MATRICES

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KEY CONCEPTS
MATRICES
USEFUL IN STUDY OF SCIENCE, ECONOMICS AND ENGINEERING

1. Definition : Rectangular array of mn numbers . Unlike determinants it has no value.

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots\dots & a_{1n} \\ a_{21} & a_{22} & \dots\dots & a_{2n} \\ : & : & : & : \\ a_{m1} & a_{m2} & \dots\dots & a_{mn} \end{bmatrix} \quad \text{or} \quad \begin{pmatrix} a_{11} & a_{12} & \dots\dots & a_{1n} \\ a_{21} & a_{22} & \dots\dots & a_{2n} \\ : & : & : & : \\ a_{m1} & a_{m2} & \dots\dots & a_{mn} \end{pmatrix}$$

Abbreviated as : $A = [a_{ij}]$ $1 \le i \le m$; $1 \le j \le n$, i denotes the row and j denotes the column is called a matrix of order $m \times n$.

2. Special Type Of Matrices :

(a) Row Matrix : $A = [a_{11}, a_{12}, \dots\dots a_{1n}]$ having one row . $(1 \times n)$ matrix.
(or row vectors)

(b) Column Matrix : $A = \begin{bmatrix} a_{11} \\ a_{21} \\ : \\ a_{m1} \end{bmatrix}$ having one column. $(m \times 1)$ matrix
(or column vectors)

(c) Zero or Null Matrix : ($A = O_{m \times n}$)
An $m \times n$ matrix all whose entries are zero .

$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ is a 3×2 null matrix & $B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is 3×3 null matrix

(d) Horizontal Matrix : A matrix of order $m \times n$ is a horizontal matrix if $n > m$.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 1 & 1 \end{bmatrix}$$

(e) Verical Matrix : A matrix of order $m \times n$ is a vertical matrix if $m > n$. $\begin{bmatrix} 2 & 5 \\ 1 & 1 \\ 3 & 6 \\ 2 & 4 \end{bmatrix}$

(f) Square Matrix : (Order n)

If number of row = number of column $\Rightarrow$ a square matrix.

Note (i) In a square matrix the pair of elements a_{ij} & a_{ji} are called **Conjugate Elements** .

e.g. $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

(ii) The elements $a_{11}, a_{22}, a_{33}, \dots\dots a_{nn}$ are called **Diagonal Elements** . The line along which the diagonal elements lie is called **"Principal or Leading"** diagonal.
The qty $\Sigma\, a_{ii}$ = trace of the matrice written as , i.e. $t_r A$

Square Matrix

Triangular Matrix

$$A = \begin{pmatrix} 1 & 3 & -2 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{pmatrix} \quad ; \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 2 & -3 & 0 \\ 4 & 3 & 3 \end{pmatrix}$$

Upper Triangular: $a_{ij} = 0 \ \forall \ i > j$

Lower Triangular: $a_{ij} = 0 \ \forall \ i < j$

Note that : Minimum number of zeros in a triangular matrix of order $n = n(n-1)/2$

Diagonal Matrix denote as $d_{dia}(d_1, d_2, ..., d_n)$ all elements except the leading diagonal are zero

diagonal Matrix

$$\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$$

Unit or Identity Matrix

$$a_{ij} = \begin{bmatrix} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{bmatrix}$$

Note: (1) If $d_1 = d_2 = d_3 = a$ Scalar Matrix

(2) If $d_1 = d_2 = d_3 = 1$ Unit Matrix

Note: Min. number of zeros in a diagonal matrix of order $n = n(n - 1)$

"It is to be noted that with square matrix there is a corresponding determinant formed by the elements of A in the same order."

3. Equality Of Matrices :

Let $A = [a_{ij}]$ & $B = [b_{ij}]$ are equal if ,

(i) both have the same order . **(ii)** $a_{ij} = b_{ij}$ for each pair of i & j.

4. Algebra Of Matrices :

Addition : $A + B = \left[a_{ij} + b_{ij}\right]$ where A & B are of the same type. (same order)

(a) Addition of matrices is commutative.

i.e. $A + B = B + A$ $\quad A = m \times n$; $B = m \times n$

(b) Matrix addition is associative .

$(A + B) + C = A + (B + C)$ **Note :** A , B & C are of the same type.

(c) Additive inverse.

If $A + B = \mathbf{O} = B + A$ $\quad A = m \times n$

5. Multiplication Of A Matrix By A Scalar :

If $A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$; $kA = \begin{bmatrix} ka & kb & kc \\ kb & kc & ka \\ kc & ka & kb \end{bmatrix}$

6. Multiplication Of Matrices : (Row by Column)

AB exists if, $A = m \times n$ & $B = n \times p$

2×3 $\quad 3 \times 3$

AB exists, but BA does not $\Rightarrow AB \neq BA$

Note : In the product AB , $\begin{cases} A = \text{pre factor} \\ B = \text{post factor} \end{cases}$

$$A = (a_1, a_2, ... a_n) \quad \& \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$1 \times n$ $\quad n \times 1$

$AB = [a_1 b_1 + a_2 b_2 + ... + a_n b_n]$

If $A = \left[a_{ij}\right]$ $m \times n$ & $B = \left[b_{ij}\right]$ $n \times p$ matrix , then $\boxed{(AB)_{ij} = \sum_{r=1}^{n} a_{ir} \cdot b_{rj}}$

Properties Of Matrix Multiplication :

1. Matrix multiplication is not commutative.

$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$; $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$; $AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$; $BA = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow$ $AB \neq BA$ (in general)

2. $AB = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\Rightarrow$ $AB = \mathbf{O}$ $\not\Rightarrow$ $A = \mathbf{O}$ or $B = \mathbf{O}$

Note: If A and B are two non- zero matrices such that $AB = \mathbf{O}$ then A and B are called the divisors of zero. Also if $[AB] = \mathbf{O} \Rightarrow |AB| \Rightarrow |A|\,|B| = 0 \Rightarrow |A| = 0$ or $|B| = 0$ but not the converse.
If A and B are two matrices such that
(i) $AB = BA$ $\Rightarrow$ A and B commute each other
(ii) $AB = -BA$ $\Rightarrow$ A and B anti commute each other

3. **Matrix Multiplication Is Associative :**
If A, B & C are conformable for the product AB & BC, then
$(A . B) . C = A . (B . C)$

4. **Distributivity :**

$\left.\begin{array}{l} A(B + C) = AB + AC \\ (A + B)C = AC + BC \end{array}\right]$ Provided A, B & C are conformable for respective products

5. **Positive Integral Powers Of A Square Matrix :**
For a square matrix A, $A^2 A = (AA)A = A(AA) = A^3$.
Note that for a unit matrix I of any order, $I^m = I$ for all $m \in N$.

6. **MATRIX POLYNOMIAL :**
If $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots\dots\dots + a_nx^0$ then we define a matrix polynomial
$f(A) = a_0A^n + a_1A^{n-1} + a_2A^{n-2} + \dots\dots + a_nI^n$
where A is the given square matrix. If f (A) is the null matrix then A is called the zero or root of the polynomial f (x).

DEFINITIONS :

(a) **Idempotent Matrix :** A square matrix is idempotent provided $A^2 = A$.
Note that $A^n = A$ $\forall$ $n \geq 2$, $n \in N$.

(b) **Nilpotent Matrix:** A square matrix is said to be nilpotent matrix of order m, $m \in N$, if $A^m = \mathbf{O}$, $A^{m-1} \neq \mathbf{O}$.

(c) **Periodic Matrix :** A square matrix is which satisfies the relation $A^{K+1} = A$, for some positive integer K, is a periodic matrix. The period of the matrix is the least value of K for which this holds true.
Note that period of an idempotent matrix is 1.

(d) **Involutary Matrix :** If $A^2 = I$, the matrix is said to be an involutary matrix.
Note that $A = A^{-1}$ for an involutary matrix.

7. **The Transpose Of A Matrix : (Changing rows & columns)**
Let A be any matrix. Then, $A = a_{ij}$ of order $m \times n$
$\Rightarrow$ A^T or $A' = [a_{ji}]$ for $1 \leq i \leq n$ & $1 \leq j \leq m$ of order $n \times m$

Properties of Transpose : If A^T & B^T denote the transpose of A and B,

(a) $(A \pm B)^T = A^T \pm B^T$; note that A & B have the same order.
IMP. (b) $(AB)^T = B^T A^T$ A & B are conformable for matrix product AB.
(c) $(A^T)^T = A$
(d) $(kA)^T = kA^T$ k is a scalar.

General : $(A_1, A_2, \dots\dots A_n)^T = A_n^T, \dots\dots\dots, A_2^T, A_1^T$ (reversal law for transpose)

8. Symmetric & Skew Symmetric Matrix :

A square matrix $A = [a_{ij}]$ is said to be,
symmetric if,

$a_{ij} = a_{ji} \quad \forall \; i \;\&\; j$ (conjugate elements are equal) **(Note $A = A^T$)**

Note: Max. number of distinct entries in a symmetric matrix of order n is $\frac{n(n+1)}{2}$.

and skew symmetric if,

$a_{ij} = -a_{ji} \quad \forall \; i \;\&\; j$ (the pair of conjugate elements are additive inverse of each other) **(Note $A = -A^T$)**

Hence If A is skew symmetric, then

$a_{ii} = -a_{ii} \Rightarrow a_{ii} = 0 \quad \forall \; i$

Thus the digaonal elements of a skew symmetric matrix are all zero, but not the converse.

Properties Of Symmetric & Skew Matrix :

P – 1 A is symmetric if $A^T = A$
A is skew symmetric if $A^T = -A$

P – 2 $A + A^T$ is a symmetric matrix
$A - A^T$ is a skew symmetric matrix.
Consider $(A + A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T$
$A + A^T$ is symmetric.
Similarly we can prove that $A - A^T$ is skew symmetric.

P – 3 The sum of two symmetric matrix is a symmetric matrix and the sum of two skew symmetric matrix is a skew symmetric matrix.
Let $A^T = A$; $B^T = B$ where A & B have the same order.
$\mathbf{(A + B)^T = A + B}$
Similarly we can prove the other

P – 4 If A & B are symmetric matrices then,
(a) $AB + BA$ is a symmetric matrix
(b) $AB - BA$ is a skew symmetric matrix.

P – 5 Every square matrix can be uniquely expressed as a sum of a symmetric and a skew symmetric matrix.

$$A = \underbrace{\frac{1}{2}(A + A^T)}_{\substack{P \\ \text{Symmetric}}} + \underbrace{\frac{1}{2}(A - A^T)}_{\substack{Q \\ \text{Skew Symmetric}}}$$

9. Adjoint Of A Square Matrix :

Let $A = [a_{ij}] = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ be a square matrix and let the matrix formed by the cofactors of $[a_{ij}]$ in determinant $|A|$ is $= \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}$.

Then $(\text{adj } A) = \begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix}$

V. Imp. Theorem : $A(\text{adj. } A) = (\text{adj. } A).A = |A|\, I_n$, If A be a square matrix of order n.

Note : If A and B are non singular square matrices of same order, then

(i) $|\text{adj } A| = |A|^{n-1}$

(ii) $\text{adj }(AB) = (\text{adj } B)(\text{adj } A)$

(iii) $\text{adj}(KA) = K^{n-1}(\text{adj } A)$, K is a scalar

Inverse Of A Matrix (Reciprocal Matrix) :

A square matrix A said to be invertible (non singular) if there exists a matrix B such that,

$$AB = I = BA$$

B is called the inverse (reciprocal) of A and is denoted by A^{-1}. Thus

$$A^{-1} = B \Leftrightarrow AB = I = BA.$$

We have, $A.(\text{adj } A) = |A|\ I_n$

$$A^{-1} A\ (\text{adj } A) = A^{-1} I_n\ |A|$$

$$I_n\ (\text{adj } A) = A^{-1}\ |A|\ I_n$$

$$\therefore \quad A^{-1} = \frac{(\text{adj } A)}{|A|}$$

Note : The necessary and sufficient condition for a square matrix A to be invertible is that $|A| \neq 0$.

Imp. Theorem : If A & B are invertible matrices ofthe same order, then $(AB)^{-1} = B^{-1} A^{-1}$. This is reversal law for inverse.

Note :

(i) If A be an invertible matrix, then A^T is also invertible & $(A^T)^{-1} = (A^{-1})^T$.

(ii) If A is invertible, **(a)** $(A^{-1})^{-1} = A$; **(b)** $(A^k)^{-1} = (A^{-1})^k = A^{-k}$, $k \in N$

(iii) If A is an Orthogonal Matrix. $AA^T = I = A^TA$

(iv) A square matrix is said to be **orthogonal** if, $A^{-1} = A^T$.

(v) $|A^{-1}| = \dfrac{1}{|A|}$

SYSTEM OF EQUATION & CRITERIAN FOR CONSISTENCY

GAUSS - JORDAN METHOD

$$x + y + z = 6$$
$$x - y + z = 2$$
$$2x + y - z = 1$$

or $\begin{pmatrix} x+y+z \\ x-y+z \\ 2x+y-z \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 1 \end{pmatrix}$$

$$AX = B \quad \Rightarrow \quad A^{-1} AX = A^{-1} B$$

$$X = A^{-1} B = \frac{(\text{adj. } A).B}{|A|}.$$

Note :

(1) If $|A| \neq 0$, system is consistent having unique solution

(2) If $|A| \neq 0$ & $(\text{adj } A) . B \neq \mathbf{O}$ (Null matrix), system is consistent having unique non–trivial solution.

(3) If $|A| \neq 0$ & $(\text{adj } A) . B = \mathbf{O}$ (Null matrix), system is consistent having trivial solution.

(4) If $\mathbf{|A| = 0}$, **matrix method fails**

If $(\text{adj } A) . B = \text{null matrix} = \mathbf{O}$ $\rightarrow$ Consistent (Infinite solutions)

If $(\text{adj } A) . B \neq \mathbf{O}$ $\rightarrow$ Inconsistent (no solution)

EXERCISE-I

Q.1 If, $E=\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}$ and $F=\begin{bmatrix}0&0&0\\1&0&0\\0&1&0\end{bmatrix}$ calculate the matrix product EF & FE and show that $E^2F+FE^2=E$.

Q.2 Find the number of 2×2 matrix satisfying
(i) a_{ij} is 1 or –1 ; (ii) $a_{11}^2+a_{12}^2=a_{21}^2+a_{22}^2=2$; (iii) $a_{11}a_{21}+a_{12}a_{22}=0$

Q.3 Find the value of x and y that satisfy the equations.

$$\begin{bmatrix}3&-2\\3&0\\2&4\end{bmatrix}\begin{bmatrix}y&y\\x&x\end{bmatrix}=\begin{bmatrix}3&3\\3y&3y\\10&10\end{bmatrix}$$

Q.4 Let $A=\begin{bmatrix}a&b\\c&d\end{bmatrix}$ and $B=\begin{bmatrix}p\\q\end{bmatrix}\neq\begin{bmatrix}0\\0\end{bmatrix}$. Such that AB = B and a + d = 5050. Find the value of (ad – bc).

Q.5 Define $A=\begin{bmatrix}0&1\\3&0\end{bmatrix}$. Find a vertical vector V such that $(A^8+A^6+A^4+A^2+I)V=\begin{bmatrix}0\\11\end{bmatrix}$ (where I is the 2×2 identity matrix).

Q.6 If, $A=\begin{bmatrix}1&0&2\\0&2&1\\2&0&3\end{bmatrix}$, then show that the maxtrix A is a root of the polynomial $f(x)=x^3-6x^2+7x+2$.

Q.7 For a non zero λ, use induction to prove that : (Only for XII CBSE)

(a) $$\begin{bmatrix}\lambda&1&0\\0&\lambda&1\\0&0&\lambda\end{bmatrix}^n=\begin{bmatrix}\lambda^n&n\lambda^{n-1}&\dfrac{n(n-1)}{2}\lambda^{n-2}\\0&\lambda^n&n\lambda^{n-1}\\0&0&\lambda^n\end{bmatrix},\ \text{for every } n\in N$$

(b) If, $A=\begin{bmatrix}0&1\\0&0\end{bmatrix}$, then $(aI+bA)^n=a^nI+na^{n-1}bA$, where I is a unit matrix of order 2, $\forall\ n\in N$.

Q.8 If the matrices $A=\begin{bmatrix}1&2\\3&4\end{bmatrix}$ and $B=\begin{bmatrix}a&b\\c&d\end{bmatrix}$ (a, b, c, d not all simultaneously zero) commute, find the value of $\dfrac{d-b}{a+c-b}$. Also show that the matrix which commutes with A is of the form $\begin{bmatrix}\alpha-\beta&2\beta/3\\\beta&\alpha\end{bmatrix}$

Q.9 If $\begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$ is an idempotent matrix. Find the value of f(a), where $f(x) = x - x^2$, when $bc = 1/4$. Hence otherwise evaluate a.

Q.10 If the matrix A is involutary, show that $\frac{1}{2}(I + A)$ and $\frac{1}{2}(I - A)$ are idempotent and $\frac{1}{2}(I+A)\cdot\frac{1}{2}(I-A)=\mathbf{O}$.

Q.11 Show that the matrix $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ can be decomposed as a sum of a unit and a nilpotent marix. Hence evaluate the matrix $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}^{2007}$.

Q.12 Given matrices $A = \begin{bmatrix} 1 & x & 1 \\ x & 2 & y \\ 1 & y & 3 \end{bmatrix}$; $B = \begin{bmatrix} 3 & -3 & z \\ -3 & 2 & -3 \\ z & -3 & 1 \end{bmatrix}$
Obtain x, y and z if the matrix AB is symmetric.

Q.13 Let X be the solution set of the equation $A^x = I$, where $A = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$ and I is the corresponding unit matrix and $x \subseteq N$ then find the minimum value of $\sum(\cos^x\theta + \sin^x\theta)$, $\theta \in R$.

Q.14 $A = \begin{pmatrix} 3 & a & -1 \\ 2 & 5 & c \\ b & 8 & 2 \end{pmatrix}$ is Symmetric and $B = \begin{pmatrix} d & 3 & a \\ b-a & e & -2b-c \\ -2 & 6 & -f \end{pmatrix}$ is Skew Symmetric, then find AB.
Is AB a symmetric, Skew Symmetric or neither of them. Justify your answer.

Q.15 Express the matrix $\begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & -6 \\ -1 & 0 & 4 \end{bmatrix}$ as a sum of a lower triangular matrix & an upper triangular matrix with zero in its leading diagonal. Also Express the matrix as a sum of a symmetric & a skew symmetric matrix.

Q.16 A is a square matrix of order n.
l = maximum number of distinct entries if A is a triangular matrix
m = maximum number of distinct entries if A is a diagonal matrix
p = minimum number of zeroes if A is a triangular matrix
If $l + 5 = p + 2m$, find the order of the matrix.

Q.17 If A is an idempotent non zero matrix and I is an identity matrix of the same order, find the value of n, $n \in N$, such that $(A + I)^n = I + 127A$.

Q.18 Consider the two matrices A and B where $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$; $B = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$. If n(A) denotes the number of elements in A such that n(XY) = 0, when the two matrices X and Y are not conformable for multiplication. If C = (AB)(B'A); D = (B'A)(AB) then, find the value of $\left(\frac{n(C)\left(|D|^2 + n(D)\right)}{n(A) - n(B)}\right)$.

EXERCISE-II

Q.1 If $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then prove that value of f and g satisfying the maxtrix equation $A^2 + fA + gI = \mathbf{O}$ are equal to $- t_r$ (A) and determinant of A respectively. Given a, b, c, d are non zero reals and $I=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$; $O=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Q.2 $A_{3\times 3}$ is a matrix such that $|A| = a$, $B = (\text{adj } A)$ such that $|B| = b$. Find the value of $(ab^2 + a^2b + 1)S$ where $\frac{1}{2}S = \frac{a}{b} + \frac{a^2}{b^3} + \frac{a^3}{b^5} + \ldots\ldots$ up to ∞, and $a = 3$.

Q.3 For the matrix $A=\begin{bmatrix} 4 & -4 & 5 \\ -2 & 3 & -3 \\ 3 & -3 & 4 \end{bmatrix}$ find A^{-2}.

Q.4 Given $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 1 \\ 2 & 3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$. Find P such that $BPA = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

Q.5 Given the matrix $A = \begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}$ and X be the solution set of the equation $A^x = A$, where $x \in N - \{1\}$. Evaluate $\prod\left(\frac{x^3+1}{x^3-1}\right)$ where the continued product extends $\forall\ x \in X$.

Q.6 If $F(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$ then show that $F(x).\ F(y) = F(x + y)$

Hence prove that $[\,F(x)\,]^{-1} = F(-x)$.

Q.7 If A is a skew symmetric matrix and I + A is non singular, then prove that the matrix $B = (I - A)(I + A)^{-1}$ is an orthogonal matrix. Use this to find a matrix B given $A=\begin{bmatrix} 0 & 5 \\ -5 & 0 \end{bmatrix}$.

Q.8 Use matrix to solve the following system of equations.

(i) $\begin{aligned} x+y+z&=3 \\ x+2y+3z&=4 \\ x+4y+9z&=6 \end{aligned}$ (ii) $\begin{aligned} x+y+z&=6 \\ x-y+z&=2 \\ 2x+y-z&=1 \end{aligned}$ (iii) $\begin{aligned} x+y+z&=3 \\ x+2y+3z&=4 \\ 2x+3y+4z&=7 \end{aligned}$ (iv) $\begin{aligned} x+y+z&=3 \\ x+2y+3z&=4 \\ 2x+3y+4z&=9 \end{aligned}$

Q.9 Let A be a 3×3 matrix such that $a_{11} = a_{33} = 2$ and all the other $a_{ij} = 1$. Let $A^{-1} = xA^2 + yA + zI$ then find the value of $(x + y + z)$ where I is a unit matrix of order 3.

Q.10 Given that $A=\begin{bmatrix} 1 & 2 & 2 \\ 2 & 2 & 3 \\ 1 & -1 & 3 \end{bmatrix}$, $C=\begin{bmatrix} 2 & 1 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, $D=\begin{bmatrix} 10 \\ 13 \\ 9 \end{bmatrix}$ and that Cb=D. Solve the matrix equation Ax=b.

Q.11 Find the matrix A satisfying the matrix equation, $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} . A . \begin{bmatrix} 3 & 2 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 3 & -1 \end{bmatrix}$.

Q.12 If $A = \begin{bmatrix} k & m \\ l & n \end{bmatrix}$ and $kn \neq lm$; then show that $A^2 - (k+n)A + (kn - lm) I = \mathbf{O}$.
Hence find A^{-1}.

Q.13 Given $A = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$; $B = \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}$. I is a unit matrix of order 2. Find all possible matrix X in the following cases.
(i) $AX = A$ (ii) $XA = I$ (iii) $XB = \mathbf{O}$ but $BX \neq \mathbf{O}$.

Q.14 Find the product of two matrices A & B, where $A = \begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix}$ & $B = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$ and use it to solve the following system of linear equations,
$x + y + 2z = 1$; $3x + 2y + z = 7$; $2x + y + 3z = 2$.

Q.15 If $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ then, find a non-zero square matrix X of order 2 such that $AX = \mathbf{O}$. Is $XA = \mathbf{O}$.
If $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$, is it possible to find a square matrix X such that $AX = O$. Give reasons for it.

Q.16 Determine the values of a and b for which the system $\begin{bmatrix} 3 & -2 & 1 \\ 5 & -8 & 9 \\ 2 & 1 & a \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b \\ 3 \\ -1 \end{bmatrix}$
(i) has a unique solution ; (ii) has no solution and (iii) has infinitely many solutions

Q.17 If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$; $B = \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}$; $C = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ and $X = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$ then solve the following matrix equation.
(a) $AX = B - I$ (b) $(B - I)X = IC$ (c) $CX = A$

Q.18 If A is an orthogonal matrix and B = AP where P is a non singular matrix then show that the matrix PB^{-1} is also orthogonal.

Q.19 Consider the matrices $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$ and let P be any orthogonal matrix and $Q = PAP^T$ and $R = P^TQ^KP$ also $S = PBP^T$ and $T = P^TS^KP$

Column I	Column II
(A) If we vary K from 1 to n then the first row first column elements at R will form	(P) G.P. with common ratio *a*
(B) If we vary K from 1 to n then the 2nd row 2nd column elements at R will form	(Q) A.P. with common difference 2
(C) If we vary K from 1 to n then the first row first column elements of T will form	(R) G.P. with common ratio *b*
(D) If we vary K from 3 to n then the first row 2nd column elements of T will represent the sum of	(S) A.P. with common difference – 2.

EXERCISE-III

Q.1 If matrix $A=\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$ where a, b, c are real positive numbers, abc = 1 and $A^TA = I$, then find the value of $a^3 + b^3 + c^3$. [JEE 2003, Mains-2 out of 60]

Q.2 If $A=\begin{bmatrix} \alpha & 2 \\ 2 & \alpha \end{bmatrix}$ and $|A^3| = 125$, then α =

(A) ± 3 (B) ± 2 (C) ± 5 (D) 0 [JEE 2004(Scr)]

Q.3 If M is a 3 × 3 matrix, where $M^TM = I$ and det (M) = 1, then prove that det (M – I) = 0. [JEE 2004, 2 out of 60]

Q.4 $A=\begin{bmatrix} a & 1 & 0 \\ 1 & b & d \\ 1 & b & c \end{bmatrix}$, $B=\begin{bmatrix} a & 1 & 1 \\ 0 & d & c \\ f & g & h \end{bmatrix}$, $U=\begin{bmatrix} f \\ g \\ h \end{bmatrix}$, $V=\begin{bmatrix} a^2 \\ 0 \\ 0 \end{bmatrix}$, $X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

If AX = U has infinitely many solution, then prove that BX = V cannot have a unique solution. If further afd ≠ 0, then prove that BX = V has no solution. [JEE 2004, 4 out of 60]

Q.5 $A=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 4 \end{bmatrix}$, $I=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $A^{-1}=\frac{1}{6}(A^2+cA+dI)$, then the value of c and d are

(A) –6, –11 (B) 6, 11 (C) –6, 11 (D) 6, – 11 [JEE 2005(Scr)]

Q.6 If $P=\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$, $A=\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $Q = PAP^T$ and $x = P^TQ^{2005}P$, then x is equal to

(A) $\begin{bmatrix} 1 & 2005 \\ 0 & 1 \end{bmatrix}$ (B) $\begin{bmatrix} 4+2005\sqrt{3} & 6015 \\ 2005 & 4-2005\sqrt{3} \end{bmatrix}$

(C) $\frac{1}{4}\begin{bmatrix} 2+\sqrt{3} & 1 \\ -1 & 2-\sqrt{3} \end{bmatrix}$ (D) $\frac{1}{4}\begin{bmatrix} 2005 & 2-\sqrt{3} \\ 2+\sqrt{3} & 2005 \end{bmatrix}$ [JEE 2005 (Screening)]

Comprehension (3 questions)

Q.7 $A=\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$, U_1, U_2 and U_3 are columns matrices satisfying. $AU_1=\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$; $AU_2=\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$, $AU_3=\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$

and U is 3 × 3 matrix whose columns are U_1, U_2, U_3 then answer the following questions

(a) The value of $|U|$ is

(A) 3 (B) – 3 (C) 3/2 (D) 2

(b) The sum of elements of U^{-1} is

(A) – 1 (B) 0 (C) 1 (D) 3

(c) The value of $\begin{bmatrix}3 & 2 & 0\end{bmatrix} U \begin{bmatrix}3\\2\\0\end{bmatrix}$ is

(A) 5 (B) 5/2 (C) 4 (D) 3/2

[JEE 2006, 5 marks each]

Q.8 Match the statements / Expression in **Column-I** with the statements / Expressions in **Column-II** and indicate your answer by darkening the appropriate bubbles in the 4 × 4 matrix given in OMR.

	Column-I		**Column-II**
(A)	The minimum value of $\frac{x^2+2x+4}{x+2}$ is	(P)	0
(B)	Let A and B be 3 × 3 matrices of real numbers, where A is symmetric, B is skew-symmetric, and $(A+B)(A-B) = (A-B)(A+B)$. If $(AB)^t = (-1)^k AB$, where $(AB)^t$ is the transpose of the matrix AB, then the possible values of k are	(Q)	1
(C)	Let $a = \log_3 \log_3 2$. An integer k satisfying $1 < 2^{(-k+3^{-a})} < 2$, must be less than	(R)	2
(D)	If $\sin\theta = \cos\phi$, then the possible values of $\frac{1}{\pi}\left(\theta \pm \phi - \frac{\pi}{2}\right)$ are	(S)	3

[JEE 2008, 6]

Comprehension (3 questions)

Q.8 Let $\mathcal{A}$ be the set of all 3 × 3 symmetric matrices all of whose entries are either 0 or 1. Five of these entries are 1 and four of them are 0.

(a) The number of matrices in $\mathcal{A}$ is

(A) 12 (B) 6 (C) 9 (D) 3

(b) The number of matrices A in $\mathcal{A}$ for which the system of linear equations

$$A\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix}$$

has a unique solution, is

(A) less than 4 (B) at least 4 but less than 7

(C) at least 7 but less than 10 (D) at least 10

(c) The number of matrices A in $\mathcal{A}$ for which the system of linear equations

$$A\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix}$$

is inconsistent, is

(A) 0 (B) more than 2 (C) 2 (D) 1 [JEE 2009, 4+4+4]

ANSWER KEY

MATRICES

EXERCISE-I

Q.1 $EF = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, FE = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Q.2 8

Q.3 $x = \frac{3}{2}, \ y = 2$

Q.4 5049

Q.5 $V = \begin{bmatrix} 0 \\ \frac{1}{11} \end{bmatrix}$

Q.8 1

Q.9 f (a) = 1/4, a = 1/2

Q.11 $\begin{bmatrix} 1 & 0 \\ 4014 & 1 \end{bmatrix}$

Q.12 $\left(-\frac{4\sqrt{2}}{3}, \frac{2}{3}, 2\sqrt{2}\right), \left(\frac{4\sqrt{2}}{3}, \frac{2}{3}, -2\sqrt{2}\right)$,(3,3, –1)

Q.13 2

Q.14 AB is neither symmetric nor skew symmetric

Q.15 $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ -1 & 0 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 2 & 5 \\ 0 & 0 & -6 \\ 0 & 0 & 0 \end{bmatrix}; \begin{bmatrix} 1 & 2 & 2 \\ 2 & 3 & -3 \\ 2 & -3 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & -3 \\ -3 & 3 & 0 \end{bmatrix}$

Q.16 4

Q.17 n = 7

Q.18 650

EXERCISE-II

Q.1 $f = -(a+d)$; $g = ad - bc$ Q.2 225 Q.3 $\begin{bmatrix} 17 & 4 & -19 \\ -10 & 0 & 13 \\ -21 & -3 & 25 \end{bmatrix}$

Q.4 $\begin{bmatrix} -4 & 7 & -7 \\ 3 & -5 & 5 \end{bmatrix}$ Q.5 $\frac{3}{2}$ Q.7 $\frac{1}{13}\begin{bmatrix} -12 & -5 \\ 5 & -12 \end{bmatrix}$

Q.8 (i) $x = 2, y = 1, z = 0$; (ii) $x = 1, y = 2, z = 3$;
(iii) $x = 2 + k, y = 1 - 2k, z = k$ where $k \in R$; (iv) inconsistent, hence no solution

Q.9 1

Q.10 $x_1 = 1, x_2 = -1, x_3 = 1$

Q.11 $\frac{1}{19}\begin{bmatrix} 48 & -25 \\ -70 & 42 \end{bmatrix}$

Q.12 $\frac{1}{kn - lm}\begin{bmatrix} n & -m \\ -l & k \end{bmatrix}$

Q.13 (i) $X = \begin{bmatrix} a & b \\ 2-2a & 1-2b \end{bmatrix}$ for $a, b \in R$; (ii) X does not exist ;

(iii) $X = \begin{bmatrix} a & -3a \\ c & -3c \end{bmatrix}$ $a, c \in R$ and $3a + c \neq 0$; $3b + d \neq 0$

Q.14 $x = 2, y = 1, z = -1$ Q.15 $X = \begin{bmatrix} -2c & -2d \\ c & d \end{bmatrix}$, where $c, d \in R - \{0\}$, NO

Q.16 (i) $a \neq -3$, $b \in R$; (ii) $a = -3$ and $b \neq 1/3$; (iii) $a = -3$, $b = 1/3$

Q.17 (a) $X = \begin{bmatrix} -3 & -3 \\ \frac{5}{2} & 2 \end{bmatrix}$, (b) $X = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$, (c) no solution

Q.19 (A) Q; (B) S; (C) P; (D) P

EXERCISE-III

Q.1 4 Q.2 A Q.5 C Q.6 A Q.7 (a) A, (b) B, (c) A

Q.8 (A) R (B) Q,S (C) R,S (D) P,R Q.9 (a) A, (b) B, (b) B