SOLUTION OF TRIANGLE

(TRIGO. PH-3)

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KEY CONCEPTS (SOLUTIONS OF TRIANGLE)

I. SINE FORMULA : In any triangle ABC, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.

II. COSINE FORMULA : **(i)** $\cos A = \frac{b^2+c^2-a^2}{2bc}$ or $a^2 = b^2 + c^2 - 2bc.\cos A$

(ii) $\cos B = \frac{c^2+a^2-b^2}{2ca}$ **(iii)** $\cos C = \frac{a^2+b^2-c^2}{2ab}$

III. PROJECTION FORMULA : **(i)** $a = b\cos C + c\cos B$ **(ii)** $b = c\cos A + a\cos C$

(iii) $c = a\cos B + b\cos A$

IV. NAPIER'S ANALOGY – TANGENT RULE : **(i)** $\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2}$

(ii) $\tan\frac{C-A}{2} = \frac{c-a}{c+a}\cot\frac{B}{2}$ **(iii)** $\tan\frac{A-B}{2} = \frac{a-b}{a+b}\cot\frac{C}{2}$

V. TRIGONOMETRIC FUNCTIONS OF HALF ANGLES :

(i) $\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$; $\sin\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}}$; $\sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$

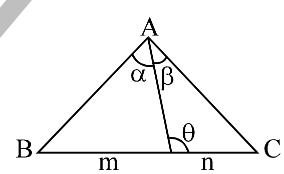
(ii) $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$; $\cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}}$; $\cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$

(iii) $\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \frac{\Delta}{s(s-a)}$ where $s = \frac{a+b+c}{2}$ & Δ = area of triangle.

(iv) Area of triangle $= \sqrt{s(s-a)(s-b)(s-c)}$.

VI. m – n RULE : In any triangle,

$$(m+n)\cot\theta = m\cot\alpha - n\cot\beta$$
$$= n\cot B - m\cot C$$



VII. $\frac{1}{2}ab\sin C = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B$ = area of triangle ABC.

$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$

Note that $R = \frac{abc}{4\Delta}$; Where R is the radius of circumcircle & Δ is area of triangle

VIII. Radius of the incircle 'r' is given by:

(a) $r = \frac{\Delta}{s}$ where $s = \frac{a+b+c}{2}$ **(b)** $r = (s-a)\tan\frac{A}{2} = (s-b)\tan\frac{B}{2} = (s-c)\tan\frac{C}{2}$

(c) $r = \frac{a\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}}$ & so on **(d)** $r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

IX. Radius of the Ex–circles r_1, r_2 & r_3 are given by :

(a) $r_1 = \frac{\Delta}{s-a}$; $r_2 = \frac{\Delta}{s-b}$; $r_3 = \frac{\Delta}{s-c}$ **(b)** $r_1 = s\tan\frac{A}{2}$; $r_2 = s\tan\frac{B}{2}$; $r_3 = s\tan\frac{C}{2}$

(c) $r_1 = \frac{a\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}$ & so on **(d)** $r_1 = 4R\sin\frac{A}{2}.\cos\frac{B}{2}.\cos\frac{C}{2}$;

$r_2 = 4R\sin\frac{B}{2}.\cos\frac{A}{2}.\cos\frac{C}{2}$; $r_3 = 4R\sin\frac{C}{2}.\cos\frac{A}{2}.\cos\frac{B}{2}$

X. LENGTH OF ANGLE BISECTOR & MEDIANS :

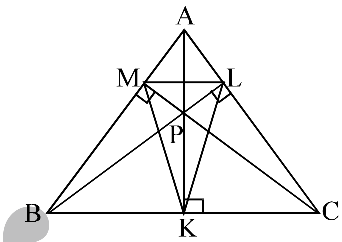
If m_a and β_a are the lengths of a median and an angle bisector from the angle A then,

$$m_a = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2} \quad \text{and} \quad \beta_a = \frac{2bc\cos\frac{A}{2}}{b+c}$$

Note that $m_a^2 + m_b^2 + m_c^2 = \frac{3}{4}(a^2 + b^2 + c^2)$

XI. ORTHOCENTRE AND PEDAL TRIANGLE :

The triangle KLM which is formed by joining the feet of the altitudes is called the pedal triangle.



– the distances of the orthocentre from the angular points of the ΔABC are $2R\cos A$, $2R\cos B$ and $2R\cos C$

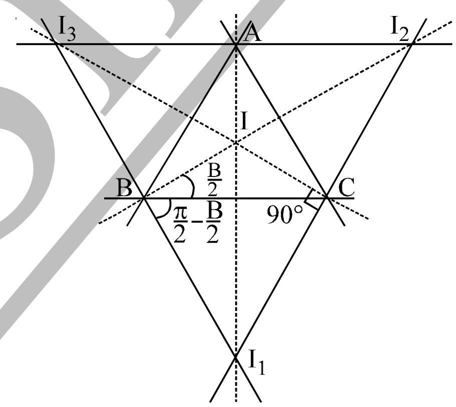
– the distances of P from sides are $2R\cos B\cos C$, $2R\cos C\cos A$ and $2R\cos A\cos B$

– the sides of the pedal triangle are $a\cos A$ ($= R\sin 2A$), $b\cos B$ ($= R\sin 2B$) and $c\cos C$ ($= R\sin 2C$) and its angles are $\pi - 2A$, $\pi - 2B$ and $\pi - 2C$.

– circumradii of the triangles PBC, PCA, PAB and ABC are equal .

XII EXCENTRAL TRIANGLE :

The triangle formed by joining the three excentres I_1, I_2 and I_3 of ΔABC is called the excentral or excentric triangle.



Note that :

– Incentre I of ΔABC is the orthocentre of the excentral $\Delta I_1I_2I_3$.

– ΔABC is the pedal triangle of the $\Delta I_1I_2I_3$.

– the sides of the excentral triangle are

$4R\cos\frac{A}{2}$, $4R\cos\frac{B}{2}$ and $4R\cos\frac{C}{2}$

and its angles are $\frac{\pi}{2} - \frac{A}{2}$, $\frac{\pi}{2} - \frac{B}{2}$ and $\frac{\pi}{2} - \frac{C}{2}$.

– $II_1 = 4R\sin\frac{A}{2}$; $II_2 = 4R\sin\frac{B}{2}$; $II_3 = 4R\sin\frac{C}{2}$.

XIII. THE DISTANCES BETWEEN THE SPECIAL POINTS :

(a) The distance between circumcentre and orthocentre is $= R\,.\sqrt{1 - 8\cos A\cos B\cos C}$

(b) The distance between circumcentre and incentre is $= \sqrt{R^2 - 2Rr}$

(c) The distance between incentre and orthocentre is $\sqrt{2r^2 - 4R^2\cos A\cos B\cos C}$

XIV. Perimeter (P) and area (A) of a regular polygon of n sides inscribed in a circle of radius r are given by

$P = 2nr\sin\frac{\pi}{n}$ and $A = \frac{1}{2}nr^2\sin\frac{2\pi}{n}$

Perimeter and area of a regular polygon of n sides circumscribed about a given circle of radius r is given by

$P = 2nr\tan\frac{\pi}{n}$ and $A = nr^2\tan\frac{\pi}{n}$

EXERCISE–I

With usual notations, prove that in a triangle ABC:

Q.1 $\frac{b-c}{r_1}+\frac{c-a}{r_2}+\frac{a-b}{r_3}=0$

Q.2 $a\cot A+b\cot B+c\cot C=2(R+r)$

Q.3 $\frac{r_1}{(s-b)(s-c)}+\frac{r_2}{(s-c)(s-a)}+\frac{r_3}{(s-a)(s-b)}=\frac{3}{r}$

Q.4 $\frac{r_1-r}{a}+\frac{r_2-r}{b}=\frac{c}{r_3}$

Q.5 $\frac{abc}{s}\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}=\Delta$

Q.6 $(r_1+r_2)\tan\frac{C}{2}=(r_3-r)\cot\frac{C}{2}=c$

Q.7 $(r_1-r)(r_2-r)(r_3-r)=4Rr^2$

Q.8 $(r+r_1)\tan\frac{B-C}{2}+(r+r_2)\tan\frac{C-A}{2}+(r+r_3)\tan\frac{A-B}{2}=0$

Q.9 $\frac{1}{r^2}+\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}=\frac{a^2+b^2+c^2}{\Delta^2}$

Q.10 $(r_3+r_1)(r_3+r_2)\sin C=2r_3\sqrt{r_2r_3+r_3r_1+r_1r_2}$

Q.11 $\frac{1}{bc}+\frac{1}{ca}+\frac{1}{ab}=\frac{1}{2Rr}$

Q.12 $\left(\frac{1}{r}-\frac{1}{r_1}\right)\left(\frac{1}{r}-\frac{1}{r_2}\right)\left(\frac{1}{r}-\frac{1}{r_3}\right)=\frac{4R}{r^2s^2}$

Q.13 $\frac{bc-r_2r_3}{r_1}=\frac{ca-r_3r_1}{r_2}=\frac{ab-r_1r_2}{r_3}=r$

Q.14 $\left(\frac{1}{r}+\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\right)^2=\frac{4}{r}\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\right)$

Q.15 $Rr(\sin A+\sin B+\sin C)=\Delta$

Q.16 $2R\cos A=2R+r-r_1$

Q.17 $\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\frac{s^2}{\Delta}$

Q.18 $\cot A+\cot B+\cot C=\frac{a^2+b^2+c^2}{4\Delta}$

Q.19 Given a triangle ABC with sides a = 7, b = 8 and c = 5. If the value of the expression $\left(\sum\sin A\right)\left(\sum\cot\frac{A}{2}\right)$ can be expressed in the form $\frac{p}{q}$ where $p, q \in N$ and $\frac{p}{q}$ is in its lowest form find the value of $(p+q)$.

Q.20 If $r_1=r+r_2+r_3$ then prove that the triangle is a right angled triangle.

Q.21 If two times the square of the diameter of the circumcircle of a triangle is equal to the sum of the squares of its sides then prove that the triangle is right angled.

Q.22 In acute angled triangle ABC, a semicircle with radius r_a is constructed with its base on BC and tangent to the other two sides. r_b and r_c are defined similarly. If r is the radius of the incircle of triangle ABC then prove that, $\frac{2}{r}=\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}$.

Q.23 Given a right triangle with $\angle A=90°$. Let M be the mid-point of BC. If the inradii of the triangle ABM and ACM are r_1 and r_2 then find the range of r_1/r_2.

Q.24 If the length of the perpendiculars from the vertices of a triangle A, B, C on the opposite sides are p_1, p_2, p_3 then prove that $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}=\frac{1}{r}=\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}$.

Q.25 Prove that in a triangle $\frac{bc}{r_1}+\frac{ca}{r_2}+\frac{ab}{r_3}=2R\left[\left(\frac{a}{b}+\frac{b}{a}\right)+\left(\frac{b}{c}+\frac{c}{b}\right)+\left(\frac{c}{a}+\frac{a}{c}\right)-3\right]$.

EXERCISE–II

Q.1 With usual notation, if in a ΔABC, $\frac{b+c}{11}=\frac{c+a}{12}=\frac{a+b}{13}$; then prove that, $\frac{\cos A}{7}=\frac{\cos B}{19}=\frac{\cos C}{25}$.

Q.2 For any triangle ABC, if $B=3C$, show that $\cos C=\sqrt{\frac{b+c}{4c}}$ & $\sin\frac{A}{2}=\frac{b-c}{2c}$.

Q.3 In a triangle ABC, BD is a median. If $l(BD)=\frac{\sqrt{3}}{4}\cdot l(AB)$ and $\angle DBC=\frac{\pi}{2}$. Determine the $\angle ABC$.

Q.4 ABCD is a trapezium such that AB, DC are parallel & BC is perpendicular to them. If angle $ADB=\theta$, $BC=p$ & $CD=q$, show that $AB=\frac{(p^2+q^2)\sin\theta}{p\cos\theta+q\sin\theta}$.

Q.5 If sides a, b, c of the triangle ABC are in A.P., then prove that
$\sin^2\frac{A}{2}\operatorname{cosec}2A;\quad \sin^2\frac{B}{2}\operatorname{cosec}2B;\quad \sin^2\frac{C}{2}\operatorname{cosec}2C$ are in H.P.

Q.6 Find the angles of a triangle in which the altitude and a median drawn from the same vertex divide the angle at that vertex into 3 equal parts.

Q.7 In a triangle ABC, if $\tan\frac{A}{2}$, $\tan\frac{B}{2}$, $\tan\frac{C}{2}$ are in AP. Show that $\cos A$, $\cos B$, $\cos C$ are in AP.

Q.8 ABCD is a rhombus. The circumradii of ΔABD and ΔACD are 12.5 and 25 respectively. Find the area of rhombus.

Q.9 In a triangle ABC if $a^2+b^2=101c^2$ then find the value of $\frac{\cot C}{\cot A+\cot B}$.

Q.10 The two adjacent sides of a cyclic quadrilateral are 2 & 5 and the angle between them is $60°$. If the area of the quadrilateral is $4\sqrt{3}$, find the remaining two sides.

Q.11 If I be the in–centre of the triangle ABC and x, y, z be the circum radii of the triangles IBC, ICA & IAB, show that $4R^3-R(x^2+y^2+z^2)-xyz=0$.

Q.12 Sides a, b, c of the triangle ABC are in H.P., then prove that
$\operatorname{cosec}A(\operatorname{cosec}A+\cot A)$; $\operatorname{cosec}B(\operatorname{cosec}B+\cot B)$ & $\operatorname{cosec}C(\operatorname{cosec}C+\cot C)$ are in A.P.

Q.13 A point 'O' is situated on a circle of radius R and with centre O, another circle of radius $\frac{3R}{2}$ is described. Inside the crescent shaped area intercepted between these circles, a circle of radius R/8 is placed. If the same circle moves in contact with the original circle of radius R, then find the length of the arc described by its centre in moving from one extreme position to the other.

Q.14 If in a Δ ABC, $a=6$, $b=3$ and $\cos(A-B)=4/5$ then find its area.

Q.15 In a Δ ABC, (i) $\frac{a}{\cos A} = \frac{b}{\cos B}$ (ii) $2 \sin A \cos B = \sin C$

(iii) $\tan^2 \frac{A}{2} + 2 \tan \frac{A}{2} \tan \frac{C}{2} - 1 = 0$, prove that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i).

Q.16 The sequence $a_1, a_2, a_3, \ldots\ldots$ is a geometric sequence.
The sequence $b_1, b_2, b_3, \ldots\ldots$ is a geometric sequence.

$b_1 = 1$; $b_2 = \sqrt[4]{7} - \sqrt[4]{28} + 1$; $a_1 = \sqrt[4]{28}$ and $\sum_{n=1}^{\infty} \frac{1}{a_n} = \sum_{n=1}^{\infty} b_n$

If the area of the triangle with sides lengths a_1, a_2 and a_3 can be expressed in the form of p/q where p and q are relatively prime, find (p + q).

Q.17 If p_1, p_2, p_3 are the altitudes of a triangle from the vertices A, B, C & Δ denotes the area of the triangle, prove that $\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_3} = \frac{2ab}{(a+b+c)\Delta} \cos^2 \frac{C}{2}$.

Q.18 Two sides of a triangle are of lengths $\sqrt{6}$ and 4 and the angle opposite to smaller side is 30^0. How many such triangles are possible ? Find the length of their third side and area.

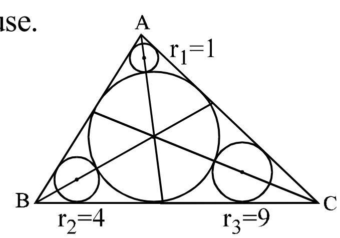
Q.19 The triangle ABC (with side lengths a, b, c as usual) satisfies
$\log a^2 = \log b^2 + \log c^2 - \log (2bc \cos A)$. What can you say about this triangle?

Q.20 With reference to a given circle, A_1 and B_1 are the areas of the inscribed and circumscribed regular polygons of n sides, A_2 and B_2 are corresponding quantities for regular polygons of 2n sides. Prove that
(1) A_2 is a geometric mean between A_1 and B_1.
(2) B_2 is a harmonic mean between A_2 and B_1.

Q.21 The sides of a triangle are consecutive integers n, $n + 1$ and $n + 2$ and the largest angle is twice the smallest angle. Find n.

Q.22 The triangle ABC is a right angled triangle, right angle at A. The ratio of the radius of the circle circumscribed to the radius of the circle escribed to the hypotenuse is, $\sqrt{2} : \left(\sqrt{3} + \sqrt{2}\right)$. Find the acute angles B & C. Also find the ratio of the two sides of the triangle other than the hypotenuse.

Q.23 ABC is a triangle. Circles with radii as shown are drawn inside the triangle each touching two sides and the incircle. Find the radius of the incircle of the ΔABC.



Q.24 Line l is a tangent to a unit circle S at a point P. Point A and the circle S are on the same side of l, and the distance from A to l is 3. Two tangents from point A intersect line l at the point B and C respectively. Find the value of (PB)(PC).

Q.25 Let ABC be an acute triangle with orthocenter H. D, E, F are the feet of the perpendiculars from A, B, and C on the opposite sides. Also R is the circumradius of the triangle ABC.
Given (AH)(BH)(CH) = 3 and $(AH)^2 + (BH)^2 + (CH)^2 = 7$. Find

(a) the ratio $\frac{\prod \cos A}{\sum \cos^2 A}$, (b) the product (HD)(HE)(HF) (c) the value of R.

EXERCISE–III

Q.1 The radii r_1, r_2, r_3 of escribed circles of a triangle ABC are in harmonic progression. If its area is 24 sq. cm and its perimeter is 24 cm, find the lengths of its sides. [REE '99, 6]

Q.2(a) In a triangle ABC, Let $\angle C = \frac{\pi}{2}$. If 'r' is the inradius and 'R' is the circumradius of the triangle, then $2(r + R)$ is equal to:

(A) a + b (B) b + c (C) c + a (D) a + b + c

(b) In a triangle ABC, $2\,a\,c\,\sin\frac{1}{2}(A - B + C) =$

(A) $a^2 + b^2 - c^2$ (B) $c^2 + a^2 - b^2$ (C) $b^2 - c^2 - a^2$ (D) $c^2 - a^2 - b^2$

[JEE '2000 (Screening) 1 + 1]

Q.3 Let ABC be a triangle with incentre 'I' and inradius 'r'. Let D, E, F be the feet of the perpendiculars from I to the sides BC, CA & AB respectively. If r_1, r_2 & r_3 are the radii of circles inscribed in the quadrilaterals AFIE, BDIF & CEID respectively, prove that

$$\frac{r_1}{r - r_1} + \frac{r_2}{r - r_2} + \frac{r_3}{r - r_3} = \frac{r_1 r_2 r_3}{(r - r_1)(r - r_2)(r - r_3)}.$$ [JEE '2000, 7]

Q.4 If Δ is the area of a triangle with side lengths a, b, c, then show that: $\Delta \le \frac{1}{4}\sqrt{(a + b + c)abc}$

Also show that equality occurs in the above inequality if and only if a = b = c. [JEE ' 2001]

Q.5 Which of the following pieces of data does NOT uniquely determine an acute–angled triangle ABC (R being the radius of the circumcircle)?

(A) a, sinA, sinB (B) a, b, c (C) a, sinB, R (D) a, sinA, R

[JEE ' 2002 (Scr), 3]

Q.6 If I_n is the area of n sided regular polygon inscribed in a circle of unit radius and O_n be the area of the polygon circumscribing the given circle, prove that

$$I_n = \frac{O_n}{2}\left(1 + \sqrt{1 - \left(\frac{2I_n}{n}\right)^2}\right)$$ [JEE 2003, Mains, 4 out of 60]

Q.7 The ratio of the sides of a triangle ABC is $1 : \sqrt{3} : 2$. The ratio A : B : C is

(A) 3 : 5 : 2 (B) $1 : \sqrt{3} : 2$ (C) 3 : 2 : 1 (D) 1 : 2 : 3

[JEE 2004 (Screening)]

Q.8(a) In ΔABC, a, b, c are the lengths of its sides and A, B, C are the angles of triangle ABC. The correct relation is

(A) $(b - c)\sin\left(\frac{B - C}{2}\right) = a\cos\left(\frac{A}{2}\right)$ (B) $(b - c)\cos\left(\frac{A}{2}\right) = a\sin\left(\frac{B - C}{2}\right)$

(C) $(b + c)\sin\left(\frac{B + C}{2}\right) = a\cos\left(\frac{A}{2}\right)$ (D) $(b - c)\cos\left(\frac{A}{2}\right) = 2a\sin\left(\frac{B + C}{2}\right)$

[JEE 2005 (Screening)]

(b) Circles with radii 3, 4 and 5 touch each other externally if P is the point of intersection of tangents to these circles at their points of contact. Find the distance of P from the points of contact.

[JEE 2005 (Mains), 2]

Q.9(a) Given an isosceles triangle, whose one angle is 120° and radius of its incircle is $\sqrt{3}$. Then the area of triangle in sq. units is

(A) $7 + 12\sqrt{3}$ (B) $12 - 7\sqrt{3}$ (C) $12 + 7\sqrt{3}$ (D) 4π

[JEE 2006, 3]

(b) Internal bisector of $\angle A$ of a triangle ABC meets side BC at D. A line drawn through D perpendicular to AD intersects the side AC at E and the side AB at F. If a, b, c represent sides of ΔABC then

(A) AE is HM of b and c (B) $AD = \frac{2bc}{b+c}\cos\frac{A}{2}$

(C) $EF = \frac{4bc}{b+c}\sin\frac{A}{2}$ (D) the triangle AEF is isosceles [JEE 2006, 5]

Q.10 Let ABC and ABC′ be two non-congruent triangles with sides AB = 4, AC = AC′ = $2\sqrt{2}$ and angle B = 30°. The absolute value of the difference between the areas of these triangles is

[JEE 2009, 5]

ANSWER KEY

SOLUTIONS OF TRIANGLE

EXERCISE–I

Q.19 107 **Q.23** $\left(\frac{1}{2}, 2\right)$

EXERCISE–II

Q.3 120° **Q.6** $\pi/6,\ \pi/3,\ \pi/2$ **Q.8** 400 **Q.9** 50 **Q.10** 3 cms & 2 cms

Q.13 $\frac{7\pi R}{12}$ **Q.14** **Q.1** 9 sq. unit **Q.16** 9

Q.18 **Q.3** 2, $\left(2\sqrt{3}-\sqrt{2}\right)$, $\left(2\sqrt{3}+\sqrt{2}\right)$, $\left(2\sqrt{3}-\sqrt{2}\right)$ & $\left(2\sqrt{3}+\sqrt{2}\right)$ sq. units

Q.19 triangle is isosceles **Q.21** 4 **Q.22** $B = \frac{5\pi}{12}$; $C = \frac{\pi}{12}$; $\frac{b}{c} = 2+\sqrt{3}$

Q.23 r = 11 **Q.24** 3 **Q.25** (a) $\frac{3}{14R}$, (b) $\frac{9}{8R^3}$, (c) $\frac{3}{2}$

EXERCISE–III

Q.1 6, 8, 10 cms **Q.2** **(a)** A, **(b)** B **Q.5** D **Q.7** D **Q.8** **(a)** B; **(b)** $\sqrt{5}$

Q.9 **(a)** C, **(b)** A, B, C, D **Q.10** 4