HYPERBOLA

Introduction

General Equation : $ax^2+2hxy+by^2+2gx+2fy+c = 0$ denotes the hyperbola if $h^2 > ab$ and $e > 1$.

Standard Equation & Basic Terminology

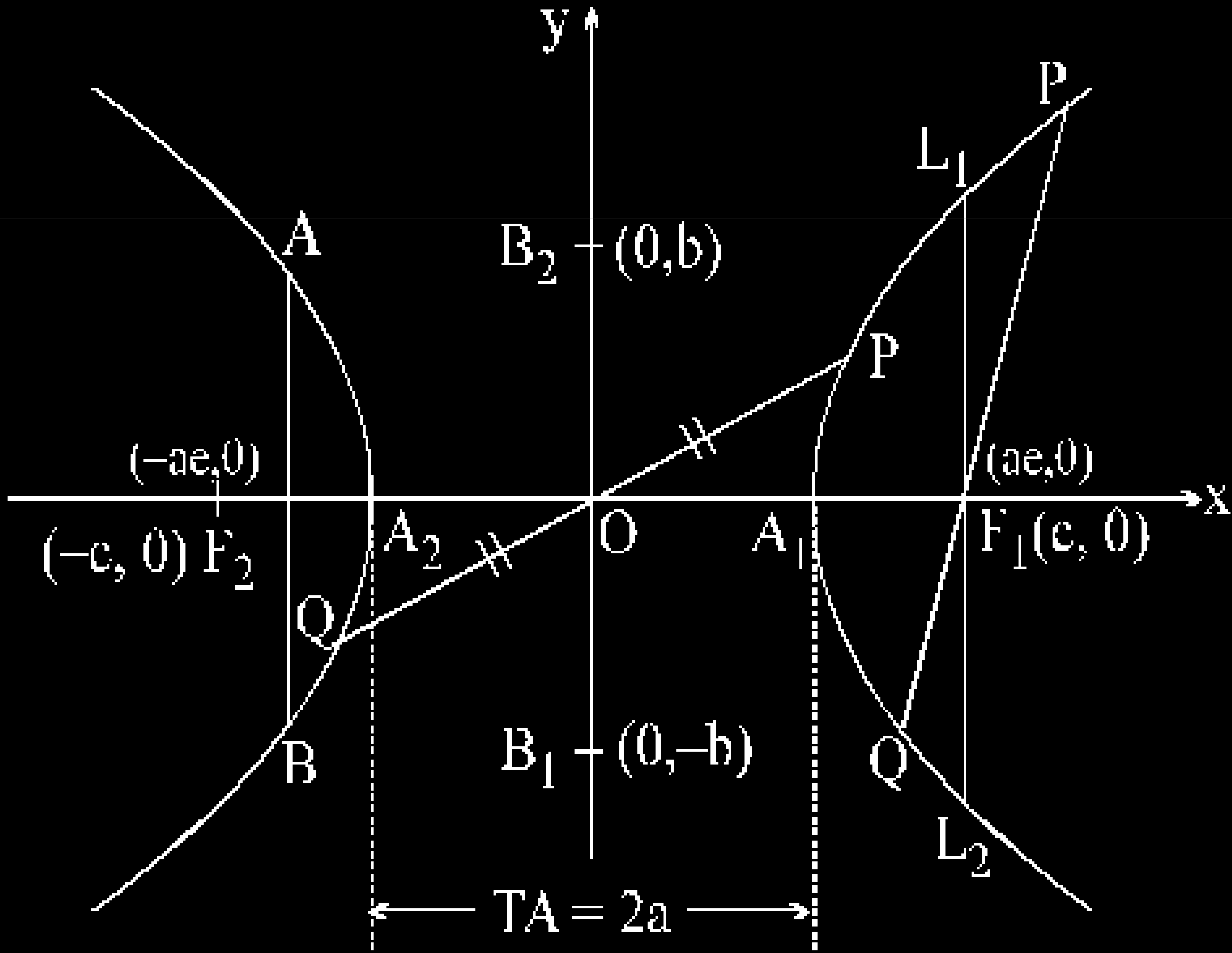
Standard equation of hyperbola is deduced using an important property of hyperbola that the difference of a point moving on it, from two fixed points is constant.

i.e. $|PF_1 - PF_2| = 2a$ ($2a < 2c$ i.e. $> a$)

i.e. $\sqrt{(x-c)^2+y^2} - \sqrt{(x+c)^2+y^2} = \pm 2a$

i.e. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ (where $c^2 - a^2 = b^2$).......(1)

Definitions



(i) Line containing the fixed point F_1 and F_2 (called Foci) is called Transverse Axis (TA) of a Focal Axis and the distance between F_1 and F_2 is called Focal Length.

(ii) The points of intersection (A_1, A_2) of the curve with the transverse axis are called vertices of the hyperbola.

(iii) The length '2a' between the vertices is called the Length of Transverse Axis.

(iv) The perpendicular bisector of transverse axis is called the Conjugate Axis (CA). The point $B_1(0,–b)$ and $B_2(0,–b)$ which have special significance, are known as the extermities of conjugate axis and the length '2b' is called the Length of conjugate axis. The point of intersection of these two axes is called the centre 'O' of the hyperbola. (Transverse axis and conjugate axis together are called the Principal Axis). Any chord passing through centre is called Diameter (PQ) and is bisected by it.

(v) Any chord passing through focus is called is Focal Chord and any chord perpendicular to the Transverse axis is called a Double Ordinate (AB).

(vi) A particular double ordinate which passes through focus or a particular focal chord passing through focus is called the Latus Rectum (L_1L_2).

Eccentricity

Defines the curvature of the hyperbola and is mathematically spelled as :

$$e = \frac{\text{distance from centre to focus}}{\text{distance from centre to vertex}}$$

$$e^2 = 1 + \frac{b^2}{a^2}$$

Remember that

(i) $a^2e^2 = a^2 + b^2$

(ii) Coordinates of foci : $(\pm ae, 0)$ and

(iii) Two hyperbolas are said to be similar if they have the same value of eccentricity.

(iv) Equation of hyperbola in terms of eccentricity can be written as $\frac{x^2}{a^2} - \frac{y^2}{a^2(e^2-1)} = 1$

(i) Extremities of latus rectum $\left(\pm ae, \pm \frac{b^2}{a}\right)$

Conjugate Hyperbola

Corresponding to every hyperbola these exist a hyperbola such that, the conjugate axis and transverse axis of one is equal to the transverse axis and conjugate axis of other, such hyperbola are known conjugate to each other.

* Hence for the hyperbola, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

The conjugate hyperbola is, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$

Q. If e_1 and e_2 are the eccentricities of a hyperbola and its conjugate respectively, then prove that $e_1^{-2} + e_2^{-2} = 1$

Note :

The foci of a hyperbola and its conjugate are concylic and form the vertices of a square.

Focal Directrix Property

Rectangular Hyperbola

If a = b, hyperbola is said to be equilateral or rectangular and has the equation $x^2 - y^2 = a^2$. Eccentricity for such a hyperbola is

l(LR) 2a (e2 – 1) = 2a = l(Ta)

* Parametric coordinates

$x = a \sec\theta$ and $y = b \tan\theta$

Illustrations on Basic Parameters

Q. On a level plain the crack of the rifle and the thud of the ball striking the target are heard at the same instant; prove that the locus of the hearer is a hyperbola.

Q. Show that the locus of the centre of a circle which touches externally two given circles is a hyperbola.

Q. Given the base of a triangle and the ratio of the tangents of half the base angles, prove that the vertex moves on a hyperbola whose foci are the extremities of the base.

Q. An ellipse and a hyperbola are confocal (have the same focus) and the conjugate axis of the hyperbola is equal to the minor axis of the ellipse, If e_1 and e_2 are the eccentricities of ellipse and hyperbola then prove that

$$\frac{1}{e_1^2}+\frac{1}{e_2^2}=2$$

Q. Find the equation of hyperbola referred to its principal axes as the coordinates axes

(a) If the distance of one of its vertices from the foci are 3 and 1.

Q. (a) Whose centre is (1, 0) ; focus is (6, 0) and transverse axis 6.

Q. (c) Whose centre is $(3, 2)$, one focus is $(5, 2)$ and one vertex is $(4, 2)$

Q. (d) Whose centre is (-3, 2), one vertex is (-3, 4) and eccentricity is 5/2.

Q. (e) Whose foci are (4, 2) and (8, 2) and eccentricity is 2.

Q. (e) Find the coordinates of the foci and the centre of the hyperbola

$$\frac{(3x-4y-12)^2}{100}-\frac{(4x+3y-12)^2}{225}1$$

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$$\frac{(3x-4y-12)^2}{100}-\frac{(4x+3y-12)^2}{225}1$$

Q. Find everything for the hyperbola

$$9x^2 - 18x - 16y^2 - 64y + 89 = 0$$

Auxiliary Circle

A circle drawn with centre C & T.A. as a diameter is called the Auxiliary Circle of the hyperbola. Equation of the auxiliary circle is $x^2 + y^2 = a^2$.

Position of A Point ‘P’ w.r.t. A Hyperbola

Line And A Hyperbola

Note

(i) For a given m, there can be two parallel tangents to the hyperbola

(ii) Passing though a given point (h, k) there can be a maximum of two tangents.

Director Circle

$x^2 + y^2 = a^2 - b^2$

(i) If l (TA) > l (CA) ; director circle is real with finite radius.

(ii) If l (TA) = l (CA) ; director circle is a point circle

(iii) If l (TA) < l (CA) ; no real circle

Examples

Q. Tangents are drawn to the hyperbola $x^2 - y^2 = a^2$ enclosing at an angle of 45°. Show that the locus of their points of intersection is $(x^2 + y^2)^2 + 4a^2 (a^2 - y^2) = 4a^4$.

Q. Find common tangent to $y^2 = 8x$ and $3x^2 - y^2 = 3$

Q. Find Tangent to $\frac{x^2}{36} - \frac{y^2}{9} = 1$ passing through (0, 4).

Q. Prove that the two tangents drawn from any point on the hyperbola $x^2 - y^2 = a^2 - b^2$ to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ make complementary angles with the axes.

Chord Of Hyperbola

$$\frac{x}{a}\cos\frac{\alpha-\beta}{2}-\frac{y}{b}\sin\frac{\alpha+\beta}{2}=\cos\frac{\alpha+\beta}{2}$$

If (1) passes through (d, 0) then

Tangents And Normals

(1) Certesian Tangent : $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$

Parametric Tangent : $\dfrac{x\sec\theta}{a} - \dfrac{y\tan\theta}{b} = 1$

Certesian Normal : $\dfrac{a^2x}{x_1}+\dfrac{b^2y}{y_1}=a^2+b^2=a^2e^2 \text{ at } (x_1, y_1)$

Parametric Normal : $\dfrac{ax}{\sec\theta}+\dfrac{by}{\tan\theta}=a^2+b^2=a^2e^2$

at (a secθ, b tanθ)

Q. For the hyperbola $x^2 - y^2 = a^2$, equations of the normal becomes

(i) $\dfrac{x}{x_1} + \dfrac{y}{y_1} = 2\,(\text{certesian})$

Q. $$\frac{x}{\sec\theta}+\frac{y}{\tan\theta}=2a \text{ (Parametric)}$$

Q. Find the equation to common tangent to the hyperbolas

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \quad \text{and} \quad \frac{y^2}{a^2}-\frac{x^2}{b^2}=1$$

Q. Perpendicular form the centre upon the tangent and normal at any point of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

meet them in Q and R. Find their loci.

* Chord of contact ; Chord with a given mid point ; Pair of tangents

Q. From points on the circle $x^2 + y^2 = a^2$ tangents are drawn to the hyperbola $x^2 - y^2 = a^2$; prove that the locus of the middle points of the chords of contact is the curve

$$(x^2 - y^2) = a^2 (x^2 + y^2)$$

Q. A point P moves such that the chord of contact of the pair of tangents from P on the parabola $y^2 = 4ax$ touches the rectangular hyperbola $x^2 - y^2 = c^2$. Show that the locus of P is the ellipse

$$\frac{x^2}{c^2} + \frac{y^2}{(2a)^2} = 1.$$

Q. Find the equation to the locus of the middle points of the chords of the hyperbola $2x^2-3y^2=1$, each of which makes an angle of 45° with the x-axis.

Q. A tangent to the hyperbola $\frac{x^2}{e^2} - \frac{y^2}{b^2} = 1$ cut the ellipse $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at P and Q. show that the locus of the mid points of PQ is

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) = \frac{x^2}{a^2} = \frac{y^2}{b^2}$$

Q. Show that the mid points of focal chords of a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ lie on another similar hyperbola.

Highlights

H-1 Locus of the feet of the perpendicular drawn from focus of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ upon any tangent is its auxiliary circle i.e. $x^2+y^2 = a^2$ & the product of the feet of these perpendiculars is b^2. (semi C.A)^2

H-2 The portion of the tangent between the point of contact & the directrix subtends a right angle at the corresponding focus.

H-3 The tangent & normal at any point of a hyperbola bisect the angle between the focal radii. This the reflection property of the hyperbola as “An incoming light ray” aimed towards one focus is reflected from the outer surface of the hyperbola towards the other focus. It follows that if an ellipse and hyperbola have the same foci, they cut at right angles at any of their common point.

Asymptotes

Definition : If the length of the perpendicular let fall from a point on a hyperbola to a straight line tends to zero as the point on the hyperbola moves to infinity along the hyperbola, then the straight line is called the Asymptote of the Hyperbola.

To find the asymptote of the hyperbola :

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$$

Particular Case

When b = a the asymptotes of the rectangular hyperbola. $x^2 - y^2 = a^2$ are, $y = \pm x$ which are at right angles.

Note

(i) Equilateral hyperbola $\Leftrightarrow$ rectangular hyperbola

(ii) If a hyperbola is equilateral then the conjugate hyperbola is also equilateral.

(iii) A hyperbola and its conjugate have the same asymptote.

(iv) The equation of the pair of asymptotes differ the hyperbola & the conjugate hyperbola by the same constant only.

(v) The asymptotes pass through the centre of the hyperbola & the bisectors of the angles between the asymptotes are the axes of the hyperbola.

(vi) The asymptotes of a hyperbola are the diagonals of the rectangle formed by the lines drawn through the extremities of each axis parallel to the other axis.

(vii) Asymptotes are the tangent to the hyperbola from the centre.

(viii) A simple method to find the coordinates of the centre of the hyperbola expressed as a general equation of degree 2 should be remembered as :

Let f(x, y) = 0 represents a hyperbola.

Find $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$. Then the point of intersection of

$\frac{\partial f}{\partial x} = 0$ & $\frac{\partial f}{\partial y} = 0$ gives the centre of the hyperbola.

Q. Find the asymptotes of the hyperbola,

$$3x^2 - 5xy - 2y^2 - 5x + 11y - 8 = 0.$$

Also find the equation of the conjugate hyperbola.

Q. Find the equation to the hyperbola whose asymptotes are the straight line $2x + 3y + 3 = 0$ and $3x + 4y + 5 = 0$ and which passes through the point $(1, -1)$. Also write the equation to the conjugate hyperbola and the coordinates of its centre.

Rectangular Hyperbola

(a) Equation is $xy = c^2$ with parametric representation $x = ct$, $y = c/t$, $t \in R - \{0\}$.

(b) Equation of a chord joining the point $P(t_1)$ & $Q(t_2)$ is $x + t_1t_2y = c\,(t_1 + t_2)$ with slope

$$m = -\frac{1}{t_1t_2}$$

(c) Equation of the tangent at P (x_1, y_1) is

$$\frac{x}{x_1} + \frac{y}{y_1} = 2 \text{ \& at P (t) is } \frac{x}{t} + ty = 2c.$$

(d) Equation of normal :

$$y-\frac{c}{t}=t^{2}(x-ct)$$

(e) Chord with a given middle points as (h, k) is $kx + hy = 2hk$.

(f) Equation of the normal at $P(t)$ is $xt^3 - yt = c(t^4 - 1)$.

(f) If a circle and the rectangular hyperbola $xy = c^2$ meet in the four parametric points t_1, t_2, t_3 & t_4, then prove $t_1 \, t_2 \, t_3 \, t_4 = 1$