Analytical Geometry

The Distance Formula

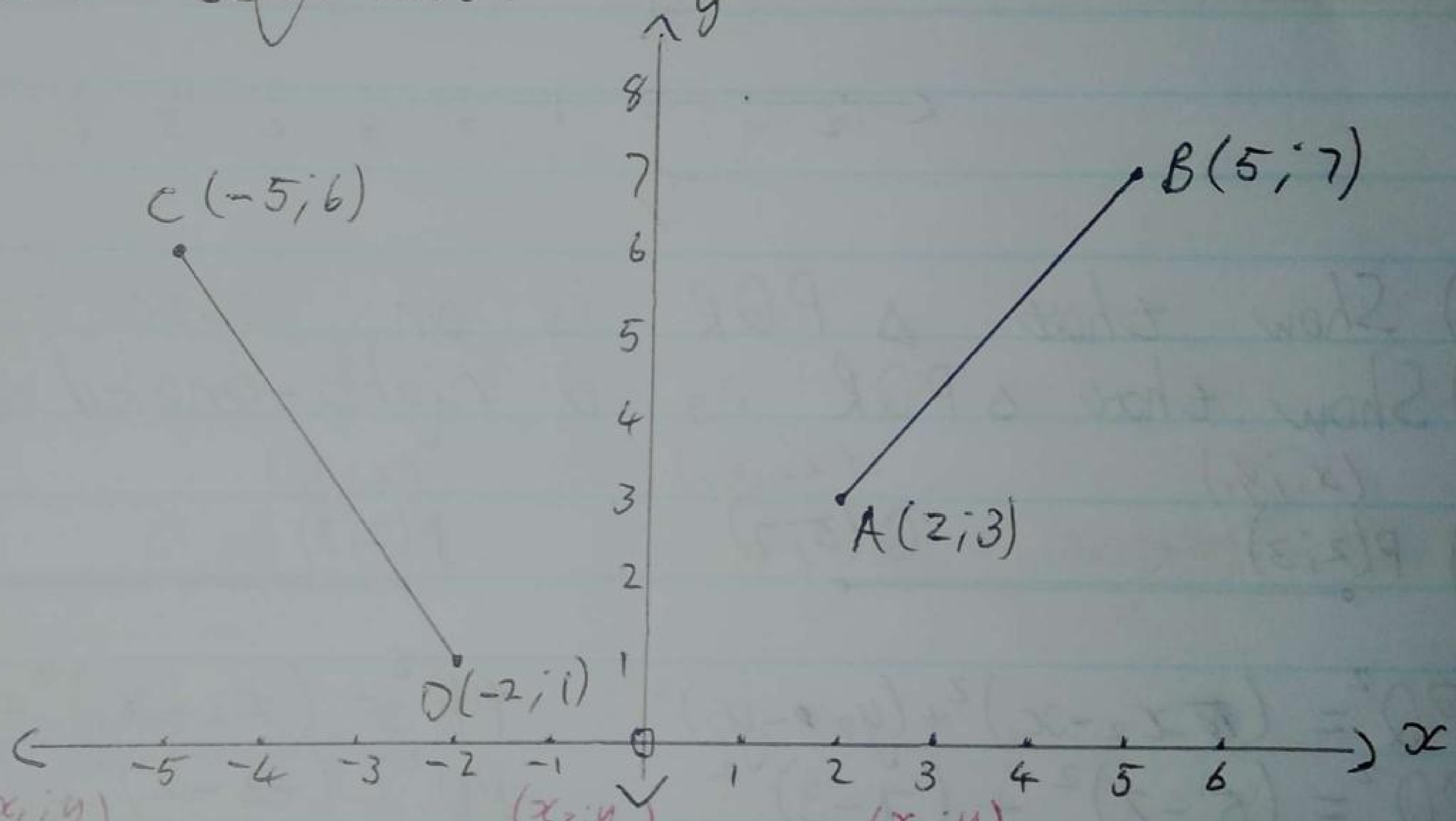
$$AB^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

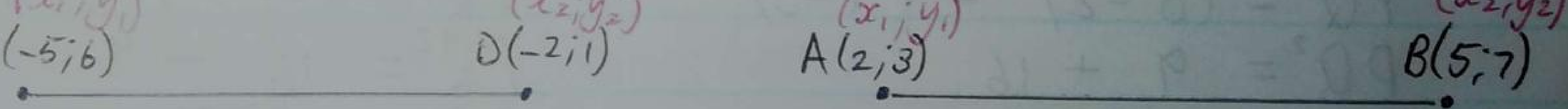
OR

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Example 1

Calculate the lengths of the following line segments:





$CD^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$
$CD^2 = (-2 - (-5))^2 + (1 - 6)^2$
$CD^2 = (3)^2 + (-5)^2$
$CD^2 = 9 + 25$
$CD = \sqrt{34}$
$CD = 5{,}8$ units

$AB^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$
$AB^2 = (5 - 2)^2 + (7 - 3)^2$
$AB^2 = (3)^2 + (4)^2$
$AB^2 = 9 + 16$
$AB = \sqrt{25}$
$AB = 5$ units

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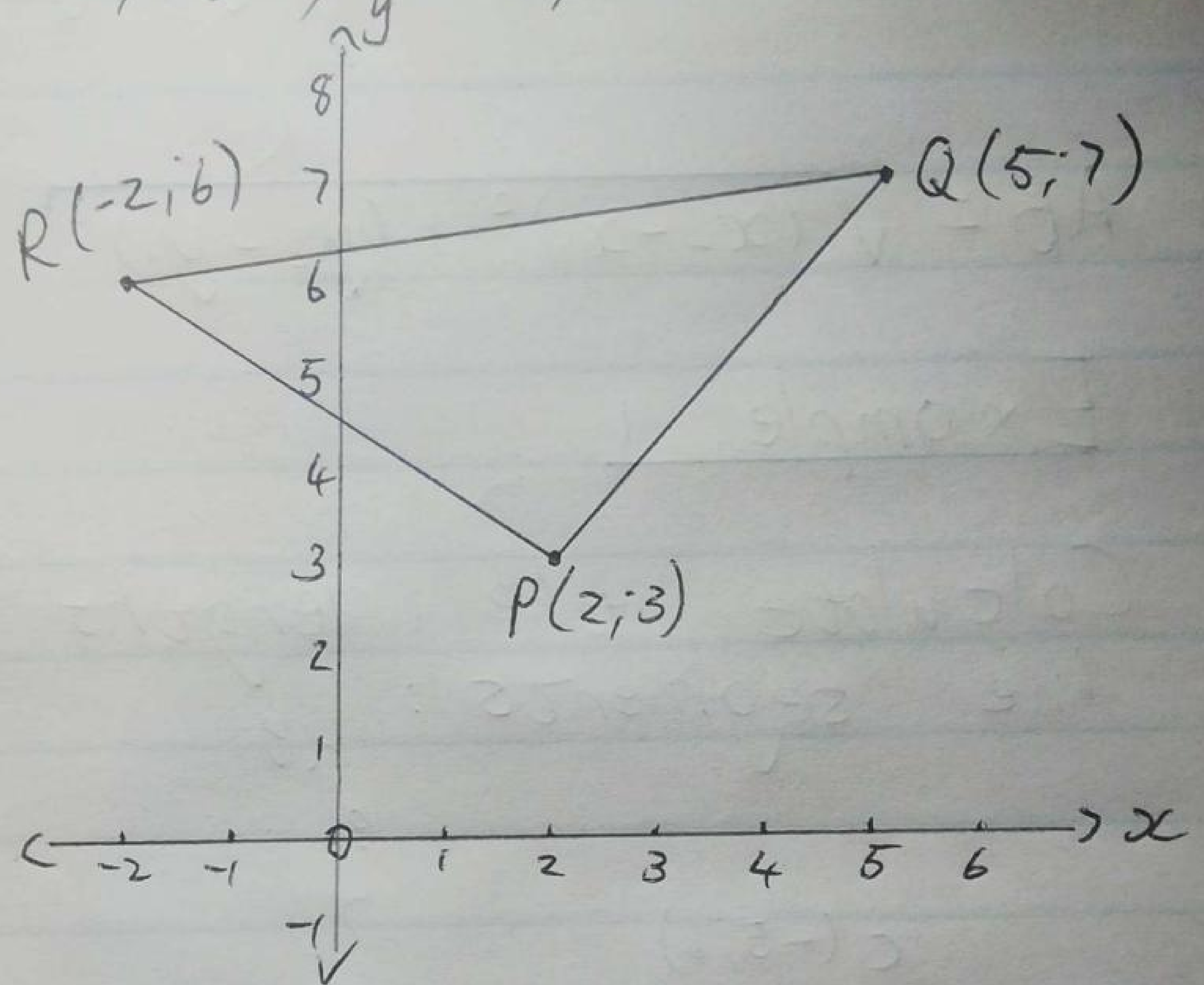
Example 2

In the diagram below, the coordinates of $\triangle PQR$ are $P(2;3)$, $Q(5;7)$ and $R(-2;6)$



a) Show that $\triangle PQR$ is an isosceles triangle.
b) Show that $\triangle PQR$ is a right-angled triangle.

a) $P(2;3)$ $(x_1;y_1)$ — $Q(5;7)$ $(x_2;y_2)$

$$PQ^2 = (x_2-x_1)^2+(y_2-y_1)^2$$
$$PQ^2 = (5-2)^2 + (7-3)^2$$
$$PQ^2 = 9 + 16$$
$$PQ = \sqrt{25}$$
$$PQ = 5 \text{ units}$$

$P(2;3)$ $(x_1;y_1)$ — $R(-2;6)$ $(x_2;y_2)$

$$PR^2 = (x_2-x_1)^2+(y_2-y_1)^2$$
$$PR^2 = (-2-2)^2 + (6-3)^2$$
$$PR^2 = 16 + 9$$
$$PR = \sqrt{25}$$
$$PR = 5 \text{ units}$$

$\therefore PQ = PR$

$\therefore \triangle PQR$ is an isosceles triangle

b) We can ~~say~~ prove that $\triangle PQR$ is a right-angled triangle by showing that the theorem of Pythagoras holds.

$$\therefore QR^2 = PR^2 + PQ^2$$
$$QR^2 = 5^2 + 5^2 \quad \text{(by (a))}$$
$$QR = \sqrt{50}$$
$$QR = 5\sqrt{2} \text{ units}$$

This is only true if $\hat{P} = 90°$. Since we don't know this, we verify the distance with the distance formula.

$$QR^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

$(x_1;y_1)$ $(x_2;y_2)$

$Q(5;7)$ $R(-2;6)$

$$QR^2 = (-2-5)^2 + (6-7)^2$$
$$QR^2 = 49 + 1$$
$$QR = \sqrt{50}$$
$$QR = 5\sqrt{2}$$

$\therefore \triangle PQR$ is a right-angled triangle

Exercise 11.2 P. 244

1. b; e; d - No sketch
2. c
3. a; e
4. 6

The Gradient Formula

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Before we look at examples, let's make a few conclusions:

Parallel Lines

Let AB//CD with $A(2;3)$, $B(4;2)$, $C(1;-1)$ and $D(5;-3)$. Find the gradient of AB and CD. What do you notice?

$(x_1;y_1)$ $(x_2;y_2)$ $(x_1;y_1)$ $(x_2;y_2)$

$A(2;3)$ $B(4;2)$ $C(1;-1)$ $D(5;-3)$

$$m_{AB} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2-3}{4-2} = -\frac{1}{2}$$

$$m_{CD} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-3-(-1)}{5-1} = -\frac{2}{4} = -\frac{1}{2}$$

$\therefore$ We make an important conclusion:

The gradients of parallel lines are equal.

Perpendicular Lines

Let $EF \perp GH$. $E(-5;2)$, $F(1;-1)$, $G(-1;3)$ and $H(-4;-3)$. Find the gradient of EF and GH. What do you notice about

(a) the signs?

(b) the fractions?

(c) the answer of $m_{EF} \times m_{GH}$?

$E(-5;2)$ $F(1;-1)$ $G(-1;3)$ $H(-4;-3)$

$$m_{EF} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-1 - 2}{1 - (-5)} = \frac{-3}{6} = -\frac{1}{2}$$

$$m_{GH} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-3 - 3}{-4 - (-1)} = \frac{-6}{-3} = \frac{2}{1}$$

(a) m_{EF} is negative and m_{GH} is positive.
$\therefore$ The signs are different.

(b) The fractions are each other's reciprocals. (The reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$).

(c) $m_{EF} \times m_{GH}$
$= -\frac{1}{2} \times \frac{2}{1}$
$= -1$

$\therefore$ We make the conclusion:

The product of two perpendicular lines is equal to -1.

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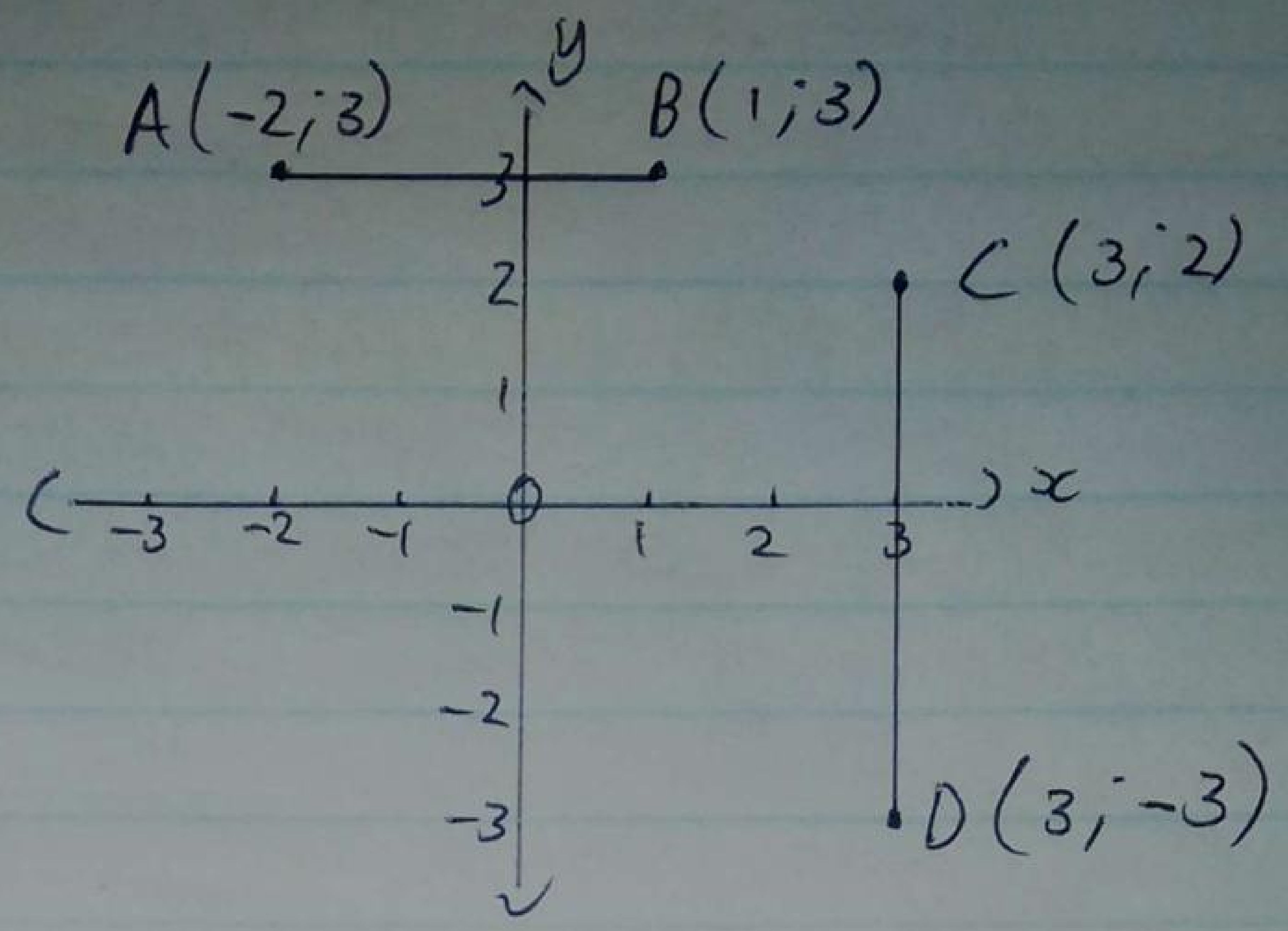
Vertical and Horizontal Lines



$$m_{AB} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3-3}{1-(-2)} = \frac{0}{3} = 0$$

$\therefore$ The gradient of a horizontal line is zero.

$$m_{CD} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-3-2}{3-3} = \frac{-5}{0} = \text{undefined}$$

$\therefore$ The gradient of a vertical line is undefined

Exercise 11.4   P. 257

1. a, c, f
2. (for a, c, f)
3. a – e
4.

Collinear Lines

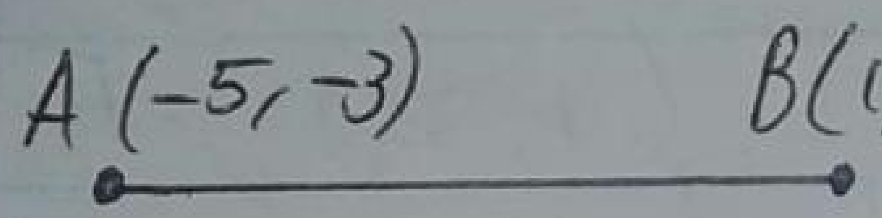
We say that three points are collinear if they lie on the line segment.

$\therefore$ A, B, C are collinear if

$$m_{AB} = m_{BC}$$

Examples

1) Show that the points $A(-5;-3)$, $B(1;0)$ and $C(3,1)$ are collinear



$$m_{AB} = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{0-(-3)}{1-(-5)}$$
$$= \frac{3}{6}$$
$$= \frac{1}{2}$$

$$m_{BC} = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{1-0}{3-1}$$
$$= \frac{1}{2}$$

$\therefore$ A, B and C are collinear

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2) Given $D(-1;y)$, $E(-2;-1)$ and $F(-4;-3)$ are collinear. Calculate the value of y.

$$m_{DE} = m_{EF}$$
$$\frac{y_E - y_D}{x_E - x_D} = \frac{y_F - y_E}{x_F - x_E}$$
$$\frac{1 - y}{-2-(-1)} = \frac{-3-1}{-4-(-2)}$$
$$\frac{1-y}{-1} = \frac{-4}{-2}$$
$$\frac{1-y}{-1} \times -1 = 2 \times -1$$
$$1-y = -2$$
$$3 = y$$

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3) Given $EF \perp GH$ with $E(1;2)$, $F(x;0)$, $G(4;4)$ and $H(2;0)$. Calculate x

$$m_{EF} \times m_{GH} = -1$$
$$\frac{y_F - y_E}{x_F - x_E} \times \frac{y_H - y_G}{x_H - x_G} = -1$$
$$\frac{0-2}{x-1} \times \frac{0-4}{2-4} = -1$$
$$\frac{-2}{x-1} \times 2 = -1$$

$$\frac{-2}{x-1} \times 2(x-1) = \frac{-1}{2} \times 2(x-1)$$
$$-4 = -x+1$$
$$x = 5$$

Exercise 11.4 P. 257

5
6
7. b

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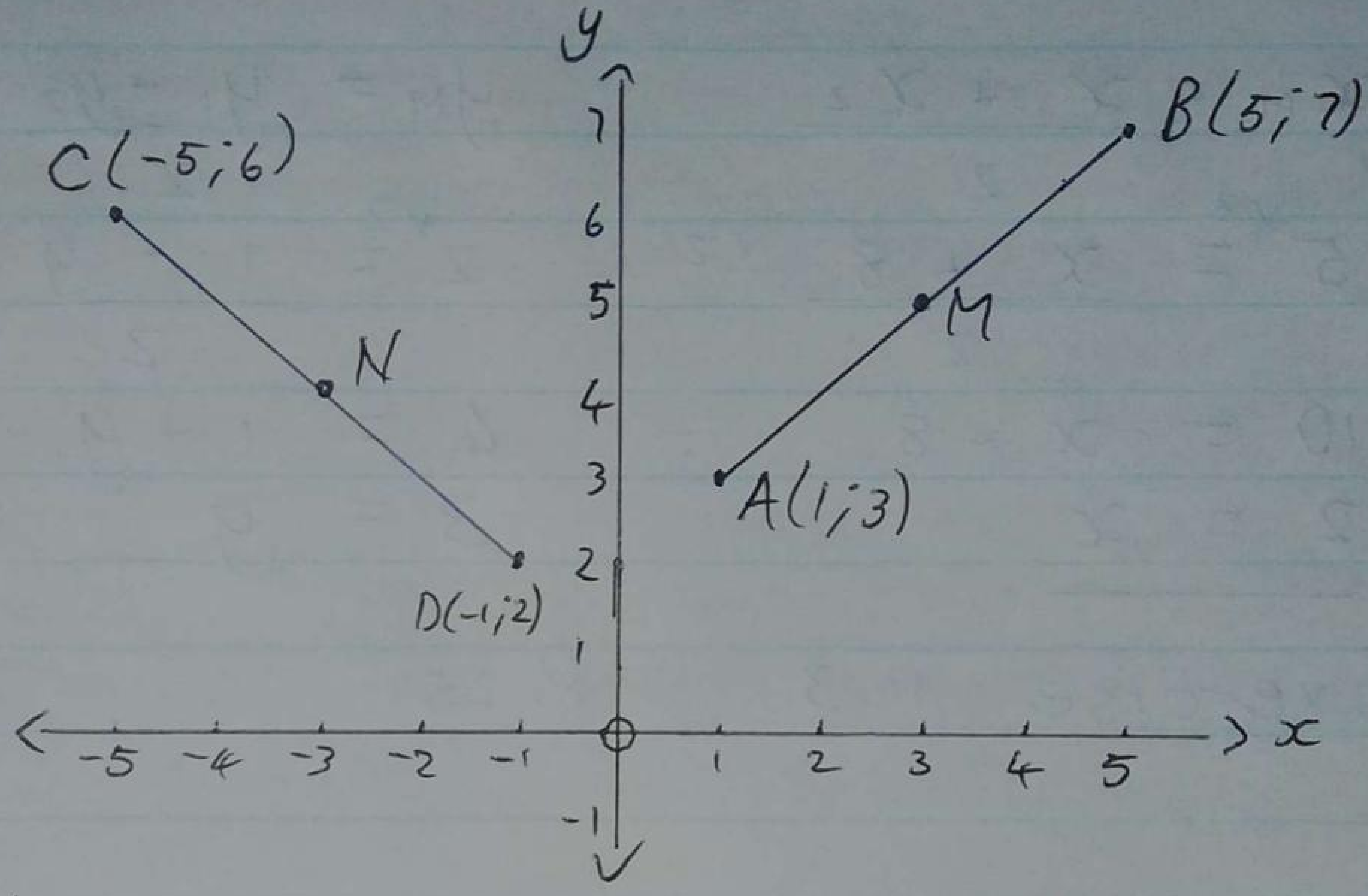
The Midpoint Formula

$$x_M = \frac{x_1 + x_2}{2} \qquad y_M = \frac{y_1 + y_2}{2}$$

Examples

1)



Find the coordinates of M and N:

$(x_1;y_1)$ $(x_M;y_M)$ $(x_2;y_2)$

C(-5;6) N(x;y) D(-1;2)

$$x_M = \frac{x_1 + x_2}{2}$$

$$x_M = \frac{-5 + (-1)}{2} = \frac{-6}{2} = -3$$

$$y_M = \frac{y_1 + y_2}{2} = \frac{6+2}{2} = \frac{8}{2} = 4$$

$\therefore N(-3;4)$

A(1;3) $M(x_M;y_M)$ B(5;7)

$$x_M = \frac{x_1 + x_2}{2} = \frac{1+5}{2} = 3$$

$$y_M = \frac{y_1 + y_2}{2} = \frac{3+7}{2} = 5$$

$\therefore M(3;5)$

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2. Determine the value of x and y if $M(5;2)$ is the midpoint of the line segment connecting $A(x;1)$ and $B(8;y)$

$(x_1;y_1)$ $\quad$ $(x_M;y_M)$ $\quad$ $(x_2;y_2)$

$A(x;1)$ $\quad$ $M(5;2)$ $\quad$ $B(8;y)$

$$x_M = \frac{x_1 + x_2}{2}$$

$$5 \overset{\times 2}{=} \frac{x + 8}{2} \quad \times 2$$

$$10 = x + 8$$

$$2 = x$$

$$y_M = \frac{y_1 + y_2}{2}$$

$$2 \overset{\times 2}{=} \frac{1 + y}{2} \quad \times 2$$

$$4 = 1 + y$$

$$3 = y$$

Exercise 11.3 $\quad$ P. 251

2

3

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