Continuity

General Introduction :

A function is said to be continuous at $x = a$ if while travelling along the graph of the function and in crossing over the point at $x = a$ either from L to R or from R to L one does not have to lift his pen.

Different type of situations which may come up at $x=a$ along the graph can be :

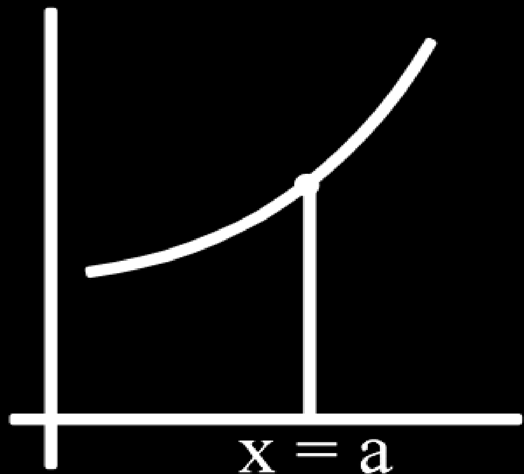
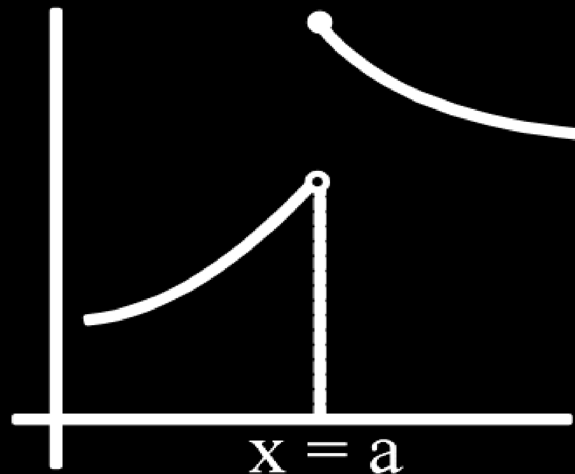


figure (1)



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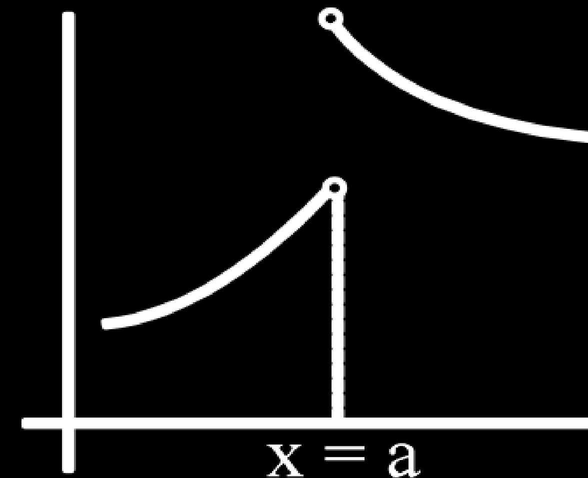

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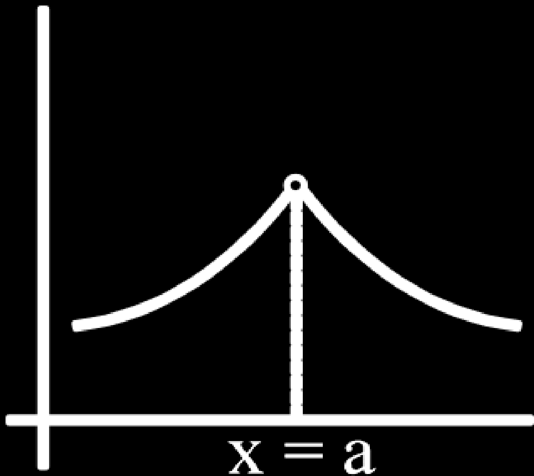

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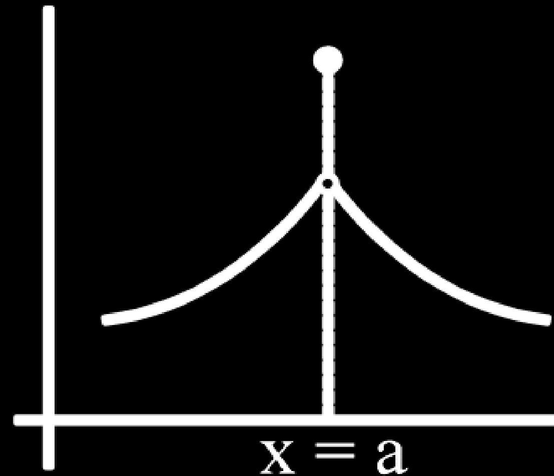

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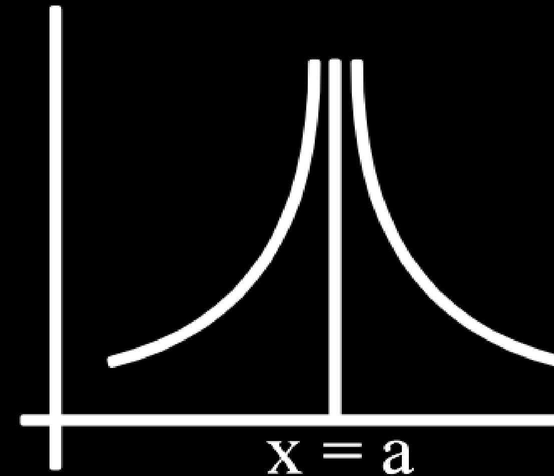


figure (6)

Formulative Definition of Continuity

A function f(x) is said to be continuous at x = a,

$\lim\limits_{x \to a} f(x)$ exists and = f (a). Symbolically f is continuous at x=a if $\lim\limits_{h \to 0} f(a-h) = \lim\limits_{h \to 0} f(a+h) = f(a) = a$ finite quantity.

Note

(1) Continuity at $x = a \Rightarrow$ existence of limit at $x=a$, but not the converse

(2) Continuity at $x = a \Rightarrow f$ is well defined at $x=a$, but not the converse

(3) Continuity is always talk in the domain of function and hence if you want to talk of discontinuity then we can say $\frac{1}{x-1}$ is discontinuous at

x = 1, $\frac{1}{x}$ is discontinuous at x = 0. All rational functions are continuous.

Point Function are continuous

Continuity In An Interval

(a) A function f is said to be continuous in (a, b) if f is continuous at each & every point $\in$(a, b).

(b) A function f is said to be continuous in a closed interval [a, b] if :

(i) f is continuous in the open interval (a, b) &

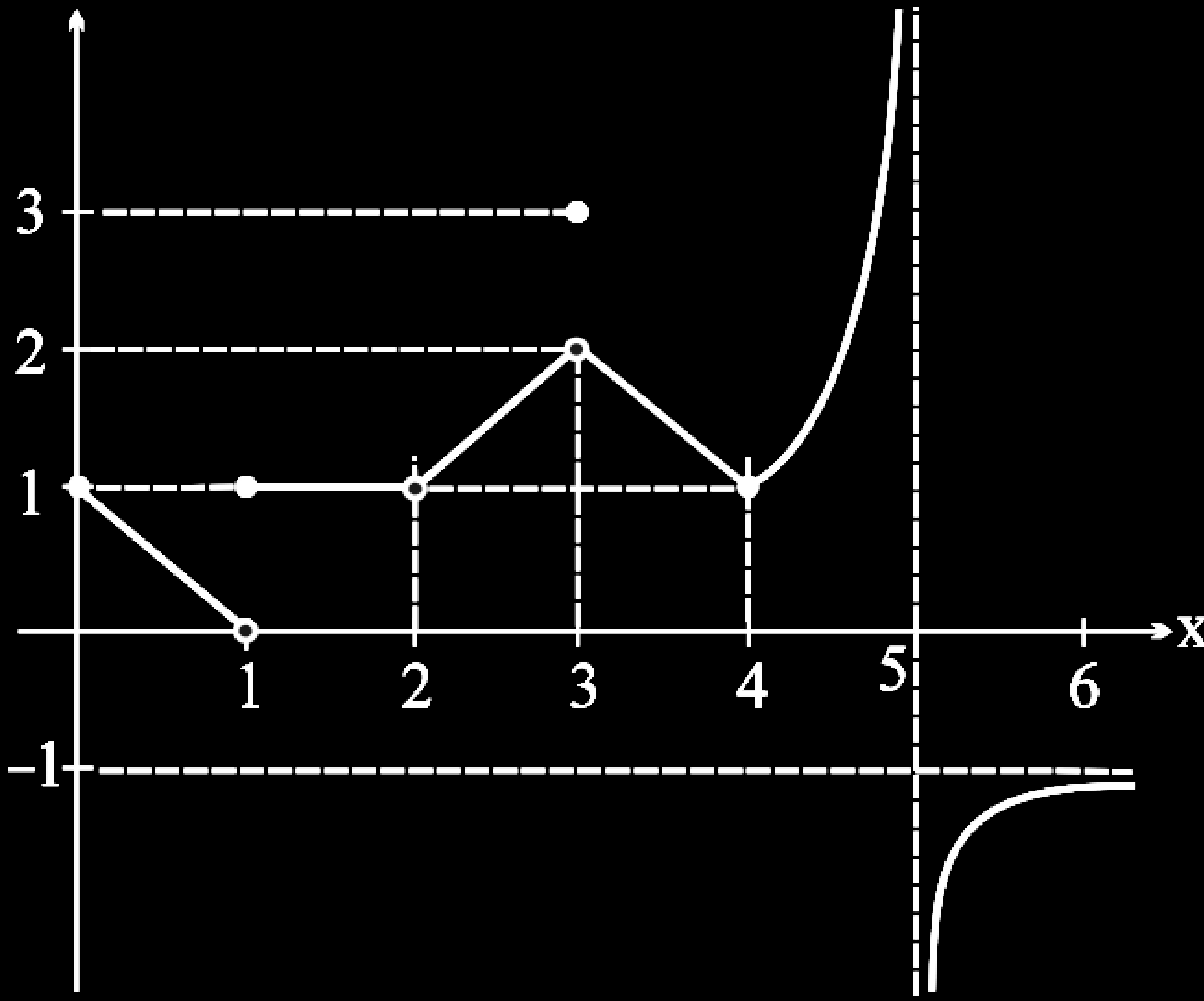
(ii) f is right continuous at '**a**'

i.e. $\lim\limits_{x \to a^+} f(x) = f(a) =$ a finite quantity.

(iii) f is left continuous at '**b**'

i.e. $\lim\limits_{x \to b^-} f(x) = f(b) =$ a finite quantity.

Consider the following graph of a function.



True/False

1. f is continuous at $x = 0$ and $x = 4$.
2. f is discontinuous at $x = 1$.
3. f is discontinuous at $x = 2$.
4. f is discontinuous at $x = 3$.
5. f is discontinuous at $x = 5$.

Note

It should be remembered that all polynomial functions, trigonometric function, exponential and logarithmic functions are continuous in their domain.

Example

Q. $y = \begin{cases} x^2 + 2x + 3 & x < 0 \\ 2x + 2 & x \geq 0 \end{cases}$

Q. $y = \begin{cases} \sin x & x \le \pi/4 \\ \cos x & x > \pi/4 \end{cases}$

Q. $f(x) = \begin{cases} (\cos x)^{\cot^2 x} & x \neq 0 \\ e^{-1/2} \text{ if} & x = 0 \end{cases}$

find whether the $f(x)$ is continuous at $x = 0$ or not.

Q. If $f(x) = \begin{cases} \dfrac{(e^x - 1)^3 \operatorname{cosec}(ax)}{\ln(1+x^2)} & x \neq 0 \\ b & x = 0 \end{cases}$

is continuous at $x=0$, find relation between a & b.

Q. Find the values of ‘a’ and ‘b’ so that the function

$$f(x) = \begin{cases} x + a\sqrt{2}\sin x & 0 \le x < \frac{\pi}{4} \\ 2x\cot x + b & \frac{\pi}{4} \le x \le \frac{\pi}{2} \\ a\cos 2x - b\sin x & \frac{\pi}{2} < x \le \pi \end{cases}$$

is continuous in $[0, \pi]$

Q. $$f(x)=\begin{cases} \dfrac{1-\cos 4x}{x^2} & \text{if } x<0 \\ a & \text{if } x=0 \\ \dfrac{\sqrt{x}}{\sqrt{16+\sqrt{x}}-4} & \text{if } x>0 \end{cases}$$

Determine ‘a’ if possible so that the function is continuous at x = 0

Q. Let $f(x) = \begin{cases} \left(1+|\sin x|\right)^{\frac{a}{|\sin x|}} & \text{for } -\frac{\pi}{6} < x < 0 \\ b & \text{for } x = 0 \\ e^{\frac{\tan 2x}{\tan 3x}} & \text{for } 0 < x < \frac{\pi}{6} \end{cases}$

Find 'a' and 'b' if f is continuous at x = 0

Q. $f(x) = \begin{cases} 3 - \left[\cot^{-1} \dfrac{2x^3 - 3}{x^2}\right] & \text{if } x > 0 \\ \{x^2\} \cos e^{\frac{1}{x}} & \text{if } x < 0 \end{cases}$

where [x] and {x} denotes greatest integer & fractional part. Can f (x) be made continuous.

Q. If $f(x) = \cos\left(x \cos \frac{1}{x}\right)$ and $g(x) = \frac{\ln(\sec^2 x)}{x \sin x}$

are both continuous at $x = 0$ then

(A) $f(0) = g(0)$ (B) $g(0) = 2f(0)$

(C) $f(0) = 2g(0)$ (D) $f(0) + g(0) = 1$

Q. Let $f(x) = \begin{cases} \dfrac{(e^{2x}+1)-(x+1)(e^{x}+e^{-x})}{x(e^{x}-1)} & \text{if } x \neq 0 \\ k & \text{if } x = 0 \end{cases}$

if f(x) is continuous at x = 0 then k is equal to

(A) ½ (B) 1 (C) 3/2 (D) 2

Q. If $H(x) = \begin{cases} (\cos x)^{(\cot x)} + (\sec x)^{\operatorname{cosec} x} & \text{if } x > 0 \\ p & \text{if } x = 0 \\ \dfrac{e^{x} + e^{-x} - 2\cos x}{x \sin x} & \text{if } x < 0 \end{cases}$

Find the value of p, if possible to make the function H (x) continuous at x = 0

Q. Discuss the continuity of $f(x) = \text{sgn}(\sin x + 2)$

Q. Discuss the continuity of $f(x) = \text{sgn}(\sin x - 1)$

Q. If $f(x) = \text{sgn}(\sin x + a)$ is continuous $\forall\ x \in R$ then find range of a.

Q. $$f(x)=\begin{cases}\dfrac{a+3\cos x}{x^2} & x<0\\ b\tan\left(\dfrac{\pi}{[x+3]}\right) & x\geq 0\end{cases}$$

where [ ] denotes greatest integer function. Find a and b for which f(x) is continuous at x = 0

Q. Find the number of points of discontinuity of

(i) $f(x) = [5x],\ x \in [0,1]$

where [ ] denotes greatest integer function

Q. Find the number of points of discontinuity of

(ii) $f(x) = [5\sin x], x \in [0,\pi]$

where [ ] denotes greatest integer function

Types of Discontinuities

Type – 1
(Removable discontinuity)

Here $\lim\limits_{x \to a} f(x)$ necessarily exists, but it is either not equal to f(a) or f(a) is not defined.

In this case, therefore it is possible to redefine the function in such a manner that $\lim\limits_{x \to a} f(x) = f(a)$

Types of Removable discontinuity

(A) Missing Point Discontinuity :

In this case, Function is not defined at $x = a$

Examples

Q. $\mathbf{f(x) = \dfrac{(x-1)(9-x^2)}{x-1}}$ at x = 1

Q. $f(x) = \dfrac{x^2 - 4}{x - 2}$ at $x = 2$

(B) Isolated Point Discontinuity :

In this case, Function is defined at $x = a$

but $\lim\limits_{x \to a} f(x) \neq f(a)$

Examples

Q. $f(x) = [x] + [-x]$

Q. $f(x) = \operatorname{sgn}(\sin x + 1)$

Type – 2
(Non removable discontinuity)

Here $\lim_{x \to a}$ f(x) does not exists and therefore it is not possible to redefined the function in any manner to make it continuous.

Types of non removable discontinuity

(A) Finite type

(Both limits finite and unequal)

(B) Infinite type

(at least one of two limit are infinity)

(C) Oscillatory

(limits oscillate between two finite quantities)

Examples of Finite Type

Q. $\displaystyle \lim_{x \to 0} \tan^{-1}\left(\frac{1}{x}\right)$

Q. $\lim\limits_{x \to 0} \dfrac{|\sin x|}{x}$

Q. $\lim\limits_{x \to 2} \frac{[x]}{x}$

Note :

In this case non negative difference between the two limits is called the Jump of discontinuity.

Examples of Infinite Type

Q. $f(x) = \dfrac{x}{1-x}$ at $x = 1$

Q. $f(x) = 2^{\tan x}$ at $x = \dfrac{\pi}{2}$

Q. $f(x) = \frac{1}{x^2}$ at $x = 0$

Examples of Oscillatory

Q. $f(x) = \sin\frac{1}{x}$ at $x = 0$

Q. $f(x) = \cos\frac{1}{x}$ at $x = 0$

Continuity of Functions Defined by Some Functional Rule

Example

Q. If $f(x + y) = f(x).\ f(y)$ for all x & y & $f(x) = 1 + g(x).\ G(x)$ where $\lim_{x\to 0} g(x) = 0$ & $\lim_{x\to 0} G(x)$ exist. Prove that f(x) is continuous for all x.

Theorems on Continuity

T–1 :

Sum, difference, product and quotient of two continuous functions is always a continuous function.

However $h(x) = \frac{f(x)}{g(x)}$ is continuous at $x = a$ only if $g(a) \neq 0$

Important Notes

(A) If $f(x)$ is continuous and $g(x)$ is discontinuous then $f(x) \pm g(x)$ is a discontinuous function.

(B) If $f(x)$ is continuous & $g(x)$ is discontinuous at $x = a$ then the product function $\phi(x)=f(x).\,g(x)$ is not necessarily be discontinuous at $x = a$.

Examples

Q. $$f(x) = \cos\left(\frac{2x-1}{2}\right)\pi$$

is continuous at x = 1 and g(x) = [x] is discontinuous at x = 1 but f (x). g (x) is continuous at x = 1.

(C) If $f(x)$ and $g(x)$ both are discontinuous at $x = a$ then the product function $\phi(x) = f(x).\ g(x)$ is not necessarily be discontinuous at $x = a$.

Intermediate Value Theorem :

If f is continuous on $[a, b]$ and $f(a) \neq f(b)$ then for some value $c \in (f(a), f(b))$, there is at least one number x_0 in (a, b) for which $f(x_0) = c$.

Examples

Q. Prove that function

$$f(x) = a\sqrt{x-1} + b\sqrt{2x-1} - \sqrt{2x^2-3x+1}$$

where $a + 2b = 3$, a & b are real number, $b \neq 0$ always has a root in $(1,5)$ $\forall\, b \in R$.

Note

A polynomial of degree odd has atleast one real root.

Q. Let f be a continuous function defined onto on [0,1] with range [0,1], show that there is some $c \in [0,1]$ such that $f(c) = 1 - c$

Functions continuous only at one point and defined everywhere (Single point continuity)

Examples

Q. $$f(x) = \begin{cases} x \text{ if } x \in Q \\ 0 \text{ if } x \notin Q \end{cases}$$

Q. $$f(x) = \begin{cases} x & \text{if } x \in Q \\ -x & \text{if } x \notin Q \end{cases}$$

Q. $$f(x) = \begin{cases} x & \text{if } x \in Q \\ 1 - x & \text{if } x \notin Q \end{cases}$$

Q. $f(x) = \begin{cases} x^2 & \text{if } x \in Q \\ 1 & \text{if } x \notin Q \end{cases}$

Some Problems on Continuity

Q. $$f(x)=\begin{cases}\dfrac{\left(x^2+3x-1\right)\tan x}{x^2+2x} & \text{if } x\neq 0\\ k & \text{if } x=0\end{cases}$$

Q. $$f(x) = \begin{cases} \dfrac{\sin x + \sin 5x}{\cos x + \cos 5x} & \text{if } x \neq -\dfrac{\pi}{4} \\ k & \text{if } x = -\dfrac{\pi}{4} \end{cases}$$

Find k if f is continuous at $x = -\dfrac{\pi}{4}$

(A) 1 (B) –1 (C) 0 (D) $-\frac{1}{\sqrt{2}}$

Q. $f(x) = \begin{cases} x+1 & x \le 1 \\ 3-ax^2 & x > 1 \end{cases}$ at x = 1

Q. What kind of discontinuity function $\frac{\cos x}{x}$ has at $x = 0$

Q. $$f(x) = \begin{cases} \dfrac{8^x - 4^x - 2^x + 1^2}{x^2} & \text{if } x > 0 \\ e^x \sin x + 4x + k \ln 4, & x \le 0 \end{cases}$$

is continuous at $x = 0$ then find k