus of P is a parabola

Case-II: $R - r_1 = PM$

$\therefore$ locus of P is a parabola.]





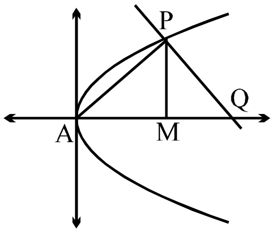
Q.4 The vertex A of the parabola $y^2 = 4ax$ is joined to any point P on it and PQ is drawn at right angles to AP to meet the axis in Q. Projection of PQ on the axis is equal to

(A) twice the latus rectum (B*) the latus rectum

(C) half the latus rectum (D) one fourth of the latus rectum

[Sol. $A(0, 0)$, $P(at^2, 2at)$, $Q(x, 0)$

Slope of AP × slope of PQ = – 1



$$\frac{2at}{at^2} \times \frac{-2at}{x_1 - at^2} = -1$$

$$(x_1 - at^2)(at^2) = 4a^2t^2$$

$$x = 4a + at^2 = AQ$$

∴ projection QM = AQ – AM = 4a = Latus rectum **Ans.]**

Q.5 Two unequal parabolas have the same common axis which is the x-axis and have the same vertex which is the origin with their concavities in opposite direction. If a variable line parallel to the common axis meet the parabolas in P and P' the locus of the middle point of PP' is

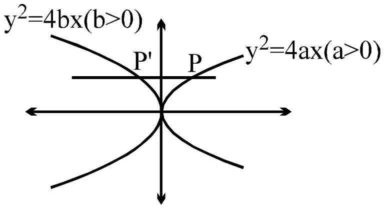
(A*) a parabola (B) a circle (C) an ellipse (D) a hyperbola

[Sol. $P(at_1^2, 2at_1)$

$P'(-bt_2^2, -2bt_2)$

Slope of PP' = 0

$at_1 + bt_2 = 0$ ....(1)



$$\text{Mid-point of PP'} = \left(\frac{at_1^2 - bt_2^2}{2}, at_1 - bt_2\right) = (h, k)$$

$$\text{put } t_2 = \frac{-at_1^2}{b}$$

$$\Rightarrow \quad \frac{k}{a} = \frac{2\left(at_1 + \frac{at_1}{b} \cdot b\right)}{at_1^2 - b\left(\frac{-at_1}{b}\right)^2} = \frac{4at_1}{a(b-a)t_1^2}; \quad \therefore \quad t_1 = \frac{4h}{k(b-a)}$$

$$\therefore \quad k = at_1 - bt_2 = at_1 + at_1 = 2a \cdot \frac{4h}{k(b-a)} \quad \Rightarrow \quad k^2 = \frac{8a}{b-a} \cdot h$$

∴ locus of (h, k) is $y^2 = \frac{8a}{b-a}x$, a parabola **Ans.]**

Q.6 The straight line $y = m(x – a)$ will meet the parabola $y^2 = 4ax$ in two distinct real points if

(A) $m \in R$ (B) $m \in [-1, 1]$

(C) $m \in (-\infty, 1] \cup [1, \infty)R$ (D*) $m \in R - \{0\}$

[Sol. $y = m(x – a)$ passes through the focus $(a, 0)$ of the parabola. Thus for this to be focal chord $m \in R - \{0\}$ **Ans.]**

Q.7 All points on the curve $y^2 = 4a\left(x + a\sin\frac{x}{a}\right)$ at which the tangent is parallel to x-axis lie on

(A) a circle (B*) a parabola (C) an ellipse (D) a line

[Sol. $y^2 = 4a\left(x + a\sin\frac{x}{a}\right)$

$$2y\frac{dy}{dx} = 4a\left(1 + a\cdot\frac{1}{a}\cos\frac{x}{a}\right)$$

$\because$ tangent is parallel to x-axis.

$\therefore$ $\frac{dy}{dx} = 0$, $\therefore$ $\cos\frac{x}{a} = -1$

$\therefore$ $\sin\frac{x}{2} = \pm\sqrt{1-\cos^2\frac{x}{a}} = 0$

$\therefore$ $y^2 = 4a\left(x + a\sin\frac{x}{a}\right) = 4ax$ $\Rightarrow$ a parabola **Ans.**]

Q.8 Locus of trisection point of any arbitrary double ordinate of the parabola $x^2 = 4by$, is

(A) $9x^2 = by$ (B) $3x^2 = 2by$ (C*) $9x^2 = 4by$ (D) $9x^2 = 2by$

[Sol. Let $A \equiv (2bt, bt^2)$, $B \equiv (-2bt, bt^2)$ be the extremities on the double ordinate AB.
If C(h, k) be it's trisection point, then
$3h = 4bt - 2bt$, $3k = 2bt^2 + bt^2$

$\Rightarrow$ $t = \frac{3h}{2b}$, $t^2 = \frac{k}{b}$ $\Rightarrow$ $\frac{k}{b} = \frac{9h^2}{4b^2}$

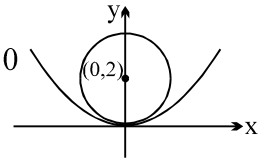
Thus locus of C is $9x^2 = 4by$ **Ans.**]

Q.9 The equation of the circle drawn with the focus of the parabola $(x-1)^2 - 8y = 0$ as its centre and touching the parabola at its vertex is :

(A) $x^2 + y^2 - 4y = 0$ (B) $x^2 + y^2 - 4y + 1 = 0$

(C) $x^2 + y^2 - 2x - 4y = 0$ (D*) $x^2 + y^2 - 2x - 4y + 1 = 0$

[Hint: Put $X^2 = 8Y$; when $x - 1 = X$ and $y = Y$
$\Rightarrow$ $(X-0)^2 + (Y-2)^2 = 4$ $\Rightarrow$ $(x-1)^2 + (y-2)^2 = 4$ $\Rightarrow$ (D)]



Q.10 The length of the latus rectum of the parabola, $y^2 - 6y + 5x = 0$ is

(A) 1 (B) 3 (C*) 5 (D) 7

[Sol. $y^2 = -6y + 5x = 0$

$\Rightarrow$ $(y-3)^2 = 9 - 5x = -5\left(x - \frac{9}{5}\right)$ $\Rightarrow$ **(C) Ans.**]

Q.11 Which one of the following equations represented parametrically, represents equation to a parabolic profile?

(A) $x = 3\cos t$; $y = 4\sin t$ (B*) $x^2 - 2 = -2\cos t$; $y = 4\cos^2\frac{t}{2}$

(C) $\sqrt{x} = \tan t$; $\sqrt{y} = \sec t$ (D) $x = \sqrt{1-\sin t}$; $y = \sin\frac{t}{2} + \cos\frac{t}{2}$

[Sol. (A) $\left(\frac{x}{3}\right)^2+\left(\frac{y}{4}\right)^2=\cos^2 t+\sin^2 t=1 \Rightarrow$ An ellipse

(B) $x^2-2=-2\cos t=-2\left(2\cos^2\frac{t}{2}-1\right) \Rightarrow$ A parabola

(C) $\sqrt{x}=\tan x$, $\sqrt{y}=\sec t \quad (x \ge 0, y \ge 0)$
$\sec^2 t-\tan^2 t=1 \quad \Rightarrow \quad y-x=1 \Rightarrow$ a line tangent

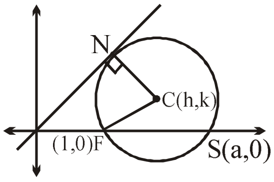
(D) $x^2=1-\sin t=2-(1+\sin t)=2-\left(\sin\frac{t}{2}+\cos\frac{t}{2}\right)^2=2-y^2 \Rightarrow$ a circle]

Q.12 The length of the intercept on y – axis cut off by the parabola, $y^2-5y=3x-6$ is
(A*) 1 (B) 2 (C) 3 (D) 5

[Sol. $y^2-5y=3x-6$
for point of intersection with y-axis, i.e. $x=0$
$\Rightarrow \quad y^2-5y+6=0$
$y=2, 3$
$\therefore \quad A(0, y_1),\ B(0, y_2)$
$\therefore \quad AB=|y_2-y_1|=|3-2|=1$ **Ans.]**

Q.13 A variable circle is described to pass through (1, 0) and touch the line y = x. The locus of the centre of the circle is a parabola, whose length of latus rectum, is

(A) 2 (B*) $\sqrt{2}$ (C) $\frac{1}{\sqrt{2}}$ (D) 1



[Sol. CF = CN Þ locus of C is a parbola with focus at (1, 0) and directrix y = x
$\Rightarrow$ length of latus rectum = 2(distance from focus to directrix)

$$=2\left(\frac{1}{\sqrt{2}}\right)=\sqrt{2}$$ **Ans.]**

Q.14 Angle between the parabolas $y^2=4b(x-2a+b)$ and $x^2+4a(y-2b-a)=0$ at the common end of their latus rectum, is

(A) $\tan^{-1}(1)$ (B*) $\cot^{-1}1+\cot^{-1}\frac{1}{2}+\cot^{-1}\frac{1}{3}$

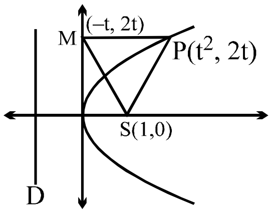
(C) $\tan^{-1}\left(\sqrt{3}\right)$ (D) $\tan^{-1}(2)+\tan^{-1}(3)$

[Hint: $y^2=4b(x-(2a-b))$ or $y^2=4bX$ where $x-(2a-b)=X$
$x^2+4a(y-(a+2b))$ or $x^2=-4aY$ where $y-(a+2b)=Y$
for $y^2=4bX$, extremities of latus rectum (b, 2b) and (b, – 2b) w.r.t. X Y axis
i.e. (2a, 2b) and (2a, – 2b) w.r.t. xy axis
for $x^2=-4aY$, extremities of latus rectum (2a, – a) and (–2a, –a) w.r.t. XY axis
i.e. (2a, 2b) and (–2a, 2b)
Hence the common end of latus rectum (2a, 2b)

now for 1st parabola $2y\frac{dy}{dx}=4b \Rightarrow \frac{dy}{dx}=\frac{2b}{y_1}=1$ at (2a, 2b)

also for 2nd parabola $2x=-4a\frac{dy}{dx}$ or $\frac{dy}{dx}=-\frac{x}{2a}=-1$ at (2a, 2b)
Hence parabolas intersect orthogonally at (2a, 2b) $\Rightarrow$ **(B)**]

Q.15 A point P on a parabola $y^2 = 4x$, the foot of the perpendicular from it upon the directrix, and the focus are the vertices of an equilateral triangle, find the area of the equilateral triangle. [Ans. $4\sqrt{3}$]



[Sol. $PM = 1 + t^2$

$PS = 1 + t^2$

$MS = 1 + t^2$

$\Rightarrow \quad 2^2 + 4t^2 = (1 + t^2)^2$

$\therefore \quad PM = 1 + t^2 = 4$

$\therefore \quad$ Area of $\Delta PMS = \dfrac{\sqrt{3}}{4}(4^2) = 4\sqrt{3}$ **Ans.]**

Q.16 Given $y = ax^2 + bx + c$ represents a parabola. Find its vertex, focus, latus rectum and the directrix.

[Ans. $\left(\dfrac{-b}{2a}, \dfrac{4ac-b^2}{4a}\right); \left(\dfrac{-b}{2a}, \dfrac{1}{4a} + \dfrac{4ac-b^2}{4a}\right); \dfrac{1}{a}, y = \dfrac{4ac-b^2}{4a} - \dfrac{1}{4a}$]

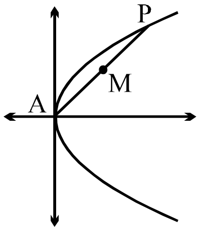
[Sol. $y = ax^2 + bx + c$

$$\Rightarrow \quad \left(x + \frac{b}{2a}\right)^2 = \frac{1}{a}\left(y - \frac{4ac-b^2}{4a}\right)$$

$\therefore \quad$ vertex : $\left(\dfrac{-b}{2a}, \dfrac{4ac-b^2}{4a}\right)$; focus : $\left(\dfrac{-b}{2a}, \dfrac{1}{4a} + \dfrac{4ac-b^2}{4a}\right)$

Latus rectum : $\dfrac{1}{a}$ and directrix : $y = \dfrac{4ac-b^2}{4a} - \dfrac{1}{4a}$]

Q.17 Prove that the locus of the middle points of all chords of the parabola $y^2 = 4ax$ passing through the vetex is the parabola $y^2 = 2ax$.



[Sol. $P(at^2, 2at)$

Mid-point of AP

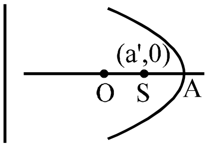
$M\left(\dfrac{a}{2}t^2, at\right) = M(h, k)$

$\therefore \quad k^2 = a^2t^2 = 2ah$

$\therefore \quad y^2 = 2ax$ **Ans.]**

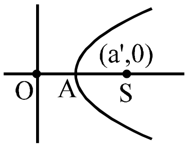
Q.18 Prove that the equation to the parabola, whose vertex and focus are on the axis of x at distances a and a' from the origin respectively, is $y^2 = 4(a' - a)(x - a)$

[Sol. Case-I:



$\therefore \quad$ equation of parabola : $y^2 = -4(a - a')(x - a) = 4(a' - a)(x - a)$ Hence proved.

Case-II:



$\therefore \quad$ equation of parabola : $y^2 = 4(a' - a)(x - a)$ Hence proved.]

Q.19 Prove that the locus of the centre of a circle, which intercepts a chord of given length 2a on the axis of x and passes through a given point on the axis of y distant b from the origin, is the curve

$$x^2 - 2yb + b^2 = a^2.$$

[Sol. $(x - x_1)^2 + (y - y_1)^2 = r^2$

x-axis intercept = 2a $\Rightarrow$ $2\sqrt{x_1^2 - (x_1^2 + y_1^2 - r^2} = 2a$

$r^2 - y_1^2 = a^2$

$\therefore$ $y_1^2 = r^2 - a^2$

Passes through (0, b)

$\therefore$ $x_1^2 + (b - y_1)^2 = r^2$ $\Rightarrow$ $x_1^2 + b^2 - 2by_1 + y_1^2 - r^2 = 0$ $\Rightarrow$ $x_1^2 + b^2 - 2by_1 - a^2 = 0$

$\therefore$ Locus of (x_1, y_1) will be $x^2 - 2by + b^2 = a^2$ Hence proved.]

Q.20 A variable parabola is drawn to pass through A & B, the ends of a diameter of a given circle with centre at the origin and radius c & to have as directrix a tangent to a concentric circle of radius 'a' (a > c) ; the axes being AB & a perpendicular diameter, prove that the locus of the focus of the parabola is the standard ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where $b^2 = a^2 - c^2$.

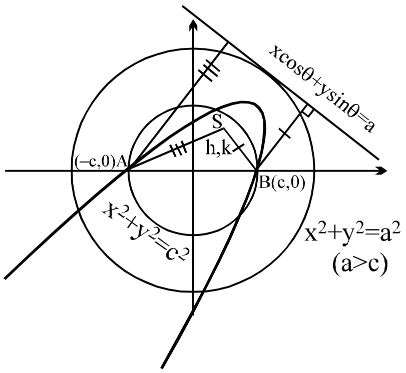
[Sol. $(h - c)^2 + k^2 = (c\cos\theta - a)^2$ ....(1)

sub. $(h + c)^2 + k^2 = (c\cos\theta + a)^2$ ....(2)

$4ch = 4ca\cos\theta \Rightarrow h = a\cos\theta$ .....(3)

add $2(c^2 + h^2 + k^2) = 2\,(c^2\cos^2\theta + a^2)$ .....(4)

put $\cos\theta = \frac{h}{a}$ in equation (4)



we get $c^2 + h^2 + k^2 = c^2 \cdot \frac{h^2}{a^2} + a^2$

$$c^2a^2 + h^2a^2 + k^2a^2 = c^2h^2 + a^4$$

$$(a^2 - c^2)h^2 + k^2a^2 = a^2(a^2 - c^2)$$

$$\frac{h^2}{a^2} + \frac{k^2}{a^2 - c^2} = 1$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$]

MC SIR

CLASS : XIII ***DPP. NO.- 2***

Q.1 If a focal chord of $y^2 = 4ax$ makes an angle α, $\alpha \in \left(0, \frac{\pi}{4}\right]$ with the positive direction of x-axis, then minimum length of this focal chord is

(A) 6a (B) 2a (C*) 8a (D) None

[Sol. Length of focal chord making an angle α with x-axis is $4a\,\text{cosec}^2\alpha$.

For $\alpha \in \left(0, \frac{\pi}{4}\right]$, it's minimum length = (4a)(2) = 8a units. **Ans.**]

Q.2 OA and OB are two mutually perpendicular chords of $y^2 = 4ax$, 'O' being the origin. Line AB will always pass through the point

(A) (2a, 0) (B) (6a, 0) (C) (8a, 0) (D*) (4a, 0)

[Sol. Let $A \equiv (at_1^2, 2at_1)$, $B \equiv (at_2^2, 2at_2)$

Thus $t_1t_2 = -4$

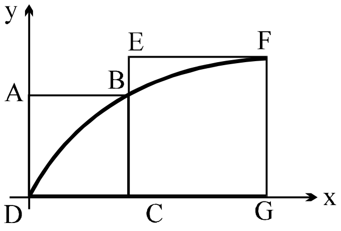
Equation of line AB is

$$y(t_1 + t_2) = 2(x + at_1t_2),$$

i.e. $y(t_1 + t_2) = 2(x - 4a)$

which clearly passes through a fixed point (4a, 0) **Ans.**]

Q.3$_{4/para}$ ABCD and EFGC are squares and the curve $y = k\sqrt{x}$ passes through the origin D and the points B and F. The ratio $\frac{FG}{BC}$ is



(A*) $\frac{\sqrt{5}+1}{2}$ (B) $\frac{\sqrt{3}+1}{2}$

(C) $\frac{\sqrt{5}+1}{4}$ (D) $\frac{\sqrt{3}+1}{4}$

[Sol. $y^2 = k^2x \quad \Rightarrow \quad y^2 = 4ax \quad$ where $k^2 = 4a$

$B = \left(at_1^2, 2at_1\right)$; $F = \left(at_2^2, 2at_2\right) \qquad \{t_1 > 0, t_2 > 0\}$

to find $\frac{FG}{BC} = \frac{2at_2}{2at_1} = \frac{t_2}{t_1}$

now $DC = BC \quad \Rightarrow \quad at_1^2 = 2at_1 \quad \Rightarrow \quad t_1 = 2$

also $at_2^2 - at_1^2 = 2at_2$

$t_2^2 - 4 = 2t_2$

$t_2^2 - 2t_2 - 4 = 0$

$t_2 = \frac{2 \pm \sqrt{4+16}}{2}$ but $t_2 > 0 \quad \therefore \quad t_2 \neq \frac{2-\sqrt{20}}{2}$

$t_2 = \left(\sqrt{5}+1\right)$

$\therefore \quad \frac{t_2}{t_1} = \frac{\sqrt{5}+1}{2}$ Ans.]

Q.4$_{13/para}$ From an external point P, pair of tangent lines are drawn to the parabola, $y^2 = 4x$. If θ_1 & θ_2 are the inclinations of these tangents with the axis of x such that, $\theta_1 + \theta_2 = \frac{\pi}{4}$, then the locus of P is :

(A) $x - y + 1 = 0$ (B) $x + y - 1 = 0$ (C*) $x - y - 1 = 0$ (D) $x + y + 1 = 0$

[Hint: $y = mx + \frac{1}{m}$

or $m^2h - mk + 1 = 0$

$m_1 + m_2 = \frac{k}{h}$; $m_1 m_2 = \frac{1}{h}$

given $\theta_1 + \theta_2 = \frac{\pi}{4}$ $\Rightarrow$ $\frac{m_1 + m_2}{1 - m_1 m_2} = 1$ $\Rightarrow$ $\frac{k}{h} = 1 - \frac{1}{h}$ $\Rightarrow$ $y = x - 1$]

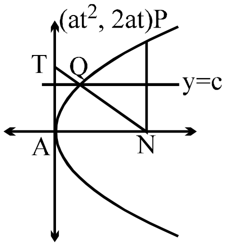
Q.5$_{14/para}$ Maximum number of common chords of a parabola and a circle can be equal to

(A) 2 (B) 4 (C*) 6 (D) 8

[Sol. A circle and a parabola can meet at most in four points. Thus maximum number of common chords in ${}^4C_2$ i.e. 6 **Ans.**]

Q.6$_{53/para}$ PN is an ordinate of the parabola $y^2 = 4ax$. A straight line is drawn parallel to the axis to bisect NP and meets the curve in Q. NQ meets the tangent at the vertex in apoint T such that AT = kNP, then the value of k is (where A is the vertex)

(A) 3/2 (B*) 2/3 (C) 1 (D) none



[Sol. Equation of PN : $x = at^2$

$y = c$ bisects PN

$\therefore$ $c = at$

which cuts the parabola at Q

$\Rightarrow$ $c^2 = 4ax$ $\therefore$ $x = \frac{c^2}{4a}$

$\therefore$ $Q\left(\frac{c^2}{4a}, c\right) = Q\left(\frac{at^2}{4}, at\right)$

$\therefore$ Equation of NQ: $y - 0 = \dfrac{at - 0}{\frac{at^2}{4} - at^2}(x - at^2)$

$y = \frac{-4}{3t}(x - at^2)$

which cuts $x = 0$ at $\left(0, \frac{4at}{3}\right)$

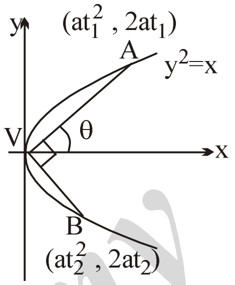
$\therefore$ $T = \frac{4at}{3}$ and $NP = 2at$

$\therefore$ $\frac{AT}{NP} = \frac{4at/3}{2at} = \frac{2}{3}$ **Ans.**]

Q.7$_{24/para}$ Let A and B be two points on a parabola $y^2 = x$ with vertex V such that VA is perpendicular to VB and θ is the angle between the chord VA and the axis of the parabola. The value of $\frac{|VA|}{|VB|}$ is

(A) $\tan\theta$ (B) $\tan^3\theta$ (C) $\cot^2\theta$ (D*) $\cot^3\theta$

[Hint: $\tan\theta = \frac{2}{t_1}$ ....(1)



Also $\frac{2}{t_1} \times \frac{2}{t_2} = -1$

$t_1t_2 = -4$

$|VA| = \sqrt{a^2t_1^4 + 4a^2t_1^2} = at_1\sqrt{t_1^2+4}$

using $t_1t_2 = -4$

$|VB| = \frac{4a}{t_1}\sqrt{\frac{16}{t_1^2}+4} = \frac{8a}{t_1^2}\sqrt{4+t_1^2}$

$\therefore \quad \frac{|VA|}{|VB|} = \frac{at_1^3\sqrt{4+t_1^2}}{8a\sqrt{4+t_1^2}} = \frac{t_1^3}{8}$

Also $\tan\theta = \frac{2}{t_1}$; $\therefore$ $t_1^3 = 8\cot^3\theta$

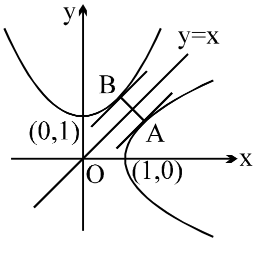
$\therefore \quad \frac{|VA|}{|VB|} = \cot^3\theta$ Ans.]

Q.8$_{25/para}$ Minimum distance between the curves $y^2 = x - 1$ and $x^2 = y - 1$ is equal to

(A*) $\frac{3\sqrt{2}}{4}$ (B) $\frac{5\sqrt{2}}{4}$ (C) $\frac{7\sqrt{2}}{4}$ (D) $\frac{\sqrt{2}}{4}$

[Sol. Both curve are symmstrical about the line y = x. If line AB is the line of shortest distance then at A and B slopes of curves should be equal to one

for $y^2 = x - 1$, $\frac{dy}{dx} = \frac{1}{2y} = 1$



$\Rightarrow \quad y = \frac{1}{2}, x = \frac{5}{4}$

$\Rightarrow \quad B \equiv \left(\frac{1}{2}, \frac{5}{4}\right)$ and $A \equiv \left(\frac{5}{4}, \frac{1}{2}\right)$

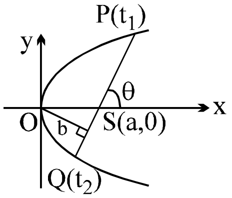
hence minimum distance AB,

$$= \sqrt{\left(\frac{5}{4}-\frac{1}{2}\right)^2 + \left(\frac{5}{4}-\frac{1}{2}\right)^2} = \frac{3\sqrt{2}}{4} \textbf{ Ans.]}$$

Q.9$_{28/\text{para}}$ The length of a focal chord of the parabola $y^2 = 4ax$ at a distance b from the vertex is c, then

(A) $2a^2 = bc$ (B) $a^3 = b^2c$ (C) $ac = b^2$ (D*) $b^2c = 4a^3$

[Sol. Equation focal chord PQ : $2x - (t_1 + t_2)y - 2a = 0$ ....(1)



l(PQ) = c

$$\text{cosec}\,\theta = \frac{a}{b}$$

now $c = 4a\,\text{cosec}^2\theta$

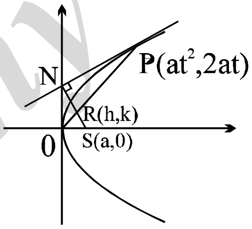
$$c = 4a \cdot \frac{a^2}{b^2}; \quad 4a^3 = b^2c$$]

Q.10$_{34/\text{para}}$ The straight line joining any point P on the parabola $y^2 = 4ax$ to the vertex and perpendicular from the focus to the tangent at P, intersect at R, then the equaiton of the locus of R is

(A) $x^2 + 2y^2 - ax = 0$ (B*) $2x^2 + y^2 - 2ax = 0$

(C) $2x^2 + 2y^2 - ay = 0$ (D) $2x^2 + y^2 - 2ay = 0$

[Sol. T : $ty = x + at^2$ ....(1)



line perpendicular to (1) through (a,0)

$tx + y = ta$ ....(2)

equation of OP : $y - \frac{2}{t}x = 0$ ....(3)

from (2) & (3) eleminating t we get locus]

Q.11$_{41/\text{para}}$ Locus of the feet of the perpendiculars drawn from vertex of the parabola $y^2 = 4ax$ upon all such chords of the parabola which subtend a right angle at the vertex is

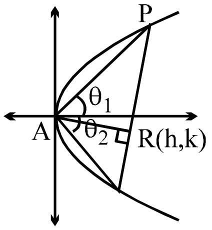
(A*) $x^2 + y^2 - 4ax = 0$ (B) $x^2 + y^2 - 2ax = 0$

(C) $x^2 + y^2 + 2ax = 0$ (D) $x^2 + y^2 + 4ax = 0$

[Hint: Chord with feet of the perpendicular as (h, k) is $hx + ky = h^2 + k^2$ .....(1)

homogenise $y^2 = 4ax$ with the help of (1) and use coefficient of x^2 + coefficient of $y^2 = 0$

Alternatively-1: $\tan\theta_1 = \frac{2}{t_1}$; $\tan\theta_2 = \frac{2}{t_2}$



$$\frac{2}{t_1}\cdot\frac{2}{t_2} = -1, \quad \therefore \quad t_1t_2 = -4$$

$\therefore$ equation of chord PQ $2x - (t_1 + t_2)y - 8a = 0$

slope of AR × slope of PQ = – 1

$$\therefore \quad \frac{k}{h}\left(\frac{2}{t_1 + t_2}\right) = -1$$

$$\therefore \quad t_1 + t_2 = \frac{-2k}{h}$$

$\therefore$ equation of chord PQ will be $2x - \left(\frac{-2k}{h}\right)y - 8a = 0$

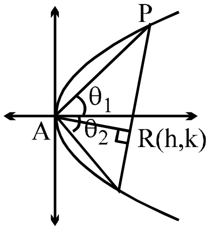
$hx + ky = 4ah$

(h, k) lies on this line

$\therefore$ $h^2 + k^2 = 4ah$

$\therefore$ locus of R(h, k) is $x^2 + y^2 = 4ax$ **Ans.**

Alternatively-2:



Slope of PQ $= \dfrac{-1}{\text{slope of OR}} = \dfrac{-h}{k}$

$\therefore$ equation of PQ will be

$$y - k = \frac{-h}{k}(x-h) \quad \Rightarrow \quad hx + ky = h^2 + k^2$$

This chord subtends a right angle at vertex O(0, 0)

$\therefore$ by homogenisation we get equation of pair of straight line OP and OQ

$$y^2 = 4ax\left(\frac{hx+ky}{h^2+k^2}\right)$$

$\because$ OP ^ OQ

$\therefore$ coefficient of x^2 + coefficient of $y^2 = 0$

$\therefore$ $h^2 + k^2 = 4ah$

$\therefore$ locus of (h, k) wil be

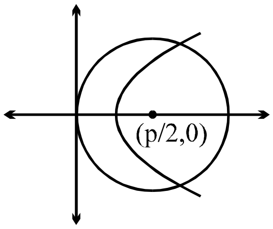
$x^2 + y^2 = 4ax$ **Ans.]**

More than one are correct:

Q.12$_{503/para}$ Consider a circle with its centre lying on the focus of the parabola, $y^2 = 2px$ such that it touches the directrix of the parabola. Then a point of intersection of the circle & the parabola is

(A*) $\left(\frac{p}{2}, p\right)$ (B*) $\left(\frac{p}{2}, -p\right)$ (C) $\left(-\frac{p}{2}, p\right)$ (D) $\left(-\frac{p}{2}, -p\right)$

[Sol. Equation of circle will be $\left(x-\frac{p}{2}\right)^2 + y^2 = p^2$



which intersects $y^2 = 2px$

$$\therefore \quad \left(x-\frac{p}{2}\right)^2 + 2px = p^2$$

$$x^2 + px - \frac{3p^2}{4} = 0$$

$$\left(x+\frac{3p^2}{4}\right)\left(x-\frac{p}{2}\right) = 0$$

$$x + \frac{3p}{4} \neq 0$$

$$\therefore \quad x = \frac{p}{2} \text{ only}$$

$$\therefore \quad y^2 = 2p\,\frac{p}{2} \quad \Rightarrow \quad y = \pm p$$

Hence $\left(\frac{p}{2}, -p\right)$ and $\left(\frac{p}{2}, p\right)$ **Ans.]**

Q.1 y-intercept of the common tangent to the parabola $y^2 = 32x$ and $x^2 = 108y$ is
(A) – 18 (B*) – 12 (C) – 9 (D) – 6

[Sol. Tangent to $y^2 = 32x$ is $y = mx + \frac{8}{m}$ and tangent to $x^2 = 108y$ is $y = mx - 27m^2$

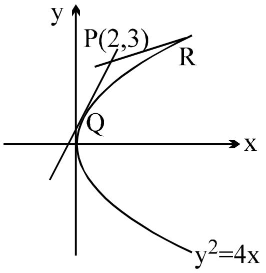
$$\therefore \quad \frac{8}{m} = -27m^2, \quad \therefore \quad m^3 = \frac{-8}{27}$$

$$\therefore \quad m = \frac{-2}{m}$$

$$\therefore \quad \text{y-intercept} = \frac{8}{m} = 8\left(\frac{-3}{2}\right) = -12 \ \textbf{Ans.]}$$

Q.2$_{9/\text{para}}$ The points of contact Q and R of tangent from the point P (2, 3) on the parabola $y^2 = 4x$ are

(A) (9, 6) and (1, 2) (B*) (1, 2) and (4, 4) (C) (4, 4) and (9, 6) (D) (9, 6) and $(\frac{1}{4}, 1)$



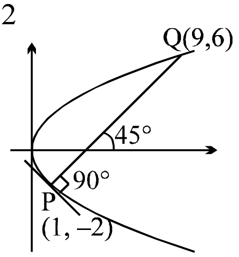
[Hint: $\left.\begin{array}{l} t_1t_2 = 2 \\ t_1 + t_2 = 3 \end{array}\right] \Rightarrow t_1 = 1$ and $t_2 = 2$

Hence point $(t_1^2, 2t_1)$ and $(t_2^2, 2t_2)$
i.e. (1, 2) and (4, 4)]

Q.3$_{18/\text{para}}$ Length of the normal chord of the parabola, $y^2 = 4x$, which makes an angle of $\frac{\pi}{4}$ with the axis of x is:

(A) 8 (B*) $8\sqrt{2}$ (C) 4 (D) $4\sqrt{2}$

[Sol. N : $y + tx = 2t + t^3$; slope of the normal is $-t$
hence $-t = 1 \Rightarrow t = -1 \Rightarrow$ coordinates of P are (1, –2)
Hence parameter at Q, $t_2 = -t_1 - 2/t_1 = 1 + 2 = 3$
$\therefore$ Coordinates at Q are (9, 6)
$\therefore$ $l\,(PQ) = \sqrt{64+64} = 8\sqrt{2}$]



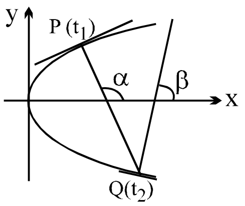
Q.4$_{21/\text{para}}$ If the lines $(y - b) = m_1(x + a)$ and $(y - b) = m_2(x + a)$ are the tangents to the parabola $y^2 = 4ax$, then
(A) $m_1 + m_2 = 0$ (B) $m_1m_2 = 1$ (C*) $m_1m_2 = -1$ (D) $m_1 + m_2 = 1$

[Sol. Clearly, both the lines passes through (–a, b) which is a point lying on the directrix of the parabola
Thus, $m_1m_2 = -1$
Because tangents drawn from any point on the directrix are always mutually perpendicular]

Q.5$_{23/\text{para}}$ If the normal to a parabola $y^2 = 4ax$ at P meets the curve again in Q and if PQ and the normal at Q makes angles α and β respectively with the x-axis then $\tan\alpha(\tan\alpha + \tan\beta)$ has the value equal to

(A) 0 (B*) – 2 (C) $-\frac{1}{2}$ (D) – 1

[Sol. $\tan\alpha = -t_1$ and $\tan\beta = -t_2$



also $t_2 = -t_1 - \frac{2}{t_1}$

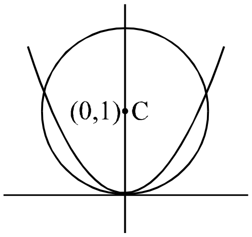
$t_1 t_2 + t_1^2 = -2$

$\tan\alpha\tan\beta + \tan^2\alpha = -2$ ⇒ (B)]

Q.6$_{43/\text{para}}$ C is the centre of the circle with centre (0, 1) and radius unity. P is the parabola $y = ax^2$. The set of values of 'a' for which they meet at a point other than the origin, is

(A) $a > 0$ (B) $a \in \left(0, \frac{1}{2}\right)$ (C) $\left(\frac{1}{4}, \frac{1}{2}\right)$ (D*) $\left(\frac{1}{2}, \infty\right)$

[Hint: put $x^2 = \frac{y}{a}$ in circle, $x^2 + (y-1)^2 = 1$, we get (Note that for $a < 0$ they cannot intersect other than origin)



$\frac{y}{a} + y^2 - 2y = 0$; hence we get $y = 0$ or $y = 2 - \frac{1}{a}$

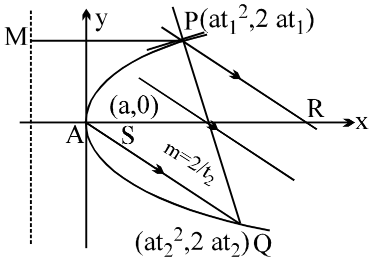
substituting $y = 2 - \frac{1}{a}$ in $y = ax^2$, we get

$ax^2 = 2 - \frac{1}{a}$; $x^2 = \frac{2a-1}{a^2} > 0$ ⇒ $a > \frac{1}{2}$]

Q.7$_{80/\text{para}}$ PQ is a normal chord of the parabola $y^2 = 4ax$ at P, A being the vertex of the parabola. Through P a line is drawn parallel to AQ meeting the x–axis in R. Then the length of AR is :

(A) equal to the length of the latus rectum
(B) equal to the focal distance of the point P
(C*) equal to twice the focal distance of the point P
(D) equal to the distance of the point P from the directrix.

[Hint : $t_2 = -t_1 - \frac{2}{t_1}$ ⇒ $t_1t_2 + t_1^2 = -2$



Equation of the line through P parallel to AQ

$y - 2at_1 = \frac{2}{t_2}(x - at_1^2)$

put $y = 0$ ⇒ $x = at_1^2 - at_1t_2$

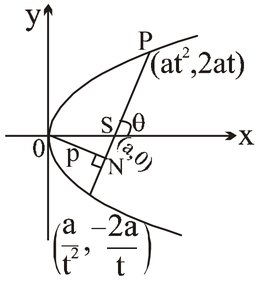
$= at_1^2 - a(-2 - t_1^2)$

$= 2a + 2at_1^2 = 2(a + at_1^2)$

= twice the focal distance of P]

Q.8$_{84/\text{para}}$ Length of the focal chord of the parabola $y^2 = 4ax$ at a distance p from the vertex is :

(A) $\frac{2a^2}{p}$ (B) $\frac{a^3}{p^2}$ (C*) $\frac{4a^3}{p^2}$ (D) $\frac{p^2}{a}$



[Hint : Length $= \sqrt{\left(2at + \frac{2a}{t}\right)^2 + \left(at^2 - \frac{a}{t^2}\right)^2} = \frac{a\left(1+t^2\right)^2}{t^2}$

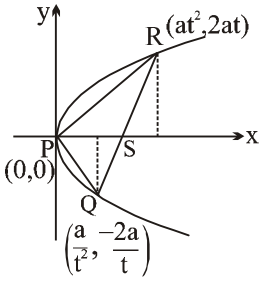
Now equation of focal chord, $2tx + y(1 - t^2) - 2at = 0$

$$\Rightarrow\ p = \left|\frac{2at}{1+t^2}\right| \ \Rightarrow\ \frac{4a^2}{p^2} = \frac{\left(1+t^2\right)^2}{t^2}.$$

Alternatively : $\operatorname{cosec}\theta = \frac{a}{p}$ $\Rightarrow$ Length of focal chord $= 4a\operatorname{cosec}^2\theta = \frac{4a^3}{p^2}$]

Q.9$_{88/\text{para}}$ The triangle PQR of area 'A' is inscribed in the parabola $y^2 = 4ax$ such that the vertex P lies at the vertex of the parabola and the base QR is a focal chord. The modulus of the difference of the ordinates of the points Q and R is :

(A) $\frac{A}{2a}$ (B) $\frac{A}{a}$ (C*) $\frac{2A}{a}$ (D) $\frac{4A}{a}$



[Hint : $d = \left|2at + \frac{2a}{t}\right| = 2a\left|t + \frac{1}{t}\right|.$

$$\text{Now } A = \frac{1}{2}\begin{vmatrix} at^2 & 2at & 1 \\ \frac{a}{t^2} & -\frac{2a}{t} & 1 \\ 0 & 0 & 1 \end{vmatrix} = a^2\left(t + \frac{1}{t}\right)$$

$$\Rightarrow\ 2a\left(t + \frac{1}{t}\right) = \frac{2A}{a}$$]

Q.10$_{127/\text{para}}$ The roots of the equation $m^2 - 4m + 5 = 0$ are the slopes of the two tangents to the parabola $y^2 = 4x$. The tangents intersect at the point

(A) $\left(\frac{4}{5}, \frac{1}{5}\right)$ (B*) $\left(\frac{1}{5}, \frac{4}{5}\right)$ (C) $\left(-\frac{1}{5}, \frac{4}{5}\right)$

(D) point of intersection can not be found as the tangents are not real

[Sol. $y = mx + \frac{1}{m}$

it passes through (h, k)

$$k = mh + \frac{1}{m} \quad\Rightarrow\quad hm^2 - km + 1 = 0 \qquad ....(1)$$

$$m_1 + m_2 = \frac{k}{h} \quad\text{and}\quad m_1m_2 = \frac{1}{h}$$

but m_1 and m_2 are the roots of $m^2 - 4m + 5 = 0$

$$\therefore\quad m_1 + m_2 = 4 = \frac{k}{h} \quad\text{and}\quad m_1m_2 = 5 = \frac{1}{h}$$

$$h = \frac{1}{5} \ \text{and}\ \therefore\ k = \frac{4}{5} \ \Rightarrow \left(\frac{1}{5}, \frac{4}{5}\right) \text{ Ans. }$$]

Q.11$_{17/para}$ Through the focus of the parabola $y^2 = 2px$ $(p > 0)$ a line is drawn which intersects the curve at $A(x_1, y_1)$ and $B(x_2, y_2)$. The ratio $\frac{y_1y_2}{x_1x_2}$ equals

(A) 2 (B) – 1 (C*) – 4 (D) some function of p

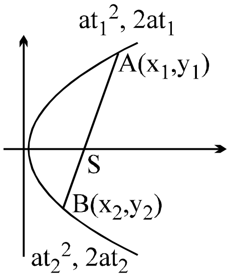
[Sol. $y^2 = 4ax$, $4a = 2p > 0$

$x_1 = at_1^2$, $y_1 = 2at_1$

$x_2 = at_2^2$, $y_2 = 2at_2$

and $t_1t_2 = -1$

ratio $= \frac{4a^2t_1t_2}{a^2t_1^2t_2^2} = -4$ Ans.]



Q.12$_{15/para}$ If the line $2x + y + K = 0$ is a normal to the parabola, $y^2 + 8x = 0$ then K =

(A) – 16 (B) – 8 (C) – 24 (D*) 24

[Sol. $m = -2$, $a = -2$

$\therefore$ equation of normal

$y = -2x - 2(-2)(-2) - (-2)(-2)^3$

$2x + y + 24 = 0$

$\therefore$ $k = 24$ **Ans.]**

Q.13$_{89/para}$ The normal chord of a parabola $y^2 = 4ax$ at the point whose ordinate is equal to the abscissa, then angle subtended by normal chord at the focus is :

(A) $\frac{\pi}{4}$ (B) $\tan^{-1}\sqrt{2}$ (C) $\tan^{-1}2$ (D*) $\frac{\pi}{2}$

[Sol. $at_1^2 = 2at_1$ $\Rightarrow$ $t_1 = 2$; P(4a , 4a)

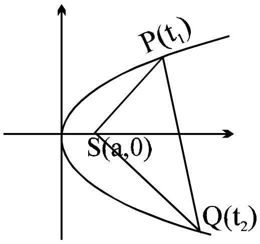
$t_2 = -t_1 - \frac{2}{t_1} = -3$

$\therefore$ Q (9a ,– 6a)

$\therefore$ $m_{SP} = \frac{4a}{4a - a} = \frac{4}{3}$

$m_{SQ} = \frac{-6a}{9a - a} = -\frac{3}{4}$

$\angle PSQ = 90^0$



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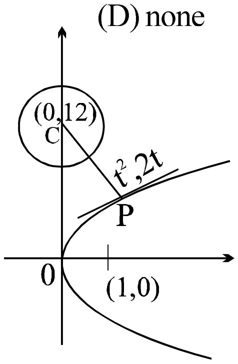
Q.14$_{90/para}$ The point(s) on the parabola $y^2 = 4x$ which are closest to the circle, $x^2 + y^2 - 24y + 128 = 0$ is/are :

(A) (0, 0) (B) $\left(2, 2\sqrt{2}\right)$ (C*) (4, 4) (D) none

[Hint : centre (0, 12) ; slope of tangent at $(t^2, 2t)$ is $1/t$, hence slope of normal is $-t$. This must be the slope of the line joining centre (0, 12) to the point $(t^2, 2t)$ $\Rightarrow$ $t = 2$]

[Sol. slope at normal at P = m_{CP}



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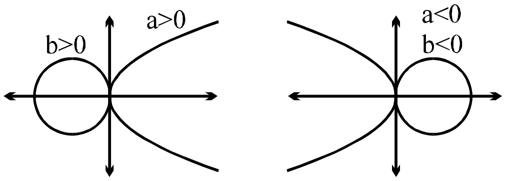
More than one are correct:

Q.15$_{508/\text{para}}$ Let $y^2 = 4ax$ be a parabola and $x^2 + y^2 + 2\,bx = 0$ be a circle. If parabola and circle touch each other externally then :

(A*) $a > 0, b > 0$ (B) $a > 0, b < 0$ (C) $a < 0, b > 0$ (D*) $a < 0, b < 0$

[Hint : For externally touching a & b must have the same sign



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Q.16$_{516/\text{para}}$ The straight line $y + x = 1$ touches the parabola :

(A*) $x^2 + 4y = 0$ (B*) $x^2 - x + y = 0$

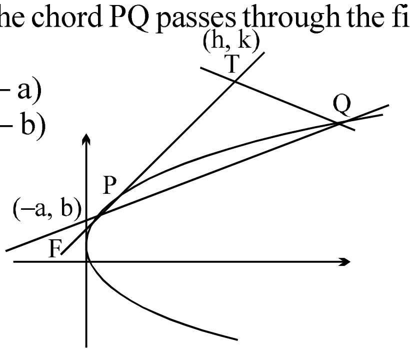
(C*) $4x^2 - 3x + y = 0$ (D) $x^2 - 2x + 2y = 0$

[Hint: put $y = 1 - x$ and see that the resulting exprassion is a perfect square]

Q.1$_{45/para}$ TP & TQ are tangents to the parabola, $y^2 = 4ax$ at P & Q. If the chord PQ passes through the fixed point $(-a, b)$ then the locus of T is :

(A) $ay = 2b(x - b)$ (B) $bx = 2a(y - a)$

(C*) $by = 2a(x - a)$ (D) $ax = 2b(y - b)$



[Hint : Chord of contact of (h, k)

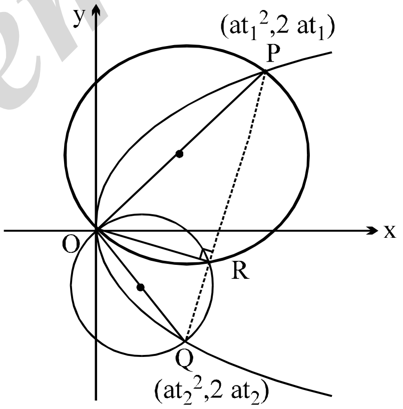
$ky = 2a(x + h)$. It passes through $(-a, b)$

$\Rightarrow bk = 2a(-a + h)$

$\Rightarrow$ Locus is $by = 2a(x - a)$]

Q.2$_{46/para}$ Through the vertex O of the parabola, $y^2 = 4ax$ two chords OP & OQ are drawn and the circles on OP & OQ as diameters intersect in R. If θ_1, θ_2 & ϕ are the angles made with the axis by the tangents at P & Q on the parabola & by OR then the value of, $\cot\theta_1 + \cot\theta_2 =$

(A*) $-2\tan\phi$ (B) $-2\tan(\pi - \phi)$ (C) 0 (D) $2\cot\phi$



[Hint : Slope of tangant at P is $\dfrac{1}{t_1}$

and at Q = $\dfrac{1}{t_2}$

$\Rightarrow \cot\theta_1 = t_1$ and $\cot\theta_2 = t_2$

Slope of PQ = $\dfrac{2}{t_1 + t_2}$

$\Rightarrow$ Slope of OR is $\left(-\dfrac{t_1 + t_2}{2}\right) = \tan\phi$

(Note angle in a semicircle is 90°)

$\Rightarrow \tan\phi = -\dfrac{1}{2}(\cot\theta_1 + \cot\theta_2) \Rightarrow \cot\theta_1 + \cot\theta_2 = -2\tan\phi$]

Q.3$_{51/para}$ If a normal to a parabola $y^2 = 4ax$ makes an angle ϕ with its axis, then it will cut the curve again at an angle

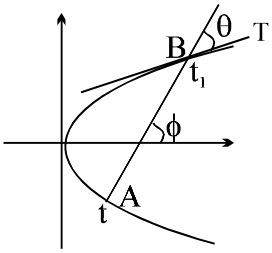
(A) $\tan^{-1}(2\tan\phi)$ (B*) $\tan^{-1}\left(\dfrac{1}{2}\tan\phi\right)$ (C) $\cot^{-1}\left(\dfrac{1}{2}\tan\phi\right)$ (D) none

[Sol. equation of normal at t : $y + tx = 2at + at^3$

$\therefore$ m_N at A = $-t = \tan\phi$

$t = -\tan\phi = m_1$

Now tangent at B $t_1y = x + at_1^2$ with $m_2 = \dfrac{1}{\tan\theta}$



also $t_1 = -t - \dfrac{2}{t}$

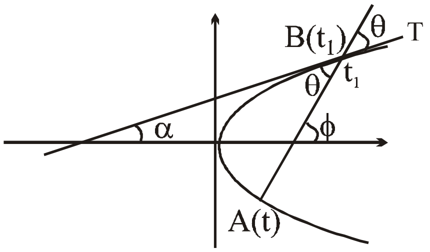
$$\therefore \quad \tan\theta = \left|\frac{\frac{1}{t_1} + t}{1 - \frac{t}{t_1}}\right| = \left|\frac{1 + tt_1}{t - t_1}\right| = \left|\frac{1 + t^2}{2\left(t + \frac{1}{t}\right)}\right| = \left|\frac{\sec^2\phi \,.\, \tan\phi}{2(\sec^2\phi)}\right| \quad [\text{As } t\,t_1 = -t^2 - 2]$$

Hence $\tan\theta = \dfrac{\tan\phi}{2} \quad \Rightarrow \theta = \tan^{-1}\left(\dfrac{\tan\phi}{2}\right)$

Alterntively: Equation of normal at A

$$y + tx = 2at + at^3$$

$\therefore$ Slope of normal at A, $m_A = -\,t \;\Rightarrow\; \tan\phi = -\,t$



Equation of tangent at B: $t_1 y = xt + at_1^2$

slope, $\tan a = \dfrac{1}{t_1}$ where $t_1 = -\,t - \dfrac{2}{t}$

$$\Rightarrow \quad \frac{1}{\tan\alpha} = \tan\phi + \frac{2}{\tan\phi} \quad \Rightarrow \quad \tan\alpha = \frac{\tan\phi}{2+\tan^2\phi}$$

$$\therefore \quad \tan\theta = \left|\frac{\tan\alpha - \tan\phi}{1+\tan\alpha\tan\phi}\right| = \left|\frac{\dfrac{\tan\phi}{2+\tan^2\phi} - \tan\phi}{1+\dfrac{\tan\phi}{2+\tan^2\phi}\tan\phi}\right| = \left|\tan\phi\left(\frac{1-2-\tan^2\phi}{2+\tan^2\phi+\tan^2\phi}\right)\right|$$

$$= \frac{\tan\phi}{2} \qquad \therefore \qquad \theta = \tan^{-1}\left(\frac{\tan\phi}{2}\right) \;]$$

Q.4$_{52/para}$ Tangents are drawn from the points on the line $x - y + 3 = 0$ to parabola $y^2 = 8x$. Then the variable chords of contact pass through a fixed point whose coordinates are :

(A) (3, 2) (B) (2, 4) (C*) (3, 4) (D) (4, 1)

[Hint: Let P (a, (a + 3)) be a point on the line and chord of contact is

$(a + 3)y = 4\,(x + a) \Rightarrow 4x - 3y + a\,(4 - y) = 0 \Rightarrow$ line passes through a fixed point (3,4)]

Q.5$_{61/para}$ If the tangents & normals at the extremities of a focal chord of a parabola intersect at (x_1, y_1) and (x_2, y_2) respectively, then :

(A) $x_1 = x_2$ (B) $x_1 = y_2$ (C*) $y_1 = y_2$ (D) $x_2 = y_1$

[Hint: $x_1 = at_1t_2,\ y_1 = a(t_1 + t_2)$; $x_2 = a(t_1^2 + t_2^2 + t_1t_2 + 2),\ y_2 = -\,at_1t_2(t_1 + t_2)$ with $t_1t_2 = -\,1$

$x_1 = -\,a,\ y_1 = a(t_1 + t_2)$; $x_2 = a(t_1^2 + t_2^2 + 1)$; $y_2 = a(t_1 + t_2)$]

Q.6$_{74/para}$ If two normals to a parabola $y^2 = 4ax$ intersect at right angles then the chord joining their feet passes through a fixed point whose co-ordinates are :

(A) (– 2a, 0) (B*) (a, 0) (C) (2a, 0) (D) none

[Hint : $t_1\,t_2 = -\,1$]

[Sol. N : $y + tx = 2at + at^3$; passes through (h, k)

Hence $at^3 + (2a - h)t + k = 0$; $t_1\,t_2\,t_3 = -\dfrac{k}{a}$; $t_1\,t_2 = -\,1$

chord joining t_1 and t_2 is $2x - (t_1 + t_2)y + 2at_1\,t_2 = 0$

$(2x - 2a) - (t_1 + t_2)y = 0 \;\Rightarrow\; x = a \;\&\; y = 0$

Alternatively: If the normal intersect at right angles then their corresponding tangents will also intersect at right angles hence the chord joining their feet must be a focal chord

$\therefore$ it will always pass through (a, 0)]

Q.7$_{75/para}$ The equation of a straight line passing through the point (3, 6) and cutting the curve $y=\sqrt{x}$ orthogonally is

(A*) $4x + y - 18 = 0$ (B) $x + y - 9 = 0$ (C) $4x - y - 6 = 0$ (D) none

[Hint: Normal to the parabola $y^2 = x$ is $y = mx - \frac{m}{2} - \frac{m^3}{4}$; pass through the point

(3, 6) $\Rightarrow$ $m^3 - 10m + 24 = 0$; $m = -4$ is a root $\Rightarrow$ required equation $4x + y - 18 = 0$

alt. (t^2, t) be a point on $y = \sqrt{x}$ $\Rightarrow$ $\frac{dy}{dx} = \frac{1}{2\sqrt{x}} = \frac{1}{2t}$ $\Rightarrow$ $\frac{t-6}{t^2-3} = -2t$ (slope of normal)

$\Rightarrow$ $2t^3 - 5t - 6 = 0$

$= (t-2)(2t^2 + 4t + 3)$ $\Rightarrow$ $t = 2$ $\Rightarrow$ slope of normal is -4]

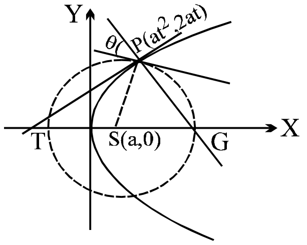
Q.8$_{121/para}$ The tangent and normal at P(t), for all real positive t, to the parabola $y^2 = 4ax$ meet the axis of the parabola in T and G respectively, then the angle at which the tangent at P to the parabola is inclined to the tangent at P to the circle passing through the points P, T and G is

(A) $\cot^{-1}t$ (B) $\cot^{-1}t^2$ (C*) $\tan^{-1}t$ (D) $\tan^{-1}t^2$

[Sol. slope of tangent $= \frac{1}{t}$ (m_1) at P on parabola

slope of PS $= \frac{2at}{a(t^2-1)} = \frac{2t}{t^2-1}$ (normal to the circle)

$\therefore$ slope of tangent at P on circle $= \frac{1-t^2}{2t}(m_2)$



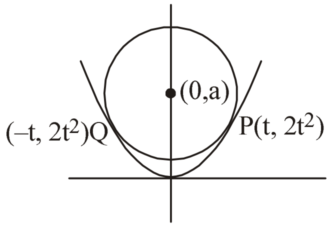
$\therefore$ $\tan\theta = \dfrac{\frac{1}{t} - \frac{1-t^2}{2t}}{1+\frac{1-t^2}{2t^2}} = \dfrac{(2-1+t^2)2t^2}{2t(1+t^2)} = t$

$\therefore$ $\theta = \tan^{-1}t$ $\Rightarrow$ (C)]

Q.9$_{131/para}$ A circle with radius unity has its centre on the positive y-axis. If this circle touches the parabola $y = 2x^2$ tangentially at the points P and Q then the sum of the ordinates of P and Q, is

(A*) $\frac{15}{4}$ (B) $\frac{15}{8}$ (C) $2\sqrt{15}$ (D) 5

[Sol. $\left.\frac{dy}{dx}\right|_P = 4t$



$\therefore$ $(4t)\left(\frac{2t^2-a}{t}\right) = -1$

$2t^2 - a = -\frac{1}{4}$ ....(1)

Also $t^2 + (2t^2 - a)^2 = 1$ [$(t, 2t^2)$ satisfies the circle $x^2 + (y-a)^2 = 1$]

$t^2 = \frac{15}{16}$ $\Rightarrow$ $4t^2 = \frac{15}{4}$ Ans.]

Q.10$_{29/\text{para}}$ Normal to the parabola $y^2 = 8x$ at the point P (2, 4) meets the parabola again at the point Q. If C is the centre of the circle described on PQ as diameter then the coordinates of the image of the point C in the line y = x are

(A*) (– 4, 10) (B) (– 3, 8) (C) (4, – 10) (D) (– 3, 10)

[Sol. $at_1^2 = 2$; $y^2 = 8x$ $\Rightarrow$ $a = 2$

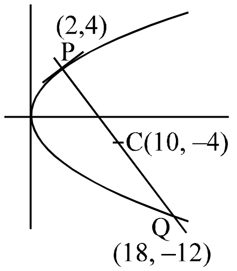
$t_1 = 1 \;\Rightarrow\; t_1 = 1$ or -1

$$t_2 = -t_1 - \frac{2}{t_1} = -3$$

Q is $\left(2(-3)^2, 2(2)(-3)\right)$ i.e. (18, – 12)

$\therefore$ C is (10, – 4)

The image is (– 4, 10)]



Q.11$_{49/\text{para}}$ Two parabolas $y^2 = 4a(x - l_1)$ and $x^2 = 4a(y - l_2)$ always touch one another, the quantities l_1 and l_2 are both variable. Locus of their point of contact has the equation

(A) $xy = a^2$ (B) $xy = 2a^2$ (C*) $xy = 4a^2$ (D) none

[Sol. $y^2 = 4a(x - l_1)$; $x^2 = 4a(y - l_2)$

$$2y\frac{dy}{dx} = 4a \quad \text{and} \quad 2x = 4a\frac{dy}{dx}$$

$$\left.\frac{dy}{dx}\right|_{(x_1, y_1)} = \frac{2a}{y_1} \quad \text{and} \quad \left.\frac{dy}{dx}\right|_{(x_1, y_1)} = \frac{x_1}{2a}$$

$$\therefore \quad \frac{2a}{y_1} = \frac{x_1}{2a} \Rightarrow x_1y_1 = 4a^2 \Rightarrow \text{R.H.]}$$

Q.12$_{98/\text{para}}$ A pair of tangents to a parabola is are equally inclined to a straight line whose inclination to the axis is α. The locus of their point of intersection is :

(A) a circle (B) a parabola (C*) a straight line (D) a line pair

[Sol. Let $P\left(at_1t_2, a(t_1 + t_2)\right) = P(h, k)$

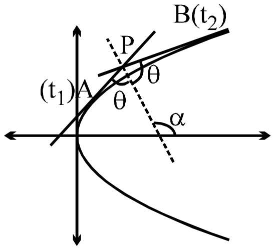
slope of tangents at A, $m_1 = \dfrac{1}{t_1}$ and at B, $m_2 = \dfrac{1}{t_2}$

Let $m = \tan\alpha$

then $\dfrac{m_1 - m}{1 + mm_1} = \dfrac{m - m_2}{1 + mm_2}$

$\tan(\theta_1 - \alpha) = \tan(\alpha - \theta_2)$

$\therefore$ $(\theta_1 - \alpha) = n\pi + 2a\alpha$



$$\therefore \quad \tan 2\alpha = \tan(\theta_1 + \theta_2) = \frac{\tan\theta_1 + \tan\theta_2}{1 - \tan\theta_1\tan\theta_2} = \frac{\frac{1}{t_1} + \frac{1}{t_2}}{1 - \frac{1}{t_1}\frac{1}{t_2}} = \frac{2(t_1 + t_2)}{t_1t_2 - 1}; \tan 2\alpha = \frac{\frac{k}{a}}{\frac{h}{a} - 1} = \frac{k}{h - a}$$

$\therefore$ locus of P(h, k) will be $y = (x - a)\tan 2\alpha$ **Ans.]**

Q.13$_{\text{110/para}}$ In a parabola $y^2 = 4ax$ the angle θ that the latus rectum subtends at the vertex of the parabola is

(A) dependent on the length of the latus rectum

(B) independent of the latus rectum and lies between $\frac{5\pi}{6}$ & π

(C) independent of the latus rectum and lies between $\frac{3\pi}{4}$ & $\frac{5\pi}{6}$

(D*) independent of the latus rectum and lies between $\frac{2\pi}{3}$ & $\frac{3\pi}{4}$

[Hint: Equation of latus rectum is $x = a \;\Rightarrow\; \frac{x}{a} = 1$

∴ angle subtended at the vertex of $y^2 = 4ax$ will be $y^2 = 4ax\frac{x}{a} \;\Rightarrow y^2 = 4x^2$

∴ slopes of OA and OB will be 2 and – 2 respectively

∴ $\tan\theta = \frac{2-(-2)}{1+2(-2)} = \frac{4}{-3}$ ∴ $\theta = \pi - \tan^{-1}\frac{4}{3} \Rightarrow$ (D)]

MC SIR

CLASS : XIII (VXYZ) **DPP. NO.- 5**

Q.1_{para} Normals are drawn at points A, B, and C on the parabola $y^2 = 4x$ which intersect at P(h, k). The locus of the point P if the slope of the line joining the feet of two of them is 2 , is

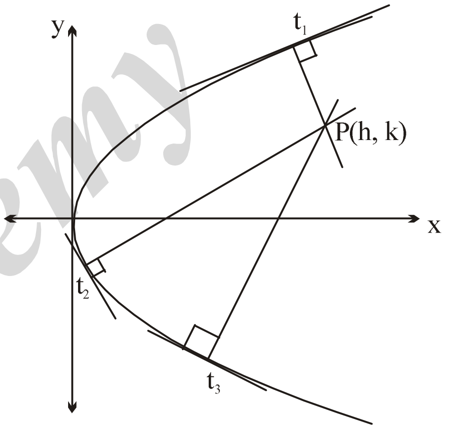
(A) $x + y = 1$ (B*) $x - y = 3$ (C) $y^2 = 2(x - 1)$ (D) $y^2 = 2\left(x - \frac{1}{2}\right)$

[Sol. The equation of normal at $(at^2, 2at)$ is **[12th, 20-12-2009, complex]**

$$y + tx = 2at + at^3 \qquad ....(1)$$

As (1) passes through P(h, k), so

$$at^3 + t(2a - h) - k = 0 \begin{cases} t_1 \\ t_2 \\ t_3 \end{cases} \quad ....(2)$$



Here $a = 1$

$$t_1 + t_2 + t_3 = 0 \qquad ....(3)$$

Also $\frac{2}{t_1 + t_2} = 2 \Rightarrow t_1 + t_2 = 1 \quad ....(4)$

From (3) and (4) $\Rightarrow t_3 = -1$

Put $t_3 = -1$ in (2), we get

$$-1 - 1\,(2 - h) - k = 0$$

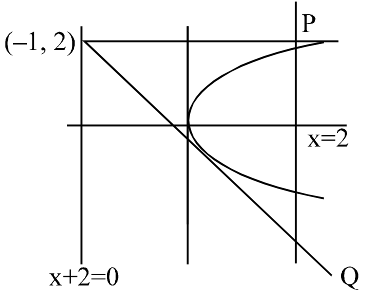
$\Rightarrow \quad -1 - 2 + h - k = 0$

$\therefore$ Locus of P(h, k), is $x - y = 3$ **Ans.**]

Q.$2_{115/para}$ Tangents are drawn from the point (– 1, 2) on the parabola $y^2 = 4x$. The length , these tangents will intercept on the line $x = 2$ is :

(A) 6 (B*) $6\sqrt{2}$ (C) $2\sqrt{6}$ (D) none of these

[Sol. $SS_1 = T^2$



$(y^2 - 4x)(y_1^2 - 4x_1) = (y y_1 - 2(x + x_1))^2$

$(y^2 - 4x)(4 + 4) = [2y - 2(x - 1)]^2 = 4(y - x + 1)^2$

$2(y^2 - 4x) = (y - x + 1)^2$;

solving with the line $x = 2$ we get,

$2(y^2 - 8) = (y - 1)^2$ or $2(y^2 - 8) = y^2 - 2y + 1$

or $y^2 + 2y - 17 = 0$

where $y_1 + y_2 = -2$ and $y_1 y_2 = -17$

Now $|y_1 - y_2|^2 = (y_1 + y_2)^2 - 4y_1 y_2$

or $|y_1 - y_2|^2 = 4 - 4(-17) = 72$

$\therefore (y_1 - y_2) = \sqrt{72} = 6\sqrt{2}$]

Q.$3_{123/para}$ Which one of the following lines cannot be the normals to $x^2 = 4y$?

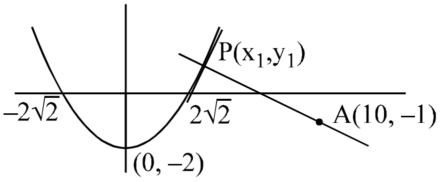
(A) $x - y + 3 = 0$ (B) $x + y - 3 = 0$ (C) $x - 2y + 12 = 0$ (D*) $x + 2y + 12 = 0$

[Hint: equation of the normal to $x^2 = 4y$ in terms of slope $y = mx + \frac{1}{m^2} + 2$]

Q.$4_{129/para}$ An equation of the line that passes through (10, –1) and is perpendicular to $y = \frac{x^2}{4} - 2$ is

(A) $4x + y = 39$ (B) $2x + y = 19$ (C) $x + y = 9$ (D*) $x + 2y = 8$

[Sol. $4y = x^2 - 8$ **[29-01-2006, 12&13]**



$$4\frac{dy}{dx} = 2x$$

$$\left.\frac{dy}{dx}\right|_{x_1,y_1} = \frac{x_1}{2}$$

$\therefore$ slope of normal $= -\frac{2}{x_1}$; but slope of normal $= \frac{y_1+1}{x_1-10}$

$\therefore$ $\frac{y_1+1}{x_1-10} = -\frac{2}{x_1}$ $\Rightarrow$ $x_1y_1 + x_1 = -2x_1 + 20 \Rightarrow$ $x_1y_1 + 3x_1 = 20$

substituting $y_1 = \frac{x_1^2-8}{4}$ (from the given equation)

$$x_1\left(\frac{x_1^2-8}{4}+3\right) = 20 \Rightarrow x_1(x_1^2 - 8 + 12) = 80 \Rightarrow x_1(x_1^2 + 4) = 80$$

$$x_1^3 + 4x_1 - 80 = 0$$

$$x_1^2(x_1 - 4) + 4x(x_1 - 4) + 20(x_1 - 4) = 0$$

$$(x_1 - 4)(x_1^2 + 4x_1 + 20) = 0$$

Hence $x_1 = 4$; $y_1 = 2$

$\therefore$ P = (4, 2)

equation of PA is

$$y + 1 = -\frac{1}{2}(x - 10) \Rightarrow 2y + 2 = -x + 10 \Rightarrow x + 2y - 8 = 0 \text{ Ans. }]$$

Paragraph for question nos. 5 to 6

Consider the parabola $y^2 = 8x$

Q.$5_{408/para}$ Area of the figure formed by the tangents and normals drawn at the extremities of its latus rectum is

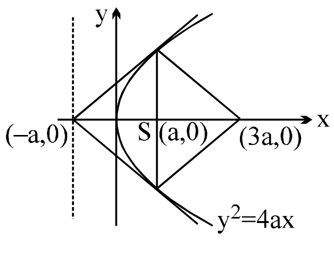
(A) 8 (B) 16 (C*) 32 (D) 64

Q.$6_{409/para}$ Distance between the tangent to the parabola and a parallel normal inclined at 30° with the x-axis, is

(A*) $\frac{16}{3}$ (B) $\frac{16\sqrt{3}}{9}$ (C) $\frac{2}{3}$ (D) $\frac{16}{\sqrt{3}}$

[Sol.

(i) For $y^2 = 4ax$



$$A = \frac{(4a)(4a)}{2} = 8a^2$$

Here a = 2

$\therefore$ A = 32 sq. units **[08-01-2006, 12th & 13th]**

(ii) $y = mx + \frac{a}{m}$

$y = mx - 2am - am^3$

$$d = \left|\frac{(a/m)+2am+am^3}{\sqrt{1+m^2}}\right| = a\left|\frac{(m^2+1)^2}{m\sqrt{1+m^2}}\right| = a\left|\frac{(1+m^2)\sqrt{1+m^2}}{m}\right| \text{ ; put } m = \tan\theta$$

$$= a\left|\frac{\sec^2\theta\cdot\sec\theta}{\tan\theta}\right| = a\,(\sec^2\theta\cdot\operatorname{cosec}\theta)$$

put $a = 2$ and $\theta = 30°$

$$d = 2\cdot\frac{4}{3}\cdot 2 = \frac{16}{3} \text{ Ans.]}$$

Paragraph for question nos. 7 to 9

Tangents are drawn to the parabola $y^2 = 4x$ from the point $P(6, 5)$ to touch the parabola at Q and R. C_1 is a circle which touches the parabola at Q and C_2 is a circle which touches the parabola at R. Both the circles C_1 and C_2 pass through the focus of the parabola.

Q.7_{para} Area of the ΔPQR equals

(A*) $\frac{1}{2}$ (B) 1 (C) 2 (D) $\frac{1}{4}$

Q.8_{para} Radius of the circle C_2 is

(A) $5\sqrt{5}$ (B*) $5\sqrt{10}$ (C) $10\sqrt{2}$ (D) $\sqrt{210}$

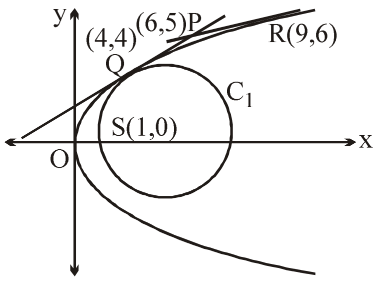
Q.9_{para} The common chord of the circles C_1 and C_2 passes through the

(A) incentre (B) circumcentre

(C*) centroid (D) orthocentre of the ΔPQR

[Sol. Equation of tangent of slope m to $y^2 = 4x$ is

$y = mx + \frac{1}{m}$ ....(1) **[12th, 03-01-2010, P-1]**



(i) As (1) passes through P(6, 5), so

$$5 = 6m + \frac{1}{m}$$

$$\Rightarrow \quad 6m^2 - 5m + 1 = 0 \quad \Rightarrow \quad m = \frac{1}{2} \text{ or } m = \frac{1}{3}$$

Points of contact are $\left(\frac{1}{m_1^2}, \frac{2}{m_1}\right)$ and $\left(\frac{1}{m_2^2}, \frac{2}{m_2}\right)$

Hence P (4, 4) and Q (9, 6)

$$\text{Area of } \Delta PQR = \frac{1}{2}\begin{vmatrix} 6 & 5 & 1 \\ 4 & 4 & 1 \\ 9 & 6 & 1 \end{vmatrix} = \frac{1}{2} \Rightarrow \textbf{(A)}$$

(ii) $y = \frac{1}{2}x + 2 \Rightarrow x - 2y + 4 = 0$ ....(2)

and $y = \frac{1}{3}x + 3 \Rightarrow x - 3y + 9 = 0$

Now equation of circle C_2 touching $x - 3y + 9 = 0$ at (9, 6), is

$(x - 9)^2 + (y - 6)^2 + \lambda(x - 3y + 9) = 0$

As above circle passes through (1, 0), so

$64 + 36 + 10\lambda = 0 \quad \Rightarrow \quad \lambda = -10$

Circle C_2 is $x^2 + y^2 - 28x + 18y + 27 = 0$ .....(3)

Radius of C_2 is

$r_2^2 = 196 + 81 - 27 = 277 - 27 = 250 \quad \Rightarrow \quad r_2 = 5\sqrt{10} \quad \Rightarrow$ **(B)**

(iii) Equation of C_1

$(x - 4)^2 + (y - 4)^2 + \lambda(x - 2y + 4) = 0$

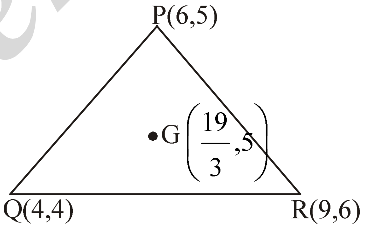
As above circle passes through (1, 0)

$9 + 16 + \lambda(5) = 0 \quad \Rightarrow \quad \lambda = -5$

Now C_1 is $x^2 + y^2 - 13x + 2y + 12 = 0$ ....(4)

$\therefore$ Common chord of (3) and (4) is

$15x - 16y - 15 = 0$ ....(5)



Also centroid (G) of ΔPQR is $\left(\frac{19}{3},5\right)$

Clearly $\left(\frac{19}{3},5\right)$ satisfies equation (5)

Hence **(C)**]

Paragraph for question nos. 10 to 12

Tangents are drawn to the parabola $y^2 = 4x$ at the point P **which is the upper end of latus rectum**.

Q.10$_{410/para}$ Image of the parabola $y^2 = 4x$ in the tangent line at the point P is

(A) $(x + 4)^2 = 16y$ (B) $(x + 2)^2 = 8(y - 2)$

(C*) $(x + 1)^2 = 4(y - 1)$ (D) $(x - 2)^2 = 2(y - 2)$

Q.11 Radius of the circle touching the parabola $y^2 = 4x$ at the point P and passing through its focus is

(A) 1 (B*) $\sqrt{2}$ (C) $\sqrt{3}$ (D) 2

Q.12 Area enclosed by the tangent line at P, x-axis and the parabola is

(A*) $\frac{2}{3}$ (B) $\frac{4}{3}$ (C) $\frac{14}{3}$ (D) none

[Sol. Point P is (1, 2) **[13th, 17-02-2008]** **[Illustration]**

Tangent is $2y = 2(x + 1)$

i.e. $y = x + 1$ ....(1)

hence image of $y^2 = 4x$ in (2) can be written as

$(x + 1)^2 = 4(y - 1) \quad \Rightarrow$ **(C)**

note find the image of $(t^2, 2t)$ in the tangent line and then eliminate *t* to get the image

now family of circle touching the parabola $y^2 = 4x$ at (1, 2)

$(x-1)^2+(y-2)^2+\lambda(x-y+1)=0$ ....(2)

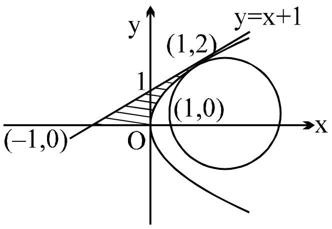
it passes through (1, 0)

$4+2\lambda=0 \quad \Rightarrow \quad \lambda=-2$

hence circle is

$x^2+y^2-4x-2y+3=0$

$r=\sqrt{4+1-3}=\sqrt{2} \quad \Rightarrow$ **(B)**



$$\text{Area}=\int_0^2\left[\frac{y^2}{4}-(y-1)\right]dy=\left.\frac{y^3}{12}-\frac{y^2}{2}+y\right]_0^2$$

$$=\frac{8}{12}-2+2=\frac{2}{3} \quad \Rightarrow$$ **(A)**]

More than one are correct:

Q.13$_{509/para}$ Let P, Q and R are three co-normal points on the parabola $y^2=4ax$. Then the correct statement(s) is/are

(A*) algebraic sum of the slopes of the normals at P, Q and R vanishes

(B*) algebraic sum of the ordinates of the points P, Q and R vanishes

(C*) centroid of the triangle PQR lies on the axis of the parabola

(D*) circle circumscribing the triangle PQR passes through the vertex of the parabola

Q.14$_{511/para}$ Variable chords of the parabola $y^2=4ax$ subtend a right angle at the vertex. Then :

(A*) locus of the feet of the perpendiculars from the vertex on these chords is a circle

(B*) locus of the middle points of the chords is a parabola

(C*) variable chords passes through a fixed point on the axis of the parabola

(D) none of these

[Hint: $A=x^2+y^2-4ax=0$; $B=y^2=2a(x-4a)$; $C\equiv(4a,0)$]

Q.15$_{517/para}$ Through a point P (– 2, 0), tangents PQ and PR are drawn to the parabola $y^2=8x$. Two circles each passing through the focus of the parabola and one touching at Q and other at R are drawn. Which of the following point(s) with respect to the triangle PQR lie(s) on the common chord of the two circles?

(A*) centroid (B*) orthocentre

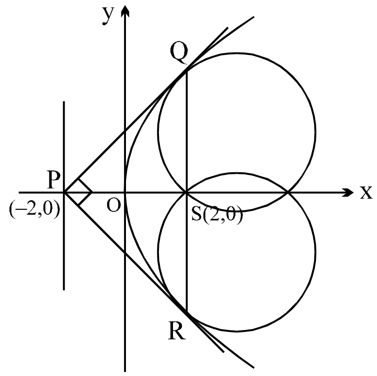
(C*) incentre (D*) circumcentre

[Sol. (–2, 0) is the foot of directrix.

Hence Q and R are the extremities of the latus rectum and angle $\angle QPR=90°$ with ΔPQR as right isosceles.

Hence by symmetric the common chord of the two circles will be the x-axis which will be the median, altitude, angle bisector and also the perpendicular bisector.

Hence centroid, orthocentre, incentre and circumcentre all will lie on it.] **[13th, 09-03-2008]**



Q.16$_{para}$ TP and TQ are tangents to parabola $y^2=4x$ and normals at P and Q intersect at a point R on the curve. The locus of the centre of the circle circumscribing ΔTPQ is a parabola whose

(A*) vertex is (1, 0). (B*) foot of directrix is $\left(\frac{7}{8},0\right)$.

(C) length of latus-rectum is $\frac{1}{4}$. (D*) focus is $\left(\frac{9}{8},0\right)$.

[Sol. We have $2h = t_3^2 + 2$ ....(1)

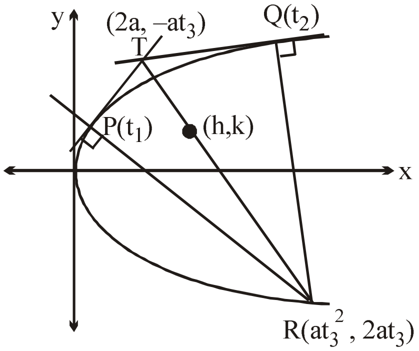
$2k = t_3$ ....(2)

$\therefore\ 2h = 4k^2 + 2$

$\therefore\ 2y^2 = x - 1$

$y^2 = \frac{1}{2}(x-1)$ (Parabola)

Now interpret.] **[12th, 20-12-2009]**



Here $a = 1$

$t_1 t_2 = 2$

$t_1 + t_2 + t_3 = 0$

Match the column:

Q.17_{71} Consider the parabola $y^2 = 12x$

	Column-I		Column-II
(A)	Tangent and normal at the extremities of the latus rectum intersect the x axis at T and G respectively. The coordinates of the middle point of T and G are	(P)	(0, 0)
(B)	Variable chords of the parabola passing through a fixed point K on the axis, such that sum of the squares of the reciprocals of the two parts of the chords through K, is a constant. The coordinate of the point K are	(Q) (R)	(3, 0) (6, 0)
(C)	All variable chords of the parabola subtending a right angle at the origin are concurrent at the point		
(D)	AB and CD are the chords of the parabola which intersect at a point E on the axis. The radical axis of the two circles described on AB and CD as diameter always passes through	(S)	(12, 0)

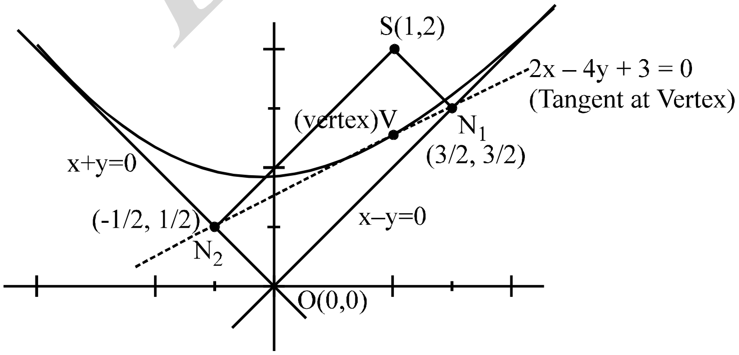
[Ans. (A) Q; (B) R; (C) S; (D) P] **[13th, 09-03-2008]**

Subjective:

Q.18_{para} Let $L_1 : x + y = 0$ and $L_2 : x - y = 0$ are tangent to a parabola whose focus is S(1, 2).

If the length of latus-rectum of the parabola can be expressed as $\frac{m}{\sqrt{n}}$ (where m and n are coprime) then find the value of (m + n). [**Ans.** 0011]

[Sol.



[12th, 20-12-2009, complex]

Feet of the perpendicular (N_1 and N_2) from focus upon any tangent to parabola lies on the tangent line at the vertex.

Now equation of SN_1 is $x + y = \lambda$ passing through (1, 2) $\Rightarrow$ $\lambda = 3$

Equation of SN_1 is $x + y = 3$

Solving $x + y = 3$ and $y = x$, we get $N_1 \equiv \left(\frac{3}{2}, \frac{3}{2}\right)$

|||ly equation of SN_2 is $x - y = \lambda$ passing through $(1, 2)$ $\Rightarrow$ $\lambda = -1$
Equation of SN_2 is $y - x = 1$

Solving $y - x = 1$ and $y = -x$, we get $N_2 \equiv \left(\frac{-1}{2}, \frac{1}{2}\right)$

Now equation of tangent line at vertex is, $2x - 4y + 3 = 0$
Distance of $S(1, 2)$ from tangent at vertex is

$$= \frac{|2-8+3|}{\sqrt{20}} = \frac{3}{2\sqrt{5}} = \frac{1}{4} \times \text{latus rectum}.$$

and hence length of latus rectum $= \frac{6}{\sqrt{5}} = \frac{m}{\sqrt{n}}$

Hence $m + n = 6 + 5 = 11$ **Ans.**]

MC SIR

CLASS : XIII (VXYZ) **DPP. NO.- 6**

Q.1$_{2/elli}$ Let 'E' be the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ & 'C' be the circle $x^2+y^2=9$. Let P & Q be the points (1 , 2) and (2, 1) respectively. Then :

(A) Q lies inside C but outside E
(B) Q lies outside both C & E
(C) P lies inside both C & E
(D*) P lies inside C but outside E.

Q.2$_{3/elli}$ The eccentricity of the ellipse $(x-3)^2+(y-4)^2=\frac{y^2}{9}$ is

(A) $\frac{\sqrt{3}}{2}$
(B*) $\frac{1}{3}$
(C) $\frac{1}{3\sqrt{2}}$
(D) $\frac{1}{\sqrt{3}}$

[Sol. $9(x-3)^2+9(y-4)^2=y^2 \quad\Rightarrow\quad 9(x-3)^2+8y^2-72y+144=0$

$9(x-3)^2+8(y^2-9y)+144=0$

$$9(x-3)^2+8\left[\left(y-\frac{9}{2}\right)^2-\frac{81}{4}\right]+144=0 \quad\Rightarrow\quad 9(x-3)^2+8\left(y-\frac{9}{2}\right)^2=162-144=18$$

$$\frac{9(x-3)^2}{18}+\frac{8\left(y-\frac{9}{2}\right)}{18}=1 \quad\Rightarrow\quad \frac{(x-3)^2}{2}+\frac{\left(y-\frac{9}{2}\right)}{9/4}=1; \qquad e^2=1-\frac{2\cdot4}{9}=\frac{1}{9}\ ;\ \therefore\ e=\frac{1}{3}$$

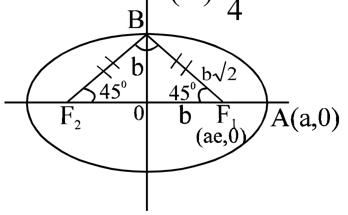
Alternatively: put $x-3=X$ and $y-4=Y$]

Q.3$_{47/ellipse}$ An ellipse has OB as a semi minor axis where 'O' is the origin. F, F′ are its foci and the angle FBF′ is a right angle. Then the eccentricity of the ellipse i

(A*) $\frac{1}{\sqrt{2}}$
(B) $\frac{1}{2}$
(C) $\frac{\sqrt{3}}{2}$
(D) $\frac{1}{4}$

[Sol. $BF_1=OA \Rightarrow b\sqrt{2}=a$

$$\therefore e^2=1-\frac{b^2}{a^2} \Rightarrow e^2=1-\frac{b^2}{2b^2}=\frac{1}{2} \Rightarrow e=\frac{1}{\sqrt{2}}$$



]

Q.4$_{10/ellipse}$ There are exactly two points on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ whose distance from the centre of the ellipse are greatest and equal to $\sqrt{\frac{a^2+2b^2}{2}}$. Eccentricity of this ellipse is equal to

(A) $\frac{\sqrt{3}}{2}$
(B) $\frac{1}{\sqrt{3}}$
(C*) $\frac{1}{\sqrt{2}}$
(D) $\sqrt{\frac{2}{3}}$

[Sol. The given distance is clearly the length of semi major axis **[11th, 25-02-2009, P-1]**

Thus, $\sqrt{\frac{a^2+2b^2}{2}} = a \quad \Rightarrow \quad 2b^2 = a^2 \quad \Rightarrow \quad 2a^2(1-e^2) = a^2$

$\Rightarrow \quad e^2 = \frac{1}{2} \Rightarrow \quad e = \frac{1}{\sqrt{2}}$ **Ans.]**

Q.5$_{12/\text{ellipse}}$ A circle has the same centre as an ellipse & passes through the foci F_1 & F_2 of the ellipse, such that the two curves intersect in 4 points. Let 'P' be any one of their point of intersection. If the major axis of the ellipse is 17 & the area of the triangle PF_1F_2 is 30, then the distance between the foci is :

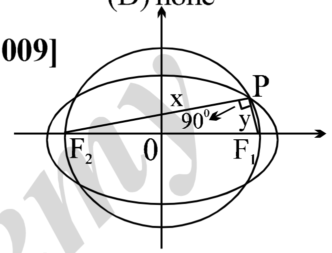
(A) 11 (B) 12 (C*) 13 (D) none

[Hint : $x + y = 17$; $xy = 60$, To find $\sqrt{x^2+y^2}$] **[11th, 14-02-2009]**

now, $x^2 + y^2 = (x + y)^2 - 2xy$

$= 289 - 120 = 169$

$\Rightarrow \quad \sqrt{x^2+y^2} = 13$]



Q.6$_{24/\text{ellipse}}$ The latus rectum of a conic section is the width of the function through the focus. The positive difference between the lengths of the latus rectum of $3y = x^2 + 4x - 9$ and $x^2 + 4y^2 - 6x + 16y = 24$ is

(A*) $\frac{1}{2}$ (B) 2 (C) $\frac{3}{2}$ (D) $\frac{5}{2}$

[Hint: $3y = (x + 2)^2 - 13$ **[12th & 13th 03-03-2007]**

$3y + 13 = (x + 2)^2$

$\therefore \quad (x + 2)^2 = 3\left(y + \frac{13}{3}\right) \Rightarrow$ Latus Rectum = 3

The other conic is, $(x - 3)^2 + 4(y^2 + 4y) = 24 + 9$

$(x - 3)^2 + 4(y + 2)^2 = 49$

$$\frac{(x-3)^2}{7^2} + \frac{(y+2)^2}{(7/2)^2} = 1 \text{ which is an ellipse}$$

$$\text{Latus Rectum} = \frac{2b^2}{a} = \frac{2\cdot 49}{4\cdot 7} = \frac{7}{2}$$

$\therefore$ positive difference $\frac{7}{2} - 3 = \frac{1}{2}$ Ans.]

Q.7$_{31/\text{ellipse}}$ Imagine that you have two thumbtacks placed at two points, A and B. If the ends of a fixed length of string are fastened to the thumbtacks and the string is drawn taut with a pencil, the path traced by the pencil will be an ellipse. The best way to maximise the area surrounded by the ellipse with a fixed length of string occurs when

I the two points A and B have the maximum distance between them.
II two points A and B coincide.
III A and B are placed vertically.
IV The area is always same regardless of the location of A and B.

(A) I (B*) II (C) III (D) IV

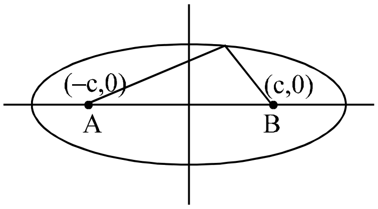
[Sol. $A = \pi ab$ 'a' is constant and b varies **[12 & 13 05-3-2006]**

$A^2 = \pi^2 a^2(a^2 - c^2)$

for A to be maximum c must be minimum; A & B → centre

as $A \to B \quad \Rightarrow \quad c \to 0$

ellipse becomes circle]



Q.8$_{37/\text{ellipse}}$ An ellipse having foci at (3, 3) and (– 4, 4) and passing through the origin has eccentricity equal to

(A) $\frac{3}{7}$ (B) $\frac{2}{7}$ (C*) $\frac{5}{7}$ (D) $\frac{3}{5}$

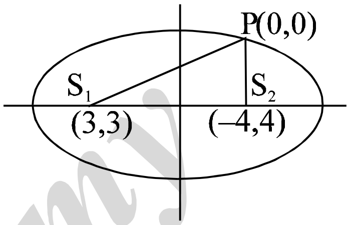
[Hint : $PS_1 + PS_2 = 2a$ **[11th, 14-02-2009]**

$3\sqrt{2} + 4\sqrt{2} = 2a$

$\therefore 2a = 7\sqrt{2}$

Also $2ae = S_1S_2 = \sqrt{1+49} = 5\sqrt{2}$

$\therefore \frac{2ae}{2a} = \frac{5\sqrt{2}}{7\sqrt{2}} = \frac{5}{7} = e \Rightarrow$ (C)]



Q.9$_{\text{ellipse}}$ Let S(5, 12) and S'(– 12, 5) are the foci of an ellipse passing through the origin.
The eccentricity of ellipse equals

(A) $\frac{1}{2}$ (B) $\frac{1}{\sqrt{3}}$ (C*) $\frac{1}{\sqrt{2}}$ (D) $\frac{2}{3}$

[Sol. We have $2ae = 13\sqrt{2}$ = focal length ...(1) **[12th, 20-12-2009, complex]**

$\because$ 2a = 26 $\Rightarrow$ a = 13 (By focus-directrix property)

$\therefore$ On putting a = 13 in equation (1), we get

$$2(13)e = 13\sqrt{2} \Rightarrow e = \frac{1}{\sqrt{2}}$$ **Ans.**]

More than one are correct:

Q.10$_{501/\text{ellipse}}$ Consider the ellipse $\frac{x^2}{\tan^2\alpha} + \frac{y^2}{\sec^2\alpha} = 1$ where $\alpha \in (0, \pi/2)$.

Which of the following quantities would vary as α varies?

(A*) degree of flatness (B*) ordinate of the vertex

(C) coordinates of the foci (D*) length of the latus rectum

[Hint: $e^2 = 1 - \frac{\tan^2\alpha}{\sec^2\alpha} = \cos^2\alpha$ (as $\sec^2\alpha > \tan^2\alpha$) **[12 & 13 05-3-2006]**

hence $e = \cos\alpha$; vertex $(0, \pm \sec\alpha)$

foci = (0, 1) ; $l(LR) = \frac{2b^2}{a} = \frac{2\tan^2\alpha}{\sec\alpha} = 2\sin\alpha \cdot \tan\alpha$]

Q.11$_{504/\text{ellipse}}$ Extremities of the latera recta of the ellipses $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (a > b) having a given major axis 2a lies on

(A*) $x^2 = a(a-y)$ (B*) $x^2= a(a+y)$ (C) $y^2 = a(a+x)$ (D) $y^2 = a(a-x)$

[Sol. $h = \pm ae$; $k = \pm\frac{b^2}{a}$

$$k = \pm a(1-e^2) = \pm a\left(1-\frac{h^2}{a^2}\right) = \pm\left(a-\frac{h^2}{a}\right)$$

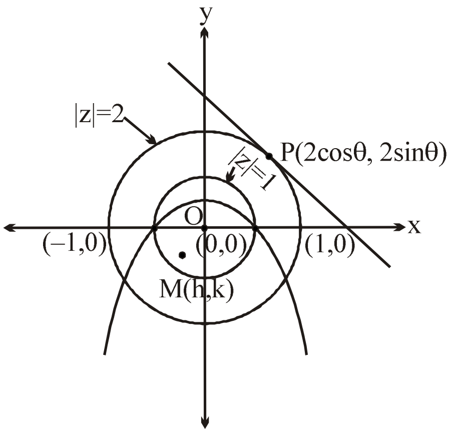
$$+\text{ve sign},\ k = a-\frac{h^2}{a} \Rightarrow \frac{h^2}{a} = a-k \Rightarrow h^2 = a(a-k) \Rightarrow (A)$$

$$-\text{ve sign},\ k = -a+\frac{h^2}{a} \Rightarrow h^2 = a(a+k) \Rightarrow (B)\ ]$$

Subjective:

Q.12$_{\text{elli}}$ Consider two concentric circles $S_1 : |z| = 1$ and $S_2 : |z| = 2$ on the Argand plane. A parabola is drawn through the points where 'S_1' meets the real axis and having arbitrary tangent of 'S_2' as its directrix. If the locus of the focus of drawn parabola is a conic C then find the area of the quadrilateral formed by the tangents at the ends of the latus-rectum of conic C. [**Ans.** 0016]

[Sol. Clearly the parabola should pass through (1, 0) and (–1,0). Let directrix of this parabola be $x\cos\theta + y\sin\theta = 2$. If M (h,k) be the focus of this parabola, then distance of (±1, 0) from 'M' and from the directrix should be same.



$\Rightarrow$ $(h-1)^2 + k^2 = (\cos\theta - 2)^2$ ....(1)

and $(h+1)^2 + k^2 = (\cos\theta + 2)^2$ ....(2)

Now (2) – (1) $\Rightarrow$ $\cos\theta = \frac{h}{2}$ ....(3)

Also (2) + (1) $\Rightarrow$ $(h^2 + k^2 + 1) = (\cos^2\theta + 4)$ ....(4)

$\therefore$ From (3) and (4), we get

$$h^2 + k^2 + 1 = 4 + \frac{h^2}{4} \Rightarrow \frac{3h^2}{4} + k^2 = 3$$

Hence locus of focus M(h, k) is $\frac{x^2}{4}+\frac{y^2}{3}=1$ (Ellipse)

Also we know that area of the quadrilateral formed by the tangents at the ends of the latus-rectum is

$\frac{2a^2}{e}$ (where e is eccentricity of ellipse) **[12th, 20-12-2009]**

$\therefore$ Requred area $=\frac{2(4)}{\frac{1}{2}} = 16$ (square units) (As $e^2 = 1-\frac{3}{4}=\frac{1}{4} \Rightarrow e=\frac{1}{2}$)]

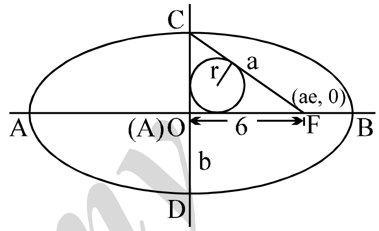
MC SIR

CLASS : XIII (VXYZ) ***DPP. NO.- 7***

Q.1$_{13/\text{ellipse}}$ Point 'O' is the centre of the ellipse with major axis AB & minor axis CD. Point F is one focus of the ellipse. If OF = 6 & the diameter of the inscribed circle of triangle OCF is 2, then the product (AB) (CD) is equal to

(A*) 65 (B) 52 (C) 78 (D) none

[Hint : $a^2 e^2 = 36 \Rightarrow a^2 - b^2 = 36$ ....(1); $4ab = ?$



Using $r = (s-a)\tan\dfrac{A'}{2}$ in Δ OCF

$1 = (s-a)\tan 45^\circ$ where a = CF

$2 = 2(s-a)$

or $2 = 2s - 2a = 2s - AB$

or $2 = (OF + FC + CO) - AB$

$$2 = 6 + \frac{AB}{2} + \frac{CD}{2} - AB$$

$$\frac{AB - CD}{2} = 4 \Rightarrow 2(a-b) = 8 \Rightarrow a - b = 4 \quad — \quad (2)$$

From (1) & (2) $a + b = 9 \Rightarrow 2a = 13$; $2b = 5 \quad \Rightarrow$ (AB) (CD) = 65]

Q.2$_{17/\text{ellipse}}$ The y-axis is the directrix of the ellipse with eccentricity e = 1/2 and the corresponding focus is at (3, 0), equation to its auxilary circle is

(A*) $x^2 + y^2 - 8x + 12 = 0$ (B) $x^2 + y^2 - 8x - 12 = 0$

(C) $x^2 + y^2 - 8x + 9 = 0$ (D) $x^2 + y^2 = 4$

[Sol. Directrix : x = 0 **[12th & 13th 19-3-2006]**

e = 1/2

Focus = (3, 0)

$$\therefore \quad \sqrt{(x-3)^2 + y^2} = \frac{1}{2}\cdot|x|$$

$$\therefore \quad (x-3)^2 + y^2 = \frac{1}{4}\cdot x^2 \quad \Rightarrow \quad 4(x-3)^2 + 4y^2 = x^2 \quad \Rightarrow \quad 3x^2 - 24x + 4y^2 + 36 = 0$$

$$\Rightarrow \quad 3(x-4)^2 + 4y^2 = 12 \quad \Rightarrow \quad \frac{(x-4)^2}{4} + \frac{y^2}{3} = 1 \quad ....(1)$$

$\therefore \quad a = 2$; $b = \sqrt{3}$; centre (4, 0) $\Rightarrow$ auxillary circle is $(x-4)^2 + y^2 = 4$ Ans.]

Q.3$_{20/\text{elli}}$ Which one of the following is the common tangent to the ellipses, $\dfrac{x^2}{a^2+b^2} + \dfrac{y^2}{b^2} = 1$ and $\dfrac{x^2}{a^2} + \dfrac{y^2}{a^2+b^2} = 1$?

(A) $ay = bx + \sqrt{a^4 - a^2b^2 + b^4}$ (B*) $by = ax - \sqrt{a^4 + a^2b^2 + b^4}$

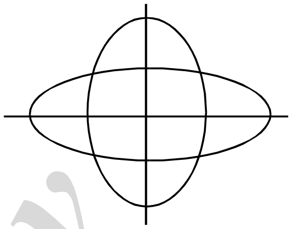
(C) $ay = bx - \sqrt{a^4 + a^2b^2 + b^4}$ (D) $by = ax + \sqrt{a^4 - a^2b^2 + b^4}$

[Sol. Equation of a tangent to $\dfrac{x^2}{a^2+b^2}+\dfrac{y^2}{b^2}=1$

$$y = mx \pm\sqrt{(a^2+b^2)m^2+b^2} \qquad ....(1)$$

If (1) is also a tangent to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{a^2+b^2}=1$ then

$$(a^2+b^2)m^2+b^2 = a^2m^2+a^2+b^2 \qquad (\text{using } c^2 = a^2m^2+b^2)$$



$$b^2m^2 = a^2 \Rightarrow m^2 = \frac{a^2}{b^2} \Rightarrow m = \pm\frac{a}{b}$$

$$y = \pm\frac{a}{b}x \pm \sqrt{(a^2+b^2)\frac{a^2}{b^2}+b^2}$$

$$by = \pm ax \pm \sqrt{a^4+a^2b^2+b^4}$$

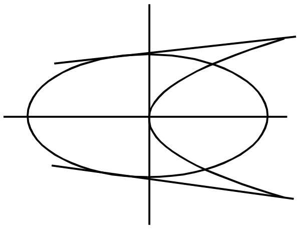
Note : Although there can be four common tangents but only one of these appears in B]

Q.4$_{26/ellise}$ $x - 2y + 4 = 0$ is a common tangent to $y^2 = 4x$ & $\dfrac{x^2}{4}+\dfrac{y^2}{b^2}=1$. Then the value of b and the other common tangent are given by :

(A*) $b=\sqrt{3}$; $x + 2y + 4 = 0$ (B) $b = 3$; $x + 2y + 4 = 0$

(C) $b=\sqrt{3}$; $x + 2y - 4 = 0$ (D) $b=\sqrt{3}$; $x - 2y - 4 = 0$

[Sol. $y = x/2 + 2$ is tangent on the ellipse then $4 = 4.(1/4) + b^2 \Rightarrow b^2 = 3$
parabola is , $y = mx + 1/m$



using condition of tangency, $\dfrac{1}{m^2} = 4m^2 + 3$
$4y^2 + 3y - 1 = 0$ (when $m^2 = y$)
$4y^2 + 4y - y - 1 = 0 \Rightarrow 4y(y + 1) - (y + 1) = 0$
$\Rightarrow y = 1/4$; $y = -1$
$m = \pm 1/2$
$y = x/2 + 2$ or $y = -x/2 - 2 \Rightarrow 2y + x + 4 = 0$ (other tangent)]

Q.5$_{33/ellipse}$ If α & β are the eccentric angles of the extremities of a focal chord of an standard ellipse, then the eccentricity of the ellipse is :

(A) $\dfrac{\cos\alpha+\cos\beta}{\cos(\alpha+\beta)}$ (B) $\dfrac{\sin\alpha-\sin\beta}{\sin(\alpha-\beta)}$ (C) $\dfrac{\cos\alpha-\cos\beta}{\cos(\alpha-\beta)}$ (D*) $\dfrac{\sin\alpha+\sin\beta}{\sin(\alpha+\beta)}$

[Hint : $\dfrac{x}{a}\cos\dfrac{\alpha+\beta}{2}+\dfrac{y}{b}\sin\dfrac{\alpha+\beta}{2}=\cos\dfrac{\alpha-\beta}{2}$; $\dfrac{ae}{a}\cos\dfrac{\alpha+\beta}{2}=\cos\dfrac{\alpha-\beta}{2}$

$$\Rightarrow \qquad e = \frac{\cos\frac{\alpha-\beta}{2}}{\cos\frac{\alpha+\beta}{2}}\cdot\frac{2\sin\frac{\alpha+\beta}{2}}{2\sin\frac{\alpha+\beta}{2}} = \frac{\sin\alpha+\sin\beta}{\sin(\alpha+\beta)} \quad]$$

Q.6$_{34/\text{ellipse}}$ An ellipse is inscribed in a circle and a point within the circle is chosen at random. If the probability that this point lies outside the ellipse is 2/3 then the eccentricity of the ellipse is :

(A*) $\frac{2\sqrt{2}}{3}$ (B) $\frac{\sqrt{5}}{3}$ (C) $\frac{8}{9}$ (D) $\frac{2}{3}$

[Hint : $\frac{2}{3} = \frac{\pi a^2 - \pi ab}{\pi a^2} = 1 - \frac{b}{a} = 1 - \sqrt{1-e^2} \Rightarrow e^2 = \frac{8}{9} \Rightarrow e = \frac{2\sqrt{2}}{3}$]

Q.7$_{41/\text{elli}}$ Consider the particle travelling clockwise on the elliptical path $\frac{x^2}{100} + \frac{y^2}{25} = 1$. The particle leaves the orbit at the point (–8, 3) and travels in a straight line tangent to the ellipse. At what point will the particle cross the y-axis?

(A*) $\left(0, \frac{25}{3}\right)$ (B) $\left(0, -\frac{25}{3}\right)$ (C) (0, 9) (D) $\left(0, \frac{7}{3}\right)$

[12th & 13th 11-3-2007]

Q.8$_{\text{ellipse}}$ The Locus of the middle point of chords of an ellipse $\frac{x^2}{16} + \frac{y^2}{25} = 1$ passing through P(0, 5) is another ellipse E. The coordinates of the foci of the ellipse E, is

(A) $\left(0, \frac{3}{5}\right)$ and $\left(0, \frac{-3}{5}\right)$ (B) (0, – 4) and (0, 1)

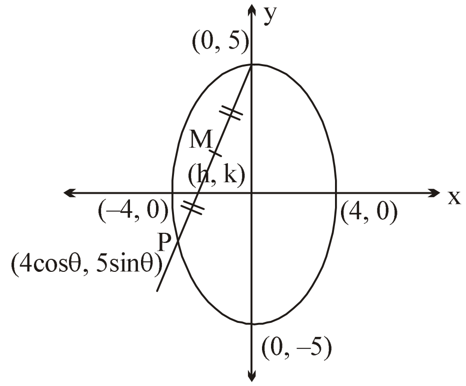
(C*) (0, 4) and (0, 1) (D) $\left(0, \frac{11}{2}\right)$ and $\left(0, \frac{-1}{2}\right)$

[Sol. We have $4\cos\theta = 2h$ and $5(1 + \sin\theta) = 2k$ **[12th, 20-12-2009, complex]**

As $\cos^2\theta + \sin^2\theta = 1$

$$\Rightarrow \frac{h^2}{4} + \left(\frac{2k}{5} - 1\right)^2 = 1 \Rightarrow \frac{x^2}{4} + \frac{4y^2}{25} - \frac{4y}{5} = 0 \Rightarrow \frac{x^2}{4} + \frac{4}{25}(y^2 - 5y) = 0$$

$$\Rightarrow \frac{x^2}{4} + \frac{4}{25}\left[\left(y - \frac{5}{2}\right)^2 - \frac{25}{4}\right] = 0$$

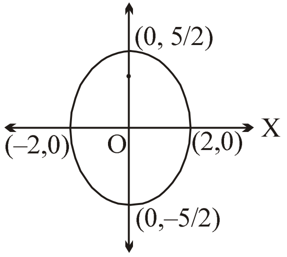


$$\Rightarrow \frac{x^2}{4} + \frac{\left(y - \frac{5}{2}\right)^2}{\frac{25}{4}} = 1 \quad ....(1)$$

Put $X = x, \quad y - \frac{5}{2} = Y$

∴ Equation (1) becomes

$$\frac{X^2}{4} + \frac{Y^2}{\frac{25}{4}} = 1 \quad \text{(Ellipse)}$$



$$e^2 = 1 - \frac{4 \cdot 4}{25} = \frac{9}{25} \Rightarrow e = \frac{3}{5}$$

$\therefore \qquad F_1 = \left(0, \frac{3}{2}\right), \; F_2 = \left(0, \frac{-3}{2}\right)$

Hence in x–y system, foci are (0, 4), (0, 1) $\qquad \Rightarrow \qquad$ **(C)**]

Paragraph for question nos. 9 to 11

Consider the curve $C : y^2 - 8x - 4y + 28 = 0$. Tangents TP and TQ are drawn on C at P(5, 6) and Q(5, – 2). Also normals at P and Q meet at R.

Q.9 The coordinates of circumcentre of ΔPQR, is

(A) (5, 3) (B*) (5, 2) (C) (5, 4) (D) (5, 6)

Q.10 The area of quadrilateral TPRQ, is

(A) 8 (B) 16 (C*) 32 (D) 64

Q.11 Angle between a pair of tangents drawn at the end points of the chord $y + 5t = tx + 2$ of curve C $\forall t \in R$, is

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{3}$ (D*) $\frac{\pi}{2}$

[Sol.(i) Given curve is a parabola $(y - 2)^2 = 8(x - 3)$ whose focus is (5, 2).

As P(5, 6) and Q (5, – 2) are the coordinates of the end points of the latus-rectum of the parabola.

$\therefore$ Normals at P & Q are perpendicular to each other and meeting on the axis of the parabola

$\therefore$ ΔPQR is right angled at R

$\Rightarrow$ Circumcentre of ΔPQR is focus of the parabola i.e. (5, 2)

(ii) Area of quadrilateral TPRQ = Area of square TPRQ $= \left(\frac{8}{\sqrt{2}}\right)\left(\frac{8}{\sqrt{2}}\right) = 32$ (square units)

(iii) Also $y + 5t = tx + 2$ is a focal chord of the given parabola

$\Rightarrow$ Angle between a pair of tangents $= \frac{\pi}{2}$.]

More than one are correct:

Q.12$_{506/ellipse}$ If a number of ellipse be described having the same major axis 2a but a variable minor axis then the tangents at the ends of their latera recta pass through fixed points which can be

(A*) (0, a) (B) (0, 0) (C*) (0, – a) (D) (a, a)

[Hint: e is a variable quantity

$$\frac{xae}{a^2} + \frac{yb^2}{ab^2} = 1 \;\Rightarrow\; ex + y = a \qquad \Rightarrow \qquad y - a + ex = 0$$

it passes through (0, a).

|||ly other point is (0, – a)]

MC SIR

CLASS : XIII (VXYZ) ***DPP. NO.- 8***

Q.1$_{21/\text{ellipse}}$ The normal at a variable point P on an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ of eccentricity e meets the axes of the ellipse in Q and R then the locus of the mid-point of QR is a conic with an eccentricity e' such that :

(A) e' is independent of e (B) $e' = 1$

(C*) $e' = e$ (D) $e' = 1/e$

Q.2$_{28/\text{ellipse}}$ The area of the rectangle formed by the perpendiculars from the centre of the standard ellipse to the tangent and normal at its point whose eccentric angle is π/4 is :

(A*) $\frac{(a^2-b^2)\,ab}{a^2+b^2}$ (B) $\frac{(a^2-b^2)}{(a^2+b^2)\,ab}$ (C) $\frac{(a^2-b^2)}{ab\,(a^2+b^2)}$ (D) $\frac{a^2+b^2}{(a^2-b^2)\,ab}$

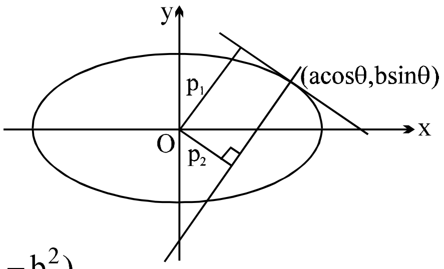
[Hint: $P\left(\frac{a}{\sqrt{2}},\frac{b}{\sqrt{2}}\right)$ $p_1=\frac{\sqrt{2}\,ab}{a^2+b^2}$; $p_2=\frac{a^2-b^2}{\sqrt{2}\,(a^2+b^2)}$ $\Rightarrow p_1p_2 = \text{result}$]

[Sol. $T : \frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$

$$p_1=\left|\frac{ab}{\sqrt{b^2\cos^2\theta+a^2\sin^2\theta}}\right| \quad ....(1)$$

$$N_1 : \frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2$$

$$p_2=\left|\frac{(a^2-b^2)\sin\theta\cos\theta}{\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}}\right| \quad ....(2)$$



$$p_1p_2=\frac{ab(a^2-b^2)}{2\left(\frac{a^2}{2}+\frac{b^2}{2}\right)} \quad \text{when } \theta=\pi/4;\ p_1p_2==\frac{ab(a^2-b^2)}{a^2+b^2} \text{ Ans]}$$

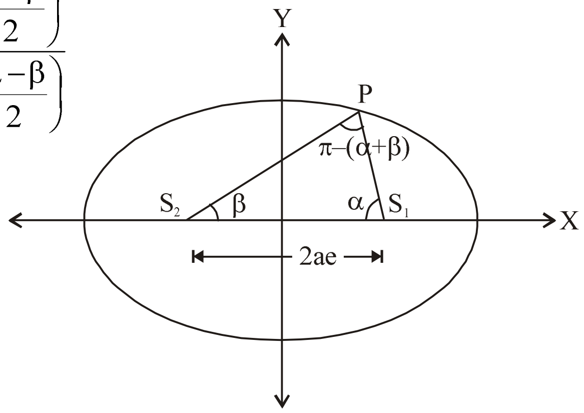
Q.3 If P is any point on ellipse with foci S_1 & S_2 and eccentricity is $\frac{1}{2}$ such that

$\angle PS_1S_2=\alpha$, $\angle PS_2S_1=\beta$, $\angle S_1PS_2=\gamma$, then $\cot\frac{\alpha}{2}$, $\cot\frac{\gamma}{2}$, $\cot\frac{\beta}{2}$ are in

(A*) A.P. (B) G.P. (C) H.P. (D) ***NOT*** A.P., G.P. & H.P.

[Sol. By sine rule in ΔPS_1S_2, we get $\frac{2ae}{\sin(\alpha+\beta)}=\frac{S_1P}{\sin\beta}=\frac{S_2P}{\sin\alpha}=\frac{2a}{\sin\alpha+\sin\beta}$

$$\Rightarrow e = \frac{\sin(\alpha+\beta)}{\sin\alpha+\sin\beta} \Rightarrow \frac{e}{1} = \frac{2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)}{2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)}$$



Now $\frac{1-e}{1+e} = \tan\frac{\alpha}{2}\ \tan\frac{\beta}{2} = \frac{1-\frac{1}{2}}{1+\frac{1}{2}} = \frac{\frac{1}{2}}{\frac{3}{2}} = \frac{1}{3}$

$\therefore \tan\frac{\alpha}{2}\ \tan\frac{\beta}{2} = \frac{1}{3}$ ... (1)

Also we know that

$$\cot\frac{\alpha}{2} + \cot\frac{\beta}{2} + \cot\frac{\gamma}{2} = \cot\frac{\alpha}{2}\ \cot\frac{\beta}{2}\ \cot\frac{\gamma}{2}$$

$\Rightarrow 2\cot\frac{\gamma}{2} = \cot\frac{\alpha}{2} + \cot\frac{\beta}{2} \Rightarrow \cot\frac{\alpha}{2}, \cot\frac{\gamma}{2}, \cot\frac{\beta}{2}$ are in A.P.]

Paragraph for question nos. 4 to 6

Consider the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ and the parabola $y^2 = 2x$. They intersect at P and Q in the first and fourth quadrants respectively. Tangents to the ellipse at P and Q intersect the x-axis at R and tangents to the parabola at P and Q intersect the x-axis at S.

Q.4$_{\text{ellipse}}$ The ratio of the areas of the triangles PQS and PQR, is

(A) 1 : 3 (B) 1 : 2 (C*) 2 : 3 (D) 3 : 4

Q.5 The area of quadrilateral PRQS, is

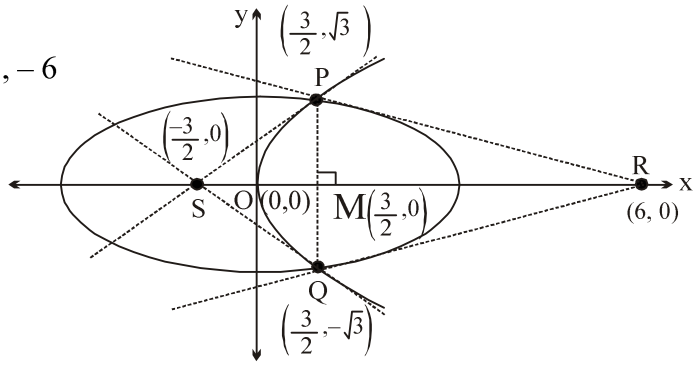
(A) $\frac{3\sqrt{15}}{2}$ (B*) $\frac{15\sqrt{3}}{2}$ (C) $\frac{5\sqrt{3}}{2}$ (D) $\frac{5\sqrt{15}}{2}$

Q.6 The equation of circle touching the parabola at upper end of its latus rectum and passing through its vertex, is

(A) $2x^2 + 2y^2 - x - 2y = 0$ (B) $2x^2 + 2y^2 + 4x - \frac{9}{2}y = 0$

(C) $2x^2 + 2y^2 + x - 3y = 0$ (D*) $2x^2 + 2y^2 - 7x + y = 0$

[Sol. Solving the curves $y^2 = 2x$ and $\frac{x^2}{9}+\frac{y^2}{4}=1$ for the points of intersection, we have


y
$\left(\frac{3}{2},\sqrt{3}\right)$
P
$\left(\frac{-3}{2},0\right)$
S
O
(0,0)
$M\left(\frac{3}{2},0\right)$
R
x
(6, 0)
Q
$\left(\frac{3}{2},-\sqrt{3}\right)$

$4x^2 + 18x - 36 = 0 \Rightarrow x = \frac{3}{2}, -6$

But from $y^2 = 2x$ we have $x > 0$

$\therefore \quad x = \frac{3}{2}$

at which $y^2 = 2 \cdot \frac{3}{2}$

$\Rightarrow \quad y = \pm\sqrt{3}$

$\therefore \quad P\left(\frac{3}{2}, \sqrt{3}\right)$ and $Q\left(\frac{3}{2}, -\sqrt{3}\right)$

Now equation of tangents at P and Q to ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ is $\frac{x}{9}\left(\frac{3}{2}\right)+\frac{y}{4}\left(\pm\sqrt{3}\right)=1$ which intersect at R(6, 0) **[12th, 03-01-2010, P-2]**

Equation of tangents at P and Q to parabola $y^2=2x$ will be $y\left(\pm\sqrt{3}\right)=x+\frac{3}{2}$ which cut x-axis $S\left(\frac{-3}{2},0\right)$

$$\therefore \quad \frac{\text{Area }\Delta PQS}{\text{Area }\Delta PQR}=\frac{\frac{1}{2}PQ\cdot MS}{\frac{1}{2}PQ\cdot MR}=\frac{MS}{MR}=\frac{\frac{3}{2}-\left(\frac{-3}{2}\right)}{6-\frac{3}{2}}=\frac{3}{\frac{9}{2}}=\frac{2}{3}$$ **Ans.(i)**

Area of quadrilateral PRQS $=\frac{1}{2}PQ(MS+MR)=\frac{1}{2}\cdot 2\sqrt{3}\,(6-(-3/2))=\frac{15\sqrt{3}}{2}$ **Ans.(ii)**

(iii) Clearly upper end of latus rectum of parabola is $\left(\frac{1}{2},1\right)$.

And equation of tangent at $\left(\frac{1}{2},1\right)$ to

$y^2=2x$ is $y=x+\frac{1}{2}$

$\therefore$ The equation of circle is

$$\left(x-\frac{1}{2}\right)^2+(y-1)^2+\lambda\left(y-x-\frac{1}{2}\right)=0$$

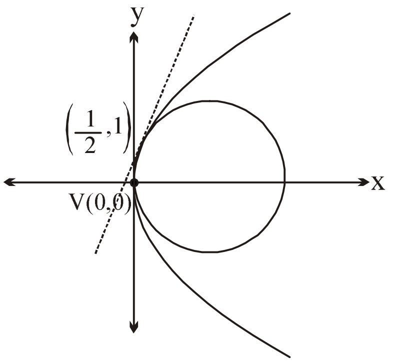
As above circle passes through V (0,0), so

$\frac{1}{4}+1-\frac{\lambda}{2}=0 \Rightarrow \lambda=\frac{5}{2}$

$\Rightarrow$ The equation of required circle is

$$\left(x-\frac{1}{2}\right)^2+(y-1)^2+\frac{5}{2}\left(y-x-\frac{1}{2}\right)=0$$

$\Rightarrow 2x^2+2y^2-7x+y=0$]



Paragraph for question nos. 7 to 11

Let the two foci of an ellipse be (– 1, 0) and (3, 4) and the foot of perpendicular from the focus (3, 4) upon a tangent to the ellipse be (4, 6).

Q.7 The foot of perpendicular from the focus (– 1, 0) upon the same tangent to the ellipse is

(A*) $\left(\frac{12}{5},\frac{34}{5}\right)$ (B) $\left(\frac{7}{3},\frac{11}{3}\right)$ (C) $\left(2,\frac{17}{4}\right)$ (D) (– 1, 2)

Q.8 The equation of auxiliary circle of the ellipse is

(A) $x^2+y^2-2x-4y-5=0$ (B*) $x^2+y^2-2x-4y-20=0$

(C) $x^2+y^2+2x+4y-20=0$ (D) $x^2+y^2+2x+4y-5=0$

Q.9 The length of semi-minor axis of the ellipse is

(A) 1 (B) $2\sqrt{2}$ (C*) $\sqrt{17}$ (D) $\sqrt{19}$

Q.10 The equations of directrices of the ellipse are

(A) $x - y + 2 = 0,\ x - y - 5 = 0$ (B) $x + y - \frac{21}{2} = 0, x + y + \frac{17}{2} = 0$

(C) $x - y + \frac{3}{2} = 0, x - y - \frac{5}{2} = 0$ (D*) $x + y - \frac{31}{2} = 0, x + y + \frac{19}{2} = 0$

Q.11 The point of contact of the tangent with the ellipse is

(A*) $\left(\frac{40}{11}, \frac{68}{11}\right)$ (B) $\left(\frac{4}{7}, \frac{8}{7}\right)$ (C) $\left(\frac{8}{5}, \frac{17}{5}\right)$ (D) $\left(\frac{41}{13}, \frac{83}{13}\right)$

[Sol.

(i) Equation tangent is $(y - 6) = -\left(\frac{4-3}{6-4}\right)(x - 4)$ i.e., $x + 2y - 16 = 0$

So, $\frac{\alpha-(-1)}{1} = \frac{\beta-0}{2} = -\frac{[1(-1)+2(0)-16]}{(1)^2+(2)^2}$

$\Rightarrow\ (\alpha, \beta) \equiv \left(\frac{12}{5}, \frac{34}{5}\right)$

(ii) Centre $\equiv \left(\frac{3-1}{2}, \frac{4+0}{2}\right) \equiv (1, 2)$ and radius $= \sqrt{(4-1)^2+(6-2)^2} = 5$

So, circle is $x^2 + y^2 - 2x - 4y - 20 = 0$

(iii) a = radius = 5. Also $2ae = \sqrt{(3+1)^2+(4-0)^2} = 4\sqrt{2}$,

So $b^2 = a^2 - a^2e^2$

$\Rightarrow$ $b^2 = 25 - \left(2\sqrt{2}\right)^2 = 17$

$\therefore$ $b^2 = 17$ gives $b = \sqrt{17}$

(iv) The directrices are at distances i.e. $\frac{a}{e} = \frac{5}{2\sqrt{2}/5} = \frac{25}{2\sqrt{2}}$ from centre (1, 2) and perpendicular to the line joining foci. Let its equation be $x + y + k = 0$, so $\frac{|1+2+k|}{\sqrt{2}} = \frac{25}{2\sqrt{2}}$ $\Rightarrow k = \frac{19}{2}, -\frac{31}{2}$

Ans.5 Let the point of contact of tangent be $P \equiv (16 - 2\beta, \beta)$. Now SP = ePM, (focus-directrix property),

$\Rightarrow\ (16 - 2\beta - 3)^2 + (\beta - 4)^2 = \left(\frac{2\sqrt{2}}{5}\right)^2 \frac{\left(16-2\beta+\beta-\frac{31}{2}\right)^2}{2}$

$\Rightarrow\ 25\,(5\beta^2 - 60\beta + 185) = 4\beta^2 - 4\beta + 1$

$\Rightarrow\ (11\beta - 68)^2 = 0\ \Rightarrow\ \beta = \frac{68}{11}$, So $16 - 2\beta = \frac{40}{11}$.]

Subjective:

Q.12 Find the number of integral values of parameter 'a' for which three chords of the ellipse $\frac{x^2}{2a^2}+\frac{y^2}{a^2}=1$ (other than its diameter) passing through the point $P\left(11a,-\frac{a^2}{4}\right)$ are bisected by the parabola $y^2=4ax$.

[Ans. 0002]

[Sol. Any point on the parabola $y^2=4ax$ is $(at^2, 2at)$. Equation of chord of the ellipse $\frac{x^2}{2a^2}+\frac{y^2}{a^2}=1$, whose mid-point is $(at^2, 2at)$ is $\frac{x\cdot at^2}{2a^2}+\frac{y\cdot 2at}{a^2}=\frac{a^2t^4}{2a^2}+\frac{4a^2t^2}{a^2}$

$\Rightarrow tx + 4y = at^3 + 8at \;\; (\because t \neq 0)$

As it passes through $\left(11a,-\frac{a^2}{4}\right)$,

$\Rightarrow 11at-4\left(\frac{a^2}{4}\right)=at^3+8at \;\Rightarrow\; at^3-3at+a^2=0$

$\Rightarrow t^3-3t+a=0 \;\; (a\neq 0)$

Now, three chords of the ellipse will be bisected by the parabola if the equation (1) has three real and distinct roots.

Let $f(t)=t^3-3t+a$

$f'(t)=3t^2-3=0 \quad \Rightarrow t=\pm 1$

So, $f(1)\,f(-1)<0$

$\Rightarrow$ $a\in(-2,2)$

But $a\neq 0$, so $a\in(-2,0)\cup(0,2)$

$\therefore$ Number of integral values of 'a' = 2.]

CLASS : XIII (VXYZ) ***DPP. NO.- 9***

Q.1 Consider the hyperbola $9x^2 - 16y^2 + 72x - 32y - 16 = 0$. Find the following:
(a) centre **(b)** eccentricity **(c)** focii **(d)** equation of directrix
(e) length of the latus rectum **(f)** equation of auxilary circle
(g) equation of director circle

[Ans. **(a)** $(-4, -1)$; **(b)** $\frac{5}{4}$; **(c)** $(1, -1), (-9, -1)$; **(d)** $5x + 4 = 0, 5x + 36 = 0$, **(e)** $\frac{9}{2}$;
(f) $(x + 4)^2 + (y + 1)^2 = 16$; **(g)** $(x + 4)^2 + (y + 1)^2 = 7$]

[Sol.

(i) The equation of the hyperbola can be written as
$9(x^2 + 8x) - 16(y^2 + 2y) = 16$ i.e. $9\{(x + 4)^2 - 16\} - 16\{(y + 1)^2 - 1\} = 16$

i.e. $9(x + 4)^2 - 16(y + 1)^2 = 144$ i.e. $\frac{(x+4)^2}{16} - \frac{(y+1)^2}{9} = 1$

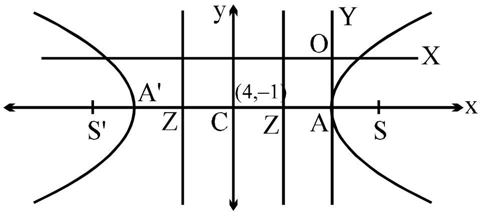
Shifting the origin to $(-4, -1)$, the equation of the hyperbola becomes $\frac{X^2}{16} - \frac{Y^2}{9} = 1$.

$\therefore$ The centre of the hyperbola is the point $(-4, -1)$ **Ans.(i)**

(ii) The semi-transverse axis $a = 4$, the semi-conjugate axis $b = 3$
$b^2 = a^2(e^2 - 1)$

$\therefore$ $9 = 16(e^2 - 1) \Rightarrow e = \frac{5}{4}$ **Ans.(ii)**

(iii) The transverse axis lies along the new x-axis and the conjugate axis lies along the new y-axis.
$CA = 4, CA' = 4$.



$CS = ae = \frac{4\times5}{4} = 5$

$\therefore$ $AS = 1$

$\therefore$ The coordinates of S are $(1, -1)$. **Ans.(iii)**
$CS' = ae = 5$

$\therefore$ The coordinates of S' are $(-9, -1)$ **Ans.(iii)**

If the directrix corresponding to S meet the transverse axis at Z,

$CZ = \frac{a}{e} = \frac{16}{5}$

$\therefore$ $AZ = 4 - \frac{16}{5} = \frac{4}{5}$

The equation of the directrix is $x = \frac{-4}{5}$ i.e. $5x + 4 = 0$ **Ans.(iv)**

|||ly the equation of the directrix orresponding to S' is $5x + 36 = 0$ **Ans.(iv)**]

Q.$2_{1/hyp}$ The area of the quadrilateral with its vertices at the foci of the conics
$9x^2 - 16y^2 - 18x + 32y - 23 = 0$ and
$25x^2 + 9y^2 - 50x - 18y + 33 = 0$, is
(A) 5/6 (B*) 8/9 (C) 5/3 (D) 16/9

[Sol. 1st is a hyperbola **[12th, 04-01-2009, P-1]**

$9(x-1)^2 - 16(y-1)^2 = 16$ with $e = 5/4$

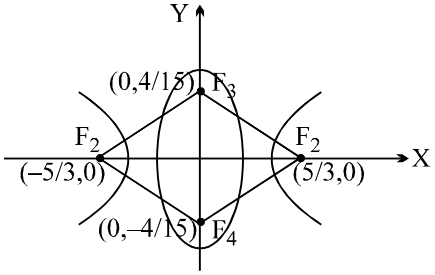
and 2nd is an ellipse

$25(x-1)^2 + 9(y-1)^2 = 1$ with $e = 4/5$

with $x - 1 = X$ and $y - 1 = Y$

$$\text{area} = \frac{1}{2} d_1 d_2 = \frac{1}{2} \cdot \frac{10}{3} \cdot \frac{8}{15} = \frac{8}{9} \text{ Ans.}$$

Note that $e_E \cdot e_H = 1$]



Q.3$_{5/hyp}$ Eccentricity of the hyperbola conjugate to the hyperbola $\frac{x^2}{4} - \frac{y^2}{12} = 1$ is

(A*) $\frac{2}{\sqrt{3}}$ (B) 2 (C) $\sqrt{3}$ (D) $\frac{4}{3}$

[Hint: $e_1^2 = 1 + \frac{b^2}{a^2} = 1 + \frac{12}{4} = 4 \Rightarrow e_1 = 2$; now $\frac{1}{e_1^2} + \frac{1}{e_2^2} = 1$ **[12th, 04-01-2009, P-1]**

$$\frac{1}{e_2^2} = 1 - \frac{1}{4} = \frac{3}{4} \quad \Rightarrow \quad e_2^2 = \frac{4}{3} \quad \Rightarrow \quad e_2 = \frac{2}{\sqrt{3}}$$]

Q.4$_{13/hyper}$ The locus of the point of intersection of the lines $\sqrt{3}x - y - 4\sqrt{3}\,t = 0$ & $\sqrt{3}tx + ty - 4\sqrt{3} = 0$ (where t is a parameter) is a hyperbola whose eccentricity is

(A) $\sqrt{3}$ (B*) 2 (C) $\frac{2}{\sqrt{3}}$ (D) $\frac{4}{3}$

[Hint: hyperbola $\frac{x^2}{16} - \frac{y^2}{48} = 1$] **[11th, 14-02-2009]**

Q.5$_{15/hyp}$ If the eccentricity of the hyperbola $x^2 - y^2 \sec^2\alpha = 5$ is $\sqrt{3}$ times the eccentricity of the ellipse $x^2 \sec^2\alpha + y^2 = 25$, then a value of α is :

(A) $\pi/6$ (B*) $\pi/4$ (C) $\pi/3$ (D) $\pi/2$

[Sol. $\frac{x^2}{5} - \frac{y^2}{5\cos^2\alpha} = 1$

$e_1^2 = 1 + \frac{b^2}{a^2} = 1 + \frac{5\cos^2\alpha}{5} = 1 + \cos^2\alpha$; |||ly eccentricity of the ellipse

$\frac{x^2}{25\cos^2\alpha} + \frac{y^2}{25} = 1$ is $e_2^2 = 1 - \frac{25\cos^2\alpha}{25} = \sin^2\alpha$; put $e_1 = \sqrt{3}\, e_2 \Rightarrow e_1^2 = 3e_2^2$

$$\Rightarrow \quad 1 + \cos^2\alpha = 3\sin^2\alpha \quad \Rightarrow \quad 2 = 4\sin^2\alpha \quad \Rightarrow \quad \sin\alpha = \frac{1}{\sqrt{2}}$$]

Q.6$_{17/hyp}$ The foci of the ellipse $\frac{x^2}{16}+\frac{y^2}{b^2}=1$ and the hyperbola $\frac{x^2}{144}-\frac{y^2}{81}=\frac{1}{25}$ coincide. Then the value of b^2 is

(A) 5 (B*) 7 (C) 9 (D) 4

[Hint: $e_H = \frac{5}{4}$; $e_E = \frac{3}{4} \Rightarrow \frac{9}{16} = 1 - \frac{b^2}{16} \Rightarrow b^2 = 7$] **[12th Test(16-1-2005)]**

More than one are correct:

Q.7$_{505hyp}$ Which of the following equations in parametric form can represent a hyperbola, where 't' is a parameter.

(A*) $x = \frac{a}{2}\left(t+\frac{1}{t}\right)$ & $y = \frac{b}{2}\left(t-\frac{1}{t}\right)$ (B) $\frac{tx}{a} - \frac{y}{b} + t = 0$ & $\frac{x}{a} + \frac{ty}{b} - 1 = 0$

(C*) $x = e^t + e^{-t}$ & $y = e^t - e^{-t}$ (D*) $x^2 - 6 = 2\cos t$ & $y^2 + 2 = 4\cos^2\frac{t}{2}$

Q.8$_{hyper}$ Let p and q be non-zero real numbers. Then the equation $(px^2 + qy^2 + r)(4x^2 + 4y^2 - 8x - 4) = 0$ represents

(A*) two straight lines and a circle, when r = 0 and p, q are of the opposite sign.

(B*) two circles, when p = q and r is of sign opposite to that of p.

(C*) a hyperbola and a circle, when p and q are of opposite sign and $r \neq 0$.

(D*) a circle and an ellipse, when p and q are unequal but of same sign and r is of sign opposite to that of p.

[Sol. $(px^2 + qy^2 + r)(4x^2 + 4y^2 - 8x - 4) = 0$ **[12th, 03-01-2010, P-1]**

$\Rightarrow \quad 4x^2 + 4y^2 - 8x - 4 = 0 \Rightarrow (x-1)^2 + y^2 = 1$

or

$px^2 + qy^2 + r = 0$ will represents

(i) two straight lines if r = 0 and p, q are of opposite sign.

(ii) a circle if p = q and r is of opposite sign that of p.

(iii) a hyperbola if p and q are of opposite sign & $r \neq 0$.

(iv) an ellipse if p and q are unequal but of same sign and r is of sign opposite to that of p.]

Match the column:

Q.9$_{70}$ Match the properties given in **column-I** with the corresponding curves given in the **column-II**.

	Column-I		**Column-II**
(A)	The curve such that product of the distances of any of its tangent from two given points is constant, can be	(P)	Circle
(B)	A curve for which the length of the subnormal at any of its point is equal to 2 and the curve passes through (1, 2), can be	(Q)	Parabola
(C)	A curve passes through (1, 4) and is such that the segment joining any point P on the curve and the point of intersection of the normal at P with the x-axis is bisected by the y-axis. The curve can be	(R)	Ellipse
(D)	A curve passes through (1, 2) is such that the length of the normal at any of its point is equal to 2. The curve can be	(S)	Hyperbola

[Ans. (A) R, S; (B) Q; (C) R; (D) P]

[Sol. **[13th, 17-02-2008]**

(A) Very important property of ellipse and hyperbola $(p_1p_2 = b^2)$ ⇒ **(R), (S)**

(B) $y\dfrac{dy}{dx} = 2 \quad \Rightarrow \quad \dfrac{y^2}{2} = 2x + C$

$x = 1, y = 2 \quad \Rightarrow \quad C = 0$

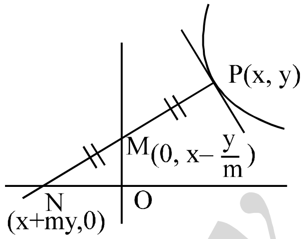
$\Rightarrow \quad y^2 = 4x \quad \Rightarrow$ parabola ⇒ **(Q)**

(C) Equation of normal at P

$$Y - y = -\frac{1}{m}(X - x)$$

$Y = 0, \; X = x + my$

$X = 0, \; Y = y - \dfrac{x}{m}$



hence $x + my + x = 0 \Rightarrow \quad 2x + y\dfrac{dy}{dx} = 0$

$2x\,dx + y\,dy = 0$

$x^2 + \dfrac{y^2}{2} = C$ passes through (1, 4)

$1 + 8 = C$

hence $x^2 + \dfrac{y^2}{2} = 9 \quad \Rightarrow \quad \dfrac{x^2}{9} + \dfrac{y^2}{18} = 1 \quad \Rightarrow$ ellipse ⇒ **(R)**

(D) length of normal

$(x + my - x)^2 + y^2 = 4$

$m^2y^2 + y^2 = 4$

$m^2 = \dfrac{4 - y^2}{y^2}; \quad \dfrac{dy}{dx} = \dfrac{\sqrt{4 - y^2}}{y}; \quad \displaystyle\int \frac{y\,dy}{\sqrt{4 - y^2}} = \int dx$

$-\sqrt{4 - y^2} = x + C$

$x = 1, y = 4 \quad \Rightarrow \quad C = -1$

$\therefore \quad (x - 1)^2 = 4 - y^2$

$(x - 1)^2 + y^2 = 4 \quad \Rightarrow$ circle ⇒ **(P)**]

MC SIR

CLASS : XIII (VXYZ) **_DPP. NO.- 10_**

Q.1$_{2/hyp}$ The magnitude of the gradient of the tangent at an extremity of latera recta of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is equal to (where e is the eccentricity of the hyperbola)

(A) be (B*) e (C) ab (D) ae

[Sol. T: $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}$; $\frac{x\cdot ae}{a^2}-\frac{y\cdot b^2}{a\cdot b^2}=1$ or $\frac{ex}{a}-\frac{y}{a}=1$ or $ex-y=a \Rightarrow m=e$ Ans.]

[12th, 04-01-2009, P-1]

Q.2$_{8/hyp}$ The number of possible tangents which can be drawn to the curve $4x^2-9y^2=36$, which are perpendicular to the straight line $5x+2y-10=0$ is :

(A*) zero (B) 1 (C) 2 (D) 4

[Hint: $y=-(5/2)x+5 \Rightarrow m=2/5 \Rightarrow a^2m^2-b^2=9\ .\ 4/25-4=(36-100)/25<0$

Note that the slope of the tangent (2/5) is less than the slope of the asymptote which is 2/3 which is not possible] **[12th, 04-01-2009, P-1]**

Q.3$_{33/hyp}$ Locus of the point of intersection of the tangents at the points with eccentric angles ϕ and $\frac{\pi}{2}-\phi$ on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is :

(A) $x=a$ (B*) $y=b$ (C) $x=ab$ (D) $y=ab$

[Sol. Tangent at ϕ, $\frac{x\sec\phi}{a}-\frac{y\tan\phi}{b}=1$

at $\frac{\pi}{2}-\phi$ $\frac{x\,\text{cosec}\phi}{a}-\frac{y\cot\phi}{b}=1$

$\therefore$ $(b\sec\phi)h-(a\tan\phi)k=ab$

$(b\,\text{cosec}\phi)h-(a\cot\phi)k=ab$

$$K=-\frac{\begin{vmatrix} b\sec\phi & ab \\ b\,\text{cosec}\phi & ab \end{vmatrix}}{\begin{vmatrix} b\sec\phi & a\tan\phi \\ b\,\text{cosec}\phi & a\cot\phi \end{vmatrix}}=\frac{b(\sec\phi-\text{cosec}\phi)}{\cot\phi\ \sec\phi-\tan\phi\ \text{cosec}\phi}=-\frac{b(\sec\phi-\text{cosec}\phi)}{\text{cosec}\phi-\sec\phi}=b \Rightarrow \text{(B)]}$$

Q.4$_{9/hyp}$ The equation $\frac{x^2}{29-p}+\frac{y^2}{4-p}=1$ $(p\neq 4, 29)$ represents

(A) an ellipse if p is any constant greater than 4.

(B*) a hyperbola if p is any constant between 4 and 29.

(C) a rectangular hyperbola if p is any constant greater than 29.

(D) no real curve if p is less than 29.

[Hint: For ellipse $29-p>0$ and $4-p>0$ $\Rightarrow$ $p<4$

for hyperbola $29-p>0$ and $4-p<0$ $\Rightarrow$ $p\in(4, 29)$]

Q.5$_{46/hyp}$ If $\dfrac{x^2}{\cos^2\alpha} - \dfrac{y^2}{\sin^2\alpha} = 1$ represents family of hyperbolas where 'α' varies then

(A*) distance between the foci is constant
(B) distance between the two directrices is constant
(C) distance between the vertices is constant
(D) distances between focus and the corresponding directrix is constant

[Hint: $d^2 = 4\,a^2\,e^2$
$= 4(a^2 + b^2) = 4 \Rightarrow d = 2 \Rightarrow$ (A)]

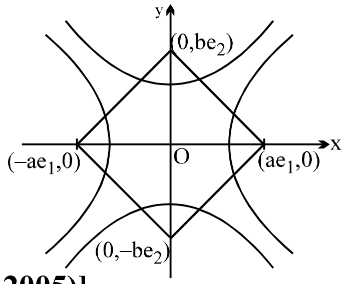
Q.6$_{35/hyp}$ Number of common tangent with finite slope to the curves $xy = c^2$ & $y^2 = 4ax$ is :
(A) 0 (B*) 1 (C) 2 (D) 4

[Hint: $m^3 = -(a^2/4c^2)$]

Q.7$_{52hyp}$ Area of the quadrilateral formed with the foci of the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ and $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = -1$ is

(A) $4(a^2 + b^2)$ (B*) $2(a^2 + b^2)$ (C) $(a^2 + b^2)$ (D) $\dfrac{1}{2}(a^2 + b^2)$

[Hint: Given hyperbolas are conjugate and the quadrilateral formed by their foci is a square



now $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ and $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = -1$

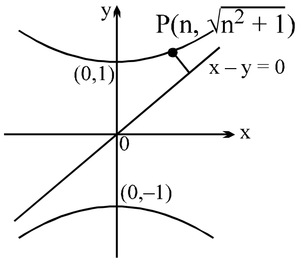
$e_1^2 = 1 + \dfrac{b^2}{a^2}$; $e_2^2 = 1 + \dfrac{a^2}{b^2}$; $e_1^2 e_2^2 = \dfrac{(a^2+b^2)^2}{a^2 b^2}$; $e_1 e_2 = \dfrac{a^2+b^2}{ab}$

$A = \dfrac{(2ae_1)(2be_2)}{2} = 2abe_1e_2 = \dfrac{2ab(a^2+b^2)}{ab}$] **[13th test (24-3-2005)]**

Q.8$_{55hyp}$ For each positive integer *n*, consider the point P with abscissa *n* on the curve $y^2 - x^2 = 1$. If d_n represents the shortest distance from the point P to the line $y = x$ then $\lim\limits_{n\to\infty}(n \cdot d_n)$ has the value equal to

(A*) $\dfrac{1}{2\sqrt{2}}$ (B) $\dfrac{1}{2}$ (C) $\dfrac{1}{\sqrt{2}}$ (D) 0

[Sol. Curve is rectangular hyperbola. **[13th, 16-12-2007]**



perpendicular distance, $d_n = \left|\dfrac{n - \sqrt{n^2+1}}{\sqrt{2}}\right|$

$$\lim_{n\to\infty}(n \cdot d_n) = \lim_{n\to\infty} \frac{n}{\sqrt{2}}\left|\left(\sqrt{n^2+1} - n\right)\right|$$

$$= \lim_{n\to\infty} \frac{n}{\sqrt{2}}\left|\frac{1}{\sqrt{n^2+1}+n}\right| = \frac{1}{2\sqrt{2}} \text{ Ans.]}$$

Paragraph for question nos. 9 to 11

The graph of the conic $x^2 - (y-1)^2 = 1$ has one tangent line with positive slope that passes through the origin. the point of tangency being (a, b). Then

Q.9$_{407/hyp}$ The value of $\sin^{-1}\left(\frac{a}{b}\right)$ is

(A) $\frac{5\pi}{12}$ (B) $\frac{\pi}{6}$ (C) $\frac{\pi}{3}$ (D*) $\frac{\pi}{4}$

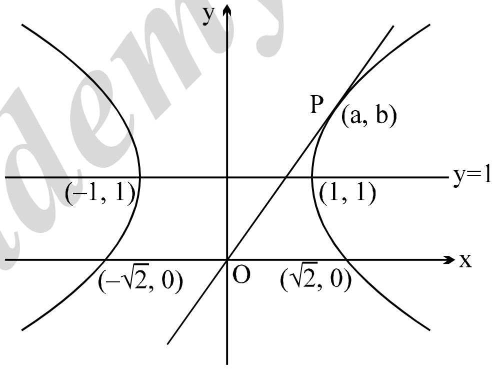
Q.10 Length of the latus rectum of the conic is

(A) 1 (B) $\sqrt{2}$ (C*) 2 (D) none

Q.11 Eccentricity of the conic is

(A) $\frac{4}{3}$ (B) $\sqrt{3}$ (C) 2 (D*) none

[**Sol.(i)** differentiate the curve **[13th test (09-10-2005)]**



$$2x - 2(y-1)\frac{dy}{dx} = 0$$

$$\left.\frac{dy}{dx}\right]_{a,b} = \frac{a}{b-1} = \frac{b}{a} \qquad \left(m_{OP} = \frac{b}{a}\right)$$

$$a^2 = b^2 - b \qquad ....(1)$$

Also (a, b) satisfy the curve

$$a^2 - (b-1)^2 = 1$$
$$a^2 - (b^2 - 2b + 1) = 1$$
$$a^2 - b^2 + 2b = 2$$

$\therefore \quad -b + 2b = 2 \;\Rightarrow\; b = 2 \quad \{\text{ putting } a^2 - b^2 = -b \text{ from (1) }\}$

$\therefore \quad a = \sqrt{2} \qquad (a \neq -\sqrt{2})$

$\therefore \quad \sin^{-1}\left(\frac{a}{b}\right) = \frac{\pi}{4}$ Ans.

Sol.(ii) Length of latus rectum = $\frac{2b^2}{a}$ = 2a = distance between the vertices = 2

(note that the hyperbola is rectangular)

Sol.(iii) Curve is a rectangular hyperbola $\Rightarrow$ e = $\sqrt{2}$ Ans.]

MC SIR

CLASS : XIII (VXYZ) **DPP. NO.- 11**

Q.$1_{44/hyp}$ If $x + iy = \sqrt{\phi + i\psi}$ where $i = \sqrt{-1}$ and ϕ and ψ are non zero real parameters then ϕ = constant and ψ = constant, represents two systems of rectangular hyperbola which intersect at an angle of

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{3}$ (C) $\frac{\pi}{4}$ (D*) $\frac{\pi}{2}$

[Hint: $x^2 - y^2 + 2xyi = \phi + i\psi$; $x^2 - y^2 = \phi$ and $xy = \psi$; which intersects at $\frac{\pi}{2}$ $\Rightarrow$ (D)]

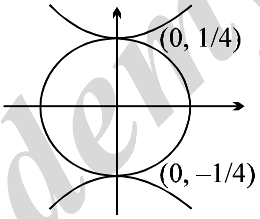
Q.$2_{12/hyp}$ Locus of the feet of the perpendiculars drawn from either foci on a variable tangent to the hyperbola $16y^2 - 9x^2 = 1$ is

(A) $x^2 + y^2 = 9$ (B) $x^2 + y^2 = 1/9$ (C) $x^2 + y^2 = 7/144$ (D*) $x^2 + y^2 = 1/16$

[Sol. $\frac{y^2}{1/16} - \frac{x^2}{1/9} = 1$ **[12th, 04-01-2009, P-1]**

Locus will be the auxilary circle

$x^2 + y^2 = 1/16$]



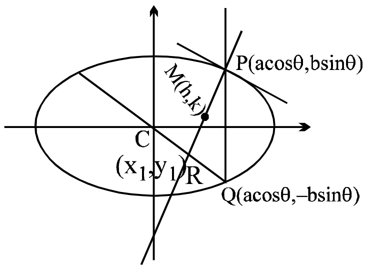
Q.$3_{45/hyp}$ PQ is a double ordinate of the ellipse $x^2 + 9y^2 = 9$, the normal at P meets the diameter through Q at R, then the locus of the mid point of PR is

(A) a circle (B) a parabola (C*) an ellipse (D) a hyperbola

[Sol. $\frac{x^2}{9} + \frac{y^2}{1} = 1$; $a = 3$, $b = 1$

Equation of PR : $\frac{a^2x}{a\cos\theta} - \frac{b^2y}{b\sin\theta} = a^2 - b^2$

$$\frac{3x}{\cos\theta} - \frac{y}{\sin\theta} = 8 \qquad ....(1)$$



Equation of CQ : $y = -\frac{b\sin\theta}{a\cos\theta}x \Rightarrow y = -\frac{\sin\theta}{3\cos\theta}x$

$$\frac{-y}{\sin\theta} = \frac{x}{3\cos\theta} \qquad ....(2) \qquad \text{put in (1)}$$

$$\frac{3x}{\cos\theta} + \frac{x}{3\cos\theta} = 8 \Rightarrow \frac{10x}{3\cos\theta} = 8 \Rightarrow x_1 = \frac{12\cos\theta}{5} \ ; \ y_1 = \frac{-4\cos\theta}{5} \qquad \text{from (2)}$$

we have, $2h = \frac{12\cos\theta}{5} + 3\cos\theta = \frac{27\cos\theta}{5} \Rightarrow \cos\theta = \frac{10h}{27}$

$$2k = \sin\theta - \frac{4\sin\theta}{5} = \frac{\sin\theta}{5} \Rightarrow \sin\theta = 10k$$

$\sin^2\theta + \cos^2\theta = 1 \Rightarrow 100y^2 + \frac{100x^2}{729} = 1 \Rightarrow$ Ellipse]

Q.4$_{40/hyp}$ With one focus of the hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$ as the centre , a circle is drawn which is tangent to the hyperbola with no part of the circle being outside the hyperbola. The radius of the circle is

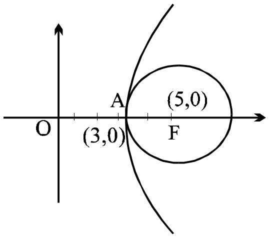
(A) less than 2 (B*) 2 (C) $\frac{11}{3}$ (D) none

[Hint : $e^2 = 1 + \frac{16}{9} = \frac{25}{9} \Rightarrow e = \frac{5}{3}$

$\therefore$ focus = (5, 0)

Use reflection property to prove that circle cannot touch at two points. It can only be tangent at the vertex

r = 5 – 3 = 2]



Q.5$_{49hyp}$ If the tangent and normal at any point of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, meets the conjugate axis at Q and R, then the circle described on QR as diameter passes through the

(A) vertices (B*) focii

(C) feet of directrices (D) ends of latera recta

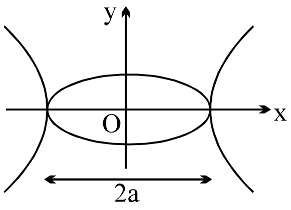
Q.6$_{28/hyp}$ Let the major axis of a standard ellipse equals the transverse axis of a standard hyperbola and their director circles have radius equal to 2R and R respectively. If e_1 and e_2 are the eccentricities of the ellipse and hyperbola then the correct relation is

(A) $4e_1^2 - e_2^2 = 6$ (B) $e_1^2 - 4e_2^2 = 2$ (C*) $4e_2^2 - e_1^2 = 6$ (D) $2e_1^2 - e_2^2 = 4$

[Sol. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ....(1); $\frac{x^2}{a^2} - \frac{y^2}{b_1^2} = 1$ ....(2)

$R = \sqrt{a^2 - b_1^2}$

$2R = \sqrt{a^2 + b^2}$



$\therefore \quad 2\sqrt{a^2 - b_1^2} = \sqrt{a^2 + b^2} \qquad \left[e_1^2 = 1 - \frac{b^2}{a^2};\ e_2^2 = 1 + \frac{b_1^2}{a^2}\right]$

$4(a^2 - b_1^2) = a^2 + b^2$

$4\left(1 - \frac{b_1^2}{a^2}\right) = 1 + \frac{b^2}{a^2}$

$4[(1 - (e_2^2 - 1))] = 1 + 1 - e_1^2$

$8 - 4e_2^2 = 2 - e_1^2$

$4e_2^2 - e_1^2 = 6$ Ans.] **[12th, 06-01-2008]**

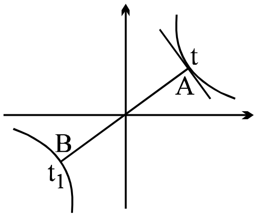
Q.7$_{39/hyp}$ If the normal to the rectangular hyperbola $xy = c^2$ at the point 't' meets the curve again at 't_1' then $t^3 t_1$ has the value equal to

(A) 1 (B*) – 1 (C) 0 (D) none

[Sol. $x = ct \Rightarrow \frac{dx}{dt} = c$

$y = \frac{c}{t} \Rightarrow \frac{dy}{dt} = -\frac{c}{t^2}$

$\frac{dy}{dx} = -\frac{1}{t^2}$



∴ $m_N = t^2$

∴ $t^2 = m_{AB} = -\frac{1}{t_1 t}$

∴ $t^3 t_1 = -1$]

Q.$8_{23/hyp}$ P is a point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, N is the foot of the perpendicular from P on the transverse axis. The tangent to the hyperbola at P meets the transverse axis at T . If O is the centre of the hyperbola, the OT. ON is equal to :

(A) e^2 (B*) a^2 (C) b^2 (D) b^2/a^2

[Hint: $OT = a\cos\theta$; $N = a\sec\theta \Rightarrow OT \,.\, ON = a^2$]

More than one are correct:

Q.$9_{513/hyp}$ Solutions of the differential equation $(1 - x^2)\frac{dy}{dx} + xy = ax$ where $a \in R$, is

(A*) a conic which is an ellipse or a hyperbola with principal axes parallel to coordinates axes.
(B*) centre of the conic is (0, a)
(C) length of one of the principal axes is 1.
(D*) length of one of the principal axes is equal to 2.

[Sol. $\frac{dy}{dx} + \frac{x}{1-x^2}y = \frac{ax}{1-x^2}$ **[12th, 06-01-2008]**

I.F. $e^{\int \frac{x}{1-x^2}dx} = e^{-\frac{1}{2}\log|1-x^2|} = \frac{1}{\sqrt{|1-x^2|}}$

$$\frac{y}{\sqrt{|1-x^2|}} = a\int \frac{x}{|1-x^2|^{3/2}}dx + C$$

let $|1 - x^2| = v^2$; $-2x\,dx = 2v\,dv$; $x\,dx = -v\,dv$

hence $\frac{y}{v} = -a\int \frac{v\,dv}{v^3} = +\frac{a}{v} + C$

$y = a + Cv$

$y = a + C\sqrt{|1-x^2|}$

$(y - a)^2 = C^2|1 - x^2| = C^2(1 - x^2)$ or $C^2(x^2 - 1)$

$(y - a)^2 + C^2x^2 = C^2$ or $(y - a)^2 - C^2x^2 = -C^2$

$\frac{(y-a)^2}{C^2} + \frac{x^2}{1} = 1$ or $\frac{(y-a)^2}{C^2} - \frac{x^2}{1} = -1 \Rightarrow$ centre (0, a)]

Q.10$_{514/hyp}$ In which of the following cases maximum number of normals can be drawn from a point P lying in the same plane

(A*) circle (B) parabola (C) ellipse (D) hyperbola

[13th, 20-01-2008]

Q.11$_{515/hyp}$ If θ is eliminated from the equations

$a \sec\theta - x\tan\theta = y$ and $b\sec\theta + y\tan\theta = x$ (a and b are constant)

then the eliminant denotes the equation of

(A) the director circle of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

(B) auxiliary circle of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

(C*) Director circle of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

(D*) Director circle of the circle $x^2 + y^2 = \frac{a^2+b^2}{2}$.

[Sol. $a\sec\theta = y + x\tan\theta$ **[13th, 10-08-2008, P-2]**

$b\sec\theta = x - y\tan\theta$

$(a^2 + b^2)\sec^2\theta = x^2(1 + \tan^2\theta) + y^2(1 + \tan^2\theta)$

$\Rightarrow \quad x^2 + y^2 = a^2 + b^2 \quad \Rightarrow$ (C) and (D)]

Q.$1_{29/hyp}$ If $P(x_1, y_1)$, $Q(x_2, y_2)$, $R(x_3, y_3)$ & $S(x_4, y_4)$ are 4 concyclic points on the rectangular hyperbola $xy = c^2$, the co-ordinates of the orthocentre of the triangle PQR are :

(A) $(x_4, -y_4)$ (B) (x_4, y_4) (C*) $(-x_4, -y_4)$ (D) $(-x_4, y_4)$

[Hint: A rectangular hyperbola circumscribing a Δ also passes through its orthocentre

if $\left(ct_i, \frac{c}{t_i}\right)$ where i = 1, 2, 3 are the vertices of the Δ then therefore orthocentre is $\left(\frac{-c}{t_1t_2t_3}, -ct_1t_2t_3\right)$,

where $t_1 t_2 t_3 t_4 = 1$. Hence orthocentre is $\left(-ct_4, \frac{-c}{t_4}\right) = (-x_4, -y_4)$]

Q.$2_{4/hyp}$ Let F_1, F_2 are the foci of the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ and F_3, F_4 are the foci of its conjugate hyperbola. If e_H and e_C are their eccentricities respectively then the statement which holds true is

(A) Their equations of the asymptotes are different.

(B) $e_H > e_C$

(C*) Area of the quadrilateral formed by their foci is 50 sq. units.

(D) Their auxillary circles will have the same equation.

[Hint: $e_H = 5/4$; $e_C = 5/3$ **[12th & 13th 11-3-2007]**

area = $\frac{d_1d_2}{2} = \frac{100}{2} = 50$

$A_C: x^2 + y^2 = 16$; $A_H = x^2 + y^2 = 9$]

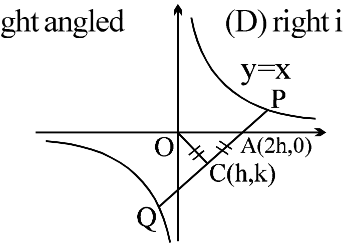
Q.$3_{31/hyp}$ The chord PQ of the rectangular hyperbola $xy = a^2$ meets the axis of x at A ; C is the mid point of PQ & 'O' is the origin. Then the Δ ACO is :

(A) equilateral (B*) isosceles (C) right angled (D) right isosceles.

[Sol. Chord with a given middle point

$$\frac{x}{h} + \frac{y}{k} = 2$$

obv. OCA is isosceles with OC = CA.]



Q.$4_{7/hyp}$ The asymptote of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ form with any tangent to the hyperbola a triangle whose area is $a^2\tan\lambda$ in magnitude then its eccentricity is :

(A*) $\sec\lambda$ (B) $\operatorname{cosec}\lambda$ (C) $\sec^2\lambda$ (D) $\operatorname{cosec}^2\lambda$

[Hint : $A = ab = a^2 \tan\lambda \Rightarrow b/a = \tan\lambda$, hence $e^2 = 1 + (b^2/a^2) \Rightarrow e^2 = 1 + \tan^2\lambda \Rightarrow e = \sec\lambda$]

Q.$5_{34/hyp}$ Latus rectum of the conic satisfying the differential equation, $x\,dy + y\,dx = 0$ and passing through the point (2, 8) is :

(A) $4\sqrt{2}$ (B) 8 (C*) $8\sqrt{2}$ (D) 16

[Sol. $\frac{dy}{y} + \frac{dx}{x} = 0 \Rightarrow \ln xy = c \Rightarrow xy = c$

passes through (2,8) $\Rightarrow$ c = 16

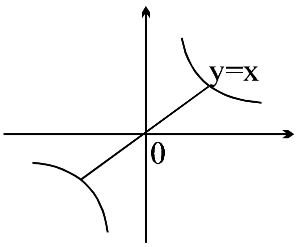
xy = 16 LR = $2a(e^2 - 1) = 2a$ (e = $\sqrt{2}$)

solving with y = x

vertex is (4, 4)

distance from centre to vertex = $4\sqrt{2}$

L.R. = length of TA = $8\sqrt{2}$ Ans]



Q.6$_{41/hyp}$ AB is a double ordinate of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ such that ΔAOB (where 'O' is the origin) is an equilateral triangle, then the eccentricity e of the hyperbola satisfies

(A) e > $\sqrt{3}$ (B) $1 < e < \frac{2}{\sqrt{3}}$ (C) $e = \frac{2}{\sqrt{3}}$ (D*) $e > \frac{2}{\sqrt{3}}$

[Sol. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ where $y = l$

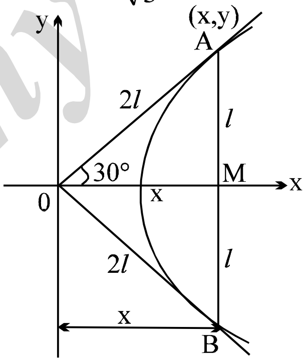
$\frac{x^2}{a^2} = 1 + \frac{l^2}{b^2} \Rightarrow x^2 = (b^2 + l^2)\frac{a^2}{b^2}$ ....(1)

now $x^2 + l^2 = 4l^2 \Rightarrow x^2 = 3l^2$ ....(2)

from (1) and (2) $\frac{a^2(b^2 + l^2)}{b^2} = 3l^2 \Rightarrow a^2b^2 + a^2l^2 = 3b^2l^2$

$l^2(3b^2 - a^2) = a^2b^2$

$l^2 = \frac{a^2b^2}{3b^2 - a^2} > 0 \Rightarrow 3b^2 - a^2 > 0 \Rightarrow \frac{b^2}{a^2} > \frac{1}{3}$; $1 + \frac{b^2}{a^2} > \frac{4}{3} \Rightarrow e^2 > \frac{4}{3} \Rightarrow e > \frac{2}{\sqrt{3}}$



Note: $\frac{b}{a} > \frac{1}{\sqrt{3}} \Rightarrow 1 + \frac{b^2}{a^2} > \frac{4}{3} \Rightarrow e^2 > \frac{4}{3} \Rightarrow e > \frac{2}{\sqrt{3}}$]

Q.7$_{47/hyp}$ The tangent to the hyperbola $xy = c^2$ at the point P intersects the x-axis at T and the y-axis at T′. The normal to the hyperbola at P intersects the x-axis at N and the y-axis at N′. The areas of the triangles PNT and PN'T' are Δ and Δ' respectively, then $\frac{1}{\Delta} + \frac{1}{\Delta'}$ is

(A) equal to 1 (B) depends on t (C*) depends on c (D) equal to 2

[Sol. Tangent : $\frac{x}{ct} + \frac{yt}{c} = 2$

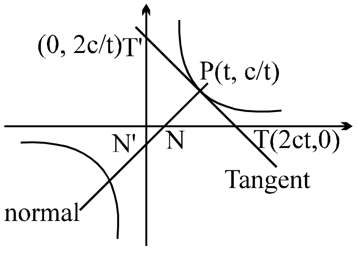
put y = 0; x = 2ct (T)

x = 0; $y = \frac{2c}{t}$ (T')

|||ly normal is $y - \frac{c}{t} = t^2(x - ct)$

put y = 0; $x = ct - \frac{c}{t^3}$ (N)

x = 0; $\frac{c}{t} - ct^3$ (N')



Area of Δ PNT $= \dfrac{c}{2t}\left(ct + \dfrac{c}{t^3}\right) \quad \Rightarrow \quad \Delta = \dfrac{c^2(1+t^4)}{2t^4}$

area of Δ PN'T' $= ct\left(\dfrac{c}{t} + ct^3\right) \quad \Rightarrow \quad \Delta' = \dfrac{c^2(1+t^4)}{2}$

$$\therefore \quad \frac{1}{\Delta} + \frac{1}{\Delta'} = \frac{2t^4}{c^2(1+t^4)} + \frac{2}{c^2(1+t^4)} = \frac{2}{c^2(1+t^4)}(t^4+1) = \frac{2}{c^2}$$

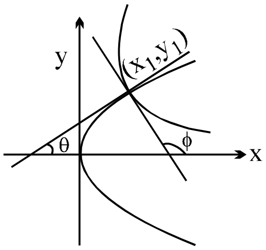
which is independent of t.]

Q.8$_{50hyp}$ At the point of intersection of the rectangular hyperbola $xy = c^2$ and the parabola $y^2 = 4ax$ tangents to the rectangular hyperbola and the parabola make an angle θ and ϕ respectively with the axis of X, then

(A*) $\theta = \tan^{-1}(-2\tan\phi)$ (B) $\phi = \tan^{-1}(-2\tan\theta)$

(C) $\theta = \dfrac{1}{2}\tan^{-1}(-\tan\phi)$ (D) $\phi = \dfrac{1}{2}\tan^{-1}(-\tan\theta)$

[Sol. Let (x_1, y_1) be the point of intersection $\Rightarrow$ $y_1^2 = 4ax_1$ and $x_1y_1 = c^2$



$y^2 = 4ax$ **$xy = c^2$**

$$\therefore \quad \frac{dy}{dx} = \frac{2a}{y} \qquad \frac{dy}{dx} = -\frac{y}{x}$$

$$\left.\frac{dy}{dx}\right|_{(x_1,y_1)} = \tan\phi = \frac{2a}{y_1} \qquad \left.\frac{dy}{dx}\right|_{(x_1,y_1)} = \tan\theta = -\frac{y_1}{x_1}$$

$$\therefore \quad \frac{\tan\theta}{\tan\phi} = \frac{-y_1/x_1}{2a/y_1} = \frac{-y_1^2}{2ax_1} = -\frac{4ax_1}{2ax_1} = -2$$

$\Rightarrow \quad \theta = \tan^{-1}(-2\tan\phi)$]

Q.9$_{19/hyp}$ Locus of the middle points of the parallel chords with gradient m of the rectangular hyperbola $xy = c^2$ is

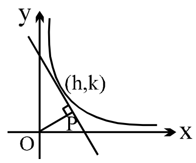
(A*) $y + mx = 0$ (B) $y - mx = 0$ (C) $my - x = 0$ (D) $my + x = 0$

[Hint : equation of chord with mid point (h, k) is $\dfrac{x}{h} + \dfrac{y}{k} = 2$; $m = -\dfrac{k}{h}$ $\Rightarrow$ $y + mx = 0$]

Q.10$_{20/hyp}$ The locus of the foot of the perpendicular from the centre of the hyperbola $xy = c^2$ on a variable tangent is :

(A) $(x^2 - y^2)^2 = 4c^2xy$ (B) $(x^2 + y^2)^2 = 2c^2xy$

(C) $(x^2 + y^2) = 4x^2xy$ (D*) $(x^2 + y^2)^2 = 4c^2xy$



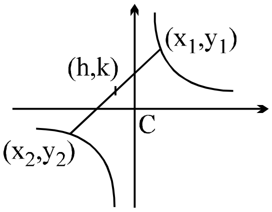
[Hint: $hx + ky = h^2 + k^2$. Solve it with $xy = c^2$ & $D = 0$

or compare these with tangent at t and eliminate 't'.]

Q.11$_{25/hyp}$ The equation to the chord joining two points (x_1, y_1) and (x_2, y_2) on the rectangular hyperbola $xy = c^2$ is :

(A*) $\frac{x}{x_1 + x_2} + \frac{y}{y_1 + y_2} = 1$ (B) $\frac{x}{x_1 - x_2} + \frac{y}{y_1 - y_2} = 1$

(C) $\frac{x}{y_1 + y_2} + \frac{y}{x_1 + x_2} = 1$ (D) $\frac{x}{y_1 - y_2} + \frac{y}{x_1 - x_2} = 1$



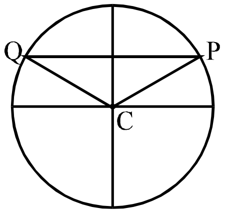
[Hint : note that chord of $xy = c^2$ whose middle point is (h, k) in $\frac{x}{h}+\frac{y}{k}=2$

further, now $2h = x_1 + x_2$ and $2k = y_1 + y_2$]

Q.12 A tangent to the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ meets its director circle at P and Q. Then the product of the slopes of CP and CQ where 'C' is the origin is

(A) $\frac{9}{4}$ (B*) $\frac{-4}{9}$ (C) $\frac{2}{9}$ (D) $-\frac{1}{4}$

[Sol. The equation of the tangent at $(3 \cos \theta, 2 \sin \theta)$ on $\frac{x^2}{9}+\frac{y^2}{4}=1$ is



$$\frac{x}{3}\cos\theta+\frac{y}{2}\sin\theta=1 \qquad \text{... (i)}$$

The equation of the director circle is

$$x^2 + y^2 = 9 + 3 = 13 \qquad \text{... (ii)}$$

The combined equation of CP and CQ is obtained by homogenising equation (ii) with (i). Thus combined equation is

$$x^2 + y^2 = 13\left(\frac{x}{3}\cos\theta+\frac{y}{2}\sin\theta\right)^2$$

$$\Rightarrow \quad \left(\frac{13}{9}\cos^2\theta-1\right)x^2+\frac{13}{3}\sin\theta\cos\theta\, xy + \left(\frac{13}{4}\sin^2\theta-1\right)y^2=0$$

∴ Product of the slopes of CP and CQ

$$\frac{\text{coefficient of } x^2}{\text{coefficient of } y^2}=\frac{\frac{13}{9}\cos^2\theta-1}{\frac{13}{4}\sin^2\theta-1} = \frac{13\cos^2\theta-9}{13\sin^2\theta-4}\times\frac{4}{9} = \frac{13\cos^2\theta-9}{9-13\cos^2\theta-4}\times\frac{4}{9}=-\frac{4}{9} \quad]$$

Q.13 The foci of a hyperbola coincide with the foci of the ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$. Then the equation of the hyperbola with eccentricity 2 is

(A) $\frac{x^2}{12}-\frac{y^2}{4}=1$ (B*) $\frac{x^2}{4}-\frac{y^2}{12}=1$ (C) $3x^2 - y^2 + 12 = 0$ (D) $9x^2 - 25y^2 - 225 = 0$

[Sol. For the ellipse, $a^2 = 25$, $b^2 = 9$

$$\therefore \quad 9 = 25(1 - e^2) \Rightarrow \quad e^2 = \frac{16}{25} \quad \Rightarrow \quad e = \frac{4}{5}$$

$\therefore$ One of the foci is (ae, 0) i.e. (4, 0)

$\therefore$ For the hyperbola

$a'e' = 4 \Rightarrow 2a' = 4 \Rightarrow a' = 2$

and $b'^2 = 4(e'^2 - 1) = 4 \times 3 = 12$

$\therefore$ equation of the hyperbola is $\frac{x^2}{4} - \frac{y^2}{12} = 1$ **Ans.**]

Paragraph for question nos. 14 to 16

From a point 'P' three normals are drawn to the parabola $y^2 = 4x$ such that two of them make angles with the abscissa axis, the product of whose tangents is 2. Suppose the locus of the point 'P' is a part of a conic 'C'. Now a circle S = 0 is described on the chord of the conic 'C' as diameter passing through the point (1, 0) and with gradient unity. Suppose (a, b) are the coordinates of the centre of this circle. If L_1 and L_2 are the two asymptotes of the hyperbola with length of its transverse axis 2a and conjugate axis 2b (principal axes of the hyperbola along the coordinate axes) then answer the following questions.

Q.14$_{404/hyp}$ Locus of P is a

(A) circle (B*) parabola (C) ellipse (D) hyperbola

Q.15 Radius of the circle S = 0 is

(A*) 4 (B) 5 (C) $\sqrt{17}$ (D) $\sqrt{23}$

Q.16 The angle $\alpha \in (0, \pi/2)$ between the two asymptotes of the hyperbola lies in the interval

(A) (0, 15°) (B) (30°, 45°) (C) (45°, 60°) (D*) (60°, 75°)

[Sol. Equation of a normal $y = mx - 2m - m^3$ **[12th & 13th 03-03-2007]**

passes through (h, k)]

$m^3 + (2 - h)m + k = 0$

$m_1 m_2 m_3 = -k$

but $m_1 m_2 = 2$

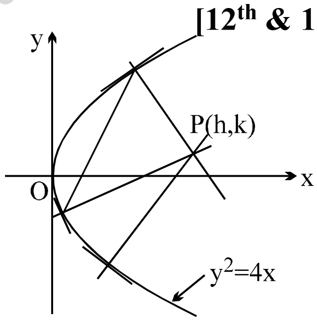
$\Rightarrow m_3 = -k/2$

this must satisfy equation (1)

$$\frac{k^3}{8} - (2 - h)\frac{k}{2} + k = 0$$

$k^3 - 4k(2 - h) + 8k = 0 \quad (k \neq 0)$

$k^2 - 8 - 4h + 8 = 0$



locus of 'P' is $y^2 = 4x$ which is a parabola **Ans.**

now chord passing through (1, 0) is the focal chord.

Given that gradient of focal chord is 1

$\therefore \frac{2}{t_1 + t_2} = 1 \Rightarrow t_1 + t_2 = 2$, Also $t_1 t_2 = -1$

equation of circle described on $t_1 t_2$ as diameter is

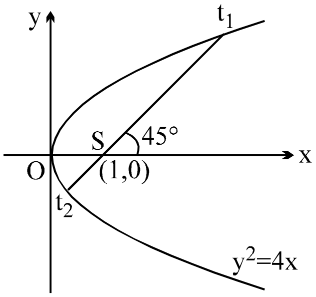
$(x - t_1^2)(x - t_2^2) + (y - 2t_1)(y - 2t_2) = 0$

$x^2 + y^2 - x(t_1^2 + t_2^2) + t_1^2 t_2^2 - 2y(t_1 + t_2) + 4t_1 t_2 = 0$

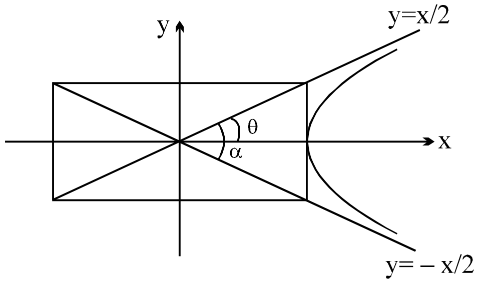
$x^2 + y^2 - x[4 + 2] + 1 - 2y(2) - 4 = 0$

$x^2 + y^2 - 6x - 4y - 3 = 0$

centre a = 3 and b = 2; r = 4 **Ans.**



now the hyperbola is $\frac{x^2}{9} - \frac{y^2}{4} = 1$



asymptotes are $y = \frac{2x}{3}$ and $y = -\frac{2x}{3}$

now $\tan\theta = 2/3$

$\therefore \quad \alpha = 2\theta$

$$\tan\alpha = \frac{2\cdot(2/3)}{1-(4/9)}; \qquad \tan\alpha = \frac{12}{5}; \qquad \alpha = \tan^{-1}\left(\frac{12}{5}\right)$$

hence $\alpha \in (60°, 75°)$ **Ans.**]

Paragraph for question nos. 17 to 19

A conic C passes through the point (2, 4) and is such that the segment of any of its tangents at any point contained between the co-ordinate axes is bisected at the point of tangency. Let S denotes circle described on the foci F_1 and F_2 of the conic C as diameter.

Q.17 Vertex of the conic C is

(A) (2, 2), (–2, – 2)
(B*) $\left(2\sqrt{2}, 2\sqrt{2}\right), \left(-2\sqrt{2}, -2\sqrt{2}\right)$
(C) (4, 4), (–4, – 4)
(D) $\left(\sqrt{2}, \sqrt{2}\right), \left(-\sqrt{2}, -\sqrt{2}\right)$

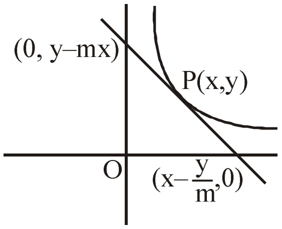
Q.18 Director circle of the conic is

(A) $x^2 + y^2 = 4$ (B) $x^2 + y^2 = 8$ (C) $x^2 + y^2 = 2$ (D*) None

Q.19 Equation of the circle S is

(A) $x^2 + y^2 = 16$ (B) $x^2 + y^2 = 8$ (C*) $x^2 + y^2 = 32$ (D) $x^2 + y^2 = 4$

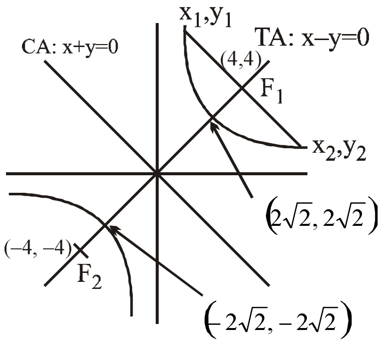
[Sol. $Y - y = m(X - x)$; if $Y = 0$ then



$X = x - \frac{y}{m}$ and if $X = 0$ then $Y = y - mx$.

Hence $x - \frac{y}{m} = 2x \Rightarrow \frac{dy}{dx} = -\frac{y}{x}$

$\int\frac{dy}{y} + \int\frac{dx}{x} = c \Rightarrow xy = c$


x_1,y_1
CA: x+y=0
TA: x–y=0
(4,4)
F_1
x_2,y_2
$\left(2\sqrt{2}, 2\sqrt{2}\right)$
(–4, –4)
F_2
$\left(-2\sqrt{2}, -2\sqrt{2}\right)$

passes through (2, 4)

$\Rightarrow$ equation of conic is $xy = 8$

which is a rectangular hyperbola with $e = \sqrt{2}$.

Hence the two vertices are $\left(2\sqrt{2}, 2\sqrt{2}\right), \left(-2\sqrt{2}, -2\sqrt{2}\right)$

focii are (4, 4) & (– 4, 4)

$\therefore$ Equation of S is $x^2 + y^2 = 32$ **Ans.**]

Assertion and Reason

Q.20 **Statement-1:** Diagonals of any parallelogram inscribed in an ellipse always intersect at the centre of the ellipse.

Statement-2: Centre of the ellipse is the only point at which two chords can bisect each other and every chord passing through the centre of the ellipse gets bisected at the centre.

(A*) Statement-1 is True, Statement-2 is True ; Statement-2 is a correct explanation for Statement-1
(B) Statement-1 is True, Statement-2 is True ; Statement-2 is NOT a correct explanation for Statement-1
(C) Statement -1 is True, Statement -2 is False
(D) Statement -1 is False, Statement -2 is True

[Sol. Statement-2 is correct as ellipse is a central conic and it also explains Statement-1.
Hence, code (A) is the correct answer.]

Q.21 **Statement-1:** The points of intersection of the tangents at three distinct points A, B, C on the parabola $y^2 = 4x$ can be collinear.

Statement-2: If a line L does not intersect the parabola $y^2 = 4x$, then from every point of the line two tangents can be drawn to the parabola.

(A) Statement-1 is True, Statement-2 is True ; Statement-2 is a correct explanation for Statement-1
(B) Statement-1 is True, Statement-2 is True ; Statement-2 is NOT a correct explanation for Statement-1
(C) Statement -1 is True, Statement -2 is False
(D*) Statement -1 is False, Statement -2 is True

[Sol. Area of the triangle made by the intersection points of tangents at point $A(t_1)$, $B(t_2)$ and $C(t_3)$ is

$$\frac{1}{2}|t_1 - t_2||t_2 - t_3||t_3 - t_1| \neq 0$$

Hence, Statement-1 is wrong. Statement-2 is correct.
Hence, code (D) is the correct answer.]

Q.22 **Statement-1:** The latus rectum is the shortest focal chord in a parabola of length 4a.
because

Statement-2: As the length of a focal chord of the parabola $y^2 = 4ax$ is $a\left(t+\frac{1}{t}\right)^2$, which is minimum when t = 1.

(A*) Statement-1 is True, Statement-2 is True ; Statement-2 is a correct explanation for Statement-1
(B) Statement-1 is True, Statement-2 is True ; Statement-2 is NOT a correct explanation for Statement-1
(C) Statement -1 is True, Statement -2 is False
(D) Statement -1 is False, Statement -2 is True

[Sol. Let $P(at^2, 2at)$ be the end of a focal chord PQ of the parabola $y^2 = 4ax$. Thus, the coordinate of the other end point Q is $\left(\frac{a}{t^2}, -\frac{2a}{t}\right)$

$$\therefore\ PQ = \sqrt{\left(at^2 - \frac{a}{t^2}\right)^2 + \left(2at + \frac{2a}{t}\right)^2} = \sqrt{\left(t^2 - \frac{1}{t^2}\right)^2 + 4\left(t + \frac{1}{t}\right)^2}$$

$$= a\left(t + \frac{1}{t}\right)\sqrt{\left(t - \frac{1}{t}\right)^2 + 4} = a\left(t + \frac{1}{t}\right)\sqrt{t^2 + \frac{1}{t^2} - 2 + 4} = a\left(t + \frac{1}{t}\right)^2$$

$\therefore$ Length of focal chord is, $a\left(t+\frac{1}{t}\right)^2$, where $\left(t+\frac{1}{t}\right)\geq 2$ for all $t\neq 0$.

$\therefore\ a\left(t+\frac{1}{t}\right)^2\geq 4a \Rightarrow PQ\geq 4a$

Thus, the length of the focal chord of the parabola is 4a which is the length of its latus rectum.
Hence, the latusrectum of a parabola is the shortest focal chord.
Thus, Statement-1 and Statement-2 is true and Statement-2 s correct explanation of Statement-1]

Q.23 **Statement-1:** If P(2a, 0) be any point on the axis of parabola, then the chord QPR, satisfy

$$\frac{1}{(PQ)^2}+\frac{1}{(PR)^2}=\frac{1}{4a^2}.$$

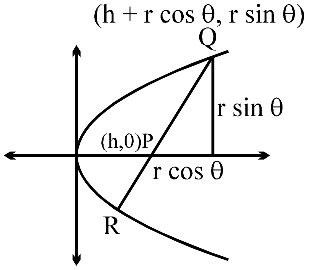
Statement-2: There exists a point P on the axis of the parabola $y^2 = 4ax$ (other than vertex), such that

$$\frac{1}{(PQ)^2}+\frac{1}{(PR)^2} = \text{constant for all chord QPR of the parabola.}$$

(A*) Statement-1 is True, Statement-2 is True ; Statement-2 is a correct explanation for Statement-1
(B) Statement-1 is True, Statement-2 is True ; Statement-2 is NOT a correct explanation for Statement-1
(C) Statement -1 is True, Statement -2 is False
(D) Statement -1 is False, Statement -2 is True

[Sol. Let P(h, 0) (where $h\neq 0$) be a point on the axis of parabola $y^2 = 4ax$ the straight line passing through P cuts the parabola at a distance r.
$\Rightarrow (r\sin\theta)^2 = 4a(h + r\cos\theta)$
$\Rightarrow r^2\sin^2\theta - (4a\cos\theta)r - 4ah = 0$ ... (i)

where, $r_1 + r_2 = \frac{4a\cos\theta}{\sin^2\theta}$ and $r_1r_2 = -\frac{4ah}{\sin^2\theta}$.



$$\therefore\ \frac{1}{PQ^2}+\frac{1}{PR^2}=\frac{1}{r_1^2}+\frac{1}{r_2^2}=\frac{r_1^2+r_2^2}{r_1^2r_2^2}=\frac{\cos^2\theta}{h^2}+\frac{\sin^2\theta}{2ah}$$

which is constant only, if $h^2 = 2ah$ i.e., $h = 2a$

$$\Rightarrow \frac{1}{PQ^2}+\frac{1}{PR^2}=\frac{\cos^2\theta}{4a^2}+\frac{\sin^2\theta}{4a^2}=\frac{1}{4a^2}$$

Thus, $\frac{1}{PQ^2}+\frac{1}{PR^2}$ = constant for all chords QPR,

if $h = 2a$.
Hence, (2a, 0) is the required point on the axis of parabola.
$\therefore$ Statement-1 and Statement-2 are true and Statement-2 is correct explanation of Statement-1]

Q.24 **Statement-1:** The quadrilateral formed by the pair of tangents drawn from the point (0, 2) to the parabola $y^2 - 2y + 4x + 5 = 0$ and the normals at the point of contact of tangents in a square.

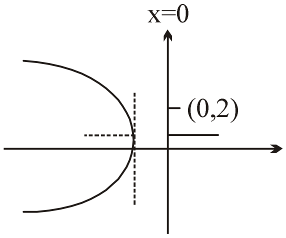
Statement-2: The angle between tangents drawn from the given point to the parabola is 90°.
(A) Statement-1 is True, Statement-2 is True ; Statement-2 is a correct explanation for Statement-1
(B) Statement-1 is True, Statement-2 is True ; Statement-2 is NOT a correct explanation for Statement-1
(C) Statement -1 is True, Statement -2 is False
(D*) Statement -1 is False, Statement -2 is True

[Sol. $(y-1)^2 = -4(x+1)$

Directrix $x + 1 = 1$

$x = 0$

If tangents are drawn from (0, 2) to the parabola (i.e. from directrix) then length of tangent will be unequal hence the quadrilateral formed by pair of tangents and normals at the point of contact is rectangle.]



More than one are correct:

Q.25$_{502hyp}$ If the circle $x^2 + y^2 = a^2$ intersects the hyperbola $xy = c^2$ in four points $P(x_1, y_1)$, $Q(x_2, y_2)$, $R(x_3, y_3)$, $S(x_4, y_4)$, then

(A*) $x_1 + x_2 + x_3 + x_4 = 0$ (B*) $y_1 + y_2 + y_3 + y_4 = 0$

(C*) $x_1 x_2 x_3 x_4 = c^4$ (D*) $y_1 y_2 y_3 y_4 = c^4$

[Sol. solving $xy = c^2$ and $x^2 + y^2 = a^2$

$$x^2 + \frac{c^4}{x^2} = a^2$$

$$x^4 - ax^3 - a^2x^2 + ax + c^4 = 0$$

$$\Rightarrow \quad \sum x_i = 0 \ ; \ \sum y_i = 0$$

$$x_1 x_2 x_3 x_4 = c^4 \quad \Rightarrow \quad y_1 y_2 y_3 y_4 = c^4$$]

Q.26$_{503hyp}$ The tangent to the hyperbola, $x^2 - 3y^2 = 3$ at the point $(\sqrt{3}, 0)$ when associated with two asymptotes constitutes :

(A) isosceles triangle (B*) an equilateral triangle

(C*) a triangles whose area is $\sqrt{3}$ sq. units (D) a right isosceles triangle .

[Hint: area of the Δ = ab sq units ; H : $x^2/3 - y^2/1 = 1$]

Q.27 The locus of the point of intersection of those normals to the parabola $x^2 = 8y$ which are at right angles to each other, is a parabola. Which of the following hold(s) good in respect of the locus?

(A*) Length of the latus rectum is 2.

(B) Coordinates of focus are $\left(0, \frac{11}{2}\right)$

(C*) Equation of a directro circle is $2y - 11 = 0$

(D) Equation of axis of symmetry $y = 0$.

[Hint: Locus is $x^2 - 2y + 12 = 0$] [REE '97, 6]

Match the column:

Q.28$_{105}$

	Column-I		Column-II
(A)	If the chord of contact of tangents from a point P to the parabola $y^2 = 4ax$ touches the parabola $x^2 = 4by$, the locus of P is	(P)	Straight line
(B)	A variable circle C has the equation $x^2 + y^2 - 2(t^2 - 3t + 1)x - 2(t^2 + 2t)y + t = 0$, where t is a parameter. The locus of the centre of the circle is	(Q)	Circle
(C)	The locus of point of intersection of tangents to an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at two points the sum of whose eccentric angles is constant is	(R)	Parabola
(D)	An ellipse slides between two perpendicular straight lines. Then the locus of its centre is	(S)	Hyperbola

[Ans. (A) S; (B) R; (C) P; (D) Q]

[Sol.

(A) $yy_1 = 2a(x + x_1)$; $x^2 = 4by = 4b[(2a/y_1)(x + x_1)] \Rightarrow y_1x^2 - 8abx - 8abx_1 = 0$;
D = 0 gives $xy = -2ab \quad \Rightarrow$ Hyperbola

(B) centre is $x = t^2 - 3t + 1 \quad ....(1)$ **[18-12-2005, 12th]**
$y = t^2 + 2t \quad ....(2)$
(2) – (1) gives $-x + y = 5t - 1$

or $\quad t = \frac{1 - x + y}{5}$

Substituting the value of t in (2)

$$y = \left(\frac{y - x + 1}{5}\right)^2 + 2\left(\frac{y - x + 1}{5}\right)$$

$25y = (y - x + 1)^2 + 10(y - x + 1)$
$25y = y^2 + x^2 + 1 - 2xy - 2x + 2y + 10y - 10x + 10$
$x^2 + y^2 - 2xy - 12x - 13y + 11 = 0$
which is a parabola
as $\Delta \neq 0$ and $h^2 = ab$]

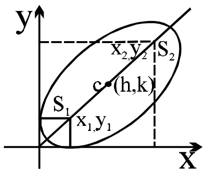
(C) $$h = \frac{a\cos\frac{\alpha + \beta}{2}}{\cos\frac{\alpha - \beta}{2}} \;;\; k = \frac{b\sin\frac{\alpha + \beta}{2}}{\cos\frac{\alpha - \beta}{2}}$$

given $\frac{\alpha + \beta}{2} = \text{constant} = C$

$$\therefore \cos\frac{\alpha - \beta}{2} = \frac{a\cos C}{h} = \frac{b\sin C}{k} \quad \Rightarrow y = \left(\frac{b}{a}\tan C\right)x$$

Locus of (h, k) is a straight line

(D) $y_1y_2 = x_1x_2 = b^2 \quad ....(1)$
and $\quad (x_2 - x_1)^2 + (y_2 - y_1)^2 = 4(a^2 - b^2) \quad ....(2)$
Also $\quad 2h = x_1 + x_2$
$2k = y_1 + y_2$
from (2) $(x_1 + x_2)^2 + (y_1 + y_2)^2 - 4(x_1x_2 + y_1y_2) = 4(a^2 - b^2)$
$\cancel{4}(h^2 + k^2) - \cancel{4}(2b^2) = \cancel{4}(a^2 - b^2)$
$\therefore x^2 + y^2 = a^2 + b^2 \Rightarrow$ Circle



Alternative: Equation of director circle with centre (h, k)

$(x-h)^2 + (y-k)^2 = a^2 + b^2$

(0, 0) lies on it $\Rightarrow \quad h^2 + k^2 = a^2 + b^2 \quad \Rightarrow \quad$ locus is $x^2 + y^2 = a^2 + b^2$]

Q.29$_{106}$ **Column-I** **Column-II**

(A) For an ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ with vertices A and A', tangent drawn at the point P in the first quadrant meets the y-axis in Q and the chord A'P meets the y-axis in M. If 'O' is the origin then $OQ^2 - MQ^2$ equals to (P) 2

(B) If the product of the perpendicular distances from any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ of eccentricity $e = \sqrt{3}$ from its asymptotes is equal to 6, then the length of the transverse axis of the hyperbola is (Q) 3

(C) The locus of the point of intersection of the lines $\sqrt{3}x - y - 4\sqrt{3}\,t = 0$ and $\sqrt{3}tx + ty - 4\sqrt{3} = 0$ (where t is a parameter) is a hyperbola whose eccentricity is (R) 4

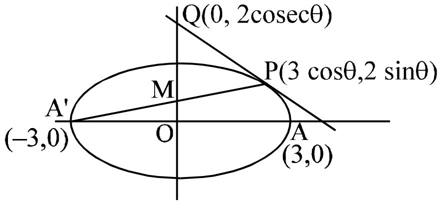
(D) If F_1 & F_2 are the feet of the perpendiculars from the foci S_1 & S_2 of an ellipse $\frac{x^2}{5} + \frac{y^2}{3} = 1$ on the tangent at any point P on the ellipse, then $(S_1F_1).(S_2F_2)$ is equal to (S) 6

[Ans. (A) R; (B) S; (C) P; (D) Q]

[Sol.**(A)** a = 3 ; b = 2

$$T : \frac{x\cos\theta}{3} + \frac{y\sin\theta}{2} = 1$$

x = 0 ; y = 2 cosecθ



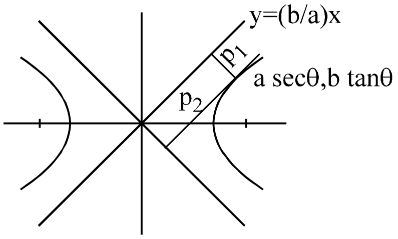
chord A'P, $\quad y = \frac{2\sin\theta}{3(\cos\theta + 1)}(x+3)$

put x = 0 $\; y = \frac{2\sin\theta}{1+\cos\theta} = OM$

Now $OQ^2 - MQ^2 = OQ^2 - (OQ - OM)^2 = 2(OQ)(OM) - OM^2 = OM\{\, 2(OQ) - (OM) \,\}$

$$= \frac{2\sin\theta}{1+\cos\theta}\left[\frac{4}{\sin\theta} - \frac{2\sin\theta}{1+\cos\theta}\right] = \frac{4\sin\theta}{1+\cos\theta}\left[\frac{2(1+\cos\theta) - (1-\cos^2\theta)}{\sin\theta(1+\cos\theta)}\right] = \frac{4(1+\cos\theta)(2-1+\cos\theta)}{(1+\cos\theta)(1+\cos\theta)} = 4$$

(B) $\quad p_1p_2 = \frac{a^2b^2}{a^2+b^2} = \frac{a^2.a^2(e^2-1)}{a^2e^2} = 6;$



$\frac{2a^2}{3} = 6 \Rightarrow a^2 = 9 \Rightarrow a = 3$

hence 2a = 6

(C) $\quad$ hyperbola $\frac{x^2}{16} - \frac{y^2}{48} = 1$

(D) $\quad$ Product of the feet of the perpendiculars is equal to the square of its semi minor axes.]