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FUNCTION & ITF

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KEY CONCEPTS (FUNCTIONS)

THINGS TO REMEMBER:

1. GENERAL DEFINITION:

If to every value (Considered as real unless other–wise stated) of a variable x, which belongs to some collection (Set) E, there corresponds one and only one finite value of the quantity y, then y is said to be a function (Single valued) of x or a dependent variable defined on the set E ; x is the argument or independent variable .

If to every value of x belonging to some set E there corresponds one or several values of the variable y, then y is called a multiple valued function of x defined on E.Conventionally the word **"FUNCTION"** is used only as the meaning of a single valued function, if not otherwise stated.

Pictorially : $\xrightarrow[\text{input}]{x}$ $\boxed{f}$ $\xrightarrow[\text{output}]{f(x)=y}$, y is called the image of x & x is the pre-image of y under f.

Every function from $A \rightarrow B$ satisfies the following conditions .

(i) $f \subset A \times B$ **(ii)** $\forall\ a \in A \Rightarrow (a, f(a)) \in f$ and

(iii) $(a, b) \in f$ & $(a, c) \in f \Rightarrow b = c$

2. DOMAIN, CO–DOMAIN & RANGE OF A FUNCTION :

Let $f : A \rightarrow B$, then the set A is known as the domain of f & the set B is known as co-domain of f . The set of all f images of elements of A is known as the range of f . Thus :

Domain of $f = \{a \mid a \in A, (a, f(a)) \in f\}$

Range of $f = \{f(a) \mid a \in A, f(a) \in B\}$

It should be noted that range is a subset of co–domain . If only the rule of function is given then the domain of the function is the set of those real numbers, where function is defined. For a continuous function, the interval from minimum to maximum value of a function gives the range.

3. IMPORTANT TYPES OF FUNCTIONS :

(i) POLYNOMIAL FUNCTION :

If a function f is defined by $f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + ... + a_{n-1} x + a_n$ where n is a non negative integer and $a_0, a_1, a_2, ..., a_n$ are real numbers and $a_0 \neq 0$, then f is called a polynomial function of degree n .

NOTE : (a) A polynomial of degree one with no constant term is called an odd linear function . i.e. $f(x) = ax$, $a \neq 0$

(b) There are two polynomial functions, satisfying the relation ; $f(x).f(1/x) = f(x) + f(1/x)$. They are :

(i) $f(x) = x^n + 1$ & **(ii)** $f(x) = 1 - x^n$, where n is a positive integer .

(ii) ALGEBRAIC FUNCTION :

y is an algebraic function of x, if it is a function that satisfies an algebraic equation of the form

$P_0(x)\, y^n + P_1(x)\, y^{n-1} + ... + P_{n-1}(x)\, y + P_n(x) = 0$ Where n is a positive integer and $P_0(x), P_1(x)$ .... are Polynomials in x.

e.g. $y = |x|$ is an algebraic function, since it satisfies the equation $y^2 - x^2 = 0$.

Note that all polynomial functions are Algebraic but not the converse. A function that is not algebraic is called **TRANSCEDENTAL FUNCTION** .

(iii) FRACTIONAL RATIONAL FUNCTION :

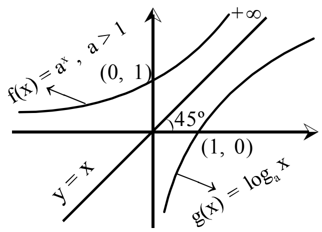
A rational function is a function of the form. $y = f(x) = \frac{g(x)}{h(x)}$, where

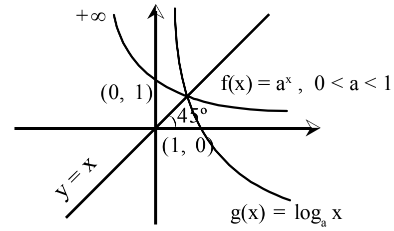
g (x) & h (x) are polynomials & $h(x) \neq 0$.

(iv) **EXPONENTIAL FUNCTION :**

A function $f(x) = a^x = e^{x\ln a}$ $(a > 0,\ a \neq 1, x \in R)$ is called an exponential function. The inverse of the exponential function is called the logarithmic function . i.e. $g(x) = \log_a x$.

Note that f(x) & g(x) are inverse of each other & their graphs are as shown .





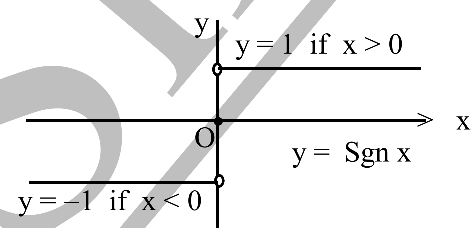
(v) **ABSOLUTE VALUE FUNCTION :**

A function $y = f(x) = |x|$ is called the absolute value function or Modulus function. It is defined as : $y = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

(vi) **SIGNUM FUNCTION :**

A function $y = f(x) = \text{Sgn}(x)$ is defined as follows :

$$y = f(x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x = 0 \\ -1 & \text{for } x < 0 \end{cases}$$



It is also written as Sgn x = |x|/ x ;
$x \neq 0$; $f(0) = 0$

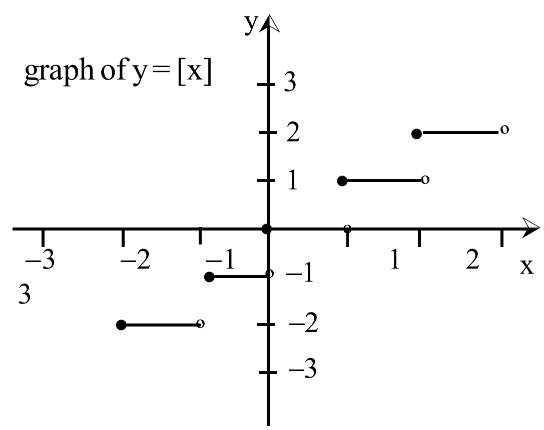
(vii) **GREATEST INTEGER OR STEP UP FUNCTION :**

The function $y = f(x) = [x]$ is called the greatest integer function where [x] denotes the greatest integer less than or equal to x . Note that for :

$-1 \leq x < 0$;	$[x] = -1$	$0 \leq x < 1$;	$[x] = 0$
$1 \leq x < 2$;	$[x] = 1$	$2 \leq x < 3$;	$[x] = 2$

and so on .

Properties of greatest integer function :



(a) $[x] \leq x < [x] + 1$ and $x - 1 < [x] \leq x$, $0 \leq x - [x] < 1$

(b) $[x + m] = [x] + m$ if m is an integer .

(c) $[x] + [y] \leq [x + y] \leq [x] + [y] + 1$

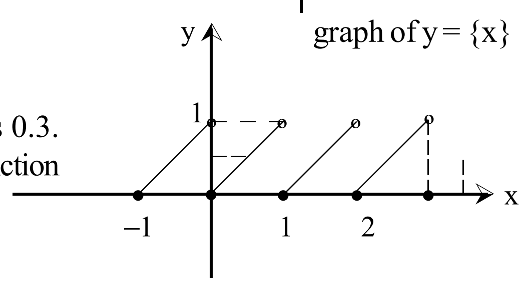
(d) $[x] + [-x] = 0$ if x is an integer
$= -1$ otherwise .

(viii) **FRACTIONAL PART FUNCTION :**

It is defined as :

$g(x) = \{x\} = x - [x]$.

e.g. the fractional part of the no. 2.1 is $2.1 - 2 = 0.1$ and the fractional part of -3.7 is 0.3. The period of this function is 1 and graph of this function is as shown .



4. DOMAINS AND RANGES OF COMMON FUNCTION :

Function ($y = f(x)$)	Domain (i.e. values taken by x)	Range (i.e. values taken by f(x))

A. Algebraic Functions

	Function	Domain	Range
(i)	x^n , $(n \in N)$	R = (set of real numbers)	R, if n is odd $R^+ \cup \{0\}$, if n is even
(ii)	$\frac{1}{x^n}$, $(n \in N)$	$R - \{0\}$	$R - \{0\}$, if n is odd R^+, if n is even
(iii)	$x^{1/n}$, $(n \in N)$	R, if n is odd $R^+ \cup \{0\}$, if n is even	R, if n is odd $R^+ \cup \{0\}$, if n is even
(iv)	$\frac{1}{x^{1/n}}$, $(n \in N)$	$R - \{0\}$, if n is odd R^+, if n is even	$R - \{0\}$, if n is odd R^+, if n is even

B. Trigonometric Functions

	Function	Domain	Range
(i)	sin x	R	$[-1, +1]$
(ii)	cos x	R	$[-1, +1]$
(iii)	tan x	$R - (2k+1)\frac{\pi}{2}, k \in I$	R
(iv)	sec x	$R - (2k+1)\frac{\pi}{2}, k \in I$	$(-\infty, -1] \cup [1, \infty)$
(v)	cosec x	$R - k\pi, k \in I$	$(-\infty, -1] \cup [1, \infty)$
(vi)	cot x	$R - k\pi, k \in I$	R

C. Inverse Circular Functions (Refer after Inverse is taught)

	Function	Domain	Range
(i)	$\sin^{-1} x$	$[-1, +1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
(ii)	$\cos^{-1} x$	$[-1, +1]$	$[0, \pi]$
(iii)	$\tan^{-1} x$	R	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
(iv)	$\text{cosec}^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$
(v)	$\sec^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$[0, \pi] - \left\{\frac{\pi}{2}\right\}$
(vi)	$\cot^{-1} x$	R	$(0, \pi)$

	Function ($y = f(x)$)	Domain (i.e. values taken by x)	Range (i.e. values taken by f(x))
D.	**Exponential Functions**		
(i)	e^x	R	R^+
(ii)	$e^{1/x}$	$R - \{0\}$	$R^+ - \{1\}$
(iii)	a^x, $a > 0$	R	R^+
(iv)	$a^{1/x}$, $a > 0$	$R - \{0\}$	$R^+ - \{1\}$
E.	**Logarithmic Functions**		
(i)	$\log_a x$, $(a > 0)(a \neq 1)$	R^+	R
(ii)	$\log_x a = \frac{1}{\log_a x}$ $(a > 0)(a \neq 1)$	$R^+ - \{1\}$	$R - \{0\}$
F.	**Integral Part Functions Functions**		
(i)	$[x]$	R	I
(ii)	$\frac{1}{[x]}$	$R - [0, 1)$	$\left\{\frac{1}{n}, n \in I - \{0\}\right\}$
G.	**Fractional Part Functions**		
(i)	$\{x\}$	R	$[0, 1)$
(ii)	$\frac{1}{\{x\}}$	$R - I$	$(1, \infty)$
H.	**Modulus Functions**		
(i)	$\|x\|$	R	$R^+ \cup \{0\}$
(ii)	$\frac{1}{\|x\|}$	$R - \{0\}$	R^+
I.	**Signum Function**		
	$\text{sgn}(x) = \frac{\|x\|}{x}, x \neq 0$; $= 0, x = 0$	R	$\{-1, 0, 1\}$
J.	**Constant Function**		
	say $f(x) = c$	R	$\{c\}$

5. EQUAL OR IDENTICAL FUNCTION :

Two functions f & g are said to be equal if :

(i) The domain of f = the domain of g.

(ii) The range of f = the range of g and

(iii) $f(x) = g(x)$, for every x belonging to their common domain. eg.

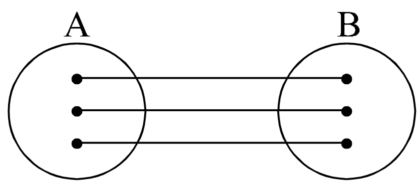
$f(x) = \frac{1}{x}$ & $g(x) = \frac{x}{x^2}$ are identical functions .

6. CLASSIFICATION OF FUNCTIONS :

One – One Function (Injective mapping) :

A function $f: A \rightarrow B$ is said to be a one–one function or injective mapping if different elements of A have different f images in B . Thus for $x_1, x_2 \in A$ & $f(x_1)$, $f(x_2) \in B$, $f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2$ or $x_1 \neq x_2 \Leftrightarrow f(x_1) \neq f(x_2)$.

Diagramatically an injective mapping can be shown as



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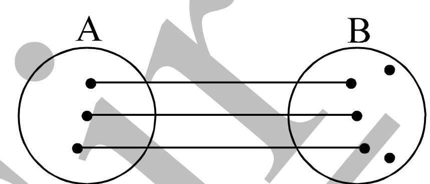


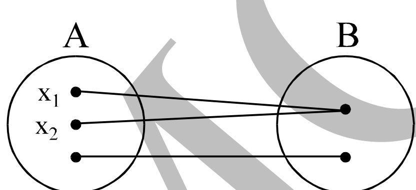
Note : (i) Any function which is entirely increasing or decreasing in whole domain, then f(x) is one–one .

(ii) If any line parallel to x–axis cuts the graph of the function atmost at one point, then the function is one–one .

Many–one function :

A function $f: A \rightarrow B$ is said to be a many one function if two or more elements of A have the same f image in B . Thus $f: A \rightarrow B$ is many one if for ; $x_1, x_2 \in A$, $f(x_1) = f(x_2)$ but $x_1 \neq x_2$.

Diagramatically a many one mapping can be shown as



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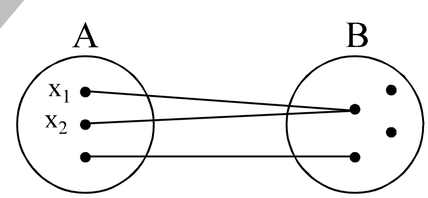


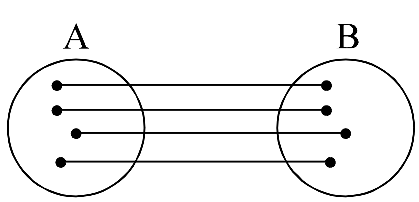
Note : (i) Any continuous function which has atleast one local maximum or local minimum, then f(x) is many–one . In other words, if a line parallel to x–axis cuts the graph of the function atleast at two points, then f is many–one .

(ii) If a function is one–one, it cannot be many–one and vice versa .

Onto function (Surjective mapping) :

If the function $f: A \rightarrow B$ is such that each element in B (co–domain) is the f image of atleast one element in A, then we say that f is a function of A 'onto' B . Thus $f: A \rightarrow B$ is surjective iff $\forall\ b \in B$, $\exists$ some $a \in A$ such that $f(a) = b$.

Diagramatically surjective mapping can be shown as



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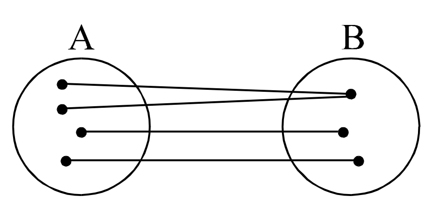
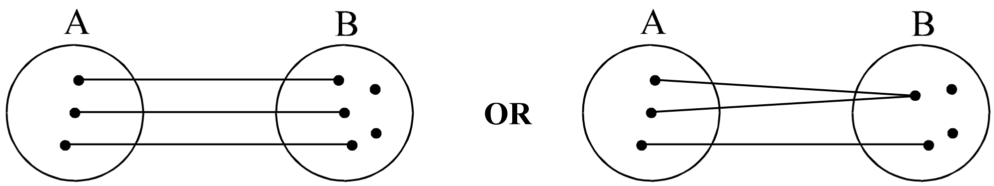


Note that : if range = co–domain, then f(x) is onto.

Into function :
If $f: A \to B$ is such that there exists atleast one element in co–domain which is not the image of any element in domain, then $f(x)$ is into .

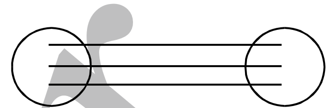
Diagramatically into function can be shown as



Note that : If a function is onto, it cannot be into and vice versa . A polynomial of degree even will always be into.

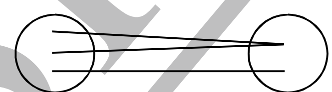
Thus a function can be one of these four types :

(a) one–one onto (injective & surjective)

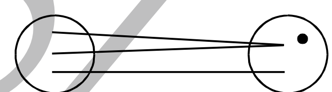


(b) one–one into (injective but not surjective)

(c) many–one onto (surjective but not injective)



(d) many–one into (neither surjective nor injective)



Note : (i) If f is both injective & surjective, then it is called a **Bijective** mapping.
The bijective functions are also named as invertible, non singular or biuniform functions.

(ii) If a set A contains n distinct elements then the number of different functions defined from $A \to A$ is n^n & out of it n ! are one one.

Identity function :
The function $f: A \to A$ defined by $f(x) = x \ \forall \ x \in A$ is called the identity of A and is denoted by I_A. It is easy to observe that identity function is a bijection .

Constant function :
A function $f: A \to B$ is said to be a constant function if every element of A has the same f image in B . Thus $f: A \to B$; $f(x) = c, \ \forall \ x \in A, \ c \in B$ is a constant function. Note that the range of a constant function is a singleton and a constant function may be one-one or many-one, onto or into .

7. ALGEBRAIC OPERATIONS ON FUNCTIONS :

If f & g are real valued functions of x with domain set A, B respectively, then both f & g are defined in $A \cap B$. Now we define $f+g$, $f-g$, $(f.g)$ & (f/g) as follows :

(i) $(f \pm g)(x) = f(x) \pm g(x)$
(ii) $(f.g)(x) = f(x).g(x)$
domain in each case is $A \cap B$

(iii) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ domain is $\{x \mid x \in A \cap B \ \text{s.t} \ g(x) \neq 0\}$.

8. COMPOSITE OF UNIFORMLY & NON-UNIFORMLY DEFINED FUNCTIONS :

Let $f: A \to B$ & $g: B \to C$ be two functions. Then the function $gof: A \to C$ defined by $(gof)(x) = g(f(x)) \ \forall \ x \in A$ is called the composite of the two functions f & g .

Diagramatically $\xrightarrow{x} \boxed{f} \xrightarrow{f(x)} \boxed{g} \longrightarrow g(f(x))$.

Thus the image of every $x \in A$ under the function gof is the g–image of the f–image of x .

Note that gof is defined only if $\forall\ x \in A$, f(x) is an element of the domain of g so that we can take its g-image. Hence for the product gof of two functions f & g, the range of f must be a subset of the domain of g.

Properties Of Composite Functions :

(i) The composite of functions is not commutative i.e. $gof \neq fog$.

(ii) The composite of functions is associative i.e. if f, g, h are three functions such that fo (goh) & (fog) oh are defined, then fo (goh) = (fog) oh .

(iii) The composite of two bijections is a bijection i.e. if f & g are two bijections such that gof is defined, then gof is also a bijection.

9. HOMOGENEOUS FUNCTIONS :

A function is said to be homogeneous with respect to any set of variables when each of its terms is of the same degree with respect to those variables .

For example $5x^2 + 3y^2 - xy$ is homogeneous in x & y . Symbolically if,

$f(tx, ty) = t^n . f(x, y)$ then f(x, y) is homogeneous function of degree n .

10. BOUNDED FUNCTION :

A function is said to be bounded if $|f(x)| \leq M$, where M is a finite quantity .

11. IMPLICIT & EXPLICIT FUNCTION :

A function defined by an equation not solved for the dependent variable is called an **Implicit Function** . For eg. the equation $x^3 + y^3 = 1$ defines y as an implicit function. If y has been expressed in terms of x alone then it is called an **Explicit Function**.

12. INVERSE OF A FUNCTION :

Let $f : A \to B$ be a one–one & onto function, then their exists a unique function $g : B \to A$ such that $f(x) = y \Leftrightarrow g(y) = x,\ \forall\ x \in A\ \&\ y \in B$. Then g is said to be inverse of f . Thus $g = f^{-1} : B \to A = \{(f(x), x)\ |\ (x, f(x)) \in f\}$.

Properties Of Inverse Function :

(i) The inverse of a bijection is unique .

(ii) If $f : A \to B$ is a bijection & $g : B \to A$ is the inverse of f, then $fog = I_B$ and $gof = I_A$, where I_A & I_B are identity functions on the sets A & B respectively.
Note that the graphs of f & g are the mirror images of each other in the line $y = x$. As shown in the figure given below a point (x ',y ') corresponding to $y = x^2\ (x \geq 0)$ changes to (y ',x ') corresponding to $y = +\sqrt{x}$, the changed form of $x = \sqrt{y}$.

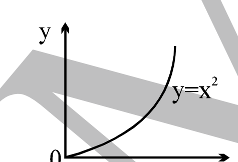


fig. 1

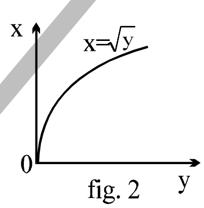


fig. 2

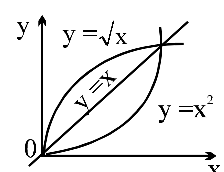


fig. 3

(iii) The inverse of a bijection is also a bijection .

(iv) If f & g are two bijections $f : A \to B$, $g : B \to C$ then the inverse of gof exists and $(gof)^{-1} = f^{-1} o\ g^{-1}$.

13. ODD & EVEN FUNCTIONS :

If $f(-x) = f(x)$ for all x in the domain of 'f' then f is said to be an even function.
e.g. $f(x) = \cos x$; $g(x) = x^2 + 3$.

If $f(-x) = -f(x)$ for all x in the domain of 'f' then f is said to be an odd function.
e.g. $f(x) = \sin x$; $g(x) = x^3 + x$.

Note : (a) $f(x) - f(-x) = 0 \Rightarrow f(x)$ is even & $f(x) + f(-x) = 0 \Rightarrow f(x)$ is odd .

(b) A function may neither be odd nor even .

(c) Inverse of an even function is not defined .

(d) Every even function is symmetric about the y–axis & every odd function is symmetric about the origin.

(e) Every function can be expressed as the sum of an even & an odd function.

e.g. $f(x) = \underbrace{\dfrac{f(x)+f(-x)}{2}}_{\text{EVEN}} + \underbrace{\dfrac{f(x)-f(-x)}{2}}_{\text{ODD}}$

(f) The only function which is defined on the entire number line & is even and odd at the same time is $f(x) = 0$.

(g) If f and g both are even or both are odd then the function f.g will be even but if any one of them is odd then f.g will be odd.

14. PERIODIC FUNCTION :

A function f(x) is called periodic if there exists a positive number T (T > 0) called the period of the function such that $f(x+T) = f(x)$, for all values of x within the domain of x.

e.g. The function sin x & cos x both are periodic over 2π & tan x is periodic over π.

NOTE : (a) $f(T) = f(0) = f(-T)$, where 'T' is the period.

(b) Inverse of a periodic function does not exist.

(c) Every constant function is always periodic, with no fundamental period.

(d) If f(x) has a period T & g(x) also has a period T then it does not mean that $f(x)+g(x)$ must have a period T. e.g. $f(x) = |\sin x| + |\cos x|$.

(e) If f(x) has a period p, then $\dfrac{1}{f(x)}$ and $\sqrt{f(x)}$ also has a period p.

(f) if f(x) has a period T then f(ax + b) has a period T/a (a > 0).

15. GENERAL :

If x, y are independent variables, then :

(i) $f(xy) = f(x) + f(y) \Rightarrow f(x) = k \ln x$ or $f(x) = 0$.

(ii) $f(xy) = f(x) \cdot f(y) \Rightarrow f(x) = x^n$, $n \in R$

(iii) $f(x+y) = f(x) \cdot f(y) \Rightarrow f(x) = a^{kx}$.

(iv) $f(x+y) = f(x) + f(y) \Rightarrow f(x) = kx$, where k is a constant.

EXERCISE–I

Q.1 Find the domains of definitions of the following functions :
(Read the symbols [*] and {*} as greatest integers and fractional part functions respectively.)

(i) $f(x) = \sqrt{\cos 2x} + \sqrt{16 - x^2}$

(ii) $f(x) = \log_7 \log_5 \log_3 \log_2 (2x^3 + 5x^2 - 14x)$

(iii) $f(x) = \ln\left(\sqrt{x^2 - 5x - 24} - x - 2\right)$

(iv) $f(x) = \sqrt{\dfrac{1-5^x}{7^{-x}-7}}$

(v) $y = \log_{10} \sin(x-3) + \sqrt{16 - x^2}$

(vi) $f(x) = \log_{100x}\left(\dfrac{2\log_{10} x + 1}{-x}\right)$

(vii) $f(x) = \dfrac{1}{\sqrt{4x^2 - 1}} + \ln x(x^2 - 1)$

(viii) $f(x) = \sqrt{\log_{\frac{1}{2}} \dfrac{x}{x^2 - 1}}$

(ix) $f(x) = \sqrt{x^2 - |x|} + \dfrac{1}{\sqrt{9 - x^2}}$

(x) $f(x) = \sqrt{(x^2 - 3x - 10) \cdot \ln^2(x-3)}$

(xi) $f(x) = \sqrt{\log_x(\cos 2\pi x)}$

(xii) $f(x) = \dfrac{\sqrt{\cos x - (1/2)}}{\sqrt{6 + 35x - 6x^2}}$

(xiii) $f(x)=\sqrt{\log_{1/3}\left(\log_4\left([x]^2-5\right)\right)}$ (xiv) $f(x)=\dfrac{[x]}{2x-[x]}$ (xv) $f(x)=\log_x \sin x$

(xvi) $f(x)=\log_2\left(-\log_{1/2}\left(1+\dfrac{1}{\sin\left(\frac{x^\circ}{100}\right)}\right)\right)+\sqrt{\log_{10}\left(\log_{10}x\right)-\log_{10}\left(4-\log_{10}x\right)-\log_{10}3}$

(xvii) $f(x)=\dfrac{1}{[x]}+\log_{1-\{x\}}(x^2-3x+10)+\dfrac{1}{\sqrt{2-|x|}}+\dfrac{1}{\sqrt{\sec(\sin x)}}$

(xviii) $f(x)=\sqrt{(5x-6-x^2)\,[\{ln\{x\}\}]}+\sqrt{(7x-5-2x^2)}+\left(ln\left(\dfrac{7}{2}-x\right)\right)^{-1}$

(xix) $f(x)=\log_{\left[x+\frac{1}{x}\right]}\left|x^2-x-6\right|+{}^{16-x}C_{2x-1}+{}^{20-3x}P_{2x-5}$

(xx) $f(x)=\log_{10}\left(\log_{|\sin x|}(x^2-8x+23)-\dfrac{3}{\log_2|\sin x|}\right)$

Q.2 Find the domain & range of the following functions.
(Read the symbols [*] and {*} as greatest integers and fractional part functions respectively.)

(i) $y=\log_{\sqrt{5}}\left(\sqrt{2}(\sin x-\cos x)+3\right)$ **(ii)** $y=\dfrac{2x}{1+x^2}$ **(iii)** $f(x)=\dfrac{x^2-3x+2}{x^2+x-6}$

(iv) $f(x)=\dfrac{x}{1+|x|}$ **(v)** $y=\sqrt{2-x}+\sqrt{1+x}$

(vi) $f(x)=\log_{(\text{cosec}\,x-1)}(2-[\sin x]-[\sin x]^2)$ **(vii)** $f(x)=\dfrac{\sqrt{x+4}-3}{x-5}$

Q.3(a) Draw graphs of the following function, where [ ] denotes the greatest integer function.

(i) $f(x)=x+[x]$ **(ii)** $y=(x)^{[x]}$ where $x=[x]+(x)$ & $x>0$ & $x\le 3$

(iii) $y=\text{sgn}\,[x]$ **(iv)** $\text{sgn}\,(x-|x|)$

(b) Identify the pair(s) of functions which are identical?
(where [x] denotes greatest integer and {x} denotes fractional part function)

(i) $f(x)=\text{sgn}\,(x^2-3x+4)$ and $g(x)=e^{[\{x\}]}$ **(ii)** $f(x)=\sqrt{\dfrac{1-\cos 2x}{1+\cos 2x}}$ and $g(x)=\tan x$

(iii) $f(x)=ln(1+x)+ln(1-x)$ and $g(x)=ln(1-x^2)$ **(iv)** $f(x)=\dfrac{\cos x}{1-\sin x}$ and $g(x)=\dfrac{1+\sin x}{\cos x}$

Q.4 Classify the following functions f(x) definzed in $R\to R$ as injective, surjective, both or none .

(a) $f(x)=\dfrac{x^2+4x+30}{x^2-8x+18}$ **(b)** $f(x)=x^3-6x^2+11x-6$ **(c)** $f(x)=(x^2+x+5)(x^2+x-3)$

Q.5 Solve the following problems from **(a)** to **(e)** on functional equation.

(a) The function $f(x)$ defined on the real numbers has the property that $f\left(f(x)\right)\cdot\left(1+f(x)\right)=-f(x)$ for all x in the domain of f. If the number 3 is in the domain and range of f, compute the value of $f(3)$.

(b) Suppose f is a real function satisfying $f(x+f(x))=4f(x)$ and $f(1)=4$. Find the value of $f(21)$.

(c) Let 'f' be a function defined from $R^+\to R^+$. If $[f(xy)]^2=x\left(f(y)\right)^2$ for all positive numbers x and y and $f(2)=6$, find the value of $f(50)$.

(d) Let $f(x)$ be a function with two properties
(i) for any two real number x and y, $f(x+y) = x + f(y)$ and **(ii)** $f(0) = 2$.
Find the value of $f(100)$.

(e) Let f be a function such that $f(3) = 1$ and $f(3x) = x + f(3x-3)$ for all x. Then find the value of $f(300)$.

(f) Suppose that f (x) is a function of the form $f(x) = \dfrac{ax^8 + bx^6 + cx^4 + dx^2 + 15x + 1}{x}$ $(x \neq 0)$. If $f(5) = 2$ then find the value of $f(-5)$.

Q.6 Suppose $f(x) = \sin x$ and $g(x) = 1 - \sqrt{x}$. Then find the domain and range of the following functions.
(a) fog (b) gof (c) fof (d) gog

Q.7 If $f(x) = \sin^2 x + \sin^2\left(x + \frac{\pi}{3}\right) + \cos x \cos\left(x + \frac{\pi}{3}\right)$ and $g\left(\frac{5}{4}\right) = 1$, then find (gof) (x).

Q.8 A function $f : R \to R$ is such that $f\left(\dfrac{1-x}{1+x}\right) = x$ for all $x \neq -1$. Prove the following.
(a) $f(f(x)) = x$ (b) $f(1/x) = -f(x),\ x \neq 0$ (c) $f(-x-2) = -f(x) - 2$.

Q.9(a) Find the formula for the function fogoh, given $f(x) = \dfrac{x}{x+1}$; $g(x) = x^{10}$ and $h(x) = x + 3$. Find also the domain of this function. Also compute (fogoh)(–1).
(b) Given $F(x) = \cos^2(x+9)$. Find the function f, g, h such that F = fogoh.

Q.10 If $f(x) = \max(x, 1/x)$ for $x > 0$ where max (a, b) denotes the greater of the two real numbers a and b. Define the function $g(x) = f(x) \cdot f(1/x)$ and plot its graph.

Q.11(a) The function $f(x)$ has the property that for each real number x in its domain, $1/x$ is also in its domain and $f(x) + f(1/x) = x$. Find the largest set of real numbers that can be in the domain of $f(x)$?
(b) Let $f(x) = \sqrt{ax^2 + bx}$. Find the set of real values of 'a' for which there is at least one positive real value of 'b' for which the domain of f and the range of f are the same set.

Q.12 $f(x) = \begin{cases} 1-x & \text{if } x \le 0 \\ x^2 & \text{if } x > 0 \end{cases}$ and $g(x) = \begin{cases} -x & \text{if } x < 1 \\ 1-x & \text{if } x \ge 1 \end{cases}$ find (fog)(x) and (gof)(x)

Q.13 Find whether the following functions are even or odd or none
(a) $f(x) = \log\left(x + \sqrt{1+x^2}\right)$ (b) $f(x) = \dfrac{x(a^x+1)}{a^x - 1}$ (c) $f(x) = \sin x + \cos x$
(d) $f(x) = x \sin^2 x - x^3$ (e) $f(x) = \sin x - \cos x$ (f) $f(x) = \dfrac{(1+2^x)^2}{2^x}$
(g) $f(x) = \dfrac{x}{e^x - 1} + \dfrac{x}{2} + 1$ (h) $f(x) = [(x+1)^2]^{1/3} + [(x-1)^2]^{1/3}$

Q.14(i) Write explicitly, functions of y defined by the following equations and also find the domains of definition of the given implicit functions :
(a) $10^x + 10^y = 10$ **(b)** $x + |y| = 2y$
(ii) The function f(x) is defined on the interval [0,1]. Find the domain of definition of the functions.
(a) $f(\sin x)$ **(b)** $f(2x+3)$
(iii) Given that $y = f(x)$ is a function whose domain is [4, 7] and range is [–1, 9]. Find the range and domain of
(a) $g(x) = \frac{1}{3} f(x)$ **(b)** $h(x) = f(x-7)$

Q.15 Compute the inverse of the functions:

(a) $f(x) = ln\left(x + \sqrt{x^2+1}\right)$ **(b)** $f(x) = 2^{\frac{x}{x-1}}$ **(c)** $y = \dfrac{10^x - 10^{-x}}{10^x + 10^{-x}}$

Q.16 Find the inverse of $f(x) = 2^{\log_{10} x} + 8$ and hence solve the equation $f(x) = f^{-1}(x)$.

Q.17 Function f & g are defined by $f(x) = \sin x, x \in R$; $g(x) = \tan x, x \in R - \left(K + \frac{1}{2}\right)\pi$

where $K \in I$. Find **(i)** periods of fog & gof. **(ii)** range of the function fog & gof.

Q.18(a) Suppose that f is an even, periodic function with period 2, and that $f(x) = x$ for all x in the interval [0, 1]. Find the value of f (3.14).

(b) Find out for what integral values of n the number 3π is a period of the function :

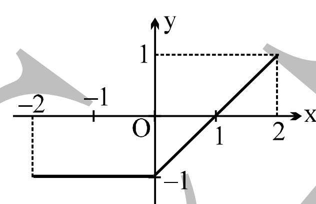
$f(x) = \cos nx \,.\, \sin(5/n)x.$

Q.19 Let $f(x) = ln\, x$ and $g(x) = x^2 - 1$

Column-I contains composite functions and column-II contains their domain. Match the entries of column-I with their corresponding answer is column-II.

Column-I		Column-II	
(A)	fog	(P)	$(1, \infty)$
(B)	gof	(Q)	$(-\infty, \infty)$
(C)	fof	(R)	$(-\infty, -1) \cup (1, \infty)$
(D)	gog	(S)	$(0, \infty)$

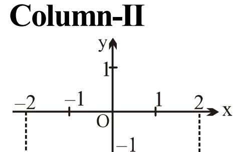
Q.20 The graph of the function $y = f(x)$ is as follows.



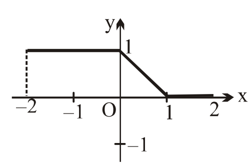
Match the ***function*** mentioned in **Column-I** with the respective ***graph*** given in **Column-II**.

Column-I **Column-II**

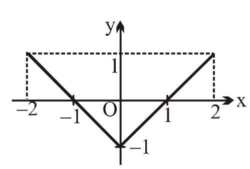
(A) $y = |f(x)|$ (P)



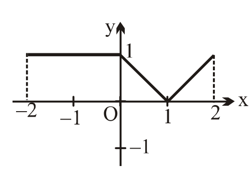
(B) $y = f(|x|)$ (Q)



(C) $y = f(-|x|)$ (R)



(D) $y = \frac{1}{2}(|f(x)| - f(x))$ (S)



EXERCISE–II

Q.1 Let f be a one–one function with domain {x,y,z} and range {1,2,3}. It is given that exactly one of the following statements is true and the remaining two are false.
$f(x) = 1$; $f(y) \neq 1$; $f(z) \neq 2$. Determine $f^{-1}(1)$

Q.2 Let $x = \log_4 9 + \log_9 28$
show that [x] = 3, where [x] denotes the greatest integer less than or equal to x.

Q.3(a) A function f is defined for all positive integers and satisfies $f(1) = 2005$ and $f(1)+f(2)+ ... + f(n) = n^2 f(n)$ for all n > 1. Find the value of $f(2004)$.

(b) If a, b are positive real numbers such that a – b = 2, then find the smallest value of the constant L for which $\sqrt{x^2 + ax} - \sqrt{x^2 + bx} < L$ for all x > 0.

(c) Let $f(x) = x^2 + kx$; k is a real number. The set of values of k for which the equation f (x) = 0 and $f(f(x)) = 0$ have same real solution set.

(d) Let $P(x) = x^6 + ax^5 + bx^4 + cx^3 + dx^2 + ex + f$ be a polynomial such that P(1) = 1 ; P(2) = 2 ; P(3) = 3; P(4) = 4; P(5) = 5 and P(6) = 6 then find the value of P(7).

(e) Let *a* and *b* be real numbers and let $f(x) = a \sin x + b\sqrt[3]{x} + 4, \forall x \in R$. If $f(\log_{10}(\log_3 10)) = 5$ then find the value of $f(\log_{10}(\log_{10} 3))$.

Q.4 **Column I** contains functions and **column II** contains their natural domains. Exactly one entry of **column II** matches with exactly one entry of **column I**.

Column I	Column II
(A) $f(x) = \sin^{-1}\left(\dfrac{x+1}{x}\right)$	(P) $(1, 3) \cup (3, \infty)$
(B) $g(x) = \sqrt{ln\left(\dfrac{x^2+3x-2}{x+1}\right)}$	(Q) $(-\infty, 2)$
(C) $h(x) = \dfrac{1}{ln\left(\dfrac{x-1}{2}\right)}$	(R) $\left(-\infty, -\dfrac{1}{2}\right]$
(D) $\phi(x) = ln\left(\sqrt{x^2+12} - 2x\right)$	(S) $[-3, -1) \cup [1, \infty)$

Q.5 Let [x] = the greatest integer less than or equal to x. If all the values of x such that the product $\left[x - \frac{1}{2}\right]\left[x + \frac{1}{2}\right]$ is prime, belongs to the set $[x_1, x_2) \cup [x_3, x_4)$, find the value of $x_1^2 + x_2^2 + x_3^2 + x_4^2$.

Q.6 Suppose p(x) is a polynomial with integer coefficients. The remainder when p(x) is divided by x – 1 is 1 and the remainder when p(x) is divided by x – 4 is 10. If r (x) is the remainder when p(x) is divided by (x – 1)(x – 4), find the value of r (2006).

Q.7 Prove that the function defined as , $f(x) = \begin{cases} e^{-\sqrt{|ln\{x\}|}} - \{x\}^{\sqrt{\frac{1}{|ln\{x\}|}}} & \text{where ever it exists} \\ \{x\} & \text{otherwise, then} \end{cases}$
f (x) is odd as well as even. (where {x} denotes the fractional part function)

Q.8 In a function $2f(x) + xf\left(\frac{1}{x}\right) - 2f\left(\left|\sqrt{2}\sin\left(\pi\left(x + \frac{1}{4}\right)\right)\right|\right) = 4\cos^2\frac{\pi x}{2} + x\cos\frac{\pi}{x}$

Prove that **(i)** f(2) + f(1/2) = 1 and **(ii)** f(2) + f(1) = 0

Q.9 A function f, defined for all $x, y \in R$ is such that $f(1) = 2$; $f(2) = 8$ & $f(x+y) - kxy = f(x) + 2y^2$, where k is some constant. Find f(x) & show that :

$$f(x+y)\, f\left(\frac{1}{x+y}\right) = k \text{ for } x + y \neq 0.$$

Q.10 Let $f: R \to R - \{3\}$ be a function with the property that there exist $T > 0$ such that $f(x+T) = \dfrac{f(x)-5}{f(x)-3}$ for every $x \in R$. Prove that $f(x)$ is periodic.

Q.11 If $f(x) = -1 + |x-2|$, $0 \le x \le 4$
$g(x) = 2 - |x|$, $-1 \le x \le 3$
Then find fog(x) & gof(x). Draw rough sketch of the graphs of fog(x) & gof(x).

Q.12 Let $f(x) = x^{135} + x^{125} - x^{115} + x^5 + 1$. If $f(x)$ is divided by $x^3 - x$ then the remainder is some function of x say g(x). Find the value of g(10).

Q.13 Let {x} & [x] denote the fractional and integral part of a real number x respectively. Solve $4\{x\} = x + [x]$

Q.14 Let $f(x) = \dfrac{9^x}{9^x + 3}$ then find the value of the sum $f\left(\frac{1}{2006}\right) + f\left(\frac{2}{2006}\right) + f\left(\frac{3}{2006}\right) + \ldots + f\left(\frac{2005}{2006}\right)$

Q.15 Let $f(x) = (x+1)(x+2)(x+3)(x+4) + 5$ where $x \in [-6, 6]$. If the range of the function is [a, b] where $a, b \in N$ then find the value of (a + b).

Q.16 Find a formula for a function g(x) satisfying the following conditions
(a) domain of g is $(-\infty, \infty)$ **(b)** range of g is [–2, 8]
(c) g has a period π and **(d)** g(2) = 3

Q.17 The set of real values of 'x' satisfying the equality $\left[\frac{3}{x}\right] + \left[\frac{4}{x}\right] = 5$ (where [ ] denotes the greatest integer function) belongs to the interval $(a, b/c]$ where $a, b, c \in N$ and b/c is in its lowest form. Find the value of $a + b + c + abc$.

Q.18 $f(x)$ and $g(x)$ are linear function such that for all x, $f(g(x))$ and $g(f(x))$ are Identity functions. If $f(0) = 4$ and $g(5) = 17$, compute f (2006).

Q.19 A is a point on the circumference of a circle. Chords AB and AC divide the area of the circle into three equal parts. If the angle BAC is the root of the equation, $f(x) = 0$ then find $f(x)$.

Q.20 If for all real values of u & v, $2f(u)\cos v = f(u+v) + f(u-v)$, prove that, for all real values of x.
(i) $f(x) + f(-x) = 2a\cos x$ **(ii)** $f(\pi - x) + f(-x) = 0$
(iii) $f(\pi - x) + f(x) = -2b\sin x$. Deduce that $f(x) = a\cos x - b\sin x$, a, b are arbitrary constants.

EXERCISE–III

Q.1 If the function $f: [1, \infty) \to [1, \infty)$ is defined by $f(x) = 2^{x(x-1)}$, then $f^{-1}(x)$ is [JEE '99, 2]

(A) $\left(\frac{1}{2}\right)^{x(x-1)}$ (B) $\frac{1}{2}\left(1 + \sqrt{1 + 4\log_2 x}\right)$ (C) $\frac{1}{2}\left(1 - \sqrt{1 + 4\log_2 x}\right)$ (D) not defined

Q.2 The domain of definition of the function, y(x) given by the equation, $2^x + 2^y = 2$ is
(A) $0 < x \le 1$ (B) $0 \le x \le 1$ (C) $-\infty < x \le 0$ (D) $-\infty < x < 1$
[JEE 2000 (Scr.), 1 out of 35]

Q.3 Given $X = \{1, 2, 3, 4\}$, find all one–one, onto mappings, $f: X \to X$ such that, $f(1) = 1$, $f(2) \neq 2$ and $f(4) \neq 4$. [REE 2000, 3 out of 100]

Q.4(a) Let $g(x) = 1 + x - [x]$ & $f(x) = \begin{cases} -1 & , \ x < 0 \\ 0 & , \ x = 0 \\ 1 & , \ x > 0 \end{cases}$. Then for all x, $f(g(x))$ is equal to

(A) x (B) 1 (C) $f(x)$ (D) $g(x)$

where [ ] denotes the greatest integer function.

(b) If $f : [1, \infty) \to [2, \infty)$ is given by, $f(x) = x + \frac{1}{x}$, then $f^{-1}(x)$ equals

(A) $\frac{x + \sqrt{x^2 - 4}}{2}$ (B) $\frac{x}{1 + x^2}$ (C) $\frac{x - \sqrt{x^2 - 4}}{2}$ (D) $1 - \sqrt{x^2 - 4}$

(c) The domain of definition of $f(x) = \frac{\log_2 (x + 3)}{x^2 + 3x + 2}$ is :

(A) $R \setminus \{-1, -2\}$ (B) $(-2, \infty)$ (C) $R \setminus \{-1, -2, -3\}$ (D) $(-3, \infty) \setminus \{-1, -2\}$

(d) Let $E = \{1, 2, 3, 4\}$ & $F = \{1, 2\}$. Then the number of onto functions from E to F is

(A) 14 (B) 16 (C) 12 (D) 8

(e) Let $f(x) = \frac{\alpha x}{x + 1}$, $x \neq -1$. Then for what value of α is $f(f(x)) = x$?

(A) $\sqrt{2}$ (B) $-\sqrt{2}$ (C) 1 (D) -1.

[JEE 2001 (Screening) $5 \times 1 = 5$]

Q.5(a) Suppose $f(x) = (x + 1)^2$ for $x \geq -1$. If $g(x)$ is the function whose graph is the reflection of the graph of $f(x)$ with respect to the line $y = x$, then $g(x)$ equals

(A) $-\sqrt{x} - 1, x \geq 0$ (B) $\frac{1}{(x+1)^2}, x \geq -1$ (C) $\sqrt{x+1}, x \geq -1$ (D) $\sqrt{x} - 1, x \geq 0$

(b) Let function $f : R \to R$ be defined by $f(x) = 2x + \sin x$ for $x \in R$. Then f is

(A) one to one and onto
(B) one to one but NOT onto
(C) onto but NOT one to one
(D) neither one to one nor onto

[JEE 2002 (Screening), 3 + 3]

Q.6(a) Range of the function $f(x) = \frac{x^2 + x + 2}{x^2 + x + 1}$ is

(A) $[1, 2]$ (B) $[1, \infty)$ (C) $\left[2, \frac{7}{3}\right]$ (D) $\left(1, \frac{7}{3}\right]$

(b) Let $f(x) = \frac{x}{1 + x}$ defined from $(0, \infty) \to [0, \infty)$ then by $f(x)$ is

(A) one- one but not onto
(B) one- one and onto
(C) Many one but not onto
(D) Many one and onto

[JEE 2003 (Scr),3+3]

Q.7 Let $f(x) = \sin x + \cos x$, $g(x) = x^2 - 1$. Thus $g(f(x))$ is invertible for $x \in$

(A) $\left[-\frac{\pi}{2}, 0\right]$ (B) $\left[-\frac{\pi}{2}, \pi\right]$ (C) $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ (D) $\left[0, \frac{\pi}{2}\right]$

[JEE 2004 (Screening)]

Q.8 If the functions $f(x)$ and $g(x)$ are defined on $R \to R$ such that

$$f(x) = \begin{cases} 0, & x \in \text{rational} \\ x, & x \in \text{irrational} \end{cases}, \quad g(x) = \begin{cases} 0, & x \in \text{irrational} \\ x, & x \in \text{rational} \end{cases}$$

then $(f - g)(x)$ is

(A) one-one and onto
(B) neither one-one nor onto
(C) one-one but not onto
(D) onto but not one-one

[JEE 2005 (Scr.)]

KEY CONCEPTS (INVERSE TRIGONOMETRY FUNCTION)

GENERAL DEFINITION(S):

1. $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$ etc. denote angles or real numbers whose sine is x , whose cosine is x and whose tangent is x, provided that the answers given are numerically smallest available . These are also written as arc sinx , arc cosx etc .

If there are two angles one positive & the other negative having same numerical value, then positive angle should be taken .

2. PRINCIPAL VALUES AND DOMAINS OF INVERSE CIRCULAR FUNCTIONS :

(i) $y = \sin^{-1}x$ where $-1 \le x \le 1$; $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ and $\sin y = x$.

(ii) $y = \cos^{-1}x$ where $-1 \le x \le 1$; $0 \le y \le \pi$ and $\cos y = x$.

(iii) $y = \tan^{-1}x$ where $x \in R$; $-\frac{\pi}{2} < x < \frac{\pi}{2}$ and $\tan y = x$.

(iv) $y = \text{cosec}^{-1}x$ where $x \le -1$ or $x \ge 1$; $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, $y \ne 0$ and $\text{cosec}\, y = x$.

(v) $y = \sec^{-1}x$ where $x \le -1$ or $x \ge 1$; $0 \le y \le \pi$; $y \ne \frac{\pi}{2}$ and $\sec y = x$.

(vi) $y = \cot^{-1}x$ where $x \in R$, $0 < y < \pi$ and $\cot y = x$.

NOTE THAT :
(a) 1st quadrant is common to all the inverse functions .
(b) 3rd quadrant is **not used** in inverse functions .
(c) 4th quadrant is used in the **CLOCKWISE DIRECTION** i.e. $-\frac{\pi}{2} \le y \le 0$.

3. PROPERTIES OF INVERSE CIRCULAR FUNCTIONS :

P–1
(i) $\sin(\sin^{-1}x) = x$, $-1 \le x \le 1$
(ii) $\cos(\cos^{-1}x) = x$, $-1 \le x \le 1$
(iii) $\tan(\tan^{-1}x) = x$, $x \in R$
(iv) $\sin^{-1}(\sin x) = x$, $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$
(v) $\cos^{-1}(\cos x) = x$; $0 \le x \le \pi$
(vi) $\tan^{-1}(\tan x) = x$; $-\frac{\pi}{2} < x < \frac{\pi}{2}$

P–2
(i) $\text{cosec}^{-1}x = \sin^{-1}\frac{1}{x}$; $x \le -1$, $x \ge 1$

(ii) $\sec^{-1}x = \cos^{-1}\frac{1}{x}$; $x \le -1$, $x \ge 1$

(iii) $\cot^{-1}x = \tan^{-1}\frac{1}{x}$; $x > 0$

$= \pi + \tan^{-1}\frac{1}{x}$; $x < 0$

P–3
(i) $\sin^{-1}(-x) = -\sin^{-1}x$, $-1 \le x \le 1$
(ii) $\tan^{-1}(-x) = -\tan^{-1}x$, $x \in R$
(iii) $\cos^{-1}(-x) = \pi - \cos^{-1}x$, $-1 \le x \le 1$
(iv) $\cot^{-1}(-x) = \pi - \cot^{-1}x$, $x \in R$

P–4
(i) $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$ $-1 \le x \le 1$
(ii) $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$ $x \in R$
(iii) $\text{cosec}^{-1}x + \sec^{-1}x = \frac{\pi}{2}$ $|x| \ge 1$

P–5 $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\dfrac{x+y}{1-xy}$ where $x > 0$, $y > 0$ & $xy < 1$

$= \pi + \tan^{-1}\dfrac{x+y}{1-xy}$ where $x > 0$, $y > 0$ & $xy > 1$

$\tan^{-1}x - \tan^{-1}y = \tan^{-1}\dfrac{x-y}{1+xy}$ where $x > 0$, $y > 0$

P–6 **(i)** $\sin^{-1}x + \sin^{-1}y = \sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right]$ where $x \ge 0$, $y \ge 0$ & $(x^2 + y^2) \le 1$

Note that : $x^2 + y^2 \le 1 \Rightarrow 0 \le \sin^{-1}x + \sin^{-1}y \le \dfrac{\pi}{2}$

(ii) $\sin^{-1}x + \sin^{-1}y = \pi - \sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right]$ where $x \ge 0$, $y \ge 0$ & $x^2 + y^2 > 1$

Note that : $x^2 + y^2 > 1 \Rightarrow \dfrac{\pi}{2} < \sin^{-1}x + \sin^{-1}y < \pi$

(iii) $\sin^{-1}x - \sin^{-1}y = \sin^{-1}\left[x\sqrt{1-y^2} - y\sqrt{1-x^2}\right]$ where $x \ge 0$, $y \ge 0$

(iv) $\cos^{-1}x \pm \cos^{-1}y = \cos^{-1}\left[xy \mp \sqrt{1-x^2}\sqrt{1-y^2}\right]$ where $x \ge 0$, $y \ge 0$

P–7 If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \tan^{-1}\left[\dfrac{x+y+z-xyz}{1-xy-yz-zx}\right]$ if, $x > 0, y > 0, z > 0$ & $xy + yz + zx < 1$

Note : (i) If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi$ then $x + y + z = xyz$

(ii) If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \dfrac{\pi}{2}$ then $xy + yz + zx = 1$

P–8 $2\tan^{-1}x = \sin^{-1}\dfrac{2x}{1+x^2} = \cos^{-1}\dfrac{1-x^2}{1+x^2} = \tan^{-1}\dfrac{2x}{1-x^2}$

Note very carefully that :

$$\sin^{-1}\frac{2x}{1+x^2} = \begin{cases} 2\tan^{-1}x & \text{if } |x| \le 1 \\ \pi - 2\tan^{-1}x & \text{if } x > 1 \\ -(\pi + 2\tan^{-1}x) & \text{if } x < -1 \end{cases}$$

$$\cos^{-1}\frac{1-x^2}{1+x^2} = \begin{cases} 2\tan^{-1}x & \text{if } x \ge 0 \\ -2\tan^{-1}x & \text{if } x < 0 \end{cases}$$

$$\tan^{-1}\frac{2x}{1-x^2} = \begin{cases} 2\tan^{-1}x & \text{if } |x| < 1 \\ \pi + 2\tan^{-1}x & \text{if } x < -1 \\ -(\pi - 2\tan^{-1}x) & \text{if } x > 1 \end{cases}$$

REMEMBER THAT :

(i) $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \dfrac{3\pi}{2} \Rightarrow x = y = z = 1$

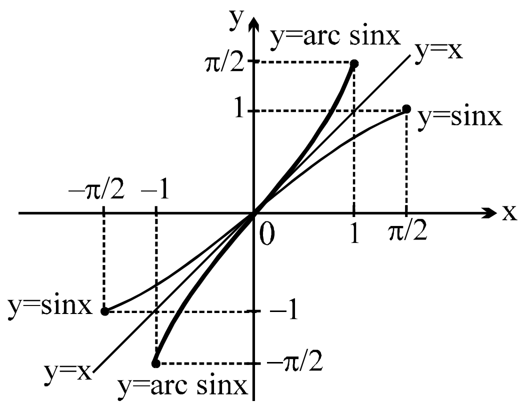
(ii) $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = 3\pi \Rightarrow x = y = z = -1$

(iii) $\tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3 = \pi$ and $\tan^{-1}1 + \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = \frac{\pi}{2}$

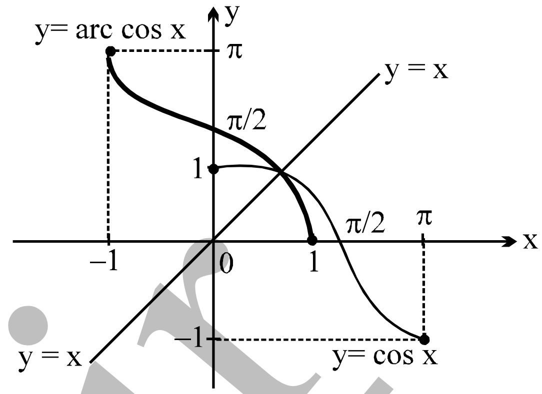
INVERSE TRIGONOMETRIC FUNCTIONS

Some Useful Graphs

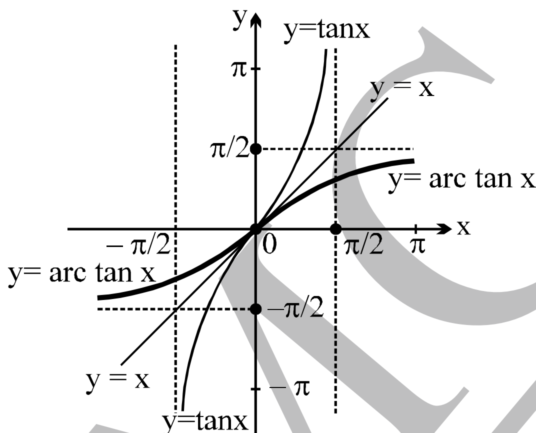
1. $y = \sin^{-1} x$, $|x| \le 1$, $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



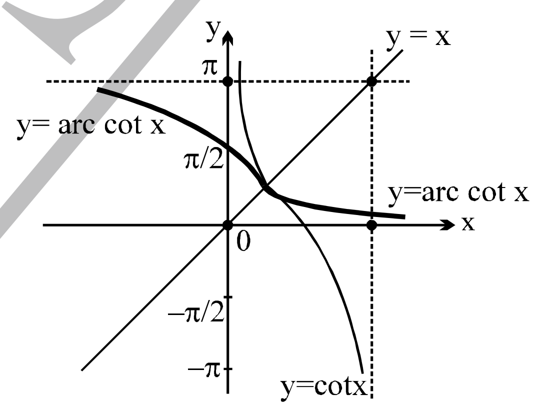
2. $y = \cos^{-1} x$, $|x| \le 1$, $y \in [0, \pi]$



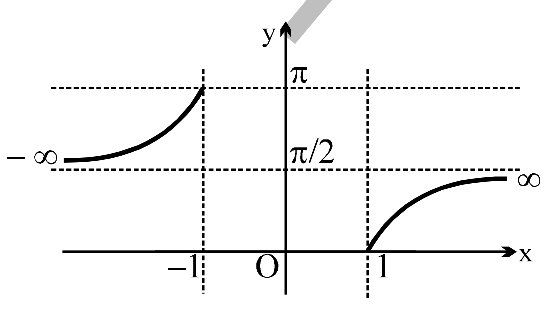
3. $y = \tan^{-1} x$, $x \in R$, $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



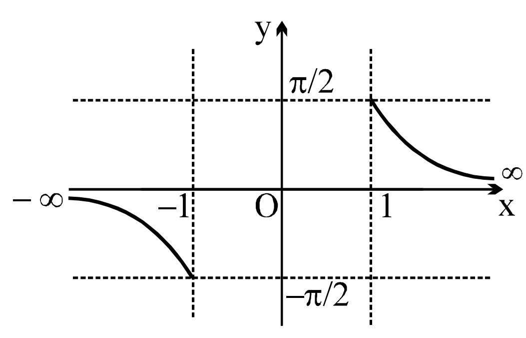
4. $y = \cot^{-1} x$, $x \in R$, $y \in (0, \pi)$



5. $y = \sec^{-1} x$, $|x| \ge 1$, $y \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$



6. $y = \text{cosec}^{-1} x$, $|x| \ge 1$, $y \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$



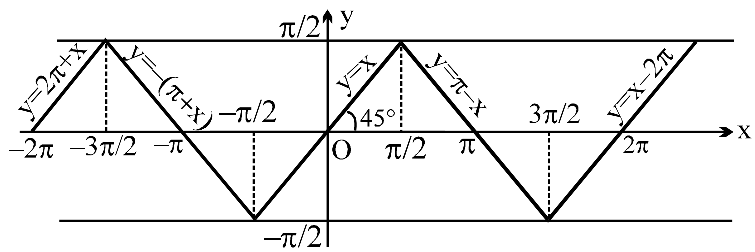
7. (a) $y = \sin^{-1}(\sin x)$, $x \in R$, $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$,

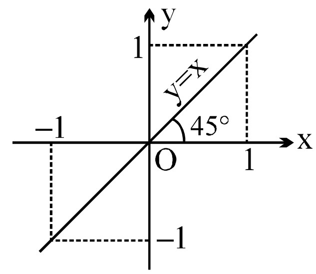
Periodic with period 2π

7.(b) $y = \sin(\sin^{-1} x)$,

$= x$

$x \in [-1, 1]$, $y \in [-1, 1]$, y is aperiodic





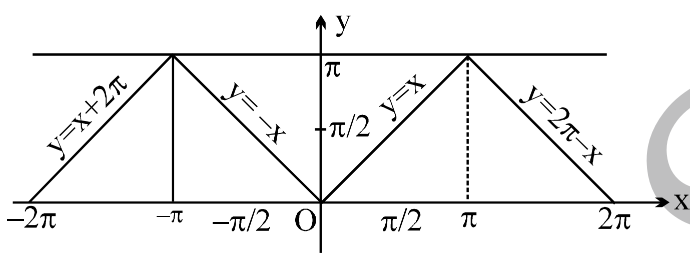
8. (a) $y = \cos^{-1}(\cos x)$, $x \in R$, $y \in [0, \pi]$, periodic with period 2π

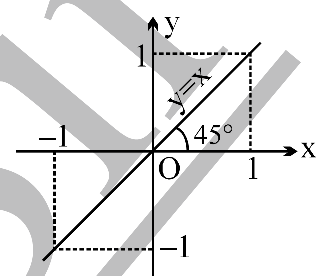
$= x$

8. (b) $y = \cos(\cos^{-1} x)$,

$= x$

$x \in [-1, 1]$, $y \in [-1, 1]$, y is aperiodic





9. (a) $y = \tan(\tan^{-1} x)$, $x \in R$, $y \in R$, y is aperiodic

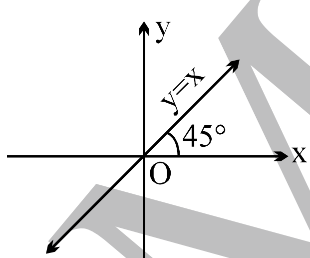
$= x$

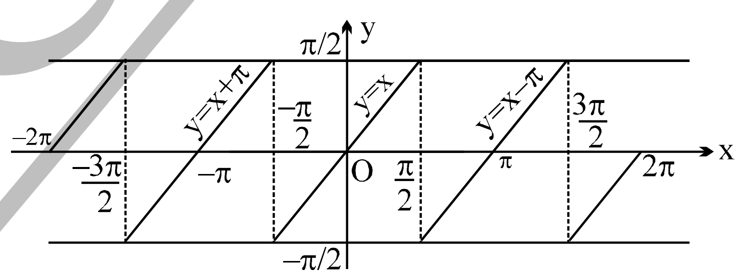
9. (b) $y = \tan^{-1}(\tan x)$,

$= x$

$x \in R - \left\{(2n-1)\frac{\pi}{2} \quad n \in I\right\}$, $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$,

periodic with period π





10. (a) $y = \cot^{-1}(\cot x)$,

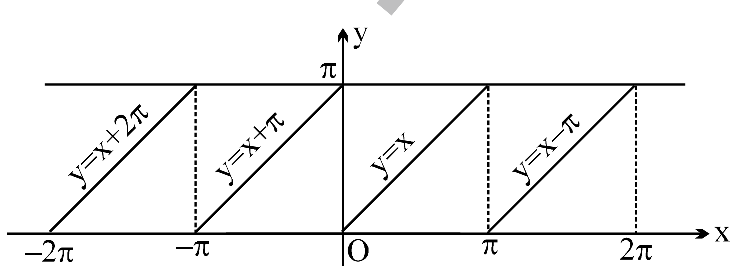
$= x$

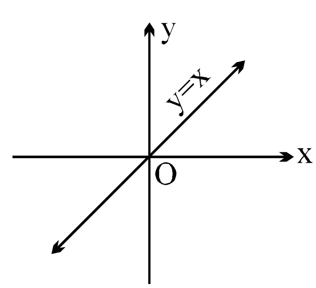
$x \in R - \{n\pi\}$, $y \in (0, \pi)$, periodic with π

10. (b) $y = \cot(\cot^{-1} x)$,

$= x$

$x \in R$, $y \in R$, y is aperiodic

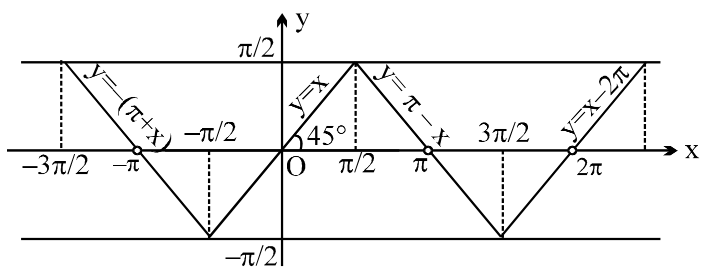




11. (a) $y = \text{cosec}^{-1}(\text{cosec}\, x)$,
$= x$

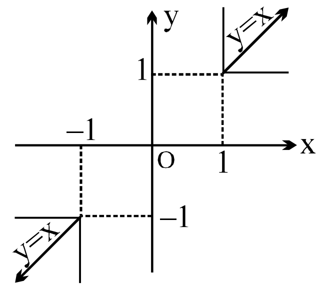
$x \in R - \{n\pi,\ n \in I\}$, $y \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$

y is periodic with period 2π



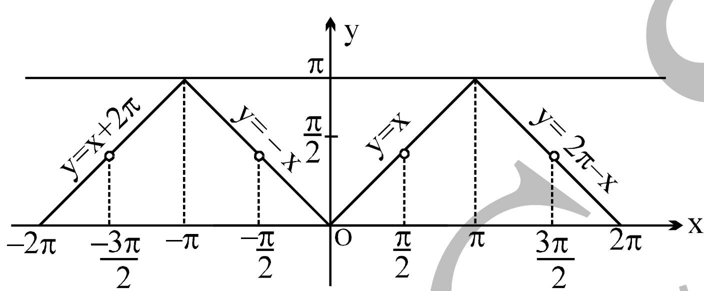
11. (b) $y = \text{cosec}(\text{cosec}^{-1} x)$,
$= x$

$|x| \geq 1$, $|y| \geq 1$, y is aperiodic

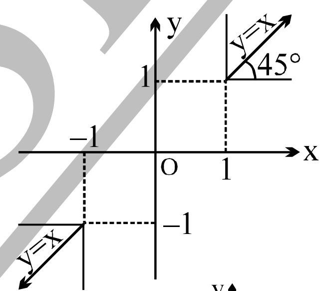


12. (a) $y = \sec^{-1}(\sec x)$,
$= x$
y is periodic with period 2π;

$x \in R - \left\{(2n-1)\frac{\pi}{2}\ \ n \in I\right\}$ $y \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$



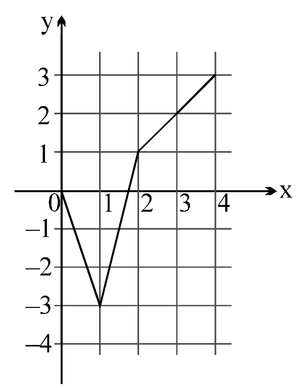
12. (b) $y = \sec(\sec^{-1} x)$,
$= x$
$|x| \geq 1$; $|y| \geq 1$], y is aperiodic



EXERCISE–I

Q.1 Given is a partial graph of an even periodic function f whose period is 8. If [*] denotes greatest integer function then find the value of the expression.

$$f(-3) + 2\,|f(-1)| + \left[f\left(\frac{7}{8}\right)\right] + f(0) + \text{arc}\cos\big(f(-2)\big) + f(-7) + f(20)$$



Q.2**(a)** Find the following

(i) $\tan\left[\cos^{-1}\frac{1}{2} + \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)\right]$ (ii) $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$

(iii) $\cos\left(\tan^{-1}\frac{3}{4}\right)$ (iv) $\tan\left(\sin^{-1}\frac{3}{5} + \cot^{-1}\frac{3}{2}\right)$

(b) Find the following :

(i) $\sin\left[\frac{\pi}{2} - \sin^{-1}\left(\frac{-\sqrt{3}}{2}\right)\right]$ (ii) $\cos\left[\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right) + \frac{\pi}{6}\right]$

(iii) $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$ (iv) $\sin\left(\frac{1}{4}\,\text{arc}\sin\frac{\sqrt{63}}{8}\right)$

Q.3 Find the domain of definition the following functions.
(Read the symbols [*] and {*} as greatest integers and fractional part functions respectively.)

(i) $f(x) = \text{arc}\cos\dfrac{2x}{1+x}$ (ii) $f(x) = \sqrt{\cos(\sin x)} + \sin^{-1}\dfrac{1+x^2}{2x}$

(iii) $f(x) = \sin^{-1}\left(\dfrac{x-3}{2}\right) - \log_{10}(4-x)$ (iv) $f(x) = \sin^{-1}(2x + x^2)$

(v) $f(x) = \dfrac{\sqrt{1-\sin x}}{\log_5(1-4x^2)} + \cos^{-1}(1-\{x\})$, where {x} is the fractional part of x .

(vi) $f(x) = \sqrt{3-x} + \cos^{-1}\left(\dfrac{3-2x}{5}\right) + \log_6\left(2|x|-3\right) + \sin^{-1}\left(\log_2 x\right)$

(vii) $f(x) = \log_{10}\left(1-\log_7(x^2-5x+13)\right) + \cos^{-1}\left(\dfrac{3}{2+\sin\frac{9\pi x}{2}}\right)$

(viii) $f(x) = e^{\sin^{-1}\left(\frac{x}{2}\right)} + \tan^{-1}\left[\dfrac{x}{2}-1\right] + \ell n\left(\sqrt{x-[x]}\right)$

(ix) $f(x) = \sqrt{\sin(\cos x)} + ln(-2\cos^2 x + 3\cos x + 1) + e^{\cos^{-1}\left(\frac{2\sin x+1}{2\sqrt{2\sin x}}\right)}$

Q.4 Identify the pair(s) of functions which are identical. Also plot the graphs in each case.

(a) $y = \tan(\cos^{-1}x)$; $y = \dfrac{\sqrt{1-x^2}}{x}$ (b) $y = \tan(\cot^{-1}x)$; $y = \dfrac{1}{x}$

(c) $y = \sin(\text{arc}\tan x)$; $y = \dfrac{x}{\sqrt{1+x^2}}$ (d) $y = \cos(\text{arc}\tan x)$; $y = \sin(\text{arc}\cot x)$

Q.5 Find the domain and range of the following functions .
(Read the symbols [*] and {*} as greatest integers and fractional part functions respectively.)

(i) $f(x) = \cot^{-1}(2x - x^2)$ (ii) $f(x) = \sec^{-1}(\log_3\tan x + \log_{\tan x}3)$

(iii) $f(x) = \cos^{-1}\left(\dfrac{\sqrt{2x^2+1}}{x^2+1}\right)$ (iv) $f(x) = \tan^{-1}\left(\log_{\frac{4}{5}}\left(5x^2 - 8x + 4\right)\right)$

Q.6 Let l_1 be the line $4x + 3y = 3$ and l_2 be the line $y = 8x$. L_1 is the line formed by reflecting l_1 across the line $y = x$ and L_2 is the line formed by reflecting l_2 across the x-axis. If θ is the acute angle between L_1 and L_2 such that $\tan\theta = a/b$, where a and b are coprime then find (a + b).

Q.7 Let $y = \sin^{-1}(\sin 8) - \tan^{-1}(\tan 10) + \cos^{-1}(\cos 12) - \sec^{-1}(\sec 9) + \cot^{-1}(\cot 6) - \text{cosec}^{-1}(\text{cosec } 7)$.
If y simplifies to $a\pi + b$ then find $(a - b)$.

Q.8 Show that : $\sin^{-1}\left(\sin\dfrac{33\pi}{7}\right) + \cos^{-1}\left(\cos\dfrac{46\pi}{7}\right) + \tan^{-1}\left(-\tan\dfrac{13\pi}{8}\right) + \cot^{-1}\left(\cot\left(-\dfrac{19\pi}{8}\right)\right) = \dfrac{13\pi}{7}$

Q.9 Let $\alpha = \sin^{-1}\left(\dfrac{36}{85}\right)$, $\beta = \cos^{-1}\left(\dfrac{4}{5}\right)$ and $\gamma = \tan^{-1}\left(\dfrac{8}{15}\right)$, find $(\alpha + \beta + \gamma)$ and hence prove that

(i) $\sum\cot\alpha = \prod\cot\alpha$, (ii) $\sum\tan\alpha\cdot\tan\beta = 1$

Q.10 Prove that : $\sin\cot^{-1}\tan\cos^{-1}x = \sin\text{cosec}^{-1}\cot\tan^{-1}x = x$ where $x \in (0,1]$

Q.11 Prove that: (a) $2\cos^{-1}\dfrac{3}{\sqrt{13}} + \cot^{-1}\dfrac{16}{63} + \dfrac{1}{2}\cos^{-1}\dfrac{7}{25} = \pi$

(b) $\cos^{-1}\left(\dfrac{5}{13}\right) + \cos^{-1}\left(-\dfrac{7}{25}\right) + \sin^{-1}\dfrac{36}{325} = \pi$ (c) $\text{arc}\cos\sqrt{\dfrac{2}{3}} - \text{arc}\cos\dfrac{\sqrt{6}+1}{2\sqrt{3}} = \dfrac{\pi}{6}$

Q.12 If α and β are the roots of the equation $x^2 + 5x - 49 = 0$ then find the value of $\cot(\cot^{-1}\alpha + \cot^{-1}\beta)$.

Q.13 If $a > b > c > 0$ then find the value of : $\cot^{-1}\left(\frac{ab+1}{a-b}\right) + \cot^{-1}\left(\frac{bc+1}{b-c}\right) + \cot^{-1}\left(\frac{ca+1}{c-a}\right)$.

Q.14 Find all values of *k* for which there is a triangle whose angles have measure $\tan^{-1}\left(\frac{1}{2}\right)$, $\tan^{-1}\left(\frac{1}{2}+k\right)$, and $\tan^{-1}\left(\frac{1}{2}+2k\right)$.

Q.15 Prove that: $\tan^{-1}\left(\frac{3\sin 2\alpha}{5+3\cos 2\alpha}\right) + \tan^{-1}\left(\frac{\tan\alpha}{4}\right) = \alpha$ (where $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$)

Q.16 Find the simplest value of

(a) $f(x) = \text{arc } \cos x + \text{arc } \cos\left(\frac{x}{2}+\frac{1}{2}\sqrt{3-3x^2}\right)$, $x \in \left(\frac{1}{2}, 1\right)$

(b) $f(x) = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$, $x \in R - \{0\}$

Q.17 Prove that the identities.

(a) $\sin^{-1}\cos(\sin^{-1}x) + \cos^{-1}\sin(\cos^{-1}x) = \frac{\pi}{2}$, $|x| \le 1$

(b) $2\tan^{-1}(\operatorname{cosec}\tan^{-1}x - \tan\cot^{-1}x) = \tan^{-1}x$ $(x \ne 0)$

(c) $\tan^{-1}\left(\frac{2mn}{m^2-n^2}\right) + \tan^{-1}\left(\frac{2pq}{p^2-q^2}\right) = \tan^{-1}\left(\frac{2MN}{M^2-N^2}\right)$ where $M = mp - nq$, $N = np + mq$,

$\left|\frac{n}{m}\right| < 1$; $\left|\frac{q}{p}\right| < 1$ and $\left|\frac{N}{M}\right| < 1$

(d) $\tan(\tan^{-1}x + \tan^{-1}y + \tan^{-1}z) = \cot(\cot^{-1}x + \cot^{-1}y + \cot^{-1}z)$

Q.18(a) Solve the inequality: $(\text{arc sec } x)^2 - 6(\text{arc sec } x) + 8 > 0$

(b) If $\sin^2x + \sin^2y < 1$ for all $x, y \in R$ then prove that $\sin^{-1}(\tan x \cdot \tan y) \in (-\pi/2, \pi/2)$.

Q.19 Let $f(x) = \cot^{-1}(x^2 + 4x + \alpha^2 - \alpha)$ be a function defined $R \to (0, \pi/2]$ then find the complete set of real values of α for which $f(x)$ is onto.

Q.20 If $S_n = \sum_{r=1}^{n} r!$ then for $n > 6$ $\left(\text{given } \sum_{r=1}^{6} r! = 873\right)$

Column-I	Column-II
(A) $\sin^{-1}\left(\sin\left(S_n - 7\left[\frac{S_n}{7}\right]\right)\right)$	(P) $5 - 2\pi$
(B) $\cos^{-1}\left(\cos\left(S_n - 7\left[\frac{S_n}{7}\right]\right)\right)$	(Q) $2\pi - 5$
(C) $\tan^{-1}\left(\tan\left(S_n - 7\left[\frac{S_n}{7}\right]\right)\right)$	(R) $6 - 2\pi$
(D) $\cot^{-1}\left(\cot\left(S_n - 7\left[\frac{S_n}{7}\right]\right)\right)$	(S) $5 - \pi$
	(T) $\pi - 4$

(where [ ] denotes greatest integer function)

EXERCISE–II

Q.1 Prove that: (a) $\tan\left[\frac{\pi}{4}+\frac{1}{2}\cos^{-1}\frac{a}{b}\right]+\tan\left[\frac{\pi}{4}-\frac{1}{2}\cos^{-1}\frac{a}{b}\right]=\frac{2b}{a}$

(b) $\cos^{-1}\frac{\cos x+\cos y}{1+\cos x\cos y}=2\tan^{-1}\left(\tan\frac{x}{2}\cdot\tan\frac{y}{2}\right)$ (c) $2\tan^{-1}\left[\sqrt{\frac{a-b}{a+b}}\cdot\tan\frac{x}{2}\right]=\cos^{-1}\left[\frac{b+a\cos x}{a+b\cos x}\right]$

Q.2 If $y=\tan^{-1}\left[\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right]$ prove that $x^2=\sin 2y$.

Q.3 If $u=\cot^{-1}\sqrt{\cos 2\theta}-\tan^{-1}\sqrt{\cos 2\theta}$ then prove that $\sin u=\tan^2\theta$.

Q.4 If $\alpha=2\arctan\left(\frac{1+x}{1-x}\right)$ & $\beta=\arcsin\left(\frac{1-x^2}{1+x^2}\right)$ for $0<x<1$, then prove that $\alpha+\beta=\pi$, what the value of $\alpha+\beta$ will be if $x>1$.

Q.5 If $x\in\left[-1,-\frac{1}{2}\right]$ then express the function $f(x)=\sin^{-1}(3x-4x^3)+\cos^{-1}(4x^3-3x)$ in the form of $a\cos^{-1}x+b\pi$, where a and b are rational numbers.

Q.6 Find the sum of the series:

(a) $\cot^{-1}7+\cot^{-1}13+\cot^{-1}21+\cot^{-1}31+\ldots\ldots$ to n terms.

(b) $\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{2}{9}+\ldots..+\tan^{-1}\frac{2^{n-1}}{1+2^{2n-1}}+\ldots..\infty$

(c) $\tan^{-1}\frac{1}{x^2+x+1}+\tan^{-1}\frac{1}{x^2+3x+3}+\tan^{-1}\frac{1}{x^2+5x+7}+\tan^{-1}\frac{1}{x^2+7x+13}$ to n terms.

(d) $\sin^{-1}\frac{1}{\sqrt{5}}+\sin^{-1}\frac{1}{\sqrt{65}}+\sin^{-1}\frac{1}{\sqrt{325}}+\ldots\ldots\infty$ terms

(e) $\sin^{-1}\frac{1}{\sqrt{2}}+\sin^{-1}\frac{\sqrt{2}-1}{\sqrt{6}}+\ldots..+\sin^{-1}\frac{\sqrt{n}-\sqrt{n-1}}{\sqrt{n(n+1)}}+\ldots\ldots\infty$

Q.7 Solve the following equations / system of equations:

(a) $\sin^{-1}x+\sin^{-1}2x=\frac{\pi}{3}$ (b) $\tan^{-1}\frac{1}{1+2x}+\tan^{-1}\frac{1}{1+4x}=\tan^{-1}\frac{2}{x^2}$

(c) $\tan^{-1}(x-1)+\tan^{-1}(x)+\tan^{-1}(x+1)=\tan^{-1}(3x)$ (d) $3\cos^{-1}x=\sin^{-1}\left(\sqrt{1-x^2}\,(4x^2-1)\right)$

(e) $\tan^{-1}\frac{x-1}{x+1}+\tan^{-1}\frac{2x-1}{2x+1}=\tan^{-1}\frac{23}{36}$ (f) $\sin^{-1}x+\sin^{-1}y=\frac{2\pi}{3}$ & $\cos^{-1}x-\cos^{-1}y=\frac{\pi}{3}$

(g) $2\tan^{-1}x=\cos^{-1}\frac{1-a^2}{1+a^2}-\cos^{-1}\frac{1-b^2}{1+b^2}$ (a>0, b>0). (h) $\cos^{-1}\frac{x^2-1}{x^2+1}+\tan^{-1}\frac{2x}{x^2-1}=\frac{2\pi}{3}$

Q.8 If α and β are the roots of the equation $x^2-4x+1=0$ $(\alpha>\beta)$ then find the value of $f(\alpha,\beta)=\frac{\beta^3}{2}\operatorname{cosec}^2\left(\frac{1}{2}\tan^{-1}\frac{\beta}{\alpha}\right)+\frac{\alpha^3}{2}\sec^2\left(\frac{1}{2}\tan^{-1}\frac{\alpha}{\beta}\right)$.

Q.9 Find the integral values of K for which the system of equations;

$$\left[\begin{array}{ll}\arccos x+(\arcsin y)^2 & =\frac{K\pi^2}{4}\\ (\arcsin y)^2\cdot(\arccos x) & =\frac{\pi^4}{16}\end{array}\right.$$

possesses solutions & find those solutions.

Q.10 Find all the positive integral solutions of, $\tan^{-1}x + \cos^{-1}\frac{y}{\sqrt{1+y^2}} = \sin^{-1}\frac{3}{\sqrt{10}}$.

Q.11 If $X = \text{cosec} . \tan^{-1} . \cos . \cot^{-1} . \sec . \sin^{-1} a$ & $Y = \sec \cot^{-1} \sin \tan^{-1} \text{cosec} \cos^{-1} a$; where $0 \le a \le 1$. Find the relation between X & Y. Express them in terms of 'a'.

Q.12

Column-I	Column-II
(A) $f(x) = \sin^{-1}\left(\frac{2}{\lvert\sin x - 1\rvert + \lvert\sin x + 1\rvert}\right)$	(P) $f(x)$ is many one
(B) $f(x) = \cos^{-1}(\lvert x-1\rvert - \lvert x-2\rvert)$	(Q) Domain of $f(x)$ is R
(C) $f(x) = \sin^{-1}\left(\frac{\pi}{\lvert\sin^{-1}x - (\pi/2)\rvert + \lvert\sin^{-1}x + (\pi/2)\rvert}\right)$	(R) Range contain only irrational number
(D) $f(x) = \cos(\cos^{-1}\lvert x\rvert) + \sin^{-1}(\sin x) - \text{cosec}^{-1}(\text{cosec } x) + \text{cosec}^{-1}\lvert x\rvert$	(S) $f(x)$ is even.

Q.13 Prove that the equation $(\sin^{-1}x)^3 + (\cos^{-1}x)^3 = \alpha\pi^3$ has no roots for $\alpha < \frac{1}{32}$ and $\alpha > \frac{7}{8}$

Q.14 Solve the following inequalities :
(a) $\text{arc } \cot^2 x - 5 \text{ arc } \cot x + 6 > 0$ (b) $\text{arc } \sin x > \text{arc } \cos x$ (c) $\tan^2(\text{arc } \sin x) > 1$

Q.15 Solve the following system of inequations
$4 \text{ arc } \tan^2 x - 8 \text{arc } \tan x + 3 < 0$ & $4 \text{ arc } \cot x - \text{arc } \cot^2 x - 3 \ge 0$

Q.16 If the total area between the curves $f(x) = \cos^{-1}(\sin x)$ and $g(x) = \sin^{-1}(\cos x)$ on the interval $[-7\pi, 7\pi]$ is A, find the value of 49A. (Take $\pi = 22/7$)

Q.17 If the sum $\sum_{n=1}^{10}\sum_{m=1}^{10} \tan^{-1}\left(\frac{m}{n}\right) = k\pi$, find the value of k.

Q.18 Show that the roots *r*, *s*, and *t* of the cubic $x(x-2)(3x-7) = 2$, are real and positive. Also compute the value of $\tan^{-1}(r) + \tan^{-1}(s) + \tan^{-1}(t)$.

Q.19 Solve for x : $\sin^{-1}\left(\sin\left(\frac{2x^2+4}{1+x^2}\right)\right) < \pi - 3$.

Q.20 Find the set of values of 'a' for which the equation $2\cos^{-1}x = a + a^2(\cos^{-1}x)^{-1}$ posses a solution.

EXERCISE–III

Q.1 The number of real solutions of $\tan^{-1}\sqrt{x(x+1)} + \sin^{-1}\sqrt{x^2+x+1} = \frac{\pi}{2}$ is :
(A) zero (B) one (C) two (D) infinite [JEE '99, 2 (out of 200)]

Q.2 Using the principal values, express the following as a single angle :
$$3\tan^{-1}\left(\frac{1}{2}\right) + 2\tan^{-1}\left(\frac{1}{5}\right) + \sin^{-1}\frac{142}{65\sqrt{5}}.$$ [REE '99, 6]

Q.3 Solve, $\sin^{-1}\frac{ax}{c} + \sin^{-1}\frac{bx}{c} = \sin^{-1}x$, where $a^2 + b^2 = c^2$, $c \ne 0$. [REE 2000(Mains), 3 out of 100]

Q.4 Solve the equation:
$$\cos^{-1}\left(\sqrt{6}x\right) + \cos^{-1}\left(3\sqrt{3}x^2\right) = \frac{\pi}{2}$$ [REE 2001 (Mains), 3 out of 100]

Q.5 If $\sin^{-1}\left(x - \frac{x^2}{2} + \frac{x^3}{4} - ....\right) + \cos^{-1}\left(x^2 - \frac{x^4}{2} + \frac{x^6}{4} - ....\right) = \frac{\pi}{2}$ for $0 < |x| < \sqrt{2}$ then x equals to

[JEE 2001(screening)]

(A) 1/2 (B) 1 (C) – 1/2 (D) – 1

Q.6 Prove that $\cos\tan^{-1}\sin\cot^{-1}x = \sqrt{\frac{x^2+1}{x^2+2}}$ [JEE 2002 (mains) 5]

Q.7 Domain of $f(x) = \sqrt{\sin^{-1}(2x) + \frac{\pi}{6}}$ is

(A) $\left(-\frac{1}{2}, \frac{1}{2}\right]$ (B) $\left[-\frac{1}{4}, \frac{3}{4}\right)$ (C) $\left[-\frac{1}{4}, \frac{1}{4}\right]$ (D) $\left[-\frac{1}{4}, \frac{1}{2}\right]$

[JEE 2003 (Screening) 3]

Q.8 If $\sin\left(\cot^{-1}(x+1)\right) = \cos(\tan^{-1}x)$, then x =

(A) $-\frac{1}{2}$ (B) $\frac{1}{2}$ (C) 0 (D) $\frac{9}{4}$

[JEE 2004 (Screening)]

Q.9 Let (x, y) be such that

$$\sin^{-1}(ax) + \cos^{-1}(y) + \cos^{-1}(bxy) = \frac{\pi}{2}$$

Match the statements in **Column I** with statements in **Column II** and indicate your answer by darkening the appropriate bubbles in the 4 × 4 matrix given in the ORS.

	Column I		**Column II**
(A)	If a = 1 and b = 0, then (x, y)	(P)	lies on the circle $x^2 + y^2 = 1$
(B)	If a = 1 and b = 1, then (x, y)	(Q)	lies on $(x^2 – 1)(y^2 – 1) = 0$
(C)	If a = 1 and b = 2, then (x, y)	(R)	lies on y = x
(D)	If a = 2 and b = 2, then (x, y)	(S)	lies on $(4x^2 – 1)(y^2 – 1) = 0$

[JEE 2007, 6]

Q.10 If 0 < x < 1, then $\sqrt{1+x^2}\,[\{x\cos(\cot^{-1}x) + \sin(\cot^{-1}x)\}^2 - 1]^{1/2} =$

(A) $\frac{x}{\sqrt{1+x^2}}$ (B) x (C) $x\sqrt{1+x^2}$ (D) $\sqrt{1+x^2}$

[JEE 2008, 3]

ANSWER KEY
FUNCTIONS
EXERCISE–I

Q 1. (i) $\left[-\frac{5\pi}{4}, \frac{-3\pi}{4}\right] \cup \left[-\frac{\pi}{4}, \frac{\pi}{4}\right] \cup \left[\frac{3\pi}{4}, \frac{5\pi}{4}\right]$ **(ii)** $\left(-4, -\frac{1}{2}\right) \cup (2, \infty)$ **(iii)** $(-\infty, -3]$

(iv) $(-\infty, -1) \cup [0, \infty)$ **(v)** $(3 - 2\pi < x < 3 - \pi)$ U $(3 < x \le 4)$ **(vi)** $\left(0, \frac{1}{100}\right) \cup \left(\frac{1}{100}, \frac{1}{\sqrt{10}}\right)$

(vii) $(-1 < x < -1/2)$ U $(x > 1)$ **(viii)** $\left[\frac{1-\sqrt{5}}{2}, 0\right) \cup \left[\frac{1+\sqrt{5}}{2}, \infty\right)$ **(ix)** $(-3, -1]$ U $\{0\}$ U $[1,3)$

(x) $\{4\} \cup [5, \infty)$ **(xi)** $(0, 1/4)$ U $(3/4, 1)$ U $\{x : x \in N, x \ge 2\}$ **(xii)** $\left(-\frac{1}{6}, \frac{\pi}{3}\right] \cup \left[\frac{5\pi}{3}, 6\right)$

(xiii) $[-3, -2) \cup [3, 4)$ **(xiv)** $R - \left\{-\frac{1}{2}, 0\right\}$

(xv) $2K\pi < x < (2K+1)\pi$ but $x \ne 1$ where K is non–negative integer

(xvi) $\{x \mid 1000 \le x < 10000\}$ **(xvii)** $(-2, -1)$ U $(-1, 0)$ U $(1, 2)$ **(xviii)** $(1, 2) \cup (2, 5/2)$;

(xix) $x \in \{4, 5\}$ **(xx)** $x \in (3, 5)$ $\{x \ne \pi, \frac{3\pi}{2}\}$

Q.2

(i) $D : x \in R$ $R : [0, 2]$ **(ii)** $D = R$; range $[-1, 1]$

(iii) $D : \{x \mid x \in R ; x \ne -3 ; x \ne 2\}$ $R : \{f(x) \mid f(x) \in R, f(x) \ne 1/5 ; f(x) \ne 1\}$

(iv) $D : R$; $R : (-1, 1)$ **(v)** $D : -1 \le x \le 2$ $R : \left[\sqrt{3}, \sqrt{6}\right]$

(vi) $D : x \in (2n\pi, (2n+1)\pi) - \left\{2n\pi + \frac{\pi}{6}, 2n\pi + \frac{\pi}{2}, 2n\pi + \frac{5\pi}{6}, n \in I\right\}$ and

$R : \log_a 2$; $a \in (0, \infty) - \{1\} \Rightarrow$ Range is $(-\infty, \infty) - \{0\}$

(vii) $D : [-4, \infty) - \{5\}$; $R : \left(0, \frac{1}{6}\right) \cup \left(\frac{1}{6}, \frac{1}{3}\right]$

Q.3 **(b)** (i), (iii) are identical

Q.4 **(a)** neither surjective nor injective **(b)** surjective but not injective

(c) neither injective nor surjective

Q.5 (a) – 3/4; (b) 64; (c) 30, (d) 102; (e) 5050; (f) 28

Q.6 **(a)** domain is $x \ge 0$; range $[-1, 1]$; **(b)** domain $2k\pi \le x \le 2k\pi + \pi$; range $[0, 1]$

(c) Domain $x \in R$; range $[-\sin 1, \sin 1]$; **(d)** domain is $0 \le x \le 1$; range is $[0, 1]$

Q.7 1

Q.9 **(a)** $\frac{(x+3)^{10}}{(x+3)^{10}+1}$, domain is R, $\frac{1024}{1025}$; **(b)** $f(x) = x^2$; $g(x) = \cos x$; $h(x) = x + 9$

Q.10 $g(x) = \begin{cases} \frac{1}{x^2} & \text{if } 0 < x \le 1 \\ x^2 & \text{if } x > 1 \end{cases}$ **Q.11** **(a)** $\{-1, 1\}$ **(b)** $a \in \{0, -4\}$

Q.12 $(gof)(x) = \begin{cases} x & \text{if } x \le 0 \\ -x^2 & \text{if } 0 < x < 1 \\ 1-x^2 & \text{if } x \ge 1 \end{cases}$; $(fog)(x) = \begin{cases} x^2 & \text{if } x < 0 \\ 1+x & \text{if } 0 \le x < 1 \\ x & \text{if } x \ge 1 \end{cases}$

Q.13 (a) odd, (b) even, (c) neither odd nor even, (d) odd, (e) neither odd nor even, (f) even, (g) even, (h) even

Q.14(i) **(a)** $y = \log(10 - 10^x)$, $-\infty < x < 1$

(b) $y = x/3$ when $-\infty < x < 0$ & $y = x$ when $0 \le x < +\infty$

(ii) **(a)** $2K\pi \le x \le 2K\pi + \pi$ where $K \in I$ **(b)** $[-3/2, -1]$

(iii) **(a)** Range : $[-1/3, 3]$, Domain = $[4, 7]$; **(b)** Range $[-1, 9]$ and domain $[11, 14]$

Q.15 **(a)** $\frac{e^x - e^{-x}}{2}$; **(b)** $\frac{\log_2 x}{\log_2 x - 1}$; **(c)** $\frac{1}{2}\log\frac{1+x}{1-x}$ **Q.16** $x = 10$; $f^{-1}(x) = 10^{\log_2(x-8)}$

Q.17 **(i)** period of fog is π, period of gof is 2π ; **(ii)** range of fog is $[-1, 1]$, range of gof is $[-\tan 1, \tan 1]$

Q.18 (a) 0.86 (b) ± 1, ± 3, ± 5, ± 15

Q.19 (A) R; (B) S; (C) P; (D) Q **Q.20** (A) S; (B) R; (C) P; (D) Q

EXERCISE–II

Q 1. $f^{-1}(1) = y$ **Q.2** 152

Q.3 (a) $\frac{1}{1002}$, (b) 1, (c) [0, 4), (d) 727, (e) 3 **Q.4** **(A)** R; **(B)** S; **(C)** P; **(D)** Q

Q.5 11 **Q.6** 6016 **Q 9.** $f(x) = 2x^2$

Q 11. $fog(x) = \begin{cases} -(1+x) & , -1 \le x \le 0 \\ x - 1 & , 0 < x \le 2 \end{cases}$; $gof(x) = \begin{cases} x+1 & , 0 \le x < 1 \\ 3-x & , 1 \le x \le 2 \\ x-1 & , 2 < x \le 3 \\ 5-x & , 3 < x \le 4 \end{cases}$;

$fof(x) = \begin{cases} x & , 0 \le x \le 1 \\ 4 - x & , 3 \le x \le 4 \end{cases}$; $gog(x) = \begin{cases} -x & , -1 \le x \le 0 \\ x & , 0 < x \le 2 \\ 4-x & , 2 < x \le 3 \end{cases}$

Q.12 21 **Q.13** $x = 0$ or $5/3$

Q.14 1002.5 **Q.15** 5049 **Q.16** $g(x) = 3 + 5\sin(n\pi + 2x - 4)$, $n \in I$

Q.17 20 **Q.18** 122 **Q.19** $f(x) = \sin x + x - \frac{\pi}{3}$

EXERCISE–III

Q.1 B **Q.2** D

Q.3 {(1, 1), (2, 3), (3, 4), (4, 2)} ; {(1, 1), (2, 4), (3, 2), (4, 3)} and {(1, 1), (2, 4), (3, 3), (4, 2)}

Q.4 **(a)** B, **(b)** A, **(c)** D, **(d)** A, **(e)** D **Q.5** **(a)** D; **(b)** A

Q.6 **(a)** D , **(b)** A **Q.7** C **Q.8** A

INVERSE TRIGONOMETRY FUNCTIONS

EXERCISE–I

Q.1 5 **Q.2** **(a)** (i) $\frac{1}{\sqrt{3}}$, (ii) $\frac{5\pi}{6}$, (iii) $\frac{4}{5}$, (iv) $\frac{17}{6}$; **(b)** (i) $\frac{1}{2}$, (ii) -1, (iii) $-\frac{\pi}{4}$, (iv) $\frac{\sqrt{2}}{4}$

Q.3 **(i)** $-1/3 \le x \le 1$ **(ii)** $\{1, -1\}$ **(iii)** $1 \le x < 4$ **(iv)** $[-(1+\sqrt{2}), (\sqrt{2}, -1)]$
(v) $x \in (-1/2, 1/2), x \ne 0$ **(vi)** $(3/2, 2]$
(vii) $\{7/3, 25/9\}$ **(viii)** $(-2, 2) - \{-1, 0, 1\}$ **(ix)** $\{x \mid x = 2n\pi + \frac{\pi}{6}, n \in I\}$

Q.4 (a), (b), (c) and (d) all are identical.

Q.5 **(i)** D : x ε R R : $[\pi/4, \pi)$

(ii) D: $x \in \left(n\pi, n\pi + \frac{\pi}{2}\right) - \left\{x \middle| x = n\pi + \frac{\pi}{4}\right\}$ $n \in I$; R : $\left[\frac{\pi}{3}, \frac{2\pi}{3}\right] - \left[\frac{\pi}{2}\right]$

(iii) D : $x \in R$ R : $\left[0, \frac{\pi}{2}\right)$ **(iv)** D : $x \in R$ R : $\left(-\frac{\pi}{2}, \frac{\pi}{4}\right]$

Q.6 57 **Q.7** 53 **Q.8** 56 **Q.9** $\pi/2$ **Q.12** 10 **Q.13** π

Q.14 $k = \frac{11}{4}$ **Q.16** **(a)** $\frac{\pi}{3}$; **(b)** $\frac{1}{2}\tan^{-1}x$

Q.18 **(a)** $(-\infty, \sec 2) \cup [1, \infty)$ **Q.19** $\frac{1 \pm \sqrt{17}}{2}$ **Q.20** (A) P; (B) Q ; (C) P ; (D) S

EXERCISE–II

Q.4 $-\pi$ **Q.5** $6\cos^{-1}x - \frac{9\pi}{2}$, so $a = 6$, $b = -\frac{9}{2}$

Q.6 **(a)** arc cot$\left[\frac{2n+5}{n}\right]$, **(b)** $\frac{\pi}{4}$, **(c)** arc tan (x + n) – arc tan x, **(d)** $\frac{\pi}{4}$, **(e)** $\frac{\pi}{2}$

Q.7 **(a)** $x = \frac{1}{2}\sqrt{\frac{3}{7}}$; **(b)** $x = 3$; **(c)** $x = 0, \frac{1}{2}, -\frac{1}{2}$; **(d)** $\left[\frac{\sqrt{3}}{2}, 1\right]$; **(e)** $x = \frac{4}{3}$; **(f)** $x = \frac{1}{2}, y = 1$; **(g)** $x = \frac{a-b}{1+ab}$

(h) $x = 2 - \sqrt{3}$ or $\sqrt{3}$

Q 8. $(\alpha^2 + \beta^2)(\alpha + \beta)$

Q 9. K = 2 ; $\cos\frac{\pi^2}{4}, 1$ & $\cos\frac{\pi^2}{4}, -1$ **Q.10** x = 1 ; y = 2 & x = 2 ; y = 7

Q.11 $X = Y = \sqrt{3 - a^2}$ **Q.12** (A) P, Q, R, S; (B) **P,** Q; (C) P, R, S; (D) **P,** R, S

Q.14 **(a)** $(\cot 2, \infty) \cup (-\infty, \cot 3)$ **(b)** $\frac{\sqrt{2}}{2}, 1$ **(c)** $\left(\frac{\sqrt{2}}{2}, 1\right) \cup \left(-1, -\frac{\sqrt{2}}{2}\right)$

Q.15 $\left(\tan\frac{1}{2}, \cot 1\right]$ Q.16 3388 **Q17.** k = 25 **Q.18** $\frac{3\pi}{4}$ **Q.19** $x \in (-1, 1)$

Q.20 $a \in [-2\pi, \pi] - \{0\}$

EXERCISE–III

Q.1 C **Q.2** π **Q.3** $x \in \{-1, 0, 1\}$ **Q.4** x = 1/3 **Q.5** B **Q.7** D **Q.8** A
Q.9 (A) P; (B) Q; (C) P; (D) S **Q.10** C