INVERSE TRIGONOMETRIC FUNCTION (ITF)

All trigonometric function are periodic and hence not invertible. To make them invertible we cut their domain.

$\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$ etc. denote angles or real numbers whose sine is x, whose cosine is x and whose tangent is x.

Principal Value Range and Domain of ITF

ITF	Domain	Range
$\sin^{-1}x$	$[-1, 1]$	$[-\pi/2, \pi/2]$
$\cos^{-1}x$	$[-1, 1]$	$[0,\pi]$
$\tan^{-1}x$	R	$(-\pi/2, \pi/2)$
$\text{cosec}^{-1}x$	$(-\infty, -1]\cup[1,\infty]$	$[-\pi/2, \pi/2]-\{0\}$
$\sec^{-1}x$	$(-\infty, -1]\cup[1,\infty]$	$[0, \pi]-\left\{\frac{\pi}{2}\right\}$
$\cot^{-1}x$	R	$(0, \pi)$

$\cos^{-1}x$
$\cot^{-1}x$
$\sec^{-1}x$

All ITF +ve

$\sin^{-1}x$
$\tan^{-1}x$
$\text{cosec}^{-1}x$

Note

(i) Ist quadrant common to all ITF.

(ii) 3rd quadrant is not used in ITF.

(iii) 4th quadrant is not used in the clock wise direction.

Note

(i) All ITF are bounded

(ii) ITF will be reflection of function about line $y = x$.

Graphs of all 6 ITF

(I) $y = \sin^{-1} x$

Highlights

(i) $\sin^{-1} x$ is Aperiodic

(ii) $\sin^{-1} x$ is bounded

(iii) $\sin^{-1} x$ is odd function

(iv) $\sin^{-1} x$ is increasing

(v) Max. value is 1 and Min. value is –1

(vi) $y = \sin^{-1} x$ $\quad \dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-x^2}}$

(vii) Vertical Tangent

Graphs of all 6 ITF

(II) $y = \cos^{-1} x$

Highlights

(i) $\cos^{-1} x$ is Aperiodic

(ii) $\cos^{-1} x$ is bounded

(iii) $\cos^{-1} x$ is neither odd nor even

(iv) $\cos^{-1} x$ is always decreasing

(v) Vertical tangent at $x = 1$ or -1

(vi) Max value is π & min value is 0

(vii) $\dfrac{dy}{dx}$

Graphs of all 6 ITF

(III) $y = \tan^{-1} x$

Highlights

(i) $\tan^{-1} x$ is Aperiodic

(ii) $\tan^{-1} x$ is bounded

(iii) $\tan^{-1} x$ is odd function

(iv) $\tan^{-1} x$ is always decreasing

(v) No vertical tangent

(vi) No maxima & no minima

(vii) $\frac{dy}{dx}$

Graphs of all 6 ITF

(IV) $y = \cot^{-1} x$

Highlights

(i) $\cot^{-1} x$ is Aperiodic

(ii) $\cot^{-1} x$ is bounded

(iii) Neither odd nor even

(iv) Decreasing

(v) No vertical tangent

(vi) No maxima & no minima

(vii) $\frac{dy}{dx}$

Note

(1) $\tan^{-1}(x)$ and $\cot^{-1}(x)$ are continuous and monotonic on R $\not\Rightarrow$ that their range is R

Note

(2) If $f(x)$ is continuous and has a range $R \not\Rightarrow$ it is monotonic.

e.g. $y = x^3 - 3x$

Examples

Q. Domain & range of $y = \sin^{-1}(e^x)$

Q. Domain & range of $\cos^{-1}[x]$

Q. Domain & range of $\cos^{-1}\{x\}$, $\sin^{-1}\{x\}$

Q. Domain & range of $\cot^{-1}(\text{sgn } x)$

Q. Domain & range of $\tan^{-1}(\log_2(x^2 - 2x + 2))$

Q. Value of

$$\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)+\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)-\tan^{-1}\left(-\sqrt{3}\right)+\cot^{-1}\left(-\frac{1}{\sqrt{3}}\right)$$

Q. Value of $\sin(\tan^{-1}(2))$

Q. Value of $\cos(\tan^{-1}(3))$

Q. Value of

$$\sin\left(2\sin^{-1}\left(\frac{3}{5}\right)\right)$$

Q. Value of $\cos(2\tan^{-1}(2))$

Q. Value of $\cos(2\tan^{-1}(3))$

Q. Value of

$$\sin\left(\arcsin\frac{3}{5}-\arccos\frac{3}{5}\right)$$

Q. Value of

$$\sin\left(\tan^{-1}\cos\cot^{-1}\left(-\frac{1}{\sqrt{3}}\right)\right)$$

Q. Show that :

$$\cos\left(\tan^{-1}\left(\sin\left(\cot^{-1}x\right)\right)\right)=\sqrt{\frac{x^2+1}{x^2+2}}$$

Q. If $\cos^{-1}x + \cos^{-1}y = 0$ find value of $x + y$

Q. If $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = 3\pi$ find value of $x+y+z$

Q. If $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = 3\pi/2$ find value of $x+y+z$

Q. True/False

$y = \operatorname{sgn}(\cot^{-1}x)$ & $y = \sin^2 x + \cos^2 x$ are identical functions.

Q. If $f : R \to [0, \pi/2)$

$f(x) = \tan^{-1}(x^2 + 2x + \alpha)$, function is onto then find α.

Q. If $f : R \to \left[\frac{-\pi}{4}, \frac{\pi}{2}\right)$,

$f(x) = \tan^{-1}(x^2 + 2x + \alpha)$ f(x) is onto find α

Properties of Inverse

Property - 1

(I) $f(x) = \sin(\sin^{-1}x)$

(II) $f(x) = \cos(\cos^{-1}x)$

(III) $f(x) = \tan(\tan^{-1} x)$

(IV) $f(x) = \cot(\cot^{-1}x)$

(V) $f(x) = \operatorname{cosec}(\operatorname{cosec}^{-1}x)$

(VI) $f(x) = \sec(\sec^{-1}x)$

(VII) $f(x) = \tan^{-1}(\tan x)$

Example

Q. $\tan^{-1}\left(\tan\frac{13\pi}{3}\right)$

Q. $\tan^{-1}\left(\tan\frac{25\pi}{4}\right)$

Q. $\tan^{-1}\left(\tan\frac{19\pi}{4}\right)$

Q. $\tan^{-1}\left(\tan\frac{37\pi}{4}\right)$

Q. $\tan^{-1}\left(\tan\frac{17\pi}{3}\right)$

Find the value of the following :

Q. $\tan^{-1}(\tan 1)$

Q. $\tan^{-1}(\tan 2)$

Q. $\tan^{-1}(\tan 3)$

Q. $\tan^{-1}(\tan 4)$

Q. $\tan^{-1}(\tan 5)$

Q. $\tan^{-1}(\tan 6)$

Q. $\tan^{-1}(\tan 7)$

Q. $\tan^{-1}(\tan 8)$

(VIII) $f(x) = \cot^{-1}(\cot x)$

Example

Q. $\cot^{-1}\cot\left(\frac{11\pi}{3}\right)$

Q. $\cot^{-1}\cot\left(\frac{17\pi}{3}\right)$

Q. $\cot^{-1}\cot\left(\frac{13\pi}{7}\right)$

Q. $\cot^{-1}\cot\left(\frac{25\pi}{4}\right)$

Q. $\cot^{-1}\cot\left(\frac{19\pi}{4}\right)$

Q. $\cot^{-1}\cot\left(\frac{37\pi}{4}\right)$

Find the value of the following :

Q. $\cot^{-1} \cot (1)$

Q. $\cot^{-1} \cot (2)$

Q. $\cot^{-1} \cot (3)$

Q. $\cot^{-1} \cot (4)$

Q. $\cot^{-1} \cot (5)$

Q. $\cot^{-1} \cot (6)$

Q. $\cot^{-1} \cot (7)$

Q. $\cot^{-1} \cot (8)$

(IX) $f(x) = \cos^{-1}(\cos x)$

Example

Find the value of the following :

Q. $\cos^{-1}\left(\cos\frac{11\pi}{3}\right)$

Q. $\cos^{-1}\left(\cos\frac{25\pi}{4}\right)$

Q. $\cos^{-1}\left(\cos\frac{7\pi}{3}\right)$

Q. $\cos^{-1}\left(\cos\frac{37\pi}{4}\right)$

Q. $\cos^{-1}\left(\cos\frac{13\pi}{4}\right)$

Find the value of the following :

Q. $\cos^{-1}(\cos 1)$

Q. $\cos^{-1}(\cos 2)$

Q. $\cos^{-1}(\cos 3)$

Q. $\cos^{-1}(\cos 4)$

Q. $\cos^{-1}(\cos 5)$

Q. $\cos^{-1}(\cos 6)$

Q. $\cos^{-1}(\cos 7)$

Q. $\cos^{-1}(\cos 8)$

(X) $f(x) = \sin^{-1}(\sin x)$

Example

Find Values :

Q. $\sin^{-1}\left(\sin\frac{13\pi}{3}\right)$

Q. $\sin^{-1}\left(\sin\frac{25\pi}{4}\right)$

Q. $\sin^{-1}\left(\sin\frac{19\pi}{4}\right)$

Q. $\sin^{-1}\left(\sin\frac{37\pi}{4}\right)$

Q. $\sin^{-1}\left(\sin\frac{17\pi}{3}\right)$

Find Values :

Q. $\sin^{-1}(\sin 1)$

Q. $\sin^{-1}(\sin 2)$

Q. $\sin^{-1}(\sin 3)$

Q. $\sin^{-1}(\sin 4)$

Q. $\sin^{-1}(\sin 5)$

Q. $\sin^{-1}(\sin 6)$

Q. $\sin^{-1}(\sin 7)$

Q. $\sin^{-1}(\sin 8)$

(XI) $f(x) = \sec^{-1}(\sec x)$

(XII) $f(x) = \operatorname{cosec}^{-1}(\operatorname{cosec} x)$

Inequalities

Q. $\log_2 \tan^{-1} x > 1$

Q. $(\cot^{-1}x)^2 - 5\cot^{-1}x + 6 > 0$

Q. Domain of $f(x) = \dfrac{1}{\sqrt{ln \cot^{-1} x}}$

Q. Integral solution of the inequality

$3x^2 + 8x < 2\sin^{-1}(\sin 4) - \cos^{-1}(\cos 4)$

Property - 2

(I) $\operatorname{cosec}^{-1} x = \sin^{-1}\frac{1}{x}$ & $\sin^{-1} x = \operatorname{cosec}^{-1}\frac{1}{x}$

(II) $\sec^{-1} x = \cos^{-1} \frac{1}{x}$ & $\cos^{-1} x = \sec^{-1} \frac{1}{x}$

(III) $\cot^{-1} x = \tan^{-1} \frac{1}{x}$; $x > 0 = \pi + \tan^{-1} \frac{1}{x}$; $x < 0$

Property - 3

(I) $\sin^{-1}(-x) = -\sin^{-1} x$

(II) $\cos^{-1}(-x) = \pi - \cos^{-1} x$ etc.

Property - 4

(I) $\sin^{-1}x + \cos^{-1}x = \dfrac{\pi}{2}$

(II) $\sec^{-1}x + \operatorname{cosec}^{-1}x = \frac{\pi}{2}$

(III) $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$

Example

Q. $tan^{-1}\left(\frac{1}{2}\right)+tan^{-1}(2)+tan^{-1}\left(\sqrt{3}\right)=?$

Q. $\sin^{-1}x + \cos^{-1}\left(x^2 - 2x + 2\right) = \frac{\pi}{2}$

Q. If $\sin^{-1}a + \cos^{-1}b = \frac{\pi}{2}$ & $\sec^{-1}a + \operatorname{cosec}^{-1}b = \frac{\pi}{2}$

Find a + b

(a) 2 (b) –2 (c) 0 (d) None

Q. $\left(\tan^{-1}x\right)^2+\left(\cot^{-1}x\right)^2=\dfrac{5\pi^2}{8}$

Q. $5\tan^{-1}x + 3\cot^{-1}x = \dfrac{7\pi}{4}$

Q. $4\sin^{-1}x + \cos^{-1}x = \dfrac{3\pi}{4}$

Q. $\cot^{-1}\left(\frac{n}{\pi}\right) > \frac{\pi}{6}$

Find maximum value of $n \in N$

Q. Maximum & Minimum values of $(\sin^{-1}x)^3 + (\cos^{-1}x)^3$

Q. Find the range of $f(x) = \sin^{-1}x + \cos^{-1}x + \tan^{-1}x$.

Property - 5

Q. $x > 0$ & $y > 0$ $\tan^{-1}x - \tan^{-1} y = \tan^{-1} \dfrac{x-y}{1+xy}$

$$\tan^{-1}x+\tan^{-1}y=\begin{cases}\tan^{-1}\dfrac{x+y}{1-xy}, x>0, y>0\; and\; xy<1 \Rightarrow 0\ \ \tan^{-1}x+\tan^{-1}y<\dfrac{\pi}{2}\\[2em] \\[2em] \pi+\tan^{-1}\dfrac{x+y}{1-xy}, x>0, y>0\; and\; xy>1 \Rightarrow \dfrac{\pi}{2}<\tan^{-1}x+\tan^{-1}y<\pi\end{cases}$$

Example

Q. $\tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3 = ?$

Q. $\tan^{-1} 1 + \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3} = ?$

Q. Show that :

$$\frac{\tan^{-1} 1 + \tan^{-1} 2 + \tan^{-1} 3}{\cot^{-1} + \cot^{-1} 2 + \cot^{-1} 3} = 2$$

Q. If $\tan^{-1}4+\tan^{-1}5 = \cot^{-1}(\lambda)$ then find λ

Q. $\alpha = \tan^{-1} 5 - \tan^{-1} 3 + \tan^{-1} \dfrac{7}{9} = \dfrac{\pi}{4}$ and

$\beta = \tan^{-1} \dfrac{2}{11} + \cot^{-1} \dfrac{24}{7} + \tan^{-1} \dfrac{1}{3} = \dfrac{\pi}{4}$, then

(A) $\alpha = \beta$ (B) $\alpha > \beta$

(C) $\alpha < \beta$ (D) $\alpha + \beta = \pi/2$

Q. Find x satisfying $\sin^{-1}\left(\frac{1}{\sqrt{5}}\right) + \cos^{-1}(x) = \frac{\pi}{4}$

Q. $\sin^{-1}\left(\frac{3}{\sqrt{73}}\right)+\cos^{-1}\left(\frac{11}{\sqrt{146}}\right)+\cot^{-1}\left(\sqrt{3}\right)$

(A) π (B) $\pi/2$ (C) $5\pi/12$ (D) $\pi/3$

Q. Which is greater

$$\cos^{-1}\frac{7}{25}+\cos^{-1}\frac{3}{5} \quad \text{or} \quad \cot^{-1}(-1)$$

Q. Find A $2\cos^{-1}\frac{3}{\sqrt{13}} = \tan^{-1} A$

Q. Find B $\frac{1}{2}\cos^{-1}\frac{7}{25} = \tan^{-1}B$

Property 6

(a) $\sin^{-1} x + \sin^{-1} y =$

$$\begin{cases} \sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right) if\ x \geq 0; y \geq 0\ and\ x^2+y^2 \leq 1 \\ \pi - \sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right) if\ x \geq 0; y \geq 0\ and\ x^2+y^2 > 1 \end{cases}$$

Q. Find whether $\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{12}{13}$ is acute/obtuse.

Property 6

(b) $\sin^{-1} x - \sin^{-1} y$

$= \sin^{-1}\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right)$, $x > 0$; $y > 0$

$$\cos^{-1} x \pm \cos^{-1} y = \cos^{-1}\left(xy \mp \sqrt{1-x^2}\sqrt{1-y^2}\right),$$

$x > 0;\ y > 0,\ x < y.$

Q. If $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$ then prove that $x^2 + y^2 + z^2 + 2xyz = 1$

Q. Show that $\cos^{-1}\sqrt{\frac{2}{3}}-\cos^{-1}\frac{\sqrt{6}+1}{2\sqrt{3}}=\frac{\pi}{6}$

Property 7

$\tan^{-1} x + \tan^{-1} y + \tan^{-1} z$

$$= \tan^{-1}\left[\frac{x+y+z-xyz}{1-(xy+yz+zx)}\right]$$

Where $x > 0$, $y > 0$, $z > 0$ and $xy + yz + zx < 1$

Simplification of Inverse Functions by Elementary Substitution

$$f(x)=\sin^{-1}\frac{2x}{1+x^2}=\begin{cases} 2\tan^{-1}x & -1\le x\le 1 \\ \pi-2\tan^{-1}x \quad \text{if} & x\ge 1 \\ -\pi-2\tan^{-1}x & x\le -1 \end{cases}$$

$$f(x) = \cos^{-1}\frac{1-x^2}{1+x^2} = \begin{cases} 2\tan^{-1} x & x \geq 0 \\ -2\tan^{-1} x & x < 0 \end{cases}$$

$$f(x) = \tan^{-1}\frac{2x}{1-x^2} = \begin{cases} \pi + 2\tan^{-1} x & x < -1 \\ 2\tan^{-1} x & -1 < x < 1 \\ 2\tan^{-1} x - \pi & x > 1 \end{cases}$$

Q. $f(x) = \sin^{-1}\dfrac{2x}{1+x^2} + 2\tan^{-1} x$, find $\cos(f(10))$

$$f(x)=\sin^{-1}\left(3x-4x^{3}\right)=\begin{cases} -\left(\pi+3\sin^{-1}x\right) & \text{if } -1\le x\le -\tfrac{1}{2} \\ 3\sin^{-1}x & \text{if } -\tfrac{1}{2}\le x\le \tfrac{1}{2} \\ \pi-3\sin^{-1}x & \text{if } \tfrac{1}{2}\le x\le 1 \end{cases}$$

$$f(x) = \cos^{-1}\left(4x^3 - 3x\right) = \begin{cases} 3\cos^{-1}x - 2\pi & \text{if } -1 \le x \le -\tfrac{1}{2} \\ 2\pi - 3\cos^{-1}x & \text{if } -\tfrac{1}{2} \le x \le \tfrac{1}{2} \\ 3\cos^{-1}x & \text{if } \tfrac{1}{2} \le x \le 1 \end{cases}$$

$$f(x) = \tan^{-1}\frac{3x - x^3}{1 - 3x^2} = \begin{cases} 3\tan^{-1}x & \text{if } -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}} \\ -\pi + 3\tan^{-1}x & \text{if } x > \frac{1}{\sqrt{3}} \\ \pi + 3\tan^{-1}x & \text{if } x < -\frac{1}{\sqrt{3}} \end{cases}$$

Identities involving inverse trigonometric functions

Q. Show that

$$2\tan^{-1}\left(\tan\left(\frac{\pi}{4}-\alpha\right)\tan\frac{\beta}{2}\right)=\cos^{-1}\left(\frac{\sin^2\alpha+\cos\beta}{1+\sin 2\alpha\cos\beta}\right)$$

Q. Show that

$$\tan^{-1} x = 2\tan^{-1} [\operatorname{cosec} (\tan^{-1}x) - \tan(\cot^{-1}x)]$$

Equations involving inverse trigonometric functions

Q. $2\cot^{-1}2 - \cos^{-1}\dfrac{4}{5} = \operatorname{cosec}^{-1}x$

Q. $\cos^{-1} x - \sin^{-1} x = \cos^{-1} x\sqrt{3}$

Q. $\sin[2\cos^{-1}\{\cot(2\tan^{-1}x)\}] = 0$

Q. $\sin^{-1} x + \sin^{-1} 2x = \dfrac{\pi}{3}$

Q. $\tan^{-1}\dfrac{x+1}{x-1}+\tan^{-1}\dfrac{x-1}{x}=\tan^{-1}(-7)$

Q. $2\cot^{-1}2 + \cos^{-1}(3/5) = \operatorname{cosec}^{-1}x$

Simultaneous equations and Inequations involving I.T.F.

Q. $\cos^{-1} x > \cos^{-1} x^2$

Q. $\sin^{-1}x > \cos^{-1}x$

Q. $\sin^{-1}x > \sin^{-1}(1 - x)$

Q. $\text{arc}\tan^2 x - 3\,\text{arc}\tan x + 2 > 0$

Q. $[\sin^{-1}x] > [\cos^{-1}x]$

Summation of series

Idea is

$$\tan^{-1} x - \tan^{-1} y = \tan^{-1} \frac{x+y}{1+xy}$$

Q. $$\tan^{-1}\frac{x}{1+(1\times 2)x^2}+\tan^{-1}\frac{x}{1+(2\times 3)x^2}+\ldots\ldots\ldots$$

$$+\tan^{-1}\frac{x}{1+n(n+1)x^2}$$

Q. $\tan^{-1}\dfrac{2}{2+1^2+1^4}+\tan^{-1}\dfrac{4}{2+2^2+2^1}+\tan^{-1}\dfrac{6}{2+3^2+3^4}+\ldots$

Q. $$S = \sum_{n=1}^{\infty} \tan^{-1}\left(\frac{4n}{n^4 - 2n^2 + 2}\right)$$

Q. $\sum_{r=1}^{\infty} \tan^{-1}\left(\frac{1}{2r^2}\right)$

Q. $\text{cosec}^{-1}\sqrt{5} + \text{cosec}^{-1}\sqrt{65} + \text{cosec}^{-1}\sqrt{325} + \ldots$

Q. $\cot^{-1}(2a^{-1} + a) + \cot^{-1}(2a^{-1} + 3a) + \cot^{-1}(2a^{-1} + 6a) + \cot^{-1}(2a^{-1} + 10a) + \ldots$