SECONDARY SCHOOL IMPROVEMENT PROGRAMME (SSIP) 2019

GRADE 12

SUBJECT: TECHNICAL MATHEMATICS

LEARNER NOTES

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SESSION NO: 1

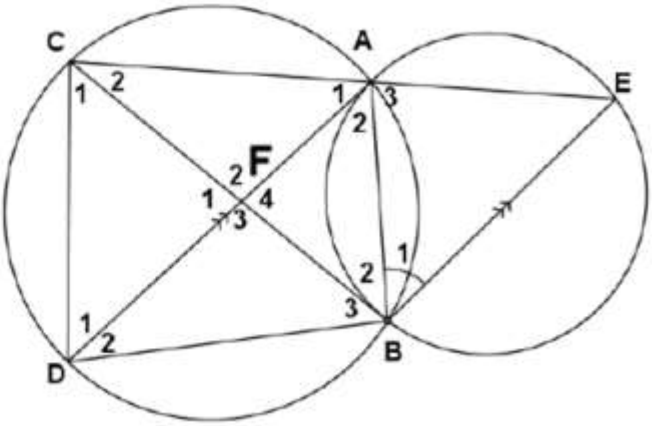
TOPIC: EUCLIDEAN GEOMETRY THEORY TO THE END OF GRADE 11

Learner note:
Euclidean Geometry builds on what was learnt in previous grades. Euclidean Geometry counts 40 to 43 of the 150 marks of Paper 2. Know how to recognise the diagrams of theorems and know how to write statements and reasons for these diagrams. They are given in **SECTION B: NOTES ON CONTENT.** The theorems in Section B have been divided into 3 groups: lines, triangles and circles.

SECTION A: RECOGNISING THEORY IN TYPICAL EXAM DIAGRAMS

QUESTION 1 (12 marks) (10 minutes) (Diagram from CAPS Grade 11 Exemplar 2017)

In the diagram below, DA is parallel to BE. DA and BE are cut by transversals AE, AB, FB and DB. Focus on the lines and triangles to answer the questions that follow.



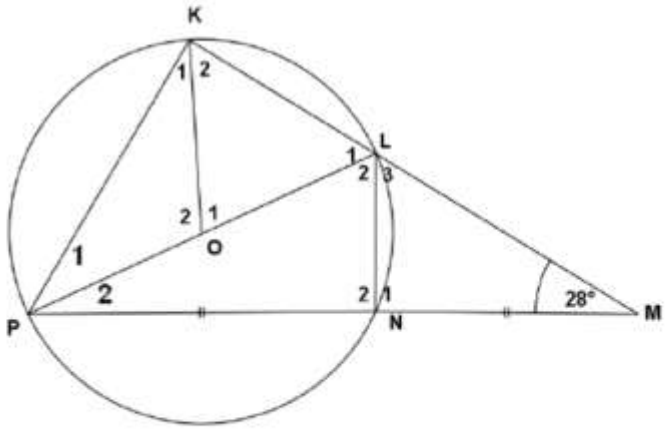
1.1 Give statements and reasons for the pairs of angles that are equal because DA||BE. (6)

1.2 Give statements and reasons for the pairs of angles that are supplementary because DA||BE. (4)

1.3 Complete the statement and give a reason: $\hat{B}_1 + \hat{E} = \cdots$ (2)

QUESTION 2 (13 marks) (10 minutes) (Diagram from CAPS Grade 11 Exemplar 2017)

Refer to the diagram below to answer the questions that follow.



2.1 PL is the diameter of the circle with centre O.

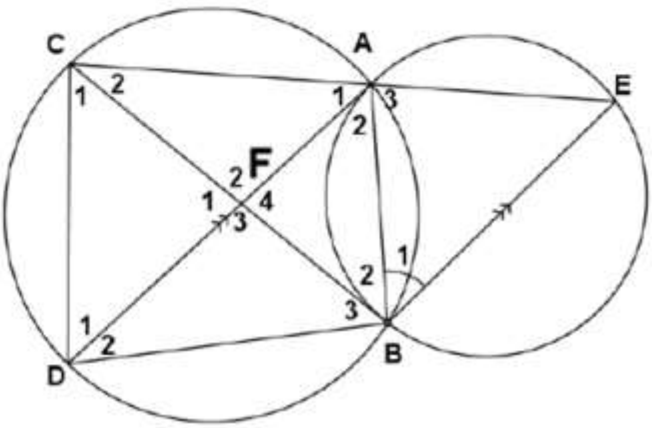
Which pairs of angles are equal and why? (3)

2.2 It is given that PN = NM. If $\widehat{N}_1 = 90°$, prove that $\hat{L}_2 = \hat{L}_3$. (5)

2.3 If LM = 12 cm and NM = x cm determine the length of LN in terms of x. (5)

QUESTION 3 (12 marks) (10 minutes) (Diagram from CAPS Grade 11 Exemplar 2017)

In the diagram below, BE is a tangent to the bigger circle at B and a chord of the smaller circle. Focus on BE and ABDC.



3.1 Give reasons for the following statements

$\hat{B}_1 = \hat{C}_2$.

$\hat{C}_2 = \hat{D}_2$. (2)

3.2 Draw a conclusion from the 2 statements above. (1)

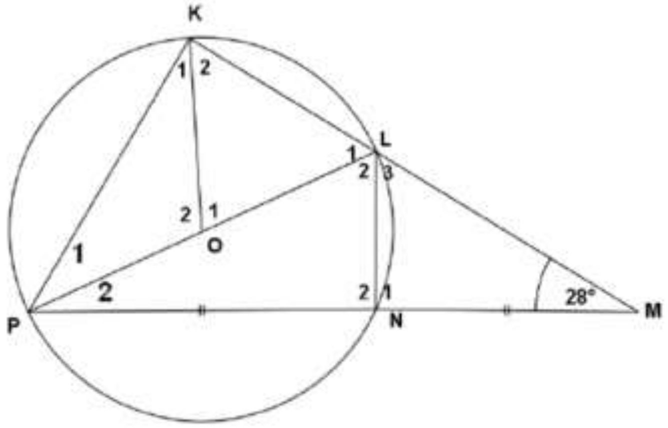
3.3 List the pairs of equal angles opposite AC, then CD then DB and give the reason why the pairs are equal. (4)

3.4 Complete the statement and give a reason: $\hat{A}_3 = \cdots$ (2)

3.5 Write statements for two pairs of angles in ABCD that are supplementary and give the reason. (3)

QUESTION 4 (16 marks) (10 minutes) (Diagram from CAPS Grade 11 Exemplar 2017)

In the diagram below:
P, K, L and N lie on the circumference of the circle with centre O and PL is the diameter of the circle with centre O.



4.1 What is the size of $P\hat{K}L$? Write a statement and give a reason. (2)

4.2 What is the size of $L\hat{N}P$? Write two different statements with reasons. (3)

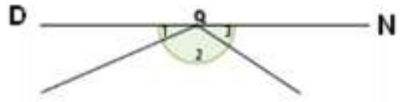
4.3 If $\hat{O}_1 = 68°$ determine the sizes of $\hat{K}_2$ and $\hat{L}_1$. (6)

4.4 If $\hat{O}_1 = 68°$ determine the size of $\hat{P}_1$. (5)

SECTION B: NOTES ON CONTENT
Given below are diagrams of theorems. There are statements about the diagram with the reason for the statement next to it, in brackets.

Theorems relating to straight lines

1. The sum of the adjacent angles on a straight line is 180°.



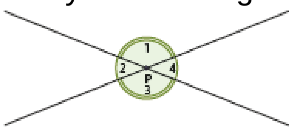
$\hat{Q}_1 + \hat{Q}_2 + \hat{Q}_3 = 180°$ (∠s on a str line)

Conversely:
If the adjacent angles are supplementary, the outer arms of these angles form a straight line.

DON is a straight line... (adj ∠s supp)

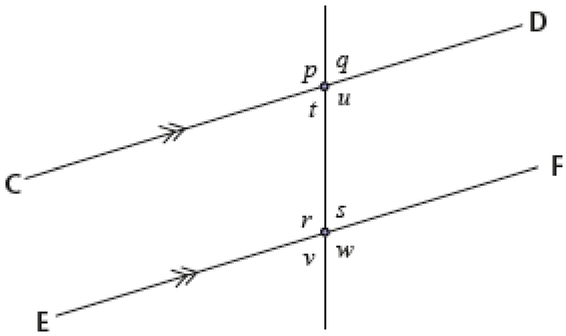
A **converse** is formed by taking the conclusion as a starting point and having the starting point as a conclusion.

2. Vertically opposite angles formed by intersecting lines are equal.



$\hat{P}_1 = \hat{P}_3$ .... (vert opp ∠s =)
$\hat{P}_2 = \hat{P}_4$ .... (vert opp ∠s =)

Theorems relating to parallel lines

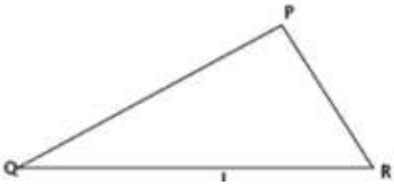


1. If two parallel lines are cut by a transversal, then corresponding angles are equal.
   $p = r, q = s, t = v$ and $u = w$... Reason for each pair:(corresp ∠s; CD||EF)
2. If two parallel lines are cut by a transversal, then alternate angles are equal.
   $t = s$ and $u = r$ ... Reason for each pair: (alt ∠s; CD||EF).
3. If two parallel lines are cut by a transversal, then the pairs of co-interior angles are supplementary (add up to $180°$)
   $t + r = 180°$ and $u + s = 180°$... Reason for each pair: (co-int ∠s; CD||EF)

Conversely: If two straight lines are cut by a transversal and a pair of corresponding angles is equal or a pair of alternate angles is equal, a pair of co interior angles is supplementary, then the lines are parallel.
CD||EF... (corresp ∠s =) or (alt ∠s =) or (coint ∠s sup)

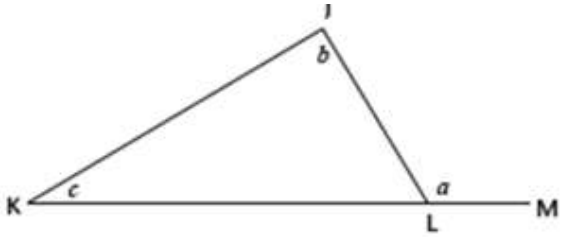
Theorems relating to triangles

1. The 3 angles of a triangle add up to 180°, or are supplementary.



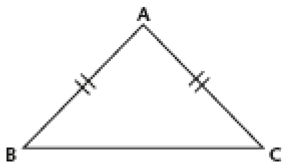
$\hat{P}+\hat{Q}+\hat{R}=180°$....
(Int ∠s Δ) **OR** (∠ sum in Δ) **OR** (sum of ∠s in Δ)

2. The exterior angle which is formed when one of the sides of the triangle is produced, is equal to the two interior opposite angles.



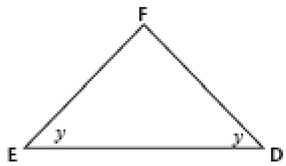
$a=b+c$... (ext ∠ of Δ)

3. If a triangle has two sides equal, the angles opposite the equal sides are also equal.



$\hat{B}=\hat{C}$ .....(∠s opp equal sides)

If a triangle has two angles equal, the sides opposite the equal angles are also equal.

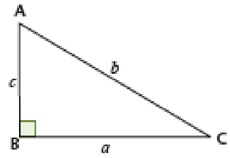


FE = FD.... (sides opp equal ∠s)

4. The three sides and the three angles of an equilateral triangle are equal.
   Each angle measures 60°.
   If we have all 3 sides equal we can conclude all 3 angles are equal and give (∠s opp equal sides) as a reason.
   If we have all 3 angles equal we can conclude all 3 sides are equal and give (sides opp equal ∠s) as a reason.

5. Pythagoras' theorem

   In a right-angled triangle, the square on the hypotenuse is equal to the sum of the squares on the other two sides.

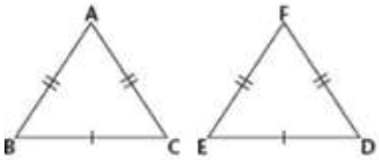


If $\hat{B} = 90°$, then

$b^2 = a^2 + c^2$ ...(Pythagoras)

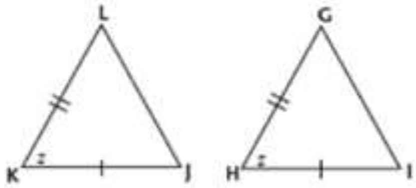
6. Congruent triangles. To prove triangles congruent we use a different format to other proofs.

- If three sides of one triangle are respectively equal to the three sides in another triangle, the two triangles are congruent



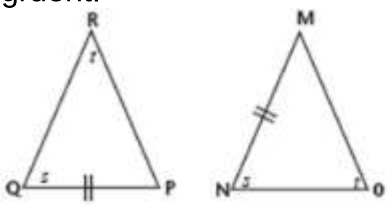
Δ ABC ≡ Δ FED...(SSS)

- If two sides and the *included angle* of one triangle are respectively equal to the two sides and the *included angle* in another triangle, the two triangles are congruent.



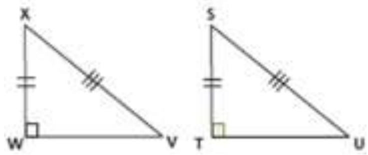
Δ JKL≡ Δ IHG...(SAS) or (S∠S)

- If two angles and a side in one triangle are respectively equal to two angles and the *corresponding side* in another triangle, the two triangles are congruent.



Δ PQR ≡ Δ ONM...(AAS) or (∠∠S)

- If the hypotenuse and a side of one right-angled triangle is respectively equal to the hypotenuse and side of another right-angled triangle, the two triangles are congruent



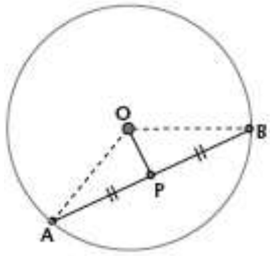
Δ XWV ≡ Δ STU...(RHS) or (90°HS)

Congruency is useful to prove sides are **equal** in length or angles are **equal** in size or both.

Theorems relating to circles

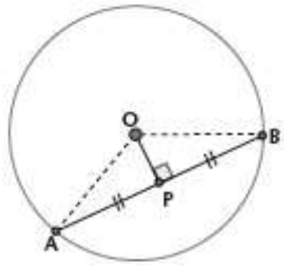
1. Line, centre, midpoint

The line drawn from the centre of a circle to the midpoint of a chord is perpendicular to the chord.



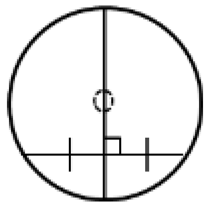
OP⊥ AB... (line from centre to midpt of chord)

The line drawn from the centre of a circle perpendicular to a chord bisects the chord.



AP = PB... (line from centre ⊥ to chord)

The perpendicular bisector of a chord passes through the centre of the circle

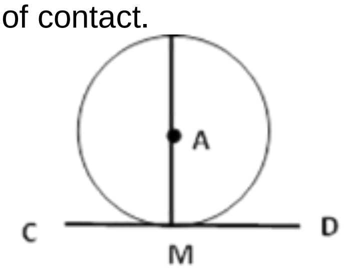


O is the centre of the circle. (perp bisector of chord)

A **chord** is a line that goes from one side of a circle to the other.

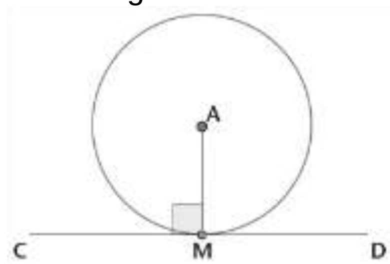
2. Tangent and radius or diameter. *Know the difference between the 3 theorems above and the 2 theorems below.*

A tangent to a circle is perpendicular to the radius/**diameter** drawn at the point of contact.



AM ⊥ AC (tan ⊥ radius) (tan ⊥ diameter)

If a line is drawn perpendicular to a radius/diameter at the point where the radius/diameter meets the circle, then the line is a tangent to the circle.



CD is a tangent (line ⊥ radius)

A **tangent** to a circle is a line which no matter how far it is extended touches the circle at one point only

A **diameter** is a chord that passes through the centre of a circle.

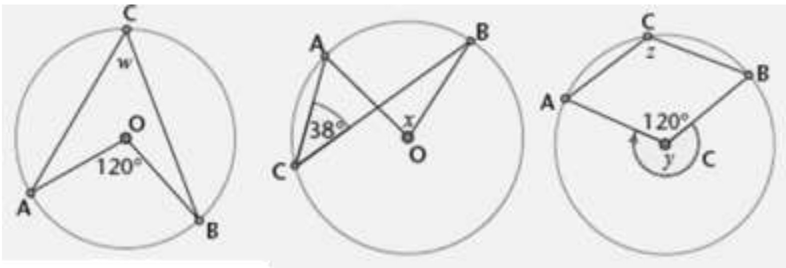
A **radius** is a straight line from the centre of a circle to the curve of the circle.

3. Angles in circles

- The angle **subtended by** an **arc** at the centre of a circle is double the size of the angle subtended by the same arc at the circle (on the same side of the chord as the centre)

  An **arc** is part of the edge (circumference) of a circle.

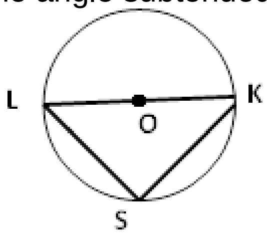
When an angle is **subtended by** an arc or chord, the angle is opposite the arc or chord.



$w = 60^0$ $x = 76^0$ $z = 120^0$

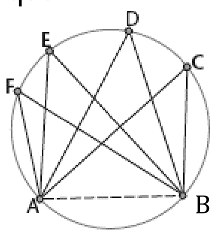
All have the same reason: (∠ at centre = 2 × ∠ at circumference)

- The angle subtended by the diameter at the circumference of the circle is 90°.



$\hat{S} = 90^0$ (∠s in semi circle) **OR** (diameter subtends right angle) **OR** ∠ in $\frac{1}{2}$⊙

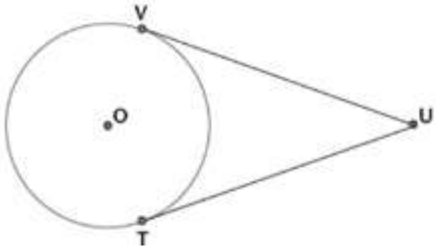
- Angles subtended by a chord of the circle, on the same side of the chord, are equal



$A\hat{F}B = A\hat{E}B = A\hat{D}B = A\hat{C}B$ (∠s in the same seg)

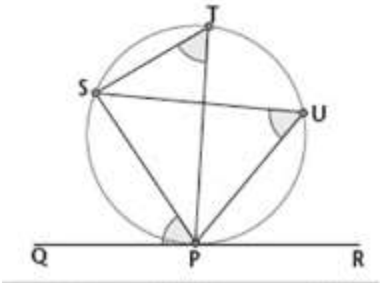
4. More tangent theorems

- Two tangents drawn to a circle from the same point outside the circle are equal in length.



UV = UT (Tans from same pt) **OR** (Tans from common pt)

- The angle between the tangent to a circle and the chord drawn from the point of contact is equal to the angle in the alternate segment.

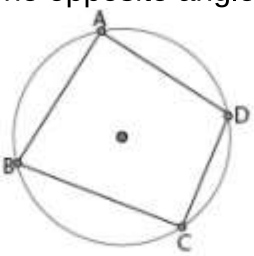


$\mathrm{Q\hat{P}S} = \mathrm{\hat{T}}$; $\mathrm{Q\hat{P}S} = \mathrm{\hat{U}}$ and $\mathrm{U\hat{P}R} = \mathrm{P\hat{S}U}$ (tan chord theorem)

5. Quadrilaterals in circles

A **cyclic quadrilateral** has all four of its corners (vertices) on the edge of a circle.

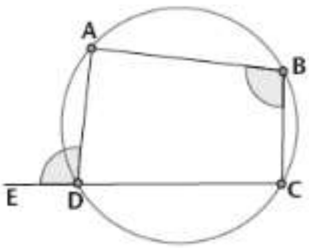
- The opposite angles of a cyclic quadrilateral are supplementary.



$\hat{A} + \hat{C} = 180°$
and
$\hat{B} + \hat{D} = 180°$...
(opp ∠s of cyclic quad)

Know the difference between the opposite angles of a parallelogram (which equal each other) and the opposite angles of a cyclic quadrilateral (which are supplementary.

- The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.



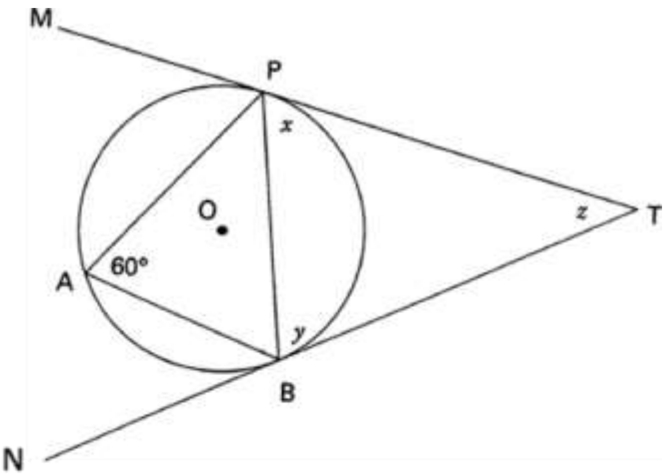
$A\hat{D}E = A\hat{B}C$
(ext ∠ of cyclic quad)

Know the difference between the exterior angle of a cyclic quadrilateral and the exterior angle of a triangle.

SECTION C: HOMEWORK QUESTIONS

QUESTION 1 (9 MARKS) (9 MINUTES) (Based on Classroom Mathematics Grade 11 p 277 Ex 10.7 no 1f)

O is the centre of the circle and MPT is a tangent at P. NT is a tangent at B.

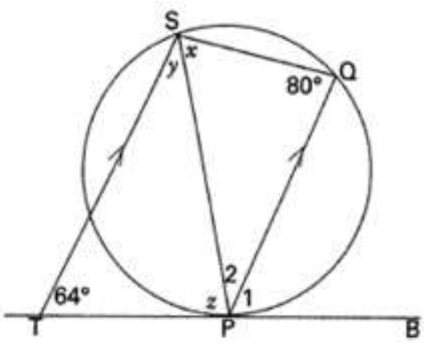


1.1 Determine the sizes of x, y and z. (7)

1.2 Write a statement about the sides of ΔPBT and give a reason. (2)

QUESTION 2 (10 MARKS) (10 MINUTES) (Based on Classroom Mathematics Grade 11 p 280 Ex 10.8 no 3)

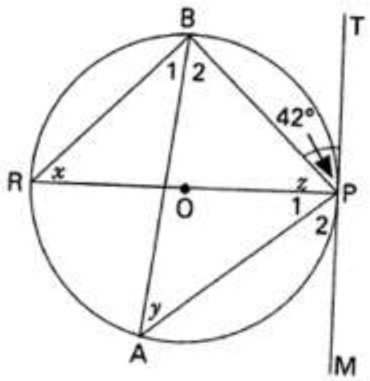
TPB is a tangent at P. ST || QP. $\hat{T} = 64°$; $\hat{Q} = 80°$.



Determine the sizes of x, y and z.. (10)

QUESTION 3 (10 MARKS) (10 MINUTES) (Based on Classroom Mathematics Grade 11 p 277 Ex 10.7 no 1c)

O is the centre of the circle and MPT is a tangent at P. $B\hat{P}T = 42°$. RPO is a diameter.



3.1 Give with reasons the size of

a) $R\hat{P}M$ (2)

b) $R\hat{B}P$ (2)

3.2 Determine the sizes of x, y and z.. (6)

SESSION NO: 2

TOPIC: EUCLIDEAN GEOMETRY EXAM TYPE QUESTIONS 1: (GRADE 11)

Learner Note: In this session we will answer the actual exam questions from the Grade 11 Exemplar 2017. HINTS: Read the given information and underline important words like centre, diameter, tangent, bisects, and so on. Fill in the given information on the diagram. Highlight centres, parallel lines and tangents on the diagram. Mark radii equal. Read the information again and read **all** the questions so you do not answer for example: question 1.2 in question 1.1. Read the information again and answer the first sub-question. Fill in sizes or lengths you have worked out on the diagram. They can help when answering the next sub- question. For each statement made in a calculation or proof, a reason must be given. Reasons are usually written in round brackets after the mathematical statement. Conclusions usually do not require a reason. Conclusions usually follow a therefore sign. The only time a reason is required in a conclusion is when a converse of a theorem is the conclusion. Proving triangles to be congruent or similar have their own methods of proof.

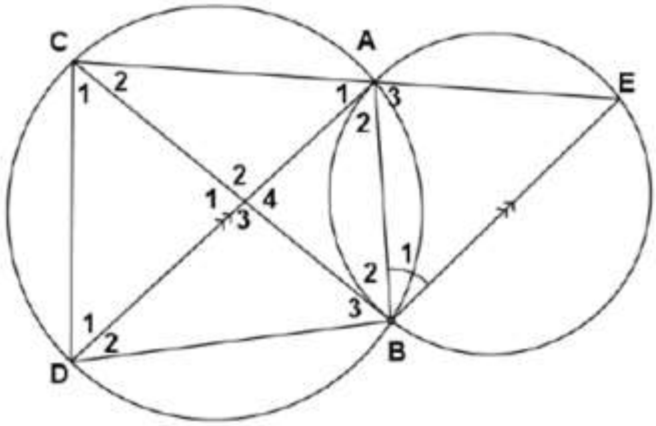
SECTION A: TYPICAL EXAM QUESTIONS 1

Try to answer Questions 2 to 4 under exam conditions:

1 Clearly show ALL calculations, diagrams, graphs, et cetera, which you have used in determining the answers.
2 An approved scientific calculator (non-programmable and non-graphical) may be used, unless stated otherwise.
3 If necessary, answers should be rounded off to TWO decimal places, unless stated otherwise.
4 Diagrams are not drawn to scale

QUESTION 1 (16 MARKS) (16 MINUTES) (CAPS Grade 11 Exemplar 2017)

In the diagram below two circles intersect at A and B. BE is a tangent to the bigger circle at B and a chord of the smaller circle. DA||BE and DA bisects $C\hat{A}B$.
$A\hat{B}E = 40°$.

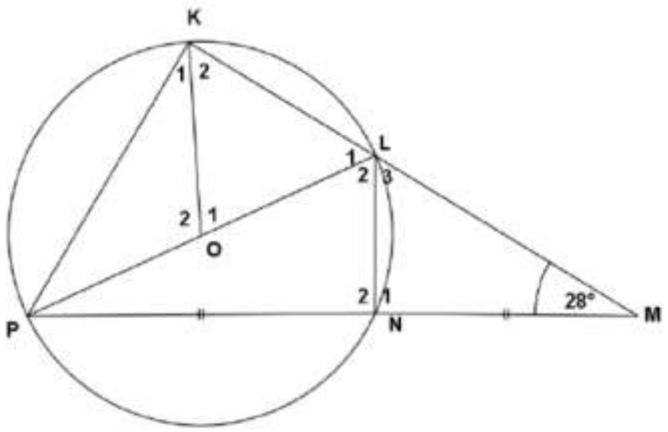


1.1 Name, with reasons, SEVEN other angles equal to $A\hat{B}E = 40°$. (14)

1.2 Determine the size of $\hat{A}_3$. (2)

QUESTION 2 (11 MARKS) (11 MINUTES) (CAPS Grade 11 Exemplar 2017)

In the diagram below PL is the diameter of the circle with centre O. Chord PN is produced to M such that PN = NM. $P\hat{M}K = 28°$.



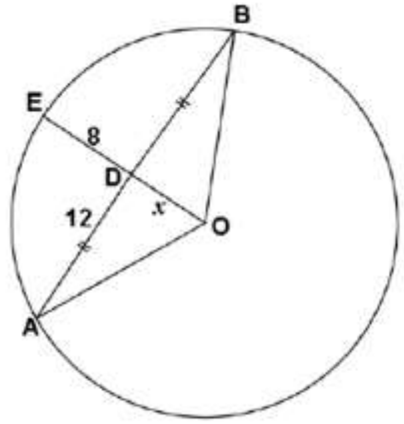
Calculate the sizes of:

2.1 $L\hat{P}N$ (6)

2.2 $K\hat{O}P$ (5)

QUESTION 3 (5 MARKS) (5 MINUTES) (CAPS Grade 11 Exemplar 2017)

AB is a chord of a circle with centre O. OE bisects AB. AD = 12cm, ED = 8cm and OD = x.

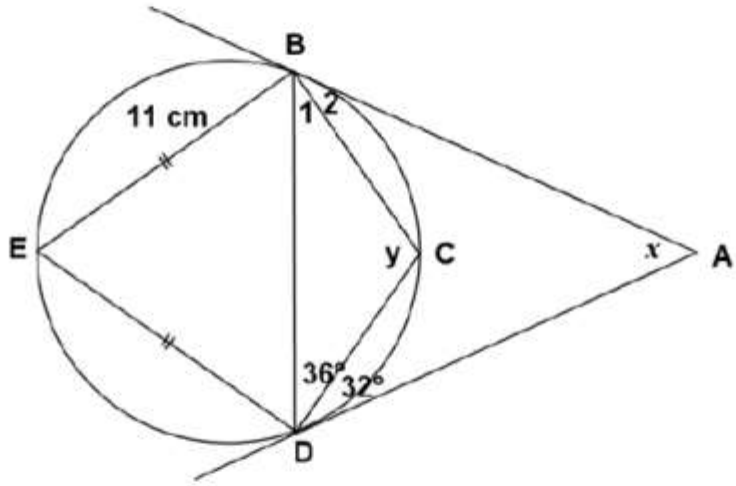


3.1 Determine the radius OB in terms of x. (1)

3.2 Hence calculate the length of the radius OB. (4)

QUESTION 4 (6 MARKS) (6 MINUTES) (CAPS Grade 11 Exemplar 2017)

In the diagram below AB and AD are tangents to the circle at B and D respectively. $B\widehat{D}C = 36°$; $A\widehat{D}C = 32°$. [Ignore EB = ED; and EB = 11cm on the diagram]



4.1 Determine the value of x. (3)

4.2 Determine the value of y. (3)

SECTION B: NOTES ON CONTENT

Below are some words or phrases that come up in the questions:

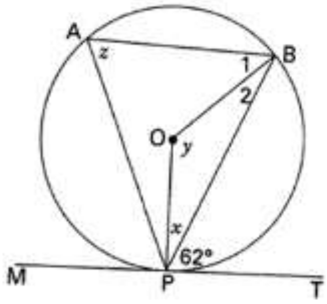
To **intersect** is to touch or cross at some point.
A **tangent** to a circle is a line which no matter how far it is extended touches the circle at one point only
A **chord** is a line that goes from one side of a circle to the other.
To **bisect** is to cut into two equal parts.
A **diameter** is a chord that passes through the centre of a circle.
Produced to is the same as extending from one point to another.
A **radius** is a straight line from the centre of a circle to the curve of the circle.
At B and D respectively means first at B then at D.
A **cyclic quadrilateral** has all four of its corners (vertices) on the edge of a circle.

SECTION C: HOMEWORK QUESTIONS

Learner note:
Try to answer these under exam conditions:

QUESTION 1 (8 MARKS) (8 MINUTES)
(Classroom Mathematics p 277 Ex 10.7 no 1 a)

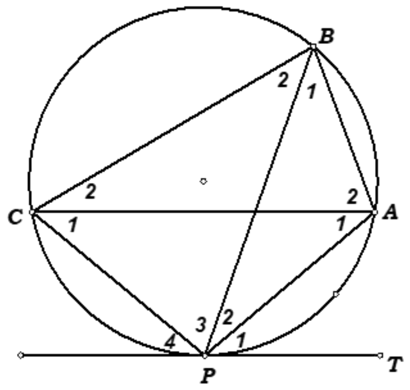
O is the centre of the circle and MPT is a tangent at P.



Find the sizes of x, y and z. (8)

QUESTION 2 (12 MARKS) (12 MINUTES) (DoE paper pre-2014)

In the diagram, BCPD is a cyclic quadrilateral. BP and CA are drawn. PT is a tangent to the circle at P, and PC = PA.

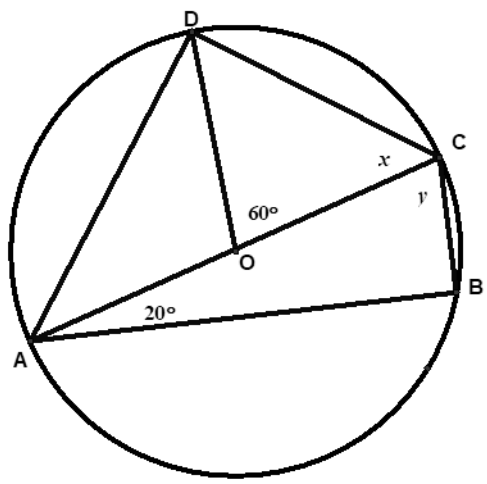


2.1 Find 5 angles equal to $\hat{P}_1$, giving reasons. (10)

2.1 Under what conditions will $\hat{P}_1 = 45^\circ$? (2)

QUESTION 3 (8 MARKS) (8 MINUTES) (RADMASTE CENTRE EDUC 0086 Book 1)

O is the centre of the circle. AOC is the diameter.



3.1 Find the values of *x* and *y.* (6)

3.2 What type of triangle is ODC? Why? (2)

SESSION NO: 3

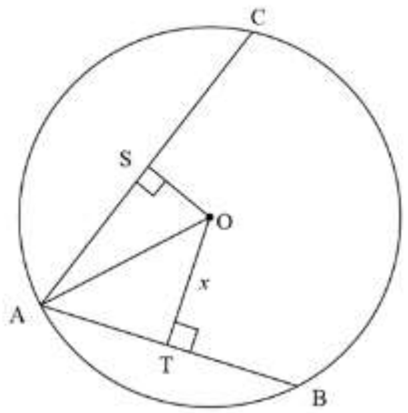
TOPIC: EUCLIDEAN GEOMETRY EXAM TYPE QUESTIONS 2: (GRADE 11)

SECTION A: TYPICAL EXAM QUESTIONS 2

1. Clearly show ALL calculations, diagrams, graphs, et cetera, which you have used in determining the answers.
2. An approved scientific calculator (non-programmable and non-graphical) may be used, unless stated otherwise.
3. If necessary, answers should be rounded off to TWO decimal places, unless stated otherwise.
4. Diagrams are not drawn to scale

QUESTION 1 (6 MARKS) (6 MINUTES) (DoE paper 2016 adapted)

A, B and C are points on the circle having centre O. S and T are points on AC and AB respectively such that OS⊥AC and OT⊥AB. AB = 40 and AC = 48.



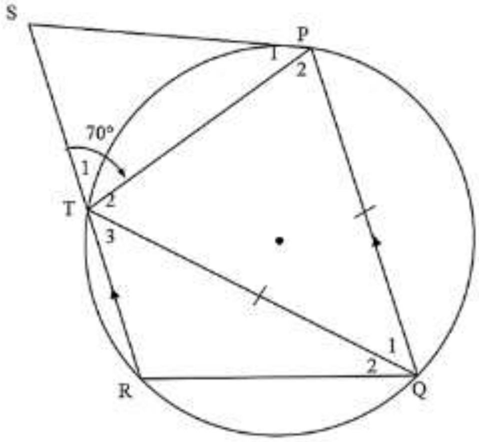
1.1 Calculate AT. (1)

1.2 If OA = 25, calculate x. (2)

1.3 Determine the value of $\frac{\text{OS}}{\text{OT}}$. (3)

QUESTION 2 (6 MARKS) (6 MINUTES) (DoE Maths paper-2016)

In the diagram below PQRT is a cyclic quadrilateral having $RT \parallel QP$. The tangent at P meets RT produced at S. $QP = QT$ and $P\hat{T}S = 70°$.



2.1 Give a reason why $\hat{P}_2 = 70°$. (1)

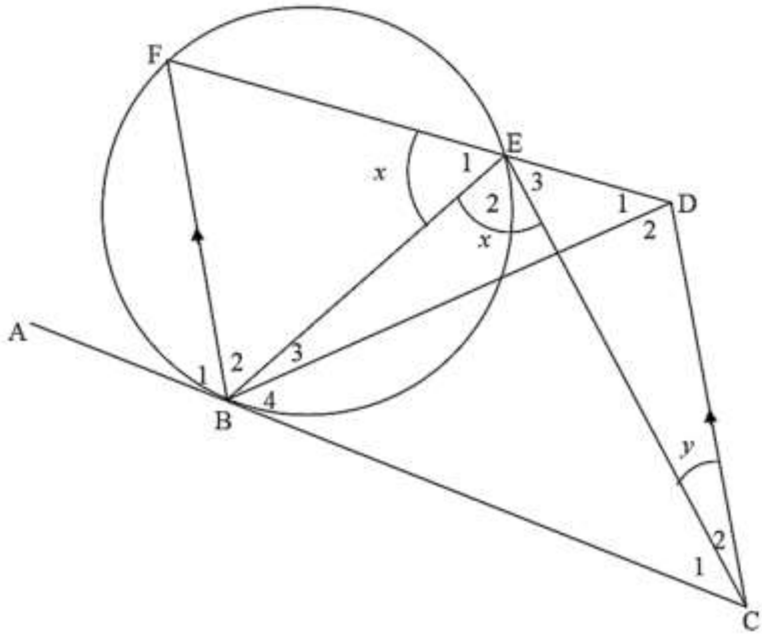
2.2 Calculate, with reasons, the size of:

(a) $\hat{Q}_1$ (3)

(b) $\hat{P}_1$ (2)

QUESTION 3 (7 MARKS) (7 MINUTES) (DoE Maths paper-2016)

ABC is a tangent to the circle BFE at B. From C a straight line is drawn parallel to BF to meet FE produced at D. EC and BD are drawn. $\hat{E}_1 = \hat{E}_2 = x$ and $\hat{C}_2 = y$.



3.1 Give a reason why EACH of the following is TRUE:

a) $\hat{B}_1 = x$ (1)

b) $B\hat{C}D = \hat{B}_1$ (1)

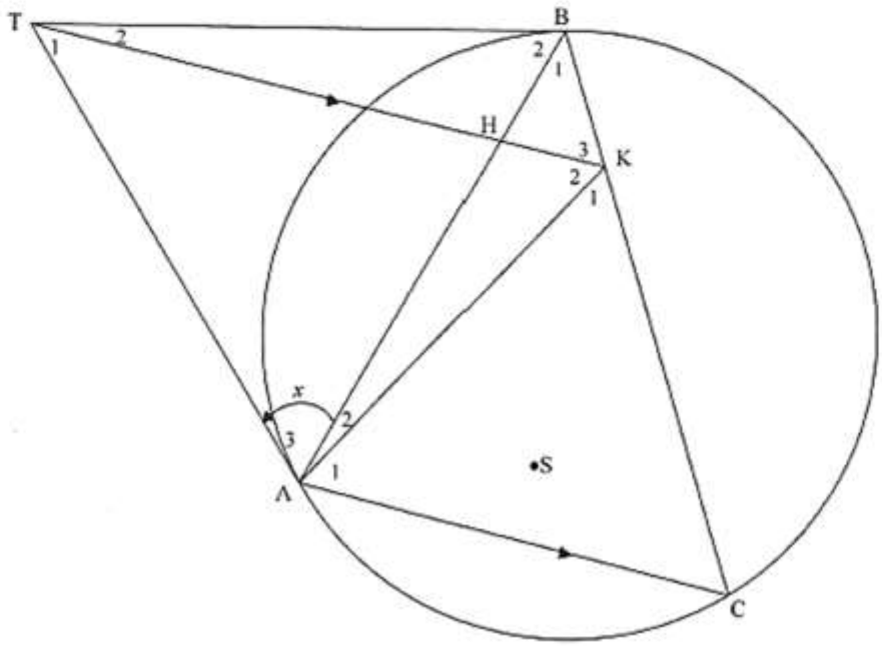
3.2

If BCDE is a cyclic quadrilateral, which TWO other angles are each equal to x? (2)

3.3 Prove that $\hat{B}_2 = \hat{C}_1$. (3)

QUESTION 4 (4 MARKS) (4 MINUTES) (DoE Maths paper-2015)

In the diagram below, ΔABC is drawn in the circle. TA and TB are tangents to the circle. The straight line THK is parallel to AC with H on BA and K on BC. AK is drawn. Let $\hat{A}_3 = x$.



Prove that $\hat{K}_3 = x$. (4)

SECTION B: NOTES ON CONTENT

Below are some symbols and phrases that come up in the questions:

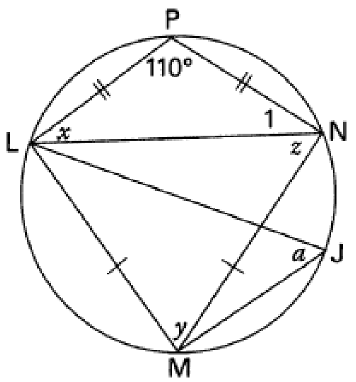
⊥: is parallel to
||: is perpendicular to
A **common tangent** is a tangent that is shared between circles.
A **double chord** is a chord that stretches over two circles.

SECTION C: HOMEWORK QUESTIONS

Learner note:
Try to answer these under exam conditions:

QUESTION 1 (9 MARKS) (9 MINUTES)
(Classroom Mathematics p 269 Ex 10.4 no 1 d)

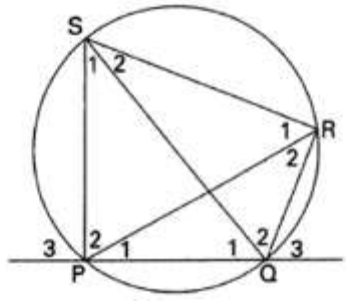
In the diagram, PL = PN; LM = MN and $\hat{P} = 110°$.



Find the size of $a; x; y$ and z. (9)

QUESTION 2 (10 MARKS) (10 MINUTES)
(Classroom Mathematics p 269 Ex 10.4 no 2 adapted)

In the diagram, $\hat{P}_3 = 80°$; $\hat{Q}_3 = 70°$ and $\hat{S}_2 = 40°$.



Determine the size of

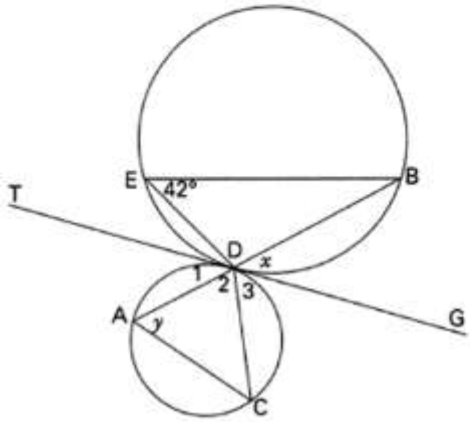
2.1 $\hat{S}_1$ (2)

2.2 $\hat{R}_1$ (4)

2.3 $\hat{P}_2$. (4)

QUESTION 3 (8 MARKS) (8 MINUTES)
(Classroom Mathematics p 280 Ex 10.4 no 5 adapted)

In the diagram, TDG is a common tangent. ADB is a double chord; $\widehat{D}_2 = 70°$ and $\hat{E} = 42°$.



Find

3.1 x (2)

3.2 y (6)

SESSION NO: 4

TOPIC: COMPLEX NUMBERS OVERVIEW

Learner note:
In this session we revise and practice the basics of complex numbers. A summary of the basics is given in **SECTION B: NOTES ON CONTENT.**
The following applies to answering all questions.

1. Clearly show ALL calculations, diagrams, graphs, et cetera, which you have used in determining the answers.
2. An approved scientific calculator (non-programmable and non-graphical) may be used, unless stated otherwise.
3. If necessary, answers should be rounded off to TWO decimal places, unless stated otherwise.
4. Diagrams are not drawn to scale

SECTION A: PRACTICING THE BASICS OF COMPLEX NUMBERS

QUESTION 1 (4 marks) (2 minutes)

1.1 For the complex number $3 + 2i$
write the a) real part
b) imaginary part (2)

1.2 For the complex number $-3i$
write the a) real part
b) imaginary part (2)

QUESTION 2 (6 marks) (3 minutes)

Simplify

2.1 $2i.3i$ (2)

2.2 $2(5i^3)$ (2)

2.3 $2i^4$ (2)

QUESTION 3 (6 marks) (6 minutes)

Express each of the following in terms of i

3.1 $\sqrt{-144}$ (2)

3.2 $\sqrt{-21}$ (2)

3.3 $\sqrt{-75}$ (2)

QUESTION 4 (8 marks) (8 minutes)

Simplify the following without a calculator and write your answer in the form $a + bi$.

4.1 $(2+i)(3-i)$ (3)

4.2 $-\sqrt{144}-\sqrt{-1}+\sqrt{-25}$ (3)

4.3 $\frac{3+2i}{-3}$ (2)

QUESTION 5 (9 marks) (9 minutes)

5.1 Write down the complex conjugate,$\bar{z}$, for

$z = 1+3i$ (1)

5.2 Simplify the given complex number to rectangular (standard) form:

$\frac{3+i}{1+3i}$ (5)

5.3 Calculate $\bar{z} \times \bar{z}$ and write the final answer in rectangular form. (3)

QUESTION 6 (4 marks) (4 minutes)

Solve for x and y if $x, y \epsilon \mathbb{R}$:

$i-(x+iy)+2(x+iy)-3i=-2+7i$ (4)

QUESTION 7 (7 marks) (7 minutes)

Show the following complex numbers on an Argand diagram.

7.1 $z_1 = -2+i$ (2)

7.2 $z_2 = 2i$ (2)

7.3 z_3 the conjugate of z_1 (3)

QUESTION 8 (6 marks) (6 minutes)

Give your answers correct to 2 decimal places.

8.1 Use a formula to calculate the modulus (r) of $z_1 = -2+i$ (2)

8.2 Use a formula to calculate the value of the argument (θ) of $z_1 = -2+i$ (2)

8.3 Show the modulus and argument of $z_1$on the Argand diagram drawn in QUESTION 6. [Check your answer using the calculator method] (2)

QUESTION 9 (7marks) (7 minutes)

9.1 Convert the complex number $z = 5\ cis\ 240°$ to rectangular form $z = a + bi$.
Give your answer in surd form. (3)

9.2 Plot the complex number $z = 2|\underline{150°}$ on an Argand diagram and find its rectangular form. Give your answer correct to 2 decimal places. (4)

SECTION B: NOTES ON CONTENT

1. Complex numbers ($\mathbb{C}$) consist of all the real numbers ($\mathbb{R}$) and all the imaginary numbers ($\mathbb{I}$).

$\mathbb{C}$

$\mathbb{R}$ and $\mathbb{I}$

$a + bi$

Real part   Imaginary part

2. Imaginary numbers ($\mathbb{I}$)are any numbers that can be written in the form bi and are the square roots of negative numbers.

$$\boxed{\text{If } i = \sqrt{-1}\text{, then } i^2 = -1}$$

3. Operations on complex numbers
   - Add and subtract like terms
   - When multiplying remember that $i^2 = -1$
   - When dividing: Use the conjugate to rationalise the denominator
     Divide the real and imaginary parts separately

4. Formats of complex numbers

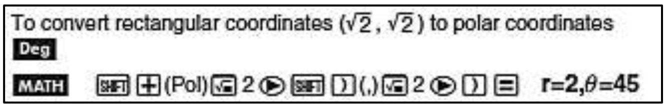
Rectangular form (or **standard form**) $z = a + bi$	where a and b are real numbers and $i = \sqrt{-1}$
Conjugate form $\bar{z} = a - bi$	$\bar{z}$ is the ***complex conjugate*** of z.
Polar forms (or **trigonometric forms**) $z = r(\cos\theta + i\sin\theta)$	$\boldsymbol{r} = \sqrt{a^2 + b^2}$ and $\theta = \tan^{-1}\left(\frac{b}{a}\right)$ use when converting from rectangular form $\sin\theta = \frac{b}{r}$ so, $b = r\sin\theta$ $\cos\theta = \frac{a}{r}$ so, $a = r\cos\theta$ use when converting to rectangular form

Abbreviations of polar forms: $z = r\ cis\ \theta$	where ***cis*** stands for $\underline{c}os\ \theta + \underline{\boldsymbol{i}}\ \underline{s}in\ \theta$
$z = r\ \lvert\underline{\theta}$	

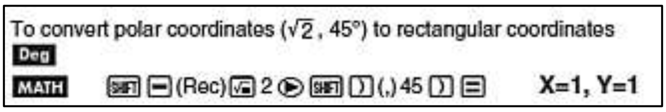
5. Using a calculator to convert between the rectangular and polar forms

USING THE CONVERSION KEY ON THE CALCULATOR TO CONVERT BETWEEN THE RECTANGULAR AND POLAR FORM OF A COMPLEX NUMBER

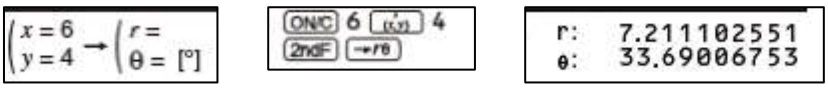
1. **CASIO *fx-82ZA PLUS***
   - **Converting from rectangular co-ordinates to polar co-ordinates:**
     Use the [Pol] key (above the [+] key, so to access it you must press [SHIFT] [+]).
     For example:



   - **Converting from polar co-ordinates to rectangular co-ordinates:**
     Use the [Rec] key (above the [–] key, so to access it you must press [SHIFT] [–]).
     For example



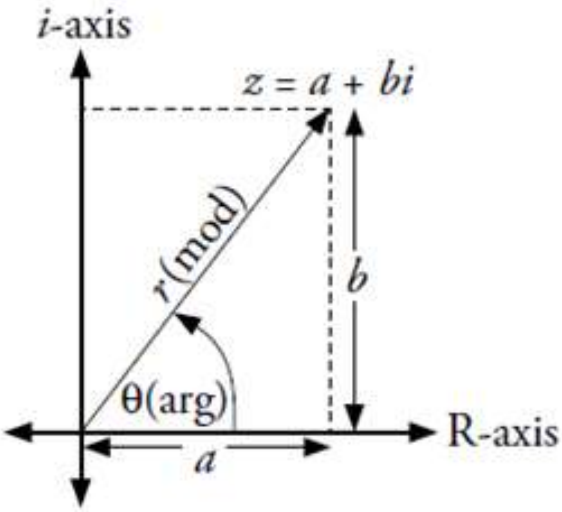
2. **SHARP EL535**
   - **Converting from rectangular co-ordinates to polar co-ordinates:**
     Use the [$\rightarrow r\theta$] and $\left[\frac{,}{(x;y)}\right]$ keys. To access these keys you must press [2nd F] each time.
     For example:



   - **Converting from polar co-ordinates to rectangular co-ordinates:**
     Use the [$\rightarrow xy$] and $\left[\frac{,}{(x;y)}\right]$ keys. To access these keys you must press [2nd F] each time.
     For example

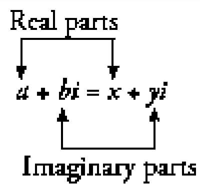
6. Plotting z in the Complex plane:

- An **Argand diagram** is a graphical representation of a complex number.
- The x-axis (horizontal axis) is used to plot the ***real part*** of the complex number. It is the *real axis.*
- The y-axis (vertical axis) is used to plot the ***imaginary part*** of the complex number. It is the *imaginary axis.*



- z is the point where the dotted lines meet Its coordinates are:
  (*x; y*) = (real part; coefficient of imaginary part)
  = (*a; b*)
- The length of the line segment (r) is called the ***modulus*** of the complex number.
- The angle from the positive real axis to the line segment (θ) is called the ***argument*** of the complex number. ***This is often written arg(z) = θ.°***

7. Solving complex equations



So, $a = x$ and $b = y$ where , $x, y \in \mathbb{R}$

Worked Example 1
Solve for x and y, $x, y \epsilon \mathbb{R}$ if $4x + 4yi - 2 + 5i = 6 - 7i$

Solution
STEP1: Write the LHS and the RHS of the equation in the form $z = a + bi$
$4x + 4yi - 2 + 5i = 6 - 7i$
$4x - 2 + 4yi + 5i = 6 - 7i$
$(4x - 2) + (4y + 5)i = 6 - 7i$
STEP 2: Compare the real and imaginary parts of the two complex numbers:
$4x - 2 = 6$ and $4y + 5 = -7$
$4x = 8$ and $4y = -12$
$\therefore x = 2$ and $\therefore y = -3$

Worked Example 2
Sometimes you will have to solve simultaneous equations to determine x and y:
$(x + 2y) + (x - 2y)i = 3 + 7i$
$x + 2y = 3$ and $x - 2y = 7$
$\therefore x = -2y + 3$ substitute into $x - 2y = 7$:
$(-2y + 3) - 2y = 7$
$-4y = 4$
$\therefore\ y = -1$ substitute into
$x = -2y + 3$
$= -2(-1) + 3$
$\therefore x = 5$

8. Applications of Complex numbers
- electrical engineering, physics and quantum mechanics.
- Equations having roots that are negative numbers, for example $x = \pm\sqrt{-16}$, have solutions in the complex number system.

See notes in Section 5.

SECTION C: HOMEWORK QUESTIONS

QUESTION 1 (4 marks) (2 minutes)

1.1 For the complex number $5 - 4i$
write the a) real part
b) imaginary part (2)

1.2 For the complex number 2
write the a) real part
b) imaginary part (2)

QUESTION 2 (10 marks) (10 minutes)

Simplify

2.1 $-2i.3i$ (2)

2.2 $-4i^2 \times i^3$ (2)

2.3 $\left(1 + \sqrt{5}i\right)^2$ (2)

2.4 $\left(1 + \sqrt{5}i\right)\left(1 - \sqrt{5}i\right)$ (2)

2.5 $\sqrt{-25} - \sqrt{-7}$ (2)

QUESTION 3 (3 marks) (3 minutes)

Simplify the following without a calculator and write your answer in the form $a + bi$.

$\frac{3+2i}{-3i}$ (3)

QUESTION 4 (3 marks) (3 minutes)

Solve for x and y if $x, y \epsilon \mathbb{R}$:

$3x + 14yi - 6 = 6 + 7i$ (3)

QUESTION 5 (4 marks) (4 minutes)

5.1 Write down the complex conjugate, $\bar{z}$, for

$z = \sqrt{15} - 2i$ (1)

5.2 Show z and $\bar{z}$ on an Argand diagram. (3)

QUESTION 6 (7 marks) (7 minutes)

Give your answers correct to 2 decimal places.

6.1 Use a formula to calculate the modulus (r) of $z = 1 + 2\sqrt{2}\,i$ (2)

6.2 Use a formula to calculate the value of the argument (θ) of $z = 1 + 2\sqrt{2}\,i$ (2)

6.3 Write $z = 1 + 2\sqrt{2}\,i$ in polar (trigonometric) form (3)

QUESTION 7 (6 marks) (6 minutes)

Given the complex number $z = 2\ cis\ 135°$

7.1 Plot $z = 2\ cis\ 135°$ on an Argand diagram (3)

7.2 Convert $z = 2\ cis\ 135°$ to rectangular form $z = a + bi$. Give your answer in surd form. (3)

SESSION NO: 5

TOPIC: TYPICAL EXAMINATION QUESTIONS ON COMPLEX NUMBERS

Learner Note: In this session we will answer questions expected in the exam, these include application questions. See **SECTION B: NOTES ON CONTENT Using complex numbers in electricity questions**

The following applies to answering all questions.

1 Clearly show ALL calculations, diagrams, graphs, et cetera, which you have used in determining the answers.

2 An approved scientific calculator (non-programmable and non-graphical) may be used, unless stated otherwise.

3 If necessary, answers should be rounded off to TWO decimal places, unless stated otherwise.

4 Diagrams are not drawn to scale

SECTION A: TYPICAL EXAMINATION QUESTIONS

QUESTION 1 (8 MARKS) (8 MINUTES)

Simplify the following complex numbers. Give answers in the form $a + bi$.

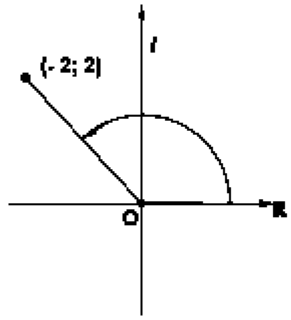
1.1 $-\sqrt{16} - \sqrt{-1} + \sqrt{-81}$ (3)

1.2 $3 + 14i^3 - 6 - (6 + 7i)$ (3)

1.3 $\frac{7+i}{2}$ (2)

QUESTION 2 (8 MARKS) (8 MINUTES)

Refer to the diagram below and answer the questions that follow.



2.1 Determine the rectangular form of the complex number z. (2)

2.2 Calculate the modulus (r) of z (2)

2.3 Calculate the argument (θ) of z (2)

2.4 Write z in polar form. (2)

QUESTION 3 (3 MARKS) (3 MINUTES)

Simplify the following to the form $a + bi$ without converting to polar form.

$(1+i)(5+3i)$ (3)

QUESTION 4 (2 MARKS) (2 MINUTES)

Express the following complex number in rectangular form:

$2\sqrt{2}(\cos 315° + i\cos 315°)$ (2)

QUESTION 5 (4 MARKS) (4 MINUTES)

Solve for x and y if $y - 4i = (2-i)(3x+i)$ and x and y are real numbers (4)

QUESTION 6 (4 MARKS) (4 MINUTES)

Impedance opposes the flow of current in an electrical circuit . Calculate the total impedance in a parallel circuit if $Z_T = \frac{Z_1 Z_2}{Z_1 + Z_2}$ and $z_1 = 2 - 3i$ and $z_2 = 1 + 4i$. (4)

SECTION B: NOTES ON CONTENT

Using complex numbers in electricity questions

In an alternating current circuit, the voltage (*E*), the current (*I*) and the impedance (*Z*) are related by the formula *E = IZ.*

Calculate the voltage in a circuit with current (1 + 4*i*) amperes, and impedance (3 – 6*i*) ohms.

Current = $I = (1 + 4i)$ amperes
Impedance $= Z = (3 - 6i)$ ohms
Voltage $= E = IZ$

$$\begin{aligned} &= (1 + 4i)(3 - 6i) \\ &= 1(3 - 6i) + 4i(3 - 6i) \\ &= 3 - 6i + 12i - 24i^2 \\ &= 3 + 6i - 24(-1) \\ &= 27 + 6i \text{ volts} \end{aligned}$$

Impedance opposes current flow.

SECTION C: HOMEWORK: TYPICAL EXAMINATION QUESTIONS

QUESTION 1 (7 MARKS) (7 MINUTES)

Simplify the following complex numbers. Give answers in in the form $a + bi$.

1.1 $\sqrt{-5} - \sqrt{-2} + \sqrt{-4}$ (2)

1.2 $(2i)^3 + 14i^2 - (9 + 2i)$ (3)

1.3 $(1 + i)(5 - 3i)$ (2)

QUESTION 2 (13 MARKS) (13 MINUTES)

21. Write down the conjugate of $z = \frac{3}{5} + \frac{4}{5}i$ (2)

2.2 Represent z and its conjugate ($\bar{z}$) on an Argand diagram. (3)

2.3 Calculate the modulus (r) of $z = \frac{3}{5} + \frac{4}{5}i$ (2)

2.4 Calculate the argument (θ) of $z = \frac{3}{5} + \frac{4}{5}i$ (2)

2.5 Indicate the r and θ just calculated on an Argand diagram (2)

2.6 Express $z = \frac{3}{5} + \frac{4}{5}i$ in polar form (2)

QUESTION 3 (4 MARKS) (4 MINUTES)

Simplify the following to the form $a + bi$ without converting to polar form.

$$\frac{3-2i}{i-3}$$ (4)

QUESTION 4 (3 MARKS) (3 MINUTES)

Express the following complex number in rectangular form:

$$5(\cos 0° + i \cos 0°)$$ (2)

QUESTION 5 (5 MARKS) (5 MINUTES)

Solve for x and y if $(1 - 2i)(x + 3i) = y - 3i$ and x and y are real numbers (4)

QUESTION 6 (4 MARKS) (4 MINUTES)

The relationship between voltage. E (volts), current, I (amps) and impedance, Z (ohms), in an alternating current circuit, is given by $E = I.Z$ If the current in a circuit is $2 + 6i$ amps and the impedance is $3 - 4i$ ohms, calculate the voltage. (4)

SESSION NO: 6

TOPIC: CIRCLES, ANGLES AND ANGULAR MOVEMENT

Learner Note: In this session we will answer questions expected in the exam, these include application questions. For homework the November 2017 paper has been given. See **SECTION B: NOTES ON CONTENT**

The following applies to answering all questions.

1 Clearly show ALL calculations, diagrams, graphs, et cetera, which you have used in determining the answers.

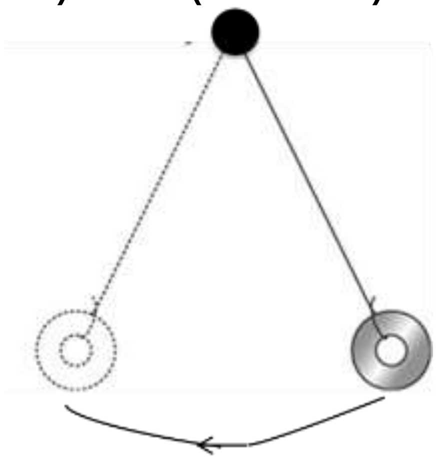
2 An approved scientific calculator (non-programmable and non-graphical) may be used, unless stated otherwise.

3 If necessary, answers should be rounded off to TWO decimal places, unless stated otherwise.

4 Diagrams are not drawn to scale

SECTION A: TYPICAL EXAM QUESTIONS

QUESTION 1 (4 MARKS) (4 MINUTES)



A pendulum swings through an angle of 1,3 radians in a single swing. The length of the arc of the sector it forms is 50 cm

1.1 Calculate the length of the radius of the circle in cm.

Give your answer in decimal form (2)

1.2 If the radius is 38 cm, calculate the area of the sector.

(Hint: Use Area of sector = $\frac{1}{2}r^2\theta$) (2)

QUESTION 2 (2 MARKS) (2 MINUTES)

Calculate the length of an arc that subtends a central angle of 32^0 in a circle with a diameter of 35 cm. (2)

QUESTION 3 (5 MARKS) (5 MINUTES)

The diameter of a circle is 25 mm. A chord of length 20 mm divides the circle into two segments Calculate the heights of the segments. (5)

QUESTION 4 (3 MARKS) (3 MINUTES)

A circle has a radius of 35 mm. The size of the angle subtended at centre of the circle is 75°. Calculate the area of the sector of this circle.
Give your answer to the nearest square millimetre. (3)

QUESTION 5 (4 MARKS) (4 MINUTES)

A clothes drier rotates at 540 revolutions per minute

5.1 Convert 540 revolutions per minute to revolutions per second. (1)

5.2 Calculate the angular velocity. (3)

QUESTION 6 (6 MARKS) (6 MINUTES) (Gr 11 Exemplar-2017)

A train takes 40 minutes to cover two-thirds of a revolution on a circular track. The diameter of the track is 8 km.	

Calculate

6.1 the peripheral velocity of a train in kilometres per hour (4)

6.2 the time it takes to complete 4 revolutions. (2)

QUESTION 7 (3 MARKS) (3 MINUTES)

A bicycle has a rev counter on its front wheel that counts the number of revolutions made from a starting point.

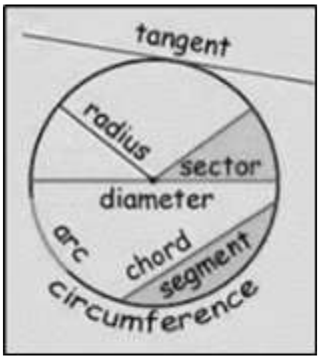
A cyclist resets the rev counter to zero before a trial race of 4 km. At the end of the trial, the rev counter reads 1 852.
What is the diameter of the front bicycle wheel in cm?

QUESTION 8 (3 MARKS) (3 MINUTES)

A toy racing car travels at 0,31 radians per second around a circular track that is 3,2 m in diameter. Find the linear velocity in metres per second. (3)

SECTION B: NOTES ON CONTENT

1. Circles and semi-circles



r the radius of the circle d the diameter of the circle

Diameter $d = 2r$ **Radius** $r = \frac{d}{2}$

Diameter $D = h + \frac{x^2}{4h}$ where x is the length of a chord and h is the height of the segment formed by the chord.

Area of the circle (A) = πr^2 or (A) = $\pi\left(\frac{d}{2}\right)^2 = \frac{\pi d^2}{4}$

Circumference of a circle (C) = $2\pi r$ or (C) = πd

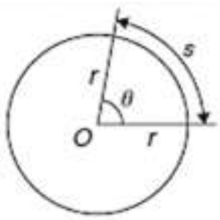
The **equation of a circle, with centre at the origin**, and with radius r is $x^2 + y^2 = r^2$

The **equation of a semi-circle, with centre at the origin** and with radius r is

$y = \sqrt{r^2 - x^2}$ in the 1st and 2nd quadrants

$y = -\sqrt{r^2 - x^2}$ in the 3rd and 4th quadrants

2. Arcs, sectors and radians



***r* is the radius of the circle**

***θ* is the angle at centre is in radians**

Length of an arc $s = r\theta$

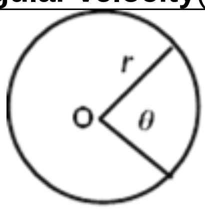
From $s = r\theta$, ***θ* radians** $= \frac{s}{r}$

- 1 radian = $\frac{180°}{\pi}$
- $1°$ = $\frac{\pi}{180}$ radians
- $2\pi = 360°$ is the number of radians in a revolution

Perimeter of an arc = (2 × radius) + arc length
= $(2 \times r) + r\theta$

Area of a sector of a circle = $\frac{1}{2}r^2\theta$

3. Angular velocity(ω) (omega)



***r* is the radius of the circle**

***θ* is the angle at centre**

Angular velocity (ω) tells us how **fast an angle is changing** with respect to time. ω measures the velocity at the **centre** of a turning wheel or disc.

Formulae:

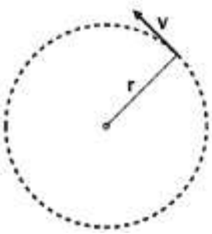
$\boldsymbol{\omega} = \frac{\boldsymbol{\theta}}{\boldsymbol{t}}$ where θ is the angular displacement in radians

Unit $\text{rad.}s^{-1}$

OR

	For 1 revolution per second $\boldsymbol{\omega} = \frac{\boldsymbol{\theta}}{\boldsymbol{t}} = \frac{\mathbf{2\pi}}{\mathbf{1\ second}} =$ $\mathbf{2\pi}$ rad.s^{-1} For n revolutions per second $\boldsymbol{\omega} = \mathbf{2\pi n}$ rad.s^{-1} • n is also called the rotational frequency Units can be in ***revolutions per second (rev/s)*** or ***radians per second (rad/s)*** Angular displacement is the angle θ through which the disc turns, i.e: $\omega = \frac{\theta \text{ measured in radians}}{t \text{ measured in seconds}}$

3. Linear (circumferential) (peripheral) velocity $(\boldsymbol{v})$

 **$\boldsymbol{r}$ is the radius of the circle** **$\boldsymbol{D}$ is the diameter of the circle**	Linear velocity tells us how fast the **object is travelling** with respect to time. v measures the velocity **along the edge of a circle** – along the tangent to the circle **Formulae:** From $v = \frac{\text{displacement}}{\text{time}} = \frac{\text{displacement along circumference}}{\text{time to travel this distance}}$ **Circumference of a circle** = $2\pi r$ = πD in metres where $\boldsymbol{D}$ is the diameter of the circle For 1 revolution per second $\boldsymbol{v} = \frac{\boldsymbol{\pi D}}{\mathbf{1}}$ m.s^{-1} For n revolutions per second $\boldsymbol{v} = \boldsymbol{\pi D n}$ m.s^{-1} For n revolutions per hour $\boldsymbol{v} = \boldsymbol{\pi D n}$ km.h^{-1} **OR** In terms of $\omega = \frac{\theta}{t}$: the linear velocity is $\boldsymbol{v} = \frac{\boldsymbol{\theta}}{\boldsymbol{t}}\ \boldsymbol{r}$ rad.s^{-1} but $\boldsymbol{\omega} = \frac{\boldsymbol{\theta}}{\boldsymbol{t}}$ so $\boldsymbol{v} = \boldsymbol{\omega r}$ **Example:** Determine the linear velocity of an object with an angular velocity of 3 rad/min around a circle whose radius is 5 m. **Solution:** $v = \omega r$ [$\omega = 3\frac{rad}{min}$; $r = 5$] = 3(5) = 15 m/min

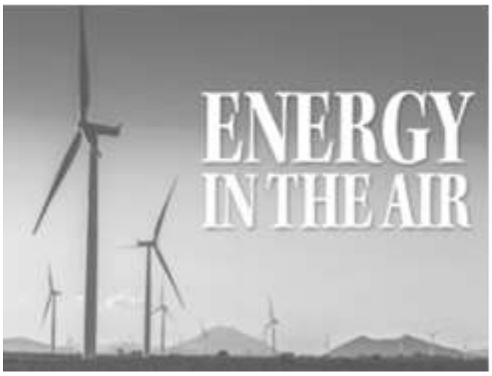
4. Note:

- If you have an angle in π format, you convert it to degrees by multiplying by $\frac{180°}{\pi}$
- If you have a length in π format, use the π button on the calculator to simplify the length.

SECTION C: HOMEWORK QUESTIONS

QUESTION 1(4 MARKS) (4 MINUTES)(2017 Grade 11 Technical Maths Paper 2)

A wind turbine with a large bladed wheel rotated by the wind to generate electricity. The turbine rotates at **45** revolutions per minute.



1.1 Convert 45 revolutions per minute to revolutions per seconds. (1)

1.2 Calculate the angular velocity. (3)

QUESTION 2 (9 MARKS) (9 MINUTES) (2017 Grade 11 Technical Maths Paper 2)

A point on the circumference of a driver wheel driving a belt moves through 2 m every second. The diameter of the driver wheel is 0,4 m. Ignore the thickness of the belt.

2.1 Calculate the distance covered by the point in 10 s. (2)

2.2 Calculate the rotational frequency of the driver wheel. (Hint: Use $\boldsymbol{v} = \pi D n$) (3)

2.3 Calculate the time taken for the driver to make 12 revolutions. (4)

QUESTION 3 (3 MARKS) (3 MINUTES) (2017 Grade 11 Technical Maths Paper 2)

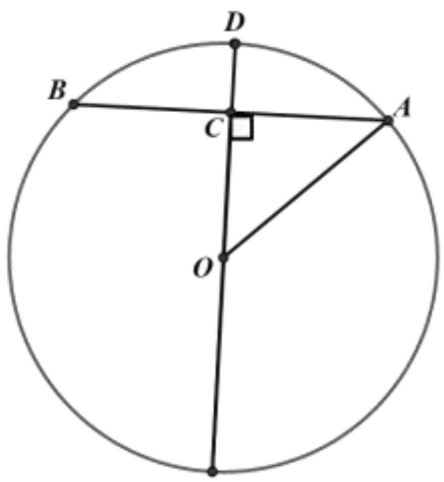
The arc of a sector is 4,3m. It subtends an angle of 2,8 radians at the centre of the circle.

3.1 Calculate the radius of the circle in cm. (1)

3.2 Calculate the area of the sector. (Hint: Use Area of sector = $\frac{\theta r^2}{2}$) (2)

QUESTION 4 (7 MARKS) (7 MINUTES) (2017 Grade 11 Technical Maths Paper 2)

In the given diagram, O is the centre of the circle. OC is perpendicular to AB. The radius OA = 10 m and OC = 6 m.



4.1 Calculate the length of CA. (2)

4.2 Calculate the height of the minor segment (Hint: Use **D** = $h + \frac{x^2}{4h}$) (5)