SCIENCE CLINIC

MATHS ESSENTIALS

GRADE 11

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Content Acknowledgement

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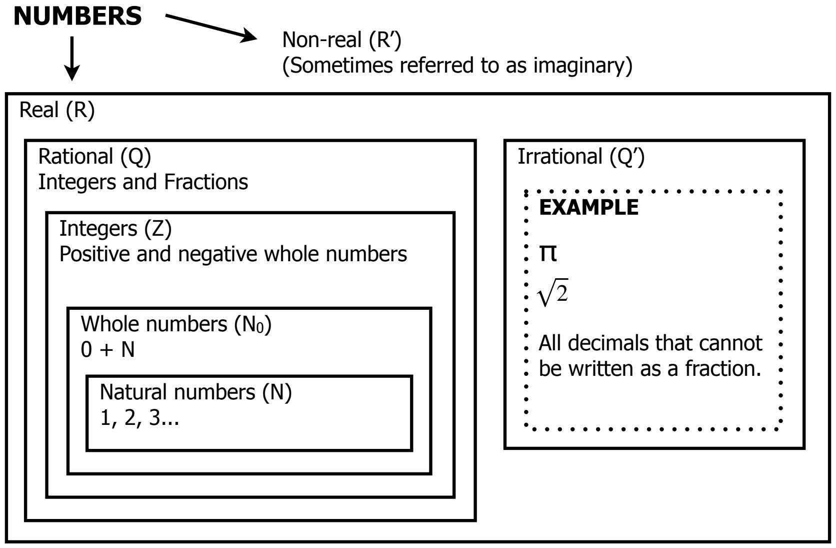
NATURE OF ROOTS



NUMBERS

Non-real (R′)
(Sometimes referred to as imaginary)

Real (R)

Rational (Q)
Integers and Fractions

Integers (Z)
Positive and negative whole numbers

Whole numbers (N_0)
0 + N

Natural numbers (N)
1, 2, 3...

Irrational (Q′)

EXAMPLE

π

$\sqrt{2}$

All decimals that cannot be written as a fraction.

QUADRATIC FORMULA

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

a = the coefficient of x^2
b = the coefficient of x
c = the constant term

Used to factorise quadratic equations.

EXAMPLE

$$3x^2 + 2x - 4 = 0$$
$$a = 3$$
$$b = 2$$
$$c = -4$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{-(2) \pm \sqrt{(2)^2 - 4(3)(-4)}}{2(3)}$$

Then solve for x

THE DISCRIMINANT

$$x = \frac{-b \pm \sqrt{\boxed{b^2 - 4ac}}}{2a}$$

Indicated by Δ.

$$\therefore \Delta = b^2 - 4ac$$

EXAMPLE

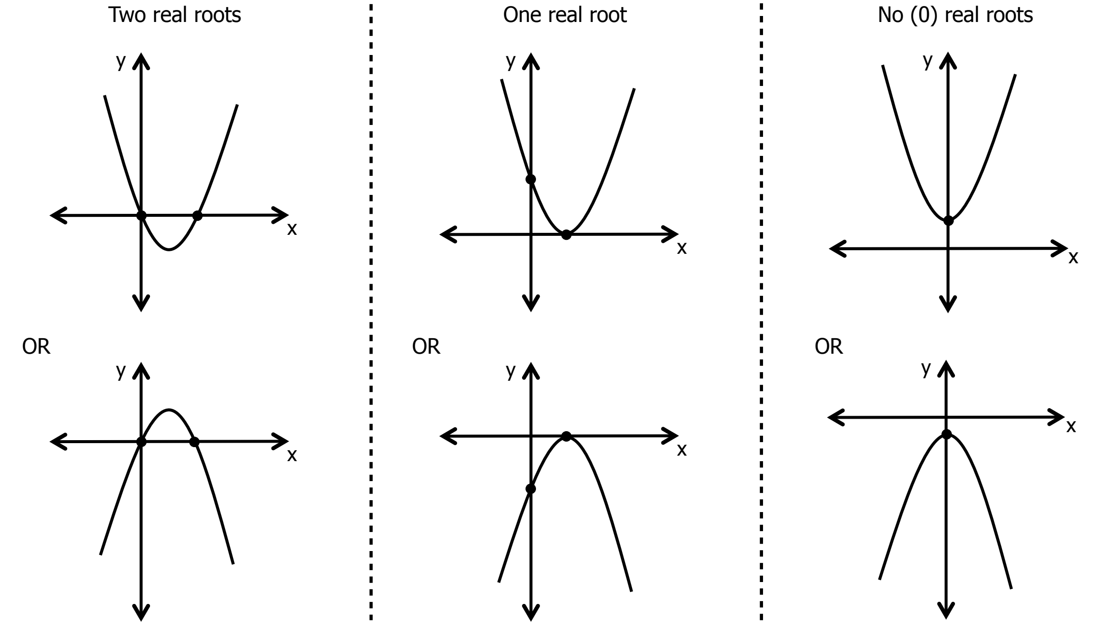
$$3x^2 + 2x - 4 = 0$$
$$a = 3$$
$$b = 2$$
$$c = -4$$

$$\Delta = b^2 - 4ac$$
$$\Delta = (2)^2 - 4(3)(-4)$$

Then solve for Δ

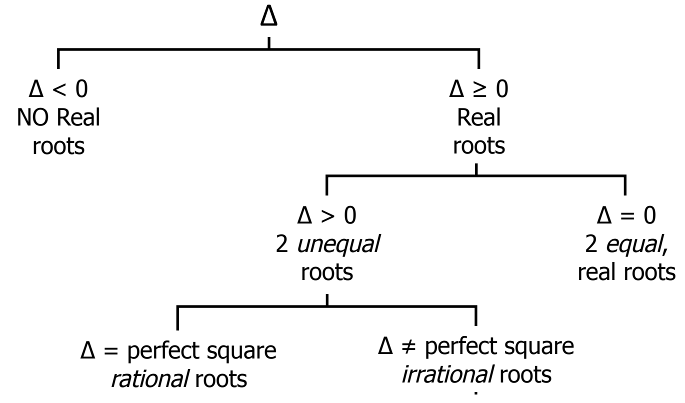
NATURE: Refers to the type of numbers that the roots are.

ROOTS: The x-intercepts/solutions/zeros of a quadratic equation.



DETERMINING THE NATURE OF THE ROOTS

The **DISCRIMINANT** is used to determine the nature of the roots.



Δ

$\Delta < 0$
NO Real
roots

$\Delta \geq 0$
Real
roots

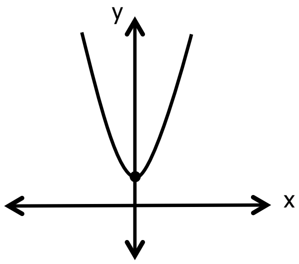
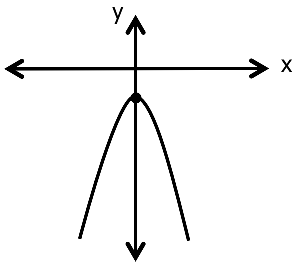
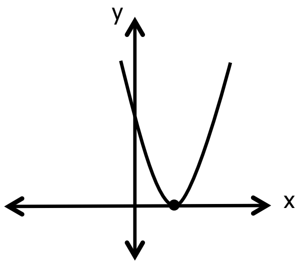
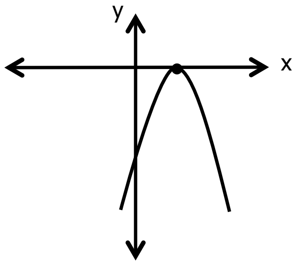
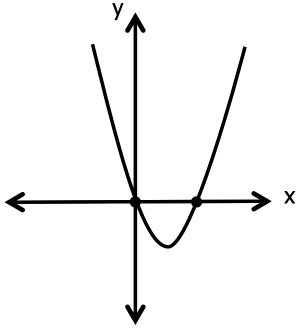
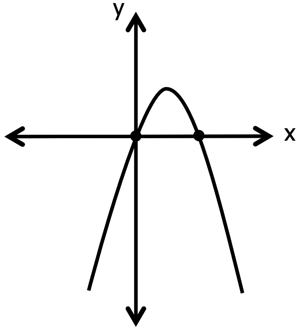
$\Delta > 0$
2 *unequal*
roots

$\Delta = 0$
2 *equal*,
real roots

Δ = perfect square
rational roots

$\Delta \neq$ perfect square
irrational roots

NATURE OF ROOTS

EXAMPLES	**DISCRIMINANT** **($\Delta = b^2 - 4ac$)**	**NATURE OF ROOTS**	**NUMBER OF REAL ROOTS**	**$b^2 - 4\textcircled{a}c$**	
				a > 0	**a < 0**
$\underset{\overset{\uparrow}{a}}{x^2} + \underset{\overset{\uparrow}{b}}{x} + \underset{\overset{\uparrow}{c}}{1} = 0$	$\Delta = b^2 - 4ac$ $= (1)^2 - 4(1)(1)$ $= 1 - 4$ $= -3$ $\Delta < 0$	Non real	0		
$\underset{\overset{\uparrow}{a}}{x^2} - \underset{\overset{\uparrow}{b}}{6x} + \underset{\overset{\uparrow}{c}}{9} = 0$	$\Delta = b^2 - 4ac$ $= (-6)^2 - 4(1)(9)$ $= 36 - 36$ $= 0$ $\Delta = 0$	Real ($\Delta = +$) Rational ($\Delta =$ perfect square) Equal ($\Delta = 0$)	1 (2 of the same)		
$\underset{\overset{\uparrow}{a}}{x^2} - \underset{\overset{\uparrow}{b}}{5x} - \underset{\overset{\uparrow}{c}}{6} = 0$	$\Delta = b^2 - 4ac$ $= (-5)^2 - 4(1)(-6)$ $= 25 + 24$ $= 49$ $\Delta > 0$ (perfect square)	Real ($\Delta = +$) Rational ($\Delta =$ perfect square) Unequal ($\Delta \neq 0$)	2		
$\underset{\overset{\uparrow}{a}}{2x^2} + \underset{\overset{\uparrow}{b}}{3x} - \underset{\overset{\uparrow}{c}}{7} = 0$	$\Delta = b^2 - 4ac$ $= (3)^2 - 4(2)(-7)$ $= 9 + 56$ $= 65$ $\Delta > 0$ (not perfect square)	Real ($\Delta = +$) Irrational ($\Delta \neq$ perfect square) Unequal ($\Delta \neq 0$)	2		

NATURE OF ROOTS

DETERMINING THE NATURE OF ROOTS WITHOUT SOLVING THE EQUATION

The roots of an equation can be determined by calculating the value of the discriminant (Δ).

Steps to determine the roots using the discriminant:

1. Put the equation in its standard form
2. Substitute the correct values in and calculate the discriminant
3. Determine the nature of the roots of the equation

EXAMPLE

Determine the nature of the roots of $x^2 = 2x + 1$ without solving the equation

1. Standard form

$x^2 = 2x + 1$
$x^2 - 2x - 1 = 0$
a b c

2. Calculate the discriminant

$\Delta = b^2 - 4ac$
$\Delta = (-2)^2 - 4(1)(-1)$
$\Delta = 4 + 4$
$\Delta = 8$

3. Determine the nature of the roots

The Roots are:
Real ($\Delta > 0$)
Unequal ($\Delta \neq 0$)
Irrational ($\Delta \neq$ perfect square)

FOR WHICH VALUES OF k WILL THE EQUATION HAVE EQUAL ROOTS?

The discriminant (Δ) can be used to calculate the unknown value of k. (e.g. Ask yourself, for which values of k will the discriminant be 0?)

Steps to determine the values of k using the discriminant:

1. Put the equation in its standard form
2. Substitute the correct values in and calculate the discriminant
3. Equate the discriminant to 0 and solve for k (quadratic equation)

EXAMPLE

For which values of k the equation will have equal roots?

REMEMBER: $\Delta = 0$ for equal roots

1. Standard form

$x^2 + 2kx = -4x - 9k$
$x^2 + \underline{2kx + 4x} + 9k = 0$
a b c

2. Calculate the discriminant

$\Delta = b^2 - 4ac$
$\Delta = (2k + 4)^2 - 4(1)(9k)$
$\Delta = 4k^2 + 16k + 16 - 36k$
$\Delta = 4k^2 - 20k + 16$

3. Equate to zero (0) and solve for k

$0 = 4k^2 - 20k + 16 \quad (\div 4)$
$0 = k^2 - 5k + 4$
$0 = (k - 1)(k - 4)$
Therefore k = 1 **or** k = 4
k needs to either be **1** or **4** to ensure that the discriminant of the equation is 0 (the discriminant **must** be 0 in order for equal roots)

PROVE THE NATURE OF THE ROOTS

The nature of the roots will be supplied and the discriminant can be used to prove the nature, with either one, or no, unknown value.

Steps to prove the nature of roots (NO unknown):

1. Put the equation in its standard form
2. Substitute the correct values in and calculate the discriminant
3. Determine the roots and confirm whether they are as supplied

EXAMPLE

Prove the equation has two, unequal, irrational roots: $x^2 = 2x + 9$

1. Standard form

$x^2 - 2x - 9 = 0$
a b c

2. Calculate the discriminant

$\Delta = b^2 - 4ac$
$\Delta = (-2)^2 - 4(1)(-9)$
$\Delta = 4 + 36$
$\Delta = 40$

3. Determine the roots

The Roots are:
Real ($\Delta > 0$)
Unequal ($\Delta \neq 0$)
Irrational ($\Delta \neq$ perfect square)

Steps to prove the nature of roots (ONE unknown):

1. Put the equation in its standard form
2. Substitute the correct values in and calculate the discriminant
3. Determine the roots and confirm whether they are as supplied

EXAMPLE

For the equation $x(6x - 7m) = 5m^2$, prove that the roots are real, rational and unequal if $m > 0$

1. Standard form

$6x^2 - 7mx - 5m^2 = 0$
a b c

2. Calculate the discriminant

$\Delta = b^2 - 4ac$
$\Delta = (-7m)^2 - 4(6)(-5m^2)$
$\Delta = 49m^2 + 120m^2$
$\Delta = 169m^2$

3. Determine the roots

The Roots are:
Real ($\Delta > 0$)
Unequal ($\Delta \neq 0$)
Rational ($\Delta =$ perfect square)

QUADRATIC EQUATIONS

Quadratic Equations are equations of the second degree (i.e. the highest exponent of the variable is 2). The degree of the equation determines the maximum number of real roots/solutions/x-intercepts/zeros. The standard form of a quadratic equation is:

$$\boxed{ax^2 + bx + c = 0 \quad where\ a \neq 0}$$

SOLVING QUADRATIC EQUATIONS

FACTORING	QUADRATIC FORMULA	DIFFERENCE OF TWO SQUARES	COMPLETE THE SQUARE	ONE ROOT	TWO ROOTS	FRACTIONS AND RESTRICTIONS
1. Put in standard form 2. Apply the zero factor law 3. List possible solutions	Substitute into the quadratic formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ where a = coefficient of x^2, b = coefficient of x, c = constant term		1. Write in standard form 2. Move C across 3. Divide both sides by A 4. Add (½ x b)^2 to both sides 5. Factorise and solve	1. Substitute the known root 2. Solve for the variable 3. Substitute the value of the variable and solve for the root	1. Substitute the roots into the equation 2. Use "FOIL" for the quadratic equation	1. Find the LCD and list restrictions 2. Solve for x 3. Check your answers against your restrictions REMEMBER: $\frac{0}{x} = 0$ BUT $\frac{x}{0} = undefined$
<u>Eg. $x^2 = -2x + 63$</u> $x^2 + 2x - 63 = 0$ Find factors of 63 so that F1 x F2 = –63 and F1 + F2 = 2 (x + 9)(x – 7) = 0 x + 9 = 0 or x – 7 = 0 x = –9 x = 7	<u>Eg. $-3x^2 = -12 + 7x$</u> $-3x^2 - 7x + 12 = 0$ a = –3; b = –7; c = 12 $x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(-3)(12)}}{2(-3)}$ $x = \frac{7 \pm \sqrt{49 + 144}}{-6}$ $x = \frac{7 \pm \sqrt{193}}{-6}$ $x = \frac{7 + \sqrt{193}}{-6} \quad or \quad x = \frac{7 - \sqrt{193}}{-6}$ Answer in surd form or can be calculated/rounded off to 2 decimals x = –3,48 OR x = 1,15	Either method may be used <u>Eg. $x^2 = 25$</u> $x^2 - 25 = 0$ (x – 5)(x + 5) = 0 x = 5 or x = –5 Therefore x = ±5 $\sqrt{x^2} = \pm\sqrt{25}$ x = ±5	<u>Eg. $x^2 + 2x = 1$</u> (½ . 2)^2 (1) $x^2 + 2x + 1 = 1 + 1$ $\sqrt{(x+1)^2} = \pm\sqrt{2}$ $x + 1 = \pm\sqrt{2}$ $x + 1 = -\sqrt{2} \quad or \quad x + 1 = \sqrt{2}$ $x = -1 - \sqrt{2} \qquad x = -1 + \sqrt{2}$	<u>Eg. $x^2 + mx - 15 = 0$, where 5 is a root.</u> $(5)^2 + (5)m - 15 = 0$ 25 + 5m – 15 = 0 5m = –10 m = –2 $x^2 - 2x - 15 = 0$ (x – 5)(x + 3) = 0 x = 5 or x = –3 (given)	<u>Eg. –9 and 7 are the roots of a quadratic equation</u> x = –9 or x = 7 x + 9 = 0 x – 7 = 0 (x + 9)(x – 7) = 0 $x^2 - 7x + 9x - 63 = 0$ $x^2 + 2x - 63 = 0$	<u>Eg.</u> $\frac{x}{x-2} = \frac{1}{x-3} - \frac{2}{2-x}$ $\frac{x}{x-2} = \frac{1}{x-3} + \frac{2}{x-2}$ LCD: $(x-2)(x-3)$ Restrictions: $x - 2 \neq 0;\ x - 3 \neq 0$ $\frac{x\cancel{(x-2)}(x-3)}{\cancel{(x-2)}} = \frac{1(x-2)\cancel{(x-3)}}{\cancel{(x-3)}} + \frac{2\cancel{(x-2)}(x-3)}{\cancel{(x-2)}}$ x(x – 3) = 1(x – 2) + 2(x – 3) $x^2 - 3x = x - 2 + 2x - 6$ $x^2 - 3x - x - 2x + 2 + 6 = 0$ $x^2 - 6x + 8 = 0$ (x – 4)(x – 2) = 0 x – 4 = 0 or x – 2 = 0 x = 4 x = 2 Check restrictions: x ≠ 2, x ≠ 3 Thus, x = 4 is the only solution.

Remember:

* LHS = Left Hand Side * RHS = Right Hand Side
* Fractions and their restrictions

Finding factors:

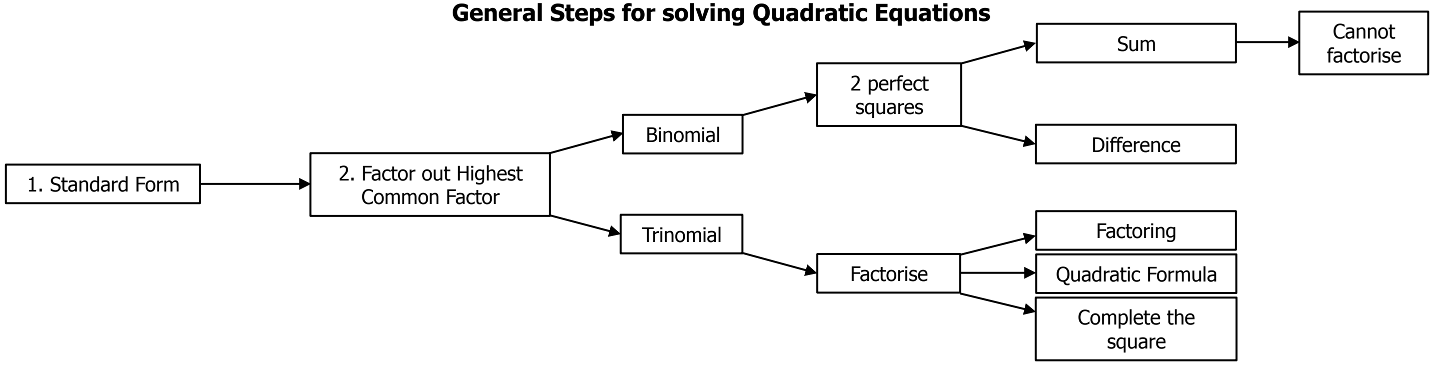
* Factor 1 **+** Factor 2 = **b** * Factor 1 **x** Factor 2 = **c**

Zero factor law:
If A x B = 0 then A, B or both = 0

EXAMPLE

x . y = 0
x = 0 **or** y = 0 **or** (x + 1)(x – 3) = 0
x + 1 = 0 **or** x – 3 = 0

General Steps for solving Quadratic Equations



FUNCTIONS AND GRAPHS

FLASHBACK: Revising Grade 10 Functions

Straight line graphs

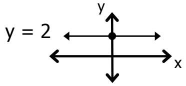
$y = mx + c \longrightarrow$ y-intercept

Gradient: 1) $\dfrac{y_2 - y_1}{x_2 - x_1}$

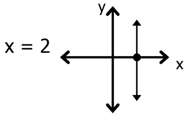
2) ∥ lines $m_1 = m_2$

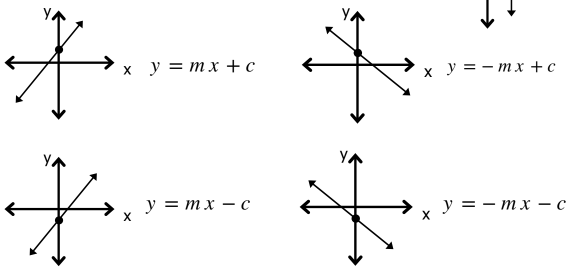
3) ⊥ lines $m_1 \times m_2 = -1$

No gradient



Undefined gradient





$y = mx + c$ | $y = -mx + c$

$y = mx - c$ | $y = -mx - c$

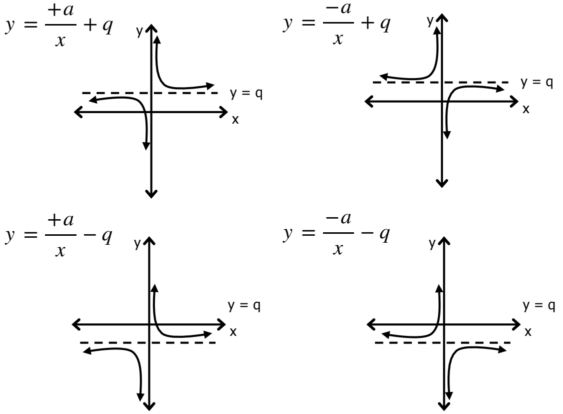
Hyperbolas

$y = \dfrac{a}{x} + q \longrightarrow$ q = asymptote

'a' determines <u>shape</u>

$y = \dfrac{+a}{x} + q$ | $y = \dfrac{-a}{x} + q$

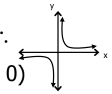
$y = \dfrac{+a}{x} - q$ | $y = \dfrac{-a}{x} - q$



EXAMPLE

Sketch the graph with the equation: $f(x) = \dfrac{2}{x} + 2$

- Shape: a > 0 ∴



- x-intercept (y = 0)

$0 = \dfrac{2}{x} + 2$

$-2 = \dfrac{2}{x}$

$-2x = 2$

$x = -1$

- y-intercept (x = 0)

$y = \dfrac{2}{0} \rightarrow$ undefined

No y-intercept

- Use a table and plot at least 2 other points

x	-2	-1	0	1	2
y	1	0	$\cancel{\mathcal{E}}$	4	3

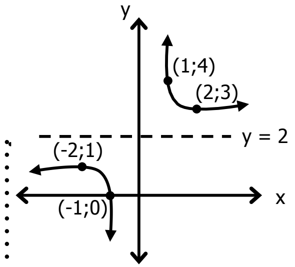
Asymptote!

- Asymptotes

$x = 0$

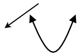
$y = 2$

NOTE:
Start with the asymptotes when sketching



Parabolas (Quadratic functions)

$y = ax^2 + q \longrightarrow$ y-intercept

a > 0  OR a < 0 

EXAMPLE

Sketch the graph with the equation:
$f(x) = x^2 - 4$

- Shape: a > 0 ∴ 
- x-intercept (y = 0)

$0 = x^2 - 4$

<u>Option 1:</u> Factorise

$0 = (x + 2)(x - 2)$

$x = -2$ OR $x = 2$

<u>Option 2:</u> Solve for x

$0 = x^2 - 4$

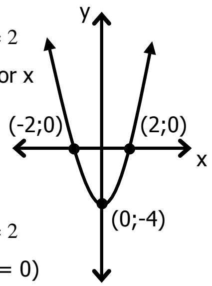
$4 = x^2$

$\pm\sqrt{4} = x$

$x = -2$ OR $x = 2$

- y-intercept (x = 0)

$y = -4$

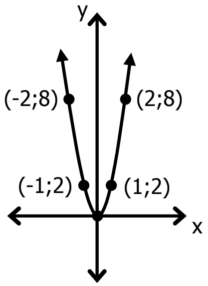


EXAMPLE

Sketch the graph with the equation:
$f(x) = 2x^2$

- Shape: a > 0 ∴ 
- Use a table (on your calculator)

x	-2	-1	0	1	2
y	8	2	0	2	8

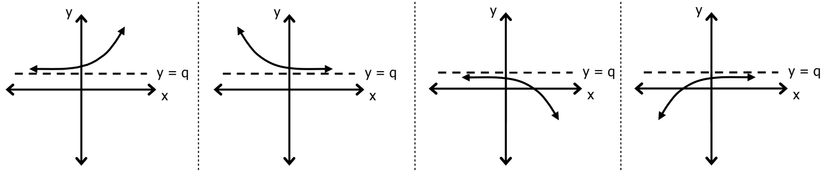


Exponential Graphs

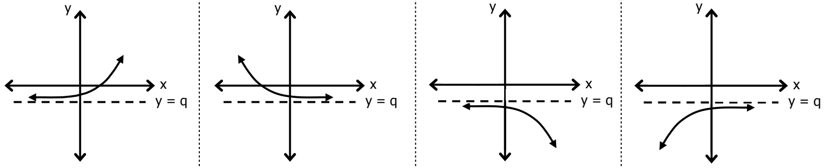
$y = a^x + q$ OR $y = a.(b^x) + q$ q = asymptote

'a' determines <u>shape</u>

$y = +a.b^x + q;\ b > 1$ | $y = +a.b^x + q;\ 0 < b < 1$ | $y = -a.b^x + q;\ b > 1$ | $y = -a.b^x + q;\ 0 < b < 1$

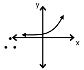


$y = +a.b^x - q; b > 1$ | $y = +a.b^x - q; 0 < b < 1$ | $y = -a.b^x - q; b > 1$ | $y = -a.b^x - q; 0 < b < 1$



EXAMPLE

Sketch the graph with the equation
$f(x) = 2^x - 1$

- Shape: a > 0 ∴ 
- x-intercept (y = 0)

$0 = 2^x - 1$

$1 = 2^x$

$2^0 = 2^x$

$0 = x$

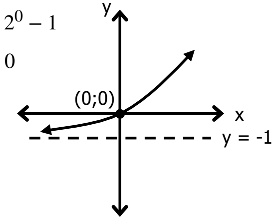
- Asymptote (y = q)

$y = -1$

- y-intercept (x = 0)

$y = 2^0 - 1$

$y = 0$



FUNCTIONS AND GRAPHS

Grade 11 Functions: Quadratic Functions

$$y = a(x - p)^2 + q$$

Shape:
- a > 0 ∪
- a < 0 ∩

Horizontal shift:
- x-value of turning pt
- axis of symmetry

Vertical shift:
- y-value of turning pt

Steps for sketching $y = a(x - p)^2 + q$

1. Determine the shape ('a')
2. Find the x- and y-intercepts
3. Find the turning point
4. Plot points and sketch graph

EXAMPLE 1

Sketch $f(x) = (x + 1)^2 - 9$

- Shape: a > 0 $\therefore$ ∪
- x-intercept (y = 0)

  $0 = (x + 1)^2 - 9$

  $9 = (x + 1)^2$

  $\pm\sqrt{9} = x + 1$

  $+3 = x + 1$ OR $-3 = x + 1$

  $2 = x$ OR $-4 = x$
- y-intercept (x = 0)

  $y = (0 + 1)^2 - 9$

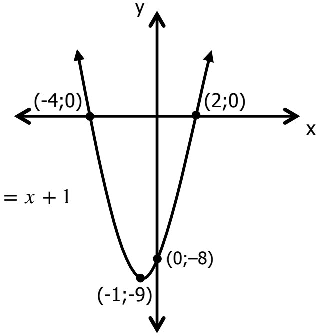
  $y = -8$
- Turning point (p;q)

  $(-1; -9)$
- Axis of symmetry

  $x = -1$
- Domain

  $x \in R$
- Range

  $y \geq -9$



Remember:
$(x - (-1))^2 - 9$
$(x - p)^2 + q$

NOTE:
$f(x) = x^2$
→ moved 1 unit to the left
→ moved 9 units down

Steps for sketching $y = ax^2 + bx + c$

1. Determine the shape ('a')
2. Find the x- and y-intercepts
3. Find the turning point $(\frac{-b}{2a})$
4. Plot points and sketch graph

EXAMPLE 2

Sketch $f(x) = x^2 - 4x + 3$

- Shape: a > 0 $\therefore$ ∪
- x-intercept (y = 0)

Option 1: $0 = x^2 - 4x + 3$

$0 = (x - 3)(x - 1)$

$x = 3$ OR $x = 1$

Option 2: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(3)}}{2(1)}$

$x = \frac{4 \pm 2}{2}$

$x = 3$ OR $x = 1$

- y-intercept (x = 0)

  $y = 3$
- Turning point (p;q)
  1) x-value of TP = $\frac{-b}{2a} = \frac{-(-4)}{2(1)}$

     $x = 2$
  2) Subst. into original eq:

     $y = (2)^2 - 4(2) + 3$

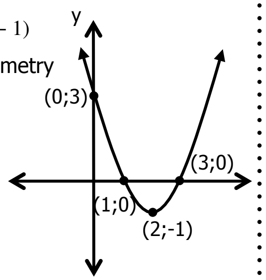
     $y = -1$

     TP $(2; -1)$
- Axis of symmetry

  $x = 2$
- Domain

  $x \in R$
- Range

  $y \geq -1$



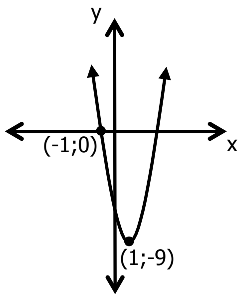
Finding the equation in the form $y = a(x - p)^2 + q$

Given the **turning point** and **another point**

1. Substitute the turning point into $y = a(x - p)^2 + q$
2. Substitute the other point into the equation to find 'a'
3. Determine the equation of the graph

EXAMPLE 1

Find the equation of the following graph:



- Turning point (p;q)

  $p = 1$ and $q = -9$

  $y = a(x - 1)^2 - 9$
- Other point

  $(-1; 0)$

  $0 = a(-1 - 1)^2 - 9$

  $9 = 4a$

  $a = \frac{9}{4}$
- Equation

  $y = \frac{9}{4}(x - 1)^2 - 9$

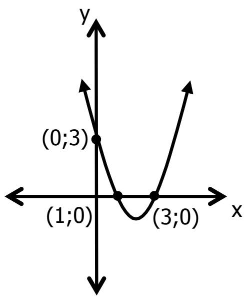
Finding the equation in the form $y = ax^2 + bx + c$

Given the **x-intercepts** and **another point**

1. Substitute the x-intercepts into $y = a(x - x_1)(x - x_2)$
2. Substitute the other point in to find 'a'
3. Write/simplify your final equation

EXAMPLE 2

Find the equation of the following graph:



- x-intercepts

  $x = 1$ OR $x = 3$

  Formula: $y = a(x - x_1)(x - x_2)$

  $y = a(x - 1)(x - 3)$
- Other point

  $(0; 3)$

  $3 = a(-1)(-3)$

  $1 = a$
- Equation

  $y = 1(x - 1)(x - 3)$

  $y = x^2 - 4x + 3$

NOTE:
If you need to write this equation in the form $y = a(x - p)^2 + q$ complete the square

$y = (x - 2)^2 + 3 - 4$

$y = (x - 2)^2 - 1$

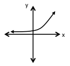
FUNCTIONS AND GRAPHS

Exponential Graphs

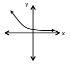
$$y = ab^{x-p} + q$$

Shape:

- b > 1



- 0 < b < 1



Vertical shift:

- asymptote

Steps for sketching $y = ab^{x-p} + q$

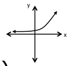
1. Determine the asymptote (‘q’)
1. Determine the shape (‘a’)
2. Find the x- and y-intercepts
3. Plot points (at least 2 others) and sketch graph

EXAMPLE 1

Sketch $f(x) = 2^{x+1} + 1$

- Asymptote

  $y = 1$
- Shape: a > 0 ∴



- x-intercept (y = 0)

  $0 = 2^{x+1} + 1$

  $-1 = 2^{x+1}$

  Not possible to solve for x

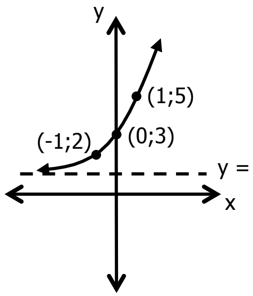
  ∴ No x-intercept
- y-intercept (x = 0)

  $y = 2^{0+1} + 1$

  $y = 3$
- Domain

  $x \in R$
- Range

  $y > 1$



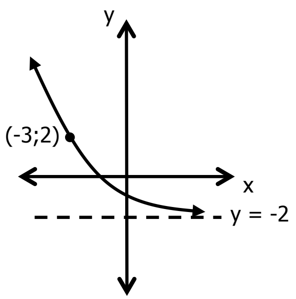
Finding the equation in the form $y = ab^{x-p} + q$

Given the **asymptote** and **another point**

1. Substitute the asymptote into the equation
2. Substitute the other point in
3. Write/simplify your final equation

EXAMPLE 2

Find the equation of the following graph given $y = b^{x+1} + q$:



- Asymptote

  $q = -2$

  $y = b^{x+1} - 2$
- Other point

  $(-3; 2)$

  $2 = b^{-3+1} - 2$

  $4 = b^{-2}$

  $4 = \frac{1}{b^2}$

  $b^2 = \frac{1}{4}$

  $b = \pm\frac{1}{2}$ $b \neq -\frac{1}{2}$

  $b = +\frac{1}{2}$
- Equation

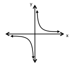
  $y = (\frac{1}{2})^{x+1} - 2$

Hyperbola

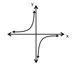
$$y = \frac{a}{x-p} + q$$

Shape:

- a > 0



- a < 0



Horizontal shift:

- asymptote

Vertical shift:

- asymptote

Steps for sketching $y = \frac{a}{x-p} + q$

1. Determine the asymptotes (y = ‘q’ and x = ‘p’)
1. Determine the shape (‘a’)
2. Find the x- and y-intercepts
3. Plot points (at least 2 others) and sketch graph

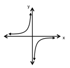
EXAMPLE 1

Sketch $f(x) = \frac{-1}{x-2} - 1$

- Asymptotes

  $x = 2$

  $y = -1$
- Shape: a < 0 ∴



- x-intercept (y = 0)

  $0 = \frac{-1}{x-2} - 1$

  $1 = \frac{-1}{x-2}$

  $x - 2 = -1$

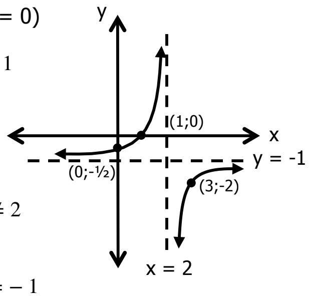
  $x = 1$
- y-intercept (x = 0)

  $y = \frac{-1}{-2} - 1$

  $y = -\frac{1}{2}$
- Domain

  $x \in R; x \neq 2$
- Range

  $y \in R; y \neq -1$



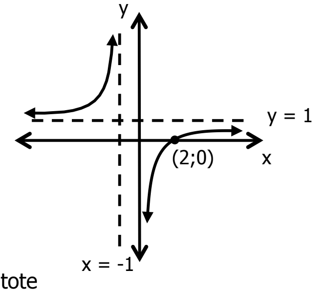
Finding the equation in the form $y = \frac{a}{x-p} + q$

Given the **asymptotes** and **another point**

1. Substitute the asymptotes into the equation
2. Substitute the other point into the equation to find ‘a’
3. Write/simplify your final equation

EXAMPLE 2

Find the equation of the following graph:



- Asymptote

  $y = 1$ and $x = -1$

  $f(x) = \frac{a}{x-(-1)} + 1$

  $f(x) = \frac{a}{x+1} + 1$
- Other point

  $(2; 0)$

  $0 = \frac{a}{2+1} + 1$

  $-1 = \frac{a}{3}$

  $-3 = a$
- Equation

  $f(x) = \frac{-3}{x+1} + 1$

Lines of Symmetry:

Use point of intersection of asymptotes. (-1;1)

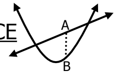
$y = x + c$	$(-1; 1)$	$y = -x + c$	$(-1; 1)$
$1 = -1 + c$		$1 = 1 + c$	
$2 = c$		$0 = c$	
$y = x + 2$		$y = -x$	

FUNCTIONS AND GRAPHS

Deductions from Graphs

DISTANCE

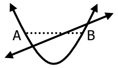
Steps for determining VERTICAL DISTANCE



1. Determine the vertical distance
   Vertical distance = top graph – (bottom graph)
2. Substitute the given x-value to derive your answer

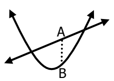
Steps for determining HORIZONTAL DISTANCE

1. Find the applicable x-values
   $AB = x_B - x_A$ (largest – smallest)



Steps for determining MAXIMUM DISTANCE

1. Determine the vertical distance
   Vertical distance = top graph – bottom graph
2. Complete the square
   $y = a(x-p)^2 + q$
3. State the maximum distance
   $y = a(x-p)^2 + (\mathbf{q}) \rightarrow$ q is the max distance

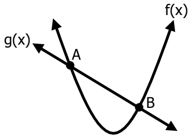


NOTE:
- Distance is <u>always</u> positive
- Distance on a graph is measured in <u>units</u>

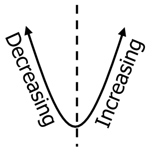
INTERSECTION OF GRAPHS

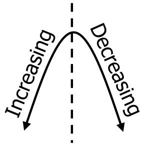
Steps for determining POINTS OF INTERCEPTION

1. Equate the two functions
   $f(x) = g(x)$
2. Solve for x (look for the applicable x-value: A or B)
3. Substitute the applicable x-value into any of the two equations to find 'y'



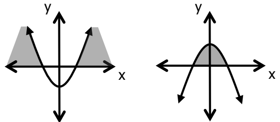
INCREASING/DECREASING





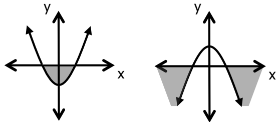
NOTATION

- $f(x) > 0$ ⊕
  (<u>above</u> the line y = 0)



(i.e. where y is positive)

- $f(x) < 0$ ⊖
  (<u>below</u> the line y = 0)



(i.e. where y is negative)

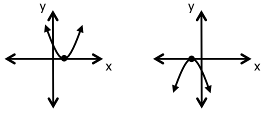
⊕ ⊖
- $f(x) \cdot g(x) \leq 0$ ⊖

⊖ ⊕
(one graph lies above y = 0 and one graph lies below y = 0)

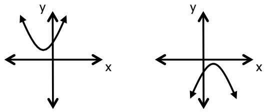
- $f(x) \geq g(x)$
  top bottom
  (i.e. f(x) lies <u>above</u> g(x))
- $f(x) = g(x)$
  (point of intersection)

ROOTS & PARABOLAS

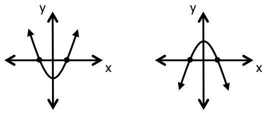
- Equal, real roots



- Non-real/ No real roots

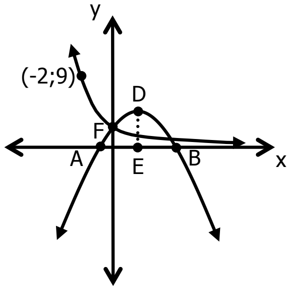


- Real, unequal roots



EXAMPLE 1

$f(x) = ax^2 + bx + c$ and $g(x) = k^x$ are sketched. D is the turning point of f(x) with the axis of symmetry at x=2. AB is 6 units.



Questions:

a. Determine the value of k.

b. Determine the x-values of A and B.

c. Show that $a = \frac{-1}{5}$ and $b = \frac{4}{5}$.

d. Determine the coordinates of D.

e. Determine the maximum distance of DE.

f. Determine the values of p for which:
$\frac{-1}{5}x^2 + \frac{4}{5}x + p < 0$

g. Determine for which values of x:

i. $f(x) \geq 0$

ii. $\frac{f(x)}{g(x)} > 0$

iii. $f(x)$ is increasing

Solutions:

a. $(-2;9)$
$9 = k^{-2}$
$9 = \frac{1}{k^2}$
$k = \pm\frac{1}{3}$
$k = \frac{1}{3}$

b. $E = (2;0)$ and $AB = 6\ units$
$A = (-1;0) \quad x = -1$
$B = (5;0) \quad x = 5$

c. $y = a(x - x_1)(x - x_2)$
$(-1;0)$ and $(5;0)$
$y = a(x+1)(x-5)$
Use $F(0;1)$
$1 = a(+1)(-5)$
$-\frac{1}{5} = a$
$y = -\frac{1}{5}(x^2 - 4x + 1)$
$y = -\frac{1}{5}x^2 + \frac{4}{5}x + 1$
$b = \frac{4}{5}$

d. $y = -\frac{1}{5}(2)^2 + \frac{4}{5}(2) + 1$
$y = \frac{9}{5} \quad \therefore\ D = (2;\frac{9}{5})$

e. $\frac{9}{5}$ units (y-value of co-ordinate D is also TP)

f. $-\frac{1}{5}x^2 + \frac{4}{5}x + p < 0$ ⊖
$p < -\frac{9}{5}$

NOTE:
- Interpret question as: How many units must the graph move for the max. value to be < 0

g.

i. $x \in [-1;5]$

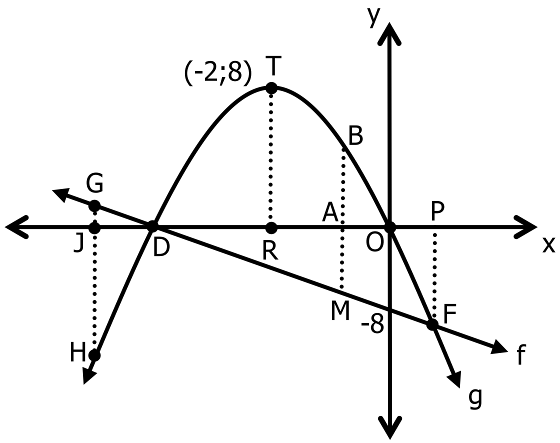
ii. $x \in (-1;5)$

iii. $x \in (-\infty;2)$

Deductions from Graphs

EXAMPLE 2

$f(x) = mx + c$ and $g(x) = a(x-p)^2 + q$ are sketched below. T is the turning point of g(x).



Questions:

Determine:

a. The value of a, p, q, m and c.

b. The length of OD.

c. The length of TR.

d. The equation of TR.

e. BM if OA = 1 unit.

f. OJ if GH = 28 units.

g. The length of FP.

h. The maximum length of BM.

i. The value of k for which $-2x^2 - 8x + k$ has two equal roots.

j. For which value(s) of x will $\frac{f(x)}{g(x)} < 0$?

Solutions:

a. $y = a(x+2)^2 + 8 \quad (0;0)$
$-8 = 4a$
$-2 = a$ and $p = -2$ and $q = 8$
$\therefore\ g(x) = -2(x+2)^2 + 8$
$D\ (-4;0)$
$\therefore\ m = \frac{8}{-4} = -2$ and $c = -8$

b. OD = 4 units

c. TR = 8 units

d. TR: $x = -2$

e. $g(x) = -2(x+2)^2 + 8$
$= -2(x^2 + 4x + 4) + 8$
$= -2x - 8x - 8 + 8$
$= -2x^2 - 8x$

BM = $g(x) - f(x)$

BM $= -2x^2 - 8x - (-2x - 8)$
$= -2x^2 - 6x + 8$
BM $= -2(-1)^2 - 6(-1) + 8$
BM = 12 units

f. $28 = -2x - 8 - (-2x^2 - 8x)$
$28 = -2x - 8 + 2x^2 + 8x$
$0 = 2x^2 + 6x - 36$
$0 = 2(x^2 + 3x - 18)$
$0 = 2(x+6)(x-3)$
$x = -6$ or $x = 3$ (NA)
$\therefore$ OJ = 6 units

g. $-2x^2 - 8x = -2x - 8$
$0 = 2x^2 + 6x - 8$
$0 = 2(x-1)(x+4)$
$x = 1$ or $x = -4$ (NA)
$y = -2(1) - 8$
$y = -10$
$\therefore$ FP = 10 units

h. Max length is given by TP of parabola ($L(x)$) given by $L(x) = g(x) - f(x)$. Find the TP by completing the square.

$\therefore$ Max BM = $g(x) - f(x)$
$= -2x^2 - 8x - (-2x - 8)$
$= -2x^2 - 6x + 8$
$= -2(x^2 + 3x - 4)$
$= -2[(x + \frac{3}{2})^2 - 4 - \frac{9}{4}]$
$= -2[(x + \frac{3}{2})^2 - \frac{25}{4}]$
$= -2(x + \frac{3}{2})^2 + \frac{25}{2}$

$\therefore$ Max of BM = $\frac{25}{2}$ units

i. k = -8

j. $x \in (-\infty; 0);\ x \neq -4$

EXPONENTS AND SURDS

WHAT ARE:

Exponents: Exponents occur when multiplying or dividing expressions/ bases/variables numerous times by similar expressions/bases/variables

Surds: A surd is the Mathematical terminology for irrational roots, when numbers are left in "root-form" as opposed to rounding them off to a decimal place.

HELPFUL HINTS FOR EQUATIONS\EXPRESSIONS

1. Express larger numbers in exponential form by prime factorising
2. Remove a common factor if two unlike terms are separated by a +/-
3. Ensure your surds are <u>always</u> expressed in their simplest form
4. Express surds in exponential form for simplification
5. Take note of the following:

   A common error, when solving for an unknown base with a fraction as an exponent, is to multiply the exponents on both sides by the unknown exponent's inverse (so that the exponent will be 1). However, if you express these fractions as surds, you will notice the following:

   a. An even power will <u>always</u> produce a positive AND negative solution

   $x^{\frac{4}{3}} = 3$

   $\sqrt[3]{x^4} = 3$

   $x^4 = 27$

   $x = \pm \sqrt[4]{27}$

   b. A negative number inside an even root <u>cannot</u> solve for a real solution

   $-2^{\frac{1}{2}} = x$

   $\sqrt{-2} = x$

   No real solution

   c. An unknown inside an even root <u>cannot</u> solve for a negative solution

   $x^{\frac{3}{4}} = -2$

   $\sqrt[4]{x^3} = -2$

   No real solution

ADDING AND SUBTRACTING LIKE-TERMS

Like terms are terms in an equation/expression that have **identical variables and exponents.** To add/subtract these, simply add/subtract their coefficients. Exponents **<u>never</u>** change when the operator is +/-

EXAMPLE

1. $3x^2y^4 - 5x^3y + 2x^2y^4 + x^3y = \mathbf{5x^2y^4 - 4x^3y}$

2. $3\sqrt{2} + 5\sqrt{3} - 8\sqrt{2} + \sqrt{3} = \mathbf{-5\sqrt{2} + 6\sqrt{3}}$

LAWS OF EXPONENTS

Laws of exponents <u>only</u> apply to multiplication, division, brackets and roots. NEVER adding or subtracting

Algebraic Notation		Exponential Notation	Exponential Law in operation
1	$16 = 2 \times 2 \times 2 \times 2$	$16 = 2^4$	When we MULTIPLY the SAME bases we ADD the exponents.
2	$\frac{64}{16} = 4$	$\frac{2^6}{2^4} = 2^2$	When we DIVIDE the SAME bases we MINUS the exponents (always top minus bottom).
3	$4^3 = 64$	$(2^2)^3 = 2^6$	When we have the exponents outside the BRACKETS we DISTRIBUTE them into the brackets.
4	$\frac{64}{64} = 1$	$\frac{2^6}{2^6} = 2^0 = 1$	Any base to the POWER OF ZERO is equal to one. (But 0^0 is undefined).
5	$\sqrt[3]{64} = 4$	$\sqrt[3]{2^6} = 2^2$	The POWER inside the root is DIVIDED by the size of the root.
6	$4 \times 9 = 36$	$2^2 \times 3^2 = 6^2$	When we have non-identical bases, but identical exponents, we keep the exponents and multiply the bases. (This same rule will also apply for division).
7	$\sqrt{2} \times \sqrt{3} = \sqrt{6}$	$2^{\frac{1}{2}} \times 3^{\frac{1}{2}} = 6^{\frac{1}{2}}$	
8	$\sqrt{2} \times \sqrt{2} = \sqrt{4} = 2$	$2^{\frac{1}{2}} \times 2^{\frac{1}{2}} = 2^1$	Any square root multiplied by itself will equal the term inside the root.

CONVERTING SURDS INTO EXPONENTIAL FORM (AND VICE VERSA)

The power inside the root becomes the NUMERATOR and the size of the root becomes the DENOMINATOR.

$$\boxed{\sqrt[D]{x^N} = x^{\frac{N}{D}}}$$

EXAMPLE

1. $\sqrt[5]{x^2}$

   $= x^{\frac{2}{5}}$

2. $x^{\frac{3}{4}}$

   $= \sqrt[4]{x^3}$

OPERATIONS WITH SURDS

<u>Steps for working with surds:</u>

1. Express the surd in its simplest surd form
2. Identify like terms (+ and −) or use Laws of Exponents (× and ÷)

Note: If you use your calculator, make sure to show the changes you made i.e. $\sqrt{50} = \sqrt{25 \times 2} = \sqrt{25} \times \sqrt{2} = 5\sqrt{2}$

EXAMPLE 1

Simplify

$\sqrt{50} + 3\sqrt{18} - \sqrt{98}$

$= 5\sqrt{2} + 9\sqrt{2} - 7\sqrt{2}$

$= 7\sqrt{2}$

EXAMPLE 2

Simplify

$$\frac{\sqrt[5]{81} \times \sqrt[4]{27}}{\sqrt[5]{9} \times \sqrt{3}}$$

$$= \frac{\sqrt[5]{3^4} \times \sqrt[4]{3^3}}{\sqrt[5]{3^2} \times \sqrt{3}}$$

$$= \frac{3^{\frac{4}{5}} \times 3^{\frac{3}{4}}}{3^{\frac{2}{5}} \times 3^{\frac{1}{2}}}$$

$$= \frac{3^{\frac{31}{20}}}{3^{\frac{9}{10}}}$$

$$= 3^{\frac{13}{20}}$$

$$= \sqrt[20]{3^{13}}$$

EXAMPLE 3

A rectangle has a length of $\sqrt{7} + 1$ and a width of $\sqrt{7} - 1$. Determine the length of the diagonal.

$(\sqrt{7} + 1)^2 + (\sqrt{7} - 1)^2 = r^2$

$7 + 2\sqrt{7} + 1 + 7 - 2\sqrt{7} + 1 = r^2$

$16 = r^2$

$4 = r$

RATIONALISING THE DENOMINATOR

The process of finding an equivalent fraction that can be expressed **without a surd in the denominator**

Steps for rationalising monomial denominators:

1. Multiply the numerator and denominator by the denominator's surd
2. Simplify

EXAMPLE 1

Express the following with rational denominators:

1. $\frac{3}{\sqrt{7}}$

$= \frac{3}{\sqrt{7}} \times \frac{\sqrt{7}}{\sqrt{7}}$

$= \frac{3\sqrt{7}}{7}$

2. $\frac{6+3\sqrt{2}}{2\sqrt{3}}$

$= \frac{6+3\sqrt{2}}{2\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$

$= \frac{6\sqrt{3}+3\sqrt{6}}{2 \times 3}$

$= \frac{6\sqrt{3}+3\sqrt{6}}{6}$

$= \frac{2\sqrt{3}+\sqrt{6}}{2}$

EXAMPLE 2

If $x = \sqrt{3}+2$, simplify: $\frac{x^2+2}{x-2}$ and express the answer with a rational denominator

1. $\frac{x^2+2}{x-2}$

$= \frac{(\sqrt{3}+2)^2+2}{(\sqrt{3}+2)-2}$

$= \frac{3+4\sqrt{3}+4+2}{\sqrt{3}}$

$= \frac{9+4\sqrt{3}}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$

$= \frac{9\sqrt{3}+4 \cdot 3}{3}$

$= 3\sqrt{3}+4$

Steps for rationalising binomial denominators:

1. Multiply numerator and denominator by the binomial in the denominator with the opposite sign (conjugate)
2. Simplify

Why do we do this?
Multiplying the binomial by itself will give us a trinomial with an irrational middle term. To avoid this, we multiply the binomial by its **conjugate** (same binomial with the opposite sign) to create a difference of two squares.

EXAMPLE 1

Express the following fractions with rational denominators:

1. $\frac{3}{5-\sqrt{7}}$

$= \frac{3}{5-\sqrt{7}} \times \frac{5+\sqrt{7}}{5+\sqrt{7}}$

$= \frac{15+3\sqrt{7}}{25-7}$

$= \frac{15+3\sqrt{7}}{18}$

$= \frac{5+\sqrt{7}}{6}$

2. $\frac{7}{\sqrt{x}-\frac{1}{\sqrt{x}}}$

$= \frac{7}{\sqrt{x}-\frac{1}{\sqrt{x}}} \times \frac{\sqrt{x}+\frac{1}{\sqrt{x}}}{\sqrt{x}+\frac{1}{\sqrt{x}}}$

$= \frac{7\sqrt{x}+\frac{7}{\sqrt{x}}}{x-\frac{1}{x}}$

$= \frac{\frac{7x+7}{\sqrt{x}}}{\frac{x^2-1}{x}}$

$= \frac{7x+7}{\sqrt{x}} \div \frac{x^2-1}{x}$

$= \frac{7\cancel{(x+1)}}{\sqrt{x}} \times \frac{x}{\cancel{(x+1)}(x-1)}$

$= \frac{7x}{\sqrt{x}(x-1)} \times \frac{\sqrt{x}}{\sqrt{x}}$

$= \frac{7\cancel{x}\sqrt{x}}{\cancel{x}(x-1)}$

$= \frac{7\sqrt{x}}{(x-1)}$

EXPONENTS AND SURDS

FACTORISING

Factorising is the **opposite** of distribution, which means that you will subtract the exponents when "taking out" factors. There are 6 different types of factorisation.

1. Common Factor:

Remove the highest common factor from the coefficients and common variables.

EXAMPLES

Factorise the following:

1. $3x^5y^4 + 9x^3y^5 - 12x^2y^4$

$= 3x^2y^4(x^3 + 3xy - 4)$

2. $\frac{4x^5}{9y^3} - \frac{8x^3}{27y^2} + \frac{16x^2}{3y}$

$= \frac{4x^2}{3y}\left(\frac{x^3}{3y^2} + \frac{2x}{9y} + 4\right)$

4. Exponential Factorising:

Similar to common factorising (1). Remove the highest common factor, in this case, a base with its exponent(s). Exponents are subtracted from the same bases.

EXAMPLES

Factorise the following:

1. $2^{x+3} - 2^{x+1}$

$= 2^x(2^3 - 2)$

$= 2^x \cdot 6$

2. $\frac{9^{x+2} - 3^{2x}}{3^x \cdot 2^3 \times 3^x \cdot 5}$

$= \frac{(3^2)^{x+2} - 3^{2x}}{3^{2x} \cdot 8 \cdot 5}$

$= \frac{3^{2x+4} - 3^{2x}}{3^{2x} \cdot 40}$

$= \frac{\cancel{3^{2x}}(3^4 - 1)}{\cancel{3^{2x}} \cdot 40}$

$= \frac{80}{40}$

$= 2$

3. $\frac{5^x - 5^{x-2}}{2 \cdot 5^x - 5^x}$

$= \frac{\cancel{5^x}(1 - 5^{-2})}{\cancel{5^x}(2 - 1)}$

$= \frac{1 - \frac{1}{25}}{1}$

$= \frac{24}{25}$

2. Difference of two squares:

Applied when there are two perfect squares separated by a '–' sign. The square root of both terms will be in both pairs of brackets, one with a + and the other with a –

A perfect square is a term whose number will not leave an irrational solution once square-rooted, and whose exponents are divisible by 2.

EXAMPLES

Factorise the following:

1. $9x^2 - 4y^6$

$= (3x + 2y^3)(3x - 2y^3)$

2. $x^4 - 16$

$= (x^2 + 4)(x^2 - 4)$

$= (x^2 + 4)(x + 2)(x - 2)$

3. $\frac{x^2 - 7}{x + \sqrt{7}}$

$= \frac{\cancel{(x + \sqrt{7})}(x - \sqrt{7})}{\cancel{x + \sqrt{7}}}$

$= x - \sqrt{7}$

4. $a^2 + 2ab + b^2 - x^2$

$= (a + b)^2 - x^2$

$= (a + b + x)(a + b - x)$

5. Grouping:

Remove the common binomial factor from the expression

EXAMPLES

Factorise the following:

1. $x(y - 4) + 3(y - 4)$

$= (y - 4)(x + 3)$

2. $a^2 + 2ab + b^2 - 3a - 3b$

$= (a + b)^2 - 3(a + b)$

$= (a + b)(a + b - 3)$

3. $5x - 15y + 9ay - 3ax$

$= 5(x - 3y) + 3a(3y - x)$

$= 5(x - 3y) - 3a(x - 3y)$

$= (x - 3y)(5 - 3a)$

3. Sum or difference of two cubes:

Applied when there are two perfect cubes separated by a +/–.The final answer will be a binomial in the one bracket and a trinomial in the other.

A perfect cube is a term whose number will not leave an irrational solution once cube-rooted, and whose exponents are divisible by 3.

EXAMPLES

Factorise the following:

1. $x^3 - 8$

$= (x - 2)(x^2 + 2x + 4)$

2. $27x^6 + 64y^9$

$= (3x^2 + 4y^3)(9x^4 - 12x^2y^3 + 16y^6)$

6. Trinomials:

Note: Ratio of exponents of term 1 to term 2 is 2:1. A combination of factors of term 1 and term 3 must give you term 2.

EXAMPLES

Factorise the following: (Q2 - Q6 are conceptually the same)

1. $3x^2 - 5x - 2$

$= (3x + 1)(x - 2)$

2. $x^2 + 3x - 10$

$= (x + 5)(x - 2)$

3. $x^4 + 3x^2 - 10$

$= (x^2 + 5)(x^2 - 2)$

4. $x^{\frac{2}{3}} + 3x^{\frac{1}{3}} - 10$

$= (x^{\frac{1}{3}} + 5)(x^{\frac{1}{3}} - 2)$

5. $5^{2x} + 3 \cdot 5^x - 10$

$= (5^x + 5)(5^x - 2)$

6. $3^{2x} + 3^{x+1} - 10$

$= 3^{2x} + 3 \cdot 3^x - 10$

$= (3^x + 5)(3^x - 2)$

EXPONENTS AND SURDS

EQUATIONS

1. Linear Equations:

Move all the variables to the one side, and the constants to the other to solve. Linear equations have only **one** solution.

EXAMPLES

Solve:

1. $3(x-2)+10=5-(x+9)$

$3x-6+10=5-x-9$

$3x+4=-x-4$

$4x=-8$

$x=-2$

2. $(x-2)^2-1=(x+3)(x-3)$

$\cancel{x^2}-4x+4-1=\cancel{x^2}-9$

$-4x+3=-9$

$-4x=-12$

$x=3$

2. Quadratic Equations:

Move everything to one side and equate to zero. By factorising the trinomial, you should find **two** solutions.

EXAMPLES

Solve: (Q3 - Q6 are the most likely exam-type questions)

1. $x^2+5=6x$

$x^2-6x+5=0$

$(x-5)(x-1)=0$

$x=5$ or $x=1$

2. $(3x-4)(5x+2)=0$

$3x=4$ or $5x=-2$

$x=\frac{4}{3}$ or $x=\frac{-2}{5}$

3. $x^4+3x^2-10=0$

$(x^2+5)(x^2-2)=0$

$x^2=-5$ or $x^2=2$

No sol. or $x=\pm\sqrt{2}$

4. $x^{\frac{2}{3}}+3x^{\frac{1}{3}}-10=0$

$(x^{\frac{1}{3}}+5)(x^{\frac{1}{3}}-2)=0$

$x^{\frac{1}{3}}=-5$ or $x^{\frac{1}{3}}=2$

$x=-125$ or $x=8$

5. $x+3\sqrt{x}-10=0$

$x+3x^{\frac{1}{2}}-10=0$

$(x^{\frac{1}{2}}+5)(x^{\frac{1}{2}}-2)=0$

$x^{\frac{1}{2}}=-5$ or $x^{\frac{1}{2}}=2$

$\sqrt{x}=-5$ or $\sqrt{x}=2$

No sol. or $x=4$

6. $2^{2x}-6\cdot 2^x-16=0$

$(2^x+2)(2^x-8)=0$

$2^x=-2$ or $2^x=8$

No sol. or $2^x=2^3$

$x=3$

3. Simultaneous Equations:

Solve for two unknowns in two different equations using the substitution method. Remember to solve for both unknowns by substituting them back into the original equation.

EXAMPLES

Solve:

1. Equation 1: $2x+3y=18$

Equation 2: $-3x+5y=11$

From 1: $2x+3y=18$

$2x=-3y+18$

$x=\frac{-3y+18}{2}$....1a

Sub 1a into 2: $-3x+5y=11$

$-3\left(\frac{-3y+18}{2}\right)+5y=11$

$\frac{9y-54}{2}+5y=11$

$9y-54+10y=22$

$19y=76$

$y=4$....3

Sub 3 into 1: $2x+3(4)=18$

$2x=6$

$x=3$

$(3;4)$

2. Equation 1: $y+3x=2$

Equation 2: $y^2-9x^2=16$

From 1: $y+3x=2$

$y=-3x+2$...1a

Sub 1a into 2: $y^2-9x^2=16$

$(-3x+2)^2-9x^2=16$

$\cancel{9x^2}-12x+4-\cancel{9x^2}=16$

$-12x=12$

$x=-1$...3

Sub 3 into 1: $y+3(-1)=2$

$y=5$

$(-1;5)$

4. Surd Equations:

Isolate the surd on the one side of the equation. Power both sides of the equation by the root. Ensure that you check your solutions by substituting your answers back into the original equation.

EXAMPLES

Solve:

1. $\sqrt{x-2}=3$

$x-2=9$

$x=9+2$

$x=11$

2. $\sqrt{x+5}-x=3$

$\sqrt{x+5}=x+3$

$x+5=x^2+6x+9$

$0=x^2+5x+4$

$0=(x+1)(x+4)$

$x=-1$ or $x\neq-4$

Check:

$LHS=\sqrt{(-1)+5}-(-1)$

$LHS=3 \qquad RHS=3$

$\therefore x=-1$

$LHS=\sqrt{(-4)+5}-(-4)$

$LHS=5 \qquad RHS=3$

$\therefore x\neq-4$

5. Exponential Equations:

Make sure that you get a term on the one side of the equation that has a base that is equal to the base with the unknown exponent. Then, drop the bases, equate the exponents and solve.

Hints:

- NEVER drop the base if the terms are separated by a + or –
  - Remove common factors until the equation is in its simplest form and then solve
- Always convert decimals to fractions and then to bases with negative exponents

EXAMPLES

1. $4^x=8$

$2^{2x}=2^3$

$2x=3$

$x=\frac{3}{2}$

2. $0{,}0625^x=64$

$\left(\frac{1}{16}\right)^x=2^6$

$\left(\frac{1}{2^4}\right)^x=2^6$

$2^{-4x}=2^6$

$-4x=6$

$x=\frac{-3}{2}$

3. $2\cdot 3^{x+1}+5\cdot 3^x=33$

$3^x(2\cdot 3+5)=33$

$3^x(11)=33$

$3^x=3^1$

$x=1$

4. $27^{3x+1}=81^{2x+5}$

$(3^3)^{3x+1}=(3^4)^{2x+5}$

$3^{9x+3}=3^{8x+20}$

$9x+3=8x+20$

$x=17$

5. $0{,}5^x\cdot\sqrt{1+\frac{9}{16}}=10$

$\left(\frac{1}{2}\right)^x\cdot\sqrt{\frac{25}{16}}=10$

$2^{-x}\cdot\frac{5}{4}=10$

$2^{-x}=8$

$2^{-x}=2^3$

$-x=3$

$x=-3$

NUMBER PATTERNS

REMINDERS:

1. Consecutive: directly follow one another
2. Common/constant difference: difference between two consecutive terms in a pattern
   $$d = T_2 - T_1$$
3. General term T_n: also referred to as the nth term.
   - General term for linear patterns:
     $T_n = dn + c$
   - General term for quadratic patterns:
     $T_n = an^2 + bn + c$
4. $T_1;\ T_2;\dots.T_{100}$: Terms indicated by T and the number of the term as a sub-script.
5. Objective:
   a. Find the values of the variables
   b. Use the values to find the general term
   c. Use the general term to calculate specific term values
   d. Use specific term values to find the term number

Patterns/ Sequences: ordered set of numbers

Linear:

Constant **first** difference between consecutive terms.

$$\boxed{T_n = dn + c}$$

T_n = general term
d = constant difference
n = number of the term

Notice how this is similar to a linear function $y = mx + c$

Steps to determine the nth term:

1. Find the constant difference
2. Substitute the constant difference (d) and the term value, along with the term number
3. Substitute the c- and d-values to define the nth term.

EXAMPLE

1. Determine the nth term of the following sequence:

T_1 T_2 T_3 T_4

2; 7; 12; 17

$7-2$ $12-7$ $17-12$

5 5 5

Using term 3 where $T_3 = 12$

$T_n = 5n + c$

$12 = 5(3) + c$

$12 = 15 + c$

$12 - 15 = c$

$-3 = c$

$\therefore\ T_n = 5n + 3$

2. Determine the 100th term

$T_{100} = 5(100) - 3$

$= 500 - 3$

$= 497$

$\therefore\ T_{100} = 497$

Quadratic:

Constant **second** difference between consecutive terms.

$$\boxed{T_n = an^2 + bn + c}$$

T_n = general term

n = number of the term

Notice how this is similar to the quadratic equation and formula for the parabola

Steps to determine the nth term:

1. Find the constant difference
2. Use the value of the second difference to find "a"
3. Use the "a" value and first difference to find "b"
4. Use "a" and "b" to find "c"

EXAMPLE

Determine the nth term of the following sequence:

T_1 T_2 T_3 T_4

Term 1 (a + b + c) → **6; 17; 34; 57**

$17-6$ $34-17$ $57-34$

First difference (3a + b) → **11 17 23**

$7-2$ $12-7$

Second difference (2a) → **6 6**

Second difference $= 2a$

$6 = 2a$

$3 = a$

First difference $= 3a + b$

$11 = 3a + b$

$11 = 3(3) + b$

$2 = b$

Term 1 $= a + b + c$

$6 = (3) + (2) + c$

$6 - 3 - 2 = c$

$1 = c$

$\therefore\ T_n = 3n^2 + 2n + 1$

NUMBER PATTERNS

Solving Quadratic Number Patterns

EXAMPLE 1

Prove that the second difference is 2a.

$T_n = an^2 + bn + c$

The terms will then be:

$T_1 = a(1)^2 + b(1) + c$

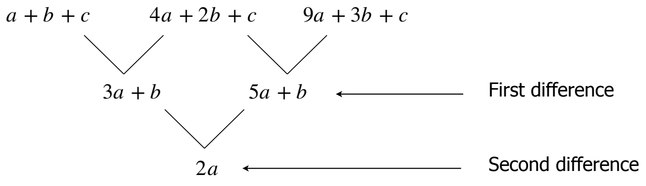
$T_1 = a + b + c$

$T_2 = a(2)^2 + b(2) + c$

$T_2 = 4a + 2b + c$

$T_3 = a(3)^2 + b(3) + c$

$T_3 = 9a + 3b + c$



EXAMPLE 2

Determine the term number for a specific value:
Which term is equal to **80** in the quadratic pattern with the general term $T_n = 2n^2 + n + 2$?

$80 = 2n^2 + n + 2$

$0 = 2n^2 + n + 2 - 80$

$0 = 2n^2 + n - 79$

$0 = (2n + 13)(n - 6)$

$2n + 13 = 0$ or $n - 6 = 0$

$2n = 13$ or $n = 6$

$n = \frac{13}{2}$ **BUT** no half terms/decimal number of terms

$\therefore\ n = 6$

Prove that n = 6

$T_6 = 2(6)^2 + (6) + 2$

$= 72 + 6 + 2$

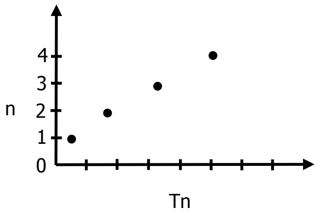
$= 80$

EXAMPLE 3

The relationship between the term value (T_n) and the term number (n) can be plotted on a graph.

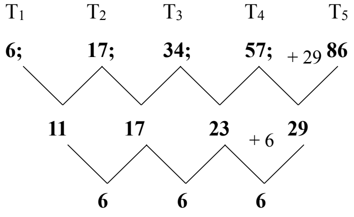
n	1	2	3	4
Tₙ	6	17	34	57

For $T_n = 3n^2 + 2n + 1$



Use the quadratic equation for the graph above ($T_n = 3n^2 + 2n + 1$) and find the next two terms.

Term 5:



OR

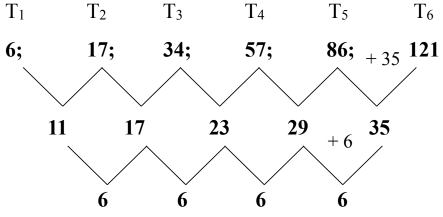
n = 5

$T_5 = 3(5)^2 + 2(5) + 1$

$= 75 + 10 + 1$

$= 86$

Term 6:



OR

n = 6

$T_6 = 3(6)^2 + 2(6) + 1$

$= 108 + 12 + 1$

$= 121$

Now find the value of the 100th term:

$T_{100} = 3(100)^2 + 2(100) + 1$

$= 30\ 000 + 200 + 1$

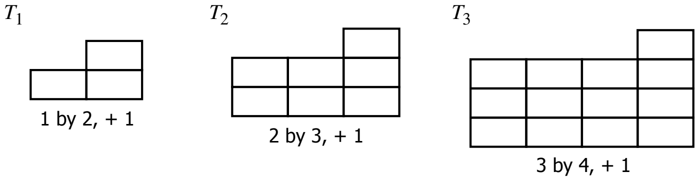
$= 30\ 201$

NUMBER PATTERNS

Solving Quadratic Number Patterns

EXAMPLE 4

Find the general term by inspecting the pictures



$T_1 = 3$: $1 \times 2 + 1$: $n(n+1)+1$

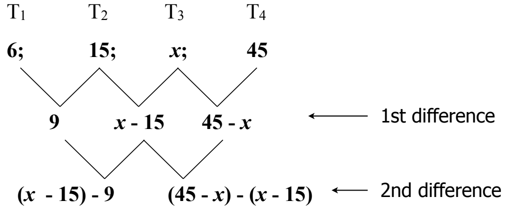
$T_2 = 7$: $2 \times 3 + 1$: $n(n+1)+1$

$T_3 = 13$: $3 \times 4 + 1$: $n(n+1)+1$

$$\therefore\ T_n = n(n+1)+1$$
$$= n^2 + n + 1$$

EXAMPLE 5

Determine the value of x if $6;\ 15;\ x;\ 45\ldots$ is a quadratic sequence



NB: Because this is a quadratic sequence, the **second difference** is equal!

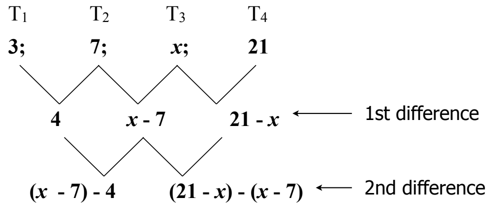
$$(x-15)-9 = (45-x)-(x-15)$$
$$x-24 = 45-x-x+15$$
$$x+2x = 45+24+15$$
$$3x = 84$$
$$x = 28$$

EXAMPLE 6

Researchers investigate the change in a new cell. Each hour they record the growth of the cell. In the table below, you can review the recorded changes. The researchers realised that the size of the cell followed a quadratic pattern.

Time	Hour monitored	Size (pm)
08:00	1	3
09:00	2	7
10:00	3	x
11:00	4	21
12:00	5	31
13:00	6	43

1. What is the size of the cell after 3 hours (x)?

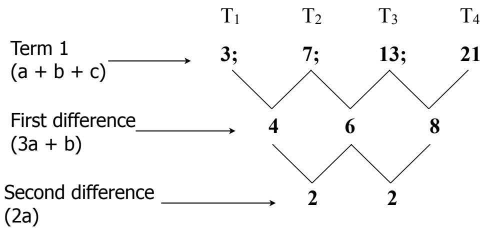


NB: Because this is a quadratic sequence, the **second difference** is equal!

$$\therefore\ (x-7)-4 = (21-x)-(x-7)$$
$$x-11 = 21-x-x+7$$
$$x+2x = 28+11$$
$$3x = 39$$
$$x = 13$$

2. Determine the size of the cell at 20:00 that evening.

Recording started at 07:00, ∴ 20:00 - 07:00 = **13 hrs**



Second difference $= 2a$
$$2 = 2a$$
$$1 = a$$
First difference $= 3a + b$
$$4 = 3(1) + b$$
$$4 - 3 = b$$
$$1 = b$$
Term 1 $= a + b + c$
$$3 = (1) + (1) + c$$
$$3 - 1 - 1 = c$$
$$1 = c$$

$$T_n = 1n^2 + 1n + 1$$
$$\therefore\ T_{13} = 1(13)^2 + 1(13) + 1$$
$$= 169 + 13 + 1$$
$$= 183$$

∴ At 20:00, the size of the cell will be 183 pm

FINANCE - SIMPLE AND COMPOUND INTEREST

REMINDERS:

1. **Inflation:**
   The rate at which prices increase over time
2. **Consumer Price Index (CPI):**
   the average prices of a basket of goods
3. **Exchange Rates:**
   The value of one currency for the purpose of conversion to another
4. **Population Growth:**
   Change of population size over time
5. **Hire Purchase:**
   Short term loan, deposit payable. Calculated using simple interest.
6. **Reducing balance loan:**
   Interest is paid on the reducing balance, the lower the balance, the less you have to pay.
7. **Nominal interest rate:**
   Quoted period and compounded period is different eg 15% per annum compounded monthly.
8. **Effective interest rate:**
   Quoted period and compound period is equal eg 0,75% per month compounded monthly.

COMPOUND PERIODS

Annually:	1 per year
Semi-annually:	2 per year
Quarterly:	4 per year
Monthly:	12 per year
Daily:	365 per year* *(excl leap years)

SIMPLE INTEREST

$$A = P(1 + in)$$

OR

$$A = P\left(1 + \frac{r}{100}n\right)$$

A = accumulated amount
P = original amount
n = number of periods
r = interest rate as a %
i = interest rate $\frac{r}{100}$

COMPOUND INTEREST

$$A = P(1 + i)^n$$

OR

$$A = P\left(1 + \frac{r}{100}\right)^n$$

EXAMPLE

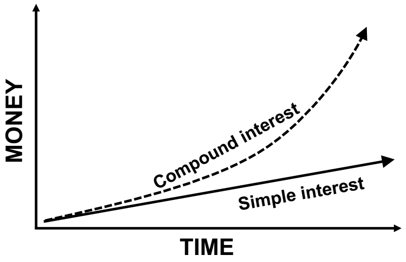
Determine the difference in the accumulated amounts when investing your savings of R 15 000 for 4 years at two different banks, both offer 6,5% however one offers simple interest and the other compound.

SIMPLE INTEREST

$$\begin{aligned} A &= P(1 + in) \\ A &= 15\ 000(1 + (0{,}065)(4)) \\ A &= R18\ 900 \end{aligned}$$

COMPOUND INTEREST

$$\begin{aligned} A &= P(1 + i)^n \\ A &= 15\ 000(1 + (0{,}065))^4 \\ A &= R19\ 296{,}99 \end{aligned}$$



HIRE PURCHASE

You buy a washing machine of R4 000 by signing a 2 year hire purchase agreement, you pay an R800 deposit. Calculate the

a) total amount you will repay if the interest rate is 12%

b) your monthly installment.

a) Deposit : R800

$$\begin{aligned} P &= 4\ 000 - 800 \\ &= 3\ 200 \\ A &= P(1 + in) \\ &= 3\ 200(1 + (0{,}12)(2)) \\ &= R3\ 968 \end{aligned}$$

b) $A = R3\ 968$

2 years = 24 equal payments

$$\frac{3\ 968}{24} = R165{,}33$$

POPULATION GROWTH

$$\boxed{P_{future} = P_{present}(1 + i)^n}$$

P_{future} = future population size
$P_{present}$ = present population size
i = average population (%)
n= number of years

The population of lions is 2 567 in 2015.
If the growth rate is 1,34%, calculate the number of lions in 2020.

$$2020 - 2015 = 5$$

$$\begin{aligned} P_{future} &= P_{present}(1 + i)^n \\ &= 2567(1 + 0{,}0134)^5 \\ &= 2743 \end{aligned}$$

(note that the number of lions will be an integer)

EXAMPLE

Calculate the future value of your investment after three years at an interest rate of 15% per annum compounded:

a) Annually

$$\begin{aligned} A &= P(1 + i)^n \\ &= 15\ 000(1 + (0{,}15))^3 \\ &= R22\ 813{,}13 \end{aligned}$$

b) Semi-annually

$$\begin{aligned} A &= P(1 + i)^n \\ &= 15\ 000\left(1 + \left(\frac{0{,}15}{2}\right)\right)^6 \\ &= R23\ 149{,}52 \end{aligned}$$

c) Quarterly

$$\begin{aligned} A &= P(1 + i)^n \\ &= 15\ 000\left(1 + \left(\frac{0{,}15}{4}\right)\right)^{12} \\ &= R23\ 331{,}81 \end{aligned}$$

d) Monthly

$$\begin{aligned} A &= P(1 + i)^n \\ &= 15\ 000\left(1 + \left(\frac{0{,}15}{12}\right)\right)^{36} \\ &= R23\ 495{,}16 \end{aligned}$$

Notice: As compounding periods increase during the year, so the accumulated amount increases.

EXAMPLE

If R13 865 is received after 6 years of being invested and the interest rate was 16% compounded annually, what was the original amount invested?

$$\begin{aligned} A &= P(1 + i)^n \\ 13\ 865 &= P(1 + 0{,}16)^6 \\ 13\ 865 &= 2{,}44P \\ \frac{13\ 865}{2{,}44} &= P \\ P &= 5\ 690{,}78 \end{aligned}$$

$\therefore$R 5 690,78 was the principal amount invested.

OR use the following formulae:

$A = P(1 + i)^n$ To find A
$P = A(1 + i)^{-n}$ To find P

FINANCE

NOMINAL VS EFFECTIVE INTEREST RATES (COMPOUND INTEREST)

Annual effective rate is equivalent to the nominal rate per annum compounded monthly, because it produces the same accumulated amount.

$$\boxed{1+i_{\text{eff}}=\left(1+\frac{i_{\text{Nom}}}{n}\right)^n}$$

i_{eff} = effective rate (annual)
i_{Nom} = nominal rate
n = number of compoundings per year

EXAMPLE

Convert a nominal rate of 18% per annum compounded monthly to an annual effective rate.

$$\begin{aligned}1+i_{\text{eff}} &= \left(1+\frac{0{,}18}{12}\right)^{12}\\ i_{\text{eff}} &= \left(1+\frac{0{,}18}{12}\right)^{12}-1\\ i_{\text{eff}} &= 0{,}196\\ \therefore i_{\text{eff}} &= 19{,}6\,\%\end{aligned}$$

EXAMPLE

You invest R25 000 at 14% per annum compounded monthly for a period of 12 months. Use the annual effective rate to show that the same accumulated amount will be obtained as when using the nominal rate.

$$\begin{aligned}1+i_{\text{eff}} &= \left(1+\frac{0{,}14}{12}\right)^{12}\\ i_{\text{eff}} &= \left(1+\frac{0{,}14}{12}\right)^{12}-1\\ i_{\text{eff}} &= 0{,}1493\\ \therefore i_{\text{eff}} &= 14{,}93\,\%\end{aligned}$$

The exponent **(12)** is calculated by noting there will be 12 compounding periods: once a month for 12 months.

$$\begin{aligned}A &= P(1+i)^n\\ &= 25\ 000\left(1+\left(\frac{0{,}14}{12}\right)\right)^{12}\\ &= R28\ 733{,}55\end{aligned}$$

$$\begin{aligned}A &= P(1+i)^n\\ &= 25\ 000(1+0{,}1493)^1\\ &= R28\ 733{,}55\end{aligned}$$

CHANGING INTEREST RATES

If the interest rate changes after a set period of time:

1. Determine the accumulated amount after the first period
2. Set the accumulated amount as the initial amount for the second period
3. Determine the accumulated amount after the second period.

EXAMPLE

R100 000 is invested for 6 years at an interest rate of 16% per annum compounded quarterly. Thereafter the accumulated amount is reinvested for 5 years at an interest rate of 14% compounded semi-annually. Calculate the value of the investment at the end of this period.

$$\begin{aligned}A &= P(1+i)^n\\ A &= 100\ 000\left(1+\frac{0{,}16}{4}\right)^{24}\\ A &= R256\ 330{,}42\end{aligned}$$

24 periods :
4 period (quarterly) p.a. over 6 years

$$\begin{aligned}A &= P(1+i)^n\\ A &= 256\ 330{,}42\left(1+\frac{0{,}14}{2}\right)^{10}\\ A &= R504\ 239{,}91\end{aligned}$$

10 periods :
2 period (semi-annual) p.a. over 6 years

EXAMPLE

R30 000 was left to you in a savings account. The interest rate for the first 4 years is 12% per annum compounded semi-annually. Thereafter the rates change to 18% per annum compounded monthly and you leave the money for another 3 years. What is the future value of the investment after the savings period.

$$\begin{aligned}A &= P(1+i)^n\\ A &= 30\ 000\left(1+\frac{0{,}12}{2}\right)^{8}\\ A &= R47\ 815{,}44\end{aligned}$$

8 periods :
2 period (semi-annual) p.a. over 4 years

$$\begin{aligned}A &= P(1+i)^n\\ A &= 47\ 815{,}44\left(1+\frac{0{,}18}{12}\right)^{36}\\ A &= R81\ 723{,}25\end{aligned}$$

36 periods :
12 period (monthly) p.a. over 3 years

Alternatively:
$$\begin{aligned}A &= P(1+i)^n\times(1+i)^n\\ &= 30\ 000\left(1+\frac{0{,}12}{2}\right)^{8}\times\left(1+\frac{0{,}18}{12}\right)^{36}\\ &= R81\ 723{,}26\end{aligned}$$

DEPRECIATION (DECAY)

Depreciation is the loss or decrease of value at a specified rate over time.

Depreciation: Loss of value over time
Book value: Value of equipment at a given time after depreciation
Scrap value: Book value of equipment at the end of its useful life

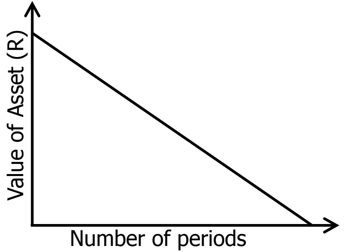
A = Book or scrap value
P = Present value
i = Depreciation rate
n = time period

LINEAR DEPRECIATION

Also known as simple decay or straight line depreciation

$$\mathbf{A = P(1 - in)}$$

Straight Line Depreciation

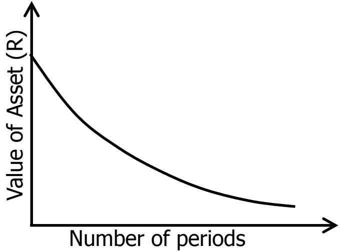


COMPOUND DEPRECIATION

Also known as depreciation on a reducing balance

$$\mathbf{A = P(1 - i)^n}$$

Reducing Balance Depreciation



EXAMPLE

My new car , to the value of R 200 000 depreciates at a rate of 9% per annum. What would the value of my car be after 6 years? Compare a linear depreciation to a reducing balance depreciation.

LINEAR DEPRECIATION

$$\begin{aligned}A &= P(1-in)\\ &= 200\ 000(1-(0{,}09)(6))\\ &= R92\ 000\end{aligned}$$

REDUCING BALANCE DEPRECIATION

$$\begin{aligned}A &= P(1-i)^n\\ &= 200\ 000(1-0{,}09)^6\\ &= R113\ 573{,}85\end{aligned}$$

EXAMPLE

The value of a piece of equipment depreciates from R15 000 to R5 000 in four years. What is the rate of depreciation calculated on the:

a) Straight line method

$$\begin{aligned}A &= P(1-in)\\ 5\ 000 &= 15\ 000((1-(x)4)\\ 5\ 000 &= 15\ 000-60\ 000x\\ -10\ 000 &= -60\ 000x\\ x &= 0{,}1667\end{aligned}$$

Depreciation rate = 16,67%

b) Reducing balance depreciation

$$\begin{aligned}A &= P(1-i)^n\\ 5\ 000 &= 15\ 000(1-i)^4\\ \frac{1}{3} &= (1-i)^4\\ \sqrt[4]{\frac{1}{3}}-1 &= -x\\ -0{,}2401... &= -i\\ i &= 0{,}2401...\times 100\\ r &= 24\,\%\end{aligned}$$

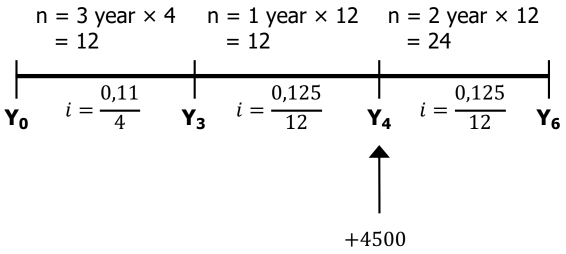
FINANCE

ADDITIONAL PAYMENTS OR WITHDRAWALS

Timelines assist in visualising and keeping track of different rates and payments. Set up each section with information about the number of terms, compound periods and interest rates.

EXAMPLE

You deposit R5 000 into a savings account and 4 years later you add R4 500. The interest for the first 3 years was 11% compounded quarterly, then it changed to 12,5% compounded monthly. Calculate the savings at the end of the 6th year.



$$\begin{aligned} Y_3 &= P(1+i)^n \\ &= 5000\left(1+\tfrac{0,11}{4}\right)^{12} \\ &= R6923,92 \end{aligned}$$

$$\begin{aligned} Y_4 &= P(1+i)^n + 4500 \\ &= 6923,92\left(1+\tfrac{0,125}{12}\right)^{12} \textcircled{+} 4500 \\ &= R12\ 340,76 \end{aligned}$$

$$\begin{aligned} Y_6 &= P(1+i)^n \\ &= 12\ 340,76\left(1+\tfrac{0,125}{12}\right)^{24} \\ &= R15\ 825,37 \end{aligned}$$

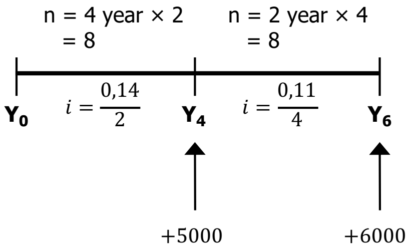
If withdrawals are made from the savings account, use subtraction instead

OR

$$\begin{aligned} Y_6 &= 5000\left[\left(1+\tfrac{0,11}{4}\right)^{12}\right]\left[\left(1+\tfrac{0,125}{12}\right)^{12} \textcircled{+} 4500\right]\left[\left(1+\tfrac{0,125}{12}\right)^{24}\right] \\ &= R15\ 825,37 \end{aligned}$$

EXAMPLE

You take out a loan to buy a new iPad. You make an additional payment of R 5 000 four years after taking out the loan. Two years later you repay the final amount of R 6 000. During the first four years of the loan the interest rate is 14% per annum compounded semi annually. For the last two years the rate changed to 11% per annum compounded quarterly. How much did you initially borrow?



Working backwards, we need the value of the the loan at Y_4:

$$\begin{aligned} P &= A(1+i)^{-n} \\ P &= 6000\left(1+\tfrac{0,11}{4}\right)^{-8} \\ P &= R4\ 829,44 \end{aligned}$$

$A = P(1+i)^n$ To find A

$P = A(1+i)^{-n}$ To find P

That means that the loan amount at the end of Y4, before the payment of R5000 was made, was **R9 829,44.**

Work backing backwards, we determine the initial amount borrowed / loan amount at Y_0.

$$\begin{aligned} P &= A(1+i)^{-n} \\ P &= 9829,44\left(1+\tfrac{0,14}{2}\right)^{-8} \\ P &= R5\ 720,82 \end{aligned}$$

OR combine the steps into a single calculation

$$\begin{aligned} P &= 6000\left[\left(1+\tfrac{0,11}{4}\right)^{-8} + 5000\right]\left[\left(1+\tfrac{0,14}{2}\right)^{-8}\right] \\ &= R5\ 720,82 \end{aligned}$$

PROBABILITY

Theoretical Probability of an event happening:

$$P(E) = \frac{\text{number of possible times event can occur}}{\text{number of possible outcomes}} = \frac{\text{n}(E)}{\text{n}(S)}$$

E = event S = sample space

Relative frequency or Experimental probability:

$$P(E) = \frac{\text{number of times the event occured}}{\text{number of trials done}}$$

Theoretical Probability of an event happening:

$S = \{\text{sample set}\}$

$A = \{\text{event A}\}$

$B = \{\text{event B}\}$

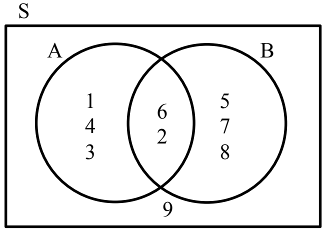
$A \cup B = \{\text{A union B}\}$ = in sets A or B

$A \cap B = \{\text{A intersection B}\}$ = in sets A and B

Addition Rule:

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

A and B are **inclusive** events as they have elements in common.



$A \cap B = \{2;6\}$

$A \cup B = \{1;\ 2;\ 3;\ 4;\ 5;\ 6;\ 7;\ 8\}$

$n(A) = 5$

$n(B) = 5$

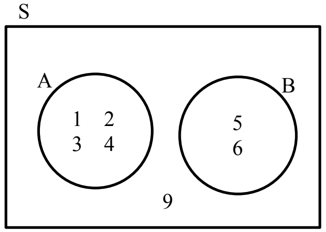
$n(A \cap B) = 2$

$n(A \cup B) = 8$

Mutually Exclusive:

$$A \cap B = \{\}$$

A and B are **mutually exclusive** events as they have no elements in common.



NOTE:

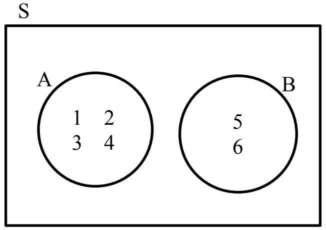
$n(A \cup B) = n(A) + n(B)$

$n(A \cap B) = 0$

$P(A \cup B) = P(A) + P(B)$

Exhaustive Events:

Events are **exhaustive** when they cover all elements in the sample set.



Complimentary Events:

Events A and B are **complimentary** events if they are mutually exclusive **and** exhaustive.

$$n(\text{not in } A) = n(A') = 1 - n(A)$$
$$P(A') = 1 - P(A) = P(B)$$

Independent Events:

Independent events are two events that do not affect each other's outcomes. E.g. choosing two coloured marbles from a bag, with replacement, thus, the first choice doesn't affect the outcome of the second choice.

Thus, the multiplication rule holds

$$P(A \text{ and } B) = P(A \cap B) = P(A) \times P(B)$$

Dependent Events:

Dependent events are when the first event (A) affects the outcome of the second event (B). E.g. choosing two coins from a wallet without replacing the first coin. The first choice affects the second choice as the coin in hand in no longer available for the second choice.

PROBABILITY

TREE DIAGRAMS

EXAMPLE 1

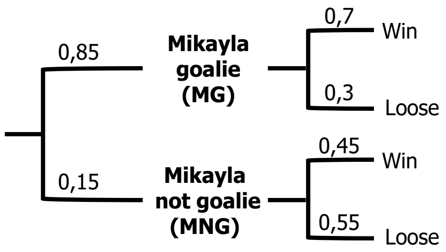
Questions:

The girls' hockey team has a match on Saturday. There is an 85% chance Makayla will play goalie. If she does there is a 70% chance the team will win but if she doesn't play there is a 45% chance they will win the game.

Draw a tree diagram to help answer the following questions:

1. What is the probability of Makayla playing goalie and the team winning?
2. What is the probability of the team losing the match on Saturday?

Solutions:



1. $\text{P(MG and win)} = 0{,}85 \times 0{,}7$

$= 0{,}6$

2. $\text{P(MG and loose)} + \text{P(MNG and loose)}$

$= 0{,}85 \times 0{,}3 + 0{,}15 \times 0{,}55$

$= 0{,}34$

EXAMPLE 2

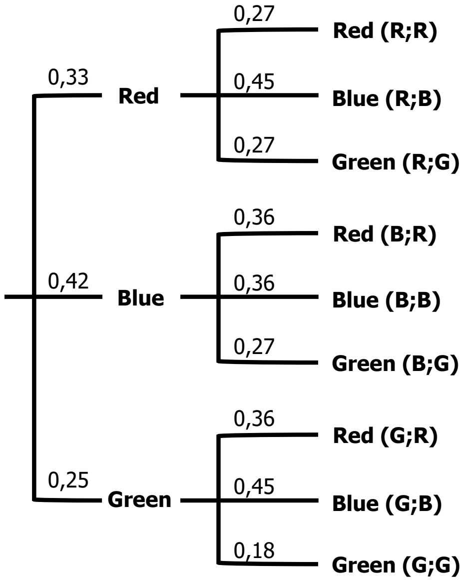
Questions:

A bag containing 4 red metal balls, 5 blue metal balls and 3 green metal balls. A ball is chosen at random and not replaced and then another ball is chosen at random again. Draw a tree diagram to list all possible events

Use the tree diagram to determine the probabilities of the following:

1. a blue metal ball in the first drawn.
2. a red metal ball then a green metal ball is drawn.
3. two metal balls of the same colour are drawn.

Solutions:



1. $\text{P(B and any)} = 0{,}42 \times 0{,}36 + 0{,}42 \times 0{,}36 + 0{,}42 \times 0{,}27$

$= 0{,}42$

2. $\text{P(R then G)} = 0{,}33 \times 0{,}27$

$= 0{,}09$

3. $\text{P(RR or BB or GG)} = 0{,}33 \times 0{,}27 + 0{,}42 \times 0{,}36 + 0{,}25 \times 0{,}18$

$= 0{,}29$

CONTINGENCY TABLE (OR TWO-WAY TABLE)

EXAMPLE 1

Questions:

In a group of 26 learners, 10 wear glasses, 8 are left handed and 6 of the students are both left-handed and wear glasses.

1. Complete the missing information in the table below.

	Left-handed	**Right-handed**	**TOTAL**
Glasses	a	b	c
No glasses	d	e	f
TOTAL	g	h	i

2. Calculate that if you pick any one of the 26 learners at random that they will be:
   a. Right handed
   b. Right handed and not wear glasses

Solutions:

1. $a = 6;\quad b = 4;\quad c = 10;\quad d = 2;\quad e = 14;\quad f = 16;\quad g = 8;\quad h = 18;\quad i = 26$

2. $\text{P(RH)} = \dfrac{18}{26} = 0{,}69$

3. $\text{P(RH and NG)} = \dfrac{14}{26} = 0{,}54$

EXAMPLE 2

Questions:

A survey was conducted asking learners which hand they use to write with and what colour ink they prefer. The results are summarised below.

		Hand used to write with		
		Left (L)	**Right (R)**	**Total**
Ink colour	**Blue (Bu)**	a	b	24
	Black (Ba)	c	d	56
	Total	50	30	80

The survey concluded that "the hand used to write with" and "Ink colour" are independent. Calculate the values of a, b and c.

Solutions:

$$\text{P(L and BU)} = \text{P(L)} \times \text{P(Bu)}$$

$$\frac{a}{80} = \frac{50}{80} \times \frac{24}{80}$$

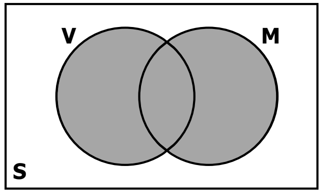
$$a = 15$$

$\therefore b = 24 - 15 = 9$

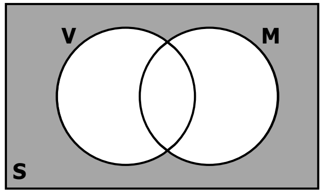
$c = 50 - 15 = 35$

PROBABILITY – VENN DIAGRAM

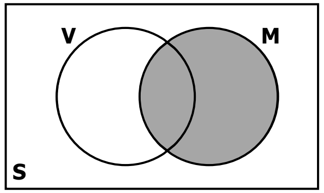
$V \cup M$



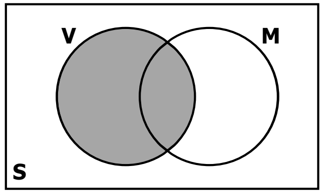
$(V \cup M)'$ or $V' \cap M'$



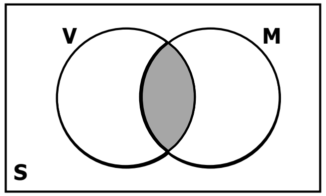
M



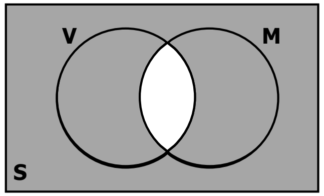
V



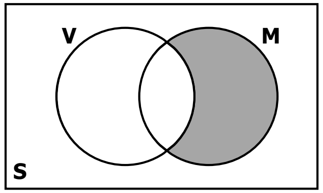
$V \cap M$



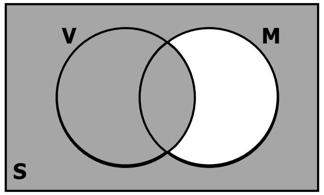
$(V \cap M)'$ or $V' \cup M'$



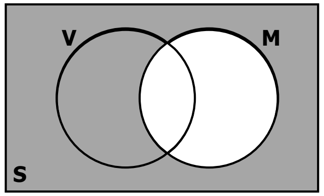
$V' \cup M$



$V \cup M'$



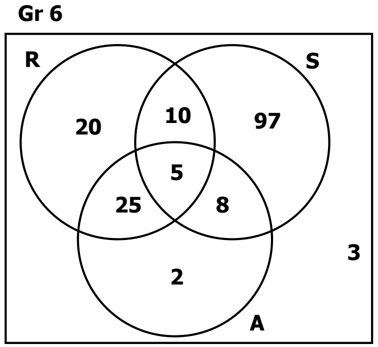
M'



EXAMPLE 1

Questions:

Calculate from the Venn diagram for a grade 6 group in which the number of equally likely ways the events (Reading(R); Sports(S) and Art(A)) can occur has been filled in:



1. P(A∩R∩S)
2. P(R and A and not S)
3. P(A or R)
4. P(S or R and not A)

Solutions:

1. $\text{P}(\text{A} \cap \text{R} \cap \text{S}) = \dfrac{5}{170} = \dfrac{1}{34}$
2. $\text{P(R and A and not S)} = \dfrac{25}{170} = \dfrac{5}{34}$
3. $\text{P(A or R)} = \dfrac{70}{170} = \dfrac{7}{17}$
4. $\text{P(S or R and not A)} = \dfrac{127}{170}$

EXAMPLE 2

Questions:

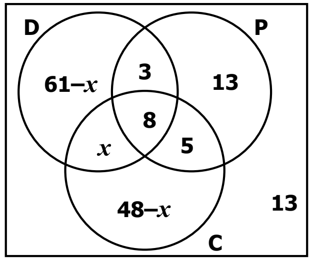
120 Gr 12 girls at Girls High where asked about their participation in the school's culture activities:

- 61 girls did drama (D)
- 29 girls did public speaking (P)
- 48 girls did choir (C)
- 8 girls did all three
- 11 girls did drama and public speaking
- 13 girls did public speaking and choir
- 13 girls did no culture activities

1. Draw a Venn diagram to represent this information.
2. Determine the number of Girls who participate in drama and choir.
3. Determine the probability that a grade 12 pupil chose at random will:
   a. only do choir.
   b. not do public speaking.
   c. participate in at least two of these activities.

Solutions:

1. **Gr 12**



2. $(61 - x) + 3 + 13 + x + 8 + 5 + (48 - x) + 13 = 120$

$-x = 120 - 151$

$\therefore x = 31$

$\therefore 61 - x = 30$ and $48 - x = 17$

3.

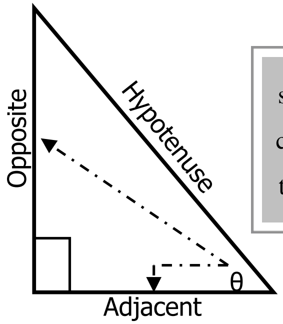
a. $\text{P(C only)} = \dfrac{17}{120} = 0{,}14$

b. $\text{P}(\text{P}') = \dfrac{30 + 31 + 17 + 13}{120} = \dfrac{91}{120} = 0{,}76$

c. $\text{P(at least 2)} = \dfrac{3 + 30 + 8 + 5}{120} = \dfrac{46}{120} = 0{,}38$

TRIGONOMETRY

BASIC DEFINITIONS



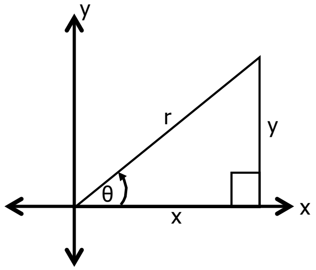
$\sin\theta = \frac{o}{h}$ soh

$\cos\theta = \frac{a}{h}$ cah

$\tan\theta = \frac{o}{a}$ toa

These are our basic trig ratios.

On the Cartesian Plane



$\sin\theta = \frac{y}{r}$

$\cos\theta = \frac{x}{r}$

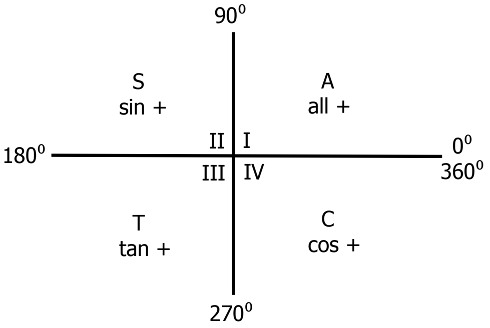
$\tan\theta = \frac{y}{x}$

Remember:

- $x^2 + y^2 = r^2$ (Pythagoras)
- Angles are measured upwards from the positive (+) x-axis (anti-clockwise) up to the hypotenuse (r).

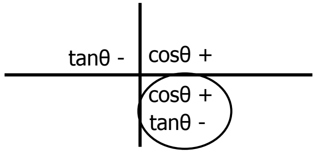
BASIC CAST DIAGRAM

Shows the quadrants where each trig ratio is +



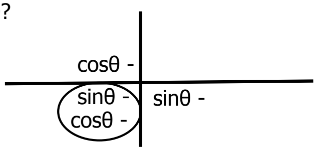
EXAMPLE

1. In which quadrant does θ lie if tanθ < 0 and cosθ > 0?



Quadrant IV

2. In which quadrant does θ lie if sinθ < 0 and cosθ < 0?



Quadrant III

FUNDAMENTAL TRIG IDENTITIES

Memorise:

$$\frac{\sin A}{\cos A} = \tan A$$

$$\sin^2 B + \cos^2 B = 1$$

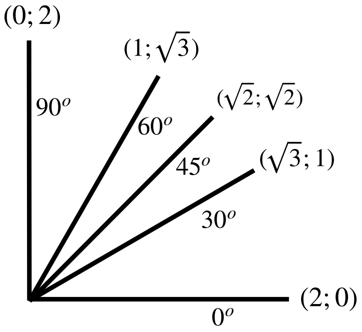
can be written as

$$\sin^2 B = 1 - \cos^2 B$$

$$\cos^2 B = 1 - \sin^2 B$$

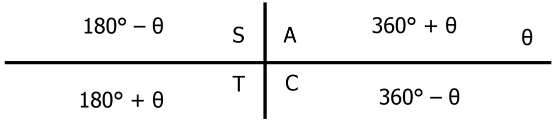
Special Angles

$r = 2$ $(x; y)$



REDUCTION FORMULAE

Reducing all angles to **acute** angles.



EXAMPLES

Reduce to an acute angle and simplify if possible (without a calculator):

1.
$$\sin 125^o$$
$$= \sin(180^o - 55^o)$$
$$= \sin 55^o$$
(QII so sin is +)

2.
$$\cos 260^o$$
$$= \cos(180^o + 80^o)$$
$$= -\cos 80^o$$
(QIII so cos is -)

3.
$$\tan 660^o$$
$$= \tan(360^o + 300^o)$$
$= \tan 300^o$ (QI so tan is +)
$$= \tan(360^o - 60^o)$$
$= -\tan 60^o$ (QIV so tan is -)
$$= -\frac{\sqrt{3}}{1}$$
$$= -\sqrt{3}$$

Remember: 60^o is a special angle

4.
$$\frac{\tan(180^o-\beta)\cos(180^o+\beta)\cos^2(360^o-\beta)}{\sin(360+\beta)} + \sin^2(180^o+\beta)$$
$$= \frac{(-\tan\beta)(-\cos\beta)(\cos\beta)^2}{\sin\beta} + (-\sin\beta)^2$$
$$= \tan\beta \,.\, \frac{\cos^3\beta}{\sin\beta} + \sin^2\beta$$
$$= \frac{\sin\beta}{\cos\beta} \,.\, \frac{\cos^3\beta}{\sin\beta} + \sin^2\beta$$
$$= \cos^2\beta + \sin^2\beta$$
$$= 1$$

Remember: Identities

Pythagoras Problems

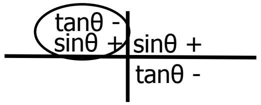
Steps:
1. Isolate the trig ratio
2. Determine the quadrant
3. Draw a sketch and use Pythagoras
4. Answer the question

EXAMPLE

If 3sinθ − 2 = 0 and tanθ < 0, determine $2\cos\theta + \frac{1}{\tan\theta}$ without using a calculator and using a diagram.

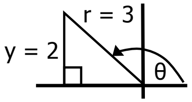
1.
$$3\sin\theta - 2 = 0$$
$$\sin\theta = \frac{2}{3} \quad \frac{y}{r}$$

2.



$\therefore$ *Quadrant II*

3.



$$x^2 + y^2 = r^2$$
$$x^2 + (2)^2 = (3)^2$$
$$x^2 = 5$$
$$x = \pm\sqrt{5}$$
$$\therefore x = -\sqrt{5}$$

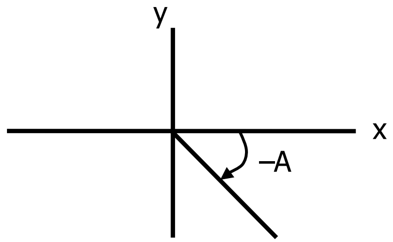
4.
$$2\cos\theta + \frac{1}{\tan\theta}$$
$$= 2(\frac{-\sqrt{5}}{3}) + \frac{1}{(\frac{2}{-\sqrt{5}})}$$
$$= \frac{-2\sqrt{5}}{3} - \frac{\sqrt{5}}{2}$$
$$= \frac{-4\sqrt{5} - 3\sqrt{5}}{6}$$
$$= \frac{-7\sqrt{5}}{6}$$

Remember:
$\cos\theta = \frac{x}{r}$
and
$\tan\theta = \frac{y}{x}$

TRIGONOMETRY

NEGATIVE ANGLES

Angles measured downwards (clockwise) from the positive x-axis, which can be seen as Quadrant IV.



Method 1: Q IV

sin(–A) = –sinA
cos(–A) = cosA
tan(–A) = –tanA

Method 2: Get rid of negative

Add 360° to the angle to make it positive.

EXAMPLES

Simplify without the use of a calculator: sin(–330°)

NB: Negative Angle

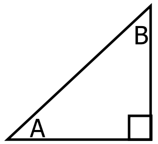
1) QIV

$\sin(-330^o)$
$= -\sin 330^o$
$= -\sin(360^o - 30^o)$
$= -(-\sin 30^o)$
$= \sin 30^o$
$= \frac{1}{2}$

2) $+360^o$

$= \sin(-330^o)$
$= \sin(360^o - 330^o)$
$= \sin 30^o$
$= \frac{1}{2}$

CO-FUNCTIONS



If A + B = 90° then sinA and cosB are known as co-functions.

sinA = sin(90^0 – B)
= cosB

EXAMPLES

1. $\sin 30^o$
$= \sin(90^o - 60^o)$
$= \cos 60^o$

2. $\cos 25^o$
$= \cos(90^o - 65^o)$
$= \sin 65^o$

NOTE:
Look at the quadrant first, **THEN** use the reduction/co-function formulae

3. $\sin(90^o - \alpha)$ $\quad QI,\ so\ \sin\ +\ 90^o\ \therefore \sin \leftrightarrow \cos$
$= \cos\alpha$

4. $\cos(90^o + \beta)$ $\quad QII,\ so\ \cos\ -\ 90^o\ \therefore \sin \leftrightarrow \cos$
$= -\sin\beta$

5. $\sin(\theta - 90^o)$ $\quad QIV,\ so\ \sin\ -\ 90^o\ \therefore \sin \leftrightarrow \cos$
$= -\cos\theta$

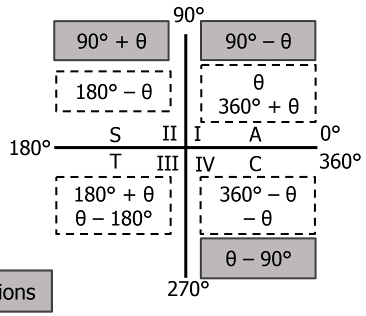
6. Simplify to a ratio of 10^0:

a) $\cos 100^o$ $\quad QII,\ so\ \cos\ -\ 90^o\ \therefore \sin \leftrightarrow \cos$
$= \cos(90^o + 10^o)$
$= -\sin 10^o$

b) $\tan 170^o$ $\quad QII,\ so\ \tan\ -\ 180^o\ \therefore reduction$
$= \tan(180^o - 10^o)$
$= -\tan 10^o$

FULL CAST DIAGRAM

Memorise the following diagram:



* Reductions Co-functions

PROVING IDENTITIES

Steps:

1. Separate LHS and RHS
2. Start on the more complex side
3. Prove that the sides are equal.

EXAMPLES

1. $\cos^2 x\,.\tan^2 x = \sin^2 x$

LHS = $\cos^2 x\,.\,\mathbf{tan^2\,x}$
$= \cos^2 x\,.\,\frac{\sin^2 x}{\cos^2 x}$
$= \sin^2 x =$ RHS

2. $1 - 2\sin x\,.\cos x = (\sin x - \cos x)^2$

RHS = $(\sin x - \cos x)^2$
$= \mathbf{sin^2\,x + cos^2\,x} - 2\sin x\,.\cos x$
$= 1 - 2\sin x\,.\cos x =$ LHS

3. $\tan x + \frac{\cos x}{1+\sin x} = \frac{1}{\cos x}$

LHS $= \mathbf{tan\,x} + \frac{\cos x}{1 + \sin x}$
$= \frac{\sin x}{\cos x} + \frac{\cos x}{1+\sin x}$
$= \frac{\sin x(1+\sin x) + \cos x(\cos x)}{\cos x(1+\sin x)}$
$= \frac{\sin x + \mathbf{sin^2\,x + cos^2\,x}}{\cos x(1+\sin x)}$
$= \frac{\cancel{\sin x + 1}}{\cos x\cancel{(1+\sin x)}}$
$= \frac{1}{\cos x} =$ RHS

PROBLEM SOLVING:

If cos25° = p, express the following in terms of p (i.e. get all angles to 25°):

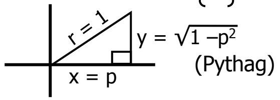
1. $\cos(-385^o)$ negative angle, so a) + 360° or b) Q IV - 385 Q I $\therefore + \cos$
$= \cos(-25^o)$
$= \cos 25^o$
$= p$

2. $\sin(65^o)$
$= \sin(90^o - 25^o)$ Q I, sin +
$= \cos(25^o)$
$= p$

3. $\sin(335^o)$
$= \sin(360^o - 25^o)$ Q IV, sin -
$= -\sin 25^o$

REMEMBER: Correct angle is 25° BUT wrong sin ratio. Thus draw sketch.

Given cos25^0 = $\frac{p}{1}\left(\frac{x}{r}\right)$



So, $-\sin 25^o = \frac{-y}{r}$
$= \frac{-\sqrt{1-p^2}}{1} = -\sqrt{1-p^2}$

4. $\tan(155^o)$
$= \tan(180^o - 25^o)$ Q II, tan -
$= -\tan 25^o$

This can be solved in two ways:

Method 1: Ratio
$= \frac{-\sin 25^o}{\cos 25^o}$
$= \frac{-\sqrt{1-p^2}}{p}$

Method 2: Sketch
$= \frac{-y}{x}$
$= \frac{-\sqrt{1-p^2}}{p}$

TRIG EQUATIONS

BASICS

Steps:

- Isolate trig ratios
- Reference angle (don't put negative into calculator)
- Choose quadrants
  - ➡ sin or cos: 2 Quadrants
  - ➡ tan: 1 Quadrant
- General solutions
  - ➡ $\sin\theta$ or $\cos\theta + k360°;\ k \in \mathbb{Z}$
  - ➡ $\tan\theta + k180°;\ k \in \mathbb{Z}$

REMEMBER: Only round off at the end

Common formulae:

$\theta = \sin^{-1} a + k360°$ or

$\theta = (180° - \sin^{-1} a) + k360°\ (k \in \mathbb{Z})$

$\theta = \pm\cos^{-1} a + k360°\ (k \in \mathbb{Z})$

$\theta = \tan^{-1} a + k180°\ (k \in \mathbb{Z})$

EXAMPLES

Solve for θ:

1. $3\sin\theta - 1 = 0$

$\sin\theta = \dfrac{1}{3}$

$\sin +$ in **QI and QII**

Reference∠ : 19,47°

QI: $\theta = 19{,}47° + k360°;\ k \in \mathbb{Z}$

QII: $\theta = 180° - 19{,}47° + k360°;\ k \in \mathbb{Z}$

$= 160{,}53° + k360°$

2. $\tan(3\theta + 30^o) + 1 = 0$

$\tan(3\theta + 30^o) = -1$

$\tan -$ in **QII**

Reference∠ : 45°

QII: $3\theta + 30° = 180° - 45° + k180°;\ k \in \mathbb{Z}$

$3\theta = 105° + k180°$

$\theta = 35° + k60°$

SQUARES

Hints:

- Do all four quadrants (± means the ratio **must** be both + **and** -)

EXAMPLE

Solve for β:

$4\sin^2\beta - 3 = 0$

$\sin^2\beta = \dfrac{3}{4}$

$\sin\beta = \pm\sqrt{\dfrac{3}{4}}$

Reference∠ : 60°

QI: $\beta = 60° + k360°;\ k \in \mathbb{Z}$

QII: $\beta = 180° - 60° + k360°$

$= 120° + k360°$

QIII: $\beta = 180° + 60° + k360°$

$= 240° + k360°$

QIV: $\beta = 360° - 60° + k360°$

$= 300° + k360°$

SINθ AND COSθ

Steps:

- sin and cos with the same angle
- Divide by cos to get tan

EXAMPLE

Solve for α:

$2\sin 2\alpha - \cos 2\alpha = 0$

$2\sin 2\alpha = \cos 2\alpha$

$\dfrac{2\sin 2\alpha}{\cos 2\alpha} = \dfrac{\cos 2\alpha}{\cos 2\alpha}$

$2\tan 2\alpha = 1$

$\tan 2\alpha = \dfrac{1}{2}$

$\tan +$ in **QI**

Reference∠ : 26,57°

QI: $2\alpha = 26{,}57° + k180°;\ k \in \mathbb{Z}$

$\alpha = 13{,}28° + k90°$

CO-FUNCTIONS

Hints:

- sin and cos with different angles
- Introduce the co-function with 90° - z
- The angle you change is the reference angle

EXAMPLES

Solve for x:

1. $\cos x = \sin(x - 10^o)$

$\cos x = \cos(90^o - (x - 10^o))$

$\cos x = \cos(100^o - x)$

Reference∠ : $100° - x$

QI: $x = 100° - x + k360°;\ k \in \mathbb{Z}$

$2x = 100° + k360°$

$x = 50° + k180°$

QII: $x = 360° - (100° - x) + k360°$

$x - x = 260° + k360°$

$0 = 260° + k360°$

No real solution

2. $\sin(x + 30^o) = \cos 2x$

$\sin(x + 30^o) = \sin(90^o - 2x)$

Reference∠ : $90° - 2x$

QI: $x + 30° = 90° - 2x + k360°;\ k \in \mathbb{Z}$

$3x = 60° + k360°$

$x = 20° + k120°$

QII: $x + 30° = 180° - (90° - 2x) + k360°$

$x + 30° = 90° + 2x + k360°$

$-x = 60° + k360°$

$x = -60° - k360°$

NOTE: Specific Solutions

If they ask for $x \in [-360°; 360°]$, choose integer values for k (...-3; -2; -1; 0; 1; 2; 3...) so that x falls in the given intervals.

$x = 30° + k120°$

$x = -330°;\ -210°;\ -90°;\ 30°;\ 150°;\ 270°$

$k = -3;\ k = -2;\ k = -1;\ k = 0;\ k = 1;\ k = 2$

OR

$x = -60° + k360°$

$x = -60°;\ 300°$

$k = 0;\ k = 1$

FACTORISING

Steps:

- Solve as you would a quadratic equation

EXAMPLES

Solve for x:

1. $\tan^2 x - 2\tan x + 1 = 0$

$(\tan x - 1)(\tan x - 1) = 0$

$\tan x = 1$

Reference∠ : 45°

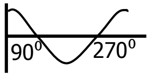
QI: $x = 45° + k180°;\ k \in \mathbb{Z}$

2. $\cos^2 x + \sin x . \cos x = 0$

$\cos x(\cos x + \sin x) = 0$

$\cos x = 0$ OR $\cos x = -\sin x$

Use trig graph:



$\dfrac{\cos x}{\cos x} = \dfrac{-\sin x}{\cos x}$

$\tan x = -1$

Reference∠ : 45°

$x = 90° + k180°;\ k \in \mathbb{Z}$

QII: $x = 135° + k180°$

3. $2\cos^2 x + 3\sin x = 0$

$2(1 - \sin^2 x) + 3\sin x = 0$

$2\sin^2 x - 3\sin x - 2 = 0$

$(2\sin x + 1)(\sin x - 2) = 0$

$\sin x = \dfrac{-1}{2}$ OR $\sin x = 2$

Reference∠ : 30°

No real solution

QIII: $x = 180° + 30° + k360°;\ k \in \mathbb{Z}$

$x = 210° + k360°$

QIV: $x = 360° - 30° + k360°;\ k \in \mathbb{Z}$

$x = 330° + k360°$

TRIG GRAPHS

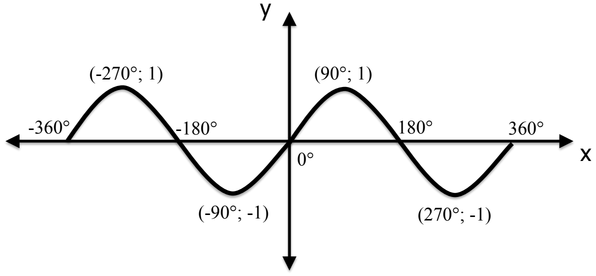
IMPORTANT!

When sketching trig graphs, you need to label the following:

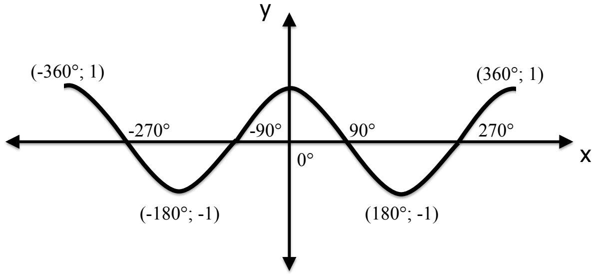
- **both axes**
- **x- and y-intercepts**
- **turning points**
- **endpoints** (if not on the axes)
- **asymptotes** (tan graph only)

BASICS

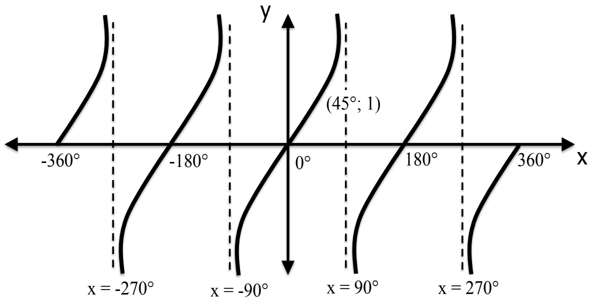
- $y = \sin x$ for $x \in [-360°; 360°]$



- $y = \cos x$ for $x \in [-360°; 360°]$



- $y = \tan x$ for $x \in [-360°; 360°]$



Notes for $\sin x$ and $\cos x$:

- ❖ Key points (intercepts/turning pts) every 90°
- ❖ Period (1 complete graph): 360°
- ❖ Amplitude (halfway between min and max): 1

Notes for $\tan x$:

- ❖ Key points every 45°
- ❖ Period (1 complete graph): 180°
- ❖ No amplitude can be defined
- ❖ Asymptotes at $x = 90^{o} + k180^{o}, k \in \mathbb{Z}$

AMPLITUDE CHANGE

- $y = a\,.\sin x$ or $y = a\,.\cos x$ or $y = a\,.\tan x$

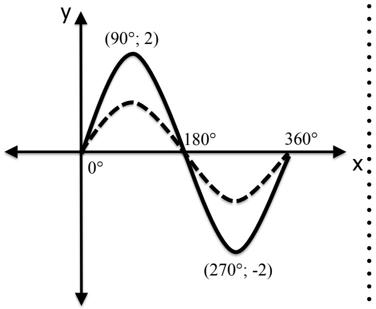
If a > 1 : stretch upwards
0 < a < 1 : compress downward
a < 0 : reflection in x-axis

EXAMPLES

1. $y = 2\sin x$ (solid line)

$y = \sin x$ (dotted line - for comparison)

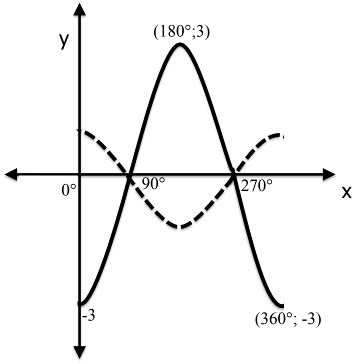
* Amplitude = 2



2. $y = -3\cos x$ (solid line)

$y = \cos x$ (dotted line - for comparison)

* Range: $y \in [-3; 3]$



VERTICAL SHIFT

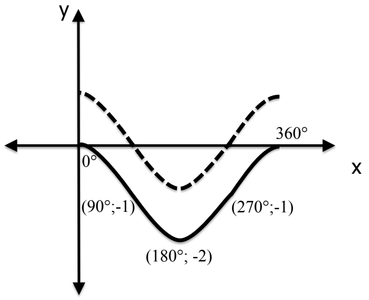
- $y = \sin x + q$ or $y = \cos x + q$ or $y = \tan x + q$

If q > 0 : upwards (e.g: $y = \sin x + 1$)
If q < 0 : downwards (e.g: $y = \cos x - 2$)

EXAMPLE

$y = \cos x - 1 \quad x \in [0°; 360°]$ (solid line)

$y = \cos x$ (dotted line - for comparison)



PERIOD CHANGE

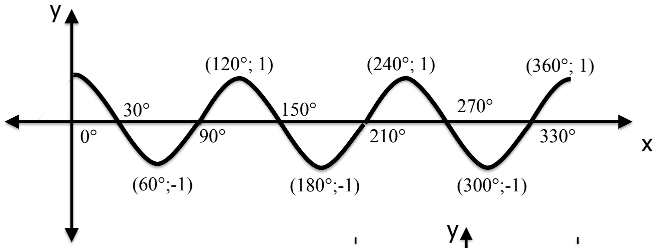
- $y = \sin bx$ or $y = \cos bx$ or $y = \tan bx$

The value of b indicates how many graphs are completed in the 'regular' period of that graph (i.e. $\sin x / \cos x$: 360° and $\tan x$: 180°)

EXAMPLES

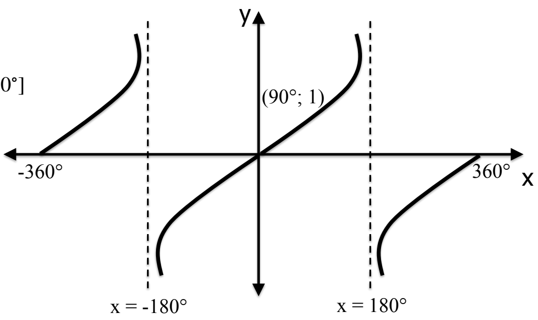
1. $y = \cos 3x \quad x \in [0°; 360°]$

* Normal period: 360°
* New period: 120° (3 graphs in 360°)
* Critical points every 90/3 = **30°**



2. $y = \tan \frac{1}{2}x \quad x \in [0°; 360°]$

* Normal period: 180°
* New period: 360° (½ graph in 180°)
* Critical points: every 45/0,5 = **90°**



TRIG GRAPHS

HORIZONTAL SHIFT

- $y = \sin(x - p)$ or $y = \cos(x - p)$ or $y = \tan(x - p)$

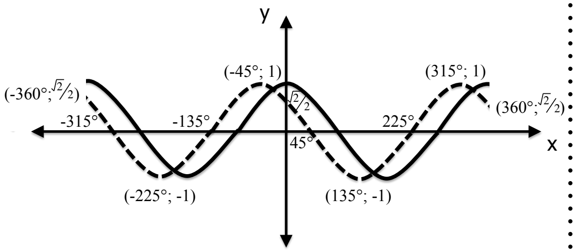
If p > 0 : shift right (e.g: $y = \sin(x - 30°)$)
p < 0 : shift left (e.g: $y = \cos(x + 45)$)

How to plot a horizontal shift:

- Plot the original curve
- Move the critical points left/right
- Label the x-cuts and turning points
- Calculate and label the endpoints and y-cut

EXAMPLES

1. $y = \cos(x + 45°)$ for $x \in [-360°; 360°]$ (dotted line)
$y = \cos x$ (solid line - for comparison)



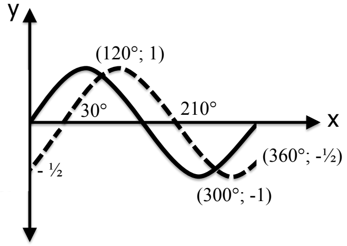
Endpoints:

$$\cos(-360° + 45°) = \frac{\sqrt{2}}{2} \quad \text{and} \quad cos(-360° + 45°) = \frac{\sqrt{2}}{2}$$

y-cut:

$$\cos(0° + 45°) = \frac{\sqrt{2}}{2}$$

2. $y = \sin(x - 30°)$ for $x \in [0°; 360°]$ (dotted line)
$y = \sin x$ (solid line - for comparison)



Endpoints:

$$\sin(0° + 45°) = -\frac{1}{2} \quad \text{and} \quad \sin(360° - 30°) = -\frac{1}{2}$$

y-cut:

The y-cut is one of the endpoints

EXAMPLE

Given $f(x) = \cos(x + 60°)$ and $g(x) = \sin 2x$

Questions:

1. Determine algebraically the points of intersection of $f(x)$ and $g(x)$ for $x \in$ [-90°;180°]
2. Sketch $f(x)$ and $g(x)$ for $x \in$ [-90°;180°]
3. State the amplitude of $f(x)$
4. Give the period of $g(x)$
5. Use the graphs to determine the values of x for which:
    a. $g(x)$ is increasing and positive
    b. $f(x)$ is increasing and positive
    c. $f(x) \geq g(x)$ - i.e. $f(x)$ is above $g(x)$
    d. $f(x) \cdot g(x) \geq 0$ - i.e. product is + or 0
6. Explain the transformation that takes $y = sin x$ to $y = \sin(2x - 60°)$

Solutions:

1. $\cos(x + 60°) = \sin 2x$

$\cos(x + 60°) = \cos(90° - 2x)$

Reference $\angle$: $90° - 2x$

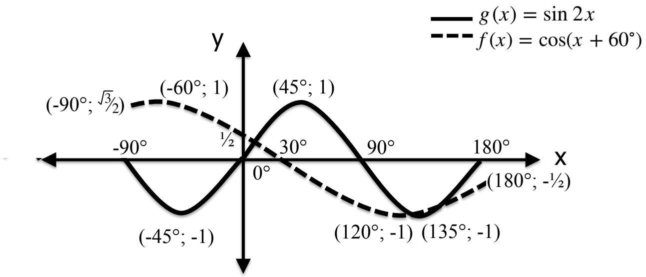
QI: $x + 60° = 90° - 2x + k360°;\ k \in \mathbb{Z}$
$3x = 30° + k360°$
$x = 10° + k120°$

QIV: $x + 60° = 360° - (90° - 2x) + k360°;\ k \in \mathbb{Z}$
$x + 60° = 270° + 2x + k360°$
$-x = 210° + k360°$
$x = -210° + k360°$

but $x \in [-90°; 180°]$

$\therefore\ x = 10°;\ 130°;\ 150°$

2.



For f(x):

Endpoints: $\cos(-90° + 60°) = \frac{\sqrt{3}}{2}$ and $\cos(180° + 60°) = -\frac{1}{2}$

y-cut: $\cos(0° + 60°) = \frac{1}{2}$

3. 1

4. $180°$

5. a. $x \in (0°; 45°)$
    b. $x \in [-90°; -60°)$
    c. $x \in [-90°; 10°] \cup (130°; 150°)$
    d. $x \in [0°; 30°] \cup [90°; 180°]$ also at $x = -90°$

6. Rewrite $y = \sin(2x - 60°)$ in the form $y = \sin b(x - p) = \sin(2(x - 30°))$
Transformation: b = 2 ∴ period is halved
p = 30 ∴ shifted 30 to the right°

TRIG GRAPHS

USING TRIG GRAPHS TO FIND RESTRICTIONS ON IDENTITIES

i.e. answering the question
"for which values of x will this identity be undefined?"

Identities are undefined if:

- the function is undefined
  $\tan x$ has asymptotes at $x = 90^\circ + k180^\circ;\ k \in \mathbb{Z}$
- any denominator is zero

Reminder:
$\frac{A}{0}$ is undefined

EXAMPLES

1. For which values of x will $\cos^2 x \cdot \tan^2 x = \sin^2 x$ be <u>defined</u>?

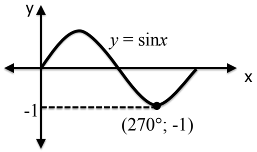
- $\tan x$ is undefined at $x = 90^\circ + k180^\circ;\ k \in \mathbb{Z}$
  $\therefore$ will be defined at $x \in \mathbb{R}$ and $x \neq 90^\circ + k180^\circ;\ k \in \mathbb{Z}$
- no denominators that could be zero

2. For which values of x will $\tan x + \frac{\cos x}{1 + \sin x} = \frac{1}{\cos x}$ be undefined?

- $\tan x$ is undefined at $x = 90^\circ + k180^\circ;\ k \in \mathbb{Z}$
- fractions are undefined if the denominator = 0

$\therefore$ if $1 + \sin x = 0$ or if $\cos x = 0$

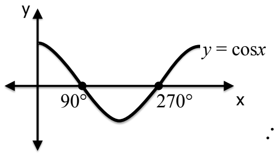
* $1 = \sin x = 0$
$\therefore \sin x = -1$



Use trig graphs for $0; \pm 1$

$\therefore x = 270^\circ + k360^\circ;\ k \in \mathbb{Z}$

* $\cos x = 0$



Use trig graphs for $0; \pm 1$

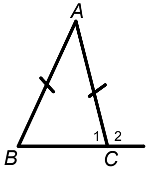
$\therefore x = 90^\circ + k180^\circ;\ k \in \mathbb{Z}$

$$\left.\begin{array}{l} x = 90^\circ + k180^\circ;\ k \in \mathbb{Z} \\ x = 270^\circ + k360^\circ;\ k \in \mathbb{Z} \\ x = 90^\circ + k180^\circ;\ k \in \mathbb{Z} \end{array}\right\}$$

can be summarised as: $x = 90^\circ + k180^\circ;\ k \in \mathbb{Z}$

EUCLIDEAN GEOMETRY

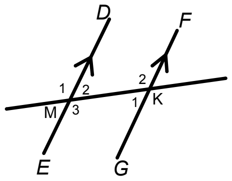
FLASHBACK: Theory from previous grades



$\hat{B} = \hat{C}_1$ (∠'s opp. = sides)

$\hat{A} + \hat{B} + \hat{C}_1 = 180°$ (sum ∠'s of Δ)

$\hat{C}_2 = \hat{A} + \hat{B}$ (ext. ∠'s of Δ)



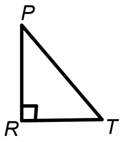
$\hat{K}_2 = \hat{M}_1$ (corres. ∠'s DE//GF)

$\hat{K}_2 = \hat{M}_3$ (alt. ∠'s DE//GF)

$\hat{K}_2 + \hat{M}_2 = 180°$ (co-int. ∠'s DE//GF)

$\hat{M}_1 = \hat{M}_3$ (vert. opp. ∠'s)

$\hat{K}_2 + \hat{K}_1 = 180°$ (∠'s on a str. line)



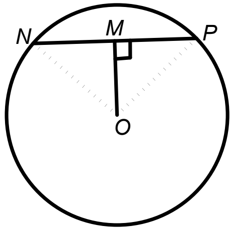
$PT^2 = PR^2 + RT^2$ (Pythag. Th.)

CIRCLE GEOMETRY

Theorem 1:

(line from centre ⊥ chord)

A line drawn from the centre of a circle perpendicular to a chord bisects the chord.



GIVEN: Circle centre O with chord $NP \perp MO$.

RTP: $NM = MP$

PROOF:

Join ON and OP

In ΔMON and ΔMOP

$N\hat{M}O = P\hat{M}O$ (OM⊥PN, given)

$ON = OP$ (radii)

$OM = OM$ (common)

$\therefore \Delta MON \equiv \Delta MOP$ (RHS)

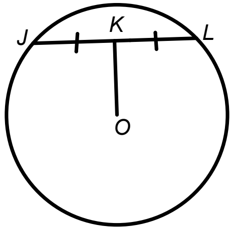
$NM = MP$

Converse of Theorem 1:

(line from centre mid-pt. chord)

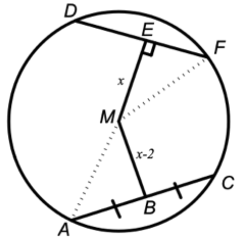
The line segment joining the centre of a circle to the midpoint of a chord is perpendicular to the chord.

If $JK = KL$, then
$OK \perp JL$



EXAMPLE

Given circle centre M with a diameter of 20 cm and chord DF of 12 cm.



Determine the length of of chord AC.

Join MF

$DE = EF = 6$ cm (line from centre ⊥ chord)

$MF = 10$ cm (radius)

$x^2 = 10^2 - 6^2$ (Pythag. Th.)

$x^2 = 64$

$x = 8$ cm

$\therefore MB = 8 - 3 = 5$ cm (given)

Join MA

$MA \perp AC$ (line from centre mid-pt. chord0

$MA = 10$ cm (radius)

$AB^2 = 10^2 - 5^2$ (Pythag. Th.)

$AB^2 = 75$

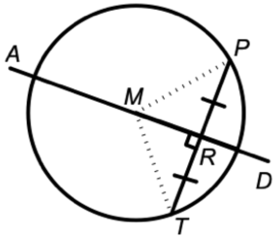
$AB = 8,66$ cm

$\therefore AC = 17,32$ cm

Converse two of Theorem 1:

(perp bisector of chord)

The perpendicular bisector of a chord passes through the centre of the circle.



GIVEN: $RT = RP$ and $MR \perp TP$

RTP: MR goes through the centre of the circle.

PROOF:

Choose any point, say M, on AD.

Join MT and MP

In ΔMRP and ΔMRT

$PR = RT$ (given)

$MR = MR$ (common)

$M\hat{R}P = M\hat{R}T = 90°$ (∠'s on a str. line)

$\Delta MRT \equiv \Delta MRP$ (SAS)

$\therefore MT = MP$

∴All points on AD are equidistant from P and T and the centre is equidistant from P and T.

∴The centre lies on AD.

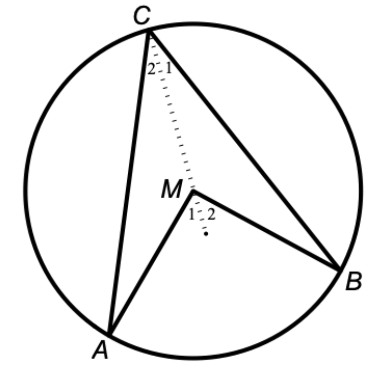
EUCLIDEAN GEOMETRY

CIRCLE GEOMETRY

Theorem 2:

(∠ at centre = 2 x ∠ at circum.)

The angle subtended by an arc at the centre of the circle is twice the angle the arc subtends at any point on the circumference of the circle.



GIVEN: Circle centre M with arc AB subtending $A\hat{M}B$ at the centre and $A\hat{C}B$ at the circumference.

RTP: $A\hat{M}B = 2 \times A\hat{C}B$

PROOF:

$AM = BM = CM$ (radii)

$\hat{A} = \hat{C}_2$ (∠ 's opp. = sides)

$\hat{B} = \hat{C}_1$ (∠ 's opp. = sides)

$\hat{M}_1 = \hat{A} + \hat{C}_2$ (ext. ∠ of Δ)

$\therefore \hat{M}_1 = 2\hat{C}_2$

$\hat{M}_2 = \hat{B} + \hat{C}_1$ (ext. ∠ of Δ)

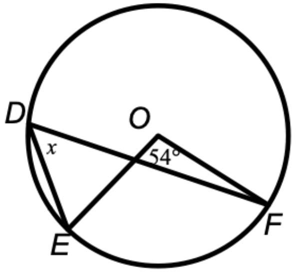
$\therefore \hat{M}_2 = 2\hat{C}_1$

$\therefore \hat{M}_1 + \hat{M}_2 = 2(\hat{C}_1 + \hat{C}_2)$

$\therefore A\hat{M}B = 2 \times A\hat{C}B$

EXAMPLE 1

Determine the value of x:

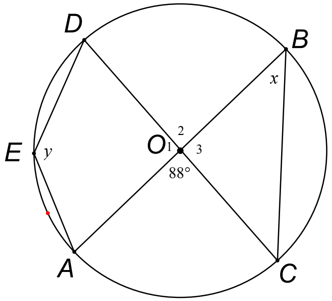


$x = 54° \div 2$ (∠ at centre = 2 x ∠ at circum.)

$\therefore x = 27°$

EXAMPLE 2

Determine the value(s) of x and y:



$x = 44°$ (∠ at centre = 2 x ∠ at circum.)

$OB = OC$ (radii)

$\hat{C} = 44°$ (∠ 's opp. = sides)

$\hat{O}_3 = 92°$ (sum ∠ 's of Δ)

$\hat{O}_2 = 88°$ (vert. opp. ∠ 's)

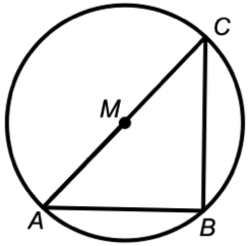
$$y = \frac{88° + 92° + 88°}{2}$$

$y = 137{,}5°$ (∠ at centre = 2 x ∠ at circum.)

Theorem 3:

(∠ in semi-circle)

The angle subtended by the diameter at the circumference of a circle is a right angle.

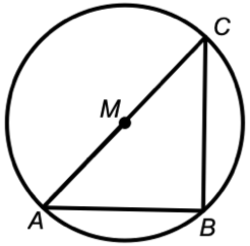


If AMC is the diameter then $\hat{B} = 90°$.

Converse Theorem 3:

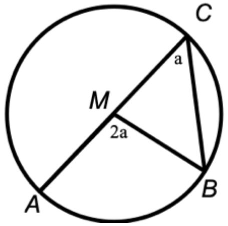
(chord subtends 90°)

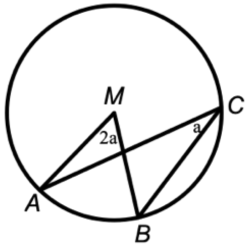
If a chord subtends an angle of 90° at the circumference of a circle, then that chord is a diameter of the circle.

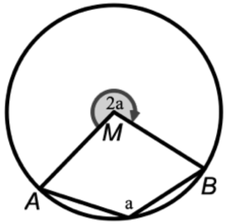


If $\hat{B} = 90°$ then AMC is the diameter.

ALTERNATIVE DIAGRAMS:

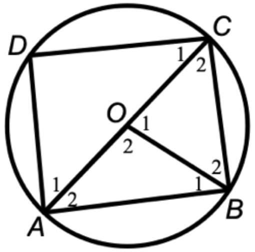






EXAMPLE

In circle O with diameter AC, $DC = AD$ and $\hat{B}_2 = 56°$. Determine the size of $D\hat{A}B$



$CO = OB$ (radii)

$\hat{C}_2 = \hat{B}_2 = 56°$ (∠ 's opp. = sides)

$\hat{O}_1 = 68°$ (sum ∠ 's of Δ)

$\hat{A}_2 = 34°$ (∠ at centre = 2 x ∠ at circum.)

$\hat{D} = 90°$ (∠ in semi-circle)

$\hat{A}_1 = \hat{C}_1$ (∠'s opp. = sides, DC = AD)

$\hat{A}_1 = 45°$ (sum ∠ 's of Δ)

$\therefore D\hat{A}B = 34° + 45° = 79°$

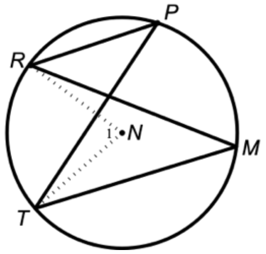
EUCLIDEAN GEOMETRY

CIRCLE GEOMETRY

Theorem 4:

(∠ in same seg.)

Angles subtended by a chord (or arc) at the circumference of a circle on the same side of the chord are equal.



GIVEN: Circle centre N with arc RT subtending $R\hat{P}T$ and $R\hat{M}T$ in the same segment.

RTP: $R\hat{P}T = R\hat{M}T$

PROOF:
Join NR and NT to form $\hat{N}_1$.

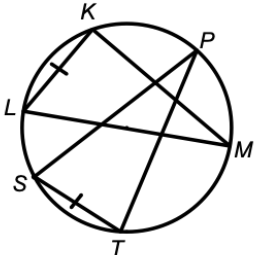
$\hat{M} = \frac{1}{2} \times \hat{N}_1$ (∠ at centre = 2 x ∠ at circum.)

$\hat{P} = \frac{1}{2} \times \hat{N}_1$ (∠ at centre = 2 x ∠ at circum.)

$\therefore R\hat{M}T = R\hat{P}T$

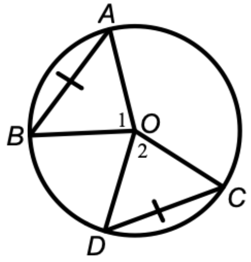
COROLLARIES:

a) Equal chords (or arcs) subtend equal angles at the circumference.



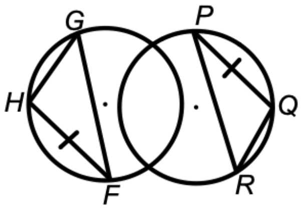
$KL = ST$ then $\hat{P} = \hat{M}$ (= chords, = ∠'s)

b) Equal chords subtend equal angles at centre of the circle.



If $AB = CD$ then $\hat{O}_1 = \hat{O}_2$ (= chords, = ∠'s)

c) Equal chords in equal circles subtend equal angles at their circumference.

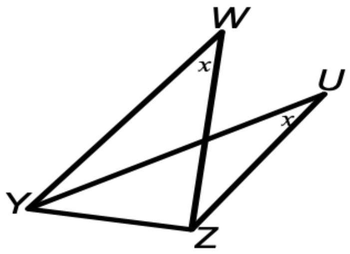


If $HF = PQ$ then $\hat{G} = \hat{R}$ (= chords, = ∠'s)

Converse Theorem 4:

(line subt. = ∠'s)

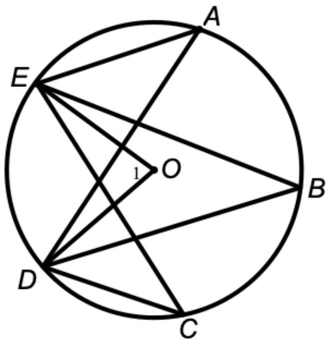
If a line segment joining two points subtends equal angles at two other points on the same side of the line segment, then these four points are concyclic (that is, they lie on the circumference of a circle.)



If $\hat{W} = \hat{U}$, then $WUZY$ is a cyclic quadrilateral.

EXAMPLE 1

Given circle centre O with $\hat{C} = 36°$



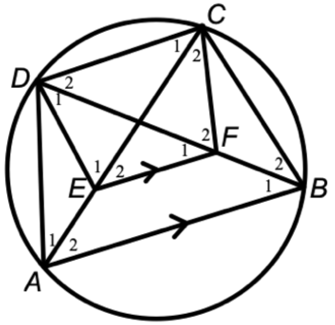
Calculate the values of angles:
$\hat{O}_1$, $\hat{A}$ and $\hat{B}$.

$\hat{O}_1 = 2 \times 36° = 72°$ (∠ at centre = 2 x ∠ at circum.)

$\hat{A} = \hat{B} = \hat{C} = 36°$ (∠'s same seg.)

EXAMPLE 2

Given circle $ABCD$ with $AB || EF$.



Questions:
a) Prove $CDEF$ is a cylindrical quad.
b) If $\hat{D}_2 = 38°$, calculate $\hat{E}_2$

Solutions**:**
a) $\hat{B}_1 = \hat{C}_1$ (∠'s same seg.)
$\hat{B}_1 = \hat{F}_1$ (corres. ∠'s, AB||EF)
$\therefore \hat{C}_1 = \hat{F}_1$
$\therefore CDEF$ cyc. quad (line subt = ∠'s)

b) $\hat{D}_2 = \hat{E}_2 = 38°$ (∠'s same seg quad CDEF)

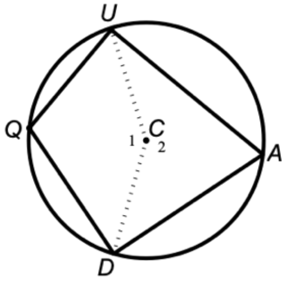
EUCLIDEAN GEOMETRY

CIRCLE GEOMETRY

Theorem 5:

(opp. ∠'s cyc. quad)

The opposite angles of a cyclic quadrilateral are supplementary.



GIVEN: Circle centre C with quad $QUAD$.

RTP: $\hat{Q} + \hat{A} = 180°$

PROOF:

Join UC and DC

$\hat{C}_1 = 2\hat{A}$ (∠ at centre = 2 x ∠ at circum.)

$\hat{C}_2 = 2\hat{Q}$ (∠ at centre = 2 x ∠ at circum.)

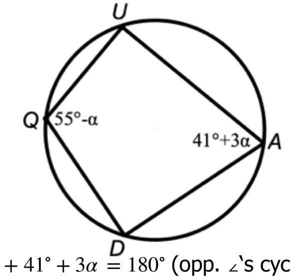
$\hat{C}_1 + \hat{C}_2 = 360°$ (∠'s around a pt.)

$\therefore 2\hat{A} + 2\hat{Q} = 360°$

$\therefore \hat{A} + \hat{Q} = 180°$

EXAMPLE 1

Calculate the value of α.



$55° - \alpha + 41° + 3\alpha = 180°$ (opp. ∠'s cyc. quad)

$2\alpha = 180° - 96°$

$2\alpha = 84°$

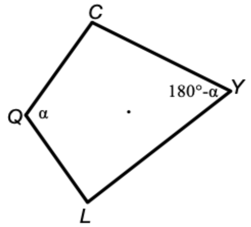
$\therefore \alpha = 42°$

Converse Theorem 5:

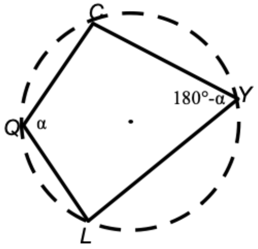
(opp. ∠'s quad supp)

If the opposite angles of a quadrilateral are supplementary, then the quadrilateral is cyclic.

If $\hat{Q} + \hat{Y} = 180°$
or $\hat{C} + \hat{L} = 180°$

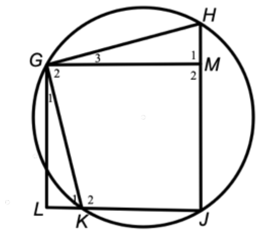


Then $QCYL$
is cyclic



EXAMPLE 2

Given circle $GHJK$ with $GM \perp HJ$ and $GL \perp LJ$. $\hat{G}_3 = 24°$



a) Is quadrilateral $GLJM$ a cyclic quad?
b) Is quadrilateral $GLJH$ a cyclic quad?

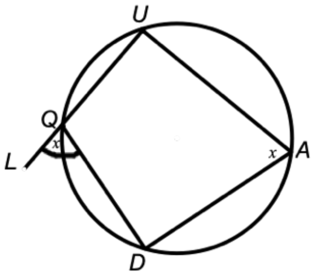
a) $\hat{M}_2 = 90°$ (Given $GM \perp HJ$)
$\hat{L} = 90°$ (Given $GL \perp LJ$)
$\therefore GLJM$ cyc quad (opp ∠'s quad suppl)

b) $\hat{H} = 180° - 24° - 90°$ (sum ∠'s of Δ)
$\hat{H} = 66°$
$GLJH$ not cyclic (opp ∠'s = 156° not 180°)

Theorem 6:

(ext. ∠ cyc quad)

The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.

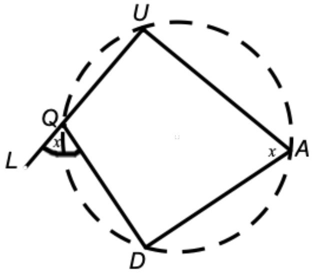


$L\hat{Q}D = \hat{A}$ (ext. ∠ cyc quad)

Converse Theorem 6:

(ext. ∠ = int. opp. ∠)

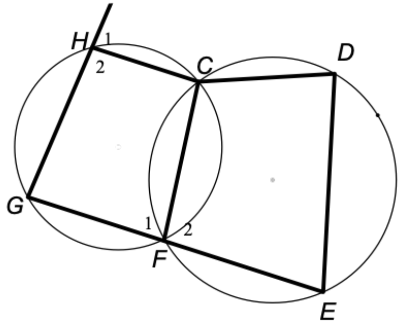
If the exterior angle of a quadrilateral is equal to the interior opposite angle, then the quadrilateral is cyclic.



If $L\hat{Q}D = \hat{A}$ then $QUAD$ is cyclic

EXAMPLE 1

GFE is a double chord and $\hat{H}_1 = 75°$



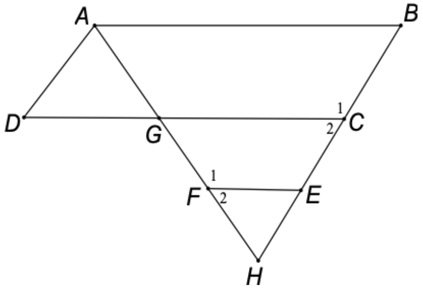
Determine the value of $\hat{D}$.

$\hat{H}_1 = \hat{F}_1 = 75°$ (ext. ∠ cyc quad)

$\hat{F}_1 = \hat{D} = 75°$ (ext. ∠ cyc quad)

EXAMPLE 2

$ABCD$ is a parallelogram and $B\hat{A}D = \hat{F}_1$.
Prove that $CEFG$ is a cyclic quad.



$B\hat{A}D = \hat{C}_1$ (opp. ∠'s parm)

$B\hat{A}D = \hat{F}_1$ (given)

$\therefore \hat{C}_1 = \hat{F}_1$

$\therefore CEFG$ is a cyc quad (ext. ∠ = int. opp. ∠)

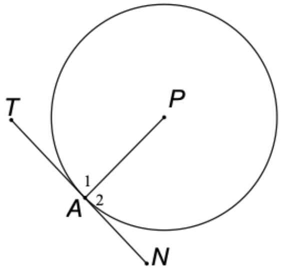
EUCLIDEAN GEOMETRY

CIRCLE GEOMETRY

Theorem 7:

(tan ⊥ radius)

A tangent to a circle is perpendicular to the radius at its point of contact.

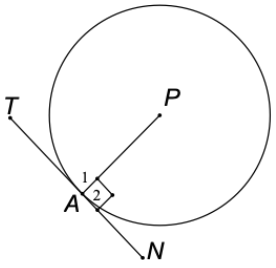


If TAN is a tangent to circle P, then $PA \perp TAN$

Converse Theorem 7:

(line seg ⊥ radius)

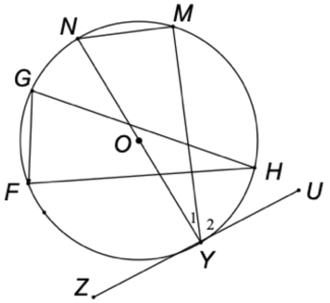
A line drawn perpendicular to the radius at the point where the radius meets the circumference is a tangent to the circle.



If $PA \perp TAN$, then TAN is a tangent to circle P.

EXAMPLE 1

Given circle centre O with tangent ZYU and $MN = FG$. If $\hat{H} = 18^\circ$ determine the size of $\hat{Y}_2$.



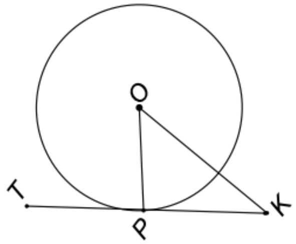
$\hat{Y}_1 = \hat{H} = 18^\circ$ (equal chords, = ∠'s)

$\hat{Y}_1 + \hat{Y}_2 = 90^\circ$ (tan ⊥ radius)

$\therefore \hat{Y}_2 = 90^\circ - 18^\circ = 72^\circ$

EXAMPLE 2

Prove that TPK is a tangent to circle centre O and radius of 8 cm, if $OK = 17$ cm and $PK = 15$ cm.



$OK^2 = 17^2 = 289$

$OP^2 + PK^2 = 8^2 + 15^2$

$= 289$

$\therefore OK^2 = OP^2 + PK^2$

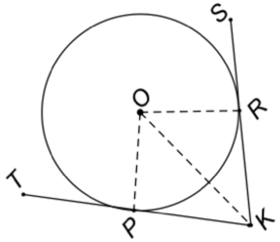
$\therefore OP \perp TPK$ (conv. Pythag. Th.)

$\therefore TPK$ is a tan to circle O (line seg ⊥ radius)

Theorem 8:

(tan from same pt.)

Two tangents drawn to a circle from the same point outside the circle are equal in length.



GIVEN: Tangents TPK and SRK to circle centre O.

RTP: $PK = RK$

PROOF:

Construct radii OR and OP and join OK.

In ΔOPK and ΔORK

$OP = OR$ (radii)

$OK = OK$ (common)

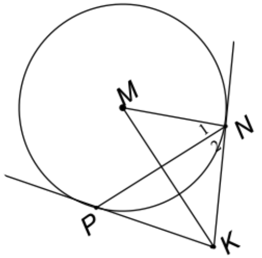
$O\hat{P}K = O\hat{R}K = 90^\circ$ (tan ⊥ radius)

$\therefore \Delta OPK \equiv \Delta ORK$ (RHS)

$\therefore PK = RK$

EXAMPLE

PK and KN are tangents to circle centre M. If $\hat{N}_1 = 24^\circ$, determine the size of $P\hat{K}N$.



$M\hat{N}K = 90^\circ$ (tan ⊥ radius)

$\therefore \hat{N}_2 = 66^\circ$

$PK = NK$ (tan from same pt.)

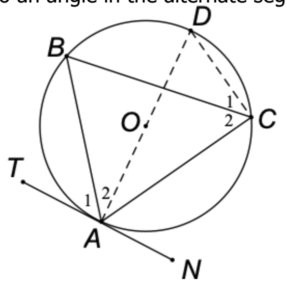
$\hat{N}_2 = N\hat{P}K = 66^\circ$ (∠ 's opp. = sides)

$\therefore P\hat{K}N = 48^\circ$ (sum ∠'s of Δ)

Theorem 9:

(tan-chord th.)

The angle between a tangent to a circle and a chord drawn from the point of contact is equal to an angle in the alternate segment.



GIVEN: Tangent TAN to circle O, and chord AC subtending $\hat{B}$.

RTP: $\hat{A}_1 = \hat{C}_2$

PROOF:

Draw in diameter AOD an join DC.

$\hat{A}_1 + \hat{A}_2 = 90^\circ$ (tan ⊥ radius)

$\hat{C}_1 + \hat{C}_2 = 90^\circ$ (∠ in semi-circle)

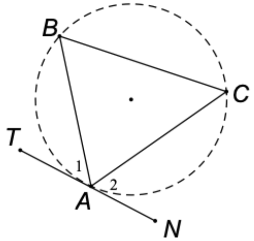
$\hat{A}_2 = \hat{C}_1$ (∠'s in same seg)

$\therefore \hat{A}_1 = \hat{C}_2$

Converse Theorem 9:

(∠ betw. line and chord)

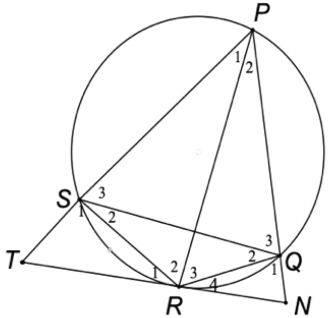
If a line is drawn through the end point of a chord, making with the chord an angle equal to an angle in the alternate segment, then the line is a tangent to the circle.



If $\hat{A} = \hat{C}$ or $\hat{A}_2 = \hat{B}$, TAN a tangent

EXAMPLE 1

TRN is a tangent at R and $SR = RQ$.
If $\hat{R}_1 = x$, find five angles equal to x.



$\hat{R}_1 = \hat{P}_1 = x$ (tan-chord th.)

$\hat{Q}_2 = x$ (tan-chord or ∠'s in same seg)

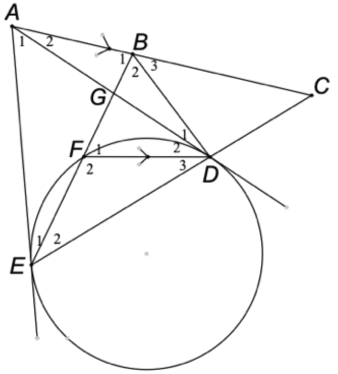
$\hat{Q}_2 = \hat{S}_2 = x$ (∠'s opp. = sides)

$\hat{S}_2 = \hat{P}_2 = x$ (∠'s same seg)

$\hat{P}_2 = \hat{R}_4 = x$ (tan-chord th.)

EXAMPLE 2

In the figure, AD and AE are tangents to the circle DEF. The straight line drawn through A, parallel to FD meets ED produced at C and EF produced at B. The tangent AD cuts EB at G.



a) Prove that $ABDE$ is a cyclic quadrilateral given $\hat{E}_2 = x$.

b) If it is further given that $EF = DF$, prove that ABC is a tangent to the circle passing through the points B, F and D.

a) $\hat{E}_2 = \hat{D}_2 = x$ (tan-chord th.)
$\hat{D}_2 = \hat{A}_2 = x$ (alt ∠'s AB||FD)
$\therefore ABDE$ a cyc quad (line seg subt. = ∠'s)

b) $\hat{E}_2 = \hat{D}_3 = x$ (∠'s opp. = sides)
$\hat{F}_1 = \hat{E}_2 + \hat{D}_3 = 2x$ (ext. ∠ of Δ)
$AE = AD$ (tan from same pt.)
$\hat{E}_1 + \hat{E}_2 = \hat{D}_2 + \hat{D}_3 = 2x$ (∠'s opp. = sides)
$\therefore \hat{B}_3 = 2x$ (ext. ∠ cyc quad)
$\hat{B}_3 = \hat{F}_1$
$\therefore ABC$ tan to circle (∠ betw. line and chord)

ALTERNATIVE

$\hat{F}_1 = \hat{B}_1$ (alt ∠'s AB||FD)
$\hat{B}_1 = \hat{D}_2 + \hat{D}_3$ (∠'s same seg)
$\hat{D}_1 = \hat{E}_1$ (∠'s same seg)
$\hat{E}_1 = \hat{D}_3$ (tan-chord th.)
$\therefore \hat{B}_1 = \hat{D}_2 + \hat{D}_1$
$\therefore ABC$ tan to circle (∠ betw. line and chord)

Hints when answering Geometry Questions

- Read the given information and mark on to the diagram if not already done.
- Never assume anything. If not given or marked on diagram is not true unless proved.
- As you prove angles equal or calculate angles mark them on to the diagram and write down statement and reason there and then.
- Make sure that by the end of the question you have used all the given information.
- If asked to prove something, it is true.
  For EXAMPLE if ask to prove ABCD a cyclic quad, then it is, but if you can't then you can use it as one in the next part of the question.

ANALYTICAL GEOMETRY

What is Analytical Geometry?

Analytical Geometry (Co-ordinate Geometry): Application of straight line functions in conjunction with Euclidean Geometry by using points on a Cartesian Plane.

FLASHBACK

Straight line parallel to the x-axis: m = 0

Straight line parallel to the y-axis: m = undefined

Straight line equation:

$y = mx + c$

Gradient formula:

$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$

Parallel gradients:

$m_1 = m_2$

Perpendicular gradients:

$m_1 \times m_2 = -1$

Distance:

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Co-linear:

$m_{AB} = m_{BC}$ OR $d_{AB} + d_{BC} = d_{AC}$

Collinear points A, B and C lie on the same line

Midpoint formula:

$M(x; y) = \left(\frac{x_2 + x_1}{2}; \frac{y_2 + y_1}{2}\right)$

Midpoint Theorem: If two midpoints on adjacent sides of a triangle are joined by a straight line, the line will be parallel to and half the distance of the third side of the triangle.

EXAMPLE

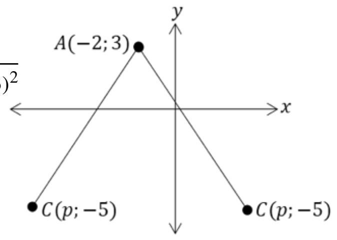
Given: $A(-2; 3)$ and $C(p; -5)$ are points on a Cartesian Plane.

1. If $AC = 10$ units determine the value(s) of p.
2. If $C(4; -5)$, determine the equation of the line AC.
3. Determine the co-ordinates of M, the midpoint of AC.
4. If $B\left(-1; \frac{5}{3}\right)$ determine if A, B and C are collinear.
5. Determine the equation of the line perpendicular to AC passing through B.

SOLUTION

1. Draw a sketch diagram. C has two potential x-coordinates for p.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$10 = \sqrt{(p - (-2))^2 + (-5 - 3)^2}$$
$$100 = (p + 2)^2 + 64$$
$$100 = p^2 + 4p + 4 + 64$$
$$0 = p^2 + 4p - 32$$
$$0 = (p + 4)(p + 8)$$
$$p = 4 \text{ or } p = -8$$



2. **Line equation** requires solving m and c.

$$m = \frac{\Delta y}{\Delta x}$$
$$m_{AC} = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{3 - (-5)}{-2 - 4}$$
$$= -\frac{4}{3}$$

$$y = mx + c$$
$$(3) = -\frac{4}{3}(-2) + c$$
$$c = \frac{1}{3}$$

$$\therefore y = -\frac{4}{3}x + \frac{1}{3}$$

3. **Midpoint formula**

$$M(x; y) = \left(\frac{x_2 + x_1}{2}; \frac{y_2 + y_1}{2}\right)$$
$$= \left(\frac{-2 + 4}{2}; \frac{3 + (-5)}{2}\right)$$
$$M(1; -1)$$

4. **Prove collinearity** by proving that the points share a common gradient.

$$m = \frac{\Delta y}{\Delta x} \qquad m = \frac{\Delta y}{\Delta x}$$
$$m_{AB} = \frac{3 - \frac{5}{3}}{-2 - (-1)} \qquad m_{BC} = \frac{\frac{5}{3} - (-5)}{-1 - 4}$$
$$m_{AB} = -\frac{4}{3} \qquad m_{BC} = -\frac{4}{3}$$

$$\therefore A, B \text{ and } C \text{ are collinear}$$

5. **Line equation** requires solving m_2 and c w.r.t. B.

$$m_{AC} \times m_2 = -1$$
$$-\frac{4}{3} \times m_2 = -1$$
$$m_2 = \frac{3}{4}$$

$$y = mx + c$$
$$\left(\frac{5}{3}\right) = \frac{3}{4}(-1) + c$$
$$c = \frac{29}{12}$$

$$\therefore y = \frac{4}{3}x + \frac{29}{12}$$

ANALYTICAL GEOMETRY

Converting gradient (m) into angle of inclination (θ)

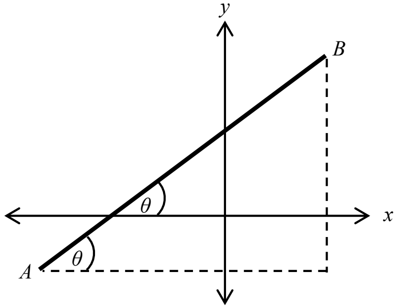
$m_{AB} = \frac{\Delta y}{\Delta x}$

and

$\tan\theta = \frac{o}{a} = \frac{\Delta y}{\Delta x}$

therefore;

$\therefore m_{AB} = \tan\theta$

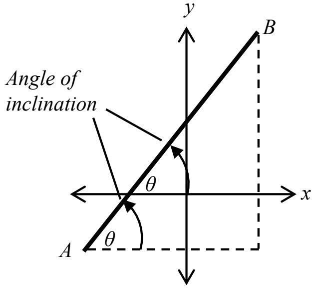


The angle of inclination (θ) is always in relation to a horizontal plane in an anti-clockwise direction.

Positive gradient:

$m > 0$
$\tan^{-1}(m) = \theta$

The reference angle is equal to the angle of inclination.

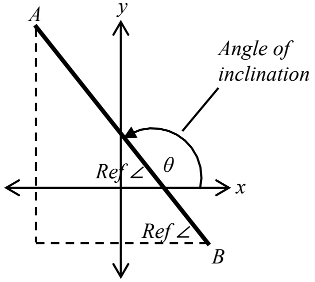


Negative gradient:

$m < 0$
$\tan^{-1}(m) = \text{ref } \angle$

Angle of inclination:
$\theta + \text{ref } \angle = 180°$ (∠'s on str. line)

The angle of inclination must be calculated from the reference angle.



Converting a positive gradient into an angle

$m > 0$
$\tan^{-1}(m) = \theta$

The reference angle is equal to the angle of inclination.

Given: $A(-1;-6)$ and $B(3;5)$ are two points on a straight line. Determine the angle of inclination.

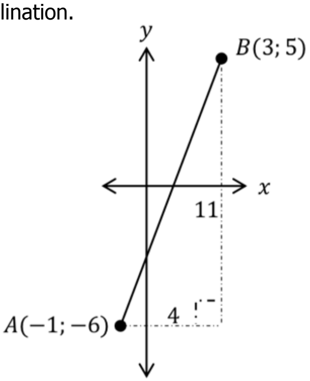
$m = \tan\theta$

$\frac{y_2 - y_1}{x_2 - x_1} = \tan\theta$

$\frac{5-(-6)}{3-(-1)} = \tan\theta$

$\tan^{-1}\left(\frac{11}{4}\right) = \theta$

$\therefore \theta = 70°$

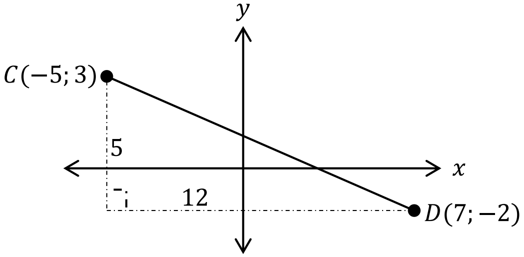


Converting a negative gradient into an angle

$m < 0$
$\tan^{-1}(m) = \text{ref } \angle$

Angle of inclination:
$\theta + \text{ref } \angle = 180°$ (∠'s on str. line)

Given: $C(-5;3)$ and $D(7;-2)$ are two points on a straight line. Determine the angle of inclination.



$m = \tan\theta$

$\frac{5}{12} = \tan\theta$

$\tan^{-1}\left(\frac{5}{12}\right) = \theta$

$\therefore \text{ref. } \angle = 22{,}6°$

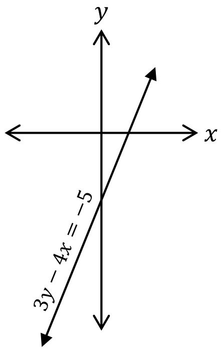
$\theta + \text{ref. } \angle = 180°$

$\theta = 180° - 22{,}6°$

$= 157{,}4°$

EXAMPLE

Given: straight line with the equation $3y - 4x = -5$. Determine the angle of inclination correct to two decimal places.

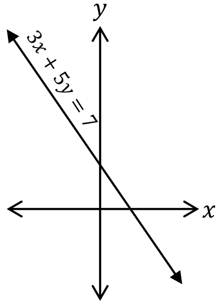


$3y - 4x = -5$ - make y the subject
$3y = 4x - 5$
$y = \frac{4}{3}x - \frac{5}{3}$ - note that m>0

$m = \tan\theta$ - sub. m and solve θ
$\frac{4}{3} = \tan\theta$
$\tan^{-1}\left(\frac{4}{3}\right) = \theta$
$\therefore \theta = 53{,}13°$ - m>0; ref. ∠ = angle of inclination

EXAMPLE

Given: straight line with the equation $3x + 5y = 7$. Determine the angle of inclination correct to two decimal places.



$3x + 5y = 7$ - make y the subject
$5y = -3x + 7$
$y = -\frac{3}{5}x - \frac{7}{5}$ - note that m<0

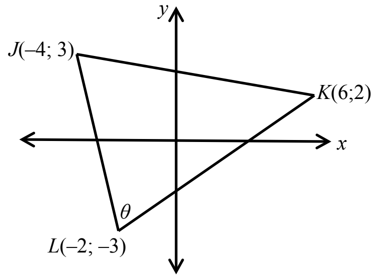
$m = \tan\theta$ - sub. m as a positive value to determine the ref. ∠
$\frac{3}{5} = \tan\theta$
$\tan^{-1}\left(\frac{3}{5}\right) = \theta$
$\therefore \text{ref. } \angle = 30{,}96°$

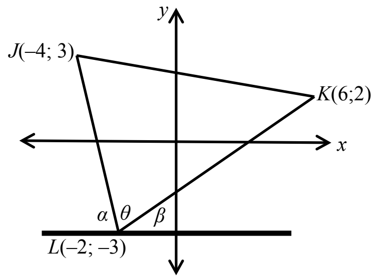
$\theta + \text{ref } \angle = 180°$ - m<0; ref. ∠ + θ = 180°
$\theta = 180° - 30{,}96°$
$\theta = 149{,}04°$

ANALYTICAL GEOMETRY

Finding an angle that is not in relation to a horizontal plane

Construct a horizontal plane, parallel to the x-axis. This will allow you to use the 'sum of adjacent angles on a straight line' in order to calculate the value of the angle.





$m_{JL} = -\frac{6}{2} = -3$

$m = tan\,\alpha$

$3 = tan\,\alpha$

$tan^{-1}(3) = \alpha$

$71{,}6° = \alpha$

$m_{KL} = \frac{5}{8}$

$m = tan\,\beta$

$\frac{5}{8} = tan\,\beta$

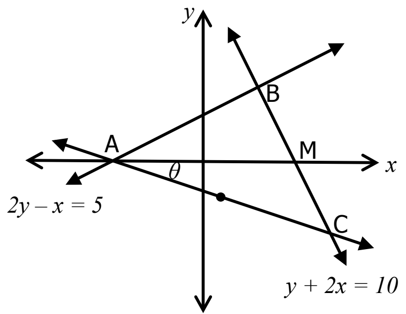
$tan^{-1}\left(\frac{5}{8}\right) = \beta$

$32° = \beta$

$$\begin{aligned}\theta &= 180° - (\alpha + \beta)\\ &= 180° - (71{,}6° + 32°)\\ &= 76{,}4°\end{aligned}$$

EXAMPLE

Given: In the diagram: Straight line with the equation $2y - x = 5$, which passes through A and B. Straight line with the equation $y + 2x = 10$, which passes through B and C. M is the midpoint of BC. A, B and C are vertices of ΔABC. $M\hat{A}C = \theta$. A and M lie on the x-axis.



Questions:

1. Determine the following:
   a. The co-ordinates of A
   b. The co-ordinates of M.
   c. The co-ordinates of B.
2. What type of triangle is ABC? Give a reason for your answer.
3. If $A(-5;0)$ and $B(3;4)$, show that $AB = BC$ (leave your answer in simplest surd form).
4. If $C(7;-4)$, determine the co-ordinate of N, the midpoint of AC.
5. Hence, or otherwise, determine the length of MN.
6. If $ABCD$ is a square, determine the co-ordinates of D.
7. Solve for θ correct to one decimal places.

Solutions:

a. $2y - x = 5$

$2y = x + 5$

$y = \frac{1}{2}x + \frac{5}{2}$

x – cut : $0 = \frac{1}{2}x + \frac{5}{2}$

$0 = x + 5$

$-5 = x$

$\therefore A(-5;0)$

b. $y + 2x + 10$

$y = -2x + 10$

x – cut : $0 = -2x + 10$

$2x = 10$

$x = 5$

$\therefore M(5;0)$

c. $\frac{1}{2}x + \frac{5}{2} = -2x + 10$

$x + 5 = -4x + 20$

$5x = 15$

$x = 3$

$y = -2(3) + 10$

$y = 4$

$\therefore B(3;4)$

2. ABC is a right-angled triangle:

$m_{AD} \times m_{BC} = -1$

$\therefore b = 90°$

3. $d_{AB} = \sqrt{(-5-3)^2 + (0-4)^2}$

$= 4\sqrt{5}$

$d_{BC} = \sqrt{(3-7)^2 + (4-(-4))^2}$

$= 4\sqrt{5}$

$\therefore AB = BC$

4. $N(x;y) = \left(\frac{-5+7}{2};\frac{0+(-4)}{2}\right)$

$N(1;-2)$

5. $MN = 2\sqrt{5}$ (Midpt theorem)

6. If $ABCD$ is a square, then AC is the diagonal, which makes N the midpoint for both diagonals $\therefore D(-3;-8)$

7. $m_{AC} = \frac{\Delta y}{\Delta x}$

$= \frac{0-(-4)}{-5-7}$

$= -\frac{1}{3}$

$m = tan\,\theta$

$-\frac{1}{3} = tan\,\theta$

$tan^{-1}\left(-\frac{1}{3}\right) = \theta$

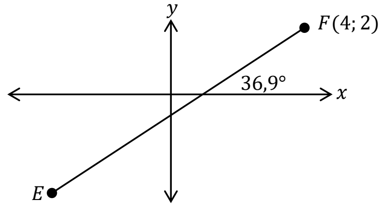
$\theta = 18{,}4°$

Converting an angle into a gradient

Sub. the ref. ∠ into $m = \tan\theta$.

Remember to add the – sign to answers for negative gradients.

Given: E and $F(4;2)$ are points on a straight line with an angle of inclination of 36,9°. Determine the value of m correct to two decimal places.



$m = \tan\theta$

$m = \tan(36{,}9°)$

$m = 0{,}75$

HELPFUL HINTS:

1. Make a quick rough sketch if you are given co-ordinates without a drawing.
2. Always make y the subject if you are given straight line equations.
3. Know your types of triangles and quadrilaterals. Proving them or using their properties is a common occurrence.
4. The angle of inclination is ALWAYS in relation to the horizontal plane.

STATISTICS

REMINDER

Discrete data: Data that can be counted, e.g. the number of people.

Continuous data: quantitative data that can be measured, e.g. temperature range.

Measures of central tendency: a descriptive summary of a dataset through a single value that reflects the data distribution.

Measures of dispersion: The dispersion of a data set is the amount of variability seen in that data set.

Cumulative frequency: The total of a frequency and all frequencies so far in a frequency distribution

Variance: measures the variability from an average or mean. a Small change in the numbers of a data set equals a very small variance

Standard Deviation: the amount the data value or class interval differs from the mean of the data set.

Outliers: Any data value that is more than 1,5 IQR to the left of Q_1 or the right of Q_3, i.e.

Outlier < $Q_1 - (1{,}5\times IQR)$ or

Outlier > $Q_3 + (1{,}5\times IQR)$

Causation: the action of causing something

Univariate: Data concerning a single variable

Bivariate: Data concerning two variables

Interpolation: an estimation of a value within two known values in a sequence of values.

Extrapolation: an estimation of a value based on extending a known sequence of values or facts beyond the area that is certainly known

REPRESENTING DATA

Ungrouped data = discrete
Grouped data = continuous

NB: Always arrange data in ascending order.

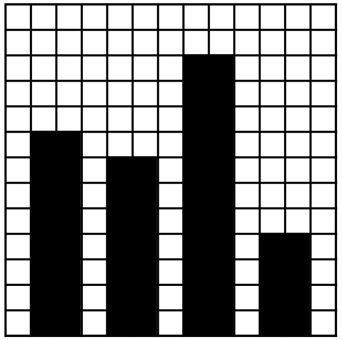
FREQUENCY TABLE

Mark	Tally	Frequency
4	||	2
5	||	2
6	||||	4
7	||||	5
8	||||	4
9	||	2
10	|	1

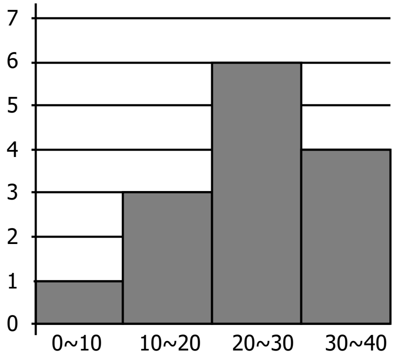
STEM AND LEAF PLOTS

Stem	Leaf
0	1, 1, 2, 2, 3, 4
1	0, 0, 0, 1, 1, 1
2	5, 5, 7, 7, 8, 8
3	0, 1, 1, 1, 2, 2
4	0, 4, 8, 9
5	2, 6, 7, 7, 8
6	3, 6

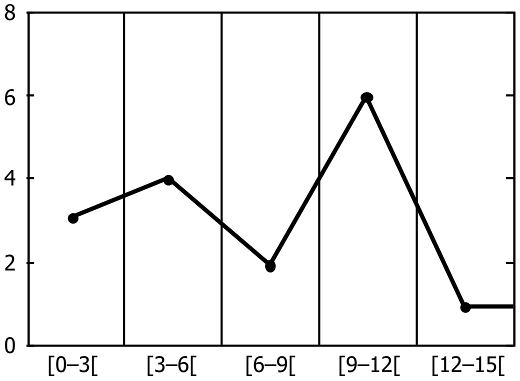
BAR GRAPH



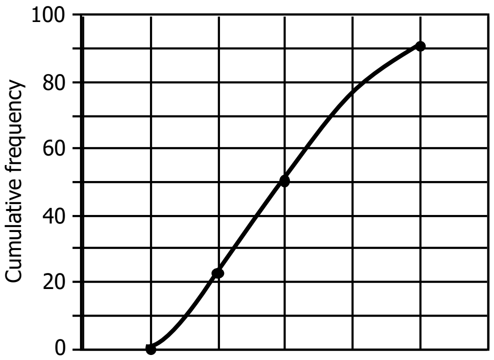
HISTOGRAM



FREQUENCY POLYGON



OGIVES



MEASURES OF DISPERSEMENT

Range

$$\text{range} = \text{max value} - \text{min value}$$

Note: range is greatly influenced by outliers

Interquartile range

$$\text{IQR} = Q_3 - Q_1$$

Note: spans 50% of the data set

Semi-Interquartile range

$$\text{semi} - \text{IQR} = \frac{1}{2}(Q_3 - Q_1)$$

Note: good measure of dispersion for skewed distribution

INDICATORS OF POSITION

Quartiles

The three quartiles divide the data into four quarters.

Q_1 = Lower quartile or first quartile
Q_2 = Second quartile or median
Q_3 = Upper quartile or third quartile

Percentiles

Indicates which percentage of data is below the specific percentile.

Q_1= 25th percentile
Q_2= 50th percentile
Q_3= 75th percentile

All other percentiles can be calculated using the formula:

$$i = \frac{p}{100}(n)$$

where;
i = the position of the p^{th} percentile
p = the value of the i^{th} position

STATISTICS

MEASURES OF CENTRAL TENDENCY FOR UNGROUPED DATA

Mean

$$\bar{x} = \frac{\text{sum of all values}}{\text{total number of values}}$$

$$\bar{x} = \frac{\Sigma x}{n}$$

where;

$\bar{x}$ = mean

Σx = sum of all values

n = number of values

Mode

The mode is the value that appears most frequently in a set of data points.

Bimodal: a data set with 2 modes

Trimodal: a data set with 3 modes

Median

The median is the middle number in a set of data points.

$$\text{position of median} = \frac{1}{2}(n+1)$$

Where;

n = number of values

If n = odd number, the median is part of the data set.

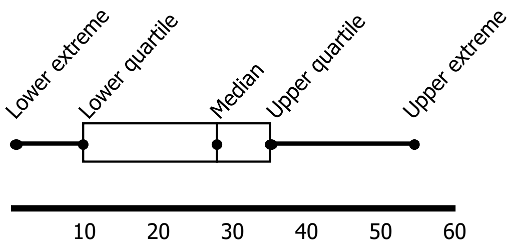
If n = even number, the median will be the average between the two middle numbers.

FIVE NUMBER SUMMARY

1. Minimum value
2. Lower quartile Q_1
3. Median
4. Upper quartile Q_3
5. Maximum value

BOX AND WHISKER PLOT

A box and whisker plot is a visual representation of the five number summary.



MEASURES OF DISPERSION AROUND THE MEAN

Variance

Variance measures the variability from an average or mean.

The variance for a population is calculated by:

1. Calculate the mean(the average).
2. Subtracting the mean from each number in the data set and then squaring the result. The results are squared to make the negatives positive. Otherwise negative numbers would cancel out the positives in the next step. It's the distance from the mean that's important, not positive or negative numbers.
3. Averaging the squared differences.

Standard deviation

Standard deviation is the amount the data value or class interval differs from the mean of the data set.

$$\sigma = \sqrt{\frac{\Sigma f(x-\bar{x})^2}{n}}$$

where;

n = number of data values

x = midpoint of interval

$\bar{x}$ = estimated mean

EXAMPLE:

Continuous data is grouped into class intervals which consist of an upper class boundary (maximum value) and lower class values (minimum value).

Class interval	frequency (f)	Midpoint $x = \frac{\text{uper class barrier} + \text{lower class barrier}}{2}$	$(f \times x)$	$(x-\bar{x})^2$	$f(x-\bar{x})^2$
$0 \leq x \leq 10$	3	$\frac{10+0}{2} = 5$	$3 \times 5 = 15$	$(5 \times \overline{15{,}71})^2 = 114{,}7$	$3(114{,}7) = 344{,}11$
$10 \leq x \leq 20$	7	$\frac{20+10}{15} = 15$	$7 \times 15 = 105$	$(15 \times \overline{15{,}71})^2 = 0{,}5$	$7(0{,}5) = 3{,}53$
$20 \leq x \leq 30$	4	$\frac{30+20}{2} = 25$	$4 \times 25 = 100$	$(25 \times \overline{15{,}71})^2 = 88{,}3$	$4(88{,}3) = 354{,}22$
total :	**14**	**14**	**220**		$\mathbf{\Sigma f(x-\bar{x})^2 = 692{,}86}$

Mean:

$$\text{mean}(\bar{x}) = \frac{\text{sum of all frequency} \times \text{mean value}}{\text{total frequency}}$$

$$= \frac{220}{14}$$

$$= 15{,}71$$

Standard deviation:

$$\sigma = \sqrt{\frac{\Sigma f(x-\bar{x})^2}{n}}$$

$$= \sqrt{\frac{692{,}86}{14}}$$

$$= 7{,}03$$

A small Standard deviation tells us the numbers are clustered closely around the mean, a larger standard deviation indicates more scattered data.

STATISTICS

MEASURES OF CENTRAL TENDENCY FOR GROUPED DATA

Estimated mean

$$\text{mean}(\bar{x}) = \frac{\text{sum of all frequency} \times \text{mean value}}{\text{total frequency}}$$

where;

$\bar{x}$ = estimated mean

n = number of values

Modal class interval

The modal class interval is the class interval that contains the greatest number of data points.

Median class interval

The median class interval is the interval that contains the middle number in a set of data points.

$$\text{position of median} = \frac{1}{2}(n+1)$$

Where;

n = number of values

If n = odd number, the median is part of the data set.

If n = even number, the median will be the average between the two middle numbers.

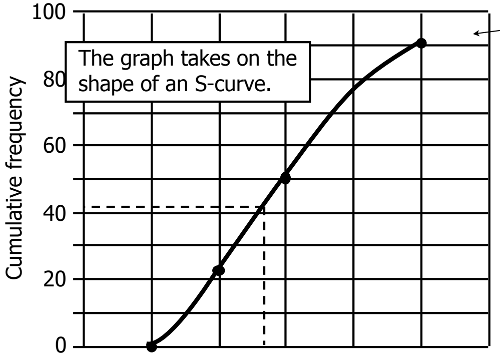
EXAMPLE:

Step 1: Determine cumulative frequencies form a frequency table.

We conduct a survey on the ages of people who visit the corner shop, 80 people partake in the survey.

Class interval	Frequency	Cumulative frequency	Interpretation	Graph points
$0 \le x < 15$	0	0	0 participants are younger than 15.	(15;0)
$15 \le x < 30$	14	0 + 14 = **14**	14 people were younger than 30.	(30;14)
$30 \le x < 45$	22	14 + 22 = **36**	36 people were younger than 45.	(45;36)
$45 \le x < 60$	30	36 + 30 = **66**	66 people were younger than 60.	(60;66)
$60 \le x < 75$	14	66+ 14 = **80**	All participants were younger than 75.	(75;80)

Step 2: Represent information on a cummulative frequency/ogive curve



Coordinates (x;y)

The x-coordinate represents the upper boundary of the class interval.

y-coordinate represents the cumulative frequency.

Interpretations from the graph:

Median

There is an even nr of data items in our set (80) so the median liesmidway between the two middle values. The median is halfway between the 40th and 41st term. Find the value on the y-axis and draw a line from that point to determine the value on the x-axis.

Quartiles

Similar to the method used to find the median you can determine the upper or lower quartiles from the graph.

Percentiles

The median and quartiles divide the data into 50% and 25% respectively, should you need to calculate a different percentile this can be done by calculation or read from the graph.

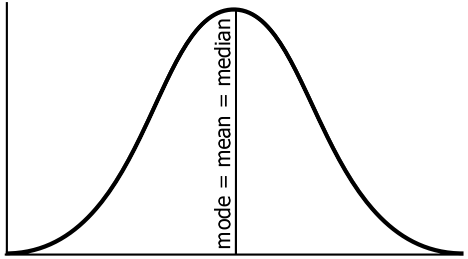
Calculation of the 90th percentile: $0{,}9 \times 80 = 72$

So 90% of the data is below the 72nd value which will be int the last class interval.

SYMMETRIC AND SKEWED DATA

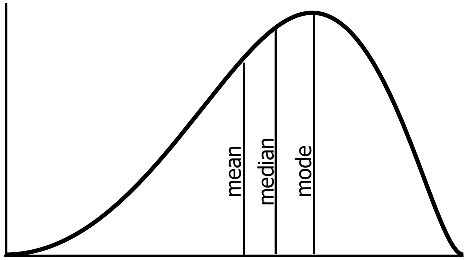
Symmetric

Symmetric data has a balanced shape, with mean, median and mode close together.

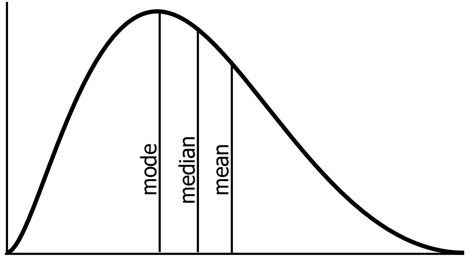


Skewed

Skewed data is data that is spread more towards one side or the other.



Skewed left: Negatively skewed if the tail extends to the left



Skewed right: Positively skewed if the tail extends to the right