Quadratic Equation

MC Sir

1. Introduction, Graphs
2. Inequality
3. Theory of Equations : Relation between Roots and Coefficients of Cubic and Higher Polynomials
4. Identity
5. Infinite Roots, Common Roots
6. Maximum and Minimum Values of Quadratic and Rational Function

Quadratic Equation

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7. General 2^0 in x and y

8. Condition for General 2^0 in x and y to be factorized in two linears

9. Location of Roots

10. Modulus Inequality

11. Logarithm Inequality

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Quadratic Equation

No. of Questions				
2008	2009	2010	2011	2012
1	--	3	2	--

Quadratic

- $y = ax^2 + bx + c \; ; \qquad a \neq 0$

a = leading coefficient

b = coefficient of linear term

c = absolute term

- $y = f(x) = ax^2 + bx + c$

  In case

  $a = 0 \Rightarrow y = bx + c$ *is linear polynomial*

  $a = c = 0 \Rightarrow y = bx$ *is odd linear polynomial*

Cubic Polynomial

- $y = ax^3 + bx^2 + cx + d$

  $a =$ *leading coefficient*

  $d =$ *absolute term*

Roots of Quadratic Equation

- $y = ax^2 + bx + c$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Where $D = b^2 - 4ac$ is called discriminant.

- $ax^2 + bx + c = 0$

  Sum of roots = $-b/a$

  Product of roots = c/a

  $D = b^2 - 4ac$

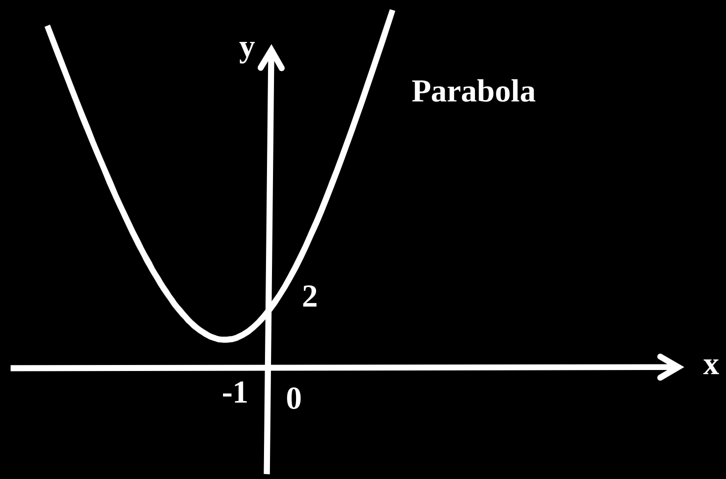
Different Graphs of Quadratic Expression

Example

Q. $y = x^2 + 2x + 2 \quad = (x+1)^2 + 1$

$D = 2^2 - 8 = -4 < 0$

For $x = -1 \quad y$ *is minimum*



x	0	1	2	3	4	-1	-2	-3	-4	-5	∞	$-\infty$
y	2	5	10	17	26	1	2	5	10	17	∞	∞

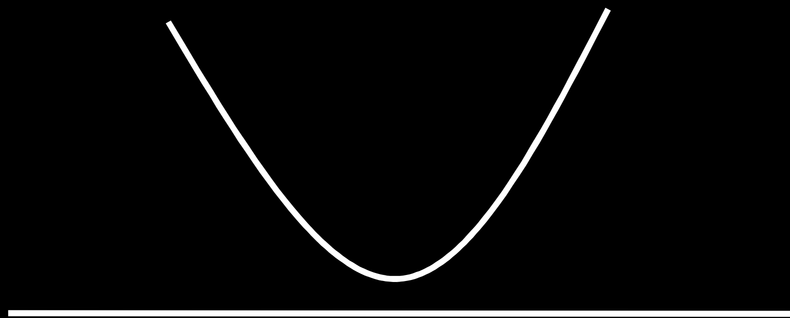
Leading coefficient > 0

In general graph of $y = ax^2 + bx + c$;

$a > 0$ Mouth facing upward

$D < 0$ Parabola don’t touch x axis (no real root)

$y > 0 \ \forall x \in R$



Example

Q. $y = x^2 - 4x + 4 = (x - 2)^2$

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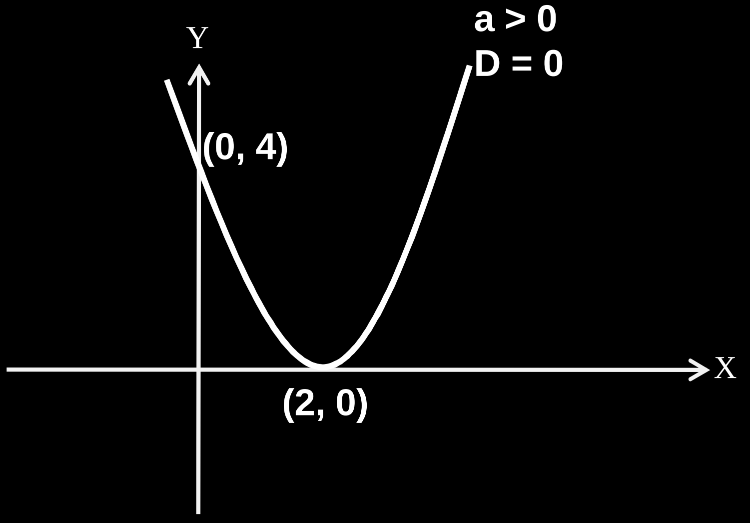


$D = 0$

x	0	1	2	3	4	5	6	−1	−2	∞	$-\infty$
y	4	1	0	1	4	9	16	9	16	∞	∞

$y \geq 0 \ \forall x \in R$

Leading Coefficient. > 0

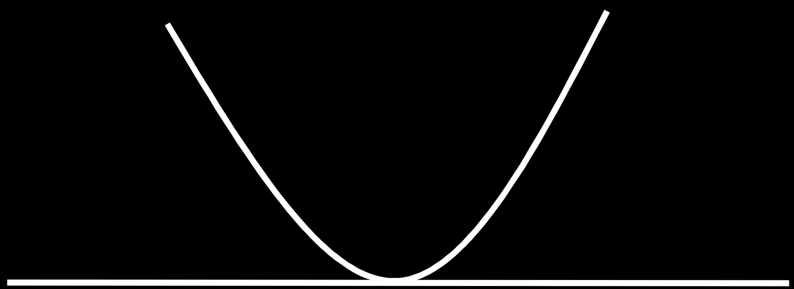
In general graph of $y = ax^2 + bx + c$;

$a > 0$ *Mouth facing upward*

$D = 0$ *(One Real Root) Parabola touch the x Axis*

$y = 0$ for only one value of x (root)

$y > 0\ \forall x \in R - \{\text{root}\}$

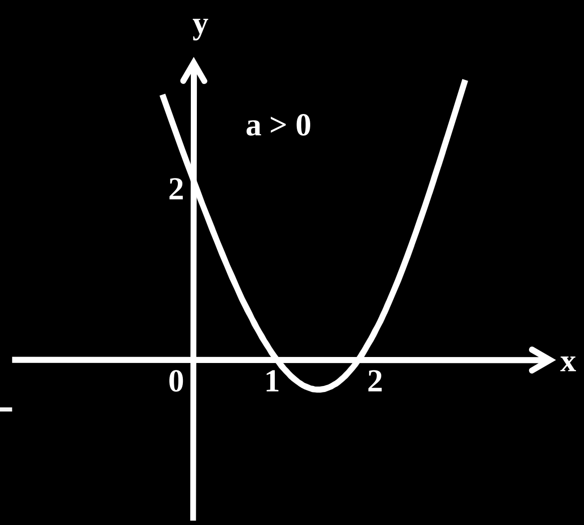


Example

Q. $y = x^2 - 3x + 2$

$D = 3^2 - 4(2) = 1 > 0$

x	0	1	2	3	4	$^3/_2$	∞	$-\infty$
y	2	0	0	2	6	$-\frac{1}{4}$	∞	∞



$y > 0 \Rightarrow x\ (-\infty, 1) \cup (2, \infty)$

$y < 0 \Rightarrow x \in (1,2)$

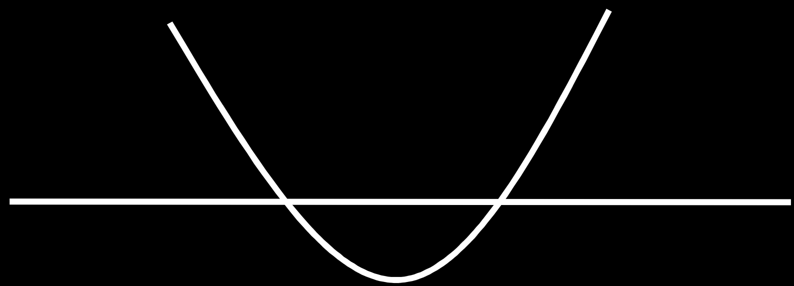
$y = 0 \Rightarrow x \in \{1,2\}$

Q. ***In General***

$$y = ax^2 + bx + c$$

$a > 0 \Rightarrow$ ***parabola mouth facing upward***

$D > 0 \Rightarrow$ ***Two distinct real root (parabola cuts the x axis at 2 distinct point)***



Example

Q. $y = -x^2 - 2x - 2 = -(x + 1)^2 - 1$

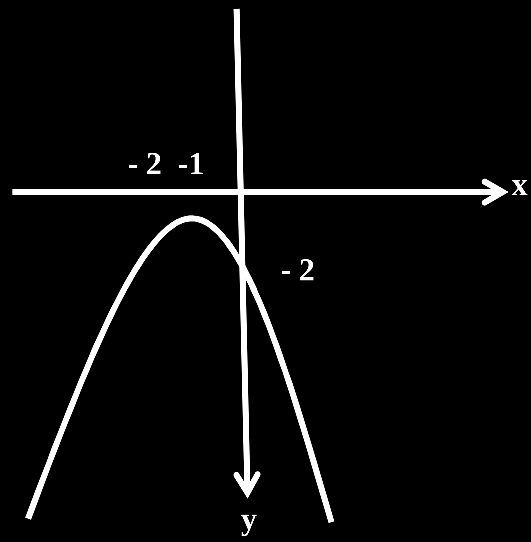
Example

Q. $y = -x^2 - 2x - 2 = -(x + 1)^2 - 1$

$D < 0$

Example

Q. $y = -x^2 - 2x - 2 = -(x+1)^2 - 1$



$D < 0$

x	0	1	2	3	-1	-2	-3	∞	$-\infty$
y	-2	-5	-10	-17	-1	-2	-5	$-\infty$	$-\infty$

Leading Coefficient < 0

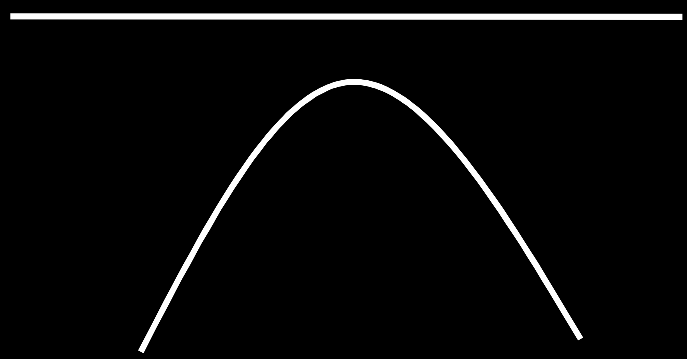
Q. ***In General***

$$y = ax^2 + bx + c$$

$a < 0 \quad \Rightarrow$ ***mouth facing downward***

$D < 0 \quad \Rightarrow$ ***no real root***

$y < 0 \qquad \forall x \in R$



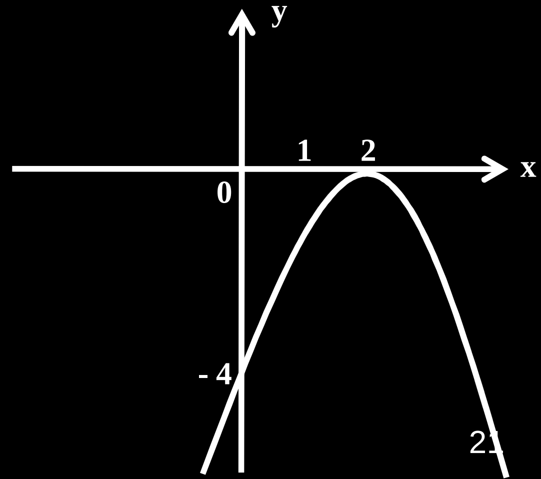
Example

Q. $y = -x^2 + 4x - 4 = -(x-2)^2$

$D = 0$

x	0	1	2	3	4	-1	∞	$-\infty$
y	-4	-1	0	-1	-4	0	$-\infty$	$-\infty$

Leading Coefficient < 0



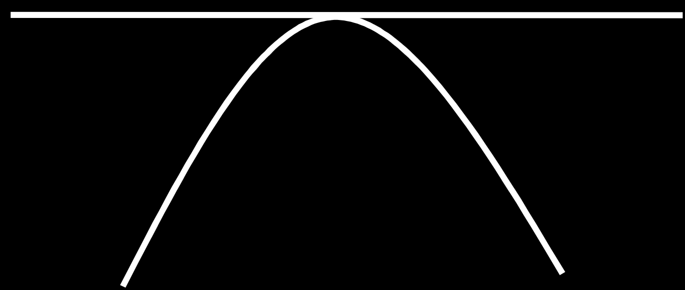
Q. ***In General***

$$y = ax^2 + bx + c$$

$a < 0$ mouth facing downward

$D = 0$ (one real root) parabola touch the x axis

$$y \leq 0 \quad \forall x \in R$$



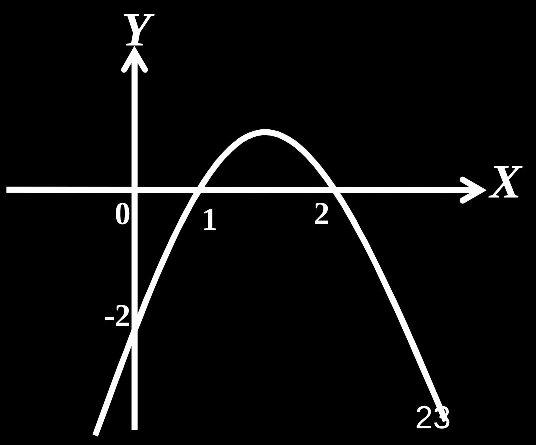
Example

Q. $y = -x^2 + 3x - 2 = -(x-1)(x-2)$

D > 0

x	0	1	2	3	4	-1	-2	∞	$-\infty$
y	-2	0	0	-2	-9	-9	-12	$-\infty$	$-\infty$

Leading Coefficient < 0

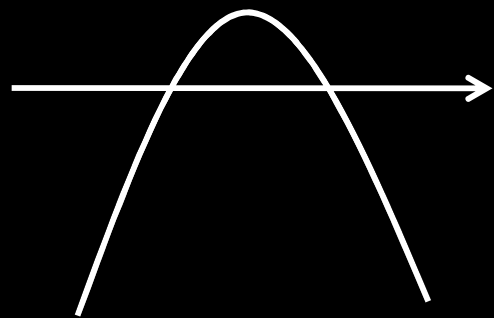


Q. ***In General***

$y = ax^2 + bx + c$

$a < 0$ ***Parabola mouth facing downward***

$D > 0$ ***Two distinct real root (Parabola cut the x-axis at two distinct points.***

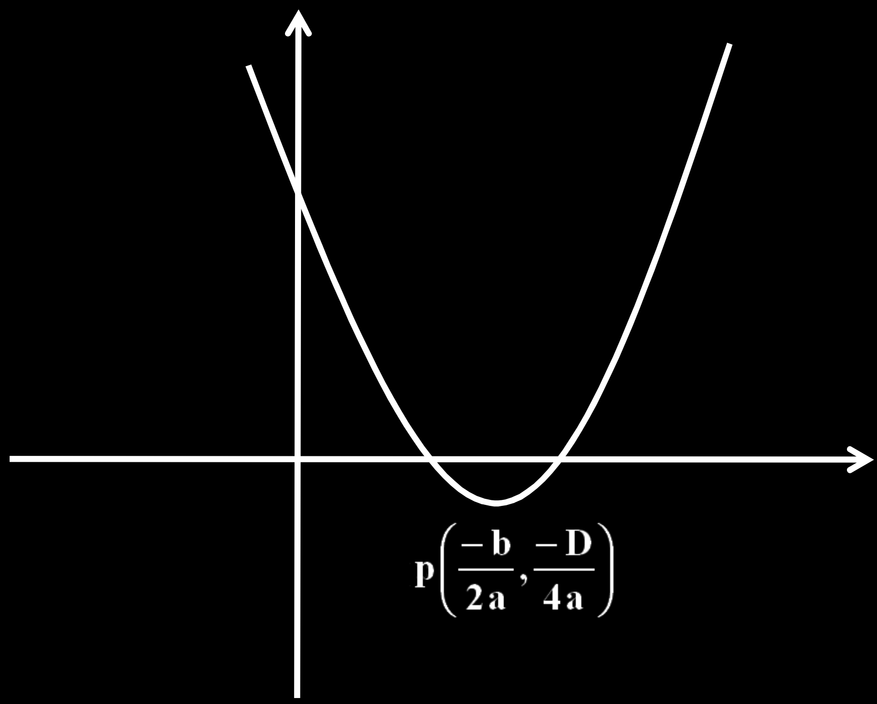


Co-ordinate of vertex

- $y = ax^2 + bx + c$

$$x = \frac{-b}{2a}$$

$$y = \frac{-D}{4a}$$



Nature of Roots

- *$D > 0 \Leftrightarrow$ roots are real & distinct (unequal)*

Nature of Roots

- $D > 0 \Leftrightarrow$ *roots are real & distinct (unequal)*

- $D = 0 \Leftrightarrow$ *roots are real & coincident (equal)*

Nature of Roots

- ***D > 0 ⇔ roots are real & distinct (unequal)***
- ***D = 0 ⇔ roots are real & coincident (equal)***
- ***D < 0 ⇔ roots are imaginary.***

Nature of Roots

Consider the quadratic equation $ax^2 + bx + c = 0$

where $a, b, c \in Q$ & $a \neq 0$ *then;*

If D is a perfect square, then roots are rational.

Note

If $\alpha = p + \sqrt{q}$ is one root in this case, (where p is rational & $\sqrt{q}$ is a surd) then other root will be $p - \sqrt{q}$

Note

If $p + iq$ is one root of a quadratic equation, then the other root must be the conjugate $p - iq$ & vice versa. ($p, q \in R$ & $i = \sqrt{-1}$).

Example

Q. Let $a > 0$, $b > 0$ and $c > 0$. Then, both the roots of the equation $ax^2 + bx + c = 0$

(a) are real and negative

(b) have negative real parts

(c) have positive real parts

(d) None of the above *[IIT-JEE 1979]*

Example

Q. Both the roots of the equation

$$(x - b)(x - c) + (x - a)(x - c) + (x - a)(x - b) = 0$$

are always

(a) positive

(b) negative

(c) real

(d) None of these

[IIT-JEE 1980]

Example

Q. The number of real solutions of the equation

$|x|^2 - 3|x| + 2 = 0$ ***is***

(a) 4 ***(b) 1***

(c) 3 ***(d) 2***

[IIT-JEE 1982]

Example

Q. Let $f(x)$ be a quadratic expression which is positive for all real values of x.

If $g(x) = f(x) + f'(x) + f''(x)$, then for any real x

(a) $g(x) < 0$ *(b) $g(x) > 0$*

(c) $g(x) = 0$ *(d) $g(x) \geq 0$*

[IIT-JEE 1990]

Example

Q. Let α, β be the roots of the equation

$(x - a)(x - b) = c, c \neq 0$

Then the roots of the equation

$(x - \alpha)(x - \beta) + c = 0$ *are*

(a) a, c (b) b, c (c) a, b (d) $a + c, b + c$

[IIT-JEE 1992]

Example

True / False

Q. If $a < b < c < d$, then the roots of the equation

$$(x - a)(x - c) + 2(x - b)(x - d) = 0$$

are real and distinct. [IIT-JEE 1984]

Example

Q. The number of points of intersection of two curves $y = 2 \sin x$ and $y = 5x^2 + 2x + 3$ is

(a) 0 (b) 1 (c) 2 (d) ∞

[IIT-JEE 1994]

Example

Q. For all x, $x^2 + 2ax + 10 - 3a > 0$,

then the interval in which a lies is

(a) $a < -5$

(b) $-5 < a < 2$

(c) $a > 5$

(d) $2 < a < 5$

[IIT – JEE 2004]

Example

Q. If b > a, then the equation $(x - a)(x - b) - 1 = 0$ has

(a) Both roots in (a, b)

(b) both roots in $(-\infty, a)$

(c) both roots in $(b, +\infty)$

(d) one root in $(-\infty, a)$ and the other in $(b, +\infty)$

[IIT-JEE 2000]

Assignment 1

Q.1 If the equation

$$\sin^4 x - (k + 2)\sin^2 x - (k + 3) = 0$$

has a solution then k must lie in the interval

(A) $(-4, -2)$ ***(B)*** $[-3, 2)$

(C) $(-4, -3)$ ***(D)*** $[-3, -2]$

Q.2 If $a, b \in R$, $a \neq 0$ and the quadratic equation $ax^2 - bx + 1 = 0$ has imaginary roots then $a + b + 1$ is :

(A) positive ***(B) negative***

(C) zero ***(D) depends on the sign of b.***

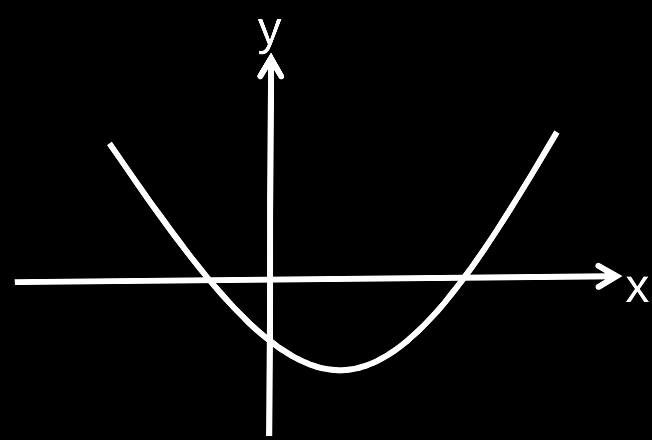
[Multiple Objective Type]

Q.3 The graph of the quadratic polynomial;

$y = ax^2 + bx + c$ is as shown in the figure . Then

(A) $b^2 - 4ac > 0$ (B) $b < 0$

(C) $a > 0$ (D) $c < 0$



Q.4 If a, b, c $\in$ R such that a + b + c = 0 and a $\neq$ c, then prove that the roots of $(b + c - a)\,x^2 + (c + a - b)\,x + (a + b - c)$ are real and distinct.

Q.5 Find the value of a for which the roots of the equation $(2a - 5)x^2 - 2(a - 1)x + 3 = 0$ are equal.

Q.6 For what values of m does the equation

$x^2 - x + m = 0$ possess no real roots ?

Q.7 For what values of m does the equation

$x^2 - x + m^2 = 0$ possess no real roots ?

Relation between root and Coefficient of Quadratic Equation

$$ax^2 + bx + c = 0 \qquad ; a \neq 0 \qquad a,b,c \in R$$

Relation between root and Coefficient of Quadratic Equation

$$ax^2 + bx + c = 0 \qquad ; a \neq 0 \qquad a,b,c \in R$$

- $ax^2 + bx + c = a\,(x - \alpha)\,(x - \beta) = 0$

Relation between root and Coefficient of Quadratic Equation

$$ax^2 + bx + c = 0 \qquad ; a \neq 0 \qquad a,b,c \in R$$

- $ax^2 + bx + c = a\,(x - \alpha)\,(x - \beta) = 0$
- $\alpha + \beta = -\dfrac{b}{a}$ & $\alpha\beta = \dfrac{c}{a}$

Relation between root and Coefficient of Quadratic Equation

$$ax^2 + bx + c = 0 \qquad ; a \neq 0 \qquad a,b,c \in R$$

- $ax^2 + bx + c = a(x - \alpha)(x - \beta) = 0$
- $\alpha + \beta = -\dfrac{b}{a}$ & $\alpha\beta = \dfrac{c}{a}$
- $x^2 + \left(\dfrac{b}{a}\right)x + \left(\dfrac{c}{a}\right) = (x - \alpha)(x - \beta)$

Formation of Quadratic Equation

$$x^2 - (\text{sum of roots})\, x + \text{product of roots} = 0$$

Example

Q. Form a Quadratic Equation with rational coefficients whose one root is $\tan 75°$

Example

Q. Form a Quadratic Equation with rational coefficients whose one root is $\cos 36°$

Example

Q. Form a Quadratic Equation with rational coefficients whose one root is tanπ/8

Inequalities

Rules :

- ***Adding positive number both-sides inequality remains same.***

Example :

$2 > 1 \Rightarrow 3 > 2$

Inequalities

Rules :

- ***Subtracting both sides by positive number inequality remains same***

Example :

$2 > 1 \Rightarrow 1 > 0$

Inequalities

Rules :

- *Multiply & divide by positive number without affecting inequality*

Example :

$4 > 2 \Rightarrow 1 > ½$

Inequalities

Rules :

- ***Multiply & divide by negative number to change sign of inequality***

Example :

$2 > 1 \Rightarrow -2 < -1$

Type – 1
Expression which can not be factorized

Example :

- $x^2 + x + 1 > 0$

Type – 1
Expression which can not be factorized

Example :

- $x^2 - 3x + 4 < 0$

Type – 1
Expression which can not be factorized

Example :

- $3x^2 - 7x + 6 > 0$

Type – 1
Expression which can not be factorized

Example :

- $-x^2 - 2x - 4 > 0$

Type – 2

Expression which can be factorized

Rules :

- ***Factorize in linear as far as possible***

Type – 2

Expression which can be factorized

Rules :

- ***Factorize in linear as far as possible***
- ***Make coefficient of x, as 1 in all linear by multiplying, dividing by appropriate number***

Type – 2

Expression which can be factorized

Rules :

- ***Factorize in linear as far as possible***
- ***Make coefficient of x, as 1 in all linear by multiplying, dividing by appropriate number***
- ***Mark zeros of linear on number line***

Type – 2

Expression which can be factorized

Rules :

- ***Factorize in linear as far as possible***
- ***Make coefficient of x, as 1 in all linear by multiplying, dividing by appropriate number***
- ***Mark zeros of linear on number line***
- ***Give sign to respective area on number line***

Type – 2
Expression which can be factorized

- $(1 - x)(4 + 2x)(x - 2)(x - 7) > 0$

Type – 2

Expression which can be factorized

- $(x^2 - x - 6)(x^2 + 6x) > 0$

Type – 2
Expression which can be factorized

- $(x + 1)(x - 3)(x - 2)(3x + 7) < 0$

Type – 3

- $(x^2 - 5x + 6)\,(x^2 - 6x + 5) \leq 0$

Type – 3

- $2 - x - x^2 \geq 0$

Type – 3

- $3x^2 - 7x + 4 \geq 0$

Type – 4
Repeated Linear Factor

Rules :

- ***Get rid of even power***

Type – 4
Repeated Linear Factor

Rules :

- ***Get rid of even power***
- ***odd power treat as linear***

Type – 4

Repeated Linear Factor

- $(x + 1)\ (x - 3)\ (x - 2)^2 > 0$

Type – 4
Repeated Linear Factor

- $x(x+6)(x+2)^2(x-3) > 0$

Type – 4
Repeated Linear Factor

- $(x - 1)^2 (x + 1)^3 (x - 4) < 0$

Type – 5
Rational Inequality

- $$\frac{(x-1)(x-2)(x-3)}{(x-4)(x-5)}<0$$

Type – 5
Rational Inequality

- $\dfrac{2x-3}{3x-7} < 0$

Type – 5
Rational Inequality

- $\dfrac{2x-3}{3x-7} \geq 0$

Type – 5
Rational Inequality

- $$\frac{x^3(2x-3)^2(x-4)^6}{(x-3)^3(3x-8)^4} \le 0$$

Type – 5
Rational Inequality

- $$\frac{x^2 - 5x + 12}{x^2 - 4x + 5} > 3$$

Type – 5
Rational Inequality

- $$\frac{x^2 - 5x + 6}{x^2 + x + 1} < 0$$

Type – 5
Rational Inequality

- $$\frac{(x-1)^2(x+1)^3}{x^4(x-2)} < 0$$

Example

Q. $$\frac{x+1}{x-1} \geq \frac{x+5}{x+1}$$

Example

Q.

$$\frac{2(x-4)}{(x-1)(x-7)} \geq \frac{1}{x-2}$$

Example

Q. $$\frac{x^2+6x-7}{|x+4|}<0$$

Example

Q. *Let* $y = \sqrt{\dfrac{(x+1)(x-3)}{(x-2)}}$

Find all the real values of x for which y takes real values. **[IIT-JEE 1980]**

Example

Q. ***Find the set of all x for which***

$$\frac{2x}{2x^2+5x+2} > \frac{1}{x+1}$$

[IIT-JEE 1987]

Example

Q. *Solve* $|x^2 + 4x + 3| + 2x + 5 = 0$ *[IIT-JEE 1988]*

Example

Q. ***Let a and b be the roots of the equation $x^2 - 10cx - 11d = 0$ and those of $x^2 - 10ax - 11b = 0$ are c, d. Then find the value of $a + b + c + d$, when $a \neq b \neq c \neq d$.***

[IIT-JEE 2006]

Example

Q. ***Let α, β be the roots of the equation $x^2 - px + r = 0$ and $\alpha/2, 2\beta$ be the roots of the equation $x^2 - qx + r = 0$. Then the value of r is***

(a) 2/9 (p – q) (2q – p) ***(b) 2/9 (q – p) (2p – q)***

(c) 2/9 (q – 2p) (2q – p) ***(d) 2/9 (2p – q) (2q – p)***

[IIT-JEE 2007]

Example

Fill in the blank :

Q. *If $2 + i\sqrt{3}$ is a root of the equation*

$x^2 + px + q = 0,$

where p and q are real, then (p, q) = (……).

[IIT–JEE 1982]

Example

Fill in the blank :

Q. ***If the products of the roots of the equation***

$x^2 - 3kx + 2e^{2 \log k} - 1 = 0$ ***is 7,***

then the roots are real for k = ….. .

[IIT-JEE 1984]

Example

Q. If x, y and z are real and different and

$u = x^2 + 4y^2 + 9z^2 - 6yz - 3zx - 2xy$*, then u*

is always

(a) non-negative *(b) zero*

(c) non-positive *(d) none of these*

[IIT-JEE 1979]

Example

Q. If one root is square of the other root of the equation $x^2 + px + q = 0$, then the relation between p and q is

(a) $p^3 - (3p - 1)\,q + q^2 = 0$

(b) $p^3 - q(3p + 1) + q^2 = 0$

(c) $p^3 + q(3p - 1) + q^2 = 0$

(d) $p^3 + q(3p + 1) + q^2 = 0$ [IIT-JEE 2004]

Assignment 2

Q.1 ***The sum of all the value of m for which the roots x_1 and x_2 of the quadratic equation $x^2 - 2mx + m = 0$ satisfy the condition***

$$x_1^3 + x_2^3 = x_1^2 + x_2^2$$

, is

(A) $\frac{3}{4}$ ***(B) 1***

(C) $\frac{9}{4}$ ***(D)*** $\frac{5}{4}$

Q.2 If α and β are the roots of the equation $ax^2 + bx + c = 0$ then the sum of the roots of the equation

$$a^2x^2 + (b^2 - 2ac)x + b^2 - 4ac = 0$$

in terms of α and β is given by

(A) $-(\alpha^2 - \beta^2)$ (B) $(\alpha + \beta)^2 - 2\alpha\beta$

(C) $\alpha^2\beta + \beta^2\alpha - 4\alpha\beta$ (D) $-(\alpha^2 + \beta^2)$

Q.3 *The set of values of 'a' for which the inequality,* $(x - 3a)(x - a - 3) < 0$ *is satisfied for all* $x \in [1, 3]$ *is :*

(A) $(1/3, 3)$ (B) $(0, 1/3)$

(C) $(-2, 0)$ (D) $(-2, 3)$

Q.4 *If α and β are the roots of $a(x^2 - 1) + 2bx = 0$ then, which one of the following are the roots of the same equation?*

(A) $\alpha + \beta, \alpha - \beta$

(B) $2\alpha + \frac{1}{\beta}, 2\beta + \frac{1}{\alpha}$

(C) $\alpha + \frac{1}{\beta}, \beta - \frac{1}{\alpha}$

(D) $\alpha + \frac{1}{2\beta}, \beta - \frac{1}{2\alpha}$

Q.5 Solve the following Inequality

- $\dfrac{1}{x} < 1$

Q.5 Solve the following Inequality

- $$\frac{x}{x+2} \le \frac{1}{x}$$

Q.5 Solve the following Inequality

- $(x-1)(3-x)(x-2)^2 > 0$

Q.5 Solve the following Inequality

- $$\frac{(x-1)(x+2)^2}{-1-x} < 0$$

Double Inequality

Example

Q. Solve the following Inequality

(i) $$1 < \frac{3x^2 - 7x + 8}{x^2 + 1} \le 2$$

Q. ***Solve the following Inequality***

(ii) $\left(x^2+3x+1\right)\left(x^2+3x-3\right)\geq 5$

Q. ***Solve the following Inequality***

(iii) $10^x(x-1)(x-2) \geq 0$

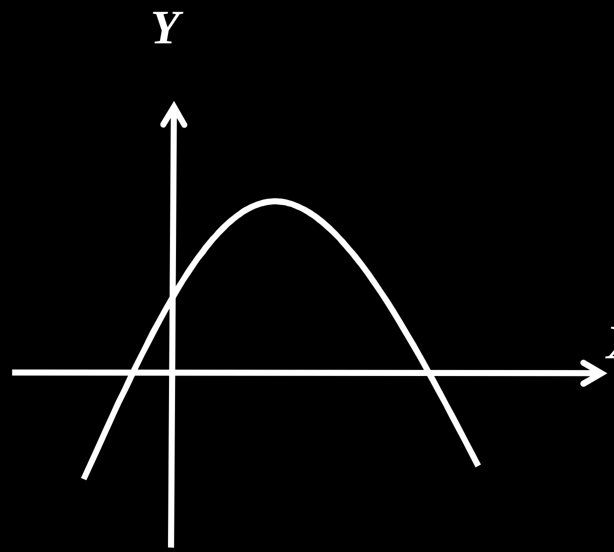
Q. ***Solve the following Inequality***

(iv) $\left(2^{x}-8\right)\left(x-7\right)\left(x+1\right)\leq 0$

True / False :

$$y = ax^2 + bx + c$$

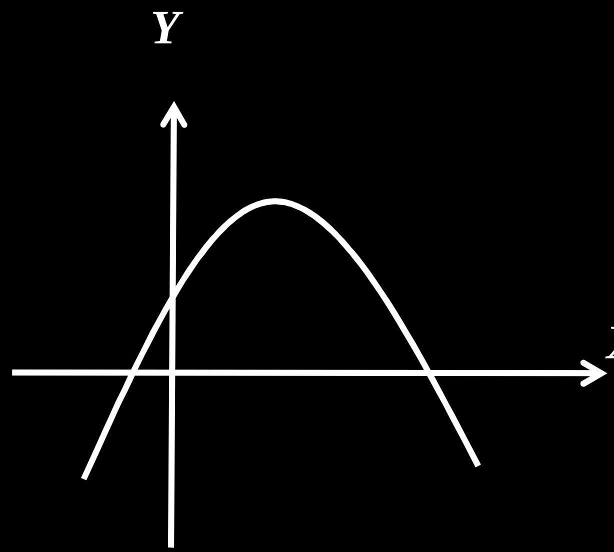
Q. $a > 0$



True / False :

$$y = ax^2 + bx + c$$

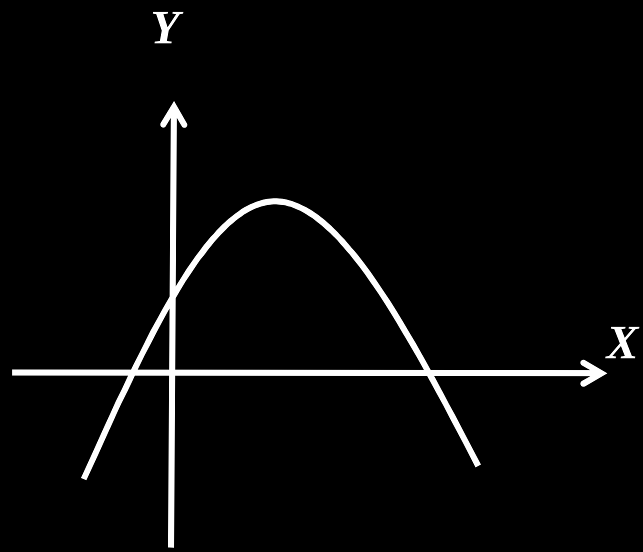
Q. $c > 0$



True / False :

$$y = ax^2 + bx + c$$

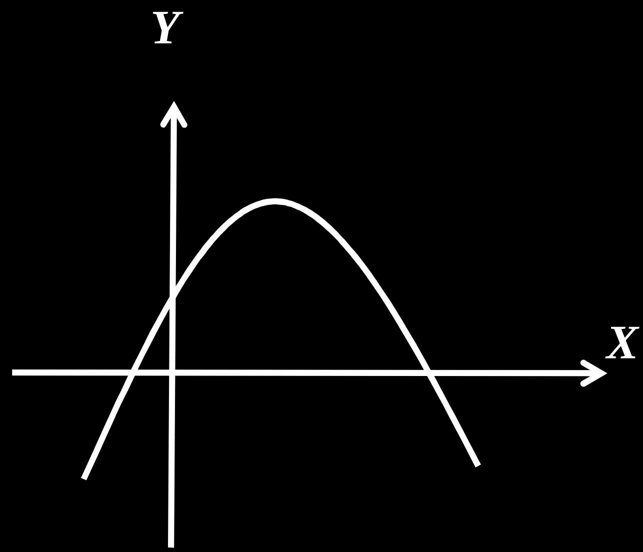
Q. D > 0



True / False :

$$y = ax^2 + bx + c$$

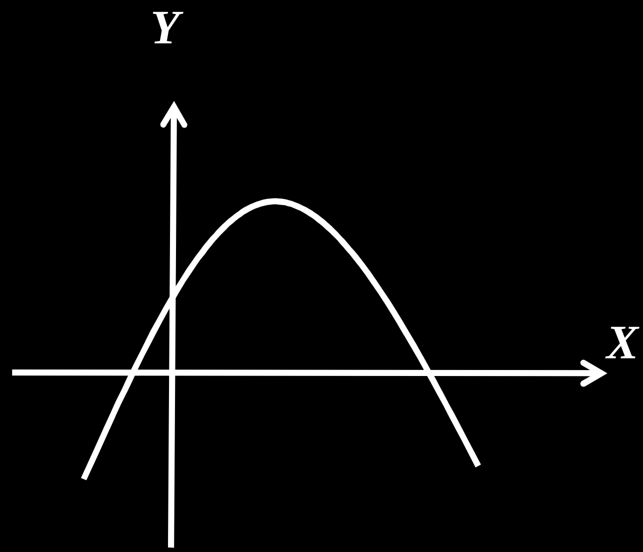
Q. $-b/a > 0$



True / False :

$$y = ax^2 + bx + c$$

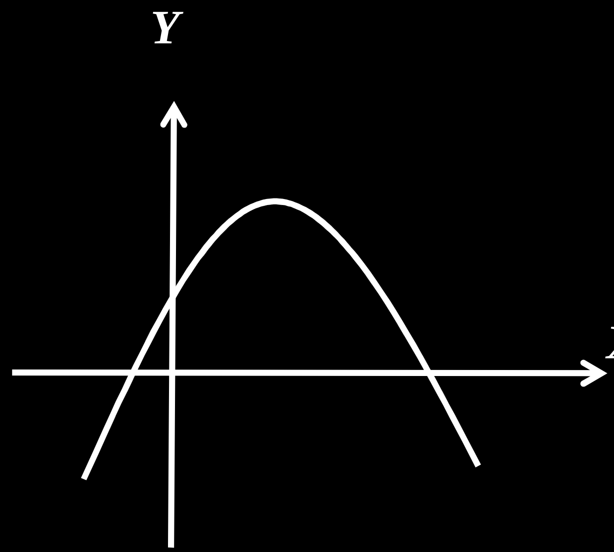
Q. c/a > 0



True / False :

$$y = ax^2 + bx + c$$

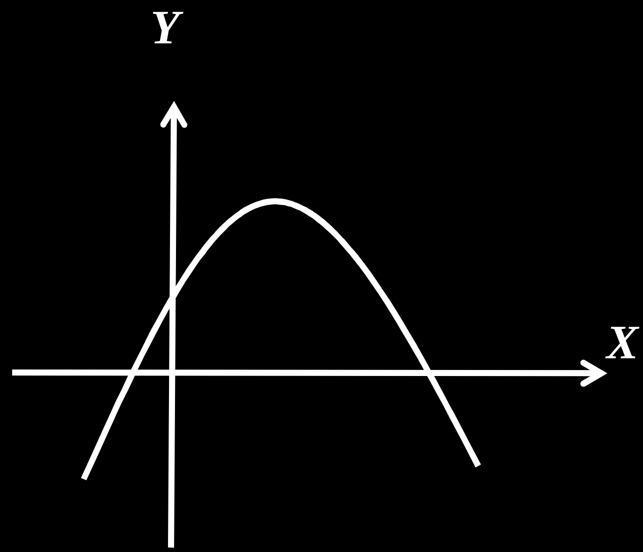
Q. $b > 0$



True / False :

$$y = ax^2 + bx + c$$

Q. $-D/4a > 0$



Q. Quadratic Equation $ax^2 + bx + c = 0$ has no real roots then show that $c(a + b + c) > 0$

Q. Find a,

$$(a-1)\,x^2-(a+1)\,x+a+1>0 \qquad \forall x \in R$$

Q. ***Find a, if*** $(a+4)x^2 - 2ax + 2a - 6 < 0 \quad \forall x \in R$

Q. *If α is root of $x^2 - 2x + 5 = 0$*

Find the value of $\alpha^3 + \alpha^2 - \alpha + 21$

Q. ***If β is root of $x^2 - 2x + 5 = 0$***

Find the value of $\beta^3 + 4\beta^2 - 7\beta + 37$

Q. ***If*** $x = 3 + \sqrt{5}$

Find the value of $x^4 + 12x^3 + 44x^2 - 48x + 17$

Q. ***If p (q – r) x^2 + q (r – p) x + r (p – q) = 0 has equal root.***

Show that : $\frac{2}{q} = \frac{1}{p} + \frac{1}{r}$

Q. ***If*** $x^2 + \dfrac{1}{x^2} = 14$; $x > 0$ ***then*** **(MCQ)**

(a) $x^3 + x^{-3} = 62$ (b) $x^3 + x^{-3} = 52$

(c) $x^5 + x^{-5} = 624$ (d) $x^5 + x^{-5} = 724$

Q. ***Find the integral solutions of the following system of inequalities***

(a) $5x - 1 < (x + 1)^2 < 7x - 3$

(b) $\dfrac{x}{2x+1} \geq \dfrac{1}{4}, \dfrac{6x}{4x-1} < \dfrac{1}{2}$

[IIT-JEE 1978]

Q. ***If l, m, n are real $l \neq m$, then the roots of the equation $(l - m) x^2 - 5 (l + m) x - 2(l - m) = 0$ are***

(a) real and equal ***(b) complex***

(c) real and unequal ***(d) none of these***

[IIT-JEE 1979]

Q. ***For what value of m, does the system of equations $3x + my = m$, $2x - 5y = 20$ has solution satisfying the conditions $x > 0$, $y > 0$***

[IIT-JEE 1980]

Q. ***Find all real values of which satisfy***

$x^2 - 3x + 2 > 0$ and $x^2 - 3x - 4 \leq 0$.

[IIT-JEE 1983]

Q. ***Let a, b, c be real numbers with $a \neq 0$ and let α, β be the roots of the equations $ax^2 + bx + c = 0$.***

Express the roots of $a^3x^2 + abc\, x + c^3 = 0$ in terms of α, β. ***[IIT-JEE 2001]***

Q. *If α and β are the roots of $x^2 + px + q = 0$ and γ, δ are the roots of $x^2 + rx + s = 0$, then evaluate $(\alpha - \gamma)(\beta - \gamma)(\alpha - \delta)(\beta - \delta)$ in terms of p, q, r and s.*

[IIT-JEE 1979]

Assignment 3

Solve the following inequalities

Q.1 $x^4 - 2x^2 - 63 \leq 0$

Solve the following inequalities

Q.2 $\dfrac{7x-5}{8x+3} > 4$

Solve the following inequalities

Q.3 $$\frac{x+7}{x-5}+\frac{3x+1}{2}\geq 0$$

Solve the following inequalities

Q.4 $$\frac{\left(2-x^{2}\right)\left(x-3\right)^{3}}{\left(x+1\right)\left(x^{2}-3x-4\right)}\geq 0$$

Solve the following inequalities

Q.5 $$\frac{20}{(x-3)(x-4)}+\frac{10}{x-4}+1>0$$

Solve the following inequalities

Q.6 $$\frac{x-1}{x^2-x-12} \le 0$$

Solve the following inequalities

Q.7 ***For what values of c does the equation***

$$(c - 2)\, x^2 + 2\,(c - 2)\, x + 2 = 0$$

possess no real roots ?

Solve the following inequalities

Q.8 For what values of a does the equation

$$x^2 + 2a\sqrt{a^2 - 3}\,x + 4 = 0$$

possess equal roots ?

Solve the following inequalities

Q.9 Find the value of k for which the curve

$y = x^2 + kx + 4$ touches the Ox axis.

Solve the following inequalities

Q.10 Find the least integral value of k for which the equation $x^2 - 2(k + 2)x + 12 + k^2 = 0$ has two different real roots.

Solve the following inequalities

Q.11 If the equation $4x^2 - 4(5x + 1) + p^2 = 0$ has one root equals to two more than the other, then the value of p is equal to

(a) $\pm \dfrac{\sqrt{236}}{3}$ ***(b)*** ± 5

(c) 5 or -1 ***(d) 4 or -3***

Solve the following inequalities

Q.12 Possible values of x simultaneously satisfying the system of inequalities

$$\left.\begin{aligned} &\frac{(x-6)(x-3)}{x+2} \geq 0 \\ &\text{and} \quad \frac{x-5}{x+1} \leq 3 \end{aligned}\right\} \text{is}$$

(A) (-1, 3] $\cup$ [6, ∞) *(B) (-2, 3] $\cup$ [6, ∞)*

(C) (-2, -1) $\cup$ (4, ∞) *(D) [3, 6]*

Identity

$$ax^2 + bx + c = 0$$

Number of roots are infinite

When $a = b = c = 0$

Note

3 distinct real root of quadratic $\Rightarrow$ infinite root

Examples

Q. ***Find the value of p for which the equation***

$$(p + 2)(p - 1)x^2 + (p - 1)(2p + 1)x + p^2 - 1 = 0$$

has infinite roots

Q. $$\frac{(x-a)(x-b)}{(c-a)(c-b)}+\frac{(x-b)(x-c)}{(a-b)(a-c)}+\frac{(x-c)(x-a)}{(b-c)(b-a)}=1$$

Prove that above is an identity

Quadratic With One Root Zero

$$ax^2 + bx + c = 0$$

$$Product\ of\ root = \frac{c}{a} = 0$$

$$c = 0$$

Quadratic With Both Root Zero

$$ax^2 + bx + c = 0$$

Sum of root = Product of root = 0

$$b = 0,\ c = 0$$

Quadratic With One Root Infinite

$$ax^2 + bx + c = 0$$

$$a = 0$$

Quadratic With Both Root ∞

$$y = ax^2 + bx + c$$

$$a = 0,\ b = 0,\ c \neq 0$$

Example

Q. ***If*** $(2p - q)\,x^2 + (p - 1)\,x + 5 = 0$ ***has both roots infinite. Find*** p ***&*** q

Symmetric Function

If $f(\alpha, \beta) = f(\beta, \alpha) \quad \forall \alpha, \beta$

Then $f(\alpha, \beta)$ *is called symmetric function of* α, β

Example

Q. ***Check if $f(\alpha, \beta)$ is symmetric or not***

(i) $f(\alpha, \beta) = \alpha^2\beta + \alpha\beta^2$

Example

Q. ***Check if*** $f(\alpha, \beta)$ ***is symmetric or not***

(i) $f(\alpha, \beta) = \alpha^2\beta + \alpha\beta^2$

(ii) $f(\alpha, \beta) = \cos(\alpha - \beta)$

Example

Q. Check if $f(\alpha, \beta)$ is symmetric or not

(i) $f(\alpha, \beta) = \alpha^2\beta + \alpha\beta^2$

(ii) $f(\alpha, \beta) = \cos(\alpha - \beta)$

(iii) $f(\alpha, \beta) = \sin(\alpha - \beta)$

Example

Q. ***Check if $f(\alpha, \beta)$ is symmetric or not***

(i) $f(\alpha, \beta) = \alpha^2\beta + \alpha\beta^2$

(ii) $f(\alpha, \beta) = \cos(\alpha - \beta)$

(iii) $f(\alpha, \beta) = \sin(\alpha - \beta)$

(iv) $f(\alpha, \beta) = (\alpha^2 - \beta)$

Condition of Common Root

Condition for both Roots Common

$$a_1x^2 + b_1x^2 + c_1 = 0$$

$$a_2x^2 + b_2x^2 + c_2 = 0$$

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

Condition for One Root Common

$$\frac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1} = \frac{b_1c_2 - b_2c_1}{a_2c_1 - a_1c_2}$$

Examples

Q. *Find k for which equations* $x^2 - 3x + 2 = 0$, $3x^2 + 4kx + 2 = 0$ *have a common root*

Q. ***Find p and q if*** $px^2 + 5x + 2 = 0$

$3x^2 + 10x + q = 0$ ***have both roots in common***

Q. ***Find the value of a & b if*** $x^2 - 4x + 5 = 0$, $x^2 + ax + b = 0$ ***have a common root where*** $a, b \in R$

Q. *If* $4x^2\sin^2\theta - (4\sin\theta)x + 1 = 0$ *&*

$a^2(b^2 - c^2)x^2 + b^2(c^2 - a^2)x^2 + c^2(a^2 - b^2) = 0$

have a common root and the second equation has equal roots find possible value of θ *where* $\theta \in (0, \pi)$

Q. *If the quadratic equation $ax^2 + bx + c = 0$ & $x^2 + cx + b = 0$ $b \neq c$ have a common root then prove that there uncommon roots are roots of the equation $x^2 + x + bc = 0$*

Q. $x^2 + ax + 12 = 0$, $x^2 + bx + 15 = 0$ &

$x^2 + (a + b)\,x + 36 = 0$

have a common positive root

Find a, b & common root of equation.

Q. If one root of quadratic equation $x^2 - x + 3a = 0$ is double the root of the equation $x^2 - x + a = 0$ find a

Q. If $Q_1(x) = x^2 + (k - 29)x - k$

$Q_2(x) = 2x^2 + (2k - 43)x + k$

both are factors of a cubic polynomial find k

Q. If $x^2 + abx + c = 0$ & $x^2 + acx + b = 0$ have only one root common then show that quadratic equation containing their other common roots is

$$a(b + c)x^2 + (b + c)x - abc = 0$$

Q. *A value of b for which the equations*

$x^2 + bx - 1 = 0, x^2 + x + b = 0$

have one root in common is

(a) $-\sqrt{2}$ (b) $-i\sqrt{3}$ (c) $i\sqrt{5}$ (d) $\sqrt{2}$

[IIT-JEE 2011]

Fill in the blank :

Q. ***If the quadratic equations $x^2 + ax + b = 0$ and $x^2 + bx + a = 0$ $(a \neq b)$ have a common root, then the numerical value of $a + b$ is …. .***

[IIT-JEE 1986]

Assignment 4

Q.1 Find value of k for which the equation $(x - 1)(x - 2) = 0$ & $2x^2 + kx - 8 = 0$ have a common root

Q.2 ***If x be the real number such that $x^3 + 4x + 8$. then the value of the expression $x^7 + 64x^2$ is***

(A) 124 (B) 125 (C) 128 (D) 132

Q.3 ***If every solution of the equation $3\cos^2 x - \cos x - 1 = 0$ is a solution of the equation $a\cos^2 2x + b\cos 2x - 1 = 0$. Then the value of $(a + b)$ is equal to***

(A) 5 (B) 9 (C) 13 (D) 14

Q.4 ***If $x^2 + 3x + 5 = 0$ & $ax^2 + bx + c = 0$ have common root/roots and $a : b, c \in N$ then find minimum value of $a + b + c$***

Q.5 Determine the values of m for which the equation $3x^2 + 4mx + 2 = 0$ and $2x^2 + 3x - 2$ may have a common root.

Q.6 Q.7 For what value of a is the difference between the roots of the equation $(a - 2)x^2 - (a - 4)x - 2 = 0$ equal to 3 ?

Q.7 Find all values of a for which the sum of the roots of the equation $x^2 - 2a(x - 1) - 1 = 0$ is equal to the sum of the squares of its roots.

Q.8 For what values of a do the equations $x^2 + ax + 1 = 0$ and $x^2 + x + a = 0$ have a root in common ?

Maximum & Minimum Value of Quadratic Equation

- $y = ax^2 + bx + c$ *attain its maximum or minimum at point where* $x = \dfrac{-b}{2a}$ *according as* $a < 0$ *or* $a > 0$.

Maximum & Minimum Value of Quadratic Equation

- *$y = ax^2 + bx + c$ attain its maximum or minimum at point where $x = \dfrac{-b}{2a}$ according as $a < 0$ or $a > 0$.*
- *Maximum and Minimum value can be obtained by making a perfect square.*

Examples

Q. *$p(x) = ax^2 + bx + 8$ is quadratic polynomial.*

Minimum value of $p(x)$ is 6 when $x = 2$

Find a & b

Q. $y = 2x^2 - 3x + 1$, ***find minimum value of y***

Q. $y = 7 + 5x - 2x^2$ ***find maximum value of y***

Q. ***For $x \geq 2$ smallest possible value of***

$$\log_{10}(x^3 - 4x^2 + x + 26) - \log_{10}(x + 2)$$

Range of Linear

$$y = ax + b \quad ;a \neq 0$$

$$y \in R$$

Example

Q. $y = f(x) = x + 1$

Range of $\frac{\text{Linear}}{\text{Linear}}$

$$y = \frac{ax+b}{cx+d}$$

$$y \in R - \left\{\frac{a}{c}\right\}$$

Examples

Q. $\quad y = \dfrac{2x+3}{x+1}$, *Find range of y*

Q. $y = \dfrac{1}{3x-1}$, ***Find range of y***

Q. $$y = \frac{x(x-1)}{x-1}$$, ***Find range of y***

Q. $$y=\frac{(x-1)(x-2)(x-3)}{(x-2)(x-3)}$$, ***Find range of y***

Range of

$$\frac{\text{Linear}}{\text{Quadratic}}, \frac{\text{Quadratic}}{\text{Quadratic}}, \frac{\text{Quadratic}}{\text{Linear}}$$

- *Assume y*

Range of $\dfrac{\text{Linear}}{\text{Quadratic}}, \dfrac{\text{Quadratic}}{\text{Quadratic}}, \dfrac{\text{Quadratic}}{\text{Linear}}$

- *Assume y*
- *Check for common roots in numerator & denominator*

Range of $\dfrac{\text{Linear}}{\text{Quadratic}}, \dfrac{\text{Quadratic}}{\text{Quadratic}}, \dfrac{\text{Quadratic}}{\text{Linear}}$

- *Assume y*
- *Check for common roots in numerator & denominator*
- *Form Quadratic Equation*

Range of $\dfrac{\text{Linear}}{\text{Quadratic}}, \dfrac{\text{Quadratic}}{\text{Quadratic}}, \dfrac{\text{Quadratic}}{\text{Linear}}$

- *Assume y*
- *Check for common roots in numerator & denominator*
- *Form Quadratic Equation*
- *Apply D ≥ 0 (since x is real)*

Range of $\dfrac{\text{Linear}}{\text{Quadratic}}, \dfrac{\text{Quadratic}}{\text{Quadratic}}, \dfrac{\text{Quadratic}}{\text{Linear}}$

- *Assume y*
- *Check for common roots in numerator & denominator*
- *Form Quadratic Equation*
- *Apply D ≥ 0 (since x is real)*
- *Solve inequality in y and hence the range*

Note

Always check for coefficient of x^2 not equal to zero

Examples

Find range of following

Q. $$\frac{x^2 + 2x - 11}{2(x-3)}$$

Find range of following

Q. $$\frac{x^2 - 3x + 4}{x^2 + 3x + 4}$$

Find range of following

Q. $$\frac{(x+1)(x-2)}{x(x+3)}$$

Find range of following

Q. $$\frac{x^2+2x-2}{x^2+2x+1}$$

Find range of following

Q. $$\frac{x^2+14x+9}{x^2+2x+3}$$

Find range of following

Q. $$\frac{x^2 - 5x + 6}{x^2 - 4x + 3}$$

Assignment 5

Q.1 Find the range of the function $f(x) = x^2 - 2x - 4$

Q.2 Find the least value of $\dfrac{6x^2-22x+21}{5x^2-18x+17}=y \quad \forall x \in R$

Q.3 Find Range $\dfrac{x^2 - x + 1}{x^2 + x + 1}$

Q.4 Find the domain and Range of

$$f(x) = \sqrt{x^2 - 3x + 2}$$

General 2° in x & y

$$f(x, y) = ax^2 + 2h\,xy + by^2 + 2gx + 2fy + c$$

Condition of General 2° in *x* & *y* to be Resolved into two linear Factors

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

Rule

Step 1 :

factorize purely 2°

Rule

Step 1 :

factorize purely 2°

Step 2 :

Add constant to both the linear

Rule

Step 1 :

factorize purely 2°

Step 2 :

Add constant to both the linear

Step 3 :

Compare coefficient of x & coefficient of y & absolute term if needed

Examples

Q. ***Prove that the Expression***

$$2x^2 + 3xy + y^2 + 2y + 3x + 1$$

can be factorized into two linear factors & find them

Q. ***Prove that the Expression***

$$x^2 - 3xy + 2y^2 - 2x - 3y - 35 = 0$$

can be factorized into two linear factors & find them

Q. ***If the equation $x^2 + 16y^2 - 3x + 2 = 0$ is satisfied by real values of x & y then show that $x \in [1, 2]$ & $y \in [-1/8, 1/8]$***

Theory of Equation

$$ax^2 + bx + c = a\,(x - \alpha)\,(x - \beta)$$

Theory of Equation

$$ax^2 + bx + c = a(x - \alpha)(x - \beta)$$

$$ax^3 + bx^2 + cx + d = a(x - \alpha)(x - \beta)(x - \gamma)$$

Sum & Product of Root taken 1 at a time

$$\alpha + \beta + \gamma = -b/a$$

$$\alpha\beta\gamma = -d/a$$

Sum of root taken 2 at a time

$$\alpha\beta + \beta\gamma + \gamma\alpha = c/a$$

Bi Quadratic

$$ax^4 + bx^3 + cx^2 + dx + e = a(x - \alpha)\ \ldots\ldots\ (x - \delta)$$

Sum of root taken 2 at a time

$$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = c/a$$

Sum of root taken 3 at a time

$$\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\beta + \beta\gamma\delta = -d/a$$

Note

$$(a + b + c)^2 = \sum a^2 + 2\sum ab$$

Examples

Q. ***Find sum of square & sum of cubes of roots of the cubic equation*** $x^3 - px^2 + qx - r = 0$

Q. Solve the cubic

$$4x^3 + 16x^2 - 9x - 36 = 0$$

Where sum of 2 root is zero

Q. *If a, b, c are roots of cubic* $x^3 - x^2 + 1 = 0$

Find $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$

Q. ***If $\alpha, \beta, \gamma, \delta$ are roots of the equation***

$$\tan\left(\frac{\pi}{4}+x\right) = 3\tan 3x$$

Find the value of $\tan\alpha + \tan\beta + \tan\gamma + \tan\delta$

Q. ***Find a cubic each of its roots is greater by unity then a root of $x^3 - 5x^2 + 6x - 3 = 0$***

Q. Find the cubic whose roots are cubes of the roots of $x^3 + 3x^2 + 2 = 0$

Q. ***The length of side of a Δ are roots of the equation $x^3 - 12x^2 + 47x - 60 = 0$***

Find Δ^2

Location of Roots

Type -1

Both roots of a quadratic equation are greater than a specified number

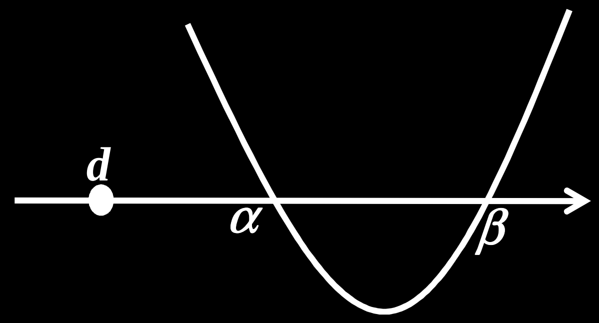
$$(\alpha, \beta) > d$$

Condition

If $y = ax^2 + bx + c$

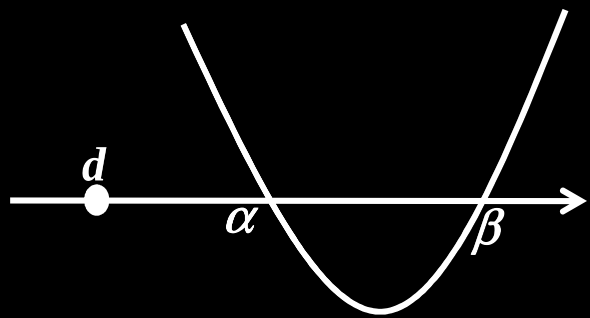
Condition

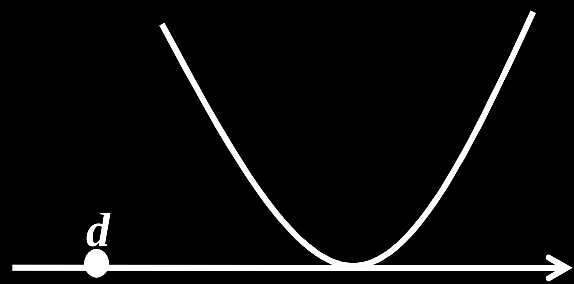
If $y = ax^2 + bx + c$



Condition

$$If\ y = ax^2 + bx + c$$

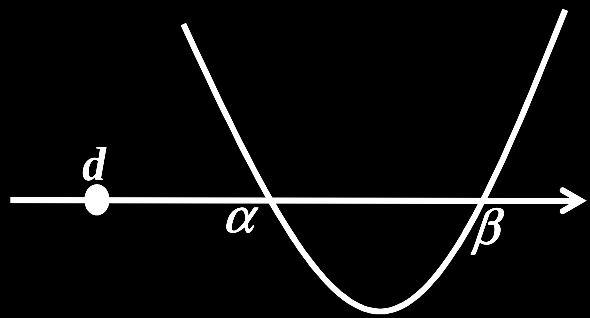


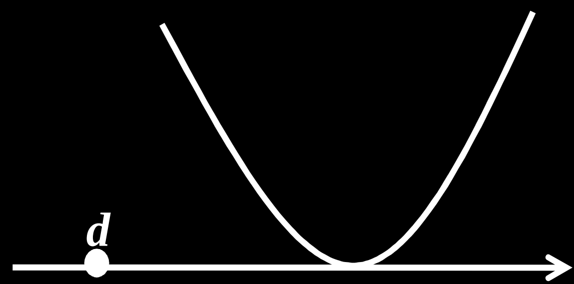


Condition

If $y = ax^2 + bx + c$

(i) $a > 0$



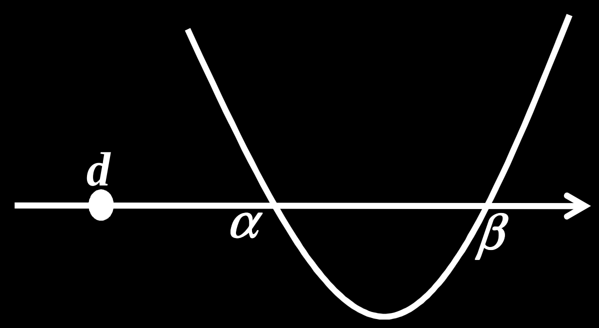


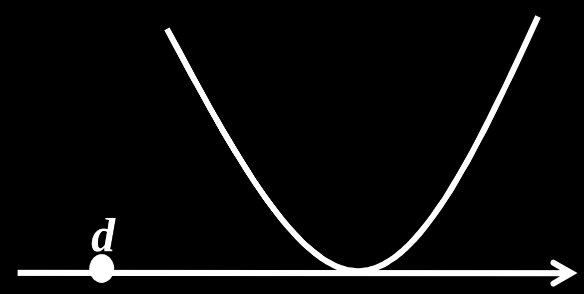
Condition

$If\ y = ax^2 + bx + c$

(i) $a > 0$

(ii) $D \geq 0$





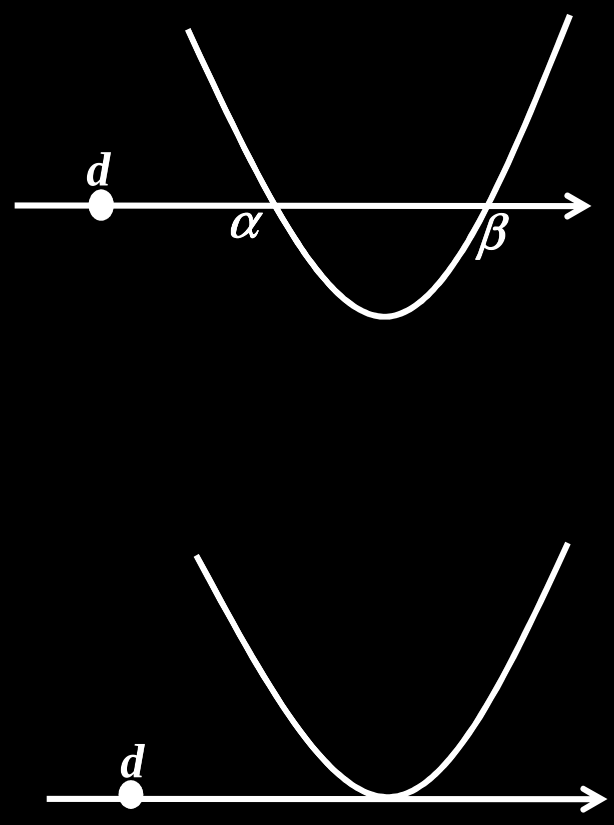
Condition

If $y = ax^2 + bx + c$

(i) $a > 0$

(ii) $D \geq 0$

(iii) $\dfrac{-b}{2a} > d$



Condition

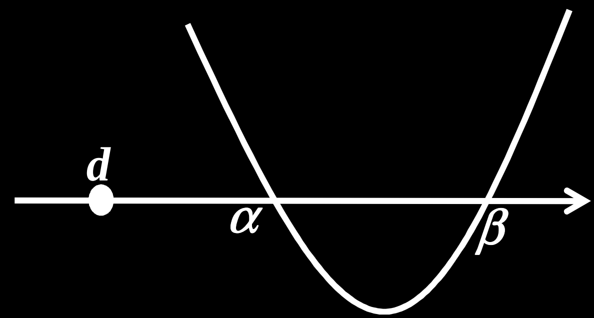
If $y = ax^2 + bx + c$

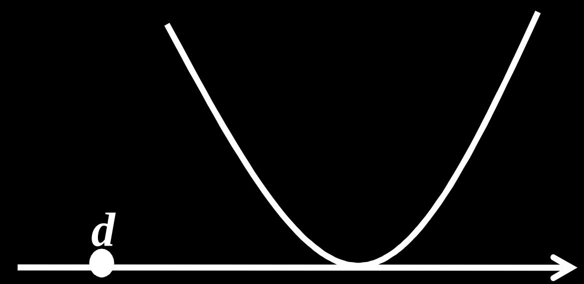
(i) $a > 0$

(ii) $D \geq 0$

(iii) $\dfrac{-b}{2a} > d$

(iv) $f(d) > 0$





Examples

Q. ***Find the value of d for which both roots of the equation*** $x^2 - 6dx + 2 - 2d + 9d^2 = 0$ ***are greater than 3***

Q. Find all the values of 'a' for which both roots of the equation $x^2 + x + a = 0$ exceed the quantity 'a'.

Type - 2

Both roots lies on either side of a fixed number

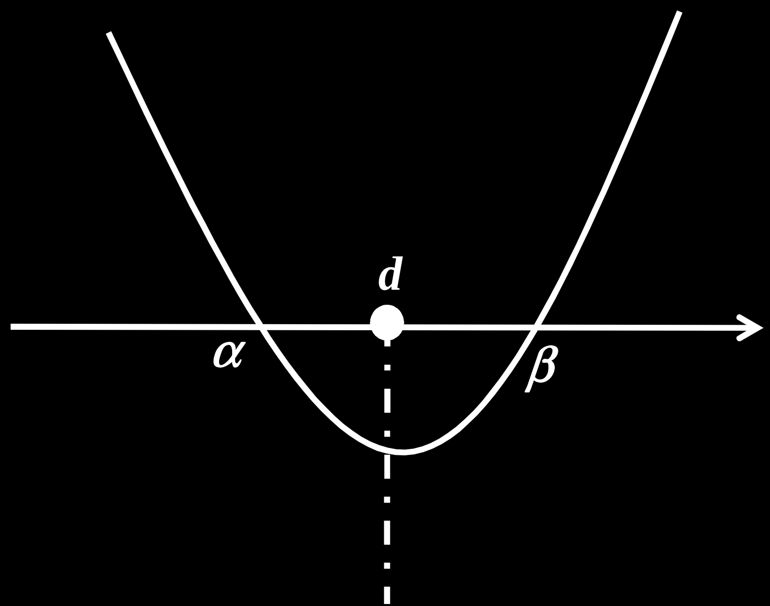
say (d)

$$\alpha < d < \beta$$

Condition

$a > 0$

$f(d) < 0$



Examples

Q. ***Find k for which 1 root of the equation is greater than 2 and other is less than 2***

$$x^2 - (k + 1)x + k^2 + k - 8 = 0$$

Q. ***Find the set of value of 'a' for which zeroes of the quadratic polynomial $(a^2 + a + 1)\, x^2 + (a - 1)x + a^2$ are located on either side of 3.***

Q. ***Find a for which one root is positive, one is negative*** $-x^2 - (3a - 2)x + a^2 + 1 = 0$

Q. ***Find a for which both root lie on either side of -1 of quadratic***

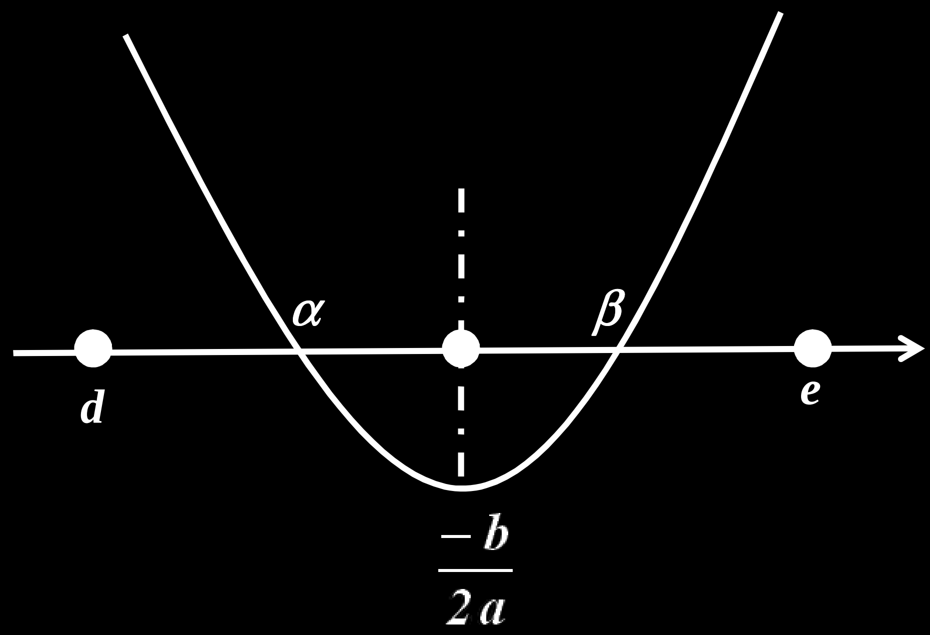
$$(a^2 - 5a + 6)\,x^2 - (a - 3)\,x + 7 = 0$$

Type - 3

Both roots lies between two fixed number

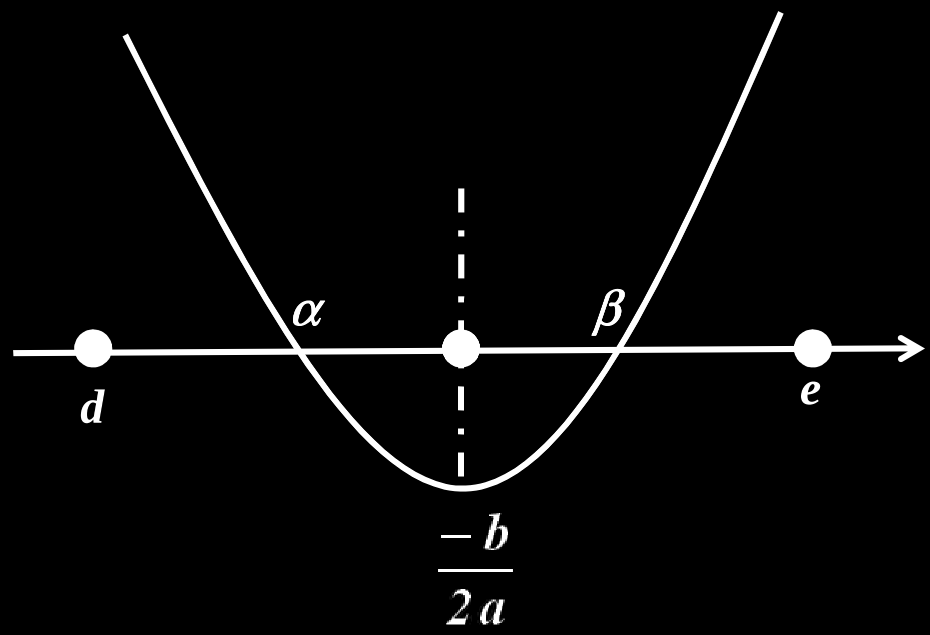
$$d < \alpha < \beta < e$$

Conditions



Conditions

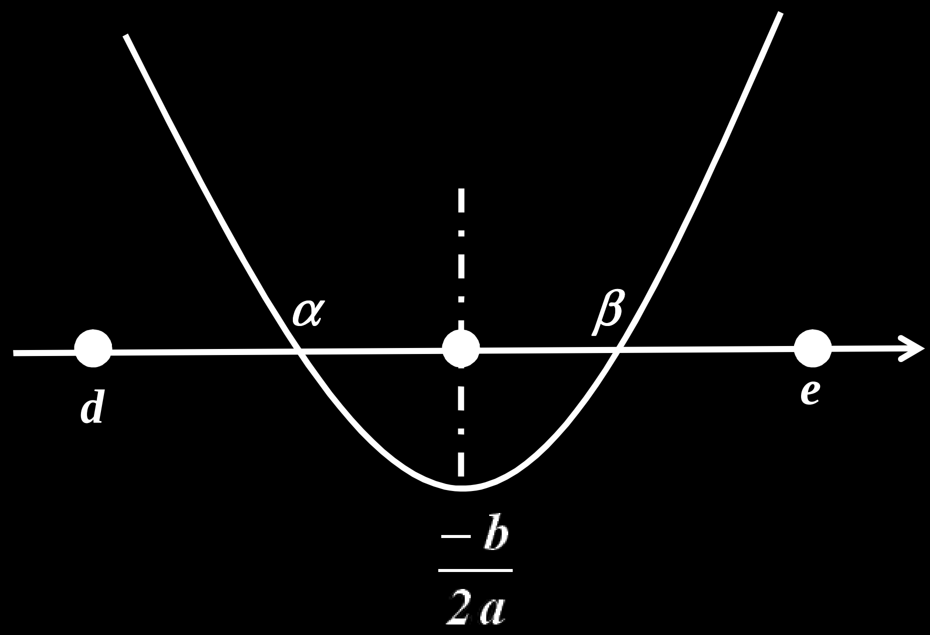
(i) $D \geq 0$



Conditions

(i) $D \geq 0$

(ii) $f(e) > 0$

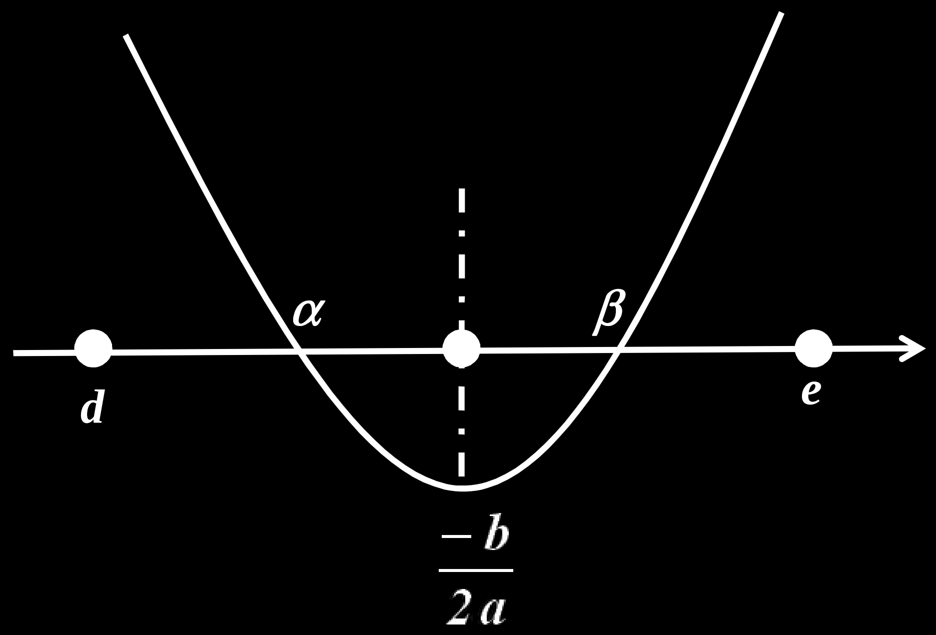


Conditions

(i) $D \geq 0$

(ii) $f(e) > 0$

(iii) $f(d) > 0$



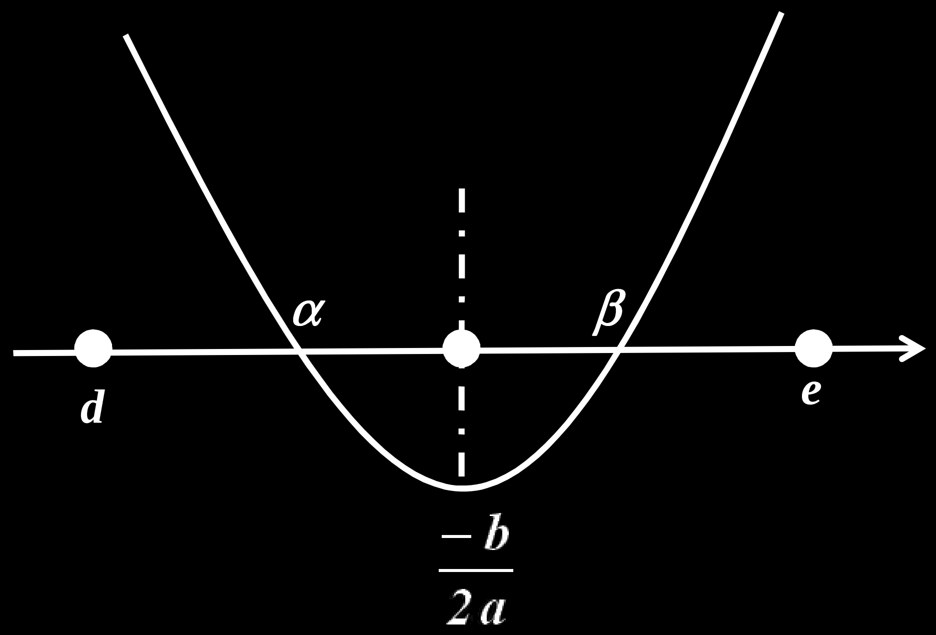
Conditions

(i) $D \geq 0$

(ii) $f(e) > 0$

(iii) $f(d) > 0$

(iv) $d < \frac{-b}{2a} < e$



Example

Q. *If $\alpha, \beta \in (-6, 1)$*

Find k for which

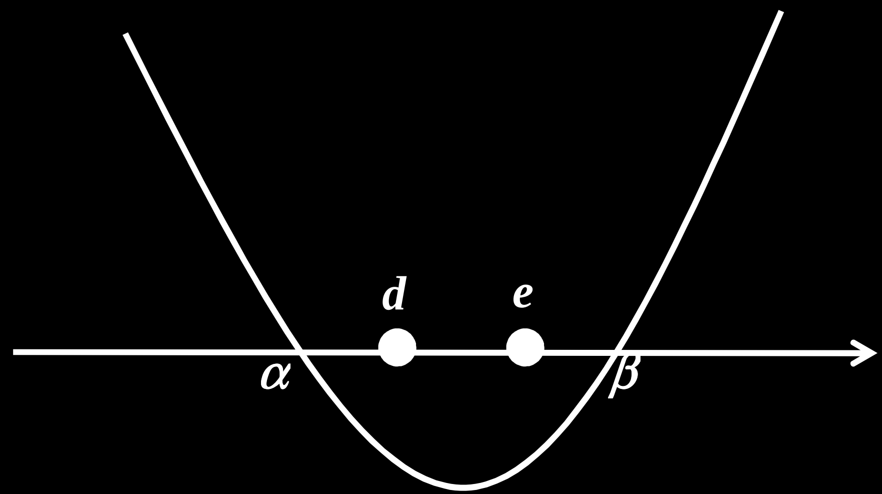
$$x^2 + 2(k - 3)x + 9 = 0$$

Type - 4

Both roots lies on either side of two fixed number

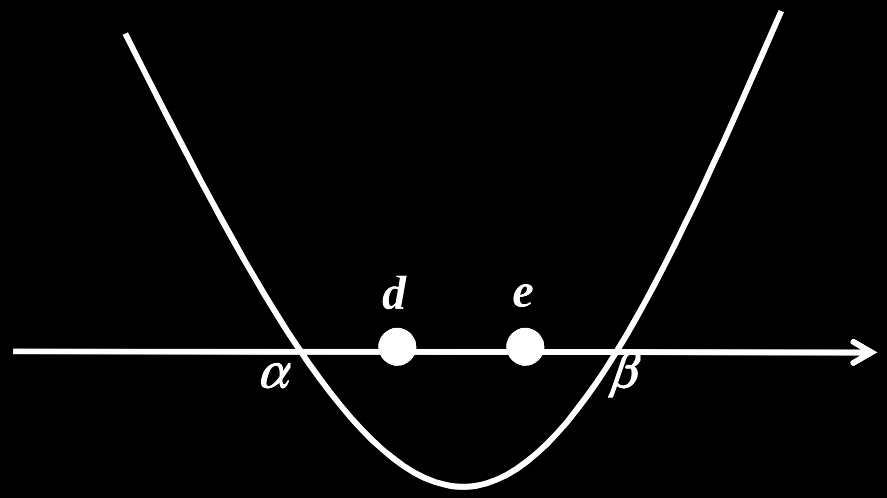
$$\alpha < d < e < \beta$$

Conditions



Conditions

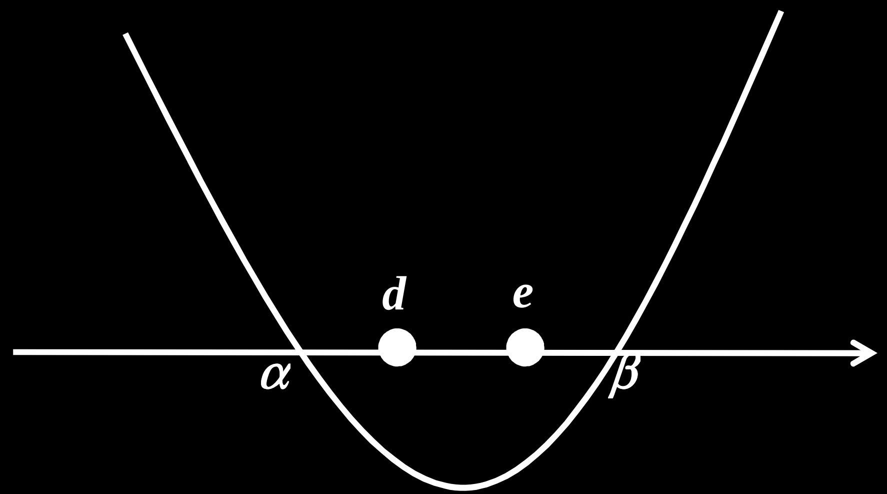
(i) $f(d) < 0$



Conditions

(i) $f(d) < 0$

(ii) $f(e) < 0$



Example

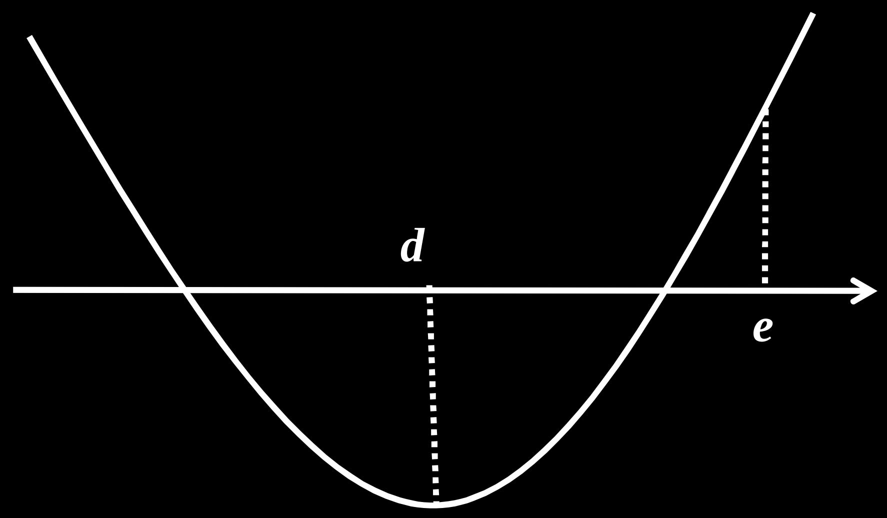
Q. ***Find k for which one root of the equation $(k-5)x^2 - 2kx + k - 4 = 0$ is smaller than 1 and the other root is greater than 2***

Type - 5

Exactly one root lies in the interval (d, e)

Type - 5

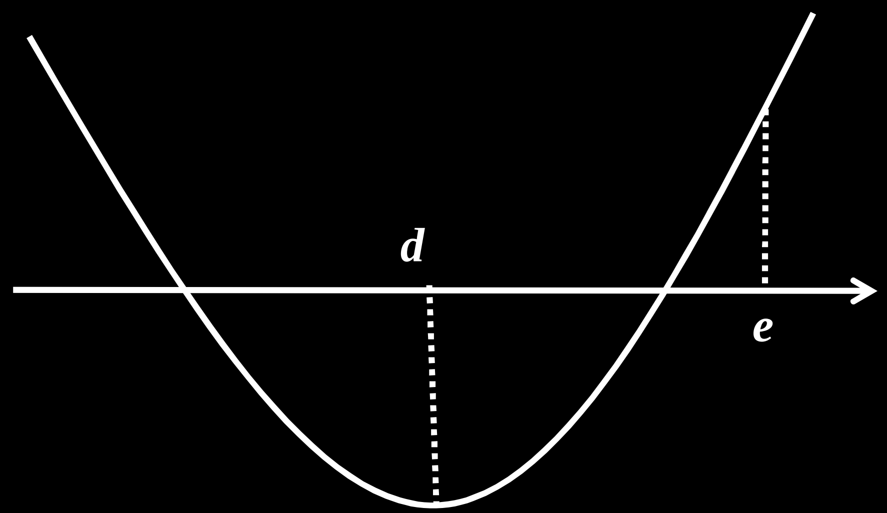
Exactly one root lies in the interval (d, e)



Type - 5

Exactly one root lies in the interval (d, e)

f(d) f(e) < 0



Examples

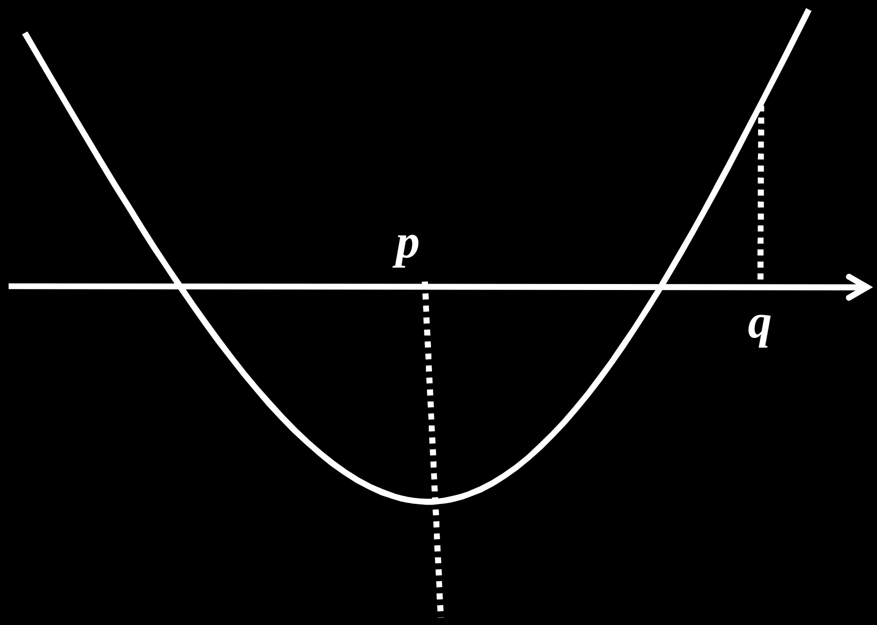
Q. ***Find the set of values of m for which exactly one root of the equation*** $x^2 + mx + (m^2 + 6m) = 0$ ***lie in*** $(-2, 0)$

Q. ***Find a for which exactly one root of the quadratic equation $x^2 - (a + 1)x + 2a = 0$ lies in (0,3)***

Type - 6

If $f(p)\,f(q) < 0$

$\Rightarrow$ *Exactly one root lies between (p, q)*



Examples

Q. If $a < b < c < d$ show that

Quadratic $(x - a)(x - c) + \lambda (x - b)(x - d) = 0$

has real root for all real values of λ

Q. Find p for which the expression $x^2 - 2px + 3p + 4 < 0$ is satisfied for at least one real x

Q. ***Find a for which expression***

$(a^2 + 3)\,x^2 + \sqrt{5a+3}\,x - \frac{1}{4} < 0$ ***is satisfied for at least one real x***

Q. Find m if $x^2 - 4x + 3m + 1 > 0$ is satisfied for all positive x

Q. ***Show that for any real value of a $(a^2 + 3)x^2 + (a + 2)x - 5 < 0$ is true for at least one negative x.***

Q. If $f(x) = 4x^2 + ax + (a - 3)$ is negative for at least one negative x, find all values of a

Q. ***Find a for which $x^2 + 2(a - 1)x + a + 5 = 0$ has at least one positive root.***

Q. ***Find p for which the least value of***

$4x^2 - 4px + b^2 - 2p + 2$ in $x \in [0,2]$ is equal to 3

Q. Find k for which the equation

$x^4 + x^2(1 - 2k) + k^2 - 1 = $ ***has***

(i) No real solution

Q. Find k for which the equation

$x^4 + x^2 (1 - 2k) + k^2 - 1 =$ has

(ii) one real solution

Q. *Find k for which the equation*

$x^4 + x^2 (1 - 2k) + k^2 - 1 =$ *has*

(iii) two real solution

Q. ***Find k for which the equation***

$x^4 + x^2 (1 - 2k) + k^2 - 1 = \text{has}$

(iv) three real solution

Q. ***Find k for which the equation***

$$x^4 + x^2 (1 - 2k) + k^2 - 1 = \text{has}$$

(v) Four real solution

Modulas Inequality

Example

Q. $|x^2 + 4x + 2| = \dfrac{5x+16}{3}$

Note

$$|\mathbf{x}| < \alpha \Rightarrow \mathbf{x} \in (-\alpha, \alpha)$$

Note

$$|\,\mathbf{x}\,| < \alpha \Rightarrow \mathbf{x} \in (-\alpha, \alpha)$$

$$|\,\mathbf{x}\,| > \beta \Rightarrow \mathbf{x} \in (-\infty, -\beta) \cup (\beta, \infty)$$

Examples

Q. $(|x - 1| - 3)(|x + 2| - 5) < 0$

Q. $|x-5| > |x^2-5x+9|$

Q. $$\left|\frac{x^2 - 5x + 4}{x^2 - 4}\right| \le 1$$