NORTH-HOLLAND
MATHEMATICS STUDIES 143

Problems in Distributions and Partial Differential Equations

C. ZUILY

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PROBLEMS IN DISTRIBUTIONS AND PARTIAL DIFFERENTIAL EQUATIONS

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INTRODUCTION

The aim of this book is to provide a comprehensive introduction to the theory of distributions through solved problems. It was originally written for undergraduate students in Mathematics but it can be used by a wider audience, engineers, physicists and also by more advanced students.

The first six chapters deal with the classical theory with special emphasis on the concrete aspect. The reader will find many examples of distributions and learn how to work with them.

The last chapter, written for more advanced readers, is a very short introduction to a very wide and important field in analysis which can be considered as the most natural application of distributions, namely the theory of partial differential equations. The reader will find exercises on the classical differential operators (Laplace, heat, wave $\bar{\partial}$, elliptic operators), on fundamental solutions, on hypoellipticity, analytic hypoellipticity, on Sobolev spaces, local solvability, on the Cauchy problem etc. At the beginning of each chapter the theoretical material used in it is briefly recalled. Moreover, the more difficult problems are indicated by one (or more) star(s).

At the end of the book the interested reader will find an index of words, an index of notations and a short bibliography where he will be able to find material for further study.

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PROGRAMME

Spaces whose topology is defined by a collection of semi-norms

Space $C^k(\Omega)$ $(0 \leq k \leq +\infty)$ of k-times differentiable functions on an open subset Ω of $\mathbf{R}^n$

Space $\mathscr{D}(\Omega)$ (or $C_0^\infty(\Omega)$) of C^∞ functions with compact support in Ω.

The Leibniz formula

The Taylor formula with integral remainder.

BASICS — CHAPTER 1

a) Notations

A multi-index $\alpha \in \mathbb{N}^n$ can be written $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_j \in \mathbb{N}$. We shall denote

$$|\alpha| = \alpha_1 + \cdots + \alpha_n; \quad \alpha! = \alpha_1! \cdots \alpha_n!; \quad \alpha - \beta = (\alpha_1 - \beta_1, \ldots, \alpha_n - \beta_n);$$

$$\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha - \beta)!}$$

If α and β are two multi-indices in $\mathbb{N}^n$ we write $\alpha \leq \beta$ if $\alpha_i \leq \beta_i$ $i = 1, \ldots, n$. For $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{N}^n$ we set $x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$. Moreover we shall set

$$\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} \text{ where } \partial_j = \frac{\partial}{\partial x_j}$$

The expression $P = \sum_{|\alpha| \leq m} a_\alpha(x)\partial^\alpha$ will be called a differential operator of order $m \in \mathbb{N}$ and the functions $a_\alpha(x)$ the coefficients of the operator.

The support of a function f, denoted by supp f, will be the closure of the set $\{x: f(x) \neq 0\}$

b) Spaces whose topology is defined by a collection of semi-norms.

Let E be a vector space on a field K ($\mathbb{R}$ or $\mathbb{C}$). A semi-norm on E is a map p from E to $\mathbb{R}_+ = \{x \in \mathbb{R}, x \geq 0\}$ such that

i) $p(\lambda x) = |\lambda| p(x) \quad \forall x \in E \quad \forall \lambda \in K$

ii) $p(x + y) \leq p(x) + p(y) \quad \forall x \in E \quad \forall y \in E$

We say that p is a norm if moreover $p(x) = 0$ implies $x = 0$.

Let I be a subset of $\mathbb{R}$ and $(p_i)_{i \in I}$ a collection of semi-norms on E. For every $x_0 \in E$, $\varepsilon > 0$ and for all finite subset F of I we set

$$V(x_0, \varepsilon, F) = \{x \in E, p_i(x - x_0) < \varepsilon, i \in F\}$$

The collection $V(x_0, \varepsilon, F)$, when $\varepsilon > 0$ and F ranges over the finite subset of I, defines a filter of neighborhoods of x_0 and thus generates a topology on E which is compatible with the linear structure on E (which means that the maps $(x, y) \mapsto x + y$ from $E \times E$ to E and $(\lambda, x) \mapsto \lambda x$ from $K \times E$ to E are continuous). We say then that E is a locally convex topological vector space (l.c.t.v.s). Let us assume that I is countable (we may take $I = \mathbb{N}$) then the topology defined by the collection $(p_i)_{i\in\mathbb{N}}$ is metrizable. Indeed if for x and y in E we set

$$d(x, y) = \sum_{i=0}^{\infty} \frac{1}{2^i} \cdot \frac{p_i(x - y)}{1 + p_i(x - y)},$$

one can show that d is a distance on E and that the topology defined by d is equivalent to the one defined by the collection $(p_i)_{i\in\mathbb{N}}$.
Let $(E, (p_i)_{i\in I})$, $(F, (q_j)_{j\in J})$ be two l.c.t.v.s. Let T be a linear map from E to F. Then T is continuous if and only if:
For every semi-norm q_j there exists a positive constant C and a semi-norm p_i such that:

$$q_j(Tx) \leqq Cp_i(x)$$

for every $x \in E$.
The reader interested in these questions may consult reference [4].

c) The spaces $C^k(\Omega)$
Let Ω be an open subset of $\mathbb{R}^n$ and $k \in \mathbb{N}$ (or $k = +\infty$). We denote by $C^k(\Omega)$ the space of functions defined in Ω with values in $\mathbb{C}$ which are k times (or infinitely) differentiable. It is equipped with the semi-norms

$$p_K(u) = \sum_{|\alpha|\le k} \sup_{x\in K} |\partial^\alpha u(x)| \text{ where } K \text{ is a compact subset of } \Omega \text{ (if } k \in \mathbb{N})$$

$$p_{K,j}(u) = \sum_{|\alpha|\le j} \sup_{x\in K} |\partial^\alpha u(x)| \text{ where } K \text{ is a compact subset of } \Omega \text{ and } j \in \mathbb{N} \text{ (if } k = +\infty)$$

They give the topology of uniform convergence, on every compact, of the derivatives of order less or equal to k (if $k \in \mathbb{N}$) and of all derivatives (if $k = +\infty$).
These topologies are metrizable and then the spaces $C^k(\Omega)$ are complete for $0 \le k \le \infty$.

d) The space $\mathscr{D}(\Omega)$ or $C_0^\infty(\Omega)$
It is the space of all C^∞ functions on Ω with compact support. To define the topology of $\mathscr{D}(\Omega)$ one has to introduce the notion of inductive limit topology. The reader may consult [4]. For the sequel it will be sufficient to know how the sequences converge. One has the following result.
A sequence $(\varphi_j)_{j\in\mathbb{N}}$ of elements of $\mathscr{D}(\Omega)$ converges to zero in $\mathscr{D}(\Omega)$ if and only if:
i) There exists a compact subset K of Ω such that for every $j \in \mathbb{N}$, $\operatorname{supp} \varphi_j \subset K$.
ii) For every $\alpha \in \mathbb{N}^n$ the sequence $(\partial^\alpha \varphi_j)_{j\in\mathbb{N}}$ converges uniformly in K to zero.

If a is a C^∞ function on Ω the maps $\varphi \mapsto a\varphi$ and $\varphi \mapsto \dfrac{\partial \varphi}{\partial x_j}$ are continuous from $\mathcal{D}(\Omega)$ to itself.

If K is a fixed compact subset of Ω we shall denote by $\mathcal{D}_K(\Omega)$ the space of all u in $\mathcal{D}(\Omega)$ such that supp $u \subset K$.

e) The Leibniz formula

Let u and v be two functions in $C^k(\Omega)$. Then for all $\alpha \in \mathbb{N}^n$ such that $|\alpha| \leq k$ one has

$$\partial^\alpha(u \cdot v) = \sum_{\beta \leqslant \alpha} \binom{\alpha}{\beta} \partial^\beta u \cdot \partial^{\alpha-\beta} v$$

f) The Taylor formula with integral remainder

Let $x_0 \in \mathbf{R}^n$ and φ be a C^∞ function in a neighborhood of x_0. Then for every $N \in \mathbb{N}$ and all x in a neighborhood of x_0 we have

$$\varphi(x) = \sum_{|\alpha| \leqslant N} \frac{1}{\alpha!} (x - x_0)^\alpha \cdot (\partial^\alpha \varphi)(x_0) +$$
$$+ \int_0^1 (1 - t)^N \sum_{|\alpha| = N+1} \frac{N + 1}{\alpha!} (x - x_0)^\alpha (\partial^\alpha \varphi)(tx + (1 - t)x_0) \, dt$$

g) The polar coordinates in $\mathbf{R}^n$.

They are defined for $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$ and $(r, \theta_1, \ldots, \theta_n) \in]0, \infty[\times]0, \pi[\times \cdots \times]0, \pi[\times]0, 2\pi[$, by the formulas

$$\begin{cases} x_1 = r \cos \theta_1 \\ x_2 = r \sin \theta_1 \cos \theta_2 \\ x_{n-1} = r \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{n-2} \cos \theta_{n-1} \\ x_n = r \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{n-1} \end{cases}$$

Then we have $dx = r^{n-1}(\sin \theta_1)^{n-2}(\sin \theta_2)^{n-3} \cdots \sin \theta_{n-2} \, dr \, d\theta_1 \cdots d\theta_{n-1}$. We shall write shortly $x = r \cdot \omega$, $\omega = (\omega_1, \ldots, \omega_n)$, and one can see that $|\omega| = 1$, which means that ω belongs to the unit sphere S^{n-1}. Then $dx = r^{n-1} \, dr \, d\omega$ where $d\omega$ is the measure on S^{n-1}. If $f \in L^1(\mathbf{R}^n)$ one can write

$$\int_{\mathbf{R}^n} f(x) \, dx = \int_0^\infty \int_{S^{n-1}} f(r \cdot \omega) r^{n-1} \, dr \, d\omega$$

STATEMENTS OF EXERCISES* CHAPTER 1

Exercise 1: Borel's theorem

Our purpose is to prove that given a sequence $(a_j)_{j\in\mathbb{N}}$ of complex numbers there exists a function $f \in C^\infty(\mathbb{R})$ such that $\left[\dfrac{d^j}{dx^j} f\right](0) = a_j, j = 0, 1, 2, \ldots$

a) Let $\varphi \in \mathscr{D}(]-2, 2[)$ such that $\varphi = 1$ for $|x| \leq 1$.
Prove that we can find a sequence $(\lambda_n)_{n\in\mathbb{N}}$ of real numbers such that if we set

$$(1)\quad f_n(x) = \frac{a_n}{n!} x^n \varphi(\lambda_n x)$$

then

$$(2)\quad \sup_{x\in\mathbb{R}} \left| \left(\frac{d}{dx}\right)^k f_n(x) \right| \leq 2^{-n} \quad \text{for } 0 \leq k \leq n - 1$$

b) Prove that the series $\sum_{n=0}^{\infty} f_n(x)$ defines a function $f(x)$ which is C^∞ and solves our original problem.

Exercise 2

Let Ω be an open set of $\mathbb{R}^n$, k and m be two positive integers such that $k \geqq m$ and $P(x, \partial) = \sum_{|\alpha|\leq m} a_\alpha(x)\partial^\alpha$ be a differential operator of order m whose coefficients are in $C^{k-m}(\Omega)$.
Prove that P is continuous from $C^k(\Omega)$ in $C^{k-m}(\Omega)$.

Exercise 3

Prove that there is no function δ in $C_c^0(\mathbb{R})$ (the space of continuous functions with compact support) (resp. in $L^1(\mathbb{R})$) such that $\delta * f = f$ for all f in $C_c^0(\mathbb{R})$ (resp. in $L^1(\mathbb{R})$).
(Hint: Use the equality $f(0) = \int f(x)\delta(-x)\,dx$).

Exercise 4

Let $\varphi \in \mathscr{D}(\mathbb{R})$ and $M > 0$ such that supp $\varphi \subset \{x \in \mathbb{R}: |x| \leq M\}$. If $n \in \mathbb{N}$ we set

$$\psi(x) = \begin{cases} \dfrac{1}{x^{n+1}}\left\{\varphi(x) - \displaystyle\sum_{j=0}^{n} \frac{x^j}{j!}\varphi^{(j)}(0)\right\} & \text{for } x \neq 0 \\[2ex] \dfrac{1}{(n+1)!}\varphi^{(n+1)}(0) & \text{for } x = 0 \end{cases}$$

*Solutions pp 18 to 24

Prove that ψ is continuous on $\mathbb{R}$ and that we can find $C > 0$ such that

$$(1)\quad \sup_{|x|\leq M} |\psi(x)| \leqq C \sup_{|x|\leq M} |\varphi^{(n+1)}(x)|$$

Exercise 5

Let $\varphi \in \mathscr{D}(\mathbb{R}^n)$, $h \in \mathbb{R}^n\backslash\{0\}$. For t in $\mathbb{R}\backslash\{0\}$ we set

$$\varphi_t(x) = \frac{\varphi(x+th) - \varphi(x)}{t}$$

a) Show that $\varphi_t \in \mathscr{D}(\mathbb{R}^n)$ for $t \neq 0$.
b) Prove that when t tends to zero, φ_t converges in $\mathscr{D}(\mathbb{R}^n)$ to a function. Compute this function.

Exercise 6: The Poincaré inequality

a) Let $\varphi \in \mathscr{D}(\mathbb{R}^n)$. Prove that for $i = 1, 2, \ldots, n$ we have

$$(1)\quad \int_{\mathbb{R}^n} |\varphi(x)|^2 \, dx = -2\,\mathrm{Re}\left(\int_{\mathbb{R}^n} x_i \varphi(x) \frac{\partial \bar{\varphi}}{\partial x_i}(x)\, dx\right)$$

b) Let Ω be a bounded set in $\mathbb{R}^n$. Show, using a), that we can find $C > 0$ such that

$$(2)\quad \int_\Omega |\varphi(x)|^2 \, dx \leq C \sum_{i=1}^n \int_\Omega \left|\frac{\partial \varphi}{\partial x_i}(x)\right|^2 dx$$

for all φ in $\mathscr{D}(\Omega)$

Exercise 7

Let Ω be an open set in $\mathbb{R}^n$, E be a subspace of $\mathscr{D}'(\Omega)$ and $k \in \mathbb{N}$. Show that the following claims are equivalent:

a) $u \in \mathscr{D}'(\Omega)$ and $\partial^\alpha \varphi u \in E$ $|\alpha| \leq k$, for all φ in $\mathscr{D}(\Omega)$.
b) $u \in \mathscr{D}'(\Omega)$ and $\psi \partial^\beta u \in E$ $|\beta| \leq k$, for all ψ in $\mathscr{D}(\Omega)$.
(Hint: For a) $\Rightarrow$ b) use an induction on k and the Leibniz Formula).

Exercise 8

Give a sequence $(\varphi_n)_{n\in\mathbb{Z}}$ of elements in $\mathscr{D}(\mathbb{R})$ such that

a) For each $x \in \mathbb{R}$, the set $\{n \in \mathbb{Z}: \varphi_n(x) \not\equiv 0\}$ is finite.
b) $\sum_{-\infty}^{\infty} \varphi_n(x) = 1 \quad \forall x \in \mathbb{R}$

SOLUTIONS OF THE EXERCISES — CHAPTER 1

Solution 1

We shall set $g^{(k)} = \left(\dfrac{\mathrm{d}}{dx}\right)^k g$, and we shall use the Leibniz formula

$$(3) \quad (u \cdot v)^{(k)} = \sum_{p=0}^{k} \binom{k}{p} u^{(p)} \cdot v^{(k-p)}$$

a) If $0 \leq k \leq n - 1$ we have, using (3):

$$f_n^{(k)}(x) = \frac{a_n}{n!} \sum_{p=0}^{k} C_p^k n(n-1) \cdots (n-p+1) x^{n-p} \lambda_n^{k-p} \varphi^{(k-p)}(\lambda_n x)$$

Since supp $\varphi \subset]-2, 2[$, $\varphi^{(k-p)}(\lambda_n x) \equiv 0$ for $|x| > \dfrac{2}{|\lambda_n|}$, then

$$\sup_{\mathbf{R}} |f_n^{(k)}(x)| \leq \frac{|a_n|}{n!} \sum_{p=0}^{k} C_p^k \frac{n!}{(n-p)!} \left(\frac{2}{|\lambda_n|}\right)^{n-p} |\lambda_n|^{k-p} \sup_{y \in]-2,2[} |\varphi^{(k-p)}(y)|$$

Let us set

$$M_n = \sum_{j=0}^{n-1} \sup_{y \in]-2,2[} |\varphi^{(j)}(y)|$$

We get

$$\sup_{\mathbf{R}} |f_n^{(k)}(x)| \leqq |a_n| \frac{M_n}{|\lambda_n|^{n-k}} \sum_{p=0}^{k} C_p^k \frac{2^{n-p}}{(n-p)!} n!$$

Since $0 \leq k \leq n - 1$ we have the following estimates:

$$C_p^k \leqq k! \leq (n-1)!, \qquad \frac{2^{n-p}}{(n-p)!} \leq 2^n, \qquad \frac{1}{|\lambda_n|^{n-k}} \leq \frac{1}{|\lambda_n|} \quad \text{if } |\lambda_n| \geq 1$$

thus

$$\sup_{\mathbf{R}} |f_n^{(k)}(x)| \leqq \frac{|a_n| M_n \cdot 2^n \cdot n!}{|\lambda_n|}$$

If we take

$$|\lambda_n| \geq \mathrm{Max}\,(1, |a_n| M_n 4^n n!)$$

we get

$$\sup_{\mathbf{R}} |f_n^{(k)}(x)| \leqq 2^{-n} \qquad 0 \leq k \leq n - 1$$

b) By question a) the series $\sum\limits_0^\infty f_n(x)$ is uniformly convergent and therefore defines a continuous function $f(x) = \sum\limits_0^\infty f_n(x)$.

Moreover if $k \in \mathbb{N}$ the series $\sum\limits_0^\infty f_n^{(k)}$ is uniformly convergent since

$$\sum_0^\infty f_n^{(k)}(x) = \sum_{n=0}^k f_n^{(k)}(x) + \sum_{n=k+1}^\infty f_n^{(k)}(x)$$

and in the second sum in the right hand side we have $k \leq n - 1$ hence

$$|f_n^{(k)}(x)| \leqq 2^{-n}$$

This proves that $f \in C^k$ and that $f^{(k)}(x) = \sum\limits_0^\infty f_n^{(k)}(x)$ for all $k \in \mathbb{N}$, therefore $f \in C^\infty$.

On the other hand

$$f^{(k)}(x) = \sum_{n=0}^k \frac{a_n}{n!}(x^n\varphi(\lambda_n x))^{(k)} + \sum_{n=k+1}^\infty \sum_{p=0}^k \frac{a_n}{n!} C_p^k (x^n)^{(p)} \lambda_n^{k-p} \varphi^{(k-p)}(\lambda_n x)$$

The second sum in the right hand side vanishes at $x = 0$ because each of its terms contains x as a factor since $n \geq k + 1$ and $p \leq k$.
Then we have

$$f^{(k)}(x) = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k \left[a_0\varphi(\lambda_0 x) + \cdots + \frac{a_{k-1}}{(k-1)!} x^{k-1}\varphi(\lambda_{k-1}x)\right]$$
$$+ \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k \left[\frac{a_k x^k}{k!} \varphi(\lambda_k x)\right] + R(x) \quad \text{where } R(0) = 0$$

The first term in the right hand side vanishes at $x = 0$ because we have $\varphi^{(j)}(0) = 0$ for all $j \geq 1$ since $\varphi = 1$ for $|x| \leq 1$. The second term is equal to

$$\sum_{p=0}^k C_p^k \frac{a_k}{k!} (x^k)^{(p)} \lambda_k^{k-p} \varphi^{(k-p)}(\lambda_k x)$$

If $k - p > 0$ then $\varphi^{(k-p)}(0) = 0$. The only non vanishing term corresponds to $p = k$ and $C_k^k \frac{a_k}{k!}(x^k)^{(k)} = \frac{a_k}{k!} \cdot k! = a_k$. Therefore $f^{(k)}(0) = a_k$.

Solution 2

The application $u \mapsto Pu$ being linear, it is sufficient to prove that for every compact $K \subset \Omega$ we can find a compact K' in Ω and a constant $C > 0$ such that

$$(1) \quad \sum_{|\beta| \leq k-m} \sup_{x \in K} |(\partial^\beta Pu)(x)| \leq C \sum_{|\gamma| \leq k} \sup_{x \in K'} |(\partial^\gamma u)(x)|$$

Now by the Leibniz formula

$$\partial^\beta Pu = \sum_{|\alpha|\le m} \sum_{\beta_1 \le \beta} C_\beta^{\beta_1} \partial^{\beta_1} a_\alpha \partial^{\beta-\beta_1} \partial^\alpha u$$

So

$$\sum_{|\beta|\le k-m} \sup_K |\partial^\beta Pu| \leq \sum_{|\beta|\le k-m} \sum_{|\alpha|\le m} \sum_{\beta_1\le\beta} C_\beta^{\beta_1} \sup_K |\partial^{\beta_1} a_\alpha| \sup_K |\partial^{\alpha+\beta-\beta_1} u|$$

since $|\beta_1| \le |\beta| \le k - m$ and $a_\alpha \in C^{k-m}(\Omega)$ we have $\sup\limits_{\substack{|\alpha|\le m\\ |\beta_1|\le k-m}} \sup\limits_K |\partial^{\beta_1} a_\alpha| = C_K$ and

$$\sum_{|\beta|\le k-m} \sup_K |\partial^\beta Pu| \leq C_K \sum_{|\beta|\le k-m} \sum_{|\alpha|\le m} \sum_{\beta_1\le\beta} \sup_K |\partial^{\alpha+\beta-\beta_1} u| \leq C'_K \sum_{|\gamma|\le k} \sup_K |\partial^\gamma u|$$

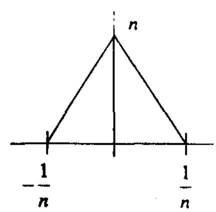
since $|\alpha + \beta - \beta_1| = |\alpha| + |\beta| - |\beta_1| \leq m + k - m = k$. Indeed we have bounded some derivatives of order $\le k$, repeated a finite number of times, by a constant which is multiplied by the sum of all derivatives of order $\le k$.)

Solution 3

Indeed we must have for all f

$$f(t) = \int f(x)\delta(t - x)\,dx \quad \text{so}$$

$$(1) \quad f(0) = \int f(x)\delta(-x)\,dx$$



Let us consider the sequence (φ_n) defined by

$$\begin{cases} \varphi_n(x) = n^2 x + n & -\frac{1}{n} \le x \le 0 \\ \varphi_n(x) = -n^2 x + n & 0 \le x \le \frac{1}{n} \\ \varphi_n(x) = 0 & |x| > \frac{1}{n} \end{cases}$$

Then $\varphi_n \in C_c^0(\mathbb{R})$ and $\int_{-1/n}^{1/n} \varphi_n(x)\,dx = 1$. By (1) we have

$$n = \varphi_n(0) = \int_{-1/n}^{1/n} \varphi_n(x)\delta(-x)\,dx$$

so

$$\int_{-1/n}^{1/n} \varphi_n(x)\delta(-x)\,dx = n \int_{-1/n}^{1/n} \varphi_n(x)\,dx$$

therefore

$$(2) \quad \int_{-1/n}^{1/n} \varphi_n(x)[n - \delta(-x)]\,dx = 0$$

If δ was in $C_c^0(\mathbb{R})$ we should have a constant $C > 0$ such that

$$\sup_{x\in\mathbb{R}} |\delta(x)| \leq C$$

For n large enough we should have

$$n - \delta(x) > 0, \qquad x \in \left]-\frac{1}{n}, \frac{1}{n}\right[$$

and

$$\varphi_n(x)[n - \delta(-x)] > 0, \qquad |x| < \frac{1}{n}$$

which is in contradiction with (2).

Let us prove that $\delta \notin L^1(\mathbb{R})$. We consider the sequence (φ_n) defined by

$$\varphi_n(x) = \begin{cases} n & |x| \leq \dfrac{1}{n} \\ \\ 0 & |x| > \dfrac{1}{n} \end{cases}$$

$n = \varphi_n(0) = \int_{-1/n}^{1/n} \varphi_n(x)\delta(-x)\,\mathrm{d}x = n\int_{-1/n}^{1/n} \delta(-x)\,\mathrm{d}x$ so

$$(3) \quad \int_{-1/n}^{1/n} \delta(x)\,\mathrm{d}x = 1$$

Now if $\delta \in L^1(\mathbb{R})$ we should have, by the Lebesgue theorem:

$$(4) \quad \lim_{n\to\infty} \int_{-1/n}^{1/n} \delta(x)\,\mathrm{d}x = 0$$

Indeed $1_{]-1/n,1/n[}\delta(x) \underset{p.p}{\to} 0$ and $|1_{]-1/n,1/n[}\delta(x)| \leq |\delta(x)| \in L^1(\mathbb{R})$.

It follows from (3) and (4) that $\delta \notin L^1(\mathbb{R})$.

Remark:

Actually we can prove that δ cannot belong to any well known space (such as L^p ...).
This fact is clarified by the distribution theory.

Solution 4

The Taylor formula, (with integral remainder) applied to the C^∞ function φ, up to the order $n + 1$, gives

$$\varphi(x) - \sum_{j=0}^{n} \frac{x^j}{j!}\varphi^{(j)}(0) = \frac{x^{n+1}}{n!}\int_0^1 (1 - t)^n \varphi^{(n+1)}(tx)\,\mathrm{d}t$$

Therefore for $x \neq 0$,

$$(2) \quad \psi(x) = \frac{1}{n!} \int_0^1 (1-t)^n \varphi^{(n+1)}(tx)\,dt$$

On the other hand we obviously have

$$\lim_{x \to 0} \psi(x) = \frac{1}{n!} \int_0^1 (1-t)^n \lim_{x \to 0} \varphi^{(n+1)}(tx)\,dt = \frac{1}{(n+1)!} \varphi^{(n+1)}(0)$$

The function ψ is therefore continuous on $\mathbb{R}$ (since it is obviously continuous for $x \neq 0$). Now (2) implies that, for every $x \neq 0$, we have

$$(3) \quad |\psi(x)| \leqq \frac{1}{(n+1)!} \sup_{\substack{|x| \leq M \\ x \neq 0}} |\varphi^{(n+1)}(x)| \leqq \frac{1}{(n+1)!} \sup_{|x| \leq M} |\varphi^{(n+1)}(x)| = A$$

Since $\psi(0) = \dfrac{1}{(n+1)!} \varphi^{(n+1)}(0)$, $|\psi(0)|$ is also bounded by A which, together with (3), gives the inequality (1).

Solution 5

a) φ having a compact support there exists $M > 0$ such that $\varphi \equiv 0$ if $|x| \geq M$. If $|x| \geq M + |th|$ then $|x| \geq M$ thus $\varphi = 0$ and $|x + th| \geq |x| - |th| \geq M$ thus $\varphi(x + th) = 0$. This implies that the support of φ_t is contained in the ball $B(0, M + |th|)$.

b) First we have to prove that, for small t, the supports of all φ_t are contained in a same compact K. For this purpose we just have to remark using a), that for $|t| \leq 1$, $\operatorname{supp} \varphi_t \subset B(0, M + |th|) \subset B(0, M + |h|)$.

Let α be a multi-index. We have

$$\partial^\alpha \varphi_t(x) = \frac{1}{t} [\partial^\alpha \varphi(x + th) - \partial^\alpha \varphi(x)]$$

By the Taylor formula at the order two applied to the function $\partial^\alpha \varphi$, we have

$$(\partial^\alpha \varphi)(x + th) - (\partial^\alpha \varphi)(x) = t \sum_{i=1}^n h_i \frac{\partial}{\partial x_i} (\partial^\alpha \varphi)(x) + \\ + t^2 \sum_{i,j=1}^n \int_0^1 (1-u) h_i h_j \left(\frac{\partial^2}{\partial x_i \partial x_j}\right) (\partial^\alpha \varphi)(z)\,du$$

where $z = x + uth$. Therefore

$$\left| (\partial^\alpha \varphi_t)(x) - \sum_{i=1}^n h_i \frac{\partial}{\partial x_i} (\partial^\alpha \varphi)(x) \right| \leq C|t| \cdot |h|^2 \sum_{|\beta| \leq |\alpha|+2} \sup_{y \in K} |\partial^\beta \varphi(y)|$$

therefore when $t \to 0$, $\partial^\alpha \varphi_t$ converges uniformly on K to the function $\sum_{i=1}^{n} h_i \left(\frac{\partial}{\partial x_i}\right)(\partial^\alpha \varphi)(x)$. This proves that φ_t converges in $\mathscr{D}(\Omega)$ to the function $x \mapsto \sum_{i=1}^{n} h_i \frac{\partial \varphi}{\partial x_i}(x)$.

Solution 6

a) Function φ having a compact support, we can use the Fubini theorem and write

$$\int_{\mathbf{R}^n} x_i \varphi(x) \frac{\partial \bar{\varphi}}{\partial x_i} \, \mathrm{d}x = \int_{\mathbf{R}} \int_{\mathbf{R}} \left(\int_{\mathbf{R}} x_i \varphi(x) \frac{\partial \bar{\varphi}}{\partial x_i} \, \mathrm{d}x_i \right) \mathrm{d}x_1 \cdots \mathrm{d}x_{i-1} \, \mathrm{d}x_{i+1} \cdots \mathrm{d}x_n$$

In the integral with respect to x_i (the other variables are considered as parameters) let us make an integration by parts. We get

$$\int_{\mathbf{R}} x_i \varphi(x) \frac{\partial \bar{\varphi}}{\partial x_i}(x) \, \mathrm{d}x_i = [x_i \varphi(x) \bar{\varphi}(x)]_{-\infty}^{+\infty} - \int_{\mathbf{R}} \frac{\partial}{\partial x_i}(x_i \varphi(x)) \cdot \bar{\varphi}(x) \, \mathrm{d}x_i$$

Since φ has a compact support, $\varphi(\mp\infty) = 0$. So we get

$$\begin{aligned} \int_{\mathbf{R}} x_i \varphi(x) \frac{\partial \bar{\varphi}}{\partial x_i}(x) \, \mathrm{d}x_i &= -\int_{\mathbf{R}} |\varphi(x)|^2 \, \mathrm{d}x_i - \int_{\mathbf{R}} x_i \frac{\partial \varphi}{\partial x_i}(x) \bar{\varphi}(x) \, \mathrm{d}x_i \\ &= -\int_{\mathbf{R}} |\varphi(x)|^2 \, \mathrm{d}x_i - \int_{\mathbf{R}} x_i \bar{\varphi}(x) \frac{\partial \varphi}{\partial x_i}(x) \, \mathrm{d}x_i \end{aligned}$$

Thus

$$-2 \operatorname{Re} \int_{\mathbf{R}} x_i \varphi(x) \frac{\partial \bar{\varphi}}{\partial x_i}(x) \, \mathrm{d}x_i = \int_{\mathbf{R}} |\varphi(x)|^2 \, \mathrm{d}x_i$$

To get (1) we just have to integrate with respect to the other variables.

b) Using (1) and the Cauchy–Schwarz inequality we get

$$\int_\Omega |\varphi(x)|^2 \, \mathrm{d}x \leqq 2 \left| \int_\Omega x_i \varphi(x) \frac{\partial \varphi}{\partial x_i}(x) \, \mathrm{d}x \right| \leq 2 \left(\int_\Omega |x_i \varphi(x)|^2 \, \mathrm{d}x \right)^{1/2} \left(\int_\Omega \left| \frac{\partial \varphi}{\partial x_i} \right|^2 \mathrm{d}x \right)^{1/2}$$

Since Ω is bounded we can find a constant $C_1 > 0$ such that $\sup_{x \in \Omega} |x_i| \leq C_1$. Hence

$$(3) \quad \int_\Omega |\varphi(x)|^2 \, \mathrm{d}x \leq 2\, C_1 \left(\int_\Omega |\varphi(x)|^2 \, \mathrm{d}x \right)^{1/2} \left(\int_\Omega \left| \frac{\partial \varphi}{\partial x_i}(x) \right|^2 \mathrm{d}x \right)^{1/2}$$

If $\int_\Omega |\varphi(x)|^2 \, \mathrm{d}x = 0$ the inequality (2) is obvious. Otherwise we just have to divide both sides of (3) by $(\int_\Omega |\varphi(x)|^2 \, \mathrm{d}x)^{1/2}$ to get (2).

Solution 7

b) ⇒ a):

$$\partial^\alpha \varphi u = \sum_{\beta\le\alpha} C_\alpha^\beta\, \partial^\beta\varphi\partial^{\alpha-\beta}u$$

since $\partial^\beta\varphi \in \mathscr{D}(\Omega)$ and $|\alpha - \beta| \le |\alpha| \le k$, using b) we get $(\partial^\beta\varphi)\cdot(\partial^{\alpha-\beta}u) \in E$ for every $\beta \le \alpha$, which implies that $\partial^\alpha(\varphi u) \in E$ for all α, $|\alpha| \le k$.

a) ⇒ b) We make an induction on k. The statement is obvious for $k = 0$ and for $k = 1$ it is a consequence of

$$\psi\frac{\partial u}{\partial x_i} = \frac{\partial}{\partial x_i}(\psi\cdot u) - \left(\frac{\partial\psi}{\partial x_i}\right)u$$

Let us suppose that a) ⇒ b) up to the order $k - 1$ and that a) is true for k; in particular a) is true for $k - 1$ and by the induction b) is true for $k - 1$. We have to prove that $\psi\partial^\beta u \in E$ for all β, $|\beta| = k$ and all $\psi \in \mathscr{D}(\Omega)$. By the Leibniz formula

$$\psi\partial^\beta u = \partial^\beta(\psi\cdot u) - \sum_{0<\gamma\le\beta} C_\beta^\gamma(\partial^\gamma\psi)\cdot\partial^{\beta-\gamma}u$$

Using a) at the order k we get $\partial^\beta\psi u \in E$. On the other side in the above sum $|\gamma| \ne 0$ so $|\beta - \gamma| \le k - 1$; since $(\partial^\gamma\psi) \in \mathscr{D}(\Omega)$ we get $(\partial^\gamma\psi)\partial^{\beta-\gamma}u \in E$ by the induction hypothesis. So $\psi\partial^\beta u \in E$, $|\beta| = k$. q.e.d.

Solution 8

Let $\theta \in \mathscr{D}(]0, 2[)$, $\theta \geqq 0$, $\theta = 1$ if $\frac{1}{2} \le |x| \le \frac{3}{2}$. For $n \in \mathbb{N}$ let us set

$$\begin{cases} \star\ \tau_n\theta(x) = \theta(x + n) \\ \star\quad \psi(x) = \sum_{n\in\mathbb{Z}} \tau_n\theta(x) \end{cases}$$

Then $\psi(x) \geqq 1$ and the sum defining ψ is locally finite so $\psi \in C^\infty(\mathbb{R})$. Therefore $\varphi(x) = \dfrac{\theta(x)}{\psi(x)}$ is a C^∞ function with compact support. Moreover

$$\varphi_n(x) = \tau_n\varphi(x) = \frac{\tau_n\theta(x)}{\tau_n\psi(x)} = \frac{\tau_n\theta(x)}{\psi(x)}$$

since $\tau_n\psi = \psi$. So we get

$$\sum_{n\in\mathbb{Z}} \varphi_n(x) = \frac{\sum\tau_n\theta(x)}{\psi(x)} = \frac{\psi(x)}{\psi(x)} = 1$$

CHAPTER 2

The distributions

PROGRAMME

Examples of distributions; order and support of a distribution

Distribution with compact support

Image of a distribution by a linear map

Product of a distribution by a C^∞ function

Division by x in $\mathscr{D}'(\mathbb{R})$

BASICS CHAPTER 2

a) Distributions on an open set in $\mathbb{R}^n$

A linear map T from $\mathscr{D}(\Omega)$ to $\mathbb{C}$ is called a distribution if for every sequence $(\varphi_j)_{j\in\mathbb{N}}$ in $\mathscr{D}(\Omega)$ converging to zero in $\mathscr{D}(\Omega)$, the sequence $T(\varphi_j)$ (which will be denoted by $\langle T, \varphi_j\rangle$) converges to zero in $\mathbb{C}$ or if:

For every compact K contained in Ω we can find $C > 0$ and $k \in \mathbb{N}$ such that

$$(1) \quad |\langle T, \varphi\rangle| \leqq C \sum_{|\alpha|\leq k} \sup_{x\in K} |\partial^\alpha \varphi(x)|$$

for every $\varphi \in \mathscr{D}(\Omega)$ such that $\operatorname{supp}\varphi \subset K$.

See exercise 9 for the equivalence of these two definitions.

The distribution is said to be of order k_0 if the inequality (1) is satisfied with a k_0 which is independant of the compact K. The space of distributions of order $\leq k_0$ is denoted by $\mathscr{D}'^{(k_0)}(\Omega)$. It is the dual of the space of functions which are C^{k_0} and have compact support. The support of a distribution T (denoted by supp T) is the closure of the set of x in Ω such that:

For every neighborhood V of x, we can find $\varphi \in \mathscr{D}(V)$ such that $\langle T, \varphi\rangle \neq 0$.

The singular support of the distribution T, denoted by sing supp T, is the complementary of the set of x in Ω in the neighborhood of which T is a C^∞ function.

Remark

It is not sufficient, as one might believe, to take, in the right hand side of the inequality (1), the supremum on the support of T. See exercise 10.

b) Distributions with compact support:

We shall denote by $\mathscr{E}'(\Omega)$ the space of distributions with compact support. One can prove that $\mathscr{E}'(\Omega)$ is the dual of the space $C^\infty(\Omega)$ (whose topology is described in chapter 1 b)).

If $T \in \mathscr{E}'(\Omega)$ and $\varphi \in C^\infty(\Omega)$ we shall denote $\langle T, \varphi\rangle$ instead of $T(\varphi)$.

Let us give a characterisation of the space $\mathscr{E}'(\Omega)$. A distribution T is in $\mathscr{E}'(\Omega)$ if and only if we can find $C > 0$, an integer $m \geq 0$ and compact K in Ω such that for every $\varphi \in C^\infty(\Omega)$

$$|\langle T, \varphi\rangle| \leqq C \sum_{|\alpha|\leq m} \sup_{x\in K} |\partial^\alpha \varphi(x)|$$

Exercise 10 shows that this compact is not, in general, the support of T.

c) Image of a distribution by a linear map
Let A be a linear bijective map from Ω_1 to Ω_2 (open sets in $\mathbb{R}^n$). Let T be in $\mathscr{D}'(\Omega_2)$; we define the image of T by the map A, which is a distribution on Ω_1 denoted by $T \circ A$, by the formula

$$(2) \quad \langle T \circ A, \varphi \rangle = \frac{1}{|\det A|} \langle T, \varphi \circ A^{-1} \rangle$$

When T is a locally integrable function, this definition agrees with the usual one which is

$$(T \circ A)(x) = T(Ax)$$

d) Product of a distribution by a C^∞ function
If $T \in \mathscr{D}'(\Omega)$ and $a \in C^\infty(\Omega)$ we define the distribution aT by

$$\langle aT, \varphi \rangle = \langle T, a\varphi \rangle$$

for all φ in $\mathscr{D}(\Omega)$.
The map $T \mapsto aT$ is continuous from $\mathscr{D}'(\Omega)$ to $\mathscr{D}'(\Omega)$.

e) Division by x in $\mathscr{D}'(\in)$.
Let a be a C^∞ function which does not vanish in Ω. Let S be in $\mathscr{D}'(\Omega)$. The unique solution of the equation $aT = S$ is $T = \frac{1}{a} S$ which makes sense since $\frac{1}{a} \in C^\infty(\Omega)$. It is not the case if a vanishes.
In the particular case where $\Omega = \mathbb{R}$ and $a(x) = x$ we have the following result:
Let S be in $\mathscr{D}'(\mathbb{R})$. The general solution in $\mathscr{D}'(\mathbb{R})$ of the equation

$$(3) \quad xT = S$$

is given by $T = T_0 + C\delta$ where C is a constant, δ the Dirac distribution and T_0 is a particular solution of (3). It can be obtained by the formula

$$\langle T_0, \varphi \rangle = \left\langle S, \frac{\varphi(x) - \varphi(0)\psi(x)}{x} \right\rangle$$

where $\psi \in \mathscr{D}(\mathbb{R})$ and $\psi(0) = 1$.

STATEMENTS OF EXERCISES* CHAPTER 2

Exercise 9

We recall that a linear form T on $\mathscr{D}(\Omega)$ is called a distribution if for every sequence (φ_j) in $\mathscr{D}(\Omega)$ which converges to zero in $\mathscr{D}(\Omega)$, the sequence $\langle T, \varphi_j\rangle$ converges to zero in $\mathbb{C}$. Prove that a linear form T on $\mathscr{D}(\Omega)$ is a distribution if and only if

$$(1)\begin{cases} \forall K \text{ compact}, \quad K \subset \Omega, \ \exists C > 0 \quad \exists k \in \mathbb{N}: \\ |\langle T, \varphi\rangle| \leq C \sum\limits_{|\alpha| \leq k} \sup\limits_{K} |\partial^\alpha \varphi(x)| \qquad \forall \varphi \in \mathscr{D}_K(\Omega) \end{cases}$$

(Hint: to show that a distribution T satisfies (1), use a contradiction argument and a sequence $(\varphi_j)_{j\in\mathbb{N}}$ in $\mathscr{D}(\Omega)$ which satisfies $|\langle T, \varphi_j\rangle| \geq j \sum\limits_{|\alpha|\leq j} \sup\limits_K |\partial^\alpha \varphi_j(x)|$).

Exercise 10

1°) Show that the application

$$\mathscr{D}(\mathbb{R}) \ni \varphi \mapsto \langle T, \varphi\rangle = \lim_{m\to\infty} \left\{ \sum_{j=1}^{m} \varphi\left(\frac{1}{j}\right) - m\varphi(0) - \operatorname{Log} m \cdot \varphi'(0) \right\}$$

is a distribution on $\mathbb{R}$

Find its support S.

2°) We consider a sequence $(\varphi_k)_{k\in\mathbb{N}}$ in $\mathscr{D}(\mathbb{R})$ such that

$$0 \leq \varphi_k(x) \leq \frac{1}{\sqrt{k}} \quad \text{for all } x \in \mathbb{R}$$

$$\varphi_k(x) = 0 \quad \text{for } x \leq \frac{1}{k+1} \text{ or } x \geq 2$$

$$\varphi_k(x) = \frac{1}{\sqrt{k}} \quad \text{for } \frac{1}{k} \leq x \leq 1$$

Show that:

a) (φ_k) converges uniformly on $\mathbb{R}$ to zero.

b) $\partial^\alpha \varphi_k = 0$ on S for all $\alpha \in \mathbb{N}$, $\alpha \neq 0$, and all $k \in \mathbb{N}$.

c) $\lim\limits_{k\to\infty} \langle T, \varphi_k\rangle = +\infty$.

3°) What can you conclude?

We recall that $\lim\limits_{m\to\infty} \left\{1 + \frac{1}{2} + \cdots \frac{1}{m} - \operatorname{Log} m\right\} = C_0 \in \mathbb{R}$.

*Solutions pp 33 to 50.

Exercise 11

Show that the following applications are distributions

a) $\mathscr{D}(\mathbb{R}) \ni \varphi \rightsquigarrow \left\langle pv\,\frac{1}{x}, \varphi\right\rangle = \lim\limits_{\varepsilon\to 0} \displaystyle\int_{|x|\geq\varepsilon} \frac{\varphi(x)}{x}\,\mathrm{d}x$

b) $\mathscr{D}(\mathbb{R}) \ni \varphi \rightsquigarrow \left\langle fp\,\frac{1}{x^2}, \varphi\right\rangle = \lim\limits_{\varepsilon\to 0} \left[\displaystyle\int_{|x|\geq\varepsilon} \frac{\varphi(x)}{x^2}\,\mathrm{d}x - 2\,\frac{\varphi(0)}{\varepsilon}\right]$

c) $\mathscr{D}(\mathbb{R}) \ni \varphi \rightsquigarrow \left\langle fp\,\frac{H}{x^2}, \varphi\right\rangle = \lim\limits_{\varepsilon\to 0} \left[\displaystyle\int_{\varepsilon}^{+\infty} \frac{\varphi(x)}{x^2}\,\mathrm{d}x - \frac{\varphi(0)}{\varepsilon} + \varphi'(0)\,\mathrm{Log}\,\varepsilon\right]$

pv = principal value; fp = finite part; H = Heaviside function

$$H(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$$

d) Define $fp\,\frac{1}{x^k}$, $fp\,\frac{H}{x^k}$ for k integer, $k \geq 3$.

Exercise 12: Singular integrals

Let u be a continuous function on $\mathbb{R}^n - \{0\}$ such that

$$(1) \quad u(tx) = t^{-n}u(x) \qquad t > 0,\ x \neq 0$$

a) Show that

$$(2) \quad \langle U, \varphi\rangle = \lim_{\varepsilon\to 0} \int_{|x|\geq\varepsilon} u(x)\varphi(x)\,\mathrm{d}x$$

exists for all $\varphi \in \mathscr{D}(\mathbb{R}^n)$ if and only if

$$(3) \quad \int_{|\omega|=1} u(\omega)\,\mathrm{d}\omega = 0$$

(Hint: Use the polar coordinates: $(r, \omega) \in\,]0, +\infty[\, \times S^{n-1}$, for $\{x\} \geq \varepsilon$).
Show that formula (2) defines a distribution on $\mathbb{R}^n$.

Exercise 13

Let f be a function which is locally integrable on $\mathbb{R}^n \setminus \{0\}$ and satisfies:

$$(1)\quad \exists C > 0\ \exists m \in \mathbb{N}\setminus\{0\}\colon\quad |f(x)| \le \frac{C}{|x|^m}\quad \forall x\colon |x| \le 1.$$

Prove that we can find a distribution $T \in \mathscr{D}'(\mathbb{R}^n)$ such that:

$$(2)\quad \langle T, \varphi\rangle = \int_{\mathbb{R}^n} f(x)\varphi(x)\,\mathrm{d}x$$

for every $\varphi \in \mathscr{D}(\mathbb{R}^n)$ with $\operatorname{supp}\varphi \subset \mathbb{R}^n\setminus\{0\}$

Exercise 14

Show that there is no distribution $T \in \mathscr{D}'(\mathbb{R})$ such that

$$(1)\quad \langle T, \varphi\rangle = \int_{\mathbb{R}} \mathrm{e}^{1/x^2}\cdot\varphi(x)\,\mathrm{d}x$$

for all φ in $\mathscr{D}(\mathbb{R}\setminus 0\})$.

(Hint: Find a sequence $(\varphi_n)_{n\in\mathbb{N}}$, in $\mathscr{D}(\mathbb{R})$, whose support is contained in $\left\{\frac{1}{n} < |x| < \frac{2}{n}\right\}$, which tends to zero in $\mathscr{D}(\mathbb{R})$ and such that $\langle T, \varphi_n\rangle$ tends to infinity.)

Exercise 15

a) Show that for complex $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > -1$ the functions

$$x_+^\lambda = \begin{cases} x^\lambda & x > 0 \\ 0 & x \le 0 \end{cases}\qquad x_-^\lambda = \begin{cases} |x|^\lambda & x < 0 \\ 0 & x \ge 0 \end{cases}$$

define distributions on $\mathbb{R}$.

b) Using that, for $\operatorname{Re}\lambda > -1$, we have

$$(1)\quad \int_0^\infty x^\lambda\varphi(x)\,\mathrm{d}x = \int_0^1 x^\lambda[\varphi(x) - \varphi(0)]\,\mathrm{d}x + \int_1^\infty x^\lambda\varphi(x)\,\mathrm{d}x + \frac{\varphi(0)}{\lambda + 1}$$

prove that we can define x_+^λ as a distribution for $\operatorname{Re}\lambda > -2$ and $\lambda \ne -1$.
Extend this procedure to the case where $\operatorname{Re}\lambda > -n - 1$, $\lambda \ne -1, \ldots, -n$.
Use the same method to define x_-^λ for these λ.

c) Let x_+^λ, x_-^λ be the distributions so defined for $\operatorname{Re}\lambda > -n - 1$, $\lambda \ne -1, -2, \ldots, -n$.
Compute $x \cdot x_+^\lambda$, $x \cdot x_-^\lambda$.

Exercise 16: Homogeneous distributions

For $\varphi \in \mathscr{D}(\mathbb{R}^n)$ and $\lambda > 0$ we set $A_\lambda = \lambda Id$ where Id is the identity matrix in $\mathbb{R}^n$. A distribution $T \in \mathscr{D}'(\mathbb{R}^n)$ is said to be homogeneous of degree $p \in \mathbb{R}$ if

$$T \circ A_\lambda = \lambda^p T$$

a) Prove that, when T is a locally integrable function, the definition (1) is equivalent to

(2) $T(\mu x) = \mu^p T(x)$ for every $\mu > 0$ and almost everywhere in x.

b) For φ in $\mathscr{D}(\mathbb{R}^n)$, λ and λ_0 in $\mathbb{R}^+$, $\lambda \neq \lambda_0$ we set $\varphi_\lambda(x) = \varphi(\lambda x)$ and

$$\psi_\lambda(x) = \frac{\varphi_\lambda(x) - \varphi_{\lambda_0}(x)}{\lambda - \lambda_0}$$

Prove that when λ goes to λ_0, ψ_λ converges in $\mathscr{D}(\mathbb{R}^n)$; compute the limit.

c) Prove that the distributions x_+^p, x_-^p, $p \in \mathbb{R}$, $p \neq -1, -2, \ldots$ (which were defined in exercise 15) are homogeneous.

Prove that it is also the case for the distribution $fp\frac{1}{x^m}$ defined by

$$\left\langle fp\frac{1}{x^m}, \varphi \right\rangle = \lim_{\varepsilon \to 0}\left[\int_{|x| \geq \varepsilon} \frac{\varphi(x)\,dx}{x^m} - \sum_{j=1}^{m-1} \frac{(-1)^{m-j} - 1}{(j-m)\varepsilon^{m-j}} \frac{\varphi^{(j-1)}(0)}{(j-1)!}\right]$$

$\varphi \in \mathscr{D}(\mathbb{R})$, $m \in \mathbb{N}^*$.

d) Prove that homogeneous distributions of different degrees are linearly independant.

Exercise 17: Distributions of infinite order

Prove that the map, $\mathscr{D}(\mathbb{R}) \ni \varphi \to \langle T, \varphi \rangle = \sum_{n=0}^{\infty} \varphi^{(n)}(n)$, defines a distribution which is not of finite order. $\left(\text{Hint: Assuming that } T \text{ is of order } k\text{, consider the sequence } (\varphi_\varepsilon) \text{ given by } \varphi_\varepsilon(x) = \varphi\left(\frac{x-k-1}{\varepsilon}\right), \varphi \in \mathscr{D}(]-1, 1[).\right)$

Exercise 18

For λ real and k in $\mathbb{N}\backslash\{0\}$ we set

$$(1) \quad \langle A_{\lambda,k}, \varphi \rangle = \int_{-\pi/2}^{\pi/2} \frac{\cos \lambda x}{x^k}\left[\varphi(x) - \sum_{i=0}^{k-1} \varphi^{(i)}(0) \cdot \frac{x^i}{i!}\right] dx$$

a) Prove that $\varphi \mapsto \langle A_{\lambda,k}, \varphi\rangle$ is a distribution of order less than or equal to k. What is its support?

b) Find all the solutions in $\mathscr{D}'(\mathbb{R})$ of the equation

$$xu = A_{1,1},$$

Exercise 19

Let T be a distribution on $\mathbb{R}^n$ and f be a C^∞ function such that $f = 0$ on the support of T. Do we have $f \cdot T = 0$?

Exercise 20

Prove that we cannot define the product of the distributions δ and $pv\frac{1}{x}$ in the sense that it cannot be commutative and distributive with respect to the multiplication by C^∞ functions.

SOLUTIONS OF THE EXERCISES CHAPTER 2

Solution 9

If T satisfies (1) it is a distribution, because if φ_j tends to zero in $\mathscr{D}(\Omega)$ then, for $j \geqq j_0$, $\operatorname{supp}\varphi_j \subset K$ and $(\partial^\alpha \varphi_j)$ tends to zero uniformly for every α, thus using (1) $\langle T, \varphi_j\rangle$ tends to zero.

Let now T be a distribution. Let us suppose that we can find a compact K such that (1) is not true for all $C > 0$ and every k. Take $C = k = j$, then we can find φ_j in $\mathscr{D}_k(\Omega)$ such that:

$$|\langle T, \varphi_j\rangle| \geqq j \sum_{|\alpha| \leq j} \sup_{x\in K} |\partial^\alpha \varphi_j(x)|$$

Let us set $\psi_j = \dfrac{\varphi_j}{|\langle T, \varphi_j\rangle|}$ then $|\langle T, \psi_j\rangle| = 1$ and:

$$1 \geqq j \sum_{|\alpha| \leq j} \sup_K |\partial^\alpha \psi_j|$$

Therefore $\sup_K |\partial^\alpha \psi_j| \leqq \dfrac{1}{j}$ for $j \geqq |\alpha|$ and $\operatorname{supp}\psi_j \subset \operatorname{supp}\varphi_j \subset K$; now for every α the sequence $(\partial^\alpha \psi_j)_j$ tends to zero in $\mathscr{D}(\Omega)$ but $\langle T, \psi_j\rangle = 1$ which is a contradiction. So if T is a distribution it satisfies (1).

Solution 10

1°) If $\varphi \in \mathscr{D}(\mathbb{R})$ we can write (see Exercise 4)

$$\varphi(x) = \varphi(0) + x\varphi'(0) + x^2\psi(x)$$

where ψ is continuous on $\mathbb{R}$ and $\sup_{\mathbb{R}} |\psi(x)| \leq C \sup_{\mathbb{R}} |\varphi''(x)|$. So we have

$$\sum_{j=1}^{m} \varphi\left(\frac{1}{j}\right) - m\varphi(0) - \operatorname{Log} m \cdot \varphi'0) = \sum_{j=1}^{m} \left\{\varphi\left(\frac{1}{j}\right) - \varphi(0)\right\} - \operatorname{Log} m\varphi'(0)$$

$$= \left(\sum_{j=1}^{m} \frac{1}{j} - \operatorname{Log} m\right)\varphi'(0) + \sum_{j=1}^{m} \frac{1}{j^2}\psi\left(\frac{1}{j}\right)$$

Now

$$\lim_{m\to\infty} \left(\sum_{j=1}^{m} \frac{1}{j} - \operatorname{Log} m\right) = C_0 \quad \text{(The Euler constant)}$$

On the other hand $\left|\psi\left(\frac{1}{j}\right)\right| \leq C^{te} \sup_{\mathbb{R}} |\varphi''(x)|$, so the series $\sum_{j=1}^{m} \frac{1}{j^2}\psi\left(\frac{1}{j}\right)$ is convergent and

$$\lim_{m\to\infty} \sum_{j=1}^{m} \frac{1}{j^2}\psi\left(\frac{1}{j}\right) = \sum_{j=1}^{\infty} \frac{1}{j^2}\psi\left(\frac{1}{j}\right)$$

Therefore

$$\langle T, \varphi\rangle = C_0\varphi'(0) + \sum_{j=1}^{\infty} \frac{1}{j^2}\psi\left(\frac{1}{j}\right)$$

The definition of $\langle T, \varphi\rangle$, given in the statement of the exercise, makes sense; moreover it is linear in φ and if K is a compact in $\mathbb{R}$ and $\varphi \in \mathscr{D}_K(\mathbb{R})$ we have

$$|\langle T, \varphi\rangle| \leq C_0 \sup_K |\varphi'(x)| + C\left(\sum_{j=1}^{\infty} \frac{1}{j^2}\right) \sup_K |\varphi''(x)|$$

thus $\varphi \mapsto \langle T, \varphi\rangle$ is a distribution of order less or equal to 2.
Let us show now that its support is

$$S = \left\{0, 1, \frac{1}{2}, \ldots, \frac{1}{m}, \ldots\right\}$$

Indeed if $\varphi \in \mathscr{D}(\mathbb{R}\backslash S)$ we have $\langle T, \varphi\rangle = 0$, so $\operatorname{supp} T \subset S$.

Now let $x = \frac{1}{p} \in S$ and V be a neighborhood of x. Let φ be in $\mathscr{D}(\mathbb{R})$ with $\operatorname{supp} \varphi \subset V \cap \left]\frac{1}{p-1}, \frac{1}{p+1}\right[$ and $\varphi = 1$ at x. We have $\varphi(0) = \varphi\left(\frac{1}{j}\right) = 0$ for $j \neq p$ so

$$\langle T, \varphi\rangle = 1$$

which proves that $x \in \operatorname{supp} T$.

Let us prove that $0 \in \operatorname{supp} T$. Indeed the support of T is a closed set which contains the points $\left(\frac{1}{j}\right)$ which tend to zero. The limit is therefore in the support of T.

2°) Let $k \in \mathbb{N}\backslash\{0\}$ and ψ_k be a function in $\mathscr{D}(\mathbb{R})$ such that

$$\psi_k(x) = \begin{cases} 1 & x \geq \dfrac{1}{k} \\ 0 & x \leq \dfrac{1}{k+1} \quad \text{and } x \geq 2 \end{cases}$$

$$0 \leq \psi_k(x) \leq 1$$

We just have to set $\varphi_k(x) = \dfrac{1}{\sqrt{k}}\psi_k(x)$

a) By definition we have

$$\sup_{x \in K} |\varphi_k(x)| \leq \frac{1}{\sqrt{k}}$$

so (φ_k) converges uniformly to zero.

b) If $\alpha \neq 0$, the support of $\partial^\alpha \varphi_k$ is contained in $\left[\dfrac{1}{k+1}, \dfrac{1}{k}\right] \cup [1,2]$ and $\partial^\alpha \varphi_k\left(\dfrac{1}{j}\right) = 0$ for $j \neq k$ and $j \neq k+1$. Moreover $\partial^\alpha \varphi_k\left(\dfrac{1}{k}\right) = \partial^\alpha \varphi_k\left(\dfrac{1}{k+1}\right) = \partial^\alpha \varphi(0)$ $= 0$ i.e. $\partial^\alpha \varphi_k = 0$ on S.

c) Let k be fixed. If m is large enough $(m \geq k+1)$ the points $\dfrac{1}{j}$ which are in the support of φ_k are $\dfrac{1}{k}, \ldots, \dfrac{1}{2}, 1$. Moreover $\varphi_k(0) = \varphi_k'(0) = 0$. So

$$\langle T, \varphi_k \rangle = \sum_{j=1}^{k} \varphi\left(\frac{1}{j}\right)$$

Since $\varphi_k(x) = \dfrac{1}{\sqrt{k}}$ for $x \geq \dfrac{1}{k}$ we have $\varphi\left(\dfrac{1}{j}\right) = \dfrac{1}{\sqrt{k}}$, thus

$$\langle T, \varphi_k \rangle = k \cdot \frac{1}{\sqrt{k}} = \sqrt{k}$$

and

$$\lim_{k \to \infty} \langle T, \varphi_k \rangle = +\infty$$

3°) We deduce from the above questions that the inequality

$$(1) \quad |\langle T, \varphi \rangle| \leqq C \sum_{|\alpha| \leq p} \sup_{x \in S} |\partial^\alpha \varphi(x)|$$

is impossible.

Indeed let us take $\varphi = \varphi_k$, then by a), b) and c) we get

$$\lim_{k\to\infty} \langle T, \varphi_k \rangle = +\infty$$

and

$$\sum_{|\alpha|\le p} \sup_{x\in S} |\partial^\alpha \varphi_k(x)| \le \sup_{x\in\mathbb{R}} |\varphi_k(x)| \to 0$$

in other words, inequality (1) is not equivalent to say that T is a distribution. Moreover if T is a distribution with compact support K we know that one can find $C > 0$, a compact K' and $k \in \mathbb{N}$ such that

$$|\langle T, \varphi \rangle| \le C \sum_{|\alpha|\le k} \sup_{x\in K'} |\partial^\alpha \varphi(x)|$$

So this problem shows that in general we cannot take $K = K'$.

Solution 11

Let φ be in $\mathscr{D}(\mathbb{R})$ and let us suppose that $K = \operatorname{supp}\varphi \subset \{x \in \mathbb{R}: |x| \le M\}$.

a) We can write (see Exercise 4)

$$\varphi(x) = \varphi(0) + x\psi(x) \quad \text{where } \psi \in C^0(\mathbb{R}) \text{ and } \sup_{|x|\le M} |\psi(x)| \le C \sup_{|x|\le M} |\varphi'(x)|$$

It follows that

$$\int_{\varepsilon\le|x|\le M} \frac{\varphi(x)\,dx}{x} = \varphi(0) \int_{\varepsilon\le|x|\le M} \frac{dx}{x} + \int_{\varepsilon\le|x|\le M} \psi(x)\,dx$$

Now

$$\int_{\varepsilon\le|x|\le M} \frac{dx}{x} = \int_{-M}^{-\varepsilon} \frac{dx}{x} + \int_{\varepsilon}^{M} \frac{dx}{x} = 0$$

On the other hand, by the Lebesgue theorem, since $\psi \in L^1_{\mathrm{loc}}(\mathbb{R})$:

$$\lim_{\varepsilon\to 0} \int_{\varepsilon\le|x|\le M} \psi(x)\,dx = \int_{|x|\le M} \psi(x)\,dx$$

So

$$\left|\left\langle pv\frac{1}{x}, \varphi \right\rangle\right| = \left|\int_{|x|\le M} \psi(x)\,dx\right| \le C \sup_{|x|\le M} |\psi(x)| \le C' \sup_{x\in K} |\varphi'(x)|$$

Therefore $pv\dfrac{1}{x}$ is a distribution of order ≤ 1.

b) We use the same method as above. We write:

$$\varphi(x) = \varphi(0) + x\varphi'(0) + x^2\psi(x),\ \psi \in C^0(\mathbb{R}),\ \sup_{|x|\le M} |\psi| \le C \sup_{|x|\le M} |\varphi''|$$

Thus

$$\left\langle fp\frac{1}{x^2}, \varphi\right\rangle = \lim_{\varepsilon\to 0}\left[\int_{\varepsilon\le|x|\le M} \frac{\varphi(0)\,dx}{x^2} + \int_{\varepsilon\le|x|\le M} \frac{\varphi'(0)}{x}dx + \int_{\varepsilon\le|x|\le M} \psi(x)\,dx - 2\frac{\varphi(0)}{\varepsilon}\right]$$

Now

$$\int_{\varepsilon\le|x|\le M} \frac{\varphi(0)}{x^2}dx = \int_{-M}^{-\varepsilon} \frac{\varphi(0)\,dx}{x^2} + \int_{\varepsilon}^{M} \frac{\varphi(0)\,dx}{x^2} = \frac{2\varphi(0)}{\varepsilon} - \frac{2\varphi(0)}{M}$$

and

$\displaystyle\int_{\varepsilon\le|x|\le M} \frac{\varphi'(0)}{x}dx = 0$ since the function $\dfrac{1}{x}$ is odd, then

$$\left\langle fp\,\frac{1}{x^2}, \varphi\right\rangle = \lim_{\varepsilon\to 0}\left[\int_{\varepsilon\le|x|\le M} \psi(x)\,dx - \frac{2\varphi(0)}{M}\right] = \int_{|x|\le M} \psi(x)\,dx - \frac{2\varphi(0)}{M}$$

So

$$\left|\left\langle fp\,\frac{1}{x^2}, \varphi\right\rangle\right| \le C_1 \sum_{j=0}^{2} \sup_{K} |\varphi^{(j)}(x)| \qquad \text{Q.E.D.}$$

c) By the same argument as in b)

$$\left\langle fp\,\frac{H}{x^2}, \varphi\right\rangle = \lim_{\varepsilon\to 0}\left[\int_{\varepsilon}^{M} \frac{\varphi(0)}{x^2}dx + \int_{\varepsilon}^{M} \frac{\varphi'(0)}{x}dx + \int_{\varepsilon}^{M} \psi(x)\,dx - \frac{\varphi(0)}{\varepsilon} + \varphi'(0)\,\mathrm{Log}\,\varepsilon\right]$$

Now

$$\int_{\varepsilon}^{M} \frac{\varphi(0)\,dx}{x^2} = -\frac{\varphi(0)}{M} + \frac{\varphi(0)}{\varepsilon}$$

$$\int_{\varepsilon}^{M} \frac{\varphi'(0)}{x}dx = \varphi'(0)\,\mathrm{Log}\,M - \varphi'(0)\,\mathrm{Log}\,\varepsilon$$

So

$$\left\langle fp\,\frac{H}{x^2}, \varphi\right\rangle = \lim_{\varepsilon\to 0}\left[\int_{\varepsilon}^{M} \psi(x)\,dx + \varphi'(0)\,\mathrm{Log}\,M - \frac{\varphi(0)}{M}\right]$$

therefore

$$\left|\left\langle fp\,\frac{H}{x^2}, \varphi\right\rangle\right| \le C \sum_{j=0}^{2} \sup_{K} |\varphi^{(j)}(x)| \qquad \text{Q.E.D.}$$

d) The method of construction of distributions in a), b) and c) is as follows. We consider the integral over $|x| \geq \varepsilon$ which in general does not have a limit when ε goes to zero, (see b) and c)) but if we remove some terms from this integral, we get an expression which has a limit when $\varepsilon \to 0$. That is why we call it a finite part of the integral. Let us extend this procedure to the case $k \geq 3$. Let us set

$$\left\langle fp\,\frac{1}{x^k}, \varphi\right\rangle = \lim_{\varepsilon \to 0}\left[\int_{|x|\geq\varepsilon} \frac{\varphi(x)}{x^k}\,\mathrm{d}x - f(\varepsilon)\right]$$

To calculate $f(\varepsilon)$ we write

$$\varphi(x) = \varphi(0) + x\varphi'(0) + \cdots + \frac{x^{k-1}}{(k-1)!}\varphi^{(k-1)}(0) + x^k\psi(x)$$

where $\psi \in C^0(\mathbb{R})$ and $\sup\limits_{|x|\leq M}|\psi| \leq C \sup\limits_{K}|\varphi^{(k)}(x)|$. We get

$$\int_{\varepsilon\leq|x|\leq M} \frac{\varphi(x)}{x^k}\,\mathrm{d}x = \varphi(0)\int_{\varepsilon\leq|x|\leq M}\frac{\mathrm{d}x}{x^k} + \varphi'(0)\int_{\varepsilon\leq|x|\leq M}\frac{\mathrm{d}x}{x^{k-1}}$$
$$+ \cdots + \frac{\varphi^{(k-1)}(0)}{(k-1)!}\int_{\varepsilon\leq|x|\leq M}\frac{\mathrm{d}x}{x} + \int_{\varepsilon\leq|x|\leq M}\psi(x)\,\mathrm{d}x$$

so:

$$\int_{\varepsilon\geq|x|\leq M} \frac{\varphi(x)}{x^k}\,\mathrm{d}x = \sum_{j=1}^{k-1}\left[\frac{(-1)^{k-j}}{\varepsilon^{k-j}} - \frac{1}{\varepsilon^{k-j}}\right]\frac{\varphi^{(j-1)}(0)}{(j-k)}\frac{1}{(j-1)!} + \sum_{j=0}^{k-2} C_j\varphi^{(j)}(0)$$
$$+ \int_{\varepsilon\leq|x|\leq M}\psi(x)\,\mathrm{d}x$$

If we set

$$f(\varepsilon) = \sum_{j=1}^{k-1}\frac{(-1)^{k-j}-1}{(j-k)\varepsilon^{k-j}}\cdot\frac{\varphi^{(j-1)}(0)}{(j-1)!}$$

$$\lim_{\varepsilon\to 0}\left[\int_{\varepsilon\leq|x|\leq M}\frac{\varphi(x)}{x^k}\,\mathrm{d}x - f(\varepsilon)\right] = \int_{|x|\leq M}\psi(x)\,\mathrm{d}x + \sum_{j=0}^{k-2} C_j\varphi^{(j)}(0)$$

(The C_j's are constants which depend only on M and are independant of ε.) So

$$\left|\left\langle fp\,\frac{1}{x^k}, \varphi\right\rangle\right| \leq C\sum_{j=0}^{k}\sup_{K}|\varphi^{(j)}(x)|$$

therefore $fp\,\dfrac{1}{x^k}$ is a distribution of order $\leq k$.

By analogy we shall set

$$\left\langle fp\,\frac{H}{x^k}, \varphi\right\rangle = \lim_{\varepsilon\to 0}\left[\int_{\varepsilon}^{+\infty}\frac{\varphi(x)}{x^k}\,\mathrm{d}x - g(\varepsilon)\right]$$

with

$$g(\varepsilon) = -\sum_{j=1}^{k-1} \frac{\varphi^{(j-1)}(0)}{(j-k)\varepsilon^{k-j}} \frac{1}{(j-1)!} - \frac{\varphi^{(k-1)}(0)}{(k-1)!} \operatorname{Log} \varepsilon$$

We get a distribution of order $\leq k$.

Solution 12

Let φ be in $\mathscr{D}(\mathbb{R}^n)$. The function $u(x)\varphi(x)$ is then integrable on $\{x\colon |x| \geq \varepsilon\}$. Let us perform the change of coordinates $x = r \cdot w$ where $r \in]0, +\infty[$ and ω belongs to the unit sphere S^{n-1} (polar coordinates).

Assuming that the support of φ is contained in $\{x\colon |x| \leq M\}$ we get:

$$\int_{\varepsilon \leq |x| \leq M} u(x)\varphi(x)\,\mathrm{d}x = \int_\varepsilon^M \int_{|\omega|=1} u(r\cdot\omega)\varphi(r\cdot\omega)r^{n-1}\,\mathrm{d}r\,\mathrm{d}\omega$$

By hypothesis (1) we have $u(r\cdot\omega) = r^{-n}u(\omega)$. So

$$\int_{\varepsilon \leq |x| \leq M} u(x)\varphi(x)\,\mathrm{d}x = \int_\varepsilon^M \frac{1}{r}\left(\int_{|\omega|=1} u(\omega)\varphi(r\cdot\omega)\,\mathrm{d}\omega\right)\mathrm{d}r$$

Using the Taylor formula

$$\varphi(x) = \varphi(0) + \sum_{i=1}^n \int_0^1 (1-t)x_i\psi_i(tx)\,\mathrm{d}t$$

where $\psi_i = \dfrac{\partial\varphi}{\partial x_i}$, we get

$$\varphi(r\omega) = \varphi(0) + \sum_{i=1}^n r\int_0^1 (1-t)\omega_i\psi_i(t\cdot r\cdot\omega)\,\mathrm{d}t$$

Now

$$(4)\quad \int_{\varepsilon \leq |x| \leq M} u(x)\varphi(x)\,\mathrm{d}x = \varphi(0)\underbrace{\int_\varepsilon^M \frac{\mathrm{d}r}{r}\int_{|\omega|=1}}_{A_1} u(\omega)\,\mathrm{d}\omega$$

$$+\sum_{i=1}^n \int_\varepsilon^M \left(\int_{|\omega|=1} \underbrace{\omega_i\cdot u(\omega)}_{A_2}\left(\int_0^1 (1-t)\psi_i(t\cdot r\cdot\omega)\,\mathrm{d}t\right)\mathrm{d}\omega\right)\mathrm{d}r$$

u being continuous in $\mathbb{R}^n\backslash\{0\}$ and the sphere S^{n-1} being compact we get:

$$\sup_{|\omega|=1} |u(\omega)| \leq C_1 < +\infty$$

On the other hand

$$(5)\quad \sup_{\substack{t\in[0,1]\\ r\in[0,M]}} \sup_{|\omega|=1} \sum_{i=1}^{n} |\omega_i \psi_i(tr\omega)| \leqq \sup_{x\in\mathbf{R}^n} \sum_{i=1}^{n} \left|\frac{\partial\varphi}{\partial x_i}(x)\right| \quad .$$

We can apply the Lebesgue theorem to the sequence

$$f_\varepsilon = 1_{\{r\geq\varepsilon\}} u(\omega) \sum_{i=1}^{n} \omega_i \psi_i(t\cdot r\cdot\omega)$$

and conclude that the second term of the right hand side of (4) has a limit, when ε tends to zero, which is:

$$\sum_{i=1}^{n} \int_0^M \int_0^1 \int_{|\omega|=1} \omega_i u(\omega)(1-t)\frac{\partial\varphi}{\partial x_i}(t\cdot r\cdot\omega)\,d\omega\,dt\,dr$$

Therefore the left hand side of (4) has a limit, when $\varepsilon\to 0$, if and only if the term A_1 has a limit.

If $\int_{|\omega|=1} u(\omega)\,d\omega = 0$ then $A_1 = 0$. Conversely if $\int_{|\omega|=1} u(\omega)\,d\omega = C \neq 0$ then $A_1 = C[\operatorname{Log} M - \operatorname{Log}\varepsilon]\varphi(0)$. For all functions φ such that $\varphi(0)\neq 0$ we have $\lim_{\varepsilon\to 0} A_1 = +\infty$. Q.E.D.

When $A_1 = 0$, using (5) we can write

$$|\langle U,\varphi\rangle| \leqq C\sum_{i=1}^{n} \sup_{x\in\mathbf{R}^n}\left|\frac{\partial\varphi}{\partial x_i}(x)\right|$$

so U, which is of course linear, is a distribution of order ≤ 1.

Solution 13

There are several ways to define T. Here is one of them. Let us set

$$(3)\quad \langle T,\varphi\rangle = \int_{|x|\geq 1} f(x)\varphi(x)\,dx + \int_{|x|\leq 1} f(x)\left[\varphi(x) - \sum_{|\alpha|\leq m-1}\frac{1}{\alpha!}x^\alpha(\partial^\alpha\varphi)(0)\right]dx = I_1 + I_2$$

By the Taylor formula we get

$$(4)\quad \varphi(x) - \sum_{|\alpha|\leq m-1}\frac{1}{\alpha!}x^\alpha(\partial^\alpha\varphi)(0) = \int_0^1 \frac{(1-t)^{m-1}}{m!}\sum_{|\alpha|=m} x^\alpha(\partial^\alpha\varphi)(tx)\,dt$$

Using (4), the inequality $|x^\alpha| \leq |x|^{|\alpha|}$ and that $\operatorname{supp}\varphi \subset \{x\colon |x|\leq M\}$ we get:

$$|I_2| \leqq C_1\int_{|x|\leq 1}\frac{|x|^m}{|x|^m}\left(\int_0^1 (1-t)^{m-1}\sum_{|\alpha|=m}|\partial^\alpha\varphi(tx)|\,dt\right)dx \leqq C_2\sum_{|\alpha|=m}\sup_{x\in\mathbf{R}}|\partial^\alpha\varphi(x)|$$

and

$$|I_1| \leq \int_{1\leq|x|\leq M} |\varphi(x)||f(x)|\,dx \leq \left(\int_{1\leq|x|\leq M} |f(x)|\,dx\right) \sup_{x\in\mathbb{R}} |\varphi(x)| \leq C_3 \sup_{x\in\mathbb{R}} |\varphi(x)|$$

Therefore formula (3) defines a linear map on $\mathscr{D}(\mathbb{R}^n)$ such that:

$$|\langle T, \varphi\rangle| \leq C_4 \sum_{|\alpha|\leq m} \sup_{x\in\mathbb{R}} |\partial^\alpha\varphi(x)|$$

thus it is a distribution of order $\leq m$.

Now for $\varphi \in \mathscr{D}(\mathbb{R}^n)$ with $\operatorname{supp}\varphi \subset \mathbb{R}^n\backslash\{0\}$ we have $\partial^\alpha\varphi(0) = 0\ \forall\alpha$ and the formula (3) reduces to (2). Q.E.D.

Remark:

If in formula (3) we cut the integral in another point we obtain another distribution T_1 such that $T - T_1 = \sum_{|\alpha|\leq m-1} C_\alpha\delta^{(\alpha)}$, where $C_\alpha \in \mathbb{C}$.

This is not surprising because if T_1 and T_2 satisfy condition (2), then $\langle T_1 - T_2, \varphi\rangle = 0$ for all φ such that $\operatorname{supp}\varphi \subset \mathbb{R}^n\backslash\{0\}$, so $\operatorname{supp}(T_1 - T_2) = \{0\}$ which implies that $T_1 - T_2 = \sum_{|\alpha|\leq N} a_\alpha\delta^{(\alpha)}$, $a_\alpha \in \mathbb{C}$, $N \in \mathbb{N}$.

Solution 14

Let φ be in $\mathscr{D}(\mathbb{R})$ such that $\operatorname{supp}\varphi \subset \{1 < x < 2\}$, $0 \leq \varphi(x) \leq 1$ and $\varphi(x) = 1$ for $a \leq x \leq b$ where $1 < a < b < 2$. If $n \in \mathbb{N}\backslash\{0\}$ we set

$$\varphi_n(x) = e^{-n}\varphi(nx)$$

then $\varphi_n \in \mathscr{D}(\mathbb{R})$, $\operatorname{supp}\varphi_n \subset \left\{\frac{1}{n} < x < \frac{2}{n}\right\}$. Moreover if $k \in \mathbb{N}$,

$$\varphi_n^{(k)}(x) = e^{-n}n^k\varphi^{(k)}(nx)$$

so

$$(2)\quad \sup_{x\in\mathbb{R}} |\varphi_n^{(k)}(x)| \leq n^k e^{-n} \sup_{1\leq y\leq 2} |\varphi^{(k)}(y)|$$

From inequality (2), since $\operatorname{supp}\varphi_n \subset [0, 2]$, we deduce that sequence (φ_n) tends to zero in $\mathscr{D}(\mathbb{R})$.

Let us assume that there exists $T \in \mathscr{D}'(\mathbb{R})$ which coincides with e^{1/x^2} in $\mathbb{R}\backslash\{0\}$ then:

$$(3)\quad \lim_{n\to\infty} \langle T, \varphi_n\rangle = 0$$

Since the support of φ_n is contained in $\mathbb{R}\backslash\{0\}$ by (1) we must have

$$\langle T, \varphi_n\rangle = \int_{\mathbb{R}} e^{1/x^2}\varphi_n(x)\,dx = \int_{1/n}^{2/n} e^{1/x^2}\varphi_n(x)\,dx$$

Now φ_n is positive and for $\frac{a}{n} < x < \frac{b}{n}$ we have $\varphi_n(x) = e^{-n}$, so

$$\langle T, \varphi_n \rangle \geqq e^{-n} \int_{a/n}^{b/n} e^{1/x^2} dx$$

Finally for $0 < x \leq \frac{b}{n}$ we have $e^{1/x^2} \geq e^{n^2/b^2}$ and $\int_{a/n}^{b/n} dx = \frac{b-a}{n}$ so

$$\langle T, \varphi_n \rangle \geqq \frac{b-a}{n} e^{n^2/b^2} e^{-n} \geq \frac{b-a}{n} e^n \text{ for } n \geq 2b^2$$

which implies that

$$\lim_{n\to\infty} \langle T, \varphi_n \rangle = +\infty$$

and is a contradiction to (3).

Remark:
Instead of e^{1/x^2} we could use any function f which is C^∞ outside the origin and which satisfies, near the origin,

$$|f(x)| > \frac{1}{|x|^m}$$

for every $m \in \mathbb{N}$.
We just have to take $\varphi_n = C_n \varphi(nx)$, $C_n = \frac{n^2}{|f(x_n)|}$ where x_n is a point in $\left[\frac{a}{n}, \frac{b}{n}\right]$ where f is minimum. Then

$$|f(x_n)| > \frac{1}{|x_n|^m} \geq \left(\frac{n}{b}\right)^m \quad \text{for all } m$$

so $n^k C_n$ tends to zero for all k in $\mathbb{N}$.
We can compare this remark with exercise 13.

Solution 15

a) For $\operatorname{Re}\lambda > -1$ the functions x_+^λ and x_-^λ are locally integrable so the formulas $\langle x_+^\lambda, \varphi \rangle = \int_0^\infty x^\lambda \varphi(x)\,dx$, $\langle x_-^\lambda, \varphi \rangle = \int_{-\infty}^0 |x|^\lambda \varphi(x)$ define distributions.

b) Formula (1) follows from the fact that for $\operatorname{Re}\lambda > -1$, $\int_0^1 x^\lambda dx = \frac{1}{1+\lambda}$.

A same formula follows for x_-^λ, for $\operatorname{Re}\lambda > -1$, from

$$\langle x_-^\lambda, \varphi \rangle = \langle x_+^\lambda, \check{\varphi} \rangle \quad \text{where } \check{\varphi}(x) = \varphi(-x)$$

The right hand side of (1) makes sense for $\operatorname{Re}\lambda > -2$, $\lambda \neq -1$.

Indeed in the first integral we have $\varphi(x) - \varphi(0) = x\psi(x)$ where $\psi \in C^0(0, 1)$ so $\int_0^1 x^\lambda[\varphi(x) - \varphi(0)]\,dx = \int_0^1 x^{\lambda+1}\psi(x)\,dx$ makes sense for $\mathrm{Re}(\lambda+1) > -1$.
The second integral is well defined for any λ since $x \geq 1$, and the third term makes sense for $\lambda \neq -1$.
Therefore for $\mathrm{Re}\,\lambda \geq -2$, $\lambda \neq -1$ we shall take (1) as a definition for the distribution x_+^λ.
In the same way, if $\mathrm{Re}\,\lambda \geq -n - 1$, $\lambda \neq -1, -2, \ldots, -n$ we have:

$$(2) \quad \int_0^\infty x^\lambda \varphi(x)\,dx = \int_0^1 x^\lambda\left[\varphi(x) - \sum_{k=1}^n \frac{x^{k-1}\varphi^{(k-1)}(0)}{(k-1)!}\right]dx + \int_0^\infty x^\lambda\varphi(x)\,dx + \sum_{k=1}^n \frac{\varphi^{(k-1)}(0)}{(\lambda+k)(k-1)!}$$

and the right hand side of equality (2) is well defined for $\mathrm{Re}\,\lambda > -n - 1$, $\lambda \neq -1, -2, \ldots, -n$. We shall take the right hand side of (2) as a definition for x_+^λ when $\mathrm{Re}\,\lambda > -n - 1$, $\lambda \neq -1, \ldots, -n$.
We deduce x_-^λ by the formula $(x_-^\lambda, \varphi) = (x_+^\lambda, \check{\varphi})$, $\check{\varphi}(x) = \varphi(-x)$ $\forall \varphi \in \mathscr{D}(\mathbb{R})$.
c) For $\mathrm{Re}\,\lambda > -n - 1$, $\lambda \neq -1, -2, \ldots, -n$ we have:

$$(xx_+^\lambda, \varphi) = (x_+^\lambda, x\varphi) = \int_0^1 x^\lambda\left[x\varphi - \sum_{k=1}^n \frac{x^{k-1}(x\varphi)^{(k-1)}(0)}{(k-1)!}\right]dx + \int_1^\infty x^{\lambda+1}\varphi(x)\,dx + \sum_{k=1}^n \frac{(x\varphi)^{(k-1)}(0)}{(\lambda+k)(k-1)!}$$

Now $(x\varphi)^{k-1}(0) = (k-1)\varphi^{(k-2)}(0)$ for $k \geq 2$ and $(x\varphi)(0) = 0$

$$\langle x_+^\lambda, x\varphi\rangle = \int_0^1 x^\lambda\left[x\varphi - \sum_{k=2}^n \frac{x^{k-1}\varphi^{(k-2)}(0)}{(k-2)!}\right]dx + \int_1^\infty x^{\lambda+1}\varphi(x)\,dx + \sum_{k=2}^n \frac{\varphi^{(k-2)}(0)}{(\lambda+k)(k-2)!}$$

$$\langle x_+^\lambda, x\varphi\rangle = \int_0^1 x^{\lambda+1}\left[\varphi(x) - \sum_{k=1}^{n-1} \frac{x^{k-1}\varphi^{(k-1)}(0)}{(k-1)!}\right]dx + \int_1^\infty x^{\lambda+1}\varphi(x)\,dx + \sum_{k=1}^{n-1} \frac{\varphi^{(k-1)}(0)}{(\lambda+k+1)(k-1)!}$$

Now if $\mathrm{Re}\,\lambda > -n - 1$, $\mathrm{Re}(\lambda + 1) > -n$ and $\lambda \neq -1, \ldots, -n + 1$.
Then, by definition, the right hand side is $\langle x_+^{\lambda+1}, \varphi\rangle$. Therefore

$$xx_+^\lambda = x_+^{\lambda+1}$$

Let us compute xx_-^λ

$$\langle xx_-^\lambda, \varphi\rangle = \langle x_-^\lambda, x\varphi\rangle = \langle x_+^\lambda, (x\check{\varphi})\rangle = -\langle x_+^\lambda, x\check{\varphi}\rangle$$

$$\langle xx_-^\lambda, \varphi\rangle = -\langle xx_+^\lambda, \check{\varphi}\rangle = -\langle x_+^{\lambda+1}, \check{\varphi}\rangle = -\langle x_-^{\lambda+1}, \varphi\rangle$$

so

$$xx_-^\lambda = -x_-^{\lambda+1}$$

Solution 16

a) If T is a homogeneous distribution given by $f \in L^1_{loc}$ we have using (1)

$$\langle T \circ A_\lambda, \varphi\rangle = \lambda^{-n}\langle T, \varphi \circ A_{1/\lambda}\rangle = \lambda^{-n}\int f(x)\varphi\left(\frac{x}{\lambda}\right)\mathrm{d}x = \lambda^p \int f(x)\varphi(x)\,\mathrm{d}x$$

Let us set $\frac{x}{\lambda} = y$ in the first integral, we get:

$$\int f(\lambda y)\varphi(y)\,\mathrm{d}y = \lambda^p \int f(x)\varphi(x)\,\mathrm{d}x$$

so

$$\int [f(\lambda x) - \lambda^p f(x)]\varphi(x)\,\mathrm{d}x = 0 \qquad \forall \varphi \in \mathscr{D}(\mathbb{R}^n)$$

therefore $f(\lambda x) = \lambda^p f(x)$ almost everywhere, since $\mathscr{D}(\mathbb{R}^n)$ is dense in $L^1_{loc}(\mathbb{R}^n)$. The converse follows easily from the lines above.

b) For λ in a neighborhood of $\lambda_0 > 0$, the supports of the ψ_λ are contained in a fixed compact K. Indeed let us assume that $\operatorname{supp}\varphi \subset \{x: |x| \le N\}$ and that $|\lambda - \lambda_0| < \varepsilon$ with $0 < \varepsilon < \lambda_0$. Then if

$$|x| > \frac{N}{\lambda_0 - \varepsilon} \text{ we have } \begin{cases} |\lambda x| > (\lambda_0 - \varepsilon)\cdot|x| > N \\ |\lambda_0 x| > (\lambda_0 - \varepsilon)\cdot|x| > N \end{cases}$$

so $\varphi(\lambda x) = \varphi(\lambda_0 x) = 0$ and $\psi_\lambda(x) = 0$. Therefore

$$\operatorname{supp}\psi_\lambda \subset \left\{x: |x| \le \frac{N}{\lambda_0 - \varepsilon}\right\}$$

We have to prove now that all the derivatives of ψ_λ converge uniformly on K. By the Taylor formula up to the order two we have for all y, y^0 in $\operatorname{supp}\psi$:

$$\varphi(y) = \varphi(y^0) + \sum_{k=1}^n (y_i - y_i^0)\frac{\partial\varphi}{\partial x_i}(y^0)$$
$$+ \sum_{i,j=1}^n \int_0^1 (1 - t)^2(y_i - y_i^0)(y_j - y_j^0)\frac{\partial^2\varphi}{\partial x_i \partial x_j}(ty + (1 - t)y^0)\,\mathrm{d}t$$

Let us take $y = \lambda x$ and $y^0 = \lambda_0 x$. We get

$$\frac{\varphi(\lambda x) - \varphi(\lambda_0 x)}{\lambda - \lambda_0} - \sum_{k=1}^{n} x_i \frac{\partial \varphi}{\partial x_i}(\lambda_0 x)$$
$$= \sum_{i,j=1}^{n} (\lambda - \lambda_0) \int_0^1 (1 - t)^2 x_i x_j \frac{\partial^2 \varphi}{\partial x_i \partial x_j}(z)\,\mathrm{d}t$$

where $z = [t\lambda + (1 - t)\lambda_0]x$.

Therefore

$$(2) \quad \sup_K \left| \frac{\varphi(\lambda x) - \varphi(\lambda_0 x)}{\lambda - \lambda_0} - \sum_{i=1}^{n} x_i \frac{\partial \varphi}{\partial x_i}(\lambda_0 x) \right| \leqq |\lambda - \lambda_0| \cdot \sum_{i,j=1}^{n} \sup_{\mathbf{R}^n} \left| \frac{\partial^2 \varphi}{\partial x_i \partial x_j}(x) \right|$$

which proves that ψ_λ converges uniformly on K to $\sum_{i=1}^{n} x_i \frac{\partial \varphi}{\partial x_i}(\lambda_0 x)$. Let us prove the same result for the derivatives of ψ_λ. We have

$$\partial^\alpha \psi_\lambda(x) = \frac{\lambda^{|\alpha|}(\partial^\alpha \varphi)(\lambda x) - \lambda_0^{|\alpha|}(\partial^\alpha \varphi)(\lambda_0 x)}{\lambda - \lambda_0}$$

$$(3) \quad \partial^\alpha \psi_\lambda(x) = \frac{\lambda^{|\alpha|} - \lambda_0^{|\alpha|}}{\lambda - \lambda_0}(\partial^\alpha \varphi)(\lambda x) + \lambda_0^{|\alpha|} \frac{(\partial^\alpha \varphi)(\lambda x) - (\partial^\alpha \varphi)(\lambda_0 x)}{\lambda - \lambda_0}$$

* $\dfrac{\lambda^{|\alpha|} - \lambda_0^{|\alpha|}}{\lambda - \lambda_0}$ converges to $|\alpha|\lambda_0^{|\alpha|-1}$ when λ tends to λ_0.
* $(\partial^\alpha \varphi)(\lambda x)$ converges uniformly on K to $(\partial^\alpha \varphi)(\lambda_0 x)$ since, by the Taylor formula up to the order one applied at λx and $\lambda_0 x$, we have:

$$|(\partial^\alpha \varphi)(\lambda x) - (\partial^\alpha \varphi)(\lambda_0 x)| \leqq |\lambda - \lambda_0| \int_0^1 (1 - t) \sum_{i=1}^{n} |x_i| \left| \frac{\partial \varphi}{\partial x_i}(t\lambda x + (1 - t)\lambda_0 x) \right| \mathrm{d}t$$

$$\sup_K |(\partial^\alpha \varphi)(\lambda x) - (\partial^\alpha \varphi)(\lambda_0 x)| \leq |\lambda - \lambda_0| C \sup_{\mathbf{R}^n} \sum_{i=1}^{n} \left| \frac{\partial \varphi}{\partial x_i}(x) \right|$$

* By the inequality (2) applied to $(\partial^\alpha \varphi)$, the second term of the right hand side of (3) converges uniformly on K to

$$\lambda_0^{|\alpha|} \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}(\partial^\alpha \varphi)(\lambda_0 x)$$

So $\partial^\alpha \psi_\lambda$ converges uniformly on K to the function

$$|\alpha| \lambda_0^{|\alpha|-1}(\partial^\alpha \varphi)(\lambda_0 x) + \lambda_0^{|\alpha|} \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}(\partial^\alpha \varphi)(\lambda_0 x) \qquad \text{Q.E.D.}$$

c) We have

$$(x_+^p, \varphi_\lambda) = \int_0^1 x^p \left[\varphi(\lambda x) - \varphi(0) - \lambda x\varphi'(0) - \cdots - \frac{\lambda^{n-1}x^{n-1}}{(n-1)!}\varphi^{(n-1)}(0)\right]\mathrm{d}x$$

$$+ \int_1^\infty x^p\varphi(\lambda x)\,\mathrm{d}x + \sum_{k=1}^n \frac{\lambda^{k-1}\varphi^{(k-1)}(0)}{(p+k)(k-1)!} =$$

$$= \lambda^{-p-1}\int_0^\lambda y^p\left[\varphi(y) - \varphi(0) - y\varphi'(0) - \cdots - \frac{y^{n-1}}{(n-1)!}\varphi^{(n-1)}(0)\right]\mathrm{d}y$$

$$+ \lambda^{-p-1}\int_\lambda^\infty y^p\varphi(y)\,\mathrm{d}y + \sum_{k=1}^n \frac{\lambda^{k-1}\varphi^{(k-1)}(0)}{(p+k)(k-1)!} =$$

$$= \lambda^{-p-1}\left[\int_0^1 y^p\left[\varphi(y) - \sum_{k=1}^n \frac{y^{k-1}\varphi^{(k-1)}(0)}{(k-1)!}\right]\mathrm{d}y + \right.$$

$$+ \int_1^\lambda y^p\left[\varphi(y) - \sum_{k=1}^n \frac{y^{k-1}\varphi^{(k-1)}(0)}{(k-1)!}\right]\mathrm{d}y +$$

$$\left. + \int_\lambda^1 y^p\varphi(y)\,\mathrm{d}y + \int_1^\infty y^p\varphi(y)\,\mathrm{d}y\right] + \sum_{k=1}^n \frac{\lambda^{k-1}\varphi^{(k-1)}(0)}{(p+k)(k-1)!}.$$

Now

$$\int_1^\lambda y^p[\varphi(y) - \sum\cdots]\,\mathrm{d}y + \int_\lambda^1 y^p\varphi(y)\,\mathrm{d}y = -\sum_{k=1}^n \int_1^\lambda y^{p+k-1}\frac{\varphi^{(k-1)}(0)}{(k-1)!}\,\mathrm{d}y$$

$$= \sum_{k=1}^n \frac{\varphi^{(k-1)}(0)}{(p+k)(k-1)!} - \sum_{k=1}^{n-1}\frac{\lambda^{p+k}\varphi^{(k-1)}(0)}{(p+k)(k-1)!}$$

so

$$\langle x_+^p, \varphi_\lambda\rangle = \lambda^{-p-1}\left[\int_0^1 y^p\left[\varphi(y) - \varphi(0) - \cdots - y^{n-1}\frac{\varphi^{(n-1)}(0)}{(n-1)!}\right]\mathrm{d}y\right.$$

$$\left. + \int_0^1 y^p\varphi(y)\,\mathrm{d}y + \sum_{k=1}^n \frac{\varphi^{(k-1)}(0)}{(p+k)(k-1)!}\right] = \lambda^{-p-1}\langle x_+^p, \varphi\rangle$$

The same calculus can be used for $\langle x_-^\lambda, \varphi\rangle$.

Let $m \in \mathbb{N}$. We have by definition:

$$\left\langle fp\,\frac{1}{x^m}, \varphi_\lambda\right\rangle = \lim_{\varepsilon\to 0}\left[\left[\int_{|x|\geq\varepsilon}\frac{\varphi(\lambda x)}{x^m}\,\mathrm{d}x - \sum_{j=1}^{m-1}\frac{(-1)^{m-j}-1}{(j-m)\varepsilon^{m-j}}\cdot\frac{\varphi^{(j-1)}(0)}{(j-1)!}\lambda^{j-1}\right]\right.$$

Let us perform the change of variable $\lambda x = y$ in the integral and let us set $\lambda\varepsilon = \alpha$; we get:

$$\left\langle fp\,\frac{1}{x^m}, \varphi_\lambda\right\rangle = \lim_{\alpha\to 0}\left[\lambda^{m-1}\int_{|y|\geq\alpha}\frac{\varphi(y)\,\mathrm{d}y}{y^m} - \sum_{j=1}^{m-1}\frac{[(-1)^{m-j}-1]}{(j-m)}\cdot\frac{\lambda^{m-j}}{\alpha^{m-j}}\cdot\frac{\varphi^{(j-1)}(0)}{(j-1)!}\lambda^{j-1}\right]$$
$$= \lambda^{m-1}\left\langle fp\,\frac{1}{x^m}, \varphi\right\rangle$$

which proves that the distribution $fp\,\dfrac{1}{x^m}$ is homogeneous of degree $-m$.

d) Let $T_1, \ldots, T_k$ be homogeneous distribution of different degrees $p_1, \ldots, p_k$. We can assume $p_1 > p_2 > \cdots > p_k$. Let $C_1, \ldots, C_k$ be constants such that

$$C_1 T_1 + \cdots + C_k T_k = 0$$

Let φ be in $\mathscr{D}(\mathbb{R}^n)$. We have

$$0 = C_1\langle T_1, \varphi_\lambda\rangle + \cdots + C_k\langle T_k, \varphi_\lambda\rangle = C_1\lambda^{-(n+p_1)}\langle T_1, \varphi\rangle + \cdots + C_k\lambda^{-(n+p_k)}\langle T_k, \varphi\rangle$$

so

$$C_1\lambda^{-p_1}\langle T_1, \varphi\rangle + \cdots + C_k\lambda^{-p_k}\langle T_k, \varphi\rangle = 0 \qquad \forall\lambda > 0, \forall\varphi\in\mathscr{D}$$

Let us multiply this equality by λ^{p_k} et let λ tend to $+\infty$. Since $p_k - p_i < 0$ for $i = 1, 2, \ldots, k-1$, we get $C_k\langle T_k, \varphi\rangle = 0$. Since $T_k \not\equiv 0$ we can find a function φ such that $\langle T_k, \varphi\rangle \neq 0$ so $C_k = 0$. In the same way $C_{k-1}, \ldots, C_1 = 0$. Q.E.D.

Solution 17

a) It is a distribution since for $\varphi\in\mathscr{D}(\mathbb{R})$ we can find $N(\varphi)\in\mathbb{N}$ such that $\operatorname{supp}\varphi \subset \{|x| \leq N(\varphi)\}$. Since $\operatorname{supp}\partial^\alpha\varphi \subset \operatorname{supp}\varphi$ for all α we have

$$\langle T, \varphi\rangle = \sum_0^{N(\varphi)}\varphi^{(n)}(n)$$

so $|\langle T, \varphi\rangle| \leq C\sum_{|\alpha|\leq N(\varphi)}\sup_{x\in\mathbf{R}}|\partial^\alpha\varphi(x)|$.

b) Let us assume that T has a finite order k. Let $\varphi\in\mathscr{D}(]-1, 1[)$ be such that $\varphi^{(k+1)}(0) = 1$. For $\varepsilon > 0$ let us set: $\varphi_\varepsilon(x) = \varphi\left(\dfrac{x-k-1}{\varepsilon}\right)$. Then for $\varepsilon < 1$

$$\operatorname{supp}\varphi_\varepsilon \subset]k+1-\varepsilon, k+1+\varepsilon[\subset]k, k+2[= K$$

$$(1)\quad \langle T, \varphi_\varepsilon\rangle = \sum_0^\infty\varphi_\varepsilon^{(n)}(n) = \varphi_\varepsilon^{(k+1)}(k+1) = \frac{1}{\varepsilon^{k+1}}\varphi^{(k+1)}(0) = \frac{1}{\varepsilon^{k+1}}$$

(All the other terms vanish since points $n \neq k+1$ do not belong to $\operatorname{supp}\varphi_\varepsilon$).

On the other hand

$$(2)\quad \sum_{|\alpha|\le k} \sup_K |\partial^\alpha \varphi_\varepsilon(x)| = \sum_{|\alpha|\le k} \frac{1}{\varepsilon^{|\alpha|}} \sup_K \left|\varphi^{(\alpha)}\left(\frac{x-k-1}{\varepsilon}\right)\right|$$

$$\leqq \sum_{|\alpha|\le k} \frac{1}{\varepsilon^{|\alpha|}} \sup_{y\in\mathbf{R}} |\varphi^{(\alpha)}(y)| \le \frac{C_1}{\varepsilon^k}$$

where C_1 is independant of ε. Since T is supposed to have a finite order k we have:

$$|\langle T, \varphi_\varepsilon\rangle| \le C_2 \sum_{|\alpha|\le k} |\partial^\alpha \varphi_\varepsilon(x)|$$

so by (1) and (2) we get $\frac{1}{\varepsilon^{k+1}} \le C' \cdot \frac{1}{\varepsilon^k}$, where C' is independant of ε. This implies that $1 \le C'\varepsilon$ for all $\varepsilon \in]0, 1[$, which is impossible.

Therefore T has infinite order.

Solution 18

a) First of all expression $\langle A_{\lambda,k}, \varphi\rangle$ makes sense since we have

$$\varphi(x) - \sum_{i=0}^{k-1} \varphi^{(i)}(0)\frac{x^i}{i!} = x^k \psi(x)$$

where ψ is continuous on $\mathbb{R}$ and $\sup_{x\in K}|\psi(x)| \le C_K \sup_{x\in K}|\varphi^{(k)}(x)|$ where $K \supset \operatorname{supp}\varphi$. So

$$\langle A_{\lambda,k}, \varphi\rangle = \int_{-\pi/2}^{\pi/2} \cos\lambda x \cdot \psi(x)\,\mathrm{d}x$$

and

$$|\langle A_{\lambda,k}, \varphi\rangle| \leqq C\pi \sup_{x\in K} |\varphi^{(k)}(x)|$$

Which proves that formula (1) defines a distribution of order $\le k$.

On the other hand $\operatorname{supp} A_{\lambda k} \subset \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ since if we had $\operatorname{supp}\varphi \subset \left]-\infty, -\frac{\pi}{2}\right[\cup \left]\frac{\pi}{2}, +\infty\right[$ it would be obvious, using (1), that $\langle A_{\lambda,k}, \varphi\rangle \equiv 0$.

Let us show that $\operatorname{supp} A_{\lambda k} = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$; we prove it by contradiction; let us assume that $\operatorname{supp} A_{\lambda,k} \subsetneqq \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. We can find $x_0 \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $x_0 \notin \operatorname{supp} A_{\lambda,k}$; since $\operatorname{supp} A_{\lambda,k}$ is closed we can find $\varepsilon > 0$ such that $V_{x_0} =]x_0 - \varepsilon, x_0 + \varepsilon[\subset \quad \operatorname{supp} A_{\lambda,k}$ $\left(\text{or } \left[-\frac{\pi}{2}, \frac{\pi}{2} + \varepsilon\right[\text{ if } x_0 = -\frac{\pi}{2}, \left[\frac{\pi}{2} - \varepsilon, \frac{\pi}{2}\right] \text{ if } x_0 = \frac{\pi}{2}\right)$.

Let $x_1 \in V_{x_0}$ be a point such that $x_1 \neq 0$ and $\cos \lambda x_1 \neq 0$ (such a point exists since the roots of $\cos \lambda x$ are isolated). Let $V_{x_1} \subset V_{x_0}$ be a neighborhood of x_1, which does not contain the origin and such that $\cos \lambda x \neq 0$ in V_{x_1}. Let $W \subset V_{x_1}$, $\varphi \in \mathscr{D}(V_{x_1})$, $\varphi \geq 0$ and $\varphi = 1$ on W. We have

$$\langle A_{\lambda,k}, \varphi\rangle = 0 = \int_{V_{x_1}} \frac{\cos \lambda x}{x^k} \varphi(x)\,\mathrm{d}x$$

In V_{x_1} the function $\dfrac{\cos \lambda x}{x_k} \varphi(x)$ has a constant sign. So we have

$$\int_{V_{x_1}} \left|\frac{\cos \lambda x \cdot \varphi(x)}{x^k}\right| \mathrm{d}x = 0 \quad \text{and} \quad \int_W \left|\frac{\cos \lambda x}{x^k}\right| \mathrm{d}x = 0$$

which implies $\cos \lambda x = 0$ in W, which is impossible. So $\operatorname{supp} A_{\lambda,k} = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

b) The general solution of the equation

$$(3) \quad xu = A_{1,1}$$

is the sum of a particular solution of (3) and of the general solution of the equation $xu = 0$ which is $C\delta_0$.

A particular solution of (3) is given by

$$\langle u, \varphi\rangle = \left\langle A_{1,1}, \frac{\varphi(x) - \varphi(0)\psi(x)}{x}\right\rangle \quad \text{where} \begin{cases}\psi \in \mathscr{D}(\mathbb{R})\\ \psi(0) = 1\end{cases}$$

$$\langle u, \varphi\rangle = \int_{-\pi/2}^{\pi/2} \frac{\cos x}{x}\left\{\frac{\varphi(x) - \varphi(0)\psi(x)}{x} - \varphi'(0)\right\}\mathrm{d}x$$

since $\displaystyle\lim_{x\to 0} \frac{\varphi(x) - \varphi(0)\psi(x)}{x} = \varphi'(0)$.

So we have

$$\langle u, \varphi\rangle = \int_{-\pi/2}^{\pi/2} \frac{\cos x}{x^2}(\varphi(x) - \varphi(0)\psi(x) - x\varphi'(0))\,\mathrm{d}x$$

$$\langle u, \varphi\rangle = \int_{-\pi/2}^{\pi/2} \frac{\cos x}{x^2}(\varphi(x) - \varphi(0) - x\varphi'(0))\,\mathrm{d}x + \int_{-\pi/2}^{\pi/2} \frac{\cos x}{x^2}\varphi(0)(1 - \psi(x))\,\mathrm{d}x$$

Let us take ψ such that $\psi'(0) = 0$. We have $\psi = 1 - x^2\psi_1(x)$ where $\psi_1 \in C^0$. So

$$\int_{-\pi/2}^{\pi/2} \frac{\cos x}{x^2}(1 - \psi(x))\,\mathrm{d}x = \int_{-\pi/2}^{\pi/2} \cos x \cdot \psi_1(x)\,\mathrm{d}x = C_1 = C^{te}$$

therefore

$$\langle u, \varphi\rangle = \langle A_{1,2}, \varphi\rangle + C_1\langle \delta, \varphi\rangle$$

so $A_{1,2} + C_1\delta$ is a particular solution and a general solution of (3) is then $u = A_{1,2} + C\delta$.

Solution 19

Of course not! Let us give an example on $\mathbb{R}$:

$$T = \delta';\ f \in C^\infty\colon f(0) = 0 \quad f'(0) = 1 \quad (f(x) = x \text{ for instance})$$

Then

$$f \cdot \delta' = -\delta$$

since if $\varphi \in \mathscr{D}(\mathbb{R})$

$$\langle f\delta', \varphi\rangle = \langle \delta', f\varphi\rangle = -\langle \delta, (f\varphi)'\rangle = -(f\varphi)'(0) = -\varphi(0) = -\langle \delta, \varphi\rangle$$

Solution 20

Indeed let us suppose that

$$\delta \cdot pv\,\frac{1}{x} = pv\,\frac{1}{x} \cdot \delta$$

For all $\alpha \in C^\alpha$ we would have

$$\alpha\left(\delta \cdot pv\,\frac{1}{x}\right) = (\alpha \cdot \delta) \cdot pv\,\frac{1}{x} = \delta\left(\alpha \cdot pv\,\frac{1}{x}\right)$$

Let us take $\alpha(x) = x$. Then

$$x\left(\delta \cdot pv\,\frac{1}{x}\right) = (x \cdot \delta) \cdot pv\,\frac{1}{x} = 0 \cdot pv\,\frac{1}{x} = 0$$

$$x\left(\delta \cdot pv\,\frac{1}{x}\right) = x \cdot \left(pv\,\frac{1}{x} \cdot \delta\right) = \left(x \cdot pv\,\frac{1}{x}\right) \cdot \delta = 1 \cdot \delta = \delta$$

which is impossible.

CHAPTER 3

Differentiation of distributions

PROGRAMME

Differentiation in the space of distributions

Distributions with support at the origin

Fundamental solutions of differential operators

Primitive of a distribution

Local structure of distributions

BASICS — CHAPTER 3

a) Derivative in $\mathscr{D}'(\Omega)$

Let T be in $\mathscr{D}'(\Omega)$ then the distribution $\dfrac{\partial T}{\partial x_j}, j = 1, 2, \ldots, n$ is defined by

$$\left\langle \frac{\partial T}{\partial x_j}, \varphi \right\rangle = -\left\langle T, \frac{\partial \varphi}{\partial x_j} \right\rangle$$

for all φ in $\mathscr{D}(\Omega)$.

Any distribution has derivatives of all orders.

b) Distributions with support at the origin

Let T be in $\mathscr{D}'(\mathbb{R}^n)$. Then these two claims are equivalent.

i) $\operatorname{supp} T = \{0\}$.

ii) $T = \sum\limits_{|\alpha| \leq m} c_\alpha \delta_0^{(\alpha)}$, $c_\alpha \in \mathbb{C}$, $m \in \mathbb{N}$.

c) Fundamental solution of a differential operator

Let $P = \sum\limits_{|\alpha| \leq m} a_\alpha(x)\partial^\alpha$ be a differential operator with C^∞ coefficients in Ω. A distribution $E \in \mathscr{D}'(\Omega)$ is called a fundamental solution of P if we have in $\mathscr{D}'(\Omega)$

(1) $PE = \delta$

A fundamental solution is in general not unique since $E + u$, where $u \in \mathscr{D}'(\Omega)$ satisfies $Pu = 0$, is another one.

d) Primitive of a distribution

Let S be in $\mathscr{D}'(\Omega)$. Then there exists an infinite number of distributions $T \in \mathscr{D}'(\Omega)$ such that $\dfrac{dT}{dx} = S$.

Two of them differ by a constant.

Distribution T is called a primitive of S.

Moreover if S is a C^∞ function in an open set ω, then T has the same property.

e) Green's Formula

Given $\varepsilon > 0$ we set $\Omega_\varepsilon = \{x \in \mathbb{R}^n : |x| > \varepsilon\}$.

For every f in $C^\infty(\mathbb{R}^n\backslash\{0\})$ and every φ in $\mathscr{D}(\mathbb{R}^n)$ we have:

$$(2)\quad \int_{\Omega_\varepsilon} f(x)\Delta\varphi(x)\,\mathrm{d}x - \int_{\Omega_\varepsilon} \Delta f(x)\varphi(x)\,\mathrm{d}x = \int_{|x|=\varepsilon}\left(\varphi(x)\frac{\partial f}{\partial r}(x) - f(x)\frac{\partial\varphi}{\partial r}(x)\right)\mathrm{d}\sigma_\varepsilon$$

Here $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$, $\frac{\partial}{\partial r}$ is the radial derivative and $d\sigma_\varepsilon$ is the measure on the sphere $|x| = \varepsilon$. Formula (2) is called Green's Formula.

f) Transpose of a differential operator

Let $P = \sum_{|\alpha|\leq m} a_\alpha(x)\partial^\alpha$, $a_\alpha \in C^\infty(\Omega)$, be a differential operator. The operator

$$T \mapsto {}^tPT = \sum_{|\alpha|\leq m} (-1)^{|\alpha|}\partial^\alpha(a_\alpha T)$$

is called the transpose of P. It satisfies

$$\langle PT, \varphi\rangle = \langle T, {}^tP\varphi\rangle$$

for all φ in $\mathscr{D}(\Omega)$.

g) Local structure of distributions

Any distribution with compact support $T \in \mathscr{E}'(\Omega)$ can be written (in a non unique way) as

$$T = \sum_{|p|\leq m} \partial^p f_p$$

where each f_p is a continuous function with compact support contained in an arbitrary small neighborhood of the support of T.

There is an analogue for general distribution.

Every distribution $T \in \mathscr{D}'(\Omega)$ is a locally finite sum of derivatives of continuous functions (see exercise 64).

STATEMENTS OF THE EXERCISES* CHAPTER 3

Exercise 21

Let p and q be in $\mathbb{N}$. Compute

$$T = x^p\delta^{(q)}$$

where $\delta^{(i)}$ is the derivative of order i of the Dirac measure on $\mathbb{R}$.

*Solutions pp. 61 to 86

Exercise 22
Let $\alpha \in \mathbb{C}$ and $j \in \mathbb{N}$. Compute

$$T = e^{\alpha x}\left(\frac{d}{dx}\right)^j \delta$$

where δ is the Dirac distribution on $\mathbb{R}$.

Exercise 23
We consider the linear map T from $\mathscr{D}(\mathbb{R}^2)$ to $\mathbb{C}$

$$\mathscr{D}(\mathbb{R}^2) \ni \varphi \mapsto \langle T, \varphi\rangle = \int_{\mathbb{R}} \varphi(x, -x)\,dx$$

a) Show that $T \in \mathscr{D}'^{(0)}(\mathbb{R}^2)$.
b) What is the support of T? Prove that T is not a continuous function on $\mathbb{R}^2$.
c) Compute, in the distribution's sense, $\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial t}\right)T$.

Exercise 24
Let $\varphi \in \mathscr{D}(\Omega)$ and $T \in \mathscr{D}'(\Omega)$. Does one of the following statements imply the other?
a) $\langle T, \varphi\rangle = 0$.
b) $\varphi T = 0$ in $\mathscr{D}'(\Omega)$.

Exercise 25
We consider the following differential operator on $\mathbb{R}$

$$P = \frac{d^2}{dx^2} + a\frac{d}{dx} + b, \qquad a, b \in \mathbb{C}$$

Let f and g be two C^2 functions which satisfy
i) $(Pf)(x) = (Pg)(x) = 0, \qquad x \in \mathbb{R}$
ii) $f(0) = g(0)$
iii) $f'(0) - g'(0) = 1$
We set

$$h(x) = \begin{cases} f(x) & x \le 0 \\ g(x) & x > 0 \end{cases}$$

and we consider the distribution T defined for $\varphi \in \mathscr{D}(\mathbb{R})$ by

$$\langle T, \varphi\rangle = -\int_{\mathbb{R}} h(x)\varphi(x)\,dx$$

Compute PT in the distribution's sense

Exercise 26
Compute the derivative in the distribution's sense of the locally integrable function $\operatorname{Log}|x|$ on $\mathbb{R}$.

Exercise 27 (see exercises 11 and 15)
Compute the derivatives, in $\mathscr{D}'(\mathbb{R})$, of the following distributions:

a) $T = pv\frac{1}{x}$; b) $T = x_+^\lambda \quad -1 < \lambda < 0$

Exercise 28
We consider, in the plane, the distribution defined by the locally integrable function.

$$E(x, t) = \begin{cases} \frac{1}{2} & \text{if } t - |x| > 0 \\ 0 & \text{if } t - |x| < 0 \end{cases}$$

We set $\Box = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}$ (wave operator). Compute in the distribution's sense $\Box E$.

Exercise 29

a) Prove that the function $f(x_1 + ix_2) = \dfrac{1}{x_1 + ix_2}$ defines a distribution on $\mathbb{R}^2$.

b) We set $\bar{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right)$ (Cauchy–Riemann operator)

Prove that $\bar{\partial} f = \pi\delta$ (Hint: use polar coordinates)

Exercise 30
We consider the function in $\mathbb{R}^2$

$$E(x, t) = \frac{H(t)}{\sqrt{4\pi t}}\exp\left[-\frac{x^2}{4t}\right]$$

where $H(t)$ is the Heaviside function: $H(t) = 1$ if $t > 0$, $H(t) = 0$ if $t \leqslant 0$.

a) Prove that E is a distribution on $\mathbb{R}^2$.

b) We set $P = \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}$ (heat operator). Prove that in $\mathscr{D}'(\mathbb{R}^2)$ we have

$$PE = \delta$$

Exercise 31

We set with $r = \left(\sum_{j=1}^{n} x_j^2\right)^{1/2} \neq 0$

$$E_n = \begin{cases} \log r & \text{if } n = 2 \\ r^{2-n} & \text{if } n \geq 3 \end{cases}$$

a) Prove that E_n belongs to $\mathscr{D}'(\mathbb{R}^n)$.

b) Let $\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$. Compute ΔE_n in the sense of distributions (Hint: Use Green's formula and polar coordinates)

Exercise 32

We work in $\mathbb{R}^3$ and we set $r = |x|$.

a) Compute Δf when f is a function of r.

b) Let $f = f(r)$ a function which satisfies, in $\mathbb{R}^3\backslash 0$, the equation $(\Delta + a^2)f = 0$ where $a \in \mathbb{R}\backslash\{0\}$. Write down the differential equation satisfied by $g(r) = rf(r)$. Deduce the expression of the solutions in $\mathbb{R}^3\backslash\{0\}$ of $(\Delta + a^2)f = 0$.

c) Let $f = f(r)$ be such a solution. Prove that if we set $l = \lim_{r \to 0} [rf(r)]$ we have in $\mathscr{D}'(\mathbb{R}^3)$

$$(\Delta + a^2)f = Cl\delta$$

where C is a constant. Compute C. (Hint: Use Green's formula and the method of exercise 31).

Exercise 33

We consider, in the interval $I =]a, b[$, two C^∞ functions f and g. Our purpose is to prove that if $T \in \mathscr{D}'(I)$ satisfies

$$(1) \quad \frac{\mathrm{d}T}{\mathrm{d}x} + fT = g$$

then T is a C^∞ function which satisfies (1) in the usual sense.

a) Find a solution u_0, of equation (1), which is C^∞ in I.

b) Write any solution of (1) as $T = u_0 + S e^{-A}$ where A is a primitive of f and S an unknown distribution and conclude.

Exercise 34* (This exercise follows exercise 16)

a) Prove that the derivatives $\partial^\alpha \delta$, of the Dirac distribution in $\mathbb{R}^n$, are homogeneous. Deduce, when $n = 1$, that the distributions $\delta, \delta', \ldots, \delta^{(k)}$ are linearly independants (Use question a) of exercise 16.)

b) Prove that a distribution $T \in \mathscr{D}'(\mathbb{R}^n)$ is homogeneous of degree $p \in \mathbb{R}$ if and only if:

$$(1) \quad \sum_{i=1}^{n} x_i \frac{\partial T}{\partial x_i} = pT$$

(Hint: Use question b) of exercise 16)

c) Using (1), question a) above, question c) of exercise 16 and the fact that the distributions solutions, outside the origin, of the equation

$$x\frac{\mathrm{d}T}{\mathrm{d}x} - pT = 0$$

are the usual C^1 solutions, prove that every homogeneous distribution on $\mathbb{R}$ is one of the following:

$$T = c_1 x_+^p + c_2 x_-^p, \qquad p \in \mathbb{R}, \qquad p \neq -1, -2, \ldots \qquad c_1, c_2 \in \mathbb{C}$$

or

$$T = c_1 fp \frac{1}{x^m} + c_2 \delta^{(m-1)}, \qquad m \in \mathbb{N}, \qquad c_1, c_2 \in \mathbb{C}$$

Exercise 35 (see exercise 13)

Let P be a differential operator with constant coefficients in $\mathbb{R}^n$ which has a fundamental solution E and such that for every open set ω of $\mathbb{R}$:

$$(H) \quad (u \in \mathscr{D}'(\omega),\ Pu = 0) \Rightarrow (u \in C^\infty(\omega))$$

Let $u \in \mathscr{D}'(\mathbb{R}^n \backslash \{0\})$ be a solution of $Pu = 0$ in $\mathbb{R}^n \backslash \{0\}$ such that

$$(1) \quad |u(x)| \leq \frac{C}{|x|^m}, \qquad 0 < |x| \leq 1, \qquad m \in \mathbb{N}$$

a) Using exercise 13, prove that we can find $T \in \mathscr{D}'(\mathbb{R}^n)$ such that $T = u$ in $\mathbb{R}^n \backslash \{0\}$.

b) Prove that we have

$$T = g + \sum_{|p| \leq N} a_p D^p E \qquad a_p \in \mathbb{C}$$

where $g \in C^\infty(\mathbb{R}^n)$ and $Pg = 0$.

Exercise 36* (see exercise 29)

1°) Let $S \in \mathscr{E}'(\mathbb{R}^n)$, $T \in \mathscr{D}'(\mathbb{R}^n)$. We set $U = \text{sing supp } S$, $V = \text{sing supp } T$ and we suppose that $U \cap V = \phi$.

Prove that we can define a bilinear bracket $\langle\langle S, T \rangle\rangle$ such that

a) $\langle\langle S, T \rangle\rangle = 0$ if $\operatorname{supp} S \cap \operatorname{supp} T = \phi$.

b) $\langle\langle S, T \rangle\rangle = \langle S, T \rangle$ if $T \in C^\infty$.

c) $\left\langle\left\langle \dfrac{\partial S}{\partial x_i}, T \right\rangle\right\rangle = -\left\langle\left\langle S, \dfrac{\partial T}{\partial x_i} \right\rangle\right\rangle \qquad i = 1, \ldots, n.$

2°) Let $D = \{z \in \mathbb{C}: |z| < 1\}$, $\bar{D}$ be its closure and $\partial D = \{z \in \mathbb{C}: |z| = 1\}$. Let $u \in C^\infty(\bar{D})$ be a holomorphic function in D. We set

$$f(x) = \begin{cases} u(x) & x \in D \\ 0 & x \notin D \end{cases} \quad \text{and} \quad \bar{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

Prove that $\bar{\partial} f$ is the measure $-\frac{1}{2} e^{i\theta} u(e^{i\theta})\,d\theta$ on ∂D.

3°) Using 1°), 2°) and the formula $\bar{\partial}\left(\dfrac{1}{\pi z}\right) = \delta$ (see exercise 29), deduce the Cauchy formula

$$u(0) = \frac{1}{2i\pi} \int_{\partial D} \frac{u(z)}{z}\,dz$$

Exercise 37

a) Let Ω be an open set in $\mathbb{R}$. What is the general solution of the equation in $\mathscr{D}'(\Omega)$

$$\left(\frac{d}{dx}\right)^p T = 0, \qquad p \in \mathbb{N}^*$$

b) Prove that the Dirac distribution on $\mathbb{R}$ cannot be equal to the derivative of some order of only one continuous function with compact support in $\mathbb{R}$.

Exercise 38

Find all the solutions of the differential equation in $\mathscr{D}'(\mathbb{R})$

$$(1) \quad x^k \frac{d^m T}{dx^m} = 0 \qquad k \in \mathbb{N}^*, m \in \mathbb{N}$$

Exercise 39

a) Show that the distribution with compact support, $\sum_{|\alpha| \leq m} a_\alpha \partial^\alpha \delta$, $a_\alpha \in \mathbb{C}$, vanishes in $\mathscr{E}'(\mathbb{R}^n)$ if and only if $a_\alpha = 0$ for all $\alpha \in \mathbb{N}^n$.

b) Find all the distributions in $\mathbb{R}^n$ with support at the origin which are invariant by the maps f_i: $(x_1, \ldots, x_i, \ldots, x_n) \mapsto (x_1, \ldots, -x_i, \ldots, x_n)$, $i = 1, 2, \ldots, n$.

Exercise 40

We consider the function F: $\mathbb{R}^2 \mapsto \mathbb{R}$ given by

$$F(y_1, y_2) = y_1 H(y_1) H(y_2) \exp y_2$$

and the differential operator

$$Q = Q\left(\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}\right) = \frac{\partial^2}{\partial y_1^2}\left(\frac{\partial}{\partial y_2} - 1\right)$$

a) Compute QF in $\mathscr{D}'(\mathbb{R}^2)$.

b) We consider the map A: $\mathbb{R}^2 \to \mathbb{R}^2$

$$A(x_1, x_2) = (y_1, y_2) \quad \text{with } y_1 = x_1 + x_2,\ y_2 = x_1 - x_2$$

Let P be the differential operator

$$P = \frac{1}{8}\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right)^2\left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} - 2\right)$$

Show that for every $u \in \mathscr{D}'(\mathbb{R}^2)$ we have:

$$P[u \circ A] = (Qu) \circ A$$

c) Compute $\delta \circ A$ and give a fundamental solution of P.

Exercise 41*

Let Ω be an open set of $\mathbb{R}^n$ and P be a differential operator with C^∞ coefficients. We shall denote by ${}^tP$ its transpose. Let us suppose that for every continuous semi norm p on $\mathscr{D}(\Omega)$ we can find a continuous semi norm q on $\mathscr{D}(\Omega)$ such that for every $\varphi \in \mathscr{D}(\Omega)$:

$$(1) \quad p(\varphi) \leqq q({}^tP\varphi)$$

Let us denote by E the space $\{\psi = {}^tP\varphi \text{ where } \varphi \in \mathscr{D}(\Omega)\}$.

a) Let $T \in \mathscr{D}'(\Omega)$. Show that the map

$$E \ni \psi = {}^tP\varphi \overset{\Phi}{\mapsto} \langle T, \varphi\rangle$$

is continuous from E, equipped with the $\mathscr{D}(\Omega)$ topology, in $\mathbb{C}$.

b) Using the Hahn–Banach theorem, deduce that P is surjective from $\mathscr{D}'(\Omega)$ to $\mathscr{D}'(\Omega)$.

SOLUTIONS OF THE EXERCISES — CHAPTER 3

Solution 21

First of all $x^p \in C^\infty$ so $x^p\delta^{(q)}$ has a meaning.
Let φ be in $\mathscr{D}(\mathbb{R})$. Then by definition

$$(1)\quad \langle x^p\delta^{(q)}, \varphi\rangle = (-1)^q\langle \delta, (x^p\varphi)^{(q)}\rangle = (-1)^q\left[\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^q(x^p\varphi)\right](0)$$

By the Leibniz formula

$$(2)\quad \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^q(x^p\varphi) = \sum_{i=0}^{q} C_i^q\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^i(x^p)\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{q-i}\varphi = \sum_{i=0}^{q} F_{i,q}(x)$$

a) <u>If $p > q$</u>

In that case the term (2) vanishes at $x = 0$. Indeed $\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^i(x^p) = C_{i,p}x^{p-i}$ and $p - i$ is strictly positive

b) <u>If $p \leq q$</u>. Then

$$(2) = \sum_{i=0}^{p-1} F_{i,q}(x) + \sum_{i=p}^{q} F_{i,q}(x)$$

The first of the two sums vanishes at the origin for the same reason as in a). The second sum can be written:

$$\sum_{i=p}^{q} C_i^q\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^i(x^p)\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{p-i}\varphi = C_p^q\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^p(x^p)\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{q-p}\varphi$$

since $\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^i(x^p) = 0$ if $i \geq p + 1$. By (1) we get

$$\langle x^p\delta^{(q)}, \varphi\rangle = (-1)^q C_p^q p!\varphi^{(q-p)}(0) = \frac{(-1)^p q!}{(q-p)!}\langle \delta^{(q-p)}, \varphi\rangle$$

Therefore we get

$$x^p\delta^{(q)} = \begin{cases} 0 & \text{if } p > q \\ \dfrac{(-1)^p q!}{(q-p)!}\delta^{(q-p)} & \text{if } p \leq q \end{cases}$$

Solution 22

Let $\varphi \in \mathscr{D}(\mathbb{R})$ then

$$\langle T, \varphi\rangle = (-1)^j\left\langle \delta, \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^j \mathrm{e}^{\alpha x}\varphi\right\rangle = (-1)^j(\mathrm{e}^{\alpha x}\varphi)^{(j)}(0)$$

Now

$$(\mathrm{e}^{\alpha x}\varphi)^{(j)} = \sum_{k=0}^{j}\binom{j}{k}(\mathrm{e}^{\alpha x})^{(k)}\varphi^{(k-j)}$$

so

$$\langle T, \varphi\rangle = (-1)^j\sum_{k=0}^{j}\binom{j}{k}\alpha^k\varphi^{(k-j)}(0) = (-1)^j\sum_{k=0}^{j}\binom{k}{j}\alpha^k(-1)^{k-j}\langle\delta^{(k-j)}, \varphi\rangle$$

Therefore

$$T = \sum_{k=0}^{j}\binom{j}{k}(-\alpha)^k\delta^{(k-j)}$$

Solution 23

a) Let K be a compact in $\mathbb{R}^2$ and $\varphi \in \mathscr{D}_K(\mathbb{R}^2)$. We have

$$|\langle T, \varphi\rangle| \leqq \left(\int_{K_1}\mathrm{d}x\right)\sup_{K_1}|\varphi(x, -x)| \leqq C_K \sup_{(x,t)\in K}|\varphi(x, t)|$$

where $K_1 = \{(x: (x, -x) \in K\}$; it is compact in $\mathbb{R}^2$ since K_1 is the projection of the intersection of K with the line $t = -x$ in $\mathbb{R}^2$.

b) Let us show that the support of T is the line $t = -x$ in $\mathbb{R}^2$. First

$$\text{(1)} \quad \operatorname{supp} T \subset \{(x, t) \in \mathbb{R}^2; t = -x\} = D$$

Indeed let $\varphi \in \mathscr{D}(\mathbb{R}^2)$ the support of which is contained in $\mathbb{R}^2\backslash D$ then $\varphi(x, -x) \equiv 0$ for all $x \in \mathbb{R}$ so $\langle T, \varphi\rangle = 0$, which means that T vanishes in $\mathbb{R}^2\backslash D$ and proves (1). Conversely let $(x_0, t_0) \in D$, $V = B((x_0, t_0), \varepsilon)$, $V_1 = B\left((x_0, t_0), \dfrac{\varepsilon}{2}\right)$. Let $\varphi \in \mathscr{D}(V)$, $\varphi \geqq 0$, $\varphi = 1$ on V_1; then

$$\langle T, \varphi\rangle = \int_V \varphi(x, -x)\,\mathrm{d}x$$

where $\tilde{V} = \{x: (x, -x) \in V\}$ and $\langle T, \varphi\rangle \neq 0$ since

$$\langle T, \varphi\rangle \geqq \int_{\tilde{V}_1} \varphi(x, -x)\,\mathrm{d}x = \text{measure}\,(\tilde{V}_1) > 0 \qquad \text{Q.E.D.}$$

If T was continuous, since it does not vanish identically, the support of T would contain an open set. (If $T(x_0) \neq 0$ then $T(x) \neq 0$ near x_0); but it is not the case since line D has an empty interior. So T is not continuous on $\mathbf{R}^2$.

c) Let φ be in $\mathscr{D}(\mathbb{R}^2)$

$$\left\langle \frac{\partial T}{\partial x} - \frac{\partial T}{\partial t}, \varphi \right\rangle = -\int_{\mathbf{R}} \left(\frac{\partial \varphi}{\partial x} - \frac{\partial \varphi}{\partial t}\right)(x, -x)\,\mathrm{d}x$$

Let us set $\psi(x) = \varphi(x, -x)$ then $\dfrac{\mathrm{d}\psi}{\mathrm{d}x}(x) = \left(\dfrac{\partial\varphi}{\partial x} - \dfrac{\partial\varphi}{\partial t}\right)(x, -x)$ and it follows that

$$\left\langle \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial t}\right) T, \varphi \right\rangle = -\int_{\mathbf{R}} \frac{\mathrm{d}\psi}{\mathrm{d}x}(x)\,\mathrm{d}x = -[\psi(x)]_{-\infty}^{\infty} = -[\varphi(x, -x)]_{-\infty}^{\infty} = 0$$

so

$$\frac{\partial T}{\partial x} - \frac{\partial T}{\partial t} = 0$$

Solution 24

Let us prove that a) does not imply b) in general. Indeed let us take $\Omega = \mathbb{R}$, $T = \delta'$, $\varphi \in \mathscr{D}(\mathbb{R})$, $\varphi = 1$ in a neighborhood of the origin. Then

$$\langle \delta', \varphi\rangle = -\langle \delta, \varphi'\rangle = -\varphi'(0) = 0$$

On the other hand for $\psi \in \mathscr{D}(\mathbb{R})$

$$\langle \varphi\delta', \psi\rangle = \langle \delta', \varphi\psi\rangle = -\langle \delta, (\varphi\psi)'\rangle = -(\varphi'(0)\cdot\psi(0) + \varphi(0)\psi'(0)) = -\psi'(0)$$

We just have to take ψ such that $\psi'(0) \neq 0$ to get

$$\langle \varphi\delta', \psi\rangle \neq 0$$

which proves that $\varphi\delta'$ does not vanish identically in $\mathscr{D}'(\mathbb{R})$.

Let us prove that b) implies a).

Let $\psi \in \mathscr{D}(\Omega)$, $\psi = 1$ on the support of φ; then $\varphi\psi = \varphi$. From b) we get

$$0 = \langle \varphi T, \psi\rangle = \langle T, \varphi\psi\rangle = \langle T, \varphi\rangle$$

which proves a).

Solution 25

We have by definition

$$\langle PT, \varphi\rangle = \left\langle \frac{d^2T}{dx^2}, \varphi \right\rangle + a\left\langle \frac{dT}{dx}, \varphi \right\rangle + b\langle T, \varphi\rangle$$

$$\langle PT, \varphi\rangle = \left\langle T, \frac{d^2\varphi}{dx^2} \right\rangle - a\left\langle T, \frac{d\varphi}{dx} \right\rangle + b\langle T, \varphi\rangle$$

$$(1) \quad \langle PT, \varphi\rangle = -\int_{\mathbf{R}} h(x)\cdot\varphi''(x)\,dx + a\int_{\mathbf{R}} h(x)\cdot\varphi'(x)\,dx$$
$$-b\int_{\mathbf{R}} h(x)\cdot\varphi(x)\,dx$$

Let us compute each term

$$\int_{\mathbf{R}} h(x)\cdot\varphi''(x)\,dx = \int_{-\infty}^{0} f(x)\cdot\varphi''(x)\,dx + \int_{0}^{\infty} g(x)\varphi''(x)\,dx$$

By integrating twice by parts, in each integral, we get:

$$\int_{\mathbf{R}} h(x)\cdot\varphi''(x)\,dx = \int_{-\infty}^{0} f''(x)\cdot\varphi(x)\,dx + \int_{0}^{\infty} g''(x)\cdot\varphi(x)\,dx$$
$$+ f(0)\varphi'(0) - f'(0)\cdot\varphi(0) - g(0)\varphi'(0) + g'(0)\varphi(0)$$

since $\varphi(\mp\infty) = 0$.

Using the hypothesis ii) and iii) we get:

$$(2) \quad \int_{\mathbf{R}} h(x)\cdot\varphi''(x)\,dx = \int_{-\infty}^{0} f''(x)\cdot\varphi(x)\,dx + \int_{0}^{\infty} g''(x)\cdot\varphi(x)\,dx - \varphi(0)$$

By integrating by parts and using ii) we find:

$$(3) \quad \int_{\mathbf{R}} h(x)\cdot\varphi'(x)\,dx = -\int_{-\infty}^{0} f'(x)\cdot\varphi(x)\,dx - \int_{0}^{\infty} g'(x)\cdot\varphi(x)\,dx$$

By (1), (2) and (3) we get

$$\langle PT, \varphi\rangle = \varphi(0) - \int_{0}^{\infty} (f''(x) + af'(x) + bf(x))\cdot\varphi(x)\,dx$$
$$-\int_{-\infty}^{0} (g''(x) + ag'(x) + bg(x))\cdot\varphi(x)\,dx$$

which implies by i) that for every $\varphi \in \mathscr{D}(\mathbb{R})$:

$$\langle PT, \varphi\rangle = \varphi(0) = \langle \delta, \varphi\rangle$$

so $PT = \delta$ in $\mathscr{D}'(\mathbb{R})$.

Solution 26

The function $\operatorname{Log}|x|$ for $x \neq 0$ is integrable in a neighborhood of the origin since for every $\varepsilon < 1$, $|x|^{\varepsilon}|\operatorname{Log}|x||$ tends to zero when $|x|$ goes to zero, therefore $|\operatorname{Log}|x|| \leq \frac{1}{|x|^{\varepsilon}}$ for non zero x in a neighborhood of the origin.

Let $\varphi \in \mathscr{D}(\mathbb{R})$ then by definition:

$$(1)\quad \left\langle \frac{\mathrm{d}}{\mathrm{d}x}\operatorname{Log}|x|, \varphi\right\rangle = -\left\langle \operatorname{Log}|x|, \frac{\mathrm{d}\varphi}{\mathrm{d}x}\right\rangle = -\int_{\mathbb{R}} \operatorname{Log}|x|\frac{\mathrm{d}\varphi}{\delta x}(x)\,\mathrm{d}x$$

We would like to integrate by parts but the derivative of $\operatorname{Log}|x|$ is the function $\frac{1}{x}$ which is not integrable near the origin.

So we use the following trick. By Lebesgue's theorem

$$(2)\quad \int_{\mathbb{R}} \operatorname{Log}|x|\cdot\varphi'(x)\,\mathrm{d}x = \lim_{\varepsilon\to 0}\int_{|x|\geq\varepsilon} \operatorname{Log}|x|\cdot\varphi'(x)\,\mathrm{d}x = \lim_{\varepsilon\to 0} I_{\varepsilon}$$

since $1_{\{|x|\geq\varepsilon\}}|\operatorname{Log}|x|\varphi(x)| \leq |\operatorname{Log}|x|\cdot\varphi(x)| \in L^1$. On the other hand

$$I_{\varepsilon} = \int_{-\infty}^{-\varepsilon} \operatorname{Log}|x|\cdot\varphi'(x)\,\mathrm{d}x + \int_{\varepsilon}^{+\infty} \operatorname{Log}|x|\cdot\varphi'(x)\,\mathrm{d}x$$

$$I_{\varepsilon} = [\operatorname{Log}|x|\cdot\varphi(x)]_{-\infty}^{-\varepsilon} + [\operatorname{Log}|x|\cdot\varphi(x)]_{\varepsilon}^{+\infty} - \int_{|x|\geq\varepsilon}\frac{\varphi(x)}{x}\,\mathrm{d}x$$

$$I_{\varepsilon} = \operatorname{Log}\varepsilon(\varphi(-\varepsilon) - \varphi(\varepsilon)) - \int_{|x|\geq\varepsilon}\frac{\varphi(x)}{x}\,\mathrm{d}x$$

But $|\varphi(\varepsilon) - \varphi(-\varepsilon)| \leq 2\cdot\varepsilon\cdot\sup_{\mathbb{R}}|\varphi'(x)|$, therefore

$$(3)\quad \lim_{\varepsilon\to 0} I_{\varepsilon} = -\lim_{\varepsilon\to 0}\int_{|x|\geq\varepsilon}\frac{\varphi(x)}{x}\,\mathrm{d}x$$

We deduce from (1), (2), (3), and from exercise 11 that

$$\frac{\mathrm{d}}{\mathrm{d}x}\operatorname{Log}|x| = vp\frac{1}{x}$$

Solution 27

a) Let $\varphi \in \mathscr{D}(\mathbb{R})$, then by definition

$$\left\langle \frac{\mathrm{d}}{\mathrm{d}x} pv\frac{1}{x}, \varphi \right\rangle = -\left\langle pv\frac{1}{x}, \varphi' \right\rangle = -\lim_{\varepsilon\to 0} \int_{|x|\geq\varepsilon} \frac{\varphi'(x)}{x}\mathrm{d}x = -\lim_{\varepsilon\to 0} I_\varepsilon$$

Let us integrate by parts in I_ε. We get

$$I_\varepsilon = \int_{|x|\geq\varepsilon} \frac{\varphi'(x)}{x}\mathrm{d}x = \int_{|x|\geq\varepsilon} \frac{\varphi(x)}{x^2}\mathrm{d}x - \frac{\varphi(-\varepsilon)}{\varepsilon} - \frac{\varphi(\varepsilon)}{\varepsilon}$$

Now $\varphi(x) = \varphi(0) + x\varphi'(0) + x^2\psi(x)$ where $\psi \in C^0(\mathbb{R})$, it follows that

$$\varphi(\varepsilon) = \varphi(0) + 2\varepsilon\varphi'(0) + \varepsilon^2\psi(\varepsilon), \qquad \varphi(-\varepsilon) = \varphi(0) - \varepsilon\varphi'(0) + \varepsilon^2\psi(-\varepsilon)$$

so

$$I_\varepsilon = \int_{|x|\geq\varepsilon} \frac{\varphi(x)}{x^2}\mathrm{d}x - 2\frac{\varphi(0)}{\varepsilon} + \varepsilon[\psi(\varepsilon) + \psi(-\varepsilon)]$$

and

$$\left\langle \frac{\mathrm{d}}{\mathrm{d}x} pv\frac{1}{x}, \varphi \right\rangle = -\lim_{\varepsilon\to 0}\left[\int_{|x|\geq\varepsilon} \frac{\varphi(x)}{x^2}\mathrm{d}x - 2\frac{\varphi(0)}{\varepsilon}\right] = -\left\langle Fp\frac{1}{x^2}, \varphi \right\rangle$$

since $\lim_{\varepsilon\to 0} \varepsilon[\psi(\varepsilon) + \psi(-\varepsilon)] = 0$. Therefore

$$\frac{\mathrm{d}}{\mathrm{d}x} pv\frac{1}{x} = -Fp\frac{1}{x^2}$$

b)

$$\left\langle \frac{\mathrm{d}}{\mathrm{d}x} x_+^\lambda, \varphi \right\rangle = -\langle x_+^\lambda, \varphi' \rangle = -\int_0^\infty x^\lambda \varphi'(x)\,\mathrm{d}x$$

Since for $\lambda \in]-1, 0[$, $x^\lambda\varphi$ is integrable it follows from Lebesgue's theorem that

$$\left\langle \frac{\mathrm{d}}{\mathrm{d}x} x_+^\lambda, \varphi \right\rangle = -\lim_{\varepsilon\to 0}\int_\varepsilon^\infty x^\lambda \varphi'(x)\,\mathrm{d}x$$

Let us make an integration by parts setting

$\varphi'(x)\,\mathrm{d}x = \mathrm{d}v$, $x^\lambda = u$. Then $v = \varphi(x) + C$, $\mathrm{d}u = \lambda x^{\lambda-1}$. If we take $C = -\varphi(0)$ we shall get

$$\left\langle \frac{\mathrm{d}}{\mathrm{d}x} x_+^\lambda, \varphi \right\rangle = \lim_{\varepsilon\to 0}\left[\lambda\int_\varepsilon^\infty x^{\lambda-1}[\varphi(x) - \varphi(0)]\,\mathrm{d}x - [x^\lambda(\varphi(x) - \varphi(0))]_\varepsilon^\infty\right]$$

On one hand, since $x^{\lambda-1}[\varphi(x) - \varphi(0)] = x^{\lambda}\psi$ (where $\psi \in C^0$) is locally integrable on $]0, +\infty[$ we get

$$\lim_{\varepsilon \to 0} \int_{\varepsilon}^{\infty} x^{\lambda-1}[\varphi(x) - \varphi(0)]\,\mathrm{d}x = \int_0^{\infty} x^{\lambda-1}[\varphi(x) - \varphi(0)]\,\mathrm{d}x$$

On the other hand

$$-[x^{\lambda}[\varphi(x) - \varphi(0)]]_{\varepsilon}^{\infty} = \varepsilon^{\lambda}[\varphi(\varepsilon) - \varphi(0)] = \varepsilon^{\lambda+1}\varphi'(\theta_{\varepsilon}) \to 0 \quad \text{since } \lambda + 1 > 0$$

It follows that

$$\left\langle \frac{\mathrm{d}}{\mathrm{d}x} x_+^{\lambda}, \varphi \right\rangle = \lambda \int_0^{\infty} x^{\lambda-1}[\varphi(x) - \varphi(0)]\,\mathrm{d}x$$

The right hand side is, by definition, what we called in exercise 15 $\lambda \langle x_+^{\lambda-1}, \varphi\rangle$ for $\lambda - 1 \in]-2, -1[$. So we get

$$\frac{\mathrm{d}}{\mathrm{d}x} x_+^{\lambda}, = \lambda x_+^{\lambda-1} \qquad \lambda \in]-1, 0[$$

Solution 28

Let $\Box = \dfrac{\partial^2}{\partial t^2} - \dfrac{\partial^2}{\partial x^2}$. We have for every $\varphi \in \mathscr{D}(\mathbb{R}^2)$

$$\begin{aligned}
\langle \Box E, \varphi \rangle &= \frac{1}{2}\int_{\mathbb{R}} \int_{|x|}^{\infty} \frac{\partial^2 \varphi}{\partial t^2}\mathrm{d}t\,\mathrm{d}x - \frac{1}{2}\int_0^{\infty}\int_{-t}^{t} \frac{\partial^2\varphi}{\partial x^2}\mathrm{d}x\,\mathrm{d}t \\
&= \frac{1}{2}\int_{\mathbb{R}} \left[\frac{\partial\varphi}{\partial t}(x,t)\right]_{t=|x|}^{\infty} \mathrm{d}x - \frac{1}{2}\int_0^{\infty} \left[\frac{\partial\varphi}{\partial x}(x, t)\right]_{x=-t}^{t} \mathrm{d}t \\
&= -\frac{1}{2}\int_{\mathbb{R}} \left(\frac{\partial\varphi}{\partial t}\right)(x, |x|)\,\mathrm{d}x - \frac{1}{2}\int_0^{\infty}\left(\frac{\partial\varphi}{\partial x}\right)(t,t)\,\mathrm{d}t + \frac{1}{2}\int_0^{\infty}\left(\frac{\partial\varphi}{\partial x}\right)(-t, t)\,\mathrm{d}t \\
&= -\frac{1}{2}\int_0^{\infty}\left(\frac{\partial\varphi}{\partial t}\right)(x, x)\,\mathrm{d}x - \frac{1}{2}\int_{-\infty}^{0}\left(\frac{\partial\varphi}{\partial t}\right)(x, -x)\,\mathrm{d}x - \frac{1}{2}\int_0^{\infty}\left(\frac{\partial\varphi}{\partial x}\right)(t, t)\,\mathrm{d}r \\
&\quad + \frac{1}{2}\int_0^{\infty}\left(\frac{\partial\varphi}{\partial x}\right)(-t, t)\,\mathrm{d}t
\end{aligned}$$

$$\begin{aligned}
\langle \Box E, \varphi \rangle &= -\frac{1}{2}\int_0^{\infty}\left(\frac{\partial\varphi}{\partial t}\right)(x, x) - \frac{1}{2}\int_0^{\infty}\left(\frac{\partial\varphi}{\partial t}\right)(-x, x)\,\mathrm{d}x \\
&\quad -\frac{1}{2}\int_0^{\infty}\left(\frac{\partial\varphi}{\partial x}\right)(t, t)\,\mathrm{d}t + \frac{1}{2}\int_0^{\infty}\left(\frac{\partial\varphi}{\partial x}\right)(-t, t)\,\mathrm{d}t
\end{aligned}$$

On the other hand for $a \in \mathbb{R}$:

$$\frac{d}{dy}[\varphi(ay, y)] = a\left(\frac{\partial\varphi}{\partial x}\right)(ay, y) + \left(\frac{\partial\varphi}{\partial t}\right)(ay, y)$$

It follows that

$$\langle \Box E, \varphi\rangle = -\frac{1}{2}\int_0^\infty \frac{d}{dy}[\varphi(y, y)]\,dy - \frac{1}{2}\int_0^\infty \frac{d}{dy}[\varphi(-y, y)]\,dy$$

$$\langle \Box E, \varphi\rangle = \frac{1}{2}\varphi(0, 0) + \frac{1}{2}\varphi(0, 0) = \varphi(0, 0) = \langle \delta, \varphi\rangle \qquad \forall\varphi \in \mathscr{D}(\mathbb{R}^2)$$

so

$$\Box E = \delta \quad \text{in } \mathscr{D}'(\mathbb{R}^2)$$

Solution 29

a) First of all $|f(x_1, x_2)| = \dfrac{1}{(x_1^2 + x_2^2)^{1/2}} = \dfrac{1}{|x|}$ where $x = (x_1, x_2)$.

We know that the function $\dfrac{1}{|x|^\alpha}$ is integrable near the origin in $\mathbb{R}^n$ *if* $\alpha < n$. Here $\alpha = 1$, $n = 2$ so $f \in L^1_{\text{loc}}(\mathbb{R}^2)$. This implies that f defines a distribution on $\mathbb{R}^2$.

b) Let $\varphi \in \mathscr{D}(\mathbb{R}^2)$. We have:

$$(1) \quad \langle \bar\partial f, \varphi\rangle = -\langle f, \bar\partial\varphi\rangle = \frac{-1}{2}\int_{\mathbb{R}^2} \frac{1}{x_1 + ix_2}\left(\frac{\partial\varphi}{\partial x_1} + i\frac{\partial\varphi}{\partial x_2}\right)dx_1\,dx_2$$

We use the polar coordinates

$$x_1 = r\cos\theta, \quad x_2 = r\sin\theta \quad \text{then } dx_1\,dx_2 = r\,dr\,d\theta$$

We have

$$\frac{\partial}{\partial x_1} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}; \qquad \frac{\partial}{\partial x_2} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}$$

$$\langle \bar\partial f, \varphi\rangle = -\frac{1}{2}\int_0^{2\pi}\int_0^\infty \frac{e^{-i\theta}}{r}\left(e^{i\theta}\frac{\partial\tilde\varphi}{\partial r} + i\frac{e^{i\theta}}{r}\frac{\partial\tilde\varphi}{\partial\theta}\right)r\,dr\,d\theta$$

where $\tilde\varphi(r, \theta) = \varphi(r\cos\theta, r\sin\theta)$. By Fubini's theorem:

$$\langle \bar\partial f, \varphi\rangle = -\frac{1}{2}\int_0^{2\pi}\left[\int_0^\infty \frac{\partial\tilde\varphi}{\partial r}dr\right]d\theta - \frac{i}{2}\int_0^\infty \frac{1}{r}\int_0^{2\pi}\frac{\partial\tilde\varphi}{\partial\theta}d\theta\,dr$$

Since $\tilde{\varphi}(0, \theta) = \varphi(0, 0)$ and since $\tilde{\varphi}(r, \theta)$ is 2π-periodic we get:

$$\langle \bar{\partial} f, \varphi \rangle = -\frac{1}{2} \times 2\pi \times (-\varphi(0, 0)) = \pi\varphi(0, 0) = \pi\langle \delta_0, \varphi \rangle$$

Therefore

$$\bar{\partial} f = \pi\delta_0$$

Solution 30

a) we have $E(x, t) \leq \dfrac{H(t)}{\sqrt{4\pi t}}$ and the function $\dfrac{H(t)}{\sqrt{4\pi t}}$ is locally integrable in $\mathbb{R}^2$; so $E \in L^1_{loc}(\mathbb{R}^2)$ and it defines a distribution on $\mathbb{R}^2$.

b) Let $\varphi \in \mathscr{D}(\mathbb{R}^2)$, we have

$$\left\langle \left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right) E, \varphi \right\rangle = -\left\langle E, \frac{\partial \varphi}{\partial t} + \frac{\partial^2 \varphi}{\partial x^2} \right\rangle = -\iint_{[0,\infty[\times\mathbb{R}} \frac{\exp\left[-\frac{x^2}{4t}\right]}{\sqrt{4\pi t}} \left(\frac{\partial \varphi}{\partial t} + \frac{\partial^2 \varphi}{\partial x^2}\right) \mathrm{d}x\, \mathrm{d}t$$

$$\int_0^\infty \int_{\mathbb{R}} \frac{\exp\left[-\frac{x^2}{4t}\right]}{\sqrt{4\pi t}} \frac{\partial \varphi}{\partial t} \mathrm{d}x\, \mathrm{d}t = \lim_{\varepsilon \to 0} \int_{\mathbb{R}} \left[\int_\varepsilon^\infty \frac{\exp\left[-\frac{x^2}{4t}\right]}{\sqrt{4\pi t}} \frac{\partial \varphi}{\partial t} \mathrm{d}t\right] \mathrm{d}x = \lim I_\varepsilon$$

by Lebesgue's theorem since

$$\left| 1_{[\varepsilon,\infty[\times\mathbb{R}} \frac{\exp[\ \ ]}{\sqrt{4\pi t}} \varphi(x, t) \right| \leq \frac{C|\varphi(x, t)|}{\sqrt{t}} \in L^1(\mathbb{R}^2)$$

We can make an integration by parts in I_ε and it follows that

$$I_\varepsilon = -\int_{\mathbb{R}} \int_\varepsilon^\infty \frac{\partial}{\partial t} \left[\frac{\exp\left(-\frac{x^2}{4t}\right)}{\sqrt{4\pi t}}\right] \varphi(x, t)\, \mathrm{d}t\, \mathrm{d}x + \int_{\mathbb{R}} \left[\frac{\exp\left(-\frac{x^2}{4t}\right)}{\sqrt{4\pi t}} \varphi(x, t)\right]_{t=\varepsilon}^{t=+\infty} \mathrm{d}x$$

Now

$$\frac{\partial}{\partial t}\left[\frac{\exp(\ \ )}{\sqrt{4\pi t}}\right] = \frac{1}{4\sqrt{\pi}} \left(\frac{x^2}{2t^{5/2}} - \frac{1}{t^{3/2}}\right) \exp(\ \ )$$

$$(*)\ I_\varepsilon = \frac{-1}{4\sqrt{\pi}} \int_{\mathbb{R}} \int_\varepsilon^\infty \left(\frac{x^2}{2t^{5/2}} - \frac{1}{t^{3/2}}\right) \exp\left(-\frac{x^2}{4t}\right) \varphi(x, t)\, \mathrm{d}t\, \mathrm{d}x - \int_{\mathbb{R}} \frac{\exp\left(-\frac{x^2}{4\varepsilon}\right)}{\sqrt{4\pi\varepsilon}} \varphi(x, \varepsilon)\, \mathrm{d}x$$

In the same way

$$\int_0^\infty \int_{\mathbf{R}} \frac{\exp(\quad)}{\sqrt{4\pi t}} \frac{\partial^2 \varphi}{\partial x^2} \mathrm{d}x\, \mathrm{d}t = \lim_{\varepsilon \to 0} \int_\varepsilon^\infty \int_{\mathbf{R}} \frac{\exp(\quad)}{\sqrt{4\pi t}} \frac{\partial^2 \varphi}{\partial x^2} \mathrm{d}x\, \mathrm{d}t = \lim_{\varepsilon \to 0} J_\varepsilon$$

Let us integrate twice by parts in J_ε. Since

$$\frac{\partial}{\partial x}\left[\frac{\exp\left(-\frac{x^2}{4t}\right)}{\sqrt{4\pi t}}\right] = -\frac{x}{4\sqrt{\pi}t^{3/2}} \exp\left(-\frac{x^2}{4t}\right)$$

$$\frac{\partial^2}{\partial x^2}\left[\frac{\exp\left(-\frac{x^2}{4t}\right)}{\sqrt{4\pi t}}\right] = \left(-\frac{1}{4\sqrt{\pi}t^{3/2}} + \frac{x^2}{8\sqrt{\pi}t^{5/2}}\right) \exp\left(-\frac{x^2}{4t}\right)$$

we get

$$J_\varepsilon = \frac{1}{4\sqrt{\pi}} \int_{\mathbf{R}} \int_\varepsilon^\infty \left(\frac{x^2}{2t^{5/2}} - \frac{1}{t^{3/2}}\right) \exp\left(-\frac{x^2}{4t}\right) \varphi(x, t)\, \mathrm{d}x\, \mathrm{d}t$$

$$+ \int_\varepsilon^\infty \left[\frac{\exp\left(-\frac{x^2}{4t}\right)}{\sqrt{4\pi t}} \frac{\partial \varphi}{\partial x}(x, t)\right]_{x=-\infty}^{x=+\infty} \mathrm{d}t + \int_\varepsilon^\infty \left[\frac{x}{4\sqrt{\pi}t^{3/2}} \exp(\quad) \varphi(x, t)\right]_{x=-\infty}^{x=+\infty} \mathrm{d}t$$

and since $\varphi(\mp\infty, t) = \frac{\partial \varphi}{\partial x}(\mp\infty, t) = 0$ we get

$$(**)\, J_\varepsilon = \frac{1}{4\sqrt{\pi}} \int_{\mathbf{R}} \int_\varepsilon^\infty \left(\frac{x^2}{2t^{5/2}} - \frac{1}{t^{3/2}}\right) \exp\left(-\frac{x^2}{4t}\right) \varphi(x, t)\, \mathrm{d}t\, \mathrm{d}x$$

It follows from (*) and (**) that

$$I_\varepsilon + J_\varepsilon = -\int_{\mathbf{R}} \frac{\exp\left(-\frac{x^2}{4\varepsilon}\right)}{\sqrt{4\pi\varepsilon}} \varphi(x, \varepsilon)\, \mathrm{d}x$$

so

$$\left\langle \left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right) E, \varphi \right\rangle = +\lim_{\varepsilon \to 0} \int_{\mathbf{R}} \frac{\exp\left(-\frac{x^2}{4\varepsilon}\right)}{\sqrt{4\pi\varepsilon}} \varphi(x, \varepsilon)\, \mathrm{d}x = \lim_{\varepsilon \to 0} K_\varepsilon$$

Let us perform the change of variables $y = \frac{x}{2\sqrt{\varepsilon}}$ in the integral. Then $\mathrm{d}x = 2\sqrt{\varepsilon}\,\mathrm{d}y$ and

$$K_\varepsilon = \frac{2\sqrt{\varepsilon}}{2\sqrt{\pi}\sqrt{\varepsilon}} \int_{\mathbf{R}} \mathrm{e}^{-y^2} \varphi(2\sqrt{\varepsilon}y, \varepsilon)\, \mathrm{d}y = \frac{1}{\sqrt{\pi}} \int_{\mathbf{R}} \mathrm{e}^{-y^2} \varphi(2\sqrt{\varepsilon}y, \varepsilon)\, \mathrm{d}y$$

Now $\varphi(2\sqrt{\varepsilon}y, \varepsilon) \to \varphi(0, 0)$ and $|e^{-y^2}\varphi(2\sqrt{\varepsilon}y, \varepsilon)| \le \sup_{\mathbb{R}} |\varphi(x)|e^{-y^2} \in L^1(\mathbb{R})$ so by Lebesgue's theorem $K_\varepsilon \to \frac{1}{\sqrt{\pi}}\left(\int_{\mathbb{R}} e^{-y^2}\,dy\right)\varphi(0, 0) = \varphi(0, 0)$. Therefore

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)E = \delta$$

Solution 31

a) It is easy to see that the function E_n is locally integrable in $\mathbb{R}^n$. Indeed if we use polar coordinates we get

$$\int_{x \le 1} |E_n(x)|\,dx = \begin{cases} -2\pi \displaystyle\int_0^1 r \operatorname{Log} r\,dr = \frac{\pi}{2} \\ 2\pi \displaystyle\int_0^1 r\,dr = \pi \end{cases}$$

b) We have $\langle \Delta E_n, \varphi \rangle = \langle E_n, \Delta\varphi \rangle \forall \varphi \in \mathscr{D}(\mathbb{R}^n)$ so

$$\langle \Delta E_n, \varphi \rangle = \int_{\mathbb{R}} E_n(x)\Delta\varphi(x)\,dx$$

Since not all the derivatives of E_n are locally integrable we cannot integrate by parts in the above integral. We shall overcome this difficulty in the following way. Since E_n is locally integrable then, by Lebesgue's theorem, we can write

$$\langle \Delta E_n, \varphi \rangle = \lim_{\varepsilon \to 0} \int_{|x| \ge \varepsilon} E_n(x)\Delta\varphi(x)\,dx = \lim_{\varepsilon \to 0} I_\varepsilon$$

Now $E_n \in C^\infty$ for $|x| = r \ge \varepsilon$ and $\varphi \in \mathscr{D}(\mathbb{R}^n)$ so we can apply formula (I) to compute I_ε. We have

$$I_\varepsilon = \int_{|x| \ge \varepsilon} \Delta E_n(x)\varphi(x)\,dx - \int_{|x| = \varepsilon} \left(E_n \frac{\partial \varphi}{\partial r} - \varphi \frac{\partial E_n}{\partial r}\right) d\sigma_\varepsilon$$

Let us compute ΔE_n in $\{x: |x| \ge \varepsilon\}$.

1) $n = 2$:

$$\frac{\partial}{\partial x} \operatorname{Log}(x^2 + y^2) = \frac{2x}{x^2 + y^2}$$

$$\frac{\partial^2}{\partial x^2} \operatorname{Log}(x^2 + y^2) = \frac{2(x^2 + y^2) - 4x^2}{(x^2 + y^2)^2} = \frac{2y^2 - 2x^2}{(x^2 + y^2)^2}$$

$$\frac{\partial^2}{\partial y^2} \operatorname{Log}(x^2 + y^2) = \frac{2x^2 - 2y^2}{(x^2 + y^2)^2}$$

So $\Delta E_2 = 0$.

2) If $n \geqq 3$:

$$\frac{\partial}{\partial x_i} r^{2-n} = \frac{2-n}{2} \cdot 2x_i \cdot r^{-n} = (2-n)x_i \cdot r^{-n}$$

$$\frac{\partial^2}{\partial x_i^2} = (2-n)r^{-n} + (2-n)x_i \cdot \frac{-n}{2} \cdot 2x_i \cdot r^{-n-2}$$

So $\Delta E_n = (2-n) \cdot nr^{-n} - n(2-n)\left(\sum_1^n x_i^2\right) r^{-n-2} = 0$ since $\sum x_i^2 = r^2$ i.e. $\Delta E_n = 0$.

Therefore

$$-I_\varepsilon = \int_{|x|=\varepsilon} \left(E_n \frac{\partial \varphi}{\partial r} - \varphi \frac{\partial E_n}{\partial r}\right) \mathrm{d}\sigma_\varepsilon$$

To compute I_ε we use polar coordinates

$$x_i = r \cdot f_i(\theta_1, \ldots, \theta_{n-1}) \qquad i = 1, \ldots, n$$

so we get

$$\mathrm{d}x = F(\theta_1, \ldots, \theta_{n-1}) r^{n-1} \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_{n-1}$$

and the measure on the sphere of radius ε is equal to

$$\mathrm{d}\sigma_\varepsilon = \varepsilon^{n-1} F(\theta_1, \ldots, \theta_{n-1}) \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_{n-1} = \varepsilon^{n-1} \mathrm{d}\sigma_1$$

where $\mathrm{d}\sigma_1 = F(\theta_1, \ldots, \theta_{n-1}) \mathrm{d}\theta_1 \ldots \mathrm{d}\theta_{n-1}$ is the measure on the unit sphere.
(We did not compute the f_i's nor F since we do not need it)
On the other hand:

$$\frac{\partial}{\partial r} = \sum_{i=1}^n \frac{\partial}{\partial x_i} \cdot \frac{\partial x_i}{\partial r} = \sum_{i=1}^n \frac{x_i}{r} \frac{\partial}{\partial x_i}$$

since $\dfrac{\partial x_i}{\partial r} = f_i(\theta_1, \ldots, \theta_{n-1}) = \dfrac{x_i}{r}$.

Let us compute now the limit of I_ε when ε goes to zero.

1) $n = 2$

$$-I_\varepsilon = \int_{|x|=\varepsilon} \left(\operatorname{Log} \varepsilon \frac{\partial \varphi}{\partial r} - \varphi \cdot \frac{1}{\varepsilon}\right) \varepsilon \, \mathrm{d}\sigma_1 = \underbrace{\int_{|x|=\varepsilon} \varepsilon \operatorname{Log} \varepsilon \frac{\partial \varphi}{\partial r} \mathrm{d}\sigma_1}_{①} - \underbrace{\int_{|x|=\varepsilon} \varphi \, \mathrm{d}\sigma_1}_{②}$$

We have $\left|\dfrac{\partial \varphi}{\partial r}\right| \leq \displaystyle\sum_{i=1}^n \left|\frac{x_i}{r}\right| \left|\frac{\partial \varphi}{\partial x_i}\right| \leq \sum_{i=1}^n \sup_{\mathbf{R}} \left|\frac{\partial \varphi}{\partial x_i}\right|$ since $\left|\dfrac{x_i}{r}\right| \leq 1$ so we get

$$① = \left|\int_{|x|=\varepsilon} \varepsilon \operatorname{Log} \varepsilon \frac{\partial \varphi}{\partial r} \mathrm{d}\sigma_1\right| \leq C|\varepsilon \operatorname{Log} \varepsilon| \cdot \int \mathrm{d}\sigma_1$$

So this term tends to zero when $\varepsilon \to 0$. For the second term we write

$$② = -\int \tilde{\varphi}(\varepsilon, \theta)\,\mathrm{d}\sigma_1 \quad \text{where } \tilde{\varphi}(r, \theta) = \varphi(r\cos\theta, r\sin\theta)$$

When $\varepsilon \to 0$, by Lebesgue's theorem $② \to -\tilde{\varphi}(0, \theta)\cdot\int \mathrm{d}\sigma_1$ and since $\tilde{\varphi}(0, \theta) = \varphi(0, 0)$ we get

$$\lim_{\varepsilon\to 0} I_\varepsilon = 2\pi\varphi(0, 0) = 2\pi\langle\delta, \varphi\rangle$$

2) $n \geq 3$

$$-I_\varepsilon = \int_{r=\varepsilon} \frac{1}{\varepsilon^{n-2}}\frac{\partial\varphi}{\partial r}\varepsilon^{n-1}\,\mathrm{d}\sigma_1 - \int_{r=\varepsilon} \tilde{\varphi}(\varepsilon, \theta_1, \ldots, \theta_{n-1})(2-n)\cdot\frac{1}{\varepsilon^{n-1}}\varepsilon^{n-1}\,\mathrm{d}\sigma_1$$

$$-I_\varepsilon = \int_{r=\varepsilon} \varepsilon\frac{\partial\varphi}{\partial r}\mathrm{d}\sigma_1 + (n-2)\int_{r=\varepsilon} \tilde{\varphi}(\varepsilon, \theta_1, \ldots, \theta_{n-1})\,\mathrm{d}\sigma_1$$

The first term tends to zero since $\left|\frac{\partial\varphi}{\partial r}\right| \leq \sum \sup_{\mathbb{R}^n}\left|\frac{\partial\varphi}{\partial x_i}\right| \leq C$.
By Lebesgue's theorem the second term tends to

$$(n-2)\varphi(0)\left\{\int \mathrm{d}\sigma_1\right\}$$

so

$$\lim_{\varepsilon\to 0} I_\varepsilon = C_n(2-n)\varphi(0) = (2-n)C_n\langle\delta, \varphi\rangle$$

where C_n is the measure of the unit sphere in $\mathbb{R}^n$.
Therefore in all cases we have $\Delta E_n = a_n\delta$ where a_n is a constant.

Solution 32

a) We have $\Delta = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}$. Let us compute Δ in the polar coordinates.
Since $f = f(r)$ it is sufficient to compute the part which contains only derivatives with respect to r.
We have

$$\begin{cases} x_1 = rf_1(\theta, \varphi) \\ x_2 = rf_2(\theta, \varphi) \\ x_3 = rf_3(\theta, \varphi) \end{cases}$$

$$\frac{\partial}{\partial x_i}f(r) = \frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial x_i} = \frac{x_i}{r}\frac{\partial}{\partial r}f(r);\ \frac{\partial^2}{\partial x_i^2}f(r) = \frac{\partial}{\partial x_i}\left(\frac{x_i}{r}\right)\frac{\partial f}{\partial r} + \frac{x_i^2}{r^2}\frac{\partial^2 f}{\partial r^2}$$

$$\frac{\partial^2}{\partial x_i^2}f(r) = \left(\frac{1}{r} - \frac{x_i^2}{r^3}\right)\frac{\partial f}{\partial r} + \frac{x_i^2}{r^2}\frac{\partial^2 f}{\partial r^2}$$

so

$$\Delta f(r) = \left(\frac{3}{r} - \frac{1}{r^3}\sum_1^3 x_i^2\right)\frac{\partial f}{\partial r} + \frac{1}{r^2}\left(\sum_1^3 x_i^2\right)\frac{\partial^2 f}{\partial r^2}$$

$$\Delta f(r) = \frac{\partial^2 f}{\partial r^2} + \frac{2}{r}\frac{\partial f}{\partial r}$$

b) Let $f = f(r)$ be such that $(\Delta + a^2)f = 0$ in $\mathbb{R}^3\backslash\{0\}$. Let us set $g = rf(r)$. We get $g'(r) = f(r) + rf'(r)$; $g''(r) = 2f'(r) + rf''(r)$.
Now by a) we have

$$f''(r) + \frac{2}{r}f'(\tau) + a^2 f(r) = 0 \quad \text{in } \mathbb{R}^3\backslash\{0\}$$

so multiplying by r we get:

$$rf''(r) + 2f'(r) + a^2 rf(r) = 0$$

therefore: $g''(r) + a^2 g(r) = 0$.
The general solution of this equation in $\mathbb{R}^3\backslash\{0\}$ is

$$g(r) = c_1 \cos ar + c_2 \sin ar$$

so the general solution of $(\Delta + a^2)f(r) = 0$ in $\mathbb{R}^3\backslash\{0\}$ is the C^∞ function

$$(1) \quad f(r) = c_1\frac{\cos ar}{r} + c_2\frac{\sin ar}{r}$$

c) With the notations used in the statement of this exercise $c_1 = l$ so

$$f(r) = l\frac{\cos ar}{r} + c_2\frac{\sin ar}{r} \in L^1_{\text{loc}}(\mathbb{R}^3)$$

Let us show now that in $\mathscr{D}'(\mathbb{R})$ we have $(\Delta + a^2)f(r) = Cl\delta_0$.

Function $c_2\dfrac{\sin ar}{r}$ in a C^∞ function of $r \in \mathbb{R}$ (while $l\dfrac{\cos ar}{r}$ is not defined at $r = 0$). So we can compute $(\Delta + a^2)\dfrac{\sin ar}{r}$ in the usual sense. Now by 2°) $\dfrac{\sin ar}{r}$ is a solution in $\mathbb{R}^3\backslash\{0\}$ of $(\Delta + a^2)f = 0$ (take $c_1 = 0$); since it is a C^∞ function we have $(\Delta + a^2)\dfrac{\sin ar}{r} = 0$ in all $\mathbb{R}$.

Let us compute in $\mathscr{D}'(\mathbb{R})$ $(\Delta + a^2)\dfrac{\cos ar}{r}$. Let $\varphi \in \mathscr{D}(\mathbb{R})$,

$$\left\langle (\Delta + a^2)\frac{\cos ar}{r}, \varphi \right\rangle = \left\langle \frac{\cos ar}{r}, (\Delta + a^2)\varphi \right\rangle = \int \frac{\cos ar}{r} \cdot (\Delta + a^2)\varphi \, \mathrm{d}x$$

As in exercise 31, since $\dfrac{\cos ar}{r} \in L^1_{\text{loc}}$, we get

$$\left\langle (\Delta + a^2)\frac{\cos ar}{r}, \varphi \right\rangle = \lim_{\varepsilon \to 0} \int_{r \geq \varepsilon} \frac{\cos ar}{r}(\Delta + a^2)\varphi(x)\,\mathrm{d}x = \lim I_\varepsilon$$

By Green's formula we have:

$$I_\varepsilon = \int_{r \geq \varepsilon} (\Delta + a^2)\left(\frac{\cos ar}{r}\right)\varphi(x)\,\mathrm{d}x - \int_{r=\varepsilon} \left[\frac{\cos ar}{r}\frac{\partial \varphi}{\partial r} - \varphi\frac{\partial}{\partial r}\left(\frac{\cos ar}{r}\right)\right]\mathrm{d}\sigma_\varepsilon$$

Now in $\mathbb{R}^3\backslash\{0\}$ $(\Delta + a^2)\dfrac{\cos ar}{r} = 0$ (see (1) with $c_2 = 0$) so

$$I_\varepsilon = \int_{r=\varepsilon} \left[\frac{\cos ar}{r}\frac{\partial \varphi}{\partial r} - \varphi\left(\frac{\partial}{\partial r}\right)\left(\frac{\cos ar}{r}\right)\right]\mathrm{d}\sigma_\varepsilon$$

Now $\mathrm{d}\sigma_\varepsilon = \varepsilon^2\,\mathrm{d}\omega$ where $\mathrm{d}\omega$ is the measure on the unit sphere; moreover

$$\frac{\partial}{\partial r}\left(\frac{\cos ar}{r}\right) = \frac{-ar\sin ar - \cos ar}{r^2}$$

Finally

$$\left|\frac{\partial \varphi}{\partial r}\right| \leq \sum \frac{|x_i|}{r}\left|\frac{\partial \varphi}{\partial x_i}\right| \leq \sup_{\mathbb{R}} \sum \left|\frac{\partial \varphi}{\partial x_i}\right| = M \quad \text{since } \frac{|x_i|}{r} \leq 1$$

$$-I_\varepsilon = \underbrace{\varepsilon\cos a\varepsilon \int_{r=\varepsilon} \frac{\partial \varphi}{\partial r}\,\mathrm{d}\omega}_{①} + \underbrace{a\varepsilon\sin a\varepsilon \int_{r=\varepsilon} \varphi\,\mathrm{d}\omega}_{②} + \underbrace{\cos a\varepsilon \int_{r=\varepsilon} \varphi\,\mathrm{d}\omega}_{③}$$

$$|①| \leq \varepsilon|\cos a\varepsilon| M \int_{|x|=1} \mathrm{d}\omega \to 0 \quad \text{where } \varepsilon \to 0$$

$$|②| \leq |a|\varepsilon|\cos a\varepsilon| \sup_{\mathbb{R}} |\varphi(x)| \cdot \int_{|x|=1} \mathrm{d}\omega \to 0 \quad \text{where } \varepsilon \to 0$$

The third term can be written as:

$$\cos a\varepsilon \int_{|x|=1} \tilde{\varphi}(\varepsilon, \theta, \varphi)\,\mathrm{d}\omega \quad \text{where } \tilde{\varphi}(r, \theta, \varphi) = \varphi(rf_1(\theta, \varphi), rf_2, rf_3)$$

By Lebesgue's theorem ③ $\to \varphi(0)\int_{|x|=1}\mathrm{d}\omega$ thus

$$\left\langle (\Delta + a^2)\frac{\cos ar}{r}, \varphi \right\rangle = \left(-\int_{|x|=1} \mathrm{d}\omega\right)\langle \delta, \varphi \rangle \qquad \forall \varphi \in \mathscr{D}(\mathbb{R})$$

therefore

$$(\Delta + a^2)f(r) = -4\pi l\delta$$

In particular the distribution $\dfrac{-\cos ar}{4\pi rl}$ is a fundamental solution of the operator $\Delta + a^2$ in $\mathbb{R}^3$.

Solution 33

a) When $g = 0$, the general solution of (1), which is C^∞, is $u_0(x) = C\mathrm{e}^{-A(x)}$ where C is a constant and $A(x)$ a primitive of f.

If $C = C(x)$ we are led to the equation $\dfrac{\mathrm{d}C}{\mathrm{d}x} = \mathrm{e}^{A(x)}g$ so the solution is

$$u_0(x) = \mathrm{e}^{-A(x)}\int_0^x \mathrm{e}^{A(t)}g(t)\,\mathrm{d}t = \int_0^x \exp\left(\int_0^t f(\sigma)\,\mathrm{d}\sigma\right)g(t)\,\mathrm{d}t$$

which is a C^∞ function on I.

b) Let us set $T = u_0 + \mathrm{e}^{-A(x)}S$ then

$$g = \frac{\mathrm{d}T}{\mathrm{d}x} + fT = \frac{\mathrm{d}u_0}{\mathrm{d}x} + fu_0 + \mathrm{e}^{-A(x)}\frac{\mathrm{d}S}{\mathrm{d}x} - fS + fS = g + \mathrm{e}^{-A(x)}\frac{\mathrm{d}S}{\mathrm{d}x}$$

so $\mathrm{e}^{-A(x)}\dfrac{\mathrm{d}S}{\mathrm{d}x} = 0$ and $\dfrac{\mathrm{d}S}{\mathrm{d}x} = 0$ since $\mathrm{e}^{-A(x)} \neq 0$. So S is a constant and it follows that

$$T = u_0 + C\mathrm{e}^{-A(x)}$$

therefore $T \in C^\infty(I)$ and satisfies (1) in the usual sense.

Solution 34

a) $\langle \partial^\alpha\delta, \varphi_\lambda\rangle = (-1)^{|\alpha|}\langle \delta, \partial^\alpha\varphi_\lambda\rangle = (-1)^{|\alpha|}\lambda^{|\alpha|}\varphi^{(\alpha)}(0) = \lambda^{|\alpha|}\langle \partial^\alpha\delta, \varphi\rangle$ so $\partial^\alpha\delta$ is homogeneous of degree p such that $-(n + p) = |\alpha|$ so $p = -n - |\alpha|$. When $n = 1$, $\delta^{(k)}$ is homogeneous of degree $-1 - k$. The distributions $\delta, \delta', \ldots, \delta^{(k)}$ are homogeneous of different degrees. They are therefore linearly independent, by question d) in exercise 16.

b) Let us suppose that T is homogeneous of degree p. By question b) in exercise 16

$$\lim_{\lambda\to\lambda_0}\left\langle T, \frac{\varphi_\lambda - \varphi_{\lambda_0}}{\lambda - \lambda_0}\right\rangle = \sum_{i=1}^n \left\langle T, x_i\left(\frac{\partial\varphi}{\partial x_i}\right)_{\lambda_0}\right\rangle$$

Now

$$\left\langle T, \frac{\varphi_\lambda - \varphi_{\lambda_0}}{\lambda - \lambda_0}\right\rangle = \frac{1}{\lambda - \lambda_0}[\langle T, \varphi_\lambda\rangle - \langle T, \varphi_{\lambda_0}\rangle] = \frac{\lambda^{-(n+p)} - \lambda_0^{-(n+p)}}{\lambda - \lambda_0}\langle T, \varphi\rangle$$

so $\left\langle T, \dfrac{\varphi_\lambda - \varphi_{\lambda_0}}{\lambda - \lambda_0}\right\rangle \to -(n + p)\lambda_0^{-(n+p+1)}\langle T, \varphi\rangle$. Therefore

$$\sum_{i=1}^{n} \left\langle T, x_i\left(\frac{\partial \varphi}{\partial x_i}\right)_{\lambda_0} \right\rangle = -(n+p)\lambda_0^{-(n+p+1)}\langle T, \varphi\rangle$$

Now it is obvious that

$$x_i\left(\frac{\partial \varphi}{\partial x_i}\right)_{\lambda_0} = \lambda_0^{-1}\left(x_i\frac{\partial \varphi}{\partial x_i}\right)_{\lambda_0}$$

so

$$\sum_{i=1}^{n} \lambda_0^{-1}\left\langle T, \left(x_i\frac{\partial \varphi}{\partial x_i}\right)_{\lambda_0} \right\rangle = -(n+p)\lambda^{-(n+p+1)}\langle T, \varphi\rangle$$

Since T is homogeneous we have:

$$(2) \quad \sum_{i=1}^{n} \lambda_0^{-(n+p+1)}\left\langle T, x_i\frac{\partial \varphi}{\partial x_i}\right\rangle = -(n+p)\lambda^{-(n+p+1)}\langle T, \varphi\rangle$$

On the other hand

$$(3) \quad x_i\frac{\partial \varphi}{\partial x_i} = \frac{\partial}{\partial x_i}(x_i\varphi) - \varphi$$

Dividing by $\lambda_0^{-(n+p+1)}$ in (2) and using (3) we get

$$\sum_{i=1}^{n} \left\langle T, \frac{\partial}{\partial x_i}(x_i\varphi)\right\rangle - n\langle T, \varphi\rangle = -(n+p)\langle T, \varphi\rangle$$

so

$$-\sum_{i=1}^{n} \left\langle x_i\frac{\partial T}{\partial x_i}, \varphi\right\rangle = -p\langle T, \varphi\rangle \qquad \forall \varphi \in \mathscr{D}(\mathbb{R}^n)$$

which proves (1).

Conversely let us suppose that $\sum_{i=1}^{n} x_i\frac{\partial T}{\partial x_i} = pT$, $p \in \mathbb{R}$. Then

$$\langle pT, \varphi_{\lambda_0}\rangle = -\sum_{i=1}^{n} \left\langle T, \frac{\partial}{\partial x_i}x_i\varphi_{\lambda_0}\right\rangle = -n\langle T, \varphi_{\lambda_0}\rangle - \sum_{i=1}^{n} \left\langle T, x_i\frac{\partial \varphi_{\lambda_0}}{\partial x_i}\right\rangle$$

Now

$$\frac{\partial \varphi_{\lambda_0}}{\partial x_i} = \lambda_0\left(\frac{\partial \varphi}{\partial x_i}\right)_{\lambda_0}$$

so $-(n+p)\langle T, \varphi_{\lambda_0}\rangle = \lambda_0\left\langle T, \sum_{i=1}^{n} x_i\left(\frac{\partial \varphi}{\partial x_i}\right)_{\lambda_0}\right\rangle$. Now by question b) in exercise 16 we have in $\mathscr{D}(\mathbb{R}^n)$

$$\lim_{\lambda\to\lambda_0} \frac{\varphi_\lambda(x) - \varphi_{\lambda_0}(x)}{\lambda - \lambda_0} = \sum_{i=1}^{n} x_i\left(\frac{\partial \varphi}{\partial x_i}\right)_{\lambda_0}$$

therefore

$$-(n+p)\langle T, \varphi_{\lambda_0}\rangle = \lambda_0 \lim_{\lambda\to\lambda_0}\left\langle T, \frac{\varphi_\lambda - \varphi_{\lambda_0}}{\lambda - \lambda_0}\right\rangle = \lambda_0 \lim_{\lambda\to\lambda_0} \frac{\langle T, \varphi_\lambda\rangle - \langle T, \varphi_{\lambda_0}\rangle}{\lambda - \lambda_0}$$

Let us consider the function $\lambda \to f(\lambda) = \langle T, \varphi_\lambda\rangle$. By the above equality it has a derivative at each $\lambda_0 \in \mathbb{R}^+$. Moreover it satisfies

$$\begin{cases} \lambda_0 f'(\lambda_0) = -(n+p)f(\lambda_0) & \text{for } \lambda_0 > 0 \\ f(1) = \langle T, \varphi\rangle \end{cases}$$

This is a differential equation which can be easily solved.

$$\frac{f'(\lambda_0)}{f(\lambda_0)} = -\frac{(n+p)}{\lambda_0} \quad \text{so } f(\lambda_0) = C\lambda_0^{-(n+p)},\ C = f(1)$$

therefore

$$\langle T, \varphi_{\lambda_0}\rangle = \lambda_0^{-(n+p)}\langle T, \varphi\rangle \qquad \text{Q.E.D.}$$

c) The homogeneous distributions on $\mathbb{R}$ are the distributions which are solutions of the equation

$$(4) \quad x\frac{\mathrm{d}T}{\mathrm{d}x} = pT$$

Outside the origin, x being $\neq 0$, the solution of (4) are the usual ones which can be obtained by the customery method.
We get

$$(5) \quad T(x) = \begin{cases} c_1 x^p & \text{if } x > 0 \\ c_2(-x)^p & \text{if } x < 0 \end{cases} \quad \text{for } p \in \mathbb{R},\ p \neq -1, -2, \ldots$$

$$(6) \quad T(x) = \frac{c}{x^m} \quad \text{if } p = -m \quad \text{and} \quad m \in \mathbb{N}$$

The general solution coincides with $c_1 x_+^p$ (for $x > 0$) and $c_2 x_-^p$ (for $x < 0$) when $p \in \mathbb{R}$, $p \neq -1, -2, \ldots$, and with $fp\dfrac{1}{x^m}$ when $p = -m$, $m \in \mathbb{N}$.

Therefore the support of $T - c_1 x_+^p - c_2 x_-^p$ $\left(\text{or } T - fp\dfrac{1}{x^m}\right)$ is at the origin so

$$T = c_1 x_+^p + c_2 x_-^p + \sum_{k=0}^{N} a_k \delta^{(k)} \qquad p \neq -1, -2, \ldots$$

$$T = Cfp\frac{1}{x^m} + \sum_{k=0}^{N} a_k \delta^{(k)} \qquad p = -m,\ m \in \mathbb{N}$$

1. If $p \neq -1, -2, \ldots$ $\delta^{(k)}$ being homogeneous of degree $-k-1$ we have by a), $a_1 = \cdots = a_N = 0$ so $T = c_1 x_+^p + c_2 x_-^p$.

2. If $p = -m$, $m \in \mathbb{N}$, the derivative of order $(m-1)$ of δ is homogeneous of degree $-m$ the others have a different degree of homogeneity so $T = c_1 fp\dfrac{1}{x^m} + c_2\delta^{(m-1)}$.

Solution 35

a) since $Pu = 0$ in $\mathbb{R}^n\backslash\{0\}$ we have $u \in C^\infty(\mathbb{R}^n\backslash\{0\})$. So on every compact contained in $\mathbb{R}^n\backslash\{0\}$ u is integrable and satisfies (1). By exercise 13 we can find $T \in \mathscr{D}'(\mathbb{R}^n)$ such that $T = u$ in $\mathscr{D}'(\mathbb{R}^n\backslash\{0\})$.

b) since $Pu = 0$ in $\mathbb{R}^n\backslash\{0\}$ we have supp $PT \subset \{0\}$ so

$$PT = \sum_{|p|\le N} a_p D^p\delta \qquad a_p \in \mathbb{C}$$

Let us set $S = \sum_{|p|\le N} a_p D^p E$. Then we have

$$P(D)[T - S] = \sum_{|p|\le N} a_p D^p\delta - \sum_{|p|\le N} a_p D^p P(D)E = 0$$

Since P satisfies condition (H) we conclude that $g = T - S \in C^\infty(\mathbb{R}^n)$ i.e.

$$T = g + \sum_{|p|\le N} a_p D^p E$$

Solution 36

1°) Let V_1 and V_2 be two open sets such that $V \subset V_1 \subset V_2$ and $V_2 \cap U = \varnothing$. Let $\alpha \in \mathscr{D}(\mathbb{R}^n)$ be such that $\alpha = 1$ on V_1, $\alpha = 0$ in $\complement V_2$. Then $\alpha S \in C_0^\infty(\mathbb{R}^n)$ and $(1-\alpha)T \in C^\infty(\mathbb{R}^n)$. Let us set

$$(1) \quad \langle\langle S, T\rangle\rangle = \langle T, \alpha S\rangle + \langle S, (1-\alpha)T\rangle$$

which has a meaning since $T \in \mathscr{D}'$, $\alpha S \in \mathscr{D}$, $S \in \mathscr{E}'$, $(1-\alpha)T \in C^\infty$.

Let us show that this definition is independant of α. Let $\beta \in \mathscr{D}(\mathbb{R}^n)$ having the same properties as α. Then

$$\langle T, \alpha S\rangle + \langle S, (1-\alpha)T\rangle - \langle T, \beta S\rangle + \langle S, (1-\beta)T\rangle$$
$$= \langle T, (\alpha - \beta)S\rangle - \langle S, (\alpha-\beta)T\rangle$$

which has a meaning since $(\alpha-\beta)S \in \mathscr{D}$ and $(\alpha-\beta)T \in C^\infty$.

Let us choose $\psi \in \mathscr{D}(\mathbb{R}^n)$ such that $\psi = 1$ on the support of $\alpha - \beta$, $\psi = 0$ on U; then $\psi(\alpha-\beta) = (\alpha-\beta)$; it is easy to see that such a ψ exists.

Since $\psi S \in \mathscr{D}$ and $(\alpha - \beta)T \in C^\infty$ we get

$$\langle T, (\alpha - \beta)S\rangle = \langle T, (\alpha - \beta)\psi S\rangle = \langle(\alpha - \beta)T, \psi S\rangle = \langle \psi S, (\alpha - \beta)T\rangle$$
$$= \langle S, \psi(\alpha - \beta)T\rangle = \langle S, (\alpha - \beta)T\rangle$$

Therefore the definition of $\langle\langle S, T\rangle\rangle$ is independant of α.

a) If $\operatorname{supp} S \cap \operatorname{supp} T = \varnothing$ we have $\langle T, \alpha S\rangle = \langle S, (1 - \alpha)T\rangle = 0$ so $\langle\langle S, T\rangle\rangle = 0$.

b) Let us assume $T \in C^\infty$ then $V = \varnothing$ and V_1 is any open set such that $V_1 \cap U = \varnothing$. Since T, αS are C^∞ we have

$$\langle T, \alpha S\rangle = \int T(x)(\alpha S)(x)\,\mathrm{d}x = \langle \alpha S, T\rangle = \langle S, \alpha T\rangle$$

so

$$\langle\langle S, T\rangle\rangle = \langle S, \alpha T\rangle + \langle S, (1 - \alpha)T\rangle = \langle S, T\rangle$$

Let us set $\partial = \dfrac{\partial}{\partial x}$

$$\langle\langle \partial S, T\rangle\rangle = \langle T, \alpha\partial S\rangle + \langle \partial S, (1 - \alpha)T\rangle$$
$$= \langle T, \partial\alpha S\rangle - \langle T, (\partial\alpha)S\rangle + \langle S, (\partial\alpha)T\rangle - \langle S, (1 - \alpha)\partial T\rangle$$

so

$$\langle\langle S, T\rangle\rangle = -\langle \partial T, \alpha S\rangle - \langle T, (\partial\alpha)S\rangle + \langle S, (\partial\alpha)T\rangle - \langle S, (1 - \alpha)\partial T\rangle$$

Now $(\partial\alpha) = 0$ on U and V so $(\partial\alpha)S \in \mathscr{D}$, $(\partial\alpha)T \in C^\infty$. Let $\psi_1 \in \mathscr{D}$, $\psi_1 = 0$ on V, $(\partial\alpha)\cdot\psi_1 \equiv (\partial\alpha)$. Then

$$\langle S, (\partial\alpha)T\rangle = \langle S, (\partial\alpha)\psi_1 T\rangle = \langle(\partial\alpha)S, \psi_1 T\rangle = \langle \psi_1 T, (\partial\alpha)S\rangle = \langle T, (\partial\alpha)S\rangle$$

$$\langle\langle \partial S, T\rangle\rangle = -[\langle \partial T, \alpha S\rangle + \langle S, (1 - \alpha)\partial T\rangle] = -\langle\langle S, \partial T\rangle\rangle \qquad \text{Q.E.D.}$$

2°) Let us compute $\bar{\partial}f$ *in* $\mathscr{D}'(\mathbb{R}^2)$. Let $\varphi \in \mathscr{D}(\mathbb{R}^2)$

$$\langle \bar{\partial}f, \varphi\rangle = -\langle f, \bar{\partial}\varphi\rangle = -\int_D u(x + iy)(\bar{\partial}\varphi)(x, y)\,\mathrm{d}x\,\mathrm{d}y$$

Let us set $x = r\cos\theta$, $y = r\sin\theta$, then $\mathrm{d}x\,\mathrm{d}y = r\,\mathrm{d}r\,\mathrm{d}\theta$ and

$$\bar{\partial} = \frac{1}{2}\left[\mathrm{e}^{i\theta}\frac{\partial}{\partial r} + i\frac{\mathrm{e}^{i\theta}}{r}\frac{\partial}{\partial\theta}\right]$$

Let us set $\tilde{\varphi}(r, \theta) = \varphi(r\cos\theta, r\sin\theta)$. We get

$$\langle \bar{\partial}f, \varphi\rangle = -\frac{1}{2}\int_0^1\int_0^{2\pi} u(r\,\mathrm{e}^{i\theta})\left\{\mathrm{e}^{i\theta}\frac{\partial\tilde{\varphi}}{\partial r} + i\frac{\mathrm{e}^{i\theta}}{r}\frac{\partial\tilde{\varphi}}{\partial\theta}\right\}r\,\mathrm{d}r\,\mathrm{d}\theta$$

$$\langle \bar{\partial}f, \varphi\rangle = -\frac{1}{2}\int_0^{2\pi}\left[\int_0^1 ru(r\,\mathrm{e}^{i\theta})\frac{\partial\tilde{\varphi}}{\partial r}\right]\mathrm{e}^{i\theta}\,\mathrm{d}\theta - \frac{i}{2}\int_0^1\left[\int_0^{2\pi} \mathrm{e}^{i\theta}u(r\,\mathrm{e}^{i\theta})\frac{\partial\tilde{\varphi}}{\partial\theta}\mathrm{d}\theta\right]\mathrm{d}r$$

Integrating by parts in each term we get easily

$$\langle \bar{\partial} f, \varphi \rangle = -\frac{1}{2}\int_0^{2\pi} u(e^{i\theta})e^{i\theta}\tilde{\varphi}(1,\theta)\,d\theta = -\frac{1}{2}\int_0^{2\pi} u(e^{i\theta})e^{i\theta}\varphi(\cos\theta, \sin\theta)\,d\theta$$

So $\bar{\partial} f$ is the measure $-\frac{1}{2}u(e^{i\theta})e^{i\theta}\,d\theta$ on ∂D.

3°) Let us set $E = \frac{1}{\pi z}$. Then $\bar{\partial} E = \delta$ (see exercise 29).
By question 1°) we can define $\langle\langle \bar{\partial} f, E\rangle\rangle$. Indeed $S = \bar{\partial} f$ is in $\mathscr{E}'(\mathbb{R}^2)$, $E \in \mathscr{D}'(\mathbb{R}^2)$ and sing supp $E = \{0\}$, since $\bar{\partial} f$ vanishes in D we have sing supp $E \cap$ sing supp $(\ldots)(\bar{\partial} f)$, $= \varnothing$.
On the other hand by 1°) c):

$$\langle\langle \bar{\partial} f, E\rangle\rangle = -\langle\langle f, \bar{\partial} E\rangle\rangle$$

Since $\bar{\partial} E = \delta$ we get

$$\langle\langle \bar{\partial} f, E\rangle\rangle = -\langle\langle f, \delta\rangle\rangle = -\langle \delta, \alpha f\rangle - \langle f, (1-\alpha)\delta\rangle = -\langle \delta, f\rangle$$

i.e.

$$(*)\quad \langle\langle \bar{\partial} f, E\rangle\rangle = -f(0) = -u(0)$$

Now, since $E \in L^1_{\text{loc}}(\mathbb{R}^2)$ and $\bar{\partial} f$ is a measure, we get

$$\langle\langle \bar{\partial} f, E\rangle\rangle = \langle E, \alpha\bar{\partial} f\rangle + \langle \bar{\partial} f, (1-\alpha)E\rangle = \int_D E\alpha\bar{\partial} f + \int_D \bar{\partial} f(1-\alpha)E$$

$$\langle\langle \bar{\partial} f, E\rangle\rangle = \int_D E\bar{\partial} f = -\frac{1}{2}\int_0^{2\pi} \frac{e^{-i\theta}}{\pi} e^{i\theta} u(e^{i\theta})\,d\theta$$

$$(**)\quad \langle\langle \bar{\partial} f, E\rangle\rangle = -\frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})\,d\theta = -\frac{1}{2i\pi}\int_{\partial D} \frac{u(z)}{z}\,dz$$

Using $(*)$ and $(**)$ we get:

$$u(0) = \frac{1}{2i\pi}\int_{\partial D} \frac{u(z)}{z}\,dz$$

Solution 37

a) We know that the general solution of equation (1) $\frac{dT}{dx} = S$ is the sum of the general solution of $\frac{dT}{dx} = 0$ and of a particular solution of equation (1).

Let $T \in \mathscr{D}'(\Omega)$ be such that $\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^p T = 0$. Then $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}^{p-1}T}{\mathrm{d}x^{p-1}}\right) = 0$ so $\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{p-1} T = C_0$ therefore $\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)\left[\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{p-2} T\right] = C_0$ so $\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{p-2} T = C_0 x + C_1$. By induction we deduce that T is a polynomial of order $p - 1$ with complex coefficients.

b) Let $f \in C_c^0(\mathbb{R})$ and $p \in \mathbb{N}\backslash\{0\}$ such that $\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^p f = \delta$ in $\mathscr{D}'(\mathbb{R})$.

In $\Omega = \mathbb{R}\backslash\{0\}$ the function f must satisfy

$$(2) \quad \frac{\mathrm{d}^p}{\mathrm{d}x^p} f = 0$$

The general solution, in $\mathscr{D}'(\mathbb{R}\backslash\{0\})$, of equation (2) is a polynomial of order $\leq p - 1$:

$$f(x) = C_0 x^{p+1} + C_1 x^{p\ 2} + \ldots + C_{p-1},\ C_i \in \mathbb{C}$$

Since f has compact support we conclude that f must vanish in $\mathbb{R}\backslash\{0\}$.
Indeed

$$\frac{f(x)}{x^{p-1}} = C_0 + \frac{C_1}{x} + \ldots + \frac{C_{p-1}}{x^{p-1}} \quad \text{for } x \in \mathbb{R}\backslash\{0\}$$

On one hand $\lim\limits_{|x| \to \infty} \frac{f(x)}{x^{p-1}} = 0$ since f vanishes for $|x|$ large

On the other hand $\lim\limits_{|x| \to \infty} \left(C_0 + \frac{C_1}{x} + \ldots + \frac{C_{p-1}}{x^{p-1}}\right) = C_0$.

Therefore $C_0 = 0$ and in the same way $C_1 = \cdots C_{p-1} = 0$.
Since f is continuous on $\mathbb{R}$ we conclude that $f \equiv 0$ in $\mathbb{R}$ which is impossible since $\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^p f = \delta$.

Remark

Here is another shorter solution of b):
Let K be the support of f and let us suppose $K \subset \{x : |x| \leq M\}$. For every $\varphi \in \mathscr{D}(\mathbb{R})$ we have

$$(1) \quad \langle \delta, \varphi \rangle = \varphi(0) = (-1)^p \int_k f(x) \cdot \frac{\mathrm{d}^p \varphi}{\mathrm{d}x^p}(x)\, \mathrm{d}x$$

If we take φ equal to 1 for $|x| \leq M$ then the left hand side of the above equality is equal to 1 but the right hand side vanishes.

Solution 38

We shall set in the following $\frac{d^jT}{dx^j} = T^{(j)}$. The general solution of the equation $x^kS = 0$ is $S = \sum_{p=0}^{k-1} C_p\delta^{(p)}$. Indeed let us use an induction on k.

It is obviously true when $k = 1$. Let us suppose that the general solution of $x^{k-1}S = 0$ is $\sum_{p=0}^{k-2} C_p\delta^{(p)}$. Equation (1) is equivalent to $x(x^{k-1}S) = 0$, and therefore equivalent to $x^{k-1}S = C\delta$. The general solution of this equation is the sum of the general solution of $x^{k-1}S = 0$ and of a particular solution of $x^{k-1}S = C\delta$.

Now $x^{k-1}\delta^{(k-1)} = \alpha_k\delta$. Indeed

$$\langle x^{k-1}\delta^{(k-1)}, \varphi\rangle = (-1)^{k-1}\langle\delta, (x^{k-1}\varphi)^{(k-1)}\rangle = (-1)^{k-1}(x^{k-1}\varphi)^{(k-1)}(0)$$
$$= (-1)^{k-1}(k-1)!\varphi(0)$$

Then the general solution of $x^{k-1}S = C\delta$ is $\sum_{p=0}^{k-2} C_p\delta^{(p)} + C_{k-1}\delta^{(k-1)} = \sum_{p=0}^{k-1} C_p\delta^{(p)}$.

Q.E.D.

Equation (1) is then equivalent to: (2) $T^{(m)} = \sum_{p=0}^{k-1} C_p\delta^{(p)}$. The general solution of this equation is the sum of the general solution of $T^{(m)} = 0$ and of a particular solution of (2). The general solution of $T^{(m)} = 0$ is a polynomial of degree $m - 1$. Let us look for a particular solution of (2).

a) $m < k - 1$: $T^{(m)} = C_0\delta + C_1\delta' + \ldots + C_{m-1}\delta^{(m-1)} + C_m\delta^{(m)} + \ldots + C_{k-1}\delta^{(k-1)}$. First of all for every $l \in \mathbb{N}$ $(x^lH)^{(l)} = a_lH$. Indeed

$$(x^lH)^{(l)} = l!H + \sum_{p=0}^{l-1} C_p^l l(l-1)\ldots(l-p+1)x^{l-p}\delta^{(l-1-p)} = l!H$$

because $x^{l-p}\delta^{(l-1-p)} = 0$ since $l - p > l - 1 - p$.

We deduce that $(x^lH)^{l+1} = a_l\delta$ so $(x^lH)^{l+p} = a_l\delta^{(p-1)}$. Therefore

$$(x^{m-1}H)^{(m)} = a_{m-1}\delta,\ (x^{m-2}H)^{(m)} = a_{m-2}\delta',\ \ldots,\ (x^0H)^{(m)} = a_0\delta^{(m-1)}$$

So

$$T_0 = b_1(x^{m-1}H) + b_2(x^{m-2}H) + \ldots + b_mH + b_{m+1}\delta + \ldots + b_{m+k-1}\delta^{(k-1-m)}$$

is a particular solution of (2) and $T = T_0 + P_{m-1}$, where P_{m-1} is a polynomial of order $m - 1$ is the general solution of (1).

(b) $m \geq k - 1$: In this case:

$$T_0 = b_1(x^{m-1}H) + b_2(x^{m-2}H) + \ldots + b_k(x^{m-k}H)$$

and the general solution of (1) is

$$T = T_0 + P_{m-1} \quad \text{where } P_{m-1} \text{ is a polynomial of degree } m - 1$$

Solution 39

Let $\alpha_0 \in \mathbb{N}^n$, $|\alpha_0| \leq m$. The function x^{α_0} being C^∞ we have

$$\langle T, x^{\alpha_0}\rangle = \sum_{|\alpha|\leq m} a_\alpha(-1)^{|\alpha|}[\partial^\alpha(x^{\alpha_0})](0) = 0$$

Now it is easy to see that

$$[\partial^\alpha(x^{\alpha_0})](0) = \begin{cases} 0 & \text{if } \alpha \neq \alpha_0 \\ \alpha_0! & \text{if } \alpha = \alpha_0 \end{cases}$$

We conclude that

$$\langle T, x^{\alpha_0}\rangle = (-1)^{|\alpha|}\alpha_0! a_{\alpha_0} = 0$$

So $a_{\alpha_0} = 0$ for all α_0, $|\alpha| \leq m$.

b) The application f_i corresponds to the matrix $A_i = \begin{pmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & \ddots & -1 & \\ & 0 & & \ddots & \\ & & & & 1 \end{pmatrix}$ which is

such that $|\det A_i| = 1$ and $A_i^{-1} = A_i$ for $i = 1 \ldots, n$.

On the other hand the support of T being at the origin we have $T = \sum_{|\alpha|\leq m} b_\alpha \partial^\alpha \delta$.
Moreover we must have

$$(1) \quad T \circ A_i = T = \sum_{|\alpha|\leq m} b_\alpha \partial^\alpha \delta;\ i = 1, \ldots n$$

Let us compute $T \circ A_i$. For $\varphi \in C^\infty(\mathbb{R}^n)$

$$\langle T \circ A_i, \varphi\rangle = \langle T, \varphi \circ A_i\rangle = \sum_{|\alpha|\leq m} b_\alpha(-1)^{|\alpha|}\partial^\alpha(\varphi \circ A_i)(0)$$

Now

$$\partial^\alpha(\varphi \circ A_i)(0) = (-1)^{\alpha_i}(\delta^\alpha \varphi)(0) = (-1)^{\alpha_i}(-1)^{|\alpha|}\langle \partial^\alpha \delta, \varphi\rangle$$

so

$$\langle T \circ A_i, \varphi\rangle = \sum_{|\alpha|\leq m} (-1)^{\alpha_i} b_\alpha \langle \partial^\alpha \delta, \varphi\rangle$$

and

$$(2) \quad T \circ A_i = \sum_{|\alpha|\leq m} (-1)^{\alpha_i} b_\alpha \partial^\alpha \delta$$

From (1) and (2) we deduce

$$\sum_{|\alpha|\leq m} \{(-1)^{\alpha_i} - 1\} b_\alpha \partial^\alpha \delta = 0$$

We deduce from question a) that for every $\alpha \in \mathbb{N}^n$ and all $i = 1, \ldots, n$

$$\{(-1)^{\alpha_i} - 1\}b_\alpha = 0$$

which implies that for α such that $b_\alpha \neq 0$, α_i must be even for $i = 1, \ldots, n$, i.e.

$$(3) \quad T = \sum b_{2k_1,\ldots,2k_n} \partial_1^{2k_1} \ldots \partial_n^{2k_n}\delta$$

Conversely by (2), every distribution given by (3) is invariant by the applications f_i.

Solution 40

a) $QF = \dfrac{\partial^2}{\partial y_1^2}(y_1 H(y_1))\left(\dfrac{\partial}{\partial y_2} - 1\right)(H(y_2)\exp y_2)$

$$\frac{\partial}{\partial y_1}(y_1 H(y_1)) = H(y_1) + y_1\delta_{y_1=0} = H(y_1) \quad \text{so} \quad \frac{\partial^2}{\partial y_1^2}(y_1 H(y_1)) = \delta_{y_1=0}$$

$$\left(\frac{\partial}{\partial y_2} - 1\right)(H(y_2)\exp y_2) = \delta_{y_2=0}\exp y_2 + H(y_2)\exp y_2 - H(y_2)\exp y_2 = \delta_{y_2=0}$$

Therefore $QF = \delta_{y_1=0} \otimes \delta_{y_2=0} = \delta_0$.

b) $\langle P(u \circ A), \varphi\rangle = \langle u \circ A, P_1\varphi\rangle = |\det A|^{-1}\langle u, (P_1\varphi) \circ A^{-1}\rangle$

where

$$P_1 = \frac{1}{8}\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right)^2\left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} + 2\right)$$

Let us compute $(P_1\varphi) \circ A^{-1}$. We have

$$\frac{\partial}{\partial x_1} = \frac{\partial}{\partial y_1}\frac{\partial y_1}{\partial x_1} + \frac{\partial}{\partial y_2}\frac{\partial y_2}{\partial x_1} = \frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2}$$

$$\frac{\partial}{\partial x_2} = \frac{\partial}{\partial y_1}\frac{\partial y_1}{\partial x_2} + \frac{\partial}{\partial y_2}\frac{\partial y_2}{\partial x_2} = \frac{\partial}{\partial y_1} - \frac{\partial}{\partial y_2}$$

so

$$(P_1\varphi) \circ A^{-1} = -\frac{1}{8}\left(2\frac{\partial}{\partial y_1}\right)^2\left(2\frac{\partial}{\partial y_2} + 2\right)(\varphi \circ A^{-1}) = -\frac{\partial^2}{\partial y_1^2}\left(\frac{\partial}{\partial y_2} + 1\right)(\varphi \circ A^{-1})$$
$$= Q_1(\varphi \circ A^{-1})$$

We deduce that

$$\langle P(u \circ A), \varphi\rangle = |\det A|^{-1}\langle u, Q_1(\varphi \circ A^{-1})\rangle = |\det A|^{-1}\langle Qu, \varphi \circ A^{-1}\rangle$$

so

$$\langle P(u \circ A), \varphi\rangle = \langle (Qu) \circ A, \varphi\rangle$$

for all $\varphi \in \mathscr{D}(\mathbb{R}^2)$; so

$$P(u \circ A) = (Qu) \circ A$$

c) $\langle \delta \circ A, \varphi \rangle = |\det A|^{-1} \langle \delta, \varphi \circ A^{-1} \rangle = \frac{1}{2}\varphi[A^{-1}(0, 0)] = \frac{1}{2}\varphi(0, 0)$

so $\delta_{y=0} \circ A = \frac{1}{2}\delta_{x=0}$.

By questions 1) and 2) we have

$$P(F \circ A) = (QF) \circ A = \delta \circ A = \frac{1}{2}\delta$$

So $P(2F \circ A) = \delta$. Let us compute $F \circ A$. We have

$(F \circ A)(x_1, x_2) = F(x_1 + x_2, x_1 - x_2) = (x_1 + x_2)H(x_1 + x_2)H(x_1 - x_2)e^{x_1 - x_2}$

So we obtain a fundamental solution of P setting

$$E = 2(x_1 + x_2)H(x_1 + x_2)H(x_1 - x_2) \exp (x_1 - x_2)$$

Solution 41

a) T is a distribution on Ω so for every continuous semi norm p on $\mathscr{D}(\Omega)$ we can find a constant $C > 0$ such that for every $\varphi \in \mathscr{D}(\Omega)$

$$(2) \quad |\langle T, \varphi \rangle| \le Cp(\varphi)$$

From (1) and (2) we deduce that for every $\varphi \in \mathscr{D}(\Omega)$

$$|\langle T, \varphi \rangle| \le Cq({}^tP\varphi) = Cq(\psi)$$

which proves a).

b) The application Φ being linear and continuous from E, subspace of $\mathscr{D}(\Omega)$, to $\mathbb{C}$, by the Hahn-Banach theorem it can be extended to a continuous linear map from $\mathscr{D}(\Omega)$ to $\mathbb{C}$ which means to a distribution on Ω. Therefore we can find $S \in \mathscr{D}'(\Omega)$ which coincides with Φ on E. Let $\varphi \in \mathscr{D}(\Omega)$, since ${}^tP\varphi \in E$ we have:

$$\langle S, {}^tP\varphi \rangle = \Phi({}^tP\varphi) = \langle T, \varphi \rangle$$

But by definition

$$\langle S, {}^tP\varphi \rangle = \langle PS, \varphi \rangle$$

so $\langle PS, \varphi \rangle = \langle T, \varphi \rangle \; \forall \varphi \in \mathscr{D}(\Omega)$ i.e.

$$PS = T \quad \text{in } \mathscr{D}'(\Omega)$$

CHAPTER 4

Convergence of distributions

PROGRAMME

Convergence of sequences in $\mathscr{D}'$ and in $\mathscr{E}'$.

BASICS — CHAPTER 4

Convergence of sequences of distributions

• Let $(T_j)_{j\in\mathbb{N}}$ be a sequence of distributions in an open set Ω of $\mathbb{R}^n$. We say that it converges in $\mathscr{D}'(\Omega)$ to a distribution $T \in \mathscr{D}'(\Omega)$ if

$$(1) \quad \lim_{j\to\infty} \langle T_j, \varphi\rangle = \langle T, \varphi\rangle$$

for all $\varphi \in \mathscr{D}(\Omega)$.

• If the distributions T_j are in $\mathscr{E}'(\Omega)$, we say that the sequence T_j converges to T in $\mathscr{E}'(\Omega)$ if we have (1) for every $\varphi \in C^\infty(\Omega)$.

• Let us note that it is not necessary to know the limit in order to say that the sequence $(T_j)_{j\in\mathbb{N}}$ converges in $\mathscr{D}'(\Omega)$ as the following result shows:
Let us suppose that for every φ in $\mathscr{D}(\Omega)$, the sequence of complex numbers $(\langle T_j, \varphi\rangle)_{j\in\mathbb{N}}$ converges in $\mathbb{C}$ then there exists a distribution $T \in \mathscr{D}'(\Omega)$ such that

$$\lim_{j\to\infty} T_j = T$$

in $\mathscr{D}'(\Omega)$.

• The limit of a sequence of distributions is unique.

• Convergence in $\mathscr{D}(\Omega)$, $C^k(\Omega)$, $L^p(\Omega)$ $(0 \le k \le \infty;\ 1 \le p \le \infty)$ implies convergence in $\mathscr{D}'(\Omega)$.

STATEMENTS OF EXERCISES — CHAPTER 4

Exercise 42

For $\varepsilon > 0$ we set $T_\varepsilon = \frac{\varepsilon}{2}|x|^{\varepsilon-1}$. Compute in $\mathscr{D}'(\mathbb{R})$

$$T = \lim_{\varepsilon\to 0} T_\varepsilon$$

Exercise 43

Determine a sequence $(T_n)_{n\ge 1}$ of distributions with support at the origin such that the sequence $(S_n)_{n\ge 1}$ defined, for $\varphi \in \mathscr{D}(\mathbb{R})$, by

$$\langle S_n, \varphi\rangle = \langle T_n, \varphi\rangle - \sum_{k=1}^{n} \varphi\left(\frac{1}{k}\right)$$

converges in $\mathscr{D}'(\mathbb{R})$

* Solutions pp. 94 to 109

Exercise 44

1°) Construct a sequence $(f_n)_{n\in\mathbb{N}}$ of functions on $\mathbb{R}$ such that

a) $(f_n(x))_n$ converges to zero almost everywhere.

b) (f_n) does not converge in $\mathscr{D}'(\mathbb{R})$.

2°) Construct a sequence $(g_n)_{n\in\mathbb{N}}$ of functions on $\mathbb{R}$ such that

a) $(g_n)_n$ converges to δ in $\mathscr{D}'(\mathbb{R})$.

b) $(g_n(x))_n$ converges to zero almost everywhere.

Exercise 45

We denote by δ_x the distribution $\mathscr{D}(\mathbb{R}) \ni \varphi \rightarrow \langle \delta_x, \varphi\rangle = \varphi(x)$.

a) Prove that for all $a \in \mathbb{R}$ the series $\sum_{n=1}^{\infty} n^a[\delta_{1/n} - \delta_{-1/n}]$ converges in $\mathscr{D}'(\mathbb{R}\backslash 0)$. We shall denote its sum by T_a.

b) Prove that this series converges in $\mathscr{D}'(\mathbb{R})$ if and only if $a < 0$.

c) For $0 \leqslant a < 1$ find $S_a \in \mathscr{D}'(\mathbb{R})$ such that $S_{a|\mathbb{R}\backslash 0} = T_a$.

Exercise 46

Let $A_{\lambda,k}$ be the distribution defined for $\lambda \in \mathbb{R}$, $k \in \mathbb{N}\backslash\{0\}$ by

$$\langle A_{\lambda,k}, \varphi\rangle = \int_{-\pi/2}^{\pi/2} \frac{\cos \lambda x}{x^k}\left\{\varphi(x) - \sum_{i=0}^{k-1} \frac{\varphi^{(i)}(0)}{i!}x^i\right\}\mathrm{d}x \qquad \varphi \in \mathscr{D}(\mathbb{R})$$

Compute in $\mathscr{D}'(\mathbb{R})$

$$\lim_{\lambda\to\infty} A_{\lambda,k} \qquad \lim_{\lambda\to 0} A_{\lambda,k}$$

Exercise 47 (see exercise 26)

For $x \in \mathbb{R}$ and $\varepsilon > 0$ we set

$$f_\varepsilon(x) = \operatorname{Log}(x + i\varepsilon) = \operatorname{Log}|x + i\varepsilon| + i \operatorname{Arg}(x + i\varepsilon)$$

a) Show that when $\varepsilon \to 0$, f_ε converges in $\mathscr{D}'(\mathbb{R})$ to the distribution f_0 given by

$$f_0 = \begin{cases} \operatorname{Log} x & x > 0 \\ \operatorname{Log}|x| + i\pi & x < 0 \end{cases}$$

b) Compute $\dfrac{\mathrm{d}f_0}{\mathrm{d}x}$ in the distributions sense. (Use exercise 26)

c) Deduce that we have in $\mathscr{D}'(\mathbb{R})$

$$\frac{1}{x+i0} = \lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} \frac{1}{x+i\varepsilon} = -i\pi\delta + pv\frac{1}{x}$$

d) Prove that $\dfrac{1}{x+i0} = \lim_{\varepsilon \to 0^-} \dfrac{1}{x+i\varepsilon} = i\pi\delta + pv\dfrac{1}{x}$

Deduce that $\lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} \dfrac{1}{\pi(x^2+\varepsilon^2)} = \delta.$

Exercise 48 (see exercise 47)
Using the following result (see exercise 47)

$$\frac{1}{x-i0} = i\pi\delta + pv\frac{1}{x}$$

show that

$$\lim_{t \to +\infty} \frac{e^{ixt}}{x-i0} = 2i\pi\delta$$

We recall that $\displaystyle\int_0^{\infty} \frac{\sin x}{x}\mathrm{d}x = \frac{\pi}{2}$

Exercise 49

Let $a \in \mathbb{R}^+$, $\delta_n = \delta_{x=n}$. Investigate the convergence of $\sum_{n \in \mathbb{N}} a^n \delta_n$ in $\mathscr{D}'(\mathbb{R})$ and in $\mathscr{E}'(\mathbb{R})$.

Exercise 50

a) Let (a_n) be a sequence of complex numbers such that

$$|a_n| \leqq Cn^p \qquad p \in \mathbb{N}$$

Prove that the sequence $T_N = \sum_{n=-N}^{N} a_n e^{2i\pi nx}$ converges in $\mathscr{D}'(\mathbb{R})$. We shall denote by $T = \sum_{-\infty}^{\infty} a_n e^{2i\pi nx}$ its limit.

b) Show that

$$\frac{\mathrm{d}T}{\mathrm{d}x} = \sum_{-\infty}^{\infty} (2i\pi n) a_n e^{2i\pi nx}$$

and that $\tau_1 T = T$ *where* $\langle \tau_1 T, \varphi\rangle = \langle T, \tau_{-1}\varphi\rangle = \langle T, \varphi(x+1)\rangle$.

c) We set $S = \sum_{-\infty}^{\infty} e^{2i\pi n x}$

Prove that $(1 - e^{2i\pi x}) S = 0$. Deduce that $S = \sum_{-\infty}^{\infty} c_n \delta_n$ and using b) prove that $c_n = c$ for all $n \in \mathbb{Z}$.

d) Let us consider the continuous function

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{1}{1+n^2} e^{2i\pi n x}$$

Investigate a differential equation in $\mathscr{D}'(\mathbb{R})$ satisfied by f. Prove that in $]0, 1[$ $f(x) = a(e^{2\pi(x-1/2)} + e^{-2\pi(x-1/2)})$.

e) Applying to f' the jumps formula and using the differential equation found in d) find a relation between a and c.

f) Compute the constant a by evaluating $\int_0^1 f(x)\,dx$ in two different ways. Prove that $c = 1$.

Exercise 51*

Let $P = \sum_{|\alpha| \le m} a_\alpha \partial^\alpha$, $a_\alpha \in \mathbb{C}$, be a differential operator with constant coefficients in an open set Ω in $\mathbb{R}^n$.

a) Let Ω_1 be a bounded open subset of Ω; we set

$$E = \{u \in L^2(\Omega_1) \colon Pu = 0 \text{ in } \mathscr{D}'(\Omega_1)\}$$

Prove that E is a closed subspace of $L^2(\Omega_1)$.

b) We assume that P satisfies the following property:

(H) For every open set ω in Ω: $u \in \mathscr{D}'(\omega)$, $Pu = 0$ in $\mathscr{D}'(\omega)$ imply that $u \in C^\infty(\omega)$.

Let Ω_2 be a relatively compact open subset of Ω_1. Prove that we can find a constant $C > 0$ such that for every u in E

$$(1) \quad \sum_{j=1}^{n} \int_{\Omega_2} \left|\frac{\partial u}{\partial x_j}\right|^2 dx \le C \int_{\Omega_1} |u|^2\, dx$$

(Hint: Show that the map $u \mapsto \frac{\partial u}{\partial x_j}$ in continuous from E, equipped with the topology of $L^2(\Omega_1)$, to $L^2(\Omega_2)$, using the closed graph theorem.)

c) Prove that if P is a differential operator with constant coefficients satisfying (H), for every sequence $(\zeta_k)_{k \in \mathbb{N}}$ in $\mathbb{C}^n$ such that

$$P(\zeta_k) = \sum_{|\alpha| \le m} a_\alpha (i\zeta_k)^\alpha = 0 \qquad \text{and} \qquad \lim_{k \to \infty} |\zeta_k| = +\infty$$

we must have: $\lim_{k \to \infty} |\operatorname{Im} \zeta_k| = +\infty$.

(Hint: Apply inequality (1) to the function $u(x) = e^{i\langle x, \zeta_k\rangle}$ where $\langle x, \zeta_k\rangle = \sum_{i=1}^{n} x_i\zeta_k^i$ for $x \in \Omega$ and $\zeta_k = (\zeta_k^1, \ldots, \zeta_k^n) \in \mathbb{C}^n$, then use the inequality

$$-M|\mathrm{Im}\,\zeta_k| \leqq -\langle x, \mathrm{Im}\,\zeta_k\rangle \leqq M|\mathrm{Im}\,\zeta_k|$$

for $x \in \Omega_1$.)

d) Give one (or several) operator which does not satisfy (H).

Remark: A differential operator with constant coefficients satisfying (H) is called hypoelliptic.

Exercise 52*

Let f be a holomorphic function in the subset of $\mathbb{C}$, $D =]a, b[\times]0, \delta[$ where $\delta < 1$. We assume that there exist two constants $C > 0$, $A > 0$ such that

$$(1) \quad |f(z)| \leqq \frac{C}{|\mathrm{Im}\,z|^A} \quad \text{for every } z \text{ such that } |\mathrm{Im}\,z| \leq \frac{\delta}{2}$$

For $y \in]0, \delta[$ we define the function f_y: $]a, b[\to \mathbb{C}$ by

$$f_y(x) = f(x + iy)$$

Our purpose is to prove that:

(∗) When y tends to zero, f_y converges in $\mathscr{D}'(]a, b[)$ to a distribution $T \in \mathscr{D}'(]a, b[)$.

a) Prove that in (1) we may assume that A is not an integer: $A = N + \alpha$, $N \in \mathbb{N}$, $\alpha \in]0, 1[$.

b) For $k \in \mathbb{N}$ and $z \in D$ we denote by γ_k the subset of $\mathbb{C}$: $\left[\frac{a+b}{2} + i\frac{\delta}{2^k}, z\right]$ and we set

$$(P_{\delta/2^k} f)(z) = \int_{\gamma_k} f(u)\,\mathrm{d}u$$

Moreover we set

$$P_0 = \text{Identity} \qquad P_1 = P_{\delta/2}, \qquad P_2 = P_{\delta/4} \circ P_{\delta/2}, \qquad P_k = P_{\delta/2^k} \circ P_{\delta/2^{k-1}} \circ \ldots \circ P_{\delta/2}$$

and

$$u(z) = (P_N f)(z), \qquad v(z) = (P_{N+1} f)(z) = (P_{\delta/2^{N+1}} u)(z)$$

Prove that $P_1 f$ is holomorphic in $|\mathrm{Im}\,z| \leq \delta$ and satisfies the same inequality as (1) for every z such that $|\mathrm{Im}\,z| \leq \frac{\delta}{4}$ and with $A - 1$ instead of A.

Prove by induction that

$$(2) \quad \exists C_1 > 0\colon \forall z\colon |\mathrm{Im}\,z| \leq \frac{\delta}{2^{N+1}}, \qquad |u(z)| \leq \frac{C_1}{|\mathrm{Im}\,z|^\alpha}$$

c) Prove that

$$\exists C_2 > 0\colon \sup_{x \in]a,b[} |v(x+iy_1) - v(x+iy_2)| \leqq C_2 |y_1^{1-\alpha} - y_2^{1-\alpha}|; \qquad |y_1|, |y_2| \leq \frac{\delta}{2^{N+2}}$$

Show that the sequence (v_y) defined by $v_y(x) = v(x+iy)$ converges in $\mathscr{D}'(]a, b[)$.

d) Show that for every $k \geq 1$, $\left(\frac{\partial}{\partial x}\right)^k (P_k f) = f$ and conclude.

SOLUTIONS OF THE EXERCISES — CHAPTER 4

Solution 42

First of all, for $\varepsilon > 0$, the function $|x|^{\varepsilon-1}$ is locally integrable so it defines a distribution on $\mathbb{R}$.

Let $\varphi \in \mathscr{D}(\mathbb{R})$, supp $\varphi \subset \{|x| \leq M\}$. We have

$$\langle T_\varepsilon, \varphi\rangle = \frac{\varepsilon}{2}\int_{\mathbb{R}} |x|^{\varepsilon-1}\varphi(x)\,dx = \frac{\varepsilon}{2}\int_0^M x^{\varepsilon-1}\varphi(x)\,dx + \frac{\varepsilon}{2}\int_{-M}^0 (-x)^{\varepsilon-1}\varphi(x)\,dx$$

so

$$\langle T_\varepsilon, \varphi\rangle = \frac{\varepsilon}{2}\int_0^M x^{\varepsilon-1}(\varphi(x) + \varphi(-x))\,dx$$

Now

$$\varphi(x) = \varphi(0) + x\psi(x)$$

with $\psi \in C^0(\mathbb{R})$ and $\sup_{|x|\leq M}|\psi| \leq C_0 \sup_{|x|\leq M}|\varphi'(x)|$. So we have

$$\varphi(x) + \varphi(-x) = 2\varphi(0) + x(\psi(x) - \psi(-x))$$

therefore

$$\langle T_\varepsilon, \varphi\rangle = \varphi(0)\cdot\varepsilon\int_0^M x^{\varepsilon-1}\,dx + \frac{\varepsilon}{2}\int_0^M x^{\varepsilon}(\psi(x) - \psi(-x))\,dx$$

Now for $\varepsilon < 1$

$$\frac{\varepsilon}{2}\left|\int_0^M x^{\varepsilon}(\psi(x) - \psi(-x))\,dx\right| \leq \varepsilon\cdot C_0 \sup(M, 1)\cdot M\cdot \sup_{|x|\leq M}|\varphi'(x)|$$

and

$$\varepsilon\int_0^M x^{\varepsilon-1}\,dx = M^{\varepsilon}$$

Therefore

$$\lim_{\varepsilon \to 0} \langle T_\varepsilon, \varphi \rangle = (\lim_{\varepsilon \to 0} M^\varepsilon) \cdot \varphi(0) = \varphi(0) = \langle \delta, \varphi \rangle$$

so $\lim_{\varepsilon \to 0} T_\varepsilon = \delta$.

Solution 43

For $\varphi \in \mathscr{D}(\mathbb{R})$ we can write

$$\varphi(x) = \varphi(0) + x\varphi'(0) + x^2\psi(x)$$

where $\psi \in C^0(\mathbb{R})$ and $\sup |\psi(x)| \leq C_0 \sup |\varphi''(x)|$

Therefore $\varphi\left(\frac{1}{k}\right) = \varphi(0) + \frac{1}{k}\varphi'(0) + \frac{1}{k^2}\psi\left(\frac{1}{k}\right)$ so

$$\langle S_n, \varphi \rangle = \langle T_n, \varphi \rangle - n\varphi(0) - \left(\sum_{k=1}^{n} \frac{1}{k}\right)\varphi'(0) - \sum_{k=1}^{n} \frac{1}{k^2}\psi\left(\frac{1}{k}\right)$$

Since $\frac{1}{k^2}\left|\psi\left(\frac{1}{k}\right)\right| \leq \frac{C}{k^2}$, the last term in the right hand side has a limit when $n \to +\infty$. Therefore if we set

$$\langle T_n, \varphi \rangle = n\varphi(0) + \left(\sum_{k=1}^{n} \frac{1}{k}\right)\varphi'(0)$$

the sequence (S_n) will converge in $\mathscr{D}'(\mathbb{R})$ when $n \to +\infty$ and the limit will be a distribution. So we must take

$$T_n = n \cdot \delta - \left(\sum_{k=1}^{n} \frac{1}{k}\right)\delta'$$

One should remark that there are infinitely many sequence (T_n) which work. One just has to take more terms in the Taylor expansion of φ.

Solution 44

1°) Let us set

$$f_n(x) = \begin{cases} 0 & |x| \geq \frac{1}{n} \\ n^2 & |x| < \frac{1}{n} \end{cases}$$

a) Then for all $x \neq 0, f_n(x)$ converges to zero. Indeed for a fixed $x_0 \neq 0$ there exists an integer n_0 such that for every $n \geq n_0$ we have $\frac{1}{n} \leq |x_0|$ so $f_n(x_0) = 0\ \forall n \geq n_0$.

b) Let us show that (f_n) does not converge in $\mathscr{D}'(\mathbb{R})$. Let φ be in $\mathscr{D}(\mathbb{R})$

$$\langle f_n, \varphi \rangle = n^2 \int_{-1/n}^{1/n} \varphi(x)\, dx$$

Let us assume that $\varphi(x) = 1$ for $|x| \leqslant 1$; then for $n \geqslant 1$ one has $\langle f_n, \varphi \rangle = 2n$.

2°) Let us set

$$g_n(x) = \begin{cases} 0 & |x| \geq \frac{1}{n} \\ \frac{n}{2} & |x| < \frac{1}{n} \end{cases}$$

a) Let φ be in $\mathscr{D}(\mathbb{R})$

$$\langle g_n, \varphi \rangle = \frac{n}{2} \int_{-1/n}^{1/n} \varphi(x)\, dx = \frac{n}{2}\frac{2}{n}\varphi(\xi_n) \quad \text{where } |\xi_n| \leq \frac{1}{n}$$

therefore when n goes to infinity, $\langle g_n, \varphi \rangle$ tends to $\varphi(0)$.

b) Of course $(g_n(x))$ converges to zero for every $x \neq 0$.

Solution 45

a) If $\varphi \in \mathscr{D}(\mathbb{R}\backslash\{0\})$ then $\varphi^{(k)}(0) = 0$, $\forall k \in \mathbb{N}$. By the Taylor formula for $|x| \leq M$, $\varphi(x) = 0(|x|^p)$, $\forall p \in \mathbb{N}$. So we have

$$\left| n^a \left[\varphi\left(\frac{1}{n}\right) - \varphi\left(-\frac{1}{n}\right) \right] \right| \leqq C_p \frac{n^a}{n^p} \leqq \frac{C}{n^2} \quad \text{if } p \geqq a + 2$$

Therefore the sequence

$$\left\langle \sum_{n=1}^{N} n^a [\delta_{1/n} - \delta_{-1/n}], \varphi \right\rangle = \sum_{n=1}^{N} n^a \left[\varphi\left(\frac{1}{n}\right) - \varphi\left(-\frac{1}{n}\right) \right]$$

converges in $\mathbb{C}$ for every $a \in \mathbb{R}$.

b) Let us assume $a < 0$. For $\varphi \in \mathscr{D}(\mathbb{R})$ we have $\varphi(x) = \varphi(0) + x\psi(x)$ where $\psi \in C^0(\mathbb{R})$, then

$$\varphi\left(\frac{1}{n}\right) = \varphi(0) + \frac{1}{n}\psi\left(\frac{1}{n}\right)$$

$$\varphi\left(-\frac{1}{n}\right) = \varphi(0) - \frac{1}{n}\psi\left(-\frac{1}{n}\right)$$

so $\left|n^a\left[\varphi\left(\frac{1}{n}\right) - \varphi\left(-\frac{1}{n}\right)\right]\right| \leqq \sup_{|x|\leq 1} |\psi(x)| \frac{1}{n^{1-a}}$. Since $a < 0$, this proves that the series converges for all $\varphi \in \mathscr{D}(\mathbb{R})$.

Let us assume $a \geq 0$, Let $\psi \in \mathscr{D}(\mathbb{R})$ be such that $\psi = 1$ for $|x| \leqslant 1$, $\psi = 0$ if $|x| \geq 2$. Let us set $\varphi(x) = x\psi(x)$ then

$$n^a\langle\delta_{1/n} - \delta_{-1/n}, \varphi\rangle = n^a\frac{2}{n}\left(\psi\left(\frac{1}{n}\right) + \psi\left(-\frac{1}{n}\right)\right) = \frac{2}{n^{1-a}}$$

Since a is non negative the series $\Sigma \frac{1}{n^{1-a}}$ diverges and the same is true for the series given in the statement.

c) For $\varphi \in \mathscr{D}(\mathbb{R})$ let us set

$$\langle S_a, \varphi\rangle = \sum_{n=1}^{\infty} n^a\left(\varphi\left(\frac{1}{n}\right) - \varphi\left(-\frac{1}{n}\right) - \frac{2}{n}\varphi'(0)\right)$$

This series is convergent in $\mathscr{D}'(\mathbb{R})$ and defines a distribution $S_a \in \mathscr{D}'(\in)$ since

$$\left|n^a\left[\varphi\left(\frac{1}{n}\right) - \varphi\left(-\frac{1}{n}\right) - \frac{2}{n}\varphi'(0)\right]\right| \leqq \frac{C}{n^{2-a}}$$

and $a \in [0, 1[$.

Moreover if $\varphi \in \mathscr{D}(\mathbb{R}\backslash\{0\})$, $\varphi'(0) = 0$ so

$$\langle S_a, \varphi\rangle = \sum_{n=1}^{\infty} n^a\left[\varphi\left(\frac{1}{n}\right) - \varphi\left(-\frac{1}{n}\right)\right] = \langle T_a, \varphi\rangle$$

i.e., $S_a|_{\mathbb{R}\backslash\{0\}} = T_a$.

Solution 46

When $\lambda \to \infty$, $A_{\lambda,k}$ tends to zero in $\mathscr{D}'(\mathbb{R})$. Indeed $\varphi(x) = \sum_{i=0}^{k-1} \frac{\varphi^{(i)}(0)}{i!} x^i + x^k\, \psi(x)$ so

$$(1)\quad \langle A_{\lambda,k}, \varphi\rangle = \int_{-\pi/2}^{\pi/2} \cos \lambda x \cdot \psi(x)\, \mathrm{d}x$$

where ψ is continuous on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Therefore it is the Fourier coefficient of an integrable function. The Riemann–Lebesgue theorem implies that (1) tends to zero when $\lambda \to \infty$.

When $\lambda \to 0$, by the Lebesgue theorem and formula (1) we have

$$\lim_{\lambda\to 0} \langle A_{\lambda,k}, \varphi\rangle = \int_{-\pi/2}^{\pi/2} \psi(x)\, \mathrm{d}x$$

Now $\psi(x) = \frac{1}{x^k}\left[\varphi(x) - \sum_{i=0}^{k-1} \varphi^{(i)}(0)\frac{x^i}{i!}\right], \qquad x \neq 0$

Therefore $A_{\lambda,k}$ tends, when $\lambda \to 0$, to the distribution

$$\varphi \to \langle T, \varphi\rangle = \int_{-\pi/2}^{\pi/2} \frac{1}{x^k}\left[\varphi(x) - \sum_{i=0}^{k-1} \varphi^{(i)}(0)\frac{x^i}{i!}\right]\mathrm{d}x$$

Solution 47

a) For $\varepsilon > 0$ f_ε is locally integrable so it defines a distribution.
If $|x + i\varepsilon| > 1$ we have

$$|\mathrm{Log}\,|x + i\varepsilon|| = \mathrm{Log}\,|x + i\varepsilon| \leq \frac{1}{2}\,\mathrm{Log}\,(x^2 + 1) \qquad \text{if } \varepsilon < 1$$

If $|x + i\varepsilon| < 1$

$$|\mathrm{Log}\,|x + i\varepsilon|| = \mathrm{Log}\,\frac{1}{|x + i\varepsilon|} \leq \mathrm{Log}\,\frac{1}{|x|} = |\mathrm{Log}\,|x||$$

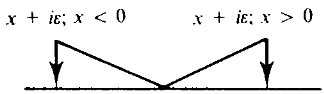
Let $\varphi \in \mathscr{D}(\mathbb{R})$. Then

$$\int f_\varepsilon(x)\varphi(x)\,\mathrm{d}x = \int \mathrm{Log}\,|x + i\varepsilon|\varphi(x)\,\mathrm{d}x + i\int \mathrm{Arg}\,(x + i\varepsilon)\varphi(x)\,\mathrm{d}x = I_1 + iI_2$$

By the above inequalities and since $\mathrm{Log}\,|x + i\varepsilon|$ converges almost everywhere to $\mathrm{Log}\,|x|$ when $\varepsilon \to 0$ we deduce that $I_1 \to \int_{\mathbb{R}} \mathrm{Log}\,|x|\varphi(x)\,\mathrm{d}x$ by the Lebesgue theorem. On the other hand:

if $x > 0$, $\lim\limits_{\varepsilon\to 0^+} \mathrm{Arg}\,(x + i\varepsilon) = 0$ and if $x < 0$, $\lim\limits_{\varepsilon\to 0^+} \mathrm{Arg}\,(x + i\varepsilon) = \pi$. Now

$$|\mathrm{Arg}\,(x + i\varepsilon)| \leq 2\pi$$



The Lebesgue theorem then implies that

$$\lim_{\varepsilon\to 0} I_2 = i\pi\int_{-\infty}^{0} \varphi(x)\,\mathrm{d}x \qquad \text{Q.E.D}$$

b) We can write $f_0 = \mathrm{Log}\,|x| + i\pi H(-x)$ where H is the Heaviside function. By exercise 26, $(\mathrm{Log}\,|x|)' = pv\,\frac{1}{x}$ and $(H(-x))' = -\delta$. Therefore

$$\frac{\mathrm{d}f_0}{\mathrm{d}x} = pv\frac{1}{x} - i\pi\delta$$

c) $\dfrac{d}{dx}$ Log $(x + i\varepsilon) = \dfrac{1}{x + i\varepsilon}$. Indeed

$$\frac{d}{dx} \operatorname{Log} |x + i\varepsilon| = \frac{x}{x^2 + \varepsilon^2}$$

and

$$\frac{d}{dx} \operatorname{Arctg} \frac{\varepsilon}{x} = -\frac{\varepsilon}{x^2} \frac{1}{1 + \dfrac{\varepsilon^2}{x^2}} = \frac{-\varepsilon}{x^2 + \varepsilon^2}$$

so

$$\frac{d}{dx} \operatorname{Log} (x + i\varepsilon) = \frac{x - i\varepsilon}{x^2 + \varepsilon^2} = \frac{1}{x + i\varepsilon}$$

Since the derivation is continuous on $\mathscr{D}'(\mathbb{R})$ we deduce from question a) and b) that

$$\lim_{\varepsilon \to 0^+} \frac{1}{x + i\varepsilon} = \lim_{\varepsilon \to 0} \frac{d}{dx} f_\varepsilon = \frac{d}{dx} f_0 = pv\frac{1}{x} - i\pi\delta$$

d) Let us set $\varepsilon = -\alpha$, $\alpha > 0$. Then $\dfrac{1}{x + i\varepsilon} = -\dfrac{1}{y + i\alpha}$ where $y \in \mathbb{R}$, $y = -x$. From c) we get

$$\lim_{\alpha \to 0^+} \frac{1}{y + i\alpha} = pv\frac{1}{y} - i\pi\delta$$

so

$$\lim_{\varepsilon \to 0} \frac{1}{x + i\varepsilon} = -pv\frac{1}{y} + i\pi\delta = pv\frac{1}{x} + i\pi\delta \quad \text{for } -pv\frac{1}{-x} = pv\frac{1}{x}$$

Therefore

$$\delta = \frac{1}{2i\pi}\left[\lim_{\varepsilon \to 0^+} \frac{1}{x - i\varepsilon} - \lim_{\varepsilon \to 0^+} \frac{1}{x + i\varepsilon}\right] = \lim_{\varepsilon \to 0^+} \frac{\varepsilon}{\pi(x^2 + \varepsilon^2)}$$

$$pv\frac{1}{x} = \frac{1}{2}\left[\lim_{\varepsilon \to 0^+} \frac{1}{x + i\varepsilon} + \lim_{\varepsilon \to 0^+} \frac{1}{x - i\varepsilon}\right] = \lim_{\varepsilon \to 0^+} \frac{x}{x^2 + \varepsilon^2}$$

Solution 48

First of all

$$(1) \quad \frac{e^{ixt}}{x - i0} = i\pi\, e^{ixt}\delta + e^{ixt} pv\frac{1}{x}$$

Now

$$\langle i\pi \mathrm{e}^{ixt}\delta, \varphi\rangle = i\pi\langle \delta, \mathrm{e}^{ixt}\varphi\rangle = i\pi\varphi(0) = i\pi\langle\delta, \varphi\rangle \quad \text{for } \varphi \in \mathscr{D}(\mathbb{R})$$

so

$$(2) \quad i\pi \mathrm{e}^{ixt}\delta = i\pi\delta$$

On the other hand for $\varphi \in \mathscr{D}(\mathbb{R})$, supp $\varphi \subset \{|x| \leq M\}$ we have

$$\left\langle \mathrm{e}^{ixt} pv\frac{1}{x}, \varphi \right\rangle = \left\langle pv\frac{1}{x}, \mathrm{e}^{ixt}\varphi \right\rangle = \lim_{\varepsilon\to 0} \int_{\varepsilon\leq|x|\leq M} \frac{\mathrm{e}^{ixt}\varphi(x)}{x}\mathrm{d}x$$

$$\int_{\varepsilon\leq|x|\leq M} \frac{\mathrm{e}^{ixt}\varphi(x)}{x}\mathrm{d}x = \int_\varepsilon^M \frac{\mathrm{e}^{ixt}\varphi(x)}{x}\mathrm{d}x - \int_\varepsilon^M \frac{\mathrm{e}^{-ixt}\varphi(-x)}{x}\mathrm{d}x$$

Since $\varphi(x) = \varphi(0) + x\psi(x)$ where ψ is continuous on $\mathbb{R}$, we get

$$\int_{\varepsilon\leq|x|\leq M} \frac{\mathrm{e}^{ixt}\varphi(x)}{x}\mathrm{d}x = \varphi(0)\int_\varepsilon^M \frac{\mathrm{e}^{ixt} - \mathrm{e}^{-ixt}}{x}\mathrm{d}x + \int_\varepsilon^M [\mathrm{e}^{ixt}\psi(x) + \mathrm{e}^{-ixt}\psi(-x)]\,\mathrm{d}x$$

$$= \underbrace{2i\varphi(0)\int_\varepsilon^M \frac{\sin xt}{x}\mathrm{d}x}_{I_\varepsilon} + \underbrace{\int_\varepsilon^M [\mathrm{e}^{ixt}\psi(x) + \mathrm{e}^{-ixt}\psi(-x)]\,\mathrm{d}x}_{J_\varepsilon}$$

$$I_\varepsilon = 2i\varphi(0)\int_\varepsilon^M \frac{\sin xt}{tx}\mathrm{d}(tx) = 2i\varphi(0)\int_{t\varepsilon}^{tM} \frac{\sin y}{y}\mathrm{d}y$$

Since the function $\dfrac{\sin y}{y}$ is continuous at the origin the Lebesgue theorem gives

$$\lim_{\varepsilon\to 0}\int_{t\varepsilon}^{tM} \frac{\sin y}{y}\mathrm{d}y = \int_0^{tM} \frac{\sin y}{y}\mathrm{d}y$$

so

$$\lim_{\varepsilon\to 0} I_\varepsilon = 2i\varphi(0)\int_0^{tM} \frac{\sin y}{y}\mathrm{d}y$$

Using once more the Lebesgue theorem we get

$$\lim_{\varepsilon\to 0} J_\varepsilon = \int_\varepsilon^M [\mathrm{e}^{ixt}\psi(x) + \mathrm{e}^{-ixt}\psi(-x)]\,\mathrm{d}x$$

Therefore

$$\left\langle \mathrm{e}^{ixt} pv\frac{1}{x}, \varphi \right\rangle = 2i\varphi(0)\int_\varepsilon^{tM} \frac{\sin y}{y}\mathrm{d}y + \int_0^M [\mathrm{e}^{ixt}\psi(x) + \mathrm{e}^{-ixt}\psi(-x)]\,\mathrm{d}x$$

When t goes to infinity we get

$$\lim_{t\to\infty} \int_0^{tM} \frac{\sin y}{y} \mathrm{d}y = \int_0^{\infty} \frac{\sin y}{y} \mathrm{d}y = \frac{\pi}{2}$$

By the Riemann–Lebesgue theorem we get

$$\lim_{t\to\infty} \int_0^{M} e^{ixt} \psi(x) \,\mathrm{d}x = 0$$

so

$$(3) \quad \lim_{t\to+\infty} \left\langle e^{ixt} pv\frac{1}{x}, \varphi \right\rangle = i\pi\varphi(0) = i\pi\langle \delta, \varphi \rangle$$

and the result follows from (1), (2), and (3).

Solution 49

- Let us set $S_N = \sum_{n=0}^{N} a^n \delta_n$. Let $\varphi \in \mathscr{D}(\mathbb{R})$, and $M > 0$ such that $\operatorname{supp} \varphi \subset \{|x| \leq M\}$.

$$\langle S_N, \varphi \rangle = \sum_{n=0}^{N} a^n \varphi(n) \quad \text{so} \lim_{N\to\infty} \langle S_N, \varphi \rangle = \sum_{n=0}^{M} a^n \varphi(n)$$

therefore S_N converges to a distribution T when N goes to infinity, for all $a \in \mathbb{R}^+$.

- If $\varphi \in C^{\infty}(\mathbb{R})$ we have

$$\langle S_N, \varphi \rangle = \sum_{n=0}^{N} a^n \varphi(n) \quad \text{so } S_N \in \mathscr{E}'$$

If $a \geq 1$ let us take $\varphi \equiv 1$ then $\langle S_N, \varphi \rangle = \sum_0^N a^n$ which tends to infinity when $N \to \infty$.

If $a < 1$ let us take $\varphi(x) = \exp\left(x \cdot \operatorname{Log} \frac{1}{a}\right) = \left(\frac{1}{a}\right)^x$. Then

$$\langle S_N, \varphi \rangle = \sum_{n=0}^{N} a^n \left(\frac{1}{a}\right)^n = N + 1$$

Therefore in all cases the series $\Sigma\, a^n \delta_n$ diverges in $\mathscr{E}'$.

Solution 50

a) Let us consider the sequence $S_N^{(p)} = \sum_{\substack{n=-N \\ n\neq 0}}^{N} \frac{a_n}{(2i\pi n)^{p+2}} e^{2i\pi n x}$. If $n \neq 0$, we have

$$\left| \frac{a_n}{(2i\pi n)^{p+2}} \right| \leq \frac{C_p}{n^2}$$

Therefore when $N \to \infty$, $S_N^{(p)}$ converges uniformly, so in $\mathscr{D}'(\mathbb{R})$, to a continuous function $S^{(p)}$. Moreover

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{p+2} S_N^{(p)} = \sum_{\substack{n=-N \\ n \neq 0}}^{N} a_n \mathrm{e}^{2i\pi n x} = T_N - a_0$$

Since the derivation is continuous on $\mathscr{D}'(\mathbb{R})$ it follows that T_N converges in $\mathscr{D}'(\mathbb{R})$ to the distribution $\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{p+2} S^{(p)} + a_0$.

b) $\frac{\mathrm{d}T_N}{\mathrm{d}x} = \sum_{-N}^{N} (2i\pi n)a_n \mathrm{e}^{2i\pi n x}$. Now $|(2i\pi n)a_n| \leq Cn^{p+1}$ so by question a) the sequence $\left(\frac{\mathrm{d}}{\mathrm{d}x}\right) T_N$ converges in $\mathscr{D}'(\mathbb{R})$ to $\frac{\mathrm{d}T}{\mathrm{d}x}$ and to $\sum_{-\infty}^{+\infty} (2i\pi n)a_n \mathrm{e}^{2i\pi n x}$.

By definition, if $\varphi \in \mathscr{D}(\mathbb{R})$ we have:

$$\langle \tau_1 T, \varphi\rangle = \langle T, \tau_{-1}\varphi\rangle = \lim_{N\to\infty} \langle T_N, \tau_{-1}\varphi\rangle = \lim_{N\to\infty} \sum_{-N}^{N} \int_{\mathbb{R}} a_n \mathrm{e}^{2i\pi n x}\varphi(x+1)\,\mathrm{d}x$$

$$= \lim_{N\to\infty} \sum_{-N}^{N} a_n \int_{\mathbb{R}} \mathrm{e}^{2i\pi n x}\varphi(x)\,\mathrm{d}x = \lim_{N\to\infty} \langle T_N, \varphi\rangle = \langle T, \varphi\rangle$$

i.e., $\tau_1 T = T$ in $\mathscr{D}'(\mathbb{R})$.

c) First of all by question a) S makes sense. On the other hand we have in $\mathscr{D}'(\mathbb{R})$:

$$\begin{aligned}(1 - \mathrm{e}^{2i\pi x})S &= \lim_{N\to\infty} (1 - \mathrm{e}^{2i\pi x})S_N \\ &= \lim_{N\to\infty} \left(\sum_{n=-N}^{N} \mathrm{e}^{2i\pi n x} - \sum_{n=-N+1}^{N+1} \mathrm{e}^{2i\pi n x}\right) \\ &= \lim_{N\to\infty} (\mathrm{e}^{-2i\pi N x} - \mathrm{e}^{2i\pi(N+1)x}) = 0\end{aligned}$$

Indeed for φ in $\mathscr{D}(\mathbb{R})$ we get:

$$\langle \mathrm{e}^{-2i\pi N x} - \mathrm{e}^{2i\pi(N+1)x}, \varphi\rangle = \int_{\mathbb{R}} \mathrm{e}^{-2i\pi N x}\varphi(x)\,\mathrm{d}x - \int_{\mathbb{R}} \mathrm{e}^{2i\pi(N+1)x}\varphi(x)\,\mathrm{d}x$$

$$\langle \mathrm{e}^{-2i\pi N x} - \mathrm{e}^{2i\pi(N+1)x}, \varphi\rangle = \frac{-1}{2i\pi N}\int_{\mathbb{R}} \mathrm{e}^{-2i\pi N x}\varphi'(x)\,\mathrm{d}x - \frac{1}{2i\pi(N+1)}\int_{\mathbb{R}} \mathrm{e}^{2i\pi(N+1)x}\varphi'(x)\,\mathrm{d}x$$

so

$$|\langle \mathrm{e}^{-2i\pi N x} - \mathrm{e}^{2i\pi(N+1)x}, \varphi\rangle| \leq \frac{C}{N}\int_{\mathbb{R}} |\varphi'(x)|\,\mathrm{d}x \to 0 \quad \text{if } N \to \infty$$

In the interval $]-N, N[$ the zeroes of the function $\mathrm{e}^{2i\pi x} - 1$ are

$$-N+1, -N+2, \ldots, 0, 1, 2, \ldots, N-1$$

If $x \in]-N, N[$ we get $e^{2i\pi x} - 1 = a(x)(x + N - 1) \cdots (x - N + 1)$ with $a(x) \neq 0$, therefore in this interval

$$(x + N - 1) \cdots (x - N + 1)S = 0.$$

Now if $a_1 \ldots a_k$ are distinct real numbers, the general solution of

$$(x - a_1) \cdots (x - a_k)S = 0$$

in the distribution $S = C_1\delta_{a_1} + C_2\delta_{a_2} + \cdots + C_k\delta_{a_k}$.
It follows that:

$$\langle S, \varphi\rangle = \sum_{-N+1}^{N-1} C_n\varphi(n) \qquad \forall\varphi \in \mathscr{D}(]-N, N[)$$

so $S = \sum_{-\infty}^{\infty} C_n\delta_n$.
On the other hand $\tau_1 S = S$, i.e. $\langle S, \tau_1\varphi\rangle = \langle S, \varphi\rangle$, $\forall\varphi \in \mathscr{D}(\mathbb{R})$.
Let $\varphi \in \mathscr{D}(]n - \frac{1}{2}, n + \frac{1}{2}[)$ be such that $\varphi(n) = 1$. We get

$$\langle S, \tau_{-1}\varphi\rangle = C_{n-1} = \langle S, \varphi\rangle = C_n \qquad \forall n \in \mathbb{Z}$$

which implies that $C_n = C$, $\forall n \in \mathbb{Z}$, so $S = C\sum_{-\infty}^{\infty} \delta_n$

d) By question b) we have in $\mathscr{D}'(\mathbb{R})$

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \sum_{-\infty}^{\infty} \frac{2i\pi n}{1 + 4\pi^2n^2}\mathrm{e}^{2i\pi nx}, \qquad \frac{\mathrm{d}^2f}{\mathrm{d}x^2} = -\sum_{-\infty}^{\infty} \frac{4\pi^2n^2}{1 + 4\pi^2n^2}\mathrm{e}^{2i\pi nx}$$

i.e., $\dfrac{\mathrm{d}^2f}{\mathrm{d}x^2} - 4\pi^2 f = -4\pi^2 S$.

Since $S = C\sum_{-\infty}^{\infty} \delta_n$, for $\varphi \in \mathscr{D}(]0, 1[)$ we have $\langle S, \varphi\rangle = 0$, so in $\mathscr{D}'(]0, 1[)$

$$\frac{\mathrm{d}^2f}{\mathrm{d}x^2} - 4\pi^2 f = 0$$

We know that the distributions solutions of this equation are C^∞ functions and that this equation can be solved by the usual methods so

$$f(x) = C_1\mathrm{e}^{2\pi x} + C_2\mathrm{e}^{-2\pi x}$$

Let us set $C_1 = a_1\mathrm{e}^{-\pi}$, $C_2 = a_2\mathrm{e}^{\pi}$ we get

$$f(x) = a_1\mathrm{e}^{2\pi(x-1/2)} + a_2\mathrm{e}^{-2\pi(x-1/2)}$$

Since f is periodic with period 1 we have $f(0) = f(1)$ so

$$a_1\mathrm{e}^{-\pi} + a_2\mathrm{e}^{\pi} = a_1\mathrm{e}^{\pi} + a_2\mathrm{e}^{-\pi}$$

and $a_1 = a_2 = a$.

e) The jump's formula applied to f in $]-1, 1[$ gives:

$$f'' = \{f''\} + \sigma_0 \cdot \delta_0$$

where f'' is the second derivative of f in $\mathscr{D}'$, $\{f''\}$ the distribution given by the function f'' in the set $]-1, 0[\cup]0, 1[$ and σ_0 is the jump of the first derivative at the origin. Now

$$f'(x) = a(2\pi\,\mathrm{e}^{2\pi(x-1/2)} - 2\pi\,\mathrm{e}^{-2\pi(x-1/2)})$$

so

$$\sigma_0 = f'(0) - f'(1) = a(2\pi\,\mathrm{e}^{-\pi} - 2\pi\,\mathrm{e}^{\pi} - 2\pi\,\mathrm{e}^{\pi} + 2\pi\,\mathrm{e}^{-\pi})$$

$$\sigma_0 = 4\pi a(\mathrm{e}^{-\pi} - \mathrm{e}^{\pi})$$

$$f'' = \{f''\} + 4\pi a(\mathrm{e}^{-\pi} - \mathrm{e}^{\pi})\delta_0$$

therefore

$$f'' - 4\pi^2 f = \{f''\} - 4\pi^2\{f\} + 4\pi a(\mathrm{e}^{-\pi} - \mathrm{e}^{\pi})\delta_0$$

Now $f'' - 4\pi^2 f = -4\pi^2 C\delta_0$ in $]-1, 1[$ and $\{f''\} - 4\pi^2\{f\} = 0$.
so

$$4\pi a(\mathrm{e}^{-\pi} - \mathrm{e}^{\pi})\delta_0 = -4\pi^2 C\delta_0$$

and

$$C = \frac{\mathrm{e}^{\pi} - \mathrm{e}^{-\pi}}{\pi}a$$

f) First of all, by the uniform convergence of the series, we have

$$\int_0^1 f(x)\,\mathrm{d}x = \sum_{-\infty}^{\infty} \frac{1}{1+n^2}\int_0^1 \mathrm{e}^{2i\pi nx}\,\mathrm{d}x = 1$$

On the other hand

$$\int_0^1 f(x)\,\mathrm{d}x = a\left(\int_0^1 \mathrm{e}^{2\pi(x-1/2)}\,\mathrm{d}x + \int_0^1 \mathrm{e}^{-2\pi(x-1/2)}\,\mathrm{d}x\right)$$

so

$$1 = a\frac{\mathrm{e}^{\pi} - \mathrm{e}^{-\pi}}{\pi} \quad \text{i.e. } a = \frac{\pi}{\mathrm{e}^{\pi} - \mathrm{e}^{-\pi}}$$

It follows that $C = \dfrac{\mathrm{e}^{\pi} - \mathrm{e}^{-\pi}}{\pi} \times \dfrac{\pi}{\mathrm{e}^{\pi} - \mathrm{e}^{-\pi}} = 1$ Q.E.D.

Solution 51

a) Let (u_k) be a sequence in E which converges to u in $L^2(\Omega_1)$. Then it converges to u in $\mathscr{D}'(\Omega_1)$. Indeed if $\varphi \in \mathscr{D}(\Omega_1)$ we have

$$|\langle u_k - u, \varphi\rangle| = \left|\int_{\Omega_1} (u_k(x) - u(x))\varphi(x)\,\mathrm{d}x\right|$$
$$\leq \left(\int_{\Omega_1} |u_k(x) - u(x)|^2\,\mathrm{d}x\right)^{1/2}\left(\int_{\Omega_1} |\varphi(x)|^2\,\mathrm{d}x\right)$$

The derivation being a continuous map in $\mathscr{D}'(\Omega_1)$, the operator P is continuous from $\mathscr{D}'(\Omega_1)$ to itself. It follows that (Pu_k) converges to Pu in $\mathscr{D}'(\Omega_1)$. Since $Pu_k = 0$, for all k, it follows that $Pu = 0$ so $u \in E$.

b) By question a) E is a complete subspace of $L^2(\Omega_1)$. If P satisfies (H) then every u in E is actually a C^∞ function in Ω_1. It follows that $\dfrac{\partial u}{\partial x_j} \in L^2(\Omega_2)$, $1 \leq j \leq n$, where $\Omega_2 \subset\subset \Omega_1$.

If we can show that the map

$$E \ni u \mapsto \frac{\partial u}{\partial x_j} \in L^2(\Omega_2)$$

has a closed graph it will follow from the closed graph theorem (which can be used here since E and $L^2(\Omega_2)$ are complete) that this map is continuous, which proves (1). Let $(u_k) \subset E$ be a sequence such that (u_k) converges to u in E and $\left(\dfrac{\partial u_k}{\partial x_j}\right)$ converges to v in $L^2(\Omega_2)$. We have to prove that $v = \dfrac{\partial u}{\partial x_j}$.
Since convergence in L^2 implies convergence in $\mathscr{D}'$ it follows that (u_k) converges to u in $\mathscr{D}'(\Omega_1)$. By the continuity of the derivation in $\mathscr{D}'$, $\left(\dfrac{\partial u_k}{\partial x_j}\right)$ converges to $\dfrac{\partial u}{\partial x_j}$ in $\mathscr{D}'(\Omega_1)$ and then in $\mathscr{D}'(\Omega_2)$. Since $\left(\dfrac{\partial u_k}{\partial x_j}\right)$ converges to v in $\mathscr{D}'(\Omega_2)$ we get $v = \dfrac{\partial u}{\partial x_j}$.

c) Let $\zeta_k = \xi_k + i\eta_k \in \mathbb{Z}^n$, $\xi_k \in \mathbb{R}^n$, $\eta_k \in \mathbb{R}^n$. Let us assume that $P(\zeta_k) = 0$. Let us set $u = \mathrm{e}^{i\langle x, \zeta_k\rangle}$ where $\langle x, \zeta\rangle = \sum_{j=1}^{n} x_j \cdot \zeta^j$. Since $u \in C^\infty(\mathbb{R}^n)$ it belongs to $L^2(\Omega_1)$ since Ω_1 is bounded. On the other hand

$$Pu = \left(\sum_{|\alpha| \leq m} a_\alpha \partial^\alpha\right)\mathrm{e}^{i\langle x, \zeta_k\rangle} = \sum_{|\alpha| \leq m} a_\alpha (i\zeta_k)^\alpha \mathrm{e}^{i\langle x, \zeta_k\rangle} = P(\zeta_k)\,\mathrm{e}^{i\langle x, \zeta_k\rangle} = 0$$

So $u \in E$. Applying inequality (1) we get:

$$(2) \quad \sum_{k=1}^{n} |\zeta_k^j|^2 \int_{\Omega_2} e^{-2\langle x, \eta_k \rangle} \mathrm{d}x \leqq C \int_{\Omega_1} e^{-2\langle x, \eta_k \rangle} \mathrm{d}x$$

since $|e^{i\langle x, \zeta_k \rangle}| = e^{-\langle x, \eta_k \rangle}$. Now by the Cauchy Schwarz inequality

$$-|x| \cdot |\eta_k| \le -\langle x, \eta_k \rangle \leqq |x| \cdot |\eta_k|$$

Since Ω_1 is bounded we have $|x| \le M$ (where M is a constant) for x in Ω_1. Then

$$(3) \quad -M \cdot |\eta_k| \le \langle x, \eta_k \rangle \le M \cdot |\eta_k|$$

It follows from (2) and (3) that

$$|\zeta_k|^2 e^{-2M \cdot |\eta_k|} \cdot \mu(\Omega_2) \le C\mu(\Omega_1) \cdot e^{2M \cdot |\eta_k|}$$

where $\mu(\Omega_i)$ is the measure of Ω_i, $i = 1, 2$. So we get

$$|\zeta_k|^2 \leqq \frac{C\mu(\Omega_1)}{\mu(\Omega_2)} e^{4M \cdot |\eta_k|}$$

therefore if $|\zeta_k| \to \infty$, $|\mathrm{Im}\, \zeta_k| = |\eta_k| \to \infty$.

d) Let us consider the differential operator in $\mathbb{R}^2$

$$P(\partial_x, \partial_t) = \frac{1}{i}\frac{\partial}{\partial t} + \frac{\partial^2}{\partial x^2} = \frac{1}{i}\frac{\partial}{\partial t} - \left(\frac{1}{i}\frac{\partial}{\partial x}\right)^2$$

The polynomial associed with this operator is

$$P(\xi, \eta) = \eta - \xi^2$$

It follows that

$$P(\xi, \xi^2) = 0 \quad \text{for all } \xi \in \mathbb{R}.$$

The sequence $\zeta_k = (\zeta_k^1, \zeta_k^2)$ in $\mathbb{C}^2$ where $\zeta_k^1 = \xi_k \in \mathbb{R}$, $\zeta_k^2 = (\xi_k)^2 \in \mathbb{R}$ satisfies $P(\zeta_k) = 0$, $|\zeta_k| \to \infty$ if $|\xi_k| \to \infty$ but $\mathrm{Im}\, \zeta_k = 0$. By question c) this operator is not hypoelliptic.

Solution 52

a) If $A \in \mathbb{N}$ one has just to remark that inequality (1) is also true with $A + \alpha$ where $\alpha \in]0, 1[$ since $\delta < 1$.

b) $P_1 f$ is holomorphic since it is a primitive of a holomorphic function. As a parametrization of γ_1 we shall take

$$x(t) = tx + (1 - t)\frac{a + b}{2}$$

$$y(t) = ty + (1 - t)\frac{\delta}{2}$$

then $\mathrm{d}u = \mathrm{d}x + i\mathrm{d}y = [x'(t) + iy'(t)]\,\mathrm{d}t = \left[\left(x - \frac{a + b}{2}\right) + i\left(y - \frac{\delta}{2}\right)\right]\mathrm{d}t$

We get

$$(P_1 f)(z) = \left[\left(x - \frac{a+b}{2}\right) + i\left(y - \frac{\delta}{2}\right)\right]\int_0^1 f\left(tx + (1 - t)\frac{a+b}{2} + i\left(ty + (1 - t)\frac{\delta}{2}\right)\right)\mathrm{d}t$$

The inequality (1) applied to f gives

$$|(P_1 f)(z)| \leqq \left|\left(x - \frac{a + b}{2}\right) + i\left(y - \frac{\delta}{2}\right)\right|\int_0^1 \frac{\mathrm{d}t}{|ty + (1 - t)\delta/2|^A}$$

Let us set

$$s = ty + (1 - t)\frac{\delta}{2}$$

we get

$$|(P_1 f)(z)| \leq \left|\left(x - \frac{a + b}{2}\right) + i\left(y - \frac{\delta}{2}\right)\right|\frac{1}{y - \delta/2}\int_{\delta/2}^{y} \frac{\mathrm{d}s}{s^A}$$

so

$$|(P_1 f)(z)| \leq \left|\left(x - \frac{a + b}{2}\right) + i\left(y - \frac{\delta}{2}\right)\right|\frac{1}{(A - 1)(\delta/2 - y)}\left[\frac{1}{y^{A-1}} - \frac{1}{\delta^{A-1}}\right]$$

Now $|x| \leq |a| + |b|$, $0 < y \leqq \frac{\delta}{4}$ then $\frac{\delta}{2} - y \geq \frac{\delta}{4}$ so

$$|(P_1 f)(z)| \leq \frac{C'}{y^{A-1}}$$

for all y such that $0 < y \leq \frac{\delta}{4}$.

Let us assume now that

$$\exists C > 0: \qquad |(P_k f)| \leq \frac{C_k}{|\mathrm{Im}\, z|^{A-k}} \quad \text{for } |\mathrm{Im}\, z| \leq \frac{\delta}{2^{k+1}}$$

In the same way as before we get

$$|(P_{k+1}f)(z)| = |P_{\delta/2^{k+1}}(P_k f)(z)| \leq C_k \frac{1}{(A - k - 1)(\delta/(2^{k+1}) - y)} \frac{1}{y^{A-k-1}}$$

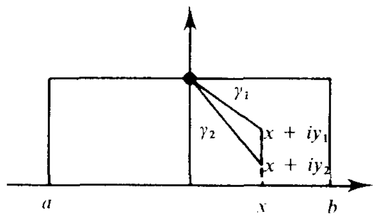
and for $0 < y \leq \frac{\delta}{2^{k+2}}$ we get

$$|(P_{k+1}f)(z)| \leq \frac{C_{k+1}}{y^{A-(k+1)}} \quad \text{for } y \leq \frac{\delta}{2^{k+2}}$$

Inequality (2) follows from the above inequality with $k = N - 1$ since $A - N = \alpha$.

c) Let y_1, y_2 be such that

$$|y_1| \leq \frac{\delta}{2^{N+2}}, \ |y_2| \leq \frac{\delta}{2^{N+2}}.$$



We have

$$v(x + iy_1) - v(x + iy_2) = \int_{\gamma_1} u(\alpha)\,\mathrm{d}\alpha - \int_{\gamma_2} u(\beta)\,\mathrm{d}\beta$$

where

$$\gamma_1 = \left[\frac{a + b}{2} + i\frac{\delta}{2^{N+1}}, x + iy_1\right], \qquad \gamma_2 = \left[\frac{a + b}{2} + i\frac{\delta}{2^{N+1}}, x + iy_2\right]$$

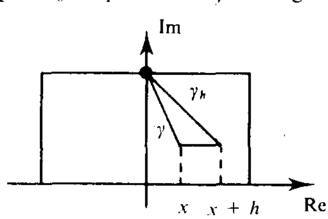
Since u is holomorphic in D we have

$$v(x + iy_1) - v(x + iy_2) = \int_{y_1}^{y_2} u(x + is)\,\mathrm{d}s$$

By (2)

$$\sup_{x \in]a,b[} |v(x + iy_1) - v(x + iy_2)| \leq \int_{y_1}^{y_2} \frac{C}{s^\alpha}\mathrm{d}s \leq C'|y_1^{1-\alpha} - y_2^{1-\alpha}|$$

Therefore the sequence (v_y) defined by $v_y(x) = v(x + iy)$ is a Cauchy sequence in the complete space $C^0(]a, b[)$. So it converges i.e. there exists $v_0 \in C^0(]a, b[)$ such that v_y converges uniformly, on every compact in $]a, b[$, to v_0. In particular v_y converges in $\mathscr{D}'(]a, b[)$ to v_0.



d) $\left(\frac{\partial}{\partial x}\right)(P_1 f)(z)$

$$= \lim_{h \to 0} \frac{1}{h}\left[\int_{\gamma_h} f(\xi)\,\mathrm{d}\xi - \int_{\gamma} f(\xi)\,\mathrm{d}\xi\right]$$

where

$$\gamma_h = \left[\frac{a+b}{2} + i\frac{\delta}{2}, x + h + iy\right], \qquad \gamma = \left[\frac{a+b}{2} + i\frac{\delta}{2}, x + iy\right]$$

Therefore

$$\left(\frac{\partial}{\partial x}\right)(P_1 f)(z) = \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t + iy)\,\mathrm{d}t = f(x + iy) = f_y(x)$$

Let us prove, by induction, that

$$\left(\frac{\partial}{\partial x}\right)^n (P_n f)(z) = f_y(x)$$

We saw that this is true for $n = 1$. Let us assume this is true for $n - 1$. Then

$$\left(\frac{\partial}{\partial x}\right)^n (P_n f)(z) = \left(\frac{\partial}{\partial x}\right)^{n-1} \left[\left(\frac{\partial}{\partial x}\right) P_{\delta/2^n}(P_{n-1} f)(z)\right]$$

In the same way we have

$$\left(\frac{\partial}{\partial x}\right) P_{\delta/2^n}(P_{n-1} f)(z) = (P_{n-1} f)(z)$$

and from the induction we get:

$$\left(\frac{\partial}{\partial x}\right)^n (P_n f)(z) = \left(\frac{\partial}{\partial x}\right)^{n-1} (P_{n-1} f)(z) = f(x + iy)$$

It follows that

$$\left(\frac{\partial}{\partial x}\right)^{N+1} (P_{N+1} f) = \left(\frac{\partial}{\partial x}\right)^{N+1} v = f$$

$$\left(\frac{\partial}{\partial x}\right)^{N+1} v_y = f_y$$

Since v_y converges to v_0 in $\mathscr{D}'(]a, b[)$ and since the derivation is continuous in $\mathscr{D}'(]a, b[)$ it follows that f_y converges to $\left(\frac{\partial}{\partial x}\right)^{N+1} v_0$ in $\mathscr{D}'(]a, b[)$ when $y \to 0$.

Q.E.D.

CHAPTER 5

Convolution of distributions

PROGRAMME

Convolution of distributions and functions

Convolution of distributions

BASICS — CHAPTER 5

a) Convolution of distributions and functions
Let $T \in \mathscr{E}'(\mathbb{R}^n)$ and $\varphi \in C^\infty(\mathbb{R}^n)$ (or $T \in \mathscr{D}'(\mathbb{R}^n)$ and $\varphi \in \mathscr{D}(\mathbb{R}^n)$). The convolution of T and φ is the C^∞ function $T * \varphi$ defined by

$$(1) \quad (T * \varphi)(x) = \langle T_t, \varphi(x - t)\rangle$$

In particular we have

$$(2) \quad (T * \varphi)(0) = \langle T, \check{\varphi}\rangle \quad \text{where} \quad \check{\varphi}(t) = \varphi(-t)$$

b) Convolution of a distribution and a distribution with compact support
Let $T \in \mathscr{E}'(\mathbb{R}^n)$, $S \in \mathscr{D}'(\mathbb{R}^n)$; the convolution $T * S$ is the distribution which is defined for $\varphi \in \mathscr{D}(\mathbb{R}^n)$ by

$$(3) \quad \langle T * S, \varphi\rangle = \langle T_x, \langle S_y, \varphi(x + y)\rangle\rangle$$

This is another definition for $T * \varphi$:

$$(4) \quad \langle T * S, \varphi\rangle = [T * (S * \check{\varphi})](0)$$

One can prove the equivalence of these two definitions.
When $S \in C^\infty(\mathbb{R}^n)$ or $T \in \mathscr{D}(\mathbb{R}^n)$, the definitions (1), (3), (4) coincide, which implies in particular that if $T \in \mathscr{E}'(\mathbb{R}^n)$ and $S \in C^\infty(\mathbb{R}^n)$, $T * S \in C^\infty(\mathbb{R}^n)$.

c) Properties of the convolution
Let S and T be two distributions, one of them with compact support.

- *Commutativity:* $S * T = T * S$
- *Associativity:* If $U \in \mathscr{E}'(\mathbb{R}^n)$ we have

$$(S * T) * U = S * (T * U)$$

- *Differentiation:* For every $\alpha \in \mathbb{N}^n$ we have

$$\partial^\alpha(S * T) = (\partial^\alpha S) * T = S * (\partial^\alpha T)$$

- *Unit element:* For all T in $\mathscr{D}'(\mathbb{R}^n)$

$$\delta * T = T$$

- *Support:* supp $(S * T) \subset$ supp $S +$ supp $T = \{x \in \mathbb{R}^n, x = y + z, y \in \text{supp } S, z \in \text{supp } T\}$
- *Singular support:* sing supp $(S * T) \subset$ sing supp $S +$ sing supp T

d) Remark:
One can define the convolution of two distributions in other cases than the one given in b), keeping the properties described in c) (See exercise 63). But we cannot define in general the convolution of two distributions keeping the properties c). See exercise 55.

STATEMENTS OF THE EXERCISES* CHAPTER 5

Exercise 53
Let $S \in \mathscr{E}'(\mathbb{R})$ and $T \in \mathscr{D}'(\mathbb{R})$. Show that for $k \in \mathbb{N}$:

$$(1)\quad x^k(S * T) = \sum_{j=0}^{k} \binom{k}{j} (x^j S) * (x^{k-j} T)$$

Exercise 54 (see exercise 21)
Let $p, q, m, n \in \mathbb{N}$. Compute

$$T = [x^p \delta^{(q)}] * [x^m \delta^{(n)}]$$

Exercise 55
Show that we cannot define the convolution of three general distributions in the sense that it cannot be associative.
(Hint: Find $u \in \mathscr{D}'$, $v \in \mathscr{E}'$, $w \in \mathscr{D}'$ such that $(u * v) * w \neq u * (v * w)$).

Exercise 56
Let A be a linear map from $\mathscr{D}(\mathbb{R}^n)$ to $C^\infty(\mathbb{R}^n)$ such that
a) If (φ_j) is a sequence in $\mathscr{D}(\mathbb{R}^n)$ which converges to zero, the sequence $(A\varphi_j)$ converges to zero in $C^\infty(\mathbb{R}^n)$.

* Solutions pp. 118–134.

b) $\tau_h A\varphi = A\tau_h\varphi$ for every $\varphi \in \mathscr{D}(\mathbb{R}^n)$ and every $h \in \mathbb{R}^n$, where $\tau_h f(x) = f(x - h)$. Show that there exists $T \in \mathscr{D}'(\mathbb{R}^n)$ such that for every $\varphi \in \mathscr{D}(\mathbb{R}^n)$.

$$(1) \quad A\varphi(x) = (T * \varphi)(x)$$

(Hint: Use the formula $\langle T, \varphi\rangle = (T * \check{\varphi})(0)$ where $\check{\varphi}(x) = \varphi(-x)$.)

Exercise 57

a) Compute in $\mathscr{D}'(\mathbb{R}^n)$

$$\lim_{p \to \infty} \frac{p^n}{\pi^{n/2}}\left(1 - \frac{|x|^2}{p}\right)^{p^3} = \lim_{p\to\infty} P_p(x)$$

b) Deduce that every distribution with compact support is a limit, in the distribution's sense, of a sequence of polynomials.

Exercise 58

a) Let $S \in \mathscr{E}'(\mathbb{R}^n)$, $T \in \mathscr{D}'(\mathbb{R}^n)$. Show that for every $a \in \mathbb{R}$ one has

$$e^{\langle a,x\rangle}(S * T) = (e^{\langle a,x\rangle}S) * (e^{\langle a,x\rangle}T)$$

b) Let $P(D) = \sum_{|\alpha|\leq 2} a_\alpha \partial^\alpha$ where $a_\alpha \in \mathbb{C}$. Find an operator Q such that

$$e^{\langle a,x\rangle}P(D)T = Q(D)[e^{\langle a,x\rangle}T] \quad \text{for every } T \in \mathscr{D}'(\mathbb{R}^n)$$

c) Assuming that $E \in \mathscr{D}'(\mathbb{R}^n)$ is a fundamental solution of $P(D)$, find a differential operator whose fundamental solution is $e^{\langle a,x\rangle}E$.

Exercise 59 (see exercise 31)

We recall (see exercise 31) that the distribution in $\mathbb{R}^n$, $n \geq 2$,

$$E_n = \begin{cases} \operatorname{Log} r & \text{if } n = 2 \\ r^{2-n} & \text{if } n \geq 3 \end{cases}$$

satisfies $\Delta E_n = \left(\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}\right)E_n = C_n\delta$ where C_n is a constant.

a) Let $\varphi \in \mathscr{D}(\mathbb{R}^n)$ be such that $\varphi = 1$ near the origin. Prove that $\psi = \Delta(\varphi E_n) - C_n\delta$ belongs to $\mathscr{D}(\mathbb{R}^n)$.

b) Show that for $i = 1, \ldots, n$, $\frac{\partial}{\partial x_i}(\varphi E_n) \in L^1(\mathbb{R}^n)$.

c) Let T be a distribution on $\mathbb{R}^n$ such that $\frac{\partial T}{\partial x_1}, \ldots, \frac{\partial T}{\partial x_n}$ are elements of $L^2(\mathbb{R}^n)$. Deduce from a) and b) that T belongs to $L^2_{\text{loc}}(\mathbb{R}^n)$.

Exercise 60

Let $P = \sum_{|\alpha| \leq m} a_\alpha \partial^\alpha$ be a differential operator with constant coefficients in $\mathbb{R}^n$ such that

(*) P has a fundamental solution which is a C^∞ function in the complement of the origin.

a) Let $\varphi \in \mathscr{D}(\mathbb{R}^n)$ be such that $\varphi = 1$ for $|x| \leq 1$. Show that $\psi = P(\varphi E) - \delta$ belongs to $\mathscr{D}(\mathbb{R}^n)$.

b) Deduce that if $u \in \mathscr{D}'(\mathbb{R}^n)$ is such that $Pu \in C^\infty(\mathbb{R}^n)$ then u is itself a C^∞ function in $\mathbb{R}^n$.

c) Give examples (from Chapter 3) of operators satisfying (*).

Exercise 61

Let $\rho \in \mathscr{D}(\mathbb{R}^n)$ be such that $\rho \geqq 0$ and $\int_{\mathbb{R}^n} \rho(x)\,\mathrm{d}x = 1$. For $\varepsilon > 0$ we set, $\rho_\varepsilon(x) = \frac{1}{\varepsilon^n}\rho\left(\frac{x}{\varepsilon}\right)$, and for $u \in \mathscr{D}'(\mathbb{R}^n)$, $u_\varepsilon = u * \rho_\varepsilon$. Show that when $\varepsilon \to 0$:

a) If $u \in \mathscr{D}'(\mathbb{R}^n)$, $u_\varepsilon \to u$ in $\mathscr{D}'(\mathbb{R}^n)$

b) If $u \in C^0_c(\mathbb{R}^n)$, $u_\varepsilon \to u$ uniformly

c) If $u \in L^p(\mathbb{R}^n)$, $1 \leq p < +\infty$, $u_\varepsilon \to u$ in $L^p(\mathbb{R}^n)$. (Hint: Use b) and prove the inequality $\|v * \rho_\varepsilon\|_{L^p} \leq \|v\|_{L^p}$, $\forall v \in L^p(\mathbb{R}^n)$.

Exercise 62

We consider the space

$$H^1(\mathbb{R}^n) = \left\{u \in L^2(\mathbb{R}^n), \frac{\partial u}{\partial x_i} \in L^2(\mathbb{R}^n), i = 1, \ldots, n\right\}$$

with the norm

$$(1) \quad \|u\|_1^2 = \|u\|^2_{L^2(\mathbb{R}^n)} + \sum_{i=1}^n \left\|\frac{\partial u}{\partial x_i}\right\|^2_{L^2(\mathbb{R}^n)}$$

a) Let $\theta \in \mathscr{D}(\mathbb{R}^n)$, $\theta = 1$ if $|x| \leq 1$, $0 \leq \theta(x) \leq 1$. We set $\theta_k(x) = \theta\left(\frac{x}{k}\right)$. Let $u \in H^1(\mathbb{R}^n)$; show that $u_k(x) = (\theta_k u)(x)$ converges to u in $H^1(\mathbb{R}^n)$. Deduce that $H^1(\mathbb{R}^n) \cap \mathscr{E}'(\mathbb{R}^n)$ is dense in $H^1(\mathbb{R}^n)$.

b) Prove that $\mathscr{D}(\mathbb{R}^n)$ is dense in $H^1(\mathbb{R}^n) \cap \mathscr{E}'(\mathbb{R}^n)$. Deduce that $\mathscr{D}(\mathbb{R}^n)$ is dense in $H^1(\mathbb{R}^n)$. (Hint: Use exercise 61, question c) with $p = 2$).

Exercise 63

We set $\mathscr{D}'_+ = \{T \in \mathscr{D}'(\mathbb{R}^n), \operatorname{supp} T \subset [0, +\infty[\}$

a) Prove that if S and T are in $\mathscr{D}'_+$ one can define the convolution of S and T by

$$\langle S * T, \varphi \rangle = \langle S_x, \langle T_y, \varphi(x + y) \rangle \rangle$$

and that $S * T \in \mathscr{D}'_+$. What is the unit element for the convolution in $\mathscr{D}'_+$?

b) If $T \in \mathscr{D}'_+$ we shall denote by T^{-1} the unique distribution X such that $T * X = \delta$. Compute H^{-1}, $(\delta')^{-1}$, $(\delta' - \lambda\delta)^{-1}$.

c) Let $P(D)$ be a differential operator with constant coefficients. What does the distribution $[P(D)\delta]^{-1}$ represent?

Let $z_1, \ldots, z_m$ be the roots of the equation $P(z) = 0$. Show that

$$[P(D)\delta]^{-1} = H\mathrm{e}^{z_1 x} * H\mathrm{e}^{z_2 x} * \cdots * H\mathrm{e}^{z_m x}$$

Deduce that every differential operator (non identical to zero) with constant coefficients on $\mathbb{R}$ has a fundamental solution.

Exercise 64

a) Find a fundamental solution for the operator $\left(\dfrac{\mathrm{d}}{\mathrm{d}x}\right)^l$, $l \in \mathbb{N}\backslash 0$.

b) Deduce a fundamental solution for the operator $P = \left(\dfrac{\partial}{\partial x_1}\right)^{l_1} \cdots \left(\dfrac{\partial}{\partial x_n}\right)^{l_n}$ where $l_j \in \mathbb{N}\backslash 0$, $j = 1, \ldots, n$. Assuming $l_1 = \cdots = l_n = k + 2$, prove that P has a fundamental solution which is a C^k function in $\mathbb{R}^n$.

c) Let $f \in \mathscr{E}'(\mathbb{R}^n)$ be a distribution with compact support of order $k \in \mathbb{N}$. Show that there exists $u \in C^0(\mathbb{R}^n)$ such that $\left(\dfrac{\partial}{\partial x_1}\right)^{k+2} \cdots \left(\dfrac{\partial}{\partial x_n}\right)^{k+2} u = f$.

d) Let $T \in \mathscr{D}'(\mathbb{R}^n)$. Using c) and a partition of unity, prove that there exist functions $u_\alpha \in C^0(\mathbb{R}^n)$ such that $T = \sum_\alpha \partial^\alpha u_\alpha$ in $\mathscr{D}'(\mathbb{R}^n)$ in the following sense:

$$\forall K \subset\subset \mathbb{R}^n, \exists N_K = N : \langle T, \varphi \rangle = \sum_{|\alpha| \leq N} (-1)^{|\alpha|} \int_{\mathbb{R}^n} u_\alpha \partial^\alpha \varphi \, \mathrm{d}x \qquad \forall \varphi \in \mathscr{D}_K(\mathbb{R}^n)$$

Exercise 65 (See exercises 47 and 61)
Let $T \in \mathscr{E}'(\mathbb{R})$ and K supp T. For $z = x + iy \in \mathbb{C}\backslash K$ we set

$$\tilde{T}(z) = \frac{1}{2i\pi}\left\langle T_t, \frac{1}{t - z}\right\rangle$$

a) Prove that $\tilde{T}$ is holomorphic in $\mathbb{C}\backslash K$.

$\left(\text{Hint: Prove that } \tilde{T} \in C^1 \text{ in } (x, y) \text{ and } \frac{\partial \tilde{T}}{\partial x} + i\frac{\partial \tilde{T}}{\partial y} = 0.\right)$

b) Prove that $\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^n \tilde{T}(z) = \left\langle T_t, \frac{n!}{(2i\pi)}(t - z)^{-n-1}\right\rangle$

c) Show that $\tilde{T}(z) = O\left(\frac{1}{|z|}\right)$ when $|z| \to \infty$

d) Let $\varphi \in \mathscr{D}(\mathbb{R})$. We set

$$\varphi^*(x + i\varepsilon) = \frac{\varepsilon}{\pi}\int_{\mathbb{R}} \frac{\varphi(t)}{(t - x)^2 + \varepsilon^2}\,\mathrm{d}t$$

Using exercise 47, d) and exercise 61, b), show that $\varphi^*(x + i\varepsilon)$ converges to φ in C^∞.

e) Let $T \in \mathscr{E}'(\mathbb{R})$ and $\varphi \in \mathscr{D}(\mathbb{R})$. We set

$$T^*(z) = [\tilde{T}(z) - \tilde{T}(\bar{z})] \quad \text{for } \operatorname{Im} z > 0$$

Show that

$$\int_{\mathbb{R}} T^*(x + i\varepsilon)\varphi(x)\,\mathrm{d}x = \langle T, \varphi^*(x + i\varepsilon)\rangle$$

f) Deduce that for all $T \in \mathscr{E}'(\mathbb{R})$ we have in $\mathscr{D}'(\mathbb{R})$

$$T = \lim_{\varepsilon \to 0}\,[\tilde{T}(x + i\varepsilon) - \tilde{T}(x - i\varepsilon)]$$

SOLUTIONS OF THE EXERCISES — CHAPTER 5

Solution 53

By definition, for $\varphi \in \mathscr{D}(\mathbb{R})$,

$$\langle x^k(S * T), \varphi\rangle = \langle S * T, x^k\varphi\rangle = \langle S_x, \langle T_y, (x + y)^k\varphi(x + y)\rangle\rangle$$

Now

$$(x + y)^k = \sum_{j=0}^{k} \binom{k}{j} x^j y^{k-j}$$

It follows that

$$\langle x^k(S * T), \varphi\rangle = \left\langle S_x, \left\langle T_y, \sum_{j=0}^{k} \binom{k}{j} x^j y^{k-j}\varphi(x + y)\right\rangle\right\rangle$$

Now

$$\left\langle T_y, \sum_{j=0}^{k} \binom{k}{j} x^j y^{k-j}\varphi(x + y)\right\rangle = \sum_{j=0}^{k} \binom{k}{j} x^j \langle T_y, y^{k-j}\varphi(x + y)\rangle$$

and

$$\langle T_y, y^{k-j}\varphi(x + y)\rangle = \langle y^{k-j}T_y, \varphi(x + y)\rangle$$

$$\langle S_x, x^j\Psi\rangle = \langle x^j S_x, \Psi\rangle \quad \text{for all } \Psi \in C^\infty(\mathbb{R})$$

It follows that

$$\begin{aligned}\langle x^k(S * T), \varphi\rangle &= \sum_{j=0}^{k} \binom{k}{j} \langle x^j S_x, \langle y^{k-j}T_y, \varphi(x + y)\rangle\rangle \\ &= \sum_{j=0}^{k} \binom{k}{j} \langle (x^j S) * (x^{k-j}T), \varphi\rangle\end{aligned}$$

so (1) is proved.

Solution 54

We know from exercise 21 that

$$x^p\delta^{(q)} = \begin{cases} 0 & \text{if } p > q \\ \dfrac{(-1)^p q!}{(q - p)!}\delta^{(q-p)} & \text{if } p \le q \end{cases}$$

It follows that

a) If $p > q$ or $m > n$ then $T = 0$.

b) Assume $p \le q$ and $m \le n$; then

$$T = A_{p,q,m,n}\delta^{(q-p)} * \delta^{(n-m)}$$

where

$$A_{p,q,m,n} = \frac{(-1)^{p+m}q!n!}{(q - p)!(n - m)!}$$

On the other hand if $T \in \mathscr{E}'$, $S \in \mathscr{D}'$ we have

$$\begin{cases} \partial^\alpha \partial^\beta (S * T) = (\partial^\alpha S) * (\partial^\beta T) \\ \delta * S = S \end{cases}$$

so

$$T = A_{p,q,m,n} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{q-p} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{n-m} (\delta * \delta) = A_{p,q,m,n} \delta^{(q+n-p-m)}$$

Solution 55

Assume that we can define the convolution of three distributions u, v, w in such a manner that $(u * v) * w = u * (v * w)$. Let us take $u = 1$, $v = \delta'$, $w = H$. Then we would have

$$u * v = 1 * \delta' = \frac{\mathrm{d}}{\mathrm{d}x}(1 * \delta) = \frac{\mathrm{d}}{\mathrm{d}x} 1 = 0$$

so

$$(u * v) * w = 0 * H = 0$$

On the other hand

$$v * w = \delta' * H = \frac{\mathrm{d}}{\mathrm{d}x}(\delta * H) = \frac{\mathrm{d}}{\mathrm{d}x} H = \delta$$

so

$$u * (v * w) = 1 * \delta = 1$$

which is impossible.

Solution 56

If T exists it must satisfy $\langle T, \varphi \rangle = (T * \check{\varphi})(0) = (A\check{\varphi})(0)$.
Now, by hypothesis a) the map from $\mathscr{D}(\mathbb{R}^n)$ to $\mathbb{C}$

$$\varphi \mapsto (A\check{\varphi})(0)$$

determines a distribution. Let us denote it by T. We have

$$(A\varphi)(0) = (T * \varphi)(0)$$

so

$$(A\varphi)(x) = \tau_{-x}(A\varphi)(0) = A(\tau_{-x}\varphi)(0) = [T * (\tau_{-x}\varphi)](0) = (T * \varphi)(x)$$

Indeed

$$(T * \tau_{-x}\varphi)(0) = \langle T_t, (\tau_{-x}\varphi)(t)\rangle = \langle T_t, \varphi(x + t)\rangle = (T * \varphi)(x)$$

which proves (1).

Solution 57

Let φ be in $\mathscr{D}(\mathbb{R}^n)$ and $M > 0$ be such supp $\varphi \subset \{x: |x| \leq M\}$. Let us set

$$I_p = \int_{|x| \leq M} \varphi(x) \frac{p^n}{\pi^{n/2}} \left(1 - \frac{|x|^2}{p}\right)^{p^3} \mathrm{d}x$$

Let us set $y = px$ so $\mathrm{d}y = p^n \mathrm{d}x$. We get

$$I_p = \frac{1}{\pi^{n/2}} \int_{|y| \leq pM} \left(1 - \frac{|y|^2}{p^3}\right)^{p^3} \varphi\left(\frac{y}{p}\right) \mathrm{d}y = \frac{1}{\pi^{n/2}} \int_{\mathbb{R}^n} 1_{\{|y| \leq pM\}} \left(1 - \frac{|y|^2}{p^3}\right)^{p^3} \varphi\left(\frac{y}{p}\right) \mathrm{d}y$$

Let $y \in \mathbb{R}^n$ be fixed. For p big enough we have

$$\left(1 - \frac{|y|^2}{p^3}\right)^{p^3} = \exp\left[p^3 \operatorname{Log}\left(1 - \frac{|y|^2}{p^3}\right)\right]$$

so

$$\begin{cases} * \lim\limits_{p \to \infty} 1_{\{|y| \leq pM\}} \left(1 - \dfrac{|y|^2}{p^3}\right)^{p^3} = \mathrm{e}^{-|y|^2} 1_{\mathbb{R}^n} \\ * \left|\left(1 - \dfrac{|y|^2}{p^3}\right)^{p^3}\right| \leq \mathrm{e}^{-|y|^2} \end{cases}$$

Since $\varphi\left(\frac{y}{p}\right) \to \varphi(0)$ and $\left|\left(1 - \frac{|y|^2}{p^3}\right)^{p^3} \varphi\left(\frac{y}{p}\right)\right| \leq C\mathrm{e}^{-|y|^2} \in L^1(\mathbb{R}^n)$ by the Lebesgue theorem we get

$$\lim_{p \to \infty} I_p = \frac{1}{\pi^{n/2}} \int_{\mathbb{R}^n} \mathrm{e}^{-|y|^2} \mathrm{d}y \cdot \varphi(0) = \varphi(0)$$

which means in $\mathscr{D}'(\mathbb{R}^n)$

$$\lim_{p \to \infty} \frac{p^n}{\pi^{n/2}} \left(1 - \frac{|x|^2}{p}\right)^{p^3} = \delta$$

b) Let $T \in \mathscr{E}'(\mathbb{R}^n)$ then $T * P_p$ is well defined and is a C^∞ function. On the other hand

$$\partial^\alpha(T * P_p) = T * \partial^\alpha P_p = 0 \quad \text{if } |\alpha| > 2p^3$$

so $T * P_p$ is a polynomial

Let us prove that $T * P_p$ converges to T in $\mathscr{D}'(\mathbb{R}^n)$.
Indeed if $U \in \mathscr{D}'$ (or $\mathscr{E}'$) and $\varphi \in \mathscr{D}$ (or C^∞) we have $\langle T, \varphi \rangle = (T * \check{\varphi})(0)$ then

$$\langle T * P_p, \varphi \rangle = [(T * P_p) * \check{\varphi}](0) = [P_p * (T * \check{\varphi})](0)$$

since $(T * P_p) \in C^\infty$ and $\varphi \in \mathscr{D}$. It follows that

$$\langle T * P_p, \varphi \rangle = \langle P_p, \overset{\vee}{\widehat{T * \check{\varphi}}} \rangle$$

and $\overset{\vee}{\widehat{T * \check{\varphi}}}$ is in $\mathscr{D}(\mathbb{R}^n)$. By question a) we have

$$\lim_{p \to \infty} \langle P_p, \overset{\vee}{\widehat{T * \check{\varphi}}} \rangle = \overset{\vee}{\widehat{T * \check{\varphi}}}(0) = T * \check{\varphi}(0)$$

so

$$\lim_{p \to \infty} \langle T * P_p, \varphi \rangle = (T * \check{\varphi})(0) = \langle T, \varphi \rangle \qquad \forall \varphi \in \mathscr{D}(\Omega)$$

It follows that in $\mathscr{D}'(\mathbb{R}^n)$ one has

$$\lim_{p \to \infty} T * P_p = T \qquad \text{Q.E.D.}$$

Solution 58

a) $\langle S * T, \varphi \rangle = \langle S_x, \langle T_y, \varphi(x + y) \rangle \rangle$, $\forall \varphi \in \mathscr{D}(\mathbb{R}^n)$. It follows that

$$\begin{aligned}
\langle (\mathrm{e}^{\langle a,x \rangle} S) * (\mathrm{e}^{\langle a,x \rangle} T), \varphi \rangle &= \langle \mathrm{e}^{\langle a,x \rangle} S_x, \langle \mathrm{e}^{\langle a,y \rangle} T_y, \varphi(x + y) \rangle \rangle \\
&= \langle S_x, \langle \mathrm{e}^{\langle a,x \rangle + \langle a,y \rangle} T_y\, \varphi(x + y) \rangle \rangle \\
&= \langle S_x, \langle T_y, \mathrm{e}^{\langle a,x+y \rangle} \varphi(x + y) \rangle \rangle \\
&= \langle S * T, \mathrm{e}^{\langle a,x \rangle} \varphi \rangle \\
&= \langle \mathrm{e}^{\langle a,x \rangle} (S * T), \varphi \rangle \qquad \forall \varphi \in \mathscr{D}(\mathbb{R}^n)
\end{aligned}$$

Q.E.D

b) We have $P(D) = \sum_{i,j=1}^{n} c_{ij} \dfrac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} d_i \dfrac{\partial}{\partial x_i} + e$ and

$$\begin{aligned}
P(D)[\mathrm{e}^{\langle a,x \rangle} T] = \mathrm{e}^{\langle a,x \rangle} P(D) T &+ \sum_{i,j=1}^{n} c_{ij} \left(a_i \frac{\partial T}{\partial x_j} + a_j \frac{\partial T}{\partial x_i} \right) \mathrm{e}^{\langle a,x \rangle} \\
&+ \left[\sum_{i=1}^{n} d_i a_i T \right] \mathrm{e}^{\langle a,x \rangle} + e\, \mathrm{e}^{\langle a,x \rangle} T
\end{aligned}$$

It follows that

$$e^{\langle a,x\rangle}P(D)T = P(D)[e^{\langle a,x\rangle}T] + \sum_{i,j=1}^{n} c_{ij}\left(a_i\frac{\partial T}{\partial x_j} + a_j\frac{\partial T}{\partial x_i}\right)e^{\langle a,x\rangle}$$
$$+ \left(\sum_{i,j=1}^{n} c_{ij}a_ia_j + \sum_{i=1}^{n} d_ia_i + e\right)e^{\langle a,x\rangle}T$$

$$e^{\langle a,x\rangle}P(D)T = P(D)[e^{\langle a,x\rangle}T] + \sum_{i,j=1}^{n} c_{ij}\left(a_i\frac{\partial}{\partial x_j} + a_j\frac{\partial}{\partial x_i}\right)[e^{\langle a,x\rangle}T]$$
$$+ \left(\sum_{i,j=1}^{n} c_{ij}a_ia_j + \sum_{i=1}^{n} d_ia_i + e - 2\sum_{i,j=1}^{n} c_{ij}a_ia_j\right)[e^{\langle a,x\rangle}T]$$

So we get

$$e^{\langle a,x\rangle}P(D)T = \left[P(D) + \sum_{i,j=1}^{n} c_{ij}\left(a_i\frac{\partial}{\partial x_j} + a_j\frac{\partial}{\partial x_i}\right) + \sum_{i=1}^{n} a_id_i - \sum_{i,j=1}^{n} c_{ij}a_ia_j + e\right]e^{\langle a,x\rangle}T$$

$$e^{\langle a,x\rangle}P(D)T = Q(D)[e^{\langle a,x\rangle}T]$$

c) Using b) we get

$$e^{\langle a,x\rangle}P(D)E = Q(D)(e^{\langle a,x\rangle}E)$$

Since $P(D)E = \delta$ and $e^{\langle a,x\rangle}\delta = \delta$ we get

$$Q(D)(e^{\langle a,x\rangle}E) = \delta$$

Solution 59

a) Let us compute $\psi = \Delta(\varphi E_n) - C_n\delta$. We have

$$\Delta(\varphi E_n) = (\Delta\varphi)E_n + \varphi(\Delta E_n) + 2\sum_{i=1}^{n} \frac{\partial\varphi}{\partial x_i}\cdot\frac{\partial E_n}{\partial x_i}$$

Now

$$\varphi(\Delta E_n) = \varphi C_n\delta = C_n\varphi(0)\delta = C_n\delta$$

since $\varphi(0) = 1$. It follows that

$$\psi = \Delta(\varphi E_n) - C_n\delta = (\Delta\varphi)E_n + 2\sum_{i=1}^{n} \frac{\partial\varphi}{\partial x_i}\cdot\frac{\partial E_n}{\partial x_i}$$

Since $\Delta\varphi$ and $\dfrac{\partial\varphi}{\partial x_i}$ vanish in a neighborhood of the origin and since E_n and $\dfrac{\partial E_n}{\partial x_i}$ are C^∞ functions outside the origin it follows that $\psi \in C^\infty$. Moreover the support of ψ is compact since it is contained in the support of φ.

b) We have

$$\frac{\partial}{\partial x_i}(\varphi E_n) = \frac{\partial \varphi}{\partial x_i} E_n + \varphi \frac{\partial E_n}{\partial x_i}$$

First of all $\frac{\partial \varphi}{\partial x_i} E_n \in \mathscr{D}(\mathbb{R}^n) \subset L^1$. Moreover

$$\frac{\partial E_n}{\partial x_i} = \begin{cases} \frac{x_i}{r^2} & \text{if } n = 2 \\ (2-n)\frac{x_i}{r^n} & \text{if } n \geq 3 \end{cases}$$

We just have to prove that the function $\varphi \frac{x_i}{r^n} \in L^1$ for $n \geq 2$. Let us use the polar coordinates $x = r \cdot \omega$, $r \in]0, +\infty[$, $\omega \in S^{n-1}$. Then $\mathrm{d}x = r^{n-1}\mathrm{d}r\,\mathrm{d}\omega$ and

$$\begin{aligned} \int |\varphi(x)| \frac{|x_i|}{r^n} \mathrm{d}x &= \int_0^\infty \int_{S^{n-1}} |\varphi(r \cdot \omega)| \frac{r \cdot |\omega_i|}{r^n} \cdot r^{n-1} \mathrm{d}r\,\mathrm{d}\omega \\ &= \int_0^\infty \int_{S^{n-1}} |\varphi(r \cdot \omega)| \cdot |\omega_i| \mathrm{d}r\,\mathrm{d}\omega < +\infty \end{aligned}$$

since $\varphi \in \mathscr{D}(\mathbb{R}^n)$ and $|\omega_i| \leq 1$.

c) Let $T \in \mathscr{D}'(\mathbb{R}^n)$. With the notations used in question a) we have

$$T = T * \delta = \frac{1}{C_n}(T * \Delta(\varphi E_n) - T * \psi)$$

First of all $\psi \in \mathscr{D}(\mathbb{R}^n)$ so $T * \psi \in C^\infty(\mathbb{R}^n) \subset L^2_{\mathrm{loc}}(\mathbb{R}^n)$.
Moreover

$$T * \Delta(\varphi E_n) = \Delta(T * \varphi E_n) = \sum_{i=1}^n \frac{\partial T}{\partial x_i} * \frac{\partial}{\partial x_i}(\varphi E_n)$$

Now $\frac{\partial T}{\partial x_i} \in L^2(\mathbb{R}^n)$ and by question b) $\frac{\partial}{\partial x_i}(\varphi E_n) \in L^1(\mathbb{R}^n)$ so

$$\sum_{i=1}^n \frac{\partial T}{\partial x_i} * \frac{\partial}{\partial x_i}(\varphi E_n) \in L^2(\mathbb{R}^n) \subset L^2_{\mathrm{loc}}(\mathbb{R}^n)$$

since the convolution of a function in L^1 and a function in L^2 is in L^2. Therefore $T \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$.

Solution 60

a) It follows from the Leibniz formula that

$$P(\varphi E) = \sum_{|\alpha|\leq m} a_\alpha \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} \partial^\beta\varphi \cdot \partial^{\alpha-\beta}E$$

so

$$(1) \quad P(\varphi E) = \varphi PE + \sum_{|\alpha|\leq m} a_\alpha \sum_{\substack{\beta\leq\alpha\\ \beta\neq 0}} \binom{\alpha}{\beta} \partial^\beta\varphi \cdot \partial^{\alpha-\beta}E$$

Since E is a fundamental solution of P we have $PE = \delta$ so

$$(2) \quad \varphi PE = \varphi\delta = \varphi(0)\delta = \delta$$

It follows from (1) and (2) that

$$\psi = P(\varphi E) - \delta = \sum_{|\alpha|\leq m} a_\alpha \sum_{\substack{\beta\leq\alpha\\ \beta\neq 0}} \binom{\alpha}{\beta} \partial^\beta\varphi \cdot \partial^{\alpha-\beta}E$$

In $\mathbb{R}^n\backslash 0$, E is a C^∞ function. Therefore $\psi \in C^\infty(\mathbb{R}^n\backslash 0)$. But for $\beta \neq 0$, $\partial^\beta\varphi = 0$ for $|x| \leqslant 1$ since $\varphi = 1$ there, so $\psi = 0$ for $|x| \leqslant 1$. It follows that $\psi \in C^\infty(\mathbb{R}^n)$. Since supp ψ is included in supp φ we have $\psi \in \mathscr{D}(\mathbb{R}^n)$.

b) Let $u \in \mathscr{D}'(\mathbb{R}^n)$ be such that $Pu \in C^\infty(\mathbb{R}^n)$. We have

$$u = \delta * u = [P(\varphi E) - \psi] * u = P(\varphi E) * u - \psi * u$$

This has a meaning since $P(\varphi E) \in \mathscr{E}'$ and $\psi \in \mathscr{D}(\mathbb{R}^n)$. Now

$$P(\varphi E) * u = (\varphi E) * Pu$$

and

$$u = (\varphi E) * Pu - \psi * u \in C^\infty(\mathbb{R}^n)$$

Indeed $\varphi E \in \mathscr{E}'(\mathbb{R}^n)$, $Pu \in C^\infty(\mathbb{R}^n)$ and $\psi \in \mathscr{D}(\mathbb{R}^n)$, $u \in \mathscr{D}'(\mathbb{R}^n)$.

c) By exercises 29, 30, 31, the operators

$$P = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} \qquad \text{in } \mathbb{R}^2$$

$$P = \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} \qquad \text{in } \mathbb{R}^2$$

$$P = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \qquad \text{in } \mathbb{R}^n \qquad n \geq 2$$

possess the property (*).

Exercises 23, 28, 51 give examples of operators which do not have the property described in question b).

Solution 61

First of all $\rho_\varepsilon \to \delta$ in $\mathscr{E}'$ when $\varepsilon \to 0$. Indeed supp $\rho \subset \{|x| \leq M\}$ and

$$\int \rho_\varepsilon(x)\varphi(x)\,\mathrm{d}x = \int_{|x|\leq M} \rho(x)\varphi(\varepsilon x)\,\mathrm{d}x \qquad \forall\varphi \in C^\infty(\mathbb{R}^n)$$

Then: • $\rho(x)\varphi(\varepsilon x) \to \rho(x)\varphi(0)$ a.e. if $\varepsilon \to 0$.

• $|1_{\{|x|\leq M\}}\rho(x)\varphi(\varepsilon x)| \leq \sup\limits_{|y|\leq M} |\varphi(y)|\rho(x) \in L^1(\mathbb{R}^n)$

The result follows from the Lebesgue theorem and from the fact that $\int \rho(x)\,\mathrm{d}x = 1$.

a) Let $u \in \mathscr{D}'(\mathbb{R}^n)$ and $\varphi \in \mathscr{D}(\mathbb{R}^n)$ so $u * \varphi \in C^\infty$ and

$$\langle u * \rho_\varepsilon, \varphi\rangle = [(u * \rho_\varepsilon) * \check{\varphi}](0) = [\rho_\varepsilon * (u * \check{\varphi})](0) = \langle \rho_\varepsilon, \overset{\vee}{\overline{u * \check{\varphi}}}\rangle$$

Now $\overset{\vee}{\overline{u * \check{\varphi}}} \in C^\infty$ and $\rho_\varepsilon \to \delta$ in $\mathscr{E}'(\mathbb{R}^n)$ so

$$\lim \langle u * \rho_\varepsilon, \varphi\rangle = \langle \delta, \overset{\vee}{\overline{u * \check{\varphi}}}\rangle = u * \check{\varphi}(0) = \langle u, \varphi\rangle \qquad \text{Q.E.D.}$$

b) Let $u \in C_c^0(\mathbb{R}^n)$

$$u * \rho_\varepsilon(x) - u(x) = \int \rho(t)u(x - \varepsilon t)\,\mathrm{d}t - \int \rho(t)\,\mathrm{d}t \cdot u(x)$$

so

$$|u * \rho_\varepsilon(x) - u(x)| \leq \int_{|t|\leq M} \rho(t)|u(x - \varepsilon t) - u(x)|\,\mathrm{d}t$$

Since u is uniformly continuous on its support we have:

$$\forall\alpha > 0, \exists\eta: \forall x, \forall y, \qquad |x - y| < \eta \Rightarrow |u(x) - u(y)| < \alpha$$

Let us take $\varepsilon < \dfrac{\eta}{M}$ *then* $|x - \varepsilon t - x| = \varepsilon|t| \leq \varepsilon M < \eta$ so $|u(x - \varepsilon t) - u(x)| < \alpha$. This implies

$$\sup_K |u * \rho_\varepsilon(x) - u(x)| < \alpha \int \rho(t)\,\mathrm{d}t = \alpha \qquad \text{Q.E.D}$$

c) Let $u \in L^p(\mathbb{R}^n)$. Since $C_c^0(\mathbb{R}^n)$ is dense in L^p, there exists a sequence (u_j) in C_c^0 such that

$$(1) \quad \forall\alpha > 0, \exists J: \qquad j \geq J \Rightarrow \|u_j - u\|_{L^p} < \frac{\alpha}{3}$$

Let j_0 be fixed, $j_0 \geqslant J$. Then

$$(2)\quad \|u * \rho_\varepsilon - u\|_{L^p} \leqq \|u * \rho_\varepsilon - u_{j_0} * \rho_\varepsilon\|_{L^p} + \|u_{j_0} * \rho_\varepsilon - u_{j_0}\|_{L^p} + \|u_{j_0} - u\|_{L^p}$$

It follows from (1)

$$(3)\quad \|u_{j_0} - u\|_{L^p(\mathbf{R}^n)} < \frac{\alpha}{3}$$

Moreover by question b) we get

$$\forall \delta > 0,\ \exists \varepsilon_0:\quad \varepsilon < \varepsilon_0 \Rightarrow \sup |u_{j_0} * \rho_\varepsilon(x) - u_{j_0}(x)| < \delta$$

Now

$$\|u_{j_0} * \rho_\varepsilon - u_{j_0}\|_{L^p} = \left(\int_K |u_{j_0} * \rho_\varepsilon(x) - u_{j_0}(x)|^p \,\mathrm{d}x\right)^{1/p}$$

$$\|u_{j_0} * \rho_\varepsilon - u_{j_0}\|_{L^p} \leqq C \sup_K |u_{j_0} * \rho_\varepsilon(x) - u_{j_0}(x)| < C\delta$$

So if $\varepsilon < \varepsilon_1$

$$(4)\quad \|u_{j_0} * \rho_\varepsilon - u_{j_0}\|_{L^p} < \frac{\alpha}{3}$$

Let us assume the following inequality has been proved

$$(5)\quad \|v * \rho_\varepsilon\|_{L^p} \leq \|v\|_{L^p} \qquad \forall v \in L^p$$

Then we shall have

$$(6)\quad \|u * \rho_\varepsilon - u_{j_0} * \rho_\varepsilon\|_{L^p} \leq \|u_{j_0} - u\|_{L^p} < \frac{\alpha}{3}$$

Using (2), (3), (4) and (6) we shall get

$$\forall \alpha > 0,\ \exists \varepsilon_1:\quad \varepsilon < \varepsilon_1 \Rightarrow \|u * \rho_\varepsilon - u\|_{L^p} < \alpha \qquad \text{Q.E.D.}$$

Let us prove (6). We have

$$(7)\quad \|v * \rho_\varepsilon\|_{L^p} = \left(\int_{\mathbf{R}^n_x} \left|\int_{\mathbf{R}^n_t} \rho_\varepsilon(t) v(x - t)\,\mathrm{d}t\right|^p \mathrm{d}x\right)^{1/p}$$

Since $\rho_\varepsilon \geq 0$ we can write $\rho_\varepsilon = \rho_\varepsilon^{1/p} \cdot \rho_\varepsilon^{1-(1/p)}$ if $1 < p < +\infty$.
By Hölder's inequality

$$\int \rho_\varepsilon(t) v(x - t)\,\mathrm{d}t \leqq \left(\int \rho_\varepsilon(t) |v(x - t)|^p \,\mathrm{d}t\right)^{1/p} \left(\int [\rho_\varepsilon(t)]^{[1-(1/p)]\cdot q} \,\mathrm{d}t\right)^{1/q}$$

where $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Since $q\left(1 - \dfrac{1}{p}\right) = 1$ and $\int \rho_\varepsilon(t)\,\mathrm{d}t = 1$ we get

$$\left|\int \rho_\varepsilon(t) v(x - t)\,\mathrm{d}t\right|^p \leqq \int \rho_\varepsilon(t) |v(x - t)|^p \,\mathrm{d}t$$

Now

$$\int\left(\int \rho_\varepsilon(t)|v(x-t)|^p \mathrm{d}t\right)\mathrm{d}x = \int \rho_\varepsilon(t)\left(\int |v(x-t)|^p \mathrm{d}x\right)\mathrm{d}t = \|v\|_{L^p}^p$$

by the Fubini's theorem. Therefore

$$\|v * \rho_\varepsilon\|_{L^p} \leqq \|v\|_{L^p} \qquad \text{Q.E.D.}$$

For $p = 1$ one has to use Fubini's theorem in (7).

Solution 62

a) Let $\theta \in \mathscr{D}(\mathbb{R}^n)$, $\theta = 1$ if $|x| \leq 1$, $\theta = 0$ if $|x| \geq 2$, $\theta(x) \in [0, 1]$. Let us set

$$\theta_k(x) = \theta\left(\frac{x}{k}\right)$$

and for $u \in H^1(\mathbb{R}^n)$: $u_k = \theta_k u$. Then $u_k \in \mathscr{E}'(\mathbb{R}^n)$ since $\theta_k = 0$ for $|x| \geq 2k$. Let us prove that $u_k \in H^1(\mathbb{R}^n)$ and converges to u in $H^1(\mathbb{R}^n)$. Indeed

$$\frac{\partial u_k}{\partial x_i} = \frac{1}{k}\frac{\partial \theta}{\partial x_i}\left(\frac{x}{k}\right)u + \theta_k \frac{\partial u}{\partial x_i}$$

so $u_k \in H^1(\mathbb{R}^n)$ since:

$$|u_k|^2 + \left|\frac{\partial u_k}{\partial x_i}\right|^2 \leqq C\left(|u|^2 + \left|\frac{\partial u}{\partial x_i}\right|^2\right)$$

since

$$\sup_{\mathbb{R}^n} |\theta_k| \leq 1, \qquad \sup_{x \in \mathbb{R}^n} \left|\frac{\partial \theta}{\partial x_i}(x)\right| \leqq C_1$$

Moreover we can write:

$$\|u_k - u\|_1^2 = \underbrace{\|\theta_k u - u\|_{L^2(\mathbb{R}^n)}}_{①} + \frac{1}{k^2}\sum_{i=1}^{n} \underbrace{\left\|\frac{\partial \theta}{\partial x_i}\left(\frac{x}{k}\right)u\right\|_{L^2(\mathbb{R}^n)}}_{②}$$

$$+ \underbrace{\sum_{i=1}^{n}\left\|\theta_k \frac{\partial u}{\partial x_i} - \frac{\partial u}{\partial x_i}\right\|_{L^2(\mathbb{R}^n)}}_{③}$$

If $v \in L^2(\mathbb{R}^n)$ we have

- $\theta_k(x)v(x) - v(x) \to 0$ a.e. if $k \to \infty$
- $|\theta_k v - v|^2 \leqq 4|v|^2 \in L^1(\mathbb{R}^n)$ since $\sup|\theta_k| \leqq 1$

It follows from Lebesgue's theorem that $\|\theta_k v - v\|_{L^2(\mathbb{R}^n)} \to 0$ when $k \to \infty$; this proves that ① and ③ converge to zero. Term ② is such that

$$\frac{1}{k^2}\sum_{i=1}^{n}\int\left|\frac{\partial\theta}{\partial x_i}\left(\frac{x}{k}\right)u(x)\right|^2 \mathrm{d}x \leqq \frac{C}{k^2}\|u\|_{L^2} \quad \text{since} \sup_{x\in\mathbb{R}^n}\left|\frac{\partial\theta}{\partial x_i}(x)\right| \leqq C_1$$

Therefore $u_k \in H^1(\mathbb{R}^n) \cap \mathscr{E}'(\mathbb{R}^n)$ and $\|u_k - u\|_1 \to 0$ when $k \to \infty$. Q.E.D.

b) Let $\varphi \in \mathscr{D}(\mathbb{R}^n)$, $\varphi \geqq 0$, be such that $\int_{\mathbb{R}^n}\varphi(x)\,\mathrm{d}x = 1$. Let us set

$$\varphi_\varepsilon = \frac{1}{\varepsilon^n}\varphi\left(\frac{x}{\varepsilon}\right)$$

and for $u \in H^1(\mathbb{R}^n) \cap \mathscr{E}'(\mathbb{R}^n)$: $u_\varepsilon = u * \varphi_\varepsilon$. Then $u_\varepsilon \in \mathscr{D}(\mathbb{R}^n) \subset H^1(\mathbb{R}^n)$ since $\varphi_\varepsilon \in \mathscr{D}(\mathbb{R}^n)$ and $u \in \mathscr{E}'(\mathbb{R}^n)$. Let us prove that $u_\varepsilon \to u$ in $H^1(\mathbb{R}^n)$.

$$\|u_\varepsilon - u\|_1^2 = \|\varphi_\varepsilon * u - u\|_{L^2(\mathbb{R}^n)}^2 + \sum_{i=1}^{n}\left\|\varphi_\varepsilon * \frac{\partial u}{\partial x_i} - \frac{\partial u}{\partial x_i}\right\|_{L^2(\mathbb{R}^n)}^2$$

It follows from exercise 61 that $\varphi_\varepsilon * v \to v$ in $L^2(\mathbb{R}^n)$ when $\varepsilon \to 0$ and $v \in L^2(\mathbb{R}^n)$ so $\|u_\varepsilon - u\|_1 \to 0$ when $\varepsilon \to 0$. Q.E.D.

Denoting by $\bar{E}$ the closure of E in $H^1(\mathbb{R}^n)$, it follows from a) and b)

$$\overline{\mathscr{D}(\mathbb{R}^n)} = H^1(\mathbb{R}^n) \cap \mathscr{E}' \quad \text{and} \quad \overline{H^1 \cap \mathscr{E}'} = H^1 \quad \text{so} \quad \overline{\mathscr{D}(\mathbb{R}^n)} = H^1(\mathbb{R}^n)$$

Solution 63

a) Indeed, for a fixed x in the support of S, $\langle T_y, \varphi(x + y)\rangle$ has a meaning since $\tau_{-x}\varphi(y) = \varphi(x + y)$ has compact support in y. Moreover function $x \mapsto \langle T_y, \varphi(x + y)\rangle$ is C^∞ with compact support. Indeed $x + y \in \operatorname{supp}\varphi$, $y \in \operatorname{supp} T$, $x \in \operatorname{supp} S$ can be written: $x \geq 0$, $y \geq 0$ and $|x + y| \leq M$ so $0 \leq x \leq x + y \leq M$ and the formula is well defined. Moreover as in the case where $S \in \mathscr{E}'$, $T \in \mathscr{D}'$ we have: $\operatorname{supp}(S * T) \subset \operatorname{supp} S + \operatorname{supp} T \subset [0, +\infty[$.
Finally $\delta \in \mathscr{D}'_+$ and $\delta * T = T, \forall T \in \mathscr{D}'_+$.

b) $X * H = \delta$ so $\dfrac{\mathrm{d}}{\mathrm{d}x}(X * H) = \dfrac{\mathrm{d}\delta}{\mathrm{d}x}$ so $X * \dfrac{\mathrm{d}H}{\mathrm{d}x} = \delta'$, i.e. $X * \delta = \delta'$ and $X = \delta'$. In the same way $(\delta')^{-1} = H$. Finally
$X * (\delta' - \lambda\delta) = \delta$ so $X * \delta' - \lambda X * \delta = \delta$, i.e.

$$\frac{\mathrm{d}X}{\mathrm{d}x} - \lambda X = \delta$$

Let us set $X = \mathrm{e}^{\lambda x}u$ then $\dfrac{\mathrm{d}u}{\mathrm{d}x} = \delta$ so $u = H$ and $X = H\mathrm{e}^{\lambda x}$.

c) $[P(D)\delta]^{-1} = X$ is such that $X * P(D)\delta = \delta$ so

$$P(D)[X * \delta] = \delta \quad \text{i.e. } P(D)X = \delta$$

Therefore $[P(D)\delta]^{-1}$ represents a fundamental solution of $P(D)$.

If $z_1, \ldots, z_m$ are the roots of $P(z) = 0$ we have

$$P(z) = a(z - z_1) \cdots (z - z_m) \quad a \neq 0$$

Since $P(D)$ is obtained from $P(z)$ by substituting $\frac{\mathrm{d}}{\mathrm{d}x}$ to z we have

$$P(D) = a\left(\frac{\mathrm{d}}{\mathrm{d}x} - z_1\right) \cdots \left(\frac{\mathrm{d}}{\mathrm{d}x} - z_m\right)$$

Then

$$P(D)\delta = a\left(\frac{\mathrm{d}}{\mathrm{d}x} - z_1\right) \cdots \left(\frac{\mathrm{d}}{\mathrm{d}x} - z_m\right)\delta = (\delta' - z_1\delta) * \cdots * (\delta' - z_m\delta)$$

Since we are in an algebra

$$[P(D)\delta]^{-1} = \frac{1}{a}(\delta' - z_m\delta)^{-1} * \cdots * (\delta' - z_1\delta)^{-1}$$

Since the convolution is commutative, using b) we get

$$[P(D)\delta]^{-1} = \frac{1}{a} H e^{z_1 x} * \cdots * H e^{z_m x}$$

and

$$P(D)\left[\frac{1}{a} H e^{z_1 x} * \cdots * H e^{z_m x}\right] = \delta \qquad \text{Q.E.D.}$$

Solution 64

a) For $l = 1$ we have $\frac{\mathrm{d}H}{\mathrm{d}x} = \delta$ where H is the Heaviside function. If $l \geqq 2$, we obtain a fundamental solution of $\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^l$ by solving the equation $\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{l-1} E = H$. A solution is given by the formula $E_l = \frac{1}{(l-1)!} x^{l-1} H(x)$. Indeed by the Leibniz formula

$$\begin{aligned}
\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{l-1} E_l &= \sum_{j=0}^{l-1} \frac{1}{(l-1)!} C_{l-1}^j (x^{l-1})^{(j)} H^{(l-1-j)} \\
&= \frac{1}{(l-1)!} C_{l-1}^{l-1} (x^{l-1})^{(l-1)} H + \sum_{j=0}^{l-2} \frac{1}{(l-1)!} C_{l-1}^j (x^{l-1})^{(j)} \delta^{(l-2-j)} \\
&= H + \sum_{j=1}^{l-1} \frac{1}{(l-1)!} C_{l-1}^{j-1} (x^{l-1})^{(j-1)} \delta^{(l-1-j)} \\
&= H + \sum_{j=1}^{l-1} a_{l,j} x^{l-j} \delta^{(l-1-j)}
\end{aligned}$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{l-1} E_l = H$$

since $x^{l-j}\delta^{(l-1-j)} = 0$.

It follows that for $l \in \mathbb{N}\backslash\{0\}$

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{l}\left[\frac{1}{(l-1)!}x^{l-1}H\right] = \delta$$

b) We deduce from a) that

$$\left(\frac{\partial}{\partial x_1}\right)^{l_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{l_n} \frac{x_1^{l_1-1}H(x_1)\cdots x_n^{l_n-1}H(x_n)}{(l_1-1)!\cdots(l_n-1)!} = \delta_{x_1=0} \otimes \cdots \otimes \delta_{x_n=0} = \delta_0$$

For $l_1 = \cdots = l_n = k+2$ the fundamental solution is

$$\frac{1}{(k+1)!}x_1^{k+1}\cdots x_n^{k+1}H(x_1)\cdots H(x_n)$$

It is a C^k function. Indeed let us compute its derivatives up to the order k. It is sufficient to compute $\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{j}(x^{k+1}H)$, for $0 \leq j \leq k$, in $\mathbb{R}$.

$$\begin{aligned}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{j}(x^{k+1}H) &= \sum_{l=0}^{j} C_j^l (x^{k+1})^{(l)}H^{(j-l)} \\ &= (x^{k+1})^{(j)}H + \sum_{l=0}^{j-1} C_j^l (x^{k+1})^{(l)}\delta^{(j-l-1)} \\ &= a_{k,j}x^{k+1-j}H + \sum_{l=0}^{j-1} b_{k,j}x^{k+1-l}\delta^{(j-l-1)}\end{aligned}$$

Since $k+1-l > j-l-1$, i.e. $k+2 > j$ it follows that $x^{k+1-l}\delta^{(j-l-1)} = 0$ so

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{j}(x^{k+1}H) = a_{k,j}x^{k+1-j}H$$

Since $k+1-j > 0$ the right hand side is a continuous function on $\mathbb{R}$. It follows that $x^{k+1}H(x) \in C^k(\mathbb{R})$ and $x_1^{k+1}\cdots x_n^{k+1}H(x_1)\cdots H(x_n) \in C^k(\mathbb{R}^n)$.

c) Let $f \in \mathscr{E}'^{(k)}$ the space of distribution with compact support of order $\leq k$. Let $P = \partial_1^{k+2}\cdots\partial_n^{k+2}$ and E be the fundamental solution of P given in b). Let us set

$$u = f * E$$

then $Pu = f * PE = f * \delta = f$. Since $f \in \mathscr{E}'^{(k)}(\mathbb{R}^n)$ which is the dual of $C^k(\mathbb{R}^n)$ the function u is given by

$$u(x) = \langle f_t, E(x-t)\rangle$$

$\langle\, , \rangle$ denoting the duality between $\mathscr{E}'^{(k)}$ and C^k. It follows that u is continuous since if $x_n \to x$, $E(x_n - t) \to E(x-t)$ in C^k so $u(x_n)$ tends to $u(x)$.

d) Let (φ_j) be a partition of unity subordinated to a cover of $\mathbb{R}^n$, i.e. $\mathbb{R}^n = \bigcup \Omega_j$, $\varphi_j \in \mathscr{D}(\Omega_j)$, and for every $x \in \mathbb{R}^n$, $\sum_j \varphi_j(x) = 1$ the sum being, for each x, finite. Let $T \in \mathscr{D}'(\mathbb{R}^n)$ then $f_j = \varphi_j T \in \mathscr{E}'(\mathbb{R}^n)$. It follows from c) that there exists $u_j \in C^0(\mathbb{R}^n)$ such that $\partial^{\alpha_j} u_j = f_j$ where $\alpha_j = (k_j + 2, \ldots, k_j + 2)$, k_j being the order of f_j.
Let $K \subset \mathbb{R}^n$ be a compact. Then K meets at most a finite number N_K of Ω_j. Moreover for $\varphi \in \mathscr{D}_K(\Omega)$:

$$\langle T, \varphi\rangle = \left\langle T, \sum_{j=1}^{N_K} \varphi_j \varphi \right\rangle = \sum_{j=1}^{N_K} \langle \varphi_j T, \varphi\rangle = \sum_{j=1}^{N_K} \langle \partial^{\alpha_j} u_j, \varphi\rangle$$

$$\langle T, \varphi\rangle = \sum_{j=1}^{N_K} (-1)^{|\alpha_j|} \langle u_j, \partial^{\alpha_j}\varphi\rangle = \sum_{j=1}^{N_K} (-1)^{|\alpha_j|} \int_{\mathbb{R}^n} u_j(x)(\partial^{\alpha_j}\varphi)(x)\,dx \qquad \text{Q.E.D.}$$

Solution 65

a) Since $z \in \mathbb{C}\backslash K$ we have $|t - z| > 0$. Let us set $f(t, z) = \dfrac{1}{t - z}$. We are going to prove that $\tilde{T} \in C^1$ in x. We have

$$(1) \quad \frac{\tilde{T}(x + h + iy) - \tilde{T}(x + iy)}{h} = \left\langle T, \frac{f(t, x + h + iy) - f(t, x + iy)}{h} \right\rangle$$

Now the sequence $h \mapsto \dfrac{f(t, x + h + iy) - f(t, x + iy)}{h}$ converges in $C^\infty(\mathbb{R}_t)$.
Indeed by the Taylor formula applied to $g = \left(\dfrac{\partial}{\partial t}\right)^k f$ we get:

$$g(t, x + h + iy) - g(t, x + iy) = h\frac{\partial g}{\partial x}(t, x + iy) +$$
$$+ \int_0^1 (1 - s)h^2 \frac{\partial^2 g}{\partial x^2}(t, s(x + h) + (1 - s)x + iy)\,ds$$

It follows, with $z = x + iy$:

$$\left|\frac{g(t, x + h + iy) - g(t, x + iy)}{h} - \frac{\partial g}{\partial x}(t, x + iy)\right| \leq C|h| \sup_{|z' - z| \leq 1} \left|\left(\frac{\partial^2 g}{\partial x^2}\right)(t, z')\right|$$

This inequality shows that $\dfrac{g(t, x + h + iy) - g(t, x + iy)}{h}$ converges uniformly on every compact to $\dfrac{\partial g}{\partial x}(t, x + iy)$, which proves our claim.
Moreover it follows from (1) that

$$(2) \quad \frac{\partial \tilde{T}}{\partial x}(x + iy) = \left\langle T, \frac{\partial f}{\partial x}(t, x + iy)\right\rangle$$

In the same way we prove that $\tilde{T}$ is C^1 in y and that $\frac{\partial \tilde{T}}{\partial y}$ in given by the same formula as (2). It follows that

$$\frac{\partial \tilde{T}}{\partial \bar{z}}(z) = \left\langle T, \frac{\partial f}{\partial \bar{z}}(t, z)\right\rangle = 0$$

since the function $f(t, z) = \frac{1}{t - z}$ is holomorphic in $\mathbb{C}\backslash K$, $\frac{\partial f}{\partial \bar{z}} = 0$.

Therefore $\tilde{T}$ is holomorphic in $\mathbb{C}\backslash K$.

b) Let us prove the formula by induction on n. The formula being true for $n = 0$ we can suppose that

$$\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{n-1}\tilde{T}(z) = \left\langle T_t, \frac{(n-1)!}{2i\pi}(t - z)^{-n}\right\rangle$$

As in question a), taking $f(t, z) = \frac{(n-1)!}{2i\pi}(t - z)^{-n}$, we deduce that $\tilde{T}^{(n-1)}$ is a C^1 function and since

$$\frac{\mathrm{d}}{\mathrm{d}z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

we get

$$T^{(n)}(z) = \frac{(n-1)!}{2i\pi}\left\langle T_t, \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(t - z)^{-n}\right\rangle$$

$$T^{(n)}(z) = \frac{n!}{2i\pi}\langle T_t, (t - z)^{-n-1}\rangle \qquad \text{Q.E.D.}$$

c) Since $T \in \mathscr{E}'(\mathbb{R})$ we can find an integer k, a compact K and $C > 0$ such that

$$\left|\left\langle T, \frac{1}{t - z}\right\rangle\right| \leqq C\sum_{p=0}^{k}\sup_{t\in K}\left|\partial_t^p\left(\frac{1}{t - z}\right)\right| \leq C \cdot p!\sum_{p=0}^{k}\sup_{t\in K}\frac{1}{|t - z|^{p+1}}$$

Since $t \in K$ we have $|t| \leq M$ and if z is big enough, $|z| \geq 2M$, we get

$$|t - z| \geq ||t| - |z|| = |z| - |t| \geq |z| - M \geq \tfrac{1}{2}|z|$$

If we take $|z| \geq \sup(2M, 2)$ we get

$$|t - z|^{p+1} \geq (\tfrac{1}{2}|z|)^{p+1} \geqq \tfrac{1}{2}|z|$$

so

$$|\tilde{T}(z)| \leqq \frac{C}{|z|} \quad \text{for } |z| \to \infty$$

d) Using question d) in exercise 47 we get

$$\lim_{\varepsilon \to \infty} \frac{\varepsilon}{\pi(x^2 + \varepsilon^2)} = \delta$$

Now for $\varphi \in \mathscr{D}(\mathbb{R})$,

$$\varphi^*(x + i\varepsilon) = \frac{\varepsilon}{\pi(x^2 + \varepsilon^2)} * \varphi \quad \text{and} \quad (\partial_x^k \varphi^*)(x + i\varepsilon) = \frac{\varepsilon}{\pi(x^2 + \varepsilon^2)} * \partial_x^k \varphi$$

From question b) in exercise 61, since $\partial_x^k \varphi \in C_c^0$ for each k, we deduce that $((\partial_x^k \varphi^*)(x + i\varepsilon))$ converges uniformly on every compact to $\partial_x^k \varphi \, \forall k$, i.e. $\varphi^*(x + i\varepsilon)$ converges in C^∞ to φ. Q.E.D.

e)
$$\tilde{T}(z) - \tilde{T}(\bar{z}) = \frac{1}{2i\pi}\left\langle T_t, \frac{1}{t - z} - \frac{1}{t - \bar{z}}\right\rangle$$

$$\tilde{T}(z) - \tilde{T}(\bar{z}) = \frac{1}{\pi}\left\langle T_t, \frac{y}{(t - x)^2 + y^2}\right\rangle \quad \text{if } z = x + iy$$

It follows that

$$T^*(x + i\varepsilon) = \langle T_t, P_\varepsilon(x - t)\rangle \quad \text{where } P_\varepsilon(x) = \frac{\varepsilon}{\pi} \cdot \frac{1}{x^2 + \varepsilon^2}$$

Since $\langle u, \psi\rangle = (u * \check{\psi})(0)$ where $\check{\psi}(x) = \psi(-x)$ we get

$$\langle T, \varphi^*\rangle = (T * \check{\varphi}^*)(0) \quad \text{now} \quad \check{\varphi}^* = (\check{\varphi})^* \quad (\text{change } t \text{ to } -t)$$

and $\check{\varphi}^*(x + i\varepsilon) = \check{\varphi} * P_\varepsilon$ so $\langle T, \varphi^*\rangle = (T * \check{\varphi} * P_\varepsilon)(0)$

$$\langle T, \varphi^*\rangle = [(T * P_\varepsilon) * \check{\varphi}](0)$$

since the convolution is commutative and associative for $T \in \mathscr{E}'$, $\varphi \in \mathscr{D}'$. It follows

$$\langle T, \varphi^*\rangle = \langle T * P_\varepsilon, \varphi\rangle$$

since $T \in \mathscr{E}'$ then $T * P_\varepsilon(x) = \langle T_t, P_\varepsilon(x - t)\rangle = T^*(x + i\varepsilon)$ so

$$\langle T, \varphi^*\rangle = \langle T^*(x + i\varepsilon), \varphi\rangle \qquad \text{Q.E.D.}$$

f) From e) we get

$$\langle \tilde{T}(x + i\varepsilon), \varphi\rangle - \langle \tilde{T}(x - i\varepsilon), \varphi\rangle = \langle T, \varphi^*(x + i\varepsilon)\rangle$$

By d) $\varphi^*(x + i\varepsilon)$ converges to φ in C^∞. Since $T \in \mathscr{E}'(\mathbb{R})$ the right hand side converges when $\varepsilon \to 0^+$, to $\langle T, \varphi\rangle$, i.e.

$$\langle T, \varphi\rangle = \lim_{\varepsilon \to 0^+} [\langle \tilde{T}(x + i\varepsilon), \varphi\rangle - \langle \tilde{T}(x - i\varepsilon), \varphi\rangle] \qquad \forall \varphi \in \mathscr{D}(\mathbb{R})$$

so

$$T = \lim_{\varepsilon \to 0^+} [\langle \tilde{T}(x + i\varepsilon) - \tilde{T}(x - i\varepsilon)] \quad \text{in } \mathscr{D}'(\mathbb{R}) \qquad \text{Q.E.D.}$$

CHAPTER 6

Fourier and Laplace Transforms of distributions

PROGRAMME

The spaces $\mathscr{S}(\mathbb{R}^n)$, $\mathscr{S}'(\mathbb{R}^n)$

Fourier transforms of tempered distributions

Fourier transforms of distributions with compact support

Laplace transforms of distributions

BASICS CHAPTER 6

a) The space $\mathcal{S}(\mathbb{R}^n)$

It is the space of all $u \in C^\infty(\mathbb{R}^n)$ such that for all α and β in $\mathbb{N}^n$ we have:

$$(1) \quad \lim_{|x|\to\infty} |x^\alpha \partial^\beta u(x)| = 0$$

The topology on $\mathcal{S}(\mathbb{R}^n)$ is defined by the denumbrable set of semi norms

$$(2) \quad p_{\alpha,\beta}(u) = \sup_{x\in\mathbb{R}^n} |x^\alpha \partial^\beta u(x)|$$

This topology is metrizable and gives to $\mathcal{S}(\mathbb{R}^n)$ the structure of a complete metric space.

We have the inclusions

$$\mathcal{D}(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n) \subset C^\infty(\mathbb{R}^n)$$

b) The space $\mathcal{S}'(\mathbb{R}^n)$ of tempered distributions

$\mathcal{S}'(\mathbb{R}^n)$ denotes the topological dual space of $\mathcal{S}(\mathbb{R}^n)$, in other terms the space of linear continuous forms on $\mathcal{S}(\mathbb{R}^n)$.

One can show that we have the inclusions

$$\mathcal{E}'(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n)$$

$$\mathcal{S}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n) \qquad 1 \le p \le \infty$$

This is a characterisation of $\mathcal{S}'(\mathbb{R}^n)$. A distribution T belongs to $\mathcal{S}'(\mathbb{R}^n)$ if and only if it is a linear form on $\mathcal{S}(\mathbb{R}^n)$ and we can find $C > 0$, $\alpha, \beta \in \mathbb{N}^n$ such that

$$(3) \quad |\langle T, \varphi\rangle| \leqq C \sup_{x\in\mathbb{R}^n} |x^\alpha \partial^\beta \varphi(x)|$$

for all φ in $\mathcal{S}(\mathbb{R}^n)$ (or $|\langle T, \varphi\rangle| \leqq C \sup\limits_{x\in\mathbb{R}^n} (1 + |x|^2)^k \sum\limits_{|\alpha|\le l} |\partial^\alpha \varphi(x)|$).

c) Structure of elements in $\mathcal{S}'(\mathbb{R}^n)$

A distribution T belongs to $\mathcal{S}'(\mathbb{R}^n)$ if and only if there exist $m \in \mathbb{N}$, $\alpha \in \mathbb{N}^n$ and a function f continuous and bounded on $\mathbb{R}^n$ such that

$$(4) \quad T = \partial^\alpha[(1 + |x|^2)^m f]$$

d) Fourier transform in $\mathscr{S}'(\mathbb{R}^n)$

- For $\varphi \in \mathscr{S}(\mathbb{R}^n)$ we shall denote by $\mathscr{F}\varphi$ or $\hat{\varphi}$ the Fourier transform of φ:

$$(5) \quad \hat{\varphi}(\xi) = \int_{\mathbb{R}^n} e^{-2i\pi\langle x, \xi\rangle} \varphi(x)\, dx$$

where $\langle x, \xi\rangle = \sum_{j=1}^{n} x_j \xi_j$.

We shall denote by $\bar{\mathscr{F}}\varphi$ the function defined by

$$\bar{\mathscr{F}}\varphi(\xi) = \hat{\varphi}(-\xi) = \int e^{2i\pi\langle x, \xi\rangle} \varphi(x)\, dx$$

Then $\mathscr{F}$ and $\bar{\mathscr{F}}$ are automorphisms of $\mathscr{S}(\mathbb{R}^n)$ and for every $\varphi \in \mathscr{S}(\mathbb{R}^n)$ we have

$$(6) \quad \mathscr{F}(\bar{\mathscr{F}}\varphi) = \bar{\mathscr{F}}(\mathscr{F}\varphi) = \varphi$$

- Let T be in $\mathscr{S}'(\mathbb{R}^n)$. We define the Fourier transform $\mathscr{F}T$ (or $\hat{T}$) of T by:

$$(7) \quad \langle \mathscr{F}T, \varphi\rangle = \langle T, \mathscr{F}\varphi\rangle \quad \text{for all } \varphi \in \mathscr{S}(\mathbb{R}^n)$$

In the same way

$$\langle \bar{\mathscr{F}}T, \varphi\rangle = \langle T, \bar{\mathscr{F}}\varphi\rangle$$

The maps $\mathscr{F}$ and $\bar{\mathscr{F}}$ are automorphisms of $\mathscr{S}'(\mathbb{R}^n)$ and

$$(8) \quad \bar{\mathscr{F}}T = \mathscr{F}(\check{T}) \text{ where } \langle \check{T}, \varphi\rangle = \langle T, \check{\varphi}\rangle \quad \text{and} \quad \check{\varphi}(\xi) = \varphi(-\xi)$$

$$(9) \quad \bar{\mathscr{F}}\mathscr{F}T = \mathscr{F}\bar{\mathscr{F}}T = T \quad \text{for all } T \in \mathscr{S}'(\mathbb{R}^n)$$

- If $f \in L^2(\mathbb{R}^n) \subset \mathscr{S}'(\mathbb{R}^n)$ then $\hat{f} \in L^2(\mathbb{R}^n)$, $\bar{\mathscr{F}}f \in L^2(\mathbb{R}^n)$ and

$$(10) \quad \|\hat{f}\|_{L^2(\mathbb{R}^n)} = \|f\|_{L^2(\mathbb{R}^n)} = \|\bar{\mathscr{F}}f\|_{L^2(\mathbb{R}^n)}$$

- If $T \in \mathscr{E}'(\mathbb{R}^n) \subset \mathscr{S}'(\mathbb{R}^n)$, the Fourier transform of T is a C^∞ function and is given by

$$(11) \quad \hat{T}(\xi) = \langle T_x, e^{-2i\pi\langle x, \xi\rangle}\rangle$$

e) The Paley–Wiener–Schwartz theorem

Let $T \in \mathscr{E}'(\mathbb{R}^n)$, the Fourier transform of T can be extended to $\mathbb{C}^n$ as an entire function given by $F(z) = \langle T_x, e^{2i\pi\langle x, z\rangle}\rangle$. Moreover:

There exist constants C, A, an integer $N \in \mathbb{N}$ such that for all $z \in \mathbb{C}^n$

$$(12) \quad |F(z)| \leq C(1 + |z|)^N e^{A|\operatorname{Im} z|}$$

Conversely for every entire function on $\mathbb{C}^n$ satisfying (12) there exists $T \in \mathscr{E}'(\mathbb{R}^n)$ such that $\hat{T}(z) = F(z)$ for all $z \in \mathbb{C}^n$.

f) Fourier transform and convolution
If $u \in \mathscr{E}'(\mathbb{R}^n)$, $v \in \mathscr{S}'(\mathbb{R}^n)$ then $u * v \in \mathscr{S}'(\mathbb{R}^n)$ and

$$(13) \quad \mathscr{F}(u * v) = \mathscr{F}u \cdot \mathscr{F}v$$

This makes sense since $\mathscr{F}u \in C^\infty(\mathbb{R}^n)$.

g) Fourier transform, derivative and product by x_j
In what follows we shall denote

$$(14) \quad D_j = \frac{1}{2i\pi} \cdot \frac{\partial}{\partial x_j}$$

and for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$:

$$D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$$

Then, for all $T \in \mathscr{S}'(\mathbb{R}^n)$

$$(15) \quad \begin{cases} \mathscr{F}(D^\alpha T) = \xi^\alpha \mathscr{F}T \\ \mathscr{F}(x^\alpha T) = (-1)^{|\alpha|} D^\alpha(\mathscr{F}T) \end{cases}$$

h) Differential operators and Fourier transform
To each polynormial $P(\xi) = \sum_{|\alpha| \leq m} a_\alpha \xi^\alpha$ of order m in $\mathbb{R}^n$, with complex coefficients we associate the differential operator with constant coefficients:

$$P(D) = \sum_{|\alpha| \leq m} a_\alpha D^\alpha$$

where $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$ and $D_j = \dfrac{1}{2i\pi} \dfrac{\partial}{\partial x_j}$.

We have

$$\widehat{(P(D)u)}(\xi) = P(\xi)\hat{u}(\xi)$$

for all u in $\mathscr{S}'(\mathbb{R}^n)$.

Remark:
Several authors define the Fourier transform of a function $\varphi \in \mathscr{S}(\mathbb{R}^n)$ by

$$\tilde{\varphi}(\xi) = \int e^{-i\langle x, \xi\rangle} \varphi(x)\, dx$$

Then $\mathscr{F}\varphi(\xi) = \tilde{\varphi}(2\pi\xi)$, and all the formulas differ from the above formulas by constants which are powers of 2π.

i) Laplace transform of a distribution:
Let T be a distribution on $\mathbb{R}$ whose support is contained in $\mathbb{R}_+ = \{x \in \mathbb{R}, x \geq 0\}$. Let us assume that there exists $\xi \in \mathbb{R}$ such that $e^{-\xi x} T \in \mathscr{S}'(\mathbb{R})$; then we define the Laplace transform $\mathscr{L}T$ of the distribution T by the formula

$$(16) \quad (\mathscr{L}T)(p) = \langle T, e^{-px} \rangle$$

which makes sense for $p \in \mathbb{C}$, $\operatorname{Re} p > \xi$. Indeed let α be a C^∞ function with support in $\mathbb{R}_+$, equal to 1 in a neighborhood of the support of T. For $\operatorname{Re} p > \xi$, the function $\alpha(x) e^{-(p-\xi)x}$ belongs to $\mathscr{S}(\mathbb{R})$ and (16) can be written

$$\langle T, e^{-px} \rangle = \langle e^{-\xi x} T, \alpha e^{-(p-\xi)x} \rangle$$

The right hand side makes sense thanks to our hypothesis.

j) Properties of the Laplace transform:
1) The Laplace transform of a distribution T is a holomorphic function in the domain where it is defined.

2) $\mathscr{L}\left[\left(\dfrac{d}{dx}\right)^m T\right](p) = p^m \mathscr{L}(T)(p) \qquad m \in \mathbb{N}.$

3) $\left[\left(\dfrac{d}{dp}\right)^m \mathscr{L}(T)\right](p) = (-p)^m \mathscr{L}(T)(p).$

4) If S and T are two distributions with support in $\{X: x \geq 0\}$ having a Laplace transform for $\operatorname{Re} p > \xi_1$ and $\operatorname{Re} p > \xi_2$, one can define $S * T$ (see exercise 63); moreover $S * T$ has a Laplace transform defined in $\operatorname{Re} p > \operatorname{Max}(\xi_1, \xi_2)$ and

$$\mathscr{L}(S * T)(p) = \mathscr{L}(S)(p) \cdot \mathscr{L}(T)(p)$$

5) If $\mathscr{L}(T)$ vanishes for $\operatorname{Re} p > \xi$ then the distribution T vanishes.

k) Inverse Laplace transform
We have the following result:
6) A necessary and sufficient condition for a function $F(p)$ to be the Laplace transform of a distribution with support in $\{x: x \geq 0\}$ is that:
* F be holomorphic in $\operatorname{Re} p > \xi$, $\xi \in \mathbb{R}$
* $|F(p)| \leq P(|p|) \quad \operatorname{Re} p > \xi$

where $P(|p|)$ is a polynomial in $|p|$.

STATEMENTS OF THE EXERCISES* CHAPTER 6

Exercise 66

Let A be a real symetric matrix such that

(1) $\exists \alpha > 0 \colon \forall x \in \mathbb{R}^n (Ax, x) \geq \alpha \|x\|^2$

1°) Show that the function $e^{-(Ax,x)}$ is in $L^1(\mathbb{R})$

2°) Our purpose is to compute the Fourier transform of $e^{-(Ax,x)}$

a) Discuss the case $n = 1$. (Hint: Integrate the holomorphic function e^{-az^2} along an appropriate contour. We recall that $\int_{\mathbb{R}} e^{-x^2}\,dx = \sqrt{\pi}$).

b) Discuss the case where A is diagonal.

c) Using an orthogonal matrix U (i.e. ${}^tU = U^{-1}$) to put A into a diagonal form, deduce from b) that

$$\mathscr{F}(e^{-(Ax,x)})(\xi) = \frac{\pi^{n/2}}{\sqrt{|\det A|}} \cdot e^{-\pi^2(A^{-1}\xi,\xi)}$$

Exercise 67

For $m \in \mathbb{N}$ we define the Hermite polynomials H_m by

(1) $$\left(\frac{d}{dx}\right)^m (e^{-2\pi x^2}) = (-1)^m \sqrt{m!}\, 2^{m-1/4} \pi^{m/2} H_m(x) e^{-2\pi x^2}$$

and the Hermite functions by

(2) $$\mathscr{H}_m(x) = e^{-\pi x^2} H_m(x)$$

1°) a) Compute $H_i(x)$ $i = 0, 1, 2, 3$.

b) Show that for $m \geq 1$ one has:

(3) $$\left(\frac{d}{dx}\right)^{m+1} (e^{-2\pi x^2}) + 4\pi x \left(\frac{d}{dx}\right)^m (e^{-2\pi x^2}) + 4m\pi \left(\frac{d}{dx}\right)^{m-1} (e^{-2\pi x^2}) = 0$$

Deduce that

(4) $$\left(\frac{d}{dx}\right)[H_m(x)] = 2\sqrt{m\pi}\, H_{m-1}(x).$$

(5) $$2\sqrt{\pi(m+1)}\, H_{m+1}(x) - 4\pi x H_m(x) + 2\sqrt{m\pi}\, H_{m-1}(x) = 0.$$

(continuation page 142)

* Solutions pp. 149–181.

c) Compute $H_m(0)$. Show that

$$(6) \quad H_{2n}(0) = 2^{1/4}\frac{(-1)^n\sqrt{(2n)!}}{n!2^n}$$

2°) a) Show that $\mathscr{H}_m$ is an element of the space $\mathscr{S}(\mathbb{R})$ and that

$$(7) \quad \int_{\mathbb{R}} \mathscr{H}_p(x)\mathscr{H}_q(x)\,\mathrm{d}x = \delta_{p,q} \text{ (i.e. } = 1 \text{ if } p = q, = 0 \text{ if } p \neq q)$$

(Hint: integrate by parts)

b) Using the value of $\mathscr{F}\mathrm{e}^{-\pi x^2}$ found in exercise 66, show that $[\mathscr{F}\mathscr{H}_m](\xi) = (-i)^m\mathscr{H}_m(\xi)$.

3°) If $T \in \mathscr{S}'(\mathbb{R})$ we shall call "developpement of T in Hermite functions" the series

$$\sum_{m=0}^{\infty} a_m(T)\mathscr{H}_m \quad \text{where } a_m(T) = \langle T, \mathscr{H}_m\rangle$$

a) Determine the development of δ in Hermite functions.

b) If $T \in \mathscr{S}'(\mathbb{R})$ we consider the transformations $\mathscr{T}_+$ and $\mathscr{T}_-$ defined by

$$\mathscr{T}_+T = \frac{\mathrm{d}T}{\mathrm{d}x} + 2\pi xT, \qquad \mathscr{T}_-T = -\frac{\mathrm{d}T}{\mathrm{d}x} + 2\pi xT$$

Show that

$$\mathscr{T}_+(\mathscr{H}_m) = 2\sqrt{m\pi}\,\mathscr{H}_{m-1} \quad \text{for } m \geq 1$$

$$\mathscr{T}_-(\mathscr{H}_m) = 2\sqrt{\pi(m+1)}\,\mathscr{H}_{m+1} \quad \text{for } m \geq 0$$

c) Using the transformations $\mathscr{T}_+$ and $\mathscr{T}_-$, show that if $\varphi \in \mathscr{S}$, the sequence $(a_m(\varphi))_{m\in\mathbb{N}}$ is rapidly decreasing.

Show that the development of δ found in a) converges in $\mathscr{S}'$ to a distribution S. Compute $\mathscr{T}_+(S)$, $\mathscr{T}_-(S)$. Deduce that $S = C\delta$ and using (7) show that $C = 1$.

Exercise 68

Give an example of a C^∞ function on $\mathbb{R}$ such that

a) There is no polynomial P in $\mathbb{R}$ such that

$$(1) \quad |f(x)| \leq |P(x)| \text{ for all } x \text{ in } \mathbb{R}.$$

b) The map $\mathscr{S}(\mathbb{R}) \ni \varphi \mapsto \int_{\mathbb{R}} f(x)\varphi(x)\,\mathrm{d}x$ determines a tempered distribution.

Exercise 69

We recall that a distribution T is said to be even (resp. odd) if $\langle T, \check{\varphi}\rangle = \langle T, \varphi\rangle$ (resp. $= -\langle T, \varphi\rangle$) for all $\varphi \in \mathscr{D}(\mathbb{R})$ where $\check{\varphi}(x) = \varphi(-x)$.
Show that if T is an even tempered distribution (resp. odd) then $\bar{\mathscr{F}}T = \mathscr{F}T$. (resp. $\bar{\mathscr{F}}T = -\mathscr{F}T$).

Exercise 70

Let A be an (n, n) real non singular matrix.

a) Let $T \in \mathscr{S}'(\mathbb{R})$. Show that $T \circ A \in \mathscr{S}'(\mathbb{R}^n)$ and that:

$$\widehat{T \circ A} = |\det A|^{-1}\hat{T} \circ ({}^tA)^{-1}$$

where ${}^tA$ is the transposed of A.

b) $T \in \mathscr{S}'(\mathbb{R})$ is said to be even (resp. odd) if $\langle T, \check{\varphi}\rangle = \langle T, \varphi\rangle$ (resp. $-\langle T, \varphi\rangle$) $\forall \varphi \in \mathscr{S}(\mathbb{R}^n)$ where $\check{\varphi}(x) = \varphi(-x)$.
Deduce from a) that if T is even (resp. odd), $\hat{T}$ is even (resp. odd)

c) $T \in \mathscr{S}'(\mathbb{R}^n)$ is said to be invariant by rotation if $T \circ A = T$ for all orthogonal matrix A. Deduce from a) that the Fourier transform of a distribution invariant by rotation is invariant by rotation.

Exercise 71

We recall that a distribution $T \in \mathscr{D}'(\mathbb{R}^n)$ is said to be homogeneous of degree $\lambda \in \mathbb{R}$ if

$$\langle T, \varphi_t\rangle = t^{-(n+\lambda)}\langle T, \varphi\rangle \qquad \forall \varphi \in \mathscr{D}(\mathbb{R}^n) \qquad \forall t > 0$$

where $\varphi_t(x) = \varphi(tx)$.
Show that the Fourier transform of a tempered distribution homogeneous of degree λ is homogeneous of degree $-n - \lambda$.

Exercise 72

a) Let f and g be two elements of $\mathscr{S}(\mathbb{R}^n)$. We assume that $f * g$ vanishes identically. Can we assert that f or g vanishes? and if $f = g$?

b) Same question for $T \in \mathscr{S}'$, $S \in \mathscr{E}'$.

Exercise 73 (see exercises 11, 70)

Compute the Fourier transform of the distribution $T = pv\frac{1}{x}$ defined in exercise 11.

Deduce $\mathcal{F}H$ and $\overline{\mathcal{F}}H$ where H is the Heaviside function.

$\left(\text{Hint: Use } x \cdot pv\frac{1}{x} = 1 \text{ and question b) of exercise 70.}\right)$

Exercise 74 (see exercises 27, 73)

Using the equality $|x| = xH(x) - xH(-x)$ (where H is the Heaviside function), the values of $\mathcal{F}H$, $\overline{\mathcal{F}}H$ found in exercise 73 and the expression of $\frac{d}{d\xi}pv\frac{1}{\xi}$ found in exercise 27, find $\mathcal{F}|x|$ and deduce $\mathcal{F}Fp\frac{1}{\xi^2}$.

Exercise 75 (see exercises 29, 71)

a) Compute the Fourier transform of the tempered distribution in $\mathbb{R}^2$, $S = z^n$ where $z = x + iy$ and $n \in \mathbb{N}$.

b) Using the formula $\frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\left(\frac{1}{\pi z}\right) = \delta$ (see exercise 29), the identity $z \cdot \frac{1}{z} = 1$ and the homogeneity of $\frac{1}{z}$ show that $\mathcal{F}\left(\frac{1}{z}\right) = -\frac{i}{\zeta}$ where $\zeta = \xi + i\eta$.

Exercise 76

We shall denote in the sequel, δ_a the Dirac measure at the point a. We define by induction the sequence of distributions $(T_k)_{k\in\mathbb{N}}$ by:

$$\begin{cases} T_1 = \frac{1}{2}(\delta_1 + \delta_{-1}) \\ T_k = T_{k-1} * T_1 \end{cases}$$

a) Write T_k as a linear combination of Dirac measures.

b) Compute the Fourier transform $\hat{T}_k$ of the distribution with compact support T_k.

c) For $k \in \mathbb{N}$, $k \geq 1$, we set:

$$f_k(\xi) = \hat{T}_k\left(\frac{\xi}{2\pi\sqrt{k}}\right)$$

Prove that the sequence $\{f_k\}$ converges in $\mathcal{D}'(\mathbb{R})$ to a function f and compute f.
(continuation p. 145)

d) We denote by g_k the distribution whose f_k is the Fourier transform. Prove that the sequence $\{g_k\}$ converges, in an appropriate sense, to a distribution g and compute g.

Exercise 77* (see exercises 15, 34, 69, 70, 71)

1°) Our purpose is to compute the Fourier transform of the distribution $|x|^\lambda$ for $\lambda \in]-1, 0[$.

1°) Let $f(x) = |x|^\lambda$, $x \in \mathbb{R}$, $\lambda \in]-1, -\frac{1}{2}[$.

a) Prove that there exist $u \in L^2(\mathbb{R})$, $v \in L^1(\mathbb{R})$ such that $f = u + v$.

b) Deduce that $\hat{f}$ is a function and, using exercise 71, that:

$$\hat{f}(\xi) = \begin{cases} C_1 \xi^{-(\lambda+1)} & \xi > 0 \\ C_2 |\xi|^{-(\lambda+1)} & \xi < 0 \end{cases}$$

c) Prove that $\hat{f}$ is even. Deduce that $C_1 = C_2 = C$. Compute C (Hint: use the fact that $\mathscr{F} e^{-\pi x^2} = e^{-\pi\xi^2}$ and the function Γ defined by $\Gamma(s) = \int_0^\infty x^{s-1} e^{-x}\,dx$).

2°) Using the inverse Fourier transform extend this result to $\lambda \in]-1, 0[$.

3°) Using exercises 15, 34, compute the Fourier transform of distribution $|x|^\lambda = x_+^\lambda + x_-^\lambda$ defined in exercise 15 for $\lambda \notin \mathbb{Z}$. (Computation of constants $M(\lambda) = \langle |x|^\lambda, e^{-\pi x^2} \rangle$ is not required.)

Exercise 78* (see exercises 33, 70)

1°) Let λ be a complex number such that $\operatorname{Re} \lambda > 0$. Compute the integral

$$I(\lambda) = \int_{\mathbb{R}} e^{-\lambda\pi x^2}\,dx$$

(Hint: Compute $I(\lambda)$ for $\lambda \in \mathbb{R}_+$ and use an analytic continuation argument).

2°) We consider function $f(x) = e^{i\pi a x^2}$ where $a \in \mathbb{R}\backslash\{0\}$.

a) Show that f determines an element of $\mathscr{S}'(\mathbb{R})$.

b) Find a differential equation in $\mathscr{S}'(\mathbb{R})$ satisfied by f?

c) Deduce that $\hat{f}$ satisfies a differential equation and show that $\hat{f}(\xi) = C e^{-i(\pi/a)\xi^2}$ where $C \in \mathbb{C}$. (Hint: Use exercise 33.)

d) Compute constant C by applying $\hat{f}$ to the function $e^{-\pi x^2}$. We recall that $\mathscr{F} e^{-\pi x^2} = e^{-\pi\xi^2}$. (Hint: use 1°)). Deduce that

$$\hat{f}(\xi) = \begin{cases} \dfrac{1}{\sqrt{a}} e^{i(\pi/4)} e^{-i(\pi/a)\xi^2} & \text{if } a > 0 \\[2ex] \dfrac{1}{\sqrt{|a|}} e^{-i(\pi/4)} e^{-i(\pi/a)\xi^2} & \text{if } a < 0 \end{cases}$$

(Continuation page 146)

3°) Let D be a real diagonal matrix $D = (\lambda_j)_{j=1,\ldots,n}$ with $\lambda_j > 0, j = 1, \ldots, k, \lambda_j < 0$, $j = k+1, \ldots, n$.
Prove that if $T = e^{i\pi\langle Dx,x\rangle}$ we have

$$\hat{T}(\xi) = \prod_{i=1}^{n} \frac{1}{\sqrt{|\lambda_j|}} e^{i(2k-n)\pi/4} e^{-i\langle D^{-1}\xi,\xi\rangle}$$

4°) Let A be a real symetric non singular matrix. Deduce from 3°) that

$$\mathscr{F}(e^{i\pi\langle Ax,x\rangle}) = |\det A|^{-1/2} e^{i(\pi/4)\sigma_A} e^{-i\pi\langle A^{-1}\xi,\xi\rangle}$$

where σ_A is the signature of A, i.e. the number of positive eigenvalues minus the number of negative eigenvalues.
(Hint: Put A into a diagonal form and use exercise 70).

Exercise 79

For $u \in \mathscr{S}'(\mathbb{R})$ we set $Du = \dfrac{1}{2i\pi}\dfrac{du}{dx}$ and for $h \in \mathbb{R}$ we define $\tau_h u \in \mathscr{S}'(\mathbb{R})$ by $\langle \tau_h u, \varphi\rangle = \langle u, \tau_{-h}\varphi\rangle$ for all $\varphi \in \mathscr{S}(\mathbb{R})$ where $\tau_{-h}\varphi(x) = \varphi(x+h)$.
We consider operator P_h: $\mathscr{S}'(\mathbb{R}) \to \mathscr{S}'(\mathbb{R})$ given by

$$P_h u = Du + \tau_h u$$

Determine the values of h for which P_h in injective and find the kernel of P_h for the other values.

Exercise 80

Let $P(\xi)$ be a polynomial in $\mathbb{R}^n$ which does not vanish identically.
We denote by $P(D)$ the corresponding differential operator with constant coefficients.
Show that if $u \in \mathscr{E}'(\mathbb{R}^n)$ satisfies $P(D)u = 0$ then $u = 0$.

Exercise 81

Let T be a distribution with compact support such that for each $\alpha \in \mathbb{N}^n$,

$$\langle T, x^\alpha\rangle = 0.$$

Prove that $T = 0$.
(Hint: Use the Paley–Wiener–Schwartz theorem and compute $\partial^\alpha F(0)$ where F is the entire function which extends $\hat{T}$.)

Exercise 82

Let $P = \sum_{|\alpha| \leq m} a_\alpha D^\alpha$ be a differential operator with constant coefficients in $\mathbb{R}^n$ such that

$$\left\{\xi \in \mathbb{R}^n \colon P(\xi) = \sum_{|\alpha| \leq m} a_\alpha \xi^\alpha = 0\right\} = \{0\}$$

Prove that the kernel of P in $\mathcal{S}'(\mathbb{R}^n)$ contains only polynomials.

Exercise 83

Let k be a strictly positive real number and $u \in \mathcal{S}'(\mathbb{R})$ such that

$$\frac{\mathrm{d}^4 u}{\mathrm{d}x^4} + ku \in L^2(\mathbb{R})$$

Prove that $\dfrac{\mathrm{d}^j u}{\mathrm{d}x^j} \in L^2(\mathbb{R})$ for $0 \leq j \leq 4$.

Exercise 84* (see exercise 52)

a) Let $T \in \mathscr{E}'(\mathbb{R})$. Prove that the formulas

$$f_+(z) = \int_0^{\infty} e^{2i\pi\xi z} \hat{T}(\xi)\, \mathrm{d}\xi \quad \left(\text{resp. } f_-(z) = \int_{-\infty}^{0} e^{2i\pi\xi z} \hat{T}(\xi)\, \mathrm{d}\xi\right)$$

define holomorphic functions in $\operatorname{Im} z > 0$ (resp. $\operatorname{Im} z < 0$) such that

$$|f_\pm(z)|| \leqq \frac{C}{|\operatorname{Im} z|^A}, \quad A \in \mathbb{R}^+ \quad \text{and} \quad |\operatorname{Im} z| \text{ small}$$

b) Using exercise 52 prove that

$$T = f_+(x + i0) + f_-(x - i0)$$

c) Deduce that for every distribution $T \in \mathscr{D}'(]a, b[$, $-\infty \leq a < b \leq +\infty$ and for each $]a_1, b_1[\subsetneq]a, b[$, there exist two functions f_+ (resp. f_-) holomorphic in $\operatorname{Im} z > 0$ (resp in $\operatorname{Im} z < 0$) such that $|f_\pm(z)| \leq \dfrac{C}{|\operatorname{Im} z|^A}$ $A \in \mathbb{R}^+$ and

$$T = f_+(x + i0) + f_-(x - i0) \quad \text{in } \mathscr{D}'(]a_1, b_1[).$$

Exercise 85

Compute the Laplace transform of the following distributions

a) $\delta^{(k)}$; b) $H(x)$; c) $H(x) \operatorname{Log} x$.

Exercise 86 (see exercise 85)

Let $k \in \mathbb{N}\backslash\{0\}$ and $Fp\dfrac{H(x)}{x^k}$ the distribution on $\mathbb{R}$ defined by

$$\left\langle Fp\frac{H(x)}{x^k}, \varphi\right\rangle = \lim_{\varepsilon\to 0}\left\{\int_0^\infty \frac{\varphi(x)}{x^k}\,\mathrm{d}x + \sum_{j=1}^{k-1}\frac{\varphi^{(j-1)}(0)}{(j-1)!(j-k)\varepsilon^{k-j}} + \frac{\varphi^{(k-1)}(0)}{(k-1)!}\operatorname{Log}\varepsilon\right\}$$

a) Compute in the distributions sense

$$\frac{\mathrm{d}}{\mathrm{d}x}(H(x)\operatorname{Log}x) \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}x}Fp\frac{H(x)}{x^k}$$

b) From the value of $\mathscr{L}(H(x)\operatorname{Log}x)$ found in exercise 85 and using question a) deduce that the Laplace transform of the distribution $Fp\dfrac{H(x)}{x^k}$ is for $\operatorname{Re}p > 0$:

$$\mathscr{L}\left(Fp\frac{H(x)}{x^k}\right)(p) = \frac{(-1)^k}{(k-1)!}p^{k-1}\left(\operatorname{Log}p + C - \sum_{l=1}^{k-1}\frac{1}{l}\right), \qquad k = 1, 2, \ldots$$

where C is the Euler constant.

Exercise 87

a) Let $\alpha \in \mathbb{C}$. Compute the Laplace transform, in $\operatorname{Re}p > \operatorname{Re}\alpha$, of the distribution

$$T = x^k e^{\alpha x}H(x), \qquad k \in \mathbb{N}$$

b) Deduce the inverse Laplace transform of the following functions:

$$F(p) = \frac{p}{p+1}, \ \operatorname{Re}p > -1; \qquad F(p) = \frac{p^2 + i}{p^3 - 3p + 2}$$

Exercise 88

For $\alpha \in \mathbb{R}$ we shall denote by δ_α the Dirac measure at α.

a) Prove that the quantity

$$T = \sum_{k=0}^{\infty} e^k \delta_k$$

determines a distribution which possesses a Laplace transform and compute this Laplace transform.

b) Deduce the Laplace transform of $T * T$ and compute $T * T$.

Exercise 89 (see exercise 87)
Find a distribution T which posseses a Laplace transform and satisfies

$$(x\,e^{x}H(x)) * T = H(x)\sin x$$

(Hint: Use question a) of exercise 87.)

SOLUTIONS OF THE EXERCISES — CHAPTER 6

Solution 66

1°) Function $e^{-\alpha\|x\|^2}$ is in $L^1(\mathbb{R})$ when α is strictly positive so, using (1), it follows that $e^{-(Ax,x)} \in L^1(\mathbb{R})$.

2°) For $f \in L^1(\mathbb{R}^n)$ let us recall that

$$\hat{f}(\xi) = \int e^{-2i\pi(x,\xi)} f(x)\,dx$$

where $(x,\xi) = \sum_{i=1}^{n} x_i\xi_i$.

a) In the case where $n = 1$, A is a positive constant a.

$$\mathcal{F}\,e^{-a|x|^2} = \int_{\mathbb{R}} e^{-2i\pi x\xi - ax^2}\,dx = \int_{\mathbb{R}} \exp\left[-a\left(x + i\frac{\pi}{a}\xi\right)^2 - \frac{\pi^2}{a}\xi^2\right]dx$$

$$(2)\quad \mathcal{F}(e^{-a|x|^2})(\xi) = e^{-(\pi^2/a)\xi^2}\int_{\mathbb{R}} e^{-a(x+i(\pi/a)\xi)^2}\,dx$$

Function e^{-az^2} is holomorphic in $\mathbb{C}$.
Its integral on contour Γ (see Fig. 1) is equal to zero.

$$\int_{\Gamma} e^{-az^2}\,dz = 0$$

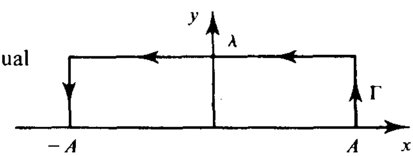


Fig. 1

so

$$\int_{[i\lambda-A,\,-A]} e^{-az^2}dz + \int_{-A}^{A} e^{-ax^2}dx + \int_{[A,\,i\lambda+A]} e^{-az^2}dz + \int_{[i\lambda+A,\,i\lambda-A]} e^{-az^2}dz = 0$$

When $A \to \infty$ the first and the third integral tend to zero. Therefore

$$(3)\quad \int_{-\infty}^{\infty} e^{-ax^2}dx = \int_{-\infty}^{\infty} e^{-a(x+i\lambda)^2}dx$$

We deduce from (2) and (3) that

$$\mathscr{F}(\mathrm{e}^{-ax^2})(\xi) = \mathrm{e}^{-(\pi^2/a)\xi^2} \cdot \int_{-\infty}^{\infty} \mathrm{e}^{-ax^2}\,\mathrm{d}x$$

Finally

$$\int_{\mathbf{R}} \mathrm{e}^{-ax^2}\,\mathrm{d}x = \frac{1}{\sqrt{a}} \int_{\mathbf{R}} \mathrm{e}^{-x^2}\,\mathrm{d}x = \sqrt{\frac{\pi}{a}}$$

so

$$\mathscr{F}(\mathrm{e}^{-ax^2})(\xi) = \sqrt{\frac{\pi}{a}}\,\mathrm{e}^{-(\pi^2/a)\xi^2}$$

b) If A is diagonal, $A = (a_j)$, $a_j > 0$, we have

$$\mathrm{e}^{-(Ax,x)} = \mathrm{e}^{-(a_1x_1^2 + \cdots + a_nx_n^2)} = \mathrm{e}^{-a_1x_1^2} \cdots \mathrm{e}^{-a_nx_n^2}$$

It follows from Fubini's theorem that

$$\mathscr{F}(\mathrm{e}^{-(Ax,x)})(\xi) = \mathscr{F}(\mathrm{e}^{-a_1x_1^2})(\xi) \cdot \cdots \cdot \mathscr{F}(\mathrm{e}^{-a_nx_n^2})(\xi_n)$$

so by question a)

$$\mathscr{F}(\mathrm{e}^{-(Ax,x)})(\xi) = \frac{\pi^{n/2}}{\sqrt{a_1 \cdot \cdots \cdot a_n}}\,\mathrm{e}^{-(\pi^2/a_1)\xi_1^2} \cdots \mathrm{e}^{-(\pi^2/a_n)\xi_n^2}$$

Now A^{-1} is the diagonal matrix $\left(\dfrac{1}{a_j}\right)$. Moreover the determinant of A is equal to $a_1 \cdots a_n$. So we can write

$$\mathscr{F}(\mathrm{e}^{-(Ax,x)})(\xi) = \frac{\pi^{n/2}}{\sqrt{\det A}}\,\mathrm{e}^{-\pi^2(A^{-1}\xi,\xi)}$$

c) Let us now discuss the general case.
We can find an orthogonal matrix U (i.e. ${}^tUU = \mathrm{Id}$) such that

$$UAU^{-1} = D$$

where D is diagonal.
Then

$$\mathrm{e}^{-(Ax,x)} = \mathrm{e}^{-(U^{-1}DUx,x)} = \mathrm{e}^{-({}^tUDUx,x)} = \mathrm{e}^{-(DUx,Ux)}$$

so

$$\mathscr{F}(\mathrm{e}^{-(Ax,x)})(\xi) = \int \mathrm{e}^{-2i\pi(x,\xi)}\,\mathrm{e}^{-(DUx,Ux)}\,\mathrm{d}x$$

Since ${}^tU \cdot U = Id$ we have

$$(x, \xi) = ({}^tU \cdot Ux, \xi) = (Ux, U\xi)$$

so

$$\mathcal{F}(\mathrm{e}^{-(Ax,x)})(\xi) = \int \mathrm{e}^{-2i\pi(Ux,\, U\xi)}\, \mathrm{e}^{-(DUx,\, Ux)}\, \mathrm{d}x$$

Let us set, in the integral, $y = Ux$. Then $dy = |\det U|\, \mathrm{d}x = \mathrm{d}x$ and

$$\mathcal{F}(\mathrm{e}^{-(Ax,x)})(\xi) = \int \mathrm{e}^{-2i\pi(y,U\xi)}\, \mathrm{e}^{-(Dy,y)}\, \mathrm{d}y = \mathcal{F}(\mathrm{e}^{-(Dx,x)})(U\xi)$$

Now we know, by the previous question, the Fourier transform of $\mathrm{e}^{-(Dx,x)}$. We have

$$\mathcal{F}(\mathrm{e}^{-(Ax,x)})(\xi) = \frac{\pi^{n/2}}{|\det D|^{1/2}} \mathrm{e}^{-\pi^2(D^{-1}U\xi,\, U\xi)} = \frac{\pi^{n/2}}{|\det D|^{1/2}} \mathrm{e}^{-\pi^2(U^{-1}D^{-1}U\xi,\, U\xi)}$$

Now $|\det D| = |\det A|$ and $D^{-1} = UA^{-1}U^{-1}$ so $U^{-1}D^{-1}U = A^{-1}$. Then

$$\mathcal{F}(\mathrm{e}^{-(Ax,x)})(\xi) = \frac{\pi^{n/2}}{|\det A|^{1/2}} \mathrm{e}^{-\pi^2(A^{-1}\xi,\xi)}$$

Solution 67

1°) a) We find easily:

$$H_0(x) = 2^{1/4}, \qquad H_1(x) = 2^{5/4}\sqrt{\pi} \cdot x, \qquad H_2(x) = 2^{-1/4}(4\pi x^2 - 1)$$

$$H_3(x) = -2^{3/4}\frac{1}{\sqrt{3}}\sqrt{\pi}(-4\pi x^3 + 3x)$$

b) The formula is obvious for $m = 1$. Let us assume it is true up to the order m and let us differentiate this formula; then we get the case $m + 1$.

Let us prove (4). By definition

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^m (\mathrm{e}^{-2\pi x^2}) = C(m) H_m(x)\, \mathrm{e}^{-2\pi x^2}$$

so

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{m+1} (\mathrm{e}^{-2\pi x^2}) = C(m)\frac{\mathrm{d}H_m}{\mathrm{d}x}\mathrm{e}^{-2\pi x^2} - 4\pi C(m) H_m(x) x\, \mathrm{e}^{-2\pi x^2}$$

Using formula (3), the definition of $H_m(x)$ and the above formula we get

$$C(m)\frac{dH_m}{dx}e^{-2\pi x^2} - 4\pi x H_m e^{-2\pi x^2} + 4\pi x C(m) H_m e^{-2\pi x^2} + \\ + 4m\pi C(m-1) H_{m-1} e^{-2\pi x^2} = 0$$

so

$$\frac{dH_m}{dx} = -\frac{4m\pi C(m-1)}{C(m)} \cdot H_{m-1}$$

Now

$$\frac{C(m-1)}{C(m)} = -\frac{1}{2\sqrt{m\pi}}$$

so

$$\left(\frac{d}{dx}\right) H_m(x) = 2\sqrt{m\pi}\, H_{m-1}(x) \quad \text{so (4).}$$

Let us prove (5). We have

$$\left(\frac{d}{dx}\right)^{m+1}(e^{-2\pi x^2}) = C(m+1)H_{m+1}(x)e^{-2\pi x^2} = -2\sqrt{\pi(m+1)}\,C(m)H_{m+1}e^{-2\pi x^2}$$

$$4\pi x\left(\frac{d}{dx}\right)^{m}(e^{-2\pi x^2}) = 4\pi x C(m) H_m(x) e^{-2\pi x^2}$$

$$4m\pi\left(\frac{d}{dx}\right)^{m-1}(e^{-2\pi x^2}) = 4m\pi C(m-1)H_{m-1}(x)e^{-2\pi x^2} = -2\sqrt{m\pi}\,C(m)H_{m-1}e^{-2\pi x^2}$$

Adding these formulas and using (3) we get (5).

c) We use an induction on m. If $p \leq m$ we assume that

$$p \text{ odd} \qquad H_p(0) = 0$$

$$p \text{ even} \qquad p = 2n \qquad H_p(0) = \frac{2^{1/4}(-1)^n\sqrt{(2n)!}}{n!2^n}$$

By formula (5)

$$H_{2n+2}(0) = -\frac{\sqrt{2n+1}}{\sqrt{2n+2}} H_{2n}(0) = -\frac{\sqrt{2n+1}\sqrt{2n+2}}{2(n+1)} \cdot \frac{(-1)^n 2^{1/4}(2n)!^{1/2}}{n!2^n}$$

$$H_{2n+2}(0) = \frac{(-1)^{n+1}2^{1/4}(2n+2)!^{1/2}}{(n+1)!2^{n+1}}$$

which prove the formula up to the order $2n + 2$. Now

$$H_{2n+1}(0) = C^{te}_{=} H_{2n-1}(0) = 0 \qquad \text{Q.E.D.}$$

2°) a) $\mathscr{H}_m(x) = H_m(x)\mathrm{e}^{-\pi x^2}$ where H_m is a polynomial. It follows easily from this fact that $\forall p, \forall q \in \mathbb{N}, \exists r \in \mathbb{N}\ \exists C > 0$:

$$\left|x^p\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^q \mathscr{H}_m\right| \leq C|x|^r \mathrm{e}^{-2\pi x^2}$$

We deduce that $\mathscr{H}_m \in \mathscr{S}(\mathbb{R})$. Let us set $I_{p,q} = \int_{-\infty}^{\infty} H_p(x)H_q(x)\mathrm{e}^{-2\pi x^2}\,\mathrm{d}x$.
Let us assume $p \neq q$, $p > q$. Then by definition

$$I_{p,q} = \frac{1}{C(p)}\int_{-\infty}^{\infty}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^p(\mathrm{e}^{-2\pi x^2})H_q(x)\,\mathrm{d}x$$

Integrating by parts we get:

$$I_{p,q} = \left[\frac{1}{C(p)}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{p-1}(\mathrm{e}^{-2\pi x^2})H_q(x)\right]_{-\infty}^{\infty} - \frac{1}{C(p)}\int_{\mathbb{R}}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{p-1}\mathrm{e}^{-2\pi x^2}\left(\frac{\mathrm{d}H_q}{\mathrm{d}x}\right)\mathrm{d}x$$

$$I_{p,q} = \left[\frac{C(p-1)}{C(p)}H_{p-1}(x)H_q(x)\mathrm{e}^{-2\pi x^2}\right]_{-\infty}^{\infty} - \frac{1}{C(p)}\int_{\mathbb{R}}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{p-1}(\mathrm{e}^{-2\pi x^2})\frac{\mathrm{d}H_q}{\mathrm{d}x}\cdot\mathrm{d}x$$

so

$$I_{p,q} = -\frac{1}{C(p)}\int_{\mathbb{R}}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{p-1}(\mathrm{e}^{-2\pi x^2})\cdot\frac{\mathrm{d}H_q}{\mathrm{d}x}\mathrm{d}x$$

Iterating $p-1$ times this integration by parts we find $I_{p,q} = 0$ for $\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^p H_q \equiv 0$ since $p > q$ and H_q is a polynomial of degree q (this follows from (4) and from the fact that H_0 is of degree zero).
If $p = q$, integrating as above we get

$$I_{p,p} = (-1)^p\int_{-\infty}^{\infty}\frac{1}{C(p)}\mathrm{e}^{-2\pi x^2}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^p H_p(x)\,\mathrm{d}x$$

Now by (4)

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)H_p = 2\sqrt{p\pi}\,H_{p-1}$$

so

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^p H_p = 2^p\pi^{p/2}\sqrt{p(p-1)\cdots 1}\,H_0 = 2^{p+1/4}\pi^{p/2}p!^{1/2}$$

It follows that

$$I_{p,p} = \frac{2^{p+1/4}\pi^{p/2}\sqrt{p!}}{\sqrt{p!}\,2^{p-1/4}\pi^{p/2}}\int_{-\infty}^{\infty}\mathrm{e}^{-2\pi x^2}\mathrm{d}x = \sqrt{2}\int_{-\infty}^{\infty}\mathrm{e}^{-2\pi x^2}\mathrm{d}x = 1$$

(we recall that $\int_{\mathbb{R}}\mathrm{e}^{-x^2}\mathrm{d}x = \sqrt{\pi}$.)

b) We recall that $\mathscr{F}(e^{-\pi x^2}) = e^{-\pi\xi^2}$.
Let us set $\mathscr{F}(\mathscr{H}_m)(\xi) = \mathscr{K}_m(\xi)$. Then

$$\mathscr{K}_{m+1} = \mathscr{F}[\mathscr{H}_{m+1}] = \mathscr{F}[e^{-\pi x^2} H_{m+1}]$$

and from (5)

$$2\sqrt{\pi(m+1)}\,\mathscr{K}_{m+1} = \mathscr{F}[4\pi x e^{-\pi x^2} H_m] - \mathscr{F}[2\sqrt{m\pi}\, e^{-\pi x^2} H_{m-1}]$$

Now

$$4\pi x e^{-\pi x^2} H_m = -2\frac{d}{dx}[e^{-\pi x^2} H_m] + 2e^{-\pi x^2}\frac{dH_m}{dx} \quad \text{and from (4)}$$

$$4\pi x e^{-\pi x^2} H_m = -2\frac{d}{dx}[e^{-\pi x^2} H_m] + 2e^{-\pi x^2}2\sqrt{m\pi}\, H_{m-1}$$

$$4\pi x e^{-\pi x^2} H_m = -2\frac{d}{dx}\mathscr{H}_m + 4\sqrt{m\pi}\,\mathscr{H}_{m-1}$$

We deduce

$$2\sqrt{\pi(m+1)}\,\mathscr{K}_{m+1} = -2\mathscr{F}\left[\frac{d\mathscr{H}_m}{dx}\right] + 4\sqrt{m\pi}\,\mathscr{F}\mathscr{H}_{m-1} - 2\sqrt{m\pi}\,\mathscr{F}\mathscr{H}_{m-1}$$

$$2\sqrt{\pi(m+1)}\,\mathscr{K}_{m+1} \; -4i\pi\xi\mathscr{K}_m + 2\sqrt{m\pi}\,\mathscr{K}_{m-1}$$

Let us divide both sides by $(-i)^{m+1}$. We get

$$2\sqrt{\pi(m+1)}\frac{\mathscr{K}_{m+1}}{(-i)^{m+1}} + 4i\pi\xi\frac{\mathscr{K}_m}{(-i)^{m+1}} - \frac{2\sqrt{m\pi}}{(-i)^{m+1}}\mathscr{K}_{m-1} = 0$$

$$2\sqrt{\pi(m+1)}\frac{\mathscr{K}_{m+1}}{(-i)^{m+1}} - 4\pi\xi\frac{\mathscr{K}_m}{(-i)^m} + 2\sqrt{m\pi}\frac{\mathscr{K}_{m-1}}{(-i)^{m-1}} = 0$$

i.e. $\dfrac{\mathscr{K}_m}{(-i)^m}$ also satisfies (5). Moreover

$$\mathscr{K}_0 = \mathscr{F}(\mathscr{H}_0) = 2^{1/4}\mathscr{F}(e^{-\pi x^2}) = 2^{1/4} = \mathscr{H}_0$$

$$\frac{\mathscr{K}_1}{-i} = i\mathscr{F}(\mathscr{H}_1) = i\mathscr{F}[e^{-\pi x^2}H_1] = i2^{5/4}\sqrt{\pi}\,\mathscr{F}[x e^{-\pi x^2}]$$

$$\frac{\mathscr{K}_1}{-i} = 2^{5/4} i\sqrt{\pi}\frac{-1}{2\pi}\mathscr{F}\left[\frac{d}{dx}e^{-\pi x^2}\right] = 2^{5/4}\sqrt{\pi}\frac{1}{2i\pi}2i\pi\xi e^{-\pi\xi^2}$$

$$\frac{\mathscr{K}_1}{-i} = 2^{5/4}\sqrt{\pi}\,\xi e^{-\pi\xi^2} = \mathscr{H}_1$$

So $\dfrac{\mathscr{K}_m(\xi)}{(-i)^m} = \mathscr{H}_m(\xi)$ i.e. $\mathscr{F}\mathscr{H}_m = (-i)^m\mathscr{H}_m(\xi)$.

3°) a) The development of δ is $\sum_{n=0}^{\infty} \mathscr{H}_{2n}(0)\mathscr{H}_{2n} = \sum_{n=0}^{\infty} H_{2n}(0)\mathscr{H}_{2n}$.

b) $\mathscr{T}_+(\mathscr{H}_m) = \frac{dH_m}{dx}e^{-\pi x^2} - 2\pi x H_m e^{-\pi x^2} + 2\pi x \mathscr{H}_m = \frac{dH_m}{dx}e^{-\pi x^2}$ and from (4) $\mathscr{T}_+(\mathscr{H}_m) = 2\sqrt{m\pi}\,\mathscr{H}_{m-1}, m \geqslant 1$.

$$\mathscr{T}_-(\mathscr{H}_m) = -\frac{d}{dx}(\mathscr{H}_m) + 2\pi x\mathscr{H}_m = -2\sqrt{m\pi}\,\mathscr{H}_{m-1} + 4\pi x\mathscr{H}_m$$

$$\mathscr{T}_-(\mathscr{H}_m) = 2\sqrt{\pi(m+1)}\,\mathscr{H}_{m+1} \quad \text{by (5)}$$

c) The functions $(\mathscr{H}_m)_m$ are orthonormal in $L^2(\mathbb{R})$; one has

$$\sum_0^{\infty} |a_m(\varphi)|^2 \le \|\varphi\|_{L^2}$$

On the other hand $\varphi \in \mathscr{S}$ so φ', $x\,\varphi \in L^2$ and $\mathscr{T}_+\varphi \in L^2$.

$$\begin{aligned} a_m(\mathscr{T}_+\varphi + \varphi) &= \langle \mathscr{T}_+\varphi + \varphi, \mathscr{H}_m\rangle = \langle \varphi, \mathscr{T}_-\mathscr{H}_m\rangle \\ &= \langle \varphi, 2\sqrt{\pi(m+1)}\,\mathscr{H}_{m+1}\rangle = 2\sqrt{\pi(m+1)}\,a_{m+1}(\varphi) \end{aligned}$$

so

$$\sum_0^{\infty} 4\pi(m+1)|a_{m+1}(\varphi)|^2 < \infty$$

In this way we can prove that $m^{\alpha}|a_m(\varphi)| \le C_{m,\alpha} \forall \alpha \in \mathbb{N}$.

Let $S = \sum_0^{\infty} a_m(\delta)\mathscr{H}_m$ and $S_N = \sum_0^{N} a_m(\delta)\mathscr{H}_m$. *Let* $\varphi \in \mathscr{S}$ then

$$\langle S_N, \varphi\rangle = \sum_0^{\infty} a_m(\delta)a_m(\varphi)$$

and

$$a_{2p}(\delta) = H_{2p}(0) = \frac{2^{1/4}(-1)^p\sqrt{(2p)!}}{p!2^p}$$

Now $(2p)! \le 2^p p!^2$ so $|a_{2p}(\delta)| \le C$ where C is independent of p and $a_m(\varphi)$ is rapidly decreasing, therefore $\langle S_N, \varphi\rangle$ has a limit when $N \to \infty$.
Moreover

$$\mathscr{T}_+(S) = \sum_1^{\infty} a_{2n}(\delta)2\sqrt{2\pi n}\,\mathscr{H}_{2n-1} = \frac{dS}{dx} + 2\pi x S$$
$$\mathscr{T}_-(S) = \sum_0^{\infty} a_{2n}(\delta)2\sqrt{\pi(2n+1)}\,\mathscr{H}_{2n+1} = -\frac{dS}{dx} + 2\pi x S$$

Therefore

$$4\pi x S = \sum_{n=0}^{\infty} [a_{2n+2}(\delta)2\sqrt{2\pi(n+1)} + a_{2n}(\delta)2\sqrt{\pi(2n+1)}]\mathscr{H}_{2n+1}$$

Now

$a_{2n+2}(\delta)\sqrt{2(n+1)} + a_{2n}(\delta)\sqrt{2n+1} = H_{2n+2}(0)\sqrt{2n+2} + H_{2n}(0)\sqrt{2n+1} = 0$

so $4\pi x S = 0$ i.e. $S = C\delta$.

Now $\langle S, \mathscr{H}_{2n}\rangle = a_{2n}(\delta) = C\mathscr{H}_{2n}(0) = \mathscr{H}_{2n}(0)$ therefore $C = 1$.

Solution 68

Let us set $f(x) = e^x \cos e^x$ for real x.

a) Let us suppose that there exists a polynomial P satisfying (1). Then for every $x \in \mathbb{R}$

$$|\cos e^x| \leqq \frac{|P(x)|}{e^x}$$

so $\lim_{x \to +\infty} |\cos e^x| = 0$ which is impossible. Indeed if

$$x_k = \operatorname{Log} 2k\pi$$

then $|\cos e^{x_k}| = 1$ and x_k tends to infinity with k.

b) Let $\varphi \in \mathscr{S}(\mathbb{R})$. An integration by parts shows that

$$-\int_{\mathbb{R}} \sin e^x \frac{d\varphi}{dx} dx = \int_{\mathbb{R}} e^x \cos e^x \varphi(x)\, dx$$

It follows that

$$\left|\int_{\mathbb{R}} e^x \cos e^x \varphi(x)\, dx\right| \leqq \int |\varphi'(x)|\, dx \leqq C \sup_{x \in \mathbb{R}} |(1 + x^2)\varphi'(x)|.$$

Solution 69

Let us assume T is even. By definition we have

$$\langle \check{\mathscr{F}} T, \varphi\rangle = \langle T, \check{\mathscr{F}}\varphi\rangle \quad \text{for all } \varphi \in \mathscr{S}(\mathbb{R}^n)$$

Now

$$(\check{\mathscr{F}}\varphi)(\xi) = \int e^{2i\pi\langle x,\xi\rangle}\varphi(x)\, dx = \int e^{-2i\pi\langle x,-\xi\rangle}\varphi(x)\, dx = \mathscr{F}\varphi(-\xi)$$

so

$$\langle \check{\mathscr{F}} T, \varphi\rangle = \langle T, \check{\mathscr{F}}\varphi\rangle = \langle T, \mathscr{F}\varphi\rangle = \langle \mathscr{F} T, \varphi\rangle \quad \text{for all } \varphi \in \mathscr{S}$$

therefore $\check{\mathscr{F}} T = \mathscr{F} T$.

If T is odd: $\langle \check{\mathscr{F}} T, \varphi\rangle = \langle T, \check{\mathscr{F}}\varphi\rangle = -\langle T, \mathscr{F}\varphi\rangle = -\langle \mathscr{F} T, \varphi\rangle$ i.e. $\check{\mathscr{F}} T = -\mathscr{F} T$.

Solution 70

Let us recall that distribution $T \circ A$ is defined by:

$$(1) \quad \langle T \circ A, \varphi \rangle = |\det A|^{-1} \langle T, \varphi \circ A^{-1} \rangle \qquad \forall \varphi \in \mathscr{S}(\mathbb{R}^n)$$

a) If $T \in \mathscr{S}'(\mathbb{R}^n)$

$$(2) \quad |\langle T, \psi \rangle| \leqq C \sup_{\mathbb{R}^n} (1 + |x|)^k \sum_{|\alpha| \leq l} |D^\alpha \psi(x)| \qquad \forall \psi \in \mathscr{S}(\mathbb{R}^n)$$

By (1), (2) applied to $\psi = \varphi \circ A^{-1}$ and since we have $\varphi \circ A^{-1} \in \mathscr{S}(\mathbb{R}^n)$ and $D^\alpha(\varphi \circ A^{-1}) = \sum_{|\beta| \leq |\alpha|} C_{\alpha\beta}(D^\beta \varphi) \circ A^{-1}$, we deduce $T \circ A \in \mathscr{S}'(\mathbb{R}^n)$.

Moreover, for every $\varphi \in \mathscr{S}(\mathbb{R}^n)$ we have by (1)

$$(3) \quad \langle \widehat{T \circ A}, \varphi \rangle = \langle T \circ A, \hat{\varphi} \rangle = |\det A|^{-1} \langle T, \hat{\varphi} \circ A^{-1} \rangle$$

Now

$$\hat{\varphi}(A^{-1}\xi) = \int e^{-2i\pi\langle x, A^{-1}\xi \rangle} \varphi(x)\, dx$$

$$\hat{\varphi} \circ A^{-1}(\xi) = \int e^{-2i\pi\langle {}^t(A^{-1})x, \xi \rangle} \varphi(x)\, dx$$

Performing the change of coordinates ${}^t(A^{-1})x = y$ we have, since ${}^t(A^{-1}) = ({}^tA)^{-1}$:

$$(4) \quad \hat{\varphi}(A^{-1}\xi) = |\det A| \widehat{\varphi \circ {}^tA}(\xi)$$

It follows from (3) and (4) that

$$\langle \widehat{T \circ A}, \varphi \rangle = \langle T, \widehat{\varphi \circ {}^tA} \rangle = \langle \hat{T}, \varphi \circ {}^tA \rangle$$

Now from (1) applied to $\hat{T}$ we get

$$\langle \hat{T}, \varphi \circ {}^tA \rangle = |\det A|^{-1} \langle \hat{T} \circ ({}^tA)^{-1}, \varphi \rangle$$

so

$$\langle \widehat{T \circ A}, \varphi \rangle = |\det A|^{-1} \langle \hat{T} \circ ({}^tA)^{-1}, \varphi \rangle \quad \text{for all } \varphi \in \mathscr{S}(\mathbb{R}^n)$$

i.e.

$$\widehat{T \circ A} = |\det A|^{-1} \hat{T} \circ ({}^tA)^{-1}$$

b) If $n = 1$, A is a real number a. Let us take $a = -1$, then T is even (resp. odd) if $T \circ A = T$ (resp. $-T$).

If T is even we have $\widehat{T \circ A} = \hat{T}$. Since $|\det A| = 1$ and $({}^tA)^{-1} = A$ we deduce from question a) that $\hat{T} = \widehat{T \circ A} = \hat{T} \circ A$ so $\hat{T}$ is even. Now if T is odd $T \circ A = -T$ so $\hat{T} = -\widehat{T \circ A} = -\hat{T} \circ A$ i.e. $\hat{T}$ is odd.

a) Let us suppose that A is orthogonal. Then ${}^tA \cdot A = Id$. If $T \circ A = T$, by question a) we can write:

$$\hat{T} = \widehat{T \circ A} = |\det A|^{-1}\hat{T} \circ ({}^tA)^{-1}$$

Now $({}^tA)^{-1} = A$ and $|\det A|^2 = 1$, therefore for all orthogonal matrices,

$$\hat{T} = \hat{T} \circ A$$

so $\hat{T}$ is invariant by rotation.

Solution 71

Let $\varphi \in \mathscr{S}(\mathbb{R}^n)$, then by definition

$$\langle \mathscr{F}T, \varphi_t \rangle = \langle T, \mathscr{F}\varphi_t \rangle$$

Now

$$\mathscr{F}\varphi_t(\xi) = \int e^{-2i\pi\langle x, \xi\rangle}\varphi(tx)\,dx = t^{-n}\int e^{-2i\pi\langle y, \xi/t\rangle}\varphi(y)\,dy$$

so

$$\mathscr{F}\varphi_t(\xi) = t^{-n}(\mathscr{F}\varphi)_{1/t}$$

Therefore

$$\langle \mathscr{F}T, \varphi_t \rangle = t^{-n}\langle T, (\mathscr{F}\varphi)_{1/t}\rangle = t^{-n}\left(\frac{1}{t}\right)^{-n-\lambda}\langle T, \mathscr{F}\varphi\rangle$$

so

$$\langle \mathscr{F}T, \varphi_t \rangle = t^{\lambda}\langle \mathscr{F}T, \varphi\rangle$$

Therefore $\mathscr{F}T$ is homogeneous of degree p where $\lambda = -(n + p)$ i.e. $p = -n - \lambda$.

Solution 72

a) The first question has a negative answer. Indeed:

Let φ and ψ two non zero elements of $\mathscr{D}(\mathbb{R}^n)$ such that $\operatorname{supp}\varphi \cap \operatorname{supp}\psi = \varnothing$. Then $\varphi \cdot \psi = 0$. Let $f = \mathscr{F}\varphi$, $g = \mathscr{F}\psi$. Then $f, g \in \mathscr{S}(\mathbb{R}^n)$ and

$$\mathscr{F}(f * g) = \mathscr{F}f \cdot \mathscr{F}g = \varphi \cdot \psi = 0$$

since the Fourier transform is an isomorphism on $\mathcal{S}$ we deduce that $f * g = 0$ but nor f neither g vanishes identically for then $\varphi \equiv 0$ or $\psi \equiv 0$.

If $f * f = 0$ then $(\mathcal{F}f)^2 \equiv 0$ so $\mathcal{F}f \equiv 0$ and $f \equiv 0$.

b) Let us take $T = 1 \in \mathcal{S}'$, $S = \delta' \in \mathcal{E}'$ then $1 * \delta' = \dfrac{d}{dx} 1 * \delta = 0$.

Solution 73

Let $\varphi \in \mathcal{D}(\mathbb{R})$. Then

$$\left\langle x \cdot pv\frac{1}{x}, \varphi \right\rangle = \left\langle pv\frac{1}{x}, x\varphi \right\rangle = \lim_{\varepsilon \to 0} \int_{|x| \geq \varepsilon} \varphi(x) = \int \varphi(x)\,dx = \langle 1, \varphi \rangle$$

by Lebesgue's theorem. So $x \cdot pv\dfrac{1}{x} = 1$. Therefore

$$\mathcal{F}\left(x \cdot pv\frac{1}{x}\right) = \frac{-1}{2i\pi}\frac{d}{d\xi}\hat{T} = \hat{1} = \delta$$

so $\dfrac{d\hat{T}}{d\xi} = -2i\pi\delta$ which implies that $\hat{T}$ is equal to $-2i\pi H + C$ where H is the Heaviside function and C a constant.

Now $T = pv\dfrac{1}{x}$ is odd for $\int_{|x| \geq \varepsilon} \varphi(-x)\,dx = -\int_{|x| \geq \varepsilon} \varphi(x)\,dx$, therefore $\hat{T}$ is an odd function so $-2i\pi + C = -C$ and $C = i\pi$. Then

$$\hat{T} = -2i\pi H + i\pi = \begin{cases} -i\pi & \xi > 0 \\ i\pi & \xi < 0 \end{cases}$$

Let us take the Fourier transform of both sides of the above equality.

$$\mathcal{F}\hat{T} = -2i\pi\mathcal{F}H + \mathcal{F}(i\pi)$$

Now $\mathcal{F}\hat{T} = \check{T}$ and $\mathcal{F}(i\pi) = i\pi\mathcal{F}1 = i\pi\delta$ so

$$\mathcal{F}H = -\frac{1}{2i\pi}\check{T} + \frac{1}{2}\delta$$

Now $T = pv\dfrac{1}{x}$ is odd so $\check{T} = -T$, so

$$\mathcal{F}H = \frac{1}{2}\delta + \frac{1}{2i\pi}pv\frac{1}{x}$$

and

$$\bar{\mathcal{F}}\mathcal{F}T = -2i\pi\bar{\mathcal{F}}H + i\pi\bar{\mathcal{F}}1 \quad \text{but} \quad \bar{\mathcal{F}}\mathcal{F} = Id \quad \text{and} \quad \bar{\mathcal{F}}1 = \delta$$

so

$$\mathcal{F}H = \frac{1}{2}\delta - \frac{1}{2i\pi}pv\frac{1}{x}$$

Solution 74

We obviously have $|x| = H(x)x - xH(-x)$ since for $x > 0$ the right hand side is equal to x and for $x < 0$ to $-x$. Therefore

$$\mathcal{F}|x| = \mathcal{F}[xH(x)] - \mathcal{F}[xH(-x)]$$

$$\mathcal{F}|x| = \frac{-1}{2i\pi}\frac{\mathrm{d}}{\mathrm{d}\xi}\mathcal{F}H(x) + \frac{1}{2i\pi}\frac{\mathrm{d}}{\mathrm{d}\xi}\mathcal{F}[H(-x)]$$

By exercise 73 $\mathcal{F}H(\xi) = \frac{1}{2}\delta + \frac{1}{2i\pi}pv\frac{1}{\xi}$.

Moreover $\mathcal{F}[H(-x)] = \mathcal{F}\check{H} = \bar{\mathcal{F}}H$. Indeed for every $\varphi \in \mathcal{S}(\mathbb{R})$

$$\langle \mathcal{F}\check{H}, \varphi\rangle = \langle \check{H}, \mathcal{F}\varphi\rangle = \langle H, \check{\widehat{\mathcal{F}\varphi}}\rangle = \langle H, \bar{\mathcal{F}}\varphi\rangle = \langle \bar{\mathcal{F}}H, \varphi\rangle$$

for

$$\check{\widehat{\mathcal{F}\varphi}}(\xi) = \mathcal{F}\varphi(-\xi) = \int \mathrm{e}^{2i\pi x\xi}\varphi(x)\,\mathrm{d}x = \bar{\mathcal{F}}\varphi(\xi)$$

Using exercise 73 we get

$$\bar{\mathcal{F}}H = \frac{1}{2}\delta - \frac{1}{2i\pi}pv\frac{1}{\xi}$$

so

$$\mathcal{F}|x| = \frac{-1}{2i\pi}\frac{\mathrm{d}}{\mathrm{d}\xi}\left[\frac{1}{2}\delta + \frac{1}{2i\pi}pv\frac{1}{\xi} - \frac{1}{2}\delta + \frac{1}{2i\pi}pv\frac{1}{\xi}\right]$$

$$\mathcal{F}|x| = \frac{1}{2\pi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}pv\frac{1}{\xi}$$

Now from exercise 27 $\frac{\mathrm{d}}{\mathrm{d}\xi}pv\frac{1}{\xi} = -Fp\frac{1}{\xi^2}$ so

$$\mathcal{F}|x| = -\frac{1}{2\pi^2}Fp\frac{1}{\xi^2}$$

Therefore

$$\mathcal{F}\mathcal{F}|x| = |x| = \frac{-1}{2\pi^2}\mathcal{F}Fp\frac{1}{\xi^2}$$

therefore

$$\mathcal{F} Fp\frac{1}{\xi^2} = -2\pi^2|x|$$

Solution 75

a) For $n = 0$, $\mathcal{F}1 = \delta$. For $n = 1$, $\mathcal{F}(x + iy) = \mathcal{F}x + i\mathcal{F}y$. Now for $T \in \mathcal{S}'$ we have:

$$\mathcal{F}(xT) = -\frac{1}{2i\pi}\frac{\partial}{\partial\xi}(\mathcal{F}T), \qquad \mathcal{F}(yT) = -\frac{1}{2i\pi}\frac{\partial}{\partial\eta}(\mathcal{F}T)$$

so

$$\mathcal{F}(x + iy) = \mathcal{F}(x \cdot 1) + i\mathcal{F}(y \cdot 1) = -\frac{1}{2i\pi}\left(\frac{\partial\delta}{\partial\xi} + i\frac{\partial\delta}{\partial\eta}\right)$$

Since z is C^∞ and slowly increasing at infinity we get:

$$(\mathcal{F}z) * (\mathcal{F}z) * \cdots * (\mathcal{F}z) = \mathcal{F}z^n$$

so

$$\mathcal{F}z^n = \left(-\frac{1}{2i\pi}\right)^n\left\{\frac{\partial\delta}{\partial\xi} + i\frac{\partial\delta}{\partial\eta}\right\} * \cdots * \left\{\frac{\partial\delta}{\partial\xi} + i\frac{\partial\delta}{\partial\eta}\right\}$$

Using the properties of the convolution we can write

$$\mathcal{F}z^n = \left(-\frac{1}{2i\pi}\right)^n\left(\frac{\partial}{\partial\xi} + i\frac{\partial}{\partial\eta}\right)^n\{\delta * \delta * \cdots \delta\} = \left(-\frac{1}{2i\pi}\right)^n\left(\frac{\partial}{\partial\xi} + i\frac{\partial}{\partial\eta}\right)^n\delta$$

b) The function $\frac{1}{z}$ is locally integrable and bounded for $|z| \geq 1$, so it determines a tempered distribution in $\mathbb{R}^2$. Moreover setting $T = \frac{1}{z}$ we have

$$z \cdot T = 1$$

so

$$\mathcal{F}(zT) = \mathcal{F}1 = \delta$$

$$(1) \quad -\frac{1}{2i\pi}\left(\frac{\partial}{\partial\xi} + i\frac{\partial}{\partial\eta}\right)\mathcal{F}T = \delta$$

A particular solution of (1) is, by exercise 29: $S_0 = -\frac{i}{\zeta}$ where $\zeta = \xi + i\eta$.

The general solution of equation (1), when the right hand side is zero, is $S = F(\zeta)$ where F is a holomorphic function of $\zeta = \xi + i\eta$. We deduce that

$$\mathcal{F}\left(\frac{1}{z}\right) = \frac{-i}{\zeta} + F(\zeta)$$

But the distribution $\frac{1}{z}$ is homogeneous of degree -1 and the same is true for its Fourier transform (see exercise 71). Since $\frac{1}{\zeta}$ is homogeneous of degree -1 the same is true for F; but F is holomorphic so $F \equiv 0$ and $\mathscr{F}\left(\frac{1}{z}\right) = \frac{-i}{\zeta}$.

Indeed if we had

$$F(\lambda\zeta) = \lambda^{-1}F(\zeta) \quad \text{for } \zeta \in \mathbb{C} \quad \text{and} \quad \lambda > 0$$

we should have

$$F(\lambda) = \lambda^{-1}F(1) \quad \text{for } \lambda > 0$$

But F is C^∞ in $\mathbb{R}^2$ while $\lim\limits_{\substack{\lambda\in\mathbb{R}\\ \lambda\to 0}} F(\lambda) = +\infty$.

Solution 76

a) First of all T_1 having a compact support the same is true for T_k since we have supp $(A * B) \subset$ supp A + supp B.
Now, by definition of the convolution, for $a, b \in \mathbb{R}$ we have:

$$\langle \delta_a * \delta_b, \varphi \rangle = \langle \delta_a, \langle \delta_b, \varphi(x + y)\rangle\rangle = \langle \delta_a, \varphi(a + y)\rangle = \varphi(a + b) = \langle \delta_{a+b}, \varphi\rangle$$

which proves that

$$(1) \quad \delta_a * \delta_b = \delta_{a+b}$$

Now, denoting by T^{*k} the distribution $T * \cdots * T$ (convolution of T k times) we have from the commutativity and the associativity of the consolution:

$$T_k = \frac{1}{2^k}(\delta_1 + \delta_{-1})^{*k} = \frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}\delta_1^{*j} * \delta_{-1}^{*(k-j)}$$

so

$$T_k = \frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}\delta_{2j-k}$$

by formula (1).

b) Since $T_k \in \mathscr{E}'$, its Fourier transform $\hat{T}_k$ is a C^∞ function which is given by

$$\hat{T}_k(\xi) = (\hat{T}_1(\xi))^k$$

since $\widehat{A * B} = \hat{A} \cdot \hat{B}$ for $A, B \in \mathscr{E}'$.

Let us compute $\hat{\delta}_a$ for $a \in \mathbb{R}$. If $\varphi \in \mathscr{S}$ we have:

$$\langle \hat{\delta}_a, \varphi \rangle = \langle \delta_a, \hat{\varphi} \rangle = \hat{\varphi}(a) = \int e^{-2i\pi a\xi}\varphi(\xi)\,d\xi = \langle e^{-2i\pi a\xi}, \varphi \rangle$$

so $\hat{\delta}_a = e^{-2i\pi a\xi} \in \mathscr{S}'$. Therefore

$$\hat{T}_1(\xi) = \frac{1}{2}(e^{-2i\pi\xi} + e^{2i\pi\xi}) = \cos 2\pi\xi$$

$$\hat{T}_k(\xi) = (\cos 2\pi\xi)^k$$

c) We have $f_k(\xi) = \left(\cos \dfrac{\xi}{\sqrt{k}}\right)^k$.

For each fixed ξ there exists a large k such that $\left|\dfrac{\xi}{\sqrt{k}}\right| < 1$ thus

$$\cos \frac{\xi}{\sqrt{k}} = \left(1 - \frac{\xi^2}{2k} + o\left(\frac{\xi^2}{k}\right)\right) \geqq 0$$

$$\operatorname{Log} f_k(\xi) = k \operatorname{Log}\left(1 - \frac{\xi^2}{2k} + o\left(\frac{\xi^2}{k}\right)\right)$$

Therefore

$$\lim_{k \to +\infty} \operatorname{Log} f_k(\xi) = -\frac{\xi^2}{2}$$

or

$$\lim_{k \to \infty} f_k(\xi) = f(\xi) = e^{-\xi^2/2}$$

On the other hand for $\varphi \in \mathscr{D}(\mathbb{R})$ (or $\varphi \in \mathscr{S}(\mathbb{R})$)

$$|f_k(\xi)\varphi(\xi)| \leq |\varphi(\xi)| \in L^1$$

so by Lebesgue's theorem

$$\lim_{k \to \infty} \int_{\mathbb{R}} f_k(\xi)\varphi(\xi)\,d\xi = \int_{\mathbb{R}} e^{-\xi^2/2}\varphi(\xi)\,d\xi$$

which means that sequence (f_k) converges in $\mathscr{D}'(\mathbb{R})$ (or in $\mathscr{S}'$) to $e^{-\xi^2/2}$.

d) Let $g_k = \bar{\mathscr{F}} f_k$. Since $f_k \in \mathscr{S}'$, for $|f_k(\xi)| \leq 1$ for all ξ, we have $g_k \in \mathscr{S}'$. Moreover (f_k) converges to f in $\mathscr{S}'$ and the Fourier transform is continuous from $\mathscr{S}'(\mathbb{R})$ to $\mathscr{S}'(\mathbb{R})$. It follows that (g_k) converges in $\mathscr{S}'(\mathbb{R})$ to $\bar{\mathscr{F}} e^{-\xi^2/2} = \sqrt{2\pi}\, e^{-2\pi^2 x^2}$.

Solution 77

a) Since $\lambda > -1$, the function $|x|^{\lambda}$ is locally integrable and determines a distribution. Let us set

$$u(x) = \begin{cases} |x|^{\lambda} & |x| < 1 \\ 0 & |x| \geq 1 \end{cases} \qquad v(x) = \begin{cases} 0 & |x| < 1 \\ |x|^{\lambda} & |x| \geq 1 \end{cases}$$

Since $\lambda \in \,]-1, -\frac{1}{2}[$, $u \in L^1(\mathbb{R})$, $v \in L^2(\mathbb{R})$ and $f = u + v$.

b) We deduce that $\hat{f} = \hat{u} + \hat{v}$ is a function for $\hat{u} \in C^0(\mathbb{R})$, $\hat{v} \in L^2(\mathbb{R})$. Since f is homogeneous of degree λ it follows from exercise 71 that $\hat{f}$ is homogeneous of degree $-(\lambda + 1)$ i.e.

$$(1) \quad \hat{f}(t\xi) = t^{-(\lambda+1)}\hat{f}(\xi), \qquad t > 0,\ \xi \in \mathbb{R}$$

Then

$$\hat{f}(\xi) = \begin{cases} C_1|\xi|^{-(\lambda+1)} & \xi > 0 \\ C_2|\xi|^{-(\lambda+1)} & \xi < 0 \end{cases}$$

Indeed let us take in (1), $\xi = 1$, $t > 0$ then

$$\hat{f}(t) = t^{-(\lambda+1)}\hat{f}(1) = C_1 t^{-(\lambda+1)}$$

If $t < 0$ let us set $t_1 = -t > 0$ and $\xi = -1$. We get

$$\hat{f}(t) = \hat{f}(-t_1) = t_1^{-(\lambda+1)}\hat{f}(-1) = C_2|t|^{-(\lambda+1)}$$

c) Since f is even it follows that $\hat{f}$ is even. Indeed

$$\langle \hat{f}, \check{\varphi}\rangle = \langle f, \mathcal{F}(\check{\varphi})\rangle = \langle f, \check{\mathcal{F}\varphi}\rangle = \langle f, \mathcal{F}\varphi\rangle = \langle \hat{f}, \varphi\rangle$$

where $\check{\varphi}(x) = \varphi(-x)$. In other words for all $\varphi \in \mathcal{S}(\mathbb{R})$

$$\int \hat{f}(\xi)\varphi(-\xi)\,\mathrm{d}\xi = \int \hat{f}(\xi)\varphi(\xi)\,\mathrm{d}\xi \quad \text{so} \quad \int [\hat{f}(\xi) - \hat{f}(-\xi)]\varphi(\xi)\,\mathrm{d}\xi = 0$$

therefore $\hat{f}(\xi) = \hat{f}(-\xi)$ a.e.
We deduce from b) that $C_1 = C_2 = C_{\lambda}$ i.e.

$$\hat{f}(\xi) = C_{\lambda}|\xi|^{-(\lambda+1)} \qquad \xi \in \mathbb{R}$$

Let us compute C. We have

$$(2) \quad \langle \hat{f}, \mathrm{e}^{-\pi\xi^2}\rangle = \langle f, \mathrm{e}^{-\pi x^2}\rangle$$

Since f and $\hat{f}$ are functions we get

$$\langle \hat{f}, \mathrm{e}^{-\pi\xi^2}\rangle = C_{\lambda}\int_{\mathbb{R}} |\xi|^{-(\lambda+1)}\mathrm{e}^{-\pi\xi^2}\,\mathrm{d}\xi = 2C_{\lambda}\int_0^{\infty} \xi^{-(\lambda+1)}\mathrm{e}^{-\pi\xi^2}\,\mathrm{d}\xi$$

Let us set $\xi = \sqrt{\dfrac{y}{\pi}}$, we get

$$\langle \hat{f}, e^{-\pi\xi^2} \rangle = 2C_\lambda \int_0^\infty [y^{-((\lambda+1)/2)}\pi^{(\lambda+1)/2}] e^{-y}\pi^{-1/2} \frac{dy}{2\sqrt{y}}$$

$$(3) \quad \langle \hat{f}, e^{-\pi\xi^2} \rangle = C_\lambda \pi^{\lambda/2} \int_0^\infty y^{-((\lambda+2)/2)} e^{-y} dy = C_\lambda \pi^{\lambda/2} \Gamma\left(-\frac{\lambda}{2}\right)$$

Moreover

$$\langle f, e^{-\pi x^2} \rangle = 2 \int_0^\infty x^\lambda e^{-\pi x^2} dx = \pi^{-((\lambda+1)/2)} \int_0^\infty y^{(\lambda-1)/2} e^{-y} dy$$

$$(4) \quad \langle f, e^{-\pi x^2} \rangle = \pi^{-((\lambda+1)/2)} \Gamma\left(\frac{\lambda+1}{2}\right)$$

It follows from (2), (3), (4) that

$$C_\lambda = \pi^{-\lambda-1/2} \frac{\Gamma\left(\dfrac{\lambda+1}{2}\right)}{\Gamma\left(-\dfrac{\lambda}{2}\right)}$$

2°) There remains the case where $-\frac{1}{2} \leq \lambda < 0$. Let us start with the case $\lambda \in]-\frac{1}{2}, 0[$. We use the inverse Fourier transform. We have $-(\lambda + 1) \in]-1, -\frac{1}{2}[$ therefore $\overline{\mathscr{F}}(|x|^{-(\lambda+1)}) = C_{-(\lambda+1)}|\xi|^\lambda = \mathscr{F}(|x|^{-(\lambda+1)})$ since the function $|x|^{-(\lambda+1)}$ is even (see exercise 69).

We deduce that

$$\overline{\mathscr{F}}(|\xi|^\lambda) = \frac{|x|^{-(\lambda+1)}}{C_{-(\lambda+1)}}$$

Now

$$C_{-(\lambda+1)} = \pi^{\lambda+1/2} \frac{\Gamma\left(-\dfrac{\lambda}{2}\right)}{\Gamma\left(\dfrac{\lambda+1}{2}\right)} = \frac{1}{C_\lambda}$$

so

$$\overline{\mathscr{F}}(|\xi|^\lambda) = C_\lambda |x|^{-(\lambda+1)} \quad \text{for } \lambda \in]-\tfrac{1}{2}, 0[$$

Let us finally examine the case $\lambda = -\frac{1}{2}$. If $\lambda \to -\frac{1}{2}$, $\lambda < -\frac{1}{2}$ then $|x|^\lambda \to |x|^{-1/2}$ in $\mathscr{S}'(\mathbb{R})$. Indeed let $\varphi \in \mathscr{S}(\mathbb{R})$.

$$\int |x|^\lambda \varphi(x)\,dx = \int_{|x|\leq 1} + \int_{|x|>1} = I_1 + I_2$$

In I_1 taking $-\frac{3}{4} < \lambda < -\frac{1}{2}$ we have

$$||x|^\lambda \varphi(x)| \leqq |x|^{-3/4}|\varphi(x)| \in L^1(]-1, 1[)$$

In I_2 since $\varphi \in \mathscr{S}(\mathbb{R})$ we have $||x|^{\lambda+2}\varphi(x)| \leq M$ for $|x| > 1$ so

$$|x|^\lambda|\varphi(x)| \leq \frac{M}{|x|^2} \in L^1 \quad \text{in } |x| > 1$$

It follows from Lebesgue's theorem that $\int |x|^\lambda \varphi(x)\,\mathrm{d}x \to \int |x|^{-1/2}\varphi(x)\,\mathrm{d}x$.
The Fourier transform being continuous in $\mathscr{S}'(\mathbb{R})$

$$\mathscr{F}(|x|^\lambda) \to \mathscr{F}(|x|^{-1/2})$$

By the first question $\mathscr{F}(|x|^\lambda) \to \pi^{1/2 - 1/2}\frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{1}{4})}|\xi|^{-1/2}$ so

$$\mathscr{F}(|x|^{-1/2}) = |\xi|^{-1/2}$$

(We compute the limit using the same method when $\lambda \to -\frac{1}{2}$ with $\lambda > -\frac{1}{2}$.)
Let $|x|^\lambda = x_+^\lambda + x_-^\lambda$ defined in exercise 15. It is a homogeneous distribution of degree λ. It follows from exercise 71 that its Fourier transform is a homogeneous distribution of degree $-(\lambda + 1)$. But exercise 34 gave the form of all homogeneous distributions of degree $-(\lambda + 1)$. Since $\lambda \notin \mathbb{Z}$, $-(\lambda + 1)$ is not a negative integer so by exercise 34

$$\mathscr{F}(|x|^\lambda) = C_1\xi_+^{-(\lambda+1)} + C_2\xi_-^{-(\lambda+1)}$$

Since $|x|^\lambda$ is even its Fourier transform is also even (see exercise 70). Therefore $C_1 = C_2 = C_\lambda$ i.e.

$$\mathscr{F}(|x|^\lambda) = C_\lambda(\xi_+^{-(\lambda+1)} + \xi_-^{-(\lambda+1)}) = C_\lambda|\xi|^{-(\lambda+1)}$$

Since $\mathscr{F}\mathrm{e}^{-\pi x^2} = \mathrm{e}^{-\pi\xi^2}$ we get:

$$\langle \mathscr{F}(|x|^\lambda), \mathrm{e}^{-\pi\xi^2}\rangle = \langle |x|^\lambda, \mathrm{e}^{-\pi x^2}\rangle$$

so

$$C_\lambda\langle |\xi|^{-(\lambda+1)}, \mathrm{e}^{-\pi\xi^2}\rangle = \langle |x|^\lambda, \mathrm{e}^{-\pi x^2}\rangle$$

Let us set $M(\mu) = \langle |\xi|^\mu, \mathrm{e}^{-\pi\xi^2}\rangle$. We get

$$C_\lambda = \frac{M(\lambda)}{M(-(\lambda + 1))}$$

i.e.

$$\mathscr{F}(|x|^\lambda) = \frac{M(\lambda)}{M(-(\lambda + 1))}|\xi|^{-(\lambda+1)}$$

Solution 78

1°) It follows from Fubini's theorem that, for real and positive λ, we have

$$I(\lambda)^2 = 4\int_0^\infty\int_0^\infty e^{-\lambda\pi(x^2+y^2)}\,\mathrm{d}x\,\mathrm{d}y = 4\int_0^{\pi/2}\int_0^\infty e^{-\pi\lambda r^2}\,r\,\mathrm{d}r\,\mathrm{d}\theta = \frac{1}{\lambda}$$

so

$$(1)\quad I(\lambda) = \frac{1}{\sqrt{\lambda}}$$

for $I(\lambda) > 0$.

Function $\lambda \to I(\lambda)$ extends to a holomorphic function in $\operatorname{Re}\lambda > 0$. Indeed let us set $\lambda = \alpha + i\beta$ with $\alpha > 0$. Thanks to the factor $e^{-\alpha x^2}$, it is easy to see that we can differentiate $I(\lambda)$ with respect to α and β and that $\left(\frac{\partial}{\partial\alpha} + i\frac{\partial}{\partial\beta}\right)I(\lambda) = 0$.

In the same way function $\lambda \to \frac{1}{\sqrt{\lambda}}$ extends to a holomorphic function in $\operatorname{Re}\lambda > 0$.
These two functions coincide, by (1), on the positive real axis. They coincide therefore for $\operatorname{Re}\lambda > 0$.

2°) a) Indeed $|f(x)| = 1$ so f defines an element of $\mathscr{S}'(\mathbb{R})$ by the formula $\mathscr{S}(\mathbb{R}) \ni \varphi \rightsquigarrow \langle f, \varphi\rangle = \int f(x)\varphi(x)\,\mathrm{d}x$.

b) $\dfrac{\mathrm{d}f(x)}{\mathrm{d}x} = 2i\pi a x f(x)$

c) Let us take the Fourier transform of both sides of the above equation. We get:

$$2i\pi\xi\hat{f} = 2i\pi a\left(-\frac{1}{2i\pi}\frac{\mathrm{d}}{\mathrm{d}\xi}\right)\hat{f}$$

so

$$(2)\quad \frac{\mathrm{d}}{\mathrm{d}\xi}\hat{f} + \frac{2i\pi}{a}\xi\hat{f} = 0$$

We know from exercise 33 that $\hat{f}$ is then a C^∞ function. Let us set $\hat{f} = e^{-i(\pi/a)\xi^2}S$. Then if $\hat{f}$ satisfies (2) we have $\dfrac{\mathrm{d}S}{\mathrm{d}\xi} = 0$ so S is a constant and

$$(3)\quad \hat{f}(\xi) = C e^{-i(\pi/a)\xi^2}$$

d) Let us apply $\hat{f}$ to the function $e^{-\pi\xi^2} \in \mathscr{S}$. We get

$$\langle \mathscr{F}f, e^{-\pi\xi^2} \rangle = \langle f, \mathscr{F} e^{-\pi\xi^2} \rangle = \langle f, e^{-\pi x^2} \rangle = \int e^{i\pi a x^2 - \pi x^2} \mathrm{d}x = I_1$$

$$\langle \mathscr{F}f, e^{-\pi\xi^2} \rangle = C\langle e^{-i(\pi/a)\xi^2}, e^{-\pi\xi^2} \rangle = C \int e^{-i(\pi/a)\xi^2 - \pi\xi^2} \mathrm{d}\xi = CI_2$$

Using the notation of question 1°) we get:

$$I_1 = I(1 - ia) = \frac{1}{\sqrt{1 - ia}}$$

$$I_2 = I\left(1 + \frac{i}{a}\right) = I\left(\frac{i}{a}(1 - ia)\right) = \frac{1}{\sqrt{\frac{i}{a}(1 - ia)}}$$

Taking the determination of the square root for which $\sqrt{\mp i} = e^{\mp i(\pi/4)}$ and using $I_1 = CI_2$ we find:

$$\begin{cases} * \text{ If } a > 0 \quad C = \dfrac{1}{\sqrt{a}} e^{i(\pi/4)} \\[2ex] * \text{ If } a < 0 \quad C = \dfrac{1}{\sqrt{|a|}} e^{-i(\pi/4)} \end{cases}$$

3°) It is easy to see that $\langle Dx, x \rangle = \sum_{j=1}^{k} \lambda_j x_j^2 + \sum_{j=k+1}^{n} \lambda_j x_j^2$ so

$$e^{i\pi\langle Dx, x\rangle} = \prod_{j=1}^{k} e^{i\pi\lambda_j x_j^2} \prod_{j=k+1}^{n} e^{i\pi\lambda_j x_j^2}$$

where each exponential depends on only one real variable. Using question 2°) we obtain

$$\mathscr{F}_x e^{i\pi\langle Dx, x\rangle} = \mathscr{F}_{x_1} e^{i\pi\lambda_1 x_1^2} \otimes \cdots \otimes \mathscr{F}_{x_k} e^{i\pi\lambda_k x_k^2} \otimes \mathscr{F}_{x_{k+1}} e^{i\pi\lambda_{k+1} x_{k+1}^2} \otimes \cdots \otimes \mathscr{F}_{x_n} e^{i\pi\lambda_n x_n^2}$$

$$= \prod_{j=1}^{k} \frac{1}{\sqrt{\lambda_j}} e^{ik(\pi/4)} e^{-i(\pi/\lambda_j)\xi_k^2} \prod_{j=k+1}^{n} \frac{1}{\sqrt{|\lambda_j|}} e^{-(n-k)i(\pi/4)} e^{-i(\pi/\lambda_j)\xi_j^2}$$

$$\mathscr{F}_x e^{i\pi\langle Dx, x\rangle} = \prod_{j=1}^{n} \frac{1}{\sqrt{|\lambda_j|}} e^{i(2k-n)(\pi/4)} e^{-i\pi\langle D^{-1}\xi, \xi\rangle}$$

since D^{-1} is the diagonal matrix $\left(\dfrac{1}{\lambda_j}\right)_{j=1,\ldots,n}$ Q.E.D.

4°) There exists an orthogonal matrix U (i.e. ${}^tU = U^{-1}$) such that

$$(4) \quad UAU^{-1} = D \quad \text{(diagonal matrix)}$$

Then

$$\langle Ax, x\rangle = \langle U^{-1}DUx, x\rangle = \langle DUx, {}^t(U^{-1})x\rangle = \langle DUx, Ux\rangle$$

Therefore

$$e^{i\pi\langle Ax,x\rangle} = T \circ U(x) \quad \text{where} \quad T = e^{i\pi\langle Dx,x\rangle}$$

We deduce from exercise 70

$$\widehat{T \circ U} = |\det U|^{-1}\hat{T} \circ ({}^tU)^{-1} = \hat{T} \circ U$$

therefore by the preceding question we have:

$$\mathscr{F}(e^{i\pi\langle Ax,x\rangle}) = \prod_{j=1}^{n} \frac{1}{\sqrt{|\lambda_j|}} e^{i(2k-n)\pi/4} e^{i\pi\langle D^{-1}U\xi, U\xi\rangle}$$

Now $\prod_{j=1}^{n} |\lambda_j|^{-1/2} = |\det D|^{-1/2} = |\det A|^{-1/2}$, $(2k - n) = \sigma_A$ (since the signature of two similar matrices is the same) and by (4)

$$\langle D^{-1}U\xi, U\xi\rangle = \langle {}^tUD^{-1}U\xi, \xi\rangle = \langle U^{-1}D^{-1}U\xi, \xi\rangle = \langle A^{-1}\xi, \xi\rangle$$

Therefore

$$\mathscr{F}(e^{i\pi\langle Ax,x\rangle}) = |\det A|^{-1/2} e^{i(\pi/4)\sigma_A} e^{i\pi\langle A^{-1}\xi,\xi\rangle}$$

Solution 79

Let us compute $\mathscr{F}(\tau_h u)$ for $h \in \mathbb{R}$ and $u \in \mathscr{S}'(\mathbb{R})$. We have for every $\varphi \in \mathscr{S}(\mathbb{R})$

$$\langle \mathscr{F}(\tau_h u), \varphi\rangle = \langle \tau_h u, \hat{\varphi}\rangle = \langle u, \tau_{-h}\hat{\varphi}\rangle$$

$$(\tau_{-h}\hat{\varphi})(\xi) = \hat{\varphi}(\xi + h) = \int e^{-2i\pi\langle x, \xi+h\rangle}\varphi(x)\,dx = \mathscr{F}[e^{-2i\pi xh}\varphi]$$

so

$$\langle \mathscr{F}(\tau_h u), \varphi\rangle = \langle u, \mathscr{F} e^{-2i\pi xh}\varphi\rangle = \langle e^{-2i\pi xh}\mathscr{F}u, \varphi\rangle \forall \varphi \in \mathscr{S}(\mathbb{R})$$

i.e. $\mathscr{F}(\tau_h u) = e^{-2i\pi xh}\mathscr{F}u$.

If $P_h u = Du + \tau_h u$ we have, taking the Fourier transform

$$x\hat{u} + e^{-2i\pi xh}\hat{u} = 0$$

i.e. $((x + \cos 2\pi xh) - i \sin 2\pi xh)\hat{u} = 0$.

Now we have $x + \cos 2\pi xh - i \sin 2\pi xh = 0$ if and only if

$$\begin{cases} \sin 2\pi xh = 0 \\ x + \cos 2\pi xh = 0 \end{cases}$$

i.e. if $2\pi xh = k\pi$, $k \in \mathbb{Z}$ and $x + \cos 2\pi xh = 0$ i.e.

$$2\pi xh = k\pi, \qquad k \in \mathbb{Z}$$

$$x + \cos 2\pi xh = x + \cos k\pi = x + (-1)^k = 0$$

so $x = (-1)^{k+1}$ and $h = (-1)^{k+1}\frac{k}{2}$. Therefore:

$$\text{If } h \neq (-1)^{k+1}\frac{k}{2} \text{ where } k \in \mathbb{Z} \quad \text{we have } (x + e^{-2i\pi xh}) \neq 0$$

so $\hat{u} = 0$ and since the Fourier transform is an isomorphism in $\mathscr{S}'(\mathbb{R})$, $u = 0$ i.e. P_h is injective. Conversely P_h injective implies $(x + e^{-2i\pi xh}) \neq 0$ so $h \neq (-1)^{k+1}\frac{k}{2}$ where $k \in \mathbb{Z}$.

Indeed let us assume $h = (-1)^{k+1}\frac{k}{2}$ where $k \in \mathbb{Z}$ then u is in the kernel of P_h if and only if

$$(x + \cos k\pi x - i \sin (-1)^{k+1} k\pi x)\hat{u} = 0$$

Now function $f(x) = x + \cos k\pi x - i \sin (-1)^{k+1} k\pi x$ vanishes only at the point $x_k = (-1)^{k+1}$ and this is a simple root since the derivative of $x + \cos k\pi x$ is equal to 1 at that point. So we can write in a neighborhood of x_k:

$$f(x) = (x - (-1)^{k+1})g(x)$$

with $g \in C^\infty$ and $g(x) \neq 0$. Dividing by $g(x)$ near x_k we obtain

$$(x - (-1)^{k+1})\hat{u} = 0$$

which implies that $\hat{u} = C\delta_{(-1)^{k+1}}$.

Therefore if $h = (-1)^{k+1}\frac{k}{2}$ the kernel of P_h is constituted by the functions

$$u = C\mathscr{F}(\delta_{(-1)^{k+1}}) = C e^{(-1)^{k+1} 2i\pi\xi}$$

- If $k = 2p$, $h = -p$, $u = C_1 e^{-2i\pi\xi}$
- If $k = 2p + 1$, $h = \frac{1}{2}(2p + 1)$, $u = C_2 e^{2i\pi\xi}$

Solution 80

The distribution u has a compact support so its Fourier transform $\hat{u}$ is a C^∞ function. Moreover since $P(D)u = 0$ we must have:

$$(1)\quad \widehat{P(D)u} = P(\xi)\hat{u}(\xi) = 0$$

Let $u \in \mathscr{S}'(\mathbb{R}^n)$ be such that $P(D)u = 0$. By Fourier transform we get:
In $\mathbb{R}^n\backslash\Sigma$ we have $P(\xi) \neq 0$; it follows from (1) that $\hat{u} = 0$ in $\mathbb{R}^n\backslash\Sigma$.
Moreover $\mathbb{R}^n\backslash\Sigma$ is dense in $\mathbb{R}^n$, since the set of zeroes of a polynomial is a closed set with empty interior. Since $\hat{u}$ is continuous then $\hat{u} = 0 \forall \xi \in \mathbb{R}^n$. By inverse Fourier transform (since $\hat{u} \in \mathscr{S}'(\mathbb{R}^n)$) we deduce that $u = 0$.

Solution 81

If $T \in \mathscr{E}'$, we know that $\hat{T}$ extends to an entire function F on $\mathbb{C}^n$ given by

$$F(z) = \langle T, e^{-2i\pi\langle x,z\rangle}\rangle \qquad , z \in \mathbb{C}^n$$

We deduce that for all $\alpha \in \mathbb{N}^n$,

$$(\partial_z^\alpha F)(z) = \langle T, \partial_z^\alpha e^{-2i\pi\langle x,z\rangle}\rangle = (-2i\pi)^{|\alpha|}\langle T, x^\alpha e^{-2i\pi\langle x,z\rangle}\rangle$$

so

$$(1)\quad (\partial_z^\alpha F)(0) = (-2i\pi)^{|\alpha|}\langle T, x^\alpha\rangle = 0$$

$\left(\text{To prove that } \left(\frac{\partial}{\partial z_j}F\right)(z) = \left\langle T, \frac{\partial}{\partial z_j} e^{-2i\pi\langle x,z\rangle}\right\rangle \text{ we write } \frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right) \text{ and we}\right.$
$\left.\text{use the method of question a) in exercise 65.}\right)$

Since F is entire

$$(2)\quad F(z) = \sum_\alpha \frac{(\partial^\alpha F)(0)}{\alpha!} z^\alpha$$

for all $z \in \mathbb{C}^n$.
From (1) and (2) we deduce that $F \equiv 0$ so $\hat{T} \equiv 0$. Finally we get in $\mathscr{S}'(\mathbb{R}^n)$

$$T = \bar{\mathscr{F}}\hat{T} = 0$$

Solution 82

Let $u \in \mathscr{S}'(\mathbb{R}^n)$ be such that $P(D)u = 0$. By Fourier transform we get:

$$(1)\quad P(\xi)\hat{u}(\xi) = 0$$

Since $P(\xi)$ is different from zero for $\xi \neq 0$ it follows from (1) that the support of $\hat{u}$ is at the origin. Therefore

$$\hat{u}(\xi) = \sum_{\text{finite}} b_\alpha \delta^{(\alpha)}, \qquad b_\alpha \in \mathbb{C}$$

By inverse Fourier transform we deduce that

$$u(x) = \sum_{\text{finite}} C_\alpha x^\alpha, \qquad C_\alpha \in \mathbb{C}$$

which means that u is a polynomial.

Solution 83

The Fourier transform in $\mathscr{S}'$ is an isomorphism from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$.

Moreover $\widehat{\dfrac{du}{dx}} = 2i\pi\xi\hat{u}$. Then, since k is positive, we have:

$$\frac{d^4u}{dx^4} + ku \in L^2(\mathbb{R}) \Leftrightarrow ((2\pi\xi)^4 + k)\hat{u} \in L^2(\mathbb{R}) \Leftrightarrow (\xi^4 + 1)\hat{u} \in L^2(\mathbb{R})$$

Now for $0 \leq j \leq 4$ we have $|\xi|^j \leq (1 + \xi^4)$. Indeed $|\xi| \leq 1$ or $|\xi| > 1$ and in this case $|\xi|^j \leq \xi^4$.

Therefore $\xi^j\hat{u} \in L^2(\mathbb{R})$ for $j = 0, \ldots, 4$. So $\dfrac{d^j u}{dx^j} \in L^2(\mathbb{R})$ for $0 \leq j \leq 4$.

Solution 84

a) If $T \in \mathscr{E}'$ then $\hat{T}$ is a C^∞ function and there exists $k \in \mathbb{N}$, $C > 0$ such that:

$$|\hat{T}(\xi)| = |\langle T, e^{-2i\pi x\xi}\rangle| \leq C \sum_{|\alpha| \leq k} |\xi^\alpha|$$

We deduce that we can find $N \in \mathbb{N}$, $C > 0$ such that for $|\xi| > R > 1$,

$$(1) \quad |\hat{T}(\xi)| \leq C|\xi|^N$$

Let us consider $f_+(z)$. We have

$$f_+(z) = \int_0^{+\infty} e^{2i\pi\xi \operatorname{Re} z} e^{-2\pi\xi \operatorname{Im} z} \hat{T}(\xi)\, d\xi, \qquad \operatorname{Im} z > 0$$

From (1) we get

$$(2) \quad e^{-2\pi\xi \operatorname{Im} z}|\hat{T}(\xi)| \leq \frac{C_N}{\xi^{N+2}(\operatorname{Im} z)^{N+2}}|\xi|^N \leq \frac{C_N}{|\operatorname{Im} z|^{N+2}} \frac{1}{|\xi|^2} \quad \text{for } |\xi| \text{ large.}$$

Therefore the function $e^{2i\pi\xi z}\hat{T}(\xi)$ is integrable on $]0, +\infty[$ for $\operatorname{Im} z > 0$ and f_+ is well defined. Let us show that it is a holomorphic function. Indeed in $\operatorname{Im} z \geqq \varepsilon$, using the Lebesgue theorem, we can differentiate f with respect to $x = \operatorname{Re} z$, $y = \operatorname{Im} z$ under the integral sign. It follows that

$$\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)f_+(z) = 2\pi \int_0^{\infty} (i\xi - i\xi)\, e^{iz\xi}\hat{T}(\xi)\,d\xi = 0$$

therefore f_+ is holomorphic in $\operatorname{Im} z > 0$.
The proof is the same for $f_-(z)$ in $\operatorname{Im} z < 0$.
From estimate (2) we get

$$|f_+(z)| < \int_0^R e^{-2\pi\xi \operatorname{Im} z}|\hat{T}(\xi)|\,d\xi + \int_R^{+\infty} e^{-2\pi\xi \operatorname{Im} z}|\hat{T}(\xi)|\,d\xi$$

$$\leqq \int_0^R |\hat{T}(\xi)|\,d\xi + \frac{C_N}{|\operatorname{Im} z|^{N+2}}\int_R^{+\infty} \frac{d\xi}{|\xi|^2} \leq C + \frac{C'}{|\operatorname{Im} z|^{N+2}}$$

and for $|\operatorname{Im} z|$ small we have $C \leq \dfrac{C'}{|\operatorname{Im} z|^{N+2}}$ so

$$|f_+(z)| \leq \frac{2C'}{|\operatorname{Im} z|^{N+2}}$$

The same is true for $f_-(z)$.

b) It follows from exercise 52 that $\lim\limits_{y\to 0^+} f_+(x + iy)$ and $\lim\limits_{y\to 0} f_-(x + iy)$ exist in $\mathscr{D}'(\mathbb{R})$, we denote them by $f_+(x + i0)$ and $f_-(x - i0)$. Let us compute them.
Let $\varphi \in \mathscr{D}(\mathbb{R})$

$$\langle f_+(x + i0), \varphi\rangle = \lim_{y\to 0^+} \int_{\mathbb{R}} \int_0^{\infty} e^{2i\pi x\xi} e^{-2\pi y\xi}\hat{T}(\xi)\varphi(x)\,d\xi\,dx$$

Function $e^{2i\pi x\xi}e^{-2\pi y\xi}\hat{T}(\xi)\varphi(x)$ is integrable, with respect to the product measure, for all fixed y. We can then apply the Fubini theorem and write

$$\langle f_+(x + i0), \varphi\rangle = \lim_{y\to 0^+} \int_0^{\infty} e^{-2\pi y\xi}\hat{T}(\xi)\left(\int_{\mathbb{R}} e^{2i\pi x\xi}\varphi(x)\,dx\right)d\xi$$

$$= \lim_{y\to 0^+} \int_0^{\infty} e^{-2\pi y\xi}\hat{T}(\xi)\hat{\varphi}(-\xi)\,d\xi$$

Now $\varphi \in \mathscr{D}(\mathbb{R}) \subset \mathscr{S}(\mathbb{R})$ so $\hat{\varphi} \in \mathscr{S}(\mathbb{R})$. Since $|\hat{T}(\xi)| \leq C|\xi|^N$ for $|\xi| > R$ the function $\hat{\varphi}(-\xi)\hat{T}(\xi)$ is integrable on $]0, +\infty[$. It follows from the Lebesgue theorem that:

$$\langle f_+(x + i0), \varphi\rangle = \int_0^{\infty} \hat{T}(\xi)\cdot\hat{\varphi}(-\xi)\,d\xi$$

In the same way we obtain

$$\langle f_-(x - i0), \varphi \rangle = \int_{-\infty}^{0} \hat{T}(\xi)\hat{\varphi}(-\xi)\,\mathrm{d}\xi$$

Therefore

$$\langle f_+(x - i0) + f_-(x - i0), \varphi \rangle = \int_{\mathbb{R}} \hat{T}(\xi)\hat{\varphi}(-\xi)\,\mathrm{d}\xi = \langle \hat{T}, \hat{\varphi}(-\xi) \rangle$$

where $\langle\, ,\, \rangle$ is the duality bracket between $\mathscr{S}$ and $\mathscr{S}'$. Moreover

$$\langle \hat{T}, \hat{\varphi}(-\xi) \rangle = \langle T, \mathscr{F}\hat{\varphi}(-\xi) \rangle = \langle T, \varphi \rangle$$

since $\mathscr{F}(\mathscr{F}\varphi)(-\xi) = \varphi(\xi)$.
This is true for all $\varphi \in \mathscr{S}(\mathbb{R})$ therefore

$$f_+(x + i0) + f_-(x - i0) = T$$

c) Let $T \in \mathscr{D}'(]a, b[)$ and $]a_1, b_1[\subset]a, b[$. Let $\psi \in \mathscr{D}(]a, b[)$, $\psi = 1$ on $]a_1, b_1[$. Then $\psi T \in \mathscr{E}'(\mathbb{R})$. Applying the above result to ψT we get:

$$\psi T = f_+(x + i0) + f_-(x - i0) \quad \text{in } \mathscr{D}'(\mathbb{R})$$

so

$$T = f_+(x + i0) + f_-(x - i0) \quad \text{in } \mathscr{D}'(]a_1, b_1[).$$

Solution 85

a) $\delta^{(k)}$ possess a Laplace transform since it has a compact support. Moreover

$$\mathscr{L}(\delta^{(k)})(p) = p^k \mathscr{L}(\delta), \qquad k \in \mathbb{N}$$

and $\mathscr{L}(\delta)(p) = \langle \delta, \mathrm{e}^{-px} \rangle = 1$ so

$$\mathscr{L}(\delta^{(k)})(p) = p^k$$

b) $H(x)$ is in $\mathscr{S}'$ and its support is contained in $\{x \geq 0\}$ so it has a Laplace transform in $\operatorname{Re} p > 0$:

$$(\mathscr{L}H)(p) = \langle H, \mathrm{e}^{-px} \rangle = \int_0^{\infty} \mathrm{e}^{-px}\,\mathrm{d}x = \frac{1}{p}$$

c) $H(x)$ Log x is a tempered distribution. We can take $\xi = 0$ and for $\operatorname{Re} p > 0$ we have

$$\mathscr{L}(H(x)\operatorname{Log} x)(p) = \langle H(x)\operatorname{Log} x, \mathrm{e}^{-px} \rangle = \int_0^{\infty} \operatorname{Log} x\, \mathrm{e}^{-px}\,\mathrm{d}x$$

First of all for real and positive p we can set $y = px$ so

$$\int_0^{\infty} \operatorname{Log} x \cdot e^{-px}\,dx = \int_0^{\infty} \operatorname{Log}\frac{y}{p} e^{-y}\frac{dy}{p} = \frac{-\operatorname{Log} p - C}{p}$$

where $C = -\int_0^{\infty} \operatorname{Log} y \cdot e^{-y}\,dy$.
Therefore for real and positive p

$$\mathscr{L}(H(x) \operatorname{Log} x)(p) = \frac{-\operatorname{Log} p - C}{p}$$

We know that $\mathscr{L}(H(x) \operatorname{Log} x)$ is holomorphic in $\operatorname{Re} p > 0$. In the same way the function $\dfrac{-\operatorname{Log} p - C}{p}$ is holomorphic in $\operatorname{Re} p > 0$ when we take the usual determination for the Logarithm. Since these two holomorphic functions coincide on the positive real axis, they coincide everywhere. So for $\operatorname{Re} p > 0$

$$\mathscr{L}(H(x) \operatorname{Log} x)(p) = \frac{-\operatorname{Log} p - C}{p}$$

Let us note that C is the Euler constant, $C = \lim\limits_{n\to\infty}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \operatorname{Log} n\right)$.

Solution 86

a) Let us set $T = H(x) \operatorname{Log} x$. For $\varphi \in \mathscr{D}(\mathbb{R})$

$$\left\langle \frac{dT}{dx}, \varphi \right\rangle = -\left\langle T, \frac{d\varphi}{dx} \right\rangle = -\int_0^{\infty} \operatorname{Log} x \cdot \varphi(x)\,dx = -\lim_{\varepsilon\to 0}\int_{\varepsilon}^{\infty} \operatorname{Log} x \cdot \varphi'(x)\,dx$$

by the Lebesgue theorem. Integrating by parts we get:

$$\left\langle \frac{dT}{dx}, \varphi \right\rangle = -\lim_{\varepsilon\to 0}\left\{-\int_{\varepsilon}^{\infty} \frac{\varphi(x)}{x}dx - \operatorname{Log}\varepsilon \cdot \varphi(\varepsilon)\right\}$$

$$(1) \quad \left\langle \frac{dT}{dx}, \varphi \right\rangle = \lim_{\varepsilon\to 0}\left\{\int_{\varepsilon}^{\infty} \frac{\varphi(x)}{x}dx + \operatorname{Log}\varepsilon \cdot \varphi(\varepsilon)\right\}$$

But we know that

$$(2) \quad \left\langle Fp\frac{H(x)}{x}, \varphi \right\rangle = \lim_{\varepsilon\to 0}\left\{\int_{\varepsilon}^{\infty} \frac{\varphi(x)}{x}dx + \varphi(0)\operatorname{Log}\varepsilon\right\}$$

Writing $\varphi(\varepsilon)\operatorname{Log}\varepsilon = \varphi(0)\operatorname{Log}\varepsilon + \varepsilon\operatorname{Log}\varepsilon \cdot \psi(\varepsilon)$, and using the fact that $\varepsilon\operatorname{Log}\varepsilon$ tends to zero with ε, we deduce from (1) and (2) that

$$(3) \quad \frac{d}{dx}H(x)\operatorname{Log} x = Fp\frac{H(x)}{x}$$

In the same way, for $\varphi \in \mathscr{D}(\mathbb{R})$

$$(4)\quad \left\langle \frac{\mathrm{d}}{\mathrm{d}x} Fp\frac{H(x)}{x^k}, \varphi \right\rangle = -\left\langle Fp\frac{H(x)}{x^k}, \varphi'(x) \right\rangle = -\lim_{\varepsilon\to 0} \left\{ \int_\varepsilon^\infty \frac{\varphi'(x)}{x^k}\mathrm{d}x + \right.$$

$$\left. + \sum_{j=1}^{k-1} \frac{\varphi^{(j)}(0)}{(j-1)!(j-k)\varepsilon^{k-j}} + \frac{\varphi^{(k)}(0)}{(k-1)!} \operatorname{Log} \varepsilon \right\} = -\lim I_\varepsilon$$

Now

$$(5)\quad \int_\varepsilon^\infty \frac{\varphi'(x)}{x^k}\mathrm{d}x = k\int_\varepsilon^\infty \frac{\varphi(x)}{x^{k+1}}\mathrm{d}x - \frac{\varphi(\varepsilon)}{\varepsilon^k}$$

$$(6)\quad \varphi(\varepsilon) = \sum_{l=0}^{k-1} \frac{\varphi^{(l)}(0)\varepsilon^l}{l!} + \varepsilon^k \psi(\varepsilon) \text{ with } \lim_{\varepsilon\to 0} \psi(\varepsilon) = \frac{\varphi^{(k)}(0)}{k!}$$

It follows from (4), (5), (6)

$$(7)\quad I_\varepsilon = k\int_0^\infty \frac{\varphi(x)}{x^{k+1}}\mathrm{d}x - \underbrace{\sum_{l=0}^{k-1} \frac{\varphi^{(l)}(0)}{l!\varepsilon^{k-l}}}_{①} - \psi(\varepsilon) +$$

$$+ \underbrace{\sum_{j=1}^{k-1} \frac{\varphi^{(j)}(0)}{(j-1)!(j-k)\varepsilon^{k-j}}}_{②} + \frac{\varphi^{(k)}(0)}{(k-1)!}\operatorname{Log}\varepsilon$$

Now

$$② - ① = \sum_{j=1}^{k-1} \frac{\varphi^{(j)}(0)}{\varepsilon^{k-j}}\left(\frac{1}{(j-1)!(j-k)} - \frac{1}{j!}\right) - \frac{\varphi(0)}{\varepsilon^k}$$

$$② - ① = \sum_{j=1}^{k-1} \frac{\varphi^{(j)}(0)}{\varepsilon^{k-j}} \cdot \frac{k}{j!(j-k)} - \frac{\varphi(0)}{\varepsilon^k}$$

We deduce that:

$$I_\varepsilon = k\left\{\int_\varepsilon^\infty \frac{\varphi(x)}{x^{k+1}}\mathrm{d}x + \sum_{j=1}^{k-1} \frac{\varphi^{(j)}(0)}{j!(j-k)\varepsilon^{k-j}} - \frac{\varphi(0)}{k\varepsilon^k} + \frac{\varphi^{(k)}(0)}{k!}\operatorname{Log}\varepsilon - \frac{1}{k}\psi(\varepsilon)\right\}$$

$$I_\varepsilon = k\left\{\int_\varepsilon^\infty \frac{\varphi(x)}{x^{k+1}}\mathrm{d}x + \sum_{j=1}^{k-1} \frac{\varphi^{(j)}(0)}{j!(j-k)\varepsilon^{k-j}} + \frac{\varphi^{(k)}(0)}{k!}\operatorname{Log}\varepsilon\right\} - \psi(\varepsilon)$$

Let us note that

$$\sum_{j=0}^{k-1} \frac{\varphi^{(j)}(0)}{j!(j-k)\varepsilon^{k-j}} = \sum_{l=1}^{k} \frac{\varphi^{(l-1)}(0)}{(l-1)!(l-k-1)\varepsilon^{k+1-l}}$$

Using the fact that $\lim_{\varepsilon \to 0} \psi(\varepsilon) = \frac{\varphi^{(k)}(0)}{k!}$ and the definition of $Fp\frac{H(x)}{x^{k+1}}$ we deduce from (4) that

$$(7) \quad \frac{\mathrm{d}}{\mathrm{d}x} Fp\frac{H(x)}{x^k} = -kFp\frac{H(x)}{x^{k+1}} + \frac{(-1)^k}{k!}\delta^{(k)}$$

By induction, using (3) and (7) we get:

$$Fp\frac{H(x)}{x^{k+1}} = \frac{(-1)^k}{k!}\left[\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{k+1}(H(x)\,\mathrm{Log}\,x) + \left(\sum_{l=1}^{k}\frac{1}{l}\right)\delta^{(k)}\right]$$

It follows that

$$\mathscr{L}\left(Fp\frac{H(x)}{x^{k+1}}\right) = \frac{(-1)^k}{k!}\left[p^{k+1}\mathscr{L}(H(x)\log x) + \left(\sum_{l=1}^{k}\frac{1}{l}\right)p^k\right]$$

$$\mathscr{L}\left(Fp\frac{H(x)}{x^{k+1}}\right)(p) = \frac{(-1)^{k+1}}{k!}p^k\left(\mathrm{Log}\,p + C - \sum_{l=1}^{k}\frac{1}{l}\right)$$

Solution 87

a) It is easy to see that T has a Laplace transform in $\mathrm{Re}\,p > \mathrm{Re}\,\alpha$ since $e^{-\mathrm{Re}\,\alpha \cdot x}T = e^{i\,\mathrm{Im}\,\alpha \cdot x}x^k H(x) \in \mathscr{S}'(\mathbb{R})$. Moreover

$$\mathscr{L}(T)(p) = \langle T, e^{-px}\rangle = \int_0^\infty x^k e^{(\alpha - p)x}\,\mathrm{d}x = I_k$$

Let us compute I_k in terms of I_{k-1} by integrating by parts for $k \geq 1$:

$$I_k = \frac{-k}{(\alpha - p)}\int_0^\infty e^{(\alpha - p)x}x^{k-1}\,\mathrm{d}x + \left[\frac{x^k}{\alpha - p}e^{(\alpha - p)x}\right]_0^\infty$$

Since $\mathrm{Re}\,(\alpha - p) < 0$, the last term in the right hand side vanishes, so

$$I_k = \frac{k}{(p - \alpha)}I_{k-1}$$

and

$$(1) \quad I_k = \frac{k!}{(p - \alpha)^k}I_0$$

Let us compute I_0. We have

$$I_0 = \int_0^x e^{(\alpha - p)x}\,dx = \frac{1}{p - \alpha} \quad \text{for Re } p > \text{Re } \alpha$$

We deduce from (1) that

$$\mathscr{L}(T)(p) = \frac{k!}{(p - \alpha)^{k+1}} \quad \text{for Re } p > \text{Re } \alpha$$

b) It follows from question a) that

$$\mathscr{L}(e^{-x}H(x)) = \frac{1}{p + 1}, \ \text{Re } p > -1$$

Moreover $\mathscr{L}\left(\dfrac{dT}{dx}\right)(p) = p\mathscr{L}(T)(p)$ so

$$\mathscr{L}\left[\frac{d}{dx}(e^{-x}H(x))\right](p) = p\mathscr{L}(e^{-x}H(x)) = \frac{p}{p + 1}$$

By the uniqueness of the Laplace transform we can write

$$\mathscr{L}^{-1}\left(\frac{p}{p + 1}\right) = \frac{d}{dx}(e^{-x}H(x))$$

Now

$$\frac{d}{dx}(e^{-x}H(x)) = -e^{-x}H(x) + e^{-x}\delta = -e^{-x}H(x) + \delta$$

and

$$\mathscr{L}^{-1}\left(\frac{p}{p + 1}\right) = \delta - e^{-x}H(x)$$

Let us use the same method for the second function. We have

$$\frac{p^2 + i}{p^2 - 3p + 2} = \frac{p^2}{p - 2} + \frac{i}{p - 2} - \frac{p^2}{p - 1} - \frac{i}{p - 1}$$

and

$$\mathscr{L}^{-1}\left(\frac{p^2+i}{p^2-3p+2}\right) = \mathscr{L}^{-1}\left(\frac{p^2}{p-2}\right) + i\mathscr{L}^{-1}\left(\frac{1}{p-2}\right) - \mathscr{L}^{-1}\left(\frac{p^2}{p-1}\right) - i\mathscr{L}^{-1}\left(\frac{1}{p-1}\right)$$

Using question a), the following formula

$$\mathscr{L}\left(\frac{d^mT}{dx^m}\right)(p) = p^m\mathscr{L}(T)(p)$$

and the uniqueness of the Laplace transform we get

$$\mathscr{L}^{-1}\left(\frac{p^2+i}{p^2-3p+2}\right)=\frac{d^2}{dx^2}(H(x)e^{-2x})+i(H(x)e^{-2x})-\frac{d^2}{dx^2}(e^{-x}H(x))-i(e^{-x}H(x))$$

Moreover

$$\frac{d^2}{dx^2}(e^{-2x}H(x)) = \delta' - 2\delta + 4e^{-2x}H(x)$$

$$\frac{d^2}{dx^2}(e^{-x}H(x)) = \delta' - \delta + e^{-x}H(x)$$

It follows that

$$\mathscr{L}^{-1}\left(\frac{p^2+i}{p^2-3p+2}\right)(p) = \delta' - 2\delta + 4e^{-2x}H(x) + ie^{-2x}H(x) - \delta' + \delta - \\ - e^{-x}H(x) - ie^{-x}H(x)$$

$$\mathscr{L}^{-1}\left(\frac{p^2+i}{p^2-3p+2}\right) = -\delta + (4+i)e^{-2x}H(x) - (1+i)e^{-x}H(x)$$

Solution 88

a) Let $\varphi \in \mathscr{D}(\mathbb{R}_+)$ then

$$\langle T, \varphi\rangle = \sum_{k=0}^{\infty} e^k\varphi(k)$$

the sum being finite. Then T is a distribution of order zero.
Moreover

$$e^{-x}T = \sum_{k=0}^{\infty} e^k e^{-x}\delta_k = \sum_{k=0}^{\infty} e^k e^{-k}\delta_k = \sum_{k=0}^{\infty} \delta_k$$

since $e^{-x}\delta_k = e^{-k}\delta_k$. Therefore $e^{-x}T \in \mathscr{S}'(\mathbb{R})$ and T has a Laplace transform in $\operatorname{Re} p > 1$. By definition, if $\lambda \in C^\infty$ and $\lambda = 1$ on supp T, we have:

$$\mathscr{L}(T)(p) = \langle e^{-x}T, \lambda(x)e^{-(p-1)x}\rangle = \sum_{k=0}^{\infty} e^{-(p-1)k}, \qquad \operatorname{Re} p > 1$$

b) We know that

$$\mathscr{L}(T * T)(p) = [\mathscr{L}(T)(p)]^2 = \left[\sum_{k=0}^{\infty} e^{-(p-1)k}\right]^2$$

But

$$\left(\sum_{k=0}^{\infty} a_k\right)^2 = \sum_{n=0}^{\infty} b_n \quad \text{where} \sum_{i+j=n} a_ia_j$$

Here

$$b_n = \sum_{i+j=n} e^{-(p-1)(i+j)} = e^{-(p-1)n} \sum_{i+j=n} 1 = (n+1)e^{-(p-1)n}$$

Therefore we have

$$\mathscr{L}(T * T)(p) = \sum_{n=0}^{\infty} (n+1)e^{-(p-1)n}$$

By inverse Laplace transform we get:

$$T * T = \sum_{n=0}^{\infty} (n+1)\mathscr{L}^{-1}(e^{-(p-1)n}) = \sum_{n=0}^{\infty} (n+1)e^n\delta_n$$

Solution 89

All the distributions described in the statement have a Laplace transform in $\operatorname{Re} p > 1$. We can write

$$\mathscr{L}[(x e^x H(x)) * T] = \mathscr{L}(H(x) \sin x)$$

But

$$\mathscr{L}[(x e^x H(x)) * T](p) = \mathscr{L}[x e^x H(x)](p)\mathscr{L}(T)(p) = \frac{1}{(p-1)^2}\mathscr{L}(T)(p), \quad \operatorname{Re} p > 1$$

by exercise 87.

Moreover

$$\mathscr{L}(H(x) \sin x) = \int_0^\infty \sin x \, e^{-px} \, dx = \frac{1}{2i}\left(\int_0^\infty e^{(i-p)x} \, dx - \int_0^\infty e^{-(i+p)x} \, dx\right)$$

$$\mathscr{L}(H(x) \sin x) = \frac{1}{2i}\left(\frac{1}{p-i} - \frac{1}{p+i}\right) = \frac{1}{p^2+1}, \quad \operatorname{Re} p > 0$$

We deduce that

$$\mathscr{L}(T)(p) = \frac{(p-1)^2}{p^2+1} = \frac{p^2+1-2p}{p^2+1} = 1 - \frac{2p}{p^2+1}$$

To find T, using the uniqueness of the Laplace transform, we just have to take the inverse Laplace transform.

Now

$$\mathscr{L}\left(\frac{dT}{dx}\right) = p\mathscr{L}(T)$$

so

$$\mathscr{L}\left(\frac{\mathrm{d}}{\mathrm{d}x}H(x)\sin x\right) = p\mathscr{L}(H(x)\sin x) = \frac{p}{p^2+1}$$

Then

$$T = \mathscr{L}^{-1}(1) - 2\mathscr{L}^{-1}\left(\frac{p}{p^2+1}\right) = \delta - 2\frac{\mathrm{d}}{\mathrm{d}x}(H(x)\sin x) = \delta - 2H(x)\cos x$$

CHAPTER 7

Applications

PROGRAMME

Sobolev spaces

Elliptic operators

Fundamental solution of differential operators with constant coefficients

Hypoellipticity and analytic hypoellipticity

Local solvability

The Cauchy problem

Singular spectrum of a distribution

Non linear first order equations

STATEMENTS OF THE EXERCISES* CHAPTER 7

Exercise 90: Sobolev spaces
For $s \in \mathbb{R}$ we denote by $H^s(\mathbb{R}^n)$ the space

$$\{u \in \mathcal{S}'(\mathbb{R}^n) : (1 + |\xi|^2)^{s/2}\hat{u} \in L^2(\mathbb{R}^n)\}$$

with the scalar product

$$(1) \quad (u, v)_s = \int_{\mathbb{R}^n} (1 + |\xi|^2)^s \hat{u}(\xi)\bar{\hat{v}}(\xi)\,\mathrm{d}\xi$$

and the norm $\|u\|_s^2 = (u, u)_s$.

a) Prove that $H^s(\mathbb{R}^n)$ is a Hilbert space and that if s_1 and s_2 are real numbers such that $s_1 \geq s_2$ we have $H^{s_1}(\mathbb{R}^n) \subset H^{s_2}(\mathbb{R}^n)$.

b) Let $m \in \mathbb{N}\backslash\{0\}$. Show that for every $\alpha \in \mathbb{N}^n$, $0 < |\alpha| \leq m$, there exists $C > 0$ such that

$$\prod_{j=1}^{n} |\xi_j|^{2\alpha_j} \leq (1 + |\xi|^2)^m \leq C\left(1 + \sum_{0<|\alpha|\leq m} \prod_{j=1}^{n} |\xi_j|^{2\alpha_j}\right) \qquad \forall \xi \in \mathbb{R}^n$$

c) Deduce that, when $s = m \in \mathbb{N}$, the space $H^m(\mathbb{R}^n)$ coincides with the space

$$E = \{u \in L^2(\mathbb{R}^n) : D^\alpha u \in L^2(\mathbb{R}^n),\ |\alpha| \leq m\}$$

and that the norm $\|u\|_m^2$ is equivalent to the norm

$$|u|_m^2 = \sum_{|\alpha|\leq m} \|D^\alpha u\|_{L^2(\mathbb{R}^n)}^2$$

Exercise 91
We recall that if $u \in L^1(\mathbb{R}^n)$ then $\mathcal{F}u$ and $\bar{\mathcal{F}}u \in C^0(\mathbb{R}^n)$.

a) Let $k \in \mathbb{N}$; prove that if $s > \frac{n}{2} + k$, $H^s(\mathbb{R}^n) \subset C^k(\mathbb{R}^n)$.

b) Let Ω be an open set of $\mathbb{R}^n$ and $s \in \mathbb{R}$. We set

$$H^s_{\mathrm{loc}}(\Omega) = \{u \in \mathcal{D}'(\Omega) : \varphi u \in H^s(\mathbb{R}^n),\ \forall \varphi \in \mathcal{D}(\Omega)\}$$

Deduce from a) that $\bigcap_{s\in\mathbb{R}} H^s_{\mathrm{loc}}(\Omega) = \bigcap_{m\in\mathbb{N}} H^m_{\mathrm{loc}}(\Omega) = C^\infty(\Omega)$.

* Solutions pp. 192–213.

Exercise 92 (see exercise 91)
Let U be an open set in $\mathbb{R}^p_x \times \mathbb{R}^q_y$ and $f\colon U \to \mathbb{C}$ a function such that:

(1) f is C^∞ separately in x and y

(2) $D^\alpha_x f$, $D^\beta_y f$ are bounded on every compact in U, for all $\alpha \in \mathbb{N}^p$ and all $\beta \in \mathbb{N}^q$

Show that f belongs to $C^\infty(U)$.
(Hint: show that $f \in H^k_{\text{loc}}(U)$ for every $k \in \mathbb{N}$ and use exercise 91.)

Exercise 93
a) Let $k \in \mathbb{N}$. Show that the space $\mathscr{E}'^{(k)}(\mathbb{R}^n)$, of distributions with compact support of order $\leq k$ is contained in $H^s(\mathbb{R}^n)$ where $s < -\dfrac{n}{2} - k$.

b) Deduce that $\mathscr{E}'(\mathbb{R}^n) \subset \bigcup\limits_{s \in \mathbb{R}} H^s(\mathbb{R}^n)$.

Exercise 94
a) Let $s \in \mathbb{R}$, prove the inequality

$$(1 + |\xi|^2)^s \leqq 4^{|s|}(1 + |\xi - \eta|^2)^s(1 + |\eta|^2)^{|s|} \qquad \forall \xi, \eta \in \mathbb{R}^n$$

b) Let $\psi \in \mathscr{D}(\mathbb{R}^n)$. Prove that the map $u \to \psi u$ is continuous from $H^s(\mathbb{R}^n)$ to $H^s(\mathbb{R}^n)$ for all $s \in \mathbb{R}$.

Exercise 95
Let $k \in \mathbb{R}\backslash\{0\}$. Show that the differential sperator $-\Delta + k^2$ is an isomorphism from $H^{s+2}(\mathbb{R}^n)$ to $H^s(\mathbb{R}^n)$, for all $s \in \mathbb{R}$. $\left(\text{Here } \Delta = \sum\limits_{i=1}^{n} \dfrac{\partial^2}{\partial x_i^2}\right)$

Exercise 96 (see exercises 7 and 91)
a) Show that there exist $F \in L^2(\mathbb{R}^n)$ and $N \in \mathbb{N}$ such that

(1) $\delta = (1 - \Delta)^N F$

b) Show that there exist functions $f_\alpha \in L^2(\mathbb{R}^n) \cap \mathscr{E}'(\mathbb{R}^n)$, $|\alpha| \le m$, such that

(2) $\delta = \sum_{|\alpha|\le m} D^\alpha f$

(Hint: Note that $\varphi \cdot \delta = \varphi$ if $\varphi \in \mathscr{D}(\mathbb{R}^n)$ and $\varphi(0) = 1$.)

c) Let $u \in \mathscr{D}(\mathbb{R}^n)$ be a distribution such that for all $f \in L^2(\mathbb{R}^n) \cup \mathscr{E}'$

(3) $(D^\alpha u) * f \in L^2_{\text{loc}}$ for all $\alpha \in \mathbb{N}^n$

Prove that $u \in C^\infty(\mathbb{R}^n)$. (Hint: use exercises 7 and 91.)

Exercise 97 (see exercise 60) **Parametrices of elliptic differential operators with constant coefficients**

Let $P(D)$ be an elliptic differential operator with constant coefficients in $\mathbb{R}^n$ which means that

$$P_m(\xi) = \sum_{|\alpha|\le m} a_\alpha \xi^\alpha \neq 0 \qquad \forall \xi \in \mathbb{R}^n\backslash\{0\}$$

a) Prove that $|P_m(\xi)| \geqq C|\xi|^m$, $\forall \xi \in \mathbb{R}^n$. Deduce that there exist $K > 0$ and $R > 0$ such that $|P(\xi)| \geqq K|\xi|^m$, $\forall \xi : |\xi| > R$.

b) Let $\chi \in \mathscr{D}(\mathbb{R}^n)$, $\chi = 1$ if $|\xi| \le R$. Show that $\dfrac{1-\chi(\xi)}{P(\xi)} \in \mathscr{S}'(\mathbb{R}^n)$. Deduce that there exists a distribution $E \in \mathscr{S}'(\mathbb{R}^n)$ and $\omega \in \mathscr{S}$ such that

$$P(D)E = \delta - \omega$$

(E is called a parametrice of P).

c) Prove that for every $k \in \mathbb{N}$ there exists $\alpha \in \mathbb{N}^n$ such that $x^\alpha E \in C^k(\mathbb{R}^n)$. (Hint: prove that $\mathscr{F}(D^\beta x^\alpha E) \in L^1(\mathbb{R}^n)$ if $|\alpha|$ is big enough and $|\beta| \le k$.)

d) By the method used in exercise 60, prove that if $u \in \mathscr{D}'(\mathbb{R}^n)$ is such that $Pu \in C^\infty(\mathbb{R}^n)$ then $u \in C^\infty(\mathbb{R}^n)$.

Exercise 98: Fundamental solutions of differential operators with constant coefficients**

The purpose of this exercise is to prove that every differential operator with constant coefficients in $\mathbb{R}^n$, non identically zero, has a fundamental solution.

1°) We shall denote $D_\rho = \{z \in \mathbb{C}, |z| < \rho\}$ and H_ρ the space of holomorphic functions in D_ρ, which are continuous in $\bar{D}_\rho$.

a) Let $f \in H_\rho$ and $q(z) = z - \lambda$ where $\lambda \in \mathbb{C}$. We assume that $g(z) = \dfrac{f(z)}{q(z)}$ is also in H_ρ. Prove that there exists a constant $C > 0$ independant of λ such that

$$(1) \quad |g(0)| \leq \frac{C}{\rho} \int_0^{2\pi} |f(\rho e^{i\theta})| \, d\theta$$

$\left(\text{Hint: start with } \rho = 1 \text{ and apply the result to the functions } f(\rho z) \text{ and } \dfrac{1}{\rho} q(\rho z).\right)$

b) Let $\varphi \in \mathscr{D}(\mathbb{R}^n)$. Prove that $\hat{\varphi}$ can be extended to an entire function (i.e. holomorphic in $\mathbb{C}^n$) which satisfies $|\hat{\varphi}(\xi + i\eta)| \leqq C e^{a|\eta|}$ where a and C are positive constants.

c) Let $P(D) = \sum_{|\alpha| \leq m} a_\alpha D^\alpha$, $a_\alpha \in \mathbb{C}$. We set $P(\xi) = \sum_{|\alpha| \leq m} a_\alpha \xi^\alpha$. Compute $\bar{\mathscr{F}}_\xi[P(\xi + i\eta)\hat{\varphi}(\xi + i\eta)]$ in terms of $P(D)$ and φ.

2°) Let us set $P_1'(\xi) = \dfrac{\partial P}{\partial \xi_1}(\xi)$.

a) Let $\varphi \in \mathscr{S}(\mathbb{R}^n)$, show that:

$$P_1'(\xi)\hat{\varphi}(\xi) = \sum_{j=1}^{m} \frac{P(\xi)\hat{\varphi}(\xi)}{\xi_1 - \lambda_j(\xi')} \qquad \text{where } \xi' = (\xi_2, \ldots, \xi_n)$$

Deduce, for $z \in \mathbb{C}$, the expression of

$$P_1'(\xi_1 + i\eta_1 + z, \xi' + i\eta')\hat{\varphi}(\xi_1 + i\eta_1 + z, \xi' + i\eta')$$

b) Prove, applying the results of questions 1° a) and 2° a) to the functions

$$f(z) = P(\xi_1 + i\eta_1 + z, \xi' + i\eta')\hat{\varphi}(\xi_1 + i\eta_1 + z, \xi' + i\eta')$$

and

$$q(z) = z + \xi_1 + i\eta_1 - \lambda_j(\xi' + i\eta')$$

that

$$\sup_{|\eta| \leq \rho/2} \int_{\mathbb{R}^n} |P_1'(\xi + i\eta)\hat{\varphi}(\xi + i\eta)|^2 \, d\xi \leqq C_\rho \sup_{|\eta| \leq \rho} \int_{\mathbb{R}^n} |P(\xi + i\eta)\hat{\varphi}(\xi + i\eta)|^2 \, d\xi$$

c) Deduce that for every $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ there exists $A_\alpha > 0$ such that

$$\sup_{|\eta| \leq \rho/A_\alpha} \int \left| \left[\left(\frac{\partial}{\partial \xi} \right)^\alpha P \right] (\xi + i\eta)\hat{\varphi}(\xi + i\eta) \right|^2 d\xi \leq C_\rho \sup_{|\eta| \leq \rho} \int |P(\xi + i\eta)\hat{\varphi}(\xi + i\eta)|^2 \, d\xi$$

(Hint: Iterate the inequality proved in question 2° b).)

(continued p. 189)

d) Using 1°) c) and 2° c) show that for every $p > 0$ one can find $C_p > 0$ such that:

$$\|\varphi\|_{L^2(\mathbb{R}^n)} \leq C_p \|e^{p|x|} P(D)\varphi\|_{L^2(\mathbb{R}^n)} \qquad \forall \varphi \in \mathscr{D}(\mathbb{R}^n)$$

3°) We set $E = \{u = e^{\rho|x|} \bar{P}(D)$ where $\varphi \in \mathscr{D}(\mathbb{R}^n)\}$ and $\bar{P}(D) = \sum_{|\alpha| \leq m} \bar{a}_\alpha D^\alpha$.

a) Let $f \in L^2(\mathbb{R}^n)$. We define a linear form G on E by

$$G(e^{\rho|x|} \bar{P}(D)\varphi) = (\varphi, f)_{L^2}$$

Prove that G is continuous on E considered as a subspace of $L^2(\mathbb{R}^n)$.

b) Using the Hahn–Banach theorem prove that we can find $h \in L^2(\mathbb{R}^n)$ such that

$$G(e^{\rho|x|} \bar{P}(D)\varphi) = (h, e^{\rho|x|} \bar{P}(D)\varphi)_{L^2(\mathbb{R}^n)}$$

c) Deduce that: $\forall \rho > 0, \forall f \in L^2(\mathbb{R}^n), \exists u$ such that $e^{\ \rho|x|} u \in L^2(\mathbb{R}^n)$ and $P(D)u = f$ in $\mathscr{D}'(\mathbb{R}^n)$.

4°) a) Prove the existence of $f \in L^2(\mathbb{R}^n)$ and of a differential operator $Q(D)$ such that $\delta = Q(D)f$. (See exercise 96.)

b) Let $P(D)$ be a differential operator with constant coefficients, non identically zero. Deduce from what precedes that there exists $E \in \mathscr{D}'(\mathbb{R}^n)$ such that

$$P(D)E = \delta$$

c) Let $g \in C_0^\infty(\mathbb{R}^n)$, prove that one can find $u \in C^\infty(\mathbb{R}^n)$ such that

$$P(D)u = g$$

Exercise 99* (see exercises 91, 93, 94)

Let $P(D) = \sum_{|\alpha| \leq m} a_\alpha D^\alpha$ be a differential operator of order m with constant coefficients in an open set Ω of $\mathbb{R}^n$. For $\beta \in \mathbb{N}^n$ we set $P^{(\beta)}(\xi) = \left(\dfrac{\partial^\beta P}{\partial \xi^\beta}\right)(\xi)$ where $P(\xi) = \sum_{|\alpha| \leq m} a_\alpha \xi^\alpha$. We shall denote by $P^{(\beta)}(D)$ the differential operator with constant coefficients associated to the polynomial $P^{(\beta)}(\xi)$.

The purpose of this problem is to prove that if the polynomial $P(\xi)$ satisfies the following condition.

(C) There exist $\mu > 0$, $R > 0$ and $C > 0$ such that for all ξ, $|\xi| > R$,

$$|P^{(\beta)}(\xi)|(1 + |\xi|^2)^{\mu/2} \leq C|P(\xi)|$$

then for every open subset ω of Ω, $u \in \mathscr{D}'(\Omega)$ and $P(D)u \in C^\infty(\omega)$ imply $u \in C^\infty(\omega)$. We shall call these operators hypoelliptic.

a) Let $\varphi \in \mathscr{D}(\Omega)$ and $v \in \mathscr{D}'(\Omega)$. Deduce from the Leibniz formula that

$$(1) \quad P(D)(\varphi v) = \varphi P(D)v + \sum_{\beta \neq 0} \frac{1}{\beta!} D^\beta \varphi \cdot P^{(\beta)}(D)v$$

In the sequel we fix $s \in \mathbb{R}$ and we want to show that if u is a distribution on Ω such that $Pu \in H^s_{\text{loc}}(\omega)$ then $u \in H^s_{\text{loc}}(\omega)$.
Let ω' be an open subset of ω such that $\bar{\omega}'$ is a compact included in ω. Then for every $N \in \mathbb{N}$ one can find open sets $\omega_0, \omega_1, \ldots, \omega_N$ included in ω such that

$$(2) \quad \begin{cases} \omega' = \omega_N \subset \omega_{N-1} \subset \cdots \ldots \subset \omega_0 = \omega \\ \bar{\omega}_j \text{ is a compact contained in } \omega_{j-1}, j = 1, 2, \ldots, N \end{cases}$$

For $j = 0 \ldots, N-1$, we shall denote by φ_j a function of $\mathscr{D}(\omega_j)$, $\varphi_j = 1$ on ω_{j+1}.

b) Using exercise 93, prove that there exists $t \in \mathbb{R}$ such that $\varphi_0 u \in H^t(\mathbb{R}^n)$. Deduce that for $\beta \neq 0$, $P^{(\beta)}(D)(\varphi_0 u) \in H^{t-(m-1)}(\mathbb{R}^n)$. (We assume $t < s$ otherwise $u \in H^s_{\text{loc}}(\omega)$.)

We define the integer N in the following way

$$(3) \quad \begin{cases} t - (m-1) + (N-1)\mu < s \\ t - (m-1) + N\mu \geq s \end{cases}$$

c) Using (1), with $\varphi = \varphi_j$, $v = \varphi_{j-1}u$, the exercise 94 then the condition (C), prove by induction that $P^{(\beta)}(\varphi_j u)$, $\beta \neq 0$ is in $H^{t-(m-1)+j\mu}(\mathbb{R}^n)$. Deduce that $u \in H^s_{\text{loc}}(\omega)$.

d) Prove the claim of the beginning of this exercise. (Hint: use exercise 91.)

e) Prove that if P is elliptic i.e. $\sum_{|\alpha|=m} a_\alpha \xi^\alpha \neq 0$ for $\xi \neq 0$, then P satisfies (C).

Prove that the heat operator $P = \dfrac{\partial}{\partial t} - \dfrac{\partial^2}{\partial x^2}$, satisfies (C) but that the operators $P = \dfrac{\partial}{\partial t} + i\dfrac{\partial^2}{\partial x^2}$ and $P = \dfrac{\partial^2}{\partial t^2} - \dfrac{\partial^2}{\partial x^2}$ do not satisfy (C). Compare with exercise 51.

Exercise 100: Analytic hypoellipiticity of the elliptic operators**

We recall that a function v which is C^∞ in an open set Ω of $\mathbb{R}^n$ is said to be analytic in Ω if:

For every compact K of Ω there exists a constant $C > 0$ such that for every $\alpha \in \mathbb{N}^n$

$$(1) \quad \sup_{x \in K} |\partial^\alpha u(x)| \leq C^{|\alpha|+1} \alpha!$$

a) Using question a) of exercise 91 prove that there exists $k_0 \in \mathbb{N}$ such that for every compact $K' \subset \Omega$ there exists $C' > 0$ such that for every $v \in \mathscr{D}(\Omega)$

$$(2) \quad \sup_{x \in K'} |v(x)| \leqq C' \sum_{|\alpha| \leq k_0} \|\partial^\alpha v\|_{L^2(\Omega)}$$

We consider in what follows a differential operator of order m with constant coefficients, which is elliptic, i.e.

$$P_m(\xi) = \sum_{|\alpha| = m} a_\alpha \xi^\alpha \neq 0 \quad \text{for all } \xi \in \mathbb{R}^n \backslash \{0\}$$

b) Prove that there exists $C_1 > 0$ such that for all $\xi \in \mathbb{R}^n$

$$|P_m(\xi)| \geq C_1 |\xi|^m$$

Deduce that one can find $C_2 > 0$ and $R > 0$ such that for all $\xi \in \mathbb{R}^n$. $|\xi| > R$

$$|P(\xi)| \geq C_2 |\xi|^m$$

c) Prove that there exists $C_3 > 0$ such that for all $v \in \mathscr{D}(\Omega)$

$$(3) \quad \|v\|_{H^m(\mathbb{R}^n)} \leqq C_3 \{\|P(D)v\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)}\}$$

Let $\tilde{\omega}$ be an open subset of Ω and u a distribution on Ω such that $P(D)u$ is an analytic function on $\tilde{\omega}$. Let ω be an open set such that $\bar{\omega} \subset \tilde{\omega}$; we know from exercise 99 that u is a C^∞ function in ω. Our purpose is to show that u is actually an analytic function in ω.

For $\varepsilon > 0$ we set $\omega_\varepsilon = \{x \in \omega : d(x, \complement \omega) > \varepsilon\}$.

d) Let $\varepsilon, \varepsilon_1 > 0$ and χ be the characteristic function of $\omega_{\varepsilon_1 + (\varepsilon/2)}$, i.e.

$$\chi(y) = 1 \text{ if } d(y, \complement \omega) > \varepsilon_1 + \frac{\varepsilon}{2}, \qquad \chi(y) = 0 \text{ if } d(y, \complement \omega) \leq \varepsilon_1 + \frac{\varepsilon}{2}$$

Let $\varphi \in \mathscr{D}(\mathbb{R}^n)$ be such that:

$$\varphi \geqq 0, \qquad \int_{\mathbb{R}^n} \varphi(x)\,\mathrm{d}x = 1, \qquad \operatorname{supp} \varphi \subset \{x : |x| \leq 1\}$$

We set

$$\varphi_{\varepsilon,\varepsilon_1}(x) = \frac{1}{\delta^n} \int_{\mathbb{R}^n} \chi(y) \varphi\left(\frac{x-y}{\delta}\right) \mathrm{d}y \quad \text{where } \delta = \frac{\varepsilon}{3}$$

Prove that $\varphi_{\varepsilon,\varepsilon_1} \in \mathscr{D}(\omega_{\varepsilon_1})$, $\varphi_{\varepsilon,\varepsilon_1} = 1$ on $\omega_{\varepsilon+\varepsilon_1}$ and:

For every $\alpha \in \mathbb{N}^n$, there exists a constant $C_\alpha > 0$ independant of $\varepsilon, \varepsilon_1$ such that

$$(4) \quad \sup_{x \in \omega_\varepsilon} |\partial^\alpha \varphi_{\varepsilon,\varepsilon_1}(x)| \leqq C_\alpha \varepsilon^{-|\alpha|}$$

e) Prove that one can find $C > 0$ such that, for every $w \in C^\infty(\omega)$, for every positive $\varepsilon, \varepsilon_1, \varepsilon \in]0, 1[$ and every $\alpha \in \mathbb{N}^n$ such that $|\alpha| \leq m$, we have:

$$(5) \quad \varepsilon^{|\alpha|}\|D^\alpha w\|_{L^2(\omega_{\varepsilon+\varepsilon_1})} \leqq C\left(\varepsilon^m\|P(D)w\|_{L^2(\omega_{\varepsilon_1})} + \sum_{|\beta|<m} \varepsilon^{|\beta|}\|D^\beta w\|_{L^2(\omega_{\varepsilon_1})}\right)$$

(Hint: Apply inequality (3) to $v = \varphi_{\varepsilon,\varepsilon_1} w$ and use the Leibniz formula.)

f) Deduce, from the analyticity of Pu in $\tilde{\omega}$, that

$$\exists M > 0 \colon \forall j \in \mathbb{N} \colon j\varepsilon \leq 1 \qquad \varepsilon^j\|D^{\alpha_0}P(D)u\|_{L^2(\omega_{j\varepsilon})} \leqq M^{j+1}, \qquad |\alpha_0| \leqq j$$

g) Prove by induction on $j \in \mathbb{N}, j \geq 1$, that there exists a constant $B > 0$ such that for every $\varepsilon \in]0, 1[$ and every $j \in \mathbb{N}$ such that $j\varepsilon \leq 1$,

$$(6)_j \quad \varepsilon^{|\gamma|}\|D^\gamma u\|_{L^2(\omega_{j\varepsilon})} \leqq B^{|\gamma|+1}, \qquad |\gamma| < m + j$$

(Hint: Note that $(6)_1$ is true if B is big enough, then increase B if necessary so that $(6)_j$ implies $(6)_{j+1}$. Use the fact that $(6)_j$ implies $(6)_{j+1}$ for $|\gamma| < m + j$ then apply (5) with $\varepsilon_1 = j\varepsilon$, $w = D^{\alpha_0}u$, $|\alpha_0| = j$.

h) Let K be a compact subset of Ω and $a \in]0, 1[$ be such that $K \subset \omega_a$. Taking $j = |\gamma|$ and $\varepsilon = \dfrac{a}{j}$ in $(6)_j$ prove that:

$$(7) \quad \|D^\gamma u\|_{L^2(\omega_a)} \leq B^{|\gamma|+1}\left(\frac{|\gamma|}{a}\right)^{|\gamma|}$$

i) Deduce from (2) and (7) that u is analytic in ω.

We recall that it follows from Stirling formula that for $p, q \in \mathbb{N}$ big enough we have $C_1 p! \leq p^p \leq C_2^p p!$ where C_1, C_2 are independant of p and $(p + q)! \leq 2^{p+q}p!q!$

Exercise 101:** (see exercises 6 and 90)

Let $P(x, D) = \sum_{|\alpha|\leq m} a_\alpha(x)D^\alpha$ be a differential operator of order $m \geqslant 2$ in an open set Ω of $\mathbb{R}^n$.

We set $p_m(x, \xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha$ and

$$\Sigma = \{(x, \xi) \in \Omega \times \mathbb{R}^n \backslash 0 \colon p_m(x, \xi) = 0\}.$$

We shall say that the operator P is of principal type if

$$(*) \quad \forall (x, \xi) \in \Sigma \qquad \exists j \in \{1, 2, \ldots, n\} \colon \frac{\partial p_m}{\partial \xi_j}(x, \xi) \neq 0.$$

The purpose of this problem is to prove the following result:
«If P is an operator of principal type with C^∞ coefficients and if $p_m(x, \xi)$ has real coefficients then P is locally solvable i.e. For every $a \in \Omega$ and every $f \in C_0^\infty$ near a there exists $u \in \mathscr{D}'(\Omega)$ such that $Pu = f$ in a neighborhood of a.»

1°) Using the homogeneity in ξ of p_m prove that P is of principal type if and only if

$$(**) \quad \forall (x, \xi) \in \Omega \times \mathbb{R}^n \backslash 0 \qquad \exists j \in \{1, 2, \ldots, n\}: \frac{\partial p_m}{\partial \xi_j}(x, \xi) \neq 0.$$

2°) Let $R > 0$, $a \in \Omega$ be such that

$$B(a, R) = \{x \in \Omega: |x - a| < R\} \subset \Omega.$$

Let $s \in \mathbb{N}^*$. Prove that there exists a positive constant $C(R)$ such that $\lim\limits_{R \to 0} C(R) = 0$ and

$$(1) \quad \|\varphi\|^2_{H^{s-1}} \leqslant C(R) \sum_{|\alpha| = s} \|D^\alpha \varphi\|^2_{L^2}$$

if R is small enough and $\varphi \in C_0^\infty(B(a, R))$.
(Hint: Discuss first the case $s = 1$, supp $\varphi \subset \{x: |x_1 - a_1| < R\}$ and write

$$\varphi(x_1, \ldots, x_n) = \int_{a_1 - R}^{x_1} \frac{\partial \varphi}{\partial x_1}(t, x_2, \ldots, x_n) \, dt.$$

3°) Assume that the coefficients of P are C^r in Ω. Let P^* be its adjoint in L^2 and $R > 0$. Show that we can find $C_1(R) > 0$, satisfying $\lim\limits_{R \to 0} C_1(R) = 0$, such that for every $\varphi \in C_0^r(B(a, R))$

$$(2) \quad \sum_{j=1}^{n} \|P_m^{(j)}(x, D)\varphi\|^2_{L^2} \leqslant C_1(R)\{\|P\varphi\|^2_{L^2} + \|P^*\varphi\|^2_{L^2} + \|\varphi\|^2_{H^{m-1}}\}$$

where $P_m^{(j)}(x, D)$ is the operator whose symbol is $\dfrac{\partial p_m}{\partial \xi_j}(x, \xi)$.
(Hint: Note that $P_m^{(j)} = i[P_m, x_j - a]$. Then write $\|P_m^{(j)}\varphi\|^2 = (P_m^{(j)}\varphi, iP_m(x_j - a)\varphi) - (P_m^{(j)}\varphi, i(x_j - a)P_m\varphi)$, and use question 2°).)

4°) Let us assume that P is of principal type. Using question 2°) prove that:

$$(3) \quad \|\varphi\|^2_{H^{m-1}} \leqslant C \sum_{j=1}^{n} \|P^{(j)}(a, D)\varphi\|^2_{L^2} \qquad \forall \varphi \in C_0^\infty(B(a, R))$$

Deduce

$$(3)' \quad \|\varphi\|^2_{H^{m-1}} \leqslant C \sum_{j=1}^{n} \|P^{(j)}(x, D)\varphi\|^2_{L^2} \qquad \forall \varphi \in C_0^\infty(B(a, R)).$$

(Hint: Use the fact that if P is of principal type then

$$\sum_{j=1}^{n} \left| \frac{\partial p_m}{\partial \xi_j}(a, \xi) \right|^2 \geqq C|\xi|^{2m-2} \qquad \forall \xi \in \mathbb{R}^n \backslash 0.)$$

5°) Let us assume moreover that p_m has real coefficients. Show that one can find a constant $M_0 > 0$ such that

$$(4) \quad \|P\varphi\|^2_{L^2} \leqq \|P^*\varphi\|^2_{L^2} + M_0\|\varphi\|^2_{H^{m-1}} \qquad \forall \varphi \in C_0^\infty(B(a, R)).$$

(Hint: $P - P^*$ is of order $m - 1$.)
Deduce from the preceding questions that there exists $C > 0$ such that

$$(5) \quad \|\varphi\|^2_{H^{m-1}} \leqslant C\|P^*\varphi\|^2_{L^2}.$$

6°) Let $f \in C_0^\infty(\Omega)$. We consider the subspace of $L^2(B(a, R))$ defined by

$$E = \{\psi = P^*\varphi \quad \text{where} \quad \varphi \in C_0^\infty(B(a, R))\}$$

and the map l: $E \to \mathbb{C}$ defined by: $\psi = P^*\varphi \to l(\psi) = \langle f, \varphi \rangle$.

Let $\hat{E}$ be the completion of E in L^2. Prove that l can be extended to a continuous linear form on $\hat{E}$. Deduce that P is locally solvable.
(Hint: Use inequality (5) and the Hahn-Banach theorem.)

7°) Give examples of differential operators which satisfy the conditions required in this problem.

Exercise 102*
Let I be an open interval in $\mathbb{R}$ and Ω be an open subset of $\mathbb{R}^n$. We shall denote by (t, x) the point in $I \times \Omega$. We shall denote by $C^k(I, \mathscr{D}'(\Omega))$ the space $\{u \in \mathscr{D}'(I \times \Omega)$: $D_t^j u \in C^0(I, \mathscr{D}'(\Omega)), 0 \leqslant j \leqslant k\}$ where $k = 0, 1, \ldots, +\infty$ and $C^0(I, \mathscr{D}'(\Omega))$ is the space of all $u \in \mathscr{D}'(I \times \Omega)$ which can be locally written as $u = \sum_{|\alpha| \leq \mu} D_x^\alpha u_\alpha$ where $u_\alpha \in C^0(I \times \Omega)$.

Let P be a second order differential operator, with constant coefficients, on the form $P = D_t^2 + \sum_{|\alpha| + j \leqslant 2} a_{j_\alpha} D_t^j D_x^\alpha$. The purpose of this problem is to prove the following claim:

$$(*) \quad u \in \mathscr{D}'(I \times \Omega),\ Pu = 0 \Rightarrow u \in C^\infty(I, \mathscr{D}'(\Omega)).$$

a) Let $u \in \mathscr{D}'(I \times \Omega)$ such that $D_t u \in C^0(I, \mathscr{D}'(\Omega))$. Prove that $u \in C^0(I, \mathscr{D}'(\Omega))$.

b) Show that $u \in C^1(I, \mathscr{D}'(\Omega))$ and $Pu = 0$ imply $u \in C^\infty(I, \mathscr{D}'(\Omega))$.

c) Show that if $u \in \mathscr{D}'(I \times \Omega)$ satisfies $Pu = 0$ then $u \in C^1(I, \mathscr{D}'(\Omega))$.

Remark:

This result is still true when P is a differential operator of order $m \geqslant 1$ with C^∞ coefficients. It implies in particular that the traces on a hyperplan $t =$ constant of the distributions solutions of $Pu = 0$ are well defined and are distributions.

Exercise 103*: The Cauchy problem for the wave equation (see exercise 70).

We shall denote, in what follows, by (t, x) the variable in $\mathbb{R} \times \mathbb{R}^n$ and $\Box = \dfrac{\partial^2}{\partial t^2} - \sum_{i=1}^{n} \dfrac{\partial^2}{\partial x_i^2}$. The Green function of the Cauchy problem for the operator $\Box$ is the distribution $G(t, x)$ in $\mathbb{R} \times \mathbb{R}^n$ solution of the problem

$$(*) \quad \Box G = 0 \text{ in } \mathbb{R} \times \mathbb{R}^n,\ G|_{t=0} = 0\ \left.\frac{\partial G}{\partial t}\right|_{t=0} = \delta_x \text{ (the Dirac measure at } x = 0)$$

The purpose of the first part of this exercise is to prove that the existence of G allows us to solve the Cauchy problem:

$$(**) \quad \Box u = f \quad in\ \mathbb{R}^{n+1},\quad u|_{t=0} = \varphi,\ \left.\frac{\partial u}{\partial t}\right|_{t=0} = \psi$$

where $f \in C_0^\infty(\mathbb{R}^{n+1})$ and φ, ψ are in $C_0^\infty(\mathbb{R}^n)$.

I. a) Show that the existence of G is sufficient to solve the problem

$$(1) \quad \Box u_0 = 0 \quad u_0|_{t=0} = 0 \quad \left.\frac{\partial u_0}{\partial t}\right|_{t=0} = \psi.$$

b) Let v be the solution of the problem

$$(1)' \quad \Box v = 0 \quad v|_{t=0} = 0 \quad \left.\frac{\partial v}{\partial t}\right|_{t=0} = \varphi$$

and $w = \partial_t v$. Find a problem for which w is a solution. Deduce from a) the solution of

$$(2) \quad \Box u_1 = 0 \quad u_1|_{t=0} = \varphi, \quad \left.\frac{\partial u_1}{\partial t}\right|_{t=0} = \psi$$

c) (Duhamel principle).
Let $\tau \in \mathbb{R}^+$ and $v(x, t, \tau)$ be the solution of the problem

$$(3) \quad \Box v = 0 \quad v|_{t=0} = 0, \quad \left.\frac{\partial v}{\partial t}\right|_{t=0} = f(x, \tau)$$

We set $u_2 = \int_0^t v(x, t - \tau, \tau)\,d\tau$. Find a problem for which u_2 is a solution and deduce the solution of the problem (**).

II. In this part we are going to compute the Green function G in $\mathbb{R} \times \mathbb{R}^3$.

a) Let $\tilde{G}(t, \xi)$ be the partial Fourier transform in x of $G(t, x)$. What differential equation does G satisfy? Deduce $\tilde{G}(t, \xi)$.

b) We consider a real number $a \in \mathbb{R}^+$ and the distribution in $\mathbb{R}^3$, $\delta(a - |x|)$ defined by $\langle \delta(a - |x|), \varphi \rangle = \int_{|x|=a} \varphi(x)\,dx$. Compute the Fourier transform of this distribution. (Hint: Use exercise 70.) Deduce $G(t, x)$.

c) Write down the solution of the Cauchy problem (**) in $\mathbb{R}^4$.

Exercise 104

I. We consider the wave operator $\Box = \dfrac{\partial^2}{\partial t^2} - \sum_{i=1}^{n} \dfrac{\partial^2}{\partial x_i^2}$ in

$$\mathbb{R}_+^{n+1} = \{(t, x) : t > 0\}$$

and we are going to prove that if u is a C^2 real solution of the Cauchy problem:

$$(1) \quad \Box u = 0 \text{ in } \mathbb{R}_+^{n+1}, \; u|_{t=0} = \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0$$

in $B_0 = \{x : |x - x_0| \leqslant t_0\}$ then u vanishes in

$$\Omega = \{(t, x) : 0 \leqslant t \leqslant t_0, |x - x_0| \leqslant t_0 - t\}.$$

a) Give a geometric interpretation in $\mathbb{R}^3$ of this result.

b) We set $B_t = \{x : |x - x_0| \leq t_0 - t\}$, $|\nabla u|^2 = \left(\dfrac{\partial u}{\partial t}\right)^2 + \sum_{j=1}^{n} \left(\dfrac{\partial u}{\partial x_j}\right)^2$ and

$E(t) = \frac{1}{2}\int_{B_t} |\nabla u|^2 \, dx$. Show that

$$\frac{dE}{dt} = \int_{B_t}\left[\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2} + \sum_{j=1}^{n}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_j \partial t}\right]dx - \frac{1}{2}\int_{\partial B_t} |\nabla u|^2 \, d\sigma$$

where ∂B_t is the boundary of B_t and $d\sigma$ the measure on it.

c) Using equation (1) and the Gauss-Green formula (*) deduce that

$$\frac{dE}{dt} = \int_{\partial B_t}\left(\frac{\partial u}{\partial t}\frac{\partial u}{\partial \nu} - \frac{1}{2}|\nabla u|^2\right)d\sigma$$

where $\frac{\partial}{\partial \nu} = \sum_{i=1}^{n} \nu_i \frac{\partial}{\partial x_i}$, $\nu = (\nu_i)$ being the unitary normal to ∂B_t.

d) Show then that $\frac{dE}{dt} \leqslant 0$, deduce that $E(t) = 0$ in Ω and conclude.

II. Let $u \in C^2(\mathbb{R}^{n+1}_+)$ be a real solution of the Cauchy problem

$$\Box u = 0 \text{ in } \mathbb{R}^{n+1}_+, \; u|_{t=0} = u_0, \; \left.\frac{\partial u}{\partial t}\right|_{t=0} = u_1$$

We set $\Gamma_0 = \operatorname{supp} u_0 \cup \operatorname{supp} u_1$. Prove that $\operatorname{supp} u \subset \Gamma = \{(x, t) : d(x, \Gamma_0) \leqslant t\}$. (Hint: Show that $\complement\Gamma \subset \complement \operatorname{supp} u$ using the result proved in the first part).

(*) We recall the Gauss-Green formula: if $f = (f_1, \ldots, f_n)$

$$\sum_{j=1}^{n}\int_{\Omega}\frac{\partial f_j}{\partial x_j}dx = \int_{\partial\Omega}\langle f, \nu\rangle \, d\sigma$$

Exercise 105

Let Ω_x and Ω_y be two open subsets of $\mathbb{R}^n$ and k a distribution on $\Omega_x \times \Omega_y$. To this distribution corresponds the operator $K: C_0^\infty(\Omega_y) \to \mathscr{D}'(\Omega_x)$ given by

$$Ku(x) = \langle k, u(y)\rangle$$

We say that k is semi-regular in x if K maps $C_0^\infty(\Omega_y)$ in $C^\infty(\Omega_x)$. We say that k is semi-regular in y if K extends to a continuous linear map from $\mathscr{E}'(\Omega_y)$ to $\mathscr{D}'(\Omega_x)$. Finally k is said to be very regular if it is semi-regular in x and y and moreover it is a C^∞ function outside the diagonal $x = y$.

Let $P(x, D_x)$ be a differential operator with C^∞ coefficients in an open set Ω of $\mathbb{R}^n$. A distribution $k \in \mathscr{D}'(\Omega \times \Omega)$ is called a parametrix of P if

$$P(x, D_x)k - \delta(x - y) \in C^\infty(\Omega \times \Omega)$$

(Here $\delta(x - y)$ is defined by $\langle \delta(x - y), \varphi(x) \rangle = \varphi(y)$).

Our purpose is to show that if ${}^tP$, the transpose of P, has a very regular parametrix then P is hypoelliptic in Ω (${}^tP$ is defined by

$$\langle {}^tPu, \varphi \rangle = \langle u, P\varphi \rangle, \ u \in \mathscr{D}'(\Omega), \ \varphi \in C_0^\infty(\Omega)).$$

Let $u \in \mathscr{D}'(\Omega)$ be such that Pu is C^∞ in a neighborhood V of a point $x_0 \in \Omega$. Let $\varphi \in C_0^\infty(V)$, $\varphi = 1$ in a neighborhood $V_1 \subset\subset V$ of x_0. Let $\rho \in C_0^\infty(\mathbb{R}^n)$, $\rho = 0$ for $|x| > \varepsilon$, $\rho = 1$ near the origin.

a) Prove that the expression

$$v(x) = \langle \rho(x - y)k, P(y, D_y)[\varphi(y)u(y)] \rangle$$

is well defined. Noting that $P(\varphi u) = \varphi Pu$ in V_1, prove that if ε is small enough, v is a C^∞ function in a neighborhood W of x_0.

b) Using the hypothesis, show that $v - \varphi u$ is a C^∞ function in Ω and conclude.

Exercise 106: Singular spectrum of a distribution

Let $u \in \mathscr{D}'(\Omega)$ and (x_0, ξ_0) a point in $\Omega \times \mathbb{R}^n \backslash 0$. We say that (x_0, ξ_0) is not in the singular spectrum of u, for short $(x_0, \xi_0) \notin ss(u)$, if there exist a neighborhood V_{x_0} of x_0, a conic neighborhood Γ_{ξ_0} of ξ_0 such that for every $\varphi \in C_0^\infty(V_{x_0})$

$$\widehat{\varphi u}(\xi) = O(|\xi|^{-N}) \qquad \forall N \in \mathbb{N}, \forall \xi \in \Gamma_{\xi_0}, \ |\xi| \to +\infty.$$

a) Let $\pi: \Omega \times \mathbb{R}^n \to \Omega$ be the projection $(x, \xi) \mapsto x$. Show that

$$\pi ss(u) = \text{sing supp } (u)$$

b) Let $\mathbb{R}^n_+ = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : x_n > 0\}$. Let χ be the characteristic function of $\mathbb{R}^n_+$. Show that $ss(\chi) = A = \{(x', x_n, \xi', \xi_n) : x_n = 0, \xi' = 0, \xi_n \neq 0\}$

Exercise 107*

Let $f \in \mathcal{S}(\mathbb{R}^{n-1})$ and $\rho \in C^\infty(\mathbb{R})$ be such that $\rho = 0$ for $\tau \leqslant 1$, $\rho = 1$ for $\tau \geqslant 2$. For $\xi = (\eta, \tau) \in \mathbb{R}^{n-1} \times \mathbb{R}$ we set $g(\xi) = f\left(\frac{\eta}{\sqrt{\tau}}\right)\rho(\tau)$.

a) Prove that $g \in \mathcal{S}'(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$.

b) Prove that g is rapidly decreasing in a conic neighborhood of the following points: (η_0, τ_0) with $\eta_0 \neq 0$ and $(0, \tau_0)$ with $\tau_0 < 0$.
(Hint: Note that in a small enough conic neighborhood of the first (resp. second) point we have $|\tau| \leqslant C|\eta|$ (resp. $\tau < 0$).

c) Show that for $|\xi|$ big enough

$$|D_\tau^k D_\eta^\alpha g(\xi) \leqslant C_{k\alpha}(1 + |\xi|)^{-k-\frac{|\alpha|}{2}}.$$

(Hint: Distinguish the cases $|\tau| \leqslant |\eta|$ and $|\eta| \leqslant |\tau|$.)

d) We consider the distribution $u = \mathcal{F}^{-1}g$ where $\mathcal{F}^{-1}$ in the inverse Fourier transform. Computing $\mathcal{F}(x^\alpha D^\beta u)$ show that u is C^∞ outside the origin.

e) Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ and $h \in L^\infty(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$. We assume that there exist a point ξ_0 and a conic neighborhood Γ_{ξ_0} of ξ_0 in which h decreases rapidly. Show that $\psi * h$ is rapidly decreasing in a cone $\tilde{\Gamma}_{\xi_0} \subset\subset \Gamma_{\xi_0}$.
(Hint: Use the fact that for $\xi \in \tilde{\Gamma}_{\xi_0}$ and $\zeta \notin \Gamma_{\xi_0}$ we have $|\xi - \zeta| \geqslant \varepsilon|\xi|$).

f) Deduce that

$$ss(u) \subset A = \{(x, \xi): x = 0, \eta = 0, \tau > 0\}.$$

g) Show that $ss(u) = A$.

Exercise 108 (see exercises 106, 94)

a) We consider the distribution on $\mathbb{R}$ defined by

$$u(x) = \int_0^{+\infty} \frac{e^{2i\pi x.\xi}}{(1 + \xi^2)^2} d\xi.$$

Prove that $ss(u) = \{x = 0, \xi > 0\}$.
(Hint: use inequality a) from exercise 94.)

b) For $(t, x) \in \mathbb{R}^2$ we set

$$v(t, x) = \int_0^{+\infty} \frac{e^{-\pi t^2 \xi} e^{2i\pi x.\xi}}{(1 + \xi^2)^2} d\xi.$$

Show that v is a C^1 function solution of the equation $Lu = \left(\frac{\partial}{\partial t} - it\frac{\partial}{\partial x}\right)v = 0.$

c) Deduce that L is not hypoelliptic.

Exercise 109

We give two C^1 functions a and u_0 from $\mathbb{R}$ to $\mathbb{R}$, u_0 with compact support. We consider the non linear Cauchy problem

$$(1) \quad \begin{cases} \dfrac{\partial u}{\partial t}(x, t) + a(u(x, t))\dfrac{\partial u}{\partial x}(x, t) = 0 \\ u(x, 0) = u_0(x) \end{cases}$$

We want to prove that this problem has a C^1 solution u in $\mathbb{R}^2_+ = \{(x, t) \in \mathbb{R}^2, t \geqslant 0\}$ if and only if

$$(2) \quad a'(u_0(x))u_0'(x) \geqslant 0 \qquad \forall x \in \mathbb{R}.$$

a) Show that every solution u of (1) is constant, equal to $u_0(x_0)$, on the curve $(y(t), t)$ where y is the solution of the differential equation

$$(3) \quad \begin{cases} \dfrac{dy}{dt} = a(u(y(t), t)) \\ y(0) = x_0 \end{cases}$$

Deduce that the solution of (3) is $y(t) = x_0 + ta(u_0(x_0))$.

b) For $t \geqslant 0$ we consider the map F_t: $\mathbb{R} \to \mathbb{R}$ defined by $x = F_t(x_0) = x_0 + ta(u_0(x_0))$. Show that this map is a C^1 diffemorphism from $\mathbb{R}$ to $\mathbb{R}$ for every $t \geqslant 0$ if and only if condition (2) is satisfied.

c) Let us assume (2) satisfied ant let $x_0 = G_t(x)$ be the inverse of F_t. We set (4) $u(x, t) = u_0(G_t(x))$. Show that u is a solution of problem (1) in $\mathbb{R}^2_+$.

d) Conversely, let us assume that (2) is not satisfied; u_0 having a compact support the function $a'(u_0(x)).\ u_0'(x)$ reaches a minimum $m < 0$ at a point y_0.

Prove that formula (4) still defines a C^1 solution u of problem (1) for $0 \leqslant t < \dfrac{-1}{m}$, such that $\left|\dfrac{\partial u}{\partial x}\right| \to +\infty$ when $t \to \dfrac{-1}{m}$.

e) Deduce that the largest T such that a C^1 solution exists for $0 \leqslant t < T$ is given by $T = \dfrac{1}{\operatorname*{Max}_{x \in \mathbb{R}} (-a'(u_0(x))u_0(x))}$.

SOLUTIONS OF THE EXERCISES — CHAPTER 7

Solution 90

a) We just have to show that $H^s(\mathbb{R}^n)$ is complete. Let $(f_j)_{j \in \mathbb{N}}$ be a Cauchy sequence in $H^s(\mathbb{R}^n)$; it follows that $\{(1 + |\xi|^2)^{s/2}\hat{f}_j\}_j$ is a Cauchy sequence in $L^2(\mathbb{R}^n)$ which is complete. Therefore $(1 + |\xi|^2)^{s/2}\hat{f}_j \to g$ in $L^2(\mathbb{R}^n)$; $g \in L^2(\mathbb{R}^n) \subset \mathscr{S}'(\mathbb{R}^n)$ and $(1 + |\xi|^2)^{-s/2}g \in \mathscr{S}'(\mathbb{R}^n)$. Then there exists $f \in \mathscr{S}'(\mathbb{R}^n)$ such that $(1 + |\xi|^2)^{-s/2}g = \hat{f}$ (isomorphism of the Fourier transform in $\mathscr{S}'(\mathbb{R}^n)$). So $f \in \mathscr{S}'(\mathbb{R}^n)$ and $(1 + |\xi|^2)^{s/2}\hat{f} = g \in L^2(\mathbb{R}^n)$, i.e. $f \in H^s(\mathbb{R}^n)$ and $(1 + |\xi|^2)^{s/2}\hat{f}_j \to (1 + |\xi|^2)^{s/2}\hat{f}$ in $L^2(\mathbb{R}^n)$, i.e. $f_j \to f$ in $H^s(\mathbb{R}^n)$.

If $s_1 \geq s_2$ we have

$$(1 + |\xi|^2)^{s_2} \leq (1 + |\xi|^2)^{s_1} \qquad \forall \xi \in \mathbb{R}^n$$

so $H^{s_1} \subset H^{s_2}$ with continuous injection.

b) Since $\alpha_1 + \cdots + \alpha_n \leqq m$ we have, setting $(1 + \xi_1^2 + \cdots + \xi_n^2) = A \geq 1$:

$$A^m \geqq A^{\alpha_1}A^{\alpha_2} \cdots A^{\alpha_n}$$

But $A \geqq \xi_i^2$ for all $i = 1, 2, \ldots, n$ so: $A^m \geqq \xi_1^{2\alpha_1} \cdots \xi_n^{2\alpha_n}$ which prove the first inequality.

By the binomial expansion

$$(1+\xi_1^2+\cdots+\xi_n^2)^m = \sum_{p=0}^{m} \sum_{p_1=0}^{p} \sum_{p_2=0}^{p-p_1} \cdots \sum_{p_{n-1}=0}^{p-p_1-\cdots-p_{n-2}} C_{p,p_1,\ldots,p_{n-1}} \xi_1^{2p_1} \cdots \xi_{n-1}^{2p_{n-1}} \xi_n^{2(p-p_1-\cdots-p_{n-1})}$$

where $C_{p,p_1,\ldots,p_{n-1}}$ are constants. Since $p_1 + p_2 + \cdots + (p - p_1 - \cdots - p_{n-1}) = p \leq m$ we have:

$$(1 + |\xi|^2)^m \leqq C\left[1 + \sum_{|\alpha|=1}^{m} \xi_1^{2\alpha_1} \cdots\cdots \xi_n^{2\alpha_n}\right]; \text{ where } \operatorname{Max} C_{p,p_1,\ldots,p_{n-1}} \leqq C$$

which proves the second inequality.

c) Let $u \in H^m(\mathbb{R}^n)$ then $|\hat{u}(\xi)|^2 \leq (1 + |\xi|^2)^m |\hat{u}(\xi)|^2$; so $\hat{u} \in L^2(\mathbb{R}^n)$ and $u \in L^2(\mathbb{R}^n)$. Moreover by b) we have:

$$\sum_{|\alpha| \leq m} \int \xi_1^{2\alpha_1} \cdots \xi_n^{2\alpha_n} |\hat{u}(\xi)|^2 \,\mathrm{d}\xi \leq C_1 \int (1 + |\xi|^2)^m |\hat{u}(\xi)|^2 \,\mathrm{d}\xi$$

Now $\mathscr{F}(D^\alpha u) = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n} \hat{v}$ so by the Parseval formula

$$\sum_{|\alpha| \leq m} \int |D^\alpha u(x)|^2 \,\mathrm{d}x \leq C_1 \|u\|_m^2$$

which proves that $H^m(\mathbb{R}^n) \subset E$ and that $|u|_m^2 \leq C_1 \|u\|_m^2$.
In the same way, by the second inequality

$$\int (1 + |\xi|^2)^m |\hat{u}(\xi)|^2 \,\mathrm{d}\xi \leq C \left[\|u\|_{L^2}^2 + \sum_{1 \leq |\alpha| \leq m} |D^\alpha u(x)|^2 \,\mathrm{d}x \right]$$

i.e. $E \subset H^m(\mathbb{R}^n)$ and $\|u\|_m^2 \leq C |u|_m^2$.

Solution 91

a) Let $u \in H^s(\mathbb{R}^n)$. It is sufficient to prove that $D^\alpha u$, derivatives in the distributions sense, are in $C^0(\mathbb{R}^n)$ for $|\alpha| \leq k$. This will be true if we prove that $D^\alpha u$, for $|\alpha| \leq k$, are the Fourier transform of functions in $L^1(\mathbb{R}^n)$. Now in $\mathscr{S}'(\mathbb{R}^n)$ we have $D^\alpha u = \mathscr{F}[\bar{\mathscr{F}}(D^\alpha u)]$. Let us prove that $\bar{\mathscr{F}}(D^\alpha u) \in L^1(\mathbb{R}^n)$. We have

$$|\bar{\mathscr{F}}(D^\alpha u)| = |\xi^\alpha \hat{u}| = |\xi^\alpha|(1 + |\xi|^2)^{-s/2}(1 + |\xi|^2)^{s/2}|\hat{u}|$$

and by the Hölder inequality

$$\int |\bar{\mathscr{F}}(D^\alpha u)(\xi)| \,\mathrm{d}\xi \leq \left(\int |\xi|^2 (1 + |\xi|^2)^{-s} \,\mathrm{d}\xi \right)^{1/2} \left(\int (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 \,\mathrm{d}\xi \right)^{1/2}$$

Now $|\xi^\alpha|^2 = \xi_1^{2\alpha_1} \cdots \xi_n^{2\alpha_n} \leq (1 + |\xi|^2)^{|\alpha|} \leq (1 + |\xi|^2)^k$ if $|\alpha| \leq k$ and we have $\dfrac{1}{(1 + |\xi|^2)^{s-k}} \in L^1(\mathbb{R}^n)$ if $2(s - k) > n$, i.e. $s > \dfrac{n}{2} + k$.

b) If $s > \dfrac{n}{2} + k$ we have $H^s_{\mathrm{loc}}(\Omega) \subset C^k(\Omega)$. Indeed let $x_0 \in \Omega$ and V_{x_0} be a neighborhood of x_0. Let $\varphi \in \mathscr{D}(\Omega)$, $\varphi = 1$ on V_{x_0} and $u \in H^s_{\mathrm{loc}}(\Omega)$. Then $\varphi u \in H^s(\mathbb{R}^n)$ and by a) $\varphi u \in C^k(\mathbb{R}^n)$; since $\varphi = 1$ on V_{x_0} $u \in C^k(V_{x_0})$ for all V_{x_0} which proves that $u \in C^k(\Omega)$. It follows that

$$\bigcap_{s \in \mathbf{R}} H^s_{\mathrm{loc}}(\Omega) \subset C^\infty(\Omega)$$

If $u \in C^\infty(\Omega)$ and $\varphi \in \mathscr{D}(\Omega)$ then $\varphi u \in \mathscr{D}(\mathbb{R}^n) \subset \mathscr{S}(\mathbb{R}^n) \subset H^s(\mathbb{R}^n)$ for all s.

Indeed if $\varphi u \in \mathcal{S}(\mathbb{R}^n)$, $(1 + |\xi|^2)^{p/2}|(\widehat{\varphi u})(\xi)| \leqq C_p$ for all $p \in \mathbb{N}$ and all $\xi \in \mathbb{R}^n$.

$$\int (1 + |\xi|^2)^s|(\widehat{\varphi u})(\xi)|^2\,d\xi \leq \int (1 + |\xi|^2)^p|(\widehat{\varphi u})(\xi)|^2(1 + |\xi|^2)^{s-p}\,d\xi$$

so

$$\|\varphi u\|_s^2 \leqq C_p^2 \int (1 + |\xi|^2)^{s-p}\,d\xi < +\infty$$

if $2(p - s) > n$, i.e. if $p > \frac{n}{2} + s$.

Therefore $C^\infty(\Omega) \subset \bigcap_{s\in\mathbf{R}} H^s_{\mathrm{loc}}(\Omega)$.

Finally $\bigcap_{s\in\mathbf{R}} H^s_{\mathrm{loc}}(\Omega) \subset \bigcap_{k\in\mathbb{N}} H^k_{\mathrm{loc}}(\Omega)$ and we have the inverse inclusion. Indeed if $u \in \bigcap_{k\in\mathbb{N}} H^k_{\mathrm{loc}}$ and $s \in \mathbb{R}$, let k_0 be an integer such that $k_0 > s$; by exercise 90 we have

$$H^{k_0}_{\mathrm{loc}}(\Omega) \subset H^s_{\mathrm{loc}}(\Omega) \quad \text{so } u \in \bigcap_{s\in\mathbf{R}} H^s_{\mathrm{loc}}(\Omega)$$

Solution 92

Let $\varphi \in \mathcal{D}(U)$ and $K = \operatorname{supp}\varphi$. Let us show first that $D^\alpha_x(\varphi f)$ and $D^\beta_y(\varphi f)$ are in $L^2(\mathbb{R}^p \times \mathbb{R}^q)$. Indeed $\operatorname{supp} D^\alpha_x(\varphi f) \subset K$, $\operatorname{supp} D^\beta_y(\varphi f) \subset K$ and:

$$D^\alpha_x(\varphi f) = \sum_{\alpha' \leq \alpha} C^{\alpha'}_\alpha D^{\alpha-\alpha'}_x \varphi D^{\alpha'}_x f \quad \text{(also for } D^\beta_y(\varphi f))$$

so

$$\sup_K |D^\alpha_x(\varphi f)| \leq C \sum_{\alpha' \leq \alpha} \sup_K |D^{\alpha'}_x f| \leq C'_\alpha$$

by the second hypothesis. Therefore $D^\alpha_x(\varphi f) \in L^\infty \cap \mathcal{E}' \subset L^2(\mathbb{R}^p \times \mathbb{R}^q)$ (also for $D^\beta_y(\varphi f)$). By Fourier transform we get for all α and β

$$\xi^\alpha(\widehat{\varphi f}) \in L^2(\mathbb{R}^p \times \mathbb{R}^q), \qquad \eta^\beta(\widehat{\varphi f}) \in L^2(\mathbb{R}^p \times \mathbb{R}^q)$$

Since $|\xi|^i$ (resp. $|\eta|^j$) are finite linear combinations of expressions such as ξ^α (resp. η^β) it follows that

(3) $|\xi|^i(\widehat{\varphi f}) \in L^2(\mathbb{R}^p \times \mathbb{R}^q)$ and $|\eta|^j(\widehat{\varphi f}) \in L^2(\mathbb{R}^p \times \mathbb{R}^q)$ for all $i, j \in \mathbb{N}$

Moreover

$$(1 + |\xi|^2 + |\eta|^2)^k = \sum_{n=0}^{k} C^n_k(|\xi|^2 + |\eta|^2)^n = \sum_{n=0}^{k}\sum_{m=0}^{n} C^n_k C^m_n |\xi|^{2m}|\eta|^{2(n-m)}$$

From the inequality $a \cdot b \leq \frac{1}{2}(a^2 + b^2)$ we get

$$(4) \quad (1 + |\xi|^2 + |\eta|^2)^k \leq \frac{1}{2}\left(\sum_{n=0}^{k} \sum_{m=0}^{n} C_k^n C_n^m |\xi|^{4m} + \sum_{n=0}^{k} \sum_{m=0}^{n} C_k^n C_n^m |\eta|^{4(n-m)} \right)$$

From (3) and (4) we deduce that $(1 + |\xi|^2 + |\eta|^2)^{k/2}(\widehat{\varphi f}) \in L^2(\mathbb{R}^p \times \mathbb{R}^q)$, so $\varphi f \in H^k(\mathbb{R}^p \times \mathbb{R}^q)$ for all $\varphi \in \mathscr{D}(U)$, i.e. $f \in H^k_{\mathrm{loc}}(U)$ for all $k \in \mathbb{N}$.
Now by exercise 91, $\bigcap_{k \in \mathbb{N}} H^k_{\mathrm{loc}}(U) = C^\infty(U)$; so $f \in C^\infty(U)$. Q.E.D.

Solution 93

a) If T has compact support, $\hat{T}$ is a C^∞ function and

$$\hat{T}(\xi) = \langle T, e^{-2i\pi\langle x,\xi\rangle} \rangle$$

Since T has finite order there exist a compact subset K of $\mathbb{R}^n$ and $C > 0$ such that

$$|\hat{T}(\xi)| \leq C \sum_{|\alpha| \leq k} \sup_K |D^\alpha e^{-2i\pi\langle x,\xi\rangle}| \leq C_\alpha \sum_{|\alpha| \leq k} |\xi^\alpha|$$

Now

$$|\xi^\alpha|^2 = \xi_1^{2\alpha_1} \cdots \xi_n^{2\alpha_n} \leqq (1 + \xi_1^2 + \cdots + \xi_n^2)^k \quad \text{since } \alpha_1 + \cdots + \alpha_n \leq k$$

It follows that

$$|\hat{T}(\xi)|^2 \leqq C\left(\sum_{|\alpha| \leq k} |\xi^\alpha| \right)^2 \leqq C' \sum_{|\alpha| \leq k} |\xi^\alpha|^2 \leqq C''(1 + |\xi|^2)^k$$

and

$$(1 + |\xi|^2)^s |\hat{T}(\xi)|^2 \leqq C''(1 + |\xi|^2)^{s+k}$$

The function $(1 + |\xi|^2)^{s+k}$ is in $L^1(\mathbb{R}^n)$ if $-2(s + k) > n$, i.e. $s < -\frac{n}{2} - k$.

b) Since every distribution with compact support has a finite order it follows from a) that for each $T \in \mathscr{E}'(\mathbb{R}^n)$ there exists $s \in \mathbb{R}$ such that $T \in H^s(\mathbb{R}^n)$ so $\mathscr{E}' \subset \bigcup_{s \in \mathbb{R}} H^s(\mathbb{R}^n)$.

Solution 94

a) First of all $\dfrac{1 + |\xi|^2}{1 + |\xi - \eta|^2} \leq 2(1 + |\eta|^2)$. Indeed

$|\xi| \leq |\xi - \eta| + |\eta|$ so $|\xi|^2 \leq |\xi - \eta|^2 + |\eta|^2 + 2|\xi - \eta| \cdot |\eta| \leq 2(|\xi - \eta|^2 + |\eta|^2)$

since $2ab \le a^2 + b^2$. It follows that

$$\frac{1+|\xi|^2}{1+|\xi-\eta|^2} \le \frac{1+2|\xi-\eta|^2+2|\eta|^2}{1+|\xi-\eta|^2} \le 2\frac{1+|\xi-\eta|^2}{1+|\xi-\eta|^2} + 2\frac{|\eta|^2}{1+|\xi-\eta|^2}$$
$$\le 2 + 2|\eta|^2 = 2(1+|\eta|^2)$$

Which proves the inequality for $s > 0$.

For $s < 0$ the proof is the same.

b) Let $\psi \in \mathscr{D}(\mathbb{R}^n)$ and $u \in H^s(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$. Then $\psi u \in L^2(\mathbb{R}^n)$ and $\widehat{\psi u} = \hat{\psi} * \hat{u}$; here $\psi \in \mathscr{S}$ so $\hat{\psi} \in L^1$, $\hat{u} \in L^2$ so the convolution is well defined. Moreover

$$\|\psi u\|^2_{H^s} = \int_{\mathbb{R}^n} (1+|\xi|^2)^s |\widehat{\psi u}(\xi)|^2 \, d\xi = \int_{\mathbb{R}^n} (1+|\xi|^2)^s \left| \int \hat{\psi}(\eta)\hat{u}(\xi-\eta)\, d\eta \right|^2 d\xi$$

$$\|\psi u\|_{H^s} \le \int_{\mathbb{R}^n} (1+|\xi|^2)^s \left[\int |\hat{\psi}(\eta)|\, d\eta \right]\left[\int |\hat{\psi}(\eta)| \cdot |\hat{u}(\xi-\eta)|^2 \, d\eta \right] d\xi$$

(We have applied the Hölder inequality to $|\hat{\psi}(\eta)|^{1/2} \cdot |\hat{\psi}(\eta)|^{1/2} \cdot |\hat{u}(\xi-\eta)|$.)
So

$$\|\psi u\|^2_{H^s} \le \|\hat{\psi}\|_{L^1} 2^s \iint (1+|\xi-\eta|^2)^s (1+|\eta|^2)^s |\hat{\psi}(\eta)| \cdot |\hat{u}(\xi-\eta)|^2 \, d\eta \, d\xi$$

Using the inequality proved in a) and the Fubini theorem which can be applied here since all functions are positive. Then

$$\|\psi u\|^2_{H^s} \le \|\hat{\psi}\|_{L^1} \cdot \left(\int (1+|\eta|^2)^s |\hat{\psi}(\eta)| \right)\left(\int (1+|\xi-\eta|^2)^s |\hat{u}(\xi-\eta)|^2 \, d\xi \right) d\eta$$

so

$$\|\psi u\|^2_{H^s} \le \|\hat{\psi}\|_{L^1} \cdot \left(\int (1+|\eta|^2)^s |\hat{\psi}(\eta)|\, d\eta \right) \cdot \|u\|^2_{H^s} \qquad \text{Q.E.D}$$

Solution 95

a) Let us prove that $-\Delta + k^2$ is a continuous operator from $H^{s+2}(\mathbb{R}^n)$ to $H^s(\mathbb{R}^n)$. Let $u \in H^{s+2}(\mathbb{R}^n) \subset \mathscr{S}'(\mathbb{R}^n)$; $\mathscr{F}(-\Delta u + k^2 u) = (4\pi^2|\xi|^2 + k^2)\hat{u} \in \mathscr{S}'(\mathbb{R}^n)$ so

$$(1) \quad (1+|\xi|^2)^{s/2} |\mathscr{F}(-\Delta u + k^2 u)| = (1+|\xi|^2)^{s/2}(4\pi^2|\xi|^2 + k^2)|\hat{u}|$$

and

$$(2) \quad (1+|\xi|^2)^{s/2}(4\pi^2|\xi|^2 + k^2) \le \operatorname{Max}(4\pi^2, k^2)(1+|\xi|^2)^{(s/2)+1}$$

Since $u \in H^{s+2}(\mathbb{R}^n)$, $(1+|\xi|^2)^{(s/2)+1}\hat{u} \in L^2(\mathbb{R}^n)$ so $(1+|\xi|^2)^{s/2}\mathscr{F}(-\Delta u + k^2 u)$ is in $L^2(\mathbb{R}^n)$, i.e. $-\Delta u + k^2 u \in H^s(\mathbb{R}^n)$. Moreover by (1) and (2)

$$\|-\Delta u + k^2 u\|^2_{H^s} \le C\|u\|^2_{H^{s+2}}$$

b) $-\Delta + k^2$ is a bijective operator from $H^{s+2}(\mathbb{R}^n)$ to $H^s(\mathbb{R}^n)$.
It is injective for $u \in H^{s+2}$ and $-\Delta u + k^2 u = 0 \Rightarrow (4\pi^2|\xi|^2 + k^2)\hat{u} = 0$ in $\mathscr{S}'(\mathbb{R}^n)$ but $(4\pi^2|\xi|^2 + k^2) \neq 0$ so $\hat{u} = 0$ and $u = 0$ since the Fourier transform is an isomorphism in $\mathscr{S}'(\mathbb{R}^n)$.
It is surjective. Indeed let $f \in H^s(\mathbb{R}^n)$ then $v = (4\pi^2|\xi|^2 + k^2)^{-1}\hat{f} \in \mathscr{S}'(\mathbb{R}^n)$, i.e. $(4\pi^2|\xi|^2 + k^2)v = \hat{f}$. By inverse Fourier transform we get

$$(-\Delta + k^2)\bar{\mathscr{F}}v = f$$

Then $u = \bar{\mathscr{F}}v \in H^{s+2}(\mathbb{R}^n)$ and $(-\Delta + k^2)u = f$. Indeed

$$(1 + |\xi|^2)^{(s/2)+1}|\hat{u}| = \frac{(1 + |\xi|^2)^{(s/2)+1}}{(4\pi^2|\xi|^2 + k^2)}|\hat{f}| \leq C(1 + |\xi|^2)^{s/2}|\hat{f}|$$

since $\dfrac{1 + |\xi|^2}{4\pi^2|\xi|^2 + k^2} \leq \operatorname{Max}\left(\dfrac{1}{4\pi^2}, \dfrac{1}{k^2}\right)$.

Therefore $-\Delta + k^2$ is a continuous bijective linear operator from $H^{s+2}(\mathbb{R}^n)$ to $H^s(\mathbb{R}^n)$ which are Banach spaces. The continuity of its inverse follows then from the Banach theorem. Therefore $-\Delta + k^2$ is an isomorphism from $H^{s+2}(\mathbb{R}^n)$ to $H^s(\mathbb{R}^n)$.

Solution 96

a) Using the Fourier transform we see that (1) is equivalent to

$$1 = (1 + 4\pi^2|\xi|^2)^N \hat{F}$$

If N is big enough $(4N > n)$ the function $\dfrac{1}{(1 + |\xi|^2)^N}$ is in $L^2(\mathbb{R}^n)$.
Then we take $F = \bar{\mathscr{F}}[(1 + 4\pi^2|\xi|^2)^{-N}] \in L^2(\mathbb{R}^n)$.

b) Let $\varphi \in \mathscr{D}(\mathbb{R}^n)$ such that $\varphi(0) = 1$. Then $\varphi \cdot \delta = \delta$ since $\langle \varphi\delta, \psi \rangle = \langle \delta, \varphi\psi \rangle = \psi(0)\varphi(0) = \psi(0) = \langle \delta, \psi \rangle$ for all $\psi \in \mathscr{D}(\mathbb{R}^n)$.
It follows from (1) that

$$\delta = \varphi \sum_{|\alpha| \leq 2N} a_\alpha D^\alpha F \text{ where } F \in L^2(\mathbb{R}^n)$$

By the Leibniz Formula we can write

$$\sum_{|\alpha| \leq 2N} a_\alpha D^\alpha(\varphi F) = \sum_{|\alpha| \leq 2N} a_\alpha \sum_{\beta \leq \alpha} C_\alpha^\beta D^\beta \varphi D^{\alpha - \beta} F$$

$$\sum_{|\alpha| \leq 2N} a_\alpha D^\alpha(F) = \varphi \sum_{|\alpha| \leq 2N} a_\alpha D^\alpha F + \sum_{|\alpha| \leq 2N} a_\alpha \sum_{\substack{\beta \leq \alpha \\ \beta \neq 0}} a_\alpha C_\alpha^\beta D^\beta \varphi D^{\alpha - \beta} F$$

So we have

$$\varphi \sum_{|\alpha| \le 2N} a_\alpha D^\alpha F = \sum_{|\alpha| \le 2N} a_\alpha D^\alpha(\varphi F) + \sum_{|\alpha| \le 2N-1} \varphi^1_\alpha D^\alpha F \quad \text{where } \varphi^1_\alpha \in \mathscr{D}(\mathbb{R}^n)$$

Now

$$\varphi^1_\alpha D^\alpha F = D^\alpha(\varphi^1_\alpha F) + \sum_{0<\beta\le\alpha} C^\beta_\alpha D^\alpha \varphi^1_\alpha D^{\alpha-\beta} F$$

so

$$\sum_{|\alpha|\le 2N-1} \varphi^1_\alpha D^\alpha F = \sum_{|\alpha|\le 2N-1} D^\alpha(\varphi^1_\alpha F) + \sum_{|\alpha|\le 2N-2} \varphi^2_\alpha D^\alpha F$$

At each step the remaining term has a lower order so at the k^{th} step we shall have

$$\delta = \sum_{|\alpha|\le 2N} a_\alpha D^\alpha(\varphi F) + \sum_{|\alpha|\le 2N-1} D^\alpha(\varphi^1_\alpha F) + \sum_{|\alpha|\le 2N-2} D^\alpha(\varphi^2_\alpha)F + \cdots + \sum_{|\alpha|\le 2N-k} \varphi^k_\alpha D^\alpha F$$

If we continue until $k = 2N$ the last term will be equal to $\varphi^{2N}_\alpha F$, so setting $a_\alpha\varphi = \varphi^0_\alpha$ we get

$$\delta = \sum_{k=0}^{2N} \sum_{|\alpha|\le 2N-k} D^\alpha(\varphi^k_\alpha F)$$

and $\varphi^k_\alpha \in \mathscr{D}(\mathbb{R}^n)$ so $\varphi^k_\alpha F \in L^2(\mathbb{R}^n) \cap \mathscr{E}'$.

c) From (2) and (3) we get

$$D^\beta u = D^\beta u * \delta = \sum_{|\alpha|\le m} (D^\beta u) * (D^\alpha f_\alpha) = \sum_{|\alpha|\le m} [(D^\alpha D^\beta u) * f_\alpha] \in L^2_{\text{loc}}$$

from which we conclude that for all $\psi \in \mathscr{D}(\mathbb{R}^n)$, $\psi D^\beta u \in L^2(\mathbb{R}^n)$. But by exercise 7, this is equivalent to say that $D^\beta \psi u \in L^2(\mathbb{R}^n)$ for all β i.e. $u \in H^k_{\text{loc}}(\mathbb{R}^n)$ for all $k \in \mathbb{N}$. By exercise 91 $\bigcap_{k\in\mathbb{N}} H^k_{\text{loc}}(\mathbb{R}^n) = C^\infty(\mathbb{R}^n)$. Q.E.D.

Solution 97

a) We have $|P_m(\xi)| \ge C$ if $|\xi| = 1$ since $\{|\xi| = 1\}$ is compact and P is a non vanishing continuous function. Let $\xi \in \mathbb{R}^n\backslash\{0\}$ then $\dfrac{\xi}{|\xi|} \in S$ so

$$\frac{1}{|\xi|^m}|P_m(\xi)| = \left|P_m\left(\frac{\xi}{|\xi|}\right)\right| \ge C$$

(If $\xi = 0$ the inequality is still true since both sides vanish).
Moreover

$$P(\xi) = P_m(\xi) + P_{m-1}(\xi) + \cdots + P_0(\xi) \quad \text{where } P_k(\xi) = \sum_{|\alpha|=k} a_\alpha \xi^\alpha$$

and

$$|P_k(\xi)| \leqq \sum_{|\alpha|=k} |a_\alpha||\xi^\alpha| \leqq \sum_{|\alpha|=k} |a_\alpha||\xi|^{|\alpha|} = C_k|\xi|^k$$

Then

$$|P(\xi)| \geqq |P_m(\xi)| - |P_{m-1}(\xi)| - \cdots - |P_0(\xi)| \geqq C|\xi|^m - C'(|\xi|^{m-1} + \cdots + 1)$$

But if $|\xi| > R \geq 1$ and $k = 0, \ldots, m - 1$ we have $|\xi|^k \leqq \frac{1}{R}|\xi|^m$ so

$$|P(\xi)| \geqq C|\xi|^m - \frac{mC'}{R}|\xi|^m \geqq \frac{C}{2}|\xi|^m$$

if $\frac{mC'}{R} \leq \frac{C}{2}$, i.e. if R is big enough.

b) Since $\chi = 1$ for $|\xi| \leq R$, function $\frac{1-\chi(\xi)}{P(\xi)}$ defines a distribution. Moreover

$$\left|\frac{1-\chi(\xi)}{P(\xi)}\right| \leqq \frac{2}{K|\xi|^m} \leqq \frac{2}{KR^m}$$

since $|\xi| > R$.

So $\frac{1-\chi}{P} \in L^\infty(\mathbb{R}^n) \subset \mathscr{S}'(\mathbb{R}^n)$. Therefore we can find $E \in \mathscr{S}'(\mathbb{R}^n)$ such that

$$\hat{E} = \frac{1-\chi(\xi)}{P(\xi)}$$

We deduce that

$$P(\xi)\hat{E} = 1 - \chi(\xi)$$

and by inverse Fourier transform

$$P(D)E = \delta - \omega$$

where $\omega = \hat{\chi} \in \mathscr{S}(\mathbb{R}^n)$ since $\chi \in \mathscr{D}(\mathbb{R}^n) \subset \mathscr{S}(\mathbb{R}^n)$.

c) Let $\beta, \alpha \in \mathbb{N}^n$. The Fourier transform of $D^\beta x^\alpha E$ is

$$\xi^\beta(-D)^\alpha \hat{E} = \xi^\beta(-D)^\alpha \frac{1-\chi(\xi)}{P(\xi)}$$

It is easy to see that

$$|\xi^\beta(-D)^\alpha \hat{E}| \leqq C|\xi|^{|\beta|} \cdot |\xi|^{-m-|\alpha|} \quad \text{for } |\xi| > R$$

(This expression vanishes for $|\xi| \leq R$ since $\chi = 1$.)
If $m + |\alpha| - |\beta| > n$ the function $\frac{1}{|\xi|^{m+|\alpha|-|\beta|}}$ is in $L^1(\mathbb{R}^n)$ for $|\xi| \geq R$.
So if $m + |\alpha| > n + |\beta|$, $\xi^\beta(-D)^\alpha \hat{E} \in L^1(\mathbb{R}^n)$ and since

$$D^\beta x^\alpha E = \mathscr{F}(\xi^\beta(-D)^\alpha \hat{E})$$

we conclude that $D^\beta x^\alpha E \in C^0(\mathbb{R}^n)$. (Fourier transform of a function in $L^1(\mathbb{R}^n)$.)
So, for a given k we choose α such that $|\alpha| + m \geq n + k$. Then we shall have $x^\alpha E \in C^k(\mathbb{R}^n)$.

d) From question c) we deduce that $E \in C^\infty(\mathbb{R}^n\backslash\{0\})$. (Indeed we just have to take α such that $x^\alpha \neq 0$ if $x \neq 0$.)
Let $\varphi \in \mathscr{D}(\mathbb{R}^n)$ such that $\varphi = 1$ for $|x| \leqslant 1$. Using the same method as in exercise 60 we show that

$$P(\varphi E) = \varphi PE + \psi$$

where $\psi \in \mathscr{D}(\mathbb{R}^n)$.
Since $PE = \delta + \omega$ and $\varphi\delta = \delta$, we get

$$P(\varphi E) = \delta + \varphi\omega + \psi$$

so

$$u = u * \delta = u * P(\varphi E) - u * (\varphi\omega) - u * \psi$$

$$u = Pu * \varphi E - u * (\varphi\omega) - u * \psi \in C^\infty$$

since $Pu \in C^\infty$, $\varphi\omega \in \mathscr{D}(\mathbb{R}^n)$ and $\psi \in \mathscr{D}(\mathbb{R}^n)$.

Solution 98

1°) a) Let $g(z) = \frac{f(z)}{z - \lambda}$. If $|\lambda| > 1$ we have $|g(0)| = \frac{|f(0)|}{|\lambda|} < |f(0)|$.
By the Cauchy formula we have, setting $\partial D = \{z \in \mathbb{C} : |z| = 1\}$:

$$f(0) = \frac{1}{2i\pi} \int_{\partial D} \frac{f(z)}{z} \mathrm{d}z = \frac{1}{2\pi} \int_0^{2\pi} f(\mathrm{e}^{i\theta}) \, \mathrm{d}\theta$$

so

$$|f(0)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(\mathrm{e}^{i\theta})| \, \mathrm{d}\theta$$

If $0 < |\lambda| \leq 1$, let us consider the following function $h \in H_1$:

$$h(z) = (1 - \bar{\lambda}z)g(z)$$

Then

$$h(0) = g(0)$$

and

$$|h(z)| = \left|\frac{1 - \bar{\lambda}z}{-\lambda}\right| |f(z)| = |f(z)| \quad \text{for } |z| = 1$$

Using the Cauchy formula we get

$$|g(0)| = |h(0)| \leq \frac{1}{2\pi}\int_0^{2\pi} |h(e^{i\theta})|\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})|\,d\theta$$

If $\lambda = 0$ we apply the Cauchy formula to g. We get

$$|g(0)| \leqq \frac{1}{2\pi}\int_0^{2\pi} |g(e^{i\theta})|\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})|\,d\theta$$

Let us assume $\rho \neq 1$ and let us consider the functions $f(\rho z)$ and $\dfrac{1}{\rho}q(\rho z)$ which are in H_1. Then

$$\rho|g(0)| \leq \frac{1}{2\pi}\int_0^{2\pi} |f(\rho\, e^{i\theta})|\,\delta\theta \qquad \text{Q.E.D.}$$

b) Let us set for $\xi + i\eta \in \mathbb{C}^n$

$$(2) \quad \hat{\varphi}(\xi + i\eta) = \int_{\mathbb{R}^n} e^{-i\langle x, \xi + i\eta\rangle}\varphi(x)\,dx = \int_{|x| \leq a} e^{-i\langle x, \xi\rangle} e^{\langle x, \eta\rangle}\varphi(x)\,dx$$

This formula makes sense for every $\xi + i\eta \in \mathbb{C}^n$ since we integrate a continuous function on a compact set. If $\eta = 0$ we recover the Fourier transform of φ. Finally we can derive under the integral sign with respect to ξ and η and we get

$$\left(\frac{\partial}{\partial \xi_j} + i\frac{\partial}{\partial \eta_j}\right)\hat{\varphi}(\xi + i\eta) = \int_{|x| \leq a} (-ix_j + ix_j)\,e^{-i\langle x, \xi\rangle} e^{\langle x, \eta\rangle}\varphi(x)\,dx = 0 \qquad j = 1, \ldots, n$$

Therefore $\hat{\varphi}$ is holomorphic in $\mathbb{C}^n$. Moreover

$$|\hat{\varphi}(\xi + i\eta)| \leq e^{a|\eta|}\int_{|x| \leq a} |\varphi(x)|\,dx = C e^{a|\eta|}$$

for $\langle x, \eta\rangle = \sum_{j=1}^{n} x_j\eta_j \leq |x| \cdot |\eta| \leq a \cdot |\eta|$.

c) We have $\widehat{P(D)\varphi}(\xi) = (P\hat{\varphi})(\xi)$. By formula (2) we get

$$P(\xi + i\eta)\hat{\varphi}(\xi + i\eta) = \int_{\mathbb{R}^n} e^{-i\langle x, \xi\rangle} e^{\langle x, \eta\rangle} P(D)\varphi(x)\,dx$$
$$= \mathscr{F}_x[e^{\langle x, \eta\rangle} P(D)\varphi]$$

Therefore

$$\bar{\mathscr{F}}_\xi[P(\xi + i\eta)\hat{\varphi}(\xi + i\eta)] = e^{\langle x, \eta\rangle} P(D)\varphi(x)$$

2°) a) We have $P(\xi) = (\xi_1 - \lambda_1(\xi'))(\xi_1 - \lambda_2(\xi')) \cdots (\xi_1 - \lambda_m(\xi'))$ since P is a polynomial in ξ_1 of order m with coefficients depending on $\xi' \in \mathbb{R}^{n-1}$.
It follows that

$$\frac{P_1'(\xi)}{P(\xi)} = \sum_{j=1}^{m} \frac{1}{\xi_1 - \lambda_j(\xi')}$$

Therefore

$$P_1'(\xi)\hat{\varphi}(\xi) = \frac{P_1'(\xi)}{P(\xi)} P(\xi)\hat{\varphi}(\xi) = \sum_{j=1}^{m} \frac{P(\xi)\hat{\varphi}(\xi)}{\xi_1 - \lambda_j(\xi')}$$

So we shall have

$$P_1'(\xi_1 + i\eta_1 + z, \xi' + i\eta')\hat{\varphi}(\xi_1 + i\eta_1 + z, \xi' + i\eta') =$$
$$= \sum_{j=1}^{m} \frac{P(\xi_1 + i\eta_1 + z, \xi' + i\eta')\hat{\varphi}(\xi_1 + i\eta_1 + z, \xi' + i\eta')}{\xi_1 + i\eta_1 + z - \lambda_j(\xi' + i\eta')}$$

For every j, $1 \leq j \leq m$, the functions.

$$f(z) = P(\xi_1 + i\eta_1 + z, \xi' + i\eta')\hat{\varphi}(\xi_1 + i\eta_1 + z, \xi' + i\eta')$$
$$q_j(z) = z + \xi_1 + i\eta_1 - \lambda_j(\xi' + i\eta')$$

$g_j(z) = \dfrac{f(z)}{g_j(z)}$ are holomorphic functions in every set $D_\rho = \{z \in \mathbb{C} : |z| < \rho\}$.
Applying, for every j, the estimate found in question 1°) a), in $D_{\rho/2}$ we get

$$|P_1'(\xi + i\eta)\hat{\varphi}(\xi + i\eta)| \leq$$
$$\leq C_\rho \int_0^{2\pi} \left| P\left(\xi_1 + i\eta_1 + \frac{\rho}{2}e^{i\theta}, \xi' + i\eta'\right)\hat{\varphi}\left(\xi_1 + i\eta_1 + \frac{\rho}{2}e^{i\theta}, \xi' + i\eta'\right)\right| d\theta$$

where the constant C is independant of ξ and η. It follows from the Cauchy–Schwarz inequality that

$$|P_1'(\xi + i\eta)\hat{\varphi}(\xi + i\eta)|^2 \leq$$
$$\leq C_\rho' \int_0^{2\pi} \left| P\left(\xi_1 + i\eta_1 + \frac{\rho}{2}e^{i\theta}, \xi' + i\eta'\right)\hat{\varphi}\left(\xi_1 + i\eta_1 + \frac{\rho}{2}e^{i\theta}, \xi' + i\eta'\right)\right|^2 d\theta$$

Integrating in ξ we get

$$\int_{\mathbb{R}^n} |P_1'(\xi + i\eta)\hat{\varphi}(\xi + i\eta)|^2 \, d\xi \leqq$$

$$\leqq C_\rho' \int_0^{2\pi} \left(\int_{\mathbb{R}^n} \left| P\left(\xi_1 + i\eta_1 + \frac{\rho}{2} e^{i\theta}, \xi' + i\eta'\right) \hat{\varphi}\left(\xi_1 + i\eta_1 + \frac{\rho}{2} e^{i\theta}, \xi' + i\eta'\right) \right|^2 d\xi \right) d\theta$$

In the integral with respect to ξ_1 in the right hand side, let us perform the change of variable $\xi_1' = \xi_1 + \frac{\rho}{2}\cos\theta$. We get

$$\int |P_1'(\xi + i\eta)\hat{\varphi}(\xi + i\eta)|^2 \, d\xi \leqq$$

$$\leqq C_\rho' \int_0^{2\pi} \int \left| P\left(\xi_1' + i\eta_1 + i\frac{\rho}{2}\sin\theta, \xi' + i\eta'\right) \hat{\varphi}\left(\xi_1' + i\eta_1 + i\frac{\rho}{2}\sin\theta, \xi' + i\eta'\right) \right|^2 d\xi \, d\theta$$

so

$$\sup_{|\eta| \leq \rho/2} \int |P_1'(\xi + i\eta)\hat{\varphi}(\xi + i\eta)|^2 \, d\xi \leq C_\rho \sup_{|\eta| \leq \rho} \int |P(\xi + i\eta)\hat{\varphi}(\xi + i\eta)|^2 \, d\xi$$

c) By the same argument as used in question 2°) c) we prove that

$$\sup_{|\eta| \leq \rho/2} \int \left| \frac{\partial P}{\partial \xi_j}(\xi + i\eta)\hat{\varphi}(\xi + i\eta) \right|^2 d\xi \leq C_\rho \sup_{|\eta| \leq \rho} \int |P(\xi + i\eta)\hat{\varphi}(\xi + i\eta)|^2 \, d\xi$$

i.e. the case $|\alpha| = 1$. Taking $\frac{\partial P}{\partial \xi_j}$ instead of P we prove the inequality with $|\alpha| = 2$ and $\frac{\rho}{4}$ instead of $\frac{\rho}{2}$ and so on. We can take $A_\alpha = 2^{|\alpha|}$.

d) There exists α, $|\alpha| = m$ such that $\left(\frac{\partial}{\partial \xi}\right)^\alpha P = C^{te} \neq 0$. For this α we obtain using 2°) c)

$$\sup_{|\eta| \leq \rho/2^m} \int |\hat{\varphi}(\xi + i\eta)|^2 \, d\xi \leqq C_\rho \sup_{|\eta| \leq \rho} \int |P(\xi + i\eta)\hat{\varphi}(\xi + i\eta)|^2 \, d\xi$$

The function $\xi \rightsquigarrow P(\xi + i\eta)\hat{\varphi}(\xi + i\eta)$ is in $\mathscr{S}(\mathbb{R}^n)$. Using the Parseval formula, the question 1°) c) and the following inequality.

$$|\hat{\varphi}(\xi)|^2 \leq \sup_{|\eta| < \rho/2^m} |\hat{\varphi}(\xi + i\eta)|^2$$

we get

$$\|\varphi\|_{L^2(\mathbb{R}^n)} \leq C_\rho \sup_{|\eta|\leq\rho} \|e^{\langle x,\eta\rangle} P(D)\varphi\|_{L^2(\mathbb{R}^n)} \leq C_\rho \| e^{\rho|x|} P(D)\varphi\|_{L^2(\mathbb{R}^n)}$$

3°) a) We have

$$|G(e^{\rho|x|}\bar{P}(D)\varphi)| = |(f, \varphi)| \leqq \|f\|_{L^2} \cdot \|\varphi\|_{L^2} \leq \|f\|_{L^2} \cdot C_\rho \|e^{\rho|x|} P(D)\varphi\|_{L^2}$$

using 2°) d) for the operator $\bar{P}(D)$.

b) G is continuous on E considered as a subspace of $L^2(\mathbb{R}^n)$. By the Hahn–Banach theorem G can be extended to a continuous linear form on $L^2(\mathbb{R}^n)$. Therefore there exists $h \in L^2(\mathbb{R}^n)$ such that $G(u) = (u, h)$, $\forall u \in L^2$. Then

$$G(e^{\rho|x|}\bar{P}(D)\varphi) = (e^{\rho|x|}\bar{P}(D)\varphi, h)$$

c) Let $\rho > 0$ and $f \in L^2(\mathbb{R}^n)$, from a) and b) we have

$$G(e^{\rho|x|}\bar{P}(D)\varphi) = (\varphi, f) = (e^{\rho|x|}\bar{P}(D)\varphi, h) = (\varphi, P(D)e^{\rho|x|}h) \qquad \forall\varphi \in \mathscr{D}$$

So we have $\langle P(D)e^{\rho|x|}h, \bar{\varphi}\rangle = \langle f, \bar{\varphi}\rangle$, $\forall\varphi \in \mathscr{D}(\mathbb{R}^n)$, where $\langle\, ,\, \rangle$ is the duality bracket between $\mathscr{D}$ and $\mathscr{D}'$. Therefore

$$P(D)e^{\rho|x|}h = f \quad \text{in } \mathscr{D}'(\mathbb{R}^n)$$

Let us set $u = e^{\rho|x|}h$, then $e^{-\rho|x|}u = h \in L^2(\mathbb{R}^n)$ so in particular $u \in \mathscr{D}'(\mathbb{R}^n)$.

4°) a) See exercise 96.

b) Since $f \in L^2(\mathbb{R}^n)$ there exists $u \in \mathscr{D}'(\mathbb{R}^n)$ such that $P(D)u = f$, then $P(D)Q(D)u = Q(D)P(D)u = Q(D)f = \delta$. Then we take $E = Q(D)u \in \mathscr{D}'(\mathbb{R}^n)$.

c) Let $g \in C_0^\infty(\mathbb{R}^n)$; let us set $u = E * g \in C^\infty(\mathbb{R}^n)$. Then

$$P(D)u = (P(D)E) * g = \delta * g = g$$

Solution 99

a) From the Leibniz formula we get

$$(4) \quad P(D)(\varphi \cdot v) = \sum_{|\alpha|\leq m} a_\alpha D^\alpha(\varphi \cdot v)$$

$$= \varphi P(D)v + \sum_{|\alpha|\leq m} a_\alpha \sum_{\substack{\beta\leq\alpha \\ \beta\neq 0}} \binom{\alpha}{\beta} D^\beta\varphi \cdot D^{\alpha-\beta}v$$

Moreover

$$P^{(\beta)}(\xi) = \sum_{\substack{|\alpha|\leq m \\ \alpha\geq\beta}} a_\alpha \frac{\alpha!}{(\alpha-\beta)!}\xi^{\alpha-\beta}$$

so the right hand side of (1) can be written

$$(5)\quad \varphi P(D)v + \sum_{\beta \neq 0} \sum_{\substack{\beta \leq \alpha \\ |\alpha| \leq m}} a_\alpha \frac{1}{\beta!} \cdot \frac{\alpha!}{(\alpha - \beta)!} D^\beta \varphi \cdot D^{\alpha - \beta} v$$

The inequality (1) follows from the fact that the sums appearing in (4) and (5) are equal.

b) Since $\varphi_0 u \in \mathscr{E}'(\mathbb{R}^n)$ there exists, by question b) of exercise 93, a real number t such that $\varphi_0 u \in H^t(\mathbb{R}^n)$.

For $\beta \neq 0$ the operator $P^{(\beta)}(D)$ is of order $\leq m - 1$, i.e.

$$P^{(\beta)}(\xi) = \sum_{|\gamma| \leq m - 1} b_{\gamma,\beta} \xi^\gamma$$

so

$$|P^{(\beta)}(\xi)| \leq \sum_{|\gamma| \leq m - 1} |b_{\gamma,\beta}| \cdot |\xi^\gamma| \leq \sum_{|\gamma| \leq m - 1} |b_{\gamma,\beta}| \cdot |\xi|^{|\gamma|}$$

since $|\xi^\gamma| \leq |\xi|^{|\gamma|}$. Moreover $|\xi|^{|\gamma|} \leq (1 + |\xi|^2)^{(m-1)/2}$ for if $|\xi| \leq 1$, $|\xi|^{|\gamma|} \leq 1$ and if $|\xi| > 1$ then $|\xi|^{|\gamma|} \leq |\xi|^{m-1}$ since $|\gamma| \leq m - 1$. So

$$(6)\quad |P^{(\beta)}(\xi)| \leq C(1 + |\xi|^2)^{(m-1)/2}$$

It follows from (6)

$$\int (1 + |\xi|^2)^{t-(m-1)} |\widehat{P^{(\beta)}(D)(\varphi_0 u)}(\xi)|^2 \, d\xi = \int (1 + |\xi|^2)^{t-(m-1)} |P^{(\beta)}(\xi) \cdot \widehat{\varphi_0 u}(\xi)|^2 \, d\xi \leq$$

$$\leq C \int (1 + |\xi|^2)^t |\widehat{\varphi_0 u}(\xi)|^2 \, d\xi = C \|\varphi_0 u\|^2_{H^t(\mathbb{R}^n)}$$

which proves that $P^{(\beta)}(D)(\varphi_0 u) \in H^{t-(m-1)}(\mathbb{R}^n)$ if $\beta \neq 0$.

c) It follows from (1)

$$(7)\quad P(D)(\varphi_1 \varphi_0 u) = \varphi_1 P(D)(\varphi_0 u) + \sum_{\substack{\beta \neq 0 \\ |\beta| \leq m}} \frac{1}{\beta!} D^\beta \varphi_1 \cdot P^{(\beta)}(D)(\varphi_0 u)$$

Now $P(D)u \in H^s_{\text{loc}}(\omega)$. Moreover since $\varphi_0 = 1$ on the support of φ_1 we have $\varphi_1 \varphi_0 = \varphi_1$ and $\varphi_1 P(D)(\varphi_0 u) = \varphi_1 P(D)u \in H^s(\mathbb{R}^n)$.

From question b), $P^{(\beta)}(D)(\varphi_0 u) \in H^{t-(m-1)}(\mathbb{R}^n)$. Since $D^\beta \varphi_1 \in \mathscr{S}(\mathbb{R}^n)$, exercise 94 shows that $(D^\beta \varphi_1) P^{(\beta)}(D)(\varphi_0 u) \in H^{t-(m-1)}(\mathbb{R}^n)$. It follows from (7) that $P(D)\varphi_1 u \in H^{t-(m-1)}(\mathbb{R}^n)$.

The condition (C) then implies that

$$\forall \beta \neq 0 \qquad P^{(\beta)}(D)(\varphi_1 u) \in H^{t-(m-1)+\mu}(\mathbb{R}^n)$$

Indeed

$$\int (1 + |\xi|^2)^{t-(m-1)+\mu} |\widehat{P^{(\beta)}(D)(\varphi_1 u)}(\xi)|^2 \, d\xi =$$

$$= \int (1 + |\xi|^2)^{t-(m-1)+\mu} |P^{(\beta)}(\xi)|^2 \cdot |\widehat{\varphi_1 u}(\xi)|^2 \, d\xi = \int_{|\xi| < R} F(\xi) \, d\xi + \int_{|\xi| > R} F(\xi) \, d\xi$$

Now

$$\int_{|\xi| \leq R} F(\xi)\,\mathrm{d}\xi \leq C_R \int_{|\xi| \leq R} |\widehat{\varphi_1 u}(\xi)|^2\,\mathrm{d}\xi < +\infty$$

Indeed

$$(1 + |\xi|^2)^{t-(m-1)+\mu}|P^{(\beta)}(\xi)|^2 \leqq (1 + R^2)^{t-(m-1)+\mu} \sum_{\gamma} |b_{\gamma,\beta}| R^{|\gamma|}$$

and $\varphi_1 u$ being a distribution with compact support, $\widehat{\varphi_1 u}$ is a C^∞ function so square integrable on every compact.
Moreover by condition (C), $(1 + |\xi|^2)^{\mu}|P^{(\beta)}(\xi)|^2 \leq C|P(\xi)|^2$, so

$$\int_{|\xi| \geq R} F(\xi)\,\mathrm{d}\xi \leq \int (1+|\xi|^2)^{t-(m-1)}|P(\xi)|^2 \cdot |\widehat{\varphi_1 u}(\xi)|^2\,\mathrm{d}\xi = \|P(D)(\varphi_1 u)\|^2_{H^{t-(m-1)}(\mathbb{R}^n)} < \infty$$

So we have proved that $P(D)(\varphi_1 u) \in H^{t-(m-1)}(\mathbb{R}^n)$ and $P^{(\beta)}(D)(\varphi_1 u) \in H^{t-(m-1)+\mu}(\mathbb{R}^n)$.
Let us assume that

$$P(D)(\varphi_j u) \in H^{t-(m-1)+(j-1)\mu}(\mathbb{R}^n) \quad \text{and} \quad P^{(\beta)}(D)(\varphi_j u) \in H^{t-(m-1)+j\mu}(\mathbb{R}^n)$$

We have

$$\begin{aligned} (8) \quad P(D)(\varphi_{j+1} u) &= P(D)(\varphi_{j+1}\varphi_j u) \\ &= \varphi_{j+1} P(D)\varphi_j u + \sum_{\beta \neq 0} \frac{1}{\beta!} D^\beta \varphi_{j+1} P^{(\beta)}(D)(\varphi_j u) \end{aligned}$$

By the same argument as before $\varphi_{j+1} P(D)\varphi_j u \in H^s(\mathbb{R}^n)$. Using the induction hypothesis the sum of the right hand side of (8) is in the space $H^{t-(m-1)+j\mu}(\mathbb{R}^n)$ (we have also used exercise 94).
Therefore $P(D)(\varphi_{j+1} u) \in H^{t-(m-1)+j\mu}(\mathbb{R}^n)$. Condition (C) implies, by the same method as above, that:

$$P^{(\beta)}(D)(\varphi_{j+1} u) \in H^{t-(m-1)+(j+1)\mu}(\mathbb{R}^n) \qquad \forall \beta \neq 0$$

which proves the next step of the induction. It follows that

$$(9) \quad P^{(\beta)}(D)(\varphi_N u) \in H^{t-(m-1)+N\mu}(\mathbb{R}^n) \subset H^s(\mathbb{R}^n)$$

Moreover since $P(\xi)$ is a polynomial there exists β, $|\beta| \leq m$, such that $P^{(\beta)}(\xi)$ is a non zero constant. Then $P^{(\beta)}(D) = C^{te} \cdot$ Identity. It follows from (9) that $\varphi_N u \in H^s(\mathbb{R}^n)$ where $\varphi_N \in \mathscr{D}(\omega')$. This implies that $u \in H^s_{\mathrm{loc}}(\omega)$. Indeed if $\varphi \in \mathscr{D}(\omega)$ then the support of φ is contained in ω' with $\bar{\omega}' \subset \omega$. So we use the method described above with $\varphi_N = \varphi$, so $\varphi u \in H^s(\mathbb{R}^n)$.

d) From exercise 91

$$\bigcap_{s \in \mathbf{R}} H^s_{\mathrm{loc}}(\omega) = C^\infty(\omega)$$

If $u \in \mathscr{D}'(\Omega)$ and $Pu \in C^\infty(\omega)$ then $Pu \in H^s_{\text{loc}}(\omega)$ for all $s \in \mathbb{R}$. We have proved in c) that $u \in H^s_{\text{loc}}(\omega)$ for all $s \in \mathbb{R}$ thus $u \in C^\infty(\omega)$.

e) If P is elliptic we have seen in exercise 97 that there exist $R > 0$ and $C > 0$ such that

$$|P(\xi)| \geqq C|\xi|^m \text{ for } |\xi| \geq R$$

Moreover for $\beta \neq 0$, $P^{(\beta)}$ is a polynomial of degree $\leq m - 1$. We have proved in question b) that $|P^{(\beta)}(\xi)| \leq C(1 + |\xi|^2)^{(m-1)/2}$ for $|\xi| > 1$.
Now for $|\xi| > 1$

$$2^{m/2} \cdot |\xi|^m \geq (1 + |\xi|^2)^{m/2}$$

Therefore if $\mu = 1$ for $|\xi|$ big enough:

$$\frac{|P^{(\beta)}(\xi)|(1 + |\xi|^2)^{\mu/2}}{|P(\xi)|} \leq C'_m \frac{(1 + |\xi|^2)^{(m-1)/2}(1 + |\xi|^2)^{1/2}}{(1 + |\xi|^2)^{m/2}} \leqq C'_m$$

Let $P = \dfrac{\partial}{\partial t} - \dfrac{\partial^2}{\partial x^2}$. Then $P(\xi, \eta) = 4\pi^2|\xi|^2 + 2i\pi \cdot \eta$.

$$|P(\xi, \eta)| = 2\pi(\eta^2 + 4\pi^2|\xi|^4)^{1/2} \sim |\eta| + |\xi|^2 \quad \text{for } |\eta| + |\xi| \text{ big enough,}$$

($a \sim b$ means that $\dfrac{a}{b}$ is bounded for big $|\eta| + |\xi|$).
Let us compute the $P^{(\beta)}$ for $|\beta| \leq 2$, $\beta \neq 0$.

$$\beta = (1, 0) \qquad P^{(\beta)} = \frac{\partial}{\partial \xi} P(\xi) = 8\pi^2\xi$$

$$\beta = (0, 1) \qquad P^{(\beta)} = 2i\pi$$

$$\beta = (2, 0) \qquad P^{(\beta)} = 8\pi^2; \qquad \beta = (0, 2) \text{ or } (1, 1) \qquad P^{(\beta)} = 0$$

Let us take $\mu = \frac{1}{2}$. Then $(1 + |\xi|^2 + |\eta|^2)^{1/4}$ is equivalent to $(|\xi| + |\eta|)^{1/2}$ for $|\xi| + |\eta|$ big enough. Then

$$\frac{|P^{(\beta)}(\xi, \eta)| \cdot (|\xi| + |\eta|)^{1/2}}{|\xi|^2 + |\eta|} \leq C\frac{|\xi|(|\xi| + |\eta|)^{1/2}}{|\xi|^2 + |\eta|} \leq C'$$

Indeed

$$|\xi|^2(|\xi| + |\eta|) \leq (|\xi|^2 + |\eta|)^2 \quad \text{for big } |\xi|, |\eta|$$

since

$$|\xi|^3 + |\eta||\xi|^2 \leqq |\xi|^4 + 2|\xi|^2|\eta| + |\eta|^2$$

for $|\xi| \geq 1$.

Let us consider $P = \frac{\partial}{\partial t} + i\frac{\partial^2}{\partial x^2}$. Then $P(\xi, \eta) = 2i\pi\eta - 4i\pi^2\xi^2$ and

$$|P(\xi, \eta)| = 2\pi|\eta - 2\pi\xi^2|$$

Let us take $\beta = (1, 0)$ the $P^{(\beta)} = 2i\pi$

$$\frac{|P^{(\beta)}(\xi)|(1 + |\xi|^2 + |\eta|^2)^{\mu/2}}{|P(\xi)|} = \frac{(1 + |\xi|^2 + |\eta|^2)^{\mu/2}}{|\eta - 2\pi\xi^2|}$$

This expression is never bounded for $\mu > 0$ since it goes to infinity when $|\eta|$, $|\xi|$ are big and $|\eta - 2\pi\xi^2|$ tends to zero.

In the same way if $P = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}$, $P(\xi, \eta) = 4\pi^2(\xi^2 - \eta^2)$ and condition (C) is not satisfied if $|\xi|$, $|\eta|$ are big but $|\xi^2 - \eta^2|$ tends to zero.

We showed in exercise 51 that these operators are not hypoelliptic.

Solution 100

a) It is proved in exercise 91 that for $k_0 \in \mathbb{N}$, $k_0 > \frac{n}{2}$,

$$H^{k_0}(\mathbb{R}^n) \subset C^0(\mathbb{R}^n)$$

with continuous injection. Thus:

for every compact subset K of Ω there exists $C > 0$ such that for every $v \in H^{k_0}(\mathbb{R}^n)$

$$\sup_{x \in K} |v(x)| \leq C\|v\|_{H^{k_0}(\mathbb{R}^n)} = C\left(\sum_{|\alpha| \leq k_0} \|\partial^\alpha v\|^2_{L^2(\mathbb{R}^n)}\right)^{1/2}$$

Indeed the semi norms which define the topology in C^0 are $p_K(u) = \sup_{x \in K} |u(x)|$, so this inequality follows from $\mathscr{D}(\Omega) \subset H^{k_0}(\mathbb{R}^n)$ and $\left(\sum_i a_i^2\right)^{1/2} \leq \sum_i a_i$, $a_i \in \mathbb{R}^+$.

b) See solution of exercise 97 question a).

c) Since $v \in \mathscr{D}(\Omega)$ we have $v \in \mathscr{S}(\mathbb{R}^n)$ and $P(D)v \in \mathscr{S}(\mathbb{R}^n)$.

$$\|v\|^2_{H^m(\mathbb{R}^n)} = \int (1 + |\xi|^2)^m |\hat{v}(\xi)|^2 \, d\xi$$

$$\int (1 + |\xi|^2)^m |\hat{v}(\xi)|^2 \, d\xi = \int_{|\xi| \leq R_1} (1 + |\xi|^2)^m |\hat{v}(\xi)|^2 \, d\xi + \int_{|\xi| > R_1} (1 + |\xi|^2)^m |\hat{v}(\xi)|^2 \, d\xi$$

where $R_1 \geq \text{Max}(1, R)$. For $|\xi| \leq R_1$, $(1 + |\xi|^2)^m \leq (1 + R_1^2)^m$ and for $|\xi| \geq R_1$, $(1 + |\xi|^2)^m \leq 2^m|\xi|^{2m}$; so

$$\int (1 + |\xi|^2)^m |\hat{v}(\xi)|^2 \, d\xi \leq (1 + R_1^2)^m \int_{|\xi| \leq R_1} |\hat{v}(\xi)|^2 \, d\xi + 2^m \int_{|\xi| > R_1} |\xi|^{2m} |\hat{v}(\xi)|^2 \, d\xi$$

By question b), for $|\xi| > r$ we have $|P(\xi)|^2 \geq C_2^2|\xi|^{2m}$; moreover we know that $\|v\|^2_{L^2(\mathbb{R}^n)} = \|\hat{v}\|^2_{L^2(\mathbb{R}^n)}$ for $v \in L^2(\mathbb{R}^n)$ (Parseval formula). So

$$\int (1 + |\xi|^2)^m |\hat{v}(\xi)|^2 \,\mathrm{d}\xi \leqq M_1 \|v\|^2_{L^2(\mathbb{R}^n)} + 2^m \frac{1}{C_2^2} \int |P(\xi)\hat{v}(\xi)|^2 \,\mathrm{d}\xi$$

Since $\widehat{P(D)v}(\xi) = P(\xi)\hat{v}(\xi)$, the Parseval formula implies that

$$\|v\|_{H^m(\mathbb{R}^n)} \leqq C_3 \{\|P(D)v\|^2_{L^2(\mathbb{R}^n)} + \|v\|^2_{L^2(\mathbb{R}^n)}\}^{1/2}$$

which proves (3) since $(a^2 + b^2)^{1/2} \leq a + b$.

d) First of all $\varphi_{\varepsilon,\varepsilon_1}$ is the convolution of χ with the function $\frac{1}{\delta^n}\varphi\left(\frac{x}{\delta}\right)$. So $\varphi_{\varepsilon,\varepsilon_1}$ is C^∞. We have

$$\varphi_{\varepsilon,\varepsilon_1}(x) = \frac{1}{\delta^n} \int \chi(y)\varphi\left(\frac{x-y}{\delta}\right) \mathrm{d}y$$

If $x \notin \omega_{\varepsilon_1}$ then $d(x, \complement\omega) \leqq \varepsilon_1$; moreover in the above integral we must have $|x - y| \leq \frac{\varepsilon}{3}$ since the support of φ is contained in $\{x: |x| \leq 1\}$. It follows that

$$d(y, \complement\omega) \leq \varepsilon_1 + \frac{\varepsilon}{3} < \varepsilon_1 + \frac{\varepsilon}{2}$$

so $\chi(y) = 0$ for all y which proves that $\varphi_{\varepsilon,\varepsilon_1}(x) = 0$, so $\operatorname{supp} \varphi_{\varepsilon,\varepsilon_1} \subset \omega_{\varepsilon_1}$.

Now if $x \in \omega_{\varepsilon+\varepsilon_1}$ we have $d(x, \complement\omega) > \varepsilon + \varepsilon_1$ and $|x - y| \leq \frac{\varepsilon}{3}$; so in the above integral we have

$$d(y, \complement\omega) \geqq d(x, \complement\omega) - d(x, y) > \varepsilon + \varepsilon_1 - \frac{\varepsilon}{3} > \varepsilon_1 + \frac{\varepsilon}{2}$$

so $\chi(y) \equiv 1$; and

$$\varphi_{\varepsilon,\varepsilon_1}(x) = \frac{1}{\delta^n} \int \varphi\left(\frac{x-y}{\delta}\right) \mathrm{d}y = \int \varphi(t)\,\mathrm{d}t = 1$$

Finally

$$\partial^\alpha \varphi_{\varepsilon,\varepsilon_1} = \chi * \partial^\alpha\left(\frac{1}{\delta}\varphi\left(\frac{x}{\delta}\right)\right)$$

$$\partial^\alpha \varphi_{\varepsilon,\varepsilon_1}(x) = \frac{1}{\delta^n}\frac{1}{\delta^{|\alpha|}} \int \chi(y)(\partial^\alpha\varphi)\left(\frac{x-y}{\delta}\right) \mathrm{d}y$$

thus

$$|\partial^\alpha \varphi_{\varepsilon,\varepsilon_1}(x)| \leqq \frac{1}{\delta^{n+|\alpha|}} \int \left|(\partial^\alpha\varphi)\left(\frac{x-y}{\delta}\right)\right| \mathrm{d}y = \frac{1}{\delta^{|\alpha|}} \int \left|(\partial^\alpha\varphi)(t)\right| \mathrm{d}t$$

setting $\dfrac{x-y}{\delta} = t$. Since $\delta = \dfrac{\varepsilon}{3}$ we get

$$\sup_{x\in\omega_{\varepsilon_1}} |\partial^\alpha \varphi_{\varepsilon,\varepsilon_1}(x)| \leq \left(\int |(\partial^\alpha\varphi)(t)|\,dt\right) 3^{|\alpha|}\varepsilon^{-|\alpha|} = C_\alpha \varepsilon^{-|\alpha|}$$

a) Let $w \in C^\infty(\omega)$; the function $\varphi_{\varepsilon,\varepsilon_1}\cdot w$ belongs to $\mathscr{D}(\omega)$, and since $\varphi_{\varepsilon,\varepsilon_1}$ is equal to one on $\omega_{\varepsilon+\varepsilon_1}$ we have for every α

$$(8)\quad \|D^\alpha w\|_{L^2(\omega_{\varepsilon+\varepsilon_1})} \leqq \|D^\alpha \varphi_{\varepsilon,\varepsilon_1} w\|_{L^2(\omega)}$$

Let us apply inequality (3) to $v = \varphi_{\varepsilon,\varepsilon_1}\cdot w$. Then

$$(9)\quad \sum_{|\alpha|\leq m} \|D^\alpha \varphi_{\varepsilon,\varepsilon_1} w\|_{L^2(\omega)} \leqq C_3\{\|P(D)\varphi_{\varepsilon,\varepsilon_1} w\|_{L^2(\omega)} + \|\varphi_{\varepsilon,\varepsilon_1} w\|_{L^2(\omega)}\}$$

Moreover

$$P(D)(\varphi_{\varepsilon,\varepsilon_1} w) = \sum_{|\alpha|\leq m}\sum_{\substack{\beta\leq\alpha\\|\beta|<m}} a_\alpha \binom{\alpha}{\beta} D^{\alpha-\beta}\varphi_{\varepsilon,\varepsilon_1}\cdot D^\beta w + \varphi_{\varepsilon,\varepsilon_1} P(D)w$$

so

$$\|P(D)\varphi_{\varepsilon,\varepsilon_1} w\|_{L^2(\omega)} \leqq \sum_{|\alpha|\leq m}\sum_{\substack{\beta\leq\alpha\\|\beta|<m}} b_{\alpha,\beta}\|(D^{\alpha-\beta}\varphi_{\varepsilon,\varepsilon_1})D^\beta\cdot w\|_{L^2(\omega)} + \|\varphi_{\varepsilon,\varepsilon_1} P(D)w\|_{L^2(\omega)}$$

But, from d)

$$\sup_{x\in\omega_{\varepsilon_1}} |D^\gamma \varphi_{\varepsilon,\varepsilon_1}| < C_\gamma \varepsilon^{-|\gamma|}$$

so

$$(10)\quad \|P(D)\varphi_{\varepsilon,\varepsilon_1} w\|_{L^2(\omega)} \leqq \sum_{|\alpha|\leq m}\sum_{\substack{|\beta|<m\\ \beta\leq\alpha}} C_{\alpha,\beta}\varepsilon^{|\beta|-|\alpha|}\|D^\beta w\|_{L^2(\omega_{\varepsilon_1})} + $$
$$+ C_0\|P(D)w\|_{L^2(\omega_{\varepsilon_1})}$$

From inequalities (8), (9), (10) we get

$$(11)\quad \|D^\alpha w\|_{L^2(\omega_{\varepsilon+\varepsilon_1})} \leqq C_4\Big\{\|P(D)w\|_{L^2(\omega_{\varepsilon_1})} +$$
$$+ \sum_{|\alpha|\leq m}\sum_{\substack{|\beta|<m\\ \beta\leq\alpha}} C_{\alpha\beta}\varepsilon^{|\beta|-|\alpha|}\|D^\beta w\|_{L^2(\omega_{\varepsilon_1})} + \|w\|_{L^2(\omega_{\varepsilon_1})}\Big\}$$

If $|\alpha| < m$, the inequality (5) is obvious for $\omega_{\varepsilon+\varepsilon_1}$ being included in ω_{ε_1} we have

$$\varepsilon^{|\alpha|}\|D^\alpha w\|_{L^2(\omega_{\varepsilon+\varepsilon_1})} \leq \varepsilon^{|\alpha|}\|D^\alpha w\|_{L^2(\omega_{\varepsilon_1})}$$

For $|\alpha| = m$, let us multiply both sides of (11) by ε^m. We get

$$\varepsilon^{|\alpha|}\|D^\alpha w\|_{L^2(\omega_{\varepsilon+\varepsilon_1})} \leqq C_4\Big\{\varepsilon^m\|P(D)w\|_{L^2(\omega_{\varepsilon_1})}$$

$$+ \sum_\alpha \sum_{|\beta|<m} C_{\alpha\beta}\varepsilon^{m-|\alpha|+|\beta|}\|D^\beta w\|_{L^2(\omega_{\varepsilon_1})} + \varepsilon^m\|w\|_{L^2(\omega_{\varepsilon_1})}\Big\}$$

Since $0 < \varepsilon < 1$ we have $\varepsilon^m < 1$ thus

$$\varepsilon^{|\alpha|}\|D^\alpha w\|_{L^2(\omega_{\varepsilon+\varepsilon_1})} \leq C_5\Big\{\varepsilon^m\|P(D)w\|_{L^2(\omega_{\varepsilon_1})} + \sum_{|\gamma|<m}\varepsilon^{|\gamma|}\|D^\gamma w\|_{L^2(\omega_{\varepsilon_1})}\Big\}$$

which proves (5).

f) Since Pu is analytic on $\tilde{\omega}$ and $\bar{\omega} \subset \tilde{\omega}$, there exists $A > 0$ such that for every j,

$$\sup_{\omega_{j\varepsilon}}|D^{\alpha_0}Pu| \leq \sup_{\bar{\omega}}|D^{\alpha_0}Pu| \leqq A^{|\alpha_0|+1}\alpha_0! \leq A^{j+1}j! \leq C^{j+1}j^j$$

So we get

$$\varepsilon^j\|D^{\alpha_0}Pu\|_{L^2(\omega_{j\varepsilon})} \leqq \left(\int_\omega \mathrm{d}x\right)\cdot\varepsilon^j\cdot\sup_{\omega_{j\varepsilon}}|D^{\alpha_0}Pu| \leq M^{j+1}(j\varepsilon)^j \leq M^{j+1}$$

since $j\varepsilon < 1$.

g) Since $u \in C^\infty(\omega)$, the inequality $(6)_1$ is true for if $|\gamma| \leq m$

$$\varepsilon^{|\gamma|}\|D^\gamma u\|_{L^2(\omega_\varepsilon)} \leqq \|D^\gamma u\|_{L^2(\omega_\varepsilon)} \leqq \sup_{|\alpha|\leqq m}\sup_{x\in\omega_1}|D^\alpha u(x)| = C_m$$

where $\omega_\varepsilon \subset \omega_1 \subset \omega$, $\bar{\omega}_1 \subset \omega$.
Let us assume that there exists B such that $(6)_j$ is true and let us prove $(6)_{j+1}$ for $|\gamma| < m + j + 1$. Let us note that $(6)_j$ implies $(6)_{j+1}$ when $|\gamma| < m + j$ for $\|D^\gamma u\|_{L^2(\omega(j+1)\varepsilon)} \leq \|D^\gamma u\|_{L^2(\omega_{j\varepsilon})}$. It remains to prove $(6)_{j+1}$ for $|\gamma| = m + j$. Let us take $\gamma = (\alpha, \alpha_0)$ with $|\alpha| = m$, $|\alpha_0| = j$. Let us apply (5) with $\varepsilon_1 = j\varepsilon$, $w = D^{\alpha_0}u$ and $|\alpha| = m$

$$\begin{aligned}\varepsilon^{|\gamma|}\|D^\gamma u\|_{L^2(\omega(j+1)\varepsilon)} &= \varepsilon^{m+j}\|D^\alpha D^{\alpha_0}u\|_{L^2(\omega(j+1)\varepsilon)}\\ &\leqq C\varepsilon^j\Big\{\varepsilon^m\|P(D)D^{\alpha_0}u\|_{L^2(\omega_{j\varepsilon})} + \sum_{|\beta|<m}\varepsilon^{|\beta|}\|D^\beta D^{\alpha_0}u\|_{L^2(\omega_{j\varepsilon})}\\ &\leqq C\Big\{\varepsilon^j\|P(D)D^{\alpha_0}u\|_{L^2(\omega_{j\varepsilon})} + \sum_{|\beta|<m}\varepsilon^{j+|\beta|}\|D^\beta D^{\alpha_0}u\|_{L^2(\omega_{j\varepsilon})}\Big\}\end{aligned}$$

Now $P(D)D^{\alpha_0}u = D^{\alpha_0}P(D)u$ and Pu is analytic in $\tilde{\omega}$. By f) we can find $M > 0$ such that for every ε and every j such that $j\varepsilon \leq 1$

$$\varepsilon^j\|P(D)D^{\alpha_0}u\|_{L^2(\omega_{j\varepsilon})} \leqq M^{j+1}$$

Moreover since $|\beta + \alpha_0| = |\beta| + |\alpha_0| = j + |\beta| < m + j$, the induction hypothesis gives

$$\varepsilon^{|\beta|+j}\|D^\beta D^{\alpha_0}u\|_{L^2(\omega_{j\varepsilon})} \leqq B^{|\beta|+j+1}$$

so

$$\varepsilon^{|\gamma|}\|D^{\gamma}u\|_{L^2(\omega_{(j+1)\varepsilon})} \leqq CM^{j+1} + C\sum_{|\beta|<m} B^{|\beta|+j+1}$$

Let us increase B in order that $B \geq 1$; then

$$\varepsilon^{\gamma}\|D^{\gamma}u\|_{L^2(\omega_{(j+1)\varepsilon})} \leq CM^{j+1} + C\left(\sum_{|\beta|<m} 1\right)B^{m+j} \quad \text{for } |\beta| \leq m-1$$

For the induction to be satisfied we must have

$$\varepsilon^{|\gamma|}\|D^{\gamma}u\|_{L^2(\omega_{(j+1)\varepsilon})} \leq B^{|\gamma|+1} = B^{m+j+1}$$

It is enough to take

$$\begin{cases} CM^{j+1} \leq \frac{1}{2}B^{m+j+1} \\ C\left(\sum_{|\beta|<m} 1\right)B^{m+j} \leq \frac{1}{2}B^{m+j+1} \quad \text{or} \quad C\left(\sum_{|\beta|<m} 1\right)' \leq \frac{1}{2}B \end{cases}$$

which is always possible. So $(6)_{j+1}$ is satisfied.

h) If we set $\varepsilon = \dfrac{a}{j}$, $j = |\gamma|$, $(6)_j$ gives

$$\left(\frac{a}{|\gamma|}\right)^{|\gamma|}\|D^{\gamma}u\|_{L^2(\omega_a)} \leqq B^{|\gamma|+1}$$

so

$$(12) \quad \|D^{\gamma}u\|_{L^2(\omega_a)} \leqq B\left(\frac{B}{a}\right)^{|\gamma|}|\gamma|^{|\gamma|} \leqq L^{|\gamma|+1}\gamma!$$

i) Let K be a compact subset of ω and $a > 0$ such that $K \subset \omega_a$; from (2) applied to $v = D^{\gamma}u$ *and* $\Omega = \omega_a$ we get

$$\sup_K |D^{\gamma}u(x)| \leqq C' \sum_{|\alpha|\leq k_0} \|D^{\alpha}D^{\gamma}u\|_{L^2(\omega_a)}$$

From (12) we have

$$\begin{aligned} \sup_K |D^{\gamma}u(x)| &\leq C' \sum_{|\alpha|\leq k_0} L^{|\gamma|+|\alpha|+1}(\gamma+\alpha)! \\ &\leqq C'L^{|\gamma|} \sum_{|\alpha|\leq k_0} L^{|\alpha|+1} \cdot 2^{|\gamma|+|\alpha|}\alpha!\gamma! \end{aligned}$$

since $(\alpha+\gamma)! \leq 2^{|\alpha|+|\gamma|}\alpha!\gamma!$. Let us set $M = \sum_{|\alpha|\leq k_0} L^{|\alpha|+1}2^{|\alpha|}\alpha!$, we get

$$\sup_K |D^{\gamma}u(x)| \leqq C' \cdot M \cdot (2L)^{|\gamma|}\gamma! \leq A^{|\gamma|+1}\gamma!$$

where $A \geqq \text{Max}\,(C' \cdot M, 2L)$. Therefore u is analytic in ω.

Solution 101

1°) Obviously we have $(**) \Rightarrow (*)$. Moreover since p_m is homogeneous of degree m in ξ we have $\sum_{j=1}^{n} \xi_j \frac{\partial p_m}{\partial \xi_j}(x, \xi) = m p_m(x, \xi)$ thus if $p_m(x, \xi) \neq 0$ there exists $j \in \{1, 2, \ldots, n\}$ such that $\frac{\partial p_m}{\partial \xi_j}(x, \xi) \neq 0$. This proves that $(*) \Rightarrow (**)$.

2°) If $\operatorname{supp} \varphi \subset \{x : |x_1 - a_1| < R\}$ we write

$$\varphi(x) = \int_{a_1 - R}^{x_1} \frac{\partial \varphi}{\partial x_1}(t, x')\,\mathrm{d}t.$$

The Cauchy-Schwarz inequality implies

$$|\varphi(x)| \leqslant \left(\int_{a_1 - R}^{x_1} \mathrm{d}t \right)^{1/2} \left(\int_{\mathbb{R}} \left| \frac{\partial \varphi}{\partial x_1}(t, x') \right|^2 \mathrm{d}t \right)^{1/2}$$

thus
$$|\varphi(x)|^2 \leqq (x_1 - a_1 + R) \int_{\mathbb{R}} \left| \frac{\partial \varphi}{\partial x_1}(t, x') \right|^2 \mathrm{d}t.$$

Since $\operatorname{supp} \varphi \subset \{x : |x_1 - a_1| < R\}$

$$\begin{aligned}\int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} |\varphi(x)|^2 \,\mathrm{d}x' \,\mathrm{d}x_1 &= \int_{a_1 - R}^{a_1 + R} \int_{\mathbb{R}^{n-1}} |\varphi(x)|^2 \,\mathrm{d}x' \,\mathrm{d}x_1 \\ &\leqslant \int_{a_1 - R}^{a_1 + R} (x_1 - a_1 + R) \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} \left| \frac{\partial \varphi}{\partial x_1}(t, x') \right|^2 \mathrm{d}x' \,\mathrm{d}t\end{aligned}$$

i.e. $\|\varphi\|_{L^2}^2 \leqq \int_0^{2R} y \,\mathrm{d}y \left\| \frac{\partial \varphi}{\partial x_1} \right\|_{L^2}^2 = 2R^2 \left\| \frac{\partial \varphi}{\partial x_1} \right\|_{L^2}^2$.

In the general case $\operatorname{supp} \varphi \subset \{|x - a| < R\}$, so $\operatorname{supp} \varphi \subset \{x : |x_1 - a_1| < R\}$. Let us apply the above inequality to $D^\alpha \varphi$ with $|\alpha| \leq m - 1$. We get

$$\begin{aligned}\sum_{|\alpha| \leq m-1} \|D^\alpha \varphi\|_{L^2}^2 &\leqq 2R^2 \sum_{|\alpha| \leq m-1} \left\| \frac{\partial}{\partial x_1} D^\alpha \varphi \right\|_{L^2}^2 \\ &\leqslant 2R^2 \left\{ \sum_{|\alpha| = m} \|D^\alpha \varphi\|_{L^2}^2 + \sum_{|\alpha| \leq m-1} \|D^\alpha \varphi\|_{L^2}^2 \right\}\end{aligned}$$

thus $\|\varphi\|_{m-1}^2 \leqq \frac{2R^2}{1 - 2R^2} \sum_{|\alpha| = m} \|D^\alpha \varphi\|_{L^2}^2$ Q.E.D

3°) We shall use the Leibniz formula

$$P_m(u \cdot v) = \sum_\alpha \frac{1}{\alpha!} D^\alpha u P_m^{(\alpha)} v.$$

Then $P_m(x_j u) = x_j P_m u + \dfrac{1}{i}\sum_{i=1}^{n} \dfrac{\partial p_m}{\partial \xi_j}(x, D)u$. This proves that

$P_m^{(j)} = i[P_m, x_j] = i[P_m, x_j - a]$. Moreover

$$\begin{aligned}
\|P_m^{(j)}\varphi\|_{L^2}^2 &= (P_m^{(j)}\varphi, P_m^{(j)}\varphi) \\
&= \underset{①}{(P_m^{(j)}\varphi, iP_m(x_j - a)\varphi)} - \underset{②}{(P_m^{(j)}\varphi, i(x_j - a)P_m\varphi)}
\end{aligned}$$

Since supp $\varphi \subset B(a, R)$ we have:

$$|②| \leqq R\,\|P_m^{(j)}\varphi\|_{L^2}\,\|P_m\varphi\|_{L^2} \leqq \frac{R}{2}(\|P_m^{(j)}\varphi\|_{L^2}^2 + \|P_m\varphi\|_{L^2}^2).$$

Moreover

$$① = \underset{③}{([P_m^*, P_m^{(j)}]\varphi, i(x_j - a)\varphi)} + \underset{④}{(P_m^* u, i[P_m^{(j)*}, x_j - a]\varphi)} + \underset{⑤}{(P^*\varphi, i(x_j - a)P_m^{(j)}\varphi)}.$$

Since $[P_m, P_m^{(j)}]$ is of order $2m - 2$ we have:

$$\begin{aligned}
|③| &\leqq \|[P_m^*, P_m^{(j)}]\varphi\|_{(m-1)}\|(x_j - a)\varphi\|_{m-1} \\
&\leqq C\|\varphi\|_{m-1}\|(x_j - a)\varphi\|_{m-1}.
\end{aligned}$$

Now $D^\alpha(x_j - a)\varphi = (x_j - a)D^\alpha\varphi + R_\alpha\varphi$ where R_α is of order $|\alpha| - 1$, thus

$$\|(x_j - a)\varphi\|_{m-1} \leqq \sum_{|\alpha| = m-1} \|(x_j - a)D^\alpha\varphi\| + C\|\varphi\|_{m-2}.$$

Then

$$\|(x_j - a)\varphi\|_{m-1} \leqq R\|\varphi\|_{m-1} + C\|\varphi\|_{m-2}.$$

From the inequality in question 2°) we get

$$\|(x_j - a)\varphi\|_{m-1} \leqq R\|\varphi\|_{m-1} + C(R)\|\varphi\|_{m-1} \leqq C_1(R)\|\varphi\|_{m-1}.$$

It follows that $|③| \leq C_2(R)\|\varphi\|_{m-1}^2$. Now

$$|④| \leqq C\|P_m^*\varphi\| \cdot \|\varphi\|_{m-2} \leqslant C_1(R)\|P_m^*\varphi\|_{L^2}\|\varphi\|_{m-1}.$$

In the same way

$$|⑤| \leqq \|P_m^*\varphi\|_{L^2} \cdot \|(x_j - a)P_m^{(j)}\varphi\|_{L^2} \leqq R\|P_m^*\varphi\| \cdot \|\varphi\|_{m-1}.$$

From these inequalities we get

$$\|P_m^{(j)}\varphi\|_{L^2}^2 \leqq \frac{R}{2}(\|P_m\varphi\|_{L^2}^2 + \|P_m^{(j)}\varphi\|_{L^2}^2) + C_2(R)(\|P_m^*\varphi\|_{L^2}^2 + \|\varphi\|_{m-1}^2).$$

Taking R small enough to absorb the expression $\frac{R}{2}\|P_m^{(j)}\varphi\|_{L^2}^2$ of the right hand side by the left hand side, we get, since $P - P_m$ and $P^* - P_m^*$ have orders at most $m - 1$

$$\|P_m^{(j)}\varphi\|_{L^2}^2 \leqq C_3(R)(\|P\varphi\|_{L^2}^2 + \|P^*\varphi\|_{L^2}^2 + \|\varphi\|_{m-1}^2)$$

where $\lim_{R\to 0} C_3(R) = 0$.

4°) First of all

$$\sum_{j=1}^{n}\left|\frac{\partial p_m}{\partial \xi_j}(a, \xi)\right|^2 \geqq C_0|\xi|^{2(m-1)}.$$

Indeed denoting by $f(\xi)$ the left hande side we have by hypothesis $f(\xi) > 0 \quad \forall\xi \neq 0$ thus we can find a positive constant C_0 such that $f(\xi) \geqslant C_0$ for $|\xi| = 1$. By homogeneity in ξ we get $f(\xi) \geqslant C_0|\xi|^{2(m-1)}$ for $\xi \in \mathbb{R}^n\backslash 0$.
Moreover $\sum_{|\alpha|\leqslant m-1} |\xi^\alpha|^2 \leqslant C(1 + |\xi|^{2(m-1)})$. Indeed if $|\xi| \leqslant 1$ this sum is bounded by a constant and if $|\xi| \geqslant 1$ we have $|\xi^\alpha| \leqslant |\xi|^{|\alpha|} \leqslant |\xi|^{m-1}$.
It follows that

$$\sum_{|\alpha|\leqslant m-1} |\xi^\alpha|^2 \leqslant C\left(1 + \sum_{j=1}^{n} |P_m^{(j)}(a, \xi)|^2\right).$$

Multiplying by $|\hat{\varphi}|^2$ and integrating over $\mathbb{R}^n$ we get

$$\|\varphi\|_{m-1}^2 \leqslant C\left(\sum_{j=1}^{n} \|P_m^{(j)}(a, D)\varphi\|_{L^2}^2 + \|\varphi\|_0^2\right).$$

From question 2°) we have $\|\varphi\|^2{}_0 \leqq C(R)\|\varphi\|_{m-1}^2$ if $m \geqslant 2$ so we can absorb $\|\varphi\|_0^2$ by

$$\|\varphi\|_{m-1}^2 \leqslant C\left(\sum_{j=1}^{n} \|P_m^{(j)}(x, D)\varphi\|_{L^2}^2 + \sum_{j=1}^{n} \|P_m^{(j)}(x, D) - P_m^{(j)}(a, D))\varphi\|_{L^2}^2\right).$$

If $Q(x, D)$ is a differential operator of order $m - 1$ with C^1 coefficients and if $\operatorname{supp}\varphi \subset B(a, R)$ we have $\|Q(x, D)\varphi - Q(a, D)\varphi\|_{L^2}^2 \leqq CR^2\|\varphi\|_{m-1}^2$. It follows that if R is small enough

$$\|\varphi\|_{m-1}^2 \leqslant C\sum_{j=1}^{n} \|P_m^{(j)}(x, D)\varphi\|^2$$

5°) If $P = \sum_{|\alpha| \leq m} a_\alpha D^\alpha$ we have $P^* = \sum_{|\alpha| \leq m} D^\alpha \bar{a}_\alpha = \bar{P}_m + R_{m-1}$. If P_m has real coefficients $P^* - P$ is of order $m - 1$ so

$$\|P\varphi\|^2_{L^2} \leq C(\|P^*\varphi\|_{L^2} + \|\varphi\|^2_{m-1}) \qquad \forall \varphi \in C_0^\infty(B(a, R))$$

It follows from questions 3°) and 4°) that

$$\|\varphi\|^2_{m-1} \leqq C\|P^*\varphi\|^2_{L^2} \qquad \forall \varphi \in C_0^\infty(B(a, R))$$

6°) Let $f \in C_0^\infty(\Omega)$, $E = \{\psi = P^*\varphi : \varphi \in C_0^\infty(B(a, R))\} \subset L^2(B(a, R))$ and $l: E \to \mathbb{C}$ given by $l(P^*\varphi) = (f, \varphi)_{L^2}$. We have

$$|l(P^*\varphi)| \leqq \|f\|_{-(m-1)} \cdot \|\varphi\|_{m-1} \leqq C\|f\|_{-(m-1)}\|P^*\varphi\|_{L^2}$$

which proves that l is a continuous antilinear form on E equipped with the topology induced by L^2. Let $\hat{E}$ be the completion of E; it is a Hilbert subspace of L^2. By the Hahn-Banach theorem l can be extended to $\hat{E}$ as a continuous antilinear form. Therefore there exists $u \in \hat{E}$ such that $l(\psi) = (u, \psi)_{L^2}$ $\forall \psi \in \hat{E}$. If $\psi = P^*\varphi \in E$ we get

$$l(\psi) = l(P^*\varphi) = (f, \varphi)_{L^2} = (u, \psi)_{L^2} = (u, P^*\varphi)_{L^2} = (Pu, \varphi)_{L^2}$$

which proves that $Pu = f$ in $\mathscr{D}'(B(a, R))$.

7°) Every elliptic operator is of principal type since the set $p_m = 0$ is then empty. The wave operator $P = \frac{\partial^2}{\partial t^2} - \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$ is of principal type. Indeed

$$\operatorname{grad} p(x, \xi) = \begin{bmatrix} -2\tau \\ 2\xi_1 \\ \vdots \\ 2\xi_n \end{bmatrix} \neq 0 \quad \text{if } \xi \neq 0.$$

Solution 102

a) By hypothesis locally $D_t u = \sum_{|\alpha| \leq \mu} D_x^\alpha f_\alpha$, $f_\alpha \in C^0(I \times \Omega)$. Let $g_\alpha \in C^1(I \times \Omega)$ be such that $D_t g_\alpha = f_\alpha$ then $D_t\left(u - \sum_{|\alpha| \leq \mu} D_x^\alpha g_\alpha\right) = 0$ thus $u = \sum_{|\alpha| \leq \mu} D_x^\alpha g_\alpha + h(x) \in C^0(I, \mathscr{D}'(\Omega))$.

b) Let us assume by induction that $u \in C^k(I, \mathscr{D}'(\Omega))$, $k \geqslant 1$. Then

$\sum_{\substack{|\alpha| + j \leq 2 \\ j < 2}} a_{\alpha j} D_t^j D_x^\alpha u \in C^{k-1}(I, \mathscr{D}'(\Omega))$ and since $Pu = 0$ we have $D_t^2 u \in C^{k-1}(I, \mathscr{D}'(\Omega))$ thus $u \in C^{k+1}(I, \mathscr{D}'(\Omega))$.

c) We can reason locally, thus assume that $u = \sum\limits_{\substack{j \leqslant \nu \\ |\alpha| \leqslant \mu}} D_t^j D_x^\alpha u_{j\alpha}$ where $u_{j\alpha} \in C^0(I \times \Omega)$.

Case I: $\nu = 0$ i.e. $u \in C^0(I, \mathscr{D}'(\Omega))$. since $Pu = 0$ we get:

$$D_t\left(D_t u + \sum_{|\alpha| = 1} a_{1\alpha} D_x^\alpha + bu\right) \in C^0(I, \mathscr{D}'(\Omega)).$$

By question a) we deduce that $D_t u + \sum\limits_{|\alpha| = 1} a_{1\alpha} D_x^\alpha u + bu \in C^0(I, \mathscr{D}'(\Omega))$ thus $D_t u \in C^0(I, \mathscr{D}'(\Omega))$ i.e. $u \in C^1(I, \mathscr{D}'(\Omega))$.

Case II: $\nu > 1$ and $u = \sum\limits_\alpha \sum\limits_{j \leqslant \nu} D_t^j D_x^\alpha u_{j\alpha}$, $u_{j\alpha} \in C^0(I, \mathscr{D}'(\Omega))$

$$D_t^2 u = \sum_\alpha \sum_{j \leqslant \nu + 1} D_t^j D_x^\alpha v_{j\alpha} = \sum_\alpha D_x^\alpha v_{0\alpha} + \sum_{1 \leqslant j \leqslant \nu + 1} D_t^j D_x^\alpha v_{j\alpha}.$$

Let $w_\alpha \in C^1(I, \mathscr{D}'(\Omega))$ be such that $D_t w_\alpha = v_\alpha$ then

$$D_t\left(D_t u - \sum_\alpha \sum_{j \leqslant \nu} D_t^j D_x^\alpha w_{j\alpha}\right) = 0$$

so

$$D_t u - \sum_\alpha \sum_{j \leqslant \nu} D_t^j D_x^\alpha w_{j\alpha} = w(x) \quad \text{i.e.}$$

$$D_t u = \sum_\alpha \sum_{j \leqslant \nu} D_t^j D_x^\alpha w_{j\alpha} + \sum D_x^\beta w_\beta,\ w_\beta \in C^0(\Omega).$$

Iterating this argument we get

$$D_t\left(u - \sum_{j \leqslant \nu - 1} \sum_\alpha D_t^j D_x^\alpha f_{j\alpha}\right) = 0$$

thus

$$u = \sum_\alpha \sum_{j \leqslant \nu - 1} D_t^j D_x^\alpha g_{j\alpha},\ g_{j\alpha} \in C^0(I, \mathscr{D}'(\Omega)).$$

Distribution u has the same form as in the beginning of case II but with $\nu - 1$ instead of ν. Therefore after a finite number of steps we shall have $\nu = 0$ but then we shall be in case I where we concluded that $u \in C^1(I, \mathscr{D}'(\Omega))$.

Solution 103

I) a) If we set $u_0(t, x) = \psi(x) \underset{x}{*} G(t, x)$, where $\underset{x}{*}$ is the convolution in x, we shall have $\Box u_0 = \psi * \Box G = 0, u|_{t=0} = \psi * G|_{t=0} = 0, \left.\dfrac{\partial u_0}{\partial t}\right|_{t=0} = \psi * \left.\dfrac{\partial G}{\partial t}\right|_{t=0} = \psi * \delta = \psi$.

b) We have $\Box w = \partial_t \Box v = 0, w|_{t=0} = \left.\dfrac{\partial v}{\partial t}\right|_{t=0} = \varphi, \left.\dfrac{\partial w}{\partial t}\right|_{t=0} = \left.\dfrac{\partial^2 v}{\partial t^2}\right|_{t=0} = \sum_{i=1}^{n} \left.\dfrac{\partial^2 v}{\partial x_i^2}\right|_{t=0}$ $= \sum_{i=1}^{n} \dfrac{\partial^2}{\partial x_i^2}(v|_{t=0}) = 0$. With the function u_0 defined in question a) let us set $u_1 = u_0 + w$, then u_1 is clearly a solution of (2).

c) Let $u_2 = \displaystyle\int_0^t v(x, t - \tau, \tau)\, d\tau$ then

$$\frac{\partial u_2}{\partial t} = v(x, 0, t) + \int_0^t \frac{\partial v}{\partial t}(x, t - \tau, \tau)\, d\tau = \int_0^t \frac{\partial v}{\partial t}(x, t - \tau, \tau)\, d\tau$$

since $v|_{t=0} = 0$. Moreover

$$\frac{\partial^2 u_2}{\partial t^2} = \frac{\partial v}{\partial t}(x, 0, t) + \int_0^t \frac{\partial^2 v}{\partial t^2}(x, t - \tau, \tau)\, d\tau$$

and

$$\frac{\partial^2 u_2}{\partial x_i^2} = \int_0^t \frac{\partial^2 v}{\partial x_i^2}(x, t - \tau, \tau)\, d\tau$$

thus

$$\Box u_2 = f(x, t) + \int_0^t \Box v(x, t - \tau, \tau)\, dt = f(x, t)$$

Since $u_2|_{t=0} = \left.\dfrac{\partial u_2}{\partial t}\right|_{t=0} = 0$, the function $u = u_0 + u_1 + u_2$ is a solution to problem (**).

II) a) $\tilde{G}(t, \xi)$ satisfies the equation $\dfrac{\partial^2 \tilde{G}}{\partial t^2} + 4\pi^2|\xi|^2\tilde{G} = 0$ with the initial conditions $\tilde{G}|_{t=0} = 0, \left.\dfrac{\partial \tilde{G}}{\partial t}\right|_{t=0} = \mathscr{F}\delta = 1$. We deduce that $\tilde{G}(t, \xi) = \dfrac{\sin 2\pi t|\xi|}{2\pi|\xi|}$.

b) The distribution $T = \delta(|x| - a)$ has a compact support and is invariant by rotation.

By exercise 70 its Fourier transform is a C^∞ function invariant by rotation, in particular $\hat{T}(\xi_1, \xi_2, \xi_3) = \hat{T}(0, 0, |\xi|)$ and we can write

$$\hat{T}(\xi) = \langle \delta(|x| - a), e^{-2i\pi\langle x, \xi\rangle} \rangle = \langle \delta(|x| - a), e^{-2i\pi x_3|\xi|} \rangle = \int_{|x|=a} e^{-2i\pi x_3|\xi|} \mathrm{d}x$$

We use the polar coordinates

$$x_1 = a \sin \theta \cos \varphi$$

$$x_2 = a \sin \theta \sin \varphi$$

$$x_3 = a \cos \theta$$

and $\mathrm{d}x = a^2 \sin \theta \,\mathrm{d}\theta$ thus

$$\hat{T}(\xi) = a^2 \int_0^{2\pi} \int_0^{\pi} e^{-2i\pi a \cos\theta |\xi|} \sin \theta \,\mathrm{d}\theta \,\mathrm{d}\varphi$$

$$\hat{T}(\xi) = 2\pi a^2 \int_0^{\pi} e^{-2i\pi a \cos\theta |\xi|} \sin \theta \,\mathrm{d}\theta.$$

Let us set $u = \cos \theta$, we get

$$\hat{T}(\xi) = 2\pi a^2 \int_{-1}^{1} e^{-2i\pi a u|\xi|} = 2\pi a^2 \left[\frac{e^{-2i\pi a u|\xi|}}{-2i\pi a|\xi|} \right]_{-1}^{1},$$

$$\hat{T}(\xi) = 2\pi a^2 \frac{e^{-2i\pi a|\xi|} - e^{2i\pi a|\xi|}}{-2i\pi a|\xi|} = 2a \frac{\sin 2\pi a|\xi|}{|\xi|}$$

We deduce from question a)

$$\frac{1}{4\pi a} \hat{T}(\xi) = \frac{\sin 2\pi a|\xi|}{2\pi|\xi|} = \tilde{G}(a, \xi)$$

By inverse Fourier transform we get $G(t, x) = \dfrac{1}{4\pi t} \delta(t - |x|)$.

c) We have

$$u_0 = G \underset{x}{*} \psi = \langle G(t, x - y), \psi(y) \rangle = \frac{1}{4\pi t} \int_{|x-y|=t} \psi(y) \,\mathrm{d}y$$

Then

$$u_1 = u_0 + w = u_0 + \frac{\partial}{\partial t}\left(\frac{1}{4\pi t} \int_{|x-y|=t} \varphi(y) \,\mathrm{d}y \right)$$

Finally

$$u_2 = \int_0^t \frac{1}{4\pi(t-\tau)} \int_{|x-y|=t-\tau} f(y,\tau)\,\mathrm{d}y\,\mathrm{d}\tau$$

therefore

$$u(x,t) = \frac{1}{4\pi t}\int_{|x-y|=t} \psi(y)\,\mathrm{d}y + \frac{\partial}{\partial t}\left(\frac{1}{4\pi t}\int_{|x-y|=t} \varphi(y)\,\mathrm{d}y\right)$$
$$+ \int_0^t \frac{1}{4\pi(t-\tau)}\int_{|x-y|=t-\tau} f(y,\tau)\,\mathrm{d}y\,\mathrm{d}\tau$$

Solution 104

a) The claimed result asserts that if $u|_{t=0}$ and $\left.\frac{\partial u}{\partial t}\right|_{t=0}$ vanish in the ball $\{x: |x-x_0| \leq t_0\}$, then u vanishes in the cone $t + |x - x_0| \leq t_0$.

b) $$\frac{\mathrm{d}E}{\mathrm{d}t}(t) = \lim_{h\to 0}\frac{1}{2h}\left(\int_{B_{t+h}} |\nabla u|^2(t+h,x)\,\mathrm{d}x - \int_{B_t} |\nabla u|^2(t,x)\,\mathrm{d}x\right)$$

But $B_t = B_{t+h} \cup C_{t,h}$ where

$$C_{t,h} = \{x: t_0 - (t+h) \leq |x - x_0| \leq t_0 - t\}$$

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \lim_{h\to 0}\frac{1}{h}\left(\int_{B_{t+h}} \frac{1}{2}[|\nabla u|^2(t+h,x) - |\nabla u|^2(t,x)]\,\mathrm{d}x - \frac{1}{2}\int_{C_{t,h}} |\nabla u(t,x)|^2\,\mathrm{d}x\right)$$

The first integral tends, by the Lebesgue theorem, to $I = \frac{1}{2}\int_{B_t} \frac{\mathrm{d}}{\mathrm{d}t}(|\nabla u|^2)\,\mathrm{d}x$.

$$I = \frac{1}{2}\int_{B_t} 2\left(\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2} + \sum_{j=1}^{n} \frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_j \partial t}\right)\mathrm{d}x.$$

To compute the limit of the second integral we use the polar coordinates, i.e. we set $x - x_0 = r\omega$, $\omega \in S^{n-1}$.

$$J_h = \frac{1}{2h}\int_{C_{t,h}} |\nabla u(t,x)|^2\,\mathrm{d}x = \frac{1}{2h}\int_{S^{n-1}}\int_{t_0-(t+h)}^{t_0-t} |\nabla u(t, x_0 + r\omega)|^2 r^{n-1}\,\mathrm{d}r\,\mathrm{d}\omega$$

Now

$$\lim_{h\to 0}\int_{t_0-(t+h)}^{t_0-t} |\nabla u(t, x_0 + r\omega)|^2 r^{n-1}\,\mathrm{d}r = |\nabla u(t, x_0 + (t_0 - t)\omega)|^2 (t_0 - t)^{n-1}$$

so

$$\lim_{h\to 0} J_h = \frac{1}{2}\int_{S^{n-1}} |\nabla u|^2(t, x_0 + (t_0 - t)\omega)|^2(t_0 - t)^{n-1}\,\mathrm{d}\omega$$

$$\lim_{h\to 0} J_h = \frac{1}{2}\int_{|x-x_0|=t_0-t} |\nabla u|^2(t, x)\,\mathrm{d}\sigma$$

and $\{|x - x_0| = t_0 - t\} = \partial B_t$, this proves the result.

c) Since $\dfrac{\partial^2 u}{\partial t^2} = \sum_{j=1}^{n} \dfrac{\partial^2 u}{\partial x_j^2}$ we get

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \sum_{j=1}^{n}\int_{B_t}\left(\frac{\partial u}{\partial t}\cdot\frac{\partial^2 u}{\partial x_j^2} + \frac{\partial u}{\partial x_j}\cdot\frac{\partial^2 u}{\partial x_j \partial t}\right)\mathrm{d}x - \frac{1}{2}\int_{\partial B_t} |\nabla u|^2\,\mathrm{d}\sigma$$

$$= \sum_{j=1}^{n}\int_{B_t}\frac{\partial}{\partial x_j}\left(\frac{\partial u}{\partial t}\frac{\partial u}{\partial x_j}\right)\mathrm{d}x - \frac{1}{2}\int_{\partial B_t} |\nabla u|^2\,\mathrm{d}\sigma$$

and using the divergence theorem we get:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \int_{\partial B_t}\left(\frac{\partial u}{\partial t}\frac{\partial u}{\partial v} - \frac{1}{2}|\nabla u|^2\right)\mathrm{d}\sigma$$

d) We have

$$\left|\frac{\partial u}{\partial t}\frac{\partial u}{\partial v}\right| \leq \frac{1}{2}\left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\sum_{j=1}^{n} v_j \frac{\partial u}{\partial x_j}\right)^2\right]$$

$$\left|\frac{\partial u}{\partial t}\frac{\partial u}{\partial v}\right| < \frac{1}{2}\left[\left(\frac{\partial u}{\partial t}\right)^2 + \sum_{j=1}^{n} v_j^2 \sum_{j=1}^{n}\left(\frac{\partial u}{\partial x_j}\right)^2\right] \leqslant \frac{1}{2}|\nabla u|^2$$

since v has norm 1. It follows from the preceding question that $\dfrac{\mathrm{d}E}{\mathrm{d}t} \leq 0$.

But $E(0) = \dfrac{1}{2}\displaystyle\int_{B_0} |\nabla u|^2\,\mathrm{d}\sigma = 0$ since $\left.\dfrac{\partial u}{\partial t}\right|_{t=0}$ and $\left.\dfrac{\partial u}{\partial x_j}\right|_{t=0} = \dfrac{\partial}{\partial x_j}(u|_{t=0})$ vanish in B_0. We conclude that $E(t) \leq 0$ so $E(t) = 0$, $0 \leq t \leq t_0$, since obviously $E(t) \geqslant 0$. This implies that $|\nabla u(x, t)| = 0$ in $\Omega = \bigcup_{0\leq t\leq t_0} B_t$ thus u is constant in Ω and since $u = 0$ in B_0 it follows that $u = 0$ in Ω.

II) Let us assume $(x_0, t_0) \notin \Gamma$. Then one can find $\varepsilon > 0$ such that the set $B_0 = \{x : |x - x_0| \leqq t_0 + \varepsilon\}$ does not intersect Γ_0. Then $u|_{t=0} = \left.\dfrac{\partial u}{\partial t}\right|_{t=0} = 0$ in B_0. By the result proved above we have $u = 0$ in $\Omega = \{(x, t) : 0 \leqslant t \leqslant t_0 + \varepsilon, |x - x_0| \leqslant t_0 + \varepsilon - t\}$. In particular $u = 0$ in a neighborhood of (x_0, t_0) therefore $(x_0, t_0) \notin \operatorname{supp} u$.

Solution 105

a) k being semi-regular in y, the same is true for $\rho(x - y)k$ and the expression which gives v is well defined since $P(y, D_y)[\varphi(y)u(y)] \in \mathscr{E}'(\Omega_y)$. Now $\varphi = 1$ on V_1 thus the Leibniz formula shows that $P(\varphi u) = \varphi Pu$ in V_1. Assume $\{y \in \Omega : |y - x_0| < \delta\} \subset V_1$. On the support of $\rho(x - y)$ we have $|x - y| < \varepsilon$. If $|x - x_0| < \alpha$ and if ε and α are small, we shall have $|y - x_0| \leq |y - x| + |x - x_0| < \varepsilon + \alpha < \delta$ so $y \in V_1$. Therefore if

$|x - x_0| < \alpha$, $v(x) \overset{\Omega}{=} \langle \rho(x - y)k, \varphi(y)P(y, D_y)u(y)\rangle$. Since $Pu \in C^\infty$ and ρk is semi-regular in x then $v \in C^\infty$ for $|x - x_0| < \alpha$.

b) We have

$$v(x) = \langle {}^tP(y, D_y)[\rho(x - y)k], \varphi(y)u(y)\rangle$$

Since k is C^∞ for $x \neq y$ and $\rho = 1$ for $x = y$ the Leibniz formula shows that $w_1(x, y) = {}^tP(y, D_y)[\rho(x - y)k] - \rho(x - y){}^tP(y, D_y)k$ is a C^∞ function of (x, y). Now, k being a parametrix of ${}^tP$ and since $\rho(0) = 1$ we have $w_2(x, y) = \rho(x - y){}^tP(y, D_y)k - \delta(x - y) \in C^\infty(\Omega \times \Omega)$. Therefore

$$v(x) = \langle \delta(y - x), \varphi(y)u(y)\rangle + \langle w_1(x, y) + w_2(x, y), \varphi(y)u(y)\rangle$$

thus

$$v(x) - \varphi(x)u(x) = \langle w_1(x, y) + w_2(x, y), \varphi(y)u(y)\rangle \in C^\infty(\Omega)$$

From question a) we deduce that $\varphi u \in C^\infty$ near x_0 and since $\varphi = 1$ in V_1, u is C^∞ in a neighborhood of x_0 thus P is hypoelliptic.

Solution 106

a) Let $x \notin$ sing supp (u) which means that u is C^∞ near x. If $\varphi \in C^\infty$ has a compact support contained in a small neighborhood of x then $\varphi u \in C_0^\infty(\mathbb{R}^n) \subset \mathscr{S}(\mathbb{R}^n)$ thus $\widehat{\varphi u}(\xi) = 0(|\xi|^{-N})$ $\forall N$, $\forall \xi \in \mathbb{R}^n$, $|\xi| \to +\infty$ so $(x, \xi) \notin ss(u)$ $\forall \xi$ i.e. $x \notin \pi ss(u)$. This proves that $\pi ss(u) \subset$ sing supp (u). Conversely let $x \notin \pi ss(u)$ i.e. $\forall \xi \in \mathbb{R}^n$, $(x, \xi) \notin ss(u)$ then: $\exists V_x$ $\exists \Gamma_\xi$ such that $\widehat{\varphi u} = 0(|\xi|^{-N})$ $\forall N$, $\forall \xi \in \Gamma_\xi$, $|\xi| \to \infty$ and $\forall \varphi \in C_0^\infty(V_x)$. The sphere in $\mathbb{R}^n_\xi$ is compact thus there exists a finite number of $\xi, \xi_1, \ldots, \xi_k$ such that $\bigcup_{i=1}^{k} \Gamma_{\xi_i} = \mathbb{R}^n_\xi$ then $\forall \varphi \in C_0^\infty(V_x)$, $\widehat{\varphi u}(\xi) = 0(|\xi|^{-N})$ $\forall N, \forall \xi \in \Gamma_{\xi_i}$ thus $\forall \xi \in \mathbb{R}^n$, $|\xi| \to +\infty$ which proves that $\varphi u \in \mathscr{S}(\mathbb{R}^n)$ thus $u \in C^\infty$ near x i.e. $x \notin$ sing supp (u).

b) We show first that $\complement A \subset \complement\, ss(\chi)$.

1°) Let $(x_0', x_n^0, \xi', \xi_n)$ be such that $x_n^0 \neq 0$. Then χ is C^∞ near x^0 thus $(x, \xi) \notin ss(u)$ for all $\xi \in \mathbb{R}^n\backslash 0$.

2°) Let $(x_0', x_n^0, \xi_0', \xi_n{}^0)$ be such that $\xi_0' \neq 0$. Let us set $\dfrac{\xi_0'}{|\xi_0|} = 2\alpha$ and consider the following conic neighborhood of ξ^0:

$$\Gamma_{\xi_0} = \left\{\xi : \left|\frac{\xi}{|\xi|} - \frac{\xi_0}{|\xi_0|}\right| < \alpha\right\}.$$

If $\xi \in \Gamma_{\xi_0}$, $\left|\dfrac{\xi'}{|\xi|} - \dfrac{\xi_0'}{|\xi_0|}\right| < \alpha$ thus

$$|\xi'|\xi_0| - \xi_0'|\xi|| < \alpha|\xi| \cdot |\xi_0|$$

so

$$|\xi'| \cdot |\xi_0| > |\xi| \cdot |\xi_0'| - \alpha|\xi||\xi_0| = (|\xi_0'| - \alpha|\xi_0|)|\xi|$$

therefore

$$(1) \quad |\xi'| > \left(\frac{|\xi_0'|}{|\xi_0|} - \alpha\right)|\xi| = \alpha|\xi| \qquad \forall \xi \in \Gamma_{\xi_0}.$$

Moreover

$$|\xi'|^{2N}\widehat{\varphi\chi}(\xi) = \int e^{-2i\pi x\cdot\xi}|\xi'|^{N}\varphi(x)\,\mathrm{d}x$$

$$= \int_0^\infty e^{-2i\pi x_n\cdot\xi_n}\int_{\mathbb{R}^{n-1}} |\xi'|^{2N} e^{-2i\pi x'\cdot\xi'}\varphi(x)\,\mathrm{d}x'\,\mathrm{d}x_n.$$

Since we have $|\xi'|^{2N} e^{-ix'\cdot\xi'} = \dfrac{(-1)^N}{(2\pi)^{2N}}\Delta_{x'}^N e^{-2i\pi x'\cdot\xi'}$, integrating by parts in the x' integral we get

$$||\xi'|^{2N}(\widehat{\varphi\chi})(\xi)| = C_N\left|\int_0^\infty \int_{\mathbb{R}^{n-1}} e^{-2i\pi x\cdot\xi}(\Delta_{x'}^N\varphi)(x)\,\mathrm{d}x'\,\mathrm{d}x_n\right|$$

thus

$$|\widehat{\varphi\chi}(\xi)| \leq C_N'|\xi'|^{-2N}$$

it follows from (1) that

$$|\widehat{\varphi\chi}(\xi)| \leqslant C_N''|\xi|^{-N} \qquad \forall \xi \in \Gamma_{\xi_0}.$$

Let us prove now the converse, i.e.

$$\{(x', 0, 0, \xi_n)\} \subset ss(u).$$

Let $\varphi(x', x_n) = \varphi_0(x_n)\psi(x')$ where $\psi \in C_0^\infty$, $\displaystyle\int_{\mathbb{R}^{n-1}} \psi(x')\,\mathrm{d}x' = 1$

and $\qquad \varphi_0 \in C^\infty, \qquad \varphi_0 = \begin{cases} 1 & \text{if } |t| \leqslant 1 \\ 0 & \text{if } |t| \geqslant 2 \end{cases}.$

$$\xi_n \widehat{\varphi\chi}(0, \xi_n) = \int_{\mathbb{R}^{n-1}} \psi(x') \int_0^{+\infty} \frac{i}{2\pi} \frac{\partial}{\partial x_n} \mathrm{e}^{-2i\pi x_n \xi_n} \varphi_0(x_n)\,\mathrm{d}x_n\,\mathrm{d}x'$$

$$\xi_n \widehat{\varphi\chi}(0, \xi_n) = \frac{-i}{2\pi} \int_{\mathbb{R}^{n-1}} \psi(x') \int_0^{+\infty} \mathrm{e}^{-2i\pi x_n \xi_n} \frac{\partial \varphi_0}{\partial x_n}(x_n)\,\mathrm{d}x_n\,\mathrm{d}x'$$
$$+ \frac{i}{2\pi} \int_{\mathbb{R}^{n-1}} \psi(x')\left[\mathrm{e}^{-2i\pi x_n \xi_n} \varphi_0(x_n)\right]_0^{+\infty} \mathrm{d}x'$$

$$\xi_n \widehat{\varphi\chi}(0, \xi_n) = \frac{-i}{2\pi} \int_{\mathbb{R}^{n-1}} \int_0^{+\infty} \mathrm{e}^{-2i\pi x_n \xi_n} \psi(x') \frac{\partial \varphi_0}{\partial x_n}(x_n)\,\mathrm{d}x_n\,\mathrm{d}x' - \frac{i}{2\pi} \int_{\mathbb{R}^{n-1}} \psi(x')\,\mathrm{d}x'.$$

Since the support of $\psi(x')\dfrac{\partial \varphi_0}{\partial x_n}$ does not contain the points $(x', 0)$, using the same argument as in the first part (i.e. multiplication by ξ_n^k and integration by parts) we see that the first integral in the right hand side is rapidly decreasing in ξ_n when $|\xi_n| \to +\infty$ therefore

$$\xi_n(\widehat{\varphi\chi})(0, \xi_n) = \frac{-i}{2\pi} + O(|\xi_n|^{-N}) \qquad \forall N$$

which proves that $\widehat{\varphi\chi}$ is not rapidly decreasing.

Solution 107

a) On the support of ρ we have $\tau \geqslant 1$ thus $f(\eta/\sqrt{\tau})$ is C^∞ in (η, τ). Moreover

$$g \in L^\infty \qquad \text{therefore} \qquad g \in \mathscr{S}'(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n).$$

b) Let Γ be a conic neighborhood of $\xi_0 = (\eta_0, \tau_0)$ with $\eta_0 \neq 0$. We have

$$\Gamma = \left\{\xi \in \mathbb{R}^n \backslash 0 : \left|\frac{\xi}{|\xi|} - \frac{\xi_0}{|\xi_0|}\right| \leqslant \varepsilon\right\}.$$

Then $\left|\tau - \frac{\tau_0|\xi|}{|\xi_0|}\right| \leqslant \varepsilon|\xi|$ so

$$|\tau| \leqslant \varepsilon|\xi| + \frac{|\tau_0|}{|\xi_0|}|\xi| \leqslant \left(\varepsilon + \frac{\tau_0}{|\xi_0|}\right)|\xi|.$$

Since $\eta_0 \neq 0$ we have $\frac{\tau_0}{|\xi_0|} = 1-\alpha$ with $0 \leqslant \alpha < 1$. If ε is so small that $-\alpha+\varepsilon = \delta<1$ we have $(1 - \delta)|\tau| \leqslant \delta|\eta|$ so $|\tau| \leqslant \frac{\delta}{1-\delta}|\eta| = C|\eta|$.

Moreover

$$|\xi|^N|g(\xi)| \leqslant \sum_{l+k=N} C_{lk}|\eta|^l|\tau|^k\left|f\left(\frac{\eta}{\sqrt{\tau}}\right)\right||\rho(\tau)|$$

$$|\eta|^l|\tau|^k = \left(\frac{\eta}{\sqrt{\tau}}\right)^l(\sqrt{\tau})^{2k+l}$$

and since in Γ_{ξ_0} we have $|\tau| \leqslant C|\eta|$, we have $\sqrt{\tau} \leqslant C\frac{|\eta|}{\sqrt{\tau}}$. It follows that

$$|\xi|^N|g(\xi)| \leqslant \sum_{l+k=N} C_{lk}\left(\frac{|\eta|}{\sqrt{\tau}}\right)^{2k+2l}\left|f\left(\frac{\eta}{\sqrt{\tau}}\right)\right||\rho(\tau)| \leqslant C_N$$

since $f \in S(\mathbb{R}^{n-1})$ and $|\rho| \leqslant 1$.

Near the points $(0, \tau_0)$ where $\tau_0 < 0$ we have $\tau < 0$. Indeed

$$\left|\tau - \frac{\tau_0}{|\tau_0|}|\xi|\right| \leqslant \varepsilon|\xi|$$

implies

$$\tau \leqslant \frac{\tau_0}{|\tau_0|}|\xi| + \varepsilon|\xi| = \left(\frac{\tau_0}{|\tau_0|} + \varepsilon\right)|\xi| < 0$$

if ε is small enough. But $\rho(\tau) = 0$ if $\tau < 0$ thus g vanishes in Γ_{ξ_0}.

c) $D_\eta^\alpha\left[f\left(\frac{\eta}{\sqrt{\tau}}\right)\right] = \tau^{-\frac{|\alpha|}{2}}(D^\alpha f)\left(\frac{\eta}{\sqrt{\tau}}\right)$ thus by the Leibniz formula

$$I = D_\tau^k D_\eta^\alpha\left[f\left(\frac{\eta}{\sqrt{\tau}}\right)\rho(\tau)\right]$$

$$= \sum_{k_1+k_2+k_3=k} c_{k,k_1,k_2,k_3,\alpha}\tau^{-\frac{|\alpha|}{2}-k_1} D_\tau^{k_2}\left[(D^\alpha f)\left(\frac{\eta}{\sqrt{\tau}}\right)\right] D_\tau^{k_3}\rho(\tau).$$

We show easily, by induction on k, that

$$D_\tau^k\left[g\left(\frac{\eta}{\sqrt{\tau}}\right)\right] = \sum_{l=1}^{k}\sum_{|\beta|=l} a_{k\beta}\tau^{-\frac{3l}{2}-(k-l)}\eta^\beta(D^\beta g)\left(\frac{\eta}{\sqrt{\tau}}\right).$$

It follows that

$$I \leqslant \sum_{k_1+k_2+k_3=k}\sum_{l=1}^{k_2}\sum_{|\beta|=l} a_{k,k_1,\alpha,\beta}|\tau|^{-\frac{|\alpha|}{2}-\frac{l}{2}-k_1-k_2}|\eta|^l\left|D^{\alpha+\beta}f\left(\frac{\eta}{\sqrt{\tau}}\right)\right||D_\tau^{k_3}\rho(\tau)|.$$

Since supp $\rho' \subset \{1 \leqq \tau \leqq 2\}$ we have $|D_\tau^{k_3}\rho(\tau)| \leqq C\tau^{-k_3}|D_\tau^{k_3}\rho(\tau)| \leqq C_{k_3}\tau^{-k_3}$. Moreover $\tau^{-\frac{l}{2}}|\eta|^l D^{\alpha+\beta}f\left(\frac{\eta}{\sqrt{\tau}}\right)\Big| \leqq C_{\alpha\beta l}$, since $f \in \mathscr{S}(\mathbb{R}^{n-1})$. Then

$$I \leqslant \sum_{k_1+k_2+k_3=k}\sum_{l=1}^{k_2}\sum_{|\beta|=l} C_{k,k_1,\alpha,\beta}\tau^{-\frac{|\alpha|}{2}-k_1-k_2-k_3} \leqq M\tau^{-\frac{|\alpha|}{2}-k}$$

1°) If $|\eta| \leqslant |\tau|$ then $1 + |\xi| \leqq C\tau$ thus $I \leqq C_1(1 + |\xi|)^{-\frac{|\alpha|}{2}-k}$.

2°) If $|\tau| \leqq |\eta|$ then $1 + |\xi| \leqq C|\eta|$.

The expression which bounds I is a sum of terms of the form $\tau^{-A}|\eta|^B\left|f^{(\alpha)}\left(\frac{\eta}{\sqrt{\tau}}\right)\right|$.

The later is bounded by

$$\frac{\tau^{N/2}}{|\eta|^{N-B}}\left(\frac{|\eta|}{\sqrt{\tau}}\right)^N|f^{(\alpha)}(\ldots)| \leqslant \frac{C_{N\alpha}}{|\eta|^{\frac{N}{2}-B}} \leqq C'_N(1 + |\xi|)^{-\frac{N}{2}-B} \qquad \forall N \geqslant 0.$$

d) $|\mathscr{F}(x^\alpha D^\beta u)| = |D_\xi^\alpha(\xi^\beta g)| \leqq \sum_{\gamma\leqslant\alpha} C_{\alpha\gamma}|D_\xi^\gamma\xi^\beta \cdot D_\xi^{\alpha-\gamma}g|$.

Using the previous question we have $|D_\xi^{\alpha-\gamma}g| \leqq C(1 + |\xi|)^{-\frac{|\alpha|-|\gamma|}{2}}$ and $D_\xi^\gamma\xi^\beta = c_{\beta\gamma}\xi^{\beta-\gamma}$ if $\gamma \leqslant \beta$ and $D_\xi^\gamma\xi^\beta = 0$ if $\gamma > \beta$. It follows that

$$|D_\xi^\alpha(\xi^\beta g)| \leqslant C\sum_{\gamma\leqslant\alpha}(1 + |\xi|)^{|\beta|-|\gamma|}(1 + |\xi|)^{-\frac{1}{2}(|\alpha|-|\gamma|)} \leqslant C'(1 + |\xi|)^{-\frac{1}{2}|\alpha|+|\beta|}.$$

Therefore if $\frac{|\alpha|}{2} - |\beta| > n$, $\mathscr{F}(x^\alpha D^\beta u) \in L^1(\mathbb{R}^n)$ thus $x^\alpha D^\beta u$ is a continuous function on $\mathbb{R}^n$ and $D^\beta u \in C^0(\mathbb{R}^n \backslash 0)$ for all β. Q.E.D.

e) $\psi * h(\xi) = \int h(\zeta)\psi(\xi-\zeta)\,d\zeta$

$$|\xi^\alpha \psi * h(\xi)| \leqq |\xi|^{|\alpha|} \underbrace{\int_{\zeta \in \Gamma_{\xi_0}} |h(\zeta)| \cdot |\psi(\xi - \zeta)|\,d\zeta}_{①} + \underbrace{|\xi|^{|\alpha|} \int_{\zeta \notin \Gamma_{\xi_0}} h(\zeta)| \cdot |\psi(\xi - \zeta)|\,d\zeta}_{②}.$$

$$① \leqq C\Big\{ \underbrace{\int_{\zeta \in \Gamma_{\xi_0}} |\zeta|^{|\alpha|}|h(\zeta)||\psi(\xi - \zeta)|\,d\zeta}_{a)} + \underbrace{\int_{\zeta \in \Gamma_{\xi_0}} |\xi - \zeta|^{|\alpha|}|h(\zeta)||\psi(\xi - \zeta)|\,d\zeta}_{b)}.$$

Term a) is bounded for h is rapidly decreasing in Γ_{ξ_0}. The same is true for b) since ψ is in $S(\mathbb{R}^{n-1})$. Concerning expression ② if $\xi \in \tilde{\Gamma}_{\xi_0} \subset \subset \Gamma_{\xi_0}$ and $\zeta \notin \Gamma_{\xi_0}$ we have $|\xi - \zeta| \geqslant c|\xi|$. Indeed we have

$$\left|\frac{\xi}{|\xi|} - \frac{\xi_0}{|\xi_0|}\right| \leqslant \varepsilon \qquad \text{and} \qquad \left|\frac{\zeta}{|\zeta|} - \frac{\xi_0}{|\xi_0|}\right| \geqslant \varepsilon'$$

thus

$$\left|\frac{\xi}{|\xi|} - \frac{\zeta}{|\zeta|}\right| \geqslant |\varepsilon' - \varepsilon| = \delta.$$

Then

$$\delta|\xi| \leqq \left|\xi - |\xi|\frac{\zeta}{|\zeta|}\right| = \left|\xi - \zeta + \zeta\frac{|\zeta| - |\xi|}{|\zeta|}\right| \leqslant |\xi - \zeta| + ||\xi| - |\zeta|| \leqslant 2|\xi - \zeta|.$$

It follows that

$$|② \leqq C \int_{\zeta \notin \Gamma_{\xi_0}} h(\zeta)| \cdot |\xi - \zeta|^{|\alpha|}|\psi(\xi - \zeta)|\,d\zeta \leqq C'$$

for ψ is in $S(\mathbb{R}^{n-1})$.

We deduce that setting $\hat{\varphi} = \psi$, $\hat{u} = h$, $\widehat{\varphi u}$ is rapidly decreasing in every cone where h is rapidly decreasing.

f) It follows from questions b), d), e) that the points (x, η, τ) such that $x \neq 0$ or $|\eta| \neq 0$ or $\tau < 0$ are not in the singular spectrum of u thus

$$ss(u) \subset \{(x, \eta, \tau) : x = 0, \eta = 0, \tau > 0\}.$$

g) Let $\varphi \in C_0^\infty(\mathbb{R}^n)$, $\varphi = 1$ near the origin. We have

$$\widehat{\varphi u}(0, 0, \lambda) = \iint \hat{\varphi}(\eta, \tau) f\left(\frac{-\eta}{\sqrt{\lambda - \tau}}\right) \rho(\lambda - \tau)\, d\eta\, d\tau.$$

Now $\rho(\lambda - \tau) = 0$ if $\lambda - \tau \leqslant 1$ thus

$$\widehat{\varphi u}(0, 0, \lambda) = \int_{\mathbb{R}^{n-1}} \int_{-\infty}^{\lambda - 1} \hat{\varphi}(\eta, \tau) f\left(\frac{-\eta}{\sqrt{\lambda - \tau}}\right) \rho(\lambda - \tau)\, d\eta\, d\tau.$$

Let us set $f_\lambda(\eta, \tau) = f\left(\dfrac{-\eta}{\sqrt{\lambda - \tau}}\right) \rho(\lambda - \tau) \hat{\varphi}(\eta, \tau) \mathbf{1}_{]-\infty, \lambda - 1]}$.

We have

- $|f_\lambda(\eta, \tau)| \leqslant M \hat{\varphi}(\eta, \tau) \in L^1$
- $f_\lambda(\eta, \tau) \to f(0)\hat{\varphi}(\eta, \tau), \quad \lambda \to +\infty.$

Thus $\widehat{\varphi u}(0, 0, \lambda) \to f(0) \displaystyle\int \hat{\varphi}(\xi)\, d\xi \neq 0$ if $f(0) \neq 0$ and $\displaystyle\int \hat{\varphi}(\xi)\, d\xi \neq 0$ which proves that $\widehat{\varphi u}(0, 0, \lambda)$ is not rapidly decreasing for $\lambda \to +\infty$ thus $(0, 0, \lambda) \in ss(u)$ for $\lambda > 0$.

Solution 108

a) Let us show that $(x_0, \xi_0) \notin ss(u)$ if $\xi_0 < 0$. Let V_{x_0} be a neighborhood of x_0 and $\varphi \in C_0^\infty(V_{x_0})$.

Since $u = \bar{\mathscr{F}}\left(\dfrac{\chi_{\mathbb{R}_+}}{(1 + \xi^2)^2}\right)$ where $\chi_{\mathbb{R}_+}$ is the characteristic function of $\mathbb{R}_+$ we have

$$\widehat{\varphi u}(\xi) = \hat{\varphi} * \hat{u}(\xi) = \int \chi_{\mathbb{R}_+}(\eta) \frac{\hat{\varphi}(\xi - \eta)}{(1 + \eta^2)^2}\, d\eta$$

$$= \int_0^{+\infty} \frac{\hat{\varphi}(\xi - \eta)}{(1 - \eta^2)^2} = \int_{-\infty}^{\xi} \frac{\hat{\varphi}(\zeta)}{(1 + |\xi - \zeta|^2)^2}$$

thus $|\widehat{\varphi u}(\xi)| \leqq \displaystyle\int_{-\infty}^{\xi} |\hat{\varphi}(\zeta)|\, d\zeta$. Since $\varphi \in \mathscr{S}(\mathbb{R})$ we get

$$|\hat{\varphi}(\xi)| \leqq C_N (1 + \xi^2)^{-N-1}.$$

If $\xi < 0$, since $\zeta \leqslant \xi$ in the integral, we have $\zeta^2 \geqslant \xi^2$ and $\dfrac{1}{1+\zeta^2} \leqslant \dfrac{1}{1+\xi^2}$. Then

$$|\varphi u(\xi)| \leqslant C_N(1+\xi^2)^{-N} \int_{-\infty}^{\xi} \frac{d\zeta}{1+\zeta^2} \leqq C'_N(1+\xi^2)^{-N} \qquad \forall N \in \mathbb{N}.$$

Let us prove that $(x_0, \xi) \notin ss(u)$ if $x_0 \neq 0$. Indeed

$$x^k u = (2i\pi)^{-k} \int_0^{+\infty} \left(\frac{\partial}{\partial\xi}\right)^k e^{2i\pi x.\xi} \frac{d\xi}{(1+\xi^2)^2}.$$

Integrating by parts we get

$$x^k u = -(2i\pi)^{-k}\left(\int_0^{+\infty} \left(\frac{\partial}{\partial\xi}\right)^{k-1} e^{2i\pi x.\xi} g(\xi)\, d\xi + \left[\left(\frac{\partial}{\partial\xi}\right)^{k-1} e^{2i\pi x\xi} \frac{1}{(1+\xi^2)^2}\right]_0^{+\infty}\right)$$

$$x^k u = -(2i\pi)^{-k}\left(\int_0^{+\infty} \left(\frac{\partial}{\partial\xi}\right)^{k-1} e^{2i\pi x\xi} g(\xi)\, d\xi - (2i\pi)^{k-1} x^{k-1}\right)$$

where
$$|g(\xi)| \leq \frac{C}{1+\xi^5}.$$

Iterating k times this integration we see that, modulo a C^∞ function, we have

$$x^k u = c_k \int_0^{+\infty} e^{2i\pi x\xi} \left(\frac{\partial}{\partial\xi}\right)^k \frac{1}{(1+\xi^2)^2}\, d\xi.$$

Since $\left(\frac{\partial}{\partial\xi}\right)^k \frac{1}{(1+\xi^2)^2} \leqslant \frac{C}{1+\xi^{k+4}}$, we see that the function $x^k u$ belongs to $C^{k+2}(\mathbb{R})$ thus $u \in C^{k+2}(\mathbb{R}\backslash 0)$, for all $k \in \mathbb{N}$. Therefore $u \in C^\infty(\mathbb{R}\backslash 0)$.

Let us show now that $(0, \xi_0) \in ss(u)$ if $\xi_0 > 0$.

Let V_0 be a neighborhood of the origin, $\psi \in C_0^\infty(V_0)$ such that $\psi(0) = 1$ and $\hat{\psi} \geqslant 0$. Let us set $\varphi = (1 - \Delta)^2 \psi$. By the previous computation and from question a) of exercise 94

$$\widehat{\varphi u}(\xi) = \int_{-\infty}^{\xi} \frac{\hat{\varphi}(\zeta)\, d\zeta}{(1+|\xi-\zeta|^2)^2} \geqslant \int_{-\infty}^{\xi} \frac{\hat{\varphi}(\zeta)\, d\zeta}{(1+\zeta^2)^2(1+\xi^2)^2}$$

$$\geqslant \int_{-\infty}^{\xi} \frac{(1+\zeta^2)^2}{(1+\zeta^2)^2} \cdot \frac{\hat{\psi}(\zeta)}{(1+\xi^2)^2}\, d\xi$$

But $\lim_{\xi\to+\infty}\int_{-\infty}^{\xi}\hat{\psi}(\zeta)\,d\zeta = \int\hat{\psi}(\zeta)\,d\zeta = \psi(0) = 1$ thus for positive and large ξ we have

$$(1+\xi^2)^2\widehat{\varphi u}(\xi) \geqslant \frac{1}{2} \quad \text{Q.E.D}$$

b) If we differentiate v with respect to t or x then a ξ appears in the numerator inside the integral. Since $e^{-\pi t^2\xi} \leqq 1$ and $\dfrac{|\xi|}{(1+\xi^2)^2} \leqq \dfrac{c}{(1+\xi^2)^{3/2}}$ it follows that $v \in C^1$ and it is easy to see that $Lv = 0$.

c) Since the singular spectrum of $v(x, 0) = u(x)$ contains points of the form $(0, \xi)$ it follows clearly that L is not hypoelliptic.

Solution 109

a) $$\frac{d}{dt}[u(y(t), t)] = \left(\frac{\partial u}{\partial x}\frac{dy}{dt} + \frac{\partial u}{\partial t}\right)(y(t), t)$$

$$= \left(\frac{\partial u}{\partial x} + a(u)\frac{\partial u}{\partial t}\right)(y(t), t) = 0$$

by (3) and (1). Thus $u(y(t), t) = u(y(0), 0) = u_0(x_0)$. It then follows from (3) that $\dfrac{dy}{dt} = a(u_0(x_0)),\ y(0) = x_0$ thus $y(t) = x_0 + ta(u_0(x_0))$.

b) The map $F_t\colon \mathbb{R} \to \mathbb{R}$ is a C^1 diffeomorphism if and only if $\dfrac{d}{dx_0}F_t \neq 0$ on $\mathbb{R}$. But $\dfrac{d}{dx_0}F_t(x_0) = 1 + ta'(u_0(x_0))u'(x_0)$. This expression does not vanish for all t and all x_0 if and only if (2) is satisfied.

c) Let $u(x, t) = u(G_t(x))$. Setting $G(x, t) = G_t(x)$, we have by definition

$$x = G(x, t) + ta(u_0(G(x, t))).$$

Differentiating both sides with respect to x and t we get

$$1 = \frac{\partial G}{\partial x}(x, t) + t(a \circ u_0)'(G(x, t)) \cdot \frac{\partial G}{\partial x}(x, t)$$

$$O = \frac{\partial G}{\partial x}(x, t) + a(u_0(G(x,t)) + t(a \circ u_0)'(G(x, t)) \cdot \frac{\partial G}{\partial t}(x, t)$$

therefore

$$\left(\frac{\partial u}{\partial t} + a(u)\frac{\partial u}{\partial x}\right)(x, t) = u_0'(G(x, t)))\left[\frac{-a(u_0(G(x, t))}{1 + ta'(u_0(G(x, t))u_0'(G(x, t))}\right] +$$

$$+ a(u_0(G(x, t)))u_0'(G(x, t)) \cdot \frac{1}{1 + ta'(u_0(G(x, t))u_0'(G(x, t))} = 0.$$

d) Let us assume that Min $(a(u_0(x_0))u_0'(x_0)) = a(u_0(y_0))u'(y_0) = m$.
We have by the previous computaton

$$(5) \quad \frac{\partial u}{\partial x}(x, t) = \frac{u_0'(y_0)}{1 + ta'(u_0(y_0))u_0'(y_0)}.$$

When $0 \leqslant t < -\frac{1}{m}$ the same function defines a C^1 solution since

$$1 + ta'(u_0(y))u_0'(y) \geqslant 1 + tm > 0$$

and from (5) $\lim_{t \to -\frac{1}{m}} \left|\frac{\partial u}{\partial x}\right| = +\infty$.

e) It follows that the largest T such that the solution exists in $[0, T]$ is given by

$$T = \frac{-1}{\operatorname{Inf}_{y\in\mathbf{R}} (a'(u_0(y))u_0'(y))} = \frac{1}{\operatorname{Max}_{y\in\mathbf{R}} (-a'(u_0(y))u_0'(y))}.$$

BIBLIOGRAPHY

[1] J. DIEUDONNÉ: *Foundations of modern analysis*. Academic Press, New York, 1969.

[2] F.G. FRIEDLANDER: *Introduction to the theory of distributions*. Cambridge University Press, 1982.

[3] L. SCHWARTZ: *Théorie des distributions*. Hermann, 1966.

[4] F. TRÈVES: *Topological vector spaces, distributions and kernels* . Academic Press, New York, 1967.

[5] F. TRÈVES: *Basic linear partial differential equations*. Academic Press, New York, 1975.

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