Quadratic Equation

MC Sir

1. Introduction, Graphs
2. Inequality
3. Theory of Equations : Relation between Roots and Coefficients of Cubic and Higher Polynomials
4. Identity
5. Infinite Roots, Common Roots
6. Maximum and Minimum Values of Quadratic and Rational Function

Quadratic Equation

MC Sir

7. General 2^0 in x and y

8. Condition for General 2^0 in x and y to be factorized in two linears

9. Location of Roots

10. Modulus Inequality

11. Logarithm Inequality

Quadratic Equation

MC Sir

No. of Questions				
2008	2009	2010	2011	2012
1	--	3	2	--

Quadratic

- $y = ax^2 + bx + c\ ;\qquad a \neq 0$

  a = leading coefficient

  b = coefficient of linear term

  c = absolute term

- $y = f(x) = ax^2 + bx + c$

  In case

  $a = 0 \Rightarrow y = bx + c$ *is linear polynomial*

  $a = c = 0 \Rightarrow y = bx$ *is odd linear polynomial*

Cubic Polynomial

- $y = ax^3 + bx^2 + cx + d$

  a = leading coefficient

  d = absolute term

Roots of Quadratic Equation

- $y = ax^2 + bx + c$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Where D = $b^2 - 4ac$ is called discriminant.

- $ax^2 + bx + c = 0$

  Sum of roots = $-b/a$

  Product of roots = c/a

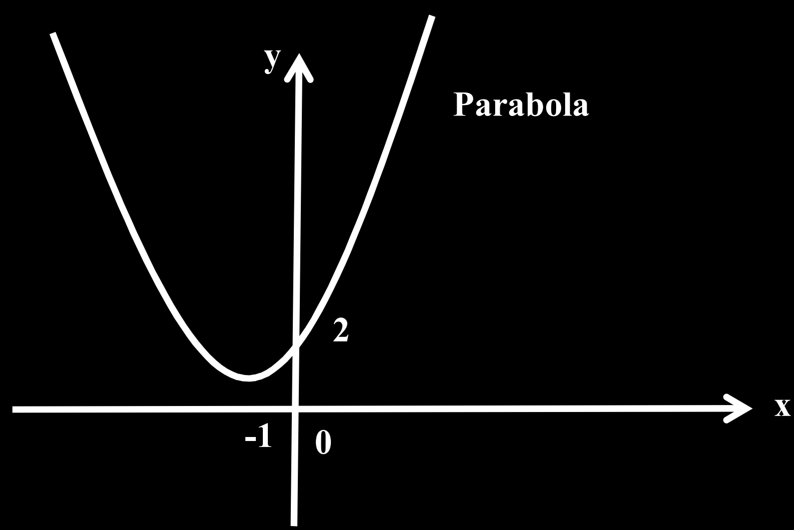
  $D = b^2 - 4ac$

Different Graphs of Quadratic Expression

Example

Q. $y = x^2 + 2x + 2 \quad = (x+1)^2 + 1$

$D = 2^2 - 8 = -4 < 0$



For $x = -1$ $\quad$ y is minimum

x	0	1	2	3	4	-1	-2	-3	-4	-5	∞	$-\infty$
y	2	5	10	17	26	1	2	5	10	17	∞	∞

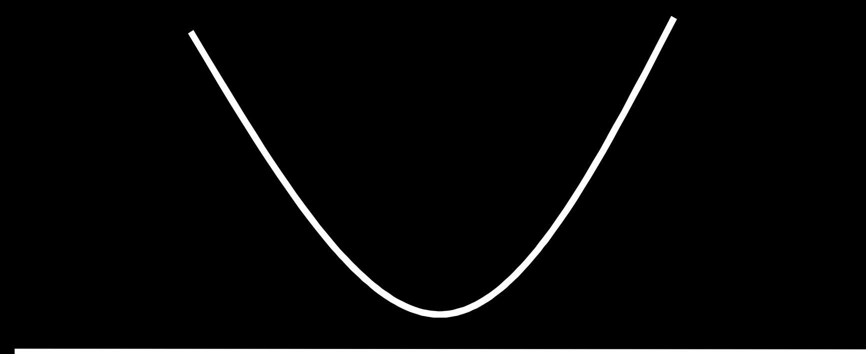
Leading coefficient > 0

In general graph of $y = ax^2 + bx + c$;

$a > 0$ *Mouth facing upward*

$D < 0$ *Parabola don't touch x axis (no real root)*

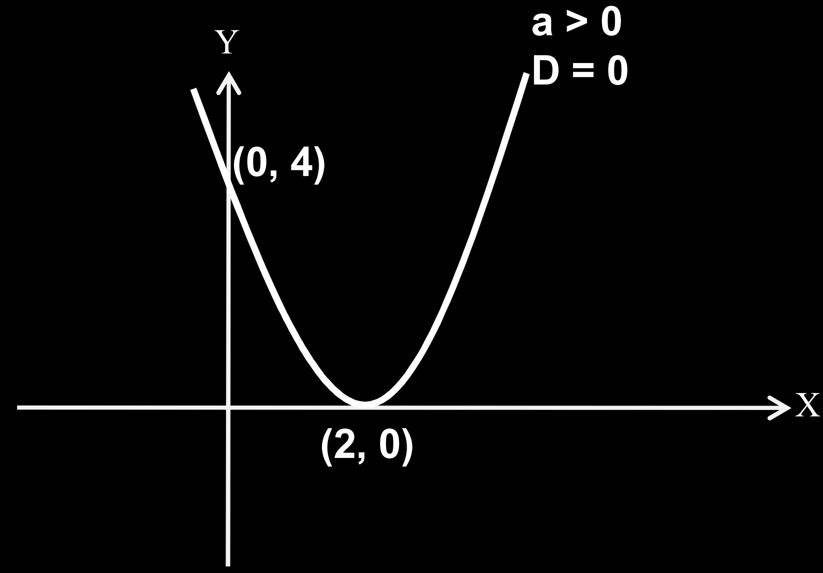
$y > 0 \; \forall x \in R$



Example

Q. $y = x^2 - 4x + 4 = (x-2)^2$

$D = 0$



x	0	1	2	3	4	5	6	–1	–2	∞	$-\infty$
y	4	1	0	1	4	9	16	9	16	∞	∞

$y \geq 0 \;\; \forall x \in R$

Leading Coefficient. > 0

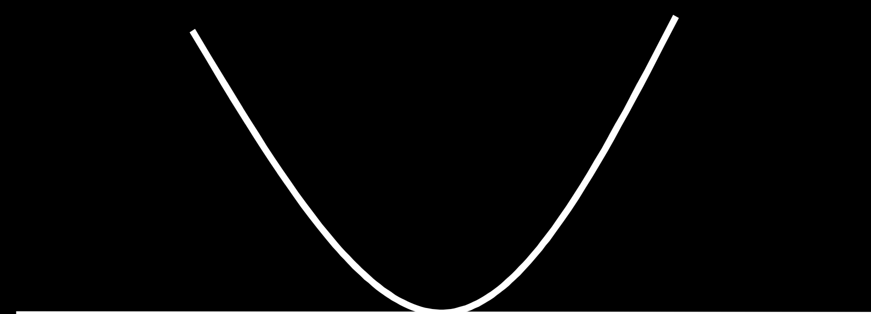
In general graph of $y = ax^2 + bx + c$;

$a > 0$ Mouth facing upward

$D = 0$ (One Real Root) Parabola touch the x Axis

$y = 0$ for only one value of x (root)

$y > 0\ \forall x \in \mathrm{R} - \{\text{root}\}$

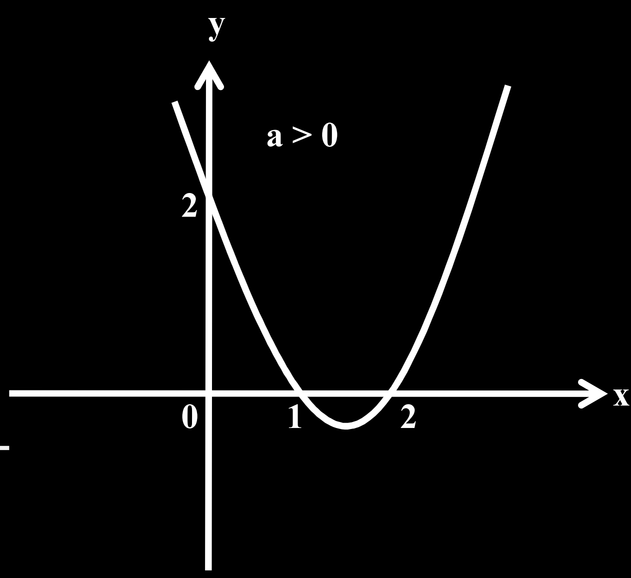


Example

Q. $y = x^2 - 3x + 2$

$D = 3^2 - 4(2) = 1 > 0$

x	0	1	2	3	4	$^3/_2$	∞	$-\infty$
y	2	0	0	2	6	$-¼$	∞	∞



$y > 0 \Rightarrow x\ (-\infty, 1) \cup (2, \infty)$

$y < 0 \Rightarrow x \in (1,2)$

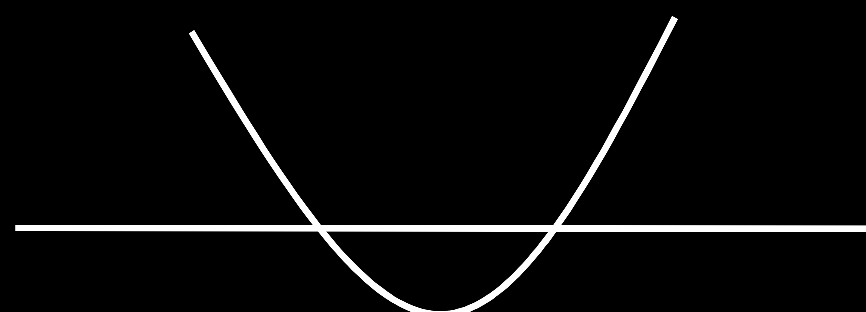
$y = 0 \Rightarrow x \in \{1,2\}$

Q. ***In General***

$$y = ax^2 + bx + c$$

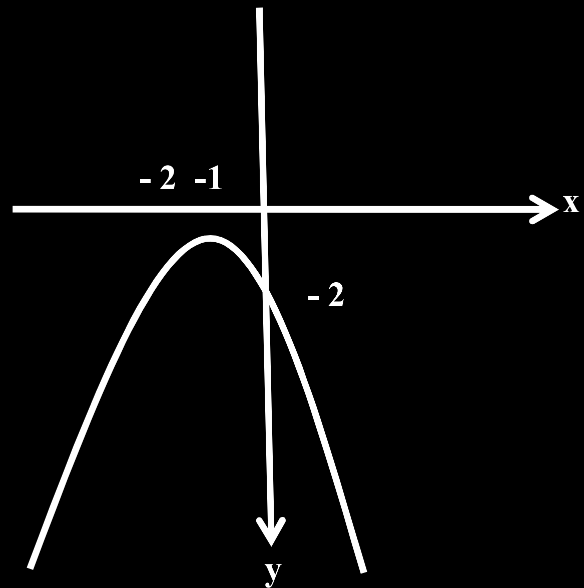
$a > 0 \Rightarrow$ *parabola mouth facing upward*

$D > 0 \Rightarrow$ *Two distinct real root (parabola cuts the x axis at 2 distinct point)*



Example

Q. $y = -x^2 - 2x - 2 = -(x+1)^2 - 1$



$D < 0$

x	0	1	2	3	-1	-2	-3	∞	$-\infty$
y	-2	-5	-10	-17	-1	-2	-5	$-\infty$	$-\infty$

Leading Coefficient < 0

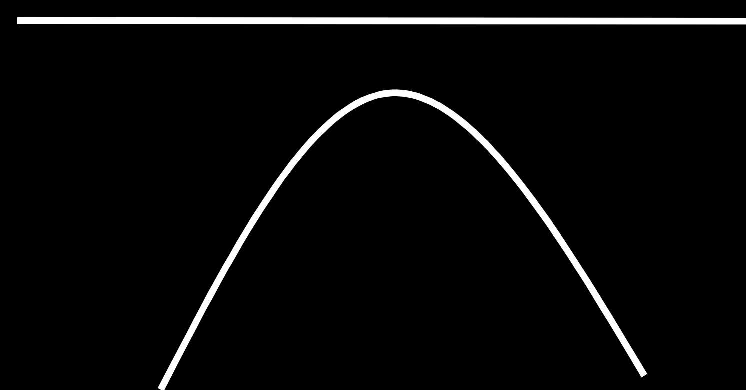
Q. In General

$$y = ax^2 + bx + c$$

$a < 0 \quad \Rightarrow$ ***mouth facing downward***

$D < 0 \quad \Rightarrow$ ***no real root***

$y < 0 \qquad \forall x \in R$



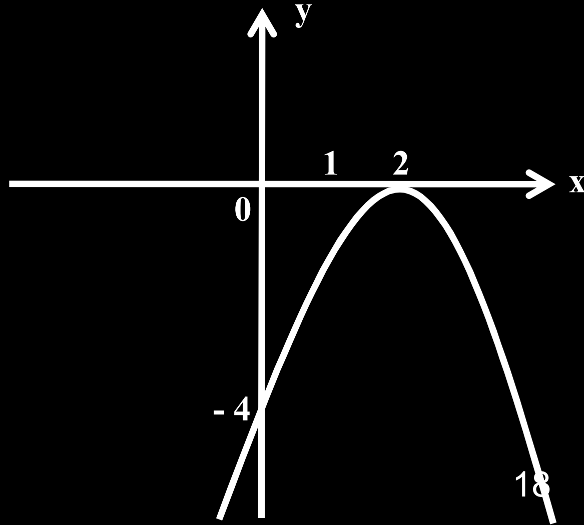
Example

Q. $y = -x^2 + 4x - 4 = -(x-2)^2$

$D = 0$

x	0	1	2	3	4	-1	∞	$-\infty$
y	-4	-1	0	-1	-4	0	$-\infty$	$-\infty$

Leading Coefficient < 0



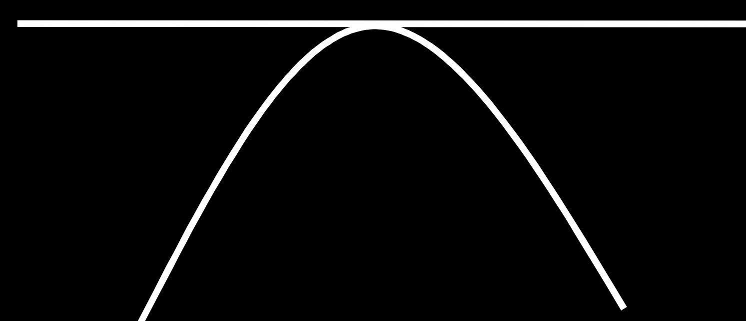
Q. In General

$y = ax^2 + bx + c$

$a < 0$ ***mouth facing downward***

$D = 0$ ***(one real root) parabola touch the x axis***

$y \leq 0 \quad \forall x \in R$



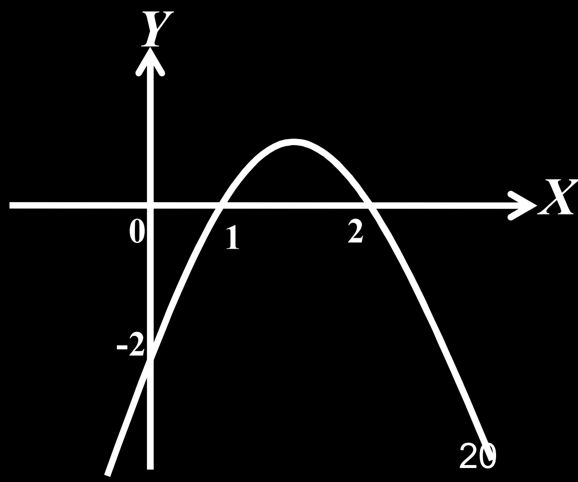
Example

Q. $y = -x^2 + 3x - 2 = -(x - 1)(x - 2)$

D > 0

x	0	1	2	3	4	–1	–2	∞	$-\infty$
y	– 2	0	0	–2	–9	–9	–12	$-\infty$	$-\infty$

Leading Coefficient < 0

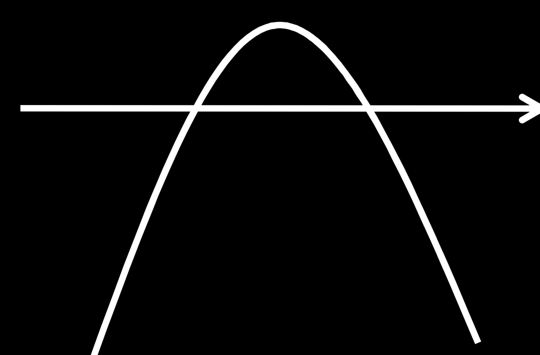


Q. ***In General***

$$y = ax^2 + bx + c$$

a < 0 ***Parabola mouth facing downward***

D > 0 ***Two distinct real root (Parabola cut the x-axis at two distinct points.***

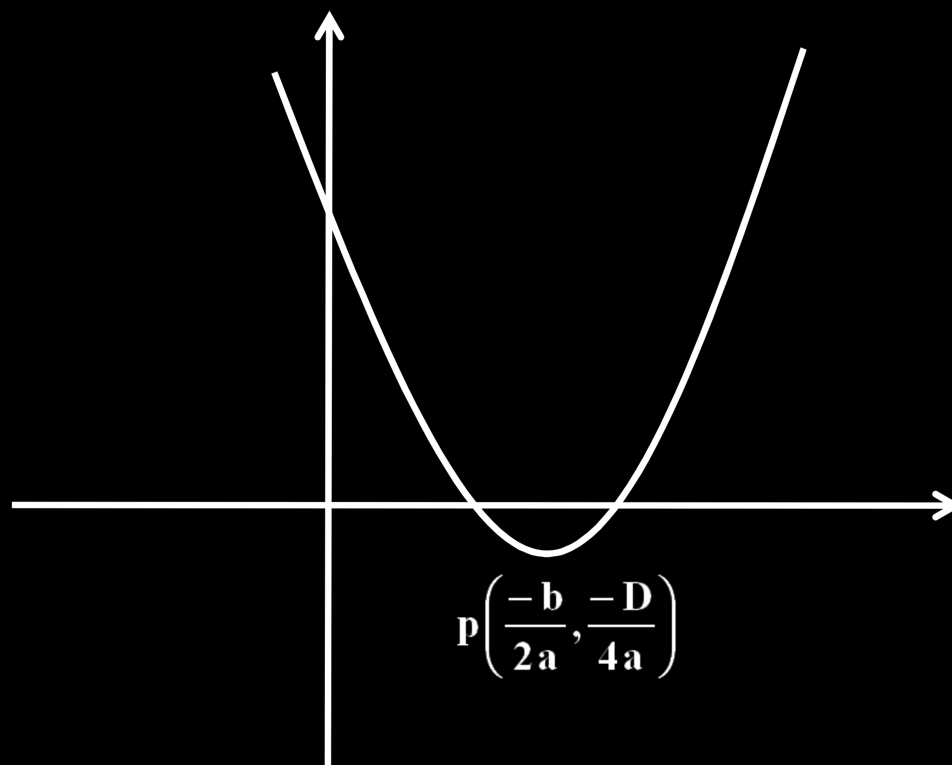


Co-ordinate of vertex

- $y = ax^2 + bx + c$

$$x = \frac{-b}{2a}$$

$$y = \frac{-D}{4a}$$



Nature of Roots

- $D > 0 \Leftrightarrow$ *roots are real & distinct (unequal)*

- $D = 0 \Leftrightarrow$ *roots are real & coincident (equal)*

- $D < 0 \Leftrightarrow$ *roots are imaginary.*

Nature of Roots

Consider the quadratic equation $ax^2 + bx + c = 0$

where $a, b, c \in Q$ *&* $a \neq 0$ *then;*

If D is a perfect square, then roots are rational.

Note

If $\alpha = p + \sqrt{q}$ *is one root in this case, (where p is rational &* $\sqrt{q}$ *is a surd) then other root will be* $p - \sqrt{q}$

Note

If $p + iq$ is one root of a quadratic equation, then the other root must be the conjugate $p - iq$ & vice versa. ($p, q \in R$ & $i = \sqrt{-1}$).

Example

Q. Let $a > 0$, $b > 0$ and $c > 0$. Then, both the roots of the equation $ax^2 + bx + c = 0$

(a) are real and negative

(b) have negative real parts

(c) have positive real parts

(d) None of the above ***[IIT-JEE 1979]***

Example

Q. Both the roots of the equation

$$(x - b)(x - c) + (x - a)(x - c) + (x - a)(x - b) = 0$$

are always

(a) positive *(b) negative*

(c) real *(d) None of these*

[IIT-JEE 1980]

Example

Q. The number of real solutions of the equation

$|x|^2 - 3|x| + 2 = 0$ *is*

(a) 4 *(b) 1*

(c) 3 *(d) 2*

[IIT-JEE 1982]

Example

Q. *Let f(x) be a quadratic expression which is positive for all real values of x.*

If $g(x) = f(x) + f'(x) + f''(x)$*, then for any real x*

(a) $g(x) < 0$ (b) $g(x) > 0$

(c) $g(x) = 0$ (d) $g(x) \geq 0$

[IIT-JEE 1990]

Example

Q. Let α, β be the roots of the equation

$(x - a)(x - b) = c, c \neq 0$

Then the roots of the equation

$(x - \alpha)(x - \beta) + c = 0$ *are*

(a) a, c *(b)* b, c *(c)* a, b *(d)* $a + c, b + c$

[IIT-JEE 1992]

Example

True / False

Q. If $a < b < c < d$, then the roots of the equation

$$(x - a)(x - c) + 2(x - b)(x - d) = 0$$

are real and distinct. [IIT-JEE 1984]

Example

Q. The number of points of intersection of two curves $y = 2\sin x$ and $y = 5x^2 + 2x + 3$ is

(a) 0 *(b) 1* *(c) 2* *(d) ∞*

[IIT-JEE 1994]

Example

Q. For all x, $x^2 + 2ax + 10 - 3a > 0$, *then the interval in which a lies is*

(a) $a < -5$

(b) $-5 < a < 2$

(c) $a > 5$

(d) $2 < a < 5$

[IIT – JEE 2004]

Example

Q. If b > a, then the equation (x – a) (x – b) – 1 = 0 has

(a) Both roots in (a, b)

(b) both roots in (-∞, a)

(c) both roots in (b, +∞)

(d) one root in (-∞, a) and the other in (b, +∞)

[IIT-JEE 2000]

Assignment 1

Q.1 If the equation

$$\sin^4 x - (k+2)\sin^2 x - (k+3) = 0$$

has a solution then k must lie in the interval

(A) (– 4, – 2) *(B) [– 3, 2)*

(C) (– 4, – 3) *(D) [– 3, – 2]*

Q.2 If $a, b \in R$, $a \neq 0$ and the quadratic equation $ax^2 - bx + 1 = 0$ has imaginary roots then $a + b + 1$ is :

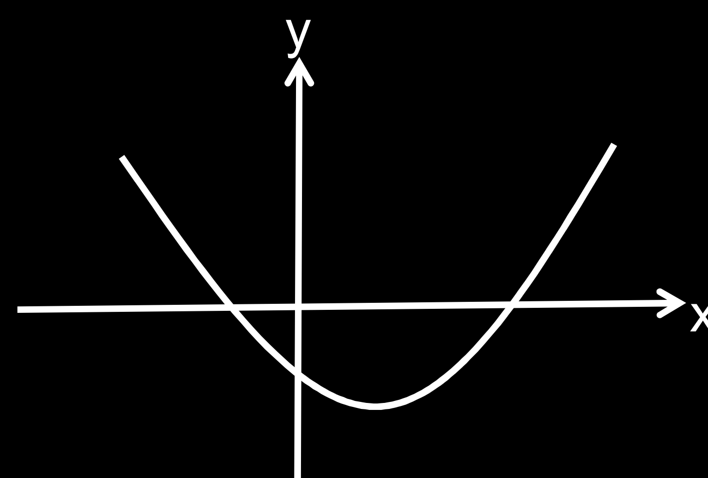
(A) positive ***(B) negative***

(C) zero ***(D) depends on the sign of b.***

[Multiple Objective Type]

Q.3 The graph of the quadratic polynomial;

$y = ax^2 + bx + c$ is as shown in the figure . Then



(A) $b^2 - 4ac > 0$ *(B)* $b < 0$

(C) $a > 0$ *(D)* $c < 0$

Q.4 ***If $a, b, c \in R$ such that $a + b + c = 0$ and $a \neq c$, then prove that the roots of $(b + c - a)x^2 + (c + a - b)x + (a + b - c)$ are real and distinct.***

Q.5 Find the value of a for which the roots of the equation $(2a - 5)\,x^2 - 2\,(a - 1)\,x + 3 = 0$ are equal.

Q.6 For what values of m does the equation $x^2 - x + m = 0$ possess no real roots ?

Q.7 For what values of m does the equation $x^2 - x + m^2 = 0$ possess no real roots ?

Relation between root and Coefficient of Quadratic Equation

$$ax^2 + bx + c = 0 \qquad ; a \neq 0 \qquad a,b,c \in R$$

- $ax^2 + bx + c = a(x - \alpha)(x - \beta) = 0$
- $\alpha + \beta = -\dfrac{b}{a}$ & $\alpha\beta = \dfrac{c}{a}$
- $x^2 + \left(\dfrac{b}{a}\right)x + \left(\dfrac{c}{a}\right) = (x - \alpha)(x - \beta)$

Formation of Quadratic Equation

$$x^2 - (\text{sum of roots})\,x + \text{product of roots} = 0$$

Example

Q. Form a Quadratic Equation with rational coefficients whose one root is $\tan 75°$

Example

Q. Form a Quadratic Equation with rational coefficients whose one root is $\cos 36°$

Example

Q. Form a Quadratic Equation with rational coefficients whose one root is tanπ/8

Inequalities

Rules :

- ***Adding positive number both-sides inequality remains same.***

Example :

$2 > 1 \Rightarrow 3 > 2$

Inequalities

Rules :

- *Subtracting both sides by positive number inequality remains same*

Example :

$2 > 1 \Rightarrow 1 > 0$

Inequalities

Rules :

- ***Multiply & divide by positive number without affecting inequality***

Example :

$4 > 2 \Rightarrow 1 > ½$

Inequalities

Rules :

- ***Multiply & divide by negative number to change sign of inequality***

Example :

$2 > 1 \Rightarrow -2 < -1$

Type – 1
Expression which can not be factorized

Example :

- $x^2 + x + 1 > 0$

Type – 1
Expression which can not be factorized

Example :

- $x^2 - 3x + 4 < 0$

Type – 1
Expression which can not be factorized

Example :

- $3x^2 - 7x + 6 > 0$

Type – 1
Expression which can not be factorized

Example :

- $-x^2 - 2x - 4 > 0$

Type – 2

Expression which can be factorized

Rules :

- ***Factorize in linear as far as possible***
- ***Make coefficient of x, as 1 in all linear by multiplying, dividing by appropriate number***
- ***Mark zeros of linear on number line***
- ***Give sign to respective area on number line***

Type – 2
Expression which can be factorized

- $(1 - x)(4 + 2x)(x - 2)(x - 7) > 0$

Type – 2
Expression which can be factorized

- $(x^2 - x - 6)(x^2 + 6x) > 0$

Type – 2

Expression which can be factorized

- $(x + 1)(x - 3)(x - 2)(3x + 7) < 0$

Type – 3

- $(x^2 - 5x + 6)\,(x^2 - 6x + 5) \leq 0$

Type – 3

- $2 - x - x^2 \geq 0$

Type – 3

- $3x^2 - 7x + 4 \geq 0$

Type – 4
Repeated Linear Factor

Rules :

- ***Get rid of even power***
- ***odd power treat as linear***

Type – 4
Repeated Linear Factor

- $(x + 1)(x - 3)(x - 2)^2 > 0$

Type – 4
Repeated Linear Factor

- $x\,(x + 6)\,(x + 2)^2\,(x - 3) > 0$

Type – 4
Repeated Linear Factor

- $(x - 1)^2 (x + 1)^3 (x - 4) < 0$

Type – 5
Rational Inequality

- $$\frac{(x-1)(x-2)(x-3)}{(x-4)(x-5)} < 0$$

Type – 5
Rational Inequality

- $\frac{2x-3}{3x-7} < 0$

Type – 5
Rational Inequality

- $\dfrac{2x-3}{3x-7} \geq 0$

Type – 5
Rational Inequality

- $$\frac{x^3\left(2x-3\right)^2\left(x-4\right)^6}{\left(x-3\right)^3\left(3x-8\right)^4}\le 0$$

Type – 5
Rational Inequality

- $\dfrac{x^2 - 5x + 12}{x^2 - 4x + 5} > 3$

Type – 5
Rational Inequality

- $\dfrac{x^2 - 5x + 6}{x^2 + x + 1} < 0$

Type – 5
Rational Inequality

- $\dfrac{(x-1)^2(x+1)^3}{x^4(x-2)} < 0$

Example

Q. $$\frac{x+1}{x-1} \geq \frac{x+5}{x+1}$$

Example

Q.

$$\frac{2(x-4)}{(x-1)(x-7)} \geq \frac{1}{x-2}$$

Example

Q. $$\frac{x^2+6x-7}{|x+4|}<0$$

Example

Q. *Let* $y = \sqrt{\dfrac{(x+1)(x-3)}{(x-2)}}$

Find all the real values of x for which y takes

real values. ***[IIT-JEE 1980]***

Example

Q. Find the set of all x for which

$$\frac{2x}{2x^2+5x+2} > \frac{1}{x+1}$$

[IIT-JEE 1987]

Example

Q. *Solve* $|x^2 + 4x + 3| + 2x + 5 = 0$ *[IIT-JEE 1988]*

Example

Q. *Let a and b be the roots of the equation $x^2 - 10cx - 11d = 0$ and those of $x^2 - 10ax - 11b = 0$ are c, d. Then find the value of $a + b + c + d$, when $a \neq b \neq c \neq d$.*

[IIT-JEE 2006]

Example

Q. *Let α, β be the roots of the equation $x^2 - px + r = 0$ and $\alpha/2, 2\beta$ be the roots of the equation $x^2 - qx + r = 0$. Then the value of r is*

(a) 2/9 (p – q) (2q – p) *(b) 2/9 (q – p) (2p – q)*

(c) 2/9 (q – 2p) (2q – p) *(d) 2/9 (2p – q) (2q – p)*

[IIT-JEE 2007]

Example

Fill in the blank :

Q. *If* $2 + i\sqrt{3}$ *is a root of the equation*

$x^2 + px + q = 0,$

where p and q are real, then (p, q) = (……).

[IIT–JEE 1982]

Example

Fill in the blank :

Q. If the products of the roots of the equation

$x^2 - 3kx + 2e^{2 \log k} - 1 = 0$ *is 7,*

then the roots are real for k = ….. .

[IIT-JEE 1984]

Example

Q. If x, y and z are real and different and $u = x^2 + 4y^2 + 9z^2 - 6yz - 3zx - 2xy$*, then u is always*

(a) non-negative *(b) zero*

(c) non-positive *(d) none of these*

[IIT-JEE 1979]

Example

Q. If one root is square of the other root of the equation $x^2 + px + q = 0$, then the relation between p and q is

(a) $p^3 - (3p - 1)\,q + q^2 = 0$

(b) $p^3 - q(3p + 1) + q^2 = 0$

(c) $p^3 + q(3p - 1) + q^2 = 0$

(d) $p^3 + q(3p + 1) + q^2 = 0$ [IIT-JEE 2004]

Assignment 2

Q.1 *The sum of all the value of m for which the roots x_1 and x_2 of the quadratic equation $x^2 - 2mx + m = 0$ satisfy the condition $x_1^3 + x_2^3 = x_1^2 + x_2^2$, is*

(A) $\frac{3}{4}$ (B) 1

(C) $\frac{9}{4}$ (D) $\frac{5}{4}$

Q.2 If α and β are the roots of the equation $ax^2 + bx + c = 0$ then the sum of the roots of the equation

$$a^2x^2 + (b^2 - 2ac)x + b^2 - 4ac = 0$$

in terms of α and β is given by

(A) $-(\alpha^2 - \beta^2)$ (B) $(\alpha + \beta)^2 - 2\alpha\beta$

(C) $\alpha^2\beta + \beta^2\alpha - 4\alpha\beta$ (D) $-(\alpha^2 + \beta^2)$

Q.3 The set of values of 'a' for which the inequality, $(x - 3a)(x - a - 3) < 0$ is satisfied for all $x \in [1, 3]$ is :

(A) (1/3, 3) ***(B) (0, 1/3)***

(C) (- 2, 0) ***(D) (- 2, 3)***

Q.4 If α and β are the roots of $a(x^2 - 1) + 2bx = 0$ then, which one of the following are the roots of the same equation?

(A) $\alpha + \beta, \alpha - \beta$

(B) $2\alpha + \frac{1}{\beta}, 2\beta + \frac{1}{\alpha}$

(C) $\alpha + \frac{1}{\beta}, \beta - \frac{1}{\alpha}$

(D) $\alpha + \frac{1}{2\beta}, \beta - \frac{1}{2\alpha}$

Q.5 Solve the following Inequality

- $\frac{1}{x} < 1$

Q.5 Solve the following Inequality

- $\dfrac{x}{x+2} \leq \dfrac{1}{x}$

Q.5 Solve the following Inequality

- $(x - 1)(3 - x)(x - 2)^2 > 0$

Q.5 Solve the following Inequality

- $$\frac{(x-1)(x+2)^2}{-1-x} < 0$$

Double Inequality

Example

Q. Solve the following Inequality

(i) $$1 < \frac{3x^2 - 7x + 8}{x^2 + 1} \le 2$$

Example

Q. Solve the following Inequality

(ii) $\left(x^2+3x+1\right)\left(x^2+3x-3\right)\geq 5$

Example

Q. Solve the following Inequality

(iii) $10^x(x-1)(x-2) \geq 0$

Example

Q. Solve the following Inequality

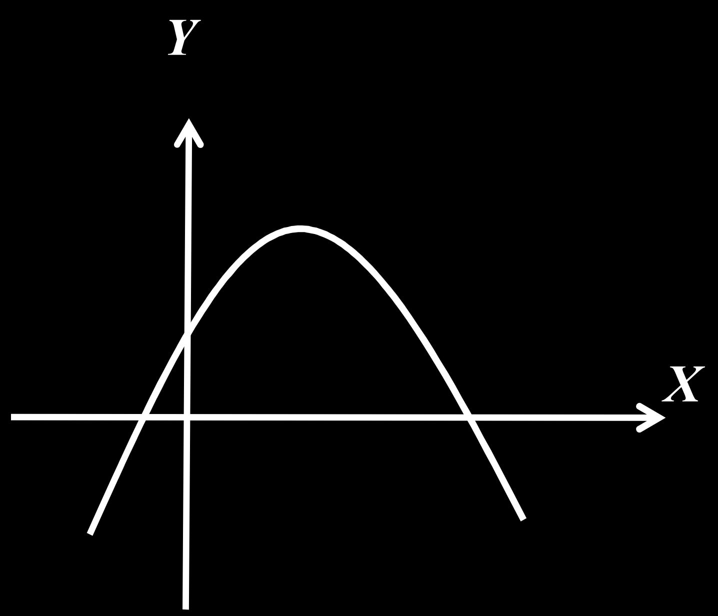
(iv) $\left(2^{x}-8\right)\left(x-7\right)\left(x+1\right)\leq 0$

Example

True / False :

$y = ax^2 + bx + c$

Q. $a > 0$

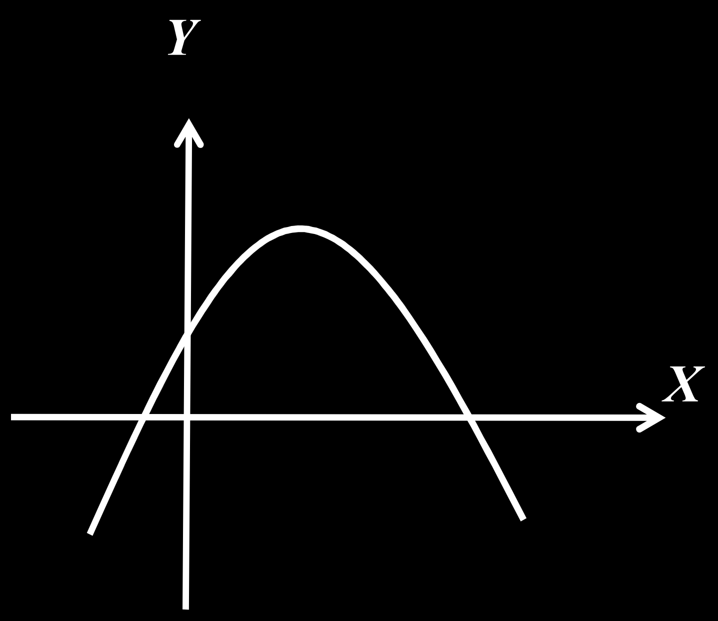


Example

True / False :

$$y = ax^2 + bx + c$$

Q. $c > 0$

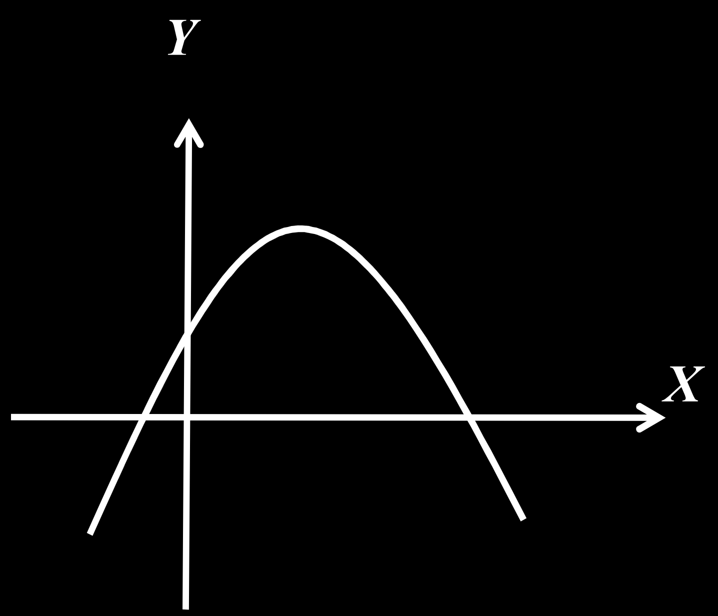


Example

True / False :

$$y = ax^2 + bx + c$$

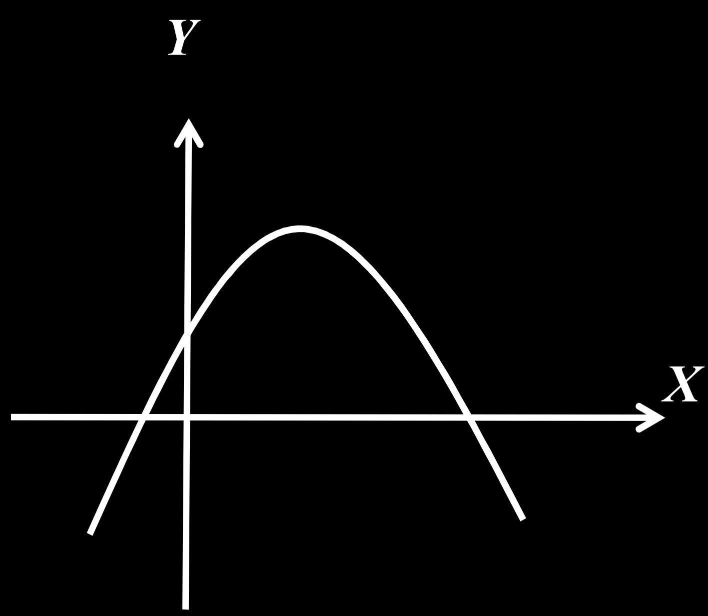
Q. $D > 0$



Example

True / False :

$$y = ax^2 + bx + c$$



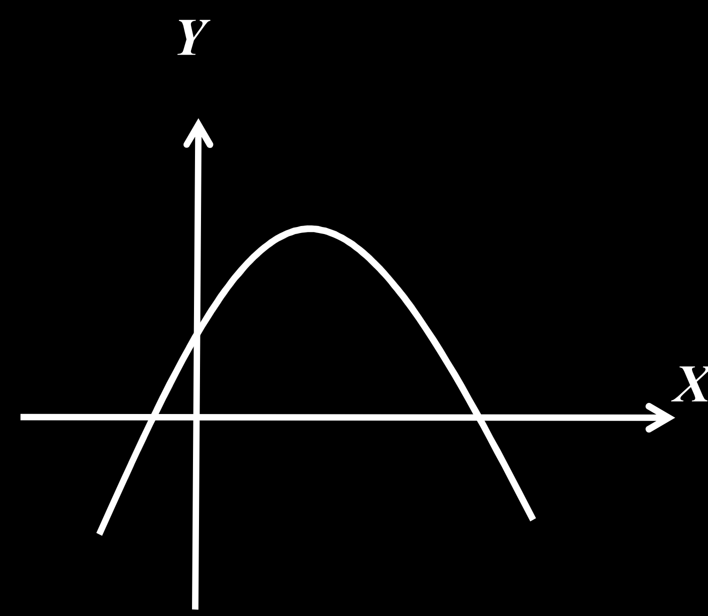
Q. $-b/a > 0$

Example

True / False :

$$y = ax^2 + bx + c$$

Q. $c/a > 0$

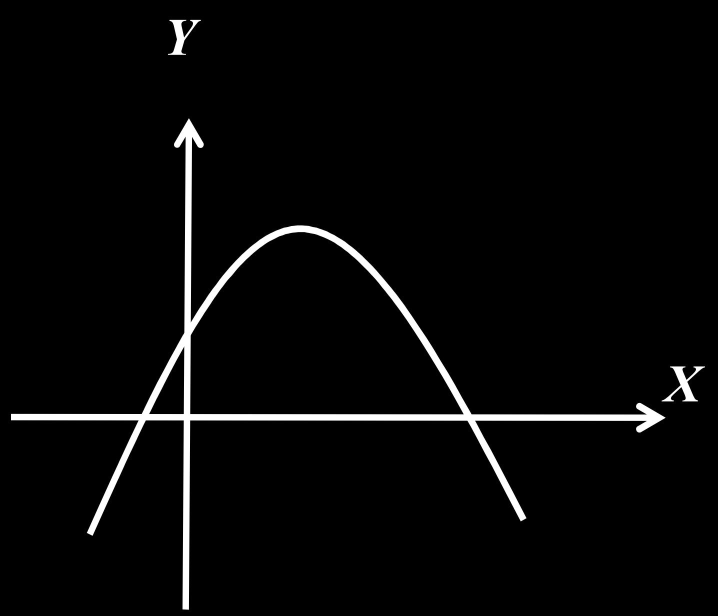


Example

True / False :

$y = ax^2 + bx + c$

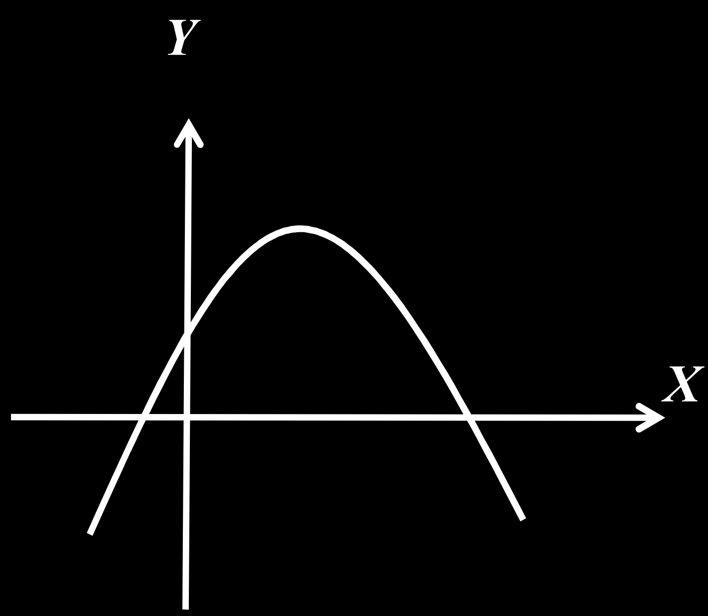
Q. $b > 0$



Example

True / False :

$y = ax^2 + bx + c$



Q. $-D/4a > 0$

Example

Q. Quadratic Equation $ax^2 + bx + c = 0$ *has no real roots then show that* $c(a + b + c) > 0$

Example

Q. Find a,

$$(a - 1)\,x^2 - (a + 1)\,x + a + 1 > 0 \qquad \forall x \in R$$

Example

Q. *Find a, if* $(a + 4)\,x^2 - 2a\,x + 2a - 6 < 0$

$\forall x \in R$

Example

Q. If α is root of $x^2 - 2x + 5 = 0$

Find the value of $\alpha^3 + \alpha^2 - \alpha + 21$

Example

Q. *If β is root of $x^2 - 2x + 5 = 0$*

Find the value of $\beta^3 + 4\beta^2 - 7\beta + 37$

Example

Q. *If* $x = 3 + \sqrt{5}$

Find the value of $x^4 + 12x^3 + 44x^2 - 48x + 17$

Example

Q. If $p(q-r)x^2 + q(r-p)x + r(p-q) = 0$ has equal root.

Show that : $\frac{2}{q} = \frac{1}{p} + \frac{1}{r}$

Example

Q. *If* $x^2 + \frac{1}{x^2} = 14$; $x > 0$ *then* *(MCQ)*

(a) $x^3 + x^{-3} = 62$ (b) $x^3 + x^{-3} = 52$

(c) $x^5 + x^{-5} = 624$ (d) $x^5 + x^{-5} = 724$

Example

Q. ***Find the integral solutions of the following system of inequalities***

(a) $5x - 1 < (x+1)^2 < 7x - 3$

(b) $\dfrac{x}{2x+1} \geq \dfrac{1}{4}, \dfrac{6x}{4x-1} < \dfrac{1}{2}$

[IIT-JEE 1978]

Example

Q. *If l, m, n are real $l \neq m$, then the roots of the equation $(l - m)x^2 - 5(l + m)x - 2(l - m) = 0$ are*

(a) real and equal *(b) complex*

(c) real and unequal *(d) none of these*

[IIT-JEE 1979]

Example

Q. For what value of m, does the system of equations $3x + my = m$, $2x - 5y = 20$ has solution satisfying the conditions $x > 0$, $y > 0$

[IIT-JEE 1980]

Example

Q. Find all real values of which satisfy

$x^2 - 3x + 2 > 0$ ***and*** $x^2 - 3x - 4 \leq 0.$

[IIT-JEE 1983]

Example

Q. *Let a, b, c be real numbers with $a \neq 0$ and let α, β be the roots of the equations*

$ax^2 + bx + c = 0$.

Express the roots of $a^3x^2 + abc\,x + c^3 = 0$ in terms of α, β. *[IIT-JEE 2001]*

Example

Q. *If α and β are the roots of $x^2 + px + q = 0$ and γ, δ are the roots of $x^2 + rx + s = 0$, then evaluate $(\alpha - \gamma)(\beta - \gamma)(\alpha - \delta)(\beta - \delta)$ in terms of p, q, r and s.*

[IIT-JEE 1979]

Assignment 3

Solve the following inequalities

Q.1 $x^4 - 2x^2 - 63 \leq 0$

Solve the following inequalities

Q.2 $\frac{7x-5}{8x+3} > 4$

Solve the following inequalities

Q.3 $$\frac{x+7}{x-5}+\frac{3x+1}{2}\geq 0$$

Solve the following inequalities

Q.4 $$\frac{\left(2-x^2\right)(x-3)^3}{(x+1)\left(x^2-3x-4\right)} \geq 0$$

Solve the following inequalities

Q.5 $$\frac{20}{(x-3)(x-4)}+\frac{10}{x-4}+1>0$$

Solve the following inequalities

Q.6 $$\frac{x-1}{x^2-x-12} \le 0$$

Solve the following inequalities

Q.7 For what values of c does the equation

$$(c-2)x^2 + 2(c-2)x + 2 = 0$$

possess no real roots ?

Solve the following inequalities

Q.8 For what values of a does the equation

$$x^2 + 2a\sqrt{a^2 - 3}\,x + 4 = 0$$

possess equal roots ?

Solve the following inequalities

Q.9 ***Find the value of k for which the curve $y = x^2 + kx + 4$ touches the Ox axis.***

Solve the following inequalities

Q.10 Find the least integral value of k for which the equation $x^2 - 2(k + 2)x + 12 + k^2 = 0$ has two different real roots.

Solve the following inequalities

Q.11 If the equation $4x^2 - 4(5x + 1) + p^2 = 0$ has one root equals to two more than the other, then the value of p is equal to

(a) $\pm \frac{\sqrt{236}}{3}$

(b) ± 5

(c) 5 or -1

(d) 4 or -3

Solve the following inequalities

Q.12 Possible values of x simultaneously satisfying the system of inequalities

$$\left.\begin{array}{l} \dfrac{(x-6)(x-3)}{x+2} \geq 0 \\ \text{and} \quad \dfrac{x-5}{x+1} \leq 3 \end{array}\right\} \text{is}$$

(A) $(-1, 3] \cup [6, \infty)$ ***(B)*** $(-2, 3] \cup [6, \infty)$

(C) $(-2, -1) \cup (4, \infty)$ ***(D)*** $[3, 6]$

Identity

$$ax^2 + bx + c = 0$$

Number of roots are infinite

When $a = b = c = 0$

Note

$$3 \text{ distinct real root of quadratic} \Rightarrow \text{infinite root}$$

Example

Q. Find the value of p for which the equation

$$(p+2)(p-1)x^2 + (p-1)(2p+1)x + p^2 - 1 = 0$$

has infinite roots

Example

Q. $$\frac{(x-a)(x-b)}{(c-a)(c-b)}+\frac{(x-b)(x-c)}{(a-b)(a-c)}+\frac{(x-c)(x-a)}{(b-c)(b-a)}=1$$

Prove that above is an identity

Quadratic With One Root Zero

$$ax^2 + bx + c = 0$$

$$Product\ of\ root = \frac{c}{a} = 0$$

$$c = 0$$

Quadratic With Both Root Zero

$$ax^2 + bx + c = 0$$

Sum of root = Product of root = 0

$$b = 0,\ c = 0$$

Quadratic With One Root Infinite

$$ax^2 + bx + c = 0$$

$$a = 0$$

Quadratic With Both Root ∞

$$y = ax^2 + bx + c$$

$$a = 0,\ b = 0,\ c \neq 0$$

Example

Q. If $(2p - q)\,x^2 + (p - 1)\,x + 5 = 0$ has both roots infinite. Find p & q

Symmetric Function

If $f(\alpha, \beta) = f(\beta, \alpha) \quad \forall \alpha, \beta$

Then $f(\alpha, \beta)$ *is called symmetric function of* α, β

Example

Q. Check if $f(\alpha, \beta)$ is symmetric or not

(i) $f(\alpha, \beta) = \alpha^2\beta + \alpha\beta^2$

(ii) $f(\alpha, \beta) = \cos(\alpha - \beta)$

(iii) $f(\alpha, \beta) = \sin(\alpha - \beta)$

(iv) $f(\alpha, \beta) = (\alpha^2 - \beta)$

Condition of Common Root

Condition for both Roots Common

$$a_1x^2 + b_1x^2 + c_1 = 0$$

$$a_2x^2 + b_2x^2 + c_2 = 0$$

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

Condition for One Root Common

$$\frac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1} = \frac{b_1c_2 - b_2c_1}{a_2c_1 - a_1c_2}$$

Example

Q. ***Find k for which equations*** $x^2 - 3x + 2 = 0$, $3x^2 + 4kx + 2 = 0$ ***have a common root***

Example

Q. *Find p and q if* $px^2 + 5x + 2 = 0$

$3x^2 + 10x + q = 0$ *have both roots in common*

Example

Q. *Find the value of a & b if* $x^2 - 4x + 5 = 0$, $x^2 + ax + b = 0$ *have a common root where* $a, b \in R$

Example

Q. If $4x^2\sin^2\theta - (4\sin\theta)\,x + 1 = 0$ &

$a^2(b^2 - c^2)\,x^2 + b^2(c^2 - a^2)\,x^2 + c^2(a^2 - b^2) = 0$

have a common root and the second equation has equal roots find possible value of θ *where* $\theta \in (0, \pi)$

Example

Q. *If the quadratic equation* $ax^2 + bx + c = 0$ & $x^2 + cx + b = 0$ $b \neq c$ *have a common root then prove that there uncommon roots are roots of the equation* $x^2 + x + bc = 0$

Example

Q. $x^2 + ax + 12 = 0$, $x^2 + bx + 15 = 0$ &

$x^2 + (a + b)x + 36 = 0$

have a common positive root

Find a, b & common root of equation.

Example

Q. *If one root of quadratic equation* $x^2 - x + 3a = 0$ *is double the root of the equation* $x^2 - x + a = 0$ *find a*

Example

Q. If $Q_1(x) = x^2 + (k - 29)x - k$

$Q_2(x) = 2x^2 + (2k - 43)x + k$

both are factors of a cubic polynomial find k

Example

Q. *If* $x^2 + abx + c = 0$ & $x^2 + acx + b = 0$ *have only one root common then show that quadratic equation containing their other common roots is*

$$a(b + c)\,x^2 + (b + c)\,x - abc = 0$$

Example

Q. *A value of b for which the equations*

$x^2 + bx - 1 = 0, x^2 + x + b = 0$

have one root in common is

(a) $-\sqrt{2}$ (b) $-i\sqrt{3}$ (c) $i\sqrt{5}$ (d) $\sqrt{2}$

[IIT-JEE 2011]

Example

Fill in the blank :

Q. *If the quadratic equations $x^2 + ax + b = 0$ and $x^2 + bx + a = 0$ $(a \neq b)$ have a common root, then the numerical value of $a + b$ is …. .*

[IIT-JEE 1986]

Assignment 4

Q.1 ***Find value of k for which the equation $(x - 1)(x - 2) = 0$ & $2x^2 + kx - 8 = 0$ have a common root***

Q.2 If x be the real number such that $x^3 + 4x + 8$. then the value of the expression $x^7 + 64x^2$ is

(A) 124 (B) 125 (C) 128 (D) 132

Q.3 If every solution of the equation $3\cos^2 x - \cos x - 1 = 0$ is a solution of the equation $a\cos^2 2x + b\cos 2x - 1 = 0$. Then the value of $(a + b)$ is equal to

(A) 5 (B) 9 (C) 13 (D) 14

Q.4 If $x^2 + 3x + 5 = 0$ & $ax^2 + bx + c = 0$ have common root/roots and $a : b, c \in N$ then find minimum value of $a + b + c$

Q.5 Determine the values of m for which the equation $3x^2 + 4mx + 2 = 0$ and $2x^2 + 3x - 2$ may have a common root.

Q.6 Q.7 For what value of a is the difference between the roots of the equation $(a - 2)x^2 - (a - 4)x - 2 = 0$ equal to 3 ?

Q.7 Find all values of a for which the sum of the roots of the equation $x^2 - 2a(x - 1) - 1 = 0$ is equal to the sum of the squares of its roots.

Q.8 For what values of a do the equations $x^2 + ax + 1 = 0$ and $x^2 + x + a = 0$ have a root in common ?

Maximum & Minimum Value of Quadratic Equation

- $y = ax^2 + bx + c$ *attain its maximum or minimum at point where* $x = \frac{-b}{2a}$ *according as* $a < 0$ *or* $a > 0$.
- *Maximum and Minimum value can be obtained by making a perfect square.*

Example

Q. $p(x) = ax^2 + bx + 8$ *is quadratic polynomial.*

Minimum value of $p(x)$ *is 6 when* $x = 2$

Find a & b

Example

Q. $y = 2x^2 - 3x + 1$, *find minimum value of y*

Example

Q. $y = 7 + 5x - 2x^2$ ***find maximum value of y***

Example

Q. *For $x \geq 2$ smallest possible value of*

$$log_{10}(x^3 - 4x^2 + x + 26) - log_{10}(x + 2)$$

Range of Linear

$$y = ax + b \quad ; a \neq 0$$

$$y \in R$$

Example

Q. $y = f(x) = x + 1$

Range of $\frac{\text{Linear}}{\text{Linear}}$

$$y = \frac{ax + b}{cx + d}$$

$$y \in R - \left\{\frac{a}{c}\right\}$$

Example

Q. $y = \dfrac{2x+3}{x+1}$, *Find range of y*

Example

Q. $y = \dfrac{1}{3x-1}$, *Find range of y*

Example

Q. $y = \dfrac{x(x-1)}{x-1}$, *Find range of y*

Example

Q. $y = \dfrac{(x-1)(x-2)(x-3)}{(x-2)(x-3)}$, ***Find range of y***

Range of $\frac{\text{Linear}}{\text{Quadratic}}, \frac{\text{Quadratic}}{\text{Quadratic}}, \frac{\text{Quadratic}}{\text{Linear}}$

- *Assume y*
- *Check for common roots in numerator & denominator*
- *Form Quadratic Equation*
- *Apply D $\geq$ 0 (since x is real)*
- *Solve inequality in y and hence the range*

Note

Always check for coefficient of x^2 not equal to zero

Example

Find range of following

Q. $$\frac{x^2+2x-11}{2(x-3)}$$

Example

Find range of following

Q. $$\frac{x^2 - 3x + 4}{x^2 + 3x + 4}$$

Example

Find range of following

Q. $$\frac{(x+1)(x-2)}{x(x+3)}$$

Example

Find range of following

Q.

$$\frac{x^2+2x-2}{x^2+2x+1}$$

Example

Find range of following

Q. $$\frac{x^2+14x+9}{x^2+2x+3}$$

Example

Find range of following

Q. $$\frac{x^2 - 5x + 6}{x^2 - 4x + 3}$$

Assignment 5

Q.1 Find the range of the function $f(x) = x^2 - 2x - 4$

Q.2 Find the least value of $\frac{6x^2 - 22x + 21}{5x^2 - 18x + 17} = y \ \forall x \in R$

Q.3 Find Range $\dfrac{x^2 - x + 1}{x^2 + x + 1}$

Q.4 Find the domain and Range of

$$f(x) = \sqrt{x^2 - 3x + 2}$$

General 2° in x & y

$$f(x, y) = ax^2 + 2h\,xy + by^2 + 2gx + 2fy + c$$

Condition of General 2° in x & y to be Resolved into two linear Factors

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

Rule

Step 1 :

factorize purely 2°

Step 2 :

Add constant to both the linear

Step 3 :

Compare coefficient of x & coefficient of y & absolute term if needed

Example

Q. *Prove that the Expression*

$$2x^2 + 3xy + y^2 + 2y + 3x + 1$$

can be factorized into two linear factors & find them

Example

Q. Prove that the Expression

$$x^2 - 3xy + 2y^2 - 2x - 3y - 35 = 0$$

can be factorized into two linear factors & find them

Example

Q. If the equation $x^2 + 16y^2 - 3x + 2 = 0$ *is satisfied by real values of x & y then show that* $x \in [1, 2]$ & $y \in [-1/8, 1/8]$

Theory of Equation

$$ax^2 + bx + c = a\,(x - \alpha)\,(x - \beta)$$

$$ax^3 + bx^2 + cx + d = a(x - \alpha)\,(x - \beta)\,(x - \gamma)$$

Sum & Product of Root taken 1 at a time

$$\alpha + \beta + \gamma = -b/a$$

$$\alpha\beta\gamma = -d/a$$

Sum of root taken 2 at a time

$$\alpha\beta + \beta\gamma + \gamma\alpha = c/a$$

Bi Quadratic

$$ax^4 + bx^3 + cx^2 + dx + e = a(x - \alpha) \ldots\ldots. (x - \delta)$$

Sum of root taken 2 at a time

$$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \boldsymbol{c/a}$$

Sum of root taken 3 at a time

$$\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\beta + \beta\gamma\delta = \boldsymbol{-d/a}$$

Note

$$(a + b + c)^2 = \sum a^2 + 2\sum ab$$

Example

Q. ***Find sum of square & sum of cubes of roots of the cubic equation*** $x^3 - px^2 + qx - r = 0$

Example

Q. Solve the cubic

$$4x^3 + 16x^2 - 9x - 36 = 0$$

Where sum of 2 root is zero

Example

Q. *If a, b, c are roots of cubic* $x^3 - x^2 + 1 = 0$

Find $\dfrac{1}{a^2} + \dfrac{1}{b^2} + \dfrac{1}{c^2}$

Example

Q. *If $\alpha, \beta, \gamma, \delta$ are roots of the equation*

$$\tan\left(\frac{\pi}{4}+x\right) = 3\tan 3x$$

Find the value of $\tan\alpha + \tan\beta + \tan\gamma + \tan\delta$

Example

Q. ***Find a cubic each of its roots is greater by unity then a root of*** $x^3 - 5x^2 + 6x - 3 = 0$

Example

Q. ***Find the cubic whose roots are cubes of the roots of*** $x^3 + 3x^2 + 2 = 0$

Example

Q. *The length of side of a* Δ *are roots of the equation* $x^3 - 12x^2 + 47x - 60 = 0$

Find Δ^2

Location of Roots

Type -1

Both roots of a quadratic equation are greater than a specified number

$$(\alpha, \beta) > d$$

Condition

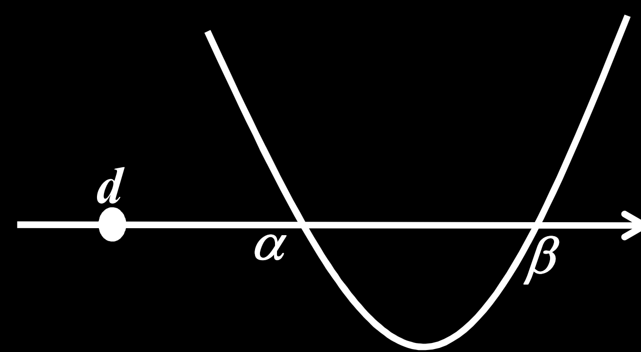
If $y = ax^2 + bx + c$

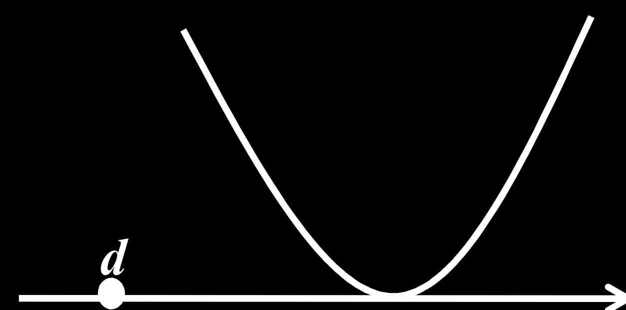
(i) $a > 0$

(ii) $D \geq 0$

(iii) $\dfrac{-b}{2a} > d$

(iv) $f(d) > 0$





Example

Q. ***Find the value of d for which both roots of the equation*** $x^2 - 6dx + 2 - 2d + 9d^2 = 0$ ***are greater than 3***

Example

Q. ***Find all the values of 'a' for which both roots of the equation $x^2 + x + a = 0$ exceed the quantity 'a'.***

Type - 2

Both roots lies on either side of a fixed number

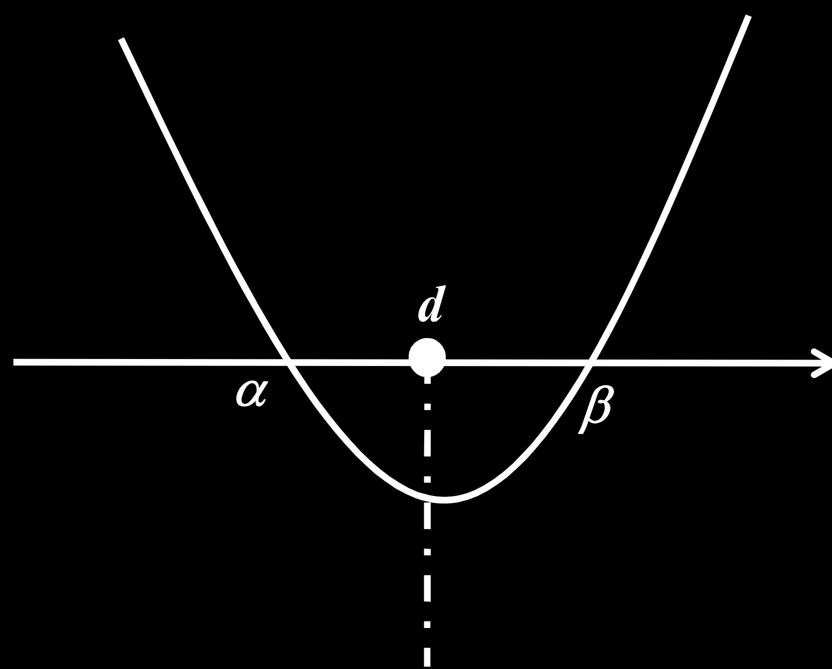
say (d)

$$\alpha < d < \beta$$

Condition

$a > 0$

$f(d) < 0$



Example

Q. ***Find k for which 1 root of the equation is greater than 2 and other is less than 2***

$$x^2 - (k + 1)\,x + k^2 + k - 8 = 0$$

Example

Q. Find the set of value of 'a' for which zeroes of the quadratic polynomial $(a^2 + a + 1)\, x^2 + (a - 1)x + a^2$ are located on either side of 3.

Example

Q. *Find a for which one root is positive, one is negative* $-x^2 - (3a - 2)x + a^2 + 1 = 0$

Example

Q. *Find a for which both root lie on either side of -1 of quadratic*

$$(a^2 - 5a + 6)x^2 - (a - 3)x + 7 = 0$$

Type - 3

Both roots lies between two fixed number

$$d < \alpha < \beta < e$$

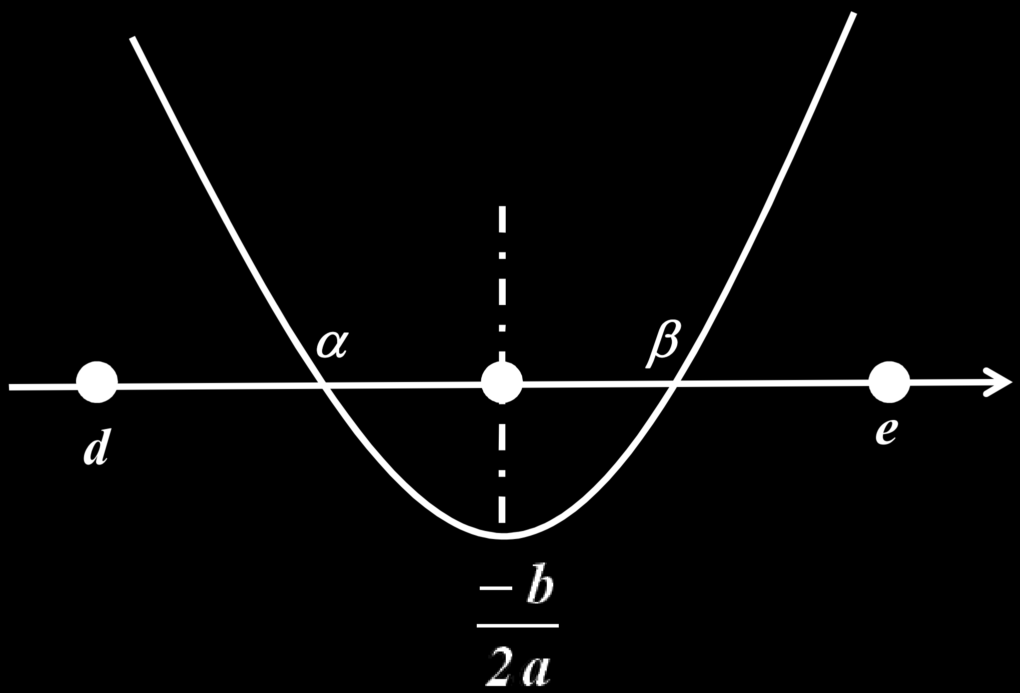
Conditions

(i) $D \geq 0$

(ii) $f(e) > 0$

(iii) $f(d) > 0$

(iv) $d < \dfrac{-b}{2a} < e$



Example

Q. *If* $\alpha, \beta \in (-6, 1)$

Find k for which

$$x^2 + 2(k - 3)x + 9 = 0$$

Type - 4

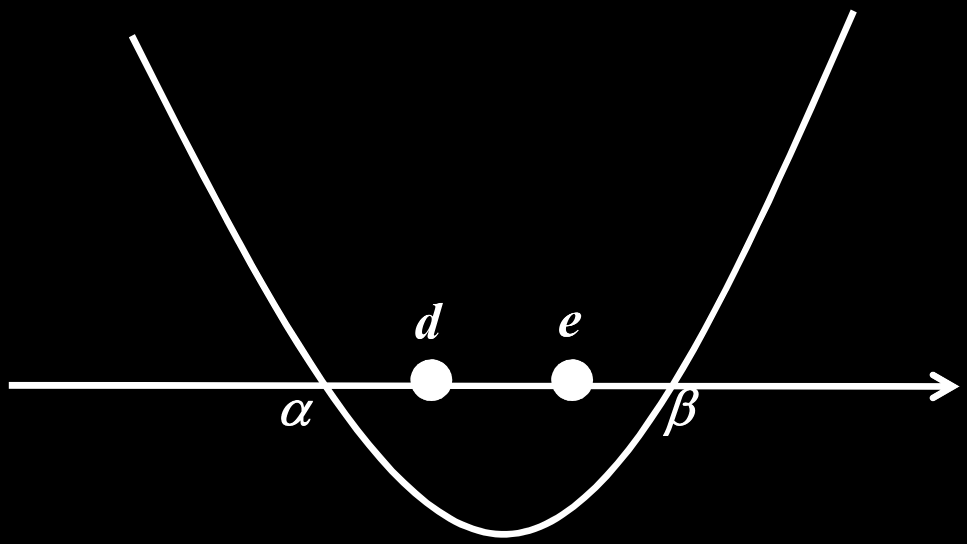
Both roots lies on either side of two fixed number

$$\alpha < d < e < \beta$$

Conditions

(i) $f(d) < 0$

(ii) $f(e) < 0$



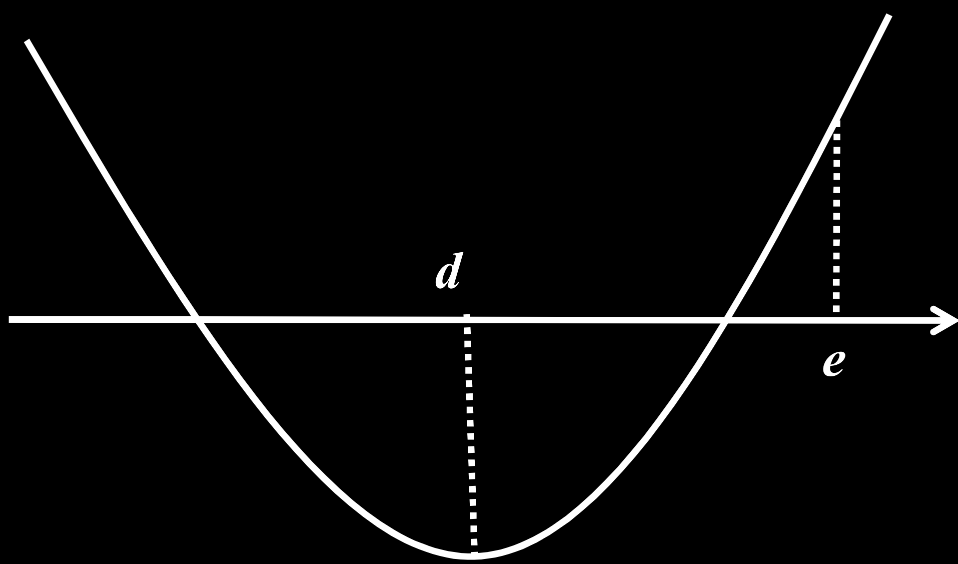
Example

Q. *Find k for which one root of the equation* $(k-5)x^2 - 2kx + k - 4 = 0$ *is smaller than 1 and the other root is greater than 2*

Type - 5

Exactly one root lies in the interval (d, e)

$f(d)\, f(e) < 0$



Example

Q. *Find the set of values of m for which exactly one root of the equation* $x^2 + mx + (m^2 + 6m) = 0$ *lie in* $(-2, 0)$

Example

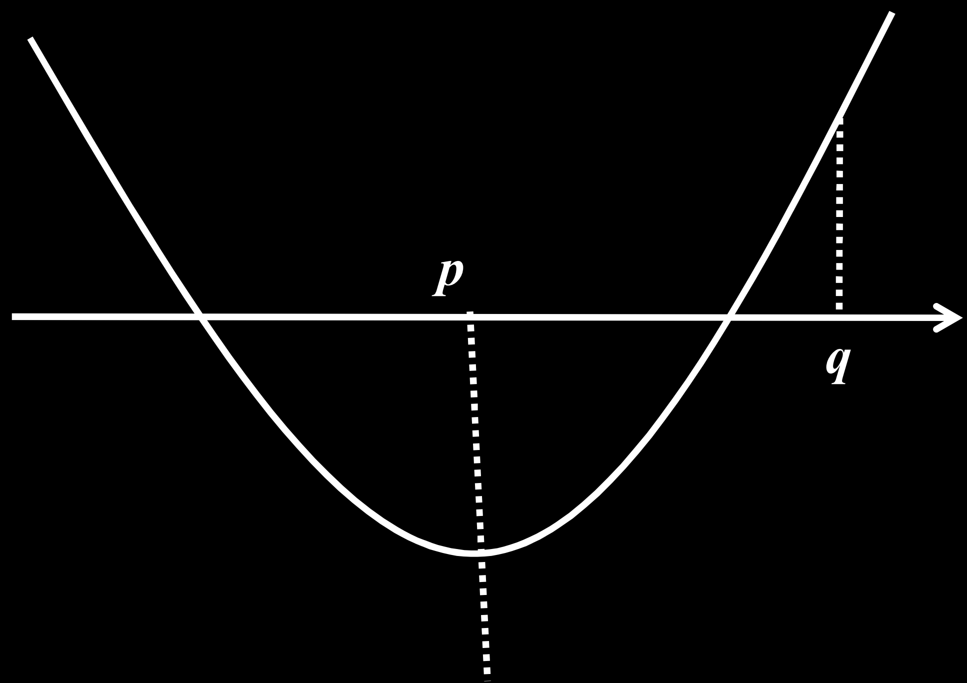
Q. *Find a for which exactly one root of the quadratic equation* $x^2 - (a + 1)x + 2a = 0$ *lies in (0,3)*

Type - 6

If $f(p)\,f(q) < 0$

$\Rightarrow$ Exactly one root

lies between (p, q)



Example

Q. If $a < b < c < d$ show that

Quadratic $(x - a)(x - c) + \lambda (x - b)(x - d) = 0$

has real root for all real values of λ

Example

Q. Find p for which the expression $x^2 - 2px + 3p + 4 < 0$ is satisfied for at least one real x

Example

Q. Find a for which expression

$(a^2 + 3)\, x^2 + \sqrt{5a+3}\, x - \frac{1}{4} < 0$ *is satisfied for at least one real x*

Example

Q. Find m if $x^2 - 4x + 3m + 1 > 0$ *is satisfied for all positive x*

Example

Q. ***Show that for any real value of a*** $(a^2 + 3)x^2 + (a + 2)x - 5 < 0$ ***is true for at least one negative x.***

Example

Q. If $f(x) = 4x^2 + ax + (a - 3)$ is negative for at least one negative x, find all values of a

Example

Q. Find a for which $x^2 + 2(a-1)x + a + 5 = 0$ *has at least one positive root.*

Example

Q. *Find p for which the least value of*

$4x^2 - 4px + b^2 - 2p + 2$ *in* $x \in [0,2]$ *is equal to 3*

Example

Q. *Find k for which the equation*

$x^4 + x^2 (1 - 2k) + k^2 - 1 = has$

(i) *No real solution*

Example

Q. *Find k for which the equation*

$x^4 + x^2 (1 - 2k) + k^2 - 1 = has$

(ii) one real solution

Example

Q. *Find k for which the equation*

$x^4 + x^2 (1 - 2k) + k^2 - 1 = has$

(iii) two real solution

Example

Q. *Find k for which the equation*

$x^4 + x^2 (1 - 2k) + k^2 - 1 = $ *has*

(iv) three real solution

Example

Q. *Find k for which the equation*

$x^4 + x^2 (1 - 2k) + k^2 - 1 = has$

(v) Four real solution

Modulas Inequality

Example

Q. $|x^2+4x+2| = \frac{5x+16}{3}$

Note

$$|\,\mathbf{x}\,| < \alpha \Rightarrow \mathbf{x} \in (-\alpha, \alpha)$$

$$|\,\mathbf{x}\,| > \beta \Rightarrow \mathbf{x} \in (-\infty, -\beta) \cup (\beta, \infty)$$

Example

Q. $(|x - 1| - 3)(|x + 2| - 5) < 0$

Example

Q. $|x - 5| > |x^2 - 5x + 9|$

Example

Q. $$\left|\frac{x^2-5x+4}{x^2-4}\right| \le 1$$