Derivability/Differentiability

Two Fold Meaning of Derivability

Geometrical meaning of derivative

Slope of the tangent drawn to the curve at x = a if it exists

Physical meaning of derivative

Instantaneous rate of change of function

Note : "Tangent at a point 'A' is the limiting case of secant through A."

Existence of Derivative

Right hand & Left hand Derivatives :

By definition :

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$ if it exists

(i) The right hand derivative of f at x = a denoted by $f'(a^+)$ is defined by :

$$f'(a^+) = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$$

provided the limit exists & is finite.

(ii) The left hand derivative of f at x = a denoted by $f'(a^-)$ is defined by :

$$f'(a^-) = \lim_{h \to 0^+} \frac{f(a-h) - f(a)}{-h}$$

provided the limit exists & is finite.

f is said to be derivable at x = a if $f'(a^+) = f'(a^-)$ = a finite quantity.

Derivability & Continuity

Theorem : If a function f is derivable at $x = a$ then f is continuous at $x = a$.

For a function f

Differentiability $\Rightarrow$ Continuity;

Non derivability $\nRightarrow$ discontinuous

Continuity $\nRightarrow$ derivability;

But discontinuity $\Rightarrow$ Non derivability

Q. Consider the function $f(x) = [x - 1] + |x - 2|$ where [ ] denotes the greatest integer function.

Statement-1 : f(x) is discontinuous at x = 2.

because :

Statement-2 : f(x) is non derivable at x = 2.

(A) Statement-1 is true, statement-2 is true and S-2 is correct explanation for S-1.

(B) Statement-1 is true, Statement-2 is true and S-2 is NOT the correct explanation for S-1.

(C) Statement-1 is true, statement-2 is false.

(D) Statement-1 is false, statement-2 is true.

Derivability Over An Interval

f(x) is said to be derivable over an open interval (a, b) if it is derivable at each & every point of the interval, f(x) is said to be derivable over the closed interval [a, b] if **:**

(i) for the point a and b, $f'(a^+)$ & $f'(b^-)$ exist &

Derivability Over An Interval

f(x) is said to be derivable over an open interval (a, b) if it is derivable at each & every point of the interval, f(x) is said to be derivable over the closed interval [a, b] if :

(ii) for any point c such that a<c<b, $f'(c^+)$ & $f'(c^-)$ exist & are equal.

Examples

Q. Find if function $f(x) = |\ln x|$ is differentiable at $x = 1$

Q. $f(x) = \ln^2 x$ at $x = 1$

Q. $f(x) = e^{-|x|}$ at $x = 0$

Q. Find L.H.D. & R.H.D. of $f(x) = |x-1|$ at $x = 1$

Q. Find L.H.D. & R.H.D. of $f(x) = |x^3|$ at $x = 0$

Q. $$f(x) = \begin{cases} \frac{\sin x^2}{x} & x \neq 0 \\ 0 & \text{at } x = 0 \end{cases}$$

Find tangent & normal at $x = 0$, if they exist.

Q. $$f(x)=\begin{cases} |1-4x^2| & 0 \le x < 1 \\ [x^2-2x] & 1 \le x \le 2 \end{cases}$$

differentiability in (0,2) where [ ] denotes greatest integer function.

Q. $y = |\sin x|$ at $x = 0$

Q. $y = x|x|$ at $x = 0$

Q. $y = x|x-1|$ at $x = 1$

Q. $y = (x - 1)\,|x - 1|$ at $x = 1$

Q. $$f(x) = \begin{cases} ax^2 + 1 & x \le 1 \\ x^2 + ax + b & x > 1 \end{cases}$$

Find a & b if f(x) is differentiable at x = 1

Q. $$f(x) = \begin{cases} ax+b & x \le -1 \\ ax^3+x+2b & x > -1 \end{cases}$$

Find a & b if f(x) is differentiable $\forall\ x \in R$

Q. $$y(x) = \begin{cases} x & x \le 1 \\ x^2 + bx + c & x > 1 \end{cases}$$

Find b & c if f(x) is differentiable at x = 1

Q. $$y = f(x) = \begin{cases} ax^2 - ax + b & x < 1 \\ x - 1 & 1 \le x \le 3 \\ cx^2 + dx + 2 & x > 3 \end{cases}$$

Find a, b, c, d so that all condition satisfy.

(a) f is continuous for all x.

(b) $f'(1)$ does not exists

(c) $f'(x)$ is continuous at x = 3

Q. $$f(x) = \begin{cases} x^3 & x^2 < 1 \\ x & x^2 \geq 1 \end{cases}$$

Check continuity & derivability of function

Q. $$f = \begin{cases} [x^2] & 0 \le x < 1 \\ |4x^2 - 1| & x \ge 1 \end{cases}$$

Then f is

(A) Continuous at x = 1

(B) Not continuous at x = 1

(C) Non differentiable at x = 1

(D) Differentiable at x = 1

Q. If $y = 2$ then find $\frac{dy}{dx}$ at $x = 3$

Q. If $y = \cos x + |\cos x|$ find $\frac{dy}{dx}$ at $x = \frac{\pi}{2}$

Q. By Graph or otherwise check if function is differentiable :

(a) $|\sin x|$

Q. By Graph or otherwise check if function is differentiable :

(b) $\sin|x|$

Q. By Graph or otherwise check if function is differentiable :

(c) $|\ln|x||$

Q. By Graph or otherwise check if function is differentiable :

(d) $\cos^{-1}(\cos x)$

Q. By Graph or otherwise check if function is differentiable :

(e) $\cos|x|$

Q. By Graph or otherwise check if function is differentiable :

(f) Max $(\sin x, \cos x)$

Q. By Graph or otherwise check if function is differentiable :

(g) Max $(1-x, 1+x, 2)$

Q. By Graph or otherwise check if function is differentiable :

(h) Max $(|x|, x^2)$

Q. By Graph or otherwise check if function is differentiable :

(i) Max (x, x^3)

Q. By Graph or otherwise check if function is differentiable :

(j) Min $(2x - 1, x^2)$

Q. By Graph or otherwise check if function is differentiable :

(k) $|x+1| + |x| + |x-1|$

Q. By Graph or otherwise check if function is differentiable :

(*l*) Min (tanx, cotx)

Q. By Graph or otherwise check if function is differentiable :

(m) Max $(\tan^{-1}x, \cot^{-1}x)$

Q. By Graph or otherwise check if function is differentiable :

(n) $\sqrt{|x|}$

Q. $$f(x) = \begin{cases} x^2 \dfrac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

then

(a) Continuous at $x = 0$

(b) Continuous and non differentiable at $x = 0$

(c) Differentiable at $x = 0$

(d) $f'(0) = 2$

Q. If $f(0) = 0$, $f'(0) = 1$ then $\lim_{x \to 0} \frac{f(x/4)}{x} = ?$

Q. If $y = \begin{cases} |x-3| & x \geq 1 \\ \dfrac{x^2}{4} - \dfrac{3x}{2} + \dfrac{13}{4} & x < 1 \end{cases}$ then

(A) Continuous at x = 1

(B) Continuous at x = 3

(C) Differentiable at x = 1

(D) Differentiable at x = 3

Q. $$f(x) = \begin{cases} \sqrt{4x^2 - 12x + 9}\,.\{x\} & \text{for } x \geq 1 \\ \cos\left(\frac{\pi}{2}(|x| - \{x\})\right) & x < 1 \end{cases}$$

where { } denotes fractional part function

Check the differentiability in [–1, 2]

Q. Let $f(x) = \text{sgn } x$ and $g(x) = x(1 - x^2)$. Investigate the composite functions $f(g(x))$ and $g(f(x))$ for continuity and differentiability.

Important Notes

Note :

(1) If f(x) and g(x) are both derivable at x = a, $f(x) \pm g(x)$; $f(x).g(x)$ and $\frac{f(x)}{g(x)}$ will also be derivable at x = a. (only if $g(a) \neq 0$)

Note :

(2) If $f(x)$ is derivable at $x = a$ and $g(x)$ is not derivable at $x = a$ then the $f(x) + g(x)$ or $f(x) - g(x)$ will not be derivable at $x = a$.

For example $f(x)=|x|$ and $g(x)=x$.

Examples

Q. $f(x) = \cos|x|$ is derivable at $x = 0$ and $g(x) = |x|$ is not derivable at $x = 0$ then $\cos|x| + |x|$ or $\cos|x| - |x|$ will not be derivable at $x = 0$

→ Nothing can be said about the product function in this case.

Examples

Q. $f(x) = x$ derivable at $x = 0$

$g(x) = |x|$ not derivable at $x = 0$ then $f(x).g(x)$ is differentiable at $x = 0$

Note :

(3) If both f(x) and g(x) are non derivable then nothing definite can be said about the sum/difference/product function.

Examples

Q. $f(x) = \sin|x|$ not derivable at $x = 0$

$g(x) = |x|$ not derivable at $x = 0$

then the function

(i) $F(x) = \sin|x| - |x|$ is derivable at $x = 0$

Q. $f(x) = \sin|x|$ not derivable at $x = 0$

$g(x) = |x|$ not derivable at $x = 0$

then the function

(ii) $G(x) = \sin|x| + |x|$ is not derivable at $x = 0$

Q. Draw graph of $y = [x] + |1-x|, -1 \leq x \leq 3$ Determine points if any where function is not differentiable

Q. $f = x^3 - x^2 + x + 1$ &

$$g = \begin{cases} \max(f(t)) \quad 0 \le t \le x & \text{for } 0 \le x \le 1 \\ 3 - x & 1 < x \le 2 \end{cases}$$

Discuss the continuity & Differentiability of g in [0,2]

Q. If f(x) is differentiable at x = a & $f'(a) = \frac{1}{4}$ then. Find

(i) $$\lim_{h\to 0} \frac{f(a+4h^2) - f(a)}{h^2}$$

Q. If f(x) is differentiable at x = a & $f'(a) = \frac{1}{4}$ then. Find

(ii) $$\lim_{h \to 0} \frac{f(a-4h^2)-f(a)}{h^2}$$

Q. If f(x) is differentiable at x = a & $f'(a) = \frac{1}{4}$ then. Find

(iii) $$\lim_{h\to 0} \frac{f(a+4h^2)-f(a-4h^2)}{h^2}$$

Determination of function which are differentiable and satisfying the given functional rule

Basic Steps :

(1) Write down the expression for

$$f'(x) \text{ as } f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Basic Steps :

(2) Manipulate f (x + h) – f (x) in such a way that the given functional rule is applicable. Now apply the functional rule and simplify the RHS to get f '(x) as a function of x along with constants if any.

Basic Steps :

(3) Integrate f ' (x) to get f (x) as a function of x and a constant of integration. In some cases a Differential Equation in formed which can be solved to get f (x).

Basic Steps :

(4) Apply the boundary value conditions to determine the value of this constant.

Examples

Q. Let f be a differentiable function satisfying

$$f\left(\frac{x}{y}\right) = f(x) - f(y) \text{ for all } x, y > 0.$$

If $f'(1) = 1$ then find $f(x)$.

Q. Suppose f is a derivable function that satisfies the equation $f(x + y) = f(x) + f(y) + x^2y + xy^2$ for all real numbers x and y. Suppose that

$$\lim_{x \to 0} \frac{f(x)}{x} = 1, \text{ find}$$

(a) $f(0)$ (b) $f'(0)$ (c) $f'(x)$ (d) $f(3)$

Q. A differentiable function satisfies the relation

$f(x + y) = f(x) + f(y) + 2xy - 1 \quad \forall\ x, y \in R$

If $f'(0) = \sqrt{3 + a - a^2}$ find $f(x)$ and prove that $f(x) > 0 \quad \forall\ x \in R$

Q. If $f(x + y) = f(x) \cdot f(y)$, $\forall\ x, y \in R$ and $f(x)$ is a differentiable function everywhere. Find $f(x)$

Q. If $f(x + y) = f(x) + f(y)$, $\forall$ $x, y \in R$ then prove that $f(kx) = k f(x)$ for $\forall$ $k, x \in R$.