Differential Equation

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KEY CONCEPTS (DIFFERENTIAL EQUATION)
DIFFERENTIAL EQUATIONS OF FIRST ORDER AND FIRST DEGREE

DEFINITIONS :

1. An equation that involves independent and dependent variables and the derivatives of the dependent variables is called a **DIFFERENTIAL EQUATION**.

2. A differential equation is said to be ordinary, if the differential coefficients have reference to a single independent variable only and it is said to be **PARTIAL** if there are two or more independent variables. We are concerned with ordinary differential equations only. eg. $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0$ is a partial differential equation.

3. Finding the unknown function is called **SOLVING OR INTEGRATING** the differential equation. The solution of the differential equation is also called its **PRIMITIVE**, because the differential equation can be regarded as a relation derived from it.

4. The order of a differential equation is the order of the highest differential coefficient occuring in it.

5. The degree of a differential equation which can be written as a polynomial in the derivatives is the degree of the derivative of the highest order occuring in it, after it has been expressed in a form free from radicals & fractions so far as derivatives are concerned, thus the differential equation :

$$f(x\,,y)\left[\frac{d^m y}{dx^m}\right]^p + \phi(x\,,y)\left[\frac{d^{m-1}(y)}{dx^{m-1}}\right]^q + ...=0 \text{ is order } m \text{ \& degree } p.$$

Note that in the differential equation $e^{y'''}-xy''+y=0$ order is three but degree doesn't apply.

6. FORMATION OF A DIFFERENTIAL EQUATION :

If an equation in independent and dependent variables having some arbitrary constant is given, then a differential equation is obtained as follows :

☞ Differentiate the given equation w.r.t. the independent variable (say x) as many times as the number of arbitrary constants in it.

☞ Eliminate the arbitrary constants .

The eliminant is the required differential equation . Consider forming a differential equation for $y^2 = 4a(x+b)$ where a and b are arbitary constant.

Note : A differential equation represents a family of curves all satisfying some common properties. This can be considered as the geometrical interpretation of the differential equation.

7. GENERAL AND PARTICULAR SOLUTIONS :

The solution of a differential equation which contains a number of independent arbitrary constants equal to the order of the differential equation is called the **GENERAL SOLUTION (OR COMPLETE INTEGRAL OR COMPLETE PRIMITIVE)** . A solution obtainable from the general solution by giving particular values to the constants is called a **PARTICULAR SOLUTION**.

Note that the general solution of a differential equation of the n^{th} order contains 'n' & only 'n' independent arbitrary constants. The arbitrary constants in the solution of a differential equation are said to be independent, when it is impossible to deduce from the solution an equivalent relation containing fewer arbitrary constants. Thus the two arbitrary constants A, B in the equation $y = A\,e^{x+B}$ are not independent since the equation can be written as $y = A\,e^B.\ e^x = C\,e^x$. Similarly the solution $y = A\sin x + B\cos(x+C)$ appears to contain three arbitrary constants, but they are really equivalent to two only.

8. Elementary Types Of First Order & First Degree Differential Equations .

TYPE–1. VARIABLES SEPARABLE : If the differential equation can be expressed as ;

$f(x)dx + g(y)dy = 0$ then this is said to be variable – separable type.

A general solution of this is given by $\int f(x)\,dx + \int g(y)\,dy = c$;

where c is the arbitrary constant . consider the example $(dy/dx) = e^{x-y} + x^2 . e^{-y}$.

Note : Sometimes transformation to the polar co–ordinates facilitates separation of variables. In this connection it is convenient to remember the following differentials.
If $x = r\cos\theta$; $y = r\sin\theta$ then,
(i) $x\,dx + y\,dy = r\,dr$ **(ii)** $dx^2 + dy^2 = dr^2 + r^2 d\theta^2$ **(iii)** $x\,dy - y\,dx = r^2 d\theta$

If $x = r\sec\theta$ & $y = r\tan\theta$ then $x\,dx - y\,dy = r\,dr$ and $x\,dy - y\,dx = r^2 \sec\theta\, d\theta$.

TYPE–2 : $\dfrac{dy}{dx} = f(ax + by + c)$, $b \neq 0$.

To solve this , substitute $t = ax + by + c$. Then the equation reduces to separable type in the variable t and x which can be solved.

Consider the example $(x + y)^2 \dfrac{dy}{dx} = a^2$.

TYPE–3. HOMOGENEOUS EQUATIONS :

A differential equation of the form $\dfrac{dy}{dx} = \dfrac{f(x,y)}{\phi(x,y)}$

where $f(x , y)$ & $\phi(x , y)$ are homogeneous functions of x & y , and of the same degree , is called **Homogeneous** . This equation may also be reduced to the form $\dfrac{dy}{dx} = g\left(\dfrac{x}{y}\right)$ & is solved by putting $y = vx$ so that the dependent variable y is changed to another variable v, where v is some unknown function, the differential equation is transformed to an equation with variables separable.

Consider $\dfrac{dy}{dx} + \dfrac{y(x+y)}{x^2} = 0$.

TYPE–4. EQUATIONS REDUCIBLE TO THE HOMOGENEOUS FORM :

If $\dfrac{dy}{dx} = \dfrac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$; where $a_1 b_2 - a_2 b_1 \neq 0$, i.e. $\dfrac{a_1}{b_1} \neq \dfrac{a_2}{b_2}$

then the substitution $x = u + h$, $y = v + k$ transform this equation to a homogeneous type in the new variables u and v where h and k are arbitrary constants to be chosen so as to make the given equation homogeneous which can be solved by the method as given in Type – 3. If

(i) $a_1 b_2 - a_2 b_1 = 0$, then a substitution $u = a_1 x + b_1 y$ transforms the differential equation to an equation with variables separable. and

(ii) $b_1 + a_2 = 0$, then a simple cross multiplication and substituting $d(xy)$ for $x\,dy + y\,dx$ & integrating term by term yields the result easily.

Consider $\dfrac{dy}{dx} = \dfrac{x - 2y + 5}{2x + y - 1}$; $\dfrac{dy}{dx} = \dfrac{2x + 3y - 1}{4x + 6y - 5}$ & $\dfrac{dy}{dx} = \dfrac{2x - y + 1}{6x - 5y + 4}$

(iii) In an equation of the form : $y f(xy)\,dx + x g(xy)dy = 0$ the variables can be separated by the substitution $xy = v$.

IMPORTANT NOTE :

(a) The function $f(x, y)$ is said to be a homogeneous function of degree n if for any real number $t\,(\neq 0)$, we have $f(tx, ty) = t^n f(x, y)$.
For e.g. $f(x, y) = ax^{2/3} + hx^{1/3}.\,y^{1/3} + by^{2/3}$ is a homogeneous function of degree 2/3 .

(b) A differential equation of the form $\frac{dy}{dx} = f(x, y)$ is homogeneous if $f(x, y)$ is a homogeneous function of degree zero i.e. $f(tx, ty) = t^{\circ} f(x, y) = f(x, y)$. The function f does not depend on x & y separately but only on their ratio $\frac{y}{x}$ or $\frac{x}{y}$.

LINEAR DIFERENTIAL EQUATIONS :

A differential equation is said to be linear if the dependent variable & its differential coefficients occur in the first degree only and are not multiplied together .
The nth order linear differential equation is of the form ;

$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + ... + a_n(x).y = \phi(x)$. Where $a_0(x)$, $a_1(x)$ ..... $a_n(x)$ are called the coefficients of the differential equation.

Note that a linear differential equation is always of the first degree but every differental equation of the first degree need not be linear. e.g. the differential equation $\frac{d^2 y}{dx^2} + \left(\frac{dy}{dx}\right)^3 + y^2 = 0$ is not linear, though its degree is 1.

TYPE – 5. LINEAR DIFFERENTIAL EQUATIONS OF FIRST ORDER :

The most general form of a linear differential equations of first order is $\frac{dy}{dx} + Py = Q$, where P & Q are functions of x .

To solve such an equation multiply both sides by $e^{\int P dx}$.

NOTE :

(1) The factor $e^{\int P dx}$ on multiplying by which the left hand side of the differential equation becomes the differential coefficient of some function of x & y , is called integrating factor of the differential equation popularly abbreviated as I. F.

(2) It is very important to remember that on multiplying by the integrating factor , the left hand side becomes the derivative of the product of y and the I. F.

(3) Some times a given differential equation becomes linear if we take y as the independent variable and x as the dependent variable. e.g. the equation ;

$(x + y + 1)\frac{dy}{dx} = y^2 + 3$ can be written as $(y^2 + 3)\frac{dx}{dy} = x + y + 1$ which is a linear differential equation.

TYPE–6. EQUATIONS REDUCIBLE TO LINEAR FORM :

The equation $\frac{dy}{dx} + py = Q \cdot y^n$ where P & Q functions of x , is reducible to the linear form by dividing it by y^n & then substituting $y^{-n+1} = Z$. Its solution can be obtained as in **Type–5**. Consider the example $(x^3y^2 + xy)\,dx = dy$.

The equation $\frac{dy}{dx} + Py = Q \cdot y^n$ is called **BERNOULI'S EQUATION**.

9. TRAJECTORIES :

Suppose we are given the family of plane curves.

$\Phi\,(x, y, a) = 0$

depending on a single parameter a.

A curve making at each of its points a fixed angle α with the curve of the family passing through that point is called an ***isogonal trajectory*** of that family ; if in particular $\alpha = \pi/2$, then it is called an ***orthogonal trajectory***.

Orthogonal trajectories : We set up the differential equation of the given family of curves. Let it be of the form

$$F\,(x, y, y') = 0$$

The differential equation of the orthogonal trajectories is of the form

$$F\left(x,\ y,\ -\frac{1}{y'}\right) = 0$$

The general integral of this equation

$$\Phi_1\,(x, y, C) = 0$$

gives the family of orthogonal trajectories.

Note : Following exact differentials must be remembered :

(i) $x\,dy + y\,dx = d(xy)$

(ii) $\frac{x\,dy - y\,dx}{x^2} = d\left(\frac{y}{x}\right)$

(iii) $\frac{y\,dx - x\,dy}{y^2} = d\left(\frac{x}{y}\right)$

(iv) $\frac{x\,dy + y\,dx}{xy} = d(\ln xy)$

(v) $\frac{dx + dy}{x + y} = d(\ln(x+y))$

(vi) $\frac{x\,dy - y\,dx}{xy} = d\left(\ln\frac{y}{x}\right)$

(vii) $\frac{y\,dx - x\,dy}{xy} = d\left(\ln\frac{x}{y}\right)$

(viii) $\frac{x\,dy - y\,dx}{x^2 + y^2} = d\left(\tan^{-1}\frac{y}{x}\right)$

(ix) $\frac{y\,dx - x\,dy}{x^2 + y^2} = d\left(\tan^{-1}\frac{x}{y}\right)$

(x) $\frac{x\,dx + y\,dy}{x^2 + y^2} = d\left[\ln\sqrt{x^2 + y^2}\right]$

(xi) $d\left(-\frac{1}{xy}\right) = \frac{x\,dy + y\,dx}{x^2y^2}$

(xii) $d\left(\frac{e^x}{y}\right) = \frac{ye^x\,dx - e^x\,dy}{y^2}$

(xiii) $d\left(\frac{e^y}{x}\right) = \frac{xe^y\,dy - e^y\,dx}{x^2}$

EXERCISE–I

[FORMATION & VARIABLES SEPARABLE]

Q.1 State the order and degree of the following differential equations:

(i) $\left[\frac{d^2x}{dt^2}\right]^3 + \left[\frac{dx}{dt}\right]^4 - xt = 0$ (ii) $\frac{d^2y}{dx^2} = \left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}$

Q.2(a) Form the differential equation of the family of curves represented by, $c\,(y + c)^2 = x^3$; where c is any arbitrary constant.

(b) Form a differential equation for the family of curves represented by $ax^2 + by^2 = 1$, where a & b are arbitary constants.

(c) Obtain the differential equation of the family of circles $x^2 + y^2 + 2gx + 2fy + c = 0$; where g , f & c are arbitary constants.

(d) Obtain the differential equation associated with the primitive,
$y = c_1e^{3x} + c_2e^{2x} + c_3\,e^x$, where c_1, c_2, c_3 are arbitrary constants.

Solve the following differential equation for Q.3 to Q.9:

Q.3 $\frac{\ell n\,(\sec x + \tan x)}{\cos x}\,dx = \frac{\ell n\,(\sec y + \tan y)}{\cos y}\,dy$ Q.4 $(1 - x^2)\,(1 - y)\,dx = xy\,(1 + y)\,dy$

Q.5 $\frac{dy}{dx} + \frac{\sqrt{(x^2 - 1)(y^2 - 1)}}{xy} = 0$ Q.6 $y - x\frac{dy}{dx} = a\left(y^2 + \frac{dy}{dx}\right)$

Q.7 $\frac{dy}{dx} = \sin\,(x + y) + \cos\,(x + y)$ Q.8 $\frac{dy}{dx} = \frac{x\,(2\ln x + 1)}{\sin y + y\cos y}$

Q.9 (a) $\frac{dy}{dx} + \sin\frac{x + y}{2} = \sin\frac{x - y}{2}$ (b) $\sin x\,.\,\frac{dy}{dx} = y\,.\,\ell ny$ if $y = e$, when $x = \frac{\pi}{2}$

Q.10 The population P of a town decreases at a rate proportional to the number by which the population exceeds 1000, proportionality constant being $k > 0$. Find

(a) Population at any time t, given initial population of the town being 2500.

(b) If 10 years later the population has fallen to 1900, find the time when the population will be 1500.

(c) Predict about the population of the town in the long run.

Q.11 It is known that the decay rate of radium is directly proportional to its quantity at each given instant. Find the law of variation of a mass of radium as a function of time if at $t = 0$, the mass of the radius was m_0 and during time t_0 α % of the original mass of radium decay.

Q.12 A normal is drawn at a point $P(x , y)$ of a curve. It meets the x– axis at Q. If PQ is of constant length k, then show that the differential equation describing such curves is, $y\frac{dy}{dx} = \pm\sqrt{k^2 - y^2}$. Find the equation of such a curve passing through (0, k).

Q.13 $\dfrac{x\,dx - y\,dy}{x\,dy - y\,dx} = \sqrt{\dfrac{1 + x^2 - y^2}{x^2 - y^2}}$

Q.14 A curve is such that the length of the polar radius of any point on the curve is equal to the length of the tangent drawn at this point. Form the differential equation and solve it to find the equation of the curve.

Q.15 Tangent is drawn at the point (x_i, y_i) on the curve $y = f(x)$, which intersects the x-axis at $(x_{i+1}, 0)$. Now again a tangent is drawn at (x_{i+1}, y_{i+1}) on the curve which intersects the x-axis at $(x_{i+2}, 0)$ and the process is repeated *n* times i.e. $i = 1, 2, 3, \ldots, n$.

(a) If $x_1, x_2, x_3, \ldots, x_n$ form an arithmetic progression with common difference equal to $\log_2 e$ and curve passes through $(0, 2)$, then find the equation of the curve.

(b) If $x_1, x_2, x_3, \ldots, x_n$ form a geometric progression with common ratio equal to 2 and the curve passes through $(1, 2)$, then find the equation of the curve.

EXERCISE–II

[HOMOGENEOUS]

Q.1 (a) $\dfrac{dy}{dx} = \dfrac{x^2 + xy}{x^2 + y^2}$ (b) $(x^3 - 3xy^2)\,dx = (y^3 - 3x^2y)\,dy$

Q.2 Find the equation of a curve such that the projection of its ordinate upon the normal is equal to its abscissa.

Q.3 The light rays emanating from a point source situated at origin when reflected from the mirror of a search light are reflected as beam parallel to the x-axis. Show that the surface is parabolic, by first forming the differential equation and then solving it.

Q.4 The perpendicular from the origin to the tangent at any point on a curve is equal to the abscissa of the point of contact. Find the equation of the curve satisfying the above condition and which passes through (1, 1).

Q.5 Show that the equation of the curve intersecting with the x- axis at the point $x = 1$ and for which the length of the subnormal at any point of the curve is equal to the arthemetic mean of the co-ordinates of this point is $(y - x)^2(x + 2y) = 1$.

Q.6 Use the substitution $y^2 = a - x$ to reduce the equation $y^3 \cdot \dfrac{dy}{dx} + x + y^2 = 0$ to homogeneous form and hence solve it.

Q.7 $\left[x\cos\dfrac{y}{x} + y\sin\dfrac{y}{x}\right]y = \left[y\sin\dfrac{y}{x} - x\cos\dfrac{y}{x}\right]x\dfrac{dy}{dx}$

Q.8 Find the curve for which any tangent intersects the y–axis at the point equidistant from the point of tangency and the origin.

Q.9 $(x - y)\,dy = (x + y + 1)\,dx$ Q.10 $\dfrac{dy}{dx} = \dfrac{x + 2y - 3}{2x + y - 3}$ Q.11 $\dfrac{dy}{dx} = \dfrac{y - x + 1}{y + x + 5}$

Q.12 $\dfrac{dy}{dx} = \dfrac{x + y + 1}{2x + 2y + 3}$ Q.13 $\dfrac{dy}{dx} = \dfrac{2(y + 2)^2}{(x + y - 1)^2}$

Q.14 $\dfrac{dy}{dx} + \dfrac{\cos x\,(3\cos y - 7\sin x - 3)}{\sin y\,(3\sin x - 7\cos y + 7)} = 0$

Q.15 Show that the curve such that the distance between the origin and the tangent at an arbitrary point is equal to the distance between the origin and the normal at the same point,

$$\sqrt{x^2 + y^2} = c\,e^{\pm\tan^{-1}\frac{y}{x}}$$

EXERCISE–III

[LINEAR]

Q.1 If solution of differential equation $\frac{dy}{dx} - y = 1 - e^{-x}$ and $y(0) = y_0$ has a finite value. When $x \to \infty$, then find y_0.

Q.2 Let $y = y(t)$ be a solution to the differential equation $y' + 2ty = t^2$, then find $\lim_{t\to\infty} \frac{y}{t}$.

Q.3 $\frac{dy}{dx} + \frac{x}{1+x^2}\, y = \frac{1}{2x(1+x^2)}$

Q.4 $(1 - x^2)\frac{dy}{dx} + 2xy = x(1 - x^2)^{1/2}$

Q.5 Find the curve such that the area of the trapezium formed by the co–ordinate axes, ordinate of an arbitrary point & the tangent at this point equals half the square of its abscissa .

Q.6 $x(x-1)\frac{dy}{dx} - (x-2)y = x^3(2x-1)$

Q.7 Find the curve possessing the property that the intercept, the tangent at any point of a curve cuts off on the y–axis is equal to the square of the abscissa of the point of tangency.

Q.8 $\sin x \frac{dy}{dx} + 3y = \cos x$

Q.9 $x(x^2+1)\frac{dy}{dx} = y(1-x^2) + x^3 . \ln x$

Q.10 $x\frac{dy}{dx} - y = 2x^2 \operatorname{cosec} 2x$

Q.11 $(1+y^2)\,dx = (\tan^{-1} y - x)dy$

Q.12 Let the function $\ln f(x)$ is defined where $f(x)$ exists for $x \geq 2$ & k is fixed positive real number, prove that if $\frac{d}{dx}(x . f(x)) \leq -k f(x)$ then $f(x) \leq A x^{-1-k}$ where A is independent of x.

Q.13 Find the differentiable function which satisfies the equation $f(x) = -\int_0^x f(t)\tan t\, dt + \int_0^x \tan(t-x)\, dt$ where $x \in (-\pi/2, \pi/2)$

Q.14 $y - x\,Dy = b(1 + x^2 Dy)$

Q.15 Find all functions $f(x)$ defined on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ with real values and has a primitive F(x) such that

$$f(x) + \cos x \cdot F(x) = \frac{\sin 2x}{(1+\sin x)^2}. \text{ Find } f(x).$$

Q.16 $2\frac{dy}{dx} - y\sec x = y^3 \tan x$

Q.17 $x^2 y - x^3 \frac{dy}{dx} = y^4 \cos x$

Q.18 $y(2xy + e^x)\,dx - e^x\,dy = 0$

Q.19 A tank contains 100 litres of fresh water. A solution containing 1 gm/litre of soluble lawn fertilizer runs into the tank at the rate of 1 lit/min, and the mixture is pumped out of the tank at the rate of 3 litres/min. Find the time when the amount of fertilizer in the tank is maximum.

Q.20 A tank with a capacity of 1000 litres originally contains 100 gms of salt dissolved in 400 litres of water. Beginning at time t = 0 and ending at time t = 100 minutes, water containing 1 gm of salt per litre enters the tank at the rate of 4 litres per minute, and the well mixed solution is drained from the tank at a rate of 2 litre/minute. Find the differential equation for the amount of salt *y* in the tank at time t.

EXERCISE–IV

(GENERAL – CHANGE OF VARIABLE BY A SUITABLE SUBSTITUTION)

Q.1 $(x - y^2)\,dx + 2xy\,dy = 0$

Q.2 $(x^3 + y^2 + 2)\,dx + 2y\,dy = 0$

Q.3 $x\dfrac{dy}{dx} + y\ln y = xye^x$

Q.4 $\dfrac{dy}{dx} - \dfrac{\tan y}{1+x} = (1 + x)\,e^x \sec y$

Q.5 $\dfrac{dy}{dx} = \dfrac{e^y}{x^2} - \dfrac{1}{x}$

Q.6 $\left(\dfrac{dy}{dx}\right)^2 - (x+y)\dfrac{dy}{dx} + xy = 0$

Q.7 $\dfrac{dy}{dx} = \dfrac{y^2 - x}{2y\,(x + 1)}$

Q.8 $(1 - xy + x^2 y^2)\,dx = x^2\,dy$

Q.9 $\dfrac{dy}{dx} = e^{x-y}\,(e^x - e^y)$

Q.10 $(x^2 + y)\dfrac{dy}{dx} = 6x$

EXERCISE–V

(MISCELLANEOUS)

Q.1 $\dfrac{dy}{dx} - y\,\ln 2 = 2^{\sin x}.\,(\cos x - 1)\,\ln 2$, y being bounded when $x \to +\infty$.

Q.2 $\dfrac{dy}{dx} = y + \displaystyle\int_0^1 y\,dx$ given $y = 1$, where $x = 0$

Q.3 Find the curve which passes through the point (2, 0) such that the segment of the tangent between the point of tangency & the y – axis has a constant length equal to 2 .

Q.4 $x\,dy + y\,dx + \dfrac{x\,dy - y\,dx}{x^2 + y^2} = 0$

Q.5 $\dfrac{y\,dx - x\,dy}{(x - y)^2} = \dfrac{dx}{2\sqrt{1-x^2}}$, given that $y = 2$ when $x = 1$

Q.6 Consider the differential equation, $\dfrac{dy}{dx} + P(x)y = Q(x)$

(i) If two particular solutions of given equation u(x) and v(x) are known, find the general solution of the same equation in terms of u(x) and v(x).

(ii) If α and β are constants such that the linear combinations $\alpha \cdot u(x) + \beta \cdot v(x)$ is a solution of the given equation, find the relation between α and β.

(iii) If w(x) is the third particular solution different from u(x) and v(x) then find the ratio $\dfrac{v(x) - u(x)}{w(x) - u(x)}$.

Q.7 Find the equation of the curve passing through the orgin if the middle point of the segment of its normal from any point of the curve to the x-axis lies on the parabola $2y^2 = x$.

Q.8 Find the continuous function which satisfies the relation, $\displaystyle\int_0^x t\,f(x - t)\,dt = \int_0^x f(t)\,dt + \sin x + \cos x - x - 1$, for all real number x.

Q.9 $(1 - x^2)^2 dy + \left(y\sqrt{1-x^2} - x - \sqrt{1-x^2}\right)dx = 0.$

Q.10 $3x^2y^2 + \cos(xy) - xy\sin(xy) + \frac{dy}{dx}\{2x^3y - x^2\sin(xy)\} = 0.$

Q.11 Given two curves $y = f(x)$ passing through the points (0, 1) & $y = \int_{-\infty}^{x} f(t)dt$ passing through the points (0, 1/2). The tangents drawn to both curves at the points with equal abscissas intersect on the x – axis. Find the curve f(x).

Q.12 Find the integral curve of the differential equation, $x(1 - x\,l n y).\frac{dy}{dx} + y = 0$ which passes through $\left(1, \frac{1}{e}\right)$.

Q.13 Find all the curves possessing the following property; the segment of the tangent between the point of tangency & the x-axis is bisected at the point of intersection with the y-axis.

Q.14 A perpendicular drawn from any point P of the curve on the x-axis meets the x-axis at A. Length of the perpendicular from A on the tangent line at P is equal to 'a'. If this curve cuts the y-axis orthogonally, find the equation to all possible curves, expressing the answer explicitly.

Q.15 A curve passing through (1 , 0) such that the ratio of the square of the intercept cut by any tangent off the y-axis to the subnormal is equal to the ratio of the product of the co-ordinates of the point of tangency to the product of square of the slope of the tangent and the subtangent at the same point. Determine all such possible curves.

Q.16 A & B are two separate reservoirs of water. Capacity of reservoir A is double the capacity of reservoir B. Both the reservoirs are filled completely with water, their inlets are closed and then the water is released simultaneously from both the reservoirs. The rate of flow of water out of each reservoir at any instant of time is proportional to the quantity of water in the reservoir at that time. One hour after the water is released, the quantity of water in reservoir A is 1.5 times the quantity of water in reservoir B. After how many hours do both the reservoirs have the same quantity of water ?

Q.17 A tank consists of 50 litres of fresh water. Two litres of brine each litre containing 5 gms of dissolved salt are run into tank per minute; the mixture is kept uniform by stirring, and runs out at the rate of one litre per minute. If 'm' grams of salt are present in the tank after t minute, express 'm' in terms of t and find the amount of salt present after 10 minutes.

Q.18 Find the curve for which the portion of y-axis cut-off between the origin and the tangent varies as cube of the absissa of the point of contact.

Q.19 Find the orthogonal trajectories for the given family of curves when 'a' is the parameter.
(i) $y = ax^2$ (ii) $\cos y = a\,e^{-x}$ (iii) $x^k + y^k = a^k$
(iv) Find the isogonal trajectories for the family of rectangular hyperbolas $x^2 - y^2 = a^2$ which makes with it an angle of 45°.

Q.20 Let $f(x, y, c_1) = 0$ and $f(x, y, c_2) = 0$ define two integral curves of a homogeneous first order differential equation. If P_1 and P_2 are respectively the points of intersection of these curves with an arbitrary line, $y = mx$ then prove that the slopes of these two curves at P_1 and P_2 are equal. (**Remember this concept**)

EXERCISE–VI
(PROBLEMS ASKED IN JEE & REE)

Q.1 A country has a food deficit of 10 %. Its population grows continuously at a rate of 3 %. Its annual food production every year is 4 % more than that of the last year. Assuming that the average food requirement per person remains constant, prove that the country will become self-sufficient in food after 'n' years, where 'n' is the smallest integer bigger than or equal to, $\dfrac{\ell n 10 - \ell n 9}{\ell n (1.04) - 0.03}$. [JEE '2000 (Mains)10]

Q.2 A hemispherical tank of radius 2 metres is initially full of water and has an outlet of 12 cm^2 cross sectional area at the bottom. The outlet is opened at some instant. The flow through the outlet is according to the law $V(t) = 0.6\sqrt{2gh(t)}$, where V(t) and h(t) are respectively the velocity of the flow through the outlet and the height of water level above the outlet at time t, and g is the acceleration due to gravity. Find the time it takes to empty the tank. [JEE '2001 (Mains) 10]

Q.3 Find the equation of the curve which passes through the origin and the tangent to which at every point (x, y) has slope equal to $\dfrac{x^4 + 2xy - 1}{1 + x^2}$. [REE '2001 (Mains) 3]

Q.4 Let f(x), $x \geq 0$, be a nonnegative continuous function, and let $F(x) = \int_0^x f(t)dt$, $x \geq 0$. If for some $c > 0$, $f(x) \leq cF(x)$ for all $x \geq 0$, then show that $f(x) = 0$ for all $x \geq 0$. [JEE 2001 (Mains) 5 out of 100]

Q.5(a) A right circular cone with radius R and height H contains a liquid which evaporates at a rate proportional to its surface area in contact with air (proportionality constant = k > 0). Find the time after which the cone is empty.

(b) If $P(1) = 0$ and $\dfrac{dP(x)}{dx} > P(x)$ for all $x \geq 1$ then prove that $P(x) > 0$ for all $x > 1$.
[JEE 2003, (Mains) 4 + 4]

Q.6(a) If $\left(\dfrac{2 + \sin x}{1 + y}\right)\dfrac{dy}{dx} = -\cos x$, $y(0) = 1$, then $y\left(\dfrac{\pi}{2}\right) =$

(A) 1 (B) 1/2 (C) 1/3 (D) 1/4
[JEE 2004 (Scr.)]

(b) A curve passes through (2, 0) and the slope of tangent at point P (x, y) equals $\dfrac{(x+1)^2 + y - 3}{(x+1)}$. Find the equation of the curve and area enclosed by the curve and the x-axis in the fourth quadrant.
[JEE 2004 (Mains)]

Q.7(a) The solution of primitive integral equation $(x^2 + y^2)dy = xy\,dx$, is $y = y(x)$. If $y(1) = 1$ and $y(x_0) = e$, then x_0 is

(A) $\sqrt{2(e^2 - 1)}$ (B) $\sqrt{2(e^2 + 1)}$ (C) $\sqrt{3}\,e$ (D) $\sqrt{\dfrac{e^2 + 1}{2}}$

(b) For the primitive integral equation $ydx + y^2dy = xdy$; $x \in R$, $y > 0$, $y = y(x)$, $y(1) = 1$, then $y(-3)$ is
(A) 3 (B) 2 (C) 1 (D) 5
[JEE 2005 (Scr.)]

(c) If length of tangent at any point on the curve $y = f(x)$ intercepted between the point and the x–axis is of length 1. Find the equation of the curve. [JEE 2005 (Mains)]

Q.8 A tangent drawn to the curve, $y = f(x)$ at $P(x, y)$ cuts the x-axis and y-axis at A and B respectively such that $BP : AP = 3 : 1$, given that $f(1) = 1$, then

(A) equation of the curve is $x\dfrac{dy}{dx} - 3y = 0$ (B) equation of curve is $x\dfrac{dy}{dx} + 3y = 0$

(C) curve passes through $(2, 1/8)$ (D) normal at $(1, 1)$ is $x + 3y = 4$ [JEE 2006, 5]

Q.9(a) Let $f(x)$ be differentiable on the interval $(0, \infty)$ such that $f(1) = 1$ and $\displaystyle\lim_{t\to x} \frac{t^2 f(x) - x^2 f(t)}{t - x} = 1$ for each $x > 0$. Then $f(x)$ is

(A) $\dfrac{1}{3x} + \dfrac{2x^2}{3}$ (B) $\dfrac{-1}{3x} + \dfrac{4x^2}{3}$ (C) $\dfrac{-1}{x} + \dfrac{2}{x^2}$ (D) $\dfrac{1}{x}$

(b) The differential equation $\dfrac{dy}{dx} = \dfrac{\sqrt{1-y^2}}{y}$ determines a family of circles with

(A) variable radii and a fixed centre at $(0, 1)$
(B) variable radii and a fixed centre at $(0, -1)$
(C) fixed radius 1 and variable centres along the x-axis.
(D) fixed radius 1 and variable centres along the y-axis. [JEE 2007, 3+3]

Q.10 Let a solution $y = y(x)$ of the differential equation, $x\sqrt{x^2-1}\,dy = y\sqrt{y^2-1}\,dx = 0$ satisfy $y(2) = \dfrac{2}{\sqrt{3}}$.

STATEMENT-1 : $y(x) = \sec\left(\sec^{-1} x - \dfrac{\pi}{6}\right)$
and

STATEMENT-2 : $y(x)$ is given by $\dfrac{1}{y} = \dfrac{2\sqrt{3}}{x} - \sqrt{1 - \dfrac{1}{x^2}}$

(A) Statement-1 is True, Statement-2 is True; statement-2 is a correct explanation for statement-1
(B) Statement-1 is True, Statement-2 is True; statement-2 is **NOT** a correct explanation for statement-1
(C) Statement-1 is True, Statement-2 is False
(D) Statement-1 is False, Statement-2 is True [JEE 2008, 3]

Q.11**(i)** Match the statements/expressions In **Column I** with the open intervals In **Column II.**

	Column I		**Column II**
(A)	Interval contained in the domain of definition of non-zero solutions of the differential equation $(x - 3)^2 y' + y = 0$	(p)	$\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$
(B)	Interval containing the value of the integral $\displaystyle\int_1^5 (x-1)(x-2)(x-3)(x-4)(x-5)dx$	(q)	$\left(0, \dfrac{\pi}{2}\right)$
(C)	Interval in which at least one of the points of local maximum of $\cos^2 x + \sin x$ lies	(r)	$\left(\dfrac{\pi}{8}, \dfrac{5\pi}{4}\right)$
(D)	Interval in which $\tan^{-1}(\sin x + \cos x)$ is increasing	(s)	$\left(0, \dfrac{\pi}{8}\right)$
		(t)	$(-\pi, \pi)$

(ii) Match the statements/expressions given in **Column I** with the values given in **Column II**.

	Column I		Column II
(A)	The number of solutions of the equation $xe^{\sin x} - \cos x = 0$ in the interval $\left(0, \frac{\pi}{2}\right)$	(p)	1
(B)	Value(s) of k for which the planes $kx + 4y + z = 0$, $4x + ky + 2z = 0$ and $2x + 2y + z = 0$ intersect in a straight line	(q)	2
(C)	Value(s) of k for which $\|x-1\| + \|x-2\| + \|x+1\| + \|x+2\| = 4k$ has integer solution(s)	(r)	3
(D)	If $y' = y + 1$ and $y(0) = 1$, then value(s) of $y(\ln 2)$	(s)	4
		(t)	5

[JEE 2009, (2+2+2+2)×2]

ANSWER KEY
DIFFERENTIAL EQUATION

EXERCISE–I

Q.1 (i) order 2 & degree 3 **(ii)** order 2 & degree 2

Q.2 **(a)** $12y\,(y')^2 = x[8(y')^3 - 27]$; **(b)** $xy\dfrac{d^2y}{dx^2} + x\left(\dfrac{dy}{dx}\right)^2 - y\dfrac{dy}{dx} = 0$; **(c)** $[1+(y')^2]\,.y''' - 3y'(y'')^2 = 0$

(d) $\dfrac{d^3y}{dx^3} - 6\dfrac{d^2y}{dx^2} + 11\dfrac{dy}{dx} - 6y = 0$

Q.3 $\ln^2(\sec x + \tan x) - \ln^2(\sec y + \tan y) = c$

Q.4 $\ln x\,(1-y)^2 = c - \dfrac{1}{2}y^2 - 2y + \dfrac{1}{2}x^2$

Q.5 $\sqrt{x^2-1} - \sec^{-1}x + \sqrt{y^2-1} = c$

Q.6 $y = c\,(1-ay)\,(x+a)$

Q.7 $\ln\left[1+\tan\dfrac{x+y}{2}\right] = x + c$

Q.8 $y\sin y = x^2\ln x + c$

Q.9 **(a)** $\ln\left|\tan\dfrac{y}{4}\right| = c - 2\sin\dfrac{x}{2}$, **(b)** $y = e^{\tan(x/2)}$

Q.10 (a) $P = 1000 + 1500e^{-kt}$ where $k = \dfrac{1}{10}\ln\left(\dfrac{5}{3}\right)$; (b) $T = 10\log_{5/3}(3)$; (c) $P = 1000$ as $t \to \infty$

Q.11 $m = m_0e^{-kt}$ where $k = -\dfrac{1}{t_0}\ln\left(1-\dfrac{\alpha}{100}\right)$

Q.12 $x^2 + y^2 = k^2$

Q.13 $\sqrt{x^2-y^2} + \sqrt{1+x^2-y^2} = \dfrac{c\,(x+y)}{\sqrt{x^2-y^2}}$

Q.14 $y = kx$ or $xy = c$

Q.15 (a) $y = 2^{1-x}$; (b) $xy = 2$

EXERCISE–II

Q.1 **(a)** $c(x-y)^{2/3}(x^2+xy+y^2)^{1/6} = \exp\left[\dfrac{1}{\sqrt{3}}\tan^{-1}\dfrac{x+2y}{x\sqrt{3}}\right]$ where $\exp x \equiv e^x$ **(b)** $y^2 - x^2 = c\,(y^2+x^2)^2$

Q.2 $\dfrac{y^2 \pm y\sqrt{y^2-x^2}}{x^2} = \ell n\left|\left(y \pm \sqrt{y^2-x^2}\right)\cdot\dfrac{c^2}{x^3}\right|$, where same sign has to be taken.

Q.4 $x^2 + y^2 - 2x = 0$

Q.6 $\dfrac{1}{2}\ln|x^2+a^2| - \tan^{-1}\left(\dfrac{a}{x}\right) = c$, where $a = x + y^2$

Q.7 $xy\cos\dfrac{y}{x} = c$

Q.8 $x^2 + y^2 = cx$

Q.9 $\text{arc}\tan\dfrac{2y+1}{2x+1} = \ln c\sqrt{x^2+y^2+x+y+\dfrac{1}{2}}$

Q.10 $(x+y-2) = c\,(y-x)^3$

Q.11 $\tan^{-1}\dfrac{y+3}{x+2} + \ln c\sqrt{(y+3)^2+(x+2)^2} = 0$

Q.12 $x + y + \dfrac{4}{3} = ce^{3(x-2y)}$

Q.13 $e^{-2\tan^{-1}\frac{y+2}{x-3}} = c\,.\,(y+2)$

Q.14 $(\cos y - \sin x - 1)^2\,(\cos y + \sin x - 1)^5 = c$

EXERCISE–III

Q.1 $-\frac{1}{2}$ **Q.2** $\frac{1}{2}$

Q.3 $y\sqrt{1+x^2} = c + \frac{1}{2} \ln \left[\tan\frac{1}{2}\arctan x\right]$ Another form is $y\sqrt{1+x^2} = c + \frac{1}{2} \ln \frac{\sqrt{1+x^2}-1}{x}$

Q.4 $y = c(1-x^2) + \sqrt{1-x^2}$ **Q.5** $y = cx^2 \pm x$ **Q.6** $y(x-1) = x^2(x^2 - x + c)$

Q.7 $y = cx - x^2$ **Q.8** $\left(\frac{1}{3}+y\right)\tan^3\frac{x}{2} = c + 2\tan\frac{x}{2} - x$

Q.9 $4(x^2+1)y + x^3(1-2\ln x) = cx$ **Q.10** $y = cx + x\ln\tan x$ **Q.11** $x = ce^{-\arctan y} + \arctan y - 1$

Q.13 $\cos x - 1$ **Q.14** $y(1+bx) = b + cx$ **Q.15** $f(x) = -\frac{2\cos x}{(1+\sin x)^2} - Ce^{-\sin x}\cdot\cos x$

Q.16 $\frac{1}{y^2} = -1 + (c+x)\cot\left(\frac{x}{2}+\frac{\pi}{4}\right)$ **Q.17** $x^3y^{-3} = 3\,\text{Sin}x + c$

Q.18 $y^{-1}e^x = c - x^2$ **Q.19** $27\frac{7}{9}$ minutes **Q.20** $\frac{dy}{dt} = 4 - \frac{y}{200+t}$

EXERCISE–IV

Q.1 $y^2 + x\ln ax = 0$ **Q.2** $y^2 = 3x^2 - 6x - x^3 + ce^{-x} + 4$ **Q.3** $x\ln y = e^x(x-1) + c$

Q.4 $\sin y = (e^x + c)(1+x)$ **Q.5** $cx^2 + 2xe^{-y} = 1$ **Q.6** $y = ce^x$; $y = c + \frac{x^2}{2}$

Q.7 $y^2 = -1 + (x+1)\ \ell n\frac{c}{x+1}$ or $x + (x+1)\ln\frac{c}{x+1}$ **Q.8** $y = \frac{1}{x}\tan(\ell n|cx|)$

Q.9 $e^y = c\,.\exp(-e^x) + e^x - 1$ **Q.10** $y = 3\ln(x^2 + y + 3) + C$

EXERCISE–V

Q.1 $y = 2^{\sin x}$ **Q.2** $y = \frac{1}{3-e}(2e^x - e + 1)$ **Q.3** $y = \pm\left[\sqrt{4-x^2} + 2\ell n\frac{2-\sqrt{4-x^2}}{x}\right]$

Q.4 $xy + \tan^{-1}\frac{y}{x} = c$ **Q.5** $\frac{\sin^{-1}x}{2} + \frac{y}{x-y} = \frac{\pi}{4} - 2$

Q.6 (i) $y = u(x) + K(u(x) - v(x))$ where K is any constant ; (ii) $\alpha + \beta = 1$; (iii) constant

Q.7 $y^2 = 2x + 1 - e^{2x}$ **Q.8** $f(x) = e^x - \cos x$ **Q.9** $y = \frac{x}{\sqrt{1-x^2}} + c\,e^{-\frac{x}{\sqrt{1-x^2}}}$

Q.10 $x(x^2y^2 + \cos xy) = c$ **Q.11** $f(x) = e^{2x}$

Q.12 $x(ey + \ln y + 1) = 1$ **Q.13** $y^2 = cx$ **Q.14** $y = \pm a\frac{e^{x/a}+e^{-x/a}}{2}$ & $y = \pm a$

Q.15 $x = e^{2\sqrt{y/x}}$; $x = e^{-2\sqrt{y/x}}$

Q.16 $T = \log_{4/3} 2$ hrs from the start

Q.17 $y = 5t\left(1+\frac{50}{50+t}\right)$ gms ; $91\frac{2}{3}$ gms

Q.18 $2y + Kx^3 = cx$

Q.19 (i) $x^2 + 2y^2 = c$, (ii) $\sin y = ce^{-x}$, (iii) $y = cx$ if $k = 2$ and $\frac{1}{x^{k-2}} - \frac{1}{y^{k-2}} = \frac{1}{c^{k-2}}$ if $k \neq 2$
(iv) $x^2 - y^2 + 2xy = c$; $x^2 - y^2 - 2xy = c$

EXERCISE–VI

Q.2 $\frac{7\pi \times 10^5}{135\sqrt{g}}$ sec.

Q.3 $y = (x - 2\tan^{-1}x)(1 + x^2)$ **Q.5** (a) $T = H/k$ **Q.6** (a) C; (b) $y = x^2 - 2x$, area = 4/3 sq. units

Q.7 **(a)** C; **(b)** A; **(c)** $\sqrt{1-y^2} + ln\left|\frac{1-\sqrt{1-y^2}}{y}\right| = \pm x + c$ **Q.8** B, C **Q.9** **(a)** A, **(b)** C

Q.10 C

Q.11 **(i)** (A) p, q, s (B) p, t (C) p, q, r, t (D) s ; **(ii)** (A) p; (B) q, s ; (C) q, r, s, t ; (D) r