TRIGONOMETRIC
PH - 2

MANOJ CHAUHAN SIR(IIT-DELHI)
EX. SR. FACULTY (BANSAL CLASSES)

THINGS TO REMEMBER :

1. If $\sin\theta = \sin\alpha \Rightarrow \theta = n\pi + (-1)^n \alpha$ where $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $n \in I$.

2. If $\cos\theta = \cos\alpha \Rightarrow \theta = 2n\pi \pm \alpha$ where $\alpha \in [0, \pi]$, $n \in I$.

3. If $\tan\theta = \tan\alpha \Rightarrow \theta = n\pi + \alpha$ where $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $n \in I$.

4. If $\sin^2\theta = \sin^2\alpha \Rightarrow \theta = n\pi \pm \alpha$.

5. $\cos^2\theta = \cos^2\alpha \Rightarrow \theta = n\pi \pm \alpha$.

6. $\tan^2\theta = \tan^2\alpha \Rightarrow \theta = n\pi \pm \alpha$. [Note : α is called the principal angle]

7. TYPES OF TRIGONOMETRIC EQUATIONS :

(a) Solutions of equations by factorising . Consider the equation ;
$(2\sin x - \cos x)(1 + \cos x) = \sin^2 x$; $\cot x - \cos x = 1 - \cot x \cos x$

(b) Solutions of equations reducible to quadratic equations. Consider the equation :
$3\cos^2 x - 10\cos x + 3 = 0$ and $2\sin^2 x + \sqrt{3}\sin x + 1 = 0$

(c) Solving equations by introducing an Auxilliary argument . Consider the equation :
$\sin x + \cos x = \sqrt{2}$; $\sqrt{3}\cos x + \sin x = 2$; $\sec x - 1 = (\sqrt{2} - 1)\tan x$

(d) Solving equations by Transforming a sum of Trigonometric functions into a product.
Consider the example : $\cos 3x + \sin 2x - \sin 4x = 0$;
$\sin^2 x + \sin^2 2x + \sin^2 3x + \sin^2 4x = 2$; $\sin x + \sin 5x = \sin 2x + \sin 4x$

(e) Solving equations by transforming a product of trigonometric functions into a sum.
Consider the equation :
$\sin 5x \cdot \cos 3x = \sin 6x \cdot \cos 2x$; $8\cos x \cos 2x \cos 4x = \frac{\sin 6x}{\sin x}$ **;** $\sin 3\theta = 4\sin\theta \sin 2\theta \sin 4\theta$

(f) Solving equations by a change of variable :

(i) Equations of the form of $a \cdot \sin x + b \cdot \cos x + d = 0$, where a, b & d are real numbers & $a, b \neq 0$ can be solved by changing $\sin x$ & $\cos x$ into their corresponding tangent of half the angle. Consider the equation $3\cos x + 4\sin x = 5$.

(ii) Many equations can be solved by introducing a new variable . eg. the equation $\sin^4 2x + \cos^4 2x = \sin 2x \cdot \cos 2x$ changes to
$2(y+1)\left(y - \frac{1}{2}\right) = 0$ by substituting , $\sin 2x \cdot \cos 2x = y$.

(g) Solving equations with the use of the Boundness of the functions $\sin x$ & $\cos x$ or by making two perfect squares. Consider the equations :
$$\sin x\left(\cos\frac{x}{4} - 2\sin x\right) + \left(1 + \sin\frac{x}{4} - 2\cos x\right) \cdot \cos x = 0 \;;$$
$$\sin^2 x + 2\tan^2 x + \frac{4}{\sqrt{3}}\tan x - \sin x + \frac{11}{12} = 0$$

8. TRIGONOMETRIC INEQUALITIES : There is no general rule to solve a Trigonometric inequations and the same rules of algebra are valid except the domain and range of trigonometric functions should be kept in mind.

Consider the examples : $\log_2\left(\sin\frac{x}{2}\right) < -1$; $\sin x\left(\cos x + \frac{1}{2}\right) \leq 0$; $\sqrt{5 - 2\sin 2x} \geq 6\sin x - 1$

EXERCISE–I

Q.1 Solve the equation for x, $5^{\frac{1}{2}} + 5^{\frac{1}{2}+\log_5(\sin x)} = 15^{\frac{1}{2}+\log_{15}\cos x}$

Q.2 Find all the values of θ satisfying the equation; $\sin\theta + \sin 5\theta = \sin 3\theta$ such that $0 \le \theta \le \pi$.

Q.3 Solve the equality: $2\sin 11x + \cos 3x + \sqrt{3}\sin 3x = 0$

Q.4 Find all value of θ, between 0 & π, which satisfy the equation; $\cos\theta \,.\, \cos 2\theta \,.\, \cos 3\theta = 1/4$.

Q.5 Find the general solution of the equation, $2 + \tan x \cdot \cot\frac{x}{2} + \cot x \cdot \tan\frac{x}{2} = 0$

Q.6 Solve for x, the equation $\sqrt{13 - 18\tan x} = 6\tan x - 3$, where $-2\pi < x < 2\pi$.

Q.7 Determine the smallest positive value of x which satisfy the equation, $\sqrt{1+\sin 2x} - \sqrt{2}\cos 3x = 0$.

Q.8 $2\sin\left(3x + \frac{\pi}{4}\right) = \sqrt{1 + 8\sin 2x \,.\, \cos^2 2x}$

Q.9 Find the number of principal solution of the equation, $\sin x - \sin 3x + \sin 5x = \cos x - \cos 3x + \cos 5x$.

Q.10 Find the general solution of the trigonometric equation $3^{\left(\frac{1}{2}+\log_3(\cos x+\sin x)\right)} - 2^{\log_2(\cos x - \sin x)} = \sqrt{2}$.

Q.11 Find all values of θ between $0°$ & $180°$ satisfying the equation;
$\cos 6\theta + \cos 4\theta + \cos 2\theta + 1 = 0$.

Q.12 Find the solution set of the equation, $\log_{\frac{-x^2-6x}{10}}(\sin 3x + \sin x) = \log_{\frac{-x^2-6x}{10}}(\sin 2x)$.

Q.13 Find the value of θ, which satisfy $3 - 2\cos\theta - 4\sin\theta - \cos 2\theta + \sin 2\theta = 0$.

Q.14 Find the general solution of the equation, $\sin \pi x + \cos \pi x = 0$. Also find the sum of all solutions in $[0, 100]$.

Q.15 Find the least positive angle measured in degrees satisfying the equation
$\sin^3 x + \sin^3 2x + \sin^3 3x = (\sin x + \sin 2x + \sin 3x)^3$.

Q.16 Find the range of y such that the equation, $y + \cos x = \sin x$ has a real solution. For $y = 1$, find x such that $0 < x < 2\pi$.

Q.17 Find the general values of θ for which the quadratic function
$(\sin\theta)\, x^2 + (2\cos\theta)x + \frac{\cos\theta + \sin\theta}{2}$ is the square of a linear function.

Q.18 Prove that the equations

(a) $\sin x \cdot \sin 2x \cdot \sin 3x = 1$ (b) $\sin x \cdot \cos 4x \cdot \sin 5x = -1/2$

have no solution.

Q.19 Let $f(x) = \sin^6 x + \cos^6 x + k(\sin^4 x + \cos^4 x)$ for some real number k. Determine

(a) all real numbers k for which $f(x)$ is constant for all values of x.

(b) all real numbers k for which there exists a real number 'c' such that $f(c) = 0$.

(c) If $k = -0.7$, determine all solutions to the equation $f(x) = 0$.

Q.20 If α and β are the roots of the equation, $a\cos\theta + b\sin\theta = c$ then match the entries of **column-I** with the entries of **column-II**.

	Column-I		**Column-II**
(A)	$\sin\alpha + \sin\beta$	(P)	$\dfrac{2b}{a+c}$
(B)	$\sin\alpha \,.\, \sin\beta$	(Q)	$\dfrac{c-a}{c+a}$
(C)	$\tan\dfrac{\alpha}{2} + \tan\dfrac{\beta}{2}$	(R)	$\dfrac{2bc}{a^2+b^2}$
(D)	$\tan\dfrac{\alpha}{2} \,.\, \tan\dfrac{\beta}{2} =$	(S)	$\dfrac{c^2-a^2}{a^2+b^2}$

EXERCISE–II

Q.1 Solve the equation : $\sin 5x = 16 \sin^5 x$.

Q.2 Find all the solutions of, $4\cos^2 x \sin x - 2\sin^2 x = 3 \sin x$.

Q.3 Solve for x, $(-\pi \le x \le \pi)$ the equation; $2(\cos x + \cos 2x) + \sin 2x (1 + 2\cos x) = 2\sin x$.

Q.4 Find the general solution of the following equation:
$2(\sin x - \cos 2x) - \sin 2x (1 + 2\sin x) + 2\cos x = 0$.

Q.5 Solve the inequality $\sin 2x > \sqrt{2}\sin^2 x + (2 - \sqrt{2})\cos^2 x$.

Q.6 Find the values of x, between 0 & 2π, satisfying the equation;

$$\cos 3x + \cos 2x = \sin\frac{3x}{2} + \sin\frac{x}{2}.$$

Q.7 Find the set of values of 'a' for which the equation, $\sin^4 x + \cos^4 x + \sin 2x + a = 0$ possesses solutions. Also find the general solution for these values of 'a'.

Q.8 Solve: $\tan^2 2x + \cot^2 2x + 2\tan 2x + 2\cot 2x = 6$.

Q.9 Solve the equation: $1 + 2\,\text{cosec}x = -\dfrac{\sec^2\dfrac{x}{2}}{2}$.

Q.10 Solve: $\tan^2 x \,.\, \tan^2 3x \,.\, \tan 4x = \tan^2 x - \tan^2 3x + \tan 4x$.

Q.11 Find the set of values of x satisfying the equality

$$\sin\left(x-\frac{\pi}{4}\right)-\cos\left(x+\frac{3\pi}{4}\right)=1 \text{ and the inequality } \frac{2\cos 7x}{\cos 3+\sin 3} > 2^{\cos 2x}.$$

Q.12 Solve: $\sin\left(\frac{\sqrt{x}}{2}\right)+\cos\left(\frac{\sqrt{x}}{2}\right)=\sqrt{2}\,\sin\sqrt{x}$.

Q.13 Find the general solution of the equation, $\sin\frac{2x+1}{x}+\sin\frac{2x+1}{3x}-3\cos^2\frac{2x+1}{3x}=0$.

Q.14 Let S be the set of all those solutions of the equation, $(1+k)\cos x\,\cos(2x-\alpha)=(1+k\cos 2x)\cos(x-\alpha)$ which are independent of k & α. Let H be the set of all such solutions which are dependent on k & α. Find the condition on k & α such that H is a non-empty set, state S. If a subset of H is $(0,\pi)$ in which $k=0$, then find all the permissible values of α.

Q.15 Solve for x & y, $\begin{array}{l} x\cos^3 y + 3x\cos y\sin^2 y = 14 \\ x\sin^3 y + 3x\cos^2 y\sin y = 13 \end{array}$

Q.16 Find the value of α for which the three elements set $S=\{\sin\alpha, \sin 2\alpha, \sin 3\alpha\}$ is equal to the three element set $T=\{\cos\alpha, \cos 2\alpha, \cos 3\alpha\}$.

Q.17 Find all values of 'a' for which every root of the equation, $a\cos 2x+|a|\cos 4x+\cos 6x=1$ is also a root of the equation, $\sin x\cos 2x=\sin 2x\cos 3x-\frac{1}{2}\sin 5x$, and conversely, every root of the second equation is also a root of the first equation.

Q.18 Solve the equations for 'x' given in **column-I** and match with the entries of **column-II**.

	Column-I		**Column-II**
(A)	$\cos 3x\,.\,\cos^3 x+\sin 3x\,.\,\sin^3 x=0$	(P)	$n\pi\pm\frac{\pi}{3}$
(B)	$\sin 3\alpha=4\sin\alpha\sin(x+\alpha)\sin(x-\alpha)$ where α is a constant $\neq n\pi$.	(Q)	$n\pi+\frac{\pi}{4}$, $n\in I$
(C)	$\lvert 2\tan x-1\rvert+\lvert 2\cot x-1\rvert=2$.	(R)	$\frac{n\pi}{4}+\frac{\pi}{8}$, $n\in I$
(D)	$\sin^{10}x+\cos^{10}x=\frac{29}{16}\cos^4 2x$.	(S)	$\frac{n\pi}{2}\pm\frac{\pi}{4}$

EXERCISE–III

Q.1 Find real values of x for which, $27^{\cos 2x} . 81^{\sin 2x}$ is minimum . Also find this minimum value. [REE 2000, 3]

Q.2 Solve the following system of equations for x and y [REE '2001(mains), 3]

$$5^{(\operatorname{cosec}^2 x - 3\sec^2 y)} = 1 \quad \text{and} \quad 2^{(2\operatorname{cosec} x + \sqrt{3}|\sec y|)} = 64.$$

Q.3 The number of integral values of k for which the equation $7\cos x + 5\sin x = 2k + 1$ has a solution is

(A) 4 (B) 8 (C) 10 (D) 12

[JEE 2002 (Screening), 3]

Q.4 $\cos(\alpha - \beta) = 1$ and $\cos(\alpha + \beta) = 1/e$, where $\alpha, \beta \in [-\pi, \pi]$, numbers of pairs of α, β which satisfy both the equations is

(A) 0 (B) 1 (C) 2 (D) 4

[JEE 2005 (Screening)]

Q.5 If $0 < \theta < 2\pi$, then the intervals of values of θ for which $2\sin^2\theta - 5\sin\theta + 2 > 0$, is

(A) $\left(0, \frac{\pi}{6}\right) \cup \left(\frac{5\pi}{6}, 2\pi\right)$ (B) $\left(\frac{\pi}{8}, \frac{5\pi}{6}\right)$ (C) $\left(0, \frac{\pi}{8}\right) \cup \left(\frac{\pi}{6}, \frac{5\pi}{6}\right)$ (D) $\left(\frac{41\pi}{48}, \pi\right)$

[JEE 2006, 3]

Q.6 The number of solutions of the pair of equations

$$2\sin^2\theta - \cos 2\theta = 0$$
$$2\cos^2\theta - 3\sin\theta = 0$$

in the interval $[0, 2\pi]$ is

(A) zero (B) one (C) two (D) four [JEE 2007, 3]

ANSWER KEY

TRIGONOMETRIC EQUATIONS AND INEQUATIONS

EXERCISE–I

Q.1 $x = 2n\pi + \frac{\pi}{6},\ n \in I$ **Q.2** $0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{6}$ & π

Q.3 $x = \frac{n\pi}{7} - \frac{\pi}{84}$ or $x = \frac{n\pi}{4} + \frac{7\pi}{48}, n \in I$

Q.4 $\frac{\pi}{8}, \frac{\pi}{3}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{2\pi}{3}, \frac{7\pi}{8}$

Q.5 $x = 2n\pi \pm \frac{2\pi}{3}, n \in I$

Q.6 $\alpha - 2\pi$; $\alpha - \pi, \alpha, \alpha + \pi$, where $\tan\alpha = \frac{2}{3}$

Q.7 $x = \pi/16$

Q.8 $x = 2n\pi + \frac{\pi}{12}$ or $2n\pi + \frac{17\pi}{12}$; $n \in I$

Q.9 10 solutions

Q.10 $x = 2n\pi + \frac{\pi}{12}$

Q.11 $30°, 45°, 90°, 135°, 150°$

Q.12 $x = -\frac{5\pi}{3}$

Q.13 $\theta = 2n\pi$ **or** $2n\pi + \frac{\pi}{2}$; $n \in I$

Q.14 $x = n - \frac{1}{4},\ n \in I$; sum = 5025

Q.15 $72°$ **Q.16** $-\sqrt{2} \le y \le \sqrt{2}$; $\frac{\pi}{2}, \pi$

Q.17 $2n\pi + \frac{\pi}{4}$ or $(2n+1)\pi - \tan^{-1}2,\ n \in I$

Q.19 (a) $-\frac{3}{2}$; (b) $k \in \left[-1, -\frac{1}{2}\right]$; (c) $x = \frac{n\pi}{2} \pm \frac{\pi}{6}$ **Q.20** (A) R; (B) S; (C) P; (D) Q

EXERCISE–II

Q.1 $x = n\pi$ or $x = n\pi \pm \frac{\pi}{6}$

Q.2 $n\pi$; $n\pi + (-1)^n \frac{\pi}{10}$ **or** $n\pi + (-1)^n \left(-\frac{3\pi}{10}\right)$

Q.3 $\frac{\pm\pi}{3}, \frac{-\pi}{2}, \pm\pi$

Q.4 $x = 2n\pi$ or $x = n\pi + (-1)^n \left(-\frac{\pi}{2}\right)$ **or** $x = n\pi + (-1)^n \frac{\pi}{6}$

Q.5 $n\pi + \frac{\pi}{8} < x < n\pi + \frac{\pi}{4}$ **Q.6** $\frac{\pi}{7}, \frac{5\pi}{7}, \pi, \frac{9\pi}{7}, \frac{13\pi}{7}$

Q.7 $\frac{1}{2}\left[n\pi + (-1)^n \sin^{-1}\left(1 - \sqrt{2a+3}\right)\right]$ where $n \in I$ and $a \in \left[-\frac{3}{2}, \frac{1}{2}\right]$

Q.8 $x = \frac{n\pi}{4} + (-1)^n \frac{\pi}{8}$ **or** $\frac{n\pi}{4} + (-1)^{n+1} \frac{\pi}{24}$ **Q.9** $x = 2n\pi - \frac{\pi}{2}$

Q.10 $\frac{(2n+1)\pi}{4}, k\pi$, where $n, k \in I$

Q.11 $x = 2n\pi + \frac{3\pi}{4},\ n \in I$ **Q.12** $x = \left(4n\pi + \frac{\pi}{2}\right)^2$ or $x = \left(\frac{4m\pi}{3} + \frac{\pi}{2}\right)^2$ where $m, n \in W$.

Q.13 $x = \dfrac{2}{6n\pi + 3\pi - 4}$ or $\dfrac{1}{3n\pi + 3(-1)^n \sin^{-1}\frac{3}{4} - 2}$ where $n \in I$

Q.14 (i) $|k \sin\alpha| \le 1$ (ii) $S = n\pi,\ n \in I$ (iii) $\alpha \in (-m\pi, 2\pi - m\pi)\ m \in I$

Q.15 $x = \pm 5\sqrt{5}$ & $y = n\pi + \tan^{-1}\dfrac{1}{2}$ **Q.16** $\dfrac{n\pi}{2} + \dfrac{\pi}{8}$

Q.17 $a = 0$ or $a < -1$ **Q.18** (A) S; (B) P; (C) Q; (D) R

EXERCISE–III

Q.1 Min. value $= 3^{-5}$ for $x = (4n-1)\dfrac{\pi}{4} - \dfrac{1}{2}\tan^{-1}\dfrac{3}{4}, n \in I$; max. value $= 3^5$ for $x = (4n+1)\dfrac{\pi}{4} - \dfrac{1}{2}\tan^{-1}\dfrac{3}{4}, n \in I$

Q.2 $x = n\pi + (-1)^n \dfrac{\pi}{6}$ and $y = m\pi \pm \dfrac{\pi}{6}$ where m & n are integers.

Q.3 B **Q.4** D **Q.5** A **Q.6** C

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