LIMIT & CONTINUITY

MANOJ CHAUHAN SIR(IIT-DELHI)
EX. SR. FACULTY (BANSAL CLASSES)

KEY CONCEPTS (LIMIT)

THINGS TO REMEMBER :

1. Limit of a function f(x) is said to exist as, $x \to a$ when

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = \text{finite quantity.}$$

2. FUNDAMENTAL THEOREMS ON LIMITS:

Let $\lim_{x \to a} f(x) = l$ & $\lim_{x \to a} g(x) = m$. If l & m exists then :

Remember
$\lim_{x \to a} \Rightarrow x \neq a$

(i) $\lim_{x \to a} f(x) \pm g(x) = l \pm m$ **(ii)** $\lim_{x \to a} f(x) \cdot g(x) = l \cdot m$

(iii) $\lim_{x \to a} \frac{f(x)}{g(g)} = \frac{\ell}{m}$, provided $m \neq 0$

(iv) $\lim_{x \to a} k\,f(x) = k \lim_{x \to a} f(x)$; where k is a constant.

(v) $\lim_{x \to a} f[g(x)] = f\left(\lim_{x \to a} g(x)\right) = f(m)$; provided f is continuous at $g(x) = m$.

For example $\lim_{x \to a} l n(f(x) = ln\left[\lim_{x \to a} f(x)\right] l n\, l\ (l > 0)$.

3. STANDARD LIMITS :

(a) $\lim_{x \to 0} \frac{\sin x}{x} = 1 = \lim_{x \to 0} \frac{\tan x}{x} = \lim_{x \to 0} \frac{\tan^{-1} x}{x} = \lim_{x \to 0} \frac{\sin^{-1} x}{x}$

[Where x is measured in radians]

(b) $\lim_{x \to 0} (1 + x)^{1/x} = e = \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$ note however there $\lim_{\substack{h \to 0 \\ n \to \infty}} (1 - h)^n = 0$

and $\lim_{\substack{h \to 0 \\ n \to \infty}} (1 + h)^n \to \infty$

(c) If $\lim_{x \to a} f(x) = 1$ and $\lim_{x \to a} \phi(x) = \infty$, then ;

$$\lim_{x \to a} [f(x)]^{\phi(x)} = e^{\lim_{x \to a} \phi(x)[f(x) - 1]}$$

(d) If $\lim_{x \to a} f(x) = A > 0$ & $\lim_{x \to a} \phi(x) = B$ (a finite quantity) then ;

$\lim_{x \to a} [f(x)]^{\phi(x)} = e^z$ where $z = \lim_{x \to a} \phi(x) . \ln[f(x)] = e^{B \ln A} = A^B$

(e) $\lim_{x \to 0} \frac{a^x - 1}{x} = ln\, a\ (a > 0)$. In particular $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$

(f) $\lim_{x \to a} \frac{x^n - a^n}{x - a} = n a^{n-1}$

4. SQUEEZE PLAY THEOREM :

If $f(x) \leq g(x) \leq h(x)\ \forall\ x$ & $\lim_{x \to a} f(x) = l = \lim_{x \to a} h(x)$ then $\lim_{x \to a} g(x) = l$.

5. INDETERMINANT FORMS :

$\frac{0}{0}, \frac{\infty}{\infty}, 0 \times \infty, 0^\circ, \infty^\circ, \infty - \infty$ and 1^∞

Note :

(i) We cannot plot ∞ on the paper. Infinity (∞) is a symbol & not a number. It does not obey the laws of elementry algebra.

(ii) $\infty + \infty = \infty$ **(iii)** $\infty \times \infty = \infty$ **(iv)** $(a/\infty) = 0$ if a is finite

(v) $\frac{a}{0}$ is not defined , if $a \neq 0$.

(vi) $a\,b = 0$, if & only if $a = 0$ or $b = 0$ and a & b are finite.

6. The following strategies should be born in mind for evaluating the limits:

(a) Factorisation

(b) Rationalisation or double rationalisation

(c) Use of trigonometric transformation ;
appropriate substitution and using standard limits

(d) Expansion of function like Binomial expansion, exponential & logarithmic expansion, expansion of sinx , cosx , tanx should be remembered by heart & are given below :

(i) $a^x = 1 + \frac{x\,\ln a}{1!} + \frac{x^2\,\ln^2 a}{2!} + \frac{x^3 \ln^3 a}{3!} + ...a > 0$

(ii) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ...\ x \in R$

(iii) $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...\text{for} -1 < x \leq 1$

(iv) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...\,x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(v) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...\,x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(vi) $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + ...\,x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(vii) $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ...$

EXERCISE–I

Q.1 $\lim\limits_{x \to 1} \frac{x^2 - \sqrt{x}}{\sqrt{x} - 1}$

Q.2 $\lim\limits_{x \to 1} \frac{\sqrt[13]{x} - \sqrt[7]{x}}{\sqrt[5]{x} - \sqrt[3]{x}}$

Q.3 $\lim\limits_{x \to 1} \frac{x^2 - x.\ln x + \ln x - 1}{x - 1}$

Q.4 $\lim\limits_{x \to 1} \frac{\left[\sum\limits_{K=1}^{100} x^k\right] - 100}{x - 1}$

Q.5 $\lim\limits_{x \to \infty} \frac{2\sqrt{x} + 3x^{1/3} + 5x^{1/5}}{\sqrt{3x - 2} + (2x - 3)^{1/3}}$

Q.6 $\lim\limits_{x \to \frac{3\pi}{4}} \frac{1 + \sqrt[3]{\tan x}}{1 - 2\cos^2 x}$

Q.7 $\lim\limits_{x \to 0} \frac{\sec 4x - \sec 2x}{\sec 3x - \sec x}$

Q.8 $\lim\limits_{x \to 1} \left(\frac{p}{1 - x^p} - \frac{q}{1 - x^q}\right)$ $p, q \in N$

Q.9 Find the sum of an infinite geometric series whose first term is the limit of the function $f(x) = \frac{\tan x - \sin x}{\sin^3 x}$ as $x \to 0$ and whose common ratio is the limit of the function $g(x) = \frac{1 - \sqrt{x}}{(\cos^{-1} x)^2}$ as $x \to 1$.

Q.10 $\lim\limits_{x\to\infty} (x - \ln \cosh x)$ where $\cosh t = \dfrac{e^t + e^{-t}}{2}$.

Q.11 **(a)** $\lim\limits_{x\to\frac{1}{\sqrt{2}}} \dfrac{\cos^{-1}\left(2x\sqrt{1-x^2}\right)}{x - \frac{1}{\sqrt{2}}}$; **(b)** $\lim\limits_{x\to\frac{\pi}{4}} \dfrac{\sqrt{1-\sqrt{\sin 2x}}}{\pi - 4x}$; **(c)** $\lim\limits_{x\to -7} \dfrac{[x]^2 + 15[x] + 56}{\sin(x+7)\sin(x+8)}$ where [ ] denotes the greatest integer function

Q.12 $\lim\limits_{x\to\frac{\pi}{4}} \dfrac{1 - \tan x}{1 - \sqrt{2}\sin x}$

Q.13 $\lim\limits_{x\to 0} \dfrac{8}{x^8}\left[1 - \cos\dfrac{x^2}{2} - \cos\dfrac{x^2}{4} + \cos\dfrac{x^2}{2}\cos\dfrac{x^2}{4}\right]$

Q.14 $\lim\limits_{\theta\to\frac{\pi}{4}} \dfrac{\sqrt{2} - \cos\theta - \sin\theta}{(4\theta - \pi)^2}$

Q.15 $\lim\limits_{x\to\frac{\pi}{2}} \dfrac{2^{-\cos x} - 1}{x\left(x - \frac{\pi}{2}\right)}$

Q.16 If $\lim\limits_{x\to 0} \dfrac{a\sin x - \sin 2x}{\tan^3 x}$ is finite then find the value of 'a' & the limit.

Q.17 (a) $\lim\limits_{x\to 0} \tan^{-1}\dfrac{a}{x^2}$, where $a \in R$; (b) Plot the graph of the function $f(x) = \lim\limits_{t\to 0}\left(\dfrac{2x}{\pi}\tan^{-1}\dfrac{x}{t^2}\right)$

Q.18 $\lim\limits_{x\to 0} [\ln(1+\sin^2 x).\cot(\ln^2(1+x))]$

Q.19 $\lim\limits_{x\to 1} \dfrac{(\ln(1+x) - \ln 2)(3.4^{x-1} - 3x)}{[(7+x)^{\frac{1}{3}} - (1+3x)^{\frac{1}{2}}].\sin(x-1)}$

Q.20 If $l = \lim\limits_{n\to\infty} \sum\limits_{r=2}^{n}\left((r+1)\sin\dfrac{\pi}{r+1} - r\sin\dfrac{\pi}{r}\right)$ then find $\{l\}$. (where { } denotes the fractional part function)

Q.21 $\lim\limits_{x\to -\infty} \dfrac{(3x^4 + 2x^2)\sin\frac{1}{x} + |x|^3 + 5}{|x|^3 + |x|^2 + |x| + 1}$

Q.22 $\lim\limits_{x\to 3} \dfrac{(x^3 + 27)\ln(x-2)}{x^2 - 9}$

Q.23 $\lim\limits_{x\to 0} \dfrac{27^x - 9^x - 3^x + 1}{\sqrt{2} - \sqrt{1+\cos x}}$

Q.24 Let $f(x) = \dfrac{x}{\sin x}$, $x > 0$ and $g(x) = x + 3$, $x < 1$

$= 2 - x$, $x \le 0$ $\qquad = x^2 - 2x - 2$, $1 \le x < 2$

$= x - 5$, $x \ge 2$

find LHL and RHL of $g(f(x))$ at $x = 0$ and hence find $\lim\limits_{x\to 0} g(f(x))$.

Q.25 Let $P_n = a^{P_{n-1}} - 1$, $\forall\, n = 2, 3, \ldots\ldots$ and Let $P_1 = a^x - 1$ where $a \in R^+$ then evaluate $\lim\limits_{x\to 0} \dfrac{P_n}{x}$.

Q.26 If the $\lim\limits_{x\to 0} \dfrac{1}{x^3}\left(\dfrac{1}{\sqrt{1+x}} - \dfrac{1+ax}{1+bx}\right)$ exists and has the value equal to l, then find the value of $\dfrac{1}{a} - \dfrac{2}{l} + \dfrac{3}{b}$.

Q.27 Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ be sequences such that
(i) $a_n + b_n + c_n = 2n + 1$; **(ii)** $a_nb_n + b_nc_n + c_na_n = 2n - 1$; **(iii)** $a_nb_nc_n = -1$; **(iv)** $a_n < b_n < c_n$
Then find the value of $\lim\limits_{n\to\infty} na_n$.

Q.28 If $n \in N$ and $a_n = 2^2 + 4^2 + 6^2 + \ldots\ldots + (2n)^2$ and $b_n = 1^2 + 3^2 + 5^2 + \ldots\ldots + (2n-1)^2$. Find the value $\lim\limits_{n\to\infty} \dfrac{\sqrt{a_n} - \sqrt{b_n}}{\sqrt{n}}$.

Q.29 At the end points A, B of the fixed segment of length L, lines are drawn meeting in C and making angles θ and 2θ respectively with the given segment. Let D be the foot of the altitude CD and let *x* represents the length of AD. Find the value of *x* as θ tends to zero i.e. $\lim_{\theta \to 0} x$.

Q.30 At the end-points and the midpoint of a circular arc AB tangent lines are drawn, and the points A and B are joined with a chord. Prove that the ratio of the areas of the two triangles thus formed tends to 4 as the arc AB decreases indefinitely.

EXERCISE–II

Q.1 $\lim_{x \to \infty} \left[\frac{2x^2+3}{2x^2+5}\right]^{8x^2+3}$

Q.2 $\lim_{x \to \infty} \left(\frac{x+c}{x-c}\right)^x = 4$ then find c

Q.3 $\lim_{x \to 0} \left[\frac{(1+x)^{1/x}}{e}\right]^{1/x}$

Q.4 $\lim_{n \to \infty} \left(\frac{\sqrt{n^2+n}-1}{n}\right)^{2\sqrt{n^2+n}-1}$

Q.5 $\lim_{x \to \infty} x^2 \sin \ell n \sqrt{\cos \frac{\pi}{x}}$

Q.6 $\lim_{x \to \infty} \left[\cos\left(2\pi\left(\frac{x}{1+x}\right)^a\right)\right]^{x^2}$ $a \in R$

Q.7 $\lim_{x \to 1} \left(\tan \frac{\pi x}{4}\right)^{\tan \frac{\pi x}{2}}$

Q.8 $\lim_{x \to 0} \left(\frac{x-1+\cos x}{x}\right)^{\frac{1}{x}}$

Q.9 $\lim_{x \to \infty} \left(\frac{a_1^{\frac{1}{x}} + a_2^{\frac{1}{x}} + a_3^{\frac{1}{x}} + \ldots + a_n^{\frac{1}{x}}}{n}\right)^{nx}$ where $a_1, a_2, a_3, \ldots\ldots a_n > 0$

Q.10 Let $f(x) = \frac{\sin^{-1}(1-\{x\}).\cos^{-1}(1-\{x\})}{\sqrt{2\{x\}}\,.\,(1-\{x\})}$ then find $\lim_{x \to 0^+} f(x)$ and $\lim_{x \to 0^-} f(x)$, where $\{x\}$ denotes the fractional part function.

Q.11 Find the values of a, b & c so that $\lim_{x \to 0} \frac{ae^x - b\cos x + ce^{-x}}{x.\sin x} = 2$

Q.12 $\lim_{x \to a} \frac{1}{(a^2-x^2)^2}\left(\frac{a^2+x^2}{ax} - 2\sin\left(\frac{a\pi}{2}\right)\sin\left(\frac{\pi x}{2}\right)\right)$ where a is an odd integer

Q.13 $\lim_{x \to 0} \frac{\tan^2 x - x^2}{x^2 \tan^2 x}$

Q.14 If $L = \lim_{x \to 1} \frac{(1-x)(1-x^2)(1-x^3)\ldots\ldots(1-x^{2n})}{[(1-x)(1-x^2)(1-x^3)\ldots\ldots\ldots(1-x^n)]^2}$ then show that L can be equal to

(a) $\prod_{r=1}^{n} \frac{n+r}{r}$ (b) $\frac{1}{n!}\prod_{r=1}^{n}(4r-2)$

(c) The sum of the coefficients of two middle terms in the expansion of $(1+x)^{2n-1}$.

(d) The coefficient of x^n in the expansion of $(1+x)^{2n}$.

Q.15 $\lim_{n \to \infty} \frac{[1.x]+[2.x]+[3.x]+\ldots\ldots+[n.x]}{n^2}$, Where [.] denotes the greatest integer function.

Q.16 Evaluate, $\lim_{x \to 1} \frac{1-x+\ln x}{1+\cos \pi x}$

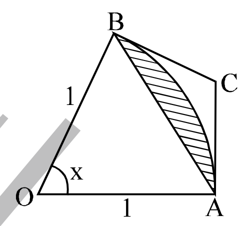
Q.17 $\lim\limits_{y\to 0}\left[\lim\limits_{x\to\infty}\dfrac{\exp\left(x\ln(1+\frac{ay}{x})\right)-\exp\left(x\ln(1+\frac{by}{x})\right)}{y}\right]$

Q.18 Let $x_0 = 2\cos\dfrac{\pi}{6}$ and $x_n = \sqrt{2+x_{n-1}}$, n = 1, 2, 3, ..., find $\lim\limits_{n\to\infty} 2^{(n+1)}\cdot\sqrt{2-x_n}$.

Q.19 $\lim\limits_{x\to 0}\left[\dfrac{\ell n\,(1+x)^{1+x}}{x^2}-\dfrac{1}{x}\right]$

Q.20 Let $L = \prod\limits_{n=3}^{\infty}\left(1-\dfrac{4}{n^2}\right)$; $M = \prod\limits_{n=2}^{\infty}\left(\dfrac{n^3-1}{n^3+1}\right)$ and $N = \prod\limits_{n=1}^{\infty}\dfrac{(1+n^{-1})^2}{1+2n^{-1}}$, then find the value of $L^{-1} + M^{-1} + N^{-1}$.

Q.21 A circular arc of radius 1 subtends an angle of x radians, $0 < x < \dfrac{\pi}{2}$ as shown in the figure. The point C is the intersection of the two tangent lines at A & B. Let T(x) be the area of triangle ABC & let S(x) be the area of the shaded region. Compute:



(a) T(x) (b) S(x) & (c) the limit of $\dfrac{T(x)}{S(x)}$ as $x \to 0$.

Q.22 Let $f(x) = \lim\limits_{n\to\infty}\sum\limits_{n=1}^{n} 3^{n-1}\sin^3\dfrac{x}{3^n}$ and $g(x) = x - 4f(x)$. Evaluate $\lim\limits_{x\to 0}\left(1+g(x)\right)^{\cot x}$.

Q.23 If $f(n,\theta) = \prod\limits_{r=1}^{n}\left(1-\tan^2\dfrac{\theta}{2^r}\right)$, then compute $\lim\limits_{n\to\infty} f(n,\theta)$

Q.24 $L = \lim\limits_{x\to 0}\dfrac{\sqrt{\dfrac{\cos 2x+(1+3x)^{1/3}}{2}}-\sqrt[3]{\dfrac{4\cos^3 x - \ln(1+x)^4}{4}}}{x}$

If $L = a/b$ where 'a' and 'b' are relatively primes find (a + b).

Q.25 $\lim\limits_{x\to\infty}\left(\dfrac{\cosh(\pi/x)}{\cos(\pi/x)}\right)^{x^2}$ where $\cosh t = \dfrac{e^t + e^{-t}}{2}$

Q.26 $f(x)$ is the function such that $\lim\limits_{x\to 0}\dfrac{f(x)}{x} = 1$. If $\lim\limits_{x\to 0}\dfrac{x(1+a\cos x)-b\sin x}{(f(x))^3} = 1$, then find the value of *a* and *b*.

Q.27 Through a point A on a circle, a chord AP is drawn & on the tangent at A a point T is taken such that AT = AP. If TP produced meet the diameter through A at Q, prove that the limiting value of AQ when P moves upto A is double the diameter of the circle.

Q.28 Using Sandwich theorem, evaluate

(a) $\lim\limits_{n\to\infty}\left(\dfrac{1}{\sqrt{n^2}}+\dfrac{1}{\sqrt{n^2+1}}+\dfrac{1}{\sqrt{n^2+2}}+..........+\dfrac{1}{\sqrt{n^2+2n}}\right)$

(b) $\lim\limits_{n\to\infty}\dfrac{1}{1+n^2}+\dfrac{2}{2+n^2}+ .... +\dfrac{n}{n+n^2}$

Q.29 Find a & b if : (i) $\lim_{x\to\infty}\left[\frac{x^2+1}{x+1}-ax-b\right]=0$ (ii) $\lim_{x\to-\infty}\left[\sqrt{x^2-x+1}-ax-b\right]=0$

Q.30 If $L=\lim_{x\to 0}\left(\frac{1}{\ln(1+x)}-\frac{1}{\ln(x+\sqrt{1+x^2})}\right)$ then find the value of $\frac{L+153}{L}$.

EXERCISE–III

Q.1 $\lim_{x\to 0}\frac{x\tan 2x-2x\tan x}{(1-\cos 2x)^2}$ is : [JEE '99, 2 (out of 200)]

(A) 2 (B) −2 (C) $\frac{1}{2}$ (D) $-\frac{1}{2}$

Q.2 For $x\in R$, $\lim_{x\to\infty}\left(\frac{x-3}{x+2}\right)^x=$ [JEE 2000, Screening]

(A) e (B) e^{-1} (C) e^{-5} (D) e^5

Q.3 $\lim_{x\to 0}\frac{\sin(\pi\cos^2 x)}{x^2}$ equals [JEE 2001, Screening]

(A) $-\pi$ (B) π (C) $\frac{\pi}{2}$ (D) 1

Q.4 Evaluate $\lim_{x\to 0}\frac{a^{\tan x}-a^{\sin x}}{\tan x-\sin x}$, $a>0$. [REE 2001, 3 out of 100]

Q.5 The integer n for which $\lim_{x\to 0}\frac{(\cos x-1)(\cos x-e^x)}{x^n}$ is a finite non-zero number is

(A) 1 (B) 2 (C) 3 (D) 4

[JEE 2002 (screening), 3]

Q.6 If $\lim_{x\to 0}\frac{\sin(nx)[(a-n)nx-\tan x]}{x^2}=0$ $(n>0)$ then the value of 'a' is equal to

(A) $\frac{1}{n}$ (B) n^2+1 (C) $\frac{n^2+1}{n}$ (D) None

[JEE 2003 (screening)]

Q.7 Find the value of $\lim_{n\to\infty}\left[\frac{2}{\pi}(n+1)\cos^{-1}\left(\frac{1}{n}\right)-n\right]$. [JEE ' 2004, 2 out of 60]

KEY CONCEPTS (CONTINUITY)

THINGS TO REMEMBER :

1. A function f(x) is said to be continuous at x = c, if $\lim_{x\to c} f(x) = f(c)$. Symbolically f is continuous at x = c if $\lim_{h\to 0} f(c-h) = \lim_{h\to 0} f(c+h) = f(c)$.

i.e. LHL at x = c = RHL at x = c equals Value of 'f' at x = c.

It should be noted that continuity of a function at x = a is meaningful only if the function is defined in the immediate neighbourhood of x = a, not necessarily at x = a.

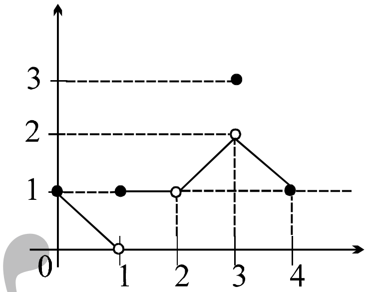
2. Reasons of discontinuity:

(i) $\lim_{x\to c} f(x)$ does not exist

i.e. $\lim_{x\to c^-} f(x) \neq \lim_{x\to c^+} f(x)$

(ii) f(x) is not defined at x= c

(iii) $\lim_{x\to c} f(x) \neq f(c)$



Geometrically, the graph of the function will exhibit a break at x= c. The graph as shown is discontinuous at x = 1 , 2 and 3.

3. Types of Discontinuities :

Type - 1: (Removable type of discontinuities)

In case $\lim_{x\to c} f(x)$ exists but is not equal to f(c) then the function is said to have a removable discontinuity or discontinuity of the first kind. In this case we can redefine the function such that $\lim_{x\to c} f(x) = f(c)$ & make it continuous at x= c. Removable type of discontinuity can be further classified as :

(a) **MISSING POINT DISCONTINUITY :** Where $\lim_{x\to a} f(x)$ exists finitely but f(a) is not defined.

e.g. $f(x) = \frac{(1-x)(9-x^2)}{(1-x)}$ has a missing point discontinuity at x = 1 , and $f(x) = \frac{\sin x}{x}$ has a missing point discontinuity at x = 0

(b) **ISOLATED POINT DISCONTINUITY :** Where $\lim_{x\to a} f(x)$ exists & f(a) also exists but ; $\lim_{x\to a} \neq f(a)$.

e.g. $f(x) = \frac{x^2-16}{x-4}$, $x \neq 4$ & f (4) = 9 has an isolated point discontinuity at x = 4.

Similarly $f(x) = [x] + [-x] = \begin{cases} 0 & \text{if } x \in I \\ -1 & \text{if } x \notin I \end{cases}$ has an isolated point discontinuity at all $x \in I$.

Type-2: (Non - Removable type of discontinuities)

In case $\lim_{x\to c} f(x)$ does not exist then it is not possible to make the function continuous by redefining it. Such discontinuities are known as non - removable discontinuity or discontinuity of the 2nd kind. Non-removable type of discontinuity can be further classified as :

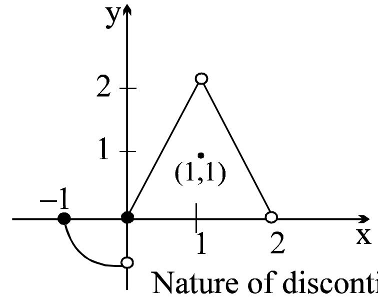
(a) Finite discontinuity e.g. $f(x) = x - [x]$ at all integral x ; $f(x) = \tan^{-1}\frac{1}{x}$ at x = 0 and $f(x) = \frac{1}{1+2^{\frac{1}{x}}}$ at x = 0 (note that $f(0^+) = 0$; $f(0^-) = 1$)

(b) Infinite discontinuity e.g. $f(x) = \frac{1}{x-4}$ or $g(x) = \frac{1}{(x-4)^2}$ at x = 4 ; $f(x) = 2^{\tan x}$ at $x = \frac{\pi}{2}$ and $f(x) = \frac{\cos x}{x}$ at x = 0.

(c) Oscillatory discontinuity e.g. $f(x) = \sin\frac{1}{x}$ at x = 0.

In all these cases the value of f(a) of the function at x= a (point of discontinuity) may or may not exist but $\lim_{x \to a}$ does not exist.

Note: From the adjacent graph note that
– f is continuous at x = – 1
– f has isolated discontinuity at x = 1
– f has missing point discontinuity at x = 2
– f has non removable (finite type) discontinuity at the origin.



Nature of discontinuity

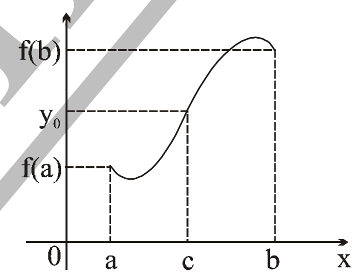
4. In case of dis-continuity of the second kind the non-negative difference between the value of the RHL at x = c & LHL at x = c is called **The Jump Of Discontinuity.** A function having a finite number of jumps in a given interval I is called a **Piece Wise Continuous** or **Sectionally Continuous** function in this interval.

5. All Polynomials, Trigonometrical functions, exponential & Logarithmic functions are continuous in their domains.

6. If f & g are two functions that are continuous at x= c then the functions defined by :
$F_1(x) = f(x) \pm g(x)$; $F_2(x) = K\, f(x)$, K any real number; $F_3(x) = f(x).g(x)$ are also continuous at x= c.

Further, if g (c) is not zero, then $F_4(x) = \frac{f(x)}{g(x)}$ is also continuous at x= c.

7. <u>**The intermediate value theorem:**</u>
Suppose f(x) is continuous on an interval I , and a and b are any two points of I. Then if y_0 is a number between f(a) and f(b) , their exists a number c between a and b such that $f(c) = y_0$.



The function f, being continuous on [a,b) takes on every value between f(a) and f(b)

Note Very Carefully That :

(a) If f(x) is continuous & g(x) is discontinuous at x = a then the product function $\phi(x) = f(x) \cdot g(x)$ is not necessarily be discontinuous at x = a. e.g.

$$f(x) = x \;\&\; g(x) = \begin{cases} \sin\frac{\pi}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

(b) If f(x) and g(x) both are discontinuous at x = a then the product function $\phi(x) = f(x) \cdot g(x)$ is not necessarily be discontinuous at x = a. e.g.

$$f(x) = -\,g(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$$

(c) Point functions are to be treated as discontinuous. eg. $f(x) = \sqrt{1-x} + \sqrt{x-1}$ is not continuous at x = 1.

(d) A Continuous function whose domain is closed must have a range also in closed interval.

(e) If f is continuous at x = c & g is continuous at x = f(c) then the composite g[f(x)] is continuous at x = c.

eg. $f(x) = \frac{x \sin x}{x^2+2}$ & $g(x) = |x|$ are continuous at x = 0 , hence the composite (gof) (x) = $\left|\frac{x \sin x}{x^2+2}\right|$ will also be continuous at x = 0 .

7. **Continuity In An Interval :**

(a) A function f is said to be continuous in (a , b) if f is continuous at each & every point $\in$(a , b).

(b) A function f is said to be continuous in a closed interval $[a,b]$ if :

(i) f is continuous in the open interval (a , b) &

(ii) f is right continuous at **'a'** i.e. $\lim_{x\to a^+} f(x) = f(a)$ = a finite quantity.

(iii) f is left continuous at **'b'** i.e. $\lim_{x\to b^-} f(x) = f(b)$ = a finite quantity.

Note that a function f which is continuous in $[a,b]$ possesses the following properties :

(i) If f(a) & f(b) possess opposite signs, then there exists at least one solution of the equation f(x) = 0 in the open interval (a , b).

(ii) If K is any real number between f(a) & f(b), then there exists at least one solution of the equation f(x) = K in the open inetrval (a , b).

8. SINGLE POINT CONTINUITY:

Functions which are continuous only at one point are said to exhibit single point continuity

e.g. $f(x) = \begin{cases} x & \text{if } x \in Q \\ -x & \text{if } x \notin Q \end{cases}$ and $g(x) = \begin{cases} x & \text{if } x \in Q \\ 0 & \text{if } x \notin Q \end{cases}$ are both continuous only at x = 0.

EXERCISE–I

Q.1 If the function $f(x) = \dfrac{3x^2 + ax + a + 3}{x^2 + x - 2}$ is continuous at x = – 2. Find $f(-2)$.

Q.2 Find all possible values of a and b so that $f(x)$ is continuous for all $x \in R$ if

$$f(x) = \begin{cases} |ax + 3| & \text{if } x \le -1 \\ |3x + a| & \text{if } -1 < x \le 0 \\ \dfrac{b \sin 2x}{x} - 2b & \text{if } 0 < x < \pi \\ \cos^2 x - 3 & \text{if } x \ge \pi \end{cases}$$

Q.3 Let $f(x) = \begin{cases} \dfrac{\ln \cos x}{\sqrt[4]{1+x^2} - 1} & \text{if } x > 0 \\ \dfrac{e^{\sin 4x} - 1}{\ln(1 + \tan 2x)} & \text{if } x < 0 \end{cases}$

Is it possible to define f(0) to make the function continuous at x = 0. If yes what is the value of f(0), if not then indicate the nature of discontinuity.

Q.4 Suppose that $f(x) = x^3 - 3x^2 - 4x + 12$ and $h(x) = \begin{cases} \dfrac{f(x)}{x - 3} & , x \ne 3 \\ K & , x = 3 \end{cases}$ then

(a) find all zeros of $f(x)$

(b) find the value of K that makes h continuous at x = 3

(c) using the value of K found in (b), determine whether h is an even function.

Q.5 Let $y_n(x) = x^2 + \dfrac{x^2}{1+x^2} + \dfrac{x^2}{(1+x^2)^2} + \ldots\ldots\ldots\ldots + \dfrac{x^2}{(1+x^2)^{n-1}}$ and $y(x) = \lim_{n\to\infty} y_n(x)$

Discuss the continuity of $y_n(x)$ $(n \in N)$ and y(x) at x = 0

Q.6 Draw the graph of the function $f(x) = x - |x - x^2|$, $-1 \le x \le 1$ & discuss the continuity or discontinuity of f in the interval $-1 \le x \le 1$.

Q.7 Let $f(x)=\begin{cases}\frac{1-\sin\pi x}{1+\cos 2\pi x}, & x<\frac{1}{2}\\ p, & x=\frac{1}{2}\\ \frac{\sqrt{2x-1}}{\sqrt{4+\sqrt{2x-1}}-2}, & x>\frac{1}{2}\end{cases}$. Determine the value of p, if possible, so that the function is continuous at x=1/2.

Q.8 Given the function $g(x)=\sqrt{6-2x}$ and $h(x)=2x^2-3x+a$. Then

(a) evaluate $h(g(2))$ (b) If $f(x)=\begin{cases}g(x), & x\le 1\\ h(x), & x>1\end{cases}$, find 'a' so that f is continuous.

Q.9 Let $f(x)=\begin{cases}1+x, & 0\le x\le 2\\ 3-x, & 2<x\le 3\end{cases}$. Determine the form of $g(x)=f[f(x)]$ & hence find the point of discontinuity of g, if any.

Q.10 Let [x] denote the greatest integer function & f(x) be defined in a neighbourhood of 2 by

$$f(x)=\begin{cases}\dfrac{\left(\exp\{(x+2)\ell n4\}\right)^{\frac{[x+1]}{4}}-16}{4^x-16}, & x<2\\ A\dfrac{1-\cos(x-2)}{(x-2)\tan(x-2)}, & x>2\end{cases}.$$

Find the values of A & f(2) in order that f(x) may be continuous at x = 2.

Q.11 The function $f(x)=\begin{cases}\left(\frac{6}{5}\right)^{\frac{\tan 6x}{\tan 5x}} & \text{if } 0<x<\frac{\pi}{2}\\ b+2 & \text{if } x=\frac{\pi}{2}\\ \left(1+|\cos x|\right)^{\left(\frac{a|\tan x|}{b}\right)} & \text{if } \frac{\pi}{2}<x<\pi\end{cases}$

Determine the values of 'a' & 'b', if f is continuous at x = π/2.

Q.12 A function $f: R\to R$ is defined as $f(x)=\lim\limits_{n\to\infty}\dfrac{ax^2+bx+c+e^{nx}}{1+c\cdot e^{nx}}$ where f is continuous on R. Find the values of a, b and c.

Q.13 Let $f(x)=\begin{cases}1+x^3, & x<0\\ x^2-1, & x\ge 0\end{cases}$; $g(x)=\begin{cases}(x-1)^{1/3}, & x<0\\ (x+1)^{1/2}, & x\ge 0\end{cases}$. Discuss the continuity of $g(f(x))$.

Q.14 Determine a & b so that f is continuous at $x=\frac{\pi}{2}$. $f(x)=\begin{cases}\frac{1-\sin^3 x}{3\cos^2 x} & \text{if } x<\frac{\pi}{2}\\ a & \text{if } x=\frac{\pi}{2}\\ \frac{b(1-\sin x)}{(\pi-2x)^2} & \text{if } x>\frac{\pi}{2}\end{cases}$

Q.15 Determine the values of a, b & c for which the function $f(x)=\begin{cases}\frac{\sin(a+1)x+\sin x}{x} & \text{for } x<0\\ c & \text{for } x=0\\ \frac{\left(x+bx^2\right)^{1/2}-x^{1/2}}{bx^{3/2}} & \text{for } x>0\end{cases}$

is continuous at x = 0.

Q.16 If $f(x)=\dfrac{\sin 3x+A\sin 2x+B\sin x}{x^5}$ $(x\ne 0)$ is cont. at x = 0. Find A & B. Also find f(0).

Q.17 Discuss the continuity of the function 'f' defined as follows: $f(x)=\begin{cases}\frac{1}{x-1} & \text{for } 0\le x\le 2\\ \frac{3}{x+1} & \text{for } 2<x\le 4\\ \frac{x+1}{x-5} & \text{for } 4<x\le 6\end{cases}$ and draw the graph of the function for $x\in[0,6]$. Also indicate the nature of discontinuities if any.

Q.18 If $f(x) = x + \{-x\} + [x]$, where $[x]$ is the integral part & $\{x\}$ is the fractional part of x. Discuss the continuity of f in $[-2, 2]$.

Q.19 Find the locus of (a, b) for which the function $f(x) = \begin{cases} ax - b & \text{for } x \le 1 \\ 3x & \text{for } 1 < x < 2 \\ bx^2 - a & \text{for } x \ge 2 \end{cases}$ is continuous at x = 1 but discontinuous at x = 2.

Q.20 Let $g(x) = \lim\limits_{n\to\infty} \dfrac{x^n f(x) + h(x) + 1}{2x^n + 3x + 3}$, $x \ne 1$ and $g(1) = \lim\limits_{x\to 1} \dfrac{\sin^2(\pi \cdot 2^x)}{\ln\left(\sec(\pi \cdot 2^x)\right)}$ be a continuous function at x = 1, find the value of $4g(1) + 2f(1) - h(1)$. Assume that f (x) and h (x) are continuous at x = 1.

Q.21 If g : [a, b] onto [a, b] is continous show that there is some $c \in [a, b]$ such that g (c) = c.

Q.22 The function $f(x) = \left(\dfrac{2 + \cos x}{x^3 \sin x} - \dfrac{3}{x^4}\right)$ is not defined at x = 0. How should the function be defined at x = 0 to make it continuous at x = 0.

Q.23 $f(x) = \dfrac{a^{\sin x} - a^{\tan x}}{\tan x - \sin x}$ for x > 0

$= \dfrac{\ln(1 + x + x^2) + \ln(1 - x + x^2)}{\sec x - \cos x}$ for x < 0, if f is continuous at x = 0, find 'a'

now if $g(x) = \ln\left(2 - \dfrac{x}{a}\right) \cdot \cot(x - a)$ for $x \ne a$, $a \ne 0$, $a > 0$. If g is continuous at x = a then show that $g(e^{-1}) = -e$.

Q.24(a) Let $f(x+y) = f(x) + f(y)$ for all x , y & if the function f(x) is continuous at x = 0 , then show that f(x) is continuous at all x.

(b) If $f(x \cdot y) = f(x) \cdot f(y)$ for all x , y and f(x) is continuous at x = 1. Prove that f(x) is continuous for all x except at x = 0. Given $f(1) \ne 0$.

Q.25 Given $f(x) = \sum\limits_{r=1}^{n} \tan\left(\dfrac{x}{2^r}\right) \sec\left(\dfrac{x}{2^{r-1}}\right)$; r , n ∈ N

$$g(x) = \underset{n\to\infty}{\text{Limit}} \frac{\ell n\left(f(x) + \tan\frac{x}{2^n}\right) - \left(f(x) + \tan\frac{x}{2^n}\right)^n \cdot \left[\sin\left(\tan\frac{x}{2}\right)\right]}{1 + \left(f(x) + \tan\frac{x}{2^n}\right)^n}$$

$= k$ for $x = \dfrac{\pi}{4}$ and the domain of g (x) is (0 , π/2).

where [ ] denotes the greatest integer function.

Find the value of k, if possible, so that g (x) is continuous at x = π/4. Also state the points of discontinuity of g (x) in (0 , π/4) , if any.

Q.26 Let $f(x) = x^3 - x^2 - 3x - 1$ and $h(x) = \dfrac{f(x)}{g(x)}$ where *h* is a rational function such that

(a) it is continuous every where except when x = – 1, (b) $\lim\limits_{x\to\infty} h(x) = \infty$ and (c) $\lim\limits_{x\to -1} h(x) = \dfrac{1}{2}$.

Find $\lim\limits_{x\to 0} \left(3h(x) + f(x) - 2g(x)\right)$

Q.27 Let f be continuous on the interval [0, 1] to R such that f (0) = f (1). Prove that there exists a point c in $\left[0, \dfrac{1}{2}\right]$ such that $f(c) = f\left(c + \dfrac{1}{2}\right)$

Q.28 Consider the function $g(x) = \begin{cases} \dfrac{1 - a^x + x a^x \ell n a}{a^x x^2} & \text{for } x < 0 \\ \dfrac{2^x a^x - x \ell n 2 - x \ell n a - 1}{x^2} & \text{for } x > 0 \end{cases}$ where $a > 0$.

find the value of 'a' & 'g(0)' so that the function g(x) is continuous at x = 0.

Q.29 Let $f(x) = \begin{cases} \dfrac{\left(\frac{\pi}{2} - \sin^{-1}(1-\{x\}^2)\right)\sin^{-1}(1-\{x\})}{\sqrt{2}\left(\{x\}-\{x\}^3\right)} & \text{for } x \neq 0 \\ \dfrac{\pi}{2} & \text{for } x = 0 \end{cases}$ where $\{x\}$ is the fractional part of x.

Consider another function g(x) ; such that

$g(x) = f(x)$ for $x \ge 0$

$= 2\sqrt{2}\ f(x)$ for $x < 0$

Discuss the continuity of the functions f(x) & g(x) at x = 0.

Q.30 Discuss the continuity of f in [0,2] where $f(x) = \begin{cases} |4x - 5|\,[x] & \text{for } x > 1 \\ [\cos \pi x] & \text{for } x \le 1 \end{cases}$; where [x] is the greatest integer not greater than x. Also draw the graph.

EXERCISE–II

(OBJECTIVE QUESTIONS)

Q.1 State whether True or False.

(i) $f(x) = \underset{n\to\infty}{\text{Limit}} \dfrac{1}{1 + n\sin^2 \pi x}$ is continuous at x = 1.

(ii) The function defined by $f(x) = \dfrac{x}{|x| + 2x^2}$ for $x \neq 0$ & f(0) = 1 is continuous at x = 0.

(iii) The function $f(x) = 2^{-2^{1/(1-x)}}$ if $x \neq 1$ & f(1) = 1 is not continuous at x = 1.

(iv) There exists a continuous function f: [0, 1] onto [0, 10], but there exists no continuous function g : [0, 1] onto (0, 10).

(v) If *f*(x) is continuous in [0 , 1] & f(x) = 1 for all rational numbers in [0 , 1] then $f\left(1/\sqrt{2}\right)$ equal to 1.

(vi) If $f(x) = \begin{cases} \dfrac{\cos \pi x + \sin(\pi x/2)}{(x-1)(3x^2 - 2x - 1)} & \text{if } x \neq 1 \\ k & \text{if } k = 1 \end{cases}$ is continuous, then the value of *k* is $\dfrac{3\pi^2}{32}$.

<u>Select the correct alternative</u> : (Only one is correct)

Q.2 *f* is a continuous function on the real line. Given that

$x^2 + (f(x) - 2)\,x - \sqrt{3} \cdot f(x) + 2\sqrt{3} - 3 = 0$. Then the value of $f(\sqrt{3})$

(A) can not be determined (B) is $2(1 - \sqrt{3})$ (C) is zero (D) is $\dfrac{2\left(\sqrt{3} - 2\right)}{\sqrt{3}}$

Q.3 If f(x) = sgn (cos 2 x – 2 sin x + 3) , where sgn ( ) is the signum function , then f(x)

(A) is continuous over its domain
(B) has a missing point discontinuity
(C) has isolated point discontinuity
(D) has irremovable discontinuity.

Q.4 Let $g(x) = \tan^{-1}|x| - \cot^{-1}|x|,\ f(x) = \dfrac{[x]}{[x+1]}\ \{x\},\ h(x) = |g(f(x))|$ where $\{x\}$ denotes fractional part and $[x]$ denotes the integral part then which of the following holds good?

(A) h is continuous at $x = 0$
(B) h is discontinuous at $x = 0$
(C) $h(0^-) = \pi/2$
(D) $h(0^+) = -\pi/2$

Q.5 Consider $f(x) = \lim\limits_{n\to\infty} \dfrac{x^n - \sin x^n}{x^n + \sin x^n}$ for $x > 0, x \neq 1,$

$f(1) = 0$

then

(A) f is continuous at $x = 1$
(B) f has an infinite or oscillatory discontinuity at $x = 1$.
(C) f has a finite discontinuity at $x = 1$
(D) f has a removable type of discontinuity at $x = 1$.

Q.6 Given $f(x) = \dfrac{[\{|x|\}]e^{x^2}\{[x+\{x\}]\}}{\left(e^{1/x^2} - 1\right)\operatorname{sgn}(\sin x)}$ for $x \neq 0$

$= 0$ for $x = 0$

where $\{x\}$ is the fractional part function; $[x]$ is the step up function and sgn(x) is the signum function of x then, f(x)

(A) is continuous at $x = 0$
(B) is discontinuous at $x = 0$
(C) has a removable discontinuity at $x = 0$
(D) has an irremovable discontinuity at $x = 0$

Q.7 Consider $f(x) = \begin{cases} x[x]^2 \log_{(1+x)} 2 & \text{for } -1 < x < 0 \\ \dfrac{ln\left(e^{x^2} + 2\sqrt{\{x\}}\right)}{\tan\sqrt{x}} & \text{for } 0 < x < 1 \end{cases}$

where [*] & {*} are the greatest integer function & fractional part function respectively, then

(A) $f(0) = ln2 \Rightarrow$ f is continuous at $x = 0$
(B) $f(0) = 2 \Rightarrow$ f is continuous at $x = 0$
(C) $f(0) = e^2 \Rightarrow$ f is continuous at $x = 0$
(D) f has an irremovable discontinuity at $x = 0$

Q.8 Consider $f(x) = \dfrac{\sqrt{1+x} - \sqrt{1-x}}{\{x\}}, x \neq 0;$

$g(x) = \cos 2x,\ -\dfrac{\pi}{4} < x < 0,$

$h(x) - \begin{cases} \dfrac{1}{\sqrt{2}} f(g(x)) & \text{for } x < 0 \\ 1 & \text{for } x = 0 \\ f(x) & \text{for } x > 0 \end{cases}$

then, which of the following holds good.
where $\{x\}$ denotes fractional part function.

(A) 'h' is continuous at $x = 0$
(B) 'h' is discontinuous at $x = 0$
(C) f(g(x)) is an even function
(D) f(x) is an even function

Q.9 The function $f(x) = [x] . \cos \frac{2x-1}{2}\pi$, where [•] denotes the greatest integer function, is discontinuous at

(A) all x
(B) all integer points
(C) no x
(D) x which is not an integer

Q.10 Consider the function defined on $[0, 1] \to R$, $f(x) = \frac{\sin x - x\cos x}{x^2}$ if $x \neq 0$ and $f(0) = 0$, then the function $f(x)$

(A) has a removable discontinuity at x = 0
(B) has a non removable finite discontinuity at x = 0
(C) has a non removable infinite discontinuity at x = 0
(D) is continuous at x = 0

Q.11 $$f(x) = \begin{cases} \sin\left(\frac{a-x}{2}\right)\tan\left[\frac{\pi x}{2a}\right] & \text{for } x > a \\ \dfrac{\left[\cos\left(\frac{\pi x}{2a}\right)\right]}{a-x} & \text{for } x < a \end{cases}$$

where [x] is the greatest integer function of x, and $a > 0$, then

(A) $f(a^-) < 0$
(B) f has a removable discontinuity at x = a
(C) f has an irremovable discontinuity at x=a
(D) $f(a^+) < 0$

Q.12 Consider the function $f(x) = \lim_{n\to\infty} \frac{\sin \pi x - x^{2n}\sin(x-1)}{1 + x^{2n+1} - x^{2n}}$, where $n \in N$

Statement-1: $f(x)$ is discontinuous at x = 1.

because

Statement-2: $f(1) = 0$.

(A) Statement-1 is true, statement-2 is true and statement-2 is correct explanation for statement-1.
(B) Statement-1 is true, statement-2 is true and statement-2 is NOT the correct explanation for statement-1.
(C) Statement-1 is true, statement-2 is false.
(D) Statement-1 is false, statement-2 is true.

Q.13 Consider the functions

$f(x) = \text{sgn}(x-1)$ and $g(x) = \cot^{-1}[x-1]$

where [ ] denotes the greatest integer function.

Statement-1 : The function $F(x) = f(x) \cdot g(x)$ is discontinuous at x = 1.

because

Statement-2 : If f (x) is discontinuous at x = a and g (x) is also discontinuous at x = a then the product function $f(x) \cdot g(x)$ is discontinuous at x = a.

(A) Statement-1 is true, statement-2 is true and statement-2 is correct explanation for statement-1.
(B) Statement-1 is true, statement-2 is true and statement-2 is NOT the correct explanation for statement-1.
(C) Statement-1 is true, statement-2 is false.
(D) Statement-1 is false, statement-2 is true.

Select the correct alternative : (More than one are correct)

Q.14 A function f is defined on an interval [a, b]. Which of the following statement(s) is/are INCORRECT?

(A) If $f(a)$ and $f(b)$, have opposite sign, then there must be a point $c \in (a, b)$ such that $f(c) = 0$.

(B) If f is continuous on [a, b], $f(a) < 0$ and $f(b) > 0$, then there must be a point $c \in (a, b)$ such that $f(c) = 0$

(C) If f is continuous on [a, b] and there is a point c in (a, b) such that $f(c) = 0$, then $f(a)$ and $f(b)$ have opposite sign.

(D) If f has no zeroes on [a, b], then $f(a)$ and $f(b)$ have the same sign.

Q.15 Which of the following functions f has/have a removable discontinuity at the indicated point?

(A) $f(x) = \dfrac{x^2 - 2x - 8}{x + 2}$ at $x = -2$ (B) $f(x) = \dfrac{x - 7}{|x - 7|}$ at $x = 7$

(C) $f(x) = \dfrac{x^3 + 64}{x + 4}$ at $x = -4$ (D) $f(x) = \dfrac{3 - \sqrt{x}}{9 - x}$ at $x = 9$

Q.16 Let 'f' be a continuous function on R. If $f(1/4^n) = (\sin e^n)e^{-n^2} + \dfrac{n^2}{n^2 + 1}$ then f(0) is :

(A) not unique (B) 1

(C) data sufficient to find f(0) (D) data insufficient to find f(0)

Q.17 Indicate all correct alternatives if, $f(x) = \dfrac{x}{2} - 1$, then on the interval $[0, \pi]$

(A) $\tan(f(x))$ & $\dfrac{1}{f(x)}$ are both continuous (B) $\tan(f(x))$ & $\dfrac{1}{f(x)}$ are both discontinuous

(C) $\tan(f(x))$ & $f^{-1}(x)$ are both continuous (D) $\tan(f(x))$ is continuous but $\dfrac{1}{f(x)}$ is not

EXERCISE–III

Q.1 The function $f(x) = [x]^2 - [x^2]$ (where [y] is the greatest integer less than or equal to y), is discontinuous at :

(A) all integers (B) all integers except 0 & 1

(C) all integers except 0 (D) all integers except 1

[JEE '99, 2 (out of 200)]

Q.2 Determine the constants a, b & c for which the function $f(x) = \begin{cases} (1+ax)^{1/x} & \text{for } x < 0 \\ b & \text{for } x = 0 \\ \dfrac{(x+c)^{1/3} - 1}{(x+1)^{1/2} - 1} & \text{for } x > 0 \end{cases}$ is continuous at x = 0. [REE '99, 6]

Q.3 Discuss the continuity of the function

$$f(x) = \begin{cases} \dfrac{e^{1/(x-1)} - 2}{e^{1/(x-1)} + 2}, & x \neq 1 \\ 1, & x = 1 \end{cases}$$

at x = 1. [REE 2001 (Mains) , 3 out of 100]

ANSWER KEY
LIMIT
EXERCISE–I

Q 1. 3 **Q 2.** $\frac{45}{91}$ **Q 3.** 2 **Q.4** 5050 **Q 5.** $\frac{2}{\sqrt{3}}$ **Q.6** $-\frac{1}{3}$

Q.7 $\frac{3}{2}$ **Q.8** $\frac{p-q}{2}$ **Q.9** $a = \frac{1}{2}$; $r = \frac{1}{4}$; $S = \frac{2}{3}$ **Q 10.** ln 2

Q.11 (a) does not exist; **(b)** does not exist; **(c)** 0 **Q 12.** 2 **Q.13** $\frac{1}{32}$ **Q.14** $\frac{1}{16\sqrt{2}}$

Q.15 $\frac{2\ln 2}{\pi}$ **Q.16** a = 2 ; limit = 1 **Q.17** (a) $\pi/2$ if $a > 0$; 0 if $a = 0$ and $-\pi/2$ if $a < 0$; (b) $f(x) = |x|$

Q.18 1 **Q 19.** $-\frac{9}{4}\ln\frac{4}{e}$ **Q.20** $\pi - 3$ **Q.21** –2 **Q.22** 9 **Q.23** $8\sqrt{2}(\ln 3)^2$

Q.24 – 3, –3, – 3 **Q.25** $(l\text{n } a)^n$ **Q.26** 72 **Q.27** – 1/2 **Q.28** $\frac{\sqrt{3}}{2}$ **Q.29** $\frac{2L}{3}$ **Q.30** 4

EXERCISE–II

Q.1 e^{-8} **Q.2** c = ln2 **Q.3** $e^{-\frac{1}{2}}$ **Q.4** e^{-1} **Q.5** $-\frac{\pi^2}{4}$ **Q.6** $e^{-2\pi^2 a^2}$ **Q.7** e^{-1}

Q.8 $e^{-1/2}$ **Q.9** $(a_1.a_2.a_3....a_n)$ **Q.10** $\frac{\pi}{2}, \frac{\pi}{2\sqrt{2}}$

Q.11 a = c = 1, b = 2 **Q.12** $\frac{\pi^2 a^2 + 4}{16a^4}$ **Q.13** $\frac{2}{3}$

Q.15 $\frac{x}{2}$ **Q.16** $-\frac{1}{\pi^2}$ **Q.17** a - b **Q.18** $\frac{\pi}{3}$ **Q.19** 1/2 **Q.20** 8

Q.21 $T(x) = \frac{1}{2}\tan^2\frac{x}{2}$. $\sin x$ or $\tan\frac{x}{2} - \frac{\sin x}{2}$, $S(x) = \frac{1}{2}x - \frac{1}{2}\sin x$, limit $= \frac{3}{2}$

Q.22 g (x) = sin x and l = e **Q.23** $\frac{\theta}{\tan\theta}$ **Q.24** 19 **Q.25** e^{π^2}

Q.26 a = – 5/2, b = – 3/2 **Q.28** (a) 2; (b) 1/2 **Q.29** **(i)** a = 1, b = –1 **(ii)** a = –1 , b = $\frac{1}{2}$ **Q.30** 307

EXERCISE–III

Q.1 C **Q.2** C **Q.3** B **Q.4** $l$na **Q.5** C

Q.6 C **Q.7** $1-\frac{2}{\pi}$

* *

CONTINUITY

EXERCISE–I

Q.1 -1 **Q.2** $a = 0,\ b = 1$

Q.3 $f(0^+) = -2$; $f(0^-) = 2$ hence $f(0)$ not possible to define

Q.4 **(a)** $-2, 2, 3$ **(b)** $K = 5$ **(c)** even

Q.5 $y_n(x)$ is continuous at $x = 0$ for all n and $y(x)$ is dicontinuous at $x = 0$

Q.6 f is cont. in $-1 \le x \le 1$ **Q.7** P not possible.

Q.8 (a) $4 - 3\sqrt{2} + a$, (b) $a = 3$

Q.9 $g(x) = 2 + x$ for $0 \le x \le 1$, $2 - x$ for $1 < x \le 2$, $4 - x$ for $2 < x \le 3$, g is discontinuous at $x = 1$ & $x = 2$

Q.10 $A = 1$; $f(2) = 1/2$ **Q.11** $a = 0$; $b = -1$ **Q.12** $c = 1,\ a, b \in R$

Q.13 gof is dis-cont. at $x = 0, 1$ & -1

Q.14 $a = 1/2,\ b = 4$ **Q.15** $a = -3/2,\ b \neq 0,\ c = 1/2$

Q.16 $A = -4$, $B = 5$, $f(0) = 1$ **Q.17** discontinuous at $x = 1, 4$ & 5

Q.18 discontinuous at all integral values in $[-2, 2]$

Q.19 locus $(a, b) \to x, y$ is $y = x - 3$ excluding the points where $y = 3$ intersects it.

Q.20 5 **Q.22** $\frac{1}{60}$

Q.25 $k = 0$; $g(x) = \begin{cases} \ell n(\tan x) & \text{if } 0 < x < \frac{\pi}{4} \\ 0 & \text{if } \frac{\pi}{4} \le x < \frac{\pi}{2} \end{cases}$. Hence $g(x)$ is continuous everywhere.

Q.26 $g(x) = 4(x + 1)$ and limit $= -\frac{39}{4}$

Q.28 $a = \frac{1}{\sqrt{2}}$, $g(0) = \frac{(\ell n 2)^2}{8}$

Q.29 $f(0^+) = \frac{\pi}{2}$; $f(0^-) = \frac{\pi}{4\sqrt{2}}$ $\Rightarrow$ f is discont. at $x = 0$;

$g(0^+) = g(0^-) = g(0) = \pi/2 \Rightarrow$ g is cont. at $x = 0$

Q.30 the function f is continuous everywhere in $[0, 2]$ except for $x = 0, \frac{1}{2}$, 1 & 2.

EXERCISE–II

Q.1 **(i)** false ; **(ii)** false ; **(iii)** true ; **(iv)** true; **(v)** true; **(vi)** True

Q.2 B **Q.3** C **Q.4** A **Q.5** C **Q.6** A **Q.7** D **Q.8** A

Q.9 C **Q.10** D; **Q.11** B **Q.12** B **Q.13** C **Q.14** A, C, D

Q.15 A, C, D **Q.16** B, C **Q.17** C, D

EXERCISE–III

Q.1 D **Q.2** $a = \ln\frac{2}{3}$; $b = \frac{2}{3}$; $c = 1$

Q.3 Discontinuous at $x = 1$; $f(1^+) = 1$ and $f(1^-) = -1$

* *