DIFFERENTIABILITY OF FUNCTION & METHOD OF DIFFERENTIATION

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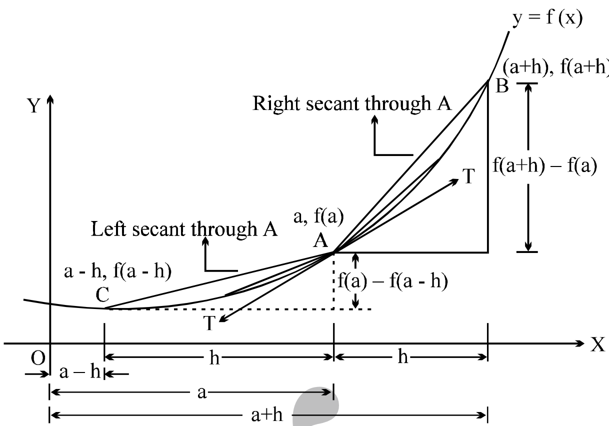
KEY CONCEPTS (DIFFERENTIABILITY)

THINGS TO REMEMBER :

1. Right hand & Left hand Derivatives ;

By definition : $f'(a) = \underset{h\to 0}{\text{Limit}} \dfrac{f(a+h)-f(a)}{h}$ if it exist



(i) The right hand derivative of f' at $x = a$ denoted by $f'(a^+)$ is defined by :

$$f'(a^+) = \underset{h\to 0^+}{\text{Limit}} \frac{f(a+h)-f(a)}{h},$$

provided the limit exists & is finite.

(ii) The left hand derivative : of f at $x = a$ denoted by $f'(a^+)$ is defined by :

$$f'(a^-) = \underset{h\to 0^+}{\text{Limit}} \frac{f(a-h)-f(a)}{-h},$$

Provided the limit exists & is finite.

We also write $f'(a^+) = f'_+(a)$ & $f'(a^-) = f'_-(a)$.

* This geomtrically means that a unique tangent with finite slope can be drawn at $x = a$ as shown in the figure.

(iii) Derivability & Continuity :

(a) If $f'(a)$ exists then $f(x)$ is derivable at $x = a \Rightarrow f(x)$ is continuous at $x = a$.

(b) If a function f is derivable at x then f is continuous at x.

For : $f'(x) = \underset{h\to 0}{\text{Limit}} \dfrac{f(x+h)-f(x)}{h}$ exists.

Also $f(x+h)-f(x) = \dfrac{f(x+h)-f(x)}{h}.h \; [h \neq 0]$

Therefore :

$$\underset{h\to 0}{\text{Limit}} [f(x+h)-f(x)] = \underset{h\to 0}{\text{Limit}} \frac{f(x+h)-f(x)}{h}.h = f'(x).0 = 0$$

Therefore $\underset{h\to 0}{\text{Limit}} [f(x+h)-f(x)] = 0 \Rightarrow \underset{h\to 0}{\text{Limit}} f(x+h) = f(x) \Rightarrow f$ is continuous at x.

Note : If $f(x)$ is derivable for every point of its domain of definition, then it is continuous in that domain.

The Converse of the above result is not true :

" IF f IS CONTINUOUS AT x , THEN f IS DERIVABLE AT x " IS NOT TRUE.

e.g. the functions $f(x) = |x|$ & $g(x) = x \sin\dfrac{1}{x}$; $x \neq 0$ & $g(0) = 0$ are continuous at $x = 0$ but not derivable at $x = 0$.

NOTE CAREFULLY :

(a) Let $f'_+(a) = p$ & $f'_-(a) = q$ where p & q are finite then :

(i) $p = q \Rightarrow f$ is derivable at $x = a \Rightarrow f$ is continuous at $x = a$.

(ii) $p \neq q \Rightarrow f$ is not derivable at $x = a$.

It is very important to note that f may be still continuous at $x = a$.

In short, for a function f :

Differentiability $\Rightarrow$ Continuity ; Continuity $\not\Rightarrow$ derivability ;

Non derivibality $\not\Rightarrow$ discontinuous ; But discontinuity $\Rightarrow$ Non derivability

(b) If a function f is not differentiable but is continuous at $x = a$ it geometrically implies a sharp corner at $x = a$.

3. DERIVABILITY OVER AN INTERVAL :

f (x) is said to be derivable over an interval if it is derivable at each & every point of the interval f(x) is said to be derivable over the closed interval [a, b] if :

(i) for the points a and b, $f'(a+)$ & $f'(b-)$ exist &

(ii) for any point c such that $a < c < b$, $f'(c+)$ & $f'(c-)$ exist & are equal.

NOTE :

1. If f(x) & g(x) are derivable at x = a then the functions f(x) + g(x), f(x) − g(x) , f(x).g(x) will also be derivable at x = a & if $g(a) \neq 0$ then the function f(x)/g(x) will also be derivable at x = a.

2. If f(x) is differentiable at x = a & g(x) is not differentiable at x = a , then the product function F(x) = f(x)· g(x) can still be differentiable at x = a e.g. f(x) = x & g(x) = $|x|$.

3. If f(x) & g(x) both are not differentiable at x = a then the product function ;
F(x) = f(x)· g(x) can still be differentiable at x = a e.g. f(x) = $|x|$ & g(x) = $|x|$.

4. If f(x) & g(x) both are non-deri. at x = a then the sum function F(x) = f(x) + g(x) may be a differentiable function. e.g. f(x) = $|x|$ & g(x) = $-|x|$.

5. If f(x) is derivable at x = a $\not\Rightarrow$ $f'(x)$ is continuous at x = a.

$$\text{e.g. } f(x) = \begin{cases} x^2 \sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

6. **A surprising result :** Suppose that the function f (x) and g (x) defined in the interval (x_1, x_2) containing the point x_0, and if f is differentiable at $x = x_0$ with $f(x_0) = 0$ together with g is continuous as $x = x_0$ then the function F (x) = f (x) · g (x) is differentiable at $x = x_0$
e.g. F (x) = sinx · $x^{2/3}$ is differentiable at x = 0.

EXERCISE–I

Q.1 Discuss the continuity & differentiability of the function f(x) = sinx + sin $|x|$, $x \in R$. Draw a rough sketch of the graph of f(x).

Q.2 Examine the continuity and differentiability of f(x) = $|x| + |x-1| + |x-2|$ $x \in R$.
Also draw the graph of f(x).

Q.3 Given a differentiable function f(x) defined for all real x, and is such that
$f(x+h) - f(x) \leq 6h^2$ for all real h and x. Show that f(x) is constant.

Q.4 A function f is defined as follows : $f(x) = \begin{cases} 1 & \text{for } -\infty < x < 0 \\ 1 + |\sin x| & \text{for } 0 \leq x < \frac{\pi}{2} \\ 2 + \left(x - \frac{\pi}{2}\right)^2 & \text{for } \frac{\pi}{2} \leq x < +\infty \end{cases}$

Discuss the continuity & differentiability at x = 0 & x = π/2.

Q.5 Examine the origin for continuity & derrivability in the case of the function f defined by
$f(x) = x \tan^{-1}(1/x)$, $x \neq 0$ and f(0) = 0.

Q.6 Let f(0) = 0 and f'(0) = 1. For a positive integer k, show that

$$\lim_{x \to 0} \frac{1}{x}\left(f(x) + f\left(\frac{x}{2}\right) + \ldots\ldots f\left(\frac{x}{k}\right) \right) = 1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots + \frac{1}{k}$$

Q.7 Let $f(x) = xe^{-\left(\frac{1}{|x|} + \frac{1}{x}\right)}$; $x \neq 0$, f(0) = 0, test the continuity & differentiability at x = 0

Q.8 If f(x)= $|x-1|$. ([x] − [−x]) , then find $f'(1^+)$ & $f'(1^-)$ where [x] denotes greatest integer function.

Q.9 If $f(x) = \begin{cases} ax^2 - b & \text{if } |x| < 1 \\ -\frac{1}{|x|} & \text{if } |x| \geq 1 \end{cases}$ is derivable at x = 1. Find the values of a & b.

Q.10 Let f(x) be defined in the interval [-2, 2] such that $f(x)=\begin{cases}-1 & , -2\le x\le 0\\ x-1 & , 0<x\le 2\end{cases}$ &

$g(x) = f(|x|) + |f(x)|$. Test the differentiability of g(x) in (−2, 2).

Q.11 Given $f(x)=\cos^{-1}\left(\text{sgn}\left(\frac{2[x]}{3x-[x]}\right)\right)$ where sgn (.) denotes the signum function & [.] denotes the greatest integer function. Discuss the continuity & differentiability of f (x) at $x = \pm 1$.

Q.12 Examine for continuity & differentiability the points x = 1 & x = 2, the function f defined by

$f(x)=\begin{cases}x[x] & , 0\le x<2\\ (x-1)[x] & , 2\le x\le 3\end{cases}$ where [x] = greatest integer less than or equal to x.

Q.13 $f(x)=x.\left(\frac{e^{[x]+|x|}-2}{[x]+|x|}\right)$, $x\ne 0$ & $f(0)=-1$ where [x] denotes greatest integer less than or equal to x. Test the differentiability of f(x) at x = 0.

Q.14 Discuss the continuity & the derivability in [0 , 2] of $f(x)=\begin{cases}|2x-3|\,[x] & \text{for } x\ge 1\\ \sin\frac{\pi x}{2} & \text{for } x<1\end{cases}$

where [ ] denote greatest integer function .

Q.15 If $f(x) = -1 + |x-1|$, $-1 \le x \le 3$; $g(x) = 2 - |x+1|$, $-2 \le x \le 2$, then calculate (fog) (x) & (gof) (x). Draw their graph. Discuss the continuity of (fog) (x) at x = −1 & the differentiability of (gof) (x) at x = 1.

Q.16 The function $f(x)=\begin{cases}ax(x-1)+b & \text{when } x<1\\ x-1 & \text{when } 1\le x\le 3\\ px^2+qx+2 & \text{when } x>3\end{cases}$

Find the values of the constants a, b, p, q so that

(i) f(x) is continuous for all x (ii) f ' (1) does not exist (iii) f'(x) is continuous at x = 3

Q.17 Examine the function, $f(x)=x.\frac{a^{1/x}-a^{-1/x}}{a^{1/x}+a^{-1/x}}$, $x\ne 0$ (a > 0) and f(0) = 0 for continuity and existence of the derivative at the origin.

Q.18 Discuss the continuity on $0 \le x \le 1$ & differentiability at x = 0 for the function.

$f(x) = x.\sin\frac{1}{x}.\sin\frac{1}{x.\sin\frac{1}{x}}$ where $x\ne 0$, $x\ne 1/r\pi$ & $f(0)=f(1/r\pi)=0$,

r = 1, 2, 3,.........

Q.19 $f(x)=\begin{cases}1-x & , (0\le x\le 1)\\ x+2 & , (1<x<2)\\ 4-x & , (2\le x\le 4)\end{cases}$ Discuss the continuity & differentiability of $y = f[f(x)]$ for $0 \le x \le 4$.

Q.20 Let *f* be a function that is differentiable every where and that has the following properties:

(i) $f(x+h)=\frac{f(x)+f(h)}{f(-x)+f(-h)}$ for all real x and h. (ii) f (x) > 0 for all real x.

(iii) $f'(0) = -1$ (iv) $f(-x)=\frac{1}{f(x)}$ and $f(x+h)=f(x)\cdot f(h)$

Use the definition of derivative to find *f*' (x) in terms of *f* (x).

Q.21 Discuss the continuity & the derivability of 'f' where f (x) = degree of $(u^{x^2} + u^2 + 2u - 3)$ at $x = \sqrt{2}$.

Q.22 Let $f(x)$ be a function defined on $(-a, a)$ with $a > 0$. Assume that $f(x)$ is continuous at $x = 0$ and $\lim\limits_{x\to 0} \dfrac{f(x) - f(kx)}{x} = \alpha$, where $k \in (0, 1)$ then compute $f'(0^+)$ and $f'(0^-)$, and comment upon the differentiability of f at $x = 0$.

Q.23 Consider the function, $f(x) = \begin{cases} x^2\left|\cos\dfrac{\pi}{2x}\right| & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

(a) Show that $f'(0)$ exists and find its value (b) Show that $f'(1/3)$ does not exist.

(c) For what values of x, $f'(x)$ fails to exist.

Q.24 Let f(x) be a real valued function not identically zero satisfies the equation, $f(x + y^n) = f(x) + (f(y))^n$ for all real x & y and $f'(0) \geq 0$ where n (> 1) is an odd natural number. Find f(10).

Q.25 A derivable function $f : R^+ \to R$ satisfies the condition $f(x) - f(y) \geq \ln\dfrac{x}{y} + x - y$ for every $x, y \in R^+$.

If g denotes the derivative of f then compute the value of the sum $\sum\limits_{n=1}^{100} g\left(\dfrac{1}{n}\right)$.

EXERCISE–II

<u>Fill in the blanks</u> :

Q.1 If f(x) is derivable at x = 3 & $f'(3) = 2$, then $\underset{h\to 0}{\text{Limit}} \dfrac{f(3+h^2) - f((3-h^2)}{2h^2}$ = ________.

Q.2 If $f(x) = |\sin x|$ & $g(x) = x^3$ then f[g(x)] is ______ & ______ at x = 0. (State continuity and derivability)

Q.3 Let f(x) be a function satisfying the condition $f(-x) = f(x)$ for all real x. If $f'(0)$ exists, then its value is _____.

Q.4 For the function $f(x) = \begin{cases} \dfrac{x}{1+e^{1/x}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$, the derivative from the right, $f'(0^+)$ = _____ & the derivative from the left, $f'(0^-)$ = _______.

Q.5 The number of points at which the function $f(x) = \max.\{a - x, a + x, b\}, -\infty < x < \infty, 0 < a < b$ cannot be differentiable is ______.

<u>Select the correct alternative</u> : (only one is correct)

Q.6 The function f(x) is defined as follows $f(x) = \begin{cases} -x & \text{if } x < 0 \\ x^2 & \text{if } 0 \leq x \leq 1 \\ x^3 - x + 1 & \text{if } x > 1 \end{cases}$ then f(x) is :

(A) derivable and continuous at x = 0
(B) derivable at x = 1 but not continuous at x = 1
(C) neither derivable nor continuous at x = 1
(D) not derivable at x = 0 but continuous at x = 1

Q.7 For what triplets of real numbers (a, b, c) with $a \neq 0$ the function

$f(x) = \begin{cases} x & x \leq 1 \\ ax^2 + bx + c & \text{otherwise} \end{cases}$ is differentiable for all real x ?

(A) $\{(a, 1-2a, a) \mid a \in R, a \neq 0\}$
(B) $\{(a, 1-2a, c) \mid a, c \in R, a \neq 0\}$
(C) $\{(a, b, c) \mid a, b, c \in R, a + b + c = 1\}$
(D) $\{(a, 1-2a, 0) \mid a \in R, a \neq 0\}$

Q.8 A function f defined as $f(x) = x[x]$ for $-1 \leq x \leq 3$ where [x] defines the greatest integer $\leq x$ is :
(A) continuous at all points in the domain of f but non-derivable at a finite number of points
(B) discontinuous at all points & hence non-derivable at all points in the domain of f(x)
(C) discontinuous at a finite number of points but not derivable at all points in the domain of f(x)
(D) discontinuous & also non-derivable at a finite number of points of f(x).

Q.9 [x] denotes the greatest integer less than or equal to x. If $f(x) = [x]\,[\sin \pi x]$ in $(-1,1)$ then f(x) is :
(A) continuous at $x = 0$ (B) continuous in $(-1, 0)$
(C) differentiable in $(-1,1)$ (D) none

Q.10 Given $f(x) = \begin{cases} \log_a \left(a\left|[x] + [-x]\right|\right)^x \left(\dfrac{a^{\left(\frac{2}{\left(\frac{[x]+[-x]}{|x|}\right)}\right)} - 5}{3 + a^{\frac{1}{|x|}}}\right) & \text{for } |x| \neq 0 \;;\; a > 1 \\ 0 & \text{for } x = 0 \end{cases}$ where [ ] represents the integral part function, then :
(A) f is continuous but not differentiable at $x = 0$
(B) f is cont. & diff. at $x = 0$
(C) the differentiability of 'f' at $x = 0$ depends on the value of a
(D) f is cont. & diff. at $x = 0$ and for $a = e$ only.

Q.11 If $f(x) = \begin{cases} x + \{x\} + x\sin\{x\} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$ where $\{x\}$ denotes the fractional part function, then :
(A) 'f' is continuous & diff. at $x = 0$ (B) 'f' is continuous but not diff. at $x = 0$
(C) 'f' is continuous & diff. at $x = 2$ (D) none of these

Q.12 The set of all points where the function $f(x) = \dfrac{x}{1+|x|}$ is differentiable is :
(A) $(-\infty, \infty)$ (B) $[0, \infty)$ (C) $(-\infty, 0) \cup (0, \infty)$ (D) $(0, \infty)$ (E) none

Q.13 Let f be an injective and differentiable function such that $f(x) \cdot f(y) + 2 = f(x) + f(y) + f(xy)$ for all non negative real x and y with $f'(0) = 0, f'(1) = 2 \neq f(0)$, then
(A) $x f'(x) - 2f(x) + 2 = 0$ (B) $x f'(x) + 2f(x) - 2 = 0$
(C) $x f'(x) - f(x) + 1 = 0$ (D) $2f(x) = f'(x) + 2$

Q.14 Let $f(x) = [n + p\sin x]$, $x \in (0, \pi)$, $n \in I$ and p is a prime number. The number of points where $f(x)$ is not differentiable is
(A) $p - 1$ (B) $p + 1$ (C) $2p + 1$ (D) $2p - 1$.
Here [x] denotes greatest integer function.

Q.15 Consider the functions $f(x) = x^2 - 2x$ and $g(x) = -|x|$

Statement-1: The composite function $F(x) = f(g(x))$ is not derivable at $x = 0$.

because

Statement-2: $F'(0^+) = 2$ and $F'(0^-) = -2$.
(A) Statement-1 is true, statement-2 is true and statement-2 is correct explanation for statement-1.
(B) Statement-1 is true, statement-2 is true and statement-2 is NOT the correct explanation for statement-1.
(C) Statement-1 is true, statement-2 is false. (D) Statement-1 is false, statement-2 is true.

Q.16 Consider the function $f(x) = x^2 - \left|x^2 - 1\right| + 2\left||x| - 1\right| + 2|x| - 7$.

Statement-1: f is not differentiable at $x = 1, -1$ and 0.

because

Statement-2: $|x|$ not differentiable at $x = 0$ and $|x^2 - 1|$ is not differentiable at $x = 1$ and -1.
(A) Statement-1 is true, statement-2 is true and statement-2 is correct explanation for statement-1.
(B) Statement-1 is true, statement-2 is true and statement-2 is NOT the correct explanation for statement-1.
(C) Statement-1 is true, statement-2 is false. (D) Statement-1 is false, statement-2 is true.

Select the correct alternative : (More than one are correct)

Q.17 $f(x) = |x[x]|$ in $-1 \le x \le 2$, where [x] is greatest integer $\le x$ then f(x) is :
(A) continuous at x = 0
(B) discontinuous x = 0
(C) not differentiable at x = 2
(D) differentiable at x = 2

Q.18 $f(x) = 1 + x.[\cos x]$ in $0 < x \le \pi/2$, where [ ] denotes greatest integer function then ,
(A) It is continuous in $0 < x < \pi/2$
(B) It is differentiable in $0 < x < \pi/2$
(C) Its maximum value is 2
(D) It is not differentiable in $0 < x < \pi/2$

Q.19 $f(x) = (\sin^{-1}x)^2 . \cos(1/x)$ if $x \ne 0$; f(0) = 0 , f(x) is :
(A) continuous no where in $-1 \le x \le 1$
(B) continuous every where in $-1 \le x \le 1$
(C) differentiable no where in $-1 \le x \le 1$
(D) differentiable everywhere in $-1 < x < 1$

Q.20 $f(x) = |x| + |\sin x|$ in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. It is :
(A) Continuous no where
(B) Continuous every where
(C) Differentiable no where
(D) Differentiable everywhere except at x = 0

Q.21 If $f(x) = 2 + |\sin^{-1} x|$, it is :
(A) continuous no where
(B) continuous everywhere in its domain
(C) differentiable no where in its domain
(D) Not differentiable at x = 0

Q.22 If $f(x) = x^2 . \sin(1/x)$, $x \ne 0$ and f(0) = 0 then ,
(A) f(x) is continuous at x = 0
(B) f(x) is derivable at x = 0
(C) $f'(x)$ is continuous at x = 0
(D) $f''(x)$ is not derivable at x = 0

Q.23 A function which is continuous & not differentiable at x = 0 is :
(A) $f(x) = x$ for $x < 0$ & $f(x) = x^2$ for $x \ge 0$
(B) $g(x) = x$ for $x < 0$ & $g(x) = 2x$ for $x \ge 0$
(C) $h(x) = x|x|$ $x \in R$
(D) $K(x) = 1 + |x|$, $x \in R$

Q.24 If $\sin^{-1}x + |y| = 2y$ then y as a function of x is :
(A) defined for $-1 \le x \le 1$
(B) continuous at x = 0
(C) differentiable for all x
(D) such that $\frac{dy}{dx} = \frac{1}{3\sqrt{1-x^2}}$ for $-1 < x < 0$

Q.25 Let f(x) = Cosx & $H(x) = \begin{cases} \text{Min}\left[f(t)/0 \le t \le x\right] & \text{for } 0 \le x \le \frac{\pi}{2} \\ \frac{\pi}{2} - x & \text{for } \frac{\pi}{2} < x \le 3 \end{cases}$, then
(A) H (x) is continuous & derivable in [0, 3]
(B) H(x) is continuous but not derivable at $x = \pi/2$
(C) H(x) is neither cont. nor deri. at $x = \pi/2$
(D) Maximum value of H(x) in [0,3] is 1

EXERCISE–III

Q.1 The function $f(x) = (x^2 - 1)|x^2 - 3x + 2| + \cos(|x|)$ is NOT differentiable at :
(A) – 1 (B) 0 (C) 1 (D) 2
[JEE'99, 2(out of 200)]

Q.2 Let $f : R \to R$ be any function. Define $g : R \to R$ by $g(x) = |f(x)|$ for all x. Then g is
(A) onto if f is onto
(B) one one if f is one one
(C) continuous if f is continuous
(D) differentiable if f is differentiable.
[JEE 2000, Screening, 1 out of 35]

Q.3 Discuss the continuity and differentiability of the function, $f(x) = \begin{cases} \frac{x}{1+|x|}, & |x| \ge 1 \\ \frac{x}{1-|x|}, & |x| < 1 \end{cases}$. [REE, 2000,3]

Q.4 [JEE 2001 (Screening)]

(a) Let f : R → R be a function defined by , f (x) = max [x , x^3]. The set of all points where f(x) is NOT differentiable is :

(A) {– 1 , 1} (B) {– 1 , 0} (C) {0 , 1} (D) {– 1 , 0 , 1}

(b) The left hand derivative of, f (x) = [x] sin (π x) at x = k , k an integer is :
where [ ] denotes the greatest function.

(A) $(-1)^k(k-1)\pi$ (B) $(-1)^{k-1}(k-1)\pi$ (C) $(-1)^k k\pi$ (D) $(-1)^{k-1} k\pi$

(c) Which of the following functions is differentiable at x = 0?

(A) $\cos(|x|) + |x|$ (B) $\cos(|x|) - |x|$ (C) $\sin(|x|) + |x|$ (D) $\sin(|x|) - |x|$

Q.5 Let α ∈ R. Prove that a function $f: R \to R$ is differentiable at α if and only if there is a function $g: R \to R$ which is continuous at α and satisfies $f(x) - f(\alpha) = g(x)(x - \alpha)$ for all x ∈ R.
[JEE 2001, (mains) 5 out of 100]

Q.6 The domain of the derivative of the function $f(x) = \begin{cases} \tan^{-1} x & \text{if } |x| \le 1 \\ \frac{1}{2}(|x|-1) & \text{if } |x| > 1 \end{cases}$ is

(A) R – {0} (B) R – {1} (C) R – {–1} (D) R – {–1, 1}
[JEE 2002 (Screening), 3]

Q.7 Let f: R → R be such that $f(1) = 3$ and $f'(1) = 6$. The $\lim_{x\to 0}\left(\frac{f(1+x)}{f(1)}\right)^{1/x}$ equals

(A) 1 (B) $e^{1/2}$ (C) e^2 (D) e^3
[JEE 2002 (Screening), 3]

Q.8 $f(x) = \begin{cases} x + a & \text{if } x < 0 \\ |x-1| & \text{if } x \ge 0 \end{cases}$ and $g(x) = \begin{cases} x + 1 & \text{if } x < 0 \\ (x-1)^2 + b & \text{if } x \ge 0 \end{cases}$

Where a and b are non negative real numbers. Determine the composite function gof. If (gof) (x) is continuous for all real x, determine the values of a and b. Further, for these values of a and b, is gof differentiable at x = 0? Justify your answer. [JEE 2002, 5 out of 60]

Q.9 If a function f : [–2a , 2a] → R is an odd function such that f (x) = f (2a – x) for x ∈ [a, 2a] and the left hand derivative at x = a is 0 then find the left hand derivative at x = – a. [JEE 2003(Mains) 2 out of 60]

Q.10**(a)** The function given by $y = \big||x| - 1\big|$ is differentiable for all real numbers except the points

(A) {0, 1, –1} (B) ± 1 (C) 1 (D) – 1
[JEE 2005 (Screening), 3]

(b) If $|f(x_1) - f(x_2)| \le (x_1 - x_2)^2$, for all $x_1, x_2 \in R$. Find the equation of tangent to the curve y = f (x) at the point (1, 2). [JEE 2005 (Mains), 2]

Q.11 If $f(x) = \min.(1, x^2, x^3)$, then

(A) f (x) is continuous ∀ x ∈ R (B) $f'(x) > 0$, ∀ x > 1

(C) f(x) is not differentiable but continuous ∀ x∈R (D) f(x) is not differentiable for two values of x
[JEE 2006, 5]

Q.12 Let $g(x) = \frac{(x-1)^n}{\ln \cos^m (x-1)}$; 0 < x < 2, *m* and *n* are integers, *m* ≠ 0, *n* > 0 and let *p* be the left hand derivative of | x – 1 | at x = 1. If $\lim_{x\to 1^+} g(x) = p$, then

(A) n = 1, m = 1 (B) n = 1, m = –1 (C) n = 2, m = 2 (D) n > 2, m = n
[JEE 2008, 3]

KEY CONCEPTS (METHOD OF DIFFERENTIATION)

1. DEFINITION :

If x and $x+h$ belong to the domain of a function f defined by $y=f(x)$, then

$\underset{h\to 0}{\text{Limit}} \dfrac{f(x+h)-f(x)}{h}$ if it exists, is called the **Derivative** of f at x & is denoted by $f'(x)$ or $\dfrac{dy}{dx}$. We have therefore, $f'(x) = \underset{h\to 0}{\text{Limit}} \dfrac{f(x+h)-f(x)}{h}$

2. The derivative of a given function f at a point $x = a$ of its domain is defined as :

$\underset{h\to 0}{\text{Limit}} \dfrac{f(a+h)-f(a)}{h}$, provided the limit exists & is denoted by $f'(a)$.

Note that alternatively, we can define $f'(a) = \underset{x\to a}{\text{Limit}} \dfrac{f(x)-f(a)}{x-a}$, provided the limit exists.

3. DERIVATIVE OF f(x) FROM THE FIRST PRINCIPLE /ab INITIO METHOD:

If f(x) is a derivable function then, $\underset{\delta x\to 0}{\text{Limit}} \dfrac{\delta y}{\delta x} = \underset{\delta x\to 0}{\text{Limit}} \dfrac{f(x+\delta x)-f(x)}{\delta x} = f'(x) = \dfrac{dy}{dx}$

4. THEOREMS ON DERIVATIVES :

If u and v are derivable function of x, then,

(i) $\dfrac{d}{dx}(u \pm v) = \dfrac{du}{dx} \pm \dfrac{dv}{dx}$ **(ii)** $\dfrac{d}{dx}(K\,u) = K\dfrac{du}{dx}$, where K is any constant

(iii) $\dfrac{d}{dx}(u\,.\,v) = u\dfrac{dv}{dx} \pm v\dfrac{du}{dx}$ known as " **Product Rule** "

(iv) $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{v\left(\frac{du}{dx}\right)-u\left(\frac{dv}{dx}\right)}{v^2}$ where $v \neq 0$ known as " **Quotient Rule** "

(v) If $y = f(u)$ & $u = g(x)$ then $\dfrac{dy}{dx} = \dfrac{dy}{du}.\dfrac{du}{dx}$ " **Chain Rule** "

5. DERIVATIVE OF STANDARDS FUNCTIONS :

(i) $D(x^n) = n.x^{n-1}$; $x \in R$, $n \in R$, $x > 0$ **(ii)** $D(e^x) = e^x$

(iii) $D(a^x) = a^x . \ln a \quad a > 0$ **(iv)** $D(\ln x) = \dfrac{1}{x}$ **(v)** $D(\log_a x) = \dfrac{1}{x}\log_a e$

(vi) $D(\sin x) = \cos x$ **(vii)** $D(\cos x) = -\sin x$ **(viii)** $D = \tan x = \sec^2 x$

(ix) $D(\sec x) = \sec x . \tan x$ **(x)** $D(\text{cosec}\,x) = -\text{cosec}\,x . \cot x$

(xi) $D(\cot x) = -\text{cosec}^2 x$ **(xii)** $D(\text{constant}) = 0$ where $D = \dfrac{d}{dx}$

6. INVERSE FUNCTIONS AND THEIR DERIVATIVES :

(a) Theorem : If the inverse functions f & g are defined by $y = f(x)$ & $x = g(y)$ & if $f'(x)$ exists & $f'(x) \neq 0$ then $g'(y) = \dfrac{1}{f'(x)}$. This result can also be written as, if $\dfrac{dy}{dx}$ exists & $\dfrac{dy}{dx} \neq 0$, then $\dfrac{dx}{dy} = 1/\left(\dfrac{dy}{dx}\right)$ or $\dfrac{dy}{dx}.\dfrac{dx}{dy} = 1$ or $\dfrac{dy}{dx} = 1/\left(\dfrac{dx}{dy}\right)$ $[\dfrac{dx}{dy} \neq 0]$

(b) Results :

(i) $D(\sin^{-1}x) = \dfrac{1}{\sqrt{1-x^2}}$, $-1 < x < 1$ **(ii)** $D(\cos^{-1}x) = \dfrac{-1}{\sqrt{1-x^2}}$, $-1 < x < 1$

(iii) $D(\tan^{-1}x) = \dfrac{1}{1+x^2}$, $x \in R$ **(iv)** $D(\sec^{-1}x) = \dfrac{1}{|x|\sqrt{x^2-1}}$, $|x| > 1$

(v) $D(\operatorname{cosec}^{-1}x) = \dfrac{-1}{|x|\sqrt{x^2-1}}$, $|x| > 1$ **(vi)** $D(\cot^{-1}x) = \dfrac{-1}{1+x^2}$, $x \in R$

Note : In general if $y = f(u)$ then $\dfrac{dy}{dx} = f'(u) \cdot \dfrac{du}{dx}$.

7. LOGARITHMIC DIFFERENTIATION : To find the derivative of :

(i) a function which is the product or quotient of a number of functions **OR**

(ii) a function of the form $[f(x)]^{g(x)}$ where f & g are both derivable, it will be found convinient to take the logarithm of the function first & then differentiate. This is called **LOGARITHMIC DIFFERENTIATION**.

8. IMPLICIT DIFFERENTIATION : $\phi(x, y) = 0$

(i) In order to find dy/dx, in the case of implicit functions, we differentiate each term w.r.t. x regarding y as a functions of x & then collect terms in dy/dx together on one side to finally find dy/dx.

(ii) In answers of dy/dx in the case of implicit functions, both x & y are present .

9. PARAMETRIC DIFFERENTIATION :

If $y = f(\theta)$ & $x = g(\theta)$ where θ is a parameter , then $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta}$.

10. DERIVATIVE OF A FUNCTION W.R.T. ANOTHER FUNCTION :

Let $y = f(x)$; $z = g(x)$ then $\dfrac{dy}{dz} = \dfrac{dy/dx}{dz/dx} = \dfrac{f'(x)}{g'(x)}$.

11. DERIVATIVES OF ORDER TWO & THREE :

Let a function y = f(x) be defined on an open interval (a, b). It's derivative, if it exists on (a, b) is a certain function $f'(x)$ [or (dy/dx) or y] & is called the first derivative of y w.r.t. x.

If it happens that the first derivative has a derivative on (a , b) then this derivative is called the second derivative of y w. r. t. x & is denoted by $f''(x)$ or (d^2y/dx^2) or y''.

Similarly, the 3rd order derivative of y w. r. t. x , if it exists, is defined by $\dfrac{d^3y}{dx^3} = \dfrac{d}{dx}\left(\dfrac{d^2y}{dx^2}\right)$ It is also denoted by $f'''(x)$ or y'''.

12. If $F(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix}$, where f, g, h, l, m, n, u, v, w are differentiable functions of x then

$$F'(x) = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ l'(x) & m'(x) & n'(x) \\ u(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u'(x) & v'(x) & w'(x) \end{vmatrix}$$

13. L' HOSPITAL'S RULE :

If f(x) & g(x) are functions of x such that :

(i) $\underset{x\to a}{\text{Limit}}\, f(x) = 0 = \underset{x\to a}{\text{Limit}}\, g(x)$ OR $\underset{x\to a}{\text{Limit}}\, f(x) = \infty = \underset{x\to a}{\text{Limit}}\, g(x)$ and

(ii) Both f(x) & g(x) are continuous at x = a &

(iii) Both f(x) & g(x) are differentiable at x = a &

(iv) Both $f'(x)$ & $g'(x)$ are continuous at x = a , Then

$\underset{x\to a}{\text{Limit}}\, \dfrac{f(x)}{g(x)} = \underset{x\to a}{\text{Limit}}\, \dfrac{f'(x)}{g'(x)} = \underset{x\to a}{\text{Limit}}\, \dfrac{f''(x)}{g''(x)}$ & so on till indeterminant form vanishes.

14. ANALYSIS AND GRAPHS OF SOME USEFUL FUNCTIONS :

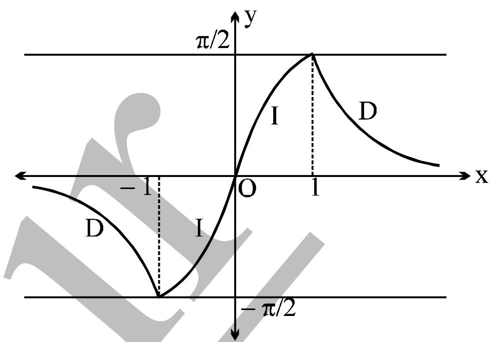
(i) $y = f(x) = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right) = \begin{bmatrix} 2\tan^{-1}x & |x| \le 1 \\ \pi - 2\tan^{-1}x & x > 1 \\ -\left(\pi + 2\tan^{-1}x\right) & x < -1 \end{bmatrix}$

HIGHLIGHTS :

(a) Domain is $x \in R$ &

range is $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

(b) f is continuous for all x but not diff. at $x = 1, -1$



(c) $\dfrac{dy}{dx} = \begin{bmatrix} \frac{2}{1+x^2} & \text{for} & |x| < 1 \\ \text{non existent} & \text{for} & |x| = 1 \\ -\frac{2}{1+x^2} & \text{for} & |x| > 1 \end{bmatrix}$

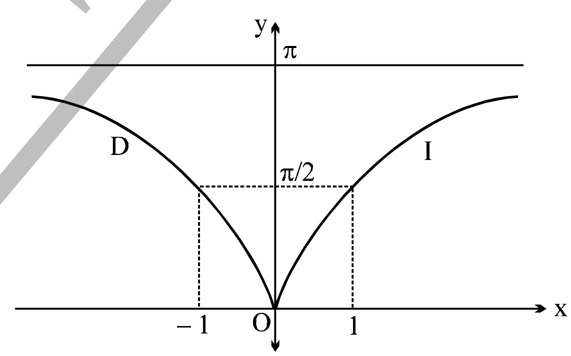
(d) I in $(-1, 1)$ & D in $(-\infty, -1) \cup (1, \infty)$

(ii) Consider $y = f(x) = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right) = \begin{bmatrix} 2\tan^{-1}x & \text{if} & x \ge 0 \\ -2\tan^{-1}x & \text{if} & x < 0 \end{bmatrix}$

HIGHLIGHTS :

(a) Domain is $x \in R$ & range is $[0, \pi)$

(b) Continuous for all x but not diff. at $x = 0$



(c) $\dfrac{dy}{dx} = \begin{bmatrix} \frac{2}{1+x^2} & \text{for} & x > 0 \\ \text{non existent} & \text{for} & x = 0 \\ -\frac{2}{1+x^2} & \text{for} & x < 0 \end{bmatrix}$

(d) I in $(0, \infty)$ & D in $(-\infty, 0)$

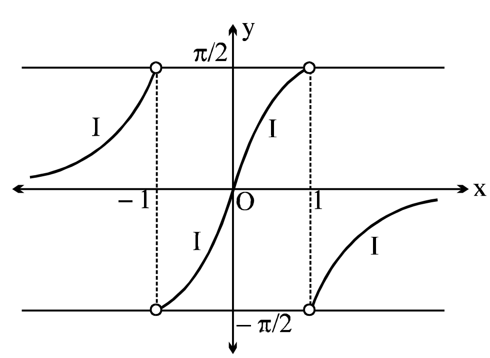
(iii) $y = f(x) = \tan^{-1}\dfrac{2x}{1-x^2} = \begin{bmatrix} 2\tan^{-1}x & |x| < 1 \\ \pi + 2\tan^{-1}x & x < -1 \\ -\left(\pi - 2\tan^{-1}x\right) & x > 1 \end{bmatrix}$

HIGHLIGHTS :

(a) Domain is $R - \{1, -1\}$ &

range is $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

(b) f is neither continuous nor diff. at $x = 1, -1$



(c) $\dfrac{dy}{dx} = \begin{bmatrix} \frac{2}{1+x^2} & |x| \ne 1 \\ \text{non existent} & |x| = 1 \end{bmatrix}$

(d) I $\forall$ x in its domain

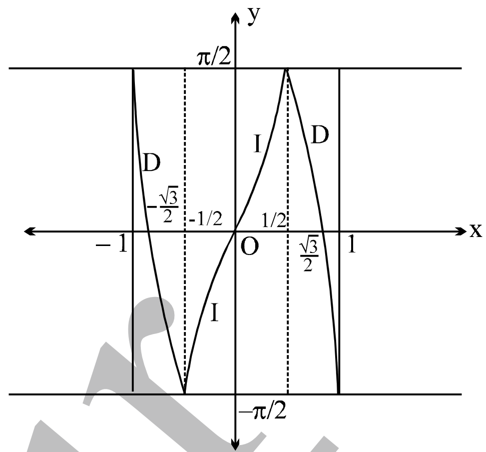
(e) It is bound for all x

(iv) $y = f(x) = \sin^{-1}(3x - 4x^3) = \begin{cases} -\left(\pi + 3\sin^{-1}x\right) & \text{if } -1 \le x \le -\frac{1}{2} \\ 3\sin^{-1}x & \text{if } -\frac{1}{2} \le x \le \frac{1}{2} \\ \pi - 3\sin^{-1}x & \text{if } \frac{1}{2} \le x \le 1 \end{cases}$

Highlights :

(a) Domain is $x \in [-1, 1]$ &

range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



(b) Not derivable at $|x| = \frac{1}{2}$

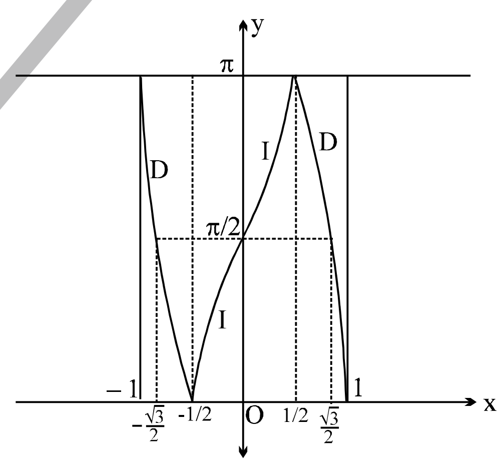
(c) $\frac{dy}{dx} = \begin{cases} \frac{3}{\sqrt{1-x^2}} & \text{if } x \in \left(-\frac{1}{2}, \frac{1}{2}\right) \\ -\frac{3}{\sqrt{1-x^2}} & \text{if } x \in \left(-1, -\frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right) \end{cases}$

(d) Continuous everywhere in its domain

(v) $y = f(x) = \cos^{-1}(4x^3 - 3x) = \begin{cases} 3\cos^{-1}x - 2\pi & \text{if } -1 \le x \le -\frac{1}{2} \\ 2\pi - 3\cos^{-1}x & \text{if } -\frac{1}{2} \le x \le \frac{1}{2} \\ 3\cos^{-1}x & \text{if } \frac{1}{2} \le x \le 1 \end{cases}$

Highlights :

(a) Domain is $x \in [-1, 1]$ &
range is $[0, \pi]$



(b) Continuous everywhere in its domain

but not derivable at $x = \frac{1}{2}, -\frac{1}{2}$

(c) I in $\left(-\frac{1}{2}, \frac{1}{2}\right)$ &

D in $\left(\frac{1}{2}, 1\right] \cup \left[-1, -\frac{1}{2}\right)$

(d) $\frac{dy}{dx} = \begin{cases} \frac{3}{\sqrt{1-x^2}} & \text{if } x \in \left(-\frac{1}{2}, \frac{1}{2}\right) \\ -\frac{3}{\sqrt{1-x^2}} & \text{if } x \in \left(-1, -\frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right) \end{cases}$

General Note :

Concavity in each case is decided by the sign of 2nd derivative as :

$\frac{d^2y}{dx^2} > 0 \Rightarrow$ Concave upwards ; $\frac{d^2y}{dx^2} < 0 \Rightarrow$ Concave downwards

D = Decreasing ; I = Increasing

EXERCISE–I

Q.1 Let f, g and h are differentiable functions. If $f(0) = 1$; $g(0) = 2$; $h(0) = 3$ and the derivatives of their pair wise products at x = 0 are

$(f g)'(0) = 6$; $(g h)'(0) = 4$ and $(h f)'(0) = 5$

then compute the value of $(fgh)'(0)$.

Q.2(a) If $y = (\cos x)^{\ln x} + (\ln x)^x$ find $\frac{dy}{dx}$.

(b) If $y = e^{x^{e^x}} + e^{x^{x^e}} + x^{e^{e^x}}$. Find $\frac{dy}{dx}$.

Q.3 Let $f(x) = x + \frac{1}{2x+}\,\frac{1}{2x+}\,\frac{1}{2x+}\,.........\infty$. Compute the value of $f(100) \cdot f'(100)$.

Q.4 If $y = \frac{x^2}{2} + \frac{1}{2}x\sqrt{x^2+1} + \ln\sqrt{x+\sqrt{x^2+1}}$ prove that $2y = xy' + \ln y'$. where ' denotes the derivative.

Q.5 If $x = \operatorname{cosec}\theta - \sin\theta$; $y = \operatorname{cosec}^n\theta - \sin^n\theta$, then show that $(x^2+4)\left(\frac{dy}{dx}\right)^2 - n^2(y^2+4) = 0$.

Q.6 If $y = \sec 4x$ and $x = \tan^{-1}(t)$, prove that $\frac{dy}{dt} = \frac{16t(1-t^4)}{(1-6t^2+t^4)^2}$.

Q.7 If $x = \frac{1+\ln t}{t^2}$ and $y = \frac{3+2\ln t}{t}$. Show that $y\frac{dy}{dx} = 2x\left(\frac{dy}{dx}\right)^2 + 1$.

Q.8 Differentiate $\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}$ w. r. t. $\sqrt{1-x^4}$.

Q.9 Find the derivative with respect to x of the function:

$(\log_{\cos x}\sin x)(\log_{\sin x}\cos x)^{-1} + \arcsin\frac{2x}{1+x^2}$ at $x = \frac{\pi}{4}$.

Q.10 If $\sqrt{1-x^6}+\sqrt{1-y^6} = a^3 . (x^3 - y^3)$, prove that $\frac{dy}{dx} = \frac{x^2}{y^2}\sqrt{\frac{1-y^6}{1-x^6}}$.

Q.11 If $y = x + \cfrac{1}{x + \cfrac{1}{x + \cfrac{1}{x + ...}}}$, prove that $\frac{dy}{dx} = \cfrac{1}{2 - \cfrac{x}{x + \cfrac{1}{x + \cfrac{1}{x + ...}}}}$.

Q.12 If $y = 1 + \frac{x_1}{x - x_1} + \frac{x_2 . x}{(x-x_1)(x-x_2)} + \frac{x_3 . x^2}{(x-x_1)(x-x_2)(x-x_3)} + ...$ upto $(n+1)$ terms then prove that

$$\frac{dy}{dx} = \frac{y}{x}\left[\frac{x_1}{x_1 - x} + \frac{x_2}{x_2 - x} + \frac{x_3}{x_3 - x} + ... + \frac{x_n}{x_n - x}\right]$$

Q.13 Suppose $f(x)=\tan\left(\sin^{-1}(2x)\right)$

(a) Find the domain and range of f.

(b) Express $f(x)$ as an algebraic function of x.

(c) Find $f'\left(1/4\right)$.

Q.14 If $y=\tan^{-1}\dfrac{u}{\sqrt{1-u^2}}$ & $x=\sec^{-1}\dfrac{1}{2u^2-1}$, $u\in\left(0,\dfrac{1}{\sqrt{2}}\right)\cup\left(\dfrac{1}{\sqrt{2}},1\right)$ prove that $2\dfrac{dy}{dx}+1=0$.

Q.15 If $y=\cot^{-1}\dfrac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}$, find $\dfrac{dy}{dx}$ if $x\in\left(0,\dfrac{\pi}{2}\right)\cup\left(\dfrac{\pi}{2},\pi\right)$.

Q.16 If $y=\tan^{-1}\dfrac{x}{1+\sqrt{1-x^2}}+\sin\left(2\tan^{-1}\sqrt{\dfrac{1-x}{1+x}}\right)$, then find $\dfrac{dy}{dx}$ for $x\in(-1,1)$.

Q.17 Let $f(x)=x^2-4x-3,\ x>2$ and let g be the inverse of f. Find the value of g' where $f(x)=2$.

Q.18 If $y=\tan^{-1}\dfrac{1}{x^2+x+1}+\tan^{-1}\dfrac{1}{x^2+3x+3}+\tan^{-1}\dfrac{1}{x^2+5x+7}+\tan^{-1}\dfrac{1}{x^2+7x+13}+\ldots\ldots$ to n terms.
Find dy/dx, expressing your answer in 2 terms.

Q.19 If $y=\ln\left(x^{e^{x}\cdot a^{y}}\right)^{y^{x}}$ find $\dfrac{dy}{dx}$.

Q.20 If $x=\tan\dfrac{y}{2}-\ln\left[\dfrac{\left(1+\tan\frac{y}{2}\right)^2}{\tan\frac{y}{2}}\right]$. Show that $\dfrac{dy}{dx}=\dfrac{1}{2}\sin y(1+\sin y+\cos y)$.

Q.21 If $\sqrt{x^2+y^2}=e^{\arcsin\frac{y}{\sqrt{x^2+y^2}}}$. Prove that $\dfrac{d^2y}{dx^2}=\dfrac{2(x^2+y^2)}{(x-y)^3}$, $x>0$.

Q.22 If $x=2\cos t-\cos 2t$ & $y=2\sin t-\sin 2t$, find the value of (d^2y/dx^2) when $t=(\pi/2)$.

Q.23 Find the value of the expression $y^3\dfrac{d^2y}{dx^2}$ on the ellipse $3x^2+4y^2=12$.

Q.24 If $f: R\to R$ is a function such that $f(x)=x^3+x^2f'(1)+xf''(2)+f'''(3)$ for all $x\in R$, then prove that $f(2)=f(1)-f(0)$.

Q.25 Let g(x) be a polynomial, of degree one & $f(x)$ be defined by $f(x)=\begin{cases} g(x), & x\le 0\\ \left(\dfrac{1+x}{2+x}\right)^{1/x}, & x>0\end{cases}$.
Find the continuous function $f(x)$ satisfying $f'(1)=f(-1)$

EXERCISE–II

Q.1 If $\sin y = x\sin(a+y)$, show that $\dfrac{dy}{dx} = \dfrac{\sin a}{1-2x\cos a + x^2}$.

Q.2 Find a polynomial function $f(x)$ such that $f(2x) = f'(x)f''(x)$.

Q.3 If $y = \text{arc}\cos\sqrt{\dfrac{\cos 3x}{\cos^3 x}}$ then show that $\dfrac{dy}{dx} = \sqrt{\dfrac{6}{\cos 2x + \cos 4x}}$, $\sin x > 0$.

Q.4 Let $y = x\sin kx$. Find the possible value of k for which the differential equation $\dfrac{d^2y}{dx^2} + y = 2k\cos kx$ holds true for all $x \in R$.

Q.5 Prove that if $|a_1\sin x + a_2\sin 2x + ... + a_n\sin nx| \le |\sin x|$ for $x \in R$, then
$$|a_1 + 2a_2 + 3a_3 + ... + na_n| \le 1$$

Q.6 The function $f: R \to R$ satisfies $f(x^2)\cdot f''(x) = f'(x)\cdot f'(x^2)$ for all real x. Given that $f(1) = 1$ and $f'''(1) = 8$, compute the value of $f'(1) + f''(1)$.

Q.7(a) Show that the substitution $z = ln\left(\tan\dfrac{x}{2}\right)$ changes the equation $\dfrac{d^2y}{dx^2} + \cot x\dfrac{dy}{dx} + 4y\,\text{cosec}^2 x = 0$ to $(d^2y/dz^2) + 4y = 0$.

(b) If the dependent variable y is changed to 'z' by the substitution $y = \tan z$ then the differential equation $\dfrac{d^2y}{dx^2} = 1 + \dfrac{2(1+y)}{1+y^2}\left(\dfrac{dy}{dx}\right)^2$ is changed to $\dfrac{d^2z}{dx^2} = \cos^2 z + k\left(\dfrac{dz}{dx}\right)^2$, then find the value of k.

Q.8 Show that $R = \dfrac{\left[1+(dy/dx)^2\right]^{3/2}}{d^2y/dx^2}$ can be reduced to the form $R^{2/3} = \dfrac{1}{(d^2y/dx^2)^{2/3}} + \dfrac{1}{(d^2x/dy^2)^{2/3}}$.

Also show that, if $x = a\sin 2\theta(1+\cos 2\theta)$ & $y = a\cos 2\theta(1-\cos 2\theta)$ then the value of R equals to $4a\cos 3\theta$.

Q.9 Let $f(x) = \dfrac{\sin x}{x}$ if $x \ne 0$ and $f(0) = 1$. Define the function $f'(x)$ for all x and find $f''(0)$ if it exist.

Q.10 Suppose f and g are two functions such that $f, g : R \to R$,

$f(x) = ln\left(1+\sqrt{1+x^2}\right)$ and $g(x) = ln\left(x+\sqrt{1+x^2}\right)$

then find the value of $x\,e^{g(x)}\left(f\left(\dfrac{1}{x}\right)\right)' + g'(x)$ at $x = 1$.

Q.11 Let $f(x) = \begin{cases} xe^x & x \le 0 \\ x + x^2 - x^3 & x > 0 \end{cases}$ then prove that

(a) f is continuous and differentiable for all x.
(b) f' is continuous and differentiable for all x.

Q.12 $f: [0, 1] \to R$ is defined as $f(x) = \begin{cases} x^3(1-x)\sin\left(\dfrac{1}{x^2}\right) & \text{if } 0 < x \le 1 \\ 0 & \text{if } x = 0 \end{cases}$, then prove that

(a) f is differentiable in [0, 1] (b) f is bounded in [0, 1] (c) f' is bounded in [0, 1]

Q.13 Let f(x) be a derivable function at $x = 0$ & $f\left(\frac{x+y}{k}\right) = \frac{f(x)+f(y)}{k}$ $(k \in R, k \neq 0, 2)$. Show that $f(x)$ is either a zero or an odd linear function.

Q.14 Let $\frac{f(x+y)-f(x)}{2} = \frac{f(y)-a}{2} + xy$ for all real x and y. If f(x) is differentiable and $f'(0)$ exists for all real permissible values of 'a' and is equal to $\sqrt{5a-1-a^2}$. Prove that f(x) is positive for all real x.

Q.15

Column-I	Column-II
(A) $f(x) = \begin{cases} \ln(1+x^3)\cdot\sin\frac{1}{x}, & \text{if } x > 0 \\ 0, & \text{if } x \le 0 \end{cases}$	(P) *continuous everywhere but not differentiable at* $x = 0$
(B) $g(x) = \begin{cases} \ln^2(1+x)\cdot\sin\frac{1}{x}, & \text{if } x > 0 \\ 0, & \text{if } x \le 0 \end{cases}$	(Q) *differentiable at* $x = 0$ *but derivative is discontinuous at* $x = 0$
(C) $u(x) = \begin{cases} \ln\left(1+\frac{\sin x}{2}\right), & \text{if } x > 0 \\ 0, & \text{if } x \le 0 \end{cases}$	(R) *differentiable and has continuous derivative*
(D) $v(x) = \lim_{t\to 0} \frac{2x}{\pi}\tan^{-1}\left(\frac{2}{t^2}\right)$	(S) *continuous and differentiable at* $x = 0$

Q.16 If $f(x) = \begin{vmatrix} (x-a)^4 & (x-a)^3 & 1 \\ (x-b)^4 & (x-b)^3 & 1 \\ (x-c)^4 & (x-c)^3 & 1 \end{vmatrix}$ then $f'(x) = \lambda . \begin{vmatrix} (x-a)^4 & (x-a)^2 & 1 \\ (x-b)^4 & (x-b)^2 & 1 \\ (x-c)^4 & (x-c)^2 & 1 \end{vmatrix}$. Find the value of λ.

Q.17 If $f(x) = \begin{vmatrix} \cos(x+x^2) & \sin(x+x^2) & -\cos(x+x^2) \\ \sin(x-x^2) & \cos(x-x^2) & \sin(x-x^2) \\ \sin 2x & 0 & \sin 2x^2 \end{vmatrix}$ then find $f'(x)$.

Q.18 If α be a repeated root of a quadratic equation $f(x) = 0$ & $A(x), B(x), C(x)$ be the polynomials of degree 3, 4 & 5 respectively, then show that $\begin{vmatrix} A(x) & B(x) & C(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$ is divisible by f(x), where dash denotes the derivative.

Q.19 Let $f(x) = \begin{vmatrix} a+x & b+x & c+x \\ \ell+x & m+x & n+x \\ p+x & q+x & r+x \end{vmatrix}$. Show that $f''(x) = 0$ and that $f(x) = f(0) + kx$ where k denotes the sum of all the co-factors of the elements in $f(0)$.

Q.20 If $Y = sX$ and $Z = tX$, where all the letters denotes the functions of x and suffixes denotes the differentiation w.r.t. x then prove that

$$\begin{vmatrix} X & Y & Z \\ X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \end{vmatrix} = X^3 \begin{vmatrix} s_1 & t_1 \\ s_2 & t_2 \end{vmatrix}$$

EXERCISE–III

Evalute the following limits using L'Hospital's Rule or otherwise :

Q.1 $\lim\limits_{x\to 0}\left[\dfrac{1}{x\sin^{-1}x}-\dfrac{1-x^2}{x^2}\right]$ Q.2 $\lim\limits_{x\to 0}\dfrac{x+ln\left(\sqrt{x^2+1}-x\right)}{x^3}$

Q.3 $\lim\limits_{x\to 0}\left[\dfrac{1}{x^2}-\dfrac{1}{\sin^2 x}\right]$ Q.4 $\lim\limits_{x\to 0^+} x^{\left(x^x-1\right)}$

Q.5 $\lim\limits_{x\to 0}\dfrac{1+\sin x-\cos x+ln(1-x)}{x\cdot\tan^2 x}$ Q.6 $\lim\limits_{x\to \frac{\pi}{2}}\dfrac{\sin x-(\sin x)^{\sin x}}{1-\sin x+ln(\sin x)}$

Q.7 Find the value of $f(0)$ so that the function $f(x)=\dfrac{1}{x}-\dfrac{2}{e^{2x}-1}$, $x\neq 0$ is continuous at $x=0$ & examine the differentiability of f(x) at $x=0$.

Q.8 $\lim\limits_{x\to 0}\dfrac{\sin(3x^2)}{ln.\cos(2x^2-x)}$

Q.9 If $\lim\limits_{x\to 0}\dfrac{a\sin x-bx+cx^2+x^3}{2x^2.ln(1+x)-2x^3+x^4}$ exists & is finite, find the values of a, b, c & the limit.

Q.10 Evaluate: $\lim\limits_{x\to 0}\dfrac{x^{6000}-(\sin x)^{6000}}{x^2\cdot(\sin x)^{6000}}$

Q.11 If $\lim\limits_{x\to 0}\dfrac{1-\cos x\cdot\cos 2x\cdot\cos 3x........\cos nx}{x^2}$ has the value equal to 253, find the value of n (where $n\in N$).

Q.12 Given a real valued function f(x) as follows:

$f(x)=\dfrac{x^2+2\cos x-2}{x^4}$ for $x<0$; $f(0)=\dfrac{1}{12}$ & $f(x)=\dfrac{\sin x-ln\left(e^x\cos x\right)}{6x^2}$ for $x>0$. Test the continuity and differentiability of $f(x)$ at $x=0$.

Q.13 Let $a_1>a_2>a_3$ .... $a_n>1$; $p_1>p_2>p_3$......... $>p_n>0$; such that $p_1+p_2+p_3+......+p_n=1$

Also $F(x)=\left(p_1a_1^x+p_2a_2^x+.......+p_na_n^x\right)^{1/x}$. Compute

(a) $\lim\limits_{x\to 0^+}F(x)$ **(b)** $\lim\limits_{x\to\infty}F(x)$ **(c)** $\lim\limits_{x\to -\infty}F(x)$

Q.14 If $\lim\limits_{x\to 0}\dfrac{1-\cos 3x\cdot\cos 9x\cdot\cos 27x.........\cos 3^n x}{1-\cos\frac{1}{3}x\cdot\cos\frac{1}{9}x\cdot\cos\frac{1}{27}x.........\cos\frac{1}{3^n}x}=3^{10}$, find the value of n.

Q.15 **Column-I** contains function defined on R and **Column-II** contains their properties. Match them.

Column-I	Column-II
(A) $\lim\limits_{n\to\infty}\left(\dfrac{1+\tan\frac{\pi}{2n}}{1+\sin\frac{\pi}{3n}}\right)^n$ equal	(P) e
(B) $\lim\limits_{x\to 0^+}\dfrac{1}{(1+\operatorname{cosec}x)^{\frac{1}{ln(\sin x)}}}$ equals	(Q) e^2
(C) $\lim\limits_{x\to 0}\left(\dfrac{2}{\pi}\cos^{-1}x\right)^{1/x}$ equals	(R) $e^{-2/\pi}$
	(S) $e^{\pi/6}$

EXERCISE–IV

Q.1 If $f(x)=\dfrac{x^2-x}{x^2+2x}$, then find the domain and the range of f. Show that f is one-one. Also find the function $\dfrac{d f^{-1}(x)}{dx}$ and its domain. [REE '99, 6]

Q.2(a) If $x^2+y^2=1$, then :

(A) $y y''-2(y')^2+1=0$ (B) $y y''+(y')^2+1=0$

(C) $y y''-(y')^2-1=0$ (D) $y y''+2(y')^2+1=0$

[JEE 2000, Screening, 1 out of 35]

(b) Suppose $p(x)=a_0+a_1x+a_2x^2+......+a_nx^n$. If $|p(x)| \le |e^{x-1}-1|$ for all $x \ge 0$ prove that $|a_1+2a_2+......+na_n| \le 1$. [JEE 2000 (Mains) 5 out of 100]

Q.3(a) If $\ln(x+y)=2xy$, then $y'(0)=$

(A) 1 (B) –1 (C) 2 (D) 0 [JEE 2004 (Scr.)]

(b) $$f(x)=\begin{cases} b\sin^{-1}\left(\dfrac{x+c}{2}\right), & -\dfrac{1}{2}<x<0 \\ \dfrac{1}{2} & \text{at } x=0 \\ \dfrac{e^{ax/2}-1}{x}, & 0<x<\dfrac{1}{2} \end{cases}.$$

If $f(x)$ is differentiable at $x=0$ and $|c|<1/2$ then find the value of 'a' and prove that $64b^2 = 4-c^2$.

[JEE 2004, 4 out of 60]

Q.4(a) If $y=y(x)$ and it follows the relation $x\cos y+y\cos x=\pi$, then $y''(0)$

(A) 1 (B) – 1 (C) π (D) $-\pi$

(b) If P(x) is a polynomial of degree less than or equal to 2 and S is the set of all such polynomials so that $P(1)=1$, $P(0)=0$ and $P'(x)>0 \ \forall x \in [0, 1]$, then

(A) $S=\phi$ (B) $S=\{(1-a)x^2+ax,\ 0<a<2$

(C) $(1-a)x^2+ax,\ a\in(0,\infty)$ (D) $S=\{(1-a)x^2+ax,\ 0<a<1$

(c) If $f(x)$ is a continuous and differentiable function and $f(1/n)=0,\ \forall n\ge 1$ and $n\in I$, then

(A) $f(x)=0,\ x\in(0,1]$ (B) $f(0)=0,\ f'(0)=0$

(C) $f'(x)=0=f''(x),\ x\in(0,1]$ (D) $f(0)=0$ and $f'(0)$ need not to be zero

[JEE 2005 (Scr.)]

(d) If $f(x-y)=f(x)\cdot g(y)-f(y)\cdot g(x)$ and $g(x-y)=g(x)\cdot g(y)+f(x)\cdot f(y)$ for all $x, y\in R$. If right hand derivative at $x=0$ exists for $f(x)$. Find derivative of $g(x)$ at $x=0$. [JEE 2005 (Mains), 4]

Q.5 For $x>0$, $\lim\limits_{x\to 0}\left((\sin x)^{1/x}+(1/x)^{\sin x}\right)$ is

(A) 0 (B) –1 (C) 1 (D) 2 [JEE 2006, 3]

Q.6 $\dfrac{d^2x}{dy^2}$ equals

(A) $\left(\dfrac{d^2y}{dx^2}\right)^{-1}$ (B) $-\left(\dfrac{d^2y}{dx^2}\right)^{-1}\left(\dfrac{dy}{dx}\right)^{-3}$ (C) $\left(\dfrac{d^2y}{dx^2}\right)\left(\dfrac{dy}{dx}\right)^{-2}$ (D) $-\left(\dfrac{d^2y}{dx^2}\right)\left(\dfrac{dy}{dx}\right)^{-3}$

[JEE 2007, 3]

Q.7(a) Let g (x) = *l*n *f* (x) where f (x) is a twice differentiable positive function on (0, ∞) such that f (x + 1) = x f (x). Then for N = 1, 2, 3

$$g''\left(N+\frac{1}{2}\right)-g''\left(\frac{1}{2}\right) =$$

(A) $-4\left\{1+\frac{1}{9}+\frac{1}{25}+.....+\frac{1}{(2N-1)^2}\right\}$ (B) $4\left\{1+\frac{1}{9}+\frac{1}{25}+.....+\frac{1}{(2N-1)^2}\right\}$

(C) $-4\left\{1+\frac{1}{9}+\frac{1}{25}+.....+\frac{1}{(2N+1)^2}\right\}$ (D) $4\left\{1+\frac{1}{9}+\frac{1}{25}+.....+\frac{1}{(2N+1)^2}\right\}$

(b) Let *f* and *g* be real valued functions defined on interval (–1, 1) such that g''(x) is continuous, g(0) ≠ 0, g'(0) = 0, g''(0) ≠ 0, and f (x) = g (x) sin x.

STATEMENT-1 : $\lim_{x\to 0}$ [g(x) cot x – g(0) cosec x] = f ''(0)

and

STATEMENT-2 : f '(0) = g(0)

(A) Statement-1 is True, Statement-2 is True ; statement-2 is a correct explanation for statement-1
(B) Statement-1 is True, Statement-2 is True ; statement-2 is NOT a correct explanation for statement-1
(C) Statement-1 is True, Statement-2 is False
(D) Statement-1 is False, Statement-2 is True [JEE 2008, 3 + 3]

DIFFERENTIABILITY

EXERCISE–I

Q 1. f(x) is conti. but not derivable at $x = 0$

Q 2. conti. $\forall\ x \in R$, not diff. at $x = 0, 1$ & 2

Q 4. conti. but not diff.at $x = 0$; diff. & conti. at $x = \pi/2$ **Q 5.** conti. but not diff. at $x = 0$

Q 7. f is cont. but not diff. at $x = 0$ **Q 8.** $f'(1^+) = 3$, $f'(1^-) = -1$

Q 9. $a = 1/2$, $b = 3/2$ **Q 10.** not derivable at $x = 0$ & $x = 1$

Q 11. f is cont. & derivable at $x = -1$ but f is neither cont. nor derivable at $x = 1$

Q 12. discontinuous & not derivable at $x = 1$, continuous but not derivable at $x = 2$

Q 13. not derivable at $x = 0$

Q 14. f is conti. at $x = 1$, $3/2$ & disconti. at $x = 2$, f is not diff. at $x = 1, 3/2, 2$

Q15. $(fog)(x) = x+1$ for $-2 \le x \le -1$, $-(x+1)$ for $-1 < x \le 0$ & $x - 1$ for $0 < x \le 2$.
(fog)(x) is cont. at $x = -1$, $(gof)(x) = x+1$ for $-1 \le x \le 1$ & $3 - x$ for $1 < x \le 3$.
(gof)(x) is not differentiable at $x = 1$

Q 16. $a \ne 1, b = 0, p = \frac{1}{3}$ and $q = -1$

Q 17. If $a \in (0, 1)$ $f'(0^+) = -1$; $f'(0^-) = 1 \Rightarrow$ continuous but not derivable
$a = 1$; $f(x) = 0$ which is constant $\Rightarrow$ continuous and derivable
If $a > 1$ $f'(0^-) = -1$; $f'(0^+) = 1 \Rightarrow$ continuous but not derivable

Q 18. conti. in $0 \le x \le 1$ & not diff. at $x = 0$

Q.19 f is conti. but not diff. at $x = 1$, disconti. at $x = 2$ & $x = 3$. cont.& diff.at all other points

Q.20 $f'(x) = -f(x)$ **Q.21** continuous but not derivable at $x = \sqrt{2}$ **Q.22** $f'(0) = \frac{\alpha}{1-k}$

Q.23 **(a)** $f'(0) = 0$, **(b)** $f'\left(\frac{1^-}{3}\right) = -\frac{\pi}{2}$ and $f'\left(\frac{1^+}{3}\right) = \frac{\pi}{2}$, **(c)** $x = \frac{1}{2n+1}$ $n \in I$

Q.24 $f(x) = x \Rightarrow f(10) = 10$ **Q.25** 5150

EXERCISE–II

Q.1 2	**Q.2** conti. & diff.	**Q.3** 0	**Q.4** $f'(0^+) = 0$, $f'(0^-) = 1$
Q.5 2	**Q.6** D	**Q.7** A	**Q.8** D
Q.9 B	**Q.10** B	**Q.11** D	**Q.12** A
Q.13 A	**Q.14** D	**Q.15** A	**Q.16** D
Q.17 A, C	**Q.18** A, B	**Q.19** B, D	**Q.20** B, D
Q.21 B, D	**Q.22** A, B , D	**Q.23** A, B, D	**Q.24** A, B, D
Q.25 A, D			

EXERCISE–III

Q.1 D

Q.2 C **Q.3** Discont. hence not deri. at $x = 1$ & -1. Cont. & deri. at $x = 0$

Q.4 **(a)** D, **(b)** A, **(c)** D **Q.6** D **Q.7** C **Q.8** $a = 1; b = 0 (gof)'(0) = 0$

Q.9 $f'(a^-) = 0$ **Q.10** **(a)** A, **(b)** $y - 2 = 0$ **Q.11** A, C **Q.12** C

* *

METHOD OF DIFFERENTIATION

EXERCISE–I

Q.1 16

Q.2 (a) $Dy = (\cos x)^{\ln x}\left[\frac{\ln(\cos x)}{x} - \tan x \ln x\right] + (\ln x)^x\left[\frac{1}{\ln x} + \ln(\ln x)\right]$;

(b) $\frac{dy}{dx} = e^{x^{e^x}}.x^{e^x}\left[\frac{e^x}{x}+e^x\ln x\right]+e^{x^{x^e}}x^{e-1}x^{x^e}[1+e\ln x]+x^{e^{e^x}}e^{e^x}\left[\frac{1}{x}+e^x\ln x\right]$

Q.3 100 **Q.8** $\frac{1+\sqrt{1-x^4}}{x^6}$ **Q.9** $\frac{32}{16+\pi^2}-\frac{8}{\ln 2}$

Q.13 (a) $\left(-\frac{1}{2},\frac{1}{2}\right)$, $(-\infty, \infty)$; (b) $f(x) = \frac{2x}{\sqrt{1-4x^2}}$; (c) $\frac{16\sqrt{3}}{9}$ **Q.15** $\frac{1}{2}$ or $-\frac{1}{2}$

Q.16 $\frac{1-2x}{2\sqrt{1-x^2}}$ **Q.17** 1/6 **Q.18** $\frac{1}{1+(x+n)^2}-\frac{1}{1+x^2}$ **Q.19** $\frac{y}{x}.\frac{x\ell nx + x\ell nx.\ell ny + 1}{\ell nx(1-x-y\ell na)}$

Q.22 $-\frac{3}{2}$ **Q.23** $-\frac{9}{4}$ **Q.25** $f(x) = \begin{cases} -\frac{2}{3}\left[\frac{1}{6}+\ln\frac{3}{2}\right]x & \text{if } x \le 0 \\ \left(\frac{1+x}{2+x}\right)^{1/x} & \text{if } x > 0 \end{cases}$

EXERCISE–II

Q.2 $\frac{4x^3}{9}$ **Q.4** k = 1, – 1 or 0 **Q.6** 6 Q.7 **(b)** k = 2

Q.9 $f'(x) = \begin{cases} \frac{x\cos x - \sin x}{x^2} & \text{if } x \ne 0 \\ 0 & \text{if } x = 0 \end{cases}$; $f''(0) = -\frac{1}{3}$ **Q.10** zero

Q.15 (A) R, S; (B) Q, S; (C) P ; (D) R, S **Q.16** 3 **Q.17** $2(1+2x).\cos 2(x+x^2)$

EXERCISE–III

Q 1. $\frac{5}{6}$ **Q 2.** $\frac{1}{6}$ **Q 3.** $-\frac{1}{3}$ **Q 4.** 1 **Q 5.** $-\frac{1}{2}$ **Q.6** 2

Q.7 $f(0) = 1$; differentiable at $x = 0$, $f'(0^+) = -(1/3)$; $f'(0^-) = -(1/3)$ **Q.8** – 6

Q.9 a = 6, b = 6, c = 0; $\frac{3}{40}$ **Q.10** 1000 **Q.11** n = 11

Q.12 f is cont. but not derivable at $x = 0$ **Q.13** **(a)** $a_1^{p_1}\cdot a_2^{p_2}.....a_n^{p_n}$; **(b)** a_1; **(c)** a_n

Q.14 n = 4 **Q.15** (A) S; (B) P ; (C) R

EXERCISE–IV

Q.1 Domain of $f(x) = R - \{-2, 0\}$; Range of $f(x) = R - \{-1/2, 1\}$; $\frac{d}{dx}[f^{-1}(x)] = \frac{3}{(1-x)^2}$

Domain of $f^{-1}(x) = R - \{-1/2, 1\}$ **Q.2 (a)** B **Q.3 (a)** A; **(b)** a = 1

Q.4 **(a)** C; **(b)** B; **(c)** B, **(d)** $g'(0) = 0$ **Q.5** C **Q.6** D **Q.7** **(a)** A, **(b)** D