PHOTON ENTANGLEMENT IN QUANTUM ČERENKOV RADIATION

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ABSTRACT

Photon entanglement calculation with the 2nd order perturbation theory.

1 State Expression & Spin-orbital Polarization Term

We define the initial states and the final states,

$$\left|\psi_i\right\rangle = \alpha\left|p_{i1}, p_{i2}\right\rangle_{\uparrow\downarrow}\left|00\right\rangle + \beta\left|p_{i1}, p_{i2}\right\rangle_{\downarrow\uparrow}\left|00\right\rangle \tag{1}$$

$$\left|\psi_f\right\rangle = \alpha\left|p_{f1}, p_{f2}\right\rangle_{\downarrow\uparrow}\left|k_1, k_2\right\rangle_{\uparrow\downarrow} + \beta\left|p_{f1}, p_{f2}\right\rangle_{\uparrow\downarrow}\left|k_1, k_2\right\rangle_{\downarrow\uparrow} \tag{2}$$

Here we assume that the entangled electrons experience spin-flip during the radiation. Meanwhile, according to the "Quantum Čerenkov Radiation: Spectral Cutoffs and the Role of Spin and Orbital Angular Momentum", the expression of matrix element (integration of density matrix) is shown as,

$$M = \frac{1}{\hbar}\int d^3xdt\left\langle p_f, k\right|j^\mu qA_\mu\left|p_i, 0\right\rangle \tag{3}$$

The final radiation ratio is expressed by the M matrix (C is the constant produced by four spatial integration, shown in PRX S18),

$$\Gamma = C * \int d^3kd^3p\left|M\right|^2 = C * \int d^3kd^3p(\left|M_{azimuthal}\right|^2 + \left|M_{radial}\right|^2) \tag{4}$$

Due to the linear relationship between M and SP, we could easily alternative M into SP as they did in PRX.

$$\Gamma = \frac{\alpha}{\pi\beta}\int sin(\theta_{ph})d\theta_{ph}\frac{\left|[SP]\right|^2}{4E_i^2\sqrt{(sin(\theta_i)sin(\theta_{CR}))^2 - (cos(\theta_{ph}) - cos(\theta_i)cos(\theta_{CR}))^2}} \tag{5}$$

Thus, we have to calculate the M based on the entangled electron pairs using the 2-nd order perturbation theory. Then, the contribution of each part of SP term corresponds to M. The SP terms are calculated in PRX.

$$SP_{azimuthal,\uparrow\uparrow} = SP^*_{azimuthal,\downarrow\downarrow} = ic\frac{p_{fr}(E_i + mc^2)e^{i(\phi_{ph} - \phi_f)} - p_{ir}(E_f + mc^2)e^{-i(\phi_{ph} - \phi_i)}}{\sqrt{E_i + mc^2}\sqrt{E_f + mc^2}} \tag{6}$$

$$SP_{azimuthal,\uparrow\downarrow} = -SP^*_{azimuthal,\downarrow\uparrow} = -ic\frac{p_{fz}(E_i + mc^2) - p_{iz}(E_f + mc^2)}{\sqrt{E_i + mc^2}\sqrt{E_f + mc^2}}e^{i\phi_{ph}} \tag{7}$$

$$
\begin{aligned}
SP_{radial,\uparrow\uparrow} = SP^*_{radial,\downarrow\downarrow} = c\frac{p_{fz}(E_i+mc^2)+p_{iz}(E_f+mc^2)}{\sqrt{E_i+mc^2}\sqrt{E_f+mc^2}}sin(\theta_{ph}) \\
-c\frac{p_{fr}(E_i+mc^2)e^{i(\phi_{ph}-\phi_f)}+p_{ir}(E_f+mc^2)e^{-i(\phi_{ph}-\phi_i)}}{\sqrt{E_i+mc^2}\sqrt{E_f+mc^2}}cos(\theta_{ph})
\end{aligned}
\tag{8}
$$

$$
\begin{aligned}
SP_{radial,\uparrow\downarrow} = -SP^*_{radial,\downarrow\uparrow} = c\frac{p_{fr}e^{i\phi_f}(E_i+mc^2)-p_{ir}e^{i\phi_i}(E_f+mc^2)}{\sqrt{E_i+mc^2}\sqrt{E_f+mc^2}}sin(\theta_{ph}) \\
+c\frac{p_{fz}(E_i+mc^2)-p_{iz}(E_f+mc^2)}{\sqrt{E_i+mc^2}\sqrt{E_f+mc^2}}e^{i\phi_{ph}}cos(\theta_{ph})
\end{aligned}
\tag{9}
$$

2 2-nd Order Perturbation Theory

In the paper "Concepts in X-ray Physics" section III, the Interaction picture is applied. From Schrodinger picture, the probability amplitude of the electron will be found in a final state f at a later time t is given by the coefficient c_{trans} as,

$$c^{(1)}_{trans} = \langle f|V|i\rangle \tag{10}$$

$$c^{(2)}_{trans} = \langle f|V|M\rangle\langle M|V|i\rangle \tag{11}$$

At this 2nd order, we have to consider the possible intermidiate states M. On the other hand, from the Feynman Diagram, we could draw the picture of one electrons scattering in Fig. 1.

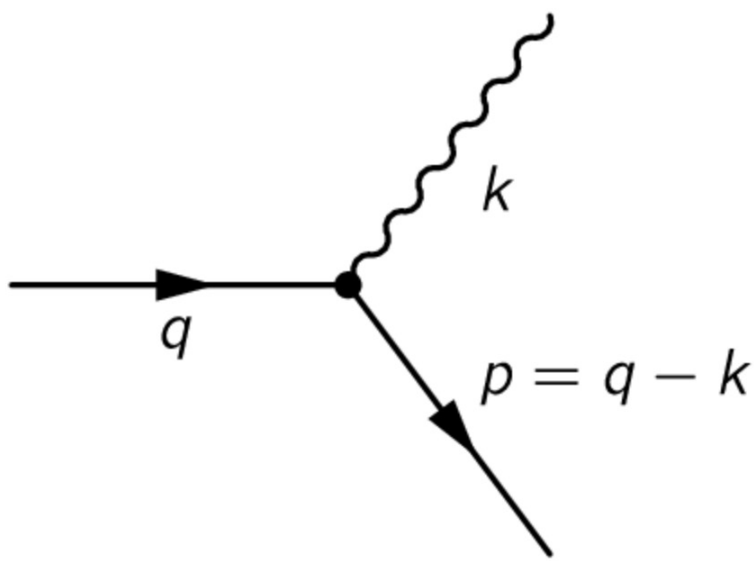

Figure 1: The Feynman Diagram gives contributions to the following processes: single electron scattering (similarly to the Compton Scattering).

For the QED interaction Lagrangian, $L_\mu = -g\psi^+(x)\gamma^0\gamma^\mu\psi(x)qA_\mu$. Therefore, according to the Dyson's formula, the S-matrix expands into perturbation series in the powers of the interaction Lagrangian L_μ. And the second order perturbation term in the S-matrix is two parts contributed by the interaction Lagrangian. We could write down the Interaction Dirac Hamiltonian as,

$$-2H^{(2)}_I = j^\mu qA_\mu j^\nu qA_\nu = \psi^+(x)\gamma^0\gamma^\mu\psi(x)qA_\mu \times \psi^+(x')\gamma^0\gamma^\nu\psi(x')qA_\nu = H_{1I}H_{2I} \tag{12}$$

Then we insert Eq. 1, Eq. 53, and Eq. 12 into Eq. 10. Similar to Eq. 3 the 1st order perturbation theory gives the matrix elements.

$$M = \frac{-1}{2\hbar}\int d^3xdt(\alpha\langle p_{f1},p_{f2}\big|_{\downarrow\uparrow}\langle k_1,k_2\big|_{\uparrow\downarrow}+\beta\langle p_{f1},p_{f2}\big|_{\uparrow\downarrow}\langle k_1,k_2\big|_{\downarrow\uparrow})H_I^{(2)} \times(\alpha\big|p_{i1},p_{i2}\rangle_{\uparrow\downarrow}\big|00\rangle+\beta\big|p_{i1},p_{i2}\rangle_{\downarrow\uparrow}\big|00\rangle) \tag{13}$$

After simplification, we get,

$$M = \frac{-1}{2\hbar}\int d^3xdt(\langle H_{1I}\rangle_{flip}\langle H_{2I}\rangle_{flip}+2\alpha\beta\langle H_{1I}\rangle_{noflip}\langle H_{2I}\rangle_{noflip}) \tag{14}$$

From the Eq. 4 and Eq. 5 we get the total radiation ratio represented by SP.

$$\begin{aligned}\Gamma = (\Gamma_{rd,\uparrow\downarrow}+\Gamma_{az,\uparrow\downarrow})_1(\Gamma_{rd,\uparrow\downarrow}+\Gamma_{az,\uparrow\downarrow})_2+4\alpha^2\beta^2(\Gamma_{rd,\uparrow\uparrow}+\Gamma_{az,\uparrow\uparrow})_1(\Gamma_{rd,\uparrow\uparrow}+\Gamma_{az,\uparrow\uparrow})_2 \\ +2\alpha\beta(\int(SP^*_{radial,\uparrow\downarrow}+SP^*_{azimuthal,\uparrow\downarrow})(SP_{radial,\uparrow\uparrow}+SP_{azimuthal,\uparrow\uparrow}))_1 \\ \times(\int(SP^*_{radial,\uparrow\downarrow}+SP^*_{azimuthal,\uparrow\downarrow})(SP_{radial,\uparrow\uparrow}+SP_{azimuthal,\uparrow\uparrow}))_2 \\ +H.C.\end{aligned} \tag{15}$$

Again, following the second order Dyson's expansion, we could write down the fourth order of the Interaction Hamiltonian based on the S-matrix expansion.

$$H_I^{(4)} = \frac{1}{4!}(j^\mu qA_\mu)^2(j^\nu qA_\nu)^2 = \frac{1}{4!}(\psi^+(x)\gamma^0\gamma^\mu\psi(x)qA_\mu)^2\times(\psi^+(x')\gamma^0\gamma^\nu\psi(x')qA_\nu)^2 = \frac{1}{4!}(H_{1I})^2(H_{2I})^2 \tag{16}$$

Along with the Eq. 11, we could get the 2nd order matrix elements.

$$M = \frac{1}{4!\hbar}\int d^3xdt(\alpha\langle p_{f1},p_{f2}\big|_{\downarrow\uparrow}\langle k_1,k_2\big|_{\uparrow\downarrow}+\beta\langle p_{f1},p_{f2}\big|_{\uparrow\downarrow}\langle k_1,k_2\big|_{\downarrow\uparrow})H_I^{(4)} \times(\alpha\big|p_{i1},p_{i2}\rangle_{\uparrow\downarrow}\big|00\rangle+\beta\big|p_{i1},p_{i2}\rangle_{\downarrow\uparrow}\big|00\rangle) \tag{17}$$

After simplification, we get

$$M = \frac{1}{4!\hbar}\int d^3xdt(\langle H_{1I}^2H_{2I}^2\rangle_{flip}+2\alpha\beta\langle H_{1I}^2H_{2I}^2\rangle_{noflip}) \tag{18}$$

2.1 Spin Flip Part A_1

Here we focus on the first component $\langle p_{f1}p_{f2}\big|_{\downarrow\uparrow}\langle k_1k_2\big|_{\uparrow\downarrow}(H_{1I})^2(H_{2I})^2\big|p_{i1}p_{i2}\big|\rangle_{\uparrow\downarrow}\big|00\rangle$. Let's insert the intermediate terms with one of electrons' spin-flip and photon emission.

$$\langle p_{f1}p_{f2}\big|_{\downarrow\uparrow}\langle k_1k_2\big|_{\uparrow\downarrow}(H_{1I}H_{2I})^2\big|p_{i1}p_{i2}\big|\rangle_{\uparrow\downarrow}\big|00\rangle = \sum_M\frac{\langle p_{f1}p_{f2}\big|_{\downarrow\uparrow}\langle k_1k_2\big|_{\uparrow\downarrow}H_{1I}H_{2I}\big|M\rangle\langle M\big|H_{1I}H_{2I}\big|p_{i1}p_{i2}\rangle_{\uparrow\downarrow}\big|00\rangle}{(E_{i1}-E_{M1}+i\sigma_1)(E_{i2}-E_{M2}+i\sigma_2)} \tag{19}$$

$$\big|M\rangle = \big|p_{f1}p_{i2}\rangle_{\downarrow\downarrow}\big|k_10\rangle_{\uparrow}\&\big|p_{i1}p_{f2}\rangle_{\uparrow\uparrow}\big|0k_2\rangle_{\downarrow} \tag{20}$$

We calculate each electron respectively. Let's take the first matrix element in Eq. 19.

$$\begin{aligned} A_1 &= \sum_M \frac{\langle p_{f1}p_{f2}|_{\downarrow\uparrow}\langle k_1k_2|_{\uparrow\downarrow}H_{1I}H_{2I}|M\rangle\langle M|H_{1I}H_{2I}|p_{i1}p_{i2}\rangle_{\uparrow\downarrow}|00\rangle}{(E_{i1}-E_{M1}+i\sigma_1)(E_{i2}-E_{M2}+i\sigma_2)} \\ &= \frac{\langle p_{f1}p_{f2}|_{\downarrow\uparrow}\langle k_1k_2|_{\uparrow\downarrow}H_{1I}H_{2I}|p_{f1}p_{i2}\rangle_{\downarrow\downarrow}|k_10\rangle_{\uparrow}\langle p_{f1}p_{i2}|_{\downarrow\downarrow}\langle k_10|_{\uparrow}H_{1I}H_{2I}|p_{i1}p_{i2}\rangle_{\uparrow\downarrow}|00\rangle}{(E_{i1}-E_{f1}+i\sigma_1)(E_{i2}-E_{i2}+i\sigma_2)} \\ &+ \frac{\langle p_{f1}p_{f2}|_{\downarrow\uparrow}\langle k_1k_2|_{\uparrow\downarrow}H_{1I}H_{2I}|p_{i1}p_{f2}\rangle_{\uparrow\uparrow}|0k_2\rangle_{\downarrow}\langle p_{i1}p_{f2}|_{\uparrow\uparrow}\langle 0k_2|_{\uparrow}H_{1I}H_{2I}|p_{i1}p_{i2}\rangle_{\uparrow\downarrow}|00\rangle}{(E_{i1}-E_{i1}+i\sigma_1)(E_{i2}-E_{f2}+i\sigma_2)} \end{aligned} \tag{21}$$

To see the equation more clearly, we define $|i_1\rangle$, $|i_2\rangle$, $|f_1\rangle$, and $|f_2\rangle$ instead. Thus, Eq. 19 becomes

$$A_1 = \frac{\langle f_1|H_{1I}|f_1\rangle\langle f_2|H_{2I}|i_2\rangle\langle f_1|H_{1I}|i_1\rangle\langle i_2|H_{2I}|i_2\rangle}{(E_{i1}-E_{f1}+i\sigma_1)i\sigma_2} + \frac{\langle f_1|H_{1I}|i_1\rangle\langle f_2|H_{2I}|f_2\rangle\langle i_1|H_{1I}|i_1\rangle\langle f_2|H_{2I}|i_2\rangle}{(E_{i2}-E_{f2}+i\sigma_2)i\sigma_1} \tag{22}$$

2.2 Spin Non-flip Part A_2

Here we focus on the second component $\langle p_{f1}p_{f2}|_{\downarrow\uparrow}\langle k_1k_2|_{\uparrow\downarrow}(H_{1I})^2(H_{2I})^2|p_{i1}p_{i2}|\rangle_{\downarrow\uparrow}|00\rangle$. Let's insert the intermediate terms with one of electrons' spin-flip and photon emission.

$$\langle p_{f1}p_{f2}|_{\downarrow\uparrow}\langle k_1k_2|_{\uparrow\downarrow}(H_{1I}H_{2I})^2|p_{i1}p_{i2}|\rangle_{\downarrow\uparrow}|00\rangle = \sum_M \frac{\langle p_{f1}p_{f2}|_{\downarrow\uparrow}\langle k_1k_2|_{\uparrow\downarrow}H_{1I}H_{2I}|M\rangle\langle M|H_{1I}H_{2I}|p_{i1}p_{i2}\rangle_{\downarrow\uparrow}|00\rangle}{(E_{i1}-E_{M1}+i\sigma_1)(E_{i2}-E_{M2}+i\sigma_2)} \tag{23}$$

$$|M\rangle = |p_{f1}p_{i2}\rangle_{\downarrow\downarrow}|k_10\rangle_{\uparrow}\&|p_{i1}p_{f2}\rangle_{\uparrow\uparrow}|0k_2\rangle_{\downarrow} \tag{24}$$

$$\begin{aligned} A_2 &= \sum_M \frac{\langle p_{f1}p_{f2}|_{\downarrow\uparrow}\langle k_1k_2|_{\uparrow\downarrow}H_{1I}H_{2I}|M\rangle\langle M|H_{1I}H_{2I}|p_{i1}p_{i2}\rangle_{\downarrow\uparrow}|00\rangle}{(E_{i1}-E_{M1}+i\sigma_1)(E_{i2}-E_{M2}+i\sigma_2)} \\ &= \frac{\langle p_{f1}p_{f2}|_{\downarrow\uparrow}\langle k_1k_2|_{\uparrow\downarrow}H_{1I}H_{2I}|p_{f1}p_{i2}\rangle_{\downarrow\downarrow}|k_10\rangle_{\uparrow}\langle p_{f1}p_{i2}|_{\downarrow\downarrow}\langle k_10|_{\uparrow}H_{1I}H_{2I}|p_{i1}p_{i2}\rangle_{\downarrow\uparrow}|00\rangle}{(E_{i1}-E_{f1}+i\sigma_1)(E_{i2}-E_{i2}+i\sigma_2)} \\ &+ \frac{\langle p_{f1}p_{f2}|_{\downarrow\uparrow}\langle k_1k_2|_{\uparrow\downarrow}H_{1I}H_{2I}|p_{i1}p_{f2}\rangle_{\uparrow\uparrow}|0k_2\rangle_{\downarrow}\langle p_{i1}p_{f2}|_{\uparrow\uparrow}\langle 0k_2|_{\uparrow}H_{1I}H_{2I}|p_{i1}p_{i2}\rangle_{\downarrow\uparrow}|00\rangle}{(E_{i1}-E_{i1}+i\sigma_1)(E_{i2}-E_{f2}+i\sigma_2)} \end{aligned} \tag{25}$$

To see the equation more clearly, we define $|i_1\rangle$, $|i_2\rangle$, $|f_1\rangle$, and $|f_2\rangle$ instead. Moreover, we add $flip$ and $non-flip$ on these notations if their spin flip while still staying on initial or final state. Thus, Eq. 19 becomes

$$\begin{aligned} A_2 = &\frac{\langle f_1|H_{1I}|f_1\rangle\langle f_2|H_{2I}|i_2\rangle\langle f_1|H_{1I}|i_1\rangle_{non}\langle i_2|H_{2I}|i_2\rangle_{flip}}{(E_{i1}-E_{f1}+i\sigma_1)i\sigma_2} \\ &+ \frac{\langle f_1|H_{1I}|i_1\rangle\langle f_2|H_{2I}|f_2\rangle\langle i_1|H_{1I}|i_1\rangle_{flip}\langle f_2|H_{2I}|i_2\rangle_{non}}{(E_{i2}-E_{f2}+i\sigma_2)i\sigma_1} \end{aligned} \tag{26}$$

Now we have to consider the eigenvalue of initial and final states (with no spin flip within the same state). According to Eq. 6-9

$$SP^{ii}_{azimuthal} = -2cp_{ir}sin(\phi_{ph}-\phi_i) \tag{27}$$

$$SP^{ii}_{radial} = 2cp_{iz}sin(\theta_{ph}) - 2cp_{ir}cos(\phi_{ph} - \phi_i)cos(\theta_{ph}) \tag{28}$$

$$SP^{ff}_{azimuthal} = -2cp_{fr}sin(\phi_{ph} - \phi_f) \tag{29}$$

$$SP^{ff}_{radial} = 2cp_{fz}sin(\theta_{ph}) - 2cp_{fr}cos(\phi_{ph} - \phi_f)cos(\theta_{ph}) \tag{30}$$

On the other hand, we calculate the eigenvalue of initial and final states (with spin flip within the same state).

$$(SP^{ii}_{radial})_{flip} = 0 \tag{31}$$

$$(SP^{ii}_{azimuthal})_{flip} = 0 \tag{32}$$

Thus, A_2 term would be omitted. And the matrix elements can be simply expressed by A_1.

$$M = \frac{1}{4!\hbar}\int d^3xdt(\left\langle H^2_{1I}H^2_{2I}\right\rangle_{flip} = \frac{1}{4!\hbar}\int d^3xdtA_1 \tag{33}$$

3 Spin Polarization term

To begin with, we have to use some relations to simplify the SP terms.

$$\begin{aligned} cp_{ir} &= \beta E_i sin(\theta_i) \\ cp_{iz} &= \beta E_i cos(\theta_i) \\ cp_{fr} &= \beta E_i sin(\theta_i)e^{-i(\phi_i-\phi_f)} - n\hbar\omega sin(\theta_{ph})e^{-i(\phi_{ph}-\phi_f)} \\ cp_{fz} &= \beta E_i cos(\theta_i) - n\hbar\omega cos(\theta_{ph}) \\ cos(\phi_{ph} - \phi_i) &= \frac{cos(\theta_{CR}) - cos(\theta_i)cos(\theta_{ph})}{sin(\theta_i)sin(\theta_{ph})} \end{aligned} \tag{34}$$

Therefore, for initial-final state transfer parts,

$$SP^2_{azimuthal,\uparrow\downarrow} = \frac{(\hbar\omega)^2}{(E_i + mc^2)(E_f + mc^2)}(\beta E_i cos(\theta_i) - n(E_i + mc^2)cos(\theta_{ph}))^2 \tag{35}$$

$$\begin{aligned} SP^2_{radial,\uparrow\downarrow} = \frac{(\hbar\omega)^2}{(E_i + mc^2)(E_f + mc^2)}\{(\beta E_i)^2[sin^2(\theta_1)sin^2(\theta_{CR}) - (cos(\theta_{ph}) - cos(\theta_i)cos(\theta_{CR}))^2] \\ +[\beta E_i cos(\theta_{CR}) - n(E_i + mc^2)]^2\} \end{aligned} \tag{36}$$

Adding up the radial and azimuthal SP term, we get

$$\begin{aligned} SP^2_{\uparrow\downarrow} = \frac{(\hbar\omega)^2}{(E_i + mc^2)(E_f + mc^2)}\{n^2(E_i + mc^2)^2[1 + cos^2(\theta_{ph})] \\ -2n\beta E_i(E_i + mc^2)[cos(\theta_i)cos(\theta_{ph}) + cos(\theta_{CR})] \\ +\beta^2 E_i^2[1 - cos^2(\theta_{ph}) + 2cos(\theta_{ph})cos(\theta_i)cos(\theta_{CR})]\} \end{aligned} \tag{37}$$

Secondly, we calculate those initial-initial parts and final-final parts (non-spin flip). Similarly, we get

$$SP^{ii} = 2\beta E_i sin(\theta_{CR}) \tag{38}$$

$$SP^{ff} = 2\beta' E_f sin(\theta'_{CR}) \tag{39}$$

where $\beta' = \frac{v_f}{c}$, and θ'_{CR} is the QCR spread angle of the same photon emission by the states. Specifically, we can write down these two QCR spread angles to compare with each other.

$$\theta_{CR} = arccos(\frac{1}{n\beta} + \frac{\omega}{\omega_C}\frac{\sqrt{1-\beta^2}}{\beta}\frac{n^2-1}{2n}) \tag{40}$$

$$\theta'_{CR} = arccos(\frac{1}{n\beta'} + \frac{\omega}{\omega_C}\frac{\sqrt{1-\beta'^2}}{\beta'}\frac{n^2-1}{2n}) \tag{41}$$

Importantly, we notice that, in the non-spin flip initial-initial and final-final situation, the SP terms do not have θ_{ph}. In the result, we could take them out directly from the θ_{ph} integration in the next section.

4 Total Emission Rate with the 2nd Order Perturbation

Based on the calculations above, we could write down the total emission rate with regard to M that contains SP term.

$$\begin{aligned}\Gamma &= C * \int d^3k d^3p \left|M\right|^2 = C * \int d^3k d^3p (\left|M_{azimuthal}\right|^2 + \left|M_{radial}\right|^2) \\ &= \frac{\alpha}{\pi\beta}\int sin(\theta_{ph}) d\theta_{ph} \frac{\left|[SP]\right|^2}{4E_i^2\sqrt{(sin(\theta_i)sin(\theta_{CR}))^2 - (cos(\theta_{ph}) - cos(\theta_i)cos(\theta_{CR}))^2}}\end{aligned} \tag{42}$$

According to A_1,

$$\begin{aligned}A_1^2 = &\frac{\langle f_1|H_{1I}|f_1\rangle^2\langle f_2|H_{2I}|i_2\rangle^2\langle f_1|H_{1I}|i_1\rangle^2\langle i_2|H_{2I}|i_2\rangle^2}{(E_{i1}-E_{f1}+i\sigma_1)^2\sigma_2^2} + \frac{\langle f_1|H_{1I}|i_1\rangle^2\langle f_2|H_{2I}|f_2\rangle^2\langle i_1|H_{1I}|i_1\rangle^2\langle f_2|H_{2I}|i_2\rangle^2}{(E_{i2}-E_{f2}+i\sigma_2)^2\sigma_1^2} \\ &+2\frac{\langle f_1|H_{1I}|f_1\rangle\langle f_2|H_{2I}|f_2\rangle\langle f_2|H_{2I}|i_2\rangle^2\langle f_1|H_{1I}|i_1\rangle^2\langle i_1|H_{1I}|i_1\rangle\langle i_2|H_{2I}|i_2\rangle}{(E_{i1}-E_{f1}+i\sigma_1)\sigma_2(E_{i2}-E_{f2}+i\sigma_2)\sigma_1}\end{aligned} \tag{43}$$

Substituting spin-polarization term in to A_1^2, we get the 2nd order total emission rate.

$$\begin{aligned}\Gamma &= \frac{\alpha}{\pi\beta}\int sin(\theta_{ph}) d\theta_{ph} \frac{\left|[SP]\right|^2}{4E_i^2\sqrt{(sin(\theta_i)sin(\theta_{CR}))^2 - (cos(\theta_{ph}) - cos(\theta_i)cos(\theta_{CR}))^2}} \\ &= C\int (***)\frac{(SP^{ff})_1^2(SP^{if})_1^2(SP^{ii})_2^2(SP^{if})_2^2}{(E_{i1}-E_{f1}+i\sigma_1)^2\sigma_2^2} + \frac{(SP^{ii})_1^2(SP^{if})_1^2(SP^{ff})_2^2(SP^{if})_2^2}{(E_{i2}-E_{f2}+i\sigma_2)^2\sigma_1^2} \\ &\quad +2\frac{(SP^{if})_1^2(SP^{if})_2^2(SP^{ff})_1(SP^{ff})_2(SP^{ii})_1(SP^{ii})_2}{(E_{i1}-E_{f1}+i\sigma_1)\sigma_2(E_{i2}-E_{f2}+i\sigma_2)\sigma_1}\end{aligned} \tag{44}$$

From the section above, we know that we could take ii and ff term out directly from the θ_{ph} integration because they do not have θ_{ph} term.

$$\begin{aligned}
\Gamma = \frac{\alpha}{\pi\beta}\int sin(\theta_{ph})d\theta_{ph}\frac{\left|[SP]\right|^2}{4E_i^2\sqrt{(sin(\theta_i)sin(\theta_{CR}))^2-(cos(\theta_{ph})-cos(\theta_i)cos(\theta_{CR}))^2}} \\
= C\times(SP^{ff})_1^2(SP^{ii})_2^2\frac{\int_1(*1*)(SP^{if})_1^2\int_2(*2*)(SP^{if})_2^2}{(E_{i1}-E_{f1}+i\sigma_1)^2\sigma_2^2} \\
+C\times(SP^{ii})_1^2(SP^{ff})_2^2\frac{\int_1(*1*)(SP^{if})_1^2\int_2(*2*)(SP^{if})_2^2}{(E_{i2}-E_{f2}+i\sigma_2)^2\sigma_1^2} \\
+2C\times(SP^{ff})_1(SP^{ff})_2(SP^{ii})_1(SP^{ii})_2\frac{\int_1(*1*)(SP^{if})_1^2\int_2(*2*)(SP^{if})_2^2}{(E_{i1}-E_{f1}+i\sigma_1)\sigma_2(E_{i2}-E_{f2}+i\sigma_2)\sigma_1}
\end{aligned} \tag{45}$$

From the supplementary of PRX, we know

$$\begin{aligned}
\Gamma_{\omega,\uparrow\downarrow} = \Gamma_{\omega,azimuthal,\uparrow\downarrow}+\Gamma_{\omega,radial,\uparrow\downarrow} \\
= \frac{\alpha}{8\beta}(\frac{\hbar\omega}{E_i})^2\{\frac{(\beta E_i)^2[1+cos^2(\theta_i)][1+cos^2(\theta_{CR})]}{(E_i+mc^2)(E_f+mc^2)} \\
-\frac{4n\beta E_i cos(\theta_{CR})}{E_f+mc^2}[1+cos^2(\theta_i)] \\
+n^2\frac{E_i+mc^2}{E_f+mc^2}[2+2cos^2(\theta_i)cos^2(\theta_{CR})+sin^2(\theta_i)sin^2(\theta_{CR})]\}
\end{aligned} \tag{46}$$

So the final expression of the total emission ratio is

$$\begin{aligned}
\Gamma = [\Gamma_{\omega_1,\uparrow\downarrow}]_1[\Gamma_{\omega_2,\uparrow\downarrow}]_2\frac{(SP^{ff})_1^2(SP^{ii})_2^2}{(E_{i1}-E_{f1}+i\sigma_1)^2\sigma_2^2} \\
+[\Gamma_{\omega_1,\uparrow\downarrow}]_1[\Gamma_{\omega_2,\uparrow\downarrow}]_2\frac{(SP^{ii})_1^2(SP^{ff})_2^2}{(E_{i2}-E_{f2}+i\sigma_2)^2\sigma_1^2} \\
+2[\Gamma_{\omega_1,\uparrow\downarrow}]_1[\Gamma_{\omega_2,\uparrow\downarrow}]_2\frac{(SP^{ff})_1(SP^{ff})_2(SP^{ii})_1(SP^{ii})_2}{(E_{i1}-E_{f1}+i\sigma_1)\sigma_2(E_{i2}-E_{f2}+i\sigma_2)\sigma_1}
\end{aligned} \tag{47}$$

5 Calculation based on Werner States

5.1 Werner State Vectors

As we known, the Werner state expression defines the summation of entangled states and non-entangled states with the probability of entanglement. Explicitly, if we assume that the two qubits are intialized in one of Bell states, $\left|\psi\right\rangle_{entangle} = \frac{1}{\sqrt{2}}(\left|\uparrow\downarrow\right\rangle - \left|\downarrow\uparrow\right\rangle)$ with probability p, or in $\left|\psi\right\rangle_{nonentangle} = \frac{1}{2}(\left|\uparrow 0\right\rangle + \left|\downarrow 0\right\rangle)(\left|0\uparrow\right\rangle + \left|0\downarrow\right\rangle)$, then the Werner state can be expressed by

$$\rho_W = (1-p)\left|\psi\right\rangle_{nonentangle}\left\langle\psi\right|_{nonentangle} + p\left|\psi\right\rangle_{entangle}\left\langle\psi\right|_{entangle} \tag{48}$$

In our situation, the entangled initial state is expressed with

$$\left|\psi_i\right\rangle = \frac{1}{\sqrt{2}}(\left|p_{i1},p_{i2}\right\rangle_{\uparrow\downarrow}\left|00\right\rangle + \left|p_{i1},p_{i2}\right\rangle_{\downarrow\uparrow}\left|00\right\rangle) \tag{49}$$

And the non-entangled initial state

$$\begin{aligned}\left|\psi_i\right\rangle_{no} &= \frac{1}{2}(\left|p_{i1},0\right\rangle_{\uparrow 0}+\left|p_{i1},0\right\rangle_{\downarrow 0})(\left|0,p_{i2}\right\rangle_{0\uparrow}+\left|0,p_{i2}\right\rangle_{0\downarrow})\left|00\right\rangle \\ &= \frac{1}{2}(\left|p_{i1},p_{i2}\right\rangle_{\uparrow\uparrow}+\left|p_{i1},p_{i2}\right\rangle_{\uparrow\downarrow}+\left|p_{i1},p_{i2}\right\rangle_{\downarrow\uparrow}+\left|p_{i1},p_{i2}\right\rangle_{\downarrow\downarrow})\left|00\right\rangle\end{aligned} \tag{50}$$

So the Werner state's density matrix can be expressed by

$$\rho_W = (1-p)\left|\psi_i\right\rangle_{no}\left\langle\psi_i\right|_{no} + p\left|\psi_i\right\rangle\left\langle\psi_i\right| \tag{51}$$

Projecting this density matrix on Bell basis,

$$\begin{aligned}\left|\psi_i\right\rangle_W &= [(1-p)\left|\psi_i\right\rangle_{no}\left\langle\psi_i\right|_{no} + p\left|\psi_i\right\rangle\left\langle\psi_i\right|]\left|\psi_i\right\rangle \\ &= (1-p)\left|\psi_i\right\rangle_{no}\left\langle\psi_i\right|_{no}\left|\psi_i\right\rangle + p\left|\psi_i\right\rangle \\ &= \frac{1}{\sqrt{2}}(1-p)\left|\psi_i\right\rangle_{no} + p\left|\psi_i\right\rangle\end{aligned} \tag{52}$$

Similarly, the final state can also be expressed as above.

$$\left|\psi_f\right\rangle = \frac{1}{\sqrt{2}}(\left|p_{f1},p_{f2}\right\rangle_{\downarrow\uparrow}\left|k_1,k_2\right\rangle_{\uparrow\downarrow}+\left|p_{f1},p_{f2}\right\rangle_{\uparrow\downarrow}\left|k_1,k_2\right\rangle_{\downarrow\uparrow}) \tag{53}$$

$$\begin{aligned}\left|\psi_f\right\rangle_{no} &= \frac{1}{2}(\left|p_{f1},0\right\rangle_{\uparrow 0}\left|k_1,0\right\rangle_{\downarrow 0}+\left|p_{f1},0\right\rangle_{\downarrow 0}\left|k_1,0\right\rangle_{\uparrow 0})(\left|0,p_{f2}\right\rangle_{0\uparrow}\left|0,k_2\right\rangle_{0\downarrow}+\left|0,p_{f2}\right\rangle_{0\downarrow}\left|0,k_2\right\rangle_{0\uparrow}) \\ &= \frac{1}{2}(\left|p_{f1},p_{f2}\right\rangle_{\uparrow\uparrow}\left|k_1,k_2\right\rangle_{\downarrow\downarrow}+\left|p_{f1},p_{f2}\right\rangle_{\uparrow\downarrow}\left|k_1,k_2\right\rangle_{\downarrow\uparrow}+\left|p_{f1},p_{f2}\right\rangle_{\downarrow\uparrow}\left|k_1,k_2\right\rangle_{\uparrow\downarrow}+\left|p_{f1},p_{f2}\right\rangle_{\downarrow\downarrow}\left|k_1,k_2\right\rangle_{\uparrow\uparrow})\end{aligned} \tag{54}$$

$$\left|\psi_f\right\rangle_W = \frac{1}{\sqrt{2}}(1-q)\left|\psi_f\right\rangle_{no} + q\left|\psi_f\right\rangle \tag{55}$$

For the 2-nd perturbation, we also have to write down the intermediate states. First of all, considering the first electron emits one photon and flip its spin while the second electron does not change its state.

$$\left|\psi_m\right\rangle^1 = \frac{1}{\sqrt{2}}(\left|p_{f1},p_{i2}\right\rangle_{\downarrow\downarrow}\left|k_1,0\right\rangle_{\uparrow 0}+\left|p_{f1},p_{i2}\right\rangle_{\uparrow\uparrow}\left|k_1,0\right\rangle_{\downarrow 0}) \tag{56}$$

$$\begin{aligned}\left|\psi_m\right\rangle^1_{no} &= \frac{1}{2}(\left|p_{f1},0\right\rangle_{\uparrow 0}\left|k_1,0\right\rangle_{\downarrow 0}+\left|p_{f1},0\right\rangle_{\downarrow 0}\left|k_1,0\right\rangle_{\uparrow 0})(\left|0,p_{i2}\right\rangle_{0\uparrow}\left|00\right\rangle+\left|0,p_{i2}\right\rangle_{0\downarrow}\left|00\right\rangle) \\ &= \frac{1}{2}(\left|p_{f1},p_{i2}\right\rangle_{\uparrow\uparrow}\left|k_1,0\right\rangle_{\downarrow 0}+\left|p_{f1},p_{i2}\right\rangle_{\uparrow\downarrow}\left|k_1,0\right\rangle_{\downarrow 0}+\left|p_{f1},p_{i2}\right\rangle_{\downarrow\uparrow}\left|k_1,0\right\rangle_{\uparrow 0}+\left|p_{f1},p_{i2}\right\rangle_{\downarrow\downarrow}\left|k_1,0\right\rangle_{\uparrow 0})\end{aligned} \tag{57}$$

$$\left|\psi_m\right\rangle^1_W = \frac{1}{\sqrt{2}}(1-a)\left|\psi_m\right\rangle^1_{no} + a\left|\psi_m\right\rangle^1 \tag{58}$$

Secondly, we consider another situation instead.

$$\left|\psi_m\right\rangle^2 = \frac{1}{\sqrt{2}}(\left|p_{i1},p_{f2}\right\rangle_{\uparrow\uparrow}\left|0,k_2\right\rangle_{0\downarrow}+\left|p_{i1},p_{f2}\right\rangle_{\downarrow\downarrow}\left|0,k_2\right\rangle_{0\uparrow}) \tag{59}$$

$$\begin{aligned}\left|\psi_m\right\rangle^2_{no} &= \frac{1}{2}(\left|p_{i1},0\right\rangle_{\uparrow 0}\left|00\right\rangle+\left|p_{i1},0\right\rangle_{\downarrow 0}\left|00\right\rangle)(\left|0,p_{f2}\right\rangle_{0\uparrow}\left|0k_2\right\rangle_{0\downarrow}+\left|0,p_{f2}\right\rangle_{0\downarrow}\left|0,k_2\right\rangle_{0\uparrow}) \\ &= \frac{1}{2}(\left|p_{i1},p_{f2}\right\rangle_{\uparrow\uparrow}\left|0,k_2\right\rangle_{0\downarrow}+\left|p_{i1},p_{f2}\right\rangle_{\uparrow\downarrow}\left|0,k_2\right\rangle_{0\uparrow}+\left|p_{i1},p_{f2}\right\rangle_{\downarrow\uparrow}\left|0,k_2\right\rangle_{0\downarrow}+\left|p_{i1},p_{f2}\right\rangle_{\downarrow\downarrow}\left|0,k_2\right\rangle_{0\uparrow})\end{aligned} \tag{60}$$

$$\left|\psi_m\right\rangle^2_W = \frac{1}{\sqrt{2}}(1-b)\left|\psi_m\right\rangle^2_{no} + b\left|\psi_m\right\rangle^2 \tag{61}$$

5.2 2nd-order Perturbation

$$\begin{aligned} M &= \frac{1}{4!\hbar}\int d^3xdt(\frac{1}{\sqrt{2}}(1-q)\langle\psi_f\big|_{no} + q\langle\psi_f\big|) \times H_I^{(4)} \times (\frac{1}{\sqrt{2}}(1-p)\big|\psi_i\rangle_{no} + p\big|\psi_i\rangle) \\ &= \frac{1}{4!\hbar}\int d^3xdt \sum_{l=1,2} \frac{\langle\psi_f\big|_W H_{1I}H_{2I}\big|\psi_m\rangle_W^l \langle\psi_m\big|_W^l H_{1I}H_{2I}\big|\psi_i\rangle_W}{(E_{i1}-E_{M1}+i\sigma_1)(E_{i2}-E_{M2}+i\sigma_2)} \end{aligned} \tag{62}$$

while $l=1$, and we represent matrix elements by spin-polarization terms.

$$\begin{aligned} \langle\psi_f\big|_W H_{1I}H_{2I}\big|\psi_m\rangle_W^1 \to \left[\frac{(1-p)(1-a)}{2} + (1-p)a + p(1-a)\right](SP_1^{ff})_{nonflip}[(SP_2^{if})_{flip} + (SP_2^{if})_{nonflip}] \\ + \frac{pa}{2}(SP_1^{ff})_{nonflip}(SP_2^{if})_{flip} \end{aligned} \tag{63}$$

$$\langle\psi_f\big|_W H_{1I}H_{2I}\big|\psi_m\rangle_W^1 \to (SP_1^{ff})_{nonflip}[C_1(SP_2^{if})_{flip} + C_2(SP_2^{if})_{nonflip}] \tag{64}$$

$$\langle\psi_m\big|_W H_{1I}H_{2I}\big|\psi_i\rangle_W^1 \to [D_1(SP_1^{if})_{flip} + D_2(SP_1^{if})_{nonflip}](SP_2^{ii})_{nonflip} \tag{65}$$

where C_1, C_2, D_1, D_2 are coefficient that replace entanglement probabilities. While $l=2$, and we represent matrix elements by spin-polarization terms.

$$\langle\psi_f\big|_W H_{1I}H_{2I}\big|\psi_m\rangle_W^2 \to [E_1(SP_1^{if})_{flip} + E_2(SP_1^{if})_{nonflip}](SP_2^{ff})_{nonflip} \tag{66}$$

$$\langle\psi_m\big|_W H_{1I}H_{2I}\big|\psi_i\rangle_W^2 \to (SP_1^{ii})_{nonflip}[F_1(SP_2^{if})_{flip} + F_2(SP_2^{if})_{nonflip}] \tag{67}$$

Substitute all the terms with spin-polarization in M.

$$\begin{aligned} M \to \int \frac{d^3xdt}{4!\hbar}\Bigg[&\frac{(SP_1^{ff})_{nonflip}(SP_2^{ii})_{nonflip}[C_1(SP_2^{if})_{flip} + C_2(SP_2^{if})_{nonflip}][D_1(SP_1^{if})_{flip} + D_2(SP_1^{if})_{nonflip}]}{(E_{i1}-E_{f1}+i\sigma_1)i\sigma_2} \\ &+ \frac{(SP_1^{ii})_{nonflip}(SP_2^{ff})_{nonflip}[F_1(SP_2^{if})_{flip} + F_2(SP_2^{if})_{nonflip}][E_1(SP_1^{if})_{flip} + E_2(SP_1^{if})_{nonflip}]}{i\sigma_1(E_{i2}-E_{f2}+i\sigma_2)}\Bigg] \end{aligned} \tag{68}$$

According to the relationship between M and total emission rate Γ, we could derive the result. (Too complicate, but without any difficulty with regard to method)

6 Graphics

6.1 Fig. 2 in PRX – Matrix Elements

In this section, we firstly rederive the Fig. 2 in PRX.

$$\begin{aligned} cp_{fr}e^{-i\phi_f} &= \beta E_i sin(\theta_i)e^{-i\phi_i} - n\hbar\omega sin(\theta_{ph})e^{-i\phi_{ph}} \\ cp_{fr}e^{i\phi_f} &= \beta E_i sin(\theta_i)e^{i\phi_i} - n\hbar\omega sin(\theta_{ph})e^{i\phi_{ph}} \\ (cp_{fr})^2 &= \beta^2E_i^2sin^2(\theta_i) + n^2\hbar^2\omega^2sin^2(\theta_{ph}) - 2n\hbar\omega\beta E_i[cos(\theta_{CR}) - cos(\theta_i)cos(\theta_{ph})] \end{aligned} \tag{69}$$

According to the relationships we have refered above, we have

$$(cp_{ir})^2 = \beta^2E_i^2sin^2(\theta_i) \tag{70}$$

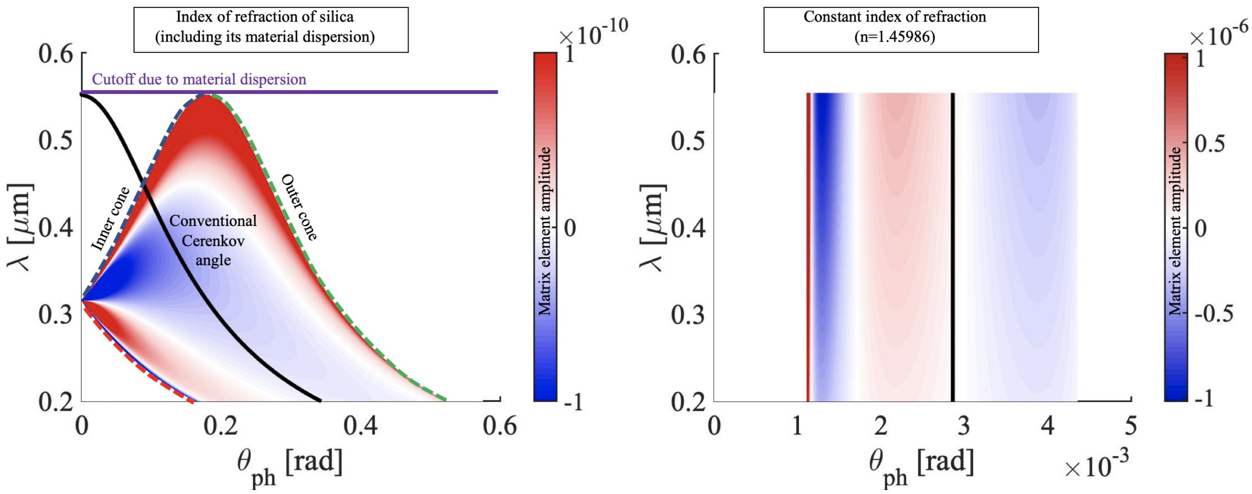

Figure 2: Matrix-element amplitudes for the ČR process, as a function of the photon wavelength λ and emission spread angle θ_{ph}. (a). Here we choose $\theta_i = 0.1798$, $l_i = 3$, $l_f = 0$, $l_{ph} = 4$, and the spin flip from $-\frac{1}{2}$ to $\frac{1}{2}$. The color map shows the spatial part of the matrix element that vanishes outside of the permitted zone, bounded by the blue, red, and green dashed curves. Along these curves, the amplitude diverges; thus, we use a saturated color scale, with darker colors corresponding to higher transition amplitudes. (b). $l_{ph} = 8$, $l_i = 7$, and $l_f = 0$
.

From the reference PRX, exactly quantifying the amplitudes requires solving a triple-Bessel integral over the cylindrical radius r, which was fortunately studied in the mathematical literature, providing us with a closed-form solution:

$$\int J_{l_i}(p_{ir}r/\hbar)J_{l_f}(p_{fr}r/\hbar)J_{l_{ph}+1=l_i-l_f}(k_r r)rdr = \frac{cos(l_i\alpha_f - l_f\alpha_i)}{2\pi S(p_{ir}/\hbar, p_{fr}/\hbar, k_r)} \tag{71}$$

where $S(p_{ir}/\hbar, p_{fr}/\hbar, k_r)$ is the area of a triangle with sides of lengths $p_{ir}/\hbar$, $p_{fr}/\hbar$, and k_r. where α_i, α_f, and α_{ph} are the angles opposite the three sides. If a triangle cannot be made, then the integral is zero, which gives another selection rule (though not a simple one) for the possible radiation emission.

$$cos(\alpha_f) = \frac{cos(\theta_{CR}) - cos(\theta_i)cos(\theta_{ph})}{sin(\theta_i)sin(\theta_{ph})} \tag{72}$$

$$cos(\alpha_i) = \frac{n\hbar\omega sin(\theta_{ph}) - \beta E_i sin(\theta_i)}{\sqrt{\beta^2 E_i^2 sin^2(\theta_i) + n^2\hbar^2\omega^2 sin^2(\theta_{ph}) - 2n\hbar\omega\beta E_i[cos(\theta_{CR}) - cos(\theta_i)cos(\theta_{ph})]}} \tag{73}$$

According to the Heron's Formula, the area of this triangle is

$$S(p_{ir}/\hbar, p_{fr}/\hbar, k_r) = \frac{\pi\beta E_i sin(\theta_i)sin(\theta_{ph})}{c\hbar\lambda}\sqrt{1 - \left[\frac{cos(\theta_{CR}) - cos(\theta_i)cos(\theta_{ph})}{sin(\theta_i)sin(\theta_{ph})}\right]^2} \tag{74}$$

The angular momentum conservation tells us that $l_i = l_f + l_{ph} \pm 1$. We assume that $l_i = 3$, $l_f = 0$, $l_{ph} = 4$, and the spin flip from $-\frac{1}{2}$ to $\frac{1}{2}$.

$$\frac{cos(l_i\alpha_f - l_f\alpha_i)}{2\pi S(p_{ir}/\hbar, p_{fr}/\hbar, k_r)} = \frac{c\hbar\lambda cos(3\alpha_f)}{2\pi^2\beta E_i sin(\theta_i)sin(\theta_{ph})\sqrt{1-\left[\frac{cos(\theta_{CR})-cos(\theta_i)cos(\theta_{ph})}{sin(\theta_i)sin(\theta_{ph})}\right]^2}}$$
$$= \frac{c\hbar\lambda\left[4(\frac{cos(\theta_{CR})-cos(\theta_i)cos(\theta_{ph})}{sin(\theta_i)sin(\theta_{ph})})^3 - 3\frac{cos(\theta_{CR})-cos(\theta_i)cos(\theta_{ph})}{sin(\theta_i)sin(\theta_{ph})}\right]}{2\pi^2\beta E_i sin(\theta_i)sin(\theta_{ph})\sqrt{1-\left[\frac{cos(\theta_{CR})-cos(\theta_i)cos(\theta_{ph})}{sin(\theta_i)sin(\theta_{ph})}\right]^2}} \quad (75)$$

Therefore, we can plot the Matrix-element amplitudes for the ČR process, as a function of the photon wavelength λ and emission spread angle θ_{ph}, Fig. 2. Plotting the amplitudes, exhibit preferred "stripes" of high amplitude at certain angles of emission θ_{ph}. Then, we try to rederive the second sub-figure in PRX's Fig. 2. In that situation, $l_{ph} = 8$. To simplify the spatial part of the matrix element, we may choose $l_i = 7$ and $l_f = 0$ so the spatial part can be written as

$$\frac{cos(l_i\alpha_f - l_f\alpha_i)}{2\pi S(p_{ir}/\hbar, p_{fr}/\hbar, k_r)} = \frac{c\hbar\lambda cos(7\alpha_f)}{2\pi^2\beta E_i sin(\theta_i)sin(\theta_{ph})sin(\alpha_f)} \quad (76)$$

We plot the Matrix-element amplitudes for the ČR process with $l_{ph} = 8$ and constant index of refraction ($n = 1.45986$) in Fig. 2. We can see that this figure is totally different from the figure in PRX. My own opinion is that while considering the lelf boundary, we just have to know the boundary of $cos(\alpha_f)$. To testify my result, we plot $\theta_{ph} - cos(\alpha_f)$, which, from the main amplitude results, should not be within -1 to 1 until θ_{ph} increases over $1 \times 10^{-3}[rad]$ (in Fig. 3).

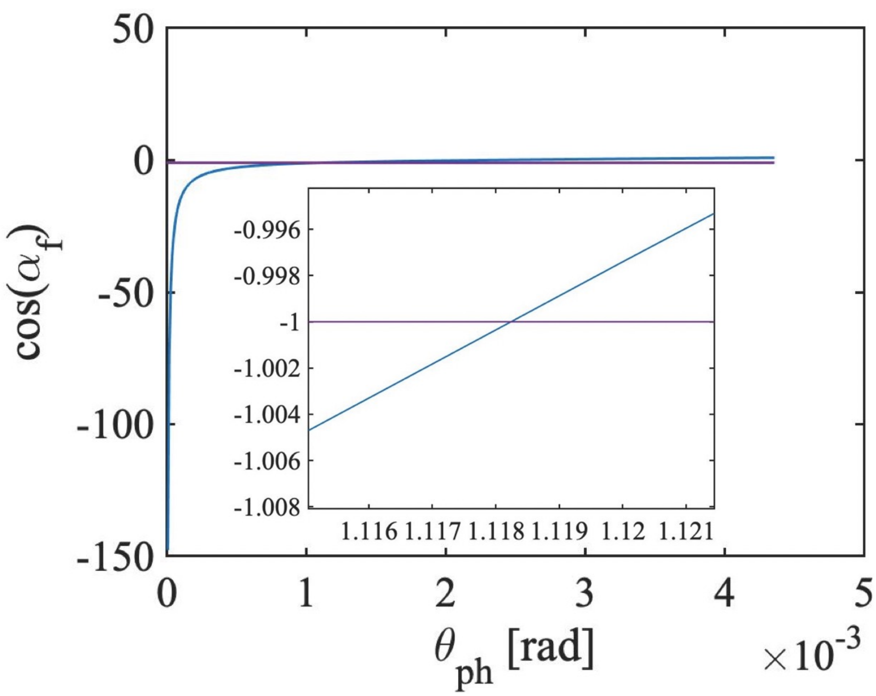

Figure 3: $cos(\alpha_f)$ could be below -1 due to its denominator $sin(\theta_{ph})$ term.

6.2 Fig. 2 in our paper – Matrix Elements

In the end of part 2, we notice that

$$M = \frac{1}{4!\hbar} \int d^3 x dt A_1 \tag{77}$$

and

$$A_1 = \frac{\langle f_1 | H_{1I} | f_1 \rangle \langle f_2 | H_{2I} | i_2 \rangle \langle f_1 | H_{1I} | i_1 \rangle \langle i_2 | H_{2I} | i_2 \rangle}{(E_{i1} - E_{f1} + i\sigma_1) i\sigma_2} + \frac{\langle f_1 | H_{1I} | i_1 \rangle \langle f_2 | H_{2I} | f_2 \rangle \langle i_1 | H_{1I} | i_1 \rangle \langle f_2 | H_{2I} | i_2 \rangle}{(E_{i2} - E_{f2} + i\sigma_2) i\sigma_1} \tag{78}$$

Therefore,

$$M \propto \tag{79}$$