GAUTENG PROVINCE
Department: Education
REPUBLIC OF SOUTH AFRICA

GRADE 12 MATHEMATICS

ANAYTICAL GEOMETRY MIND MAP

ANALYTICAL GEOMETRY

1. REVISION

   Distance
   Midpoint
   Gradient
   Straight line
   inclination

2. EQUATION OF A CIRCLE

   Circle centred at the origin
   Circle centred off the origin

3. EQUATION OF THE TANGENT TO A CIRCLE

SECTION A: NOTES ON GRADE 11 REVISION

DISTANCE BETWEEN TWO POINTS

The distance formula can be used to determine the length of a line segment between two points or the coordinates of a point when the length is known.

The formula to calculate the length of a line segment between two points $A(x_A ; y_A)$ and $B(x_B ; y_B)$ is given by the formula:

$AB^2 = (x_B - x_A)^2 + (y_B - y_A)^2$ or

$AB = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}$

MIDPOINT OF A LINE SEGMENT

The formula for point M, the midpoint of a line segment AB joining the points $A(x_A ; y_A)$ and $B(x_B ; y_B)$ is given by the formula:

$M(x_M ; y_M) = M\left(\dfrac{x_B + x_A}{2} ; \dfrac{y_B + y_A}{2}\right)$

GRADIENT OF A LINE

The gradient of a line between any two points on the line is the ratio:

$m = \dfrac{\text{change in } y\text{-values}}{\text{change in } x\text{-values}}$

A formula to calculate the gradient of a line joining two points $A(x_A ; y_A)$ and $B(x_B ; y_B)$ is given by the formula:

The gradient of line AB: $m_{AB} = \dfrac{y_B - y_A}{x_B - x_A}$

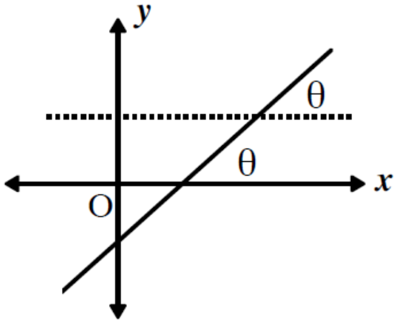
COLLINEAR POINTS

Points that are **collinear** lie on the same line. The gradient between each pair of points is the same. For example, if the points A, B and C are collinear, then:

$\text{Gradient}_{AB} = \text{Gradient}_{BC} = \text{Gradient}_{AC}$

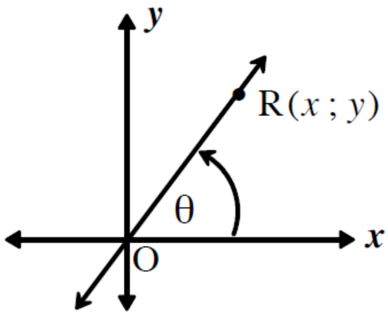
INCLINATION OF A LINE

The inclination of a line is the angle formed with the horizontal in an anti-clockwise direction. On the Cartesian plane, the inclination of a line is calculated by finding the **angle formed at the x-axis** measured in anti-clockwise direction. θ is the angle of inclination of line AB.



Formula for finding the angle of inclination of a line

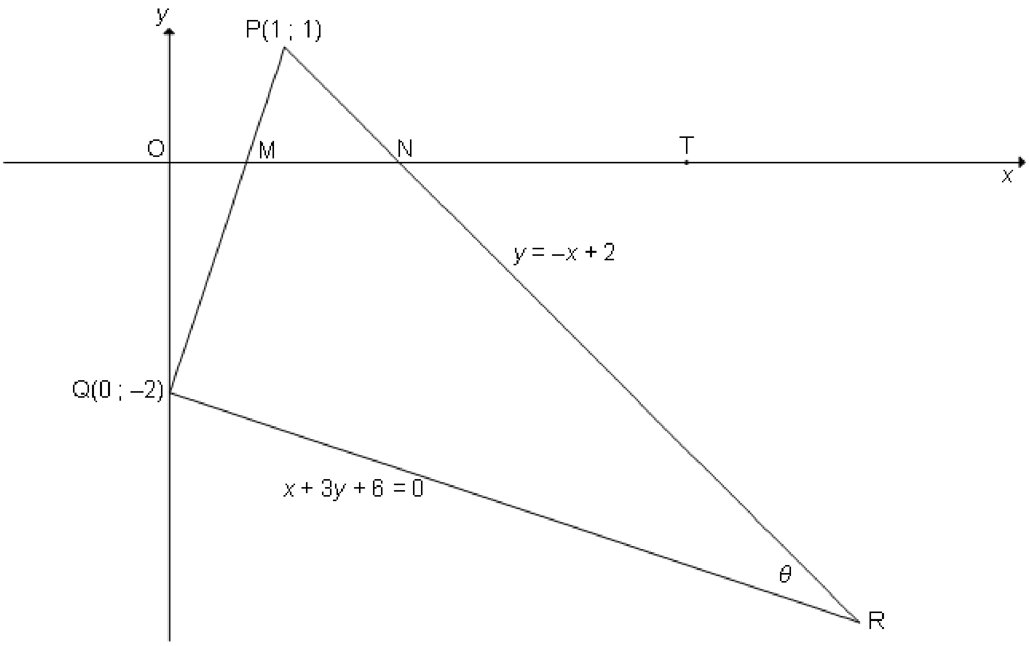
If $R(x;y)$ is a point on the terminal arm of θ, then by definition, $\tan\theta = \frac{y}{x}$. But with $O(0;0)$, the gradient of line OR $= \frac{y-0}{x-0} = \frac{y}{x}$.



$$\boxed{\tan\theta = \text{Gradient}_{OR}\text{, where } \theta \text{ is the angle of inclination of line OR.}}$$

EXAMPLE 1

In the diagram below, P(1 ; 1), Q(0 ; –2) and R are the vertices of a triangle and $P\hat{R}Q = \theta$. The x-intercepts of PQ and PR are M and N respectively. The equations of the sides PR and QR are $y = -x + 2$ and $x + 3y + 6 = 0$ respectively. T is a point on the x-axis, as shown.



1.1 Determine the gradient of QP.

Solution:

$$m_{\mathrm{PQ}} = \frac{1-(-2)}{1-0}$$
$$= 3$$

1.2 Prove that $\mathrm{P\hat{Q}R} = 90°$.

Solution:

QR: $y = -\frac{1}{3}x - 2$

$$\therefore m_{\mathrm{QR}} = -\frac{1}{3}$$

$$m_{\mathrm{PQ}} \times m_{\mathrm{QR}} = 3 \times -\frac{1}{3}$$
$$= -1$$

$\therefore \mathrm{PQ} \perp \mathrm{QR}$ $\quad \therefore \mathrm{P\hat{Q}R} = 90°$

1.3 Determine the coordinates of R.

Solution:

$$-\frac{1}{3}x - 2 = -x + 2$$
$$\frac{2}{3}x = 4$$
$$x = 6$$
$$y = -4$$
$$\therefore \mathrm{R}(6\ ;\ -4)$$

1.4 Calculate the length of PR. Leave your answer in surd form.

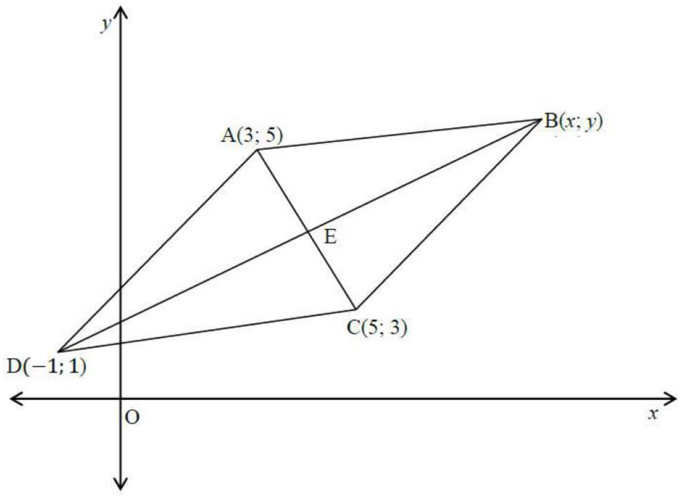
Solution

$$\mathrm{PR} = \sqrt{(1-6)^2 + (1-(-4))^2}$$
$$= \sqrt{50} = 5\sqrt{2}$$

ACTIVITIES: TYPICAL EXAM QUESTIONS

Question 1

In the figure A(3 ; 5), B(x; y), C(5 ; 3) and D(1; l) are the vertices of parallelogram ABCD. AC and BD, the diagonals of the parallelogram, intersect at E.



1.1 Determine:

1.1.1 The co-ordinates of E (2)

1.1.2 The co-ordinates of B (3)

1.1.3 The co-ordinates of the midpoint F, of CD and hence the equation of the line passing through F, parallel to AD. (5)

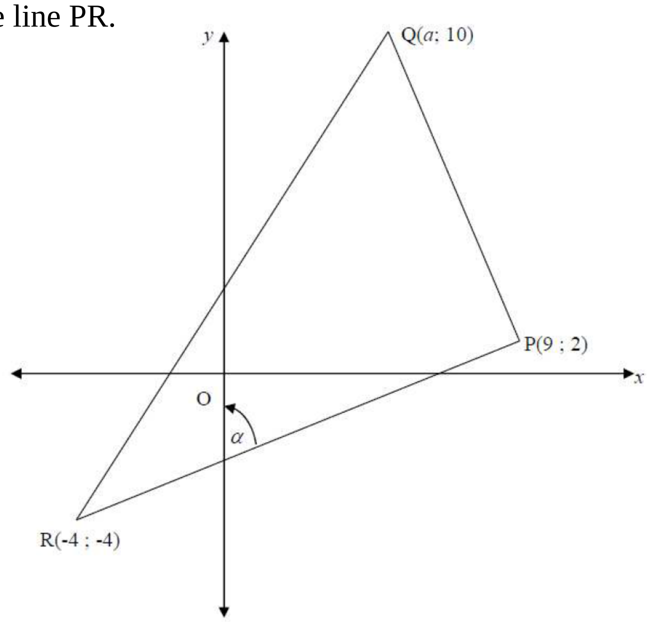
1.2 The points G(t + l ; 2,5) ,D(l; l) and E(4 ; 4) and are collinear. Calculate the value of t. (4)

1.3 Determine, by calculations, whether ABCD is a rhombus or not. Give a reason for your answer. (5)

[19]

Question 2

In the diagram below, P(9;2); Q(a;10) and R(-4;-4) are the vertices of ΔPQR. α is the angle between y-axis and the line PR.



2.1 Determine the gradient of PR. (3)

2.2 Calculate the size of a. angle between y-axis and the line PR. (4)

2.3 Show that the value of a = 5 if PQ $= 4\sqrt{5}$ units and Q (a ; 10). (6)

2.4 Determine the equation Of a line parallel PR and passing through Q. (3)

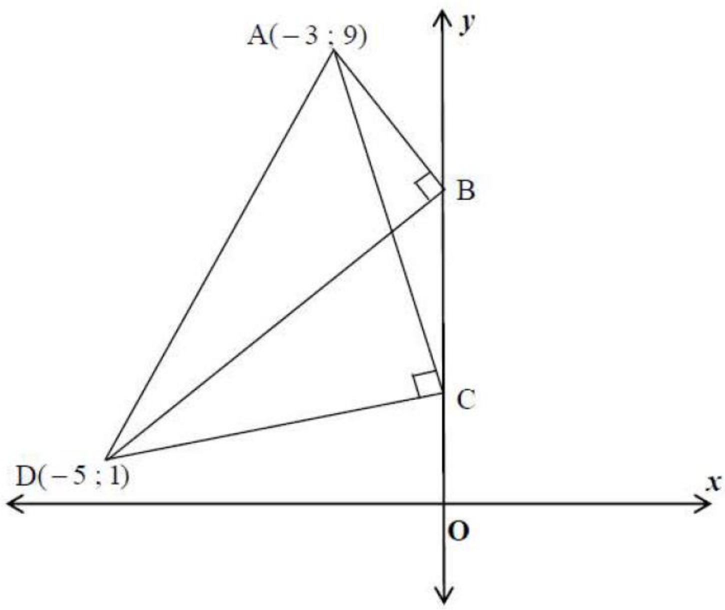
2.5 Calculate the co-ordinates of S(x ; y), if PQSR is a parallelogram and S is a point in the second quadrant. (4)

[20]

Question 3

In the diagram, $A(-3\ ;\ 9)$ and $D(-5\ ;\ 1)$ are points on ΔABD and ΔACD.

B and C are points on the y-axis such that $A\hat{B}D = A\hat{C}D = 90°$



3.1 Calculate the coordinates of M, the midpoint of AD. (2)

3.2 Calculate the length of the radius of the circle passing through A, B and D. (2)

3.3 Will point C lie on circle ABD? Give a reason for your answer. (2)

3.4 Calculate the coordinates of B. (5)

3.5 Determine the equation of the straight line passing through D and which is parallel to AB. (3)

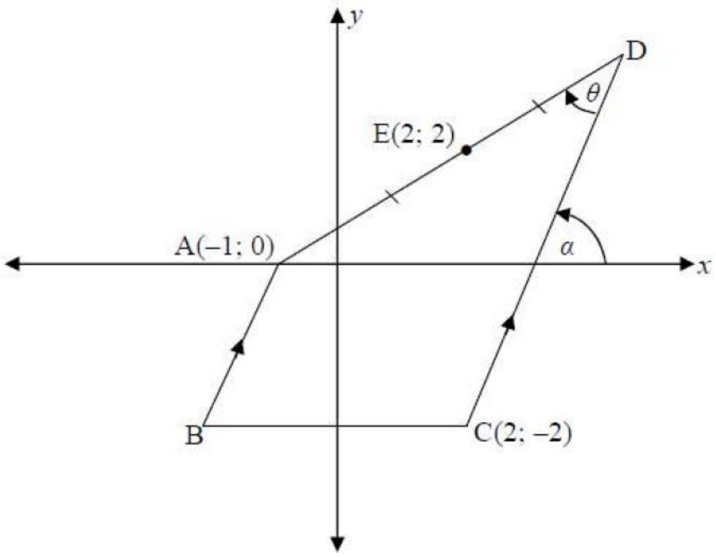
3.6 Calculate the size of $B\hat{D}A$. Round off the answer to the nearest degree. (6)

[20]

Question 4

In the diagram below, A(-1; 0), B. C(2 ; -2) and D are the vertices of a trapezium having AB // DC.

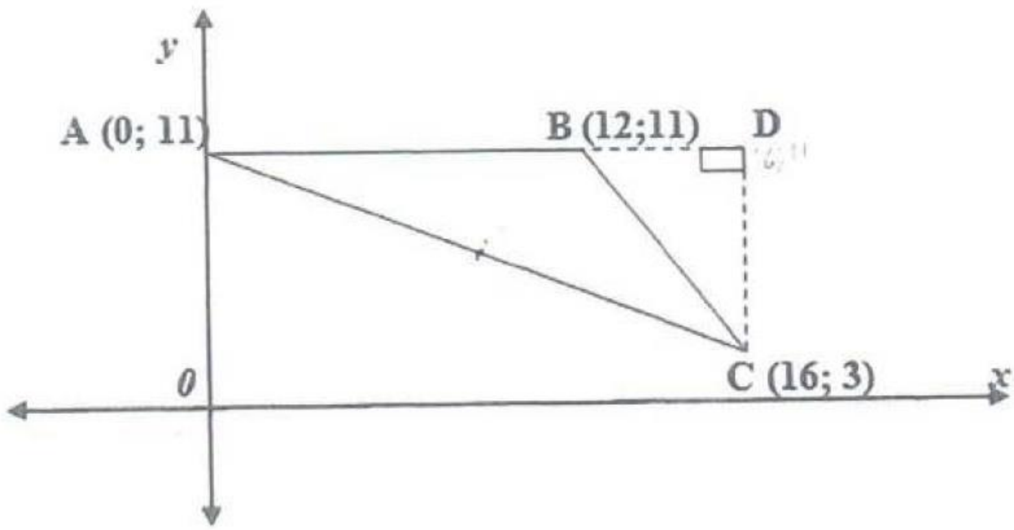
The length of DC is three times the length of AB (i.e. DC = 3AB). $A\hat{D}C = \theta$. E(2 ; 2) is the midpoint of AD. The angle of inclination of DC is α.



4.1	Determine the coordinates of D.	(2)
4.2	Calculate the size of α, correct to ONE decimal place.	(3)
4.3	Determine the equation of AB in the form y = mx + c.	(3)
4.4	Calculate the size of θ, correct to ONE decimal place.	(3)
4.5	Calculate the coordinates of B.	(5)
		[16]

Question 5

In the diagram below A(0;11), B(12;11) and C(16;3) are the vertices of ΔABC, with height CD



5.1 Write down the equation and the length of line AB. (3)

5.2 Write down the coordinates of point D. (2)

5.3 Determine the coordinates of M, the midpoint of AC. (2)

5.4 Determine the equation of the perpendicular bisector of AC. (4)

5.5 Does the line in 2.5 pass through B? Justify your answer with relevant calculations. (2)

5.6 Determine the equation of the line parallel to AC, passing through D. (3)

5.7 Calculate the area of ΔABC. (3)
[19]

SECTION B: EQUATIONS OF CIRCLES

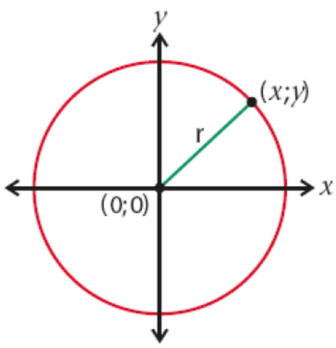
Now consider the general formula for a circle.

Circles that are centred at the origin

Let the origin be the centre of the circle, and point $(x; y)$ be any point on the circumference of the circle with radius r.

According to the distance formula:
$r = \sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2 + y^2}$

$\therefore \sqrt{x^2 + y^2} = r$
$\therefore x^2 + y^2 = r^2$



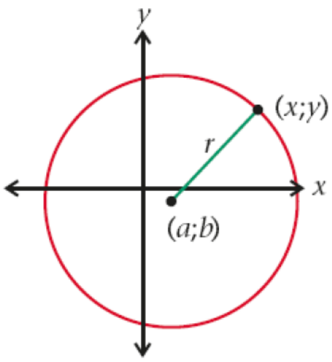
Therefore, the equation of a circle with centre (0; 0) and radius r is:
$x^2 + y^2 = r^2$

Circles that are centred off the origin

Let the centre of the circle be at point $(a; b)$, and point $(x;y)$ be any point on the circumference of the circle with radius r.

According to the distance formula:

$\sqrt{(x-a)^2 + (y-b)^2} = r$
$\therefore (x-a)^2 + (y-b)^2 = r^2$



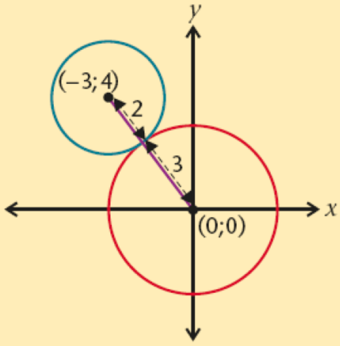
Therefore, the equation of the circle with centre $(a; b)$ and radius r is: $(x-a)^2 + (y-b)^2 = r^2$

To find the equation of a circle, we must first know the centre and the radius of the circle.

WORKED EXAMPLE 3

Show that the circle $(x + 3)^2 + (y - 4)^2 = 4$ touches the circle $x^2 + y^2 = 9$.

SOLUTION



Circle $(x + 3)^2 + (y - 4)^2 = 4$ has centre $(-3;4)$ and circle $x^2 + y^2 = 9$ has centre $(0;0)$.

The distance from $(-3;4)$ to $(0; 0)$

$= \sqrt{(-3)^2 + (4)^2} = \sqrt{25} = 5$

Radius of circle $(x + 3)^2 + (y - 4)^2 = 4$ is 2

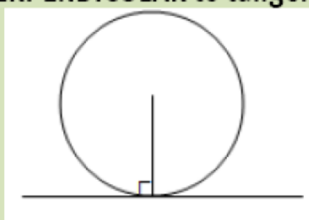
and radius of circle $x^2 + y^2 = 9$ is 3

The distance between the centres of the circles is

$2 + 3 = 5$

So the circles will touch.

Note: Circles will touch when the distance between their centres is equal to the sum of their radii.

Equation of tangent to circle at given point	**NB: Radius is PERPENDICULAR to tangent**  Determine the equation of the tangent to the circle $(x+1)^2 + (y-3)^2 = 16$ through the point $(2;5)$. The steps are: • Determine centre of circle • Determine gradient of RADIUS • Determine gradient of TANGENT: Remember: $m_{rad} \times m_{tan} = -1$ • Determine equation of tangent by substituting point of tangency The centre of circle $(x+1)^2 + (y-3)^2 = 16$ is $(-2;3)$. Radius joins centre $(-1;3)$ with point of tangency $(2;5)$. $m_{rad} = \frac{5-3}{2-(-2)} = \frac{2}{4} = \frac{1}{2}$ $\therefore m_{tangent} = -2$ Equation of tangent: $y - 5 = -2(x-2)$ $y = -2x + 9$

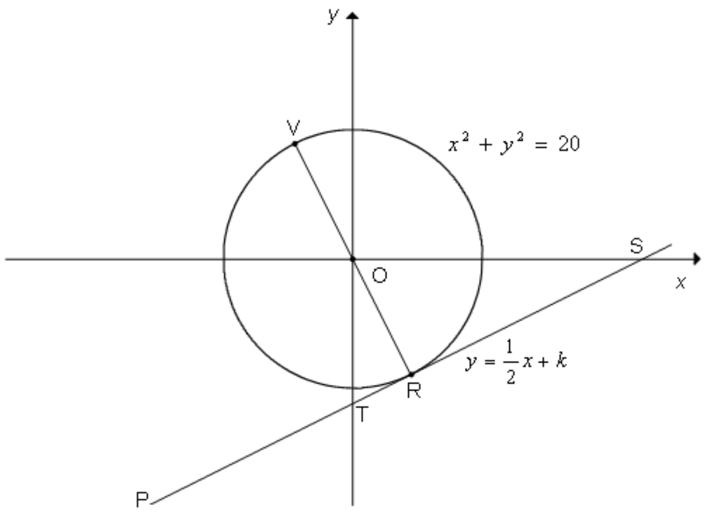
EQUATION OF TANGENT

You should now have enough knowledge to determine the equation of a tangent to any circle:

- A tangent is a straight line.
- A tangent is perpendicular to a radius of a circle at the point of contact.
- You can determine the equation of the tangent passing through a given point $(x_1; y_1)$:
  - using the formula $y - y_1 = m(x - x_1)$ where $(x_1; y_1)$ represents the point of contact of the tangent and the circle (or any other point that is given on the tangent)
  - substituting the given point of contact into the formula $y = mx + c$ to solve for c, then writing the equation in the form $y = mx + c$
  - remembering to look out for horizontal or vertical tangents, where the tangent is parallel to the x or y axis. The equation of the tangent will then be $y = y_1$ or $x = x_1$.

EXAMPLE 1

In the diagram below, the equation of the circle with centre O is $x^2 + y^2 = 20$. The tangent PRS to the circle at R has the equation $y = \frac{1}{2}x + k$. PRS cuts the y-axis at T and the x-axis at S.



1.1.1 Determine, giving reasons, the equation of OR in the form $y = mx + c$.

: Solution:

OR $\perp$ TR [radius $\perp$ tangent/*raakl*]

$\therefore\ m_{TR} \times m_{OR} = -1$

$\therefore\ m_{OR} = -2$

$\therefore\ y = -2x$

1.1.2 Determine the coordinates of R.

Solution:

$x^2 + (-2x)^2 = 20$

$x^2 + 4x^2 = 20$

$5x^2 - 20 = 0$

$x^2 - 4 = 0$

$(x+2)(x-2) = 0$

$\therefore x = 2$

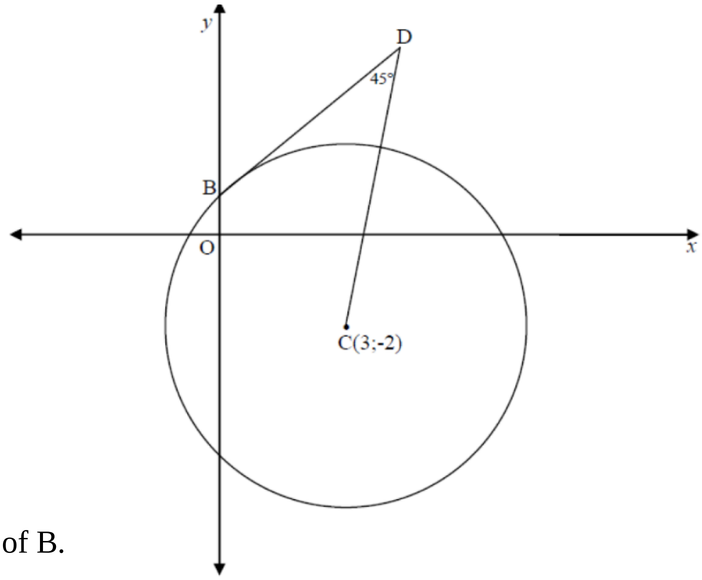
$y = -2(2) = -4$

$\therefore$ R(2 ; –4)

ACTIVITIES: TYPICAL EXAM QUESTIONS

Question 1

In the diagram below, BD is a tangent to the circle at point B, which lies on the y-axis. The centre of the circle is C(3;-2). The equation of tangent BD is given by 3x—4y +8 = 0. BDC 45^0



1.1 Determine the coordinates of B. (2)

1.2 Show that $x^2 - 6x + y^2 + 4y - 12 = 0$ is the equation of the circle. (4)

1.3 Determine the value(s) of t if $x + t = 0$ is the equation of a tangent to the circle. (4)

1.4.1 Write down the length of BD. (1)

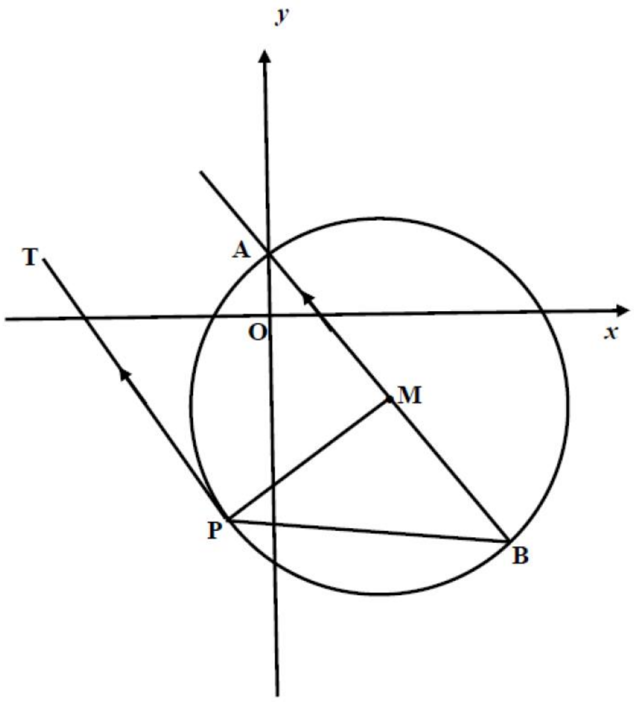
1.4.2 Hence, calculate the coordinates of D. (6)

1.5 Determine whether points E(2; –9) , C and D collinear. (3)

[20]

Question 2

In the diagram below, M is the centre of the circle $x^2 - 2x + Y^2 + 4y - 5 = 0$. Line AB passes through M, the centre of the circle. The equation of radius PM is $3y - x + 7 = 0$. PT is a tangent to the circle at P and PT // AB



2.1 Determine the coordinates of M. (4)

2.2 Write down, with reasons, the size of P MB (3)

2.3 Determine the equation of line AB. (4)

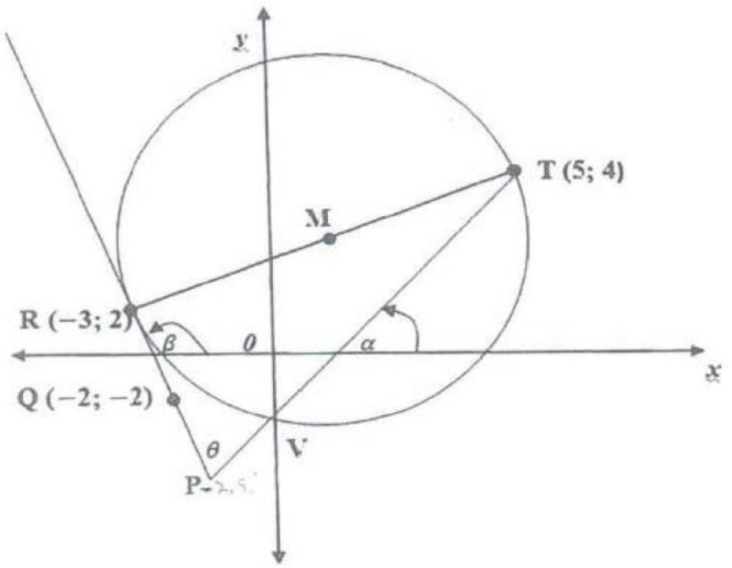
2.4 Determine the coordinates of A. (2)

2.5 If $TM = \sqrt{80}$, determine the length of the tangent PT. (3)

[16]

Question 3

In the diagram, the circle with centre M passes through points V, R$(-3;2)$ and T$(5;4)$. Q is the point $(-2;-2)$ and the lines through RQ and TV meet at P. The inclination angle of PT is α and the angle of inclination of PR is β.
V is the y-intercept of both the circle and line TP.



3.1 Determine the equation of the circle with centre M. (5)

3.2 Show, using analytical methods, that PR is a tangent to the circle at R. (3)

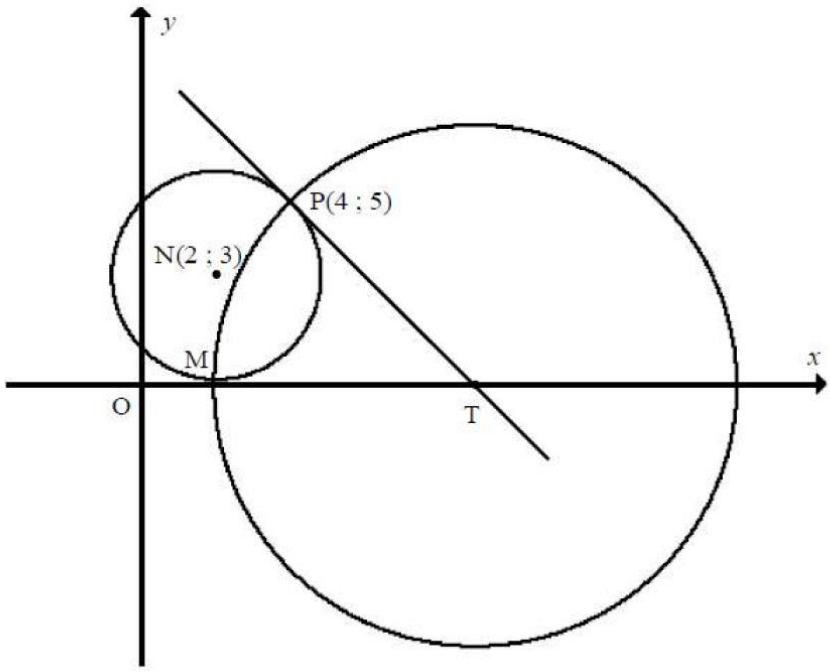
3.3 Determine the coordinates of V. (4)

3.4 If $R\hat{P}T = \theta$, calculate θ to ONE decimal place. (6)

[18]

Question 4

In the figure below, T is a point on the x-axis. A circle having T as its centre intersects a circle having N(2 ; 3) as its centre at P(4 ; 5) and M. TP is a tangent to the circle centre N at P.



4.1 Determine the equation of circle centre N in the form: $(x-a)^2+(y-b)^2=r^2$. (4)

4.2 Calculate the coordinates of T, the x-intercept of PT. (6)

4.3 Calculate the length of PT. Leave your answer in surd form. (2)

4.4 Calculate the area of circle centre T. Give your answer rounded off to the nearest integer. (2)

4.5 Calculate the size of $N\hat{T}P$, correct to ONE decimal place. (2)

4.6 Prove that MNPT is a kite. (3)

4.7 Calculate the size of $M\hat{N}P$, correct to ONE decimal place. (4)

[23]