HANDBOOK OF DIFFERENTIAL EQUATIONS

Ordinary Differential Equations

VOLUME 3

Edited by

A. Cañada

P. Drábek

A. Fonda

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VOLUME III

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Preface

This is the third volume in a series devoted to self contained and up-to-date surveys in the theory of ordinary differential equations, written by leading researchers in the area. All contributors have made an additional effort to achieve readability for mathematicians and scientists from other related fields, in order to make the chapters of the volume accessible to a wide audience. These ideas faithfully reflect the spirit of this multi-volume and the editors hope that it will become very useful for research, learning and teaching. We express our deepest gratitude to all contributors to this volume for their clearly written and elegant articles.

This volume consists of seven chapters covering a variety of problems in ordinary differential equations. Both, pure mathematical research and real word applications are reflected pretty well by the contributions to this volume. They are presented in alphabetical order according to the name of the first author. The paper by Andres provides a comprehensive survey on topological methods based on topological index, Lefschetz and Nielsen numbers. Both single and multivalued cases are investigated. Ordinary differential equations are studied both on finite and infinite dimensions, and also on compact and noncompact intervals. There are derived existence and multiplicity results. Topological structures of solution sets are investigated as well. The paper by Bonheure and Sanchez is dedicated to show how variational methods have been used in the last 20 years to prove existence of heteroclinic orbits for second and fourth order differential equations having a variational structure. It is divided in 2 parts: the first one deals with second order equations and systems, while the second one describes recent results on fourth order equations. The contribution by De Coster, Obersnel and Omari deals with qualitative properties of solutions of two kinds of scalar differential equations: first order ODEs, and second order parabolic PDEs. Their setting is very general, so that neither uniqueness for the initial value problems nor comparison principles are guaranteed. They particularly concentrate on periodic solutions, their localization and possible stability. The paper by Han is dedicated to the theory of limit cycles of planar differential systems and their bifurcations. It is structured in three main parts: general properties of limit cycles, Hopf bifurcations and perturbations of Hamiltonian systems. Many results are closely related to the second part of Hilbert's 16th problem which concerns with the number and location of limit cycles of a planar polynomial vector field of degree n posed in 1901 by Hilbert. The survey by Hartung, Krisztin, Walther and Wu reports about the more recent work on state-dependent delayed functional differential equations. These equations appear in a natural way in the modelling of evolution processes in very different fields: physics, automatic control, neural networks, infectious diseases, population growth, cell biology, epidemiology, etc. The authors emphasize on particular models and on the emerging theory from the dynamical systems point of view. The paper by Korman is devoted to two point nonlinear boundary value problems depending on a parameter λ. The main question is the precise number of solutions of the problem and how these solutions change with the parameter. To study the problem, the author uses bifurcation theory based on the implicit function theorem (in Banach spaces) and on a well known theorem by Crandall and Rabinowitz. Other topics he discusses involve pitchfork bifurcation and symmetry breaking, sign changing solutions, etc. Finally, the paper by Rachůnková, Staněk and Tvrdý is a survey on the solvability of various nonlinear singular boundary value problems for ordinary differential equations on the compact interval. The nonlinearities in differential equations may be singular both in the time and space variables. Location of all singular points need not be known.

With this volume we end our contribution as editors of the Handbook of Differential Equations. We thank the staff at Elsevier for efficient collaboration during the last three years.

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CHAPTER 1

Topological Principles for Ordinary Differential Equations

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HANDBOOK OF DIFFERENTIAL EQUATIONS
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1. Introduction

The classical courses of ordinary differential equations (ODEs) start either with the Peano existence theorem (see, e.g., [54]) or with the Picard–Lindelöf existence and uniqueness theorem (see, e.g., [71]), both related to the Cauchy (initial value) problems

$$\begin{cases} \dot{x} = f(t,x), \\ x(0) = x_0, \end{cases} \tag{1.1}$$

where $f \in C([0,\tau] \times \mathbb{R}^n, \mathbb{R}^n)$, and

$$\left| f(t,x) - f(t,y) \right| \leqslant L|x-y|, \quad \text{for all } t \in [0,\tau] \text{ and } x,y \in \mathbb{R}^n, \tag{1.2}$$

in the latter case.

In fact, if f satisfies the Lipschitz condition (1.2), then "*uniqueness implies existence*" even for boundary value problems with linear conditions that are "close" to $x(0) = x_0$, as observed in [53]. Moreover, *uniqueness implies* in general (i.e. not necessarily, under (1.2)) *continuous dependence of solutions on initial values* (see, e.g., [54, Theorem 4.1 in Chapter 4.2]), and subsequently the *Poincaré translation operator* $T_\tau : \mathbb{R}^n \to \mathbb{R}^n$, at the time $\tau > 0$, along the trajectories of $\dot{x} = f(t,x)$, defined as follows:

$$T_\tau(x_0) := \big\{ x(\tau) \mid x(.) \text{ is a solution of (1.1)} \big\}, \tag{1.3}$$

is a homeomorphism (cf. [54, Theorem 4.4 in Chapter 4.2]).

Hence, besides the existence, uniqueness is also a very important problem. W. Orlicz [92] showed in 1932 that the set of continuous functions $f : U \to \mathbb{R}^n$, where U is an open subset relative to $[0,\tau] \times \mathbb{R}^n$, for which problem (1.1) with $(0, x_0) \in U$ is not uniquely solvable, is meager, i.e. a set of the first Baire category. In other words, the generic continuous Cauchy problems (1.1) are solvable in a unique way. Therefore, no wonder that the first example of nonuniqueness was constructed only in 1925 by M.A. Lavrentev (cf. [71] and, for more information, see, e.g., [1]). The same is certainly also true for Carathéodory ODEs, because the notion of a *classical* (C^1-) *solution* can be just replaced by the *Carathéodory solution*, i.e. absolutely continuous functions satisfying (1.1), almost everywhere (a.e.). The change is related to the application of the Lebesgue integral, instead of the Riemann integral.

On the other hand, H. Kneser [80] proved in 1923 that the sets of solutions to continuous Cauchy problems (1.1) are, at every time, continua (i.e. compact and connected). This result was later improved by M. Hukuhara [75] who proved that the solution set itself is a continuum in $C([0,\tau], \mathbb{R}^n)$. N. Aronszajn [41] specified in 1942 that these continua are R_δ-sets (see Definition 2.3 below), and as a subsequence, multivalued operators T_τ in (1.3) become *admissible* in the sense of L. Górniewicz (see Definition 2.5 below).

Obviously if, for $f(t,x) \equiv f(t+\tau,x)$, operator T_τ admits a *fixed point*, say $\hat{x} \in \mathbb{R}^n$, i.e. $\hat{x} \in T_\tau(\hat{x})$, then $\hat{x}$ determines a τ-periodic solution of $\dot{x} = f(t,x)$, and vice versa. This is one of stimulations why to study the fixed point theory for multivalued mappings in order to obtain periodic solutions of nonuniquely solvable ODEs. Since the regularity of (multivalued) Poincaré's operator T_τ is the same (see Theorem 4.17 below) for differential inclusions $\dot{x} \in F(t, x)$, where F is an upper Carathéodory mapping with nonempty, convex and compact values (see Definition 2.10 below), it is reasonable to study directly such differential inclusions with this respect. Moreover, initial value problems for differential inclusions are, unlike ODEs, typically nonuniquely solvable (cf. [42]) by which Poincaré's operators are multivalued.

In this context, an interesting phenomenon occurs with respect to the Sharkovskii cycle coexistence theorem [95]. This theorem is based on a new ordering of the positive integers, namely

$$\begin{aligned}&3 \rhd 5 \rhd 7 \rhd \cdots \rhd 2\cdot 3 \rhd 2\cdot 5 \rhd 2\cdot 7 \rhd \cdots \rhd 2^2\cdot 3 \rhd 2^2\cdot 5 \rhd 2^2\cdot 7 \rhd \cdots\\ &\quad \rhd 2^n\cdot 3 \rhd 2^n\cdot 5 \rhd 2^n\cdot 7 \rhd \cdots \rhd 2^{n+1}\cdot 3 \rhd 2^{n+1}\cdot 5 \rhd 2^{n+1}\cdot 7 \rhd \cdots\\ &\quad \rhd 2^{n+1} \rhd 2^n \rhd \cdots \rhd 2^2 \rhd 2 \rhd 1,\end{aligned}$$

saying that *if a continuous function* $g : \mathbb{R} \to \mathbb{R}$ *has a point of period* m *with* $m \rhd k$ (in the above Sharkovskii ordering), *then it has also a point of period* k.

By a period, we mean the least period, i.e. a point $a \in \mathbb{R}$ is a *periodic point of period* m if $g^m(a) = a$ and $g^j(a) \neq a$, for $0 < j < m$.

Now, consider the scalar ODE

$$\dot{x} = f(t, x), \quad f(t, x) \equiv f(t + \tau, x), \tag{1.4}$$

where $f : [0, \tau] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

Since

$$T_\tau^m = \underbrace{T_\tau \circ \cdots \circ T_\tau}_{m \text{ times}} = T_{m\tau}$$

holds for the Poincaré translation operator T_τ along the trajectories of Eq. (1.4), defined in (1.3), there is (in the case of uniqueness) an apparent one-to-one correspondence between m-periodic points of T_τ and (subharmonic) $m\tau$-periodic solutions of (1.4). Nevertheless, the analogy of classical Sharkovskii's theorem does not hold for subharmonics of (1.4). In fact, we only obtain an empty statement, because every bounded solution of (1.4) is, under the uniqueness assumption, either τ-periodic or asymptotically τ-periodic (see, e.g., [94, pp. 120–122]).

This handicap is due to the assumed uniqueness condition. On the other hand, in the lack of uniqueness, the multivalued operator T_τ in (1.3) is admissible (see Theorem 4.17 below) which in $\mathbb{R}$ means (cf. Definition 2.5 below) that T_τ is upper semicontinuous (cf. Definition 2.4 below) and the sets of values consist either of single points or of compact intervals. In a series of our papers [16,29,36], we developed a version of the Sharkovskii cycle coexistence theorem which applies to (1.4) as follows:

THEOREM 1.1. *If Eq.* (1.4) *has an* $m\tau$*-periodic solution*, *then it also admits a* $k\tau$*-periodic solution*, *for every* $k \lhd m$, *with at most two exceptions*, *where* $k \lhd m$ *means that* k *is less*

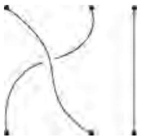

Fig. 1. Braid σ.

than m in the above Sharkovskii ordering of positive integers. In particular, if $m \neq 2^k$, *for all* $k \in \mathbb{N}$, *then infinitely many* (*subharmonic*) *periodic solutions of* (1.4) *coexist*.

REMARK 1.1. As pointed out, Theorem 1.1 holds only in the lack of uniqueness; otherwise, it is empty. On the other hand, the right-hand side of the given (multivalued) ODE can be a (multivalued upper) Carathéodory mapping with nonempty, convex and compact values (see Definition 2.10 below).

REMARK 1.2. Although, e.g., a 3τ-periodic solution of (1.4) implies, for every $k \in \mathbb{N}$, with a possible exception for $k = 2$ or $k = 4, 6$, the existence of a $k\tau$-periodic solution of (1.4), it is very difficult to prove that such a solution exists. Observe that a 3τ-periodic solution of (1.4) implies the existence of at least two more 3τ-periodic solutions of (1.4).

The Sharkovskii phenomenon is essentially one-dimensional. On the other hand, it follows from T. Matsuoka's results in [87–89] that three (harmonic) τ-periodic solutions of the planar (i.e. in $\mathbb{R}^2$) system (1.4) imply "generically" the coexistence of infinitely many (subharmonic) $k\tau$-periodic solutions of (1.4), $k \in \mathbb{N}$. "Genericity" is this time understood in terms of the Artin braid group theory, i.e. with the exception of certain simplest braids, representing the three given harmonics.

The following theorem was presented in [8], on the basis of T. Matsuoka's results in papers [87–89].

THEOREM 1.2. *Assume that a uniqueness condition is satisfied for planar system* (1.4). *Let three* (*harmonic*) τ*-periodic solutions of* (1.4) *exist whose graphs are not conjugated to the braid* σ^m *in* B_3/Z, *for any integer* $m \in \mathbb{N}$, *where* σ *is shown in Fig*. 1, B_3/Z *denotes the factor group of the Artin braid group* B_3 *and* Z *is its center* (*for definitions*, *see*, *e.g.*, [22, Chapter III.9]). *Then there exist infinitely many* (*subharmonic*) $k\tau$*-periodic solutions of* (1.4), $k \in \mathbb{N}$.

REMARK 1.3. In the absence of uniqueness, there occur serious obstructions, but Theorem 1.2 still seems to hold in many situations; for more details see [8].

REMARK 1.4. The application of the Nielsen theory considered in Section 3.2 below might determine the desired three harmonic solutions of (1.4). More precisely, it is more realistic to detect two harmonics by means of the related Nielsen number (see again Section 3.2 below), and the third one by means of the related fixed point index (see Section 3.3 below).

For $n > 2$, statements like Theorem 1.1 or Theorem 1.2 appear only rarely. Nevertheless, if $f = (f_1, f_2, \ldots, f_n)$ has a special triangular structure, i.e.

$$f_i(x) = f_i(x_1, \ldots, x_n) = f_i(x_1, \ldots, x_i), \quad i = 1, \ldots, n, \tag{1.5}$$

then Theorem 1.1 can be extended to hold in $\mathbb{R}^n$ (see [35]).

THEOREM 1.3. *Under assumption* (1.5), *the conclusion of Theorem* 1.1 *remains valid in* $\mathbb{R}^n$.

REMARK 1.5. Similarly to Theorem 1.1, Theorem 1.3 holds only in the lack of uniqueness. Without the special triangular structure (1.5), there is practically no chance to obtain an analogy to Theorem 1.1, for $n \geqslant 2$.

There is also another motivation for the investigation of *multivalued ODEs*, i.e. differential inclusions, because of the strict connection with

(i) optimal control problems for ODEs,
(ii) Filippov solutions of discontinuous ODEs,
(iii) implicit ODEs, etc.

ad (i): Consider a *control problem* for

$$\dot{x} = f(t, x, u), \quad u \in U, \tag{1.6}$$

where $f : [0, \tau] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $u \in U$ are control parameters such that $u(t) \in \mathbb{R}^n$, for all $t \in [0, \tau]$. In order to solve a control problem for (1.6), we can define a multivalued map $F(t, x) := \{f(t, x, u)\}_{u \in U}$. The solutions of (1.6) are those of

$$\dot{x} \in F(t, x), \tag{1.7}$$

and the same is true for a given control problem. For more details, see, e.g., [27,79].

ad (ii): If function f is discontinuous in x, then Carathéodory theory cannot be applied for solving, e.g., (1.1). Making, however, the *Filippov regularization* of f, namely

$$F(t, x) := \bigcap_{\delta > 0} \bigcap_{\substack{r \subset [0, \tau] \times \mathbb{R}^n \\ \mu(r) = 0}} \overline{\operatorname{conv}}\, f\big(O_\delta\big((t, x) \setminus r\big)\big), \tag{1.8}$$

where $\mu(r)$ denotes the Lebesgue measure of the set $r \subset \mathbb{R}^n$ and

$$O_\delta(y) := \big\{z \in [0, \tau] \times \mathbb{R}^n \mid |y - z| < \delta\big\},$$

multivalued F is well known (see [60]) to be again upper Carathéodory with nonempty, convex and compact values (cf. Definition 2.10 below), provided only f is measurable and satisfies $|f(t, x)| \leqslant \alpha + \beta|x|$, for all $(t, x) \in [0, \tau] \times \mathbb{R}^n$, with some nonnegative constants α, β. Thus, by a *Filippov solution* of $\dot{x} = f(t, x)$, it is so understood a Carathéodory solution of (1.7), where F is defined in (1.8). As an example from physics, dry friction problems (see, e.g., [84,91]) can be solved in this way.

ad (iii): Let us consider the *implicit differential equation*

$$\dot{x} = f(t, x, \dot{x}), \tag{1.9}$$

where $f : [0, \tau] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a compact (continuous) map and the solutions are understood in the sense of Carathéodory. We can associate with (1.9) the following two differential inclusions:

$$\dot{x} \in F_1(t, x) \tag{1.10}$$

and

$$\dot{x} \in F_2(t, x), \tag{1.11}$$

where $F_1(t, x) := \text{Fix}(f(t, x, \cdot))$, i.e. the (nonempty, see [22, p. 560]) fixed point set of $f(t, x, \cdot)$ w.r.t. the last variable, and $F_2 \subset F_1$ is a (multivalued) lower semicontinuous (see Definition 2.4 below) selection of F_1. The sufficient condition for the existence of such a selection F_2 reads (see, e.g., [22, Chapter III.11, pp. 558–559]):

$$\dim \text{Fix}\big(f(t, x, \cdot)\big) = 0, \quad \text{for all } (t, x) \in [0, \tau] \times \mathbb{R}^n, \tag{1.12}$$

where dim denotes the topological (covering) dimension.

Denoting by $S(f)$, $S(F_1)$, $S(F_2)$ the sets of all solutions of initial value problems to (1.9), (1.10), (1.11), respectively, one can prove (see [22, p. 560]) that, under (1.12), $S(f) = S(F_1) \subset S(F_2) \neq \emptyset$. For more details, see [19] (cf. [22, Chapter III.11]).

Although there are several monographs devoted to multivalued ODEs (see, e.g., [22,42, 45,58,61,74,79,91,96,97]), topological principles were presented mainly for single-valued ODEs (besides [22,45,58] and [61] for differential inclusions, see, e.g., [62,64,65,82,83, 90]). Hence our main object will be topological principles for (multivalued) ODEs; whence the title. We will consider without special distinguishing differential equations as well as inclusions; both in Euclidean and Banach spaces. All solutions of problems under our consideration (even in Banach spaces) will be understood at least in the sense of Carathéodory. Thus, in view of the indicated relationship with problems (i)–(iii), many obtained results can be also employed for solving optimal control problems, problems for systems with variable structure, implicit boundary value problems, etc.

The reader exclusively interested in single-valued ODEs can simply read "continuous", instead of "upper semicontinuous" or "lower semicontinuous", and replace the inclusion symbol $\in$ by the equality $=$, in the given differential inclusions. If, in the single-valued case, the situation simplifies dramatically or if the obtained results can be significantly improved, then the appropriate remarks are still supplied.

We wished to prepare an as much as possible self-contained text. Nevertheless, the reader should be at least familiar with the elements of nonlinear analysis, in particular of fixed point theory, in order to understand the degree arguments, or so. Otherwise, we recommend the monographs [69] (in the single-valued case) and [22] (in the multivalued case).

Furthermore, one is also expected to know several classical results and notions from the standard courses of ODEs, functional analysis and the theory of integration like the Gronwall inequality, the Arzelà–Ascoli lemma, the Mazur Theorem, the Bochner integral, etc.

We will study mainly existence and multiplicity of bounded, periodic and anti-periodic solutions of (multivalued) ODEs. Since our approach consists in the application of the fixed point principles, these solutions will be either determined by, (e.g., τ-periodic solutions $x(t)$ by the initial values $x(0)$ via (1.3)) or directly identified (e.g., solutions of initial value problems (1.1)) with fixed points of the associated (Cauchy, Hammerstein, etc.) operators.

Although the usage of the relative degree (i.e. the fixed point index) arguments is rather traditional in this framework, it might not be so when the maps, representing, e.g., problems on noncompact intervals, operate in nonnormable Fréchet spaces. This is due to the unpleasant locally convex topology possessing bounded subsets with an empty interior. We had therefore to develop with my colleagues our own fixed point index theory. The application of the Nielsen theory, for obtaining multiplicity criteria, is very delicate and quite rare, and the related problem is named after Jean Leray who posed it in 1950, at the first International Congress of Mathematics held after World War II in Cambridge, Mass. We had also to develop a new multivalued Nielsen theory suitable for applications in this field. Before presenting general methods for solvability of boundary value problems in Section 4, we therefore make a sketch of the applied fixed point principles in Section 3. Hence besides Section 4, the main results are contained in Section 5 (Existence results) and Section 6 (Multiplicity results). The reference sources to our results and their comparison with those of other authors are finally commented in Section 7 (Remarks and comments).

2. Preliminaries

2.1. *Elements of ANR-spaces*

In the entire text, all topological spaces will be metric and, in particular, all topological vector spaces will be at least Fréchet. Let us recall that by a *Fréchet space*, we understand a complete (metrizable) locally convex space. Its topology can be generated by a countable family of seminorms. If it is normable, then it becomes *Banach*.

DEFINITION 2.1. A (metrizable) space X is an *absolute neighbourhood retract* (ANR) if, for each (metrizable) Y and every closed $A \subset Y$, each continuous mapping $f : A \to X$ is extendable over some neighbourhood of A.

PROPOSITION 2.1.

(i) *If X is an ANR, then any open subset of X is an ANR and any neighbourhood retract of X is an ANR.*

(ii) *X is an ANR if and only if it is a neighbourhood retract of every* (*metrizable*) *space in which it is embedded as a closed subset.*

(iii) *X is an ANR if and only if it is a neighbourhood retract of some normed linear space*, *i.e. if and only if it is a retract of some open subset of a normed space.*

(iv) *If X is a retract of an open subset of a convex set in a Fréchet space, then it is an ANR.*
(v) *If X_1, X_2 are closed ANRs such that $X_1 \cap X_2$ is an ANR, then $X_1 \cup X_2$ is an ANR.*
(vi) *Any finite union of closed convex sets in a Fréchet space is an ANR.*
(vii) *If each $x \in X$ admits a neighbourhood that is an ANR, then X is an ANR.*

DEFINITION 2.2. A (metrizable) space X is an *absolute retract* (AR) if, for each (metrizable) Y and every closed $A \subset Y$, each continuous mapping $f : A \to X$ is extendable over Y.

PROPOSITION 2.2.
(i) *X is an AR if and only if it is a contractible (i.e. homotopically equivalent to a one point space) ANR.*
(ii) *X is an AR if and only if it is a retract of every (metrizable) space in which it is embedded as a closed subset.*
(iii) *If X is an AR and A is a retract of X, then A is an AR.*
(iv) *If X is homeomorphic to Y and X is an AR, then so is Y.*
(v) *X is an AR if and only if it is a retract of some normed space.*
(vi) *If X is a retract of a convex subset of a Fréchet space, then it is an AR.*
(vii) *If X_1, X_2 are closed ARs such that $X_1 \cap X_2$ is an AR, then $X_1 \cup X_2$ is an AR.*

Furthermore, it is well known that every ANR X is *locally contractible* (i.e. for each $x \in X$ and a neighbourhood U of x, there exists a neighbourhood V of x that is contractible in U) and, as follows from Proposition 2.2(i) that every AR X is *contractible* (i.e. if $\mathrm{id}_X : X \to X$ is homotopic to a constant map).

DEFINITION 2.3. X is called an *R_δ-set* if, there exists a decreasing sequence $\{X_n\}$ of compact, contractible sets X_n such that $X = \bigcap\{X_n \mid n = 1, 2, \ldots\}$.

Although contractible spaces need not be ARs, X is an R_δ-set if and only if it is an intersection of a decreasing sequence of compacts ARs. Moreover, every R_δ-set is *acyclic* w.r.t. any continuous theory of homology (e.g., the Čech homology), i.e. homologically equivalent to a one point space, and so it is in particular nonempty, compact and connected.

The following hierarchies hold for metric spaces:

$$
\begin{array}{ccc}
 & \text{contractible} \subset \text{acyclic} & \\
 & \cup & \\
\text{convex} \subset & \text{AR} \subset \text{ANR}, &
\end{array}
$$

compact + convex $\subset$ compact AR $\subset$ compact + contractible $\subset R_\delta \subset$ compact + acyclic, and all the above inclusions are proper.

For more details, see [47] (cf. also [22,67,69]).

2.2. *Elements of multivalued maps*

In what follows, by a multivalued map $\varphi : X \multimap Y$, i.e. $\varphi : X \to 2^Y \backslash \{0\}$, we mean the one with at least nonempty, closed values.

DEFINITION 2.4. A map $\varphi : X \multimap Y$ is said to be *upper semicontinuous* (u.s.c.) if, for every open $U \subset Y$, the set $\{x \in X \mid \varphi(x) \subset U\}$ is open in X. It is said to be *lower semicontinuous* (l.s.c.) if, for every open $U \subset Y$, the set $\{x \in X \mid \varphi(x) \cap U \neq \emptyset\}$ is open in X. If it is both u.s.c. and l.s.c., then it is called *continuous*.

Obviously, in the single-valued case, if $f : X \to Y$ is u.s.c. or l.s.c., then it is continuous. Moreover, the compact-valued map $\varphi : X \multimap Y$ is continuous if and only if it is *Hausdorff-continuous*, i.e. continuous w.r.t. the metric d in X and the Hausdorff-metric d_H in $\{B \subset Y \mid B$ is nonempty and bounded$\}$, where $d_H(A, B) := \inf\{\varepsilon > 0 \mid A \subset O_\varepsilon(B)$ and $B \subset O_\varepsilon(A)\}$ and $O_\varepsilon(B) := \{x \in X \mid \exists y \in B\colon\ d(x, y) < \varepsilon\}$. Every u.s.c. map $\varphi : X \multimap Y$ has a closed graph Γ_φ, but not vice versa. Nevertheless, if the graph Γ_φ of a compact map $\varphi : X \multimap Y$ is closed, then φ is u.s.c.

The important role will be played by the following class of admissible maps in the sense of L. Górniewicz.

DEFINITION 2.5. Assume that we have a diagram $X \overset{p}{\Longleftarrow} \Gamma \overset{q}{\longrightarrow} Y$ (Γ is a metric space), where $p : \Gamma \Rightarrow X$ is a continuous *Vietoris map*, namely

(i) p is onto, i.e. $p(\Gamma) = X$,
(ii) p is proper, i.e. $p^{-1}(K)$ is compact, for every compact $K \subset X$,
(iii) $p^{-1}(x)$ is acyclic, for every $x \in X$, where acyclicity is understood in the sense of the Čech homology functor with compact carriers and coefficients in the field $\mathbb{Q}$ of rationals,

and $q : \Gamma \to Y$ is a continuous map. The map $\varphi : X \multimap Y$ is called *admissible* if it is induced by $\varphi(x) = q(p^{-1}(x))$, for every $x \in X$. We, therefore, identify the admissible map φ with the pair (p, q) called an *admissible* (*selected*) *pair*.

DEFINITION 2.6. Let $X \overset{p_0}{\Longleftarrow} \Gamma_0 \overset{q_0}{\longrightarrow} Y$ and $X \overset{p_1}{\Longleftarrow} \Gamma_1 \overset{p_1}{\longrightarrow} Y$ be two admissible maps, i.e. $\varphi_0 = q_0 \circ p_0^{-1}$ and $\varphi_1 = q_1 \circ p_1^{-1}$. We say that φ_0 is *admissibly homotopic* to φ_1 (written $\varphi_0 \sim \varphi_1$ or $(p_0, q_0) \sim (p_1, q_1)$) if there exists an admissible map $X \times [0, 1] \overset{p}{\Longleftarrow} \Gamma_0 \overset{q}{\longrightarrow} Y$ such that the following diagram is commutative:

$$\begin{array}{ccccc} X & \overset{p_i}{\Longleftarrow} & \Gamma_i & \overset{q_i}{\longrightarrow} & Y \\ \downarrow{\scriptstyle k_i} & & \downarrow{\scriptstyle f_i} & \nearrow{\scriptstyle q} & \\ X \times [0,1] & \underset{p}{\Longleftarrow} & \Gamma & & \end{array}$$

for $k_i(x) = (x, i)$, $i = 0, 1$, and $f_i : \Gamma_i \to \Gamma$ is a homeomorphism onto $p^{-1}(X \times i)$, $i = 0, 1$, i.e. $k_0 p_0 = p f_0$, $q_0 = q f_0$, $k_1 p_1 = p f_1$ and $q_1 = q f_1$.

Thus, admissible maps are always u.s.c. with nonempty, compact and connected values. Moreover, their class is closed w.r.t. finite compositions, i.e. a finite composition of admissible maps is also admissible. In fact, a map is admissible if and only if it is a finite composition of *acyclic maps* with compact values, i.e. u.s.c. maps with acyclic and compact values.

The class of admissible maps so contains u.s.c. maps with convex and compact values, u.s.c. maps with contractible and compact values, R_δ*-maps* (i.e. u.s.c. maps with R_δ-values), acyclic maps with compact values and their compositions.

The class of compact admissible maps $\varphi : X \multimap Y$, i.e. $\overline{\varphi(X)}$ is compact, will be denoted by $\mathbb{K}(X,Y)$, or simply by $\mathbb{K}(X)$, provided φ is a self-map (an endomorphism). If the admissible homotopy in Definition 2.6 is still compact, then we say that $\varphi_0 \in \mathbb{K}(X,Y)$ and $\varphi_1 \in \mathbb{K}(X,Y)$ are *compactly admissibly homotopic*.

Another important class of admissible maps are condensing admissible maps denoted by $\mathbb{C}(X,Y)$. For this, we need to recall the notion of a measure of noncompactness (MNC).

Let E be a Fréchet space endowed with a countable family of seminorms $\|.\|_s$, $s \in S$ (S is the index set), generating the locally convex topology. Denoting by $\mathcal{B} = \mathcal{B}(E)$ the set of nonempty, bounded subsets of E, we can give

DEFINITION 2.7. The family of functions $\alpha = \{\alpha_s\}_{s\in S} : \mathcal{B} \to [0,\infty)^S$, where $\alpha_s(B) := \inf\{\delta > 0 \mid B \in \mathcal{B}$ admits a finite covering by the sets of $\mathrm{diam}_s \leqslant \delta\}$, $s \in S$, for $B \in \mathcal{B}$, is called the *Kuratowski measure of noncompactness* and the family of functions $\gamma = \{\gamma_s\}_{s\in S} : \mathcal{B} \to [0,\infty)^S$, where $\gamma_s(B) := \inf\{\delta > 0 \mid B \in \mathcal{B}$ has a finite ε_s-net$\}$, $s \in S$, for $B \in \mathcal{B}$, is called the *Hausdorff measure of noncompactness*.

These MNC are related as follows:

$$\gamma(B) \leqslant \alpha(B) \leqslant 2\gamma(B), \quad \text{i.e.} \quad \gamma_s(B) \leqslant \alpha_s(B) \leqslant 2\gamma_s(B), \quad \text{for each } s \in S.$$

Moreover, they satisfy the following properties:

PROPOSITION 2.3. *Assume that* $B, B_1, B_2 \in \mathcal{B}$. *Then we have* (*component-wise*):

(μ_1) (regularity) $\mu(B) = 0 \Leftrightarrow \overline{B}$ *is compact*,
(μ_2) (nonsingularity) $\{b\} \in \mathcal{B} \Rightarrow \{b\} \cup B \in \mathcal{B}$ *and* $\mu(\{b\} \cup B) = \mu(B)$,
(μ_3) (monotonicity) $B_1 \subset B_2 \Rightarrow \mu(B_1) \leqslant \mu(B_2)$,
(μ_4) (closed convex hull) $\mu(\overline{\mathrm{conv}}\, B) = \mu(B)$,
(μ_5) (closure) $\mu(\overline{B}) = \mu(B)$,
(μ_6) (Kuratowski condition) *decreasing sequence of closed sets* $B_n \in \mathcal{B}$ *with*

$$\lim_{n\to\infty} \mu(B_n) = 0 \quad \Longrightarrow \quad \bigcap\{B_n \mid n = 1, 2, \ldots\} \neq \emptyset,$$

(μ_7) (semiadditivity) $\mu(B_1 + B_2) \leqslant \mu(B_1) + \mu(B_2)$,
(μ_8) (union) $\mu(B_1 \cup B_2) = \max\{\mu(B_1), \mu(B_2)\}$,
(μ_9) (intersection) $\mu(B_1 \cap B_2) = \min\{\mu(B_1), \mu(B_2)\}$,
(μ_{10}) (seminorm) $\mu(\lambda B) = |\lambda|\mu(B)$, *for every* $\lambda \in \mathbb{R}$, *and* $\mu(B_1 \cup B_2) \leqslant \mu(B_1) + \mu(B_2)$,

where μ denotes either α or γ.

DEFINITION 2.8. A bounded mapping $\varphi : E \supset U \multimap E$, i.e. $\varphi(B) \in \mathcal{B}$, for $\mathcal{B} \ni B \subset U$, is said to be *$\mu$-condensing* (shortly, *condensing*) if $\mu(\varphi(B)) < \mu(B)$, whenever $\mathcal{B} \ni B \subset U$ and $\mu(B) > 0$, or equivalently, if $\mu(\varphi(B)) \geqslant \mu(B)$ implies $\mu(B) = 0$, whenever $\mathcal{B} \ni B \subset U$, where $\mu = \{\mu_s\}_{s\in S} : \mathcal{B} \to [0,\infty)^S$ is a family of functions satisfying at least conditions (μ_1)–(μ_5). Analogously, a bounded mapping $\varphi : E \supset U \multimap E$ is said to be a *k-set contraction* w.r.t. $\mu = \{\mu_s\}_{s\in S} : \mathcal{B} \to [0,\infty)^S$ satisfying at least conditions (μ_1)–(μ_5) (shortly, a *k-contraction* or a *set-contraction*) if $\mu(\varphi(B)) \leqslant k\mu(B)$, for some $k \in [0,1)$, whenever $\mathcal{B} \ni B \subset U$.

Obviously, any set-contraction is condensing and both α-condensing and γ-condensing maps are μ-condensing. Furthermore, compact maps or contractions with compact values (in vector spaces, also their sum) are well known to be (α,γ)-set-contractions, and so (α,γ)-condensing.

Besides semicontinuous maps, measurable and semi-Carathéodory maps will be also of importance. Hence, assume that Y is a separable metric space and $(\Omega, \mathcal{U}, \nu)$ is a *measurable space*, i.e. a set Ω equipped with σ-algebra $\mathcal{U}$ of subsets and a countably additive measure ν on $\mathcal{U}$. A typical example is when Ω is a bounded domain in $\mathbb{R}^n$, equipped with the Lebesgue measure.

DEFINITION 2.9. A map $\varphi : \Omega \multimap Y$ is called *strongly measurable* if there exists a sequence of step multivalued maps $\varphi_n : \Omega \multimap Y$ such that $d_H(\varphi_n(\omega), \varphi(\omega)) \to 0$, for a.a. $\omega \in \Omega$, as $n \to \infty$. In the single-valued case, one can simply replace multivalued step maps by single-valued step maps and $d_H(\varphi_n(\omega), \varphi(\omega))$ by $\|\varphi_n(\omega) - \varphi(\omega)\|$.

A map $\varphi : \Omega \multimap Y$ is called *measurable* if $\{\omega \in \Omega \mid \varphi(\omega) \subset V\} \in \mathcal{U}$, for each open $V \subset Y$.

A map $\varphi : \Omega \multimap Y$ is called *weakly measurable* if $\{\omega \in \Omega \mid \varphi(\omega) \subset V\} \in \mathcal{U}$, for each closed $V \subset Y$.

Obviously, if φ is strongly measurable, then it is measurable and if φ is measurable, then it is also weakly measurable. If φ has compact values, then the notions of measurability and weak measurability coincide. In separable Banach spaces Y, the notions of strong measurability and measurability coincide for multivalued maps with compact values as well as for single-valued maps (see [78, Theorem 1.3.1 on pp. 45–49]). If Y is a not necessarily separable Banach space, then a strongly measurable map $\varphi : \Omega \multimap Y$ with compact values has a single-valued strongly measurable selection (see, e.g., [58, Proposition 3.4(b) on pp. 25–26]). Furthermore, if Y is a separable complete space, then every measurable $\varphi : \Omega \multimap Y$ has, according to the Kuratowski–Ryll-Nardzewski theorem (see, e.g., [22, Theorem 3.49 in Chapter I.3]), a single-valued measurable selection.

Now, let $\Omega = [0,a]$ be equipped with the Lebesgue measure and X, Y be Banach.

DEFINITION 2.10. A map $\varphi : [0,a] \times X \multimap Y$ with nonempty, compact and convex values is called *u-Carathéodory* (resp. *l-Carathéodory*, resp. *Carathéodory*) if it satisfies

(i) $t \multimap \varphi(t,x)$ is strongly measurable, for every $x \in X$,

(ii) $x \multimap \varphi(t,x)$ is u.s.c. (resp. l.s.c., resp. continuous), for almost all $t \in [0,a]$,

(iii) $\|y\|_Y \leqslant r(t)(1+\|x\|_X)$, for every $(t,x) \in [0,a] \times X$, $y \in \varphi(t,x)$, where $r:[0,a] \to [0,\infty)$ is an integrable function.

For $X = \mathbb{R}^m$ and $Y = \mathbb{R}^n$, one can state

PROPOSITION 2.4.
(i) *Carathéodory maps are product-measurable* (*i.e. measurable as the whole* $(t,x) \multimap \varphi(t,x)$), *and*
(ii) *they possess a single-valued Carathéodory selection.*

It need not be so for u-Carathéodory or l-Carathéodory maps. Nevertheless, for u-Carathéodory maps, we have at least (again $X = \mathbb{R}^m$ and $Y = \mathbb{R}^n$).

PROPOSITION 2.5. *u-Carathéodory maps* (*in the sense of Definition* 2.10) *are weakly superpositionally measurable*, *i.e. the composition* $\varphi(t,q(t))$ *admits*, *for every* $q \in C([0,a],\mathbb{R}^m)$, *a single-valued measurable selection. If they are still product-measurable*, *then they are also superpositionally measurable*, *i.e. the composition* $\varphi(t,q(t))$ *is measurable, for every* $q \in C([0,a],\mathbb{R}^m)$.

REMARK 2.1. If X, Y are separable Banach spaces and $\varphi : X \multimap Y$ is a Carathéodory mapping, then φ is also superpositionally measurable, i.e. $\varphi(t,q(t))$ is measurable, for every $q \in C([0,a],X)$ (see [78, Theorem 1.3.4 on p. 56]). Under the same assumptions, Proposition 2.4 can be appropriately generalized (see [73, Proposition 7.9 on p. 229 and Proposition 7.23 on pp. 234–235]).

If $\varphi : X \multimap Y$ is only u-Carathéodory and X, Y are (not necessarily separable) Banach spaces, then φ is weakly superpositionally measurable, i.e. $\varphi(t,q(t))$ admits a single-valued measurable selection, for every $q \in C([0,a],X)$ (see, e.g., [58, Proposition 3.5 on pp. 26–27] or [78, Theorem 1.3.5 on pp. 57–58]).

For more details, see [22,40,58,67,73,78].

2.3. *Some further preliminaries*

Assume we have again a diagram (see Definition 2.5)

$$X \overset{p}{\Longleftarrow} \Gamma \overset{q}{\longrightarrow} Y$$

where $p : \Gamma \Longrightarrow X$ is a Vietoris map and $q : \Gamma \longrightarrow Y$ is continuous.

Taking $\varphi(x) = q(p^{-1}(x))$, for every $x \in X$, and denoting as

$$\operatorname{Fix}(p,q) = \operatorname{Fix}(\varphi) := \{x \in X \mid x \in \varphi(x)\},$$
$$C(p,q) := \{z \in \Gamma \mid p(z) = q(z)\}$$

the sets of *fixed points* and *coincidence points* of the admissible pair (p,q), it is clear that $p(C(p,q)) = \mathrm{Fix}(p,q)$, and so

$$\mathrm{Fix}(p,q) \neq \emptyset \iff C(p,q) \neq \emptyset.$$

The following Aronszajn–Browder–Gupta-type result (see [21, Theorem 3.15]; cf. [22, Theorem 1.4]) is very important in order to say something about the topological structure of $\mathrm{Fix}(\varphi)$.

PROPOSITION 2.6. *Let X be a metric space, E a Fréchet space, $\{U_k\}$ a base of open convex symmetric neighbourhoods of the origin in E, and let $\varphi : X \multimap E$ be a u.s.c proper map with compact values. Assume that there is a convex symmetric subset C of E and a sequence of compact, convex-valued u.s.c. proper maps $\varphi_k : X \multimap E$ such that*

(i) *$\varphi_k(x) \subset \varphi(O_{1/k}(x)) + U_k$, for every $x \in X$, where*

$$O_{1/k}(x) = \left\{ y \in X \mid d(x,y) < \tfrac{1}{k} \right\},$$

(ii) *for every $k \geqslant 1$, there is a convex, symmetric set $V_k \subset \overline{U}_k \cap C$ such that V_k is closed in E and $0 \in \varphi(x)$ implies $\varphi_k(x) \cap V_k \neq \emptyset$,*

(iii) *for every $k \geqslant 1$ and every $u \in V_k$, the inclusion $u \in \varphi_k(x)$ has an acyclic set of solutions.*

Then the set $S = \{x \in X \mid \varphi(x) \cap \{0\} \neq \emptyset\}$ is compact and acyclic.

Now, let us assume that E is a Fréchet space, C is a convex subset of E, U is an open subset of C, $\mu : \mathcal{B} \to [0,\infty)^S$ is a measure of noncompactness satisfying at least conditions (μ_1)–(μ_5) in Proposition 2.3 (see Definitions 2.7 and 2.8).

If $\varphi \in \mathbb{C}(U,C)$, then $\mathrm{Fix}(\varphi)$ can be proved relatively compact. We can say more about $\mathrm{Fix}(\varphi)$.

DEFINITION 2.11. Let $(p,q) \in \mathbb{C}(U,C)$. A nonempty, compact, convex set $S \subset C$ is called a *fundamental set* if:

(i) $q(p^{-1}(U \cap S)) \subset S$,

(ii) if $x \in \overline{\mathrm{conv}}(\varphi(x) \cup S)$, then $x \in S$.

For a homotopy $\chi \in \mathbb{C}(U \times [0,1], C)$, $S \subset C$ is called fundamental if it is fundamental to $\chi(.,\lambda)$, for each $\lambda \in [0,1]$.

PROPOSITION 2.7. *Assume $(p,q) \in \mathbb{C}(U,C)$.*

(i) *If S is a fundamental set for (p,q), then* $\mathrm{Fix}(p,q) \subset S$.

(ii) *Intersection of fundamental sets, for (p,q), is also fundamental, for (p,q).*

(iii) *The family of all fundamental sets for (p,q) is nonempty.*

(iv) *If S is a fundamental set for $\chi \in \mathbb{C}(U \times [0,1], C)$ and $P \subset S$, then the set $\overline{\mathrm{conv}}(\chi((U \cap S) \times [0,1]) \cup P)$ is also fundamental.*

For more details, see [22,67], and the references therein.

3. Applied fixed point principles

3.1. *Lefschetz fixed point theorems*

We start with the Lefschetz theory, because it is a base for our further investigation. More precisely, the generalized Lefschetz number can be used for the definition of essential classes in the Nielsen theory as well as the possible normalization property of the fixed point index. We restrict ourselves only to the presentation of necessary facts.

Consider a multivalued map $\varphi : X \multimap X$ and assume that

(i) X is a (metric) ANR-space, e.g., a retract of an open subset of a convex set in a Fréchet space,

(ii) φ is a compact (i.e. $\overline{\varphi(X)}$ is compact) composition of an R_δ-map $p^{-1} : X \multimap \Gamma$ and a continuous (single-valued) map $q : \Gamma \to X$, namely $\varphi = q \circ p^{-1}$, where Γ is a metric space.

Then an integer $\Lambda(\varphi) = \Lambda(p, q)$, called the *generalized Lefschetz number* for $\varphi \in \mathbb{K}(X)$, is well-defined (see, e.g., [12; 22, Chapter I.6; 67]) and $\Lambda(\varphi) \neq 0$ implies that

$$\text{Fix}(\varphi) := \{x \in X \mid x \in \varphi(x)\} \neq \emptyset.$$

Moreover, Λ is a homotopy invariant, namely if φ is compactly homotopic (in the same class of maps) with $\widetilde{\varphi} : X \multimap X$, then $\Lambda(\varphi) = \Lambda(\widetilde{\varphi})$.

In order to define the generalized Lefschetz number, one should be familiar with the elements of algebraic topology, in particular, of homology theory. Therefore, we only briefly sketch this definition without proofs. For more details, we recommend [51,68] (in the single-valued case) and [12,22,67] (in the multivalued case).

At first, we recall some algebraic preliminaries. In what follows, all vector spaces are taken over $\mathbb{Q}$. Let $f : E \to E$ be an endomorphism of a finite-dimensional vector space E. If $v_1, \dots, v_n$ is a basis for E, then we can write

$$f(v_i) = \sum_{j=1}^{n} a_{ij} v_j, \quad \text{for all } i = 1, \dots, n.$$

The matrix $[a_{ij}]$ is called the matrix of f (with respect to the basis $v_1, \dots, v_n$). Let $A = [a_{ij}]$ be an $(n \times n)$-matrix; then the trace of A is defined as $\sum_{i=1}^{n} a_{ii}$. If $f : E \to E$ is an endomorphism of a finite-dimensional vector space E, then the trace of f, written $\text{tr}(f)$, is the trace of the matrix of f with respect to some basis for E. If E is a trivial vector space then, by definition, $\text{tr}(f) = 0$. It is a standard result that the definition of the trace of an endomorphism is independent of the choice of the basis for E.

Hence, let $E = \{E_q\}$ be a graded vector space of a finite type.

If $f = \{f_q\}$ is an endomorphism of degree zero of such a graded vector space, then the (*ordinary*) *Lefschetz number* $\lambda(f)$ of f is defined by

$$\lambda(f) = \sum_q (-1)^q \, \text{tr}(f_q).$$

Let $f : E \to E$ be an endomorphism of an arbitrary vector space E. Denote by $f^n : E \to E$ the nth iterate of f and observe that the kernels

$$\ker f \subset \ker f^2 \subset \cdots \subset \ker f^n \subset \cdots$$

form an increasing sequence of subspaces of E. Let us now put

$$N(f) = \bigcup_n \ker f^n \quad \text{and} \quad \widetilde{E} = E/N(f).$$

Clearly, f maps $N(f)$ into itself and, therefore, induces the endomorphism $\widetilde{f} : \widetilde{E} \to \widetilde{E}$ on the factor space $\widetilde{E} = E/N(f)$.

Let $f : E \to E$ be an endomorphism of a vector space E. Assume that $\dim \widetilde{E} < \infty$. In this case, we define the generalized trace $\mathrm{Tr}(f)$ of f by putting $\mathrm{Tr}(f) = \mathrm{tr}(\widetilde{f})$.

LEMMA 3.1. *Let $f : E \to E$ be an endomorphism. If* $\dim E < \infty$, *then* $\mathrm{Tr}(f) = \mathrm{tr}(f)$.

For the proof, see [22].

Let $f = \{f_q\}$ be an endomorphism of degree zero of a graded vector space $E = \{E_q\}$. We say that f is a *Leray endomorphism* if the graded vector space $\widetilde{E} = \{\widetilde{E}_q\}$ is of finite type. For such an f, we define the (*generalized*) *Lefschetz number* $\Lambda(f)$ of f by putting

$$\Lambda(f) = \sum_q (-1)^q \, \mathrm{Tr}(f_q).$$

It is immediate from Lemma 3.1 that

LEMMA 3.2. *Let $f : E \to E$ be an endomorphism of degree zero, i.e., $f = \{f_q\}$ and $f_q : E_q \to E_q$ is a linear map. If E is a graded vector space of finite type, then* $\Lambda(f) = \lambda(f)$.

Now, the *Lefschetz number* will be defined for admissible compact mappings. For our needs in the sequel, it is enough to consider only the compact compositions of R_δ-maps and continuous single-valued maps as above (by which Lefschetz sets simplify into Lefschetz numbers). Let $\varphi : E \multimap E$ be an admissible compact map and $(p, q) \subset \varphi$ be a selected pair of φ. Then the induced homomorphism $q_* \circ p_*^{-1} : H_*(E) \to H_*(E)$ is an endomorphism of the graded vector space $H_*(E)$ into itself. So, we can define the Lefschetz number $\Lambda(p, q)$ of the pair (p, q) by putting $\Lambda(p, q) = \Lambda(q_* \circ p_*^{-1})$, provided the Lefschetz number $\Lambda(q_* \circ p_*^{-1})$ is well-defined.

It allows us to define the Lefschetz set Λ of φ as follows:

$$\Lambda(\varphi) = \big\{\Lambda(p, q) \mid (p, q) \subset \varphi\big\}.$$

In what follows, we say that the Lefschetz set $\Lambda(\varphi)$ of φ is *well-defined* if, for every $(p, q) \subset \varphi$, the Lefschetz number $\Lambda(p, q)$ of (p, q) is defined.

Moreover, from the homotopy property of Λ, we get:

LEMMA 3.3.

(i) *If* $\varphi, \psi : E \multimap E$ *are compactly homotopic* ($\varphi \sim \psi$), *then*: $\Lambda(\varphi) \cap \Lambda(\psi) \neq \emptyset$.

(ii) *If* $\varphi : E \multimap E$ *is admissible and* E *is acyclic, then the Lefschetz set* $\Lambda(\varphi)$ *is well-defined and* $\Lambda(\varphi) = \{1\}$.

It is useful to formulate

THEOREM 3.1 (Coincidence theorem). *Let* U *be an open subset of a finite dimensional normed space* E. *Consider the following diagram*:

$$U \overset{p}{\Longleftarrow} \Gamma \overset{q}{\longrightarrow} U$$

in which q *is a compact map. Then the Lefschetz number* $\Lambda(p, q)$ *of the pair* (p, q), *given by the formula*

$$\Lambda(p, q) = \Lambda\big(q_* \circ p_*^{-1}\big),$$

is well-defined, and $\Lambda(p, q) \neq 0$ *implies that* $p(y) = q(y)$, *for some* $y \in \Gamma$.

Theorem 3.1 can be reformulated in terms of multivalued mappings as follows.

Let $U \subset E$ be the same as in Theorem 3.1 and let $\varphi : U \multimap U$ be a compact, admissible map, i.e., $\varphi \in \mathbb{K}(U)$. We let $\Lambda(\varphi) = \{\Lambda(p, q) \mid (p, q) \subset \varphi\}$, where $\Lambda(p, q) = \Lambda(q_* \circ p_*^{-1})$. Then we have:

THEOREM 3.2.

(i) *The set* $\Lambda(\varphi)$ *is well-defined, i.e. for every* $(p, q) \subset \varphi$, *the generalized Lefschetz number* $\Lambda(p, q)$ *of the pair* (p, q) *is well-defined, and*

(ii) $\Lambda(\varphi) \neq \{0\}$ *implies that the set* $\operatorname{Fix}(\varphi) := \{x \in U \mid x \in \varphi(x)\}$ *is nonempty.*

Theorem 3.2 can be generalized, by means of the Schauder-like approximation technique (for more details, see [22]), for compact admissible maps $\varphi \in \mathbb{K}$ on ANR-spaces, e.g., on retracts of open subsets of convex sets in Fréchet spaces, as follows.

THEOREM 3.3 (The Lefschetz fixed point theorem). *Let* X *be an ANR-space*, *e.g.*, *a retract of an open subset* U *of a convex set in a Fréchet space. Assume*, *furthermore*, *that* $\varphi \in \mathbb{K}(X)$. *Then*:

(i) *the Lefschetz set* $\Lambda(\varphi)$ *of* φ *is well-defined*,

(ii) *if* $\Lambda(\varphi) \neq \{0\}$, *then* $\operatorname{Fix}(\varphi) \neq \emptyset$.

REMARK 3.1. If admissible map $\varphi \in \mathbb{K}(X)$ is a composition of an R_δ-map and a continuous single-valued map, then $\Lambda(\varphi)$ is an integer. If, in particular, X is an AR-space, then $\Lambda(\varphi) = 1$.

REMARK 3.2. The definition of a generalized Lefschetz number for condensing maps is far from to be obvious, and so it can not be used as a normalization property for the related fixed points index. Roughly speaking, it requires to assume additionally the existence of a compact attractor or to impose some additional restrictions on the set X like to be a special neighbourhood retract of a Fréchet space (cf. [68]).

3.2. *Nielsen fixed point theorems*

The standard Nielsen theory allows us to obtain the lower estimate of the number of fixed points. More precisely, if $f : X \to X$ is a compact (continuous) map on a (metric) ANR-space X, then a nonnegative integer $N(f)$, called the *Nielsen number* of f, is defined such that

- $N(f) \leqslant \#\mathrm{Fix}(f) := \mathrm{card}\{\hat{x} \in X \mid f(\hat{x}) = \hat{x}\}$,
- $N(f) = N(\widetilde{f})$, for any compact $\widetilde{f} : X \to X$ which is *compactly homotopic* to f, i.e. if there is a compact map $h : X \times [0,1] \to X$ such that $h_0 = f$, $h_1 = \widetilde{f}$, where $h_t(x) := h(x,t)$, for $t \in [0,1]$.

Given a compact $f : X \to X$ on $X \in \mathrm{ANR}$, we say that $x, y \in \mathrm{Fix}(f)$ are *Nielsen related* if there exists a path $u : [0,1] \to X$ such that $u(0) = x$, $u(1) = y$, and u, $f(u)$ are homotopic keeping the endpoints fixed. Since the Nielsen relation is an equivalence, $\mathrm{Fix}(f)$ splits into fixed point classes. Since the classes are open and f is compact, we have a finite number of fixed point classes.

If, for a Nielsen class $\mathcal{N} \subset \mathrm{Fix}(f)$, we have $\mathrm{ind}(\mathcal{N}, f) \neq 0$, i.e. if the associated fixed point index is nontrivial, then $\mathcal{N}$ is called *essential*. The *Nielsen number* $N(f)$ is then defined to be the number of essential Nielsen classes. For more details, see, e.g., [77].

To compute $N(f)$ can be a difficult task. In the multivalued case, the situation is even more delicate, because the above definition can not be directly generalized. Thus, we only indicate this subtle definition again. Nevertheless, in the single-valued case, these definitions are equivalent.

Consider a multivalued map $\varphi : X \multimap X$ and assume that

(i) X is a connected ANR-space, e.g., a connected retract of an open subset of a convex set in a Fréchet space,
(ii) X has a finitely generated abelian fundamental group,
(iii) φ is a compact (i.e. $\overline{\varphi(X)}$ is compact) composition of an R_δ-map $p^{-1} : X \multimap \Gamma$ and a continuous (single-valued) map $q : \Gamma \to X$, namely $\varphi = q \circ p^{-1}$, where Γ is a metric space.

Then a nonnegative integer $N(\varphi) = N(p,q)$,[1] called the *Nielsen number* for $\varphi \in \mathbb{K}$, exists (see [24] and [22, Chapter I.10] or [12]) such that $N(\varphi) \leqslant \#C(\varphi)$, where

$$\#C(\varphi) = \#C(p,q) := \mathrm{card}\big\{z \in \Gamma \mid p(z) = q(z)\big\}$$

and $N(\varphi_0) = N(\varphi_1)$, for compactly homotopic maps $\varphi_0 \sim \varphi_1$.

REMARK 3.3. Condition (ii) is satisfied, provided X is the torus $\mathbb{T}^n$ ($\pi_1(\mathbb{T}^n) = \mathbb{Z}^n$) and it can be avoided if X is compact and $q = \mathrm{id}$ is the identity (cf. [5]).

[1] We should write more correctly $N_H(\varphi) = N_H(p,q)$, because it is in fact (mod H)-Nielsen number, as can be seen below. For the sake of simplicity, we omit the index H in the following sections.

REMARK 3.4 (Important). We have a counter-example in [24] (cf. [22, Example 10.1 in Chapter I.10]) that, under the above assumptions (i)–(iii), the Nielsen number $N(\varphi)$ is rather the topological invariant for the number of essential classes of coincidences than of fixed points. On the other hand, for a compact X and $q = \mathrm{id}$, $N(\varphi)$ gives even without (ii) a lower estimate of the number of fixed points of φ (see [5]), i.e. $N(\varphi) \leqslant \#\mathrm{Fix}(\varphi)$, where $\mathrm{Fix}(\varphi) := \mathrm{card}\{x \in X \mid x \in \varphi(x)\}$. We have conjectured in [38] that if $\varphi = q \circ p^{-1}$ assumes only simply connected values, then also $N(\varphi) \leqslant \#\mathrm{Fix}(\varphi)$.

The following sketch demonstrates how subtle is the definition of the Nielsen number for multivalued maps. Let

$$X \overset{p_0}{\Longleftarrow} \Gamma \overset{q_0}{\longrightarrow} Y \quad \text{and} \quad X \overset{p_1}{\Longleftarrow} \Gamma \overset{q_1}{\longrightarrow} Y$$

be two admissible maps.

If $(p_0, q_0) \sim (p_1, q_1)$, i.e. if (p_0, q_0) is admissibly homotopic to (p_1, q_1) (see Definition 2.6), and $h : Y \to Z$ is a continuous map, then we write $(p_0, hq_0) \sim (p, hq)$. We say that a multivalued map $X \overset{p}{\Longleftarrow} \Gamma \overset{q}{\longrightarrow} Y$ *represents a single-valued map* $\rho : X \to Y$ if $q = p\rho$. Now, we assume that $X = Y$ and we are going to estimate the cardinality of the coincidence set

$$C(p, q) := \bigl\{z \in \Gamma \mid p(z) = q(z)\bigr\}.$$

We begin by defining a Nielsen-type relation on $C(p, q)$. This definition requires the following conditions on $X \overset{p}{\Longleftarrow} \Gamma \overset{q}{\longrightarrow} Y$:

(i′) X, Y are connected, locally contractible metric spaces (observe that then they admit universal coverings),

(iii′) $p : \Gamma \Longrightarrow X$ is a Vietoris map,

(iii″) for any $x \in X$, the restriction $q_1 = q|_{p^{-1}(x)} : p^{-1}(x) \to Y$ admits a lift $\widetilde{q}_1$ to the universal covering space $(p_Y : \widetilde{Y} \to Y)$:

$$\begin{array}{ccc} & & \widetilde{Y} \\ & \overset{\widetilde{q}_1}{\nearrow} & \Big\Downarrow p_Y \\ p^{-1}(x) & \xrightarrow[q_1]{} & Y \end{array}$$

Let us note that the following implications hold: (i) ⇒ (i′), (iii) ⇒ (iii′), (iii″).

Consider a single-valued map $\rho : X \to Y$ between two spaces admitting universal coverings $p_X : \widetilde{X} \Rightarrow X$ and $p_Y : \widetilde{X} \Rightarrow Y$. Let $\theta_X = \{\alpha : \widetilde{X} \to \widetilde{X} \mid p_X\alpha = p_X\}$ be the group of natural transformations of the covering p_X. Then the map ρ admits a lift $\widetilde{\rho} : \widetilde{X} \to \widetilde{Y}$. We can define a homomorphism $\widetilde{\rho}_! : \theta_X \to \theta_Y$ by the equality

$$\widetilde{q}(\alpha \cdot \widetilde{x}) = \widetilde{q}_!(\alpha)\widetilde{q}(\widetilde{x}) \quad \bigl(\alpha \in \theta_X, \widetilde{x} \in \widetilde{X}\bigr).$$

It is well known that there is an isomorphism between the fundamental group $\pi_1(X)$ and θ_X which may be described as follows. We fix points $x_0 \in X$, $\widetilde{x} \in \widetilde{X}$ and a loop $\omega\colon I \to X$ based at x_0. Let $\widetilde{\omega}$ denote the unique lift of ω starting from $\widetilde{x}_0$. We subordinate to $[\omega] \in \pi_1(X, x_0)$ the unique transformation from θ_X sending $\widetilde{\omega}(0)$ to $\widetilde{\omega}(1)$. Then the homomorphism $\widetilde{\rho}_!\colon \theta_X \to \theta_Y$ corresponds to the induced homomorphism between the fundamental groups $\rho_\#\colon \pi_1(X, x_0) \to \pi_1(Y, \rho(x_0))$.

It can be shown that, under the assumptions (i′), (iii′), (iii″), a multivalued map (p, q) admits a lift to a multivalued map between the universal coverings. These lifts will split the coincidence set $C(p, q)$ into Nielsen classes. Besides that the pair (p, q) induces a homomorphism $\theta_X \to \theta_Y$ giving the Reidemeister set.

We start with the following lemma.

LEMMA 3.4. *Suppose we are given Y, a locally contractible metric space, Γ a metric space, $\Gamma_0 \subset \Gamma$ a compact subspace, $q\colon \Gamma \to Y$, $\widetilde{q}_0\colon \Gamma_0 \to Y$ continuous maps for which the diagram*

$$\begin{array}{ccc} \Gamma_0 & \xrightarrow{\widetilde{q}_0} & \widetilde{Y} \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p_Y} \\ \Gamma & \xrightarrow[q]{} & Y \end{array}$$

commutes (here, $p_Y\colon \widetilde{Y} \to Y$ denotes the universal covering). In other words, $\widetilde{q}_0$ is a partial lift of q. Then $\widetilde{q}_0$ admits an extension to a lift onto an open neighbourhood of Γ_0 in Γ.

Consider again a multivalued map $X \xleftarrow{p} \Gamma \xrightarrow{q} Y$ satisfying (i′). Define (a pullback)

$$\widetilde{\Gamma} = \big\{(\widetilde{x}, z) \in \widetilde{X} \times \Gamma \mid p_X(\widetilde{x}) = p(z)\big\}.$$

Now, we can apply Lemma 3.4 to the multivalued map $\widetilde{X} \overset{\widetilde{p}}{\Longleftarrow} \widetilde{\Gamma} \xrightarrow{qp_\Gamma} Y$, and so we get a lift $\widetilde{q}\colon \widetilde{\Gamma} \to \widetilde{Y}$ such that the diagram

$$\begin{array}{ccccc} \widetilde{X} & \overset{\widetilde{p}}{\Longleftarrow} & \widetilde{\Gamma} & \xrightarrow{\widetilde{q}} & \widetilde{Y} \\ {\scriptstyle p_X}\downarrow & & {\scriptstyle p_\Gamma}\downarrow & & \downarrow{\scriptstyle p_Y} \\ X & \underset{p}{\Longleftarrow} & \Gamma & \xrightarrow[q]{} & Y \end{array}$$

is commutative, where $\widetilde{p}(\widetilde{x}, z) = \widetilde{x}$ and $p_\Gamma(\widetilde{x}, z) = z$. Let us note that the lift $\widetilde{p}$ is given by the above formula, but $\widetilde{q}$ is not precised. We fix such a $\widetilde{q}$.

Observe that $p : \Gamma \Longrightarrow X$ and the lift $\widetilde{p}$ induce a homomorphism $\widetilde{p}^! : \theta_X \to \theta_\Gamma$ by the formula $\widetilde{p}^!(\alpha)(\widetilde{x}, z) = (\alpha\widetilde{x}, z)$. It is easy to check that the homomorphism $\widetilde{p}^!$ is an isomorphism (any natural transformation of $\widetilde{\Gamma}$ is of the form $\alpha \cdot (\widetilde{x}, z) = (\alpha\widetilde{x}, z)$) and that $\widetilde{p}^!$ is inverse to $\widetilde{p}_!$. Recall that the lift $\widetilde{q}$ defines a homomorphism $\widetilde{q}_! : \theta_\Gamma \to \theta_Y$ by the equality $\widetilde{q}(\lambda) = \widetilde{q}_!(\lambda)\widetilde{q}$.

In the sequel, we will consider the composition $\widetilde{q}_!\widetilde{p}^! : \theta_X \to \theta_Y$.

LEMMA 3.5. *Let a multivalued map (p, q) satisfying* (i′) *represent a single-valued map ρ, i.e. $q = \rho p$. Let $\widetilde{\rho}$ be the lift of ρ which satisfies $\widetilde{q} = \widetilde{\rho}\widetilde{p}$. Then $\widetilde{\rho}_!\widetilde{p}^! = \rho_!$.*

Now, we are in a position to define the Nielsen classes. Consider a multivalued self-map $X \overset{p}{\Longleftarrow} \Gamma \overset{q}{\longrightarrow} X$ satisfying (i′). By the above consideration, we have a commutative diagram

$$\begin{array}{ccc} \widetilde{\Gamma} & \xrightarrow{\widetilde{p},\widetilde{q}} & \widetilde{X} \\ {\scriptstyle p_\Gamma}\downarrow & & \downarrow{\scriptstyle p_X} \\ \Gamma & \xrightarrow{p,q} & X \end{array}$$

Following the single-valued case (see, e.g., [77]), we can prove (see [24] and cf. [22]) the following lemma.

LEMMA 3.6.
- $C(p, q) = \bigcup_{\alpha\in\theta_X} p_\Gamma C(\widetilde{p}, \alpha\widetilde{q})$,
- *if $p_\Gamma C(\widetilde{p}, \alpha\widetilde{q}) \cap p_\Gamma C(\widetilde{p}, \beta\widetilde{q})$ is not empty, then there exists a $\gamma \in \theta_X$ such that $\beta = \gamma \circ \alpha \circ (\widetilde{q}_!\widetilde{p}^!\gamma)^{-1}$,*
- *the sets $p_\Gamma C(\widetilde{p}, \alpha\widetilde{q})$ are either disjoint or equal.*

Define an action of θ_X on itself by the formula $\gamma \circ \alpha = \gamma\alpha(\widetilde{q}_! p^! \gamma)$. The quotient set will be called the *set of Reidemeister classes* and will be denoted by $R(p, q)$. The above lemma defines an injection:

$$\text{set of Nielsen classes} \to R(p, q),$$

given by $A \to [\alpha] \in R(p, q)$, where $\alpha \in \theta_X$ satisfies $A = p_\Gamma(C(\widetilde{p}, \alpha\widetilde{q}))$.

One can prove that our definition does not depend on $\widetilde{q}$.

Let us recall that the homomorphism $\widetilde{q}_! : \theta_\Gamma \to \theta_Y$ is defined by the relation $\widetilde{q}\alpha = \widetilde{q}_!(\alpha)\widetilde{q}$, for $\alpha \in \theta_\Gamma$. If $\widetilde{q}' = \gamma\widetilde{q}$ is another lift of q ($\widetilde{\gamma} \in \theta_\Gamma$), then the induced homomorphism $\widetilde{q}'_! : \theta_\Gamma \to \theta_Y$ is defined by the relation $\widetilde{q}'\alpha = \widetilde{q}'_!(\alpha)\widetilde{q}'$.

One can also show that the Reidemeister sets obtained by different lifts of q are canonically isomorphic. That is why we write $R(p, q)$ omitting tildes.

PROPOSITION 3.1. *If $X \times [0, 1] \overset{p}{\longleftarrow} \Gamma \overset{q}{\longrightarrow} Y$ is a homotopy satisfying* (i′), (iii′), (iii″), *then the homomorphism $\widetilde{q}_{t!}\widetilde{p}^!_t : \theta_X \to \theta_Y$ does not depend on $t \in [0, 1]$, where the lifts used* *in the definitions of these homomorphisms are restrictions of some fixed lifts p, q of the given homotopy.*

REMARK 3.5. If (p,q) represents a single-valued map $\rho : X \to Y$ $(q = \rho p)$, then $\widetilde{q}_! \widetilde{p}^!$ equals $\widetilde{\rho}_!$ (here the chosen lifts satisfy $\widetilde{q} = \widetilde{\rho}\widetilde{p}$).

Let us point out that the above theory can be modified into the relative case (i.e. modulo a normal subgroup $H \subset \theta_X$). The index H will denote the relative modification.

Assuming $X = Y$, we can give

LEMMA 3.7.

(i) $C(p,q) = \bigcup_{\alpha \in \theta_{XH}} p_{\Gamma H} C(\widetilde{p}_H, \alpha \widetilde{q}_H)$,

(ii) *if* $p_{\Gamma H} C(\widetilde{p}_H, \alpha \widetilde{q}_H) \cap p_{\Gamma H} C(\widetilde{p}_H, \beta \widetilde{q}_H)$ *is not empty, then there exists a* $\gamma \in \theta_{XH}$ *such that* $\beta = \gamma \circ \alpha \circ (\widetilde{q}_{H!} \widetilde{p}^!_H \gamma)^{-1}$,

(iii) *the sets* $p_{\Gamma H} C(\widetilde{p}_H, \alpha \widetilde{q}_H)$ *are either disjoint or equal.*

Hence, we get the splitting of $C(p,q)$ into the H-Nielsen classes and the natural injection from the set of H-Nielsen classes into the set of Reidemeister classes modulo H, namely, $R_H(p,q)$.

Now, we would like to exhibit the classes which do not disappear under any compact (admissible) homotopy. For this, we need however (besides (i′), (iii′), (iii″)) the following two assumptions on the pair $X \overset{p}{\Longleftarrow} \Gamma \overset{q}{\longrightarrow} Y$:

(i″) Let X be a connected ANR-space, e.g., a connected retract of an open subset of (a convex set in) a Fréchet space, p is a Vietoris map and $\mathrm{cl}(q(\Gamma)) \subset X$ is compact, i.e. q is a compact map.

(ii′) There exists a normal subgroup $H \subset \theta_X$ of a finite index satisfying $\widetilde{q}_! \widetilde{p}^!(H) \subset H$.

Let us note that the following implications hold: (i), (ii) ⇒ (ii′), (i), (iii) ⇒ (i″).

DEFINITION 3.1. We call a pair (p,q) *N-admissible* if it satisfies (i′), (i″), (ii′), (iii′), (iii″) (⇐ (i)–(iii)).

Let us recall that, under the assumption (ii′), the Lefschetz number $\Lambda(p,q) \in \mathbb{Q}$ is defined (see Section 3.1). This is a homotopy invariant (with respect to the homotopies satisfying (ii′)) and $\Lambda(p,q) \neq 0$ implies $C(p,q) \neq \emptyset$ (cf. Section 3.1).

Let $A = p_{\Gamma H} C(\widetilde{p}, \alpha \widetilde{q})$ be a Nielsen class of an N-admissible pair (p,q). We say that (the N-Nielsen class) A is *essential* if $\Lambda(\widetilde{p}, \alpha \widetilde{q}) \neq 0$. This definition it correct, i.e. does not depend on the choice of α.

DEFINITION 3.2. Let (p,q) be an N-admissible multivalued map (for a subgroup $H \subset \theta_X$). We define the *Nielsen number modulo H* as the number of essential classes in θ_{X_H}. We denote this number by $N_H(p,q)$.

The following theorem is an easy consequence of the homotopy invariance of the Lefschetz number.

THEOREM 3.4. *$N_H(p,q)$ is a homotopy invariant (with respect to N-admissible homotopies*

$$X \times [0,1] \overset{p}{\Longleftarrow} \Gamma \overset{q}{\longrightarrow} X).$$

Moreover, (p,q) has at least $N_H(p,q)$ coincidences.

The following theorem shows that the above definition is consistent with the classical Nielsen number for single-valued maps.

THEOREM 3.5. *If an N-admissible map (p,q) is N-admissibly homotopic to a pair (p',q'), representing a single-valued map p (i.e. $q' = \rho p'$), then (p,q) has at least $N_H(\rho)$ coincidences (here H denotes also the subgroup of $\pi_1 X$ corresponding to the given $H \subset \theta_X$ in* (ii$'$)).

Although in the general case the theory requires special assumptions on the considered pair (p,q), in the case of multivalued self-maps on a torus it is enough to assume that this pair satisfies only (i$''$), i.e. it is admissible. This is due to the fact that any pair satisfying (i$''$) is homotopic to a pair representing a single-valued map.

THEOREM 3.6. *Any multivalued self-map (p,q) on the torus satisfying* (i$''$) *is admissibly homotopic to a pair representing a single-valued map.*

THEOREM 3.7. *Let $\mathbb{T}^n \overset{p}{\Longleftarrow} \Gamma \overset{q}{\longrightarrow} \mathbb{T}^n$ be such that p is a Vietoris map. Let $\rho : \mathbb{T}^n \to \mathbb{T}^n$ be a single-valued map representing a multivalued map homotopic to (p,q) (according to Theorem* 3.6, *such a map always exists). Then (p,q) has at least $N(\rho)$ coincidences.*

REMARK 3.6. Let us also recall that, on the torus $\mathbb{T}^n$, $N(\rho) = |\Lambda(\rho)| = |\det(I - A)|$, where A is an integer $(n \times n)$-matrix representing the induced homotopy homomorphism $\rho_\# : \pi_1 \mathbb{T}^n \to \pi_1 \mathbb{T}^n$. Moreover, if $\det(I - A) \neq 0$, then

$$\operatorname{card}\bigl(\pi_1\bigl(\mathbb{T}^n\bigr)/\operatorname{Im}(\rho_\#)\bigr) = \bigl|\det(I - A)\bigr|.$$

In particular, for $\rho = \mathrm{id}$, we have

$$N(\mathrm{id}) = \bigl|\Lambda(\mathrm{id})\bigr| = \bigl|\chi\bigl(\mathbb{T}^n\bigr)\bigr| = |\det O| = 0,$$

while for $\rho = -\mathrm{id}$, we have $N(-\mathrm{id}) = |\Lambda(-\mathrm{id})| = |\det 2I| = 2^n$.

For more details, see [22,12].

3.3. *Fixed point index theorems*

Consider a multivalued map $\varphi : X \multimap X$ and assume, similarly as in Section 3.1, that

(i) X is ANR-space, e.g., a retract of an open subset of a convex set in a Fréchet space,

(ii) φ is a compact composition of an R_δ-map $\Phi: X \multimap Y$ and a continuous single-valued map $f: Y \to X$, namely $\varphi = f \circ \Phi$, where Y is an ANR-space.

Let $D \subset X$ be an open subset of X with no fixed points of φ on its boundary ∂D. Then and integer $\mathrm{ind}(\varphi, X, D)$, called the *fixed point index* over X w.r.t. D exists such that the following proposition holds (see, e.g., [12,22,44]).

PROPOSITION 3.2. *Let* $\varphi: X \multimap X$ *be a map satisfying* (i), (ii). *Then* $\mathrm{ind}(\varphi, X, D) \in \mathbb{Z}$ *is well-defined satisfying the following properties*:

- (Existence) *If* $\mathrm{ind}(\varphi, X, D) \neq 0$, *then* $\mathrm{Fix}(\varphi) \neq \emptyset$.
- (Localization) *If* $D_1 \subset D$ *are open subsets of* X *such that* $\mathrm{Fix}(\varphi) \subset D_1 \subset D$, *then*
$$\mathrm{ind}(\varphi, X, D) = \mathrm{ind}(\varphi, X, D_1).$$
- (Additivity) *If* D_j, $j = 1, \ldots, n$, *are open disjoint subsets of* D *and all fixed points of* $\varphi|_D$ *are located in* $\bigcup_{j=1}^n D_j$, *then* $\mathrm{ind}(\varphi, X, D_j)$, $j = 1, \ldots, n$, *are well-defined satisfying*
$$\mathrm{ind}(\varphi, X, D) = \sum_{j=1}^n \mathrm{ind}(\varphi, X, D_j).$$
- (Homotopy) *If there is a compact homotopy* $\chi: X \times [0, 1] \multimap X$ (*in the same class of maps under consideration*) *with* $\chi(\cdot, 0) = \varphi$, $\chi(\cdot, 1) = \psi$, *and* ∂D *is fixed point free w.r.t.* χ, *then*
$$\mathrm{ind}(\varphi, X, D) = \mathrm{ind}(\psi, X, D).$$
- (Multiplicity) *If* $\psi: \widetilde{X} \multimap \widetilde{X}$ *satisfies* (i), (ii) *and an open* $\widetilde{D} \subset \widetilde{X}$ *is fixed point free w.r.t.* ψ, *then*
$$\mathrm{ind}\big(\varphi \times \psi, X \times \widetilde{X}, D \times \widetilde{D}\big) = \mathrm{ind}(\varphi, X, D) \cdot \mathrm{ind}\big(\psi, \widetilde{X}, \widetilde{D}\big).$$
- (Contraction) *If* $X' \subset X$ *are ANR-spaces such that* $\varphi(X) \subset X'$ *and* $\varphi|_{X'}$ *satisfies* (ii) *with* $\mathrm{Fix}(\varphi|_{X'}) \cap \partial(D \cap X') = \emptyset$, *then*
$$\mathrm{ind}(\varphi, X, D) = \mathrm{ind}\big(\varphi|_{X'}, X', D \cap X'\big).$$
- (Normalization) *If* $X = D$, *then*
$$\mathrm{ind}(\varphi, X, D) = \mathrm{ind}(\varphi, X, X) = \Lambda(\varphi).$$

Because of possible applications, it is very useful to formulate sufficiently general continuation principles.

For compact admissible maps from open subsets of a neighbourhood retract of a Fréchet space E into E, the fixed point index was just indicated.

Now, we will apply Proposition 3.2 to formulating the appropriate continuation principles. We restrict ourselves only to a particular class of admissible maps, namely to R_δ-maps $\Phi : D \multimap E$ (written here $\Phi \in J(D, E)$), i.e. u.s.c. maps with R_δ-values.

We often need to study fixed points for maps defined on sufficiently fine sets (possibly with an empty interior), but with values out of them. Making use of the previous results, we are in position to make the following construction.

Assume that X is a retract of a Fréchet space E and D is an open subset of X. Let $\Phi \in J(D, E)$ be locally compact, $\mathrm{Fix}(\Phi)$ be compact and let the following condition hold:

$$\forall x \in \mathrm{Fix}(\Phi)\ \exists U_x \ni x,\ U_x \text{ is open in } D \text{ such that } \Phi(U_x) \subset X. \tag{A}$$

The class of locally compact J-maps from D to E with the compact fixed point set and satisfying (A) will be denoted by the symbol $J_A(D, E)$. We say that $\Phi, \Psi \in J_A(D, E)$ are *homotopic in* $J_A(D, E)$ if there exists a homotopy $H \in J(D \times [0, 1], E)$ such that $H(\cdot, 0) = \Phi$, $H(\cdot, 1) = \Psi$, for every $x \in D$, there is an open neighbourhood V_x of x in D such that $H|_{V_x \times [0,1]}$ is compact, and

$$\forall x \in D\ \forall t \in [0, 1]$$
$$\big[x \in H(x, t) \Rightarrow \exists U_x \ni x,\ U_x \text{ is open in } D,\ H\big(U_x \times [0, 1]\big) \subset X\big]. \tag{A_H}$$

Note that the condition (A_H) is equivalent to the following one:

- If $\{x_j\}_{j\geqslant 1} \subset D$ converges to $x \in H(x, t)$, for some $t \in [0, 1]$, then $H(\{x_j\} \times [0, 1]) \subset X$, for j sufficiently large.

Let $\Phi \in J_A(D, E)$. Then $\mathrm{Fix}(\Phi) \subset \bigcup\{U_x \mid x \in \mathrm{Fix}(\Phi)\} \cap V =: D' \subset D$ and $\Phi(D') \subset X$, where V is a neighbourhood of the set $\mathrm{Fix}(\Phi)$ such that $\Phi|_V$ is compact (by the compactness of $\mathrm{Fix}(\Phi)$ and local compactness of Φ) and U_x is a neighbourhood of x as in (A). Define

$$\mathrm{Ind}_A(\Phi, X, D) = \mathrm{ind}\big(\Phi|_{D'}, X, D'\big),$$

where $\mathrm{ind}(\Phi|_{D'}, X, D')$ is defined as in Proposition 3.2. This definition is independent of the choice of D'.

In the following theorem, we give some properties of Ind_A which will be used in the proof of the continuation Theorem 3.8. The simple proof is omitted.

PROPOSITION 3.3.

(i) (Existence) *If* $\mathrm{Ind}_A(\Phi, X, D) \neq 0$, *then* $\mathrm{Fix}(\Phi) \neq \emptyset$.

(ii) (Localization) *If* $D_1 \subset D$ *are open subsets of a retract* X *of a space* E, $\Phi \in J_A(D, E)$ *is compact*, *and* $\mathrm{Fix}(\Phi)$ *is a compact subset of* D_1, *then*

$$\mathrm{Ind}_A(\Phi, X, D) = \mathrm{Ind}_A(\Phi, X, D_1).$$

(iii) (Homotopy) *If* H *is a homotopy in* $J_A(D, E)$, *then*

$$\mathrm{Ind}_A\big(H(\cdot, 0), X, D\big) = \mathrm{Ind}_A\big(H(\cdot, 1), X, D\big).$$

(iv) (Normalization) *If* $\Phi \in J(X)$ *is a compact map*, *then* $\mathrm{Ind}_A(\Phi, X, X) = 1$.

THEOREM 3.8 (Continuation principle). *Let* X *be a retract of a Fréchet space* E, D *be an open subset of* X *and* H *be a homotopy in* $J_A(D, E)$ *such that*

(i) $H(\cdot, 0)(D) \subset X$,

(ii) *there exists* $H' \in J(X)$ *such that* $H'|_D = H(\cdot, 0)$, H' *is compact and* $\mathrm{Fix}(H') \cap (X \setminus D) = \emptyset$.

Then there exists $x \in D$ *such that* $x \in H(x, 1)$.

PROOF. Applying the localization property (ii), we obtain

$$\mathrm{Ind}_A\big(H(\cdot, 0), X, D\big) = \mathrm{Ind}_A\big(H(\cdot, 0), X, X\big).$$

By the normalization property (iv), $\mathrm{Ind}_A(H(\cdot, 0), X, X) = 1$. Thus, by the homotopy property (iii), $\mathrm{Ind}_A(H(\cdot, 0), X, D) = \mathrm{Ind}_A(H(\cdot, 1), X, D) = 1$, which implies by (i) that $H(\cdot, 1)$ has a fixed point. □

COROLLARY 3.1. *Let* X *be a retract of a Fréchet space* E *and* H *be a homotopy in* $J_A(X, E)$ *such that* $H(x, 0) \subset X$, *for every* $x \in X$, *and* $H(\cdot, 0)$ *is compact. Then* $H(\cdot, 1)$ *has a fixed point.*

COROLLARY 3.2. *Let* X *be a retract of a Fréchet space* E, D *be an open subset of* X *and* H *be a homotopy in* $J_A(D, E)$. *Assume that* $H(x, 0) = x_0$, *for every* $x \in D$. *Then there exists* $x \in D$ *such that* $x \in H(x, 1)$.

COROLLARY 3.3. *Let* X *be a retract of a Fréchet space* E *and* $\Phi \in J(X)$ *be compact. Then* Φ *has a fixed point.*

REMARK 3.7. If $E = X$ is a Banach space, then it follows from Proposition 3.2 that the "pushing" condition (A_H), related to $J_A(D, E)$, can be reduced to $\mathrm{Fix}(\varphi) \cap \partial D \neq \emptyset$.

Some applications motivate us to consider weaker than (A_H) condition on H. Unfortunately, we cannot use the fixed point index technique described above. The proof of the following theorem is based on a Schauder-type approximation technique (for more details, see [19,22]).

THEOREM 3.9 (Continuation principle). *Let* X *be a closed*, *convex subset of a Fréchet space* E *and let* $H \in J(X \times [0, 1], E)$ *be compact. Assume that*

(i) $H(x, 0) \subset X$, *for every* $x \in X$,

(ii) *for any* $(x, t) \in \partial X \times [0, 1)$ *with* $x \in H(x, t)$, *there exist open neighbourhoods* U_x *of* x *in* X *and* I_t *of* t *in* $[0, 1)$ *such that* $H((U_x \cap \partial X) \times I_t) \subset X$.

Then there exists a fixed point of $H(\cdot, 1)$.

REMARK 3.8. Note that the convexity of X in Theorem 3.9 is essential only in the infinite-dimensional case. For the proof, we have namely to intersect X with a finite-dimensional subspace L.

Now, we would like to consider condensing maps. Hence, let this time $\varphi: X \multimap X$ be a multivalued map such that

(I) X is a closed, convex subset of a Fréchet space E,

(II) φ is a condensing composition of an R_δ-map $\Phi: X \multimap Y$ and a continuous single-valued map $f: Y \to X$, namely $\varphi = f \circ \Phi$, where Y is an ANR-space.

Assume that $\mathrm{Fix}(\varphi) \cap \partial D \neq \emptyset$, for some open subset $D \subset X$. Since φ is condensing, it has a nonempty compact fundamental set T (see Definition 2.11 and Proposition 2.7(iii)). Let $\mathrm{ind}(\varphi, X, D) = 0$, whenever $\mathrm{Fix}(\varphi) = \emptyset$. Since T is an AR-space, we may choose a retraction $r: X \to T$ in order to define the fixed point index, for the composition

$$\widetilde{\varphi}: \overline{D} \cap T \overset{\Phi}{\multimap} Y \overset{r \circ f}{\multimap} T,$$

by putting

$$\mathrm{ind}(\varphi, X, D) := \mathrm{ind}\big(\widetilde{\varphi}, \widetilde{X}, D \cap T\big),$$

where ind on the right-hand side is defined as in Proposition 3.2. This correct definition is independent of the chosen fundamental set, and so the index has all the appropriate properties as in Proposition 3.2, but (see Remark 3.2 and cf. [37]) the normalization property. Instead of it, a weak normalization property can be formulated as follows:

- (Weak normalization) If f in $\varphi = f \circ \Phi$ is a constant map, i.e. $f(y) = a \notin \partial D$, for each $y \in Y$, then

$$\mathrm{ind}(\varphi, X, D) = \begin{cases} 1, & \text{for } a \in D, \\ 0, & \text{for } a \notin D. \end{cases}$$

As already pointed out above, in the applications of the fixed point theory, we often need to consider maps with values in a Fréchet space and not in a closed convex set. We will also extend our theory to this case.

Again, let E be a Fréchet space and X be a closed and convex subset of E. Let $U \subset X$ be open and consider the map $\varphi \in J_A(U, E)$, where the symbol $J_A(U, E)$ is again reserved for J-maps from U to E satisfying condition (A). The notion of homotopy in J_A will be understood analogously. Thus, $\mathrm{Fix}(\varphi)$ is compact and φ has a compact fundamental set T (see Section 2.3). Set $\mathrm{Ind}_A(\varphi, X, U) = 0$, whenever $\mathrm{Fix}(\varphi) = \emptyset$. Otherwise, let $x_1, \ldots, x_n \in \mathrm{Fix}(\varphi)$ such that $\mathrm{Fix}(\varphi) \subset \bigcup_{i=1}^n U_{x_i} =: V$, where U_{x_i} are neighbourhoods of x_i such that $\overline{U}_{x_i} \subset U$ and satisfy condition (A). Then $\varphi|_V: V \multimap X$ is a J-map with compact fundamental set T and satisfies $\mathrm{Fix}(\varphi) \cap \partial V = \emptyset$. Thus, we can define

$$\mathrm{Ind}_A(\varphi, X, U) := \mathrm{ind}(\varphi|_{\overline{V}}, X, V).$$

The independence of this definition of the chosen set V follows from the additivity property. Furthermore, if $\varphi: \overline{U} \multimap X$ has a compact fundamental set and $\mathrm{Fix}(\varphi) \cap \partial U = \emptyset$, then $\mathrm{Ind}_A(\varphi, X, U)$ is defined and

$$\mathrm{Ind}_A(\varphi, X, U) = \mathrm{ind}(\varphi, X, U).$$

The following proposition easily follows from the above argumentation.

PROPOSITION 3.4.

(i) (Existence) *If* $\mathrm{Ind}_A(\varphi, X, U) \neq 0$, *then* $\mathrm{Fix}(\varphi) \neq \emptyset$.

(ii) (Additivity) *Let* $\mathrm{Fix}(\varphi) \subset U_1 \cup U_2$, *where* U_1, U_2 *are open disjoint subsets of* U. *Then*

$$\mathrm{Ind}_A(\varphi, X, U) = \mathrm{Ind}_A(\varphi|_{U_1}, X, U_1) + \mathrm{Ind}_A(\varphi|_{U_2}, X, U_2).$$

(iii) (Homotopy) *Let* $\psi : U \multimap E$ *be homotopic in* J_A *to the map* φ. *Assume that the homotopy* $\chi : U \times [0, 1] \multimap E$ *has a compact fundamental set and the set*

$$\Sigma := \big\{(x, t) \in U \times [0, 1] \mid x \in \chi(x, t)\big\}$$

is compact. Then

$$\mathrm{Ind}_A(\varphi, X, U) = \mathrm{Ind}_A(\psi, X, U).$$

(iv) (Weak normalization) *Assume that* $\varphi : U \to E$ *is a constant map* $\varphi(x) = a \in E$, *for all* $x \in U$. *Then*

$$\mathrm{Ind}_A(\varphi, X, U) = \begin{cases} 1, & \text{for } a \in U, \\ 0, & \text{for } a \notin U. \end{cases}$$

Using Proposition 3.4, we can easily formulate a continuation principle which is convenient for various applications.

THEOREM 3.10 (Continuation principle). *Let* X *be a closed, convex subset of a Fréchet space* E, *let* $U \subset X$ *be open and let* $\chi : U \times [0, 1] \multimap E$ *be a homotopy in* J_A *such that* Σ (*see* (iii) *above*) *is compact. Let* χ *be condensing and assume that there is a condensing* $\varphi \in J(X)$ *such that* $\varphi|_U = \chi(\cdot, 0)$ *and* $\mathrm{Fix}(\varphi) \cap (X \setminus U) = \emptyset$. *Then* $\chi(\cdot, 1)$ *has a fixed point.*

PROOF. The proof follows, in view of the existence property (i) in Proposition 3.4, from the following equations:

$$\mathrm{Ind}_A\big(\chi(\cdot, 1), X, U\big) = \mathrm{Ind}_A\big(\chi(\cdot, 0), X, U\big),$$

by the homotopy property (iii),

$$\mathrm{Ind}_A\big(\chi(\cdot, 0), X, U\big) = \mathrm{Ind}_A(\varphi|_U, X, U) = \mathrm{Ind}_A(\varphi, X, X),$$

by the additivity property (ii). Finally, we see that

$$\mathrm{Ind}_A(\varphi, X, X) = \mathrm{ind}(\varphi, X, X) = 1. \qquad \square$$

COROLLARY 3.4. *Let $\chi : X \times [0, 1] \multimap E$ be a condensing homotopy in J_A such that $\chi(x, 0) \subset X$, for every $x \in X$. Then $\chi(\cdot, 1)$ has a fixed point.*

REMARK 3.9. If $E = X$ is a Banach space, then it follows from the properties of the above fixed point index $\operatorname{ind}(\chi, X, D)$ that the "pushing" condition (A_H), related to $J_A(D, E)$, can be reduced to $\operatorname{Fix}(\chi) \cap \partial D \neq \emptyset$. If, in particular, $U \subset E$ is an open convex subset such that $\chi(\cdot, 0) = \varphi|_U : U \multimap U$, then the same is true even without requiring $\operatorname{Fix}(\varphi) \cap (X \setminus U) = \emptyset$.

4. General methods for solvability of boundary value problems

4.1. *Continuation principles to boundary value problems*

In this part, fixed point principles in Section 3 will be applied to differential equations and inclusions.

At first, we are interested in the existence problems for ordinary differential equations and inclusions in Euclidean spaces on not necessarily compact intervals. Let us start with some definitions.

Let J be an interval in $\mathbb{R}$. We say that a map $x : J \to \mathbb{R}^n$ is *locally absolutely continuous* if x is absolutely continuous on every compact subset of J. The set of all locally absolutely continuous maps from J to $\mathbb{R}^n$ will be denoted by $\mathrm{AC}_{\mathrm{loc}}(J, \mathbb{R}^n)$.

Consider the inclusion

$$\dot{x} \in F(t, x), \tag{4.1}$$

where F is a set-valued *u-Carathéodory map*, i.e. it has i.a. the following properties:

- the set of values of F is nonempty, compact and convex, for all $(t, x) \in J \times \mathbb{R}^n$,
- the map $F(t, \cdot)$ is u.s.c., for almost all $t \in J$,
- the map $F(\cdot, x)$ is measurable, for all $x \in \mathbb{R}^n$.

By a *solution* of the inclusion (4.1), we mean a locally absolutely continuous function x such that (4.1) holds, for almost all $t \in J$.

We recall two known results which are needed in the sequel.

PROPOSITION 4.1 (Cf. [42, Theorem 0.3.4] and Lemma 4.4 below). *Assume that the sequence of absolutely continuous functions $x_k : K \to \mathbb{R}^n$ (K is a compact interval) satisfies the following conditions*:

- *the set $\{x_k(t) \mid k \in \mathbb{N}\}$ is bounded, for every $t \in K$,*
- *there is an integrable function (in the sense of Lebesgue) $\alpha : K \to \mathbb{R}$ such that*

$$\left|\dot{x}_k(t)\right| \leqslant \alpha(t), \quad \textit{for a.a. } t \in K \textit{ and for all } k \in \mathbb{N}.$$

Then there exists a subsequence (denoted just the same) $\{x_k\}$ convergent to an absolutely continuous function $x : K \to \mathbb{R}^n$ in the following sense:

(i) *$\{x_k\}$ uniformly converges to x, and*
(ii) *$\{\dot{x}_k\}$ weakly converges in $L^1(K, \mathbb{R}^n)$ to $\dot{x}$.*

The second one is the well-known (see, e.g., [22, Theorem 1.33 in Chapter I.1]) Mazur theorem.

The following result is crucial.

PROPOSITION 4.2. *Let* $G : J \times \mathbb{R}^n \times \mathbb{R}^m \multimap \mathbb{R}^n$ *be a u-Carathéodory map and let* S *be a nonempty subset of* $\mathrm{AC_{loc}}(J, \mathbb{R}^n)$. *Assume that*

(i) *there exists a subset* Q *of* $C(J, \mathbb{R}^n)$ *such that, for any* $q \in Q$, *the set* $T(q)$ *of all solutions of the boundary value problem*

$$\begin{cases} \dot{x}(t) \in G\big(t, x(t), q(t)\big), & \text{for a.a. } t \in J, \\ x \in S \end{cases} \tag{4.2}$$

is nonempty,

(ii) $T(Q)$ *is bounded in* $C(J, \mathbb{R}^n)$,

(iii) *there exists a locally integrable function* $\alpha : J \to \mathbb{R}$ *such that*

$$\big|G\big(t, x(t), q(t)\big)\big| = \sup\big\{|y| \mid y \in G\big(t, x(t), q(t)\big)\big\} \leqslant \alpha(t), \quad \text{a.e. in } J,$$

for any pair $(q, x) \in \Gamma_T$, *where* Γ_T *denotes the graph of* T.

Then $T(Q)$ *is a relatively compact subset of* $C(J, \mathbb{R}^n)$. *Moreover, under the assumptions* (i)–(iii), *the multivalued operator* $T : Q \multimap S$ *is u.s.c. with compact values if and only if the following condition is satisfied*:

(iv) *given a sequence* $\{(q_k, x_k)\} \subset \Gamma_T$, *if* $\{(q_k, x_k)\}$ *converges to* (q, x) *with* $q \in Q$, *then* $x \in S$.

PROOF. For the relative compactness of $T(Q)$, it is sufficient to show that all elements of $T(Q)$ are equicontinuous.

By (iii), for every $x \in T(Q)$, we have $|\dot{x}(t)| \leqslant \alpha(t)$, for a.a. $t \in J$, and

$$\big|x(t_1) - x(t_2)\big| \leqslant \left|\int_{t_1}^{t_2} \alpha(s)\,\mathrm{d}s\right|.$$

This implies an equicontinuity of all $x \in T(Q)$.

We show that the set Γ_T is closed (cf. Section 2.1).

Let $\Gamma_T \supset \{(q_k, x_k)\} \to (q, x)$. Let K be an arbitrary compact interval such that α is integrable on K. By conditions (ii) and (iii), the sequence $\{x_k\}$ satisfies the assumptions of Proposition 4.1.

Thus, there exists a subsequence (denoted just the same) $\{x_k\}$, uniformly convergent to x on K (because the limit is unique) and such that $\{\dot{x}_k\}$ weakly converges to $\dot{x}$ in L^1. Therefore, $\dot{x}$ belongs to the weak closure of the set $\operatorname{conv}\{\dot{x}_m \mid m \geqslant k\}$, for every $k \geqslant 1$. By the mentioned Mazur theorem, $\dot{x}$ also belongs to the strong closure of this set. Hence, for every $k \geqslant 1$, there is $z_k \in \operatorname{conv}\{\dot{x}_m \mid m \geqslant k\}$ such that $\|z_k - \dot{x}\|_{L^1} \leqslant 1/k$. This implies that there exists a subsequence $z_{k_l} \to \dot{x}$ a.e. in K.

Let $s \in K$ be such that

$$G(s, \cdot, \cdot) \text{ is u.s.c.}, \quad \lim_{l \to \infty} z_{k_l}(s) = \dot{x}(s), \quad \dot{x}_k(s) \in G\big(s, x_k(s), q_k(s)\big).$$

Let $\varepsilon > 0$. There is $\delta > 0$ such that $G(s, z, p) \subset N_\varepsilon(G(s, x(s), q(s)))$, whenever $|x(s) - z| < \delta$ and $|q(s) - p| < \delta$. But we know that there exists $N \geqslant 1$ such that $|x(s) - x_m(s)| < \delta$ and $|q(s) - q_m(s)| < \delta$, for every $m \geqslant N$. Hence,

$$\dot{x}_k(s) \in G\big(s, x_k(s), q_k(s)\big) \subset N_\varepsilon\big(G\big(s, x(s), q(s)\big)\big).$$

By the convexity of $G(s, x(s), q(s))$, for $k_l \geqslant N$, we have

$$z_{k_l}(s) \in N_\varepsilon\big(G\big(s, x(s), q(s)\big)\big).$$

Thus, $\dot{x}(s) \in \overline{N_\varepsilon(G(s, x(s), q(s)))}$, for every $\varepsilon > 0$. This implies

$$\dot{x}(s) \in G\big(s, x(s), q(s)\big).$$

Since K was arbitrary, $\dot{x}(t) \in G(t, x(t), q(t))$, a.e. in J. □

We can now state one of the main results of this subsection.

THEOREM 4.1. *Consider the boundary value problem*

$$\begin{cases} \dot{x}(t) \in F\big(t, x(t)\big), & \text{for a.a. } t \in J, \\ x \in S, \end{cases} \tag{4.3}$$

where J is a given real interval, $F : J \times \mathbb{R}^n \multimap \mathbb{R}^n$ is a u-Carathéodory map and S is a subset of $\mathrm{AC_{loc}}(J, \mathbb{R}^n)$.

Let $G : J \times \mathbb{R}^n \times \mathbb{R}^n \times [0, 1] \multimap \mathbb{R}^n$ be a u-Carathéodory map (*cf. Definition* 2.10) *such that*

$$G(t, c, c, 1) \subset F(t, c), \quad \text{for all } (t, c) \in J \times \mathbb{R}^n.$$

Assume that

(i) *there exist a retract Q of $C(J, \mathbb{R}^n)$ and a closed bounded subset S_1 of S such that the associated problem*

$$\begin{cases} \dot{x}(t) \in G\big(t, x(t), q(t), \lambda\big), & \text{for a.a. } t \in J, \\ x \in S_1 \end{cases} \tag{4.4}$$

is solvable with an R_δ-set of solutions, for each $(q, \lambda) \in Q \times [0, 1]$,

(ii) *there exists a locally integrable function $\alpha : J \to \mathbb{R}$ such that*

$$\big|G\big(t, x(t), q(t), \lambda\big)\big| \leqslant \alpha(t), \quad \text{a.e. in } J,$$

for any $(q, \lambda, x) \in \Gamma_T$, where T denotes the set-valued map which assigns to any $(q, \lambda) \in Q \times [0, 1]$ the set of solutions of (4.4),

(iii) $T(Q \times \{0\}) \subset Q$,

(iv) *if* $Q \ni q_j \to q \in Q$, $q \in T(q,\lambda)$, *then there exists* $j_0 \in \mathbb{N}$ *such that, for every* $j \geqslant j_0$, $\theta \in [0,1]$ *and* $x \in T(q_j,\theta)$, *we have* $x \in Q$.

Then problem (4.3) *has a solution.*

PROOF. Consider the set

$$Q' = \{ y \in C(J,\mathbb{R}^{n+1}) \mid y(t) = (q(t),\lambda),\ q \in Q,\ \lambda \in [0,1] \}.$$

By Proposition 4.2, we obtain that the set-valued map $T : Q \times [0,1] \multimap S_1$ is u.s.c., and so it belongs to the class $J(Q \times [0,1], C(J,\mathbb{R}^n))$. Moreover, it has a relatively compact image. Assumption (iv) implies that T is a homotopy in $J_A(Q, C(J,\mathbb{R}^n))$. Corollary 3.1 in Section 3 now gives the existence of a fixed point of $T(\cdot,1)$. However, by the hypothesis, it is a solution of (4.3). □

Note that the conditions (iii) and (iv) in the above theorem hold if $S_1 \subset Q$.

COROLLARY 4.1. *Consider the boundary value problem* (4.3).
Let $G : J \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ *be a u-Carathéodory map such that*

$$G(t,c,c) \subset F(t,c), \quad \text{for all } (t,c) \in J \times \mathbb{R}^n.$$

Assume that

(i) *there exists a retract* Q *of* $C(J,\mathbb{R}^n)$ *such that the associated problem*

$$\begin{cases} \dot{x}(t) \in G(t,x(t),q(t)), & \text{for a.a. } t \in J, \\ x \in S \cap Q \end{cases} \tag{4.5}$$

has an R_δ*-set of solutions, for each* $q \in Q$,

(ii) *there exists a locally integrable function* $\alpha : J \to \mathbb{R}$ *such that*

$$|G(t,x(t),q(t))| \leqslant \alpha(t), \quad \text{a.e. in } J,$$

for any $(q,x) \in \Gamma_T$,

(iii) $T(Q)$ *is bounded in* $C(J,\mathbb{R}^n)$ *and* $\overline{T(Q)} \subset S$.

Then problem (4.3) *has a solution.*

Making use of the special case of Theorem 3.3 and modifying appropriately the proof of Theorem 4.1, we can easily obtain the following:

COROLLARY 4.2. *Consider problem* (4.3) *and assume that all the assumptions of Corollary* 4.1 *hold with the convex closed set* Q *and nonempty acyclic sets of solutions* (4.5). *Then the problem* (4.3) *has a solution.*

Let us note that in applications solution sets are, in fact, R_δ-sets.

If, in particular, $J = [a,b]$ (i.e. compact), then Theorem 4.1 can be easily reformulated (see Remark 3.7) as follows.

COROLLARY 4.3. *Consider the boundary value problem* (4.3), *where* $J = [a, b]$ *is a compact interval*, $F : J \times \mathbb{R}^n \multimap \mathbb{R}^n$ *is a u-Carathéodory map and* $S \subset \mathrm{AC}(J, \mathbb{R}^n)$.

Let $G : J \times \mathbb{R}^n \times \mathbb{R}^n \times [0, 1] \multimap \mathbb{R}^n$ *be a u-Carathéodory map such that* $G(t, c, c, 1) \subset F(t, c)$, *for all* $(t, c) \in J \times \mathbb{R}^n$. *Assume that*

(i) *there exist a* (*bounded*) *retract* Q *of* $C(J, \mathbb{R}^n)$ *such that* $Q \setminus \partial Q$ *is nonempty* (*open*) *and a closed bounded subset* S_1 *of* S *such that the associated problem* (4.4) *is solvable with an* R_δ*-set of solutions*, *for each* $(q, \lambda) \in Q \times [0, 1]$, *and conditions* (ii) *and* (iii) *in Theorem* 4.1 *hold true*,

(ii) *the solution map* T (*defined in condition* (ii) *of Theorem* 4.1) *has no fixed points on the boundary* ∂Q *of* Q, *for every* $(q, \lambda) \in Q \times [0, 1]$.

Then problem (4.3) *has a solution*.

REMARK 4.1. In the (single-valued) case of Carathéodory ODEs, we can only assume in Theorem 4.1(i), Corollary 4.1(i), Corollary 4.2 and Corollary 4.3(i) that the related linearized problems are uniquely solvable.

Since $C^{(n-1)}(J)$ can be considered as a subspace of $C(J, \mathbb{R}^n)$, we can also apply the previous results to nth-order scalar differential equations and inclusions. To solve an existence problem, one should check suitable a priori bounds for all the derivatives up to the order $n - 1$. Our technique simplifies a work. Let us describe it below.

We need the following lemma [52, Lemma 2.1] related to the Banach space $H^{n,1}(I)$:[2]

LEMMA 4.1. *Let* I *be a compact real interval and let* $a_0, a_1, \ldots, a_{n-1} : I \times \mathbb{R}^n \to \mathbb{R}$ *be u-Carathéodory functions. Given any* $q \in C^{(n-1)}(I)$, *consider the following linear nth-order differential operator* $L_q : H^{n,1}(I) \to L^1(I)$:

$$L_q(x)(t) = x^{(n)}(t) + \sum_{i=0}^{n-1} a_i\bigl(t, q(t), \ldots, q^{(n-1)}(t)\bigr)x^{(i)}(t).$$

Assume there exists a subset Q *of* $C^{(n-1)}(I)$ *and an* L^1*-function* $\beta : I \to \mathbb{R}$ *such that*, *for any* $q \in Q$ *and any* $i = 0, 1, \ldots, n - 1$, *we have*

$$\bigl|a_i\bigl(t, q(t), \ldots, q^{(n-1)}(t)\bigr)\bigr| \leqslant \beta(t), \quad \text{a.e. in } I.$$

Then the following two norms are equivalent in $H^{n,1}(I)$:

$$\|x\| = \sum_{i=0}^{n-1} \sup_{t\in I}\bigl|x^{(i)}(t)\bigr| + \int_I \bigl|x^{(n)}(t)\bigr|\,\mathrm{d}t,$$

$$\|x\|_Q = \sup_{t\in I}\bigl|x(t)\bigr| + \sup_{q\in Q}\int_I \bigl|L_q(x)(t)\bigr|\,\mathrm{d}t.$$

[2]By $H^{n,1}(I)$, we denote the Banach space of all $C^{(n-1)}$-functions $x : I \to \mathbb{R}$, where I is a compact interval, with absolutely continuous $(n - 1)$th derivatives.

COROLLARY 4.4. *Consider the scalar problem*

$$\begin{cases} x^{(n)}(t) \in \displaystyle\sum_{i=0}^{n-1} a_i\big(t, x(t), \dots, x^{(n-1)}(t)\big)x^{(i)}(t) \\ \qquad\qquad + F\big(t, x(t), \dots, x^{(n-1)}(t)\big), \quad \textit{for a.a. } t \in J, \\ x \in S, \end{cases} \tag{4.6}$$

where $J \subset \mathbb{R}$, $S \subset C(J)$ and a_i, F are u-Carathéodory maps on $J \times \mathbb{R}^n$.

Suppose that there exists a u-Carathéodory map $G : J \times \mathbb{R}^n \times \mathbb{R}^n \times [0, 1] \multimap \mathbb{R}^n$ such that, for every $c \in \mathbb{R}^n$ and $\lambda \in [0, 1]$, $G(t, c, c, 1) \subset F(t, c)$, a.e. in J. Then problem (4.6) *has a solution, provided the following conditions are satisfied*:

(i) *there is a retract Q of the space $C^{(n-1)}(J)$ such that, for every $(q, \lambda) \in Q \times [0, 1]$, the following problem,*

$$\begin{cases} x^{(n)}(t) \in \displaystyle\sum_{i=0}^{n-1} a_i\big(t, q(t), \dots, q^{(n-1)}(t)\big)x^{(i)}(t) \\ \qquad\qquad + G\big(t, x(t), \dots, x^{(n-1)}(t), q(t), \dots, q^{(n-1)}(t), \lambda\big), \quad \textit{for a.a. } t \in J, \\ x \in S \cap Q, \end{cases} \tag{4.7}$$

has an R_δ-set of solutions,

(ii) *there is a locally integrable function $\alpha : J \to \mathbb{R}$ such that, for every $i = 0, \dots, n-1$:*

$$\big|a_i\big(t, q(t), \dots, q^{(n-1)}(t)\big)\big| \leqslant \alpha(t), \quad \textit{a.e. in } J,$$

and

$$\big|G\big(t, x(t), \dots, x^{(n-1)}(t), q(t), \dots, q^{(n-1)}(t), \lambda\big)\big| \leqslant \alpha(t), \quad \textit{for a.e. } t \in J,$$

for each $(q, \lambda, x) \in Q \times [0, 1] \times C^{(n-1)}(J)$ satisfying (4.7),

(iii) *$T(Q \times \{0\}) \subset Q$, where T denotes the set-valued map which assigns to any $(q, \lambda) \in Q \times [0, 1]$ the set of solutions of* (4.7),

(iv) *the set $T(Q \times [0, 1])$ is bounded in $C(J)$ and its $C^{(n-1)}$-closure is contained in S (in particular, this holds if $S \cap C^{(n-1)}(J)$ is closed in $C^{(n-1)}(J)$),*

(v) *if $\{q_j\} \subset Q$ converges to $q \in Q$, $q \in T(q, \lambda)$ in $C^{(n-1)}(J)$, then there exists $j_0 \in \mathbb{N}$ such that, for every $j \geqslant j_0$, $\theta \in [0, 1]$ and $x \in T(q_j, \theta)$, we have $x \in Q$.*

PROOF. We construct a new problem in the following way:

Define $\widetilde{F} : J \times \mathbb{R}^n \multimap \mathbb{R}^n$,

$$\widetilde{F}\big(t, x(t), \dots, x^{(n-1)}(t)\big) = F\big(t, x(t), \dots, x^{(n-1)}(t)\big) + \sum_{i=0}^{n-1} a_i\big(t, x(t), \dots, x^{(n-1)}(t)\big)x^{(i)}(t).$$

Denote $\overline{x}(t) = (x(t), \ldots, x^{(n-1)}(t)) \in \mathbb{R}^n$ and define $F' : J \times \mathbb{R}^n \multimap \mathbb{R}^n$,

$$F'\big(t, \overline{x}(t)\big) = \big\{\big(\dot{x}(t), \ldots, x^{(n-1)}(t), y\big) \mid y \in \widetilde{F}\big(t, x(t), \ldots, x^{(n-1)}(t)\big)\big\}.$$

So, we have a problem

$$\begin{cases} \dot{\overline{x}}(t) \in F'\big(t, \overline{x}(t)\big), & \text{for a.a. } t \in J, \\ \overline{x} \in \overline{S}, \end{cases} \tag{4.8}$$

where $\overline{S}$ is an image of $S \cap C^{(n-1)}(J)$ via the inclusion $i : C^{(n-1)}(J) \to C(J, \mathbb{R}^n)$.

Analogously, we find the associated problem

$$\begin{cases} \dot{\overline{x}}(t) \in G'\big(t, \overline{x}(t), \overline{q}(t), \lambda\big), & \text{for a.a. } t \in J, \\ \overline{x} \in \overline{S} \cap \overline{Q}. \end{cases} \tag{4.9}$$

Notice that

(1) $G'(t, \overline{x}(t), \overline{q}(t), 1) \subset F'(t, \overline{x}(t))$,

(2) the set $\overline{Q} = i(Q)$ is a retract of $C(J, \mathbb{R}^n)$,

(3) $\overline{S} \subset \mathrm{AC}_{\mathrm{loc}}(J, \mathbb{R}^n)$,

(4) for every $(q, \lambda) \in Q \times [0, 1]$, the sets of solutions of the problems (4.7) and (4.9) are the same,

(5) $\overline{\overline{T}(\overline{Q} \times [0, 1])} \subset \overline{S}$, where $\overline{T}$ is a suitable map corresponding to T

and

$$\begin{aligned} \big|G'\big(t, \overline{x}(t), \overline{q}(t), \lambda\big)\big| &\leqslant \big|G\big(t, x(t), \ldots, x^{(n-1)}, q(t), \ldots, q^{(n-1)}(t), \lambda\big)\big| \\ &\quad + \sum_{i=0}^{n-1} \big|a_i\big(t, q(t), \ldots, q^{(n-1)}(t)\big)\big| \big|x^{(i)}(t)\big| \\ &\leqslant \alpha(t) + \alpha(t) \sum_{i=0}^{n-1} \big|x^{(i)}(t)\big|. \end{aligned}$$

Since $T(Q \times [0, 1])$ is bounded in $C(J)$, there exists a positive continuous function $m : J \to \mathbb{R}$ such that $|x(t)| \leqslant m(t)$, for all $t \in J$ and any $x \in T(Q \times [0, 1])$. We will show that $T(Q \times [0, 1])$ is also bounded in $C^{(n-1)}(J)$. It is sufficient to prove that, for any compact subinterval I in J, there is a constant $M > 0$ such that

$$p_I(x) = \sum_{i=0}^{n-1} \sup \big|x^{(i)}(t)\big| \leqslant M, \quad \text{for all } x \in T\big(Q \times [0, 1]\big).$$

Let $I \subset J$ be an arbitrary compact interval. Using the notation in Lemma 4.1, we see that $p_I(x) \leqslant \|x\|$ and, by the equivalence of norms,

$$\|x\| \leqslant c\|x\|_Q \leqslant c\left(\max_{t \in I} m(f) + \int_I \alpha(t)\, \mathrm{d}t\right) \leqslant M.$$

We conclude that $T(Q \times [0,1])$ is bounded in $C^{(n-1)}(J)$ which implies that $\overline{T}(\overline{Q} \times [0,1])$ is bounded in $C(J, \mathbb{R}^n)$. Moreover, there exists a continuous function $\varphi : J \to \mathbb{R}$ such that

$$\left|G'\big(t, \overline{x}(t), \overline{q}(t), \lambda\big)\right| \leqslant \alpha(t)\big(1 + \varphi(t)\big).$$

Obviously, the right-hand side of the above inequality is a locally integrable function.

Finally, an easy computation shows that the condition (iv) in Theorem 4.1 holds for $\overline{Q}$ and $\overline{T}$. By Theorem 4.1, there exists a solution of (4.8) as well as the one of (4.6). □

The same argument as in Corollary 4.1 shows how to modify Corollary 4.4 for the following scalar problem, namely

$$\begin{cases} x^{(n)}(t) \in \displaystyle\sum_{i=0}^{n-1} a_i\big(t, x(t), \ldots, x^{(n-1)}(t)\big) x^{(i)}(t) \\ \qquad\qquad + F\big(t, x(t), \ldots, x^{(n-1)}(t)\big), \quad \text{for a.a. } t \in J, \\ x \in S, \end{cases} \tag{4.10}$$

where $J \subset \mathbb{R}$, $S \subset C(J)$ and a_i, F are u-Carathéodory maps on $J \times \mathbb{R}^n$, by means of the following linearized problem

$$\begin{cases} x^{(n)}(t) \in \displaystyle\sum_{i=0}^{n-1} a_i\big(t, q(t), \ldots, q^{(n-1)}(t)\big) x^{(i)}(t) \\ \qquad\qquad + G\big(t, x(t), \ldots, x^{(n-1)}(t), q(t), \ldots, q^{(n-1)}(t)\big), \quad \text{for a.a. } t \in J, \\ x \in S \cap Q, \end{cases} \tag{4.11}$$

where Q is a retract of the space $C^{(n-1)}(J)$.

Theorem 3.9 in Section 3.3 gives similar consequences as those of Theorem 3.8 in Section 3.3. Unfortunately, the weakness of the assumption on solutions causes that we have to assume the convexity of the set Q. In spite of it, the results given below are important because of the applications.

THEOREM 4.2. *Consider the boundary value problem* (4.3), *where J is a given real interval, $F : J \times \mathbb{R}^n \multimap \mathbb{R}^n$ is a u-Carathéodory map and S is a subset of* $\mathrm{AC}_{\mathrm{loc}}(J, \mathbb{R}^n)$.

Let $G : J \times \mathbb{R}^n \times \mathbb{R}^n \times [0,1] \multimap \mathbb{R}^n$ be as in Theorem 4.1. *Assume that the assumptions* (i)–(iii) *of Theorem* 4.1 *hold, with the convexity of the set Q, and*

(iv) *if $\partial Q \times [0,1] \supset \{(q_j, \lambda_j)\}$ converges to $(q, \lambda) \in \partial Q \times [0,1]$, $q \in T(q, \lambda)$, then there exists $j_0 \in \mathbb{N}$ such that, for every $j \geqslant j_0$, and $x_j \in T(q_j, \lambda_j)$, we have $x_j \in Q$.*

Then problem (4.3) *has a solution.*

The proof can be obtained immediately by using our continuation principle presented in Theorem 3.9 in Section 3.3.

REMARK 4.2. In the (single-valued) case of Carathéodory ODEs, we can again only assume in Corollary 4.4(i) and Theorem 4.2 that the linearized problems are uniquely solvable. If the associated problem (4.4) for G is so uniquely solvable, for every $(q, \lambda) \in Q \times [0, 1]$, then, by continuity of T, we can reformulate the above condi tion (iv) as follows:

(iv′) if $\{(x_j, \lambda_j)\}$ is a sequence in $S_1 \times [0, 1]$, with $\lambda_j \to \lambda \in [0, 1)$ and x_j is converging to a solution $x \in Q$ of (4.4) (corresponding to (x, λ)), then x_j belongs to Q, for j sufficiently large.

Now, we are interested in the existence of several solutions of problem (4.3). For this, the Nielsen theory developed in Section 3.2 will be applied. It will be convenient to use the following definition.

DEFINITION 4.1. We say that the mapping $T : Q \multimap U$ is *retractible onto* Q, where U is an open subset of $C(J, \mathbb{R}^n)$ containing Q, if there is a (continuous) retraction $r : U \to Q$ and $p \in U \setminus Q$ with $r(p) = q$ implies that $p \notin T(q)$.

Its advantage consists in the fact that, for a retractible mapping $T : Q \multimap U$ onto Q with a retraction r in the sense of Definition 4.1, its composition with r, $r|_{T(Q)} \circ T : Q \multimap Q$, has a fixed point $\widehat{q} \in Q$ if and only if $\widehat{q}$ is a fixed point of T.

The following principal statement characterizes the matter.

THEOREM 4.3. *Let $G : J \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ be u-Carathéodory map (cf. Definition* 2.10) *and assume that*

(i) *there exists a closed, connected subset Q of $C(J, \mathbb{R}^n)$ with a finitely generated abelian fundamental group such that, for any $q \in Q$, the set $T(q)$ of all solutions of the linearized problem* (4.2) *is R_δ,*

(ii) *$T(Q)$ is bounded in $C(J, \mathbb{R}^n)$ and $\overline{T(Q)} \subset S$,*

(iii) *there exists a locally integrable function $\alpha : J \to \mathbb{R}$ such that*

$$\big|G\big(t, x(t), q(t)\big)\big| := \sup\big\{|y| \mid y \in G\big(t, x(t), q(t)\big)\big\} \leqslant \alpha(t), \quad \text{a.e. in } J,$$

for any pair $(q, x) \in \Gamma_T$, where Γ_T denotes the graph of T.

Assume, furthermore, that

(iv) *the operator $T : Q \multimap U$, related to* (4.2), *is retractible onto Q with a retraction r in the sense of Definition* 4.1.

At last, let

$$G(t, c, c) \subset F(t, c) \tag{4.12}$$

for a.a. $t \in J$ and any $c \in \mathbb{R}^n$. Then the original problem (4.3) *admits at least $N(r|_{T(Q)} \circ T)$ solutions belonging to Q, where N stands for the Nielsen number defined in Definition* 3.2 *in Section* 3.2.

PROOF. By the hypothesis, Q is a connected (metric) ANR-space with a finitely generated abelian fundamental group and $T(q)$ is an R_δ-mapping. Since T is also, according to

Proposition 4.2, u.s.c. and such that $\overline{T(Q)}$ is compact, $r \circ T$ is compact and admissible. This follows from the commutativity of the following diagram:

$$\begin{array}{ccccc} Q & \xrightarrow{\;T\;} & U & \xrightarrow{\;r\;} & Q \\ & {\scriptstyle p_T}\nwarrow & \uparrow {\scriptstyle q_T} & \nearrow {\scriptstyle r\circ q_T} & \\ & & \Gamma_T & & \end{array}$$

where (p_T, q_T) is a pair of natural projections of the graph Γ_T and p_T is Vietoris. Therefore, according to Theorem 3.4 in Section 3.2, $(p_T, r|_{T(Q)} \circ q_T)$ admits at least $N(r|_{T(Q)} \circ T(\cdot))$ coincidence points. Because of Definition 4.1, they represent the solutions of problem (4.2) and, in view of (4.12), they also satisfy the original problem (4.3). □

REMARK 4.3. In the (single-valued) case of Carathéodory ODEs, we can only assume in Theorem 4.3(i) that the linearized problem (4.2) is uniquely solvable. Moreover, the requirement that the fundamental group $\pi(Q)$ of Q to be finitely generated and abelian can be then omitted (see Section 3.2).

Furthermore, we will consider boundary value problems on arbitrary (possibly infinite) intervals for differential inclusions in Banach spaces. We start with some definitions.

Let E be a Banach space with the norm $\|\cdot\|$. Denote by $C(J, E)$ the space of all continuous functions $x : J \to E$ with the locally convex topology generated by the uniform convergence on compact subintervals of J (possibly, the whole $\mathbb{R}$). This topology is completely metrizable, and thus $C(J, E)$ is a Fréchet space.

Recall that a mapping $x : J \to E$ is locally absolutely continuous if x is absolutely continuous on every compact subinterval of J. Unfortunately, in general, on each interval $[a, b] \subset J$, there need not exist $\dot{x}(t)$ (in the sense of Fréchet), for almost all (a.a.) $t \in [a, b]$ with $\dot{x} \in L^1([a, b], E)$ (the set of all Bochner integrable functions $[a, b] \to E$) and so need not be

$$x(t) = x_0 + \int_a^t \dot{x}(s)\,\mathrm{d}s.$$

It is so if E satisfies the Radon–Nikodym property, in particular, if E is reflexive. Moreover, we have the following result (cf. [48]).

LEMMA 4.2. *Suppose $x : [a, b] \to E$ is absolutely continuous, $\dot{x}$ exists a.e., and*

$$\|\dot{x}(t)\| \leqslant y(t), \quad \textit{a.e., for some } y \in L^1([a, b], \mathbb{R}).$$

Then $\dot{x} \in L^1([a, b], E)$ and

$$\int_\tau^t \dot{x}(s)\,\mathrm{d}s = x(t) - x(\tau) \quad \big(t, \tau \in [a, b]\big). \tag{4.13}$$

The set of all locally absolutely continuous functions from J to E, satisfying all the above properties, will be denoted by $\mathrm{AC}_{\mathrm{loc}}(J, E)$.

Consider now the differential inclusion

$$\dot{x} \in F(t, x), \tag{4.14}$$

where $F : J \times E \multimap E$ is a u-Carathéodory map, i.e.

(C1) $F(t, x)$ is nonempty, compact and convex, for every $(t, x) \in J \times E$,

(C2) $F(t, \cdot)$ is u.s.c., for a.a. $t \in J$,

(C3) $F(\cdot, x)$ is strongly measurable (cf. Definition 2.9), on every compact interval $[a, b]$, for each $x \in E$.

By a *solution* of this differential inclusion we mean again a map $x \in \mathrm{AC}_{\mathrm{loc}}(J, E)$ satisfying (4.14), for a.a. $t \in J$.

To a u-Carathéodory map F, we associate the Nemytskiĭ (or superposition) operator $N_F : C(J, E) \multimap L^1_{\mathrm{loc}}(J, E)$ given by

$$N_F(x) := \left\{ f \in L^1_{\mathrm{loc}}(J, E) \mid f(t) \in F\big(t, x(t)\big), \text{ a.e. on } J \right\},$$

for each $x \in C(J, E)$.

In the sequel, we will need the following lemma (see [100, p. 88] and cf. Remark 2.1).

LEMMA 4.3. *Let $[a, b]$ be a compact interval. Let $F : [a, b] \times E \multimap E$ be a u-Carathéodory mapping and assume in addition that, for every nonempty bounded set $\Omega \subset E$, there exists $\nu = \nu(\Omega) \in L^1([a, b])$ such that*

$$\big\| F(t, x) \big\| := \sup\big\{ \|z\| \mid z \in F(t, x) \big\} \leqslant \nu(t),$$

for a.e. $t \in [a, b]$ and every $x \in \Omega$. Then the Nemytskiĭ operator

$$N_F : C\big([a, b], E\big) \multimap L^1\big([a, b], E\big)$$

has nonempty, convex values. Moreover, given sequences $\{x_n\} \subset C([a, b], E)$ and $\{f_n\} \subset L^1([a, b], E)$, $f_n \in N_F(x_n)$, $n \geqslant 1$, such that $x_n \to x$ in $C([a, b], E)$ and $f_n \to f$ weakly in $L^1([a, b], E)$, then $f \in N_F(x)$.

The following lemma extends Proposition 4.1 to infinite-dimensional spaces (see again [42, Theorem 0.3.4]).

LEMMA 4.4. *Assume that a sequence $\{x_k \mid [a, b] \to E\}$ of AC-maps satisfies the following conditions:*

(i) *$\{x_k(t)\}$ is relatively compact, for each $t \in [a, b]$,*

(ii) *there exists $\alpha \in L^1([a, b])$ such that $\|\dot{x}_k(t)\| \leqslant \alpha(t)$, for a.a. $t \in [a, b]$.*

Then there exists a subsequence (again denoted by $\{x_k\}$) that converges to an absolutely continuous map $x : [a, b] \to E$ in the following sense:

(iii) *$x_k \to x$ in $C([a, b], E)$,*

(iv) $\dot{x}_k \to \dot{x}$ *weakly in* $L^1([a,b], E)$.

PROPOSITION 4.3. *Let* $G : J \times E \times E \multimap E$ *be a u-Carathéodory map and let* S *be a nonempty subset of* $\mathrm{AC}_{\mathrm{loc}}(J, E)$. *Assume that*:

(i) *there exists a closed* $Q \subset C(J, E)$ *such that, for any* $q \in Q$, *the boundary value problem*

$$\begin{cases} \dot{x}(t) \in G\big(t, x(t), q(t)\big), & \text{for a.a. } t \in J, \\ x \in S, \end{cases}$$

has a solution. Denote by $T : Q \multimap S$ *the solution mapping.*

(ii) *There exist* $\alpha, \beta, \gamma \in L^1_{\mathrm{loc}}(J)$ *such that*

$$\big\|G(t, x, y)\big\| \leqslant \alpha(t) + \beta(t)\|x\| + \gamma(t)\|y\|,$$

for a.a. $t \in J$ *and every* $(x, y) \in E^2$.

(iii) *If* $\{(q_n, x_n)\}$ *is a sequence in the graph of* T *and* $(q_n, x_n) \to (q, x)$, *then* $x \in S$.

Then $T : Q \multimap S$ *has a closed graph* (S *is endowed with the topology of* $C(J, E)$).

PROOF. Let $\{(q_n, x_n)\}$ be an arbitrary sequence in the graph of T, i.e. $x_n \in T(q_n)$, for every $n \in \mathbb{N}$, and assume that $(q_n, x_n) \to (q_0, x_0)$. Thus, we see that

$$\dot{x}_n(t) \in G\big(t, x_n(t), q_n(t)\big), \quad \text{for a.a. } t \in J,$$

and $x_n \in S$. Then $q_0 \in Q$ and, by assumption (iii), $x_0 \in S$.

Now, let $[a, b]$ be an interval in J. Using assumption (ii), we see that the sequence $\{x_n\}$ satisfies the assumptions of Lemma 4.4. Thus, $\{x_n\}$ converges uniformly on $[a, b]$ to x_0 (because this limit is unique) and $\{\dot{x}_n\}$ converges to $\dot{x}_0$, weakly in $L^1([a, b], E)$. Using Lemma 4.3, it follows that $\dot{x}_0(t) \in G(t, x_0(t), q_0(t))$, for a.a. $t \in [a, b]$. Since $[a, b]$ was arbitrary, we see that indeed $\dot{x}_0(t) \in G(t, x_0(t), q_0(t))$, for a.a. $t \in J$ and $x_0 \in T(q_0)$. □

As another of the main results of this subsection, we can formulate the following continuation principle.

THEOREM 4.4. *Consider the boundary value problem* (*e.g., in a reflexive Banach space* E; *cf. Lemma* 4.2):

$$\begin{cases} \dot{x}(t) \in F\big(t, x(t)\big), & \text{for a.a. } t \in J, \\ x \in S, \end{cases} \tag{4.15}$$

where $F : J \times E \multimap E$ *is a u-Carathéodory map and* S *is a subset of* $\mathrm{AC}_{\mathrm{loc}}(J, E)$. *Let* $G : J \times E \times E \times [0, 1] \multimap E$ *be a u-Carathéodory map* (*cf. Definition* 2.10) *such that*

$$G(t, c, c, 1) \subset F(t, c), \quad \text{for all } (t, c) \in J \times E. \tag{4.16}$$

Assume that:

(i) *there exists a closed, convex* $Q \subset C(J, E)$ *and a closed subset* S_1 *of* S *such that the problem*

$$\begin{cases} \dot{x}(t) \in G\big(t, x(t), q(t), \lambda\big), & \text{for a.a. } t \in J, \\ x \in S_1 \end{cases}$$

is solvable with an R_δ*-set* $T(q, \lambda)$, *for each* $(q, \lambda) \subset Q \times [0, 1]$.

(ii) *There exist* $\alpha, \beta, \gamma \in L^1_{\text{loc}}(I)$ *such that*

$$\big\| G(t, x, y, \lambda) \big\| \leqslant \alpha(t) + \beta(t)\|x\| + \gamma(t)\|y\|,$$

for a.a. $t \in J$, *every* $(x, y) \in E^2$ *and every* $\lambda \in [0, 1]$.

(iii) T *is quasi-compact, i.e.* T *maps compact subsets onto compact subsets, and there exists a measure of noncompactness* μ *in the sense of Definitions* 2.7 *and* 2.8 *in Section* 2.2 *such that, for each* $\Omega \subset Q$, *if*

$$\mu\big(T\big(\Omega \times [0, 1]\big)\big) \geqslant \mu(\Omega),$$

then Ω *is relatively compact.*

(iv) $T(Q \times \{0\}) \subset Q$.

(v) *For each* $\lambda_0 \in [0, 1]$ *and* $q \in T(q_0, \lambda_0)$, *if* $q_n \to q_0$ *in* Q, *then there is* $n_0 \in \mathbb{N}$ *such that, for each* $n \geqslant n_0$, $\lambda \in [0, 1]$ *and* $x \in T(q_n, \lambda)$, *we have* $x \in Q$.

Then problem (4.15) *has a solution.*

PROOF. Using Proposition 4.3, we see that the map $T : Q \times [0, 1] \multimap S_1$ has a closed graph. Since T is also quasi-compact (assumption (iii)), we can easily derive that T is indeed an u.s.c. set-valued map (see, e.g., [78, Theorem 1.1.12]). From assumption (i), we get therefore that $T \in J(Q \times [0, 1], C(\mathbb{R}, E))$ and assumption (iii) implies that T is also μ-condensing. By (v), we finally see that T is a homotopy in J_A, and thus Corollary 3.4 in Section 3.3 implies the existence of a fixed point of $T(\cdot, 1)$. However, by the inclusion (4.16), it is a solution of (4.15). □

REMARK 4.4. As we can see, Theorem 4.4 extends Theorem 4.1 into the infinite-dimensional setting, when replacing $\mathbb{R}^n$ by a real Banach space. On the other hand, this is possible with some loss, namely Q is only convex and the solution operator T is assumed to be quasi-compact, additionally. Because of those restrictions, we are unfortunately unable to establish a full infinite dimensional analogy of Theorem 4.1.

If, in particular, $J = [a, b]$ (i.e. compact), then Theorem 4.4 can be simplified, in view of Remark 3.9, similarly as Corollary 4.3 w.r.t. Theorem 4.1, as follows.

COROLLARY 4.5. *Consider the problem*

$$\begin{cases} \dot{x}(t) \in F\big(t, x(t)\big), & \text{for a.a. } t \in [a, b], \\ x \in S, \end{cases} \tag{4.17}$$

where $F:[a,b]\times E \multimap E$ *is a u-Carathéodory map and* S *is a subset of absolutely continuous functions* $x:[a,b]\to E$, *all in a reflexive Banach space* E. *Let* $G:[a,b]\times E\times E\times[0,1]\multimap E$ *be a u-Carathéodory map such that*

$$G(t,c,c,1)\subset F(t,c),\quad \text{for all } (t,c)\in[a,b]\times E. \tag{4.18}$$

Furthermore, assume that

(i) *there exists a convex, bounded subset* $Q\subset C([a,b],E)$ *such that* $Q\setminus\partial Q$ *is nonempty and a closed subset* S_1 *of* S *such that the problem*

$$\begin{cases}\dot{x}(t)\in G\big(t,x(t),q(t),\lambda\big), & \text{for a.a. } t\in[a,b],\\ x\in S_1\end{cases}$$

is solvable with R_δ*-set* $T(q,\lambda)$, *for each* $(q,\lambda)\in Q\times[0,1]$.

(ii) *There exists* $\alpha,\beta,\gamma\in Ł^1([a,b])$ *such that*

$$\big\|G(t,x,y,\lambda)\big\|\leqslant\alpha(t)+\beta(t)\|x\|+\gamma(t)\|y\|,$$

for a.a. $t\in[a,b]$, *every* $(x,y)\in E^2$ *and every* $\lambda\in[0,1]$.

(iii) T *is quasi-compact, i.e.* T *maps compact subsets onto compact subsets, and there exists a measure of noncompactness* μ (*see Definitions* 2.7 *and* 2.8) *such that, for each* $\Omega\subset Q$, *if*

$$\mu\big(T\big(\Omega\times[0,1]\big)\big)\geqslant\mu(\Omega),$$

then Ω *is relatively compact.*

(iv) $T(Q\times\{0\})\subset Q$.

(v) *The map* T *has no fixed points on the boundary* ∂Q *of* Q, *for every* $(q,\lambda)\in Q\times[0,1]$.

Then problem (4.17) *has a solution.*

PROOF. We can proceed quite analogously as in the proof of the foregoing Theorem 4.4. The only difference consists of modifying Corollary 3.4 in the sense of Remark 3.9, both in Section 3.3. □

REMARK 4.5. In the (single-valued) case of Carathéodory ODEs, we can again only assume in Theorem 4.4(i) and Corollary 4.5(i) that the linearized problems are uniquely solvable.

Sometimes it is convenient to consider the asymptotic problems sequentially. For this purpose, it can be useful to employ

PROPOSITION 4.4. *Let* $J_1\subset J_2\subset\cdots$ *be compact intervals such that* $J=\bigcup_{m=1}^{\infty}J_m$ *and* $t_0\in J_1$. *Let* $F:J\times E\multimap E$ *be a u-Carathéodory mapping with nonempty, compact and convex values. Assume, furthermore, that*

(i) *There are* $\alpha, \beta \in L^1_{\mathrm{loc}}(J, \mathbb{R})$ *with*

$$\big\|F(t,x)\big\| \leqslant \alpha(t) + \beta(t)\|x\|.$$

(ii) *There is some Carathéodory mapping* $g : J \times [0,\infty) \to [0,\infty)$ (*called a Kamke function*) *such that the only nonnegative measurable solution of*

$$x(t) \leqslant \left| \int_{t_0}^{t} g\big(s, x(s)\big)\, \mathrm{d}s \right|$$

is 0 (*a.e.*), *and such that, for a.a.* $t \in J$, $\gamma(F(\{t\} \times C)) \leqslant g(t, \gamma(C))$, *for countable, bounded subsets* $C \subset E$, *where* γ *denotes the Hausdorff measure of noncompactness.*

(iii) E *has the so called retraction property in the sense of [99], e.g.,* E *is separable or reflexive.*

If $x_m \in \mathrm{AC}(J_m, E)$ *satisfies*

$$\dot{x}_m(t) \in F\big(t, x_m(t)\big), \quad \textit{for a.a. } t \in J_m,\ m \in \mathbb{N},$$

and $\{x_m(t_0) \mid m \in \mathbb{N}\}$ *is a relatively compact set, then there is a solution* $x \in \mathrm{AC}_{\mathrm{loc}}(J, E)$ *of the inclusion* $\dot{x}(t) \in F(t, x(t))$, *for a.a.* $t \in J$, *such that, for some subsequence,*

$$x_{m_k} \to x, \quad \textit{uniformly on each } J_m,$$

and

$$\dot{x}_{m_k} \to \dot{x}, \quad \textit{weakly in } L^1(J, E).$$

If still

(iv) $\sup\{\|x_m(t)\| \mid m \in \mathbb{N},\ t \in J_m\} < \infty$

and the values of x_m, $m \in \mathbb{N}$, *are located in a closed subdomain* $\mathcal{D}$ *of* E, *then there exists an entirely bounded solution* x *on* J *with* $x(t) \in \mathcal{D}$, *for all* $t \in \mathbb{R}$.

PROOF. By (i) and the well-known Gronwall inequality (see, e.g., [72]), we get the a priori estimates

$$\big\|x_m(t)\big\| \leqslant \widetilde{x}(t) \quad \text{and} \quad \big\|\dot{x}_m(t)\big\| \leqslant \widetilde{x}(t),$$

for some $\widetilde{x} \in L^1_{\mathrm{loc}}(J, \mathbb{R})$.

We claim that $\{x_m(t) \mid m \geqslant m_t\}$ is a relatively compact set, for a.a. $t \in J$, where $m_t = \min\{m \mid t \in J_m\}$. To show it, put $h(t) := \gamma(\{x_m(t) \mid m \geqslant m_t\})$. Then h is measurable (for more details, see Proposition 11.12 in [99]). Moreover, by means of Proposition 11.12 in [99] and (ii), we obtain

$$h(t) = \gamma\big(\big\{x_m(t) - x_m(t_0) \mid m \geqslant m_t\big\}\big) = \gamma\left(\left\{\int_{t_0}^{t} \dot{x}_m(s)\, \mathrm{d}s \;\middle|\; m \geqslant m_t\right\}\right)$$

$$\leqslant \left| \int_{t_0}^{t} \gamma\big(\{\dot{x}_m(s) \mid m \geqslant m_t\}\big)\,\mathrm{d}s \right| \leqslant \left| \int_{t_0}^{t} g\big(s, h(s)\big)\,\mathrm{d}s \right|.$$

Applying (ii) again, we arrive at $h(t) = 0$, for a.a. $t \in J$, as claimed.

Since $F(t, \cdot)$ maps compact sets into compact sets, $\{\dot{x}_m(t) \mid m \geqslant m_t\}$ becomes relatively compact as well, for a.a. $t \in J$. An application of the standard diagonalization argument implies, jointly with Lemma 4.4, the existence of a subsequence such that $x_{m_k} \to x$, uniformly on each J_m, and $\dot{x}_{m_k} \to \dot{x}$, weakly in $L^1(J, E)$, where $x \in \mathrm{AC}_{\mathrm{loc}}(J, E)$.

It follows from Lemma 4.3 that $\dot{x}(t) \in F(t, x(t))$. Since the remaining part of the assertion is implied by the foregoing one (just proved) and (iv), the proof is completed. □

REMARK 4.6. Let E be a Banach space and assume that $F : \mathbb{R} \times E \multimap E$ is a u-Carathéodory mapping (cf. Definition 2.10) such that

$$\mu\big(F(t, B)\big) \leqslant k(t)\mu(B), \quad \text{for bounded subsets } B \subset E,\ t \in \mathbb{R},$$

where $k \in L^1_{\mathrm{loc}}(\mathbb{R})$ and μ denotes either the Kuratowski MNC α or the Hausdorff MNC γ. Then it is well known (see, e.g., [58, Theorem 9.2 and Remark 9.5.4 in Chapter 4.9.3]) that the initial value problem

$$\begin{cases} \dot{x}(t) \in F\big(t, x(t)\big), & \text{for a.a. } t \in [-m, m],\ m \in \mathbb{N}, \\ x(0) = x_0, \end{cases}$$

admits a solution $x_m \in \mathrm{AC}([-m, m])$, for each $m \in \mathbb{N}$, i.e. $x \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R}, E)$. If, in particular, the values of x_m, $m \in \mathbb{N}$, are located in a given bounded, closed subdomain $\mathcal{D}$ of E, then there exists an entirely bounded solution $x \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R}, E)$ on $\mathbb{R}$, $x(0) = x_0$, with values in $\mathcal{D}$, provided E is separable or reflexive. It is namely enough to apply Proposition 4.4, for $t_0 = 0$ and $g(t, x) := 2k(t)x$. Such special g is a Kamke function by means of the Gronwall inequality.

If in particular, $E = \mathbb{R}^n$, then (ii) and (iii) hold automatically. Hence, Proposition 4.4 can be then simplified as follows.

PROPOSITION 4.5. *Let $F : \mathbb{R} \times \mathbb{R}^n \multimap \mathbb{R}^n$ be a u-Carathéodory mapping with nonempty, compact and convex values, satisfying* (i) *in Proposition* 4.4, *for $J = (-\infty, \infty)$. Then, for every $x_0 \in \mathbb{R}^n$, there exists a solution $x \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^n)$ of the Cauchy problem*

$$\begin{cases} \dot{x}(t) \in F\big(t, x(t)\big), & \text{for a.a. } t \in (-\infty, \infty), \\ x(0) = x_0. \end{cases}$$

Let $\{x_m(t)\}$ be a sequence of absolutely continuous functions such that

(i) *For every $m \in \mathbb{N}$, $x_m \in \mathrm{AC}([-m, m], \mathbb{R}^n)$ is a solution of*

$$\dot{x}(t) \in F\big(t, x(t)\big), \quad \text{for a.a. } t \in [-m, m],$$

(ii) $\sup\{|x_m(t)| \mid m \in \mathbb{N},\ t \in [-m, m]\} := M < \infty$ *and* $x_m(t) \in \mathcal{D} \subset \mathbb{R}^n$, *for every* $t \in [-m, m]$.

Then there exists an entirely bounded solution $x \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^n)$ *of the inclusion*

$$\dot{x}(t) \in F\big(t, x(t)\big), \quad \textit{for a.a. } t \in (-\infty, \infty),$$

such that

$$\sup_{t\in\mathbb{R}} \big|x(t)\big| \leqslant M\ (< \infty) \quad \textit{and} \quad x(t) \in \mathcal{D}, \quad \textit{for all } t \in \mathbb{R}.$$

4.2. *Topological structure of solution sets*

In this part, various methods for investigating the topological structure of solution sets, required in statements of the foregoing subsection, will be presented. Both initial and boundary value problems will be considered.

The classical result, due to F.S. De Blasi and J. Myjak in [57], deals with Cauchy problems for the u-Carathéodory differential inclusions in Euclidean spaces:

$$\begin{cases} \dot{x} \in F(t, x), \\ x(0) = x_0, \end{cases} \tag{4.19}$$

where $F : J \times \mathbb{R}^n \multimap \mathbb{R}^n$ is a u-Carathéodory mapping, i.e. a multivalued mapping, satisfying conditions from the beginning of Section 4.1 (cf. Definition 2.10), and such that

$$\big|F(t, x)\big| \leqslant \alpha + \beta|x|, \quad \text{for all } t \in J,\ x \in \mathbb{R}^n,$$

where α, β are nonnegative constants.

THEOREM 4.5. *Problem* (4.19), *where* J *is a compact interval*, *has under the above assumptions an* R_δ*-set of solutions.*

We omit the proof of this theorem, because below we will prove its generalized version (cf. Theorem 4.9).

We recall that a multivalued mapping $F : J \times \mathbb{R} \multimap \mathbb{R}^n$ is said to be *integrably bounded* (resp. *locally integrably bounded*) if there exists an integrable (resp. locally integrable) function $\mu : J \to [0, \infty)$ such that $|y| \leqslant \mu(t)$, for every $x \in \mathbb{R}^n$, $t \in J$ and $y \in F(t, x)$. We say that F has *at most a linear growth* (resp. *a local linear growth*) if there exist integrable (resp. locally integrable) functions $\mu, \nu : J \to [0, \infty)$ such that

$$|y| \leqslant \mu(t)|x| + \nu(t),$$

for every $x \in \mathbb{R}^n$, $t \in J$ and $y \in F(t, x)$.

It is obvious that F has at most a linear growth if there exists an integrable function $\mu : J \to [0, \infty)$ such that $|y| \leqslant \mu(t)(|x| + 1)$, for every $x \in \mathbb{R}^n$, $t \in J$ and $y \in F(t, x)$.

Let us also recall that a single-valued map $f : J \times \mathbb{R}^n \to \mathbb{R}^n$ is said to be *measurable-locally Lipschitz* (mLL) if, for every $x \in \mathbb{R}^n$, there exists a neighbourhood V_x of x in $\mathbb{R}^n$ and an integrable function $L_x : J \to [0, \infty)$ such that

$$\bigl|f(t, x_1) - f(t, x_2)\bigr| \leqslant L_x(t)|x_1 - x_2|, \quad \text{for every } t \in J \text{ and } x_1, x_2 \in V_x,$$

where $f(\cdot, x)$ is measurable, for every $x \in \mathbb{R}^n$.

Now, for the considerations below, fix J as the halfline $[0, \infty)$ and assume that $F : J \times \mathbb{R}^n \multimap \mathbb{R}^n$ is again a multivalued u-Carathéodory map. Consider the Cauchy problem (4.19). By $S(F, 0, x_0)$, we denote the set of solutions of (4.19). For the characterization of the topological structure of $S(F, 0, x_0)$, it will be useful to recall the following well-known uniqueness criterion (see, e.g., [60, Theorem 1.1.2]).

THEOREM 4.6. *If f is a single-valued, integrably bounded, measurable-locally Lipschitz map, then the set $S(f, 0, x_0)$ is a singleton, for every $x_0 \in \mathbb{R}^n$.*

The following result will be employed as well.

THEOREM 4.7. *If F is locally integrably bounded, mLL-selectionable (i.e. if there exists a measurable-locally Lipschitz single-valued selection), then $S(F, 0, x_0)$ is contractible, for every $x_0 \in \mathbb{R}^n$.*

PROOF. Let $f \subset F$ be measurable-locally Lipschitz. By Theorem 4.6, the following Cauchy problem

$$\begin{cases} \dot{x} = f(t, x), \\ x(t_0) = x_0, \end{cases} \tag{4.20}$$

has exactly one solution, for every $t_0 \in J$ and $x_0 \in \mathbb{R}^n$. For the proof, it is sufficient to define a homotopy $h : S(F, 0, x_0) \times [0, 1] \to S(F, 0, x_0)$ such that

$$h(x, s) = \begin{cases} x, & \text{for } s = 1 \text{ and } x \in S(F, 0, x_0), \\ \widetilde{x}, & \text{for } s = 0, \end{cases}$$

where $\widetilde{x} = S(f, 0, x_0)$ is exactly one solution of the problem (4.20).

Define $\gamma : [0, 1) \to [0, \infty)$, $\gamma(s) = \tan(\pi s/2)$ and put

$$h(x, s)(t) = \begin{cases} x(t), & \text{for } 0 \leqslant t \leqslant \gamma(s),\ s < 1, \\ S\bigl(f, \gamma(s), x\bigl(\gamma(s)\bigr)\bigr)(t), & \text{for } \gamma(s) \leqslant t < \infty,\ s < 1, \\ x(t), & \text{for } 0 \leqslant t < \infty,\ s = 1. \end{cases}$$

Then h is a continuous homotopy, contracting $S(F, 0, x_0)$ to the point $S(f, 0, x_0)$. □

Analogously, we can get the following result.

THEOREM 4.8. *If F is locally integrably bounded, Ca-selectionable (i.e. if there exists a Carathéodory single-valued selection), or in particular c-selectionable (i.e. if there exists a continuous single-valued selection), then $S(F, 0, x_0)$ is R_δ-contractible, for every $x_0 \in \mathbb{R}^n$.*

Observe that, if $F : J \times \mathbb{R}^n \multimap \mathbb{R}^n$ is an intersection of the decreasing sequence of $F_k : J \times \mathbb{R}^n \multimap \mathbb{R}^n$, $F(t, x) = \bigcap_{k=1}^{\infty} F_k(t, x)$ and $F_{k+1}(t, x) \subset F_k(t, x)$, for almost all $t \in J$ and for all $x \in \mathbb{R}^n$, then

$$S(F, 0, x_0) = \bigcap_{k=1}^{\infty} S(F_k, 0, x_0). \tag{4.21}$$

From Theorems 4.7 and 4.8, we obtain

PROPOSITION 4.6. *Let $F : J \times \mathbb{R}^n \multimap \mathbb{R}^n$ be a multivalued map with nonempty, closed values.*

(i) *If F is σ-mLL-selectionable (i.e. it is an intersection of a decreasing sequence of mLL-selectionable mappings), then the set $S(F, 0, x_0)$ is an intersection of a decreasing sequence of contractible sets,*
(ii) *if F is σ-Ca-selectionable, i.e., it is an intersection of a decreasing sequence of Ca-selectionable mappings, then the set $S(F, 0, x_0)$ is an intersection of a decreasing sequence of R_δ-contractible sets.*

Before formulating the following important theorem, recall that, for two metric spaces X, Y and the interval J, the multivalued map $F : J \times X \multimap Y$ is *almost upper semi-continuous* (a.u.s.c.), if for every $\varepsilon > 0$ there exists a measurable set $A_\varepsilon \subset J$ such that $m(J \setminus A_\varepsilon) < \varepsilon$ and the restriction $F|_{A_\varepsilon \times X}$ is u.s.c., where m stands for the Lebesgue measure.

It is clear that every a.u.s.c. map is u-Carathéodory. In general, the reverse is not true. The following Scorza–Dragoni type result describing possible regularizations of Carathéodory maps (see, e.g., [76]) will be employed.

PROPOSITION 4.7. *Let X be a separable metric space and J be an interval. Suppose that $F : J \times X \multimap \mathbb{R}^n$ is a nonempty, compact, convex valued u-Carathéodory map. Then there exists an a.u.s.c. map $\psi : J \times X \multimap \mathbb{R}^n$ with nonempty compact convex values and such that:*

(i) *$\psi(t, x) \subset F(t, x)$, for every $(t, x) \in J \times X$,*
(ii) *if $\Delta \subset J$ is measurable, $u : \Delta \to \mathbb{R}^n$ and $v : \Delta \to X$ are measurable maps and $u(t) \in F(t, v(t))$, for almost all $t \in \Delta$, then $u(t) \in \psi(t, v(t))$, for almost all $t \in \Delta$.*

The proof of the following statement can be found in [67].

PROPOSITION 4.8. *Let E, E_1 be two separable Banach spaces, J be an interval and $F : J \times E \multimap E$ be an a.u.s.c. map with compact convex values. Then F is σ-Ca-selectionable (i.e. it is an intersection of a decreasing sequence of Ca-selectionable mappings $F_k : J \times E \multimap E_1$). The maps $F_k : J \times E \multimap E_1$ are a.u.s.c., and we have $F_k(t, e) \subset$*

$\overline{\text{conv}}(\bigcup_{x\in E} F(t,x))$, *for all* $(t,e) \in J \times E$. *Moreover, if* F *is integrably bounded, then* F *is* σ*-mLL-selectionable, i.e., it is an intersection of a decreasing sequence of mLL-selectionable mappings.*

Now, we are ready to give

THEOREM 4.9. *If* $F: J \times \mathbb{R}^n \multimap \mathbb{R}^n$ *is a u-Carathéodory map with compact convex values having at most the linear growth, then* $S(F,0,x_0)$ *is an* R_δ*-set, for every* $x_0 \in \mathbb{R}^n$.

PROOF. By the hypothesis, there exists an integrable function $\mu: J \to [0,\infty)$ such that $\sup\{|y| \mid y \in F(t,x)\} \leqslant \mu(t)(|x|+1)$, for every $(t,x) \in J \times \mathbb{R}^n$. By means of the well-known Gronwall inequality (see [71]), we obtain that $|x(t)| \leqslant (|x_0|+\gamma)\exp(\gamma) = M$, where $x \in S(F,0,x_0)$ and $\gamma = \int_0^\infty \mu(s)\,\mathrm{d}s$.

Take $r > M$ and define $F: J \times \mathbb{R}^n \multimap \mathbb{R}^n$ as follows:

$$\widetilde{F}(t,x) = \begin{cases} F(t,x), & \text{if } |x| \leqslant r, \\ F\left(t, r\dfrac{x}{|x|}\right), & \text{if } |x| > r. \end{cases}$$

One can see that $\widetilde{F}$ is an integrably bounded u-Carathéodory map and

$$S\big(\widetilde{F},0,x_0\big) = S(F,0,x_0).$$

By Proposition 4.7, there exists an a.u.s.c. map $G: J \times \mathbb{R}^n \multimap \mathbb{R}^n$ with nonempty, convex, compact values such that $S(G,0,x_0) = S(\widetilde{F},0,x_0)$. Applying Proposition 4.8 to the map G, we obtain the sequence of maps G_k. As in Proposition 4.6, we see that $S(G,0,x_0)$ is an intersection of the decreasing sequence $S(G_k,0,x_0)$ of contractible sets. By the well-known Arzelà–Ascoli lemma and Theorem 4.6, we obtain that, for every $k \in \mathbb{N}$, the set $S(G_k,0,x_0)$ is compact and nonempty, which completes the proof. □

Using the above results and the unified approach to the u.s.c. and l.s.c. case due to A. Bressan (cf. [49,50]), we can obtain the following result.

PROPOSITION 4.9. *Let* $G: J \times \mathbb{R}^n \multimap \mathbb{R}^n$ *be a l.s.c. bounded map with nonempty closed values. Then there exists a u.s.c. map* $F: J \times \mathbb{R}^n \multimap \mathbb{R}^n$ *with compact convex values such that, for any* $x_0 \in \mathbb{R}^n$, *the set* $S(G,0,x_0)$ *contains an* R_δ*-set* $S(F,0,x_0)$ *as a subset.*

REMARK 4.7. In [22], topological structure of solution sets is also treated, provided F is not necessarily convex-valued. However, the absence of convexity seems to be a big handicap, because to prove the connectedness and compactness of the related solution set can be a difficult task (see, e.g., [22, Example 2.18 on p. 258]).

REMARK 4.8. It follows from the result in [98] (cf. also [55] or [78, Corollary 5.3.1], where mild solutions were considered for semilinear differential inclusions) that, in a real separable Banach space E, the solution sets to initial value problems are R_δ, provided conditions in Remark 4.6 hold. In general Banach spaces E, it is at least so when $F(t,.) : E \multimap E$ is completely continuous (cf. [58, Corollary 9.1(b) on pp. 118–119]).

We can also say something about the covering (topological) dimension of solution sets to the Cauchy problem (4.19).

Let Ω be an open set in $\mathbb{R}^{n+1}$ such that $[t_0, t_0 + h] \times B(x_0, r) \subset \Omega$, where B denotes the closed ball centered at x_0 and with the radius r. Assume that $F : \Omega \multimap \mathbb{R}^n$ satisfies the following conditions:

(C1) the set of values of F is nonempty, compact and convex, for all $(t, x) \in \Omega$,

(C2) $F(t, \cdot) : B(x_0, r) \multimap \mathbb{R}^n$ is continuous, for a.a. $t \in [t_0, t_0 + h]$,

(C3) $F(\cdot, x) : [t_0, t_0 + h] \to \mathbb{R}^n$ is measurable, for all $x \in B(x_0, r)$,

(C4) there exist Lebesgue-integrable nonnegative functions $\alpha, \beta : [t_0, t_0 + h] \to [0, \infty)$ such that, for any $x \in B(x_0, r)$, $|F(t, x)| \leqslant \alpha(t) + \beta(t)|x|$, for a.a. $t \in [t_0, t_0 + h]$, where $|F(t, x)| \leqslant \sup\{|y| \mid y \in F(t, x)\}$.

Denote by $S([t_0, t_0 + d], x_0)$ the set of solutions $x \in \mathrm{AC}([t_0, t_0 + d], \mathbb{R}^n)$ of (4.19) on the interval $[t_0, t_0 + d]$, $0 < d \leqslant h$.

The following two theorems are due to B.D. Gel'man [66] (cf. [22, Theorems 2.60 and 2.61 in Chapter III.2]).

THEOREM 4.10. *Let the assumptions* (C1)–(C4) *be satisfied. Assume that the set*

$$A = \big\{t \in [t_0, t_0 + h] \mid \dim\big(F(t, x)\big) \geqslant 1, \textit{ for any } x \in B(x_0, r)\big\}$$

is measurable and

$$\lim_{h \to 0} \frac{\mu(A \cap [t_0, t_0 + h])}{h} > 0,$$

where $\dim(\cdot)$ *denotes the covering dimension and* $\mu(\cdot)$ *stands for the Lebesgue measure. Then there exists a number* d_0 *such that, for any* $0 < d \leqslant d_0$, *we have* $S = S([t_0, t_0 + d], x_0) \neq \{\emptyset\}$ *and* $\dim(S) = \infty$.

THEOREM 4.11. *Let the assumptions of Theorem* 4.10 *be satisfied jointly with*

(C2′) $F(t, \cdot) : B(x_0, r) \multimap \mathbb{R}^n$ *is Lipschitz-continuous.*

Then there exists a number d_0 *such that, for any* $0 < d \leqslant d_0$, *any* $\varepsilon > 0$ *and any solution* $x \in S([t_0, t_0 + d], x_0)$ $(\neq \{\emptyset\})$, *we have* $\dim(S_{x,\varepsilon}) = \infty$, *where* $S_{x,\varepsilon} = \{y \in S \mid \|x - y\| \leqslant \varepsilon\}$.

Now, we shall study the reverse Cauchy problem when, instead of the origin, the value of solutions is prescribed at infinity, namely

$$\begin{cases} \dot{x}(t) \in F\big(t, x(t)\big), & \text{for a.a. } t \in [0, \infty), \\ \lim\limits_{t \to \infty} x(t) = x_\infty \in \mathbb{R}^n, \end{cases} \tag{4.22}$$

where $F : [0, \infty) \times \mathbb{R}^n \multimap \mathbb{R}^n$ is a u-Carathéodory map, i.e.

(i) values of F are nonempty, compact and convex, for all $(t, x) \in [0, \infty) \times \mathbb{R}^n$,
(ii) $F(t, \cdot)$ is upper semicontinuous, for a.a. $t \in [0, \infty)$,
(iii) $F(\cdot, x)$ is measurable, for all $x \in \mathbb{R}^n$.

We will prove acyclicity of the solution set of problem (4.22).

Recalling that any contractible set is acyclic, we can give

THEOREM 4.12. *Consider the target (terminal) problem* (4.22), *where* $F : [0, \infty) \times \mathbb{R}^n \multimap \mathbb{R}^n$ *is a u-Carathéodory map and* $x_\infty \in \mathbb{R}^n$ *is arbitrary. Assume that there exists a globally integrable function* $\nu : [0, \infty) \to [0, \infty)$, *where* $\int_0^\infty \nu(t)\,\mathrm{d}t = E < 1$, *such that*

$$d_H\big(F(t, x), F(t, y)\big) \leqslant \nu(t)|x - y|, \quad \textit{for all } t \in [0, \infty) \textit{ and } x, y \in \mathbb{R}^n. \tag{4.23}$$

Moreover, assume that $d_H(F(\cdot, 0), 0)$ *can be absolutely estimated by some globally integrable function. If* E *is a sufficiently small constant, then the set of solutions to problem* (4.22) *is compact and acyclic, for every* $x_\infty \in \mathbb{R}^n$.

PROOF. Observe that condition (4.23) implies the existence a globally integrable function $\alpha : [0, \infty) \to [0, \infty)$ and a positive constant B such that

$$\big|F(t, x)\big| \leqslant \alpha(t)\big(B + |x|\big), \quad \text{for every } x \in \mathbb{R}^n \text{ and a.a. } t \in [0, \infty), \tag{4.24}$$

where $|F(t, x)| = \sup\{|y| \mid y \in F(t, x)\}$. Thus, problem (4.22) can be equivalently replaced by the problem

$$\begin{cases} \dot{x}(t) \in G\big(t, x(t)\big), & \text{for a.a. } t \in [0, \infty), \\ \lim\limits_{t \to \infty} x(t) = x_\infty \in \mathbb{R}^n, \end{cases} \tag{4.25}$$

where G is a suitable Carathéodory map which can be estimated by a sufficiently large positive constant M, i.e.

$$\big|G(t, x)\big| \leqslant M, \quad \text{for every } x \in \mathbb{R}^n \text{ and a.a. } t \in [0, \infty),$$

and which satisfies condition (4.23) as well. In other words, the solution set $\mathcal{S}$ for problem (4.22) is the same as for problem (4.25), where

$$\mathcal{S} = \Big\{x \in C\big([0, \infty), \mathbb{R}^n\big) \;\Big|\; \dot{x}(t) \in F\big(t, x(t)\big), \\ \text{for a.a. } t \in [0, \infty) \text{ and } x(\infty) = x_\infty\Big\}.$$

For the structure of $\mathcal{S}$, we will modify the approach from above. Observe that, under the above assumptions, F as well G are well known to be product-measurable (see Proposition 2.4), and subsequently having a Carathéodory selection $g \subset G$ which is Lipschitzian with a not necessarily same, but again sufficiently small constant (see, e.g., [73, pp. 101–103]). By the sufficiency we mean that, besides others,

$$\big|g(t, x) - g(t, y)\big| \leqslant \gamma(t)|x - y|$$

holds, for all $x, y \in \mathbb{R}^n$ and a.a. $t \in [0, \infty)$, with a Lebesgue integrable function $\gamma : [0, \infty) \to [0, \infty)$ such that $\int_0^\infty \gamma(t)\,\mathrm{d}t < 1$.

Considering the single-valued problem ($g \subset G$)

$$\begin{cases} \dot{x}(t) = g\bigl(t, x(t)\bigr), & \text{for a.a. } t \in [0, \infty), \\ \lim\limits_{t \to \infty} x(t) = x_\infty, \end{cases} \tag{4.26}$$

we can easily prove the existence of a unique solution $\overline{x}(t)$ of problem (4.26). The uniqueness can be verified in a standard manner by the contradiction, when assuming the existence of another solution $\overline{y}(t)$ of that problem, because so we would arrive at the false inequality

$$\begin{aligned} \sup_{t \in [0,\infty)} \bigl|\overline{x}(t) - \overline{y}(t)\bigr| &= \sup_{t \in [0,\infty)} \left| \int_\infty^t g\bigl(s, \overline{x}(s)\bigr)\,\mathrm{d}s - \int_\infty^t g\bigl(s, \overline{y}(s)\bigr)\,\mathrm{d}s \right| \\ &\leqslant \int_0^\infty \bigl|g\bigl(s, \overline{x}(s)\bigr) - g\bigl(s, \overline{y}(s)\bigr)\bigr|\,\mathrm{d}t \\ &\leqslant \int_0^\infty \gamma(t) \sup_{t \in [0,\infty)} \bigl|\overline{x}(t) - \overline{y}(t)\bigr|\,\mathrm{d}t \\ &\leqslant \sup_{t \in [0,\infty)} \bigl|\overline{x}(t) - \overline{y}(t)\bigr| \int_0^\infty \gamma(t)\,\mathrm{d}t \\ &< \sup_{t \in [0,\infty)} \bigl|\overline{x}(t) - \overline{y}(t)\bigr|. \end{aligned}$$

Hence, according to the definition of contractibility in Section 2.1, it is sufficient to show that the solution set $\mathcal{S}$ of problem (4.25) is homotopic to a unique solution $\overline{x}(t)$ of problem (4.26), which is at the same time a solution of problem (4.25) as well. The desired homotopy reads ($\lambda \in [0, 1]$)

$$h(x, \lambda)(t) = \begin{cases} x(t), & \text{for } t \geqslant 1/\lambda - \lambda,\ \lambda \neq 0, \\ \overline{z}(t), & \text{for } 0 < t \leqslant 1/\lambda - \lambda,\ \lambda \neq 0, \\ \overline{x}(t), & \text{for } \lambda = 0, \end{cases}$$

where $\overline{z}$ is a unique solution to the reverse Cauchy problem

$$\begin{cases} \dot{x}(t) = g\bigl(t, z(t)\bigr), & \text{for a.a. } t \in [0, 1/\lambda - \lambda], \\ z(1/\lambda - \lambda) = x(1/\lambda - \lambda), \end{cases}$$

for each $\lambda \in [0, 1]$. Then h is a continuous homotopy such that $h(x, 0) = \overline{x}$, $h(x, 1) = x$, as required, and subsequently, the set $\mathcal{S}$ is acyclic. Using the convexity assumption on values of F, we can prove by the standard manner [22, Mazur's Theorem 1.33 in Chapter I.1] that $\mathcal{S}$ is closed in $C([0, \infty), \mathbb{R}^n)$. By Arzelà–Ascoli's lemma, this set is compact, and the proof is complete. □

Now, we shall consider the boundary value problem

$$\begin{cases} \dot{x}(t) + A(t)x(t) \in F\big(t, x(t)\big), & \text{for a.a. } t \in [0, T], \\ Lx = r, \end{cases} \tag{4.27}$$

where

(i) $A : [0, T] \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ is a measurable linear operator such that $|A(t)| \leqslant \gamma(t)$, for all $t \in [0, T]$ and some integrable function $\gamma : [0, T] \to [0, \infty)$,

(ii) the associated homogeneous problem

$$\begin{cases} \dot{x}(t) + A(t)x(t) = 0, & \text{for a.a. } t \in [0, T], \\ Lx = 0 \end{cases}$$

has only the trivial solution,

(iii) $F : [0, T] \times \mathbb{R}^n \multimap \mathbb{R}^n$ has nonempty, compact, convex values,

(iv) $F(\cdot, x)$ is measurable, for every $x \in \mathbb{R}^n$,

(v) there is a constant $M \geqslant 0$ such that

$$d_H\big(F(t, x), F(t, y)\big) \leqslant M|x - y|, \quad \text{for all } x, y \in \mathbb{R}^n \text{ and a.a. } t \in [0, T],$$

where d_H stands for the Hausdorff metric,

(vi) there are two nonnegative Lebesgue-integrable functions $\delta_1, \delta_2 : [0, T] \to [0, \infty)$ such that,

$$\big|F(t, x)\big| \leqslant \delta_1(t) + \delta_2(t)|x|, \quad \text{for a.a. } t \in [0, T] \text{ and all } x \in \mathbb{R}^n,$$

where $|F(t, x)| = \sup\{|y| \mid y \in F(t, x)\}$.

In [43], the authors have proved the functional generalization of following theorem.

THEOREM 4.13. *Under the assumptions* (i)–(vi), *a certain "critical" value* λ *exists such that if* $M < \lambda$, *then the set of solutions of* (4.27) *is a* (*nonempty*) *compact AR-space*. *Moreover, if the Lebesgue measure of the set* $\{t \mid \dim F(t, x) < 1$, *for some* $x \in \mathbb{R}\}$ *is still zero*, *then the set of solutions of* (4.27) *is an infinite dimensional compact AR-space*, *where* $\dim X$ *denotes the covering* (*topological*) *dimension of a space* X.

REMARK 4.9. Observe that for $A \equiv 0$ and $Lx = x(0)$, the related Cauchy problem can have, under the assumptions of Theorem 4.13, infinitely many linearly independent solutions on the whole interval $[0, T]$.

REMARK 4.10. The first assertion of Theorem 4.13 can be still improved (see [22, Theorem 3.13 in Chapter III.3]), namely that, for the problem

$$\begin{cases} \dot{x}(t) + A(t)x(t) \in \alpha F\big(t, x(t)\big), & \text{for a.a. } t \in [0, T], \\ Lx = \theta, \end{cases} \tag{4.28}$$

where $\alpha \leqslant \lambda$, when λ is the critical value in Theorem 4.13, the set of solutions of (4.28) is nonempty, compact and acyclic.

REMARK 4.11. In the case of ODEs, the solution set in Theorem 4.13 consists, unlike in the critical case for $\alpha = \lambda$ in Remark 4.10, of a unique solution. The same is true for Theorem 4.12.

In view of Remark 4.11, a nontrivial structure of solution sets to single-valued boundary problems can be seen as a delicate problem. The following result in this field in [46] is rather rare.

THEOREM 4.14. *Consider the Floquet problem*

$$\begin{cases} \dot{x}(t) = f\big(t, x(t)\big), & \text{for a.a. } t \in [a, b], \\ x(a) + \lambda x(b) = \xi & \big(\lambda > 0,\ \xi \in \mathbb{R}^n\big), \end{cases} \tag{4.29}$$

where $f : [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$ is a bounded Carathéodory function. Assume, furthermore, that f satisfies

$$\big|f(t, x) - f(t, y)\big| \leqslant p(t)|x - y|, \quad \text{for a.a. } t \in [a, b] \text{ and } x, y \in \mathbb{R}^n, \tag{4.30}$$

where $p : [a, b] \to [0, \infty)$ is a Lebesgue-integrable function such that

$$\int_a^b p(t)\,\mathrm{d}t \leqslant \sqrt{\pi^2 + \ln^2 \lambda}. \tag{4.31}$$

Then the set of solution to (4.29) *is an R_δ-set.*

REMARK 4.12. As pointed out in [46], if the sharp inequalities take place in (4.31), then problem (4.29) has a unique solution. On the other hand, for equalities (4.31), problem (4.29) can possess more solutions, respectively.

Unlike in the above theorems, the following problems can be regarded as those with "limiting" boundary conditions. In [53], the following result has been proved (as Theorem 3.1) for the boundary value problem

$$\begin{cases} \dot{x} = f(t, x), \\ Lx = r, \end{cases} \tag{4.32}$$

where $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function, $L : C^1(I, \mathbb{R}^n) \to \mathbb{R}^n$ is a linear operator and $I = [a, b]$ is a compact interval.

PROPOSITION 4.10. *Let $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ be a fixed continuous function such that, for every $t_0 \in I$ and $x_0 \in \mathbb{R}^n$, there exists a unique* (*smooth*) *solution $x(t)$ of the equation $\dot{x} = f(t, x)$, satisfying $x(t_0) = x_0$.*

Let $\mathcal{U}$ be an open (*in the norm topology*) *subset of the Banach space of all continuous linear operators $L : C_1^0(I, \mathbb{R}^n) \to \mathbb{R}^n$, where C_1^0 denotes the set $C^1(\mathbb{R}^n) \subset C^0(I, \mathbb{R}^n)$, topologized by the induced topology of $C^0(I, \mathbb{R}^n)$* $(:= C(I, \mathbb{R}))$.

If, for every $L \in \mathcal{U}$ and $r \in \mathbb{R}^n$, the boundary value problem (4.32) *has at most one solution, then, for every $L \in \mathcal{U}$ and $r \in \mathbb{R}^n$, problem* (4.32) *has exactly one solution.*

Our aim is to prove, by means of Proposition 2.6 in Section 2.3, the following theorem.

THEOREM 4.15. *Let $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ be a fixed continuous function and $p : I \to \mathbb{R}^n$ be a continuous function such that, for every $t_0 \in I$, $x_0 \in \mathbb{R}^n$ and $p \in C(I, \mathbb{R}^n)$ with $\|p\| \leqslant 1$, there exists a unique solution $x(t)$ of*

$$\dot{x} = f(t, x) + p(t), \tag{4.33}$$

satisfying $x(t_0) = x_0$.

Let $\mathcal{U}$ be an open (*in the norm topology*) *subset of the Banach space of all continuous linear operators $L : C_1^0(I, \mathbb{R}^n) \to \mathbb{R}^n$, where C_1^0 has the same meaning as in Proposition* 4.10.

Assume that, for every $L \in \mathcal{U}$, $r \in \mathbb{R}^n$ and $p \in C^1(\mathbb{R}^n)$ with $\|p\| \leqslant 1$, the boundary value problem

$$\begin{cases} \dot{x} = f(t, x) + p(t), \\ Lx = r, \end{cases} \tag{4.34}$$

has at most one solution and that, for every $L \in \overline{\mathcal{U}}$, all solutions of problem (4.32) *are uniformly* (*i.e. independently of $L \in \overline{\mathcal{U}}$*) *a priori bounded, where $\overline{\mathcal{U}}$ denotes the closure of $\mathcal{U}$ in the C_0^1-topology.*

Then, for every $L \in \partial\mathcal{U}$ and $r \in \mathbb{R}^n$, where $\partial\mathcal{U}$ denotes the boundary of $\mathcal{U}$ in the C_1^0-topology, problem (4.32) *has an R_δ-set of solutions.*

PROOF. Since all assumptions of Proposition 4.10 are satisfied, problem (4.34) is solvable, for every $L \in \mathcal{U}$, $r \in \mathbb{R}^n$ and $p \in C(I, \mathbb{R}^n)$ with $\|p\| \leqslant 1$.

Furthermore, since $\overline{\mathcal{U}}$ is a closed subset of the Banach space of all continuous linear operators, each element $\widetilde{L} \in \partial\mathcal{U}$ can be regarded as a uniform limit of a suitable sequence $\{L_k\}$ such that $\widetilde{L} = \lim_{k\to\infty} L_k$, where $L_k \in \mathcal{U}$ ($= \operatorname{int}\mathcal{U}$), for every $k \in \mathbb{N}$.

Fix such an $\widetilde{L} \in \partial\mathcal{U}$ and consider the compact operators Φ_k, $\Phi : \mathcal{B} \to C_1^0(I, \mathbb{R}^n)$:

$$\Phi_k(x)(t) = x(a) + L_k x - r + \int_a^t f\big(s, x(s)\big)\, \mathrm{d}s$$

and

$$\Phi(x)(t) = x(0) + \widetilde{L}x - r + \int_a^t f\big(s, x(s)\big)\, \mathrm{d}s,$$

where $\mathcal{B} \subset C^0(I, \mathbb{R}^n)$ is a suitable closed ball centered at the origin, which is implied by the assumption of a uniform a priori boundedness of solutions. The compactness of operators follows directly by means of the well-known Arzelà–Ascoli lemma.

One can readily check the one-to-one correspondence between the fixed points of Φ and the solutions of problem (4.32) as well as those of Φ_k and the solutions of the equation $\dot{x} = f(t, x(t))$, satisfying

$$L_k x = r.$$

Thus, one can associate to Φ_k and Φ the proper maps $\varphi_k = \mathrm{id} - \Phi_k$ and $\varphi = \mathrm{id} - \Phi$, respectively, where id denotes the identity, namely

$$\varphi_k(x)(t) = x(t) - x(a) - L_k x + r - \int_a^t f\big(s, x(s)\big)\,\mathrm{d}s$$

and

$$\varphi(x)(t) = x(t) - x(a) - \widetilde{L}x + r - \int_a^t f\big(s, x(s)\big)\,\mathrm{d}s.$$

So, the nonempty kernel $\varphi^{-1}(0)$ of φ corresponds to the fixed points of Φ, and subsequently to solutions of (4.32), i.e.

$$x \in \Phi(x) \quad \Longleftrightarrow \quad 0 \in x - \Phi(x) = (\mathrm{id} - \Phi)(x) = \varphi(x).$$

We can assume without any loss of generality that, for a sufficiently large $k \in \mathbb{N}$, we have

$$\big|\varphi_k(x)(t) - \varphi(x)(t)\big| = \big|L_k x - \widetilde{L}x\big| = \big|\big(L_k - \widetilde{L}\big)x\big| \leqslant \frac{1}{k}, \tag{4.35}$$

because, otherwise, we can obviously select a subsequence with this property.

Since $\|\varphi_k(x)(t) - \varphi(x)(t)\| \leqslant 1/k$ holds, for every $x \in \mathcal{B}$, condition (i) of Proposition 2.6 in Section 2.3 is satisfied.

In order to prove (ii) in Proposition 2.6, it is sufficient to verify the following inequalities

$$\big|\varphi_k(x)(t)\big| \leqslant \frac{1}{k} \quad \text{and} \quad \big|\big(\varphi_k(x)\big)\dot{}(t)\big| \leqslant \frac{1}{k}, \quad k \in \mathbb{N},$$

for every x with $\varphi(x) = 0$.

However, since $(\varphi_k(x))\dot{}(t) = \dot{x}(t) - f(t, x(t)) = 0$, $k \in \mathbb{N}$, and the first inequality follows from (4.35), we are done.

In order to verify (iii) in Proposition 2.6, we should realize that, for any $u \in V_k = \{u \in C^1(I, \mathbb{R}^n) : \|u\|_{C^0} \leqslant 1/k$ and $\|\dot{u}\|_{C^0} \leqslant 1/k\}$, for some $k \in \mathbb{N}$, $x(t)$ is a solution of the equation $u(t) = \varphi_k(x)(t)$, i.e.

$$u(t) = x(t) - x(a) - L_k x + r - \int_a^t f\big(s, x(s)\big)\,\mathrm{d}s,$$

if and only if it satisfies

$$\begin{cases} \dot{x} = \dot{u}(t) + f(t,x), \\ L_k x = r - u(a). \end{cases} \tag{4.36}$$

By the hypothesis, problem (4.36) has a unique solution, for every $L_k \in \mathcal{U}$, $r \in \mathbb{R}^n$ and $u \in C^1(I, \mathbb{R}^n)$ with $\|u\| \leqslant 1$, as required. Therefore, applying Proposition 2.6 in Section 2.3, the set $\{\varphi(0)\}$ is R_δ. In other words, the solution set of the original problem (4.32) is R_δ as well. □

REMARK 4.13. One can observe that the sole existence can be easily proved by means of the well-known Schauder fixed point theorem.

EXAMPLE 4.1. According to Example 2 in [85], problem

$$\begin{cases} \dot{x}_i = f_i(t, x_1, x_2) + p_i(t), \quad i = 1, 2, \\ a x_1(0) + x_2(0) = r_1, \quad b x_1(1) + x_2(1) = r_2, \end{cases} \tag{4.37}$$

is uniquely solvable, for every $a^2 < 1$, $b^2 > 1$, $r_i \in \mathbb{R}$ $(i = 1, 2)$ and $p = (p_1, p_2) \in C([0, 1], \mathbb{R})$, provided $f_i \in C^1([0, 1], \mathbb{R})$, $i = 1, 2$, and

$$\frac{\partial f_1}{\partial x_1} u_1^2 + \frac{\partial f_1}{\partial x_2} u_1 u_2 - \frac{\partial f_2}{\partial x_1} u_1 u_2 - \frac{\partial f_2}{\partial x_2} u_2^2 \geqslant 0 \quad (i = 1, 2),$$

for each triple $(t, x_1, x_2) \in [0, 1] \times \mathbb{R}^2$ and each double $(u_1, u_2) \in \mathbb{R}^2$.

Therefore, according to Theorem 4.15 (more precisely, according to its modified version, where the set of all continuous linear operators can be restricted (see [85]) to the set of all real $(n \times n)$-matrices), problem

$$\begin{cases} \dot{x}_i = f_i(t, x_1, x_2), \quad i = 1, 2, \\ a x_1(0) + x_2(0) = r_1, \quad b x_1(1) + x_2(1) = r_2, \end{cases} \tag{4.38}$$

has an R_δ-set of solutions, for certain $(a, b) \in \mathbb{R}^2$ in a closed subset of $\mathbb{R}^2$ with $a^2 = 1$, $b^2 \geqslant 1$ or $a^2 \leqslant 1$, $b^2 = 1$ (r_1, r_2 can be arbitrary), whenever all solutions of problem (4.38) are uniformly a priori bounded, for such $a^2 \leqslant 1$, $b^2 \geqslant 1$.

This can be achieved for $a \neq b$, i.e. particularly with the exception of $a = b = 1$ or $a = b = -1$, and $b^2 \leqslant b_*^2$, for some $b_* > 1$, when, e.g.,

$$\left| f(t, x) \right| \leqslant \alpha |x| + \beta, \quad \text{for all } (t, x) \in [0, 1] \times \mathbb{R}^2, \tag{4.39}$$

where α, β are suitable nonnegative constants (α must be sufficiently small as below) and $x = (x_1, x_2)$, $f = (f_i, f_2)$.

Indeed. Since the linear homogeneous problem

$$\begin{cases} \dot{x}_i = 0, \quad i = 1, 2, \\ a x_1(0) + x_2(0) = 0, \quad b x_1(1) + x_2(1) = 0, \end{cases}$$

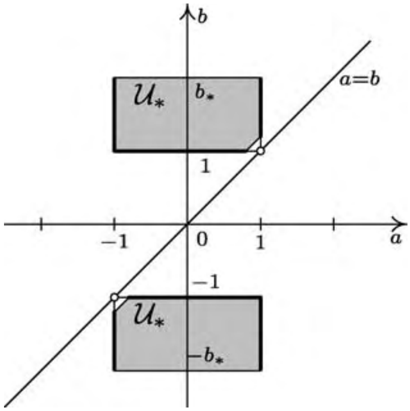

Fig. 2.

has for $a \neq b$ obviously only a trivial solution, every solution $x(t) = (x_1(t), x_2(t))$ of (4.38) takes the form (see, e.g., [22, Lemma 5.136 in Chapter III.5])

$$x_i(t) = \int_0^1 G_i(t, s, a, b) f_i\big(s, x_1(s), x_2(s)\big)\,\mathrm{d}s + \widetilde{x}_i, \quad i = 1, 2,$$

where $G = (G_1, G_2)$ is the related Green function of the linearized problem (4.38), namely

$$\begin{cases} \dot{x}_i = f_i(t, x_1, x_2), \quad i = 1, 2, \\ ax_1(0) + x_2(0) = 0, \quad bx_1(1) + x_2(1) = 0, \end{cases}$$

i.e.

$$G(t, s, a, b) = \begin{cases} \dfrac{1}{b-a}\begin{pmatrix} b & 1 \\ -ab & -a \end{pmatrix}, & \text{for } 0 \leqslant t \leqslant s \leqslant 1, \\ \dfrac{1}{b-a}\begin{pmatrix} a & 1 \\ -ab & -b \end{pmatrix}, & \text{for } 0 \leqslant s \leqslant t \leqslant 1, \end{cases}$$

and $\widetilde{x} = (\widetilde{x}_1, \widetilde{x}_2)$ is a unique solution of the problem

$$\begin{cases} \dot{x}_i = 0, \quad i = 1, 2, \\ ax_1(0) + x_2(0) = r_1, \quad bx_1(1) + x_2(1) = r_2, \end{cases}$$

i.e. $\widetilde{x}_1 = (r_2 - r_1)/(b-a)$, $\widetilde{x}_2 = r_1 - a(r_2 - r_1)/(b-a)$.

Let us fix (a, b) at the boundary $\partial\mathcal{U} = \{(a, b) \in \mathbb{R}^2 \mid a^2 = 1,\ b^2 \geqslant 1 \text{ or } a^2 \leqslant 1,\ b^2 = 1\}$ with $a \neq b$, for which we intend to get the result and cut off appropriately the corners with $a = b$, jointly with those (a, b) with $b^2 > b_*^2$ for some $b_* > 1$, as in Fig. 2. The bold curve in Fig. 2 so indicates the part of the boundary of our interest.

Denoting $g = \max_{(a,b)\in\overline{\mathcal{U}}_*} g_{a,b}$, $g_{a,b} = \max_{t,s\in[0,1]} |G(t,s,a,b)|$, where $\overline{\mathcal{U}}_*$ is indicated in Fig. 2 by the shaded region (observe that since $\overline{\mathcal{U}}_*$ is compact, g certainly exists), we obtain by means of (4.39) that

$$\|x(t)\| \leqslant g\big(\alpha\|x(t)\| + \beta\big) + \max_{(a,b)\in\overline{\mathcal{U}}_*} |\widetilde{x}|,$$

i.e.

$$\|x(t)\| \leqslant \frac{\beta g + \max_{(a,b)\in\overline{\mathcal{U}}_*} |\widetilde{x}|}{1-\alpha g},$$

whenever $\alpha < g^{-1}$, as claimed. This completes the example.

REMARK 4.14. One can easily check that, for fixed values of $(a,b) \in \partial\mathcal{U}$ with $a \neq b$, the condition $\alpha < g^{-1}$ can take, e.g., the form

$$\alpha < \frac{|b-a|}{|b| + \max(1,|ab|)}.$$

A continuous function $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ can be approximated with an arbitrary accuracy by locally Lipschitzian (in the second variable) functions (see, e.g., [22, Theorem 3.37 in Chapter I.3] or [71]), say $(f + \varepsilon_k)$, $k \in \mathbb{N}$, such that $\lim_{k\to\infty} \|\varepsilon_k\| = 0$. Therefore, applying at first, for fixed $k \in \mathbb{N}$, Theorem 4.15 to the system

$$\dot{x} = f(t,x) + \varepsilon_k(t,x), \tag{4.40}$$

we can still avoid (for more details, see [22, Chapter III.3]) the uniqueness assumption in Theorem 4.15 as follows.

THEOREM 4.16. *Let* $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ *be a fixed continuous function and* $\varepsilon_k : I \times \mathbb{R}^n \to \mathbb{R}^n$, $k \in \mathbb{N}$, *be continuous functions with* $\|\varepsilon_k\| \leqslant \varepsilon$ (ε *– a sufficiently small constant*) *such that* $(f + \varepsilon_k)(t,\cdot) : \mathbb{R}^n \to \mathbb{R}^n$, $k \in \mathbb{N}$, *are locally Lipschitzian, for every* $t \in I$, *and* $p : I \to \mathbb{R}^n$ *be a continuous function with* $\|p\| \leqslant 1$.

Let $\mathcal{U}$ *be an open* (*in the norm topology*) *subset of the Banach space of all continuous linear operators* $L : C_1^0(I,\mathbb{R}^n) \to \mathbb{R}^n$.

Assume that, for every $L \in \mathcal{U}$, $r \in \mathbb{R}^n$, $k \in \mathbb{N}$ *and* $p \in C(I,\mathbb{R}^n)$ *with* $\|p\| \leqslant 1$, *the boundary value problem*

$$\begin{cases} \dot{x} = f(t,x) + \varepsilon_k(t,x) + p(t), \\ Lx = r \end{cases}$$

has at most one solution and that, for every $L \in \overline{\mathcal{U}}$ *and* $k \in \mathbb{N}$, *all solutions of the problem*

$$\begin{cases} \dot{x} = f(t,x) + \varepsilon_k(t,x), \\ Lx = r \end{cases}$$

are uniformly (*i.e. independently of* $L \in \overline{\mathcal{U}}$) *a priori bounded, where* $\overline{\mathcal{U}}$ *denotes the closure of* $\mathcal{U}$ *in the* C_1^0*-topology.*

Then, for every $L \in \partial\mathcal{U}$ *and* $r \in \mathbb{R}^n$, *where* $\partial\mathcal{U}$ *denotes the boundary of* $\mathcal{U}$ *in the* C_1^0*-topology, the problem* (4.32), *i.e.*

$$\begin{cases} \dot{x} = f(t,x), \\ Lx = r \end{cases}$$

has a (*nonempty*) *compact acyclic set of solutions.*

4.3. *Poincaré's operator approach*

By the Poincaré operators, we mean the *translation operators along the trajectories* of the associated differential systems. The translation operator is sometimes also called as *Poincaré–Andronov* or *Levinson* or, simply, *T-operator*.

In the classical theory, these operators are defined to be single-valued, when assuming among other things, the uniqueness of the initial value problems. At the absence of uniqueness, one usually approximates the right-hand sides of the given systems by the locally Lipschitzian ones (implying already uniqueness), and then applies the *standard limiting argument* (for more details, see, e.g., [71,81]).

On the other hand, set-valued analysis allows us to handle directly with multivalued Poincaré operators which become, under suitable natural restrictions imposed on the right-hand sides of given differential systems, admissible in the sense of Definition 2.5 in Section 2.2.

Hence, consider the u-Carathéodory system

$$\dot{x} \in F(t,x), \quad x \in \mathbb{R}^n, \tag{4.41}$$

where $F : [0,\tau] \times \mathbb{R}^n \multimap \mathbb{R}^n$ satisfies all conditions in Definition 2.10.

By a *solution* $x(t)$ of (4.41), we mean an absolutely continuous function $x(t) \in \mathrm{AC}([0,\tau],\mathbb{R}^n)$ satisfying (4.41), for a.a. $t \in [0,\tau]$, i.e. the one in the sense of Carathéodory, such solutions of (4.41) exist on $[0,\tau]$.

Hence, if $x(t,x_0) := x(t,0,x_0)$ is a solution of (4.41) with $x(0,x_0) = x_0 \in \mathbb{R}^n$, then the translation operator $T_\tau : \mathbb{R}^n \multimap \mathbb{R}^n$ at the time $\tau > 0$ along the trajectories of (4.41) is defined as follows:

$$T_\tau(x_0) := \bigl\{x(\tau,x_0) \mid x(\cdot,x_0) \text{ is a solution of (4.41) with } x(0,x_0) = x_0\bigr\}. \tag{4.42}$$

More precisely, T_τ can be considered as the composition of two maps, namely $T_\tau = \psi \circ \varphi$,

$$\mathbb{R}^n \overset{\varphi}{\multimap} \mathrm{AC}\bigl([0,\tau],\mathbb{R}^n\bigr) \xrightarrow{\psi} \mathbb{R}^n,$$

where $\varphi(x_0): x_0 \multimap \{x(t, x_0) \mid x(t, x_0)$ is a solution of (4.41) with $x(0, x_0) = x_0\}$ is well known to be an R_δ-mapping (see Theorem 4.5) and $\psi(y): y \to y(\tau)$ is obviously a continuous (single-valued) evaluation mapping.

In other words, we have the following commutative diagram:

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{\ \varphi\ }\!\!\circ & \mathrm{AC}\big([0,\tau],\mathbb{R}^n\big) \\ & \ddots & \Big\downarrow \psi \\ & \xrightarrow{\ T_\tau\ }\!\!\circ & \mathbb{R}^n \end{array}$$

The following characterization of T_τ has been proved on various levels of abstraction in several papers (see, e.g., [22, Theorem 4.3 in Chapter III.4] and the references therein).

THEOREM 4.17. *T_τ defined by* (4.42) *is admissible and admissibly homotopic* (*see Definitions* 2.5 *and* 2.6) *to identity. More precisely, T_τ is a composition of an R_δ-mapping and a continuous* (*single-valued*) *evaluation mapping.*

PROOF. According to Theorem 4.5, the mapping φ has an R_δ-set of values. We will show that it is u.s.c. by proving the closedness of the graph Γ_φ of φ (cf. Section 2.2).

Let $(x_n, y_n) \in \Gamma_\varphi$, i.e. $y_n \in \varphi(x_n)$, and $(x_n, y_n) \to (x, y)$ as $n \to \infty$. Since the functions y_n are absolutely continuous on $[0, \tau]$, the application of the well-known Gronwall inequality (see, e.g., [71]) leads to the estimates (cf. (iii) in Definition 2.10)

$$\|y_n\| \leqslant M := \sup_{n\in\mathbb{N}}\big(|x_n| + \gamma\tau\big)\exp(\gamma\tau) \quad \text{and} \quad \|\dot{y}_n\| \leqslant \gamma(1 + M),$$

where $\gamma = \max\{\alpha, \beta\}$. It follows that $\{y_n\}$ are equibounded.

Proposition 4.1 guarantees the existence of a sequence $\{y_n\}$ such that $y_n \to y$, uniformly, and $\dot{y}_n \to \dot{y}$, weakly in $L^1([0, \tau], \mathbb{R}^n)$. According to Mazur's theorem (see, e.g., [22, Theorem 1.33 in Chapter I.1]), $\dot{y}$ belongs to the strong closure $\dot{y} \in \overline{\mathrm{conv}}\{\dot{y}_n \mid n \geqslant l\}$, for all $l \geqslant 1$. Thus, there also exists a subsequence $\{z_l\}$ such that $z_l \to \dot{y}$, in the L^1-topology, where $z_l \in \overline{\mathrm{conv}}\{\dot{y}_n \mid n \geqslant l\}$. Moreover, there exists a subsequence (for the simplicity, denoted again by $\{z_l\}$) satisfying $z_l \to \dot{y}$, a.e. on $[0, \tau]$.

Let $I \subset [0, \tau]$ be a set of a full measure on $[0, \tau]$, i.e. $\mu(I) = \tau$, where μ denotes the Lebesgue measure, such that $z_l \to \dot{y}$ as $l \to \infty$, for all $t \in I$. It follows from the definition of z_l that $z_l(t) \in \sum_i \lambda_i F(t, y_{n_i}(t))$, where $\sum_i \lambda_i = 1$.

Since $F(t, \cdot)$ is u.s.c., for a.a. $t \in [0, \tau]$, and $y_{n_i}(t)$ is sufficiently close to $x(t)$ as well as $z_l(t)$ to $\dot{x}(t)$, we obtain

$$\dot{x}(t) \in \sum_i \lambda_i F\big(t, x(t)\big) + \varepsilon B$$

for an arbitrary $\varepsilon > 0$, where B is an open unit ball. This already means that $\dot{x}(t) \in F(t, x(t))$, and subsequently the graph Γ_φ of φ is closed. Since the arbitrary closed set

$\{(x, y), (x_1, y_1), \ldots, (x_n, y_n), \ldots\}$ is, according to the well-known Arzelà–Ascoli lemma, compact, φ is u.s.c.

For the remaining part of the proof, it is sufficient to consider the admissible homotopy $T_{\lambda\tau}$, $\lambda \in [0, 1]$. □

REMARK 4.15. Since a composition of admissible maps is admissible as well (see Section 2.2), T_τ can be still composed with further admissible maps ϕ such that $\phi \circ T_\tau$ becomes an (admissible) self-map on a compact ENR-space (i.e. homeomorphic to ANR in $\mathbb{R}^n$), for computation of the well-defined (cf. Section 3.1) generalized Lefschetz number:

$$\Lambda(\phi \circ T_\tau) = \Lambda(\phi).$$

T_τ considered on ENRs can be even composed, e.g., with suitable homeomorphisms $\mathcal{H}$ (again considered on ENRs), namely $\mathcal{H} \circ T_\tau$, for computation of the well-defined (cf. Section 3.3) fixed point index:

$$\operatorname{ind}(\mathcal{H} \circ T_\tau) = \operatorname{ind} \mathcal{H},$$

provided the fixed point set of $\mathcal{H} \circ T_{\lambda\tau}$ is compact, for $\lambda \in [0, 1]$.

REMARK 4.16. In [10] (cf. also [22, Chapter III.4]), translation operators are also studied, e.g., for systems with constraints, systems in Banach spaces, for directionally semicontinuous systems, etc. In particular, in real separable Banach spaces, one can check that, under the conditions in Remark 4.6 (cf. also Remark 4.8 and [58, Corollary 9.1 in Chapter 9.4]), the related translation operator T_τ is like in Theorem 4.17. In order T_τ to be also condensing, one should however impose some further restrictions. Since these restrictions are rather technical (cf. [78, Theorem 6.3.1] or [22, Theorem 4.16 in Chapter III.4]), and so this Poincaré's translation operator will not be more employed, we omit them here.

5. Existence results

5.1. *Existence of bounded solutions*

We start with the application of Theorem 4.4. Hence, let $(E, \|\cdot\|)$ be a reflexive Banach space and let $\mathcal{L}(E)$ be the space of all linear continuous transformations in E. The Hausdorff measure of noncompactness (MNC) will be denoted by γ.

We are interested in the existence of a bounded solution to the semilinear differential inclusion

$$\dot{x}(t) + A(t)x(t) \in F\big(t, x(t)\big), \quad \text{for a.a. } t \in \mathbb{R} \tag{5.1}$$

with $A(t) \in \mathcal{L}(E)$ and a set-valued transformation F.

Our assumptions concerning the inclusion (5.1) will be the following:

(A1) $A : \mathbb{R} \to \mathcal{L}(E)$ is strongly measurable (cf. Definition 2.9) and Bochner integrable, on every compact interval $[a, b]$.

(A2) Assume that

$$\dot{x} + A(t)x = 0 \tag{5.2}$$

admits a regular exponential dichotomy (cf. Remark 5.3 below; for more details see, e.g., [56]). Denote by G the principal Green's function for (5.2).

(F1) Let $F : \mathbb{R} \times E \multimap E$ be a u-Carathéodory set-valued map (cf. Definition 2.10) such that

$$\|F(t, x)\| \leqslant m(t), \quad \text{for a.a. } t \in \mathbb{R},\ x \in E.$$

Here $m \in L^1_{\text{loc}}(\mathbb{R})$ is such that, for a constant M,

$$\sup\left\{ \int_t^{t+1} m(s)\,\mathrm{d}s \;\Big|\; t \in \mathbb{R} \right\} < M.$$

(F2) Assume that

$$\gamma\big(F(t, \Omega)\big) \leqslant g(t)h\big(\gamma(\Omega)\big), \quad \text{for a.a. } t \in \mathbb{R}$$

and each bounded $\Omega \subset E$, where g, h, are positive functions, g is measurable, h is nondecreasing such that

$$L := \sup\left\{ \int_{\mathbb{R}} \|G(t, s)\|_{\mathcal{L}(E)} g(s)\,\mathrm{d}s \;\Big|\; t \in \mathbb{R} \right\} < \infty$$

and $qh(t)L < t$, for each $t > 0$, with a constant $q = 1$, if E is separable, and $q = 2$, in the general case.

THEOREM 5.1. *Under the assumptions* (A1), (A2), (F1), (F2), *the semilinear differential inclusion* (5.1) *admits a bounded solution on* $\mathbb{R}$.

The main obstruction in the application of Theorem 4.4 will be the estimation of a suitably chosen MNC. For this purpose, we recall the following rule of taking the MNC under the sign of the integral (see [78, Corollary 4.2.5]).

LEMMA 5.1. *Let* $\{f_n\} \subset L^1([a, b], E)$ *be a sequence of functions such that*

(i) $\|f_n(t)\| \leqslant \nu(t)$, *for all* $n \in \mathbb{N}$ *and a.a.* $t \in [a, b]$, *where* $\nu \in L^1([a, b])$,

(ii) $\gamma(\{f_n(t)\}) \leqslant c(t)$, *for a.a.* $t \in [a, b]$, *where* $c \in L^1([a, b])$.

Then we have the estimate

$$\gamma\left(\left\{ \int_a^t f_n(s)\,\mathrm{d}s \right\}\right) \leqslant q \int_a^t c(s)\,\mathrm{d}s,$$

for each $t \in [a, b]$, *with* $q = 1$, *if* E *is separable, and* $q = 2$, *in general case.*

PROOF OF THEOREM 5.1. We carry out the proof in several steps.

(i) Let

$$Q := \Big\{ x \in C(\mathbb{R}, E) \;\Big|\; \|x(t)\| \leqslant K, \text{ for each } t \in \mathbb{R}, \; \|x(t_1) - x(t_2)\| \\ \leqslant K \int_{t_1}^{t_2} \|A(s)\|_{\mathcal{L}(E)} \,\mathrm{d}s + \int_{t_1}^{t_2} m(s)\,\mathrm{d}s, \text{ for all } t_1, t_2 \in \mathbb{R}, \; t_1 \leqslant t_2 \Big\}$$

with a constant K to be specified below. Clearly Q is a closed convex subset of $C(\mathbb{R}, E)$.

For a given $q \in Q$, we are interested in bounded solutions to the differential inclusion

$$\dot{x}(t) + A(t)x(t) \in F\big(t, q(t)\big), \quad \text{for a.a. } t \in \mathbb{R}. \tag{5.3}$$

Take $f \in N_F(q)$ (recall that such f exists, in view of Lemma 4.3), where N_F denotes the Nemytskiĭ operator. Since A admits an exponential dichotomy, we know that the problem

$$\dot{x}(t) + A(t)x(t) = f(t), \quad \text{for a.a. } t \in \mathbb{R},$$

has a unique, entirely bounded solution given by

$$x(f) = \int_{\mathbb{R}} G(t,s) f(s)\,\mathrm{d}s$$

(cf. [56,86]). Thus, problem (5.3) has a nonempty set of solutions $T(q)$. Using Lemmas 4.3 and 4.4, it is also clear that this set is closed convex, and since its compactness will become clear in the subsequent steps of the proof, it is in fact an R_δ-set.

(ii) We will show that, for each $q \in Q$, we actually have $T(q) \subset Q$. Let $x \in T(q)$. Then, for suitable $f \in N_F(q)$, we have

$$\begin{aligned}
\|x(t)\| &\leqslant \int_{\mathbb{R}} \|G(t,s)\|_{\mathcal{L}(E)} \|f(s)\|\,\mathrm{d}s \\
&\leqslant k \int_{-\infty}^{t} \mathrm{e}^{-\mu(t-s)} m(s)\,\mathrm{d}s + k \int_{t}^{\infty} \mathrm{e}^{-\mu(s-t)} m(s)\,\mathrm{d}s \\
&= k \sum_{j=0}^{\infty} \int_{j}^{j+1} \mathrm{e}^{-\mu\sigma} m(t-\sigma)\,\mathrm{d}\sigma + k \sum_{j=0}^{\infty} \int_{j}^{j+1} \mathrm{e}^{-\mu\sigma} m(t+\sigma)\,\mathrm{d}\sigma \\
&\leqslant k \sum_{j=0}^{\infty} \mathrm{e}^{-\mu j} \int_{j}^{j+1} m(t-\sigma)\,\mathrm{d}\sigma + k \sum_{j=0}^{\infty} \mathrm{e}^{-\mu j} \int_{j}^{j+1} m(t+\sigma)\,\mathrm{d}\sigma \\
&\leqslant k \sum_{j=0}^{\infty} \mathrm{e}^{-\mu j} \int_{t-j-1}^{t-j} m(s)\,\mathrm{d}s + k \sum_{j=0}^{\infty} \mathrm{e}^{-\mu j} \int_{t+j}^{t+j+1} m(s)\,\mathrm{d}s \\
&\leqslant 2kM \sum_{j=0}^{\infty} \mathrm{e}^{-\mu j} = 2kM\big(1 - \mathrm{e}^{-\mu}\big)^{-1} =: K.
\end{aligned} \tag{5.4}$$

In this estimation, we have used the fact that, by assumption (A2), there exist positive constants k, μ such that

$$\left\|G(t,s)\right\|_{\mathcal{L}(E)} \leqslant k\mathrm{e}^{-\mu|t-s|}.$$

Now, let $t_1, t_2 \in \mathbb{R}$. Then

$$\begin{aligned}\left\|x(t_1) - x(t_2)\right\| &\leqslant \int_{t_1}^{t_2} \left\|\dot{x}(s)\right\| \mathrm{d}s \\ &\leqslant \int_{t_1}^{t_2} \left\|A(s)\right\|_{\mathcal{L}(E)} \left\|x(s)\right\| \mathrm{d}s + \int_{t_1}^{t_2} \left\|f(s)\right\| \mathrm{d}s \\ &\leqslant K \int_{t_1}^{t_2} \left\|A(s)\right\|_{\mathcal{L}(E)} \mathrm{d}s + \int_{t_1}^{t_2} m(s)\,\mathrm{d}s.\end{aligned}$$

Consequently, $T(q) \subset Q$.

(iii) Let $\mathcal{M}$ be the power set of Q and define, for each $\Omega \in \mathcal{M}$, the real-valued MNC ψ by

$$\psi(\Omega) := \max_{D\in\mathcal{D}}(\Omega)\Big(\sup_{t\in\mathbb{R}} \gamma\big(D(t)\big)\Big),$$

where $\mathcal{D}(Q)$ denotes the collection of all denumerable subsets of Ω and $D(t) = \{d(t) \mid d \in D\} \subset E$. Then ψ is well-defined and from the corresponding properties of γ it is clear that ψ has monotone and nonsingular properties of measure of noncompactness (see Proposition 2.3 in Section 2.2). Finally, observe that ψ is regular in view of the Arzelà–Ascoli lemma.

We wish to show that the mapping T given in step (i) is condensing w.r.t. the MNC ψ.

Take $\Omega \in \mathcal{M}$. Considering $T(\Omega)$, we see that by the definition of ψ there exists a sequence $\{x_n\} \subset T(\Omega)$ such that

$$\psi\big(T(\Omega)\big) = \sup_{t\in\mathbb{R}} \gamma\big(\{x_n(t)\}\big).$$

Thus, for each $n \in \mathbb{N}$, there is $z_n \in \Omega$ and $f_n \in N_F(z_n)$ such that

$$x_n(t) = \int_{\mathbb{R}} G(t,s) f_n(s)\,\mathrm{d}s. \tag{5.5}$$

Let $\varepsilon > 0$ be fixed. Choose a number $a > 0$ such that $K\mathrm{e}^{-\mu a} < \varepsilon$. Analogously to the estimation (5.4), one shows that

$$\left\|\int_{-\infty}^{t-a} G(t,s) f_n(s)\,\mathrm{d}s + \int_{t+a}^{\infty} G(t,s) f_n(s)\,\mathrm{d}s\right\| \leqslant \varepsilon,$$

for every $n \in \mathbb{N}$. Using (5.5), we thus infer for an arbitrary $t \in \mathbb{R}$ that

$$\gamma\big(\{x_n(t)\}\big) \leqslant \varepsilon + \gamma\left(\left\{\int_{t-a}^{t+a} G(t,s) f_n(s)\,\mathrm{d}s\right\}\right).$$

Now, from assumption (F1), we get that

$$\big\|G(t,s) f_n(s)\big\| \leqslant \big\|G(t,s)\big\|_{\mathcal{L}(E)} m(s),$$

for a.a. $s \in \mathbb{R}$ and each $n \in \mathbb{N}$. Furthermore, using assumption (F2) and properties of γ (see Proposition 2.3 in Section 2.2), we see that the following estimate holds, namely

$$\begin{aligned}\gamma\big(\{G(t,s) f_n(s)\}\big) &\leqslant \big\|G(t,s)\big\|_{\mathcal{L}(E)} \gamma\big(\{f_n(s)\}\big)\\ &\leqslant \big\|G(t,s)\big\|_{\mathcal{L}(E)} g(s) h\big(\gamma\big(\{z_n(s)\}\big)\big)\\ &\leqslant \big\|G(t,s)\big\|_{\mathcal{L}(E)} g(s) h\big(\psi(\Omega)\big),\end{aligned}$$

for a.a. $s \in \mathbb{R}$. Hence, an application of Lemma 5.1 gives us

$$\gamma\big(\{x_n(t)\}\big) \leqslant \varepsilon + q h\big(\psi(\Omega)\big) \int_{t-a}^{t+a} \big\|G(t,s)\big\|_{\mathcal{L}(E)} g(s)\,\mathrm{d}s.$$

It follows that

$$\psi\big(T(\Omega)\big) \leqslant \varepsilon + q h\big(\psi(\Omega)\big) L,$$

and subsequently, since $\varepsilon > 0$ was arbitrary,

$$\psi\big(T(\Omega)\big) \leqslant q h\big(\psi(\Omega)\big) L. \tag{5.6}$$

Let us now assume that Ω is not relatively compact. Then $\psi(\Omega) > 0$ and so, by assumption (F2) and (5.6), we obtain

$$\psi\big(T(\Omega)\big) \leqslant \psi(\Omega).$$

Finally, observe that the estimate (5.6) also implies the quasi-compactness of the mapping T which subsequently justifies the compactness of the solution set to (5.3), as claimed.

Hence, we have verified all the assumptions of Theorem 4.4 (cf. also Definition 2.8) and we can establish the existence of a bounded solution to problem (5.1). □

REMARK 5.1. One can easily check that the assumption (F1) can be replaced by a weaker one, namely

$$\big\|F(t,x)\big\| \leqslant m(t) + K\|x\|, \quad \text{for a.a. } t \in \mathbb{R},\ x \in E, \tag{5.7}$$

where $K \geqslant 0$ is a sufficiently small constant and m is the same as above.

REMARK 5.2. If $A : E \to E$ is a linear, bounded operator whose spectrum does not intersect the imaginary axis, then the constant K in (5.7) can be easily taken as $K < 1/C(A)$, where

$$\sup_{t\in\mathbb{R}}\left|\int_{-\infty}^{\infty}\|G(t,s)\|\,\mathrm{d}s\right| \leqslant C(A), \quad G(t,s) = \begin{cases} \mathrm{e}^{-A(t-s)}P_-, & \text{for } t > s, \\ \mathrm{e}^{-A(t-s)}P_+, & \text{for } t < s \end{cases} \tag{5.8}$$

and P_-, P_+ stand for the corresponding spectral projections to the invariant subspaces of A.

REMARK 5.3. For $E = \mathbb{R}^n$ ($\Rightarrow$ (F2) holds automatically), condition (A2) is satisfied, provided there exists a projection matrix P ($P = P^2$) and constants $k > 0$, $\lambda > 0$ such that

$$\begin{cases} \left|X(t)PX^{-1}(s)\right| \leqslant k\exp\bigl(-\lambda(t-s)\bigr), & \text{for } s \leqslant t, \\ \left|X(t)(I-P)X^{-1}(s)\right| \leqslant k\exp\bigl(-\lambda(s-t)\bigr), & \text{for } t \leqslant s, \end{cases} \tag{5.9}$$

where $X(t)$ is the fundamental matrix of (5.2), satisfying $X(0) = I$, i.e., the unit matrix.

If A in (A1) is a piece-wise continuous and periodic, then it is well known that (5.9) takes place, whenever all the associated Floquet multiplies lie off the unit cycle. If A in (A1) is (continuous and) almost-periodic, then it is enough (see [93, p. 70]) that (5.9) holds only on a half-line $[t_0, \infty)$ or even on a sufficiently long finite interval.

Now, the information concerning the topological structure of solution sets in Section 4.2 will be employed for obtaining existence criteria, on the basis of general methods established in Section 4.1.

EXAMPLE 5.1. Consider the system

$$\begin{cases} \dot{x}_1 \in F_1(t,x_1,x_2)x_1 + F_2(t,x_1,x_2)x_2 + E_1(t,x_1,x_2), \\ \dot{x}_2 \in -F_2(t,x_1,x_2)x_1 + F_1(t,x_1,x_2)x_2 + E_2(t,x_1,x_2), \end{cases} \tag{5.10}$$

where $E_1, E_2, F_1, F_2 : [0,\infty) \times \mathbb{R}^2 \multimap \mathbb{R}^2$ are product-measurable u-Carathéodory maps.

Assume, furthermore, the existence of positive constants $\widetilde{E}_1$, $\widetilde{E}_2$, $\widetilde{F}_1$, $\widetilde{F}_2$, λ such that

$$\operatorname*{ess\,sup}_{t\in[0,\infty)}\Bigl[\sup_{|x_i|\leqslant D, i=1,2} F_1(t,x_1,x_2)\Bigr] \leqslant -\lambda, \tag{5.11}$$

$$\operatorname*{ess\,sup}_{t\in[0,\infty)}\Bigl[\sup_{|x_i|\leqslant D, i=1,2} \bigl|F_1(t,x_1,x_2)\bigr|\Bigr] \leqslant \widetilde{F}_1, \tag{5.12}$$

$$\operatorname*{ess\,sup}_{t\in[0,\infty)}\Bigl[\sup_{|x_i|\leqslant D, i=1,2} \bigl|F_2(t,x_1,x_2)\bigr|\Bigr] \leqslant \widetilde{F}_2, \tag{5.13}$$

$$\operatorname*{ess\,sup}_{t\in[0,\infty)}\Bigl[\sup_{|x_i|\leqslant D, i=1,2} \bigl|E_1(t,x_1,x_2)\bigr|\Bigr] \leqslant \widetilde{E}_1, \tag{5.14}$$

$$\operatorname*{ess\,sup}_{t\in[0,\infty)}\Big[\sup_{|x_i|\leqslant D, i=1,2}\big|E_2(t,x_1,x_2)\big|\Big]\leqslant \widetilde{E}_2, \tag{5.15}$$

where $D = 1/\lambda(\widetilde{E}_1, \widetilde{E}_2)$. Observe that, under the assumptions (5.12)–(5.15), we have

$$\operatorname*{ess\,sup}_{t\in[0,\infty)}\big|\dot{x}_i(t)\big|\leqslant D', \quad i=1,2, \tag{5.16}$$

where $D' = (\widetilde{F}_1 + \widetilde{F}_2)D + \max(\widetilde{E}_1, \widetilde{E}_2)$, so long as the solution $(x_1(t), x_2(t))$ of (5.10) satisfies

$$\sup_{t\in[0,\infty)}\big|x_i(t)\big|\leqslant D, \quad i=1,2. \tag{5.17}$$

Our aim is to prove, under the assumptions (5.11)–(5.15), the existence of a solution $x(t) = (x_1(t), x_2(t))$ satisfying

$$x(0)=0 \quad \text{and} \quad \sup_{t\in[0,\infty)}\big|x_i(t)\big|\leqslant D, \quad i=1,2. \tag{5.18}$$

In order to apply Corollary 4.1 for this goal, define two sets

$$Q := \Big\{r(t) = \big(r_1(t), r_2(t)\big)\in C\big([0,\infty)\times[0,\infty),\mathbb{R}^2\big) \;\big|\; \sup_{t\in[0,\infty)}\big|r_i(t)\big|\leqslant D,$$
$$i=1,2\Big\},$$
$$S := \big\{s(t) = \big(s_1(t), s_2(t)\big)\in C\big([0,\infty)\times[0,\infty),\mathbb{R}^2\big)\cap Q \;\big|\; \big|s_i(t)\big|\leqslant D't,$$
$$i=1,2\big\}$$

(observe that $s(0)=0$), where Q is a closed convex subset of $C([0,\infty)\times[0,\infty),\mathbb{R}^2)$ and S is a bounded closed subset of Q.

For $q(t) = (q_1(t), q_2(t))\in Q$, consider still the family of systems

$$\begin{cases} \dot{x}_1 = p_1(t)x_1 + p_2(t)x_2 + r_1(t), \\ \dot{x}_2 = -p_2(t)x_1 + p_1(t)x_2 + r_2(t), \end{cases} \tag{5.19}$$

where $p_1(t)\subset F_1(t,q(t))$, $p_2(t)\subset F_2(t,q(t))$, $r_1(t)\subset E_1(t,q(t))$, $r_2(t)\subset E_2(t,q(t))$ are measurable selections (see Proposition 2.5).

To show the solvability of (5.10) and (5.18) by means of Corollary 4.1, we need to verify that, for each $q\in Q$, the linearized system

$$\begin{cases} \dot{x}_1 \in F_1\big(t,q(t)\big)x_1 + F_2\big(t,q(t)\big)x_2 + E_1\big(t,q(t)\big), \\ \dot{x}_2 \in -F_2\big(t,q(t)\big)x_1 + F_1\big(t,q(t)\big)x_2 + E_2\big(t,q(t)\big) \end{cases} \tag{5.20}$$

has an R_δ-set of solutions in S.

It is well known that the general solution $x(t,0,\xi)$ of (5.19), where $\xi=(\xi_1,\xi_2)\in\mathbb{R}^2$, reads as follows:

$$\begin{aligned}
x_1(t,0,\xi) &= \left[\xi_1\cos\left(\int_0^t p_2(s)\,\mathrm{d}s\right)+\xi_2\sin\left(\int_0^t p_2(s)\,\mathrm{d}s\right)\right]\exp\int_0^t p_1(s)\,\mathrm{d}s\\
&\quad+\int_0^t\left[r_1(s)\exp\int_s^t p_1(w)\,\mathrm{d}w\cos\left(\int_s^t p_2(w)\,\mathrm{d}w\right)\right]\mathrm{d}s\\
&\quad+\int_0^t\left[r_2(s)\exp\int_s^t p_1(w)\,\mathrm{d}w\sin\left(\int_s^t p_2(w)\,\mathrm{d}w\right)\right]\mathrm{d}s,\\
x_2(t,0,\xi) &= \left[-\xi_1\sin\left(\int_0^t p_2(s)\,\mathrm{d}s\right)+\xi_2\cos\left(\int_0^t p_2(s)\,\mathrm{d}s\right)\right]\exp\int_0^t p_1(s)\,\mathrm{d}s\\
&\quad-\int_0^t\left[r_1(s)\exp\int_s^t p_1(w)\,\mathrm{d}w\sin\left(\int_s^t p_2(w)\,\mathrm{d}w\right)\right]\mathrm{d}s\\
&\quad+\int_0^t\left[r_2(s)\exp\int_s^t p_1(w)\,\mathrm{d}w\cos\left(\int_s^t p_2(w)\,\mathrm{d}w\right)\right]\mathrm{d}s.
\end{aligned}$$

Because of (5.11), (5.14) and (5.15), we get

$$\begin{aligned}
&\sup_{t\in[0,\infty)}\left|\int_0^t\left[r_i(s)\,\mathrm{d}s\exp\int_s^t p_1(w)\,\mathrm{d}w\cos\left(\int_s^t p_2(w)\,\mathrm{d}w\right)\right]\mathrm{d}s\right|\\
&\quad\leqslant\widetilde{E}_i\sup_{t\in[0,\infty)}\int_0^t\exp\left[-\int_s^t\left|p_1(w)\right|\mathrm{d}w\right]\mathrm{d}s\leqslant\frac{\widetilde{E}_i}{\lambda},\\
&\sup_{t\in[0,\infty)}\left|\int_0^t\left[r_i(s)\,\mathrm{d}s\exp\int_s^t p_1(w)\,\mathrm{d}w\sin\left(\int_s^t p_2(w)\,\mathrm{d}w\right)\right]\mathrm{d}s\right|\\
&\quad\leqslant\widetilde{E}_i\sup_{t\in[0,\infty)}\int_0^t\exp\left[-\int_s^t\left|p_1(w)\right|\mathrm{d}w\right]\mathrm{d}s\leqslant\frac{\widetilde{E}_i}{\lambda},
\end{aligned}$$

for $i=1,2$, and subsequently we arrive at

$$\sup_{t\in[0,\infty)}\left|x_i(t,0,\xi)\right|\leqslant|\xi_1|+|\xi_2|+D,\quad i=1,2,\tag{5.21}$$

and $x(0,0,\xi)=\xi$.

According to Theorem 4.9 (see also Proposition 2.5 and estimate (5.21)), problem (5.17) $\cap$ (5.20) has the R_δ-set of solutions $x(t,0,0)$. Moreover, in view of the indicated implication ((5.17) $\Rightarrow$ (5.16)), these solutions $x(t,0,0)$ belong obviously to S, for every $q\in Q$, as required.

Thus, it follows from Corollary 4.1 that problem (5.10) $\cap$ (5.18) has, under the assumptions (5.11)–(5.15), at least one solution.

If the inequality (5.11) or both inequalities (5.14) and (5.15) are sharp, then the same conclusion is true for $x(0) = 0$ in (5.18) replaced by $x(0) = \xi$, where $|\xi|$ is sufficiently small. For bigger values of $|\xi|$, the above assumptions can be appropriately modified as well.

THEOREM 5.2. *Consider the target problem*

$$\begin{cases} \dot{x}(t) \in F\big(t, x(t)\big), & \textit{for a.a. } t \in [0, \infty), \\ \lim\limits_{t\to\infty} x(t) = x_\infty \in \mathbb{R}^n \end{cases} \tag{5.22}$$

and assume that $F : [0, \infty) \times \mathbb{R}^n \multimap \mathbb{R}^n$ *is a product-measurable u-Carathéodory map. Let, furthermore, there exist a globally integrable function* $\alpha : [0, \infty) \to [0, \infty)$ *and a positive constant* β *such that, for every* $x \in \mathbb{R}^n$ *and for a.a.* $t \in [0, \infty)$, *we have* $|F(t, x)| \leqslant \alpha(t)(\beta + |x|)$, *where* $|F(t, x)| = \sup\{|y| \mid y \in F(t, x)\}$, *and* $\int_0^\infty \alpha(t)\,\mathrm{d}t < \infty$. *Then problem* (5.22) *admits a* (*bounded*) *solution, for every* $x_\infty \in \mathbb{R}^n$.

PROOF. It is convenient to consider, instead of problem (5.22), the equivalent problem

$$\begin{cases} \dot{x}(t) \in G\big(t, x(t)\big), & \text{for a.a. } t \in [0, \infty), \\ \lim\limits_{t\to\infty} x(t) = x_\infty \in \mathbb{R}^n, \end{cases} \tag{5.23}$$

where

$$G(t, x) = \begin{cases} F(t, x), & \text{for } |x| \leqslant D \text{ and } t \in [0, \infty), \\ F\left(t, D\dfrac{x}{|x|}\right), & \text{for } |x| \geqslant D \text{ and } t \in [0, \infty), \end{cases}$$

$$D \geqslant \big(|x_\infty| + AB\big)\exp A, \quad A = \int_0^\infty \alpha(t)\,\mathrm{d}t < \infty.$$

Moreover, there certainly exists a positive constant γ such that

$$|x_0| + \big|G(t, x)\big| \leqslant |x_0| + A(B + D) \leqslant \gamma,$$
$$\text{for all } x \in \mathbb{R}^n \text{ and a.a. } t \in [0, \infty). \tag{5.24}$$

Besides problem (5.23), consider still a one-parameter family of linear problems

$$\begin{cases} \dot{x}(t) \in G\big(t, q(t)\big), & \text{for a.a. } t \in [0, \infty),\ q \in Q, \\ x \in Q \cap S, \end{cases} \tag{5.25}$$

where

$$S = \left\{x \in C\big([0, \infty), \mathbb{R}^n\big) \;\Big|\; \lim_{t\to\infty} x(t) = x_\infty\right\},$$

$$Q = \left\{q \in C\big([0, \infty), \mathbb{R}^n\big) \;\Big|\; \big|q(t)\big| \leqslant |x_\infty| + A(B + D),\ \text{for } t \geqslant 0\right\}.$$

Consider the set

$$S_1 = \left\{ x \in Q \,\middle|\, \left|x(t) - x_\infty\right| \leqslant (B + D) \int_t^\infty \alpha(s)\,\mathrm{d}s, \text{ for } t \geqslant 0 \right\} \subset S.$$

It is evident that S_1 is a closed subset of S and all solutions to problem (5.25) belong to S_1.

At first, we assume that $G = g$ is single-valued. Then we have a single-valued continuous operator

$$T(q) = x_\infty + \int_\infty^t g\big(s, q(s)\big)\,\mathrm{d}s, \quad \text{for every } q \in Q.$$

Thus, to apply Corollary 4.2, only the condition $T(Q) \subset Q$ should be verified. But this follows immediately from (5.24), because

$$\begin{aligned} \sup_{t\in[0,\infty)} \left| x_\infty + \int_\infty^t G\big(s, q(s)\big)\,\mathrm{d}s \right| &\leqslant |x_\infty| + \int_0^\infty \left|G\big(t, q(t)\big)\right|\mathrm{d}t \\ &\leqslant |x_\infty| + (B + D)\int_0^\infty \alpha(t)\,\mathrm{d}t \\ &= |x_\infty| + A(B + D) < \infty. \end{aligned} \tag{5.26}$$

By Corollary 4.2, we obtain a solution to the problem with g as a right-hand side. This existence result can be used jointly with Theorem 4.12 which is needed to prove our statement in a general case. In fact, in view of the (just proved) existence result and Theorem 4.12 (cf. also Proposition 2.5), the map T which assigns to every $q \in Q$ the set of solutions to the linear problem (5.25), has nonempty, acyclic sets of values. Once more, we use Corollary 4.2, obtaining a solution to problem (5.22), and the proof is complete. □

5.2. *Solvability of boundary value problems with linear conditions*

Now, we shall deal with boundary value problems of the type

$$\begin{cases} \dot{x}(t) + A(t)x(t) \in F\big(t, x(t)\big), & \text{for a.a. } t \in [0, \tau], \\ Lx = \Theta, \end{cases} \tag{5.27}$$

where

(i) $A : [0, \tau] \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ is a measurable linear operator such that $|A(t)| \leqslant \gamma(t)$, for all $t \in [0, \tau]$ and some integrable function $\gamma : [0, \tau] \to [0, \infty)$,

(ii) the associated homogeneous problem

$$\begin{cases} \dot{x}(t) + A(t)x(t) = 0, & \text{for a.a. } t \in [0, \tau], \\ Lx = 0 \end{cases}$$

has only the trivial solution,

(iii) $F:[0,\tau]\times\mathbb{R}^n \multimap \mathbb{R}^n$ is a u-Carathéodory mapping with nonempty, compact and convex values (cf. Definition 2.10),

(iv) there are two nonnegative Lebesgue-integrable functions $\delta_1, \delta_2 : [0,\tau] \to [0,\infty)$ such that

$$\big|F(t,x)\big| \leqslant \delta_1(t) + \delta_2(t)|x|, \quad \text{for a.a. } t \in [0,\tau] \text{ and all } x \in \mathbb{R}^n,$$

where $|F(t,x)| = \sup\{|y| \mid y \in F(t,x)\}$.

Applying Theorem 4.13 (cf. also Proposition 2.5) to replace condition (i) in Corollary 4.3 for (5.27), we can immediately give

PROPOSITION 5.1. *Consider problem* (5.27) *with* (i)–(iv) *above and let* $G:[0,\tau]\times\mathbb{R}^n\times\mathbb{R}^n\times[0,1]\to\mathbb{R}^n$ *be a product-measurable u-Carathéodory map* (*cf. Definition* 2.10) *such that*

$$G(t,c,c,1) \subset F(t,c), \quad \textit{for all } (t,c) \in [0,\tau]\times\mathbb{R}^n.$$

Assume, *furthermore*, *that*

(v) *there exists a* (*bounded*) *retract* Q *of* $C([0,\tau],\mathbb{R}^n)$ *such that* $Q \setminus \partial Q$ *is nonempty* (*open*) *and such that* $G(t,x,q(t),\lambda)$ *is Lipschitzian in* x *with a sufficiently small Lipschitz constant* (*see Theorem* 4.13), *for a.a.* $t\in[0,\tau]$ *and each* $(q,\lambda)\in Q\times[0,1]$,

(vi) *there exists a Lebesgue integrable function* $\alpha:[0,\tau]\to[0,\infty)$ *such that*

$$\big|G\big(t,x(t),q(t),\lambda\big)\big| \leqslant \alpha(t), \quad \textit{a.e. in } [0,\tau],$$

for any $(x,q,\lambda)\in\Gamma_T$ (*i.e. from the graph of* T), *where* T *denotes the set-valued map which assigns*, *to any* $(q,\lambda)\in Q\times[0,1]$, *the set of solutions of*

$$\begin{cases} \dot{x}(t) + A(t)x(t) \in G\big(t,x(t),q(t),\lambda\big), & \textit{for a.a. } t\in[0,1],\\ Lx = \Theta, \end{cases}$$

(vii) $T(Q\times\{0\})\subset Q$ *holds and* ∂Q *is fixed point free w.r.t.* T, *for every* $(q,\lambda)\in Q\times[0,1]$.

Then problem (5.27) *has a solution.*

REMARK 5.4. Rescaling t in (5.27), the interval $[0,\tau]$ can be obviously replaced in Proposition 5.1 by any compact interval J, e.g., $J=[-m,m]$, $m\in\mathbb{N}$. Therefore, the second part of Proposition 4.5 can be still applied for obtaining an entirely bounded solution.

EXAMPLE 5.2. Consider problem (5.27). Assume that conditions (i)–(iv) are satisfied. Taking (for a product-measurable $F:[0,\tau]\times\mathbb{R}^n\multimap\mathbb{R}^n$)

$$G\big(t,q(t)\big) = F\big(t,q(t)\big), \quad \text{for } q\in Q,$$

where $Q = \{\mu \in C([0,\tau],\mathbb{R}^n) \mid \max_{t\in[0,\tau]} |\mu(t)| \leqslant D\}$ and $D > 0$ is a sufficiently big constant which will be specified below, we can see that (v) holds trivially. Furthermore, according to (iv), we get

$$\bigl|G\bigl(t, q(t)\bigr)\bigr| \leqslant \delta_1(t) + \delta_2(t)D, \quad \text{for a.a. } t \in [0,\tau], \tag{5.28}$$

i.e. (vi) holds as well with $\alpha(t) = \delta_1(t) + \delta_2(t)D$. At last, the associated linear problem

$$\begin{cases} \dot{x}(t) + A(t)x(t) \in F\bigl(t, q(t)\bigr), & \text{for a.a. } t \in [0,\tau], \\ Lx = \Theta \end{cases}$$

has, according to Theorem 4.13, for every $q \in Q$, an R_δ-set of solutions of the form

$$T(q) = \int_0^\tau H(t,s) f\bigl(s, q(s)\bigr)\,\mathrm{d}s,$$

where H is the related Green function and $f \subset F$ is a measurable selection (see again Proposition 2.5).

Therefore, in order to apply Proposition 5.1 for the solvability of (5.27), we only need to show (cf. (vii)) that $T(Q) \subset Q$ (and that ∂Q is fixed point free w.r.t. T, for every $q \in Q$, which is, however, not necessary here). Hence, in view of (5.28), we have that

$$\begin{aligned} \max_{t\in[0,\tau]} \bigl|T(q)\bigr| &= \max_{t\in[0,\tau]} \left| \int_0^\tau H(t,s) f\bigl(s, q(s)\bigr)\,\mathrm{d}s \right| \\ &\leqslant \max_{t\in[0,\tau]} \int_0^\tau \bigl|H(t,s)\bigr| \bigl(\delta_1(s) + \delta_2(s)D\bigr)\,\mathrm{d}s \\ &= \max_{t,s\in[0,\tau]} \bigl|H(t,s)\bigr| \left[\int_0^\tau \delta_1(t)\,\mathrm{d}t + D \int_0^\tau \delta_2(t)\,\mathrm{d}t \right], \end{aligned}$$

and subsequently the above requirement holds for

$$D \geqslant \frac{\max_{t,s\in[0,\tau]} |H(t,s)| \int_0^\tau \delta_1(t)\,\mathrm{d}t}{1 - \max_{t,s\in[0,\tau]} |H(t,s)| \int_0^\tau \delta_2(t)\,\mathrm{d}t},$$

provided

$$\int_0^\tau \delta_2(t)\,\mathrm{d}t < \frac{1}{\max_{t,s\in[0,\tau]} |H(t,s)|}.$$

(Observe that for D strictly bigger than the above quantity, ∂Q becomes fixed point free.)

5.3. *Existence of periodic and anti-periodic solutions*

Now, consider the following special cases of Floquet boundary value problems in a reflexive Banach space E:

$$\begin{cases} \dot{x}(t) + A(t)x(t) \in F\big(t, x(t)\big), & \text{for a.a. } t \in [0, \tau], \\ x(\tau) = Mx(0), & \text{where } M = \mathrm{id} \text{ or } M = -\,\mathrm{id}. \end{cases} \tag{5.29}$$

Besides (A1), (A2), assume still

$$A(t) \equiv A(t+\tau) \quad \text{and} \quad F(t,x) \equiv F(t+\tau, x) \text{ or } F(t,x) \equiv -F(t+\tau, -x)$$
$$\text{for some } \tau > 0, \tag{5.30}$$

and, instead of (F1), that only

$$\big\| F(t,x) \big\| \leqslant \alpha_0(t) + \alpha_1(t)\|x\| \tag{5.31}$$

holds for a u-Carathéodory map $F : [0, \tau] \times E \multimap E$, for a.a. $t \in [0, \tau]$ and every $x \in E$, where $\alpha_0, \alpha_1 \in L^1([0, \tau])$. Then one can check, as in the proof of Theorem 5.1, that the solution operator $T : Q \times [0, 1] \multimap E$, where $Q = \{q \in C(\mathbb{R}, E) \mid q(t) \equiv q(t+\tau) \text{ or } q(t) \equiv -q(t+\tau)\}$, associated with the fully linearized problem

$$\begin{cases} \dot{x}(t) + A(t)x(t) \in \lambda F\big(t, q(t)\big), & \text{for a.a. } t \in [0, \tau],\ \lambda \in [0, 1], \\ x(\tau) = Mx(0), & \text{where } M = \mathrm{id} \text{ or } M = -\,\mathrm{id}, \end{cases} \tag{5.32}$$

is condensing and that the set of (bounded, after τ-periodic or 2τ-periodic prolongation) solutions is convex and compact, provided an analogy of (F2) holds.

Therefore, taking $G(t, c, c, 0) \equiv -A(t)c$, Corollary 4.5 can be simplified as follows.

COROLLARY 5.1. *Assume that conditions* (5.30) *and* (5.31) *hold*, *jointly with*

(i) $A : [0, \tau] \to \mathcal{L}(E)$ *is strongly measurable* (*cf. Definition* 2.9) *and Bochner integrable on the interval* $[0, \tau]$.

(ii) *The linear equation* $\dot{x}(t) + A(t)x(t) = 0$ *admits a regular exponential dichotomy. Denote by G the related principal Green's function.*

(iii) $F : [0, \tau] \times E \multimap E$ *is a u-Carathéodory map with nonempty*, *compact and convex values.*

Assume, *furthermore*, *that there exists a nonempty*, *bounded*, *closed*, *convex subset* Q *of* $\{q \in C(\mathbb{R}, E) \mid q(t) \equiv q(t+\tau)$ *or* $q(t) \equiv -q(t+\tau)\}$ *with nonempty interior* $\operatorname{int} Q$ *such that*

(iv) $\gamma(F(t, \Omega)) \leqslant g(t)h(\gamma(\Omega))$, *for a.a.* $t \in [0, \tau]$, *and each* $\Omega \subset \{q(t) \in E \mid t \in [0, \tau],\ q \in Q\}$, *where* g, h *are positive functions*, g *is measurable*, h *is nondecreasing such that*

$$L := \sup\left\{ \int_0^\tau \big\| G(t,s) \big\|_{\mathcal{L}(E)} g(s)\, \mathrm{d}s \;\middle|\; t \in [0, \tau] \right\} < \infty$$

and $gh(t)L < t$, *for each* $t \in [0, \tau]$, *with a constant* $q = 1$, *if* E *is still separable*, *and* $q = 2$, *in general case*.

(v) $\{0\} \subset \operatorname{int} Q$ *and the boundary* ∂Q *of* Q *is fixed point free w.r.t.* T, *for every* $(q, \lambda) \in Q \times (0, 1]$, *where* T *is the map assigning*, *to any* $(q, \lambda) \in Q \times [0, 1]$, *the set of solutions of the fully linearized problem* (5.32).

Then problem (5.29) *admits a solution*.

REMARK 5.5. Obviously, under the assumptions (i), (ii), (iii), (5.7), (5.30) and (F2) in Section 5.1, problem (5.29) admits a solution, provided K in (5.7) is sufficiently small.

For periodic and anti-periodic problems, Proposition 5.1 can be easily simplified as follows (cf. Remark 5.3).

COROLLARY 5.2. *Consider problem*

$$\begin{cases} \dot{x}(t) + A(t)x(t) \in F\big(t, x(t)\big), & \text{for a.a. } t \in [0, \tau], \\ x(0) = x(\tau), \end{cases}$$

where $F(t, x) \equiv F(t + \tau, x)$ *satisfies conditions* (iii) *and* (iv) *in Proposition* 5.1. *Let* $G : [0, \tau] \times \mathbb{R}^n \times \mathbb{R}^n \times [0, 1] \multimap \mathbb{R}^n$ *be a product-measurable u-Carathéodory map such that*

$$G(t, c, c, 1) \subset F(t, c), \quad \text{for all } (t, c) \in [0, \tau] \times \mathbb{R}^n.$$

Assume that A *is a piece-wise continuous* (*single-valued*) *bounded* τ*-periodic* $(n \times n)$*-matrix whose Floquet multipliers lie off the unit cycle*, *jointly with* (v)–(vii) *in Proposition* 5.1, *where* $Lx = x(0) - x(\tau)$ *and* $\Theta = 0$. *Then the inclusion* $\dot{x} + A(t)x \in F(t, x)$ *admits a* τ*-periodic solution*.

COROLLARY 5.3. *Consider problem*

$$\begin{cases} \dot{x}(t) \in F\big(t, x(t)\big), & \text{for a.a. } t \in [0, \tau], \\ x(0) = -x(\tau), \end{cases}$$

where $F(t, x) \equiv -F(t + \tau, -x)$ *satisfies conditions* (iii) *and* (iv) *in Proposition* 5.1. *Let* $G : [0, \tau] \times \mathbb{R}^n \times \mathbb{R}^n \times [0, 1] \multimap \mathbb{R}^n$ *be a product-measurable u-Carathéodory map such that*

$$G(t, c, c, 1) \subset F(t, c), \quad \text{for all } (t, c) \in [0, \tau] \times \mathbb{R}^n.$$

Assume that (v)–(vii) *in Proposition* 5.1 *hold*, *where* $Lx = x(0) + x(\tau)$ *and* $\Theta = 0$. *Then the inclusion* $\dot{x} \in F(t, x)$ *admits a* 2τ*-periodic solution*.

In the case of ODEs, Corollary 5.3 can be still improved, in view of Theorem 4.14, where $\lambda = 1$ and $\xi = 0$, as follows.

COROLLARY 5.4. *Consider problem*

$$\begin{cases} \dot{x}(t) = f\big(t, x(t)\big), & \text{for a.a. } t \in [0, \tau], \\ x(0) = -x(\tau), \end{cases}$$

where $f(t, x) \equiv -f(t+\tau, -x)$ *is a Carathéodory function. Let* $g : [0, \tau] \times \mathbb{R}^n \times \mathbb{R}^n \times [0, 1] \to \mathbb{R}^n$ *be a Carathéodory function such that*

$$g(t, c, c, 1) = f(t, c), \quad \text{for all } (t, c) \in [0, \tau] \times \mathbb{R}^n.$$

Assume that

(i) *there exists a bounded retract* Q *of* $C([0, \tau], \mathbb{R}^n)$ *such that* $Q \setminus \partial Q$ *is nonempty* (*open*) *and such that* $g(t, x, q(t), \lambda)$ *satisfies*

$$\big|g\big(t, x, q(t), \lambda\big) - g\big(t, y, q(t), \lambda\big)\big| \leqslant p(t)|x - y|, \quad x, y \in \mathbb{R}^n$$

for a.a. $t \in [0, \tau]$ *and each* $(q, \lambda) \in Q \times [0, 1]$, *where* $p : [0, \tau] \to [0, \infty)$ *is a Lebesgue integrable function with* (*see* (4.31))

$$\int_0^\tau p(t)\, \mathrm{d}t \leqslant \pi,$$

(ii) *there exists a Lebesgue integrable function* $\alpha : [0, \tau] \to [0, \infty)$ *such that*

$$\big|g\big(t, x(t), q(t), \lambda\big)\big| \leqslant \alpha(t), \quad \text{a.e. in } [0, \tau],$$

for any $(x, q, \lambda) \in \Gamma_T$, *where* T *denotes the set-valued map which assigns*, *to any* $(q, \lambda) \in Q \times [0, 1]$, *the set of solutions of*

$$\begin{cases} \dot{x}(t) = g\big(t, x(t), q(t), \lambda\big), & \text{for a.a. } t \in [0, \tau], \\ x(0) = -x(\tau), \end{cases}$$

(iii) $T(Q \times \{0\}) \subset Q$ *holds and* ∂Q *is fixed point free w.r.t.* T, *for every* $(q, \lambda) \in Q \times [0, 1]$.

Then the equation $\dot{x} = f(t, x)$ *admits a* 2τ*-periodic solution.*

REMARK 5.6. Since in Corollaries 5.2 and 5.3 the associated homogeneous problems (cf. (ii) at the beginning of Section 5.2) have obviously only the trivial solution, the requirement $T(Q \times \{0\}) \subset Q$ reduces to $\{0\} \subset Q$, provided $G(t, x, q, \lambda) = \lambda G(t, x, \lambda)$, $\lambda \in [0, 1]$.

REMARK 5.7. The requirement concerning a fixed point free boundary ∂Q of Q in Proposition 5.1, Corollaries 5.1, 5.2, 5.3 and 5.4 can be verified by means of bounding (Liapunov-like) functions (see [11,32–34] and cf. [22, Chapter III.8]).

EXAMPLE 5.3. Consider the anti-periodic problem

$$\begin{cases} \dot{x} \in F_1(t,x) + F_2(t,x), \\ x(a) = -x(b), \end{cases} \tag{5.33}$$

where $x = (x_1, \ldots, x_n)$, $F = F_1 + F_2 = (f_{11}, \ldots, f_{1n}) + (f_{21}, \ldots, f_{2n})$, $F_1, F_2 : [a,b] \times \mathbb{R}^n \multimap \mathbb{R}^n$ are globally upper semicontinuous multivalued functions with nonempty, convex, compact values which are bounded in $t \in [a,b]$, for every $x \in \mathbb{R}^n$, and linearly bounded in $x \in \mathbb{R}^n$, for all $t \in [a,b]$.

Assume, furthermore, that there exist positive constants R_i, $i = 1, \ldots, n$ such that

$$\left| f_{1i}\big(t, x(\pm R_i)\big) \right| > \max_{\substack{t \in [a,b] \\ x \in \overline{K}}} \left| f_{2i}(t,x) \right|, \quad i = 1, \ldots, n,\ t \in (a,b),$$

where $x(\pm R_i) = (x_1, \ldots, x_{i-1}, \pm R_i, x_{i+1}, \ldots, x_n)$, $|x_j| \leqslant R_j$ and $K = \{x \in \mathbb{R}^n \mid |x_i| < R_i, i = 1, \ldots, n\}$,

$$\big[f_{1i}\big(a, x(\pm R_i)\big) + f_{2i}(a,y) \big] \cdot \big[f_{1i}\big(b, -x(\pm R_i)\big) + f_{2i}(b,z) \big] < 0,$$
$$i = 1, \ldots, n,$$

where $x, y, z \in \overline{K}$,

$F_1(t, \cdot)$ is Lipschitzian with a sufficiently small constant L,

for every $t \in [a,b]$; in the single-valued case, when $F \in C([a,b] \times \mathbb{R}^n, \mathbb{R}^n)$, it is enough to take $L \leqslant \pi/(b-a)$ (see condition (i) in Corollary 5.4).

It can be checked (see [32]; cf. [22, Example 8.40 in Chapter III.8]) that all assumptions of Corollary 5.3 (in the single-valued case, of Corollary 5.4) are satisfied, and so problem (5.33) admits a solution.

6. Multiplicity results

6.1. *Several solutions of initial value problems*

Let us recall that in order to apply Theorem 4.3, the following main steps have to be taken:

(i) the R_δ-structure of the solution set to (4.2) must be verified,

(ii) the inclusion $\overline{T(Q)} \subset S$ or, most preferably, $\overline{T(Q)} \subset Q \cap S$ must be guaranteed, together with the retractibility of T in the sense of Definition 4.1,

(iii) $N(r|_{T(Q)} \circ T)$ must be computed.

For initial value problems, condition (i) can be easily verified, provided G is still product-measurable. In fact, since u-Carathéodory inclusions (cf. Definition 2.10) with product-measurable right-hand sides G possess (according to Theorem 4.9; cf. also Proposition 2.5), for each $q \in Q \subset C(I, \mathbb{R}^n)$, an R_δ-set of solutions $x(\cdot, x_0)$ with $x(0, x_0) = x_0$, for every $x_0 \in \mathbb{R}^n$, such a requirement can be, in Theorem 4.3 with $S := \{x \in \mathrm{AC}_{\mathrm{loc}}(I, \mathbb{R}^n) \mid$

$x(0, x_0) = x_0\}$, simply avoided. Moreover, if Q is still compact and such that $\overline{T(Q)} \subset Q \cap S$, then (see Remark 4.3) $\pi_1(Q)$ need not be abelian and finitely generated.

Thus, Theorem 4.3 simplifies, for initial value problems, as follows:

COROLLARY 6.1. *Let $G : I \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ be a product-measurable u-Carathéodory mapping, where $I = [0, \infty)$ or $I = [0, \tau]$, $\tau \in (0, \infty)$. Assume, furthermore, that there exists a (nonempty) compact, connected subset $Q \subset C(I, \mathbb{R}^n)$ which is a neighbourhood retract of $C(I, \mathbb{R}^n)$ such that $|G(t, x(t), q(t))| \leqslant \mu(t)(|x| + 1)$ holds, for every $(t, x, q) \in I \times \mathbb{R}^n \times Q$. Let the initial value problem*

$$\begin{cases} \dot{x}(t) \in G\big(t, x(t), q(t)\big), & \text{for a.a. } t \in I, \\ x(0) = x_0 \end{cases}$$

have, for each $q \in Q$, a nonempty set of solutions $T(q)$ such that $\overline{T(Q)} \subset Q \cap S$, where $S := \{x \in \mathrm{AC}_{\mathrm{loc}}(I, \mathbb{R}^n) \mid x(0) = x_0\}$. Then the original initial value problem

$$\begin{cases} \dot{x}(t) \in F\big(t, x(t)\big), & \text{for a.a. } t \in I, \\ x(0) = x_0 \end{cases}$$

admits at least $N(T)$ solutions, provided $G(t, c, c) \subset F(t, c)$ holds a.e. on I, for any $c \in \mathbb{R}^n$.

EXAMPLE 6.1. Consider the scalar ($n = 1$) initial value problem with $x_0 = 0$ and $I = [0, \tau]$, $\tau > 0$. Letting

$$Q := \big\{q \in \mathrm{AC}\big([0, \tau], \mathbb{R}\big) \mid q(0) = 0 \text{ and } \delta_2 \leqslant \dot{q}(t) \leqslant \delta_1 \\ \text{or } -\delta_1 \leqslant \dot{q}(t) \leqslant -\delta_2, \text{ for a.a. } t \in [0, \tau]\big\},$$

where $0 < \delta_2 < \delta_1$ are suitable constants, Q can be easily verified to be a disjoint (!) union of two convex, compact sets, and consequently Q is a compact ANR, i.e. also a neighbourhood retract of $C([0, \tau], \mathbb{R})$. Unfortunately, Q is disconnected which excludes the direct application of Corollary 6.1.

Nevertheless, e.g., the inclusion

$$\dot{x}(t) \in \delta \,\mathrm{Sgn}\big(x(t)\big), \quad \text{for a.a. } t \in [0, \tau], \ \delta > 0, \tag{6.1}$$

where

$$\mathrm{Sgn}(x) = \begin{cases} -1, & \text{for } x \in (-\infty, 0), \\ [-1, 1], & \text{for } x = 0, \\ 1, & \text{for } x \in (0, \infty), \end{cases}$$

admits obviously two classical solutions $x_1(t) = \delta t$ with $x_1(0) = 0$ and $x_2(t) = -\delta t$ with $x_2(0) = 0$, satisfying the given inclusion everywhere.

The linearized inclusion

$$\dot{x}(t) \in \delta \operatorname{Sgn}\bigl(q(t)\bigr), \quad \text{for a.a. } t \in [0, \tau],\ \delta > 0,$$

possesses, for each $q \in Q$, either the solution $x_1(t) = \delta t$ with $x_1(0) = 0$ or $x_2(t) = -\delta t$ with $x_2(0) = 0$, depending on $\operatorname{sgn}(q(t))$, provided $\delta_2 \leqslant \delta \leqslant \delta_1$. Observe that there are no more solutions, for each $q \in Q$. Thus, we also have $\overline{T(Q)} \subset Q \cap S$ (i.e. condition (ii)), where $S := \{x \in \mathrm{AC}_{\mathrm{loc}}(I, \mathbb{R}^n) \mid x(0) = 0\}$.

The only handicap is related to the mentioned disconnectedness of Q. However, since $T : Q \to Q$, where

$$T(q) = \begin{cases} \delta t, & \text{for } q \geqslant 0, \\ -\delta t, & \text{for } q \leqslant 0, \end{cases}$$

is obviously single-valued, the application of the multivalued Nielsen theory, in the proof of Theorem 4.3 (and subsequently of Corollary 6.1), can be replaced by the single-valued one, where $Q \in \mathrm{ANR}$ can be disconnected (see the definition of the Nielsen number at the beginning of Section 3.2). We can, therefore, conclude, on the basis of the appropriately modified Corollary 6.1, that the original inclusion (6.1) admits at least $N(T) = 2$ solutions $x(t)$ with $x(0) = 0$, as observed by the direct calculations. In fact, it must therefore have, according to Theorem 4.5, a nontrivial R_δ-set of infinitely many piece-wise linear solutions $x(t)$ with $x(0) = 0$. The computation of $N(T) = 2$ (i.e. condition (iii)) is trivial, because $Q = Q^+ \cup Q^-$, where

$$\begin{aligned} (\mathrm{AR} \ni)\ Q^+ := \bigl\{ & q \in \mathrm{AC}\bigl([0, \tau], \mathbb{R}\bigr) \mid q(0) = 0 \text{ and} \\ & \delta_2 \leqslant \dot{q}(t) \leqslant \delta_1, \text{ for a.a. } t \in [0, \tau] \bigr\}, \\ (\mathrm{AR} \ni)\ Q^- := \bigl\{ & q \in \mathrm{AC}\bigl([0, \tau], \mathbb{R}\bigr) \mid q(0) = 0 \text{ and} \\ & -\delta_1 \leqslant \dot{q}(t) \leqslant -\delta_2, \text{ for a.a. } t \in [0, \tau] \bigr\}, \end{aligned}$$

and so for the computation of the generalized Lefschetz numbers we have $\Lambda(T|_{Q^+}) = \Lambda(T|_{Q^-}) = 1$, where $T|_{Q^+} : Q^+ \to Q^+$ and $T|_{Q^-} : Q^- \to Q^-$.

The same is obviously true for the inclusion

$$\dot{x}(t) \in \bigl[\delta + f\bigl(t, x(t)\bigr)\bigr] \operatorname{Sgn}\bigl(x(t)\bigr), \quad \text{for a.a. } t \in [0, \tau],\ \delta > 0,$$

where $f : [0, \tau] \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory and locally Lipschitz function in x, for a.a. $t \in [0, \tau]$, such that $\delta_2 \leqslant \delta + f(t, x) \leqslant \delta_1$, for some $0 < \delta_2 < \delta_1$, because again $T : Q \to Q$.

Of course, we could arrive at the same conclusion even without an explicit usage of the Nielsen theory arguments, just through double application (separately on Q^+ and Q^-) of Theorem 3.3 (i.e. of the Lefschetz theory arguments).

REMARK 6.1. In view of Example 6.1, it is more realistic to suppose in Corollary 6.1 that (at least, for $n = 1$) the solution operator T is single-valued and that Q can be disconnected and not necessarily compact. Naturally, the first requirement seems to be rather associated with differential equations than inclusions.

6.2. *Several periodic and bounded solutions*

Now, Theorem 4.3 will be applied to boundary value problems associated with semilinear differential inclusions.

Hence, consider the problem

$$\begin{cases} \dot{x} + A(t)x \in F(t,x), \\ Lx = \Theta. \end{cases} \tag{6.2}$$

Since the composed multivalued function $F(t,q(t))$, where $F: J \times \mathbb{R}^n \multimap \mathbb{R}^n$ is a product-measurable u-Carathéodory mapping with nonempty, compact and convex values and $q \in C(J,\mathbb{R}^n)$, is, according to Proposition 2.5, measurable, we can also employ Theorem 4.13 to the associated linearized system

$$\begin{cases} \dot{x} + A(t)x \in F\big(t,q(t)\big), \\ Lx = \Theta, \end{cases} \tag{6.3}$$

provided

$$\big|F(t,x)\big| \leqslant \mu(t)\big(|x|+1\big), \tag{6.4}$$

where $\mu: J \to [0,\infty)$ is a suitable (locally) Lebesgue integrable bounded function.

We can immediately give

THEOREM 6.1. *Consider boundary value problem* (6.2) *on a compact interval J. Assume that $A: J \to \mathbb{R}^{n^2}$ is a single-valued continuous $(n \times n)$-matrix and $F: J \times \mathbb{R}^n \multimap \mathbb{R}^n$ is a product-measurable u-Carathéodory mapping with nonempty, compact and convex values satisfying* (6.4). *Furthermore, let $L: C(J,\mathbb{R}^n) \to \mathbb{R}^n$ be a linear operator such that the homogeneous problem*

$$\begin{cases} \dot{x} + A(t)x = 0, \\ Lx = 0 \end{cases}$$

has only the trivial solution on J. Then the original problem (6.2) *has at least $N(r|_{T(Q)} \circ T(\cdot))$ solutions* (*for the definition of the Nielsen number N, see Definition* 3.2 *in Section* 3.2), *provided there exists a closed connected subset Q of $C(J,\mathbb{R}^n)$ with a finitely generated abelian fundamental group such that*

(i) *$T(Q)$ is bounded*,

(ii) *$T(q)$ is retractible onto Q with a retraction r in the sense of Definition* 4.1,

(iii) $\overline{T(Q)} \subset \{x \in \mathrm{AC}(J,\mathbb{R}^n) \mid Lx = \Theta\}$,

where $T(q)$ denotes the set of (*existing*) *solutions to* (6.3).

REMARK 6.2. In the single-valued case, we can obviously assume the unique solvability of the associated linearized problem. Moreover, Q need not then have a finitely generated abelian fundamental group (see Remark 4.3). In the multivalued case, the latter is true, provided Q is compact and $\overline{T(Q)} \subset Q$ (see again Remark 4.3).

Before presenting a nontrivial example, it will be convenient to have the following reduction property (see [6] and cf. [22, Lemma 6.6 in Chapter III.6]).

LEMMA 6.1 (Reduction). *Let X and its closed subset Y be ANR-spaces. Assume that $f: X \to X$ is a compact map, i.e. $\overline{f(X)}$ is compact, such that $f(X) \subset Y$. Denoting by $f': Y \to Y$ the restriction of f, we have*
(i) $\mathrm{Fix}(f') = \mathrm{Fix}(f)$,
(ii) *the Nielsen relations coincide*,
(iii) $\mathrm{ind}(C, f') = \mathrm{ind}(C, f)$, *for any Nielsen class $C \subset \mathrm{Fix}(f)$.*
Thus, $N(f') = N(f)$.

Consider the u-Carathéodory system (the functions e, f, g, h have the same regularity as in Theorem 6.1)

$$\begin{cases} \dot{x} + ax \in e(t, x, y)y^{(1/m)} + g(t, x, y), \\ \dot{y} + by \in f(t, x, y)x^{(1/n)} + h(t, x, y), \end{cases} \tag{6.5}$$

where a, b are positive numbers and m, n are odd integers with $\min(m, n) \geqslant 3$. Let suitable positive constants E_0, F_0, G, H exist such that

$$\begin{aligned} &\bigl|e(t, x, y)\bigr| \leqslant E_0, \qquad &\bigl|f(t, x, y)\bigr| \leqslant F_0, \\ &\bigl|g(t, x, y)\bigr| \leqslant G, \qquad &\bigl|h(t, x, y)\bigr| \leqslant H, \end{aligned}$$

hold, for a.a. $t \in (-\infty, \infty)$ and all $(x, y) \in \mathbb{R}^2$.

Furthermore, assume the existence of positive constants e_0, f_0, δ_1, δ_2 such that

$$0 < e_0 \leqslant e(t, x, y), \tag{6.6}$$

for a.a. t, all x and $|y| \geqslant \delta_2$, jointly with

$$0 < f_0 \leqslant f(t, x, y), \tag{6.7}$$

for a.a. t, $|x| \geqslant \delta_1$ and all y.

As a constraint S, consider at first the periodic boundary condition

$$\bigl(x(0), y(0)\bigr) = \bigl(x(\omega), y(\omega)\bigr). \tag{6.8}$$

More precisely, we take $S = Q = Q_1 \cap Q_2 \cap Q_3$, where

$$Q_1 = \Bigl\{q(t) \in C\bigl([0, \omega], \mathbb{R}^2\bigr) \mid \\ \bigl\|q(t)\bigr\| := \max\Bigl\{\max_{t \in [0,\omega]} \bigl|q_1(t)\bigr|, \max_{t \in [0,\omega]} \bigl|q_2(t)\bigr|\Bigr\} \leqslant D\Bigr\},$$

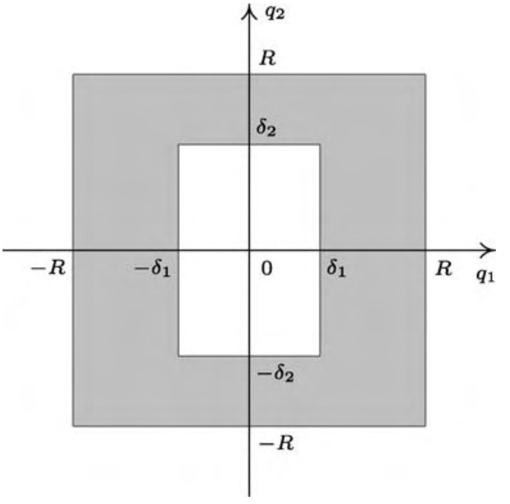

Fig. 3.

$$Q_2 = \Big\{ q(t) \in C\big([0,\omega], \mathbb{R}^2\big) \mid \min_{t\in[0,\omega]} \big|q_1(t)\big| \geqslant \delta_1 > 0 \\ \text{or } \min_{t\in[0,\omega]} \big|q_2(t)\big| \geqslant \delta_2 > 0 \Big\},$$

$$Q_3 = \big\{ q(t) \in C\big([0,\omega], \mathbb{R}^2\big) \mid q(0) = q(\omega) \big\},$$

the constants δ_1, δ_2, D will be specified below. For $(Q_1 \cap Q_2) \cap \mathbb{R}^2$, the situation is schematically sketched in Fig. 3.

Important properties of the set Q can be expressed as follows.

LEMMA 6.2. *The set Q defined above satisfies*:

(i) *Q is a closed connected subset of $C([0,\omega], \mathbb{R}^2)$*,

(ii) *$Q \in ANR$*,

(iii) *$\pi_1(Q) = \mathbb{Z}$*.

PROOF. Since Q is an intersection of closed sets Q_1, Q_2, Q_3, we conclude that Q is a closed subset of $C([0,\omega], \mathbb{R}^2)$ as well. The connectedness follows from the proof of (iii) below.

For (ii), it is enough to realize (see [69, Corollary 4.4 on p. 284]) that Q is the union of four closed, convex sets in the Banach space $C([0,\omega], \mathbb{R}^2)$, namely $Q = (Q_1 \cap Q_{21} \cap Q_3) \cup (Q_1 \cap Q_{22} \cap Q_3) \cup (Q_1 \cap Q_{23} \cap Q_3) \cup (Q_1 \cap Q_{24} \cap Q_3)$, where

$$Q_{21} = \Big\{ q(t) \in C\big([0,\omega], \mathbb{R}^2\big) \mid \min_{t\in[0,\omega]} q_1(t) \geqslant \delta_1 > 0 \Big\},$$

$$Q_{22} = \Big\{ q(t) \in C\big([0,\omega], \mathbb{R}^2\big) \mid \min_{t\in[0,\omega]} q_2(t) \geqslant \delta_2 > 0 \Big\},$$

$$Q_{23} = \Big\{q(t) \in C\big([0,\omega],\mathbb{R}^2\big) \;\Big|\; \min_{t\in[0,\omega]} q_1(t) \leqslant -\delta_1 < 0\Big\},$$

$$Q_{24} = \Big\{q(t) \in C\big([0,\omega],\mathbb{R}^2\big) \;\Big|\; \min_{t\in[0,\omega]} q_2(t) \leqslant -\delta_2 < 0\Big\}.$$

At last, we will show (iii). It is obvious that $\pi_1(A) = \mathbb{Z}$, where

$$A = \big\{(x,y) \in \mathbb{R}^2 \;\big|\; \max\big(|x|,|y|\big) \leqslant D \text{ and } \big[|x| \geqslant \delta_1 \text{ or } |y| \geqslant \delta_2\big]\big\}.$$

At the same time, $A = Q \cap \mathbb{R}^2$, when regarding $\mathbb{R}^2$ as a subspace of constant functions of Q_3. For (iii), it is sufficient to show that A is a deformation retract of Q.

We define $\rho : Q \times [0,1] \to A$ by the formula

$$\rho(q,\lambda) = \big(\lambda q_1 + (1-\lambda)\overline{q_1}, \lambda q_2 + (1-\lambda)\overline{q_2}\big),$$

where $q = (q_1, q_2) \in Q$ and $\overline{q_1} = q_1(0)$, $\overline{q_2} = q_2(0)$. One can readily check that ρ is a deformation retraction, which completes the proof of our lemma. □

Besides (6.5) consider still its embedding into

$$\begin{cases} \dot{x} + ax \in \big[(1-\mu)e_0 + \mu e(t,x,y)\big]y^{1/m} + \mu g(t,x,y), \\ \dot{y} + by \in \big[(1-\mu)f_0 + \mu f(t,x,y)\big]x^{1/n} + \mu h(t,x,y), \end{cases} \tag{6.9}$$

where $\mu \in [0,1]$ and observe that (6.9) reduces to (6.5), for $\mu = 1$.

The associated linearized system to (6.9) takes, for $\mu \in [0,1]$, the form

$$\begin{cases} \dot{x} + ax \in \big[(1-\mu)e_0 + \mu e\big(t,q_1(t),q_2(t)\big)\big]q_2(t)^{1/m} + \mu g\big(t,q_1(t),q_2(t)\big), \\ \dot{y} + by \in \big[(1-\mu)f_0 + \mu f\big(t,q_1(t),q_2(t)\big)\big]q_1(t)^{1/n} + \mu h\big(t,q_1(t),q_2(t)\big), \end{cases} \tag{6.10}$$

or, equivalently,

$$\begin{cases} \dot{x} + ax = \big[(1-\mu)e_0 + \mu e_t\big]q_2(t)^{1/m} + \mu g_t, \\ \dot{y} + by = \big[(1-\mu)f_0 + \mu f_t\big]q_1(t)^{1/n} + \mu h_t, \end{cases} \tag{6.11}$$

where $e_t \subset e(t,q_1(t),q_2(t))$, $f_t \subset f(t,q_1(t),q_2(t))$, $g_t \subset g(t,q_1(t),q_2(t))$, $h_t \subset h(t,q_1(t),q_2(t))$ are measurable selections. These exist, because the u-Carathéodory functions e, f, g, h are weakly superpositionally measurable (see Proposition 2.5).

It is well known that problem (6.11) ∩ (6.8) has, for each $q(t) \in Q$ and every fixed quadruple of selections e_t, f_t, g_t, h_t, a unique solution $X(t) = (x(t), y(t))$, namely

$$X(t) = \begin{cases} x(t) = \displaystyle\int_0^\omega G_1(t,s)\big[\big((1-\mu)e_0 + \mu e_s\big)q_2(s)^{1/m} + \mu g_s\big]\,\mathrm{d}s, \\ y(t) = \displaystyle\int_0^\omega G_2(t,s)\big[\big((1-\mu)f_0 + \mu f_s\big)q_1(s)^{1/n} + \mu h_s\big]\,\mathrm{d}s, \end{cases}$$

where

$$G_1(t,s) = \begin{cases} \dfrac{e^{-a(t-s+\omega)}}{1-e^{-a\omega}}, & \text{for } 0 \leqslant t \leqslant s \leqslant \omega, \\ \dfrac{e^{-a(t-s)}}{1-e^{-a\omega}}, & \text{for } 0 \leqslant s \leqslant t \leqslant \omega, \end{cases}$$

$$G_2(t,s) = \begin{cases} \dfrac{e^{-b(t-s+\omega)}}{1-e^{-b\omega}}, & \text{for } 0 \leqslant t \leqslant s \leqslant \omega, \\ \dfrac{e^{-b(t-s)}}{1-e^{-b\omega}}, & \text{for } 0 \leqslant s \leqslant t \leqslant \omega. \end{cases}$$

In order to verify that $\overline{T_\mu(Q)} \subset S = Q$, where $T_\mu(\cdot)$ is the solution operator to (6.10) $\cap$ (6.8), it is just sufficient to prove that $T_\mu(Q) \subset Q$, $\mu \in [0,1]$, because $S = Q$ is closed. Hence, the Nielsen number $N(T_\mu)$ is well-defined, for every $\mu \in [0,1]$, provided only product-measurability of e, f, g, h and $T_\mu(Q) \subset Q$.

Since $X(0) = X(\omega)$, i.e. $T_\mu(Q) \subset Q_3$, it remains to prove that $T_\mu(Q) \subset Q_1$ as well as $T_\mu(Q) \subset Q_2$. Let us consider the first inclusion. In view of

$$\min_{t,s\in[0,\omega]} G_1(t,s) \geqslant \frac{e^{-a\omega}}{1-e^{-a\omega}} > 0 \quad \text{and} \quad \min_{t,s\in[0,\omega]} G_2(t,s) \geqslant \frac{e^{-b\omega}}{1-e^{-b\omega}} > 0,$$

we obtain, for the above solution $X(t)$, that

$$\begin{aligned} \max_{t\in[0,\omega]} \left|x(t)\right| &\leqslant \max_{t\in[0,\omega]} \int_0^\omega \left|G_1(t,s)\right| \left[\left[(1-\mu)e_0 + \mu e_s\right] q_2(s)^{1/m} + \mu g_s\right] ds \\ &\leqslant \left[(e_0 + E_0) D^{1/m} + G\right] \int_0^\omega G_1(t,s)\, ds \\ &= \frac{1}{a}\left[(e_0 + E_0) D^{1/m} + G\right] \end{aligned}$$

and

$$\begin{aligned} \max_{t\in[0,\omega]} \left|y(t)\right| &\leqslant \max_{t\in[0,\omega]} \int_0^\omega \left|G_2(t,s)\right| \left[\left[(1-\mu)f_0 + \mu f_s\right] q_1(s)^{1/n} + \mu h_s\right] ds \\ &\leqslant \left[(f_0 + F_0) D^{1/n} + H\right] \int_0^\omega G_2(t,s)\, ds \\ &= \frac{1}{b}\left[(f_0 + F_0) D^{1/n} + H\right]. \end{aligned}$$

Because of

$$\left\|X(t)\right\| = \max\left\{\max_{t\in[0,\omega]} \left|x(t)\right|, \max_{t\in[0,\omega]} \left|y(t)\right|\right\}$$

$$\leqslant \max\left\{\frac{1}{a}\big[(e_0+E_0)D^{1/m}+G\big], \frac{1}{b}\big[(f_0+F_0)D^{1/n}+H\big]\right\},$$

a sufficiently large constant D certainly exists such that $\|X(t)\| \leqslant R$, i.e. $T_\mu(Q) \subset Q_1$, independently of $\mu \in [0,1]$ and e_t, f_t, g_t, h_t.

For the inclusion $T_\mu(Q) \subset Q_2$, we proceed quite analogously.

Assuming that $q(t) \in Q_2$, we have

$$\text{either} \quad \min_{t\in[0,\omega]} \big|q_1(t)\big| \geqslant \delta_1 > 0 \quad \text{or} \quad \min_{t\in[0,\omega]} \big|q_2(t)\big| \geqslant \delta_2 > 0.$$

Therefore, we obtain for the above solution $X(t)$ that (see (6.6))

$$\begin{aligned}\min_{t\in[0,\omega]} \big|x(t)\big| &= \min_{t\in[0,\omega]} \int_0^\omega \big|G_1(t,s)\big|\big[\big[(1-\mu)e_0+\mu e_s\big]q_2(s)^{1/m}+\mu g_s\big]\,\mathrm{d}s \\ &\geqslant \big|e_0\delta_2^{1/m}-G\big| \int_0^\omega G_1(t,s)\,\mathrm{d}s = \frac{1}{a}\big|e_0\delta_2^{1/m}-G\big| > 0,\end{aligned}$$

provided $G < e_0\delta_2^{1/m}$, for $|q_2| \geqslant \delta_2$, or (see (6.7))

$$\begin{aligned}\min_{t\in[0,\omega]} \big|y(t)\big| &= \min_{t\in[0,\omega]} \int_0^\omega \big|G_2(t,s)\big|\big[\big[(1-\mu)f_0+\mu f_s\big]q_1(s)^{1/n}+\mu h_s\big]\,\mathrm{d}s \\ &\geqslant \big|f_0\delta_1^{1/n}-H\big| \int_0^\omega G_2(t,s)\,\mathrm{d}s = \frac{1}{b}\big|f_0\delta_1^{1/n}-H\big| > 0,\end{aligned}$$

provided $H < f_0\delta_1^{1/n}$, for $|q_1| \geqslant \delta_1$.

So, in order to prove that $X(t) \in Q_2$, we need to fulfill simultaneously the following inequalities:

$$\begin{cases} (1/a)\big|e_0\delta_2^{1/m}-G\big| \geqslant \delta_1 > (H/f_0)^n \\ (1/b)\big|f_0\delta_1^{1/m}-H\big| \geqslant \delta_2 > (G/e_0)^m. \end{cases} \tag{6.12}$$

Let us observe that the "amplitudes" of the multivalued functions g, h must be sufficiently small. On the other hand, if e_0 and f_0 are sufficiently large (for fixed quantities a, b, G, H), then we can easily find δ_1, δ_2 satisfying (6.12).

After all, if there exist constants δ_1, δ_2 obeying (6.12), then we arrive at $X(t) \in Q_2$, i.e., $T_\mu(Q) \subset Q_2$, independently of $\mu \in [0,1]$ and e_t, f_t, g_t, h_t. This already means that $T_\mu(Q) \subset Q$, $\mu \in [0,1]$, as required.

Now, since all the assumptions of Theorem 6.1 are satisfied, problem (6.9) ∩ (6.8) possesses at least $N(T_\mu(\cdot))$ solutions belonging to Q, for every $\mu \in [0,1]$. In particular, problem (6.5) ∩ (6.8) has $N(T_1(\cdot))$ solutions, but according to the invariance under homotopy, $N(T_1(\cdot)) = N(T_0(\cdot))$. So, it remains to compute the Nielsen number $N(T_0(\cdot))$ for the operator $T_0 : Q \to Q$, where

$$T_0(q) = \left(e_0 \int_0^\omega G_1(t,s)q_2(s)^{1/m}\,\mathrm{d}s, f_0 \int_0^\omega G_2(t,s)q_1(s)^{1/n}\,\mathrm{d}s\right). \tag{6.13}$$

Hence, besides (6.13), consider still its embedding into the one-parameter family of operators

$$T^{\nu}(q) = \nu T_0(q) + (1-\nu) r \circ T_0(q), \quad \nu \in [0,1],$$

where $r(q) := (r(q_1), r(q_2))$ and

$$r(q_i) = q_i(0), \quad \text{for } i = 1, 2.$$

One can readily check that $r : Q \to Q \cap \mathbb{R}^2$ is a retraction and $T_0(\overline{q}) : Q \cap \mathbb{R}^2 \to Q$ is retractible onto $Q \cap \mathbb{R}^2$ with the retraction r in the sense of Definition 4.1. Thus, $r \circ T_0(\overline{q}) : Q \cap \mathbb{R}^2 \to Q \cap \mathbb{R}^2$ has a fixed point $\widehat{q} \in Q \cap \mathbb{R}^2$ if and only if $\widehat{q} = T_0(\widehat{q})$. Moreover, $r \circ T_0(q) : Q \to Q \cap \mathbb{R}^2$ has evidently a fixed point $\widehat{q} = Q \cap \mathbb{R}^2$ if and only if $\widehat{q} = T_0(\widehat{q})$. So, the investigation of fixed points for $T^0(q) = r \circ T_0(q)$ turns out to be equivalent with the one for $T^0(\overline{q}) : Q \cap \mathbb{R}^2 \to Q \cap \mathbb{R}^2$.

Since, in view of invariance under homotopy, we have

$$N\big(T_1(\cdot)\big) = N\big(T_0(\cdot)\big) = N\big(T^1(\cdot)\big) = N\big(T^0(\cdot)\big),$$

where

$$T^0(q) = \left(\frac{e_0 \mathrm{e}^{-a\omega}}{1 - \mathrm{e}^{-a\omega}} \int_0^{\omega} \mathrm{e}^{as} q_2(s)^{1/m} \, \mathrm{d}s, \frac{f_0 \mathrm{e}^{-b\omega}}{1 - \mathrm{e}^{-b\omega}} \int_0^{\omega} \mathrm{e}^{bs} q_1(s)^{1/n} \, \mathrm{d}s \right)$$

and

$$T^0(\overline{q}) = \left(\frac{e_0}{a} \overline{q_2}^{(1/m)}, \frac{f_0}{b} \overline{q_1}^{(1/n)} \right),$$
$$\text{for } \overline{q} = (\overline{q_1}, \overline{q_2}) = \big(q_1(0), q_2(0)\big) \in Q \cap \mathbb{R}^2,$$

it remains to estimate $N(T^0(\cdot))$. It will be useful to do it by passing to a simpler finite-dimensional analogy, namely by the direct computation of fixed points of the operator

$$T^0(\overline{q}) : Q \cap \mathbb{R}^2 \to Q \cap \mathbb{R}^2,$$

belonging to different Nielsen classes.

There are two fixed points $\widehat{q}_+ = (\widehat{q}_1, \widehat{q}_2)$ and $\widehat{q}_- = (-\widehat{q}_1, -\widehat{q}_2)$ in $Q \cap \mathbb{R}^2$, where

$$\widehat{q}_1 = \left(\frac{e_0}{a}\right)^{mn/(mn-1)} \left(\frac{f_0}{b}\right)^{1/(mn-1)},$$
$$\widehat{q}_2 = \left(\frac{e_0}{a}\right)^{m/(mn-1)} \left(\frac{f_0}{b}\right)^{mn/(mn-1)}.$$

These fixed points belong to different Nielsen classes, because any path u connecting them in $Q \cap \mathbb{R}^2$ and its image $T^0(u)$ are not homotopic in the space $Q \cap \mathbb{R}^2$, as it is schematically

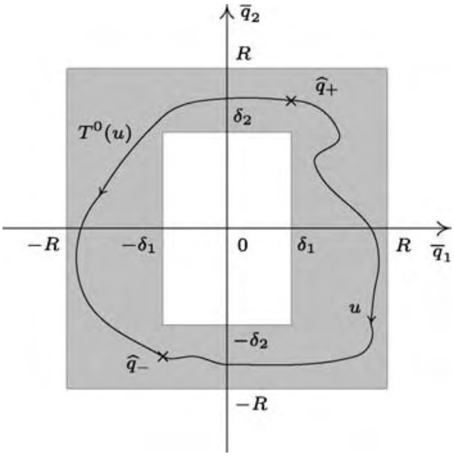

Fig. 4.

sketched in Fig. 4. Thus, according to the equivalent definition of the Nielsen number from the beginning of Section 3.2, $N(T^0(\overline{q})) = 2$. By means of the reduction property which is true here (see Lemma 6.1), we have, moreover,

$$N\big(T_1(\cdot)\big) = N\big(T^0(\cdot)\big) = N\big(T^0(\overline{q})\big) = 2$$

and so, according to Theorem 6.1, system (6.5) admits at least two solutions belonging to Q, provided suitable positive constants δ_1, δ_2 exist satisfying (6.12) and e, f, g, h are product-measurable.

In fact, system (6.5) possesses at least three solutions satisfying (6.8), when the sharp inequalities appear in (6.12), by which the lower boundary of Q becomes fixed point free. Indeed. Since

$$\Lambda\big(T_1(\cdot), Q\big) = \Lambda\big(T^0(\cdot), Q\big) = \lambda\big(T^0(\overline{q}), Q \cap \mathbb{R}^2\big)$$

holds for the generalized and ordinary Lefschetz numbers (see Section 3.1 and cf. [22, Chapter I.6]) and one can easily check that

$$\big|\lambda\big(T^0(\overline{q}), Q \cap \mathbb{R}^2\big)\big| = 2,$$

we obtain

$$\big|\Lambda\big(T_1(\cdot), Q\big)\big| = 2.$$

Furthermore, since for the self-map $T_1(\cdot)$ on the convex set $Q_1 \cap Q_3$ such that $\overline{T_1(Q_1 \cap Q_3)}$ is compact we have (see Remark 3.1)

$$\Lambda\big(T_1(\cdot), Q_1 \cap Q_3\big) = 1,$$

it follows from the additivity, contraction and existence properties of the fixed point index (see Proposition 3.2 in Section 3.3 and cf. [22, Chapter I.8]) that the mapping $T_1(\cdot)$ has the third coincidence point in $\overline{Q_1 \cap Q_3 \setminus Q}$ representing a solution of problem (6.5) $\cap$ (6.8) and belonging to $Q_1 \setminus Q$.

As we could see, problem (6.5) $\cap$ (6.8) admits at least two solutions in $Q_1 \cap Q_2$, for an arbitrary $\omega > 0$. Furthermore, because of rescaling (6.5), when replacing t by $t + (\omega/2)$, there are also two solutions of (6.5) satisfying $X(-\omega/2) = X(\omega/2)$, for an arbitrary $\omega > 0$, and belonging to $Q_1 \cap Q_2$.

Therefore, according to Proposition 4.5 and by obvious geometrical reasons, related to the appropriate subdomains of $Q_1 \cap Q_2$, system (6.5) possesses at least two entirely bounded solutions in $Q_1 \cap Q_2$.

Of course, because of replacing t by $(-t)$, the same result holds for (6.5) with negative constants a, b as well.

Finally, let us consider again system (6.5), where a, b, m, n are the same, but e, f, g, h are this time l.s.c. in (x, y), for a.a. $t \in (-\infty, \infty)$, multivalued functions with nonempty, convex, compact values and with the same estimates as above. Since each such mapping e, f, g, h has, under our regularity assumptions including the product-measurability, a Carathéodory selection (see, e.g., [22,73]), the same assertion must be also true in this new situation.

So, after summing up the above conclusions, we can give finally:

THEOREM 6.2. *Let suitable positive constants δ_1, δ_2 exist such that the inequalities*

$$\begin{cases} \dfrac{1}{|a|}\left|e_0\delta_2^{1/m} - G\right| \geqslant \delta_1 > \left(\dfrac{H}{f_0}\right)^n, \\ \dfrac{1}{|b|}\left|f_0\delta_2^{1/n} - H\right| \geqslant \delta_2 > \left(\dfrac{G}{e_0}\right)^m \end{cases} \tag{6.14}$$

are satisfied for constants e_0, f_0, G, H estimating the product-measurable u-Carathéodory or l-Carathéodory multivalued functions (*with nonempty, convex and compact values*) *e, f, g, h as above, for constants a, b with $ab > 0$ and for odd integers m, n with $\min(m, n) \geqslant 3$. Then system* (6.5) *admits at least two entirely bounded solutions. In particular, if multivalued functions e, f, g, h are still ω-periodic in t, then system* (6.5) *admits at least tree ω-periodic solutions, provided the sharp inequalities appear in* (6.14).

REMARK 6.3. Unfortunately, because of the invariance (w.r.t. the solution operator T_1) of the subdomains

$$\left\{q(t) \in C\left(\left[-\frac{\omega}{2}, \frac{\omega}{2}\right], \mathbb{R}^2\right) \,\middle|\, 0 < \delta_1 \leqslant q_1(t) \leqslant R \wedge 0 < \delta_2 \leqslant q_2(t) \leqslant R\right\}$$

and

$$\left\{q(t) \in C\left(\left[-\frac{\omega}{2}, \frac{\omega}{2}\right], \mathbb{R}^2\right) \,\middle|\, -R \leqslant q_1(t) \leqslant -\delta_1 < 0 \wedge -R \leqslant q_2(t) \leqslant -\delta_2 < 0\right\},$$

for each $\omega \in (-\infty, \infty)$, the same result can also be obtained, for example, by means of the fixed point index.

In order to avoid the handicap in Remark 6.3, let us still consider the planar system of integro-differential inclusions. For the sake of transparency, our presentation will be as simple as possible. Thus, the right-hand sides can apparently take a more general form and the multiplicity criteria in terms of inequalities can be improved.

Hence, let $x_i : [0, \omega] \to \mathbb{R}$, for $i = 1, 2$, $x = (x_1, x_2)$, $\varphi \in [0, \frac{\pi}{4}]$, $a > 0$ and consider the following system of integro-differential inclusions

$$\dot{x}_1 + ax_1 \in \sqrt[3]{p_2(x)} \cos\varphi - \sqrt[3]{p_1(x)} \sin\varphi + \varphi e, \tag{6.15}$$

$$\dot{x}_2 + ax_2 \in \sqrt[3]{p_1(x)} \cos\varphi + \sqrt[3]{p_2(x)} \sin\varphi + \varphi e, \tag{6.16}$$

where $e : [0, \omega] \times \mathbb{R}^2 \multimap \mathbb{R}$ is a product-measurable u-Carathéodory map with nonempty, convex and compact values with $|e(t, x)| \leqslant E$, for a.a. $t \in [0, \omega]$ and all $x \in \mathbb{R}^2$, and

$$p_i(x) = \frac{1}{\omega} \int_0^\omega x_i(s)\,\mathrm{d}s - B\left(\frac{1}{\omega} \int_0^\omega x_i(s)\,\mathrm{d}s - x_i\right),$$

with $B > 0$. For $\varphi = \frac{\pi}{4}$, the system takes the form

$$\dot{x}_1 + ax_1 \in \frac{\sqrt{2}}{2}\left(\sqrt[3]{p_2(x)} - \sqrt[3]{p_1(x)}\right) + \frac{\pi}{4} e, \tag{6.17}$$

$$\dot{x}_2 + ax_2 \in \frac{\sqrt{2}}{2}\left(\sqrt[3]{p_1(x)} + \sqrt[3]{p_2(x)}\right) + \frac{\pi}{4} e, \tag{6.18}$$

while, for $\varphi = 0$, it reduces to

$$\dot{x}_1 + ax_1 = \sqrt[3]{p_2(x)}, \qquad \dot{x}_2 + ax_2 = \sqrt[3]{p_1(x)}.$$

We shall be again looking for the lower estimate of the number of ω-periodic solutions to (6.17), (6.18).

Let us define sets $S = Q \subset C([0, \omega], \mathbb{R}^2)$ as follows. Function $q = (q_1, q_2)$ belongs to Q if the following conditions are satisfied:

(i) $q(0) = q(\omega)$ (ω-periodicity),

(ii) $|q(t)| \leqslant R$, for all $t \in [0, \omega]$ (boundedness),

(iii) $|q_1(t)| \geqslant \delta$ or $|q_2(t)| \geqslant \delta$, for all $t \in [0, \omega]$ (uniform boundedness of one component from below),

(iv) $q(t) = \overline{q} + \widetilde{q}(t)$, where $\overline{q} := \frac{1}{\omega} \int_0^\omega q(s)\,\mathrm{d}s$ is the integral average of q on $[0, \omega]$ (thus, $\frac{1}{\omega} \int_0^\omega \widetilde{q}(s)\,\mathrm{d}s = 0$) and $|\widetilde{q}(t)| \leqslant \varepsilon$, for all $t \in [0, \omega]$ (function q differs from its integral average by less than ε).

Values of a and ω are given, we shall specify the values of B, δ, R, E and ε in the subsequent parts.

Set Q is again a union of four closed, convex sets in the Banach space $C([0,\omega],\mathbb{R}^2)$, namely $Q = Q_1^+ \cup Q_2^+ \cup Q_1^- \cup Q_2^-$, where

$$Q_i^{\pm} = \Big\{ q(t) \in C\big([0,\omega],\mathbb{R}^2\big) \mid q \text{ satisfies (i), (ii), (iv) and } \min_{t\in[0,\omega]} \pm q_i(t) \geqslant \delta > 0 \Big\}, \quad i = 1, 2.$$

Thus, as in the proof of Lemma 6.2, it is a closed connected ANR-space such that $\pi_1(Q) = \mathbb{Z}$.

For homotopic parameter $\varphi = 0$, system (6.15), (6.16) reduces to a simpler case, which can be easily handled (in fact, we can explicitly compute two constant fixed points). For $\varphi = \frac{\pi}{4}$, the situation becomes non-trivial. We shall show that set Q is invariant under the solution operator, which takes a parameter $q \in Q$ to the solution x of the linearized inclusion. In this case, no obvious or easily detectable subset of Q can be recognized to be separately invariant.

In order to apply a slightly modified special case of Theorem 6.1 (cf. [18]), we use again the method of Schauder linearization. Let us take an arbitrary $q \in Q$. The system of fully linearized inclusions takes the form:

$$\dot{x}_1 + a x_1 \in \sqrt[3]{p_2(q)} \cos\varphi - \sqrt[3]{p_1(q)} \sin\varphi + \varphi e(t,q), \tag{6.19}$$

$$\dot{x}_2 + a x_2 \in \sqrt[3]{p_1(q)} \cos\varphi + \sqrt[3]{p_2(q)} \sin\varphi + \varphi e(t,q), \tag{6.20}$$

where

$$p_i(q) = \frac{1}{\omega} \int_0^\omega q_i(s)\,\mathrm{d}s - B\left(\frac{1}{\omega} \int_0^\omega q_i(s)\,\mathrm{d}s - q_i \right), \tag{6.21}$$

for $i = 1, 2$.

Denoting $\overline{q} := \frac{1}{\omega} \int_0^\omega q(s)\,\mathrm{d}s$ the integral average of q on $[0,\omega]$, we can write $p : Q \subset C([0,\omega],\mathbb{R}^2) \to C([0,\omega],\mathbb{R}^2)$ in the form

$$p(q) = \overline{q} - B(\overline{q} - q). \tag{6.22}$$

For $B = 1$, operator p reduces to identity. For $B < 1$, the operator "shrinks" function q closer to its integral average. Indeed, if $q = \overline{q} + \widetilde{q}$, where $|\widetilde{q}(t)| \leqslant \varepsilon$, for all $t \in [0,\omega]$, then operator p takes q to $p(q) = \overline{q} - B\widetilde{q}$.

The fully linearized system (6.19), (6.20) possesses, for any $q \in Q$, and any Lebesgue integrable (single-valued) selection $e_0 \subset \{e(t,q)\}$, $t \in [0,\omega]$, a unique solution $x(t)$ which is given by the known convolution with the Green operator

$$x_i(t) = \int_0^\omega G(t,s) f_i(s)\,\mathrm{d}s, \tag{6.23}$$

where

$$f_1(s) := \sqrt[3]{p_2(q)}\cos\varphi - \sqrt[3]{p_1(q)}\sin\varphi + \varphi e_0, \tag{6.24}$$

$$f_2(s) := \sqrt[3]{p_1(q)}\cos\varphi + \sqrt[3]{p_2(q)}\sin\varphi + \varphi e_0. \tag{6.25}$$

Let us denote by T_φ the solution operator which takes $q \in Q$ to the solutions x, given by (6.23), of the linearized system (6.19), (6.20). We shall prove that Q is invariant under T_φ, namely that $T_\varphi(q) \subset Q$, for each $q \in Q$.

Let us take $q \in Q$ arbitrary. Operator p defined by (6.22) takes q to $p(q)$ such that $p(q) = \overline{q} + \widetilde{p}$, where $|\widetilde{p}(t)| \leqslant B\varepsilon$. Substituting this $p(q)$ into (6.24) and (6.25), we obtain $f_i(t) = F_i(t) + \widetilde{f}(t)$, where

$$F_1 := \sqrt[3]{\overline{q}_2}\cos\varphi - \sqrt[3]{\overline{q}_1}\sin\varphi, \qquad F_2 := \sqrt[3]{\overline{q}_1}\cos\varphi + \sqrt[3]{\overline{q}_2}\sin\varphi, \tag{6.26}$$

and $|\widetilde{f}_i(t)| \leqslant 3\sqrt[3]{B\varepsilon} + \frac{\pi E}{4}$ for all $t \in [0,\omega]$.

This estimate can be shown as follows. For $|\overline{q}_i| \geqslant 1$, one can get by direct calculation that $|\sqrt[3]{\overline{q}_i + \widetilde{p}_i} - \sqrt[3]{\overline{q}_i}| \leqslant \sqrt[3]{B\varepsilon}$, provided that $|\widetilde{p}_i| \leqslant B\varepsilon \leqslant 1$. For $|\overline{q}_i| \leqslant 1$, a careful examination of function $|\sqrt[3]{\overline{q}_i + \widetilde{p}_i} - \sqrt[3]{\overline{q}_i}|$ reveals that it is bounded from above by the value $2^{2/3}\sqrt[3]{B\varepsilon}$, provided again $|\widetilde{p}_i| \leqslant B\varepsilon \leqslant 1$. Altogether, f_i differs from F_i not more than by

$$2\frac{\sqrt{2}}{2}2^{2/3}\sqrt[3]{B\varepsilon} + \frac{\pi E}{4} \leqslant 3\sqrt[3]{B\varepsilon} + \frac{\pi E}{4},$$

as claimed.

The Green's function G in (6.23) takes the form

$$G(t,s) = \begin{cases} \dfrac{1}{1-\mathrm{e}^{-a\omega}}\mathrm{e}^{-at}\mathrm{e}^{as}, & \text{for } 0 \leqslant s \leqslant t, \\ \dfrac{1}{1-\mathrm{e}^{-a\omega}}\mathrm{e}^{-at}\mathrm{e}^{-a\omega}\mathrm{e}^{as}, & \text{for } t \leqslant s \leqslant \omega. \end{cases} \tag{6.27}$$

Substituting (6.26) and (6.27) into (6.23), we obtain $x_i(t) = \frac{F_i}{a} + \widetilde{x}_i(t)$, where $\widetilde{x}_i$ satisfies the inequality

$$\left|\widetilde{x}_i(t)\right| \leqslant \frac{3\sqrt[3]{B\varepsilon} + \frac{\pi}{4}E}{a},$$

for all $t \in [0,\omega]$. We can now take B and E small enough to fulfill

$$\frac{\sqrt{2}(3\sqrt[3]{B\varepsilon} + \frac{\pi}{4}E)}{a} \leqslant \frac{\varepsilon}{2}, \tag{6.28}$$

for example

$$B \leqslant \left(\frac{a}{12}\right)^3 \frac{\varepsilon^2}{\sqrt{2}} \quad \text{and} \quad E \leqslant \frac{a\varepsilon}{\pi\sqrt{2}}.$$

This means that function x differs from a constant function by less than $\frac{\varepsilon}{2}$, which implies that it differs from its integral average by less than ε. The above calculations ensure that the solution x satisfies condition (iv) of the definition of the parameter set Q, independently on $q \in Q$ and $\varphi \in [0, \frac{\pi}{4}]$. Condition (i) is trivially satisfied by the form of the Green function G.

We must further ensure that, for each $q \in Q$, function x satisfies conditions (ii) and (iii). Since both q and x differ from their integral averages by less than ε, let us first deal with constant functions. This is easy, because the solution mapping T_φ takes constant functions to functions that differ from a constant function by less than $\frac{\pi E}{4a}$ and it is a composition of

- reflection $(\overline{q}_1, \overline{q}_2) \to (\overline{q}_2, \overline{q}_1)$,
- re-scaling $(\overline{q}_1, \overline{q}_2) \to \frac{1}{a}(\sqrt[3]{\overline{q}_1}, \sqrt[3]{\overline{q}_2})$,
- rotation $(\overline{q}_1, \overline{q}_2) \to (\overline{q}_1 \cos\varphi - \overline{q}_2 \sin\varphi, \overline{q}_2 \cos\varphi + \overline{q}_1 \sin\varphi)$ by angle φ in the anti-clockwise direction.

The re-scaling part of the composition ensures that constants R and δ can be specified so that the solution operator T_φ takes constant functions satisfying (ii) and (iii) to functions that again satisfy (ii) and (iii). Since functions in Q differ from their integral averages by less than ε, we need to find R, δ and ε such that the following conditions are satisfied:

$$\frac{1}{a}\sqrt[3]{R} \leqslant R - \varepsilon - \frac{\pi E}{4a} \quad \text{and} \quad \frac{\sqrt{2}}{2a}\sqrt[3]{\delta} \geqslant \delta + \sqrt{2}\varepsilon + \frac{\pi E}{4a}. \tag{6.29}$$

Inequalities (6.29) guarantee that x satisfies conditions (ii) and (iii) of the definition of the parameter set Q. Taking further $\varepsilon \leqslant \frac{\delta}{2}$ ensures that Q is a non-trivial ANR-space (leaving the "hole" inside).

Starting from $a > 0$ and $\omega > 0$, we have specified constants R, δ, B, E and ε such that set Q becomes invariant under the solution operator T_φ which takes any $q \in Q$ to the solutions x of the linearized problem (6.19), (6.20), for $\varphi \in [0, \frac{\pi}{4}]$. Moreover, observe that, for $\varphi = \frac{\pi}{4}$, there are no easily detectable subdomains of Q separately invariant under operator $T_{\pi/4}$. Figure 5 shows how operator $T_{\pi/4}$ treats constant functions in Q, for a particular choice of R and δ and helps understanding why we can not easily detect any subinvariant domains of Q.

Hence, a slightly modified special case of Theorem 6.1 (cf. [18]) ensures that system (6.15), (6.16) admits at least $N(T_{\pi/4})$ solutions. Since $T_{\pi/4}$ is compactly admissibly homotopic to T_0, we have $N(T_{\pi/4}) = N(T_0)$.

Let us further consider the retraction $r : Q \to Q$ which sends a function $q \in Q$ to its integral average $\overline{q}$. Let us define the homotopy $T^\mu : [0, 1] \times Q \to Q$ by

$$T^\mu(q) := \mu T_0(q) + (1 - \mu) r\big(T_0(q)\big).$$

This compact homotopy guarantees that $N(T^1) = N(T_0)$ equals to $N(T^0) = N(r \circ T_0)$. We can thus restrict ourselves to the computation of $N(r \circ T_0)$. Let us denote by $\overline{Q}$ the subset of Q consisting of constant functions. Since $r \circ T_0 : Q \to \overline{Q}$, all the fixed points of $r \circ T_0$ have to belong to $\overline{Q}$. Let us therefore deal with the restriction $L := r \circ T_0|_{\overline{Q}} : \overline{Q} \to \overline{Q}$, which can be explicitly written in the form

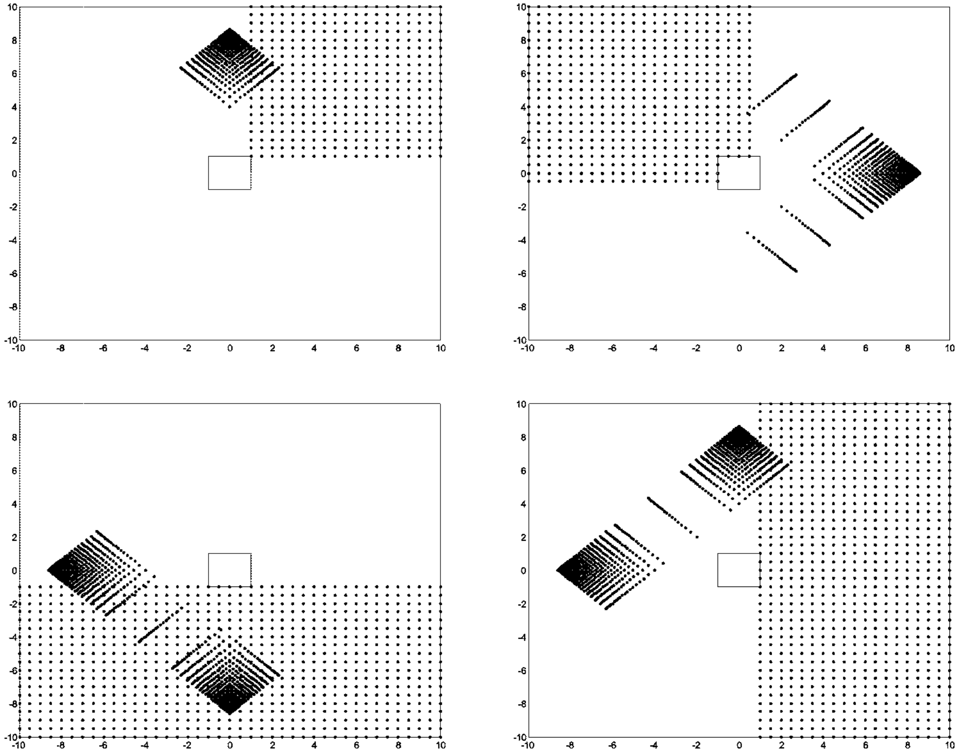

Fig. 5. Behaviour of $T_{\pi/4}$ on constant functions on Q for $R = 10$, $\delta = 1$ and $a = \frac{\sqrt{2}}{4}$. Rectangular grids of points represent constant functions $q \in Q$, the irregular grid represents their images under $T_{\pi/4}$. For simplicity, we take here $E = 0$, so that the images of constant functions become constant again. No easily detectable regions of the domain are subinvariant.

$$L(\overline{q}_1, \overline{q}_2) := \frac{1}{a}\left(\sqrt[3]{\overline{q}_2}, \sqrt[3]{\overline{q}_1}\right).$$

One can easily check by an explicit computation that L has two fixed-points in $\overline{Q}$ which belong to different Nielsen classes. Therefore, according to (reduction) Lemma 6.1, $N(L) = N(r \circ T_0) = 2$. This finally shows that system (6.17), (6.18) admits at least two ω-periodic solutions.

We are in the position to formulate the multiplicity criterion for ω-periodic solutions to system (6.17), (6.18).

THEOREM 6.3. *Let the following inequalities be satisfied*:

$$\delta < \frac{0.247}{a^{3/2}}, \quad \frac{\sqrt[3]{R}}{a} \leqslant R - \frac{5\delta}{8}, \quad E \leqslant \frac{a\delta}{2\sqrt{2}\pi}, \quad B \leqslant \left(\frac{a}{12}\right)^3 \frac{\delta^2}{4\sqrt{2}}, \tag{6.30}$$

and take $\varepsilon = \frac{\delta}{2}$. *Then system* (6.17), (6.18) *admits at least three* ω*-periodic solutions*.

PROOF. The assumptions guarantee that inequalities (6.28) and (6.29) are satisfied. Thus, two ω-periodic solutions have been already obtained by means of the Nielsen number, as above. The third can be proved quite analogously as in the proof of Theorem 6.2, by the additivity property of the fixed point index (see Proposition 3.2). □

REMARK 6.4. The inequalities in Theorem 6.3 are satisfied, e.g., for $a = \frac{\sqrt{2}}{4}$, $\delta = 1$, $R = 10$, $B \leqslant \frac{1}{2^4 3^3}$, and $E \leqslant \frac{1}{4\pi}$, as in Fig. 5, where $E = 0$.

REMARK 6.5. If the equality appears for δ in (6.30), then in the lack of periodicity, at least two entirely bounded solutions can be proved as in Theorem 6.2.

6.3. *Several anti-periodic solutions*

The following approach is via the Poincaré translation operator treated in Section 4.3. Hence, consider the system of differential inclusions

$$\dot{x} \in F(t, x), \tag{6.31}$$

where $F : \mathbb{R}^{n+1} \multimap \mathbb{R}^n$ is a u-Carathéodory mapping with nonempty, compact and convex values, satisfying (6.4). Then all solutions of (6.31) exist in the sense of Carathéodory, namely they are locally absolutely continuous and satisfy (6.31) a.e.

If $x(t, x_0) := x(t, 0, x_0)$ is a solution of (6.31) with $x(0, x_0) = x_0$, then we can define the Poincaré map (translation operator at the time $T > 0$) along the trajectories of (6.31) as follows:

$$\Phi_T : \mathbb{R}^n \multimap \mathbb{R}^n, \quad \Phi_T(x_0) := \bigl\{x(T, x_0) \mid x(t, x_0) \text{ satisfies } (6.31)\bigr\}. \tag{6.32}$$

The goal is to represent the admissible (see Theorem 4.17) map Φ_T in terms of an admissible pair (see Definition 2.5 in Section 2.2). We let $\varphi : \mathbb{R}^n \multimap C([0, T], \mathbb{R}^n)$, where

$$\varphi(X) := \bigl\{x \in C\bigl([0, T], \mathbb{R}^n\bigr) \bigm| x(0) = X \text{ and } x \text{ satisfies } (6.31)\bigr\}$$

and $C([0, T], \mathbb{R}^n)$ is the Banach space of continuous maps. According to Theorem 4.17, φ is an R_δ-mapping.

Now, we let

$$e_T : C\bigl([0, T), \mathbb{R}^n\bigr) \to \mathbb{R}^n, \quad e_T(x) = x(T),$$

where e_T is evidently continuous.

One can readily check that $\Phi_T = e_T \circ \varphi$. Moreover, $\mathcal{H} \circ \Phi_T = \mathcal{H} \circ \varphi \circ e_T$ is admissible for any homeomorphism $\mathcal{H} : \mathbb{R}^n \to \mathbb{R}^n$ (see Remark 4.15).

In fact, we have the diagram

$$\mathbb{R}^n \xleftarrow{p_\varphi} \Gamma_\varphi \xrightarrow{q_\varphi} C\bigl([0, a], \mathbb{R}^n\bigr) \xrightarrow{e_T} \mathbb{R}^n \xrightarrow{\mathcal{H}} \mathbb{R}^n,$$

where p_φ, q_φ are natural projections.

In what follows, we identify the Poincaré map Φ_T or its composition with $\mathcal{H}$, i.e. $\mathcal{H} \circ \Phi_T$, with the admissible pair $(p_\varphi, e_T \circ q_\varphi)$ or $(p_\varphi, \mathcal{H} \circ e_T \circ q_\varphi)$, i.e. we let

$$\Phi_T = (p_\varphi, e_T \circ q_\varphi) \quad \text{or} \quad \mathcal{H} \circ \Phi_T = (p_\varphi, \mathcal{H} \circ e_T \circ q_\varphi), \tag{6.33}$$

respectively.

Let $C(\mathcal{H} \circ \Phi_T)$ denote the set of coincidence points of the pair $(p_\varphi, \mathcal{H} \circ e_T \circ q_\varphi)$, while $\mathrm{Fix}(\mathcal{H} \circ \Phi_T) = \{X \in \mathbb{R}^n \mid X \in \mathcal{H}(e_T(q_\varphi(p_\varphi^{-1}(X))))\}$.

A pair for Φ_T can be easily shown to be homotopic to the identity map, so that the pair $(p_\varphi, \mathcal{H} \circ e_T \circ q_\varphi)$ is homotopic to $\mathcal{H}$ (see Theorem 4.17). We have a one-to-one correspondence between coincidence points and solutions. Since a coincidence point of $(p_\varphi, \mathcal{H} \circ e_T \circ q_\varphi)$ gives us in this way a solution $x(t)$ of (6.31) such that $x(0) = \mathcal{H}(x(T))$, the following proposition is self-evident.

PROPOSITION 6.1. *If* $\#C(p_\varphi, \mathcal{H} \circ e_T \circ q_\varphi) = \operatorname{card} C(\mathcal{H} \circ \Phi_T) \geqslant k$, *then system* (6.31) *has at least* k *solutions* $x_1(t), \dots, x_k(t)$ *such that* $x_i(0) = \mathcal{H}(x_i(T))$, $i = 1, \dots, k$.

The following lemma immediately follows from Theorem 4.17.

LEMMA 6.3. *Assume that* $Y \subset \mathbb{R}^n$ *is a compact connected ANR-space such that* $\Phi_s(Y) \subset Y$, *for every* $s \in [0, T]$ *and* $\mathcal{H}(Y) \subset Y$. *Then the pair* $(p_\varphi, \mathcal{H} \circ e_T \circ q_\varphi)$ *restricted to* Y *is admissibly homotopic to* $\mathcal{H}|_Y$.

As a consequence of Theorems 3.4 and 3.7 in Section 3.2, Proposition 6.1 and Lemma 6.3, we have the following

PROPOSITION 6.2. *Assume that* Φ_s, *for every* $s \in [0, T]$ *in* (6.32) *is a self-map on the torus* $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. *Then system* (6.31) *has at least* $N(\mathcal{H})$ *solutions* $x(t)$ *such that* $x(0) = \mathcal{H}(x(T))$ *on* $\mathbb{T}^n$, *where* $N(\mathcal{H})$ *denotes the Nielsen number of a homeomorphism* $\mathcal{H} : \mathbb{T}^n \to \mathbb{T}^n$.

PROOF. The proof follows directly from Theorems 3.4 and 3.7 in Section 3.2, Proposition 6.1, Lemma 6.3 and the properties of the Nielsen number mentioned in Section 3.2. □

If in particular $\mathcal{H} = \mathrm{id}$, then according to Remark 3.6,

$$N(\mathrm{id}) = |\lambda(\mathrm{id})| = |\chi(\cdot)|$$

holds on tori, and consequently the problem considered in Proposition 6.2 should have at least $|\chi(\cdot)|$ solutions, where λ is the Lefschetz number and χ denotes the Euler–Poincaré characteristic of $\mathbb{T}^n$. Since, unfortunately, $\chi(\cdot) = 0$ for tori, this is not a suitable case for applications. In other words, we are not able to establish several T-periodic solutions in this way.

On the other hand, as the simplest application of Proposition 6.2, we can give immediately

THEOREM 6.4. *Assume that*

$$F(t,\dots,x_j+1,\dots)\equiv F(t,\dots,x_j,\dots),\quad \textit{for } j=1,\dots,n, \tag{6.34}$$

where $x=(x_1,\dots,x_n)$ *and consider system* (6.31) *on the set* $[0,\infty)\times\mathbb{T}^n$, *where* $\mathbb{T}^n=\mathbb{R}^n/\mathbb{Z}^n$. *Then system* (6.31) *admits, for every positive constant* T, *at least* $N(\mathcal{H})$ *solutions* $x(t)$ *such that*

$$x(0)=\mathcal{H}\big(x(T)\big)\pmod 1,$$

where $\mathcal{H}$ *is a continuous self-map on* $\mathbb{T}^n$ *and* $N(\mathcal{H})$ *denotes the associated Nielsen number.*

As a consequence, we obtain for $\mathcal{H}=-\mathrm{id}$ easily:

COROLLARY 6.2. *If, in addition to the assumptions of Theorem* 6.4,

$$F(t+T,-x)\equiv -F(t,x),$$

then system (6.31) *admits at least* 2^n *anti-*T*-periodic* (*or* $2T$*-periodic*) *solutions* $x(t)$ *on* $\mathbb{T}^n$, *namely* $x(t+T)\equiv -x(t)\pmod 1$.

PROOF. According to Theorem 6.4, system (6.31) admits at least $N(-\mathrm{id})$ anti-T-periodic solutions on $\mathbb{T}^n$. On $\mathbb{T}^n$, the following formula holds (see Remark 3.6), $N(-\mathrm{id})=|\lambda(-\mathrm{id})|=|\det 2I|=2^n$, which completes the proof. □

7. Remarks and comments

7.1. *Remarks and comments to general methods*

Theorems 4.1, 4.2 and Corollaries 4.1, 4.2, 4.3 are taken from [19]. Corollaries 4.1 and 4.4 generalize many single-valued situations, e.g., in [52] and [63], where the parameter set Q was only convex. Corollary 4.3 was employed for the first time in [32,33]. Corollary 4.4 generalizes the single-valued case in [39].

Theorem 4.3, for the lower estimate of the number of solutions, comes from [23]. Its slightly modified version (cf. Remark 4.3) was presented in [5]. As far as we know, there are no other general methods, using the Nielsen number, as our Theorem 4.3.

Theorem 4.4, extending Theorem 4.1 to Banach spaces, was published in [14]. Corollary 4.5 is formally new. Intuitively clear Propositions 4.4 and 4.5 (see also Remark 4.6) are contained in [22, Chapter III.1].

In view of Remark 4.6, one can check by means of the Gronwall inequality (cf. [71]) that differential inclusion $\dot{x}\in F(t,x)$ admits an entirely bounded solution $x\in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R},E)$ with $x(0)=x_0$ such that

$$\|x(t)\|\leqslant\left(\|x_0\|+\int_{-\infty}^{\infty}r(t)\,\mathrm{d}t\right)\exp\int_{-\infty}^{\infty}r(t)\,\mathrm{d}t,\quad t\in\mathbb{R},$$

provided E is a separable or reflexive Banach space and $F:\mathbb{R} \times E \multimap E$ is a u-Carathéodory mapping (cf. Definition 2.10) such that:

(i) $\mu(F(t,B)) \leqslant k(t)\mu(B)$, for bounded subsets $B \subset E$, $t \in \mathbb{R}$, where $\mu = \alpha$ or $\mu = \gamma$, and $k \in L^1_{\text{loc}}(\mathbb{R})$,

(ii) $\|y\| \leqslant r(t)(1 + \|x\|)$, for every $(t,x) \in \mathbb{R} \times E$, $y \in F(t,x)$, where $r:\mathbb{R} \to [0,\infty)$ is an integrable function such that $\int_{-\infty}^{\infty} r(t)\,\mathrm{d}t < \infty$.

Many alternative continuation principles for ODEs can be found, e.g., in [45,58,61,62, 64,65,82,83,90], and the references therein.

Theorems 4.6, 4.8, 4.9 and Proposition 4.9, dealing with the topological structure of solution sets are taken from [19]. Theorem 4.9 extends Theorem 4.5 in [57] for arbitrary (possibly infinite) intervals. In [20,21], we have developed and applied a powerful inverse limit method for the investigation of the topological structure of the solution sets. In these papers, the structure was also systematically studied for less regular right-hand sides of multivalued ODEs than those in Section 4.2.

Theorems 4.10 and 4.11 concerning the topological dimensions of solution sets come from [66]. An extension to multivalued ODEs in Banach spaces was recently published in [45]. For further results concerning the solution sets to initial value problems, see, e.g., [42,55,58,59,67,74,78,97].

Unlike for initial value problems, there are only several results concerning the topological structure of solution sets to boundary value problems. One of the most important is Theorem 4.13 in [43] which was improved by us (cf. Remark 4.10) in [21]. Theorem 4.12 comes from [20], Theorem 4.14 from [46] and Theorems 4.15, 4.16 from [7]. In [22, Chapter III.3], we collected for the first time practically all results in this field (cf. also [59, Chapter 6]). Paper [43] contains also the information about the topological dimension of the solution sets.

Theorem 4.17 about admissibility of Poincaré's operators appeared on various levels of abstraction in many papers (cf. [22, Chapter III.4 and the related comments on pp. 592–593]). For some generalizations and extensions, see [2,10,28], and the references therein.

7.2. *Remarks and comments to existence results*

Theorem 5.1 from [14] can be regarded as an infinite-dimensional extension of our earlier results in [3,4], where under suitable additional restrictions also almost-periodic solutions were detected. In [15], the methods applied to Theorem 5.1 were modified for obtaining almost-periodic solutions with values in separable Banach spaces. Theorem 5.2 is contained in [20].

Proposition 5.1, for solvability of a rather general class of boundary value problems with linear conditions, was presented for the first time in [3].

There is an enormous amount of results about the existence of periodic and anti-periodic solutions (see, e.g., [22,62,64,65,70,74,78,81,82,90]). Our Corollary 5.1 is formally new. Corollaries 5.2, 5.3, 5.4 can be found in [22, Chapter III.5]. As pointed out in Remark 5.7, the requirement concerning a fixed point free boundary of a parameter set Q were satisfied in [11,32–34] by means of bound sets. An alternative approach for satisfying this requirement can be found, e.g., in [30,31], where canonical domains or upper and lower solutions technique were applied, in the single-valued case (cf. also [65,70,81,82,90]).

7.3. *Remarks and comments to multiplicity results*

Our multiplicity results are based on the application of the Nielsen theory in Section 3.2, eventually combined with the additivity property of the fixed point index in Section 3.3. For further Nielsen theories which can be also used here, see [12,22–26,38], and the references therein. Practically all results obtained in this way were collected in [13].

Corollary 6.1 is from [5]. Variants of Theorems 6.1 and 6.2 can be found in [5,6,9], and [23], while Theorem 6.3 is a multivalued generalization of a single-valued version in [17]. Theorem 6.4 and Corollary 6.2 were presented for the first time in [24].

As pointed out in Section 1, the delicate problem of application of the Nielsen theory to differential equations is associated with the name of J. Leray.

For further multiplicity results obtained by different methods, see, e.g., [22, Chapter III.6; 62; 82, Chapter 6], and the references therein.

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CHAPTER 2

Heteroclinic Orbits for Some Classes of Second and Fourth Order Differential Equations

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Contents

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1. Introduction

In qualitative theory of differential equations, a prominent role is played by special classes of solutions, like periodic solutions or solutions to some kind of boundary value problem. When a system of ordinary differential equations has equilibria (i.e. constant solutions) whose stability properties are known, it becomes significant to study the connections between them by trajectories of solutions of the given system. These are called *homoclinic* or *heteroclinic* solutions (sometimes *pulses* or *kinks*) according to whether they just describe a loop based at one single equilibrium or they "start" and "end" at two distinct equilibria.

In addition to their intrinsic mathematical interest (for autonomous second order systems, for instance, homoclinics and heteroclinics act in the phase portrait as separatrices between regions where solutions behave differently; when they appear as connections between saddles they prevent structural stability, see, e.g., [38]), homoclinic and heteroclinic solutions appear in the context of a number of mathematical models for problems arising in Mechanics, Chemistry and Biology.

The study of existence of homoclinic and heteroclinic solutions has a long history. Besides phase plane analysis, whose applicability is confined to second order autonomous equations, the study of such solutions has been often made with resource to the geometric theory of ordinary differential equations and dynamical systems techniques. In the nineteenth century, Poincaré [64] already studies perturbed time periodic systems. Poincaré's results have been the outbreak of many works. In particular, Melnikov's theory provides instruments for the analysis of how homoclinics and heteroclinics are affected by perturbations on an Hamiltonian system. The main idea is that existence of loops or connections at some rest points can be proved by analyzing the intersection properties of the stable and unstable manifolds through those equilibrium points. In the 60's, Melnikov [54] proves by analytical method the existence of homoclinics for non conservative perturbations, leading to chaos. Smale [81,82] then shows that in presence of a transverse homoclinic point, the Poincaré map admits a Bernouilli shift structure. Similar ideas are present in works of Birkhoff [15]. We refer to Moser [57] and Wiggins [92] for further developments.

However, starting mainly in the 80's, a functional analytic approach added powerful tools to the research in this field. Variational methods, combining classical ideas with modern critical point theory, have thus provided a wealth of new results. A large number of contributions are devoted to this topic, the main developments being due to Ambrosetti, Bolotin, Coti Zelati, Ekeland, Rabinowitz and Séré [16,66,27,28,17,9,79,67]. The advantage of this approach comes from the fact that we can often bypass the question of transversality of the stable and unstable manifolds whose verification is delicate in practice, or obtain weaker nondegeneracy conditions. Moreover, it leads to results of a global nature. Some comparison with geometric methods and variational interpretation of Melnikov and Smale–Birkhoff theorems are studied by Ambrosetti and Badiale [7,8], see also [56,86].

This monograph is devoted mainly to the existence of heteroclinics in several types of differential equations and systems of the second and fourth orders. We divide our survey in two parts. Part 1 deals with second order equations and systems, while Part 2 concerns several important types of fourth order equations.

Part 1 starts with the analysis of classical conservative systems arising in Mechanics, like

$$u'' = \nabla V(u), \tag{1.1}$$

where $V \in C^1(\mathbb{R}^N, \mathbb{R})$ is a potential (according to the usual classical mechanics terminology, $-V$ should be called the potential) with several equilibria at the same (minimum) level. We have selected a basic set of results for this and similar problems, where V may also depend on time and be periodic in each variable. We present them with some detail. These results are due mainly to P.H. Rabinowitz and to T.O. Maxwel. Further results are mentioned without proofs. Roughly speaking, the common underlying idea is the following: suppose that V has two minima, say ξ and η; we look for heteroclinic connections between ξ and η as minimizers of the *action functional*

$$\int_{-\infty}^{+\infty} \left(\frac{|u'(t)|^2}{2} + V\big(u(t)\big) \right) \mathrm{d}t \tag{1.2}$$

in an appropriate class of functions u defined in $\mathbb{R}$ so that $u(-\infty) = \xi$ and $u(+\infty) = \eta$. Then u appears as a solution of (1.1) since this is the Euler–Lagrange equation of (1.2).

We then proceed to a different kind of problem. To state it in a simple form, we recall the Fisher–Kolmogorov type equation,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + g(u), \tag{1.3}$$

proposed as a model of diffusion in Biomathematics [34]. Here g is a positive, continuous function in $[0, 1]$ such that $g(0) = g(1) = 0$. In the original model, $g(u) = u(1 - u)$. A significant feature of the dynamics of (1.3) is the existence of *travelling wave fronts*, which are solutions of the form

$$u(x, t) = U(x - ct) \tag{1.4}$$

for some constant $c > 0$. The meaning of c is the speed of wave propagation. The *front profile* $U(s)$ solves then the ordinary differential equation

$$U'' + c\,U' + g(U) = 0, \tag{1.5}$$

where we look at c as a parameter. It is required that the profile satisfies $U(-\infty) = 1$, $U(+\infty) = 0$. Hence this is a problem of existence of heteroclinics for (1.5) and it is a classical one with a very rich literature. Its mathematical treatment began with Kolmogorov, Petrovski and Piskounov [45] and reached a highlight with the paper by Aronson and Weinberger [11]. However, it has continually attracted the interest of many mathematicians up to nowadays. We give an account of the most basic features of this problem dealing with more general equations than (1.5). We present a proof of the existence of a *threshold speed* c^* such that the heteroclinic exists if and only if $c \geqslant c^*$. Following an approach similar to that of Malaguti and Marcelli [48] and introducing some variational procedures like those developed in Arias, Campos, Robles-Pérez and Sanchez [10] we essentially reduce the problem to a first order ordinary differential equation and use the elementary theory of the Cauchy problem. We show that the same method is efficient in the treatment of equations in combustion theory, which differ from (1.3) in that $g(u)$ changes sign in [0, 1] while keeping its integral positive; in this case, it is curious that the heteroclinic appears for just one value of c.

In the second part, we consider a class of fourth order autonomous differential equations of the form

$$u'''' - \beta u'' + u^3 - u = 0, \tag{1.6}$$

where β is a real parameter. This equation admits three equilibrium states: $-1, 0$ and $+1$ and we focus on the existence of heteroclinic solutions going from -1 to $+1$. The set of bounded solutions of (1.6) has been the object of much research in the past ten years. In an impressive series of papers [59–62], Peletier and Troy have performed a systematic study of periodic, homoclinic, heteroclinic and chaotic solutions of the model equation (1.6) for the parameter range $\beta \geqslant 0$. In Section 4 we present an overview of results obtained concerning heteroclinic solutions. These include the pioneer works of Peletier and Troy and later results of Kalies and VanderVorst [43], Kalies, Kwapisz and VanderVorst [42] and van den Berg [87]. We give a first insight of methods that can be used to track heteroclinic orbits. It is convenient for simplicity to first restrict our attention to the simple model (1.6). The existence of heteroclinics for (1.6) for any $\beta \geqslant 0$ has been first proved by Peletier and Troy using a shooting method. We briefly describe their arguments in Section 4.1.

Equation (1.6) has also a variational structure. It is an elementary fact that heteroclinic solutions of (1.6) correspond to critical points of the functional

$$\mathcal{F}_\beta(u) := \int_{-\infty}^{+\infty} \left(\frac{1}{2}\left(u''^2 + \beta u'^2\right) + \frac{1}{4}\left(u^2 - 1\right)^2 \right) \mathrm{d}x$$

in an appropriate functional space. The existence of heteroclinics via variational arguments was first investigated by Peletier, Troy and VanderVorst [58] and Kalies and VanderVorst [43]. For $\beta \geqslant 0$, $\mathcal{F}_\beta$ is a positive functional. It is therefore natural to look for heteroclinics as minimizers of $\mathcal{F}_\beta$. Section 4.2 deals with a global minimization procedure which works fine for all $\beta \geqslant 0$.

When $0 \leqslant \beta \leqslant \sqrt{8}$, more heteroclinics are obtained via local minimization in well-chosen homotopy classes. Kalies, Kwapisz and VanderVorst [42] have defined precise types of functions that describe the complex structure of those solutions. In Section 4.3, we describe these homotopy classes and the profiles of the corresponding minimizers.

Many of the arguments used in the minimization process (both in the whole space and in homotopy classes) rely on the positivity of the Lagrangian

$$L\left(u, u', u''\right) = \frac{1}{2}\left(u''^2 + \beta u'^2\right) + \frac{1}{4}\left(u^2 - 1\right)^2$$

and therefore on the positivity of the parameter β. A first attempt to consider changing sign Lagrangian is made in [20,37]. The authors consider the more general functional

$$\mathcal{F}_g(u) := \int_{-\infty}^{+\infty} \left(\frac{1}{2}\big(u''^2 + g(u)u'^2\big) + f(u) \right) \mathrm{d}x \tag{1.7}$$

whose Euler–Lagrange equation is given by

$$u'''' - g(u)u'' - \frac{1}{2}g'(u)u'^2 + f'(u) = 0. \tag{1.8}$$

Here f is assumed to be a double-well potential with bottoms at ± 1 and g is neither necessarily constant nor positive. The main idea of [37] is to impose a condition on g to ensure a lower bound on the action $\mathcal{F}_g(u)$. In Section 5.1 and Section 5.2, we use the ideas of [19,20,37] to prove the existence of a global minimizer of $\mathcal{F}_g$ under convenient assumptions.

In Section 6, we discuss the case of an equation similar to (1.8) where f is a triple-well potential and address the following question. Does the dynamics possess a heteroclinic orbit connecting the extremal equilibria? Let us take a look at a classical mechanics analogy. Consider a moving particle in a potential characterized by three hills of equal height. To fix the ideas, suppose the tops of the peaks are located at -1, 0 and $+1$ and consider a motion starting from -1 at time $t \to -\infty$. As the potential energy is identical at the top of each hill, the law of energy conservation implies that the particle needs an infinite amount of time to reach the top of the second hill and therefore cannot pass through the middle-equilibrium. For the fourth order equation (1.8), the energy reads

$$E\big(u, u', u'', u'''\big) = u'''u' - \frac{1}{2}u''^2 + \frac{g(u)}{2}u'^2 + f(u).$$

For this equation of motion, a particle does not need to come at rest at the top of the second hill as the constant of motion can be satisfied with a non-zero u' due to the presence of the new terms $u'''u' - \frac{1}{2}u''^2$. In other words, the intersection of the zero energy manifold and the space $u = 0$ does not reduce to a point.

We answer positively the above question at least when the middle-equilibrium is of *saddle-focus* type. We mention that this question is relevant in the study of Ginzburg–Landau models of amphiphilic systems [36]. We refer to Section 6 for a brief description of these models.

We come back to equations with a double-well potential in Section 7. We present an existence result of multi-transition connections for Eq. (1.8). These solutions are obtained as local minimizers in classes of functions with prescribed profiles. We describe these subsets and explain briefly how the local minimization process works.

Each section is followed by some remarks and complementary results that we have chosen not to consider in details. We also give references for some extensions of results presented here in a simplified version, we refer to open questions, problems for which less is known or simply mention some related topics that did not find their way in this monograph.

Part 1. Second Order Equations and Systems

2. Second order Hamiltonian systems

2.1. *Heteroclinics for some scalar second order differential equations*

Recall that a good description of the various kinds of solutions of the mathematical pendulum equation

$$u'' + a\sin u = 0 \quad (a > 0) \tag{2.1}$$

is provided by the representation of the corresponding trajectories in the phase plane (u, u'). These are level curves of the energy, that is, they are a locus of points (u, u') of the form

$$\frac{u'^2}{2} + a(1 - \cos u) = k \tag{2.2}$$

for some constant $k \in \mathbb{R}^+$. That the energy function

$$E\big(u, u'\big) := \frac{u'^2}{2} + a(1 - \cos u)$$

is constant along solutions of (2.1) is an elementary consequence of multiplying Eq. (2.1) by u'.

When $k = 0$ in (2.2), we obtain the stable equilibria $u = 2k\pi$ $(k \in \mathbb{Z})$. If $0 < k < 2a$, the trajectories are closed curves corresponding to periodic solutions. For $k > 2a$, we obtain unbounded trajectories corresponding to solutions with periodic derivative. Finally, $k = 2a$ in (2.2) yields a locus consisting of the unstable equilibria ($u = (2k + 1)\pi$, $k \in \mathbb{Z}$) and the graphs of the functions

$$u' = \pm\sqrt{2a(1 + \cos u)}, \quad u \neq (2k + 1)\pi \text{ for all } k \in \mathbb{Z}. \tag{2.3}$$

For instance, the solutions having as trajectories the graph of this function in $]-\pi, \pi[$ have the property that

$$\lim_{t\to+\infty} u(t) = \pi, \qquad \lim_{t\to-\infty} u(t) = -\pi$$

and

$$\lim_{t\to\pm\infty} u'(t) = 0$$

or the same conditions with the roles of $+\infty$ and $-\infty$ reversed. Hence these trajectories connect two distinct (consecutive) unstable equilibria. They are called *heteroclinics* and

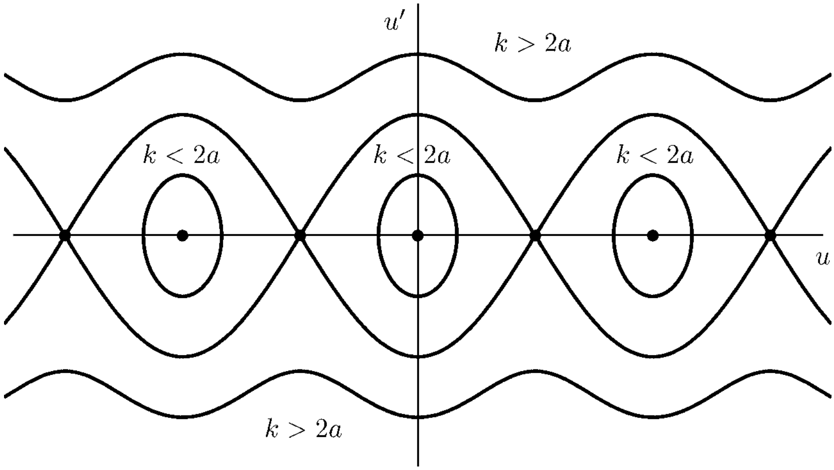

Fig. 1. The pendulum phase-plane.

any underlying solution is called a *heteroclinic solution* of (2.1). In this example it is apparent that they separate regions of the (u, u')-plane where the solutions of (2.1) have a different nature, see Fig. 1.

Of course, given the physical meaning of (2.1), it is sometimes preferable to depict trajectories not in a plane but in a cylinder (which is a plane where the points (u, u') and (v, v') are identified if and only if $u \equiv v \pmod{2\pi}$ and $u' = v'$. Then $(-\pi, 0)$ and $(\pi, 0)$ are in fact the same equilibrium and a trajectory connecting these points in the plane becomes a trajectory with equal limits at $\pm\infty$. We would then rather speak of a *homoclinic*. However, if one forgets the 2π-periodicity of the potential $a(1 - \cos u)$, or, for that matter, if one modifies it outside, say, the interval $]-\pi, \pi[$, the consideration of heteroclinics is meaningful.

Let us consider a more general autonomous scalar equation

$$u'' = f(u), \tag{2.4}$$

where $f \in C(\mathbb{R}, \mathbb{R})$ is a function such that

(A1) $f(\pm 1) = 0$;

(A2) there exists a primitive F of f such that $F(-1) = F(1) = 0$ and $F(u) > 0$ for all $u \in]-1, 1[$.

Hence Eq. (2.4) has two equilibria, $u = \pm 1$, at the (same) zero level of the potential. As for solutions of (2.4) energy is conserved, that is

$$\frac{u'^2}{2} - F(u) = K \tag{2.5}$$

for some constant K, it makes sense to look for heteroclinic solutions connecting -1 and 1, i.e. solutions such that

$$u(\pm\infty) := \lim_{x\to\pm\infty} u(x) = \pm 1 \quad \text{and} \quad u'(\pm\infty) := \lim_{x\to\pm\infty} u'(x) = 0$$

or the same properties with the roles of the $+\infty$ and $-\infty$ reversed. In fact, for such solutions we must have $K = 0$ in (2.5) and the corresponding phase plane trajectories are given explicitly by

$$u' = \pm\sqrt{2F(u)}, \quad -1 < u < 1.$$

These are equations with separable variables. With the $+$ sign, for example, integrating we obtain

$$\int_0^u \frac{\mathrm{d}v}{\sqrt{2F(v)}} = t + C, \tag{2.6}$$

for some constant C. In particular, the equilibria ± 1 are not reached in finite time whenever

$$\int_0^{\pm 1} \frac{\mathrm{d}v}{\sqrt{F(v)}}$$

diverges. This condition is obviously satisfied provided that f is locally Lipschitz. More generally, it also holds if F is at most quadratic near its minima, that is if there exists $c > 0$ so that

$$F(u) \leqslant c(u \pm 1)^2$$

in a neighborhood of -1 and $+1$, respectively.

Now we look for a heteroclinic of (2.4) from another angle. Instead of using elementary integration techniques, we show that such a solution can be characterized by a variational property. Needless to say, it may seem cumbersome to treat that simple problem in such an involved way, but since the variational method has an important role to play in the search of heteroclinics for systems and non-autonomous equations, it is worth to grasp the main ideas in an uncomplicated case.

Formally, (2.4) is the Euler–Lagrange equation of the functional

$$\mathcal{I}(u) := \int_{-\infty}^{+\infty} \left(\frac{u'^2}{2} + F(u) \right) \mathrm{d}t, \tag{2.7}$$

where $F(u)$ is the primitive of f as given in (A2). We look for the heteroclinics of (2.4) as minimizers of $\mathcal{I}$ in the functional space

$$\mathcal{E} := \left\{ u \in H^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}) \mid u(\pm\infty) = \pm 1 \right\}.$$

In fact it will be clear that we may confine ourselves to functions taking values in $[-1,1]$ by simply assuming that F is extended by 0 on $]-\infty,-1[\cup]+1,+\infty[$. We therefore make this hypothesis from now on.

FACT 1. *Let* $\varepsilon \in]0,1[$ *and*

$$\beta_\varepsilon := \min\left\{F(z) \;\middle|\; 1-\varepsilon \leqslant z \leqslant 1-\frac{\varepsilon}{2} \text{ or } -1+\frac{\varepsilon}{2} \leqslant z \leqslant -1+\varepsilon\right\}.$$

If $u \in \mathcal{E}$ *has the property that there exist* $t_1, t_2 \in \mathbb{R}$ *such that* $u(t_1) = 1-\frac{\varepsilon}{2}$ *and* $u(t_2) = 1-\varepsilon$ *(or* $u(t_1) = -1+\frac{\varepsilon}{2}$ *and* $u(t_2) = -1+\varepsilon$*), then we have*

$$\mathcal{I}(u) \geqslant \left|\int_{t_1}^{t_2}\left(\frac{u'^2}{2}+F(u)\right)\mathrm{d}t\right| \geqslant \frac{\varepsilon\sqrt{\beta_\varepsilon}}{\sqrt{2}}.$$

PROOF. We may assume $t_1 < t_2$ and $1-\varepsilon \leqslant u(t) \leqslant 1-\frac{\varepsilon}{2}$ or $-1+\frac{\varepsilon}{2} \leqslant u(t) \leqslant -1+\varepsilon$ for $t \in [t_1,t_2]$. Using the positivity of the integrand and Schwarz's inequality we write

$$\mathcal{I}(u) \geqslant \int_{t_1}^{t_2}\left(\frac{u'^2}{2}+F(u)\right)\mathrm{d}t \geqslant \frac{\varepsilon^2}{8(t_2-t_1)} + \beta_\varepsilon(t_2-t_1)$$

and the conclusion follows from the elementary inequality $\frac{a^2}{x} + b^2 x \geqslant 2ab$ which holds for all $x \geqslant 0$. □

With these preliminaries we can state and prove:

THEOREM 2.1. *If* $f \in C([-1,1],\mathbb{R})$ *satisfies the assumptions* (A1), (A2) *and is extended by* 0 *outside the interval* $]-1,1[$, *the functional* $\mathcal{I}$ *defined by* (2.7) *attains a minimum in* $\mathcal{E}$. *A minimizer is a heteroclinic solution of* (2.4) *connecting* -1 *and* 1.

PROOF. Let $(u_n)_n \subset \mathcal{E}$ be a minimizing sequence that is $\mathcal{I}(u_n) \to \inf_{\mathcal{E}} \mathcal{I}$. By passing to $v_n = \sup(-1, \inf(u_n, 1))$ and observing that $\mathcal{I}(v_n) \leqslant \mathcal{I}(u_n)$, we may assume without loss of generality $-1 \leqslant u_n \leqslant 1$.

For each $\varepsilon > 0$, we find an interval $[s_n, t_n]$ such that $u_n(s_n) = -1+\varepsilon$, $u_n(t_n) = 1-\varepsilon$ and

$$-1+\varepsilon \leqslant u_n(t) \leqslant 1-\varepsilon \quad \text{for all } t \in [s_n, t_n].$$

If u_n takes values greater than $-1+\varepsilon$ in $]-\infty, s_n]$, we may choose $s_n' < s_n$ so that $u_n(s_n') = -1+\varepsilon$ and $-1 \leqslant u_n \leqslant -1+\varepsilon$ in $]-\infty, s_n']$. In the same way we find $t_n' \geqslant t_n$ so that $u_n(t_n') = 1-\varepsilon$ and $1-\varepsilon \leqslant u_n \leqslant 1$ in $[t_n', +\infty[$. Define a new function

$$U_n(t) = \begin{cases} u_n(t-s_n+s_n') & \text{if } t \leqslant s_n, \\ u_n(t) & \text{if } s_n \leqslant t \leqslant t_n, \\ u_n(t+t_n'-t_n) & \text{if } t \geqslant t_n. \end{cases}$$

By the translation invariance of the integrals and the fact that the integrand is positive it is clear that

$$\mathcal{I}(U_n) \leqslant \mathcal{I}(u_n)$$

so that $(U_n)_n \subset \mathcal{E}$ is again a minimizing sequence which in addition satisfies $-1 \leqslant U_n \leqslant -1+\varepsilon$ in $]-\infty, s_n]$ and $1-\varepsilon \leqslant U_n \leqslant 1$ in $[t_n, +\infty[$. The sequence $t_n - s_n$ is bounded since

$$\mathcal{I}(U_n) \geqslant \Big(\min_{-1+\varepsilon \leqslant z \leqslant 1-\varepsilon} F(z)\Big)(s_n - t_n).$$

Observe also that as $\mathcal{I}$ is translation invariant, we may assume that $s_n = 0$.

Now, since $\sup_n \|U_n\|_{L^\infty} \leqslant 1$ and $\sup_n \|U_n'\|_{L^2}$ is bounded, we may apply the diagonal procedure to extract a subsequence that we still denote by $(U_n)_n$, and we obtain a function $u \in H^1_{\mathrm{loc}}(\mathbb{R})$ such that

$$U_n \overset{C_{\mathrm{loc}}(\mathbb{R})}{\longrightarrow} u$$

i.e. uniformly in compact intervals and

$$U_n' \overset{L^2(\mathbb{R})}{\rightharpoonup} u'.$$

In addition, we have $t_n \to \bar{t} > 0$.

Combining weak lower semicontinuity of the L^2-norm and Fatou's lemma we infer

$$\begin{aligned}\mathcal{I}(u) &\leqslant \liminf_{n\to\infty} \int_{-\infty}^{+\infty} \frac{U_n'^2}{2}\,\mathrm{d}t + \liminf_{n\to\infty} \int_{-\infty}^{+\infty} F(U_n)\,\mathrm{d}t \\ &\leqslant \lim_{n\to\infty} \mathcal{I}(U_n) = \inf_{\mathcal{E}} \mathcal{I}.\end{aligned} \tag{2.8}$$

By uniform convergence, we have $-1 \leqslant u(t) \leqslant -1+\varepsilon$ for $t \leqslant 0$ and $1-\varepsilon \leqslant u(t) \leqslant 1$ for $t > \bar{t}$. On the other hand, the fact that $\int_{-\infty}^{+\infty} F(u)\,\mathrm{d}t < +\infty$ implies that

$$\liminf_{t\to-\infty} u(t) = -1 \quad \text{and} \quad \limsup_{t\to+\infty} u(t) = 1.$$

In fact we have $\lim_{t\to\pm\infty} u(t) = \pm 1$. Otherwise there exist $\delta > 0$ and infinitely many disjoint intervals $[t_1, t_2]$ in the conditions of Fact 1 (with $\varepsilon = \delta$), implying $\mathcal{I}(u) = +\infty$ and contradicting (2.8). It follows that $u \in \mathcal{E}$ and therefore

$$\mathcal{I}(u) = \min_{\mathcal{E}} \mathcal{I}.$$

The fact that u is a solution of (2.4) follows by the usual argument that gives the Euler–Lagrange equation of a functional. For each $\varphi \in C_c^1(\mathbb{R})$, and $\tau \in \mathbb{R}$, the function $u + \tau\varphi$ belongs to $\mathcal{E}$; we compute

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\bigg|_{\tau=0} \mathcal{I}(u + \tau\varphi) = \int_{-\infty}^{+\infty} \big(u'\varphi' + f(u)\varphi\big)\,\mathrm{d}t = 0$$

and, by the Du Bois–Reymond lemma, u satisfies (2.4).

Finally, since u satisfies (2.5) for some K and there exist sequences $t_n \to \pm\infty$ with $u'(t_n) \to 0$, we infer that $K = 0$ and we then conclude that $u'(\pm\infty) = 0$. □

We next apply the above method to a less trivial situation: let us consider the second order non-autonomous differential equation

$$u'' = a(t) f(u), \tag{2.9}$$

where $f \in C(\mathbb{R}, \mathbb{R})$ satisfies the assumptions (A1), (A2) and $a \in L^\infty(\mathbb{R}, \mathbb{R})$ is such that

(A3) there exist $a_1, a_2 \in \mathbb{R}$ so that $0 < a_1 \leqslant a(t) \leqslant a_2$ for all $t \in \mathbb{R}$.

We look for a heteroclinic connection between the equilibria -1 and $+1$. In the absence of a conservation law, the variational argument appears as a natural device. So we now consider the functional

$$\mathcal{J}(u) := \int_{-\infty}^{+\infty} \left(\frac{u'^2}{2} + a(t)F(u)\right)\mathrm{d}t \tag{2.10}$$

and seek conditions that allow to minimize it in $\mathcal{E}$. If $a(t)$ is a T-periodic function then the proof of Theorem 2.1 can be easily adapted. While we cannot completely mimic the powerful modification arguments used in the autonomous case, the periodicity of a allows to localize a transition from a small neighborhood of -1 to a small one of $+1$. Indeed, keeping the notations of the proof of Theorem 2.1, the sequence of left endpoints of the intervals $[s_n, t_n]$ can be placed in the interval $[0, T]$ by time translations whose length is a multiple of T. The remaining of the proof is then similar.

If a does not possess any symmetry or periodicity property then we have to face a possible loss of compactness as the interval $[s_n, t_n]$ could escape to $+\infty$ or $-\infty$ so that the weak limit of u_n could either be one equilibrium or a homoclinic solution. The natural way to avoid such behaviours is to impose some coercivity assumption on the function a at $\pm\infty$. In some sense this penalizes transitions tending to $\pm\infty$.

Note that Fact 1 proved above has an obvious extension for this new functional: under condition (A3), it suffices to replace, in the right-hand side of the inequality, $\frac{\varepsilon\sqrt{\beta_\varepsilon}}{\sqrt{2}}$ with $\frac{\varepsilon\sqrt{a_1\beta_\varepsilon}}{\sqrt{2}}$, namely we have:

FACT 1′. *Under the assumptions of Fact* 1, *we have*

$$\mathcal{J}(u) \geqslant \left|\int_{t_1}^{t_2} \left(\frac{u'^2}{2} + a(t)F(u)\right)\mathrm{d}t\right| \geqslant \frac{\varepsilon\sqrt{a_1\beta_\varepsilon}}{\sqrt{2}}.$$

The following two facts are useful to ensure that the quasi-minimizers stay close to ± 1 once they enter a sufficiently small neighborhood of ± 1.

FACT 2. *Let* $\varepsilon \in \,]0,1[$ *and*

$$\mathcal{E}_{\bar{t},1-\varepsilon} := \left\{u \in C\left(\left[\bar{t},+\infty\right[,\mathbb{R}\right) \;\middle|\; u\left(\bar{t}\right) = 1-\varepsilon,\ u(+\infty) = 1,\ u' \in L^2\left(\left[\bar{t},+\infty\right[\right)\right\}.$$

Then, setting

$$\mathcal{J}_{[\bar{t},+\infty[}(u) := \int_{\bar{t}}^{+\infty} \left(\frac{u'^2}{2} + a(t)F(u)\right) \mathrm{d}t,$$

there exists $v \in \mathcal{E}_{\bar{t},1-\varepsilon}$ *so that* $1-\varepsilon \leqslant v(t) \leqslant 1$ *for all* $t \geqslant \bar{t}$ *and*

$$\mathcal{J}_{[\bar{t},+\infty[}(v) \leqslant \frac{\varepsilon^2}{2} + a_2 \max_{z\in[1-\varepsilon,1]} F(z).$$

PROOF. It suffices to compute $\mathcal{J}_{[\bar{t},+\infty[}(v)$ where

$$v(t) = \begin{cases} 1-\varepsilon+\varepsilon\left(t-\bar{t}\right), & \text{if } \bar{t} \leqslant t \leqslant \bar{t}+1, \\ 1, & \text{if } t \geqslant \bar{t}+1. \end{cases}$$ □

Analogously we have:

FACT 3. *Let* $\varepsilon \in \,]0,1[$ *and*

$$\mathcal{E}_{\bar{t},-1+\varepsilon} := \left\{u \in C\left(\left]-\infty,\bar{t}\right],\mathbb{R}\right) \;\middle|\; u\left(\bar{t}\right) = -1+\varepsilon,\ u(-\infty) = -1,\right.$$
$$\left. u' \in L^2\left(\left]-\infty,\bar{t}\right]\right)\right\}.$$

Then, setting

$$\mathcal{J}_{]-\infty,\bar{t}]}(u) := \int_{-\infty}^{\bar{t}} \left(\frac{u'^2}{2} + a(t)F(u)\right) \mathrm{d}t,$$

there exists $v \in \mathcal{E}_{\bar{t},-1+\varepsilon}$ *such that* $-1 \leqslant v(t) \leqslant -1+\varepsilon$ *for all* $t \leqslant \bar{t}$ *and*

$$\mathcal{J}_{]-\infty,\bar{t}]}(v) \leqslant \frac{\varepsilon^2}{2} + a_2 \max_{z\in[-1,-1+\varepsilon]} F(z).$$ □

The family of functionals

$$\mathcal{I}_\alpha(u) := \int_{-\infty}^{+\infty} \left(\frac{u'^2}{2} + \alpha F(u)\right) \mathrm{d}t,$$

where $\alpha > 0$ is constant, is useful in what follows. According to Theorem 2.1, $\mathcal{I}_\alpha$ has a minimum in $\mathcal{E}$. Hence we may define $u_\alpha \in \mathcal{E}$ such that

$$\varphi(\alpha) := \min_{u\in\mathcal{E}} \mathcal{I}_\alpha(u) = \mathcal{I}_\alpha(u_\alpha).$$

FACT 4. *The function φ is a strictly increasing, continuous function in* $]0, +\infty[$.

PROOF. The first statement is obvious. The second is a consequence of the inequalities

$$\varphi(\alpha) + (\beta - \alpha)\int_{-\infty}^{+\infty} F(u_\beta)\,\mathrm{d}t \leqslant \varphi(\beta) \leqslant \varphi(\alpha) + (\beta - \alpha)\int_{-\infty}^{+\infty} F(u_\alpha)\,\mathrm{d}t$$

together with the fact that, when β is close to α, $\int_{-\infty}^{+\infty} F(u_\beta)\,\mathrm{d}t$ is uniformly bounded from above. □

We can now state the following theorem.

THEOREM 2.2. *Assume that $f \in C(\mathbb{R}, \mathbb{R})$, $a \in L^\infty(\mathbb{R}, \mathbb{R})$ satisfy* (A1)–(A3). *If, in addition,*

$$\lim_{|t|\to\infty} a(t) = a_2$$

and $a(t) < a_2$ in some set of nonzero measure, then (2.9) *has a heteroclinic solution from* -1 *to* 1. *This solution takes values in* $[-1, 1]$.

PROOF. Observe first that we clearly have $\inf_{\mathcal{E}} \mathcal{J} < \varphi(a_2)$ and using Fact 4, we may find $\alpha \in\,]0, a_2[$ so that $\inf_{\mathcal{E}} \mathcal{J} < \varphi(\alpha)$.

Fix $\bar\varepsilon \in\,]0, 1[$ and then choose $\varepsilon > 0$ sufficiently small to satisfy $\varepsilon < \bar\varepsilon/2$ and

$$\left(1 + \frac{a_2}{a_1}\right)\left(\frac{\varepsilon^2}{2} + a_2\mu_\varepsilon\right) < \min\left(\frac{\bar\varepsilon\sqrt{a_1\beta_{\bar\varepsilon}}}{\sqrt{2}}, \frac{\varphi(\alpha) - \inf_{\mathcal{E}} \mathcal{J}}{2}\right) \tag{2.11}$$

where $\beta_{\bar\varepsilon}$ is defined as in Fact 1 and

$$\mu_\varepsilon := \max_{z\in[-1,-1+\varepsilon]\cup[1-\varepsilon,1]} F(z).$$

Let $u_n \in \mathcal{E}$ be a minimizing sequence for $\mathcal{J}$. Define the interval $[s_n, t_n]$ with respect to u_n and ε as in the proof of Theorem 2.1. According to the choice of ε, Fact 1′, Fact 2 and Fact 3 and modifying $u_n(t)$ in $]-\infty, s_n] \cup [t_n, +\infty[$ if necessary, we may assume that

$$-1 \leqslant u_n(t) \leqslant -1 + \bar\varepsilon \quad \text{if } t \leqslant s_n, \qquad 1 - \bar\varepsilon \leqslant u_n(t) \leqslant 1 \quad \text{if } t \geqslant t_n$$

and

$$\sup\bigl(\mathcal{J}_{]-\infty,s_n]}(u_n),\ \mathcal{J}_{[t_n,+\infty[}(u_n)\bigr) \leqslant \frac{\varepsilon^2}{2} + a_2\mu_\varepsilon.$$

As in the proof of Theorem 2.1, $t_n - s_n$ is bounded. We claim that s_n is bounded from above and t_n is bounded from below. Let us prove this for s_n, the proof for t_n being similar. If the claim is not true there exists $s_0 \in \mathbb{R}$ such that $a(t) \geqslant \alpha$ whenever $t \geqslant s_0$ so that $a(t) \geqslant \alpha$ whenever $t \geqslant s_n$ for large n. We then compute

$$\begin{aligned}\mathcal{J}(u_n) &\geqslant \int_{s_n}^{+\infty} \left(\frac{u_n'^2}{2} + \alpha F(u_n)\right) \mathrm{d}t \\ &\geqslant \varphi(\alpha) - \frac{a_2}{a_1}\left(\frac{\varepsilon^2}{2} + a_2\mu_\varepsilon\right)\end{aligned}$$

and by our choice of ε in (2.11) we infer that

$$\mathcal{J}(u_n) \geqslant \frac{\varphi(\alpha) + \inf_{\mathcal{E}} \mathcal{J}}{2},$$

a contradiction with the assumption $\mathcal{J}(u_n) \to \inf_{\mathcal{E}} \mathcal{J}$.

It now follows that along some subsequence

$$u_n \overset{C_{\mathrm{loc}}(\mathbb{R})}{\longrightarrow} u, \quad u_n' \overset{L^2(\mathbb{R})}{\rightharpoonup} u', \quad s_n \to \bar{s}, \quad t_n \to \bar{t}$$

and in particular

$$-1 \leqslant u(t) \leqslant -1 + \varepsilon \quad \text{for all } t \leqslant \bar{s}, \qquad 1 - \varepsilon \leqslant u(t) \leqslant 1 \quad \text{for all } t \geqslant \bar{t}.$$

Arguing as in the proof of Theorem 2.1, it is easily shown that $u \in \mathcal{E}$, $\mathcal{J}(u) = \inf_{\mathcal{E}} \mathcal{J}$ and u is a solution of (2.9).

It remains to show that $u'(\pm\infty) = 0$. Assume, by contradiction, that there exist $\tau_n \to \infty$ and $\Delta > 0$ with $|u'(\tau_n)| \geqslant \Delta$. As from (2.9) we have $u''(\pm\infty) = 0$, we may assume that, for n sufficiently large, $|u'(t)| \geqslant \Delta/2$ for all $t \in [\tau_n, \tau_n + 1]$. But then $|u(\tau_n) - u(\tau_n + 1)| \geqslant \Delta/2$, which is impossible since $u(\pm\infty) = \pm 1$. □

2.2. *Autonomous Hamiltonian systems*

In this section we consider a basic situation for systems, adapted from the results of Rabinowitz [66]. We concentrate on the autonomous system

$$u'' = \nabla V(u), \tag{2.12}$$

where $u = (u_1, \ldots, u_N)$ and $V \in C^1(\mathbb{R}^N, \mathbb{R})$ is a non-negative potential with several isolated equilibria at minimum level. Precisely, we assume

(A4) $V \geqslant 0$, $\mathcal{M} := V^{-1}(0)$ contains at least two points and

$$\inf\bigl\{|\xi_1 - \xi_2| \mid \xi_1, \xi_2 \in V^{-1}(0) \text{ and } \xi_1 \neq \xi_2\bigr\} > 0;$$

(A5) for each $\varepsilon > 0$, $\inf\{V(u) \mid \operatorname{dist}(u, \mathcal{M}) \geqslant \varepsilon\} > 0$.

As in the previous section, we investigate the existence of heteroclinic connections between elements of $\mathcal{M}$. A solution $u(t)$ of (2.12) such that there exist $\xi, \eta \in \mathcal{M}$, $\xi \neq \eta$ and

$$u(-\infty) = \xi, \quad u(+\infty) = \eta, \quad u'(\pm\infty) = 0$$

is called a *heteroclinic solution of* (2.12) *from ξ to η or a heteroclinic connection between ξ and η*. We say also that u *starts* at ξ and *ends* at η.

Clearly, if $u(t)$ is a heteroclinic of (2.12) from ξ to η, for every $c \in \mathbb{R}$ $u(t+c)$ is also a heteroclinic from ξ to η, and $u(-t)$ is a heteroclinic from η to ξ.

Let us fix an element in $\mathcal{M}$ which we may assume without loss of generality to be 0. Then we search for heteroclinics from 0 to some $\xi \in \mathcal{M} \setminus \{0\}$. They appear as suitable minimizers of the functional

$$\mathcal{J}(u) := \int_{-\infty}^{+\infty} \left(\frac{|u'|^2}{2} + V(u) \right) \mathrm{d}t. \tag{2.13}$$

Given $\xi \in \mathcal{M} \setminus \{0\}$, it is natural to consider the class of vector functions

$$\Gamma(\xi) := \left\{ u \in H^1_{\mathrm{loc}}\left(\mathbb{R}, \mathbb{R}^N\right) \mid u(-\infty) = 0,\ u(+\infty) = \xi \right\}.$$

Let us set

$$c(\xi) := \inf_{u \in \Gamma(\xi)} \mathcal{J}(u).$$

The following fact is proved with a computation quite similar to the proof of Fact 3 in Section 2.1.

FACT 1. *Assume that* (A5) *holds. Let* $\varepsilon > 0$, *and* $z \in H^1((t_1, t_2), \mathbb{R}^N)$ *be such that*

$$\operatorname{dist}\big(u(t), \mathcal{M}\big) \geqslant \varepsilon \quad \textit{for all } t \in [t_1, t_2].$$

Then

$$\int_{t_1}^{t_2} \left(\frac{|u'|^2}{2} + V(u) \right) \mathrm{d}t \geqslant \sqrt{2\alpha_\varepsilon} \big| u(t_2) - u(t_1) \big|,$$

where $\alpha_\varepsilon := \inf\{V(u) \mid \operatorname{dist}(u, \mathcal{M} \geqslant \varepsilon)\} > 0$.

FACT 2. *Assume that V satisfies* (A4) *and* (A5). *Let* $u \in H^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^N)$ *be such that* $\mathcal{J}(u) < +\infty$. *Then* $u(-\infty)$, $u(+\infty)$ *exist and belong to* $\mathcal{M}$.

PROOF. As $\int_{-\infty}^{+\infty} V(u(t))\,\mathrm{d}t < +\infty$, we have

$$\liminf_{t \to -\infty} V\big(u(t)\big) = \liminf_{t \to +\infty} V\big(u(t)\big) = 0.$$

By assumption (A5), there exist sequences $t_n \to -\infty,\ s_n \to +\infty$ so that

$$\lim_{n\to\infty} \operatorname{dist}\bigl(u(t_n), \mathcal{M}\bigr) = \lim_{n\to\infty} \operatorname{dist}\bigl(u(s_n), \mathcal{M}\bigr) = 0.$$

Let us prove that $\lim_{t\to+\infty} u(t)$ exists (a similar argument applies at $-\infty$): if the limit does not exist we are able to find

$$0 < \varepsilon < \gamma := \frac{1}{3} \inf\bigl\{|\xi_1 - \xi_2| \mid \xi_1 \neq \xi_2 \text{ and } \xi_1, \xi_2 \in \mathcal{M}\bigr\} \tag{2.14}$$

and disjoint intervals $[\tau_n, \sigma_n]$ such that $\operatorname{dist}(u(\tau_n), \mathcal{M}) = \varepsilon, \operatorname{dist}(u(\sigma_n), \mathcal{M}) = 2\varepsilon$ and $\operatorname{dist}(u(t), \mathcal{M}) \geqslant \varepsilon$ for all $t \in [\tau_n, \sigma_n]$. But then, by virtue of Fact 1, we infer that

$$\begin{aligned}\mathcal{J}(u) &\geqslant \sum_n \int_{\tau_n}^{\sigma_n} \left(\frac{|u'|^2}{2} + V\bigl(u(t)\bigr)\right) \mathrm{d}t \\ &\geqslant \sum_n \sqrt{2\alpha_\varepsilon}\, \min(\varepsilon, \gamma - \varepsilon) = +\infty\end{aligned}$$

which is a contradiction. □

FACT 3. *If V has infinitely many minima, then*

$$\lim_{|\xi|\to+\infty} c(\xi) = +\infty.$$

PROOF. It suffices to prove that, given $M \geqslant 0$, there exists another number $N = N(M)$ so that $\mathcal{J}(u) \leqslant M$ and $u \in \Gamma(\xi)$ implies $|\xi| \leqslant N$. Fix $\varepsilon > 0$ as in (2.14). For u as stated, let us call $[a, b]$ a transition interval from $\xi_a \in \mathcal{M}$ to $\xi_b \in \mathcal{M}$ ($\xi_a \neq \xi_b$) if $|u(a) - \xi_a| = \varepsilon = |u(b) - \xi_b|$ and for all $t \in [a, b]$, $\operatorname{dist}(u(t), \mathcal{M}) \geqslant \varepsilon$. By definition of $\Gamma(\xi)$ we find (disjoint) transition intervals $[t_i, s_i]$, from ξ_i to η_i, $i = 1, \ldots, k$, so that $\xi_1 = 0$, $\xi_{j+1} = \eta_j$ and the $\{\xi_j\}$ are all distinct. Fact 1 implies that

$$\mathcal{J}(u) \geqslant \sum_{i=1}^{k} \sqrt{2\alpha_\varepsilon}\,\gamma = k\sqrt{2\alpha_\varepsilon}\,\gamma.$$

Hence the number k of such intervals has an upper bound, $k \leqslant M/(\sqrt{2\alpha_\varepsilon}\,\gamma)$. Since $u(t)$ reaches $B(\xi, \varepsilon)$, we may assume that $\eta_k = \xi$. The length of each transition interval has an upper bound that depends only on M, since

$$M \geqslant \int_{t_i}^{s_i} V\bigl(u(t)\bigr)\, \mathrm{d}t \geqslant \alpha_\varepsilon (s_i - t_i).$$

We then infer that

$$\bigl|u(t_i) - u(s_i)\bigr| \leqslant M\sqrt{\frac{2}{\alpha_\varepsilon}}.$$

Therefore

$$|\xi| < 2\varepsilon + 2\varepsilon(k-2) + (k-1)M\sqrt{\frac{2}{\alpha_\varepsilon}}$$

and we obtain an upper bound for $|\xi|$ depending on M. □

THEOREM 2.3. *Let $V \in C^1(\mathbb{R}^N, \mathbb{R})$ satisfy the assumptions* (A4) *and* (A5), *with $0 \in \mathcal{M}$. Then the system* (2.12) *has a heteroclinic solution starting at* 0, *and another one ending at* 0.

PROOF. By reversing time, it suffices to show that the system (2.12) possesses a heteroclinic starting at 0. By the assumption (A4) and Fact 3 (if $\mathcal{M}$ is not finite) it is clear that

$$c := \min_{\zeta \in \mathcal{M}\setminus\{0\}} c(\zeta)$$

exists, since a bounded domain intersects $\mathcal{M}$ in a finite set. Hence there exists $\xi \in \mathcal{M} \setminus \{0\}$ such that $c = c(\xi)$. Now let (u_n) be a minimizing sequence, so that

$$u_n \in \Gamma(\xi) \quad \text{and} \quad \mathcal{J}(u_n) \to c(\xi) = c.$$

By translation invariance we may assume (with ε as in (2.14))

$$\left|u_n(0)\right| = \varepsilon, \quad \left|u_n(t)\right| \leqslant \varepsilon \quad \text{for all } t \leqslant 0 \text{ and all } n \in \mathbb{N}. \tag{2.15}$$

From the boundedness of $\mathcal{J}(u_n)$ it follows immediately that $(u_n')_n$ is bounded for the L^2-norm, and the arguments of the proof of Fact 3 imply that $\|u_n\|_{L^\infty}$ is also bounded. Hence for each finite interval $[a,b]$, $\|u_n\|_{H^1(a,b)}$ is bounded and, by the diagonal procedure we may extract a subsequence, also labelled $(u_n)_n$, and find a function $u \in H^1_{\text{loc}}(\mathbb{R}, \mathbb{R}^N)$ so that

$$u_n \overset{C_{\text{loc}}(\mathbb{R},\mathbb{R}^N)}{\longrightarrow} u \quad \text{and} \quad u_n' \overset{L^2(\mathbb{R},\mathbb{R}^N)}{\rightharpoonup} u'.$$

The weak lower semicontinuity of the norm and Fatou's lemma imply

$$\mathcal{J}(u) \leqslant \liminf_{n\to\infty} \mathcal{J}(u_n) = c.$$

By Fact 2, $u(\pm\infty)$ exist and belong to $\mathcal{M}$. From (2.15), using local uniform convergence we have $u(-\infty) = 0$. We now prove that $u(+\infty) = \xi$. This follows from the following two claims.

CLAIM 1. $u(+\infty) \neq 0$. Suppose by contradiction that $u(+\infty) = 0$. Fix $\delta > 0$ so that $4\delta < \varepsilon$ and

$$2\delta^2 + \sup_{B_{2\delta}(0)} V < \frac{\varepsilon}{4}\sqrt{2\alpha_{\varepsilon/2}}, \tag{2.16}$$

where $\alpha_{\varepsilon/2}$ has been defined in Fact 1. There exists t_δ so that $u(t) \in B_\delta(0)$ for all $t \geqslant t_\delta$, and therefore $u_m(t_\delta) \in B_{2\delta}(0)$ for all m sufficiently large. As $|u_m(0)| = \varepsilon$, Fact 1 implies that

$$\mathcal{J}(u_m) \geqslant \frac{\varepsilon}{2}\sqrt{2\alpha_{\varepsilon/2}} + \int_{t_\delta}^{+\infty}\left(\frac{|u_m'|^2}{2} + V(u_m)\right)\mathrm{d}t. \tag{2.17}$$

Define a new function $U_m \in \Gamma(\xi)$ by

$$U_m(t) := \begin{cases} 0 & \text{if } t \leqslant t_\delta - 1, \\ (t - t_\delta + 1)u_m(t_\delta) & \text{if } t_\delta - 1 \leqslant t \leqslant t_\delta, \\ u_m(t) & \text{if } t \geqslant t_\delta. \end{cases}$$

We clearly have the estimate

$$\mathcal{J}(U_m) \leqslant \frac{1}{2}(2\delta)^2 + \sup_{B_{2\delta}(0)} V + \int_{t_\delta}^{+\infty}\left(\frac{|u_m'|^2}{2} + V(u_m)\right)\mathrm{d}t. \tag{2.18}$$

From (2.16)–(2.18) we conclude that

$$\mathcal{J}(u_m) - \mathcal{J}(U_m) \geqslant \frac{\varepsilon}{4}\sqrt{2\alpha_{\varepsilon/2}},$$

which yields a contradiction since

$$c \leqslant \limsup_{m\to\infty} \mathcal{J}(U_m) \leqslant c - \frac{\varepsilon}{4}\sqrt{2\alpha_{\varepsilon/2}} < c.$$

CLAIM 2. If $\eta \in \mathcal{M} \setminus \{0, \xi\}$, $u(+\infty) \neq \eta$. Suppose, by contradiction, that $u(+\infty) = \eta \in \mathcal{M} \setminus \{0, \xi\}$. We then fix $\delta > 0$ so that $2\delta < \varepsilon$ and

$$2\delta^2 + \sup_{B_{2\delta}(\eta)} V < \frac{\gamma}{2}\sqrt{2\alpha_\varepsilon}.$$

Let t_δ be such that $u_m(t_\delta) \in B_{2\delta}(\eta)$ for all m sufficiently large. We introduce the new function $U_m \in \Gamma(\eta)$ defined by

$$U_m(t) := \begin{cases} u_m(t) & \text{if } t \leqslant t_\delta, \\ (1 - t + t_\delta)u_m(t_\delta) + (t - t_\delta)\eta & \text{if } t_\delta \leqslant t \leqslant t_\delta + 1, \\ \eta & \text{if } t \geqslant t_\delta + 1. \end{cases}$$

As $u_m \in \Gamma(\xi)$, there exists $\underline{t}_m < \bar{t}_m$ with $t_\delta < \underline{t}_m$, so that $|u_m(\underline{t}_m) - \eta| = \varepsilon$, $\operatorname{dist}(u_m(\bar{t}_m), \mathcal{M}) = \varepsilon$ and $\operatorname{dist}(u_m(t), \mathcal{M}) \geqslant \varepsilon$ for all $t \in [\underline{t}_m, \bar{t}_m]$. Hence we have

$$\mathcal{J}(u_m) \geqslant \int_{-\infty}^{t_\delta}\left(\frac{|u_m'|^2}{2} + V(u_m)\right)\mathrm{d}t + \gamma\sqrt{2\alpha_\varepsilon}.$$

On the other hand, we infer that

$$\mathcal{J}(U_m) \leqslant 2\delta^2 + \sup_{B_{2\delta}(\eta)} V + \int_{-\infty}^{t_\delta} \left(\frac{|u_m'|^2}{2} + V(u_m) \right) \mathrm{d}t$$

and we therefore conclude that

$$\mathcal{J}(u_m) - \mathcal{J}(U_m) \geqslant \frac{\gamma}{2}\sqrt{2\alpha_\varepsilon}.$$

It now follows that

$$\limsup_{m\to\infty} \mathcal{J}(U_m) \leqslant c - \frac{\gamma}{2}\sqrt{2\alpha_\varepsilon} < c,$$

which again contradicts the definition of the level c.

Having shown that $u \in \Gamma(\xi)$, it follows that $c = \mathcal{J}(u)$ is a minimum. The usual elementary argument of the Calculus of Variations shows that u is a solution of (2.12). At last we must check that $u'(\pm\infty) = 0$. For the autonomous system , this is particularly simple. Indeed, since u is a solution of (2.12), it satisfies the energy identity

$$\frac{|u'|^2}{2} + V(u) = K$$

for some constant K, and it is easy to see that $K = 0$. □

In presence of symmetries something else can be said. Let us consider the important case where V is periodic in each coordinate. For definiteness, we assume the period is the same for all coordinates and that the minimizers of V are the translates of 0:

(A6) $V \in C^1(\mathbb{R}^N, \mathbb{R})$ is a potential periodic in each variable u_i with period 1, $\min_{\mathbb{R}^N} V = 0$ and $\mathcal{M} = V^{-1}(0) = \mathbb{Z}^N$.

Of course, the condition (A6) implies that (A4) and (A5) hold, so that Theorem 2.3 applies to this class of potentials.

THEOREM 2.4. *If $V \in C^1(\mathbb{R}^N, \mathbb{R})$ satisfies* (A6) *then for each $\beta \in \mathcal{M}$ there are at least $2N$ heteroclinic solutions of* (2.12) *starting from β and at least $2N$ heteroclinic solutions of* (2.12) *ending at β.*

IDEA OF THE PROOF. We may assume that $\beta = 0$ and we keep the notation introduced before. Whenever $c(\xi)$ is attained for some $\xi \in \mathcal{M} \setminus \{0\}$, the minimizer is a heteroclinic from 0 to ξ. Let G be the subgroup of $\mathbb{Z}^N$ spanned by the elements $\xi \in \mathcal{M} \setminus \{0\}$ so that $c(\xi)$ is attained. It follows from Theorem 2.3 that $G \neq 0$. If $G \neq \mathbb{Z}^N$, we can select $\zeta \in \mathbb{Z}^N \setminus G$ so that

$$c(\zeta) = \min_{\zeta' \in \mathbb{Z}^N \setminus G} c(\zeta').$$

Then we prove that $c(\zeta)$ is attained, by mimicking the proof of Theorem 2.3. The main difference is in the proof of Claim 2. When we take $\eta \in \mathcal{M} \setminus \{0, \zeta\}$ we have to allow the possibility that $\eta \in G$. But this implies that $\zeta - \eta \in \mathbb{Z}^N \setminus G$. The minimizing sequence (u_m) is then modified to a sequence in $\Gamma(\zeta - \eta)$, given by

$$U_m(t) := \begin{cases} 0 & \text{if } t \leqslant t_\delta - 1, \\ (t - t_\delta + 1)\big(u_m(t_\delta) - \eta\big) & \text{if } t_\delta - 1 \leqslant t \leqslant t_\delta, \\ u_m(t) - \eta & \text{if } t \geqslant t_\delta. \end{cases}$$

The appropriate choice of δ and the fact that $V(u_m(t) - \eta) = V(u_m(t))$ lead to the usual contradiction with the choice of ζ. Therefore we conclude that $G = \mathbb{Z}^N$. This yields N $\mathbb{Z}$-independent elements ξ so that there exist heteroclinics u form 0 to ξ. For each such ξ, $u(-t) - \xi$ is a heteroclinic from 0 to $-\xi$. □

2.3. *Periodic Hamiltonian systems: Multiplicity. Multibump solutions*

In this section we are interested in non-autonomous systems of the form

$$u'' = \nabla_u V(t, u), \tag{2.19}$$

where the potential is periodic in the time variable as well as in each spatial variable u_i $(i = 1, \ldots, N)$. For simplicity, the period is supposed to be the same for all the variables. The results of the previous section carry over to this kind of systems provided the assumptions on the potential are adequately rephrased. Namely, $V(t, \cdot)$ must have equilibria at the minimum level of the potential that are independent of t.

Let us state the assumptions on V:

(A7) $V \in C^2(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ is 1-periodic in each variable t, u_i, $i = 1, \ldots, n$;

(A8) $V(t, 0) = 0 < V(t, u)$ for all $t \in \mathbb{R}$, $u \in \mathbb{R}^N \setminus \mathbb{Z}^N$.

Clearly, we may consider different given periods in each variable of V by rescaling. Note also that the assumptions (A7) and (A8) imply that $\mathcal{M} := \{u \in \mathbb{R}^N\colon\ V(t, u) = 0\} = \mathbb{Z}^N$ independently of t.

We easily recognize that Fact 1 and Fact 2 of the preceding section extend to the class of potentials we are considering by now. We only have to replace $V(u)$ by $V(t, u)$ and define

$$\alpha_\varepsilon := \inf\big\{V(t, u) \mid t \in \mathbb{R},\ \operatorname{dist}(u, \mathcal{M}) \geqslant \varepsilon\big\}. \tag{2.20}$$

As before, we make use of the functional $\mathcal{J}$ given by

$$\mathcal{J}(u) := \int_{-\infty}^{+\infty} \left(\frac{|u'|^2}{2} + V(t, u)\right) \mathrm{d}t. \tag{2.21}$$

For each given pair of elements $\xi, \eta \in \mathcal{M}$, consider the class of functions

$$\Gamma(\xi, \eta) := \big\{u \in C\big(\mathbb{R}, \mathbb{R}^N\big) \bigm| u(-\infty) = \xi,\ u(+\infty) = \eta \text{ and } u' \in L^2(\mathbb{R})\big\}.$$

Obviously, $\Gamma(\xi,\eta) \subset H^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^N)$. Set

$$c(\xi,\eta) := \inf_{u\in\Gamma(\xi,\eta)} \mathcal{J}(u). \tag{2.22}$$

We cannot guarantee, in general, that $c(\xi,\eta)$ is attained at a heteroclinic from ξ to η. However it is attained at a *heteroclinic chain* between ξ and η in the sense of the following statement which, according to Rabinowitz [70], was first obtained by Strobel in his Ph.D. dissertation.

THEOREM 2.5. *Assume $V \in C^2(\mathbb{R}\times\mathbb{R}^N, \mathbb{R})$ satisfies* (A7) *and* (A8). *For each pair $\xi,\eta \in \mathcal{M}$ ($\xi \neq \eta$) there exists a finite family $\{w_1,\dots,w_j\} \subset \mathcal{M}$ with $w_1 = \xi$, $w_j = \eta$, and a corresponding family of heteroclinics v^i of* (2.19), *with*

$$\mathcal{J}(v^i) = c(w_i, w_{i+1}), \quad v^i \in \Gamma(w_i, w_{i+1}),\ i = 1,\dots,j-1$$

and

$$\sum_{i=1}^{j-1} \mathcal{J}(v^i) = c(\xi,\eta).$$

It is easily seen by (A7)–(A8) that the v^i's may be assumed to be *basic heteroclinics* by which we mean that none of the heteroclinic chains, with more than one element, joining w_i and w_{i+1}, achieve the value $\mathcal{J}(v^i)$.

Variational methods have proved to be fruitful in investigating the existence of multiple heteroclinics for Hamiltonian systems of the type (2.19). Different approaches have been proposed by several authors but all of them share a common feature: at the start these always require some nondegeneracy assumption, like stating that some solution of (2.19) is in a sense isolated. This kind of assumption naturally excludes autonomous systems whereas it is commonly conjectured it is generically fulfilled when V depends on t although it is quite complicate to check it on concrete examples.

We shall take advantage from the existence of a basic heteroclinic chain from ξ to η in order to describe, following [70], how true heteroclinics from ξ to η may be obtained. We shall see that we can impose these solutions to spend arbitrarily large amounts of time near the "vertices" w_i. Hence there are infinitely many such solutions. This is one of several multiplicity results where solutions are distinguished by means of their behaviour with respect to the equilibria of the system (2.19): one therefore refers to "*multibump*" solutions.

Before stating the theorem, we first introduce a nondegeneracy condition. Given $\xi \neq \eta \in \mathcal{M}$, set

$$\mathcal{S}(\xi,\eta) := \{q(0) \in \mathbb{R}^N \mid q \in \Gamma(\xi,\eta),\ \mathcal{J}(q) = c(\xi,\eta)\}.$$

As $q(0) \in \mathcal{S}(\xi,\eta)$ implies $q(k) \in \mathcal{S}(\xi,\eta)$ for any integer k, we infer that ξ and η belong to $\bar{\mathcal{S}}(\xi,\eta)$. Denote respectively by $C_\xi(\xi,\eta)$ and $C_\eta(\xi,\eta)$ the connected components of ξ and η in $\bar{\mathcal{S}}(\xi,\eta)$. We assume

$$C_{w_i}(w_i, w_{i+1}) = \{w_i\} \quad \text{and} \quad C_{w_{i+1}}(w_i, w_{i+1}) = \{w_{i+1}\},$$
$$\text{for } i = 1, \dots, j-1. \tag{N}$$

It is shown in [70] that when these conditions fail then

$$C_{w_i}(w_i, w_{i+1}) = C_{w_{i+1}}(w_i, w_{i+1}),$$

which can be interpreted as a strongly degenerate situation.

THEOREM 2.6. *Assume* $V \in C^2(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ *satisfies* (A7) *and* (A8). *Let there exist a basic heteroclinic chain of* (2.19), *consisting in the basic heteroclinics* v^i *with vertices* $\xi = w_1, \dots, w_j = \eta$, *as described in Theorem* 2.5. *Then if* (N) *holds, the system* (2.19) *has infinitely many heteroclinic solutions connecting* ξ *and* η. *The time a solution spends in a neighbourhood of the vertices may be prescribed to exceed any arbitrary positive number.*

PROOF OF THEOREM 2.6. If I is an interval in $\mathbb{R}$, we write

$$\mathcal{J}_I(u) := \int_I \left(\frac{|u'|^2}{2} + V(t, u) \right) \mathrm{d}t.$$

By the nondegeneracy condition (N), given $\rho > 0$ we can fix open neighbourhoods A_i of w_i, B_{i+1} of w_{i+1} with diameter smaller than ρ and so that

$$\partial(A_i \cup B_{i+1}) \cap \bar{S}(w_i, w_{i+1}) = \emptyset, \quad \text{for } 1 \leqslant i \leqslant j-1. \tag{2.23}$$

We now consider a vector $z \in \mathbb{Z}^{2j-2}$ with coordinates

$$z_1 < z_2' < z_2 < z_3' < \cdots < z_j'$$

and define

$$X_z = \left\{ q \in \Gamma(\xi, \eta) \;\middle|\; q(z_i) \in \bar{A}_i, q\left(z_{i+1}'\right) \in \bar{B}_{i+1}, i = 1, \dots, j-1 \right\}.$$

The solutions we look for will be solutions of the following minimization problem:

$$b_z = \inf_{u \in X_z} \mathcal{J}(u) \tag{2.24}$$

provided that ρ is sufficiently small and the differences between the successive coordinates of z are sufficiently large. By the way, observe that the coordinates of z may be translated by any integer without affecting the variational problem.

CLAIM 1. *If $0 < r < \frac{1}{3}$ and $\rho > 0$ is sufficiently small, b_z is attained at a function $u_z \in X_z$ such that $|u_z(t) - w_i| \leqslant r$ for all $t \in I_i$, where $I_i = [z_i', z_i]$, $i = 2, \dots, j-1$, $I_1 =]-\infty, z_1]$ and $I_j = [z_j', \infty[$.*

Taking a minimizing sequence $(u_n)_n \subset X_z$ for (2.24), the claim follows from arguments similar to those we used in the proof of Theorem 2.3 that enable us to extract a convergent subsequence, still denoted by $(u_n)_n$, such that

$$u_n \overset{C_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^N)}{\longrightarrow} u_z \quad \text{and} \quad u_n' \overset{L^2(\mathbb{R},\mathbb{R}^N)}{\rightharpoonup} u_z',$$

together with the fact that we may assume that the minimizing sequence satisfies the inequalities in the above statement. To see this last fact with respect to I_1, let $K(s,t;x,y)$ denote the affine map

$$\tau \to x + \frac{\tau - s}{t - s}(y - x), \quad s \leqslant \tau \leqslant t,$$

and consider the modified sequence $(U_n)_n \subset X_z$ defined by

$$U_n(t) = \begin{cases} w_1(t) & \text{if } t \leqslant z_1 - 1, \\ K\big(z_1 - 1, z_1; w_1, u_n(z_1)\big)(t) & \text{if } z_1 - 1 < t < z_1, \\ u_n(t) & \text{if } t \geqslant z_1. \end{cases}$$

A straightforward computation yields $\mathcal{J}_{I_1}(U_n) = o(\rho)$ as $\rho \to 0$. However if u_n reaches the boundary of $B_r(w_1)$ at some instant $t \in I_1$, the adaptation of Fact 1 of Section 2.2 to the present situation allows to prove $\mathcal{J}_{I_1}(u_n)$ is bounded from below by a constant depending only on r. Hence, if ρ is chosen sufficiently small, we may substitute U_n for u_n and the inequality follows. The same argument enables us to deal with the remaining intervals. This proves Claim 1.

CLAIM 2. *Let $0 < \sigma < \rho$. If the differences $z_i - z_i'$ are large enough, there exist subintervals $J_i = [a_i, a_i + 2] \subset I_i$, such that $|u_z(t) - w_i| \leqslant \sigma$ for all $t \in J_i$ and $2 \leqslant i \leqslant j-1$. In addition $\mathcal{J}_{J_i}(u_z) = o(\sigma)$ as $\sigma \to 0$.*

By Claim 1, taking $z_i - z_i'$ large, we cannot have $|u_z(t) - w_i| \geqslant \sigma/2$ for all $t \in I_i$, since otherwise

$$\alpha_{\sigma/2}\big(z_i - z_i'\big) \leqslant \mathcal{J}_{I_i}(u_z) = o(\rho) \quad \text{for } \rho \to 0,$$

where $\alpha_{\sigma/2}$ is defined according to (2.20). In fact, the same argument shows that a subinterval $L \subset I_i$ with length greater than a certain $l > 0$ must contain at least one instant t such that $|u_z(t) - w_i| < \sigma/2$.

Suppose the first statement of Claim 2 is false. Then, each subinterval $J_i = [a_i, a_i + 2] \subset I_i$ containing an instant t where $|u_z(t) - w_i| < \sigma/2$ also contains an instant t' where $|u_z(t') - w_i| \geqslant \sigma$. If k is the number of such disjoint intervals, applying again the arguments of Fact 1 of Section 2.2 and the computations in the proof of Claim 1, we deduce a bound on k. Namely, we have

$$k\frac{\sigma}{2}\sqrt{2\alpha_{\sigma/2}} < o(\rho), \quad \text{for } \rho \to 0.$$

But, by the above remark, if $z_i - z_i'$ grows to infinity, so does k. This shows that the first statement of Claim 2 is true.

The second statement of Claim 2 is checked by noting that

$$\mathcal{J}_{J_i}(u_z) \leqslant \mathcal{J}_{J_i}(\psi) = o(\sigma), \quad \text{as } \sigma \to 0,$$

where ψ is defined by

$$\psi(t) := \begin{cases} u_z(t) & \text{if } t \in]-\infty, a_i[\,\cup\,]a_i+2, \infty[, \\ K\big(a_i, a_i+2, u_z(a_i), u_z(a_i+2)\big)(t) & \text{if } t \in [a_i, a_i+2]. \end{cases}$$

Let us now complete the proof. It is clear that $u_z(t)$ is a solution of (2.19) for $t \neq z_i,\ z_i'$. To prove that it is indeed a solution, we have to show that $u(z_i) \in A_i$ and $u(z_{i+1}') \in B_{i+1}$, for $1 \leqslant i \leqslant j-1$. To this purpose, an auxiliary variational problem is introduced: define

$$\Lambda(w_i, w_{i+1}) := \big\{u \in \Gamma(w_i, w_{i+1}) \mid u(0) \in \partial(A_i \cup B_{i+1})\big\}$$

and

$$d(w_i, w_{i+1}) := \inf_{u \in \Lambda(w_i, w_{i+1})} \mathcal{J}(u).$$

It is easily seen that this infimum is attained at some $S \in \Lambda(w_i, w_{i+1})$ and

$$d(w_i, w_{i+1}) > c(w_i, w_{i+1}).$$

Indeed, if there holds an equality, then $S(0) \in \partial(A_i \cup B_{i+1}) \cap \bar{S}(w_i, w_{i+1})$, contradicting (2.23).

Arguing by contradiction, assume that for some $i \in \{1, \dots, j-1\}$, we have, say,

$$u_z\big(z_{i+1}'\big) \in \partial B_{i+1}. \tag{2.25}$$

The other case may be handled in a similar way. By integer translation, we may suppose $z_{i+1}' = 0$. Also, we can translate the time variable in the basic heteroclinic v^i thus obtaining a new basic heteroclinic V^i such that $V^i(t) \in \bar{A}_i$ for $t \leqslant z_i$ and $V^i(t)$ leaves $\bar{A}_i$ at some time in $[z_i, z_{i+1}]$. Taking large differences $z_i - z_i'$ we may assume $V^i(t) \in \bar{B}_{i+1}$ for $t \geqslant z_{i+1}'$ as well. Hence $V^i(t) \in \bar{A}_i$ for $t \in J_i$ and $V^i(t) \in \bar{B}_{i+1}$ for $t \in J_{i+1}$.

Let us call U_z the modification of u_z outside $]-\infty, a_i]\cup[a_{i+1}+2, \infty[$ obtained by gluing together the following pieces: $K(a_i, a_i+1; u_z(a_i), w_i)$; $K(a_i+1, a_i+2; w_i, V^i(a_i+2))$;

the restriction of V^i to the interval $[a_i+2, a_{i+1}]$; $K(a_{i+1}, a_{i+1}+1; V^i(a_{i+1}), w_{i+1})$ and $K(a_{i+1}+1, a_{i+1}+2; w_{i+1}, u_z(a_{i+1}+2))$.

Finally we compute the difference

$$\mathcal{J}(u_z) - \mathcal{J}(U_z) = \mathcal{J}_{[a_i, a_{i+1}+2]}(u_z) - \mathcal{J}_{[a_i, a_{i+1}+2]}(U_z)$$

in the following way. Define an auxiliary curve γ_z by gluing together the constant w_i on $]-\infty, a_i]$; $K(a_i, a_i+1; w_i, u_z(a_i+1))$; u_z restricted to $[a_i+1, a_{i+1}]$; $K(a_{i+1}, a_{i+1}+1; u_z(a_{i+1}), w_{i+1})$ and the constant w_{i+1} in $[a_{i+1}+1, \infty[$. Note that $\gamma_z \in \Lambda(w_i, w_{i+1})$ by our assumption (2.25). Then we write

$$\begin{aligned}\mathcal{J}_{[a_i, a_{i+1}+2]}(u_z) - \mathcal{J}_{[a_i, a_{i+1}+2]}(U_z) &= \big(\mathcal{J}_{[a_i, a_{i+1}+2]}(u_z) - \mathcal{J}_{[a_i, a_{i+1}+2]}(\gamma_z)\big) \\ &\quad + \big(\mathcal{J}_{[a_i, a_{i+1}+2]}(\gamma_z) - \mathcal{J}_{[a_i, a_{i+1}+2]}(U_z)\big).\end{aligned}$$

Using Claim 2, we may check that the first difference on the right-hand side is an $o(\sigma)$ as $\sigma \to 0$. On the other hand, we clearly have

$$\mathcal{J}_{[a_i, a_{i+1}+2]}(\gamma_z) \geqslant b_z$$

and

$$\mathcal{J}_{[a_i, a_{i+1}+2]}(U_z) \leqslant \mathcal{J}_{[a_i, a_{i+1}+2]}\big(V^i\big) + o(\sigma), \quad \text{as } \sigma \to 0.$$

Hence, we finally deduce that

$$\begin{aligned}\mathcal{J}(u_z) - \mathcal{J}(U_z) &= \mathcal{J}_{[a_i, a_{i+1}+2]}(u_z) - \mathcal{J}_{[a_i, a_{i+1}+2]}(U_z) \\ &\geqslant b(w_i, w_{i+1}) - c(w_i, w_{i+1}) + o(\sigma), \quad \text{as } \sigma \to 0,\end{aligned}$$

leading to a contradiction with the minimizing property of u_z.

Conclusion. From what precedes, we conclude that for each $0 < r < \frac{1}{3}$ and each vector $(z_1, z'_2, z_2, z'_3, \ldots, z'_j) \in \mathbb{Z}^{2j-2}$ such that the distances $z_i - z'_i$, $2 \leqslant i \leqslant j-1$, are large enough, there exists a heteroclinic solution u_z of (2.19) with the particularity that

$$\big|u_z(t) - w_i\big| \leqslant r, \quad \text{if } t \in \big[z'_i, z_i\big], \quad \text{for } i = 2, \ldots, j-1$$

and

$$\big|u_z(t) - \xi\big| \leqslant r \quad \text{if } t \leqslant z_1, \qquad \big|u_z(t) - \eta\big| \leqslant r \quad \text{if } t \geqslant z'_j. \qquad \square$$

Several methods adapted to the search of multibump solutions may be found in the literature. For instance, in [67] Rabinowitz considers a system with two equilibria 0 and ξ and, starting from a pair of heteroclinics, v from 0 to ξ and w from ξ to 0, constructs heteroclinics that oscillate an arbitrary number of times between neighbourhoods of the equilibria. The method uses the functional $u \to \mathcal{J}(v+u)$ defined in $H^1(\mathbb{R}, \mathbb{R}^N)$. The concentration of Palais–Smale sequences with respect to homoclinics and heteroclinics is a crucial tool.

In [72], Coti Zelati and Rabinowitz give a variant of the result of Strobel for potentials of the form $V(t,u) = a(t)W(u)$ where W is periodic, a is slowly oscillating and not necessarily periodic.

The almost periodic case has been considered by Alessio, Bertotti and Montechiari [3] for fairly general Lagrangian systems. In particular, they show that, in presence of a slowly oscillating extra term, chaotic multibump dynamics arises.

2.4. *Notes and further comments*

1. In Theorem 2.1, the minimizer u takes values in $[-1, 1]$. If (2.4) has uniqueness or if $F(u)$ lies below some quadratic function $c(u \pm 1)^2$ in the neighborhood of ∓ 1, we have $u(t) \in\,]-1, 1[$ for all $t \in \mathbb{R}$. The heteroclinic of (2.4) is essentially unique (up to translation) as a consequence of (2.5) with $k = 0$.

2. If (2.9) has uniqueness of solutions for the Cauchy problem, the heteroclinics assume values in $]-1, 1[$. In addition, if f has only one zero in $]-1, 1[$, it can be checked that the minimizers of the functional $\mathcal{J}$ defined by (2.10) are strictly increasing. On the other hand, the presence of the function $a(t)$ in the functional rules out an easier argument which in the autonomous case shows that the elements of a minimizing sequence may assumed to be increasing functions.

3. In comparison with Theorem 2.2, the next theorem illustrates what can be said about second order systems where the potential depends on time and has no particular symmetry properties. For definiteness, consider the system

$$u'' = \nabla_u V(t,u), \tag{2.26}$$

where $u : \mathbb{R} \to \mathbb{R}^N$, $V \in C^2(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ and ∇_u is the gradient with respect to the variable $u \in \mathbb{R}^N$. Assume that

- (A9) $V(t,u) \geqslant 0$ for all $t \in \mathbb{R}$ and all $u \in \mathbb{R}^N$, and there exist $\xi \neq \eta$ such that $V(t,u) = 0$ if and only if $u \in \{\xi, \eta\}$;
- (A10) there exist constants $a_1, a_2 > 0$ and $\varepsilon > 0$ such that $a_1|u - z|^2 \leqslant V(t,u) \leqslant a_2|u - z|^2$ if $z \in \{\xi, \eta\}$ and $|u - z| < \varepsilon$;
- (A11) $\liminf_{|u| \to \infty} V(t,u) > 0$ uniformly in $t \in \mathbb{R}$.

Then one can prove:

THEOREM 2.7. *If* $V \in C^2(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ *satisfies the assumptions* (A9)–(A11) *and*

$$t \frac{\partial V}{\partial t}(t,u) > 0$$

whenever $t \neq 0$ *and* $u \notin \{\xi, \eta\}$, *then* (2.26) *has a heteroclinic solution from* ξ *to* η.

This theorem is a particular case of the results proved by Chen and Tzeng [26]. In their paper, further examples of existence, and results on multiplicity of heteroclinics, may be found. Thus in this respect the non-autonomous equation behaves quite differently from the autonomous one. We also mention that the system considered there may have an infinite set of equilibria.

4. It is, however, interesting to note that, in contrast to Theorems 2.2 and 2.7, Korman and Lazer [46] consider a special scalar equation

$$u'' = a(t)\big(|u|^{p-1}u - u\big), \quad p > 1, \tag{2.27}$$

and they find a heteroclinic from -1 to 1 assuming that $a(t)$ is even, $a'(t) < 0$ if $t < 0$ and $a(\infty) > 0$. They exploit the symmetry of (2.27), solve the approximate two point problem with boundary conditions $u(0) = 0$, $u(T) = 1$ and then they pass to the limit as $T \to +\infty$.

5. Many other properties of heteroclinics of Hamiltonian systems have been established by means of variational methods. Let us list a few of them:

(i) *Heteroclinics to periodics*. Consider a system of the form

$$u'' = \nabla_u V(t, u) + f(t), \tag{2.28}$$

where V is a function of class C^2, 1-periodic in t and in the spacial variables, and f is continuous, 1-periodic and of zero mean value. Assume in addition that the system is *reversible*, namely that V and f are *even* in t. Then it is known that (2.28) has periodic solutions at the minimum level of the action functional corresponding to the periodic boundary value problem for (2.28). In [68] Rabinowitz gives conditions for the existence of heteroclinics *connecting two such periodic solutions*. In [53] Maxwell proves that there are heteroclinic chains between any two periodic solutions, the "vertices" of the chain being also periodic solutions.

Calanchi and Serra [24] use a constrained minimization approach for the existence of connections between consecutive periodic solutions. For ordinary differential equation, Bosetto and Serra [21] obtain multi-bump heteroclinics between consecutive periodic motions without any reversibility assumption.

(ii) *Heteroclinics to almost periodic solutions*. Alessio, Carminati and Montecchiari [4] prove the existence of heteroclinic connections joining almost periodic solutions of a Lagrangian system.

(iii) *Heteroclinics with an endpoint at infinity*. For potentials $V \in C^1(\mathbb{R}^N, \mathbb{R})$ with one zero at the origin and vanishing at infinity like some power $|u|^{-\alpha}$, $\alpha > 0$, the system (2.12) has at least one "heteroclinic at infinity", i.e. a solution $u(t)$ such that $u(-\infty) = 0$, $|u(+\infty)| = +\infty$ and $u'(\pm\infty) = 0$. We refer to Serra [80]. A similar result holds for a potential with a finite singularity when $N \geqslant 3$.

(iv) *Almost periodic systems*. Bertotti and Montecchiari [14] and Alessio, Bertotti and Montecchiari [3] consider almost periodic Lagrangian systems. Under suitable conditions, they obtain infinitely many heteroclinic solutions connecting possibly degenerate equilibria.

(v) *Singular potentials*. Many authors consider homoclinic solutions to equilibrium points of Hamiltonian systems with a singular potential, see [85,69,13,25] and the references therein. In case of a potential in $\mathbb{R}^2$ singular at some unique point, the basic result is the existence of many homoclinics classified according to their winding number around the singularity. It seems that only few attention was paid to heteroclinic connections with such settings. In [25] an autonomous system in $\mathbb{R}^2$ is considered. The origin is a minimum of the potential and a strong force at the singularity ξ is assumed. The authors obtain a certain periodic solution $\bar{u}$ with index 1 with respect to ξ, homoclinics with arbitrarily large indexes with respect to ξ and a heteroclinic from 0 to $\bar{u}$ which is built upon an infinite sequence of homoclinics. We already mentioned the result of Serra [80] about heteroclinics at infinity which holds in some singular frameworks. But up to our knowledge, no multiplicity results were obtained for heteroclinic solutions in the spirit of those concerning homoclinics.

(vi) *Heteroclinic connections between minima at different levels of the potential*. V. Coti Zelati and Rabinowitz [29] give conditions for the existence of a heteroclinic from χ to η for a system of the form

$$u'' = a(t)\nabla V(u),$$

where a is periodic and bounded away from zero and χ, η are isolated minima of V at *different* levels.

(vii) *Spatial heteroclinics*. Consider the PDE

$$\Delta u = a(x, y) f(u) \quad \text{in } \mathbb{R}^2, \tag{2.29}$$

where a is periodic in both variables and f is a non-negative smooth function such that $f(0) = f(1) = 0$. There are solutions $u(x, y)$ of (2.29) periodic in y and "heteroclinic in x" from 0 to 1, i.e. approaching respectively 0 and 1 as $x \to -\infty$ and $x \to +\infty$. This and a lot more of related results can be consulted in Rabinowitz and Stredulinsky [73, 74] and Rabinowitz [71]. See also the papers by Alessio, Jeanjean and Montecchiari [5, 6].

3. Heteroclinics as front wave profiles in reaction diffusion equations

An efficient model for many chemical and biological phenomena is provided by the partial differential equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} H(u) = \frac{\partial}{\partial x}\left[p(u)\frac{\partial u}{\partial x}\right] + g(u), \tag{3.1}$$

where $H, p \in C^1(\mathbb{R})$, $p > 0$ and following the terminology used in [49], g is a *function of type* A in [0, 1] by which we mean that

$$\begin{array}{l} g \text{ is continuous in } [0,1], \quad g(0) = g(1) = 0 \quad \text{and} \\ g(u) > 0 \text{ if } u \in]0,1[. \end{array} \tag{Type A}$$

Hence the constants $u = 0$ and $u = 1$ are solutions of (3.1).

The first term in the right-hand side represents density dependent diffusion, while the term $\frac{\partial}{\partial x} H(u)$ accounts for convection effects.

An important problem related to this equation is that of finding positive travelling wave solutions, that is, positive solutions of the form $u(t,x) = U(x - ct)$ for some $c > 0$. Here c is the propagation speed of the wave. It is in addition required that the wave front $U(s)$ is defined in $]-\infty, +\infty[$ and satisfies $U(-\infty) = 1$, $U(+\infty) = 0$. This amounts to look at the solutions of the second order ordinary differential equation

$$\big(p(u)u'\big)' + \big(c - H'(u)\big)u' + g(u) = 0 \tag{3.2}$$

satisfying the limit conditions

$$u(-\infty) = 1, \quad u(+\infty) = 0.$$

We are therefore close to the problem of finding a heteroclinic between the equilibria 1 and 0.

The simplest and most classical case corresponds to $H \equiv 0$, $p \equiv 1$, the associated equation being then

$$u'' + cu' + g(u) = 0. \tag{3.3}$$

This (or the corresponding evolution equation) is usually referred as Fisher's equation, although in Fisher's original model $g(u)$ is the function $u(1-u)$. Among the vast literature on the existence of heteroclinics for (3.3) at least two contributions should be singled out: the pioneering work of Kolmogorov, Petrovsky and Piskounoff [45] and the paper by Aronson and Weinberger [11]—the latter concerning also diffusion in n-dimensional space. It is shown in [11] that, for $g \in C^1[0,1]$, there exists a heteroclinic connection of (3.3) between 1 and 0 if and only if $c \geqslant c^*$, where c^* is a positive number such that

$$2\sqrt{g'(0)} \leqslant c^* \leqslant 2\sqrt{\sup_{0<u<1} \frac{g(u)}{u}}.$$

The lower bound is clear, since linearizing (3.3) around $u = 0$ shows that for $c < 2\sqrt{g'(0)}$ the origin cannot act as an attractor of *positive* solutions. This result has been recently extended by Malaguti and Marcelli [48,49] for the more general problem (3.2) (in fact for even more general equations).

In this monograph we shall concentrate on the existence of heteroclinics and some bounds for the minimal speed c^*. Many other interesting problems concerning the equation (3.3) or systems of equations of this type have been dealt with in the literature. Some examples will be mentioned at the end of the section.

3.1. *Reduction to a first order equation*

Maybe the simplest way to obtain existence results for heteroclinic solutions of (3.2) or (3.3) consists in studying an equivalent first-order problem. This device may be found in [76] and has been extensively used in [49,50]. We modify it slightly, in the spirit of the approach given in [10].

Let us consider the equation (3.2) written as

$$\big(p(u)u'\big)' + h(u)u' + g(u) = 0 \tag{3.4}$$

and assume

(B1) $p \in C^1([0, 1])$ and $p(u) > 0$ for all $u \in [0, 1]$;

(B2) $h \in C[0, 1]$ and $h(u) > 0$, for all $u \in [0, 1]$;

(B3) g is a function of type A in [0, 1].

We first notice that monotone solutions of (3.4) such that $0 < u(s) < 1$ have no critical points. Indeed, if $u'(t_0) = 0$ and $0 < u(t_0) < 1$, then (3.4) implies $u''(t_0) < 0$. On the other hand, we observe that any decreasing solution $u(s)$ of (3.4) such that u is defined in $]-\infty, +\infty[$, $u(-\infty) = 1$ and $u(+\infty) = 0$ has the property

$$\lim_{s\to\pm\infty} u'(s) = 0, \tag{3.5}$$

so that u is indeed a heteroclinic between the equilibria 1 and 0. To prove (3.5) first remark that $\limsup_{s\to\pm\infty} u'(s) = 0$ by the boundedness of $u(s)$. Suppose, by contradiction, that $\liminf_{s\to+\infty} u'(t) < 0$, the limit at $-\infty$ is handled in a similar way. Let $t_n \to +\infty$ be a sequence so that $u'(t_n) \to 0$. Integrating (3.4) in $[0, t_n]$ shows that $\int_0^{t_n} g(u(s))\,\mathrm{d}s$ is bounded, whence by the positivity of g, we deduce that $\int_0^{+\infty} g(u(s))\,\mathrm{d}s$ is finite. Now let $t_n \to +\infty$ and $s_n \to +\infty$ be sequences so that $t_n < s_n < t_{n+1}$, $u'(t_n) \to 0$ and $u'(s_n) \to -\delta < 0$. From (3.4) we derive, for some $\tau_n \in [t_n, s_n]$

$$\big[p\big(u(s_n)\big)u'(s_n) - p\big(u(t_n)\big)u'(t_n)\big] + h\big(u(\tau_n)\big)\big(u(s_n) - u(t_n)\big) + \int_{t_n}^{s_n} g\big(u(s)\big)\,\mathrm{d}s = 0.$$

In the above equation the term in brackets has limit $-p(0)\delta$, while the remaining terms tend to 0, leading to a contradiction.

Now let $u = U(t)$ be a monotone decreasing heteroclinic of (3.4) and let $]t_-, t_+[$ be the maximal interval where

$$0 < U(t) < 1.$$

Then we have $U'(t) < 0$ for $t \in]t_-, t_+[$ and thus we can define $t(u)$, the inverse function of $u = U(t)$. Set

$$\varphi(u) := p\big[U\big(t(u)\big)\big]U'\big(t(u)\big). \tag{3.6}$$

Then φ is a C^1 function in $]0, 1[$ which can be continuously extended to $[0, 1]$ with $\varphi(0) = \varphi(1) = 0$. Moreover, φ satisfies

$$\varphi'(u)\varphi(u) + h(u)\varphi(u) + p(u)g(u) = 0.$$

Observe that the boundedness of h and the positivity of p and g implies that

$$\big|\varphi(u)\big| \leqslant ku$$

for some positive constant k. From Gronwall's inequality we then infer this rules out the possibility that $t_+ < +\infty$. Since $\varphi < 0$, we find that $\psi(u) := \varphi(u)^2$ is a solution of the first order differential equation

$$\psi'(u) = 2h(u)\sqrt{\psi(u)} - 2p(u)g(u) \tag{3.7}$$

in $[0, 1]$, together with the endpoint conditions

$$\psi(0) = \psi(1) = 0. \tag{3.8}$$

In particular, ψ is of type A in $[0, 1]$.

Conversely, if (3.7) has a solution ψ of type A in $[0, 1]$, we use it to define the solution $u(t)$ of the Cauchy problem

$$u' = -\frac{\sqrt{\psi(u)}}{p(u)}, \qquad u(0) = \frac{1}{2}. \tag{3.9}$$

The domain of this solution is $]t_-, t_+[$, where

$$t_- = -\int_{1/2}^{1} \frac{p(u)\,\mathrm{d}u}{\sqrt{\psi(u)}}, \qquad t_+ = \int_0^{1/2} \frac{p(u)\,\mathrm{d}u}{\sqrt{\psi(u)}}.$$

The boundedness of h and (3.7) imply that in a neighborhood of $u = 0$, we have

$$\sqrt{\psi(u)} \leqslant ku$$

for some positive constant k. Therefore it follows from assumption (B1) that $t_+ = +\infty$. If we assume in addition that

(B4) there exists $\ell > 0$ such that for all $u \in [0, 1]$, $g(u) \leqslant \ell(1 - u)$, then we infer from (3.7) that there exists some constant $k_1 > 0$ so that $\psi(u) \leqslant k_1(1-u)^2$ in a neighborhood of $u = 1$. Hence under the condition (B4) we also have $t_- = -\infty$.

A straightforward computation shows that $u(t)$ satisfies (3.4) in $]t_-, t_+[$, and it is clear that

$$\lim_{t \to t_-} u(t) = 1, \qquad \lim_{t \to t_+} u(t) = 0, \qquad \lim_{t \to t_\pm} u'(t) = 0.$$

Therefore we have shown:

PROPOSITION 3.1. *Assume* (B1)–(B4) *hold and let* φ *be the function defined by* (3.6). *Then* $u = U(t)$ *is a* (*strictly*) *decreasing heteroclinic solution of* (3.4) *between* 1 *and* 0 *if and only if* $\varphi(u)^2$ *is a solution of type A of* (3.7) *in* [0, 1].

REMARK. Under weaker hypotheses, namely if we allow a singularity at zero, i.e. $p(0) = 0$, or in the absence of (B4), t_- or t_+, or both, may be finite. In these cases, we still obtain a heteroclinic by trivially extending $U(t)$ with the value 1 to the left and with the value 0 to the right.

Proposition 3.1 reduces the existence of heteroclinics for (3.4) to the existence of type A solutions of (3.7). Therefore it is useful to have criteria that ensure existence of such solutions.

PROPOSITION 3.2. *Assume* (B1)–(B3) *hold.*

(i) *Suppose that* $s(u)$ *is a* C^1 *function in* [0, 1] *such that* $s(0) = 0$, $s(u) > 0$ *if* $u \in]0, 1[$ *and for all* $u \in [0, 1]$,

$$s'(u) \leqslant 2h(u)\sqrt{s(u)} - 2p(u)g(u). \tag{3.10}$$

Then Eq. (3.7) *has a solution of type A.*

(ii) *Equation* (3.7) *has at most one solution of type A.*

PROOF. (i) The assumption (3.10) which means s is a lower solution of the initial value problem

$$\begin{aligned} &\psi' = 2h(u)\sqrt{\psi} - 2p(u)g(u), \\ &\psi(0) = 0, \end{aligned} \tag{3.11}$$

implies, as is well known (see, e.g., [91]), that (3.11) has a solution $\psi(u)$ such that $\psi(u) \geqslant s(u)$. If $\psi(1) = 0$ then ψ is a type A solution of (3.7) and we are done.

If $\psi(1) > 0$, we consider the solution $\bar{\psi}$ of the initial value problem

$$\begin{aligned} &\bar{\psi}' = 2h(u)\sqrt{|\bar{\psi}|} - 2p(u)g(u), \\ &\bar{\psi}(1) = 0. \end{aligned} \tag{3.12}$$

It is clear that we may assume $\bar{\psi} \geqslant 0$ in $[0, 1]$, since 0 is a lower solution for the initial value problem (3.12) in $[0, 1]$. We claim that $0 < \bar{\psi}(u) < \psi(u)$ for all $u \in]0, 1[$. In fact, if u_0 is the largest zero of $\bar{\psi}$ in $]0, 1[$, then (3.12) implies $\bar{\psi}'(u_0) < 0$, which is impossible. If $u_1 \in]0, 1[$ is such that $\bar{\psi}(u_1) = \psi(u_1)$, then $\bar{\psi} \equiv \psi$, which is also impossible. Hence the claim is proved and by continuity, $\bar{\psi}(0) = 0$ so that $\bar{\psi}$ is the desired solution of type A.

(ii) Assume by contradiction that ψ_1 and ψ_2 are distinct type A solutions of (3.7). By uniqueness of the solution of the Cauchy problem, these two solutions are ordered, say $\psi_1(u) < \psi_2(u)$ for all $u \in]0, 1[$. But then (3.7) shows that $\psi_2 - \psi_1$ is increasing and this contradicts $\psi_1(1) = \psi_2(1) = 0$. □

The results we have established have straightforward consequences:

PROPOSITION 3.3.
(i) *The decreasing heteroclinic of* (3.4), *if it exists*, *is unique up to translation*.
(ii) *Consider two equations*

$$\big(p_i(u)u'\big)' + h_i(u)u' + g_i(u) = 0, \quad i = 1, 2, \tag{$*$)$_i$}$$

where p_i, h_i, g_i *satisfy assumptions* (B1)–(B4). *Then if* $h_1 \leqslant h_2$, $p_1 \geqslant p_2$ *and* $g_1 \leqslant g_2$, *and* $(*)_1$ *has a decreasing heteroclinic*, *then so does* $(*)_2$.

REMARK. If $c_1 < c_2$ the heteroclinics u_1, u_2 of (3.3) with respectively $c = c_1, c_2$ satisfy $\psi_2 < \psi_1$ by the above ordering argument.

EXAMPLES. (1) Suppose H' in (3.2) is continuous. Then there exists a number $c^* \in \mathbb{R}$ such that (3.2) admits a decreasing solution if and only if $c \geqslant c^*$. In particular, for (3.3) we have the classical estimate

$$0 < c^* \leqslant 2\sqrt{\sup_{0<u<1} \frac{g(u)}{u}}. \tag{3.13}$$

In fact, if there exists $M > 0$ so that for all $u \in [0, 1]$, $g(u) \leqslant Mu$, then we can choose a constant $\beta > 0$ so that $s(u) = \beta u^2$ is a lower solution of (3.11) with $h(u) \equiv c$, $p(u) \equiv 1$, provided that $c^2 \geqslant 4M$.

(2) Consider the autonomous case

$$u'' + cu' + g(u) = 0$$

in which there exists $M > 0$ and $\alpha \geqslant 2$ so that

$$g(u) \leqslant Mu^\alpha(1 - u).$$

(This type of nonlinear term appears in many applications.) It is easy to see that $s = \beta u^\alpha(1 - u)^2$ is a lower solution of (3.11) with $h(u) \equiv c$, $p(u) \equiv 1$, if for all $u \in [0, 1]$,

$$\alpha\beta u^{\alpha/2-1} + \big(2M - \beta(\alpha + 2)\big)u^{\alpha/2} \leqslant 2c\sqrt{\beta}.$$

Elementary but tedious computations show that such a number $\beta > 0$ exists if $c \geqslant \sqrt{2M}\big(\frac{\alpha-1}{\alpha+2}\big)^{(\alpha-1)/2}$. We thus obtain

$$c^* \leqslant \sqrt{2M}\left(\frac{\alpha-1}{\alpha+2}\right)^{(\alpha-1)/2},$$

an estimate which improves, for small values of α (e.g., $\alpha \leqslant 4$) that allowed by (3.13).

3.2. *Fast solutions and heteroclinics*

In this section we consider the simple model

$$u'' + cu' + g(u) = 0 \tag{3.14}$$

with g of type A. Following [10] we say that a solution $u(t)$ of (3.14) is a *fast solution* if it is defined in some interval $[t_0, +\infty]$ and satisfies the integrability condition

$$\int_{t_0}^{+\infty} \mathrm{e}^{ct} u'(t)^2 \,\mathrm{d}t < +\infty.$$

In order to look for this kind of solutions we introduce the Hilbert space

$$H_c = \left\{ u \in H^1_{\mathrm{loc}}(0, +\infty) \,\middle|\, \int_0^{+\infty} \mathrm{e}^{ct} u'(t)^2 \,\mathrm{d}t < +\infty \text{ and } u(+\infty) = 0 \right\}$$

with the norm

$$\|u\| = \left(\int_0^{+\infty} \mathrm{e}^{ct} u'(t)^2 \,\mathrm{d}t \right)^{1/2}.$$

Note that $u(+\infty)$ exists whenever the above integral is finite, since by Schwarz's inequality

$$\big|u(T) - u(S)\big| \leqslant \left(\frac{|\mathrm{e}^{-cS} - \mathrm{e}^{-cT}|}{c} \int_S^T \mathrm{e}^{ct} u'(t)^2 \,\mathrm{d}t \right)^{1/2}.$$

On the other hand, taking limits in the above inequality as $T \to +\infty$ it turns out that for all $u \in H_c$

$$\|u\|_{L^\infty(S,+\infty)} \leqslant \frac{\mathrm{e}^{-cS/2}}{\sqrt{c}} \|u\|. \tag{3.15}$$

PROPOSITION 3.4. *For all $u \in H_c$, we have*

$$\frac{c^2}{4} \int_0^{+\infty} \mathrm{e}^{ct} u(t)^2 \,\mathrm{d}t \leqslant \int_0^{+\infty} \mathrm{e}^{ct} u'(t)^2 \,\mathrm{d}t. \tag{3.16}$$

PROOF. In Hardy's inequality (see [40])

$$\int_0^{+\infty} \frac{v(s)^2}{s^2}\,\mathrm{d}s \leqslant 4\int_0^{+\infty} v'(s)^2\,\mathrm{d}s,$$

where v is absolutely continuous in $[0,+\infty[$ and $v(0)=0$, make the change of variables $v(s)=u(t)$, $s=\mathrm{e}^{-ct}$ to obtain

$$c\int_{-\infty}^{+\infty} \mathrm{e}^{ct}u(t)^2\,\mathrm{d}t \leqslant \frac{4}{c}\int_{-\infty}^{+\infty} \mathrm{e}^{ct}u'(t)^2\,\mathrm{d}t.$$

Then choose $u(t)$ to be constant for $t \leqslant 0$ and ignore, in the left-hand side of the estimate, the integral over $]-\infty,0]$ which is positive. □

Assume that g satisfies
(B5) $\sup_{0<u<1} \frac{g(u)}{u} < +\infty$
and extend $g(u)$ to $]-\infty,\infty[$ with the value 0 outside $[0,1]$. Solutions of (3.14) defined in $[0,+\infty[$ and taking values in $[0,1]$ can thus be identified with critical points of the functional $\mathcal{F}: H_c \to \mathbb{R}$ defined by

$$\mathcal{F}(u) = \int_0^{+\infty} \mathrm{e}^{ct}\left(\frac{u'^2}{2} - G(u)\right)\mathrm{d}t,$$

where $G(s) = \int_0^s g(\tau)\,\mathrm{d}\tau$. In fact, one can check that under the assumption (B5), $\mathcal{F}$ is a functional of class C^1 in H_c and (3.14) (with the extended g) is the Euler–Lagrange equation for $\mathcal{F}$.

Let us introduce one more assumption:
(B6) there exists a constant k such that $k \in]0, c^2/4[$ and

$$G(u) \leqslant \frac{ku^2}{2}, \qquad \text{for } 0 \leqslant u \leqslant 1.$$

PROPOSITION 3.5. *Assume* (B3), (B5) *and* (B6) *hold. Then* $\mathcal{F}$ *attains a minimum in the set*

$$M = \{u \in H_c \mid u(0) = 1\}.$$

The minimizer u *is a solution of* (3.14) *in* $[0,+\infty[$ *such that for all* $t \geqslant 0$, $u'(t) \leqslant 0$ *and* $0 \leqslant u(t) \leqslant 1$.

PROOF. We write $\mathcal{F}$ in the form

$$\mathcal{F}(u) = \int_0^{+\infty} \mathrm{e}^{ct}\left(\frac{u'^2}{2} - k\frac{u^2}{2}\right)\mathrm{d}t + \int_0^{+\infty} \mathrm{e}^{ct} H(u)\,\mathrm{d}t, \tag{3.17}$$

where by (B6),

$$H(u) = k\frac{u^2}{2} - G(u) \geqslant 0.$$

Let $(u_n)_n \subset M$ be a minimizing sequence for $\mathcal{F}$. Since $G(u) = G(\max(0, \min(1, u)))$, it is clear that arguing as in the proof of Theorem 2.1, we may assume $0 \leqslant u_n \leqslant 1$. Because of (3.16), the first integral in (3.17) is the square of an equivalent norm in H_c. Hence $(u_n)_n$ is bounded and, passing to a subsequence we may assume that there exists $u \in H_c$ so that $u_n \rightharpoonup u$ in H_c and $u_n \to u$ uniformly in compact intervals. In particular $u \in M$ and, by the lower semicontinuity of the norm and Fatou's lemma, we infer from (3.17) that

$$\mathcal{F}(u) \leqslant \liminf_{n\to\infty} \mathcal{F}(u_n).$$

Hence u is a minimizer of $\mathcal{F}$ in M. Since $0 \leqslant u \leqslant 1$, $u(t)$ is a solution of (3.14) with $u(0) = 1$. Writing (3.14) as

$$\left(\mathrm{e}^{ct}u'\right)' + \mathrm{e}^{ct}g(u) = 0 \tag{3.18}$$

we see that $\mathrm{e}^{ct}u'$ is decreasing, hence for all $t \geqslant 0$ we have $u'(t) \leqslant 0$. □

THEOREM 3.6. *Assume* (B3), (B5) *and* (B6) *hold. Then* (3.14) *has a heteroclinic connecting* 1 *and* 0.

PROOF. Let U be the solution found in the preceding proposition. If $U'(0) = 0$, then extending U with value 1 on $]-\infty, 0]$ we obtain a heteroclinic solution. Otherwise, we have $U'(t) < 0$, $0 < U(t) \leqslant 1$ for all $t \geqslant 0$ and we may define $\varphi(u) = U'(t(u))$ for $0 \leqslant u \leqslant 1$ as in (3.6). We thus obtain a solution $\psi(u) = \varphi(u)^2$ of (3.7) (with $p \equiv 1$, $h \equiv c$), such that $\psi(0) = 0$ and $\psi(u) > 0$ if $0 < u \leqslant 1$. It now follows from Proposition 3.2 that (3.7) has a solution of type A and Proposition 3.1 allows to conclude. □

We then have the straightforward corollary:

COROLLARY 3.7. *For a given function g of type A satisfying* (B5), *the threshold speed of Eq.* (3.14) *satisfies*

$$c^* \leqslant 2 \sup_{0<u<1} \sqrt{\frac{2G(u)}{u^2}}.$$

To end this section we illustrate the use of variational and comparison arguments to obtain a heteroclinic for a nonautonomous equation (see also [50] for further results relative to the non-autonomous case). We consider

$$u'' + p(t)u' + g(u) = 0, \tag{3.19}$$

where p is a continuous function in $\mathbb{R}$ such that

(B7) there exist $0 < c < d < +\infty$ such that for any $t \in \mathbb{R}$, $c \leqslant p(t) \leqslant d$.

As above, we make use of a weighted space of functions:

$$X_\tau = \left\{ u \in H^1_{\mathrm{loc}}(\tau, +\infty) \;\Big|\; \int_\tau^{+\infty} \mathrm{e}^{P(t)} u'(t)^2 \,\mathrm{d}t < +\infty \text{ and } u(+\infty) = 0 \right\},$$

where τ is taken negative and $P(t) = \int_0^t p(s)\,\mathrm{d}s$. Arguing as in the proof of Proposition 3.4 and assuming p satisfy (B7), we derive the inequality

$$\frac{c^2}{4} \int_\tau^{+\infty} \mathrm{e}^{P(t)} u(t)^2 \,\mathrm{d}t \leqslant \int_\tau^{+\infty} \mathrm{e}^{P(t)} u'(t)^2 \,\mathrm{d}t, \tag{3.20}$$

which is valid for any $u \in X_\tau$. On the other hand, it is easily seen that for any $u \in X_\tau$ we have the estimate

$$\|u\|_{L^\infty(0,+\infty)} \leqslant C \left(\int_0^{+\infty} \mathrm{e}^{P(t)} u'(t)^2 \,\mathrm{d}t \right)^{1/2} \tag{3.21}$$

for some positive constant C.

THEOREM 3.8. *Assume* (B3) *holds*, $p \in C(\mathbb{R})$ *satisfies* (B7), g *is Lipschitz in* $[0, 1]$ *and there exists* $k < c^2/4$ *such that*

$$g(u) \leqslant ku, \quad \text{for } u \in [0, 1]. \tag{3.22}$$

Then, *given* $a \in]0, 1[$, (3.19) *has a strictly decreasing heteroclinic connecting* 1 *and* 0 *and satisfying* $u(0) = a$.

PROOF. We outline the proof in four steps.

Step 1. We claim that the linear equation

$$\varphi'' + p(t)\varphi' + k\varphi = 0 \tag{3.23}$$

has a solution φ_τ such that $\varphi_\tau(\tau) = 1$, $0 < \varphi_\tau < 1$ in $]\tau, +\infty[$ and $\varphi_\tau \in X_\tau$.

Indeed, it suffices to take φ_τ as the minimizer of the quadratic functional

$$\mathcal{F}_\tau(u) = \int_\tau^{+\infty} \mathrm{e}^{P(t)} \left(\frac{u'^2}{2} - k\frac{u^2}{2} \right) \mathrm{d}t$$

in the convex set $\{u \in X_\tau \colon u(\tau) = 1\}$. It is easy to see, on the basis of (3.20) which remains valid with τ substituted by any $\xi \geqslant \tau$, that in fact $\varphi_\tau > 0$. Observe also that, multiplying the self-adjoint form of (3.23) by φ_τ and integrating by parts, we infer that $\mathrm{e}^{P(t)}\varphi'(t)\varphi(t)$ has limit as $t \to +\infty$. It then follows from (3.20) and the fact that $\varphi_\tau > 0$ that this limit is negative. This in turn implies that $\varphi_\tau'(\tau) < 0$ and $\varphi_\tau < 1$ in $]\tau, +\infty[$. Hence this step is completed.

Step 2. We claim $\lim_{\tau\to-\infty} \varphi_\tau(0) = 0$.

Given $\beta > 0$, we first evaluate $\mathcal{F}_\tau$ on the test function $u(t) = \mathrm{e}^{-(\beta/2)P(t-\tau)}$ and obtain

$$\mathcal{F}_\tau(\varphi_\tau) = \inf_{X_\tau} \mathcal{F}_\tau \leqslant \frac{\beta^2 d^2}{8} \int_0^{+\infty} \mathrm{e}^{P(\tau+t)-\beta P(t)}\,\mathrm{d}t. \tag{3.24}$$

From the inequality (3.20), we come out with the estimate

$$\mathcal{F}_\tau(\varphi_\tau) \geqslant \frac{1}{2}\left(1 - \frac{4k}{c^2}\right) \int_\tau^{+\infty} \mathrm{e}^{P(t)} \varphi_\tau'(t)^2\,\mathrm{d}t. \tag{3.25}$$

It then follows from the inequalities (3.24), (3.25) and $k < c^2/4$ that

$$\int_0^{+\infty} \mathrm{e}^{P(t)} \varphi_\tau'(t)^2\,\mathrm{d}t \leqslant C \int_0^{+\infty} \mathrm{e}^{P(\tau+t)-\beta P(t)}\,\mathrm{d}t$$

for some $C \geqslant 0$ and consequently we deduce from Step 1 and (3.21) that there exists a number $L \geqslant 0$ such that for all $\tau < 0$,

$$0 < \varphi_\tau(0) \leqslant L \left(\int_0^{+\infty} \mathrm{e}^{P(\tau+t)-\beta P(t)}\,\mathrm{d}t \right)^{1/2}.$$

If we take $\beta > d/c$, we may apply the dominated convergence theorem to the integral in the right-hand side and conclude that

$$\lim_{\tau\to-\infty} \varphi_\tau(0) = 0.$$

Step 3. Let $\tau < 0$ be such that $\varphi_\tau(0) < a$. From (3.22) it follows that φ_τ is an upper solution of (3.19) with respect to the two-point boundary conditions $u(\tau) = 1$, $u(0) = 0$. Since 0 is a lower solution, we obtain a solution v of (3.19) in $[\tau, 0]$ such that $v(\tau) = 1$, $v(0) = 0$, $0 \leqslant v \leqslant \varphi_\tau$. This in turn is a lower solution in $[\tau, 0]$ with respect to the boundary conditions $u(\tau) = 1$, $u(0) = a$. Since 1 is an upper solution, there is a solution u_τ of (3.19) such that

$$u_\tau(\tau) = 1, \qquad u_\tau(0) = a, \qquad v \leqslant u_\tau \leqslant 1 \quad \text{in } [\tau, 0].$$

We claim that the solution u_τ satisfies for all $t \geqslant 0$,

$$u_\tau(t) \geqslant \varphi_\tau(t) > 0.$$

Otherwise, since $u_\tau(\tau) = \varphi(\tau)$ and $u_\tau(0) > \varphi_\tau(0)$, there exist t_0, t_1 such that $\tau \leqslant t_0 < 0$, $t_0 < t_1 \leqslant +\infty$, $u_\tau(t_i) = \varphi_\tau(t_i)$ for $i = 0, 1$, $u_\tau > \varphi_\tau$ in $]t_0, t_1[$. Multiplying the self-adjoint forms of (3.19) and (3.23) respectively by φ_τ and u_τ, and integrating by parts, we obtain

$$\left[e^{P(t)}\left(u_\tau'(t)\varphi_\tau(t) - u_\tau(t)\varphi_\tau'(t)\right)\right]_{t_0}^{t_1} + \int_{t_0}^{t_1} e^{P(t)}\left(g(u_\tau)\varphi_\tau - ku_\tau\varphi_\tau\right) dt = 0. \tag{3.26}$$

On the other hand, since g satisfies a Lipschitz condition in $[0, 1]$, we must have $u_\tau'(t_0) > \varphi_\tau'(t_0)$ and $u_\tau'(t_1) \leqslant \varphi_\tau'(t_1)$. But then using (3.22), we conclude that the left-hand side of (3.26) is strictly negative, yielding a contradiction.

Step 4. From the last two steps it follows that we may define a sequence $\tau_n \to -\infty$ and a corresponding sequence $u_n = u_{\tau_n}$ of solutions of (3.19) such that

$$u_n(\tau_n) = 1, \qquad u_n(0) = a \qquad \text{and} \qquad 0 < u_n < 1 \quad \text{in }]\tau_n, +\infty[.$$

A diagonal argument, based on simple estimates on the sequence $(u_n)_n$, shows that some subsequence converges to a solution u of (3.19) in $]-\infty, +\infty[$ such that $u(0) = a$ and $0 < u < 1$. It is easy to see that $u' < 0$, $u(-\infty) = 1$, $u(+\infty) = 0$ and indeed u is the desired heteroclinic solution. □

3.3. *Heteroclinics in a combustion model*

In this section we consider the differential equation

$$u'' + cu' + f(u) = 0, \tag{3.27}$$

where $f \in C([0, 1], \mathbb{R})$ has the following behaviour:

(B8) $f(0) = f(1) = 0$;
(B9) there exists $a \in]0, 1[$ such that $f(u) < 0$ if $u \in]0, a[$;
(B10) $\int_0^1 f(t)\, dt > 0$.

Equation (3.27) with this type of nonlinearity arises as the front wave equation for some combustion models. We refer the reader to [12,33,52] for a study in depth of related problems. Here we confine ourselves to look for solutions of (3.27) which are monotone heteroclinics connecting the equilibria 1 and 0.

Note that by virtue of assumptions (B9) and (B10), Eq. (3.27) has at least one more equilibrium between 0 and 1. The behavior of solutions of (3.27) is quite different from those of (3.14). The main result describing the existence of heteroclinics in terms of the parameter c is the following.

THEOREM 3.9. *Assume* (B8)–(B10) *hold and suppose in addition that* $f(u) > 0$ *for* $\xi < u < 1$, *where* ξ *is the smallest number in* $]0, 1[$ *with the property that* $\int_0^\xi f(t)\, dt = 0$. *Then there exists a unique* $c^* > 0$ *such that* (3.27) *has a heteroclinic solution* $u(t)$, *connecting* 1 *and* 0, *with the property that* $u'(t) < 0$ *for all* $t \in \mathbb{R}$. *Moreover, this heteroclinic is unique up to translation.*

PROOF. In order to prove this theorem we proceed as in Section 3.1, obtaining the desired heteroclinic by means of a solution to the first-order problem (3.9) where $\psi(u)$ is a positive solution of

$$\psi'(u) = 2c\sqrt{\psi(u)} - 2f(u) \tag{3.28}$$

in [0, 1], $\psi(0) = 0 = \psi(1)$ and $p \equiv 1$. Let us therefore study the Cauchy problem

$$\begin{aligned} &\psi' = 2c\sqrt{\psi_+} - 2f(u),\\ &\psi(0) = 0, \end{aligned} \tag{3.29}$$

where $\psi_+(u) = \max(\psi(u), 0)$.

Step 1. Given $c \geqslant 0$, (3.29) has a unique solution in [0, 1].

If $c = 0$ the solution is $\psi_0(u) = -2F(u) := -2\int_0^u f(t)\,\mathrm{d}t$. Now assume $c > 0$. It is clear that any solution ψ of (3.29) satisfies $\psi(u) > \psi_0(u)$ if $0 < u \leqslant 1$. In particular we have $\psi(u) > 0$ in $]0, a[$. Write $\psi = \theta - 2F(u)$, so that in a right neighborhood of zero θ is a solution of

$$\begin{aligned} &\theta' = 2c\sqrt{\theta - 2F(u)},\\ &\theta(0) = 0. \end{aligned} \tag{3.30}$$

If ψ_1, ψ_2 are distinct positive solutions of (3.29), then they are ordered in a neighborhood of 0 that is $\psi_1(u) < \psi_2(u)$ and so are the corresponding solutions of (3.30): $\theta_1(u) < \theta_2(u)$. Since these are strictly increasing near zero we obtain, for their inverse functions $u_i(\theta)$ $(i = 1, 2)$ in some interval $0 < \theta \leqslant b$,

$$\frac{\mathrm{d}u_i}{\mathrm{d}\theta} = \frac{1}{2c\sqrt{\theta - 2F(u_i)}}$$

and $u_i(0) = 0$. However, since $-2F$ is strictly increasing in a neighborhood of zero, we may assume that $-2F(u_2) < -2F(u_1)$ and therefore $u_2 - u_1$ is increasing in $[0, b]$. This contradicts $u_1(0) = 0 = u_2(0)$. Hence the solution ψ of (3.29) is unique at least as long as it is strictly positive. Assume that $\psi(\tau) = 0$ for some $0 < \tau < 1$. We clearly have $\tau > \xi$ and, in view of our assumptions, $\psi'(\tau) < 0$. But then (3.29) implies $\psi(u) = -2\int_\tau^u f(t)\,\mathrm{d}t$ for $\tau \leqslant u \leqslant 1$.

Step 2. There exists $c > 0$ such that the solution $\psi_c(u)$ of (3.29) satisfies $\psi_c(1) > 0$.

It suffices to note that, if we set

$$M = \sup_{0<u<1} \frac{f(u)}{u}$$

and choose $c > 1 + M$, then u^2 is a lower solution of (3.29).

Step 3. Uniqueness and the fact that solutions of (3.29) are uniformly bounded when c runs over bounded sets, imply that the solution $\psi_c(u)$ of (3.29) depends continuously on c. By assumption *(B10)*, we infer that $\psi_0(1) < 0$. By Step 2 we conclude that there exists $c^* > 0$ such that $\psi_{c^*}(1) = 0$. Moreover, by uniqueness and a standard comparison theorem, $\psi_c(1)$ is strictly increasing as a function of c, so that c^* is unique.

To complete the proof, if suffices to observe that a solution of (3.29) with $\psi(1) = 0$ satisfies $\psi(u) > 0$ if $0 < u < 1$ and is therefore a solution of (3.28). But this is a straightforward consequence of the argument used at the end of Step 1. □

3.4. *Notes and further comments*

1. Elementary approaches to the existence of the travelling front solutions, for an equation or a system with two equilibria, can be found in [2,77,78] (in [78] a class of non-autonomous systems is considered). Front waves and their minimal speed are studied in [32] for two-dimensional systems with four equilibria.

2. Comparison techniques are used in [44] to give analytic approximation of the front solutions.

3. We might have enlarged the scope of the results described here by considering the presence of singularities, namely $p(0) = 0$ in (3.1). Such singularities are important in applications. In [51] Malaguti and Marcelli study the problem

$$\big(p(u)u'\big)' + cu' + g(u) = 0, \tag{3.31}$$

where g is of type A and p is a C^1 function with $p(0) = 0$ and $p'(0) > 0$. In addition to decreasing heteroclinics between 1 and 0 there appear *sharp type* solutions, by which one understands decreasing solutions $u(t)$ of (3.31) in $]-\infty, 0]$ such that $u(-\infty) = 1$, $u(0) = 0$ and $u'(0) = -c/p'(0)$. It turns out that a threshold speed c^* exists such that (3.31) has a decreasing heteroclinic from 1 to 0 if $c > c^*$, a sharp type solution if $c = c^*$ and no solution of either type if $c < c^*$.

4. Hamel and Nadirashvili [39] use front wave solutions of

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + g(u),$$

with g concave of type A, to construct an infinite dimensional manifold of entire solutions.

5. A wealth of results concerning wavefronts and their admissible speeds can be found in the recent monograph by Gilding and Kersner [35].

Part 2. Fourth Order Equations

4. The Extended Fisher–Kolmogorov equation: An overview

The class of fourth order differential equations considered in this survey is related to semi-linear evolution equations of the form

$$\frac{\partial u}{\partial t} + \frac{\partial^4 u}{\partial x^4} - \beta \frac{\partial^2 u}{\partial x^2} + u^3 - u = 0, \tag{4.1}$$

where β is a real parameter. This equation serves as a model in studies of pattern formation in many physical, chemical or biological systems. It also arises in the study of phase transitions.

When $\beta > 0$, it is related to the so-called *Extended Fisher–Kolmogorov* equation which was proposed in 1988 by Dee and van Saarloos [30] as a higher-order model equation for *bistable* systems. The term bistable indicates that the states $u = \pm 1$ are stable for the homogenized equation $-u' = u^3 - u$. When suitably scaled, solutions of (4.1) solve the equation

$$\frac{\partial v}{\partial t} + \gamma \frac{\partial^4 v}{\partial x^4} - \frac{\partial^2 v}{\partial x^2} + v^3 - v = 0, \tag{4.2}$$

where the positive parameter γ is related to β by the formula $\beta = 1/\sqrt{\gamma}$ and $v(x,t) = u(\sqrt{\beta} x, t)$. When $\gamma = 0$ in (4.2), the equation is a second order evolution equation (of Fisher type) which enters into the framework of Section 3. It is often referred to as the Fisher–Kolmogorov equation.

In this section, we summarize recent results concerning the existence of heteroclinic solutions for the stationary Extended Fisher–Kolmogorov equation

$$u'''' - \beta u'' + u^3 - u = 0. \tag{4.3}$$

Heteroclinic solutions of (4.3) connecting -1 to $+1$ in the phase-space satisfy the following conditions

$$\lim_{x \to \pm\infty} \big(u, u', u'', u'''\big)(x) = (\pm 1, 0, 0, 0). \tag{4.4}$$

Of course, we can also consider connections from $+1$ to -1 by reversing the role of $\pm\infty$ in the above condition.

The parameter β plays a central role in the analysis of the behaviour of solutions of (4.3). Indeed, the nature of the equilibria ± 1 changes at two critical values $\beta = \pm\sqrt{8}$. The linearization of (4.3) at $u = \pm 1$ reads

$$v'''' - \beta v'' + 2v = 0, \tag{4.5}$$

where v stands respectively for $u-1$ and $u+1$. The eigenvalues of the associated characteristic equation

$$\lambda^4 - \beta\lambda^2 + 2 = 0$$

are

$$\lambda = \pm\sqrt{\frac{\beta \pm \sqrt{\beta^2 - 8}}{2}}. \tag{4.6}$$

When $\beta \geqslant \sqrt{8}$, the four eigenvalues are real so that $u = \pm 1$ are *saddle-nodes*. For $\beta \in (-\sqrt{8}, \sqrt{8})$, they are all complex with non vanishing real part. The equilibria are then called *saddle-foci*. When β passes below $-\sqrt{8}$, the eigenvalues become purely imaginary and therefore $u = \pm 1$ are *centers*.

The behaviour of the solutions of the linearization of (4.3) around the equilibria provides important informations for the solutions of the nonlinear equation. Indeed, when no eigenvalue has a vanishing real part, it is well known that under some smoothness assumptions, the nonlinear flow and the flow defined by the linearization are conjugate in a neighborhood of the equilibria, see [41]. Consequently, when $\beta \geqslant -\sqrt{8}$, the solutions of (4.3) inherit some properties of the small solutions of (4.5) when they are close to $u = \pm 1$ (with small derivatives up to third order). For example, we easily obtain a qualitative description of the shape of any heteroclinic at $\pm\infty$. Indeed, when ± 1 are saddle-nodes, the solutions of (4.5) that have limit 0 at $+\infty$ or $-\infty$ are monotone while in the saddle-foci case, they oscillate around zero.

We focus on the model equation (4.3) to present different approaches which have been considered in the literature. We assume throughout this section that β is positive. We first look at the shooting method developed by Peletier and Troy. Their results are given without proof as the arguments are not used further in this monograph. We just mention some central ideas showing again the important role of the parameter β.

As previously mentioned, Eq. (4.3) can be written

$$\frac{1}{\beta^2} u'''' - u'' + u^3 - u = 0 \tag{4.7}$$

after a suitable rescaling of u. For large β, Eq. (4.7) is a fourth-order perturbation of the Fisher–Kolmogorov equation

$$-u'' + u^3 - u = 0$$

which can be analyzed using elementary methods. It is easily seen that $u^{\pm}(x) = \pm\tanh(\frac{x+a}{\sqrt{2}})$, $a \in \mathbb{R}$, are the only heteroclinics of the Fisher–Kolmogorov equation and these are monotone. We therefore expect that for large β the heteroclinic solutions of (4.3) are monotone. It turns out that a topological shooting method adapted to track monotone heteroclinics works fine for instance for $\beta \geqslant \sqrt{8}$. For this range of β, it is also remarkable that the uniqueness (up to translations and symmetry) of the heteroclinic solution of the Fisher–Kolmogorov equation extends to the fourth-order equation, see Note 1 below and [89].

For $|\beta| < \sqrt{8}$, the oscillatory behaviour of the solutions of (4.3) close to the equilibria makes the shooting method much more tricky. However, two families of heteroclinics can be singled out thanks to a careful analysis.

The remaining of the section is concerned with variational methods and more particularly with minimization. We expose in Section 4.2 a global minimization process. Here we develop the arguments in detail as they will be confronted with a more general framework in Section 5. We conclude our overview of the model equation (4.3) with the nice local minimization method developed by Kalies, Kwapisz and VanderVorst. Their method handles perfectly oscillatory graphs and is therefore efficient when $0 < \beta < \sqrt{8}$. In Section 4.3, we define an homotopy type which allows to consider convenient subsets for local minimization and describe the multi-transition profiles of the local minimizers. We do not present the results in detail but we emphasize that parts of the arguments are developed and used later on in Sections 6 and 7.

4.1. *A shooting method*

One of the basic tools that can be used to analyze the solutions of (4.3) is a topological shooting method developed by Peletier and Troy in [59–62], see also [63]. The main idea of a classical shooting method is to look the way solutions change with respect to initial conditions (taken as parameters) at some fixed initial point. The success of the method for second order ordinary differential equation comes from the number of parameters that is usually reduced to one. Here we reduce the number of parameters by restricting our attention to odd solutions. The reversibility of (4.3) and the oddness of the nonlinear term $u^3 - u$ allows indeed to look at the problem

$$\begin{cases} u''''(x) - \beta u''(x) + u^3(x) - u(x) = 0, & x \geqslant 0, \\ u(0) = u''(0) = 0, \\ \lim_{x\to+\infty} \big(u, u', u'', u'''\big)(x) = (+1, 0, 0, 0). \end{cases} \tag{4.8}$$

If u solves (4.8), then the odd extension of u,

$$u^*(x) = \begin{cases} u(x) & \text{for } x \geqslant 0, \\ -u(-x) & \text{for } x < 0, \end{cases}$$

is an odd heteroclinic solution of (4.3) connecting -1 to $+1$. To find a solution of (4.8), we consider the Cauchy problem

$$\begin{cases} u''''(x) - \beta u''(x) + u^3(x) - u(x) = 0, & x \geqslant 0, \\ u(0) = u''(0) = 0, \\ u'(0) = \mu, \\ u'''(0) = \nu, \end{cases} \tag{S}$$

where $\mu \in \mathbb{R}^+$ and $\nu \in \mathbb{R}$. According to classical theory of ordinary differential equations, this Cauchy problem defines a unique local solution $u(\mu, \nu, \cdot)$ for every $\mu,\ \nu \in \mathbb{R}$. The shooting method then consists in finding the right initial conditions $\mu,\ \nu \in \mathbb{R}$ such that $u(\mu, \nu, \cdot)$ is globally defined and solves (4.8). A priori, we thus have to handle a two dimensional topological shooting. However, we can take benefit of a first integral to link the parameters μ and ν. In fact it is easily checked that the Hamiltonian

$$H\big(u, u', u'', u'''\big) = u'''u' - \frac{1}{2}u''^2 - \frac{\beta}{2}u'^2 + \frac{(u^2-1)^2}{4} \tag{4.9}$$

is constant along solutions, i.e. for every solution u of (4.3), there exists a constant E such that for all $x \in \mathbb{R}$, $H(u(x), u'(x), u''(x), u'''(x)) = E$. This first integral is referred to as the energy of the solution (by analogy with second order equations). Observe that any solution of (4.8) belongs to the level of energy $E = 0$. Indeed, this follows from the limit value

$$\lim_{x\to+\infty} H\big(u(x), u'(x), u''(x), u'''(x)\big) = 0.$$

We then infer that any solution of (4.8) satisfies

$$u'''(0) = \frac{\beta}{2}u'(0) - \frac{1}{4u'(0)}$$

taking also into account that $u'(0) \neq 0$ for such a solution. We therefore impose the conditions $\mu > 0$ and

$$\nu = \nu(\mu) := \frac{\beta}{2}\mu - \frac{1}{4\mu}$$

in the Cauchy problem (S). Summing up, the shooting method amounts to finding $\mu \in \mathbb{R}^+$ such that $u(x) := u(\mu, \nu(\mu), x)$ satisfies

$$\lim_{x\to+\infty} \big(u(x), u'(x), u''(x), u'''(x)\big) = (+1, 0, 0, 0).$$

For $\beta \geqslant \sqrt{8}$, the method is efficient to obtain a monotone heteroclinic.

THEOREM 4.1. *Let $\beta \in [\sqrt{8}, +\infty[$. Then there exists a monotone solution of* (4.8). *Moreover, its odd extension is a heteroclinic solution of* (4.3) *connecting* -1 *to* $+1$.

The proof goes as follows. For $\mu > 0$, we define

$$\xi(\mu) := \sup\big\{x > 0 \mid u'\big(\mu, \nu(\mu), \cdot\big) > 0 \text{ on } (0, x)\big\}$$

and

$$\mu^* := \sup\big\{\hat{\mu} \mid u\big(\mu, \nu(\mu), \xi(\mu)\big) < 1 \text{ for } \mu \in]0, \hat{\mu}[\big\}.$$

It turns out that $\xi(\mu^*) = +\infty$ and the orbit of $u(\mu^*, \nu(\mu^*), \cdot)$ tends to $(+1, 0, 0, 0)$ in the phase-space. We refer to [59,63] for a complete proof.

We think it is worth pointing the role of the assumption $\beta \geqslant \sqrt{8}$ in the preceding theorem. As we already mentioned, the roots of the characteristic equation associated to the linearization of (4.3) around the equilibria are real if $\beta \geqslant \sqrt{8}$. The squares of these roots are

$$\tau_\pm = \frac{\beta \pm \sqrt{\beta^2 - 8}}{2}.$$

Now observe that the linear operator $D^4 - \beta D^2 + 2I$ can be factorized as $(D^2 - \tau_+ I)(D^2 - \tau_- I)$. Letting $v = 1 - u$, Eq. (4.3) written in terms of v yields

$$v'''' - \beta v'' + 2v = v^2(3 - v). \tag{4.10}$$

Therefore, as long as $\beta \geqslant \sqrt{8}$, the solutions (v, w) of the second order system

$$\begin{cases} v'' - \tau_+ v = w, \\ w'' - \tau_- w = v^2(3 - v), \end{cases} \tag{4.11}$$

lead to solutions v of (4.10). This formulation of (4.3) as a system turns out to be very powerful to obtain estimates on v (and therefore on u) that allow to conclude that $\xi(\mu^*) = +\infty$. These estimates rely on a repeated application of the *Strong Maximum Principle* (see [65]) which can be used once we know the sign of the right hand sides in each equation of the system. The Strong Maximum Principle is also the main tool to prove the uniqueness of the heteroclinic of Theorem 4.1, see Note 1.

When $0 < \beta < \sqrt{8}$, the preceding argument involving the Maximum Principle is no more at hand. Anyway, Theorem 4.1 cannot hold in this parameter range. Indeed, as stressed in the introduction of this section, the linearization around the equilibria display oscillatory solutions so that any solution of (4.8) oscillates around $+1$ in its tail, i.e. when $x \to +\infty$. Therefore, the shooting method must now analyze carefully the location and value of the successive local extrema of $u(\mu, \nu(\mu), \cdot)$. One of the greatest difficulties when dealing with oscillatory solution graphs is to prove the convergence at infinity. The following result is the key to overcome this problem.

PROPOSITION 4.2. *Let $\beta \geqslant 0$ and assume u is a solution of* (4.3) *such that $H(u, u', u'', u''') = 0$. Suppose that for some $a \in \mathbb{R}$ and $M > 1$,*

$$\frac{1}{\sqrt{3}} \leqslant u(x) \leqslant M \quad \text{for } x > a.$$

Then

$$\lim_{x \to +\infty} \big(u, u', u'', u'''\big)(x) = (1, 0, 0, 0).$$

Basically, the proposition states that the equilibrium attracts all the solutions that stay in a strip around it. The lower bound of the strip has the particularity that the nonlinear term $u^3 - u$ is monotone increasing for all $u \geqslant \frac{1}{\sqrt{3}}$. This makes the functional

$$\mathcal{G}(u) := \frac{1}{2}u''^2 + \frac{1}{4}\left(u^2 - 1\right)^2$$

convex in the strip. This property is extensively used to prove Proposition 4.2, see [60,63].

Using sharp estimates on the critical points of $u(\mu, \nu(\mu), \cdot)$ for which we refer to [60, 63], the following can be proved. We denote respectively by ξ_k and η_k the local maxima and the local minima of the solutions of (4.8).

THEOREM 4.3. *Let β satisfy $0 < \beta < \sqrt{8}$. There exist two solutions u_1, u_2 of* (4.8) *that have the following properties*:

$$\frac{1}{\sqrt{3}} < u_1(\eta_k) < 1 < u_1(\xi_k) < \sqrt{2} \quad \textit{for } k = 1, 2, \ldots,$$

$$-1 < u_2(\eta_1) < 0, \quad 1 < u_2(\xi_1) < \sqrt{2} \quad \textit{and}$$

$$\frac{1}{\sqrt{3}} < u_2(\eta_k) < 1 < u_2(\xi_k) < \sqrt{2} \quad \textit{for } k = 2, 3, \ldots.$$

The odd extensions of u_1, u_2 are heteroclinic solutions of (4.3).

The solution u_1 is usually called the *principal heteroclinic* as it is monotone increasing in some interval $[-T, T]$ and it oscillates around -1 in $(-\infty, -T)$ and around $+1$ in $(T, +\infty)$. Also this heteroclinic is the first of two families of odd heteroclinic solutions which have $2n + 1$ zeros on $\mathbb{R}$. The two families differ by the amplitude of the oscillations. The first family consists of so-called *multi-transition* solutions as all the successive local extrema between the zeros are outside the region $[-1, 1]$. Thus the profiles of these solutions display $2n + 1$ jumps from -1 to $+1$ and two oscillatory tails around -1 and $+1$. In the second family, the amplitude of the oscillations is smaller than 1 so that the corresponding solutions are *single-transition* heteroclinics.

THEOREM 4.4. *Let β satisfy $0 < \beta < \sqrt{8}$. For each $n \in \mathbb{N}_0$, there exist two solutions u_1, u_2 of* (4.8) *that have the following properties*:

$$\textit{if } n \textit{ is even}, \quad \begin{cases} \textit{for } k \leqslant n/2, & u_1(\eta_k) < -1, \quad u_1(\xi_k) > 1, \\ & -1 < u_2(\eta_k) < 0 < u_2(\xi_k) < 1, \\ \textit{for } k > n/2, & 1/\sqrt{3} < u_i(\eta_k), \quad 1 < u_i(\xi_k), \quad i = 1, 2, \end{cases}$$

$$\textit{if } n \textit{ is odd}, \quad \begin{cases} \textit{for } k \leqslant (n-1)/2, & u_1(\eta_k) < -1, \quad u_1(\xi_k) > 1, \\ & -1 < u_2(\eta_k) < 0 < u_2(\xi_k) < 1, \\ \textit{for } k = (n+1)/2, & u_1(\eta_k) < -1, \quad u_1(\xi_k) > 1, \\ & -1 < u_2(\eta_k) < 0, \quad u_2(\xi_k) > 1, \\ \textit{for } k > (n+1)/2 & 1/\sqrt{3} < u_i(\eta_k), \quad u_i(\xi_k) > 1, \quad i = 1, 2. \end{cases}$$

Moreover, for all $n \geqslant 1$, $\|u_i\|_\infty < \sqrt{2}$ *and* $u_i'(0) \neq 0$ *for* $i = 1, 2$.

We refer to [60,63] for the proof.

4.2. *Minimization*

Heteroclinic solutions of (4.3) are critical points of the action functional

$$\mathcal{F}_\beta(u) = \int_{\mathbb{R}} \left(\frac{1}{2}\left(u''^2 + \beta u'^2\right) + \frac{1}{4}\left(u^2 - 1\right)^2 \right) \mathrm{d}x. \tag{4.12}$$

This functional is well defined for functions u having first and second square integrable derivatives and being such that the potential is integrable. Taking into account conditions (4.4) which are satisfied by heteroclinics connecting -1 to $+1$, we can define $\mathcal{F}_\beta$ in the space

$$\left\{u : \mathbb{R} \to \mathbb{R} \mid u + 1 \in H^2\left(\mathbb{R}^-\right),\ u - 1 \in H^2\left(\mathbb{R}^+\right)\right\}.$$

Indeed, if $u + 1 \in H^1(\mathbb{R}^-)$, then $\lim_{x\to-\infty} u(x) = -1$ see for example [22] and u is bounded in every compact interval $[-T, 0]$. A similar observation can be made on $\mathbb{R}^+$ so that we infer

$$\begin{aligned}\int_{\mathbb{R}} \frac{1}{4}\left(u^2 - 1\right)^2 \mathrm{d}x &= \int_{\mathbb{R}^-} \frac{1}{4}\left(u^2 - 1\right)^2 \mathrm{d}x + \int_{\mathbb{R}^+} \frac{1}{4}\left(u^2 - 1\right)^2 \mathrm{d}x \\ &\leqslant \frac{C}{4}\left(\|u + 1\|^2_{L^2(\mathbb{R}^-)} + \|u - 1\|^2_{L^2(\mathbb{R}^+)}\right)\end{aligned}$$

for some positive constant $C = C(\|u\|_{L^\infty(\mathbb{R})})$ and therefore $\mathcal{F}_\beta(u)$ is finite.

For $\beta \geqslant 0$, $\mathcal{F}_\beta$ is positive and hence bounded from below. The a priori simplest way to find critical points of $\mathcal{F}_\beta$ is therefore to search for minimizers. The next theorem confirms the efficiency of a minimization approach. To avoid a loss of compactness due to the invariance under translations, we look for a minimizer in the space

$$\mathcal{E} := \left\{u : \mathbb{R} \to \mathbb{R} \mid u(0) = 0,\ u + 1 \in H^2\left(\mathbb{R}^-\right),\ u - 1 \in H^2\left(\mathbb{R}^+\right)\right\}. \tag{4.13}$$

Of course, this is not restrictive.

THEOREM 4.5. *For all* $\beta \geqslant 0$, *the functional* $\mathcal{F}_\beta$ *has a global minimizer* $\varphi_\beta \in \mathcal{E}$ *which is a heteroclinic solution of* (4.3). *Furthermore, any minimizer is odd and positive in* $]0, +\infty[$.

This theorem is the analogous of Theorem 2.1. Remember that for the functional $\mathcal{J}$ of Theorem 2.1, minimizing sequences $(u_n)_n$ can be taken in such a way that $-1 \leqslant u_n \leqslant 1$. Indeed, $v_n = \sup(-1, \inf(u_n, 1))$ is still a minimizing sequence. Using similar modification arguments it is easily seen that the minimizers of $\mathcal{J}$ are monotone.

One of the first differences that we encounter when considering the functional $\mathcal{F}_\beta$ in $\mathcal{E}$, comes from the fact that these modifications which keep functions in $H^1_{\mathrm{loc}}(\mathbb{R})$ do not necessarily produce functions of class C^1 and therefore, in general, the modified functions do not belong to $H^2_{\mathrm{loc}}(\mathbb{R})$. By the way, as already mentioned, it is not true in general that minimizers of $\mathcal{F}_\beta$ in $\mathcal{E}$ are monotone. When dealing with $\mathcal{J}$, modification arguments are also used to ensure that quasi-minimizers stay close to ± 1 outside a fixed compact interval which can always be centered around zero by translation invariance.

To substitute these rather simple arguments, we take benefit of the symmetry of the functional $\mathcal{F}_\beta$. The original proof of Theorem 4.5 is due to Peletier, Troy and VanderVorst [58]. We give here a proof which is closer to Kalies and VanderVorst [43].

PROOF OF THEOREM 4.5. We introduce the spaces

$$\mathcal{E}^+ := \{u : \mathbb{R}^+ \to \mathbb{R} \mid u(0) = 0,\ u - 1 \in H^2(\mathbb{R}^+)\},$$
$$\mathcal{E}^- := \{u : \mathbb{R}^- \to \mathbb{R} \mid u(0) = 0,\ u + 1 \in H^2(\mathbb{R}^-)\}$$

and define the restricted functional $\mathcal{F}^\pm_\beta : \mathcal{E}^\pm \to \mathbb{R}$ by

$$\mathcal{F}^\pm_\beta(u) := \int_{\mathbb{R}^\pm} L(u, u', u'')\,\mathrm{d}x,$$

where $L(u, u', u'')$ is the Lagrangian defined by

$$L(u, u', u'') := \frac{1}{2}(u''^2 + \beta u'^2) + \frac{1}{4}(u^2 - 1)^2. \tag{4.14}$$

Let us consider the values

$$c := \inf_{\mathcal{E}} \mathcal{F}_\beta \quad \text{and} \quad c^\pm := \inf_{\mathcal{E}^\pm} \mathcal{F}^\pm_\beta.$$

Since $\mathcal{F}_\beta$ is symmetric, it is easily seen that for all $u^+ \in \mathcal{E}^+$, $\mathcal{F}^+_\beta(u^+) = \mathcal{F}^-_\beta(u^-)$ where $u^- \in \mathcal{E}^-$ is defined by $u^-(x) = -u^+(-x)$. Therefore, $c^+ = c^- = c/2$.

Step 1. The variational problem

$$\inf\{\mathcal{F}^+_\beta(u) \mid u \in \mathcal{E}^+\}$$

has a positive solution.

Let $(v_n)_n \subset \mathcal{E}^+$ be a minimizing sequence, i.e. $\mathcal{F}^+_\beta(v_n) \to c^+$. For each $n \geqslant 0$, we define

$$x_n := \sup\{x \geqslant 0 \mid v_n(x) = 0\}.$$

Since $\lim_{x\to+\infty} v_n(x) = 1$, $x_n < +\infty$ for all $n \geqslant 0$. We now consider the positive sequence $(v_n^+)_n \in \mathcal{E}^+$ where

$$v_n^+(x) := v_n(x + x_n)$$

for $x \geqslant 0$. We observe that

$$\int_0^{x_n} L\big(v_n, v_n', v_n''\big)\,\mathrm{d}x \geqslant 0$$

so that $\mathcal{F}_\beta^+(v_n^+) \leqslant \mathcal{F}_\beta^+(v_n)$ which implies that $(v_n^+)_n$ is also a minimizing sequence for $\mathcal{F}_\beta^+$.

As the sequence $\mathcal{F}_\beta^+(v_n^+)$ is uniformly bounded, we deduce a uniform estimate for $\|v_n^+ - 1\|_{H^2(\mathbb{R}^+)}$. Indeed, the L^2-bounds for $(v_n^+)'$ and $(v_n^+)''$ follow easily from the bound on $\mathcal{F}_\beta^+(v_n^+)$ while we infer from the positivity of v_n^+ that

$$\int_0^{+\infty} \frac{(v_n^+ - 1)^2}{4}\,\mathrm{d}x \leqslant \int_0^{+\infty} \frac{(v_n^{+2} - 1)^2}{4}\,\mathrm{d}x \leqslant \mathcal{F}_\beta^+\big(v_n^+\big).$$

We now deduce that going to a subsequence if necessary, there exists $v^+ \in H^2(\mathbb{R}^+) + 1$ such that

$$v_n^+ - 1 \overset{H^2(\mathbb{R}^+)}{\rightharpoonup} v^+ - 1$$

and

$$v_n^+ \overset{C^1_{\mathrm{loc}}(\mathbb{R}^+)}{\longrightarrow} v^+.$$

As the two first terms in $\mathcal{F}_\beta^+$ are the square of seminorms and Fatou's Lemma is applicable to the last one, it follows that

$$\mathcal{F}_\beta^+\big(v^+\big) \leqslant \liminf_{n\to+\infty} \mathcal{F}_\beta^+\big(v_n^+\big) = c^+.$$

As the convergence is uniform on compact intervals, we conclude that $v^+(0) = 0$ so that $v^+ \in \mathcal{E}^+$ and $\mathcal{F}_\beta^+(v^+) = c^+$. Observe that v^+ is positive on $]0, +\infty[$ otherwise we could proceed as above to construct a positive function having smaller action.

Step 2. If $v \in \mathcal{E}^+$ is such that $(\mathcal{F}_\beta^+)'(v) = 0$, then $v''(0) = 0$, $v^* \in \mathcal{E}$ defined by

$$v^*(x) := \begin{cases} v(x) & \text{if } x \geqslant 0, \\ -v(-x) & \text{if } x < 0, \end{cases}$$

is a minimizer of $\mathcal{F}_\beta$ in $\mathcal{E}$ and v^* is a heteroclinic solution of (4.3).

We first compute

$$\big(\mathcal{F}_\beta^+\big)'(v)(h) = \int_0^{+\infty} \big(v''h'' + \beta v'h' + \big(v^3 - v\big)h\big)\,\mathrm{d}x \tag{4.15}$$

for all $h \in H^2(\mathbb{R}^+) \cap H_0^1(\mathbb{R}^+)$. Starting with $h \in C_c^2(\mathbb{R}^+)$ and using Du Bois–Reymond Lemma, we easily find that in fact $v \in C^4(\mathbb{R}^+)$ and v solves (4.3) for $x \geqslant 0$. As Eq. (4.3) expresses v'''' in terms of v'' and $v-1$, we deduce that $v'''' \in L^2(\mathbb{R}^+)$. Indeed, $v'' \in L^2(\mathbb{R}^+)$ and

$$\int_0^{+\infty} \bigl(v^3(x) - v(x)\bigr)^2 \,\mathrm{d}x \leqslant \bigl(\|v\|_\infty^2 + \|v\|_\infty\bigr)\|v-1\|_{L^2}^2 < \infty.$$

It now follows by interpolation that

$$\|v'''\|_{L^2}^2 \leqslant C\bigl(\|v''\|_{L^2} + \|v''''\|_{L^2}\bigr) < \infty.$$

Hence $v - 1 \in H^4(\mathbb{R}^+)$. Observe also that the L^2-integrability of the derivatives implies that

$$\lim_{x\to+\infty} v(x) = +1 \quad \text{and} \quad \lim_{x\to+\infty} v^{(n)}(x) = 0 \quad \text{for } n = 1, 2, 3. \tag{4.16}$$

Integrating (4.15) by parts, we obtain for all $h \in H^2(\mathbb{R}^+) \cap H_0^1(\mathbb{R}^+)$,

$$\bigl[v''h'\bigr]_0^{+\infty} - \bigl[v'''h\bigr]_0^{+\infty} - \beta\bigl[v'h\bigr]_0^{+\infty} = 0$$

as

$$\int_0^{+\infty} \bigl(v'''' - \beta v'' + v^3 - v\bigr)(x)h(x)\,\mathrm{d}x = 0.$$

Taking (4.16) into account, we now deduce that

$$v''(0)h'(0) = 0$$

for all $h \in H^2(\mathbb{R}^+) \cap H_0^1(\mathbb{R}^+)$ which of course implies that $v''(0) = 0$. The function $v^* : \mathbb{R} \to \mathbb{R}$ defined by

$$v^*(x) = \begin{cases} v(x) & \text{if } x \geqslant 0, \\ -v(-x) & \text{if } x < 0, \end{cases}$$

is therefore of class C^4 and straightforward arguments allow to conclude that v^* is a critical point of $\mathcal{F}_\beta$ which is a heteroclinic solution of (4.3), actually of class C^∞. Also, $\mathcal{F}_\beta(v^*) = 2\mathcal{F}_\beta^+(v) = 2c^+ = c$ so that v^* is a minimizer of $\mathcal{F}_\beta$ in $\mathcal{E}$.

Step 3. If $u \in \mathcal{E}$ minimizes $\mathcal{F}_\beta$, then u is odd.

Let us write $u^\pm = u_{|\mathbb{R}^\pm}$. As $\mathcal{F}_\beta(u) = c$, we have $\mathcal{F}_\beta^+(u^+) = \mathcal{F}_\beta^-(u^-) = c/2$ otherwise the odd extension of u^+ or u^- would have a lower action than c. From Step 2, we know that $u(0) = u''(0) = 0$. Hence $v^+(x) := -u^-(-x)$ satisfies $\mathcal{F}_\beta^+(v^+) = c^+$ and $v^+(0) = (v^+)''(0) = 0$. Thus both u^+ and v^+ are minimizers of $\mathcal{F}_\beta^+$ in $\mathcal{E}^+$. As

$(v^+)'(0) = (u^-)'(0) = (u^+)'(0)$ and $(v^+)'''(0) = (u^-)'''(0) = (u^+)'''(0)$, the functions v^+ and u^+ solve the same Cauchy problem

$$\begin{aligned} &u''''(x) - \beta u''(x) + u^3(x) - u(x) = 0, \quad x \geqslant 0, \\ &u(0) = 0, \quad u'(0) = \big(u^+\big)'(0), \\ &u''(0) = 0, \quad u'''(0) = \big(u^+\big)'''(0). \end{aligned}$$

By uniqueness, this implies $u^+(x) = v^+(x)$ for all $x \in \mathbb{R}^+$, i.e. $u^+(x) = -u^-(-x)$ for all $x \in \mathbb{R}^+$. □

4.3. *Homotopy classes of multi-transition solutions*

When $\beta < \sqrt{8}$, the equilibrium solution ± 1 of (4.3) are saddle-foci. It is known that heteroclinic connections between such kind of equilibria can exhibit a complex structure, see [23, 31,60]. In [43], Kalies and VanderVorst construct for (4.3) so-called *multi-bump* solutions i.e solutions with multiple oscillations separated by large distances. The usual methods used to obtain such solutions are rather tricky and require a careful study of the stable and unstable manifolds or a certain kind of nondegeneracy condition on a primary connection whose well separated copies are then glued together, see [23,28,79]. In [42], Kalies, Kwapisz and VanderVorst introduce a direct method to find multi-transition solutions. As we already mentioned, the term multi-transition refers to the fact that the graph of such solutions consists of multiple jumps from one equilibrium to the other. One jump is then called a transition. Also such solutions are qualitatively different from multi-bump solutions as the distance between two successive transition is not necessarily large. The method of Kalies et al. consists in minimizing the action functional (4.12) in specific subspaces of functions having the desired number of transitions.

When projected in the configuration plane (u, u'), a heteroclinic orbit yields a curve connecting the points $(\pm 1, 0)$. Choosing two oriented loops e_1 and e_2 around respectively $(-1, 0)$ and $(+1, 0)$ (see Fig. 2), an homotopy type can be associated to any curve connecting $(\pm 1, 0)$ (see the precise definition below). This homotopy type records the number of transitions between these points and the oscillations around them between the transitions. Hence, every orbit connecting $(-1, 0)$ to $(+1, 0)$ can be represented by a word of the form

$$e_1^{\theta_{2m}} \cdot e_2^{\theta_{2m-1}} \cdot \cdots \cdot e_1^{\theta_2} \cdot e_2^{\theta_1},$$

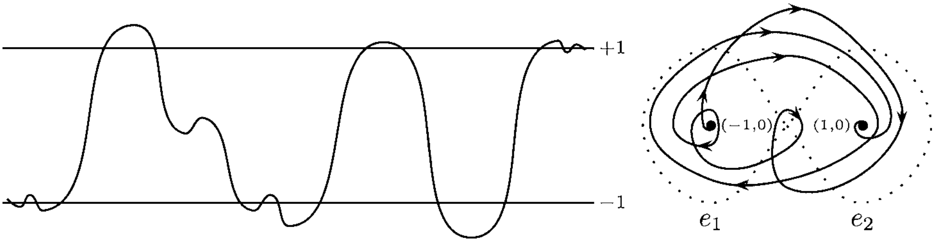

Fig. 2. An orbit with homotopy type $e_1 e_2 e_1^2 e_2$.

where $\theta(u) = (\theta_1, \ldots, \theta_{2m}) \in \mathbb{N}^{2m}$. The vector $\theta(u)$ is called the *winding* vector associated to u. Observe that the word starts with the first visit at $(+1, 0)$ and stops recording after the last visit around $(-1, 0)$. Also the vector $g(u) = 2\theta(u)$ contains exactly the number of crossings that u makes with ± 1 between the transitions. The vector $g = 0$ is therefore associated to functions with a single transition. For each vector $g \in 2\mathbb{N}^{2m} \cup \{0\}$, the homotopy class $M(g)$ consisting of functions of $\mathcal{E}$ having winding vector $g/2$. The following more precise definition is introduced in [42].

DEFINITION 4.6. A function $u \in \mathcal{E}$ is in $M(g)$ if there are nonempty sets $\{A_i\}_{i=0}^{2m+1}$ such that

(i) $u^{-1}(\pm 1) = \bigcup_{i=0}^{2m+1} A_i$,
(ii) $\# A_i = g_i$ for $i = 1, \ldots, 2m$,
(iii) $\max A_i < \min A_{i+1}$ for $i = 0, \ldots, 2m$,
(iv) $u(A_i) = (-1)^{i+1}$,
(v) $\{\max A_0\} \cup (\bigcup_{i=1}^{2m} A_i) \cup \{\min A_{2m+1}\}$ consists of transverse crossings of ± 1.

Under these conditions $M(g)$ is an open subset of $\mathcal{E}$. As the functional $\mathcal{F}_\beta$ is positive, we can define

$$\inf_{u \in M(g)} \mathcal{F}_\beta(u). \tag{4.17}$$

If the infimum is achieved by a function in the interior of $M(g)$, then a local minimizer of $\mathcal{F}_\beta$ with the corresponding properties solves the Euler–Lagrange equation (4.3). The next theorem is a particular case of Theorem 1.3 in [42].

THEOREM 4.7. *Let* $0 < \beta < \sqrt{8}$. *Then for all* $g \in 2\mathbb{N}^{2m} \cup \{0\}$, *the functional* $\mathcal{F}_\beta$ *defined by* (4.12) *has a local minimizer* u_g *in the homotopy class* $M(g)$. *Moreover, the function* u_g *is a heteroclinic solution of* (4.3) *displaying* $2m + 1$ *transitions between* ± 1.

The proof of Theorem 4.7 is carried out in [42]. The main difficulty is to show that minimizing sequences in $M(g)$ have weak limits in the interior of $M(g)$. Indeed, minimizing sequences can approach the boundary of $M(g)$ so that the limit function could gain or lose complexity. For example, tangential crossings of ± 1 could appear due to a coalescence of two or more crossings or to a new spurious oscillation around one equilibrium. The oscillatory behaviour of solutions in a neighborhood of a saddle-focus equilibrium plays then a crucial role to control the minimizing sequences. Efficient tools were developed by Kalies et al. to adjust functions of the boundary of $M(g)$. Basically these tools rely on a cut and paste technique which allows to delete spurious oscillations or replace pieces of functions tangent to ± 1 with pieces of small oscillating orbits around $(\pm 1, 0, 0, 0)$ in the phase-space.

Another source of troubles in the minimization process comes from a possible lack of compactness when passing to a weak limit. Indeed the distance between two transitions or two crossings of either -1 or $+1$ could grow to infinity. Here again the oscillatory properties of the orbits close to a saddle-focus prevent from these losses of complexity in the limit.

4.4. *Notes and further comments*

1. As already mentioned, in case $\beta \in [\sqrt{8}, +\infty[$, the heteroclinic solution obtained in Theorem 4.1 is unique (up to translation and symmetry). The first result in this direction is due to Peletier and Troy [59]. They prove that the heteroclinic obtained via the shooting method is unique in the class of monotone antisymmetric functions and they conjecture that it is actually unique in the class of all functions. Kwapisz [47] and van den Berg [89] have then confirmed the conjecture. The proof of Kwapisz relies on the use of a twist-map while the arguments of van den Berg are based on the analysis of the phase-space and more precisely of its projection into the configuration plane (u, u'). The analysis of van den Berg applies to the model equation

$$u'''' - \beta u'' + f'(u) = 0, \tag{4.18}$$

where f is a double-well potential of class C^2. Considering the set of bounded functions

$$\mathcal{B}(a, b) := \{u \in C^4(\mathbb{R}) \mid u(x) \in [a, b] \text{ for all } x \in \mathbb{R}\},$$

and the value

$$\omega(a, b) = \max\Big\{0, \max_{u\in[a,b]} -f''(u)\Big\},$$

his key result states that when $\beta \geqslant 2\sqrt{\omega(a, b)}$, bounded solutions of (4.18) in $\mathcal{B}(a, b)$ do not cross in the configuration plane (u, u'). Observing then that for $\beta \geqslant \sqrt{8}$, any bounded solution u of (4.3) satisfies $\|u\|_\infty \leqslant 1$, he deduces that the uniqueness property in the configuration plane holds for the parameter range $\beta \geqslant \sqrt{8}$. The a priori bound is obtained by applying the maximum principle twice to the factorization (4.11). The uniqueness of the heteroclinic solution then follows from the energy ordering of the bounded solutions in the configuration plane (u, u'). As stressed in Section 4.1, the Hamiltonian

$$H(u, u', u'', u''') = u'''u' - \frac{1}{2}u''^2 - \frac{\beta}{2}u'^2 + f(u)$$

is constant along solutions. It turns out that for $\beta \geqslant 2\sqrt{\omega(a, b)}$ and solutions of (4.18) in $\mathcal{B}(a, b)$, the energy $E = H(u, u', u'', u''')$ is a parameter that orders the solutions in the configuration plane.

It is worth mentioning that the results of [89] also imply that the unique heteroclinic of Theorem 4.1 is asymptotically stable in the space of bounded uniformly continuous functions.

2. The global minimization approach of Section 4.2 can be extended to the functional

$$\mathcal{J}(u) := \int_{-\infty}^{\infty} \Big(\frac{1}{2}\big(u''^2 + g(u)u'^2\big) + f(u)\Big)\,dx,$$

where $g:\mathbb{R}\to\mathbb{R}$ is a positive function and $f:\mathbb{R}\to\mathbb{R}$ is a double well potential with nondegenerate minima at the same level of energy, see Section 5. If f and g are even then the equivalent of Theorem 4.5 holds. When this is not the case, the conclusion that any minimizer is odd is false and we cannot ensure that minimizers are still single-transition solutions.

3. The results of Section 4.3 hold in a more general framework. The potential $\frac{(u^2-1)^2}{4}$ may be replaced by a function $f\in C^2(\mathbb{R})$ that has exactly two nondegenerate global minima at $u=\pm1$ and grows superquadratically at $\pm\infty$. If the parameter β satisfies $\beta^2<4f''(\pm1)$ then $u=\pm1$ are saddle-focus equilibria for the equation

$$u'''' - \beta u'' + f'(u) = 0.$$

If f is even then the equivalent of Theorem 4.7 holds, while without this symmetry assumption it is not clear that the infima are attained in the interior of each homotopy class. However, when $g_i=2$ for all i or if the g_i's are large enough (i.e. the profile is then similar to multi-bump solutions) then local minimizers exist in the corresponding class $M(g)$, see [42].

4. Not much is known about the role of the heteroclinic solutions for the evolution problem

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial^4 u}{\partial x^4} - \beta \dfrac{\partial^2 u}{\partial x^2} + u^3 - u = 0, & 0<x<L,\ t>0,\\ u(t,0)=u(t,L)=0, & t>0,\\ \dfrac{\partial^2 u}{\partial x^2}(t,0) = \dfrac{\partial^2 u}{\partial x^2}(t,L)=0, & t>0,\\ u(0,x)=u_0(x), & 0<x<L. \end{cases}$$

When $\beta \geqslant \sqrt{8}$, the stability of the heteroclinic has been proved by van den Berg [89].

Another class of solutions, namely travelling wave solutions, is important for the dynamics of the evolution equation. These solutions are waves that evolve at constant speed. Thus they can be written $u(t,x)=U(x-ct)$ for some constant c and solve the equation

$$U'''' - \beta U'' - cU' + f'(U) = 0.$$

For this last equation, the heteroclinic solutions connecting -1 to $+1$ or 0 to -1 or $+1$ are of particular interest. The problem of existence of such solutions is not completely solved. Partial results exist for similar models with an asymmetric nonlinear term like $f_a'(u)=(u+a)(u^2-1)$, $a\neq 0$. The case $a=0$ can also be treated when dealing with heteroclinics connecting 0 to -1 or $+1$. For further details, we refer to [75,88,90].

5. It was observed in [58] that the arguments used in the proof of Theorem 4.5 can be carried over to treat sixth order bistable equations of the form

$$u^{(6)} + Au^{(4)} + Bu^{(2)} + u - u^3 = 0$$

provided that $A^2 < 4B$. The associated functional is then coercive and weakly lower-semicontinuous.

5. Sign changing Lagrangians

The success of the arguments of Section 4.2 depends on the positivity and the symmetry of the Lagrangian

$$L(u, u', u'') = \frac{1}{2}\left(u''^2 + \beta u'^2\right) + \frac{1}{4}\left(u^2 - 1\right)^2.$$

The symmetry of the potential allows to consider a restricted functional on $\mathbb{R}^+$ which makes the minimization process very simple. However it is easily seen that a more tedious proof works fine without exploiting the symmetry of the functional. On the other hand, the positivity of the Lagrangian seems much more crucial. Indeed, it gives a simple lower bound on the functional. Another great advantage of the Lagrangian L is that the potential $f(u) = \frac{1}{4}(u^2 - 1)^2$ is superquadratic at $\pm\infty$ and has nondegenerate minima i.e. $f''(\pm 1) \neq 0$. This last property implies that for u close to $+1$ (respectively -1), $f(u)$ behaves like the square of the L^2-norm of $u - 1$ (respectively $u + 1$).

In this section, we consider a larger class of Lagrangians

$$L_g\left(u, u', u''\right) := \frac{1}{2}\left(u''^2 + g(u)u'^2\right) + f(u). \tag{5.1}$$

Here the function g is not assumed to be positive and f is a positive two-well potential with bottoms at ± 1. We consider in a convenient space (which is made precise below) the functional

$$\mathcal{F}(u) := \int_{-\infty}^{+\infty} \left(\frac{1}{2}\left(u''^2 + g(u)u'^2\right) + f(u)\right) \mathrm{d}x \tag{5.2}$$

whose Euler–Lagrange equation is given by

$$u'''' - g(u)u'' - \frac{1}{2}g'(u)u'^2 + f'(u) = 0. \tag{5.3}$$

The main idea is then to impose a condition on g to ensure a lower bound on the action $\mathcal{F}(u)$.

The results of Section 5.1 are based on [20,37]. The main assumption is as follows:

(C1) there exist a function $\tilde{g} \in C(\mathbb{R})$ and some $k < 1$ such that for all $u \in \mathbb{R}$,

$$g(u) \geqslant \tilde{g}(u) \quad \text{and } \left|\tilde{G}(u)\right| \leqslant k\sqrt{8f(u)},$$

where

$$\tilde{G}(u) := \int_0^u \tilde{g}(s)\,\mathrm{d}s.$$

This condition can be seen as a good balance between the possible negativity of g and the positivity of the potential. Though assumption (C1) allows g to take negative values, it rules out a negative constant. Indeed, $\tilde{g}$ must satisfy

$$\int_{-1}^{0} \tilde{g}(s)\,\mathrm{d}s = \int_{0}^{+1} \tilde{g}(s)\,\mathrm{d}s = 0.$$

Some regularity assumptions have to be made on f and g but it should be stressed that the approach does not require the nondegeneracy of the equilibria ± 1 as implicitly assumed when considering the model potential in $\mathcal{F}_\beta$. This rather weak assumption on the potential does not allow to define the functional $\mathcal{F}$ in an affine translate of $H^2(\mathbb{R})$ so that a slight modification of the functional settings has to be performed.

Section 5.2 reviews the results of [19]. Here more assumptions are in order but on the other hand g can be negative everywhere. Mainly, we only deal with symmetric functionals with an even potential having nondegenerate minima. Also, we do assume that the functional is bounded from below. We prove that this is the case when g^- is small.

5.1. *Functionals with sign changing acceleration coefficient*

We consider a two-well potential $f \in C^1(\mathbb{R})$ such that

(C2) for some $0 < a < 1$ and $\alpha > 0$,

$$\frac{f(u)}{(u-1)^2} \leqslant \alpha, \quad \text{for } |u-1| < a,$$
$$\frac{f(u)}{(u+1)^2} \leqslant \alpha, \quad \text{for } |u+1| < a;$$

(C3) $f(u) = 0$ if and only if $u = \pm 1$;

(C4) $f(u) \geqslant 0$ for all $u \in \mathbb{R}$ and

$$\liminf_{|u|\to\infty} f(u) > 0.$$

We also assume that $g \in C^1(\mathbb{R})$ satisfies (C1).

Notice that the assumption (C2) is automatically satisfied when ± 1 are nondegenerate minima. In this case, if f is of class C^2 in a neighborhood of ± 1, $g(\pm 1) < 0$ or (C1) holds with $g = \tilde{g}$ then ± 1 are saddle-foci. Indeed, by l'Hospital's rule, we obtain

$$\frac{\tilde{g}^2(\pm 1)}{4f''(\pm 1)} = \lim_{u\to\pm 1} \frac{\tilde{G}^2(u)}{8f(u)} < 1,$$

so that

$$g(\pm 1)^2 \leqslant \tilde{g}(\pm 1)^2 < 4f''(\pm 1).$$

In order to find a solution u of (5.3) that satisfies

$$\lim_{x\to\pm\infty} u(x) = \pm 1,$$

we minimize the functional $\mathcal{F}$ in a convenient functional space. As the assumptions on f are quite weak and g can vanish close to ± 1, a function u that satisfies $\mathcal{F}(u) < \infty$ does not necessarily belong to an affine translate of $H^2(\mathbb{R})$ as it was the case for heteroclinic solutions of (4.3). In fact it is sufficient to search a minimizer of $\mathcal{F}$ in the set

$$\mathcal{H} := \Big\{u \in C^1(\mathbb{R}),\ u'' \in L^2(\mathbb{R}),\ u' \in L^\infty(\mathbb{R}) \text{ and } \lim_{x\to\pm\infty} u(x) = \pm 1\Big\}. \tag{5.4}$$

We first observe that under assumption (C1), the functional $\mathcal{F}$ is bounded from below in $\mathcal{H}$. This justifies our minimization procedure.

LEMMA 5.1. *If f, $g \in C(\mathbb{R})$ satisfy* (C1), *then there exists a constant $s > 0$ such that for all $u \in \mathcal{H}$*

$$\mathcal{F}(u) \geqslant s \int_{-\infty}^{+\infty} \left(\frac{u''^2}{2} + f(u)\right) \mathrm{d}x.$$

PROOF. Let k be given by the assumption (C1). For $c \in\,]k, 1[$, we compute

$$\begin{aligned}\mathcal{F}(u) &\geqslant \int_{-\infty}^{+\infty} \left(\frac{1}{2}\big(u''^2 + \tilde{g}(u)u'^2\big) + f(u)\right) \mathrm{d}x \\ &\geqslant \int_{-\infty}^{+\infty} \left(\frac{1}{2}\big(1-c^2\big)u''^2 + \frac{1}{2}\left(cu'' - \frac{\tilde{G}(u)}{2c}\right)^2 + \left(f(u) - \frac{\tilde{G}(u)^2}{8c^2}\right)\right) \mathrm{d}x,\end{aligned}$$

where we have performed an integration by parts and used the fact that u' is bounded and $\tilde{G}(u(\pm\infty)) = \tilde{G}(\pm 1) = 0$ to obtain

$$-\int_{-\infty}^{+\infty} \tilde{G}(u)u''\,\mathrm{d}x = \int_{-\infty}^{+\infty} \tilde{g}(u)u'^2\,\mathrm{d}x.$$

Hence by our assumption, we obtain the inequality

$$\mathcal{F}(u) \geqslant \int_{-\infty}^{+\infty} \left(\frac{1}{2}\big(1-c^2\big)u''^2 + \left(1 - \frac{k^2}{c^2}\right) f(u)\right) \mathrm{d}x$$

so that the conclusion follows. □

We now state the main theorem of the section.

THEOREM 5.2. *Suppose that $f, g \in C^1(\mathbb{R})$ satisfy* (C1)–(C4). *Then, there exists a minimizer u of $\mathcal{F}$ in $\mathcal{H}$ which is a solution of* (5.3).

To prove this theorem, we need sharp estimates on minimizing sequences. The key arguments are summarized in the next proposition.

PROPOSITION 5.3. *Suppose that* $f, g \in C^1(\mathbb{R})$ *satisfy* (C1)–(C4). *Then there exist* $L > 0$, $T > 0$ *and a sequence* $(u_n)_{n\in\mathbb{N}} \subset \mathcal{H}$ *such that* $\mathcal{F}(u_n) \to \inf_{\mathcal{H}} \mathcal{F}$ *and for all* $n \in \mathbb{N}$,

(i) $\|u_n\|_{C^1} \leqslant L$,
(ii) $|u_n(x) + 1| \leqslant a$ *for all* $x \leqslant -T$ *and* $|u_n(x) - 1| \leqslant a$ *for all* $x \geqslant T$.

Observe that the first property of the minimizing sequence $(u_n)_n$ allows to choose a converging subsequence in C^1_{loc}. The second property then prevents from the lack of compactness of the functional space $\mathcal{H}$ due to the invariance of $\mathcal{F}$ under translations. The proof of Theorem 5.2 easily follows from Proposition 5.3.

PROOF OF THEOREM 5.2. Let $(u_n)_n \subset \mathcal{H}$ be a minimizing sequence satisfying the conclusions of Proposition 5.3.

Step 1. *Convergence*. According to Lemma 5.1, the sequence u_n'' is uniformly bounded in L^2. Together with the uniform bound on u_n in C^1 (property (i) of the minimizing sequence), this implies that $(u_n)_n$ has a subsequence (still written $(u_n)_n$ for simplicity) such that for some function u

$$u_n \xrightarrow{C^1_{\mathrm{loc}}(\mathbb{R})} u, \quad u_n'' \overset{L^2(\mathbb{R})}{\rightharpoonup} u''.$$

We now infer from Lemma 5.1 and Fatou's Lemma that

$$\int_0^{+\infty} f\big(u(x)\big)\,\mathrm{d}x < +\infty. \tag{5.5}$$

Step 2. $u \in \mathcal{H}$. Observe first that $u' \in L^\infty(\mathbb{R})$. Indeed, as u_n' converges uniformly to u' on every compact subset of $\mathbb{R}$, it follows from the uniform bound on $\|u_n'\|_\infty$ that u' is bounded.

We next prove that

$$\lim_{x\to+\infty} u(x) = 1.$$

From the uniform convergence on compact sets, it is clear that $|u(x) - 1| \leqslant a$ for $x \geqslant T$. We therefore deduce that

$$1 - a \leqslant \liminf_{x\to+\infty} u(x) \leqslant \limsup_{x\to+\infty} u(x) \leqslant 1 + a.$$

Assume by contradiction that $\limsup_{x\to+\infty} u(x) > 1$. This means that for some $0 < \varepsilon < a$, there exist infinitely many disjoint intervals $[a_i, b_i] \subset \mathbb{R}^+$, $i \in \mathbb{N}$ such that

$$1 + \frac{\varepsilon}{2} \leqslant u(x) \leqslant 1 + \varepsilon \quad \text{for all } x \in [a_i, b_i].$$

We can suppose without loss of generality that $u(b_i) - u(a_i) = \pm\varepsilon/2$. As $\|u'\|_\infty \leqslant L$, we infer that

$$\frac{\varepsilon}{2} = \left|u(b_i) - u(a_i)\right| = \left|\int_{a_i}^{b_i} u'(s)\,\mathrm{d}s\right| \leqslant L(b_i - a_i)$$

which implies that

$$(b_i - a_i) \geqslant \frac{\varepsilon}{2L}.$$

Now, let $m_\varepsilon > 0$ be such that $f(u) \geqslant m_\varepsilon$ for all $u \in [1 + \frac{\varepsilon}{2}, 1 + \varepsilon]$. We then compute

$$\int_0^{+\infty} f\big(u(x)\big)\,\mathrm{d}x \geqslant \sum_{i=0}^{\infty} \int_{a_i}^{b_i} f\big(u(x)\big)\,\mathrm{d}x \geqslant \sum_{i=0}^{\infty} \frac{\varepsilon\, m_\varepsilon}{2L} = +\infty$$

which contradicts (5.5). If $\liminf_{x\to+\infty} u(x) < 1$, we derive the same contradiction so that

$$\liminf_{x\to+\infty} u(x) = \limsup_{x\to+\infty} u(x) = 1.$$

As we can argue similarly if $\limsup_{x\to-\infty} u(x) > -1$ or $\liminf_{x\to-\infty} u(x) < -1$, we also infer that

$$\lim_{x\to-\infty} u(x) = -1.$$

Summing-up, we come to the conclusion that $u \in \mathcal{H}$.
Conclusion. To see that u is a minimizer of $\mathcal{F}$, arguing as in Lemma 5.1 we compute

$$\begin{aligned}
\mathcal{F}(u_n) &= \int_{-\infty}^{\infty} \left(\frac{1}{2}\big(u_n''^2 + \tilde{g}(u_n)u_n'^2\big) + f(u_n)\right)\mathrm{d}x \\
&\quad + \int_{-\infty}^{\infty} \frac{1}{2}\big(g(u_n) - \tilde{g}(u_n)\big)u_n'^2\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} \frac{1}{2}\left(u_n'' - \frac{\tilde{G}(u_n)}{2}\right)^2 \mathrm{d}x + \int_{-\infty}^{\infty} \left(f(u_n) - \frac{\tilde{G}(u_n)^2}{8}\right)\mathrm{d}x \\
&\quad + \int_{-\infty}^{\infty} \frac{1}{2}\big(g(u_n) - \tilde{g}(u_n)\big)u_n'^2\,\mathrm{d}x.
\end{aligned} \tag{5.6}$$

As all the terms of the right-hand side are positive, we deduce that

$$\sup_{n\in\mathbb{N}} \left\| u_n'' - \frac{\tilde{G}(u_n)}{2} \right\|_{L^2} < +\infty.$$

Going to a subsequence if necessary, we may assume that $u_n'' - \frac{\tilde{G}(u_n)}{2}$ converges weakly in L^2 and therefore

$$\int_{-\infty}^{\infty} \frac{1}{2}\left(u'' - \frac{\tilde{G}(u)}{2}\right)^2 \mathrm{d}x \leqslant \liminf_{n\to\infty} \int_{-\infty}^{\infty} \frac{1}{2}\left(u_n'' - \frac{\tilde{G}(u_n)}{2}\right)^2 \mathrm{d}x.$$

Now observe that the integrands of the two last terms of (5.6) converge for each $x \in \mathbb{R}$ and are positive. Fatou's lemma is then applicable and we obtain

$$\int_{-\infty}^{+\infty} \left(f(u) - \frac{\tilde{G}(u)^2}{8}\right) \mathrm{d}x \leqslant \liminf_{n\to\infty} \int_{-\infty}^{+\infty} \left(f(u_n) - \frac{\tilde{G}(u_n)^2}{8}\right) \mathrm{d}x$$

and

$$\int_{-\infty}^{+\infty} \frac{1}{2}\big(g(u) - \tilde{g}(u)\big)u'^2 \,\mathrm{d}x \leqslant \liminf_{n\to\infty} \int_{-\infty}^{+\infty} \frac{1}{2}\big(g(u_n) - \tilde{g}(u_n)\big)u_n'^2 \,\mathrm{d}x.$$

From Step 2, we know that $u \in \mathcal{H}$. The equality (5.6) thus also holds with u_n replaced by u and we finally deduce that

$$\mathcal{F}(u) \leqslant \liminf_{n\to\infty} \mathcal{F}(u_n) = \inf_{\mathcal{H}} \mathcal{F}$$

which by Step 2 implies

$$\mathcal{F}(u) = \inf_{\mathcal{H}} \mathcal{F}.$$

This completes the proof. □

We devote the remaining of the section to the proof of Proposition 5.3. We first need some preliminary results. Our next purpose is to complete the functional settings by proving that the first derivative of any function $u \in \mathcal{H}$ that satisfies $\mathcal{F}(u) < \infty$, vanishes at $\pm\infty$. To this aim, we next state the following useful estimate.

LEMMA 5.4. *Given an interval* $[a, b] \subset \mathbb{R}$ *and a function* $u \in H^2(a, b)$ *such that* $u(a) = A,\ u(b) = B,\ u'(a) = A_1,\ u'(b) = B_1$, *the following inequality holds*:

$$\int_a^b u''^2 \,\mathrm{d}x \geqslant \frac{4}{b-a}\left((B_1 - A_1)^2 + 3\left(\frac{B-A}{b-a} - A_1\right)\left(\frac{B-A}{b-a} - B_1\right)\right),$$

and equality holds if and only if u is a third degree polynomial.

PROOF. Denote by P the third degree polynomial that coincides in H^2 with u at points a and b. Writing $u = P + w$, we compute

$$\int_a^b u''^2 \,\mathrm{d}x = \int_a^b P''^2 \,\mathrm{d}x + \int_a^b w''^2 \,\mathrm{d}x + 2\int_a^b P''w'' \,\mathrm{d}x. \tag{5.7}$$

Integrating $P''w''$ by parts and using the fact that $w(a) = w(b) = w'(a) = w'(b) = 0$, we see that the last integral in (5.7) is actually zero. We thus obtain the inequality

$$\int_a^b u''^2 \, \mathrm{d}x \geqslant \int_a^b P''^2 \, \mathrm{d}x$$

and the conclusion now follows by computing the integral of P''^2. □

LEMMA 5.5. *Let $u \in \mathcal{H}$ be such that $\mathcal{F}(u) < \infty$. Then*

$$\lim_{|x|\to\infty} u'(x) = 0.$$

PROOF. Let $\varepsilon > 0$ and $u \in \mathcal{H}$ be given. Suppose by contradiction that our conclusion is false. Assume for example that $u'(x_n) \geqslant \varepsilon$ for some sequence $x_n \to +\infty$, $n \in \mathbb{N}$. Let $\delta > 0$ be such that

$$\delta < \frac{s\varepsilon^3}{16\mathcal{F}(u)},$$

where s is given by Lemma 5.1. Then, as $\lim_{x\to+\infty} u(x) = 1$, there exists $R > 0$ such that for all $x \geqslant R$, $|u(x) - 1| \leqslant \delta/2$. Let $x_0 > R$ be such that $u'(x_0) = \varepsilon$.

We claim that we can find $x_1 > x_0$ such that $u'(x_1) = \varepsilon/2$, $u(x_1) \leqslant u(x_0) + \delta$ and $x_1 - x_0 \leqslant 2\delta/\varepsilon$. Indeed, if $x > x_0$, we have $|u(x) - u(x_0)| \leqslant \delta$ and as $\lim_{x\to+\infty} u(x) = 1$, there exists $x > x_0$ such that $u'(x) \leqslant \varepsilon/2$. We can therefore choose x_1 such that $u'(x) \geqslant \varepsilon/2$ for all $x \in [x_0, x_1]$. We then have

$$\frac{\varepsilon}{2}(x_1 - x_0) \leqslant \int_{x_0}^{x_1} u'(s) \, \mathrm{d}s = u(x_1) - u(x_0) \leqslant \delta.$$

Now, letting $m = \frac{u(x_1) - u(x_0)}{x_1 - x_0}$, we infer from Lemma 5.4 that

$$\int_{x_0}^{x_1} u''^2 \, \mathrm{d}x \geqslant \frac{2\varepsilon}{\delta}\left(\left(\frac{\varepsilon}{2}\right)^2 + 3(m - \varepsilon)\left(m - \frac{\varepsilon}{2}\right)\right) \geqslant \frac{\varepsilon^3}{8\delta}.$$

Hence, we obtain a contradiction with the choice of δ since Lemma 5.1 implies that

$$2\mathcal{F}(u) \geqslant s \int_{x_0}^{x_1} u''^2 \, \mathrm{d}x \geqslant \frac{s\varepsilon^3}{8\delta}.$$

Similar arguments hold in the other cases, in particular if $\lim_{x\to-\infty} u'(x) \neq 0$. □

We now turn to the proof of Proposition 5.3.

PROOF. Proof of Proposition 5.3 We divide the proof in two parts. In the first part we prove the a priori bound (property (i)) and in the second one we prove the localization of at least one minimizing sequence (property (ii)).

Let $(u_n)_n \subset \mathcal{H}$ be a minimizing sequence of $\mathcal{F}$. As $\mathcal{F}(u_n)$ is a converging sequence, $\mathcal{F}(u_n)$ is uniformly bounded. Notice that we can assume without loss of generality that $\mathcal{F}(u_n) \leqslant C \leqslant \inf_{\mathcal{H}} \mathcal{F} + 1$.

Part 1. There exists $L > 0$ such that for all $n \in \mathbb{N}$, $\|u_n\|_{C^1} \leqslant L$.

Let s be given by Lemma 5.1. According to the assumptions on f there exist $K > 0$ and $b > 0$ such that

$$\frac{3K^2 s}{8b^3} > C \quad \text{and} \quad s\, f(u) > \frac{C}{b} \quad \text{for all } |u| \geqslant \frac{K}{2}.$$

We claim that $\|u_n\|_\infty \leqslant K$. Otherwise, either the set $\{x\colon\ |u_n(x)| \geqslant K/2\}$ has measure greater than b and

$$\mathcal{F}(u_n) \geqslant s \int_{-\infty}^{+\infty} f(u_n)\,\mathrm{d}x > s\, b \frac{C}{b\, s} = C,$$

a contradiction, or we can pick up an interval (c, d) such that $d - c < b$, $|u_n(c)| = |u_n|_\infty$, $|u_n(d)| = \frac{K}{2}$, $|u_n(x)| \geqslant \frac{K}{2}$, for all $x \in (c, d)$. It then follows using Lemma 5.4 that

$$\begin{aligned}
\mathcal{F}(u_n) &\geqslant s \int_c^d \frac{u_n''^2}{2}\,\mathrm{d}x \\
&\geqslant \frac{2s}{d-c}\left(u_n'(d)^2 + 3\frac{u_n(d) - u_n(c)}{d-c}\left(\frac{u_n(d) - u_n(c)}{d-c} - u_n'(d)\right)\right) \\
&\geqslant \frac{3s}{2(d-c)}\left(\frac{u_n(d) - u_n(c)}{d-c}\right)^2 \\
&\geqslant \frac{3K^2 s}{8b^3} > C,
\end{aligned}$$

leading again to a contradiction. Hence the bound on $\|u_n\|_\infty$ is established.

Next, choosing M such that

$$M > 4K \quad \text{and} \quad M^2 > \frac{8C}{s},$$

we show that u_n' cannot attain the value M. Assume by contradiction that $u_n'(x_0) = M$. Then there exists $x_1 \in (x_0, x_0 + 1)$ such that $u_n'(x_1) = \frac{M}{2}$. Hence, denoting again $m = \frac{u_n(x_1) - u_n(x_0)}{x_1 - x_0}$, it turns out that

$$\mathcal{F}(u_n) \geqslant s \int_{x_0}^{x_1} \frac{u_n''^2}{2}\,\mathrm{d}x$$

$$\geqslant \frac{2s}{x_1 - x_0}\left(\left(\frac{M}{2}\right)^2 + 3(m-M)\left(m - \frac{M}{2}\right)\right)$$
$$\geqslant \frac{M^2 s}{8} > C,$$

which is impossible. We show in a similar way that u_n' cannot attain the value $-M$. We thus deduce that the conclusion of Part 1 holds with $L = \max(K, M)$.

Part 2. There exist $T > 0$ and a minimizing sequence $(v_n)_n \subset \mathcal{H}$ such that for all $n \in \mathbb{N}$,

$$\left|v_n(x) + 1\right| \leqslant a \quad \text{for all } x \leqslant -T$$

and

$$\left|v_n(x) - 1\right| \leqslant a \quad \text{for all } x \geqslant T.$$

Let $\varepsilon > 0$ be given. For each $n \in \mathbb{N}$, we define

$$x_1 := \sup\left\{x \in \mathbb{R} \text{ such that } \left|u_n(x) + 1\right| \leqslant \varepsilon \text{ and } \left|u_n'(x)\right| \leqslant \varepsilon\right\},$$

and

$$x_2 := \inf\left\{x \in \mathbb{R} \text{ such that } \left|u_n(x) - 1\right| \leqslant \varepsilon \text{ and } \left|u_n'(x)\right| \leqslant \varepsilon\right\}.$$

Observe that as $u_n \in \mathcal{H}$ and $\mathcal{F}(u_n) < \infty$, x_1 and x_2 are real numbers. Basically, this part of the proof consists in showing that the length $x_2 - x_1$ can be controlled uniformly in $n \in \mathbb{N}$. Then choosing $\varepsilon > 0$ sufficiently small, pieces of orbits close to ± 1 on $]-\infty, x_1[$ and $]x_2, +\infty[$ can be glued to $u_{n|[x_1,x_2]}$ without increasing the action above $\mathcal{F}(u_n)$. This rather simple argument all the same requires some technical adjustments. We first focus on the estimate of the length $x_2 - x_1$.

Step 1. For each $0 < \varepsilon < 1$, there exists $T_\varepsilon > 0$ such that for all $n \in \mathbb{N}$, $x_2 - x_1 \leqslant 2T_\varepsilon$.

Let us define $N_\varepsilon^\pm :=]\pm 1 - \varepsilon, \pm 1 + \varepsilon[$, $N_\varepsilon := N_\varepsilon^- \cup N_\varepsilon^+$ and consider the set

$$Z := \left\{x \in [x_1, x_2] \,\middle|\, u_n(x) \in N_\varepsilon\right\}.$$

Observe that Z is a union of intervals I_i on which $|u_n'| \geqslant \varepsilon$. In the sequel, we assume that these intervals I_i are of maximal length. As $|u_n'(x)| \geqslant \varepsilon$ on any interval I_i, we infer that

$$|I_i|\varepsilon \leqslant \left|\int_{I_i} u_n'(x)\,\mathrm{d}x\right| \leqslant 2\varepsilon$$

so that $|I_i| \leqslant 2$. Further except maybe for the last one, each interval I_i is followed by an interval $J_i = [c_i, d_i]$ that we also suppose to be of maximal length and which is so that one of the following conditions holds for all $x \in [c_i, d_i]$:

$$\begin{array}{llll} \text{(a)} & u_n(x) \geqslant 1+\varepsilon, & u_n'(c_i) \geqslant \varepsilon, & u_n'(d_i) \leqslant -\varepsilon, \\ \text{(b)} & -1+\varepsilon \leqslant u_n(x) \leqslant 1-\varepsilon, & |u_n'(c_i)| \geqslant \varepsilon, & |u_n'(d_i)| \geqslant \varepsilon, \\ \text{(c)} & u_n(x) \leqslant -1-\varepsilon, & u_n'(c_i) \leqslant -\varepsilon, & u_n'(d_i) \geqslant \varepsilon. \end{array}$$

Consider an interval $J_i = [c_i, d_i]$ such that (a) or (c) hold. We then obtain the estimate

$$2\varepsilon \leqslant |u_n'(d_i) - u_n'(c_i)| = \left| \int_{c_i}^{d_i} u_n''(x)\,\mathrm{d}x \right| \leqslant \|u_n''\|_{L^2(c_i,d_i)} (d_i - c_i)^{1/2}$$

and we thus infer that

$$\int_{c_i}^{d_i} \left(\frac{1}{2} (u_n'')^2 + f(u_n) \right) \mathrm{d}x \geqslant \frac{2\varepsilon^2}{d_i - c_i} + r_\varepsilon (d_i - c_i) \geqslant 2\varepsilon \sqrt{2 r_\varepsilon}, \tag{5.8}$$

where $r_\varepsilon := \inf\{ f(u) \mid u \notin N_\varepsilon \}$. Notice that for an interval J_i of type (b) with $u_n'(c_i)u_n'(d_i) < 0$, the same inequality holds. At last, assume that J_i is of type (b) and such that $u_n'(c_i)u_n'(d_i) > 0$. Then we obtain

$$2 - 2\varepsilon = \left| \int_{c_i}^{d_i} u_n'(x)\,\mathrm{d}x \right| \leqslant L(d_i - c_i)$$

and

$$\int_{c_i}^{d_i} \left(\frac{1}{2} (u_n'')^2 + f(u_n) \right) \mathrm{d}x \geqslant r_\varepsilon (d_i - c_i) \geqslant \frac{r_\varepsilon (2 - 2\varepsilon)}{L}, \tag{5.9}$$

where L is the uniform bound on $\|u_n'\|_\infty$ obtained in Part 1. Let us denote by k the number of intervals J_i. Taking the estimates (5.8) and (5.9) into account, we infer from Lemma 5.1 that

$$\begin{aligned} C \geqslant \mathcal{F}(u_n) &\geqslant s \sum_{i=1}^{k} \int_{c_i}^{d_i} \left(\frac{1}{2} (u_n'')^2 + f(u_n) \right) \mathrm{d}x \\ &\geqslant sk \min\left(2\varepsilon \sqrt{2 r_\varepsilon}, \frac{r_\varepsilon (2 - 2\varepsilon)}{L} \right) \end{aligned} \tag{5.10}$$

so that k is uniformly bounded with respect to $n \in \mathbb{N}$.

We are now able to conclude our first step. Indeed, setting $\tilde{Z} = [x_1, x_2] \setminus Z$, we have

$$C \geqslant \mathcal{F}(u_n) \geqslant s \int_{\tilde{Z}} f(u_n(x))\,\mathrm{d}x \geqslant s r_\varepsilon |\tilde{Z}|$$

and since Z is the union of at most $k+1$ intervals of length smaller than 2, we finally deduce that

$$x_2 - x_1 = |Z| + \left|\tilde{Z}\right| \leqslant 2(k+1) + \frac{C}{s\, r_\varepsilon} =: 2T_\varepsilon.$$

Step 2. Modification of u_n in $]-\infty, x_1]$ and $[x_2, +\infty[$.

We consider the modification in $]-\infty, x_1]$. Define

$$\mathcal{F}_{]-\infty,x_1]}(u) := \int_{-\infty}^{x_1} \left(\frac{1}{2}\left(u''^2 + g(u)u'^2\right) + f(u) \right) \mathrm{d}x$$

having as domain

$$\mathcal{H}^n_{]-\infty,x_1]} := \left\{ u \in \mathcal{H} \mid u = u_n \text{ on } [x_1, +\infty[\right\}.$$

Using integration by parts and arguing as in Lemma 5.1, we see that for any function $u \in \mathcal{H}^n_{]-\infty,x_1]}$

$$\mathcal{F}_{]-\infty,x_1]}(u) \geqslant s \int_{-\infty}^{x_1} \left(\frac{u''^2}{2} + f(u) \right) \mathrm{d}x + \frac{1}{2}\tilde{G}\big(u(x_1)\big)u'(x_1). \tag{5.11}$$

We also consider the set

$$\mathcal{D}^n_{]-\infty,x_1]} := \left\{ u \in \mathcal{H}^n_{]-\infty,x_1]} \;\middle|\; \text{for all } x \leqslant x_1,\ u(x) \in [-1-a, -1+a] \right\}.$$

Let us fix again the notation $N_a^- :=]-1-a, -1+a[$. We then define

$$\gamma := \sup\left\{ g(u) \mid u \in N_a^- \right\},$$
$$\eta := \inf\left\{ f(u) \mid u \in N_a^- \setminus N_{a/2}^- \right\} > 0,$$
$$\delta := \sup\left\{ \left|\tilde{G}(u)\right| \mid u \in N_a^- \right\} \geqslant 0.$$

We first derive a lower estimate on the action of any function $u \in \mathcal{H}^n_{]-\infty,x_1]}$ whose graph does not stay in the strip $]-\infty, x_1] \times N_a^-$.

CLAIM 1. *If* $u \in \mathcal{H}^n_{]-\infty,x_1]} \setminus \mathcal{D}^n_{]-\infty,x_1]}$ *and* $\|u'\|_\infty \leqslant L$ *then*

$$\mathcal{F}_{]-\infty,x_1]}(u) \geqslant \frac{s\eta a}{2L} - \delta\varepsilon. \tag{5.12}$$

If there exists $x \leqslant x_1$ such that $u(x) \notin N_a^-$, either there exist $s_1 \leqslant s_2 \leqslant x_1$ so that

$$u(s_1) = -1 + \frac{a}{2}, \quad u(s_2) = -1 + a$$

and for all $x \in [s_1, s_2]$,

$$u(x) \in \left[-1 + \frac{a}{2}, -1 + a\right]$$

or there exist $s_3 \leqslant s_4 \leqslant x_1$ so that

$$u(s_3) = -1 - a, \qquad u(s_4) = -1 - \frac{a}{2}$$

and for all $x \in [s_3, s_4]$,

$$u(x) \in \left[-1 - a, -1 - \frac{a}{2}\right].$$

Let us for instance consider the first possibility, the second being similar. We then have

$$L(s_2 - s_1) \geqslant \int_{s_1}^{s_2} u'(x)\,\mathrm{d}x = u(s_2) - u(s_1) = \frac{a}{2}$$

and

$$\int_{-\infty}^{x_1} \left(\frac{u''^2}{2} + f(u)\right) \mathrm{d}x \geqslant \int_{s_1}^{s_2} f(u)\,\mathrm{d}x \geqslant \frac{\eta a}{2L}.$$

On the other hand, we have

$$\left|\tilde{G}\big(u(x_1)\big)u'(x_1)\right| \leqslant \delta\varepsilon,$$

so that Claim 1 now follows from (5.11).

CLAIM 2. *There exists $R > 0$ so that for all $\varepsilon > 0$,*

$$\inf_{\mathcal{D}^n_{]-\infty,x_1]}} \mathcal{F}_{]-\infty,x_1]} \leqslant R\varepsilon^2.$$

For a function $u \in \mathcal{D}^n_{]-\infty,x_1]}$, we have by virtue of (C2)

$$\mathcal{F}_{]-\infty,x_1]}(u) \leqslant \int_{-\infty}^{x_1} \left(\frac{1}{2}\big(u''^2 + \gamma u'^2\big) + \alpha(u+1)^2\right) \mathrm{d}x.$$

Let us define the functional

$$\mathcal{J}_{]-\infty,x_1]}(u) := \int_{-\infty}^{x_1} \left(\frac{1}{2}\big(u''^2 + \gamma u'^2\big) + \alpha(u+1)^2\right) \mathrm{d}x$$

on $\mathcal{H}^n_{]-\infty,x_1]}$. Let P be the third degree polynomial satisfying $P(x_1-1)=-1$, $P'(x_1-1)=0$, $P(x_1)=u_n(x_1)$ and $P'(x_1)=u_n'(x_1)$. The function $v:]-\infty,x_1]\to\mathbb{R}$ defined by

$$v(x) := \begin{cases} 0 & \text{if } x < x_1 - 1 \\ P(x) & \text{if } x_1 - 1 \leqslant x \leqslant x_1. \end{cases}$$

belongs to $\mathcal{H}^n_{]-\infty,x_1]}$ and in fact to $\mathcal{D}^n_{]-\infty,x_1]}$ if ε is taken sufficiently small. A straightforward computation then shows that there is a constant $C(\gamma,\alpha)$ such that

$$\mathcal{J}_{]-\infty,x_1]}(v) \leqslant C(\gamma,\alpha)\big(\big(u_n(x_1)+1\big)^2 + u_n'(x_1)^2\big) \leqslant 2C(\gamma,\alpha)\varepsilon^2$$

so that we obtain the estimate

$$\inf_{\mathcal{D}^n_{]-\infty,x_1]}} \mathcal{F}_{]-\infty,x_1]} \leqslant \inf_{\mathcal{D}^n_{]-\infty,x_1]}} \mathcal{J}_{]-\infty,x_1]} \leqslant 2C(\gamma,\alpha)\varepsilon^2.$$

Conclusion of Step 2. Let us choose $\varepsilon > 0$ sufficiently small in order to have

$$\mathcal{F}_{]-\infty,x_1]}(u) \geqslant \frac{s\eta a}{4L}$$

for $u \in \mathcal{H}^n_{]-\infty,x_1]} \setminus \mathcal{D}^n_{]-\infty,x_1]}$ satisfying $\|u'\|_\infty \leqslant L$ and

$$\inf_{\mathcal{D}^n_{]-\infty,x_1]}} \mathcal{F}_{]-\infty,x_1]} < \frac{s\eta a}{4L}.$$

If $u_n \notin \mathcal{D}^n_{]-\infty,x_1]}$, we infer from the above estimates that we can replace u_n by $v_n \in \mathcal{D}^n_{]-\infty,x_1]}$ such that $\mathcal{F}(v_n) \leqslant \mathcal{F}(u_n)$.

If $|u_n(x)-1| \nleqslant a$ for $x \geqslant x_2$ we proceed in the same way to modify u_n for $x \geqslant x_2$.

We are now in position to complete the proof of Part 2. Indeed, we deduce from Step 1 and Step 2 that there exist $T > 0$ and a minimizing sequence $(v_n)_n$ such that for all $n \in \mathbb{N}$, there exist $x_1 < x_2$ satisfying $x_2 - x_1 \leqslant 2T$ and

$$\big|v_n(x)+1\big| \leqslant a \quad \text{for all } x \leqslant x_1,$$

$$\big|v_n(x)-1\big| \leqslant a \quad \text{for all } x \geqslant x_2.$$

Translating v_n if necessary, we can assume $[x_1,x_2] \subset [-T,T]$ so that the conclusion of the second part follows. Observe that since $\mathcal{F}(v_n) \leqslant \mathcal{F}(u_n)$, the sequence $(v_n)_n$ satisfies the a priori bound derived in the first part. This ends the proof. □

5.2. *Functionals of Swift–Hohenberg type*

When β is negative, the model equation

$$u'''' - \beta u'' + u^3 - u = 0$$

is related to the *Swift–Hohenberg* equation

$$\frac{\partial u}{\partial t} - \kappa u + \left(1 + \frac{\partial^2}{\partial x^2}\right)^2 u + u^3 = 0, \tag{5.13}$$

where $\kappa \in \mathbb{R}$. This equation was proposed by Swift and Hohenberg [84] as a model for studies of Rayleigh–Bénard convection. When $\kappa > 1$, after rescaling, (5.13) can be written as

$$\frac{\partial u}{\partial t} + (\kappa - 1)^{3/2}\left(\frac{\partial^4 u}{\partial x^4} - \beta \frac{\partial^2 u}{\partial x^2} + u^3 - u\right) = 0$$

with $\beta = -2/\sqrt{\kappa - 1}$. The compatibility condition (C1) between f and g allows g to take negative values but it then implies that g takes positive values as well. Therefore, the stationary Swift–Hohenberg equation is not covered by the result of the preceding section.

When stronger assumptions are imposed on the potential, we can deal with negative functions g. Namely we assume that $f, g \in C^1(\mathbb{R})$ are even functions such that $f(1) = 0$ and

(C5) for some $k > 0$, $\beta < \sqrt{8k}$ and all $u \geqslant 0$,

$$f(u) \geqslant k(u-1)^2 \quad \text{and} \quad g(u) \geqslant -\beta.$$

We look then at the minimizers of the functional $\mathcal{F}$ defined by (5.2) in the space

$$\mathcal{E} = \left\{u \mid u(0) = 0,\ u + 1 \in H^2(\mathbb{R}^-),\ u - 1 \in H^2(\mathbb{R}^+)\right\}.$$

Our first observation is that under the assumption (C5), for small $\beta > 0$,

$$\inf_{u \in \mathcal{E}} \mathcal{F}(u) > -\infty.$$

LEMMA 5.6. *Let f and $g \in C(\mathbb{R})$ be even functions such that* (C5) *holds. Then there exists $\beta_1 > 0$ such that for $\beta \leqslant \beta_1$, we have*

$$\inf_{u \in \mathcal{E}} \mathcal{F}(u) \geqslant 0.$$

PROOF. Let us first recall the following interpolation inequality, see for example [1]. For all $\ell > 0$, there exists $C > 0$ such that for any interval I with $|I| \geqslant \ell$ and all $u \in H^2(I)$,

$$\|u'\|^2_{L^2(I)} \leqslant C\left(\|u\|^2_{L^2(I)} + \|u''\|^2_{L^2(I)}\right). \tag{5.14}$$

Let u be any function in $\mathcal{E}$ and consider an interval $[a, b]$ where u is nonnegative and $u(a) = u(b) = 0$. Suppose that $b - a \geqslant 1$. Then using the inequality (5.14) applied to $u - 1$, we obtain

$$\int_a^b \left(\frac{1}{2}\left(u''^2 + g(u)u'^2\right) + f(u)\right) \mathrm{d}x \geqslant \int_a^b \left(\frac{1}{2}\left(u''^2 - \beta u'^2\right) + k(u-1)^2\right) \mathrm{d}x$$

$$\geqslant \frac{1-\beta C}{2} \left\|u''\right\|^2_{L^2(a,b)} + \left(k - \frac{\beta C}{2}\right) \|u-1\|^2_{L^2(a,b)}. \tag{5.15}$$

If $b - a < 1$, we compute

$$\left\|u'\right\|_{L^2(a,b)} \leqslant \frac{b-a}{\sqrt{2}} \left\|u''\right\|_{L^2(a,b)}$$

and

$$\int_a^b \left(\frac{1}{2}\left(u''^2 - g(u)u'^2\right) + f(u)\right) \mathrm{d}x \geqslant \int_a^b \left(\frac{1}{2}\left(u''^2 - \beta u'^2\right) + f(u)\right) \mathrm{d}x$$

$$\geqslant \frac{1}{2}\left(1 - \frac{\beta}{2}\right) \left\|u''\right\|^2_{L^2(a,b)}.$$

In both cases, we obtain

$$\int_a^b \left(\frac{1}{2}\left(u''^2 + g(u)u'^2\right) + f(u)\right) \mathrm{d}x \geqslant 0$$

if $\beta \leqslant \min(2, 1/C, 2k/C)$. A similar argument holds on intervals $[a, b]$ where u is non positive. At last, if u is nonnegative on the interval $[T, +\infty[$, we deduce as in (5.15) that

$$\int_a^{+\infty} \left(\frac{1}{2}\left(u''^2 + g(u)u'^2\right) + f(u)\right) \mathrm{d}x \geqslant 0.$$

The conclusion follows. □

Next, assuming that the functional $\mathcal{F}$ is bounded from below, we prove the existence of a minimizer. We deduce from the previous lemma that the result holds at least when $g \geqslant -\beta$ with $\beta > 0$ small.

THEOREM 5.7. *Let $f \in C^1(\mathbb{R})$ and $g \in C^1(\mathbb{R})$ be even functions such that $f(1) = 0$ and* (C5) *holds. Assume further that*

$$\inf_{\mathcal{E}} \mathcal{F} > -\infty.$$

Then $\mathcal{F}$ has a minimizer which is a heteroclinic solution of Eq. (5.3) *having exactly one zero. Moreover, any minimizer is odd.*

PROOF. *Step* 1. We first prove the existence of an odd minimizing sequence $(u_n)_n$ such that $u_n(x) > 0$ for all $x > 0$. Let $(v_n)_n \subset \mathcal{E}$ be a minimizing sequence. Observe that if v_n is not odd, the appropriate symmetrization of v_n restricted to $\mathbb{R}^+$ or $\mathbb{R}^-$ decreases the action. Assume now that v_n has a positive zero and define

$$x_n := \sup\{x > 0 \mid v_n(x) = 0\}.$$

We claim that

$$\mathcal{F}_{[-x_n,x_n]}(v_n) := \int_{-x_n}^{x_n} \left(\frac{1}{2}\left(v_n''^2 + g(v_n)v_n'^2\right) + f(v_n)\right) \mathrm{d}x \geqslant 0.$$

Indeed, suppose that $\mathcal{F}_{[-x_n,x_n]}(v_n) = -C < 0$ and take $j \in \mathbb{N}$ so that $\mathcal{F}(v_n) - 2jC < \inf_{\mathcal{E}} \mathcal{F}$. Then defining the odd function $u_n^* \in \mathcal{E}$ by

$$u_n^*(x) := \begin{cases} v_n(x - 2ix_n) & \text{if } x \in \left[2ix_n, (2i+1)x_n\right[, i = 0, \dots, j, \\ -v_n(2ix_n - x) & \text{if } x \in \left[(2i-1)x_n, 2ix_n\right[,\ i = 1, \dots, j, \\ v_n\left(x - 2(j+1)x_n\right) & \text{if } x \in \left[(2j+1)x_n, +\infty\right[, \end{cases}$$

we obtain a contradiction because

$$\mathcal{F}\left(u_n^*\right) = \mathcal{F}(v_n) + 2j\mathcal{F}_{[-x_n,x_n]}(v_n) = \mathcal{F}(v_n) - 2jC < \inf_{\mathcal{E}} \mathcal{F}.$$

Now, let $u_n \in \mathcal{E}$ be the odd function defined by $u_n(x) = v_n(x + x_n)$ for $x \geqslant 0$. This function u_n vanishes only at 0 and since $\mathcal{F}_{[-x_n,x_n]}(v_n) \geqslant 0$, the sequence $(u_n)_n$ is also a minimizing sequence.

Step 2. As there exists an odd minimizing sequence, it is sufficient to minimize the restricted functional

$$\mathcal{F}^+(u) := \int_0^{+\infty} \left(\frac{1}{2}\left(u''^2 + g(u)u'^2\right) + f(u)\right) \mathrm{d}x$$

in the space of functions

$$\mathcal{E}^+ = \left\{u : \mathbb{R}^+ \to \mathbb{R} \mid u(0) = 0,\ u - 1 \in H^2\left(\mathbb{R}^+\right)\right\}.$$

We first claim that $u_n - 1$ is uniformly bounded in $H^2(\mathbb{R}^+)$. To prove this, we compute for $u \in \mathcal{E}^+$,

$$\begin{aligned} &\int_0^{+\infty} \left(\frac{1}{2}\left(u''^2 - \beta u'^2\right) + k(u-1)^2\right) \mathrm{d}x \\ &\quad = \varepsilon \|u - 1\|_{H^2}^2 + \frac{1 - 2\varepsilon}{2} \int_0^{+\infty} \left(u''^2 - \beta_\varepsilon u'^2 + \frac{1}{4}\beta_\varepsilon^2 (u-1)^2\right) \mathrm{d}x \\ &\qquad + \left(k - \varepsilon - \frac{(\beta + 2\varepsilon)^2}{8(1 - 2\varepsilon)}\right) \int_0^{+\infty} (u-1)^2 \, \mathrm{d}x, \end{aligned}$$

where $\beta_\varepsilon := \frac{\beta+2\varepsilon}{1-2\varepsilon}$. Notice that we may choose ε small enough in order to have $k-\varepsilon-\frac{(\beta+2\varepsilon)^2}{8(1-2\varepsilon)} \geqslant 0$. For the second term of the right-hand side, we have

$$\int_0^{+\infty}\left(u''^2-\beta_\varepsilon u'^2+\frac{\beta_\varepsilon^2}{4}(u-1)^2\right)\mathrm{d}x$$
$$=\int_0^{+\infty}\left(u''+\frac{\beta_\varepsilon}{2}(u-1)\right)^2\mathrm{d}x-\beta_\varepsilon u'(0)\geqslant-\beta_\varepsilon u'(0).$$

As u_n is positive, we infer that

$$\mathcal{F}^+(u_n)\geqslant\int_0^{+\infty}\left(\frac{1}{2}\big(u_n''^2-\beta u_n'^2\big)+k(u_n-1)^2\right)\mathrm{d}x$$
$$\geqslant\varepsilon\|u_n-1\|_{H^2}^2-\beta_\varepsilon u_n'(0).$$

We now deduce from the continuous injection of $H^2(0,\infty)$ into $\mathcal{C}^1([0,\infty[)$ that for some $C>0$

$$\mathcal{F}^+(u_n)\geqslant\varepsilon\|u_n-1\|_{H^2(0,\infty)}^2-C\|u_n-1\|_{H^2(0,\infty)}.$$

Hence, the claim follows from the uniform bound on $\mathcal{F}(u_n)$.

We now infer that at least for a subsequence (that we still denote by u_n)

$$u_n-1\overset{H^2(\mathbb{R}^+)}{\rightharpoonup}u-1\quad\text{and}\quad u_n\overset{C^1_{\mathrm{loc}}(\mathbb{R}^+)}{\longrightarrow}u$$

for some function $u\in\mathcal{E}^+$. In order to see that u is a minimizer, we write

$$\mathcal{F}^+(u_n)=\frac{1}{2}\int_0^{+\infty}\left(u_n''^2-\beta u_n'^2+\frac{\beta^2}{4}(u_n-1)^2\right)\mathrm{d}x$$
$$+\frac{1}{2}\int_0^{+\infty}\big(\big(g(u_n)+\beta\big)u_n'^2\big)\,\mathrm{d}x+\int_0^{+\infty}\left(f(u_n)-\frac{\beta^2}{8}(u_n-1)^2\right)\mathrm{d}x$$
$$=\frac{1}{2}\int_0^{+\infty}\left(u_n''+\frac{\beta}{2}(u_n-1)\right)^2\mathrm{d}x-\frac{\beta}{2}u_n'(0)$$
$$+\frac{1}{2}\int_0^{+\infty}\big(\big(g(u_n)+\beta\big)u_n'^2\big)\,\mathrm{d}x+\int_0^{+\infty}\left(f(u_n)-\frac{\beta^2}{8}(u_n-1)^2\right)\mathrm{d}x.$$

Observe then that in the last equality, the first integral is convex and that Fatou's Lemma is applicable to the last two so that

$$\mathcal{F}^+(u)\leqslant\varliminf_{n\to+\infty}\mathcal{F}^+(u_n)=\inf_{\mathcal{E}^+}\mathcal{F}^+.$$

Now, as $u \in \mathcal{E}^+$, we conclude that $\mathcal{F}^+(u) = \inf_{\mathcal{E}^+} \mathcal{F}^+$ and still denoting by u its odd extension on $\mathbb{R}$, we have $\mathcal{F}(u) = \inf_{\mathcal{E}} \mathcal{F}$.

The proof that any minimizer is odd follows the lines of the third step of the proof of Theorem 4.5. □

In the following theorem, we observe that when ± 1 are saddle-foci, the minimum of $\mathcal{F}$ is non-negative.

THEOREM 5.8. *Let f and $g \in C^2(\mathbb{R})$ be even functions such that $f(1) = 0$, $g(1)^2 < 4f''(1)$ and assume that* (C5) *holds. Then if u is a minimizer in $\mathcal{E}$ of $\mathcal{F}$, $\mathcal{F}(u) \geqslant 0$.*

In order to prove Theorem 5.8, we first consider the following lemma which states that if $+1$ is a saddle-focus stationary point, then the solutions of (5.3) that converge to $+1$ in the phase-space, do oscillate around the equilibrium in their tails. To simplify the notation, we assume the equilibria has been translated to 0.

LEMMA 5.9. *Let f and $g \in C^2(\mathbb{R})$ and $f(0) = 0$. Let 0 be a saddle-focus equilibria for the linearization of Eq.* (5.3), *i.e. $g(0)^2 < 4f''(0)$. Then, there exists $\Delta > 0$ such that if $\hat{u}$ is a nontrivial solution of* (5.3) *that satisfies*

$$\lim_{x \to +\infty} \big(\hat{u}(x), \hat{u}'(x), \hat{u}''(x), \hat{u}'''(x)\big) = (0, 0, 0, 0), \tag{5.16}$$

$\hat{u}$ changes sign in any interval $[x_0, x_0 + \Delta]$ for sufficiently large x_0.

PROOF. Let $\hat{u}$ be a solution of (5.3) that satisfies (5.16). Consider the linearization of (5.3)

$$z'''' - g(0)z'' + f''(0)z = 0. \tag{5.17}$$

Notice that the characteristic values of this linear equation read $\pm\rho \pm i\omega$. We then choose $\Delta = 2\pi/\omega$. Let $x_0 \in \mathbb{R}$ and $z_0 \in \mathbb{R}^4$. By the choice of Δ, the solution z of (5.17) with initial condition

$$\big(z(x_0), z'(x_0), z''(x_0), z'''(x_0)\big) = z_0$$

satisfies

$$\max_{[x_0, x_0+\Delta]} z, \quad \max_{[x_0, x_0+\Delta]} (-z) \geqslant c|z_0|$$

and

$$\|z\|_{\mathcal{C}^2([x_0, x_0+\Delta])} \leqslant M|z_0|$$

for some $c > 0$ and $M > 0$ depending only on Δ, $g(0)$ and $f''(0)$. On the other hand, we can also find $N > 0$ so that the solutions of

$$w'''' - g(0)w'' + f''(0)w = h(x),$$
$$\big(w(x_0), w'(x_0), w''(x_0), w'''(x_0)\big) = 0,$$

satisfy

$$\|w\|_{\mathcal{C}^2([x_0,x_0+\Delta])} \leqslant N\|h\|_{L^\infty(x_0,x_0+\Delta)}.$$

Next, we take $\delta > 0$ so that $c - \frac{MN\delta}{1-N\delta} > 0$.

To fix the ideas we denote by $u(x; x_0, u_0)$ the solution of (5.3) with initial conditions $x_0 \in \mathbb{R}$, $u_0 := (u(x_0), u'(x_0), u''(x_0), u'''(x_0))$. Let $\tilde{u}(x;\lambda) = u(x; x_0, \lambda\hat{u}_0)$, with $\hat{u}_0 = (\hat{u}(x_0), \hat{u}'(x_0), \hat{u}''(x_0), \hat{u}'''(x_0))$ and define

$$p(x) = -g\big(\tilde{u}(x;\lambda)\big),$$
$$q(x) = -g'\big(\tilde{u}(x;\lambda)\big)\tilde{u}'(x;\lambda),$$

and

$$r(x) = -g'\big(\tilde{u}(x;\lambda)\big)\tilde{u}''(x;\lambda) - \frac{1}{2}g''\big(\tilde{u}(x;\lambda)\big)\tilde{u}'^2(x;\lambda) + f''\big(\tilde{u}(x;\lambda)\big).$$

Let us fix now x_0 large enough so that $|\hat{u}_0|$ is small enough to have, for $0 \leqslant \lambda \leqslant 1$,

$$\sup_{x_0\leqslant x\leqslant x_0+\Delta}\big|p(x) + g(0)\big|, \sup_{x_0\leqslant x\leqslant x_0+\Delta}\big|q(x)\big|, \sup_{x_0\leqslant x\leqslant x_0+\Delta}\big|r(x) - f''(0)\big| \leqslant \delta.$$

Observe also that $|\hat{u}_0| \neq 0$ as a nonzero solution cannot reach 0 in the phase-space in a finite time. We then write

$$\hat{u}(x) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\lambda} u(x; x_0, \lambda\hat{u}_0)\,\mathrm{d}\lambda.$$

The function

$$\varphi(x; x_0, \hat{u}_0, \lambda) = \frac{\mathrm{d}}{\mathrm{d}\lambda} u(x; x_0, \lambda\hat{u}_0)$$

satisfies the Cauchy problem

$$\varphi'''' + p(x)\varphi'' + q(x)\varphi' + r(x)\varphi = 0,$$
$$\big(\varphi(x_0), \varphi'(x_0), \varphi''(x_0), \varphi'''(x_0)\big) = \hat{u}_0,$$

and we can write

$$\varphi = w + z.$$

Here w solves the equation

$$w'''' - g(0)w'' + f''(0)w \\ = -\big(g(0) + p(x)\big)\varphi''(x) - q(x)\varphi'(x) + \big(f''(0) - r(x)\big)\varphi(x),$$

together with the initial values

$$\big(w(x_0), w'(x_0), w''(x_0), w'''(x_0)\big) = 0$$

and z is a solution of

$$z'''' - g(0)z'' + f''(0)z = 0, \\ \big(z(x_0), z'(x_0), z''(x_0), z'''(x_0)\big) = \hat{u}_0.$$

We next choose $\overline{x} \in [x_0, x_0 + \Delta]$ so that

$$z(\overline{x}) \geqslant c|\hat{u}_0|$$

and compute

$$\hat{u} = z + \int_0^1 w \, \mathrm{d}\lambda.$$

Notice that

$$\|w\|_{\mathcal{C}^2([x_0,x_0+\Delta])} \leqslant N\delta \|\varphi\|_{\mathcal{C}^2([x_0,x_0+\Delta])}.$$

We then obtain the estimates,

$$\|w\|_{\mathcal{C}^2([x_0,x_0+\Delta])} \leqslant \frac{N\delta}{1 - N\delta} \|z\|_{\mathcal{C}^2([x_0,x_0+\Delta])} \leqslant \frac{MN\delta}{1 - N\delta} |\hat{u}_0|.$$

At last, we have

$$\hat{u}(\overline{x}) \geqslant c|\hat{u}_0| - \|w\|_\infty \geqslant \left(c - \frac{MN\delta}{1 - N\delta}\right)|\hat{u}_0| > 0.$$

Arguing in a similar way, we find $\underline{x} \in [x_0, x_0 + \Delta]$ so that $\hat{u}(\underline{x}) < 0$. □

We now come back to the proof of Theorem 5.8.

PROOF OF THEOREM 5.8. Let $u \in \mathcal{E}$ be such that $\mathcal{F}(u) = \inf_{\mathcal{E}} \mathcal{F} = -C$ for some $C > 0$. As u is a minimizer, it satisfies Eq. (5.3) and we also have

$$\lim_{x \to +\infty} \big(u(x), u'(x), u''(x), u'''(x)\big) = (1, 0, 0, 0).$$

Observe that Theorem 5.7 implies that u is odd so that

$$\mathcal{F}(u) = 2\int_0^{+\infty}\left(\frac{1}{2}\left(u''^2 + g(u)u'^2\right) + f(u)\right)\mathrm{d}x = -C.$$

As a consequence of Lemma 5.9, u oscillates around $+1$ at $+\infty$. Hence, we can find $T > 0$ large enough so that

$$\int_0^{T}\left(\frac{1}{2}\left(u''^2 + g(u)u'^2\right) + f(u)\right)\mathrm{d}x \leqslant -C/4,$$

and $u'(T) = 0$. Defining the odd function $v^* \in \mathcal{E}$ as follows:

$$v^*(x) = \begin{cases} u(x) & \text{if } x \in [0, T), \\ u(2T - x) & \text{if } x \in [T, 2T), \\ -u(x - 2T) & \text{if } x \in [2T, 3T], \\ -u(4T - x) & \text{if } x \in [3T, 4T], \\ u(x - 4T) & \text{if } x \geqslant 4T, \end{cases}$$

we have

$$\mathcal{F}(v^*) = 8\int_0^{T}\left(\frac{1}{2}\left(u''^2 + g(u)u'^2\right) + f(u)\right)\mathrm{d}x + 2\int_0^{+\infty}\left(\frac{1}{2}\left(u''^2 + g(u)u'^2\right) + f(u)\right)\mathrm{d}x < -2C.$$

This contradicts the definition of u. Hence, we have $\inf_{\mathcal{E}} \mathcal{F} \geqslant 0$. □

We next state an obvious corollary of Theorem 5.8.

COROLLARY 5.10. *Let f and $g \in C^2(\mathbb{R})$ be even functions such that $f(1) = 0$ and* (C5) *holds. Assume*, *moreover*, *that $g(1)^2 < 4f''(1)$. If there exists $u \in \mathcal{E}$ such that $\mathcal{F}(u) < 0$, then* $\inf_{\mathcal{E}} \mathcal{F} = -\infty$.

PROOF. Suppose by contradiction that the conclusion of the corollary is false. Then Theorem 5.7 implies the existence of a minimizer which has a negative action. This contradicts Theorem 5.8. □

Let us consider the stationary Swift–Hohenberg equation

$$u'''' + \beta u'' + u^3 - u = 0. \tag{5.18}$$

We know from Theorem 5.7 and Lemma 5.6 that the corresponding functional

$$\mathcal{F}_\beta(u) = \int_{-\infty}^{+\infty}\left(\frac{1}{2}\left(u''^2 - \beta u'^2\right) + \frac{1}{4}\left(u^2 - 1\right)^2\right)\mathrm{d}x$$

has a minimum in $\mathcal{E}$ for β lower than some small positive constant. If β becomes too large, the minimum no longer exists. Indeed, it is shown in [55] that taking the function $u(x) = A(\beta)\sin(\omega(\beta)x)$ where $A(\beta) = \sqrt{\frac{\beta^2+4}{3}}$ and $\omega(\beta) = \sqrt{\frac{\beta}{2}}$, we have

$$\int_0^{2\pi/\omega(\beta)} \left(\frac{1}{2}\left(u''^2 - \beta u'^2\right) + \frac{1}{4}\left(u^2 - 1\right)^2\right) \mathrm{d}x < 0 \tag{5.19}$$

if $\beta > \beta^* = \sqrt{2\sqrt{6} - 4} = 0.9481\ldots$. It follows that the functional is unbounded from below whenever $\beta > \beta^*$. Therefore, there exists $0 < \beta_0 \leqslant \beta^*$ such that for $\beta > \beta_0$, the functional is unbounded from below. We give in the next theorem a characterization of β_0, which shows also that the minimum of $\mathcal{F}_\beta$ exists whenever $\beta \in (-\infty, \beta_0]$.

THEOREM 5.11. *Let β_0 be defined by*

$$\beta_0 = \sup\left\{\beta > 0 \mid \inf_{\mathcal{E}} \mathcal{F}_\beta \geqslant 0\right\}.$$

Then $\beta_0 \in\ ]0, \beta^*\,]$ and for all $\beta \leqslant \beta_0$, $\mathcal{F}_\beta$ has a minimum which is an odd heteroclinic solution of (5.18) *with exactly one zero. Moreover, for all $\beta > \beta_0$, $\mathcal{F}_\beta$ is unbounded from below.*

PROOF. For $u \geqslant 0$, we have $(u^2 - 1)^2/4 \geqslant (u - 1)^2/4$ so that Theorem 5.7 is valid for $\beta < \sqrt{2}$. It follows from Lemma 5.6 and Theorem 5.7 that for small $\beta > 0$, there exists $u \in \mathcal{E}$ such that $\mathcal{F}_\beta(u) = \inf_{\mathcal{E}} \mathcal{F}_\beta$. Also we deduce from (5.19) that $\beta_0 \leqslant \beta^*$.

Notice now that $\inf_{\mathcal{E}} \mathcal{J}_{\beta_0} \geqslant 0$. If it were not the case, the same would hold true for β near enough β_0, which contradicts the definition of β_0. Next, as $\beta_0 \leqslant \beta^* < \sqrt{2}$, we infer from Theorem 5.7 that for any $\beta \leqslant \beta_0$, $\mathcal{F}_\beta$ has an odd minimizer having exactly one zero.

It follows also from Corollary 5.10 that if $\beta_0 < \beta < \sqrt{2}$, $\mathcal{F}_\beta$ is unbounded from below. In case $\sqrt{2} \leqslant \beta$ the conclusion follows from (5.19). □

Numerical computations [87] seems to indicate that heteroclinic solutions of (5.18) exist at least until $\beta \approx 2.32$. Theorem 5.11 shows that for $\beta > \beta_0$, these critical points are not global minima.

5.3. *Notes and further comments*

1. The solution u obtained in Theorem 5.2 can be seen as a heteroclinic solution in a weak sense as it does not satisfy in general the usual conditions at $\pm\infty$, namely

$$\lim_{x \to \pm\infty} \left(u, u', u'', u'''\right)(x) = (\pm 1, 0, 0, 0).$$

However the limits for u and u' hold and it is easily seen that u is in the manifold of zero energy which is the case for classical heteroclinics. Indeed, the conservation of the Hamiltonian

$$H\big(u, u', u'', u'''\big) = u'''u' - \frac{1}{2}u''^2 - \frac{1}{2}g(u)u'^2 + f(u),$$

implies $H(u(x)\,u'(x), u''(x), u'''(x)) = E$ for any $x \in \mathbb{R}$ and some $E \in \mathbb{R}$. Assume $E \neq 0$ and let $(x_n)_n \subset \mathbb{R}^+$ be such that $x_n \to +\infty$ and $(u''(x_n))_n \subset \mathbb{R}$ is a bounded sequence. Then we have

$$\int_0^{x_n} \left(u'''u' - \frac{1}{2}u''^2 - \frac{1}{2}g(u)u'^2 + f(u)\right) \mathrm{d}x = Ex_n.$$

Integrating by parts the first term of the left hand side, we notice that the integral is bounded contradicting the equality.

If in addition to the hypotheses of Theorem 5.2 we suppose that f satisfies for some $0 < b < 1/2$ and $\beta > 0$,

$$\frac{f(u)}{(u-1)^2} \geqslant \beta, \quad \text{for all } u \in (1-b, 1+b),$$
$$\frac{f(u)}{(u+1)^2} \geqslant \beta, \quad \text{for all } u \in (-1-b, -1+b),$$

then it is easily seen that a minimizer u of $\mathcal{F}$ in $\mathcal{H}$ satisfies $u + 1 \in L^2(\mathbb{R}^-)$ and $u - 1 \in L^2(\mathbb{R}^+)$. Now, as we also have $u'' \in L^2(\mathbb{R})$, we infer by interpolation that $u' \in L^2(\mathbb{R})$ and thus the minimizer u actually satisfies $u + 1 \in H^2(\mathbb{R}^-)$ and $u - 1 \in H^2(\mathbb{R}^+)$. Using Eq. (5.3), we conclude that $u'''' \in L^2(\mathbb{R})$ which in turn by interpolation implies $u - 1 \in H^4(\mathbb{R}^+)$ and $u + 1 \in H^4(\mathbb{R}^-)$. The limits at $\pm\infty$ for u'' and u''' follow now easily.

Notice that the above additional assumptions hold if -1 and $+1$ are nondegenerate minima of f.

2. Theorem 5.7 can be extended to a nonsymmetric functional, assuming for instance that for some $k_1,\ k_2 > 0,\ \beta_1,\ \beta_2 \in \mathbb{R}$,

$$f(u) \geqslant k_1(u-1)^2, \quad g(u) \geqslant -\beta_1 \quad \text{for } u \geqslant 0,$$

$$f(u) \geqslant k_2(u+1)^2, \quad g(u) \geqslant -\beta_2 \quad \text{for } u < 0$$

and $\beta_i < \sqrt{8k_i}$ for $i = 1, 2$. However, the proof is much more involved and requires the cutting method we introduce in the following section.

3. A sharp estimate of the critical parameter β_0 introduced in Theorem 5.11 is still missing. A very rough estimate can be obtained via Lemma 5.6 and an estimate of the best constant for the interpolation inequality used therein. However, this estimate is far from the numerical observation of van den Berg who claims $\beta_0 \approx 0.92$, see Conjecture 7 in [87].

4. Let us define $\mathcal{E}_0^+$ as the cone of positive functions in $\mathcal{E}^+$, i.e.

$$\mathcal{E}_0^+ := \{u \in \mathcal{E}^+ \mid u(x) > 0 \text{ for } x > 0\}$$

and $\mathcal{F}_\beta^+ : \mathcal{E}^+ \to \mathbb{R}$ the restricted functional

$$\mathcal{F}_\beta^+(u) = \int_0^{+\infty} \left(\frac{1}{2}(u''^2 - \beta u'^2) + \frac{1}{4}(u^2 - 1)^2\right) \mathrm{d}x.$$

It follows from Theorem 5.11 that $\mathcal{F}_\beta^+$ has a minimizer in $\mathcal{E}^+$ for every $\beta \leqslant \beta_0$. Moreover this minimizer belongs to the interior of $\mathcal{E}_0^+$. Indeed, as mentioned in Section 4.1, the Hamiltonian

$$H(u, u', u'', u''') = u'''u' - \frac{1}{2}u''^2 - \frac{\beta}{2}u'^2 + \frac{(u^2-1)^2}{4}$$

is constant along solutions. As for any minimizer φ, $H(\varphi, \varphi', \varphi'', \varphi''') = 0$ and $\varphi''(0) = 0$, we deduce from the conservation of the Hamiltonian that $\varphi'(0) > 0$. On the other hand, we deduce from Step 2 of the proof of Theorem 5.7 that

$$\inf_{\mathcal{E}_0^+} \mathcal{F}_\beta^+ > -\infty$$

for every $\beta < \sqrt{2}$. Hence it is natural to ask if $\mathcal{F}_\beta^+$ has a local minimizer in the interior of $\mathcal{E}_0^+$ for $\beta_0 \leqslant \beta < \sqrt{2}$. To our knowledge, this question remains open. However, we can observe that an argument of continuity implies that the answer is positive at least for β close to β_0. Indeed, it is easily seen that $\mathcal{F}_\beta^+$ has a local minimizer in $\mathcal{E}_0^+$ for every $\beta < \sqrt{2}$. For $\beta \leqslant \beta_0$, we can prove that the minimizers cannot be on the boundary of $\mathcal{E}_0^+$ so that

$$\mathcal{F}_{\beta_0}^+(\varphi_{\beta_0}) < \inf_{\partial \mathcal{E}_0^+} \mathcal{F}_{\beta_0}^+, \tag{5.20}$$

where φ_{β_0} is a minimizer of $\mathcal{F}_{\beta_0}^+$. On the other hand, we have for $\beta_1 < \sqrt{2}$ and $\beta \in [\beta_0, \beta_1]$,

$$\mathcal{F}_\beta^+(u) = \mathcal{F}_{\beta_1}^+(u) + \frac{\beta_1 - \beta}{2} \int_0^{+\infty} u'^2 \, \mathrm{d}x \geqslant -C + \frac{\beta_1 - \beta}{2} \int_0^{+\infty} u'^2 \, \mathrm{d}x.$$

We now infer from this last estimate that any function $u \in \mathcal{E}_0^+$ satisfying

$$\mathcal{F}_\beta^+(u) = \inf_{\partial \mathcal{E}_0^+} \mathcal{F}_\beta^+$$

is a priori bounded in $\mathcal{E}^+$ for β in any compact subinterval of $[\beta_0, \beta_1[$. It follows that the inequality (5.20) holds true for β in a right neighborhood of β_0, i.e.

$$\mathcal{F}_\beta^+(\varphi_{\beta_0}) < \inf_{\partial \mathcal{E}_0^+} \mathcal{F}_\beta^+.$$

Since φ_{β_0} is in the interior of $\mathcal{E}_0^+$, this means that the infimum of $\mathcal{F}_\beta^+$ in $\mathcal{E}_0^+$ cannot be achieved on the boundary.

6. A Ginzburg–Landau model for ternary mixtures: Connections between nonconsecutive equilibria

A good example of phase transition phenomena is provided by the mixing–demixing transitions of a fluid. It is well known that oil and water do not mix so that a binary fluid composed of oil and water has two separated homogeneous phases usually called oil-rich and water-rich phases. Assuming that the essence of the transitions can be described in terms of the concentration difference between oil and water, we introduce a scalar order parameter ϕ which locally measures this difference. A Ginzburg–Landau model [36] then yields for the free energy a functional of the form

$$\mathcal{F}(\phi) = \int_{R^3} \big(g_0(\nabla\phi)^2 + f(\phi) - \mu\phi\big)\,\mathrm{d}r. \tag{6.1}$$

The function f is the free-energy density and μ is the chemical potential difference between oil and water. The free-energy density f is usually approximated by an even (due to the symmetry under the interchange of the two components) fourth order polynomial

$$f(\phi) = a_2\phi^2 + a_4\phi^4.$$

Thermodynamic stability of homogeneous phases requires that $a_4 > 0$. When $a_2 < 0$, there is coexistence of water-rich and oil-rich phases at $\mu = 0$, i.e. the two phases are at the same level of energy.

The addition of an amphiphile into the fluid can provoke the formation of wealth of complex self-assembled structures. An amphiphilic molecule consists of a polarizable head which prefers the highly polarizable water environment and a hydrocarbon tail which prefers the oil. The energy of the amphiphile is lowest when it can find or create surfaces between oil and water at which it can adsorb the other two components into various structures. Adding a small amount of amphiphile to the fluid will not in general modify the two-phase coexistence. The added amphiphile will partition itself between the two phases. On the other hand, if for example the amphiphile is more present in the water phase, a third phase can be made to appear by changing an external field such as the temperature or the chemical potential of a fourth component. This new phase contains more of the amphiphile and less of water and oil than the other two. Its density will be intermediate between that of the water-rich and the oil-rich phases and will be therefore physically located between them. Hence it is called the middle-phase.

Let us come back for a while to the free energy functional (6.1) of the binary fluid oil-water. Thermodynamic quantities and correlation functions can be obtained by approximation methods in which the functional $\mathcal{F}$ is minimized in the space of functions which spatially connect the two phases. The minimizer ϕ is then the interfacial profile between the oil-rich and the water-rich phase at coexistence.

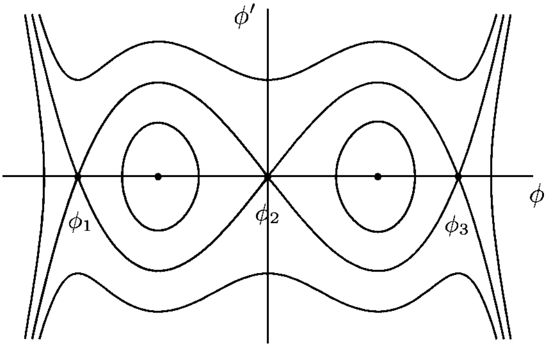

Fig. 3. The 3-stable system phase plane.

If we want to adapt the Ginzburg–Landau model (6.1) to describe the oil-water-amphiphile mixtures and keep a single order parameter which still measures the density difference between oil and water, we need to consider three-phase coexistence at $\mu = 0$ so that f must have three minima at $\phi_1 < \phi_2 < \phi_3$ corresponding respectively to the bulk phases oil-rich, middle and water-rich. Choosing their common value to be zero and assuming only one spatial dependence, the interfacial profile, $\phi(x)$, between the oil-rich phase which extends to $x \to -\infty$ and the water-rich phase which extends to $x \to +\infty$, then solves the equation

$$2g_0\phi'' - f'(\phi) = 0 \tag{6.2}$$

and the first integral

$$g_0\phi'(x)^2 - f\big(\phi(x)\big) = 0. \tag{6.3}$$

From a simple phase-plane analysis, see Fig. 3, it is easily seen that the only trajectory starting form ϕ_1 at $x \to -\infty$ and going to ϕ_3 at $x \to +\infty$ spends an infinite amount of time in ϕ_2, hence Eq. (6.2) has no solution ϕ satisfying $\phi(-\infty) = \phi_1$ and $\phi(+\infty) = \phi_3$. In the ternary mixture problem, this corresponds to an infinite thickness of middle phase, ϕ_2, between the oil-rich and water-rich phases. In other words, the model predicts the middle phase will always wet the interface between the oil- and water-rich phases, a prediction contrary to the results of experiment.

A simple way proposed by Gompper and Schick [36] to overcome this consequence of the model and yet consider a scalar order-parameter theory is to add a second order term in the Lagrangian, considering therefore the functional

$$\mathcal{F}(\phi) = \int_{R^3} \Big(c\big(\nabla^2\phi\big)^2 + g(\phi)(\nabla\phi)^2 + f(\phi) - \mu\phi\Big)\,\mathrm{d}r.$$

The function g, which quantifies the properties of the amphiphile, is negative close to the middle-phase as it tends to create interfaces and positive in the oil and water phases. The parameter c is positive and stabilizes the system.

The aim of this section is to prove that this last functional does not suffer the defect of the classical Ginzburg–Landau model (6.1) at least under some hypotheses on the nature of the middle-phase equilibrium. In the sequel, we only consider a scalar order-parameter ϕ with one spatial direction.

We consider a potential $f \in C^2(\mathbb{R})$ and a function $g \in C^2(\mathbb{R})$ satisfying (C1), (C4) and

(C2′) for some $0 < a < 1/2$ and $\alpha > 0$,

$$\frac{f(u)}{(u-1)^2} \leqslant \alpha, \quad \text{for } |u-1| < a,$$

$$\frac{f(u)}{(u+1)^2} \leqslant \alpha, \quad \text{for } |u+1| < a;$$

(C3′) $f(u) = 0$ if and only if $u = 0$ or $u = \pm 1$.

We also introduce the additional condition

$$g(0)^2 < 4f''(0).$$

This last condition implies that the trivial solution is a saddle-focus equilibrium for the linear equation

$$u'''' - g(0)u'' + f''(0)u = 0.$$

As observed in Section 5.1, if $g(0) < 0$ or (C1) holds with $g = \tilde{g}$, we just need f to be C^2 in a neighborhood of 0 and $f''(0) \neq 0$ as the inequality $g(0)^2 < 4f''(0)$ then holds.

Under the previous assumptions, it makes sense to consider the functional

$$\mathcal{F}(u) = \int_{-\infty}^{+\infty} \left(\frac{1}{2}\left(u''^2 + g(u)u'^2\right) + f(u) \right) \mathrm{d}x \tag{6.4}$$

in the space

$$\mathcal{H} := \left\{ u \in C^1(\mathbb{R}),\ u'' \in L^2(\mathbb{R}),\ u' \in L^\infty(\mathbb{R}) \text{ and } \lim_{x \to \pm\infty} u(x) = \pm 1 \right\}$$

and the associated Euler–Lagrange equation is given by

$$u'''' - g(u)u'' - \frac{1}{2}g'(u)u'^2 + f'(u) = 0. \tag{6.5}$$

Arguing as in Section 5, we observe that the functional (6.4) is bounded from below in $\mathcal{H}$.

LEMMA 6.1. *If $f, g \in C(\mathbb{R})$ satisfy* (C1), *then there exists a constant $s > 0$ such that for all $u \in \mathcal{H}$*

$$\mathcal{F}(u) \geqslant s \int_{-\infty}^{+\infty} \left(\frac{u''^2}{2} + f(u) \right) \mathrm{d}x.$$

The main theorem of the section goes as follows.

THEOREM 6.2. *Suppose that* $f, g \in C^2(\mathbb{R})$ *satisfy* (C1), (C2′), (C3′) *and* (C4). *If* $g(0)^2 < 4f''(0)$, *there exists a minimizer* u *of* $\mathcal{F}$ *in* $\mathcal{H}$ *which is a solution of* (6.5).

In comparison with Theorem 5.2, the additional condition $g(0)^2 < 4f''(0)$ allows to consider a potential f with a third bottom at 0. We do not know if Theorem 6.2 holds without this assumption.

The idea of the proof of Theorem 6.2 is identical to that of Theorem 5.2. We thus search for a control on the time it takes for a quasi-minimizer to travel from a neighborhood of $(-1, 0)$ to a neighborhood of $(+1, 0)$ in the uu'-plane. Observe that the presence of the third zero of the potential rules out some of the arguments used in the proof of Theorem 5.2. On the other hand, we easily obtain a control on time intervals from $(-1, 0)$ to $(0, 0)$ and $(0, 0)$ to $(+1, 0)$. To complete the arguments, we just need a control on the time quasi-minimizers spend close to 0. This control is obtained thanks to the additional condition that 0 is a saddle-focus. Basically, we show in Section 6.1 that due to the nature of the equilibrium, the local minimizers of (6.4) close to 0 (in the phase space) change sign in every interval of length larger than a fixed constant. We then introduce in Section 6.2 a *clipping* procedure which roughly speaking ensures that the quasi-minimizers are monotonic close to 0. This technique was first used by Kalies et al. [42]. The local analysis close to 0 then implies that quasi-minimizers do not spend much time close to 0. We give the proof of Theorem 6.2 in Section 6.3.

6.1. *Local analysis of a saddle-focus equilibrium*

We now focus on the local minimizers close to a saddle-focus equilibrium. To fix the ideas and to simplify the notation, we assume that f is a potential for which 0 is a nondegenerate global minimum and g is such that 0 is a saddle focus equilibrium of the linear equation

$$u'''' - g(0)u'' + f''(0)u = 0, \tag{6.6}$$

i.e. $g(0)^2 < 4f''(0)$. Basically, the following lemma shows that the minimizers of the functional

$$\mathcal{F}_{[a,b]}(u) := \int_a^b \left(\frac{1}{2}\left(\left(u''^2\right) + g(u)u'^2\right) + f(u)\right) \mathrm{d}x \tag{6.7}$$

on the set

$$\mathcal{H}_{a,b} := \left\{v \in H^2(a, b) \;\middle|\; \left(u(a), u'(a)\right) = y_0 \text{ and } \left(u(b), u'(b)\right) = y_1\right\}$$

are small (for the C^3-norm) whenever y_0 and y_1 are small. It then follows that the oscillatory behaviour of the solutions of the linearization (6.6) around the equilibrium extends to these minimizers. The following lemma is adapted from Theorem 4.1 [42] where it is assumed that g is positive.

LEMMA 6.3. *Let f and $g \in C^2(\mathbb{R})$ be such that $f(u) \geqslant 0$ for all $u \in \mathbb{R}$, $f(0) = f'(0) = 0$ and assume* (C1) *holds. Assume moreover that $f''(0) \neq 0$. Then, there exist $\delta_0 > 0$ and $S > 1$ such that if $b - a \geqslant 1$, $\|y_0\| \leqslant \delta_0$, $\|y_1\| \leqslant \delta_0$ and u minimizes $\mathcal{F}_{[a,b]}$ on $\mathcal{H}_{a,b}$, we have*

$$\|u\|_{C^3([a,b])} \leqslant S \max\bigl(\|y_0\|, \|y_1\|\bigr). \tag{6.8}$$

PROOF. As 0 is a nondegenerate minimum of f, there exist $\delta_0 > 0$, $\eta > 0$ and $\zeta > 0$ such that $|f'(u)| \leqslant 2\eta|u|$ and $\zeta u^2 \leqslant f(u) \leqslant \eta u^2$ for $|u| \leqslant \delta_0$. Notice also that using integration by parts and arguing as in Lemma 5.1, we see that there exists $s > 0$ such that for any function $u \in H^2(a, b)$,

$$\mathcal{F}_{[a,b]}(u) \geqslant s \int_a^b \left(\frac{u''^2}{2} + f(u)\right) \mathrm{d}x + \frac{1}{2}\bigl(\tilde{G}\bigl(u(b)\bigr)u'(b) - \tilde{G}\bigl(u(a)\bigr)u'(a)\bigr), \tag{6.9}$$

so that $\mathcal{F}_{[a,b]}$ is bounded from below on $\mathcal{H}_{a,b}$. The existence of a minimizer follows by standard arguments. Moreover, if u is a minimizer, u solves (6.5) on $[a, b]$ and satisfies the boundary conditions $(u(a), u'(a)) = y_0$ and $(u(b), u'(b)) = y_1$.

CLAIM 1. *There exists $C_1 > 0$ and $\delta_1 > 0$ such that if $\|y_0\| \leqslant \delta \leqslant \delta_1$ and $\|y_1\| \leqslant \delta \leqslant \delta_1$, then $\inf_{\mathcal{H}_{a,b}} \mathcal{F}_{[a,b]} \leqslant C_1\delta^2$.*

Define $P(x)$ as follows:

$$P(x) = \begin{cases} P_0(x) & \text{if } a \leqslant x \leqslant a + \frac{1}{2}, \\ 0 & \text{if } a + \frac{1}{2} < x \leqslant b - \frac{1}{2}, \\ P_1(x) & \text{if } b - \frac{1}{2} < x \leqslant b, \end{cases}$$

where P_i, $i = 0, 1$, are the third degree polynomials satisfying $(P_0(a), P_0'(a)) = y_0$, $(P_1(b), P_1'(b)) = y_1$ and $(P_0(a + \frac{1}{2}), P_0'(a + \frac{1}{2})) = (P_1(b - \frac{1}{2}), P_1'(b - \frac{1}{2})) = (0, 0)$. Observe that there exists $0 < \delta_1 \leqslant \delta_0$ such that if $\|y_0\| \leqslant \delta \leqslant \delta_1$ and $\|y_1\| \leqslant \delta \leqslant \delta_1$, then $\|P\|_{L^\infty} \leqslant \delta_0$. It then follows from an easy computation that

$$\inf_{\mathcal{H}_{a,b}} \mathcal{F}_{[a,b]} \leqslant \mathcal{F}_{[a,b]}(P) \leqslant C_1\delta^2,$$

where $C_1 > 0$ essentially depends on η and $\|g\|_{L^\infty(-\delta_0,\delta_0)}$.

CLAIM 2. *There exists $\delta_2 > 0$ such that if u is a minimizer in $\mathcal{H}_{a,b}$ with $\|y_0\| \leqslant \delta_2$ and $\|y_1\| \leqslant \delta_2$, then $\|u\|_\infty \leqslant \delta_0$.*

The ideas we use to prove the claim are already included in the proof of Proposition 5.3. Observe that the minimizers of $\mathcal{F}_{[a,b]}$ in $\mathcal{H}_{a,b}$ are a priori bounded in $C^1([a, b])$. Indeed, this follows easily arguing as in the first part of the proof of Proposition 5.3. We denote by L the C^1-bound. Consider next the set

$$\mathcal{D}_{a,b} := \{u \in \mathcal{H}_{a,b} \mid \text{ for all } x \in [a,b], |u(x)| \leqslant \delta_0\}.$$

Let us fix the notation $N_{\delta_0} := [-\delta_0, \delta_0]$. We then define

$$\mu := \min\{f(u) \mid u \in N_{\delta_0} \setminus N_{\delta_0/2}\} > 0,$$
$$\nu := \max\{|\tilde{G}(u)| \mid u \in N_{\delta_0}\} \geqslant 0.$$

As in Part 2—Step 2 of the proof of Proposition 5.3, we can derive a lower estimate on the action of functions $u \in \mathcal{H}_{a,b}$ whose graphs do not stay in the strip $[a,b] \times N_{\delta_0}$. Indeed, suppose $u \in \mathcal{H}_{a,b} \setminus \mathcal{D}_{a,b}$ minimizes $\mathcal{F}_{[a,b]}$ in $\mathcal{H}_{a,b}$ with $\|y_0\| \leqslant \delta \leqslant \delta_1$ and $\|y_1\| \leqslant \delta \leqslant \delta_1$. Then

$$\mathcal{F}_{[a,b]}(u) \geqslant \frac{s\mu\delta_0}{2L} - \nu\delta. \tag{6.10}$$

On the other hand, we infer from Claim 1 that

$$\mathcal{F}_{[a,b]}(u) \leqslant C_1\delta^2.$$

Choosing $\delta_2 > 0$ small enough and $0 < \delta \leqslant \delta_2$, the estimate (6.10) yields a contradiction so that $u \in \mathcal{D}_{a,b}$ and the claim follows.

CLAIM 3. *There exists $C_2 > 0$ such that if u is a minimizer in $\mathcal{H}_{a,b}$ with $\|y_0\| \leqslant \delta \leqslant \delta_2$ and $\|y_1\| \leqslant \delta \leqslant \delta_2$, then $\|u\|_{C^3} \leqslant C_2\delta$.*

Assuming that $\|y_0\| \leqslant \delta \leqslant \delta_2$ and $\|y_1\| \leqslant \delta \leqslant \delta_2$, we deduce from the estimate (6.9) that

$$\|u\|_{L^2} \leqslant C\delta \quad \text{and} \quad \|u''\|_{L^2} \leqslant C\delta.$$

Indeed, the L^2-bound for u'' follows easily while for the bound on u, it is sufficient to observe that f is bounded from below by the parabola ζu^2 for $|u| \leqslant \delta_0$. We now deduce by interpolation that

$$\|u'\|_{L^2} \leqslant C(\|u\|_{L^2} + \|u''\|_{L^2}) \leqslant C\delta.$$

The constant C can be chosen independent of the length $b-a$ as far as $b-a \geqslant 1$, see [1]. Using the continuous injection of $H^2(a,b)$ into $C^1([a,b])$, we deduce that

$$\|u\|_{C^1} \leqslant C\delta.$$

Observe that we can still choose a constant C that does not depend on the length of $[a,b]$ as $b-a \geqslant 1$. The differential equation (6.5) then yields the estimate

$$\|u''''\|_{L^2} \leqslant C\delta$$

so that by interpolation we also have

$$\|u'''\|_{L^2} \leqslant C\delta.$$

Now, the bound in C^3 follows from the bound in H^4 and the continuous injection in C^3. □

LEMMA 6.4. *Suppose that the assumptions of Lemma* 6.3 *hold and assume furthermore that* $g(0)^2 < 4f''(0)$. *Then there exist* $\delta_0 > 0$ *and* $\tau_0 > 0$ *such that if* $b - a \geqslant 1$, $\|y_0\| \leqslant \delta_0$, $\|y_1\| \leqslant \delta_0$, $\max(\|y_0\|, \|y_1\|) > 0$ *and* u *minimizes* $\mathcal{F}_{[a,b]}$ *on* $\mathcal{H}_{a,b}$, u *changes sign on every subinterval of* $[a, b]$ *having length larger than* τ_0.

The proof is included in Theorem 4.2 in [42]. A slight modification of Lemma 5.9 also leads to the conclusion. It is easily seen that the assumption on the convergence of the solution to the equilibrium in Lemma 5.9 can be replaced by a condition ensuring that the solution remains close (for the C^3-norm) to 0. Here, thanks to Lemma 6.3, the smallness of the boundary conditions on u and u' suffices to obtain such a control.

6.2. *Clipping*

Next, we recall the *clipping* procedure as introduced in [42]. When minimizing a functional in a certain space, we often want to be able to modify locally any function by another one being in the same space, having better properties and lower action. When dealing with a second order equation and its associated functional, we usually only have to worry about keeping functions continuous so that pieces of graph can be easily discarded. As our functional $\mathcal{F}$ requires square integrable second order derivative, things are more complicated. For example, any modification has to keep the functions at least C^1. Let us describe admissible cutoffs.

DEFINITION 6.5. Let $u \in C^1[a, b]$. If $u(\alpha) = u(\beta)$ and $u'(\alpha) = u'(\beta)$ for some $\alpha < \beta$ in $[a, b]$, we say that the interval $[\alpha, \beta]$ can be *clipped out* meaning that we can define a C^1 function $\hat{u}$ on the interval $[a, b - (\beta - \alpha)]$ which coincides with u and the $\beta - \alpha$ translate of $u_{|[\beta,b]}$ respectively on the intervals $[a, \alpha]$ and $[\alpha, b - (\beta - \alpha)]$. The function $\hat{u}$ is defined by

$$\hat{u}(x) := \begin{cases} u(x) & \text{if } x \in [a, \alpha], \\ u(x + \beta - \alpha) & \text{if } x \in \bigl[\alpha, b - (\beta - \alpha)\bigr]. \end{cases}$$

We say that $\hat{u}$ is a *clip* of u.

Observe that in case the Lagrangian

$$L_g\bigl(u, u', u''\bigr) = \frac{1}{2}\bigl(u''^2 + g(u)u'^2\bigr) + f(u)$$

is positive, the clipping process has the nice property that if $u \in \mathcal{H}$ and $\hat{u}$ is a clip of u, then

$$\mathcal{F}(\hat{u}) \leqslant \mathcal{F}(u).$$

When $L_g(u, u', u'')$ changes sign, this is no more true. However, we show below in Lemma 6.7 that under the assumption (C1), the clipping process decreases the action.

The following lemma gives the basic tool for clipping functions. It generalizes Lemma 3.1 in [42].

LEMMA 6.6. *Let $s_1 < s_2 < s_3 < s_4$ and let $u \in C^1(s_1, s_4)$ be such that*

$$u(s_1) = u(s_3), \qquad u(s_2) = u(s_4), \qquad \big(u'(s_1) - u'(s_3)\big)\big(u'(s_2) - u'(s_4)\big) \leqslant 0.$$

Assume, moreover, that

$$u(s_1) < u(s) < u(s_2) \quad \text{for all } s \in]s_1, s_2[,$$

and

$$u(s_3) < u(s) < u(s_4) \quad \text{for all } s \in]s_3, s_4[.$$

Then there exist $\alpha \in [s_1, s_2]$, $\beta \in [s_3, s_4]$ such that the interval $[\alpha, \beta]$ can be clipped out.

PROOF. Consider the set

$$E := \big\{(x, y) \in [s_1, s_2] \times [s_3, s_4] \,\big|\, u(x) = u(y)\big\}.$$

It follows from degree arguments that there exists a connected set $H \subset E$ that contains (s_1, s_3) and (s_2, s_4). Next we define the function $\varphi : H \to \mathbb{R}$ by

$$\varphi(x, y) = u'(x) - u'(y).$$

Since

$$\varphi(s_1, s_3)\varphi(s_2, s_4) \leqslant 0,$$

the continuity of φ leads to the conclusion. □

A typical example where Lemma 6.6 applies is displayed in Fig. 4.

As we already mentioned, the clipping procedure plays a key role to control and modify minimizing sequences. We therefore need to check that any clip has a lower action than the original function. This is the case when (C1) holds.

LEMMA 6.7. *Suppose that $f \in C(\mathbb{R})$ is a nonnegative function and $g \in C(\mathbb{R})$ satisfies* (C1). *If $u \in H^2(a, b)$ and $\hat{u} \in H^2(a, b - (\beta - \alpha))$ is a clip of u, then*

$$\mathcal{F}_{[a,b-(\beta-\alpha)]}(\hat{u}) \leqslant \mathcal{F}_{[a,b]}(u),$$

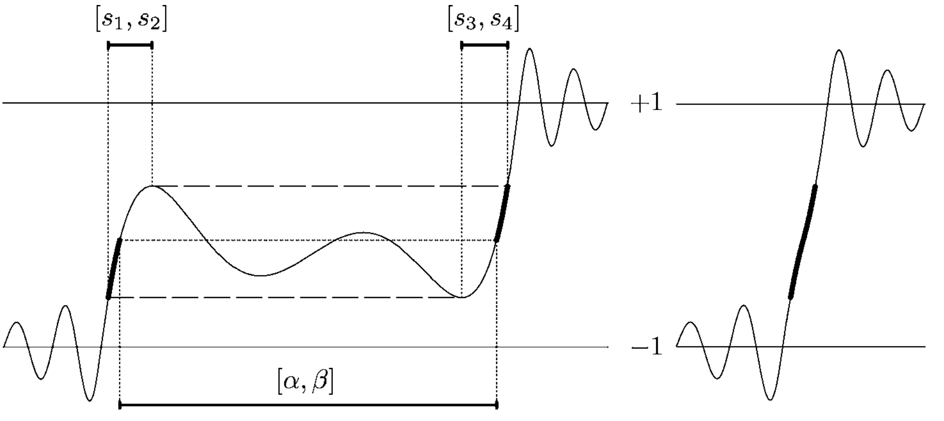

Fig. 4. *The clipping process*. Lemma 6.6 applies with $s_1 < s_2 < s_3 < s_4$ so that a subinterval $[\alpha, \beta]$ can be clipped out.

where $\mathcal{F}_{[a,b]}$ *is defined by* (6.7).

PROOF. We compute

$$\mathcal{F}_{[a,b-(\beta-\alpha)]}(\hat{u}) = \mathcal{F}_{[a,\alpha]}(u) + \mathcal{F}_{[\beta,b]}(u) = \mathcal{F}_{[a,b]}(u) - \mathcal{F}_{[\alpha,\beta]}(u)$$

and arguing as in Lemma 5.1, we infer that

$$\mathcal{F}_{[\alpha,\beta]}(u) \geqslant s \int_\alpha^\beta \left(\frac{u''^2}{2} + f(u)\right) \mathrm{d}x + \frac{1}{2}\big(\tilde{G}\big(u(\beta)\big)u'(\beta) - \tilde{G}\big(u(\alpha)\big)u'(\alpha)\big).$$

Since $u(\alpha) = u(\beta)$, $u'(\alpha) = u'(\beta)$, we obtain the inequality

$$\mathcal{F}_{[\alpha,\beta]}(u) \geqslant 0$$

so that the result follows. □

6.3. *Existence of a minimizer*

As for Theorem 5.2, the proof of Theorem 6.2 relies on the control and the localization of a minimizing sequence.

PROPOSITION 6.8. *Suppose that* $f, g \in C^2(\mathbb{R})$ *satisfy* (C1), (C2′), (C3′) *and* (C4). *Then there exists* $L > 0$, $T > 0$ *and a sequence* $(u_n)_{n\in\mathbb{N}} \subset \mathcal{H}$ *such that* $\mathcal{F}(u_n) \to \inf_{\mathcal{H}} \mathcal{F}$ *and for all* $n \in \mathbb{N}$,

(i) $\|u_n\|_{C^1} \leqslant L$,

(ii) $|u_n(x)+1| \leqslant a$ *for all* $x \leqslant -T$ *and* $|u_n(x)-1| \leqslant a$ *for all* $x \geqslant T$.

The proof of property (i) follows exactly the lines of the proof of Proposition 5.3, Part 1. We divide the proof of the second statement in three main steps. In Step 1, we approximate the minimum of $\mathcal{F}$ with functions that stay close to ± 1 at $\pm\infty$. Step 2 estimates the time for a function $u \in \mathcal{H}$ to travel in the (u, u')-plane from neighborhoods of $(\pm 1, 0)$ to a neighborhood of $(0, 0)$. Finally, in Step 3, we show that functions in $\mathcal{H}$ that stay close to 0 with small velocity can be replaced, using the clipping procedure, by functions that spend in such a neighborhood a time which is a priori bounded.

PROOF. The proof of the first statement has been worked out in the proof of Proposition 5.3. We therefore restrict our attention to the proof of the conclusion (ii).

Let $(u_n)_n \subset \mathcal{H}$ be a minimizing sequence for $\mathcal{F}$. We can assume without loss of generality that $\mathcal{F}(u_n) \leqslant C \leqslant \inf_{\mathcal{H}} \mathcal{F} + 1$. Let $\varepsilon > 0$ be fixed and such that $S\varepsilon \leqslant \delta_0$, where $\delta_0 > 0$ and $S > 1$ are given by Lemma 6.3 and Lemma 6.4. For each $n \in \mathbb{N}$, we define

$$x_1 := \sup\bigl\{x \mid \bigl|u_n(x)+1\bigr| \leqslant \varepsilon \text{ and } \bigl|u_n'(x)\bigr| \leqslant \varepsilon\bigr\},$$

and

$$x_4 := \inf\bigl\{x \mid \bigl|u_n(x)-1\bigr| \leqslant \varepsilon \text{ and } \bigl|u_n'(x)\bigr| \leqslant \varepsilon\bigr\}.$$

Step 1. Modification of u_n in $]-\infty, x_1]$ and $[x_4, \infty[$. For all $n \in \mathbb{N}$, there exists a function $v_n \in \mathcal{H}$ such that for all $x \leqslant x_1$, $|v_n(x)+1| \leqslant a$, for all $x \geqslant x_4$, $|v_n(x)-1| \leqslant a$ and $\mathcal{F}(v_n) \leqslant \mathcal{F}(u_n)$.

This step is similar to Part 2—Step 2 of the proof of Proposition 5.3 so that we skip it.

Step 2. Estimates on time intervals. Define for all $n \in \mathbb{N}$,

$$x_2 := \inf\bigl\{x \geqslant x_1 \mid \bigl|v_n(x)\bigr| \leqslant \varepsilon \text{ and } \bigl|v_n'(x)\bigr| \leqslant \varepsilon\bigr\}$$

and

$$x_3 := \sup\bigl\{x \leqslant x_4 \mid \bigl|v_n(x)\bigr| \leqslant \varepsilon \text{ and } \bigl|v_n'(x)\bigr| \leqslant \varepsilon\bigr\}.$$

Notice that x_2 and x_3 need not exist. Arguing as in Part 2—Step 1 of the proof of Proposition 5.3, we obtain a bound T_1 on $x_2 - x_1$ and $x_4 - x_3$, or on $x_4 - x_1$ if x_2 and x_3 do not exist.

Step 3. Modification in $[x_2, x_3]$. If x_2 and x_3 do not exist, this step can of course be skipped.

CLAIM 1. *For all* $n \in \mathbb{N}$, *there exists a function* w_n *which minimizes* $\mathcal{F}$ *in*

$$\mathcal{H}_{2,3} := \bigl\{w \in \mathcal{H} \mid w = v_n \text{ on } \mathbb{R} \setminus [x_2, x_3]\bigr\}.$$

Indeed, we can replace $v_n$by a minimizer of $\mathcal{F}$ in $\mathcal{H}_{2,3}$. Such a function w_n exists and satisfies $\mathcal{F}(w_n) \leqslant \mathcal{F}(v_n)$. Further, it solves the differential equation (6.5) on $[x_2, x_3]$ together with the boundary conditions

$$w_n(x_2) = v_n(x_2), \quad w_n'(x_2) = v_n'(x_2), \quad w_n(x_3) = v_n(x_3), \quad w_n'(x_3) = v_n'(x_3).$$

CLAIM 2. *For each $n \in \mathbb{N}$, if $x_3 - x_2 \geqslant \max(1, 8\tau_0)$, where τ_0 is given from Lemma* 6.4, *the function w_n can be replaced by a clip $\hat{w}_n \in \mathcal{H}$ so that after clipping the interval* $[x_1, x_4]$ *is transformed into an interval of length smaller than* $(x_2 - x_1) + \max(1, 8\tau_0) + (x_4 - x_3)$. *Further, we have $\mathcal{F}(\hat{w}_n) \leqslant \mathcal{F}(w_n)$.*

Define

$$x_2' := \max\{x \leqslant x_2 \mid |w_n(x)| = \delta_0\} \quad \text{and} \quad x_3' := \min\{x \geqslant x_3 \mid |w_n(x)| = \delta_0\}.$$

It follows from Claim 1 and Lemma 6.3 that $|w_n(x)| \leqslant S\varepsilon \leqslant \delta_0$ for all $x \in [x_2, x_3]$ and therefore also for all $x \in [x_2', x_3']$.

Suppose first that $w_n(x_2') = -\delta_0$ and $w_n(x_3') = \delta_0$. Define

$$\begin{aligned} s_2 &:= \min\{x \in [x_2', x_3'] \mid w_n'(x) = 0 \text{ and } w_n(x) \geqslant 0\}, \\ s_4 &:= \max\{x \in [x_2', x_3'] \mid w_n(x) = w_n(s_2)\}, \\ s_3 &:= \max\{x \in [s_2, s_4] \mid w_n'(x) = 0\} \end{aligned}$$

and take

$$s_1 := \max\{x \in [x_2', s_2] \mid w_n(x) = w_n(s_3)\}.$$

Observe that Lemma 6.4 ensures the existence of s_2 and moreover $s_2 \in [x_2', x_2 + 2\tau_0]$. Also, $s_3 \in [x_3 - 2\tau_0, x_3']$. Further, we can apply Lemma 6.6 to the function w_n on $[s_1, s_4]$ and discard the restriction of w_n to some interval $[\alpha, \beta]$ containing $[s_2, s_3]$. We denote by $\hat{w}_n$ the clip of w_n. Namely, we define

$$\hat{w}_n(x) := \begin{cases} w_n(x) & \text{if } x \in [a, \alpha], \\ w_n(x + \beta - \alpha) & \text{if } x \in (\alpha, b - (\beta - \alpha)). \end{cases}$$

Letting $x_4^* := x_4 - (\beta - \alpha)$, we have

$$x_4^* - x_1 \leqslant (x_4 - x_3) + 4\tau_0 + (x_2 - x_1).$$

If $w_n(x_2') = \delta_0$ and $w_n(x_3') = -\delta_0$, we use the same argument.

Assume now that $w_n(x_2') = -\delta_0$ and $w_n(x_3') = -\delta_0$. Let $q \in [x_2', x_3']$ be such that $\max_{x \in [x_2', x_3']} w_n(x) = w_n(q)$. Notice that Lemma 6.4 implies $w_n(q) > 0$ and hence we can apply the preceding argument to each of the intervals $[x_2', q]$ and $[q, x_3']$. Here, denoting by x_4^{**} the point into which x_4 is transformed, we have after clipping, $x_4^{**} - x_1 \leqslant (x_4 - x_3) + 8\tau_0 + (x_2 - x_1)$.

The case $w_n(x_2') = \delta_0$ and $w_n(x_3') = \delta_0$ is handled similarly.

Conclusion of Step 3. For each $n \in \mathbb{N}$, there exists a function $\hat{w}_n \in \mathcal{H}$ ($\hat{w}_n = w_n$ if $x_3 - x_2 \leqslant \max(1, 8\tau_0)$) which satisfies $\mathcal{F}(\hat{w}_n) \leqslant \mathcal{F}(w_n)$ and

$$\begin{aligned} \left|\hat{w}_n(x) + 1\right| &\leqslant a \quad \text{for all } x \leqslant x_1, \\ \left|\hat{w}_n(x) - 1\right| &\leqslant a \quad \text{for all } x \geqslant x_4', \end{aligned}$$

for some $x_4' \in \mathbb{R}$ such that

$$x_4' - x_1 \leqslant 2T_1 + \max(1, 8\tau_0),$$

where T_1 is defined in Step 2.

Conclusion. Taking

$$T = T_1 + \frac{1}{2}\max(1, 8\tau_0)$$

and using a time-translation if necessary, we finally construct a minimizing sequence $(z_n)_n \subset \mathcal{H}$ that satisfies the second statement of Proposition 6.8. Notice that in case x_2 and x_3 are not defined, we can choose

$$T = \frac{T_1}{2} \geqslant \frac{1}{2}(x_4 - x_1). \qquad \square$$

Now that we have at hand a minimizing sequence satisfying Proposition 6.8, the proof of Theorem 6.2 follows the lines of the proof of Theorem 5.2.

6.4. *Notes and further comments*

1. As discussed in the first note of Section 5, the minimizer obtained in Theorem 6.2 is a heteroclinic solution in a weak sense.

2. Other Ginzburg–Landau models for ternary mixtures have been proposed in the literature, including two-order-parameter and three-order-parameter models, see [36]. In these models, a second scalar parameter ψ which describes the local concentration of amphiphile is introduced. Simple cases have been studied numerically by physicists but up to our knowledge the corresponding type of functional has not been treated mathematically.

7. Multi-transition connections

We consider again the case of a bi-stable equation. We already mentioned in Section 4.3 that when ± 1 are saddle-focus equilibria, the extended Fisher–Kolmogorov equation (4.3) has infinitely many solutions that can be classified according to their homotopy type. In this section, we focus on multi-transition solutions which correspond to solutions of type $g = (g_1, \dots, g_n)$, $n \in \mathbb{N}_0$, with $g_i = 2$ for all $i = 1, \dots, n$.

Considering again the functional

$$\mathcal{F}(u) = \int_{-\infty}^{+\infty} \left(\frac{1}{2}\bigl(u''^2 + g(u)u'^2\bigr) + f(u) \right) \mathrm{d}x \tag{7.1}$$

whose Euler–Lagrange equation is given by

$$u'''' - g(u)u'' - \frac{1}{2}g'(u)u'^2 + f'(u) = 0, \tag{7.2}$$

we assume that the function f is a positive symmetric double-well potential with bottoms at ± 1 and g is an even function which is not necessarily constant. Since f and g are even, arguing as in Section 4.2, we can restrict our attention to odd solutions. We thus look at the critical points of the functional

$$\mathcal{F}^+(u) = \int_{0}^{+\infty} \left(\frac{1}{2}\bigl(u''^2 + g(u)u'^2\bigr) + f(u) \right) \mathrm{d}x \tag{7.3}$$

in the space

$$\mathcal{E}^+ = \bigl\{ u \mid u - 1 \in H^2\bigl(\mathbb{R}^+\bigr), u(0) = 0 \bigr\}. \tag{7.4}$$

If $\mathcal{F}^+$ has a minimizer u then $u''(0) = 0$ and extending u on $\mathbb{R}$ by

$$u^*(x) = \begin{cases} -u(-x) & \text{if } x < 0, \\ u(x) & \text{if } x \geqslant 0 \end{cases} \tag{7.5}$$

we obtain an odd solution of (7.2).

We expect multi-transition solutions when the equilibria ± 1 are saddle-foci, i.e. when $g(1)^2 < 4f''(1)$. We obtain these solutions by odd extension of local minima of the functional $\mathcal{F}^+$ in appropriate subsets of $\mathcal{E}^+$. Basically, these subsets correspond to classes of functions having the desired number of transitions. We define for each $n \geqslant 0$, the subset $\mathcal{E}_n^+ \subset \mathcal{E}^+$ consisting of functions whose odd extensions on $\mathbb{R}$ make $2n + 1$ transitions. More precisely, a function $u \in \mathcal{E}^+$ belongs to the subclass $\mathcal{E}_n^+$ if there exist $0 = x_0 < x_1 < \cdots < x_n < x_{n+1} = \infty$ such that

$$u(x)(-1)^{i+n} > 0 \quad \text{for } x \in (x_i, x_{i+1}),$$

$$\max_{(x_i, x_{i+1})} u(x)(-1)^{i+n} > 1.$$

We prove that $\mathcal{F}^+$ has a local minimum in each of these subspaces in the following situation.

THEOREM 7.1. *Let f and $g \in C^2(\mathbb{R})$ satisfy* (C1), (C3) *and* (C4). *Assume moreover that $g(1)^2 < 4f''(1)$. Then, for every $n \in \mathbb{N}$, $\mathcal{F}^+$ has in each subspace $\mathcal{E}_n^+$, a local minimizer u_n whose odd extension on $\mathbb{R}$ is a heteroclinic solution of* (1.8) *having exactly* $2n+1$ *zeros.*

For a function $u \in \mathcal{E}_n^+$, we denote by I_i the intervals (x_i, x_{i+1}) for $i = 0, \dots, n$ where by convention $x_0 = 0$ and $x_{n+1} = \infty$. The main idea in the proof of Theorem 7.1 is to show the existence of a minimizing sequence $(u_p)_p \subset \mathcal{E}_n^+$ that has the following properties:

(a) there exists $I > 0$ such that for all u_p, $|I_i| \leqslant I$ for all $i = 0, \dots, n-1$;

(b) for all $\varepsilon > 0$, there exists $T > 0$ such that for all u_p, $|u_p(x) - 1| \leqslant \varepsilon$ for $x \geqslant T$.

These two properties are closely related as they prevent from a loss of compactness when extracting a weakly converging subsequence. We do not enter into the details of the proof of the existence of a minimizing sequence having properties (a) and (b). Indeed, the details are rather technical and do not bring new arguments with respect to the previous sections. The control on the length of the intervals I_i is obtained thanks to the clipping procedure discussed in Section 6.2 and the oscillatory behaviour of minimizers close to the equilibria ± 1 described in Section 6.1. The scheme of the arguments is as follows. Let us suppose to fix the ideas that u_p is positive on the interval I_i. As the assumptions of Lemma 5.1 hold, the second derivative and the potential are the significant terms in $\mathcal{F}^+$. An analysis similar to that of Section 5.1 then shows that the only way for the I_i's to grow to infinity is that the u_p's stay on growing intervals in a neighborhood of $+1$. But in this case, arguing as in Step 3 of the proof of Proposition 6.8, we can locally replace u_p by a minimizer close to $+1$ and clip this minimizer to obtain a function which stays close to $+1$ in an a priori bounded interval. Once the minimizing sequence satisfies (a), the property (b) follows from the arguments of Section 5.1.

We now turn to the proof of Theorem 7.1. The main difficulty is to ensure that the minimizing sequence does not loose or gain complexity in the limit.

PROOF OF THEOREM 7.1.

Step 1. *Convergence*. Let $(u_p)_p \subset \mathcal{E}_n^+$ be a minimizing sequence for $\mathcal{F}^+$ that satisfies the properties (a) and (b). Arguing as in the previous sections, we infer that u_p converges in C^1_{loc} to some function $u \in \mathcal{E}^+$ which is such that

$$\mathcal{F}^+(u) \leqslant \inf_{\mathcal{E}_n^+} \mathcal{F}^+.$$

We denote the extremities of the intervals $I_i^{u_p}$ by x_i^p, $i = 0, \dots, n$. It is clear by uniform convergence that up to a subsequence, for all $i = 1, \dots, n$, x_i^p converges to some $x_i < T$. Remember that by convention, we set $x_0^p = x_0 = 0$ and $x_{n+1}^p = x_{n+1} = \infty$. We call I_i the intervals (x_i, x_{i+1}), $i = 0, \dots, n$. We also deduce from the convergence in $C^1([0, T])$ and the convergence in $C^1_{\mathrm{loc}}([T, +\infty[)$ that

$$u(x)(-1)^{i+n} \geqslant 0 \quad \text{for } x \in (x_i, x_{i+1}),$$
$$\max_{(x_i, x_{i+1})} u(x)(-1)^{i+n} \geqslant 1.$$

Step 2. *Elimination of the zeros of u after x_n*. If u has zeros after x_n, we first modify it to keep only one of those zeros. So, suppose that u vanishes at least two times after x_n. We then define

$$a_1 := \min\{x > x_n \mid u(x) = 0\} \quad \text{and} \quad a_2 := \max\{x > x_n \mid u(x) = 0\}.$$

Since $u(a_1) = u(a_2) = u'(a_1) = u'(a_2) = 0$ by convergence in C^1_{loc}, the interval $[a_1, a_2]$ can be clipped out and the clip has only one zero after x_n. Of course this modification decreases the action.

Assume now that u vanishes at some point $\xi > x_n$. We then have $u'(\xi) = 0$. Now as $u(x_n) = u(\xi)$, there exists at least one critical point y between x_n and ξ such that $u(y) > 0$. Here, we have two possibilities, either y can be taken in such a way that $u(y) \leqslant 1$ or [0, 1] does not contain any critical value of $u_{|[x_n,\xi]}$.

Suppose first that we can find $y \in (x_n, \xi)$ such that $0 < u(y) \leqslant 1$ and $u'(y) = 0$. As $u \in \mathcal{E}^+$,

$$\lim_{x\to\infty} \big(u(x), u'(x)\big) = (1, 0).$$

Hence, by an argument similar to that of the proof of Lemma 6.4, we see that $u(x)$ oscillates around 1 for x large enough. Therefore, we can clip out an interval containing $[y, \xi]$ in such a way that the clip of u does not vanish after x_n.

In the second case, we can find $y \in (x_n, \xi)$ such that $u(y) > 1$ and if $x \in (x_n, \xi)$ satisfies $u'(x) = 0$, then $u(x) > 1$. We now define $v \in \mathcal{E}^+$ by

$$v(x) := \begin{cases} -u(x + x_1) & \text{if } 0 \leqslant x \leqslant \xi - x_1, \\ u(x + x_1) & \text{if } x > \xi - x_1. \end{cases}$$

Observe that since $\min_{[x_n - x_1, \xi - x_1]} v(x) < -1$ and v is negative in $(x_n - x_1, \xi - x_1)$, v has the right number of transitions. Also, v does not vanish after $\xi - x_1$. On the other hand, we deduce from Lemma 5.1 that

$$\int_{x_0}^{x_1} \left(\frac{1}{2}\big(u''^2 + g(u)u'^2\big) + f(u)\right) \mathrm{d}x \geqslant 0$$

so that $\mathcal{F}^+(v) \leqslant \mathcal{F}^+(u)$.

Step 3. *Elimination of the zeros of v in the bumps*. We still denote by $0 = x_0 < x_1 < \cdots < x_n$, the extremities of the intervals I_i associated to v (actually, these are the intervals I_i which have possibly been translated in Step 2). Suppose that there exists $\xi \in v^{-1}(0)$ so that $\xi \neq x_i$ for any $i = 0, \ldots, n$. Hence, ξ lies in the interior of an interval I_i. To fix the ideas, we assume that v is nonnegative therein and denoting by $v(\bar{x}_i)$ the maximum of v over this interval we assume that ξ is at the left of $\bar{x}_i$. Next, define $\xi_1 = \min\{x > x_i \mid v(x) = 0\}$ and $\xi_2 = \max\{x \in [\xi_1, \bar{x}_i] \mid v(x) = 0\}$. It is easily seen that an interval containing $[\xi_1, \xi_2]$ can be clipped out so that the zeros can be deleted.

Step 4. *Elimination of the tangencies with* ± 1. The last condition that we have to check to be sure that $v \in \mathcal{E}_n^+$ is that

$$\max_{x\in I_i}\left|v(x)\right| > 1$$

for all $i = 0, \dots, n$. Assume that this condition fails to be true in one of the intervals I_i. In this interval, we thus have $\max_{x\in I_i} |v(x)| = 1$. Let $\tau \in I_i$ be such that $|v(\tau)| = 1$ and $v'(\tau) = 0$. To fix the ideas, assume that $v(\tau) = 1$, the second case being treated in the same way. Let τ_0 and δ_0 be given by Lemma 6.4. As the action of the function 1 is zero, we can modify v without increasing its action by stretching the point τ to an interval of arbitrary length and gluing the function 1 to both extremities, see [42]. Now, we take a_1, (respectively a_2) at the left (respectively at the right) of τ in such a way that $0 < \max_{i=1,2} \|(v(a_i) - 1, v'(a_i))\| \leqslant \delta_0$ and stretch τ to an interval of length τ_0. We still call v the function obtained after gluing 1 at τ and $\tau + \tau_0$. It follows from Lemma 6.4 that the minimizers of

$$\int_{a_1}^{a_2+\tau_0}\left(\frac{1}{2}\left(u''^2 + g(u)u'^2\right) + f(u)\right)\mathrm{d}x$$

on the set of functions $u \in H^2(a_1, \tau_0 + a_2)$ that satisfy $u(a_1) = v(a_1)$, $u'(a_1) = v'(a_1)$, $u(a_2 + \tau_0) = v(a_2)$ and $u'(a_2 + \tau_0) = v'(a_2)$ oscillate around 1. If we replace v locally by a minimizer, we obtain a new function w such that

$$\max_{x\in I_i}\left|w(x)\right| > 1$$

and $\mathcal{F}^+(w) \leqslant \mathcal{F}^+(v)$.

Conclusion. It follows from the previous steps that we can construct $w \in \mathcal{E}_n^+$ such that $\mathcal{F}^+(w) \leqslant \mathcal{F}^+(u)$. Consequently, we have $\mathcal{F}^+(w) = \inf_{\mathcal{E}_n} \mathcal{F}^+$. Now observe that for all $h \in H^2(\mathbb{R}^+)$ such that $h(0) = 0$, $\mathcal{F}^+(w) \leqslant \mathcal{F}^+(w + th)$ for t sufficiently small. Indeed, assume that there exists a sequence $(t_n)_n$ tending to 0 such that $\mathcal{F}^+(w) > \mathcal{F}^+(w + t_n h)$. If w is in the interior of the class $\mathcal{E}_n^+$, this is obviously a contradiction. In the case where w is on the boundary of $\mathcal{E}_n^+$ i.e. if for some points x_i, $w(x_i) = w'(x_i) = 0$, then even for t small, $w + th$ can have more than one zero close to the x_i's so that it does not belong necessarily to $\mathcal{E}_n^+$. However for t small enough, $w + th$ has the right number of transitions and the oscillations close to the points x_i can be erased using the clipping procedure. Therefore, for n large enough, modifying $w + t_n h$ close to the x_i's if necessary, we obtain a function in $\mathcal{E}_n^+$ whose action is strictly smaller than $\mathcal{F}^+(w)$. This contradicts the definition of w and therefore w is a critical point of $\mathcal{F}^+$ in $\mathcal{E}_n^+$. Finally, we infer from the conservation of the Hamiltonian

$$H\left(u, u', u'', u'''\right) = u'''u' - \frac{1}{2}u''^2 - \frac{1}{2}g(u)u'^2 + f(u)$$

that each minimizer is actually in the interior of $\mathcal{E}_n^+$, i.e. each crossing with zero is transverse. □

7.1. *Notes and further comments*

1. Theorem 7.1 can be seen as a partial extension of the results of [42] to Lagrangians that can take either signs. To deal with such Lagrangians, we only require an a priori lower bound on the action along admissible functions. In Theorem 7.1, this lower bound follows from assumption (C1). The conclusion of Theorem 7.1 holds also with the settings of Theorem 5.7, see [18]. In this case we explicitly assume that the functional is bounded from below.

2. Multi-transition solutions can also be obtained in the case of multi-well potentials. We can for example consider the framework of Section 6 assuming then that each equilibrium is of saddle-focus type.

3. The abundance of local minima suggests the existence of infinitely many other critical points of minimax type. However the lack of a compactness property of the Palais–Smale sequences makes the investigation of such solution very complicate. It would seem rather natural that a solution obtained from a minimax principle based on deformations from one local minimum to another one, behaves like the single-transition solutions obtained in Theorem 4.4 for $\beta \geqslant 0$. Even for this range of β, the variational nature of the single-transition solutions is unknown. D. Smets and J.B. van den Berg have used in [83] a modified functional and mountain-pass arguments to catch a homoclinic solution of the Swift–Hohenberg equation for almost every $\beta \in]-\sqrt{8}, 0[$.

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CHAPTER 3

A Qualitative Analysis, via Lower and Upper Solutions, of First Order Periodic Evolutionary Equations with Lack of Uniqueness

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Preface

The method of lower and upper solutions is an elementary but powerful tool in the existence theory of initial and periodic problems for semilinear differential equations for which a maximum principle holds, even in cases where no special structure is assumed on the nonlinearity (see, e.g., [150,118,112,39,41,117,98]). The aim of this work is to show that this method is also quite effective for investigating the qualitative properties of solutions, at the same extent of generality for which the existence theory is developed. Indeed, our main purpose is to work out, under a minimal set of basic assumptions, a qualitative theory for two sample classes of scalar periodic differential equations: the first order ordinary differential equation

$$u' = f(t, u) \tag{1}$$

and the second order parabolic problem

$$\begin{aligned} &\partial_t u + A(x, t, \partial_x)u = f(x, t, u, \nabla_x u) && \text{in } \Omega \times I, \\ &u = 0 && \text{on } \partial\Omega \times I. \end{aligned} \tag{2}$$

Here $I \subseteq \mathbb{R}$ is an interval, $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $\partial_t u + A(x, t, \partial_x)u$ is a uniformly parabolic differential operator, with time-periodic coefficients, and the functions f are time-periodic and satisfy suitable Carathéodory conditions. We must stress that under these assumptions neither uniqueness for the initial value problem associated with (1) and (2), nor validity of comparison principles are guaranteed. Accordingly, the qualitative analysis of these equations cannot be performed applying the theory of order preserving dynamical systems as given, for instance, in [60]. In this general setting we will actually ascertain, already for the easy-looking equation (1) and unlike the regular case, the occurrence of very complicated dynamics, so that even the explicit knowledge of a distinguished class of solutions (such as equilibria, or periodic solutions) is generally not sufficient to get a global qualitative portrait.

We are especially interested here in studying, with the aid of lower and upper solutions, the following three basic questions about (1) and (2):

(i) existence of periodic solutions of (1), or (2), and their localization;
(ii) qualitative properties of periodic solutions of (1), or (2), with special reference to their stability or instability;
(iii) asymptotic behaviour of solutions of the initial value problem associated with (1), or (2).

This program is pursued in the first part of this work for (1) and in the second part for (2); however, as it is quite predictable, exhaustive results can be obtained only for (1). Problems (1) and (2) share relevant similarities from the qualitative point of view. In fact Eq. (2) looks like a natural extension of (1), which in turn represents a simple, but nontrivial, paradigm of the patterns that can be possibly displayed by the dynamics of (2). Nevertheless, they exhibit several differences, which appear already evident developing the existence theory for the initial and the periodic problems in the presence of lower and upper solutions. Hence the two equations require separate treatments. We chose to describe here just basic results and to select a small number of examples or applications; further material can be found in [112] for (1) and in [42] for (2).

Although the method of lower and upper solutions applies to a larger class of boundary value problems, we decided to restrict our attention to the above two ones only. In particular, a remarkable topic which is completely left out here concerns the second order periodic ordinary differential equation

$$u'' = f\big(t, u, u'\big) \quad \text{on } I. \tag{3}$$

In this context a theory of the stability of periodic solutions, exploiting lower and upper solutions, started in [72] and has been recently developed in [116,36,111,109]. This omission is however not only dictated by the necessity of keeping our work within a reasonable length, but it is also motivated by the fact that the study of (3) relies on somewhat different techniques from the ones we use here; further, more regularity is required on f and at the moment the results are less complete.

1. First Order Periodic ODEs

1.1. *Introduction*

In this part we consider the first order scalar ordinary differential equation

$$x' = f(t, x). \tag{1.1}$$

We assume that $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is periodic in t with period $T > 0$ and satisfies the L^1-Carathéodory conditions. Associated with (1.1), we consider the Cauchy problem

$$x' = f(t, x), \quad x(t_0) = x_0, \tag{1.2}$$

where $t_0, x_0 \in \mathbb{R}$ are given, and the periodic problem on $[0, T]$

$$x' = f(t, x), \quad x(0) = x(T). \tag{1.3}$$

Solutions of (1.1), (1.2), (1.3) are intended in the Carathéodory, i.e. $W^{1,1}_{\text{loc}}$-, sense. Since any solution of (1.3) on $[0, T]$ extends to a T-periodic solution of (1.1), it will also be referred to as T-periodic.

Existence, multiplicity, localization and qualitative properties of solutions of (1.3) have been investigated using various approaches (see, e.g., [91,122,147,99,95,148,94,108,28, 142,113] and the included references). In particular, when f is a polynomial in x with coefficients T-periodic in t, the interest of the results that have been obtained stems from their connections with Hilbert Sixteenth Problem about the number of limit cycles of autonomous plane polynomial systems (see, e.g., [87,88,85,135,4,62,63,24]), or from their applications to the study of single species population models in a periodically fluctuating environment (see, e.g., [74,131,31,30,43,129,132,14,96,146,19,97]).

The knowledge of the T-periodic solutions of (1.1) is also the basic step in the realization of the Poincaré program, that is the determination of the qualitative portrait of all solutions of (1.1). If the property of uniqueness of solutions holds for the Cauchy problem (1.2), the behaviour of the solutions of scalar T-periodic differential equations such as (1.1) is, at least conceptually, nearly as simple as the behaviour of autonomous equations. Indeed, in this case the dynamics can be reduced to that of the Poincaré map, which assigns to every initial datum the value of the solution after one period, and, since this is strictly monotone, the orbit structure can be classified in an elementary way (cf. [57, Chapter 4]). In the Carathéodory setting, although generic (cf. [121, Chapter 3]), uniqueness in the future, or in the past, for the solutions of (1.2) is of course not anymore guaranteed. This fact yields evident technical complications also due to the possible failure of comparison principles for the solutions of (1.1). On the other hand, the lack of uniqueness entails, as we shall see, a rich variety of dynamics, such as the possible occurrence of homoclinic, subharmonic and almost periodic solutions. In addition to these theoretical features, that may justify by themselves a close analysis, further motivations for studying (1.1) in the absence of uniqueness come from some models in population dynamics related to those proposed by G.F. Gause in [53, Chapter VI].

The main purpose in this part of our work is to show that a quite satisfactory description of the dynamics of (1.1) can be performed in the Carathéodory setting by a systematic use of lower and upper solutions, i.e. functions satisfying suitable differential inequalities. The lower and upper solutions method is a classical tool for the study of (1.2) and (1.3) and it has already been employed by several authors since the pioneering work of G. Peano [119] (see [120,102,143,101,150,79,2,94,110,38,52,89,28] and the included references). Yet, while this approach has been largely used in the discussion of existence, multiplicity and approximation by monotone iteration, the stability properties of the solutions of (1.3) have been generally detected by other techniques, often based on linearization (see the recent papers [10,11,20,21] and the references therein). Partial exceptions are [95,96,133], where a global qualitative portrait of (1.1) was obtained, but only for a restricted class of functions f satisfying some convexity assumptions with respect to the x-variable, which in turn imply uniqueness of the solutions of (1.2). Of course, these methods do not apply to the general context we wish to discuss here, where Lipschitz or differentiability conditions on f are not required. Our work is therefore in the spirit of [147] and [112], whose declared aim was to pave the way towards a theory of periodic solutions of Eq. (1.1) when no special structure is considered.

This part is organized as follows. In Section 1.2 we formalize some definitions concerning the first order scalar ordinary differential equation (1.1) as well as we settle some notations.

In Section 1.3 we collect some basic facts concerning the solvability in the Carathéodory setting of the Cauchy problem (1.2), when a pair of possibly discontinuous lower and upper solutions is given and no special ordering between them is assumed. Existence and localization of solutions, as well as the structure of the solution set, are discussed in this frame.

In Section 1.4 we initiate the qualitative study of solutions of (1.1). We first discuss a form of monotonicity over periods, named T-monotonicity, which, unlike the case where uniqueness holds for the Cauchy problem (1.2), is not shared by all solutions of (1.1). We prove, however, that T-monotonicity is enjoyed by every solution of (1.1) that is comparable with any possible solution of (1.3). This property represents an essential tool for the description of the asymptotic behaviour of solutions of (1.1). In particular, it allows to obtain an extension of the existence part of the classical Massera Convergence Theorem [91] to the case where uniqueness for (1.2) fails; that is, the existence of a bounded solution x of (1.1) defined on an unbounded interval implies the existence of a T-periodic solution u of (1.1). A different proof of this fact is given in [147], while a discussion of various extensions of the Massera Theorem can be found in [104]. Yet, in the absence of uniqueness for (1.2), x does not generally converge to u: an example where the convergence does not take place is explicitly produced. Afterwards, we prove a technical result which describes the dynamics of solutions stemming from a lower or an upper solution of (1.3). This result plays a central role in the subsequent study of the stability of T-periodic solutions; it resembles the Monotone Convergence Criterion in the theory of monotone maps and may have an independent interest, since it guarantees the convergence to a T-periodic solution of a bounded solution emanating from and lying above a lower solution or emanating from and lying below an upper solution of (1.3). Founded on these results, we prove the main existence and localization theorem for (1.2), under the assumption of the existence of a pair of possibly discontinuous and unordered lower and upper solutions. We also point out by an example how the type of discontinuous lower and upper solutions of (1.3) here considered can be successfully used, in some cases, to get precise information on the localization of branches of T-periodic solutions.

Section 1.5 deals with the description of the dynamics of solutions of (1.1), which lie between two strictly ordered T-periodic solutions, or above the maximum T-periodic solution, or below the minimum one, if they exist. In particular, we show the existence of heteroclinic solutions connecting pairs of ordered T-periodic solutions of (1.1), when there is no further T-periodic solution in between. These results are based on the T-monotonicity of solutions of (1.1), which are comparable with any possible solution of (1.3), and they yield information about the stability and instability of solutions of (1.3), also with reference to the existence of repulsivity and attractivity basins.

Section 1.6, which represents the core of this part, is devoted to the study of the stability properties of the T-periodic solutions of (1.1) by means of lower and upper solutions. We begin with a discussion of the mutual relations of three notions of one-sided stability of a T-periodic solution we believe appropriate to be considered when uniqueness for (1.2) may fail: Lyapunov stability, order stability and weak stability. The second one is standard in the frame of order preserving discrete-time semidynamical systems (cf. [58, 60]). The third one has been used in the context of multivalued semiflows (cf. [130,50]) and is related to the notion of weak invariance (cf. [105,152,140]). If uniqueness in the future, or in the past, holds for (1.2), then Lyapunov stability and order stability are equivalent concepts. Whereas, we show here that order stability and weak stability are always equivalent: thus weak stability yields an alternative and more direct interpretation of order stability, in terms of the dynamics of solutions of (1.1). Using these three notions we can describe in a precise way the stability properties of a T-periodic solution, relating them to the existence of a lower or an upper solution close to it. Hence, when a pair of lower and upper solutions is given, we can discuss the stability or instability of the minimum and the maximum T-periodic solutions v and w wedged between them. Afterwards, we turn to investigate the behaviour of the solutions lying between v and w. If the lower and the upper solutions satisfy the standard ordering condition, i.e. the lower solution is below the upper solution, we find between v and w a weakly stable T-periodic solution; whereas, in the complementary case, Lyapunov unstable T-periodic solutions always exist. We further study the existence and the properties of nondegenerate continua of T-periodic solutions in $C^0([0,T])$. In particular, we see that if two T-periodic solutions are not strictly ordered, they give rise to a nondegenerate continuum of unstable T-periodic solutions. This fact may occur in the presence of a pair of unordered lower and upper solutions of (1.3). On the contrary, the existence of a pair of lower and upper solutions of (1.3) satisfying the standard ordering condition always guarantees the existence of a totally ordered continuum of weakly stable solutions. More generally, we investigate the topological structure of the set of all solutions of (1.3). This set is the union of a family of mutually ordered connected components. In the absence of uniqueness for (1.2) we show that such components may have arbitrarily large (Čech–Lebesgue) dimension. Moreover, when a continuum $\mathcal{K}$ is generated by two T-periodic solutions which are not strictly ordered, the dynamics of the solutions of (1.1), which lie between the minimum and the maximum elements in $\mathcal{K}$, is fairly complicated. Loosely speaking, any discrete, even chaotic, dynamics can be realized (see [114]). In particular, we can find homoclinics, subharmonic solutions of any order and almost periodic solutions which are not periodic of any period. As it is well known [91], this phenomenon cannot take place if the uniqueness property holds for (1.2).

In Section 1.7 we revisit the Massera Convergence Theorem. We have already noticed in Section 1.4 that, even when uniqueness for (1.2) fails, the existence of a solution u of (1.3) is always guaranteed in the presence of a bounded solution x of (1.1) defined on an unbounded interval, in spite of the fact that x does not generally converge to u. In this section we make this statement more precise, providing a complete extension of the Massera Convergence Theorem to the case where the uniqueness assumption for solutions of (1.2) is dropped. Namely, we see that either x converges to u, or the set of all solutions u of (1.3), whose range contains ω-limit points of x, forms a nondegenerate closed connected set in $C^0([0,T])$.

1.2. *Preliminaries*

In this section we formalize some basic definitions concerning the first order scalar ordinary differential equation (1.1). Let $T > 0$ be a fixed number. The following condition is assumed:

(C) $f :]0,T[\times\mathbb{R} \to \mathbb{R}$ satisfies the L^1-Carathéodory conditions, i.e. for every $x \in \mathbb{R}$, $f(\cdot,x)$ is measurable on $]0,T[$; for a.e. $t \in]0,T[$, $f(t,\cdot)$ is continuous on $\mathbb{R}$; for each $\rho > 0$, there exists $\gamma \in L^1(0,T)$ such that $|f(t,x)| \leqslant \gamma(t)$, for a.e. $t \in]0,T[$ and every $x \in [-\rho,\rho]$.

REMARK 1.1. The function f is identified with its T-periodic extension onto $\mathbb{R}\times\mathbb{R}$.

Solutions and lower and upper solutions

DEFINITION 1.1.

- A *solution of* (1.1) *on* $[t_1, t_2]$, with $t_1, t_2 \in \mathbb{R}$ and $t_1 < t_2$, is a function $x \in W^{1,1}(t_1, t_2)$ such that

$$x'(t) = f\big(t, x(t)\big) \quad \text{a.e. on }]t_1, t_2[.$$

- A *solution of* (1.1) *on* J, where J is a noncompact interval having endpoints t_1, t_2, with $-\infty \leqslant t_1 < t_2 \leqslant +\infty$, is a function x such that, for every compact interval $K \subset J$, $u|_K$ is a solution of (1.1) on K.

DEFINITION 1.2. Let $I \subseteq \mathbb{R}$ be an interval having endpoints ω_-, ω_+, with $-\infty \leqslant \omega_- < \omega_+ \leqslant +\infty$. A solution $x : I \to \mathbb{R}$ of (1.1) is said *right-nonextendible* if either $\omega_+ = +\infty$, or $\omega_+ < +\infty$ and there is no solution $\hat{x}$ of (1.1) on $I \cup \{\omega_+\}$ such that $\hat{x}|_I = x$. Similarly, a solution $x : I \to \mathbb{R}$ of (1.1) is said *left-nonextendible* if either $\omega_- = -\infty$, or $\omega_- > -\infty$ and there is no solution $\hat{x}$ of (1.1) on $I \cup \{\omega_-\}$ such that $\hat{x}|_I = x$. A solution $x :]\omega_-, \omega_+[\to \mathbb{R}$ of (1.1) is said *nonextendible* if it is both right-nonextendible and left-nonextendible.

We now introduce a notion of possibly discontinuous lower and upper solutions of (1.1), which have been extensively used in [38,52,86,28,112].

NOTATION 1.3. For a function $x : [t_1, t_2] \to \mathbb{R}$, we write $x|_{]t_1,t_2[} \in W^{1,1}(t_1, t_2)$ if there exists $\widetilde{x} \in W^{1,1}(t_1, t_2) \cap C^0([t_1, t_2])$ such that $\widetilde{x} = x$ on $]t_1, t_2[$.

DEFINITION 1.4.

- A *regular lower solution of* (1.1) *on* $[t_1, t_2]$, with $t_1, t_2 \in \mathbb{R}$ and $t_1 < t_2$, is a function $\alpha : [t_1, t_2] \to \mathbb{R}$, such that $\alpha|_{]t_1,t_2[} \in W^{1,1}(t_1, t_2)$,

$$\alpha'(t) \leqslant f\big(t, \alpha(t)\big) \quad \text{a.e. on } [t_1, t_2],$$

$$\lim_{t \to t_1^+} \alpha(t) \leqslant \alpha(t_1), \qquad \alpha(t_2) \leqslant \lim_{t \to t_2^-} \alpha(t).$$

- A *lower solution of* (1.1) *on* $[t_1, t_2]$ is a function $\alpha : [t_1, t_2] \to \mathbb{R}$ for which there are points $t_1 = \tau_0 < \tau_1 < \cdots < \tau_N = t_2$ such that, for $i = 0, \ldots, N-1$, $\alpha|_{[\tau_i, \tau_{i+1}]}$ is a regular lower solution on $[\tau_i, \tau_{i+1}]$.
- A *lower solution* (respectively a *regular lower solution*) *of* (1.1) *on* J, where J is a noncompact interval having endpoints t_1, t_2 with $-\infty \leqslant t_1 < t_2 \leqslant +\infty$, is a function α such that, for any compact interval $K \subset J$, $\alpha|_K$ is a lower solution (respectively a regular lower solution) of (1.1) on K.
- A *regular upper solution of* (1.1) *on* $[t_1, t_2]$, is a function $\beta : [t_1, t_2] \to \mathbb{R}$, such that $\beta|_{]t_1,t_2[} \in W^{1,1}(t_1, t_2)$,

$$\beta'(t) \geqslant f\big(t, \beta(t)\big) \quad \text{a.e. on } [t_1, t_2],$$

$$\lim_{t \to t_1^+} \beta(t) \geqslant \beta(t_1), \qquad \beta(t_2) \geqslant \lim_{t \to t_2^-} \beta(t).$$

- An *upper solution of* (1.1) *on* $[t_1, t_2]$ is a function $\beta : [t_1, t_2] \to \mathbb{R}$ for which there are points $t_1 = \sigma_0 < \sigma_1 < \cdots < \sigma_N = t_2$ such that, for $i = 0, \ldots, N-1$, $\beta|_{[\sigma_i, \sigma_{i+1}]}$ is a regular upper solution on $[\sigma_i, \sigma_{i+1}]$.
- An *upper solution* (respectively a *regular upper solution*) *of* (1.1) *on* J, where J is a noncompact interval having endpoints t_1, t_2 with $-\infty \leqslant t_1 < t_2 \leqslant +\infty$, is a function β such that, for any compact interval $K \subset J$, $\beta|_K$ is an upper solution (respectively a regular upper solution) of (1.1) on K.

REMARK 1.2. Notice that a regular lower solution α is (absolutely) continuous on $]t_1, t_2[$, but it may be discontinuous at the endpoints of the interval $[t_1, t_2]$. A similar observation holds for a regular upper solution.

Orderings

DEFINITION 1.5. Given functions $x, y : I \to \mathbb{R}$, with $I \subseteq \mathbb{R}$ an interval, we write

- $x \leqslant y$ if $x(t) \leqslant y(t)$ on I;
- $x < y$ if $x \leqslant y$ and $x \neq y$;
- $x \ll y$ if $\inf(y - x)|_K > 0$ for any compact interval $K \subseteq I$.

DEFINITION 1.6. Given functions $x, y : I \to \mathbb{R}$, with $I \subseteq \mathbb{R}$ an interval and $x < y$, we define the *order intervals*

$$[x, y] = \{z : I \to \mathbb{R} \mid x \leqslant z \leqslant y\},$$

$$[x, +\infty[= \{z : I \to \mathbb{R} \mid x \leqslant z\},$$

$$]-\infty, y] = \{z : I \to \mathbb{R} \mid z \leqslant y\}.$$

DEFINITION 1.7. Let $\mathcal{S}$ be a given set of solutions of (1.1).

- We say that a solution z of (1.1), with $z \in \mathcal{S}$, is a *maximal solution of* (1.1) *in* $\mathcal{S}$ (respectively a *minimal solution of* (1.1) *in* $\mathcal{S}$) if there is no solution x of (1.1), with $x \in \mathcal{S}$, such that $x > z$ (respectively $x < z$).
- We say that a solution z of (1.1), with $z \in \mathcal{S}$, is the *maximum solution of* (1.1) *in* $\mathcal{S}$ (respectively the *minimum solution of* (1.1) *in* $\mathcal{S}$) if every solution x of (1.1), with $x \in \mathcal{S}$, is such that $x \leqslant z$ (respectively $x \geqslant z$).

1.3. *The initial value problem*

In this section we collect some basic facts concerning the solvability of the Cauchy problem (1.2), when a pair of possibly discontinuous lower and upper solutions is given and no special ordering between them is assumed. Existence and localization of solutions, as well as the structure of the solution set, are discussed in this setting.

REMARK 1.3. Since the study of a terminal value problem may be reduced to the study of an initial value problem by reversing time, we shall mainly concentrate on the latter problem. Occasionally we shall need to refer to a terminal value problem; in that case we shall freely use definitions, notations and results about it without explicit references.

Solutions and lower and upper solutions

DEFINITION 1.8. A *solution of the initial value problem* (1.2) on I, with $I \subseteq \mathbb{R}$ an interval containing a right neighbourhood of t_0, is a solution x of (1.1) on I such that $x(t_0) = x_0$.

DEFINITION 1.9.

- A *lower solution* (respectively a *regular lower solution*) of (1.2) on I, with $I \subseteq \mathbb{R}$ an interval containing a right neighbourhood of t_0, is a function $\alpha : I \to \mathbb{R}$, which is a lower solution (respectively a regular lower solution) of (1.1) on I and satisfies $\alpha(t_0) \leqslant x_0$.
- An *upper solution* (respectively a *regular upper solution*) of (1.2) on I, with $I \subseteq \mathbb{R}$ an interval containing a right neighbourhood of t_0, is a function $\beta : I \to \mathbb{R}$, which is an upper solution (respectively a regular upper solution) of (1.1) on I and satisfies $\beta(t_0) \geqslant x_0$.

The following simple observation concerning pairs of unordered lower and upper solutions will be used in the sequel.

PROPOSITION 1.1. *Let α be a lower solution and β be an upper solution of* (1.2) *on* $[t_0, t_1]$ *with $\alpha \not\leqslant \beta$. Then there exists $\hat{t} \in [t_0, t_1]$ such that $\alpha(\hat{t}) = \beta(\hat{t})$.*

PROOF. Let us set $\gamma = \beta - \alpha$. According to Definition 1.9, γ satisfies the following condition: there are points $t_0 = \rho_0 < \rho_1 < \cdots < \rho_K = t_1$ such that

- for $i = 0, \ldots, K-1$, $\gamma|_{]\rho_i, \rho_{i+1}[} \in W^{1,1}(\rho_i, \rho_{i+1})$,
- for $i = 1, \ldots, K-1$, $\lim_{t \to \rho_i^-} \gamma(t) \leqslant \gamma(\rho_i) \leqslant \lim_{t \to \rho_i^+} \gamma(t)$,
- $\lim_{t \to t_0^+} \gamma(t) \geqslant \gamma(t_0) \geqslant 0$ and $\gamma(t_1) \geqslant \lim_{t \to t_1^-} \gamma(t)$.

We may assume $\gamma(t_0) > 0$. Since $\alpha \not\leqslant \beta$, there is a point $\xi \in]t_0, t_1]$ such that $\gamma(\xi) < 0$. Setting $\hat{t} = \sup\{t \in]t_0, \xi[\mid \gamma(s) > 0 \text{ on } [t_0, t]\}$, the properties of γ imply that $\gamma(\hat{t}) = 0$. □

Existence of solutions and the Hukuhara–Kneser property

We start with a basic existence and localization result for solutions of (1.1).

LEMMA 1.2. *Assume* (C). *Let $\alpha, \beta \in W^{1,1}(t_0, t_1)$ satisfy*

$$\alpha'(t) \leqslant f\big(t, \alpha(t)\big) \quad \text{and} \quad \beta'(t) \geqslant f\big(t, \beta(t)\big) \quad \text{a.e. on } [t_0, t_1]. \tag{1.4}$$

- *If $\alpha(t_0) \leqslant \beta(t_0)$, then for every $x_1 \in [\alpha(t_0), \beta(t_0)]$ there exists a solution $x : [t_0, t_1] \to \mathbb{R}$ of the initial value problem*

$$x' = f(t, x), \quad x(t_0) = x_1, \tag{1.5}$$

satisfying

$$\min\{\alpha, \beta\} \leqslant x \leqslant \max\{\alpha, \beta\} \quad \textit{on } [t_0, t_1]. \tag{1.6}$$

- *If* $\alpha(t_1) \geqslant \beta(t_1)$, *then for every* $x_2 \in [\beta(t_1), \alpha(t_1)]$ *there exists a solution* $x : [t_0, t_1] \to \mathbb{R}$ *of the terminal value problem*

$$x' = f(t, x), \quad x(t_1) = x_2, \tag{1.7}$$

satisfying (1.6).
- *If* $\alpha(t_0) \leqslant \beta(t_0)$ *and* $\alpha(t_1) \geqslant \beta(t_1)$, *then for every* $x_1 \in [\alpha(t_0), \beta(t_0)]$ *and* $x_2 \in [\beta(t_1), \alpha(t_1)]$ *there exists a solution* $x : [t_0, t_1] \to \mathbb{R}$ *of both* (1.5) *and* (1.7) *satisfying* (1.6).

PROOF. We first consider the case where α and β are ordered.

CLAIM 1. *Let* $\alpha, \beta \in W^{1,1}(t_0, t_1)$ *satisfy* (1.4). *Assume that* $\alpha \leqslant \beta$ *on* $[t_0, t_1]$. *Then for each* $x_1 \in [\alpha(t_0), \beta(t_0)]$ *there exists a solution* $x : [t_0, t_1] \to \mathbb{R}$ *of* (1.5) *with* $\alpha \leqslant x \leqslant \beta$ *on* $[t_0, t_1]$.

Set $\gamma(t, x) = \max\{\alpha(t), \min\{x, \beta(t)\}\}$, $\bar{f}(t, x) = f(t, \gamma(t, x))$, and consider the problem

$$x' = \bar{f}(t, x), \quad x(t_0) = x_1. \tag{1.8}$$

Let $S : C^0([t_0, t_1]) \to C^0([t_0, t_1])$ be the Volterra operator defined by $S(x)(t) = x_1 + \int_{t_0}^{t} \bar{f}(s, x(s))\, ds$. The operator S is continuous and has a relatively compact range. Hence, the existence of a solution x of (1.8) defined on $[t_0, t_1]$ follows from Schauder Theorem. We claim that $\alpha \leqslant x \leqslant \beta$ on $[t_0, t_1]$. Indeed, assuming that there are points s_1, s_2, with $t_0 \leqslant s_1 < s_2 \leqslant t_1$, such that $x(s_1) = \alpha(s_1)$ and $x(s) < \alpha(s)$ on $]s_1, s_2]$, yields $x'(s) - \alpha'(s) \geqslant 0$ a.e. on $]s_1, s_2]$, which leads to a contradiction. Therefore we have $\alpha \leqslant x$. Similarly we obtain $x \leqslant \beta$. Consequently x is a solution of (1.5) as well and the claim is proved.

CLAIM 2. *Let* $\alpha, \beta \in W^{1,1}(t_0, t_1)$ *satisfy* (1.4). *Assume that* $\alpha \geqslant \beta$ *on* $[t_0, t_1]$. *Then for each* $x_2 \in [\beta(t_1), \alpha(t_1)]$ *there exists a solution* $x : [t_0, t_1] \to \mathbb{R}$ *of* (1.7) *satisfying* $\alpha \geqslant x \geqslant \beta$ *on* $[t_0, t_1]$.

Let us consider the problem obtained from (1.7) by reversing time; namely, set $\hat{f}(t, x) = -f(t_0 + t_1 - t, x)$, $\hat{\alpha}(t) = \beta(t_0 + t_1 - t)$, $\hat{\beta}(t) = \alpha(t_0 + t_1 - t)$ and observe that $\hat{\alpha}'(t) \leqslant \hat{f}(t, \hat{\alpha}(t))$, $\hat{\beta}'(t) \geqslant \hat{f}(t, \hat{\beta}(t))$, a.e. on $[t_0, t_1]$, and that $\hat{\alpha}(t_0) \leqslant \hat{\beta}(t_0)$. By Claim 1 there exists a solution $\hat{x} : [t_0, t_1] \to \mathbb{R}$ of

$$x' = \hat{f}(t, x), \quad x(t_0) = x_2,$$

satisfying $\hat{\alpha} \leqslant \hat{x} \leqslant \hat{\beta}$ on $[t_0, t_1]$. The function $x(t) = \hat{x}(t_0 + t_1 - t)$ is then a solution of (1.7) satisfying $\alpha \geqslant x \geqslant \beta$ on $[t_0, t_1]$.

We now prove the three statements of this lemma.

Case 1: $\alpha(t_0) \leqslant \beta(t_0)$. Pick any $x_1 \in [\alpha(t_0), \beta(t_0)]$. Set $A = \{t \in [t_0, t_1] \mid \alpha(t) \leqslant \beta(t)\}$ and $B = [t_0, t_1] \setminus A$. We can write A as union of countably many disjoint closed intervals (possibly reduced to points) $[a_i, b_i]$, with $a_i \leqslant b_i$ and $a_0 = t_0$, and B as union of countably many disjoint open intervals $]c_j, d_j[$, with $c_j < d_j$. If $a_0 < b_0$, by Claim 1 we can find a solution x_0 of

$$x' = f(t, x), \quad x(t_0) = x_1,$$

satisfying $\alpha(t) \leqslant x_0(t) \leqslant \beta(t)$ on $[t_0, b_0]$ and, for every $i \geqslant 1$ such that $a_i < b_i$, a solution $x_i : [a_i, b_i] \to \mathbb{R}$ of

$$x' = f(t, x), \quad x(a_i) = \alpha(a_i),$$

satisfying $\alpha(t) \leqslant x_i(t) \leqslant \beta(t)$ on $[a_i, b_i]$. Notice that $x_i(b_i) = \alpha(b_i) = \beta(b_i)$ whenever $b_i \neq t_1$. For every j, by Claim 2 we can find a solution $y_j : [c_j, d_j] \to \mathbb{R}$ of

$$x' = f(t, x), \quad x(d_j) = \beta(d_j),$$

satisfying $\beta(t) \leqslant y_j(t) \leqslant \alpha(t)$ on $[c_j, d_j]$. Let us define $x : [t_0, t_1] \to \mathbb{R}$ by $x(t) = x_i(t)$ on $[a_i, b_i]$ and $x(t) = y_j(t)$ on $]c_j, d_j[$. Notice that $x(t_0) = x_1$, $x'(t) = f(t, x)$ a.e. on $[t_0, t_1]$ and (1.6) is satisfied. We only need to verify that x is absolutely continuous. Since x is bounded, by the L^1-Carathéodory conditions, there exists $\gamma \in L^1(t_0, t_1)$ such that $|f(t, x(t))| \leqslant \gamma(t)$ for all $t \in [t_0, t_1]$. Fix $\varepsilon > 0$ and pick points $r_k < s_k$ in $[t_0, t_1]$ with $k = 1, \dots, n$. Then $\sum_{k=1}^n |x(s_k) - x(r_k)|$ may be written as a series with terms of the form $|x_i(p_{k_i}) - x_i(q_{k_i})| \leqslant \int_{q_{k_i}}^{p_{k_i}} \gamma(s)\,\mathrm{d}s$ or $|y_j(p_{k_j}) - y_j(q_{k_j})| \leqslant \int_{q_{k_j}}^{p_{k_j}} \gamma(s)\,\mathrm{d}s$, whose sum is smaller than ε whenever $\sum_{k=1}^n |s_k - r_k|$ is sufficiently small.

Case 2: $\alpha(t_1) \geqslant \beta(t_1)$. We can argue as in Claim 2 applying our previous conclusion to a reversed time problem.

Case 3: $\alpha(t_0) \leqslant \beta(t_0)$ and $\alpha(t_1) \geqslant \beta(t_1)$. If either $\alpha(t_0) = \beta(t_0)$ or $\alpha(t_1) = \beta(t_1)$ the claim immediately follows from the previous cases. Otherwise, by Proposition 1.1 there exists a point $s_0 \in]t_0, t_1[$ such that $\alpha(s_0) = \beta(s_0)$. The assertion then follows by applying our preceding conclusions to problem (1.5) on $[t_0, s_0]$ and to problem (1.7) on $[s_0, t_1]$. □

We use Lemma 1.2 to recover an existence and localization result established in [28] under a slightly more restrictive notion of lower and upper solutions. We add here some information on the topological structure of the solution set, namely we prove the Hukuhara–Kneser property. In order to show that it is (arcwise) connected, we exploit the following property of compact subsets of $C^0([t_0, t_1])$ which are dense-in-itself with respect to the order.

DEFINITION 1.10. A subset $\mathcal{S}$ of $C^0([t_0, t_1])$ is said *dense-in-itself with respect to the order* if for any $u_1, u_2 \in \mathcal{S}$, with $u_1 < u_2$, there exists $u_3 \in \mathcal{S}$ with $u_1 < u_3 < u_2$.

LEMMA 1.3. *Let $\mathcal{S} \subseteq C^0([t_0, t_1])$ be a compact set which is dense-in-itself with respect to the order. Let $\mathcal{T} \subseteq \mathcal{S}$ be a maximal nondegenerate totally ordered subset of $\mathcal{S}$. Then $\mathcal{T}$ is homeomorphic to a nondegenerate compact interval of $\mathbb{R}$.*

PROOF. We start by verifying that $\mathcal{T}$ is closed in $\mathcal{S}$. Let $(z_n)_n$ be a sequence in $\mathcal{T}$ converging to some function $z \in \mathcal{S}$. Let us show that $z \in \mathcal{T}$, that is, for each $u \in \mathcal{T}$, either $u \leqslant z$ or $u \geqslant z$. Assume by contradiction that there exists $u \in \mathcal{T}$ such that $u \nleqslant z$ and $u \ngeqslant z$, i.e. $\min\{\|(u-z)^+\|_\infty, \|(u-z)^-\|_\infty\} > 0$. Take n such that $\|z_n - z\|_\infty < \min\{\|(u-z)^+\|_\infty, \|(u-z)^-\|_\infty\}$ and suppose, for instance, that $z_n \geqslant u$. We have $(z_n - z)^+ \geqslant (u-z)^+$ and hence $\|(u-z)^+\|_\infty \leqslant \|(z_n-z)^+\|_\infty \leqslant \|z_n - z\|_\infty < \|(u-z)^+\|_\infty$, which is a contradiction. Therefore, we conclude that $z \in \mathcal{T}$.

Moreover, as $\mathcal{T}$ is a maximal totally ordered subset of $\mathcal{S}$, $\mathcal{T}$ is dense-in-itself with respect to the order. Let $\varphi : C^0([t_0, t_1]) \to \mathbb{R}$ be a continuous linear functional such that $\varphi(u) > 0$ for every $u > 0$ (e.g., $\varphi(u) = \int_{t_0}^{t_1} u$). Since $\mathcal{T}$ is compact and $\varphi|_{\mathcal{T}}$ is one-to-one, $\varphi|_{\mathcal{T}}$ is a homeomorphism between $\mathcal{T}$ and $\varphi(\mathcal{T}) \subseteq \mathbb{R}$. Notice that $\varphi(\mathcal{T})$ is also dense-in-itself with respect to the ordering of $\mathbb{R}$ and hence it is a nondegenerate compact interval. □

THEOREM 1.4. *Assume* (C). *Let α be a lower solution and β be an upper solution of* (1.2) *on* $[t_0, t_1]$. *Then there exist the minimum solution v and the maximum solution w in* $[\min\{\alpha, \beta\}, \max\{\alpha, \beta\}]$ *of* (1.2) *on* $[t_0, t_1]$. *Further, the set*

$$\mathcal{K} = \{x : [t_0, t_1] \to \mathbb{R} \mid x \text{ is a solution of } (1.2) \text{ with } \min\{\alpha, \beta\} \leqslant x \leqslant \max\{\alpha, \beta\}\}$$

is a continuum, i.e. a compact and connected set, in $C^0([t_0, t_1])$.

PROOF. In this proof we use the following notations: if I is an interval, with endpoints a and b, and $u : I \to \mathbb{R}$, then we set $u(a^+) = \lim_{t \to a^+} u(t)$ and $u(b^-) = \lim_{t \to b^-} u(t)$, whenever the limits exist. The proof is divided into three steps.

Step 1. *Existence of a solution.* Let $\tau_0, \tau_1, \dots, \tau_N$ and $\sigma_0, \sigma_1, \dots, \sigma_M$ be the points of $[t_0, t_1]$ defining α and β as in Definition 1.9 and relabel them as $\rho_0 < \rho_1 < \cdots < \rho_P$. For each $n = 1, \dots, P$ denote by α_n, β_n the continuous extensions on $[\rho_{n-1}, \rho_n]$ of $\alpha|_{]\rho_{n-1}, \rho_n[}$ and $\beta|_{]\rho_{n-1}, \rho_n[}$, respectively. We will prove the thesis by recursion on P.

- *There exists a solution* y_1 *of* (1.2) *such that* $\min\{\alpha, \beta\} \leqslant y_1 \leqslant \max\{\alpha, \beta\}$ *on* $[t_0, \rho_1]$. *Further, if* $\alpha(\rho_1^-) \geqslant \beta(\rho_1^-)$, *then for any* x_1 *between* $\min\{\alpha(\rho_1^-), \beta(\rho_1)\}$ *and* $\max\{\alpha(\rho_1), \beta(\rho_1^-)\}$, *the solution* y_1 *can be chosen so that* $y_1(\rho_1) = x_1$.

As $\alpha_1(t_0) = \alpha(t_0^+) \leqslant \alpha(t_0) \leqslant x_0 \leqslant \beta(t_0) \leqslant \beta(t_0^+) = \beta_1(t_0)$, the claim follows by Lemma 1.2.

- *Let* $n \leqslant P - 1$. *Assume there exists a solution* y_n *of* (1.2) *such that* $\min\{\alpha, \beta\} \leqslant y_n \leqslant \max\{\alpha, \beta\}$ *on* $[t_0, \rho_n]$ *and, in case* $\alpha(\rho_n^-) \geqslant \beta(\rho_n^-)$, *for all* x_n *between* $\min\{\alpha(\rho_n^-), \beta(\rho_n)\}$ *and* $\max\{\alpha(\rho_n), \beta(\rho_n^-)\}$ *the solution* y_n *can be chosen so that* $y_n(\rho_n) = x_n$. *Then there exists a solution* y_{n+1} *of* (1.2) *such that* $\min\{\alpha, \beta\} \leqslant y_{n+1} \leqslant \max\{\alpha, \beta\}$ *on* $[t_0, \rho_{n+1}]$ *and, in case* $\alpha(\rho_{n+1}^-) \geqslant \beta(\rho_{n+1}^-)$, *for all* x_{n+1} *between* $\min\{\alpha(\rho_{n+1}^-), \beta(\rho_{n+1})\}$ *and* $\max\{\alpha(\rho_{n+1}), \beta(\rho_{n+1}^-)\}$ *the solution* y_{n+1} *can be chosen so that* $y_{n+1}(\rho_{n+1}) = x_{n+1}$.

To prove this we need to distinguish several cases.

(i) Assume $\alpha(\rho_n^-) \leqslant \beta(\rho_n^-)$. Hence $\alpha(\rho_n) \leqslant \beta(\rho_n)$. Let y_n be a solution as in the assumption. As $\alpha_{n+1}(\rho_n) \leqslant y_n(\rho_n) \leqslant \beta_{n+1}(\rho_n)$, by Lemma 1.2 there exists a solution z of (1.1) such that $z(\rho_n) = y_n(\rho_n)$, $\min\{\alpha_{n+1}, \beta_{n+1}\} \leqslant z \leqslant \max\{\alpha_{n+1}, \beta_{n+1}\}$ on $[\rho_n, \rho_{n+1}]$ and, in case $\alpha_{n+1}(\rho_{n+1}) \geqslant \beta_{n+1}(\rho_{n+1})$, for all $x_{n+1} \in [\beta_{n+1}(\rho_{n+1}), \alpha_{n+1}(\rho_{n+1})]$ such a solution can be chosen so that $z(\rho_{n+1}) = x_{n+1}$. Since in this case the interval $[\beta_{n+1}(\rho_{n+1}), \alpha_{n+1}(\rho_{n+1})]$ is contained in the interval with endpoints $\min\{\alpha(\rho_{n+1}^-), \beta(\rho_{n+1})\}$ and $\max\{\alpha(\rho_{n+1}), \beta(\rho_{n+1}^-)\}$, the function y_{n+1} defined by $y_{n+1}(t) = y_n(t)$ on $[t_0, \rho_n]$ and $y_{n+1}(t) = z(t)$ on $[\rho_n, \rho_{n+1}]$ is as required.

(ii) Assume $\alpha(\rho_n^-) > \beta(\rho_n^-)$ and $\alpha(\rho_n) \leqslant \beta(\rho_n)$. Then $\max\{\alpha(\rho_n), \beta(\rho_n^-)\} \leqslant \min\{\alpha(\rho_n^-), \beta(\rho_n)\}$. Let $x_n \in [\max\{\alpha(\rho_n), \beta(\rho_n^-)\}, \min\{\alpha(\rho_n^-), \beta(\rho_n)\}]$. By assumption there exists a solution y_n of (1.2) such that $\min\{\alpha, \beta\} \leqslant y_n \leqslant \max\{\alpha, \beta\}$ on $[t_0, \rho_n]$ and $y_n(\rho_n) = x_n$. As $\alpha_{n+1}(\rho_n) \leqslant y_n(\rho_n) \leqslant \beta_{n+1}(\rho_n)$, arguing as in (i) we get the existence of the solution y_{n+1} with all the required properties.

(iii) Assume $\alpha(\rho_n) > \beta(\rho_n)$ (hence $\alpha(\rho_n^-) > \beta(\rho_n^-)$) and $\alpha(\rho_n^+) \leqslant \beta(\rho_n^+)$. Let $x_n \in [\alpha(\rho_n^+), \beta(\rho_n^+)] \cap [\beta(\rho_n), \alpha(\rho_n)]$ and argue as in (ii).

(iv) Assume $\alpha(\rho_n^+) > \beta(\rho_n^+)$ (hence $\alpha(\rho_n) > \beta(\rho_n)$ and $\alpha(\rho_n^-) > \beta(\rho_n^-)$) and $\alpha(\rho_{n+1}^-) > \beta(\rho_{n+1}^-)$. For any x_{n+1} between $\min\{\alpha(\rho_{n+1}^-), \beta(\rho_{n+1})\}$ and $\max\{\alpha(\rho_{n+1}), \beta(\rho_{n+1}^-)\}$ we have $\alpha_{n+1}(\rho_{n+1}) \geqslant x_{n+1} \geqslant \beta_{n+1}(\rho_{n+1})$ and hence, by Lemma 1.2, there exists a solution z of (1.1) such that $z(\rho_{n+1}) = x_{n+1}$ and $\min\{\alpha_{n+1}, \beta_{n+1}\} \leqslant z \leqslant \max\{\alpha_{n+1}, \beta_{n+1}\}$ on $[\rho_n, \rho_{n+1}]$. As $\beta(\rho_n) \leqslant \beta(\rho_n^+) \leqslant z(\rho_n) \leqslant \alpha(\rho_n^+) \leqslant \alpha(\rho_n)$, by assumption there exists a solution y_n of (1.2) such that $\min\{\alpha, \beta\} \leqslant y_n \leqslant \max\{\alpha, \beta\}$ on $[t_0, \rho_n]$ and $y_n(\rho_n) = z(\rho_n)$. The function y_{n+1} defined by $y_{n+1}(t) = y_n(t)$ on $[t_0, \rho_n]$ and $y_{n+1}(t) = z(t)$ on $[\rho_n, \rho_{n+1}]$ is as required.

(v) Assume $\alpha(\rho_n^+) > \beta(\rho_n^+)$ (hence $\alpha(\rho_n) > \beta(\rho_n)$ and $\alpha(\rho_n^-) > \beta(\rho_n^-)$) and $\alpha(\rho_{n+1}^-) \leqslant \beta(\rho_{n+1}^-)$. By continuity there is a point $\hat{t} \in \,]\rho_n, \rho_{n+1}[$ such that $\alpha(\hat{t}) = \beta(\hat{t})$. Arguing as in (iv), by Lemma 1.2 for any x_{n+1} between $\min\{\alpha(\rho_{n+1}^-), \beta(\rho_{n+1})\}$ and $\max\{\alpha(\rho_{n+1}), \beta(\rho_{n+1}^-)\}$ there exists a solution z_2 of (1.1) such that $z_2(\hat{t}) = \alpha(\hat{t})$, $z_2(\rho_{n+1}) = x_{n+1}$ and $\min\{\alpha_{n+1}, \beta_{n+1}\} \leqslant z_2 \leqslant \max\{\alpha_{n+1}, \beta_{n+1}\}$ on $[\hat{t}, \rho_{n+1}]$. Further there exists a solution z_1 of (1.1) such that $z_1(\hat{t}) = \alpha(\hat{t})$ and $\min\{\alpha_{n+1}, \beta_{n+1}\} \leqslant z_1 \leqslant \max\{\alpha_{n+1}, \beta_{n+1}\}$ on $[\rho_n, \hat{t}]$. As $\beta(\rho_n) \leqslant \beta(\rho_n^+) \leqslant z_1(\rho_n) \leqslant \alpha(\rho_n^+) \leqslant \alpha(\rho_n)$, by assumption there exists a solution y_n of (1.2) such that $\min\{\alpha, \beta\} \leqslant y_n \leqslant \max\{\alpha, \beta\}$ on $[t_0, \rho_n]$ and $y_n(\rho_n) = z_1(\rho_n)$. The function y_{n+1} defined by $y_{n+1}(t) = y_n(t)$ on $[t_0, \rho_n]$, $y_{n+1}(t) = z_1(t)$ on $[\rho_n, \hat{t}]$, $y_{n+1}(t) = z_2(t)$ on $[\hat{t}, \rho_{n+1}]$ is as required.

Step 2. *Existence of extremal solutions.* We have just seen that $\mathcal{K}$ is nonempty. Let us show that $\mathcal{K}$ has a minimum and a maximum. Since the Volterra operator $S : C^0([t_0, t_1]) \to C^0([t_0, t_1])$ associated with (1.2) is completely continuous and the set of its fixed points lying between $\min\{\alpha, \beta\}$ and $\max\{\alpha, \beta\}$ is precisely $\mathcal{K}$, we have that $\mathcal{K}$ is a nonempty compact set. For each $x \in \mathcal{K}$ define the closed set $\mathcal{C}_x = \{z \in \mathcal{K} \mid z \leqslant x\}$. The family $\{\mathcal{C}_x \mid x \in \mathcal{K}\}$ has the finite intersection property since, if $x_1, x_2 \in \mathcal{K}$, then $\min\{x_1, x_2\} \in \mathcal{K}$. By the compactness of $\mathcal{K}$ there exists $v \in \bigcap_{x \in \mathcal{K}} \mathcal{C}_x$; clearly, v is the minimum solution we are looking for. The maximum solution w can be found in a similar way.

Step 3. $\mathcal{K}$ is a continuum. We have only to show that $\mathcal{K}$ is connected. Let us prove that $\mathcal{K}$ is dense-in-itself with respect to the order. Take $x_1, x_2 \in \mathcal{K}$ with $x_1 < x_2$. Let $s_0 \in \,]t_0, t_1[$ be such that $x_1(s_0) < x_2(s_0)$ and pick a real number $\hat{x}_0$ with $x_1(s_0) < \hat{x}_0 < x_2(s_0)$. By Lemma 1.2 there exists a solution $x_3 : [t_0, t_1] \to \mathbb{R}$ of (1.1), satisfying $x_3(t_0) = x_0$ and $x_3(s_0) = \hat{x}_0$, such that $x_1 \leqslant x_3 \leqslant x_2$; hence $x_3 \in \mathcal{K}$ and $x_1 < x_3 < x_2$. Let us show that $\mathcal{K}$ is arcwise connected. For any $x_1, x_2 \in \mathcal{K}$, with $x_1 \neq x_2$, let $\mathcal{T}_1$ be a maximal totally ordered subset of $\mathcal{K}$ containing x_1 and $\min\{x_1, x_2\}$ and let $\mathcal{T}_2$ be a maximal totally ordered subset of

$\mathcal{K}$ containing x_2 and $\min\{x_1, x_2\}$. By Lemma 1.3, $\mathcal{T}_1$ and $\mathcal{T}_2$ are homeomorphic to compact real intervals. Since $\min\{x_1, x_2\} \in \mathcal{T}_1 \cap \mathcal{T}_2$, we find an arc connecting x_1 to x_2. □

In the sequel we shall also use the following partial extension of Theorem 1.4 to non-compact intervals. Related results can be found in [90].

COROLLARY 1.5. *Assume* (C). *Let α be a lower solution and β be an upper solution of* (1.2) *on* $[t_0, \tau[$, *with* $t_0 < \tau \leqslant +\infty$, *and assume that* $\alpha \leqslant \beta$. *Then there exists the minimum solution v and the maximum solution w in* $[\alpha, \beta]$ *of* (1.2) *on* $[t_0, \tau[$. *Further, the set*

$$\mathcal{K} = \big\{x : [t_0, \tau[\to \mathbb{R} \mid x \text{ is a solution of } (1.2) \text{ with } \alpha \leqslant x \leqslant \beta\big\}$$

is a continuum in $C^0([t_0, \tau[)$, *endowed with the topology of uniform convergence on compact intervals.*

PROOF. Let $(\tau_n)_n$, with $\tau_0 = t_0$, be a strictly increasing sequence converging to τ.

Step 1. *Existence of extremal solutions.* In order to prove the existence of the minimum solution v in $[\alpha, \beta]$ of (1.2) on $[t_0, \tau[$, we apply recursively Theorem 1.4 on each interval $[t_0, \tau_n]$, with $n \geqslant 1$ Let us denote by v_n the minimum solution v in $[\alpha, \beta]$ of (1.2) on $[t_0, \tau_n]$. By the minimality of v_n, we have $v_{n+1}|_{[t_0,\tau_n]} \geqslant v_n$. On the other hand, as $\alpha|_{[\tau_n,\tau_{n+1}]}$ and $v_{n+1}|_{[\tau_n,\tau_{n+1}]}$ are, respectively, a lower and an upper solution of the initial value problem $x' = f(t, x)$, $x(\tau_n) = v_n(\tau_n)$, we can continue v_n to a solution $\hat{v}_{n+1}$ of (1.2) on $[t_0, \tau_{n+1}]$ satisfying $\alpha \leqslant \hat{v}_{n+1} \leqslant v_{n+1}$. By the minimality of v_{n+1}, we conclude that $\hat{v}_{n+1} = v_{n+1}$ and hence $v_{n+1}|_{[t_0,\tau_n]} = v_n$. Then we define $v : [t_0, \tau[\to \mathbb{R}$ by setting $v(t) = v_n(t)$ on $[t_0, \tau_n]$. We have that v is the minimum solution in $[\alpha, \beta]$ of (1.2) on $[t_0, \tau[$, because, if x were a solution of (1.2) on $[t_0, \tau[$ with $x \geqslant \alpha$ and $x \ngeqslant v$, then it should follow $x|_{[t_0,\tau_n]} \ngeqslant v_n$ for some n, thus contradicting the minimality of v_n. Similarly, we prove the existence of the maximum solution w in $[\alpha, \beta]$ of (1.2) on $[t_0, \tau[$.

Step 2. *$\mathcal{K}$ is a continuum.* For each $x \in \mathcal{K}$ and $n \geqslant 1$, set $x_n = x|_{[t_0,\tau_n]}$. We denote by $\mathcal{K}_n$ the set of all solutions $x : [t_0, \tau_n] \to \mathbb{R}$ of (1.2) such that $\alpha \leqslant x \leqslant \beta$ on $[t_0, \tau_n]$. By Theorem 1.4, $\mathcal{K}_n$ is a continuum in $C^0([t_0, \tau_n])$. For every $m < n$, let also $\pi^n_m : \mathcal{K}_n \to \mathcal{K}_m$ be the restriction map on $[t_0, \tau_m]$, i.e. $\pi^n_m(x) = x|_{[t_0,\tau_m]}$ for all $x \in \mathcal{K}_n$. Let us define now a function $\chi : \mathcal{K} \to \prod_{n=1}^{+\infty} \mathcal{K}_n$, by setting $\chi(x) = (x_n)_n$. Observe that χ is a homeomorphism of $\mathcal{K}$ into $\prod_{n=1}^{+\infty} \mathcal{K}_n$, when $\prod_{n=1}^{+\infty} \mathcal{K}_n$ is endowed with the Tychonoff product topology, and its range $\chi(\mathcal{K})$ is the set of all sequences $(x_n)_n \in \prod_{n=1}^{+\infty} \mathcal{K}_n$ such that, for all $m < n$, $\pi^n_m(x_n) = x_m$, i.e. $\chi(\mathcal{K})$ is the inverse limit of the sequence $(\mathcal{K}_n)_n$ with bonding maps π^n_m (cf. [47]). As the inverse limit of a sequence of continua is a continuum [47, Theorem 6.1.20] we deduce that $\chi(\mathcal{K})$ is a continuum as well. Since $\mathcal{K}$ is homeomorphic to $\chi(\mathcal{K})$, the conclusion is achieved. □

1.4. *The periodic problem*

In this section we discuss existence and localization of solutions of the periodic problem (1.3), in the presence of a pair of possibly discontinuous and unordered lower and upper solutions. With the aim of obtaining a global portrait of solutions of Eq. (1.1), we also introduce the notion of T-monotonicity. This concept plays a central role in the qualitative study of solutions of (1.1), as it helps to get a thorough classification of their asymptotic behaviour. Therefore we develop some criteria for detecting T-monotonicity. However, it should be stressed that also the study of solutions that are not T-monotone has an interest, since their existence gives rise to T-periodic solutions. All these facts are then used to get an extension of the existence part of the classical Massera Convergence Theorem to the Carathéodory setting.

Solutions and lower and upper solutions

DEFINITION 1.11. A *solution of* (1.3) is a solution u of (1.1) on $\mathbb{R}$ which is T-periodic, i.e. $u(t+T) = u(t)$ on $\mathbb{R}$.

We introduce some notions of possibly discontinuous lower and upper solutions for the periodic problem (1.3).

DEFINITION 1.12.

- A *lower solution of* (1.3) is a T-periodic function $\alpha : \mathbb{R} \to \mathbb{R}$ which is a lower solution of (1.1) on $\mathbb{R}$.
- A *regular lower solution of* (1.3) is a lower solution α of (1.3) such that $\alpha|_{[0,T]}$ is a regular lower solution of (1.1) on $[0, T]$.
- An *upper solution of* (1.3) is a T-periodic function $\beta : \mathbb{R} \to \mathbb{R}$ which is an upper solution of (1.1) on $\mathbb{R}$.
- A *regular upper solution of* (1.3) is an upper solution β of (1.3) such that $\beta|_{[0,T]}$ is a regular upper solution of (1.1) on $[0, T]$.
- A lower solution of (1.3) (respectively an upper solution of (1.3)) is *proper* if it is not a solution of (1.3).
- A proper lower solution α of (1.3) is *strict* if every solution x of (1.3), with $x > \alpha$, is such that $x \gg \alpha$. Similarly, a proper upper solution of (1.3) is *strict* if every solution x of (1.3), with $x < \beta$, is such that $x \ll \beta$.

REMARK 1.4. We notice that even regular lower and upper solutions of (1.3) may be discontinuous at the endpoints of the interval $[0, T]$.

REMARK 1.5. Sometimes we speak of solutions of (1.3) with reference to functions defined on $[t_0, t_0 + T]$ for some $t_0 \in \mathbb{R}$. We also speak of lower and upper solutions of (1.3) with reference to functions defined on $[t_0, t_0 + T[$ for some $t_0 \in \mathbb{R}$. In these cases it is understood that their T-periodic extensions to $\mathbb{R}$ have to be considered.

In the sequel we shall need the following observation, whose proof is a slight modification of that of Proposition 1.1.

PROPOSITION 1.6. *Let α be a lower solution and β be an upper solution of* (1.3), *with $\alpha \not\leq \beta$ and $\alpha \not\geq \beta$. Then there exists $\hat{t} \in [0, T]$ such that $\alpha(\hat{t}) = \beta(\hat{t})$.*

T-monotonicity

The notion of T-monotonicity plays an important role in the qualitative study of solutions of (1.1). When uniqueness of solutions of (1.2) holds, then any solution is T-monotone and this, in turn, corresponds to the monotonicity of the Poincaré operator. Of course, in the Carathéodory setting this is not anymore true.

DEFINITION 1.13.

- A function $x : I \to \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an interval, is said *T-increasing* (respectively, *T-decreasing*) if $x(t) \leqslant x(t+T)$ (respectively $x(t) \geqslant x(t+T)$) for each $t \in I$ such that $t + T \in I$. A function is said *T-monotone* if it is either T-increasing or T-decreasing.
- A function $x : I \to \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an interval unbounded from above, is said *eventually T-increasing* (respectively, *eventually T-decreasing*) if there exists $t_0 \in I$ such that $x|_{]t_0,+\infty[}$ is T-increasing (respectively, T-decreasing).

REMARK 1.6. Let x be a solution of (1.1) on $[t_0, +\infty[$. If x is T-increasing then the sequence $(\alpha_n)_n$, defined by setting, for each $n \geqslant t_0/T$, $\alpha_n(t) = x(t + nT)$ on $[0, T[$ gives rise to an increasing sequence of regular lower solutions of (1.3). Similarly, any T-decreasing solution of (1.1) on $[t_0, +\infty[$, gives rise to a decreasing sequence $(\beta_n)_n$ of regular upper solutions of (1.3).

The Massera Convergence Theorem for T-monotone solutions

The following simple result may be considered as a first step towards an extension of the classical Massera Convergence Theorem for scalar ordinary differential equations [91] to the case where the right-hand side f of (1.1) is a Carathéodory function.

PROPOSITION 1.7. *Assume* (C). *Let $x :]t_0, +\infty[\to \mathbb{R}$ be a bounded, T-increasing* (*respectively T-decreasing*) *solution of* (1.1). *Then there exists a solution u of* (1.3) *such that $u \geqslant x$* (*respectively $u \leqslant x$*) *on $]t_0, +\infty[$ and*

$$\lim_{t\to+\infty} \big(x(t) - u(t)\big) = 0. \tag{1.9}$$

REMARK 1.7. Symmetrically, we have that, if $x :]-\infty, t_0[\to \mathbb{R}$ is a bounded, T-increasing (respectively T-decreasing) solution of (1.1), then there exists a solution u of (1.3) such that $u \leqslant x$ (respectively $u \geqslant x$) on $]-\infty, t_0[$ and

$$\lim_{t\to-\infty} \big(x(t) - u(t)\big) = 0.$$

PROOF. Assume, without loss of generality, that $t_0 = n_0 T$, for some $n_0 \in \mathbb{N}$, and suppose that x is T-increasing. Define a sequence $(x_n)_n$ of functions by setting, for every $n \geqslant n_0$ and every $t \in [0, T]$, $x_n(t) = x(t + nT)$. Then x_n is a regular lower solution of (1.3), satisfying

$$x_{n+1}(0) = x_n(T) \tag{1.10}$$

and $x_n \leqslant x_{n+1}$, on $[0, T]$. By assumption, the sequence $(x_n)_n$ is bounded in $C^0([0, T])$ and hence, by monotonicity, it converges pointwise on $[0, T]$ to a function u, which by (1.10) satisfies $u(0) = u(T)$. Moreover, the L^1-Carathéodory conditions imply that $(x_n)_n$ is equicontinuous. Therefore, by Arzelà–Ascoli theorem and monotonicity, $(x_n)_n$ converges uniformly to u. Finally, using the integral representation of solutions of (1.1) and the L^1-Carathéodory conditions again, we conclude that u is a solution and that the convergence takes place in $W^{1,1}(0, T)$.

Let us show now that $\lim_{t\to+\infty}(x(t) - u(t)) = 0$. Indeed, since $(x_n)_n$ converges uniformly to u, given $\varepsilon > 0$ there is $\bar{n}$ such that, for any $n \geqslant \bar{n}$, $|x_n(t) - u(t)| < \varepsilon$ for all $t \in [0, T]$. Hence, if we take $t \geqslant \bar{n}T$, with $t \in [nT, (n+1)T[$ for some $n \geqslant \bar{n}$, we obtain, by the T-periodicity of u, $|x(t) - u(t)| = |x_n(t - nT) - u(t - nT)| < \varepsilon$. □

A Monotone Convergence Criterion

We now prove a technical result, which will often be used in the sequel. It describes the qualitative behaviour of solutions of (1.1), emanating from and lying above a lower solution of (1.3). Among these we single out a special solution $\widetilde{\alpha}$ which is T-increasing and hence, if bounded, does converge to a T-periodic solution. This statement is reminiscent of the Monotone Convergence Criterion in the theory of monotone maps (cf. [60, Section 5]) and essentially reduces to it when uniqueness of solutions of (1.2) holds.

PROPOSITION 1.8. *Assume* (C).

(i) *Let* α *be a lower solution of* (1.3) *and let* $t_0 \in \mathbb{R}$. *Then there exist* $\omega \in \,]t_0, +\infty]$ *and a* T*-increasing solution* $\widetilde{\alpha} : [t_0, \omega[\to \mathbb{R}$ *of* (1.2), *with* $x_0 = \alpha(t_0)$, *satisfying* $\widetilde{\alpha} \geqslant \alpha$ *on* $[t_0, \omega[$. *Further, every solution* $x : [t_0, \tau[\to \mathbb{R}$ *of* (1.1), *with* $x \geqslant \alpha$, *is such that* $\tau \leqslant \omega$ *and* $x \geqslant \widetilde{\alpha}$ *on* $[t_0, \tau[$. *Finally, we have that either every right-nonextendible solution* $x : [t_0, \tau[\to \mathbb{R}$ *of* (1.1), *with* $x \geqslant \alpha$, *is such that*

$$\limsup_{t\to\tau} x(t) = +\infty$$

and (1.3) *has no solution in* $[\alpha, +\infty[$, *or there exists the minimum solution* v *in* $[\alpha, +\infty[$ *of* (1.3); *in the latter case* $\omega = +\infty$, $v \geqslant \widetilde{\alpha}$ *and*

$$\lim_{t\to+\infty} \big(\widetilde{\alpha}(t) - v(t)\big) = 0.$$

(ii) *Let* β *be an upper solution of* (1.3) *and let* $t_0 \in \mathbb{R}$. *Then there exist* $\omega \in \,]t_0, +\infty]$ *and a* T*-decreasing solution* $\widetilde{\beta} : [t_0, \omega[\to \mathbb{R}$ *of* (1.2), *with* $x_0 = \beta(t_0)$, *satisfying* $\widetilde{\beta} \leqslant \beta$ *on* $[t_0, \omega[$. *Further, every solution* $x : [t_0, \tau[\to \mathbb{R}$ *of* (1.1), *with* $x \leqslant \beta$, *is such that* $\tau \leqslant \omega$ *and* $x \leqslant \widetilde{\beta}$ *on* $[t_0, \tau[$. *Finally, we have that either every right-nonextendible solution* $x : [t_0, \tau[\to \mathbb{R}$ *of* (1.1), *with* $x \leqslant \beta$, *is such that*

$$\liminf_{t\to\tau} x(t) = -\infty$$

and (1.3) *has no solution in* $]-\infty, \beta]$, *or there exists the maximum solution* w *in* $]-\infty, \beta]$ *of* (1.3); *in the latter case* $\omega = +\infty$, $w \leqslant \widetilde{\beta}$ *and*

$$\lim_{t \to +\infty} \big(\widetilde{\beta}(t) - w(t)\big) = 0.$$

REMARK 1.8. If $\omega < +\infty$, then $\lim_{t\to\omega} \widetilde{\alpha}(t) = +\infty$. Whereas, it may happen that, if $\omega = +\infty$, $\limsup_{t\to\omega} \widetilde{\alpha}(t) = +\infty$ and $\liminf_{t\to\omega} \widetilde{\alpha}(t) < +\infty$. A simple example is found by taking, in (1.1), $f(t,x) = x^2 \sin t$ if $|t| \leqslant \pi$, $f(t,x) = -x \sin t$ if $\pi < t < 2\pi$, and extending f by 3π-periodicity with respect to the t-variable. A similar remark holds for $\widetilde{\beta}$.

PROOF. We only prove statement (i), as the proof of (ii) is similar.

Let $\omega \in]t_0, +\infty]$ be the supremum of all $\tau > t_0$ such that there exists a solution $x : [t_0, \tau] \to \mathbb{R}$ of (1.2), with $x_0 = \alpha(t_0)$, satisfying $x \geqslant \alpha$ on $[t_0, \tau]$. The existence of such solutions x follows applying Theorem 1.4 to a modified problem. Take a strictly increasing sequence $(\tau_n)_n$ such that $\tau_n \to \omega$ and let $(x_n)_n$ be a corresponding sequence of solutions. Since x_1 is an upper solution and $\alpha|_{[t_0,\tau_1]}$ is a lower solution of (1.2), with $x_0 = \alpha(t_0)$, Theorem 1.4 yields the existence of the minimum solution v_1 in $[\alpha, x_1]$ of (1.2) on $[t_0, \tau_1]$. Actually, v_1 is the minimum solution in $[\alpha, +\infty[$ of (1.2), with $x_0 = \alpha(t_0)$, on $[t_0, \tau_1]$. Since $x_2|_{[\tau_1,\tau_2]}$ is an upper solution and $\alpha|_{[\tau_1,\tau_2]}$ is a lower solution of

$$x' = f(t,x), \quad x(\tau_1) = v_1(\tau_1), \tag{1.11}$$

on $[\tau_1, \tau_2]$, there is the minimum solution v_2 in $[\alpha, x_2]$ of (1.11) on $[\tau_1, \tau_2]$. Actually, v_2 is the minimum solution in $[\alpha, +\infty[$ of (1.11) on $[\tau_1, \tau_2]$. Proceeding in this way, we construct a sequence $(v_n)_n$ such that, for each n, v_n is the minimum solution in $[\alpha, +\infty[$ of

$$x' = f(t,x), \quad x(\tau_{n-1}) = v_{n-1}(\tau_{n-1}),$$

on $[\tau_{n-1}, \tau_n]$. Now, we define a function $\widetilde{\alpha} : [t_0, \omega[\to \mathbb{R}$ by setting

$$\widetilde{\alpha}(t) = v_n(t) \quad \text{if } t \in [\tau_{n-1}, \tau_n[.$$

It is clear that $\widetilde{\alpha}$ is a right-nonextendible solution of (1.2), with $x_0 = \alpha(t_0)$, satisfying $\widetilde{\alpha} \geqslant \alpha$ on $[t_0, \omega[$. Moreover, from the minimality of each v_n we deduce that every solution $x : [t_0, \tau[\to \mathbb{R}$ of (1.2), with $x_0 = \alpha(t_0)$, satisfying $x \geqslant \alpha$, is such that $\tau \leqslant \omega$ and $x \geqslant \widetilde{\alpha}$ on $[t_0, \tau[$. The same holds if $x_0 > \alpha(t_0)$, as $\min\{x, \widetilde{\alpha}\}$ is a solution of (1.2) with $x_0 = \alpha(t_0)$.

If $\omega \leqslant t_0 + T$ the lemma is proved. Suppose next that $t_0 + mT < \omega \leqslant t_0 + (m+1)T$ for some integer $m \geqslant 1$. In this case we may assume that $\tau_k = t_0 + kT$ for $k = 1, \ldots, m$, and the T-monotonicity of $\widetilde{\alpha}$ follows from the minimality of the solutions v_n on $[\tau_{n-1}, \tau_n]$. Finally suppose that $\omega = +\infty$. In this case we may assume that $\tau_n = t_0 + nT$ for all n and the T-monotonicity of $\widetilde{\alpha}$ follows as in the previous case. Suppose further that $\limsup_{t\to+\infty} \widetilde{\alpha}(t) < +\infty$. Since $\widetilde{\alpha}$ is T-increasing, Proposition 1.7 implies that there exists a solution v of (1.3) such that $\widetilde{\alpha} \leqslant v$ on $[t_0, +\infty[$ and

$$\lim_{t \to +\infty} \big(\widetilde{\alpha}(t) - v(t)\big) = 0.$$

Note that v is the minimum solution in $[\alpha, +\infty[$ of (1.3), because the existence of a solution u of (1.3), such that $\alpha \leqslant u < v$, would contradict the minimality properties of $\widetilde{\alpha}$. □

REMARK 1.9. The following version of Proposition 1.8 for a terminal value problem holds.

(j) Let β be an upper solution of (1.3) and let $t_0 \in \mathbb{R}$. Then there exist $\omega \in [-\infty, t_0[$ and a T-decreasing solution $\widetilde{\beta} :]\omega, t_0] \to \mathbb{R}$ of the terminal value problem (1.2), with $x_0 = \beta(t_0)$, satisfying $\widetilde{\beta} \geqslant \beta$ on $]\omega, t_0]$. Further, every solution $x :]\tau, t_0] \to \mathbb{R}$ of (1.1), with $x \geqslant \beta$, is such that $\tau \geqslant \omega$ and $x \geqslant \widetilde{\beta}$ on $]\tau, t_0]$. Finally, we have that either every left-nonextendible solution $x :]\tau, t_0] \to \mathbb{R}$ of (1.1), with $x \geqslant \beta$, is such that

$$\limsup_{t \to \tau} x(t) = +\infty$$

and (1.3) has no solution in $[\beta, +\infty[$, or there exists the minimum solution v in $[\beta, +\infty[$ of (1.3); in the latter case $\omega = -\infty$, $v \geqslant \widetilde{\beta}$ and

$$\lim_{t \to -\infty} \bigl(\widetilde{\beta}(t) - v(t)\bigr) = 0.$$

(jj) Let α be a lower solution of (1.3) and let $t_0 \in \mathbb{R}$. Then there exist $\omega \in [-\infty, t_0[$ and a T-increasing solution $\widetilde{\alpha} :]\omega, t_0] \to \mathbb{R}$ of the terminal value problem (1.2), with $x_0 = \alpha(t_0)$, satisfying $\widetilde{\alpha} \leqslant \alpha$ on $]\omega, t_0]$. Further, every solution $x :]\tau, t_0] \to \mathbb{R}$ of (1.1), with $x \leqslant \alpha$, is such that $\tau \geqslant \omega$ and $x \leqslant \widetilde{\alpha}$ on $]\tau, t_0]$. Finally, we have that either every left-nonextendible solution $x :]\tau, t_0] \to \mathbb{R}$ of (1.1), with $x \leqslant \alpha$, is such that

$$\liminf_{t \to \tau} x(t) = -\infty,$$

and (1.3) has no solution in $]-\infty, \alpha]$, or there exists the maximum solution w in $]-\infty, \alpha]$ of (1.3); in the latter case $\omega = -\infty$, $w \leqslant \widetilde{\alpha}$ and

$$\lim_{t \to -\infty} \bigl(\widetilde{\alpha}(t) - w(t)\bigr) = 0.$$

Existence of solutions

The counterpart of Theorem 1.4 for the periodic problem (1.3) is the following theorem.

THEOREM 1.9. *Assume* (C). *Let α be a lower solution and β be an upper solution of* (1.3). *Then there exist the minimum solution v and the maximum solution w in* $[\min\{\alpha, \beta\}, \max\{\alpha, \beta\}]$ *of* (1.3).

PROOF. We distinguish three cases.

Case 1: $\alpha \leqslant \beta$. Let $\widetilde{\alpha}$ be the T-increasing solution of (1.2), with $x_0 = \alpha(t_0)$, whose existence is guaranteed by Proposition 1.8. Corollary 1.5, together with the minimality property of $\widetilde{\alpha}$, imply that $\widetilde{\alpha}$ is defined on $[t_0, +\infty[$ and $\widetilde{\alpha} \leqslant \beta$. Since $\widetilde{\alpha}$ is bounded, there exists the minimum solution v in $[\alpha, \beta]$ of (1.3). A symmetric argument proves the existence of the maximum solution w in $[\alpha, \beta]$ of (1.3).

Case 2: $\alpha > \beta$. We can reduce this case to the previous one by reversing time, as we did in Claim 2 of Lemma 1.2.

Case 3: $\alpha \not\leqslant \beta$ *and* $\beta \not\leqslant \alpha$. By Proposition 1.6 there exists $t_0 \in [0, T]$ such that $\alpha(t_0) = \beta(t_0)$. The result follows then immediately by Theorem 1.4 applied to problem (1.2), with $x_0 = \alpha(t_0) = \beta(t_0) = \alpha(t_0 + T) = \beta(t_0 + T)$, on the interval $[t_0, t_0 + T]$. □

We produce a simple example showing how Theorem 1.9 can be effectively applied to detect the existence of branches of T-periodic solutions by an appropriate choice of lower and upper solutions.

EXAMPLE 1.1. Let us consider the equation

$$x' = \sqrt{|x|} + h(t), \tag{1.12}$$

with $h:\mathbb{R} \to \mathbb{R}$ the 1-periodic function defined by $h(t) = -\frac{1}{8}|t|$ on $[-\frac{1}{2}, \frac{1}{2}[$. For any $p \in]-2^{-10}, 2^{-10}[$, define $\alpha_p : [-\frac{1}{2}, \frac{1}{2}[\to \mathbb{R}$, by setting $\alpha_p(-\frac{1}{2}) = p$ and $\alpha_p(t) = 2^{-4}\operatorname{sgn}(t)t^2$ on $]-\frac{1}{2}, \frac{1}{2}[$, and $\beta_p : [-\frac{1}{2}, \frac{1}{2}[\to \mathbb{R}$, by setting $\beta_p(-\frac{1}{2}) = p$ and $\beta_p(t) = -2^{-8}\operatorname{sgn}(t)t^2$ on $]-\frac{1}{2}, \frac{1}{2}[$. Then α_p and β_p are respectively a lower and an upper solution of the 1-periodic problem associated with (1.12). By Theorem 1.9, Eq. (1.12) has, for each $p \in]-2^{-10}, 2^{-10}[$, a 1-periodic solution u_p such that $u_p(-\frac{1}{2}) = p$.

A criterion of T-monotonicity

In the next theorem we show that T-monotonicity is a property shared by any solution of (1.1) which is comparable with all T-periodic solutions of (1.1).

THEOREM 1.10. *Assume* (C). *Let $x :]\omega_-, \omega_+[\to \mathbb{R}$ be a nonextendible solution of* (1.1) *such that, if u is a solution of* (1.3), *then either $x > u$ or $x < u$. Then x is T-monotone.*

PROOF. We assume that $\omega_+ - \omega_- > T$, because otherwise there is nothing to prove. Let us suppose by contradiction that x is not T-monotone, i.e. there are points $t_1, t_2 \in]\omega_-, \omega_+[$ such that

$$x(t_1) < x(t_1 + T) \quad \text{and} \quad x(t_2) > x(t_2 + T).$$

We can assume that $t_1 < t_2$, otherwise we set $y(t) = -x(t)$ and replace equation (1.1) with $y' = -f(t, -y)$.

Step 1. Assume $\omega_+ = +\infty$. We distinguish two cases.

• Suppose that there exists $n \in \mathbb{N}^+$ such that $t_2 = t_1 + nT$ and $x(t_1 + kT) \leqslant x(t_1 + (k + 1)T)$ for each $0 \leqslant k \leqslant n - 1$. Let $m \in \mathbb{N}$ be such that $x(t_1 + mT) < x(t_1 + (m + 1)T) = x(t_1 + nT)$. Since $\max\{x(t_1 + mT), x(t_1 + (n + 1)T)\} < x(t_1 + nT)$, we can pick $p \in]\max\{x(t_1 + mT), x(t_1 + (n + 1)T)\}, x(t_1 + nT)[$. We define a function $\alpha : [t_1 + mT, t_1 + (m + 1)T[\to \mathbb{R}$ by $\alpha(t) = x(t)$ on $]t_1 + mT, t_1 + (m + 1)T[$ and $\alpha(t_1 + mT) = p$, and a function $\beta:[t_1+nT, t_1+(n+1)T[\to\mathbb{R}$ by $\beta(t)=x(t)$ on $]t_1+nT, t_1+(n+1)T[$ and $\beta(t_1+nT)=p$. The T-periodic extensions to $\mathbb{R}$ of α and β are respectively a lower and an upper solution of (1.3). Hence Theorem 1.9 yields, for any $p\in]\max\{x(t_1+mT), x(t_1+(n+1)T)\}, x(t_1+nT)[$, the existence of a solution u of (1.3) such that $\min\{\alpha,\beta\}\leqslant u\leqslant\max\{\alpha,\beta\}$. Since $u(t_1+mT)=p$ and $x(t_1+mT)<p<x(t_1+nT)$, we conclude that $u\nleqslant x$ and $u\ngeqslant x$, which is a contradiction.

• Suppose that there exists $n\in\mathbb{N}$ such that $t_1+nT<t_2<t_1+(n+1)T$ and $x(t_1+kT)\leqslant x(t_1+(k+1)T)$ for each $k\in\mathbb{N}$. Set $\varphi(t)=x(t+T)-x(t)$ on $]\omega_-,+\infty[$. As $\varphi(t_2)<0$ and $\varphi(t_1+(n+1)T)\geqslant 0$, there exists $s_1\in]t_2, t_1+(n+1)T]$ such that $\varphi(s_1)=0$, i.e. $x(s_1+T)=x(s_1)$. Let $v:\mathbb{R}\to\mathbb{R}$ be the T-periodic extension to $\mathbb{R}$ of $x|_{[s_1,s_1+T]}$; v is a solution of (1.3). Since $v(t_2)=v(t_2+T)=x(t_2+T)<x(t_2)$ and $v(t_1+nT)=v(t_1+(n+1)T)=x(t_1+(n+1)T)\geqslant x(t_1+nT)$, there exists $s_2\in[t_1+nT, t_2[$ such that $v(s_2)=x(s_2)$. Note that $x(s_2)=x(s_2+T)$. Let $w:\mathbb{R}\to\mathbb{R}$ be the T-periodic extension to $\mathbb{R}$ of $x|_{[s_2,s_2+T]}$; w is a solution of (1.3). Since $v(t_2)=x(t_2+T)<x(t_2)=w(t_2)$, we can pick $p\in]v(t_2), w(t_2)[$. Then we define a function $\alpha:[s_2, s_2+T[\to\mathbb{R}$, by $\alpha(t)=w(t)$ on $]s_2,t_2[$, $\alpha(t_2)=p$ and $\alpha(t)=v(t)$ on $]t_2, s_2+T[$, and a function $\beta:[s_2,s_2+T[\to\mathbb{R}$, by $\beta(t)=v(t)$ on $]s_2,t_2[$, $\beta(t_2)=p$ and $\beta(t)=w(t)$ on $]t_2,s_2+T[$. The T-periodic extensions to $\mathbb{R}$ of α and β are respectively a lower and an upper solution of (1.3). Hence Theorem 1.9 yields, for any $p\in]v(t_2), w(t_2)[$, the existence of a solution u of (1.3) such that $\min\{\alpha,\beta\}\leqslant u\leqslant\max\{\alpha,\beta\}$. Since $u(t_2)=p$ and $x(t_2+T)<p<x(t_2)$, we conclude that $u\nleqslant x$ and $u\ngeqslant x$, which is a contradiction.

Step 2. Assume $\omega_-=-\infty$. We set $y(t)=-x(-t)$ and we replace Eq. (1.1) with $y'=f(-t,-y)$. Thus we are reduced to the situation discussed in Step 1.

Step 3. Assume $\omega_-,\omega_+\in\mathbb{R}$. Set $\varphi(t)=x(t+T)-x(t)$ on $]\omega_-,\omega_+-T[$. We essentially distinguish two cases.

• Suppose that

$$\lim_{t\to\omega_-}x(t)=-\infty\quad\text{and}\quad\lim_{t\to\omega_+}x(t)=+\infty;\tag{1.13}$$

the case where $\lim_{t\to\omega_-}x(t)=+\infty$ and $\lim_{t\to\omega_+}x(t)=-\infty$ being treated similarly. Since $\varphi(t_1)>0>\varphi(t_2)$, there exists $t_0\in]t_1,t_2[$ such that $\varphi(t_0)=0$, i.e. $x(t_0+T)=x(t_0)$. Let $u:\mathbb{R}\to\mathbb{R}$ be the T-periodic extension to $\mathbb{R}$ of $x|_{[t_0,t_0+T]}$; u is a solution of (1.3). Since (1.13) implies that $u\nleqslant x$ and $u\ngeqslant x$, a contradiction follows.

• Suppose that

$$\lim_{t\to\omega_-}x(t)=\lim_{t\to\omega_+}x(t)=+\infty;\tag{1.14}$$

the case where $\lim_{t\to\omega_-}x(t)=\lim_{t\to\omega_+}x(t)=-\infty$ being treated similarly. Since $\lim_{t\to\omega_-}\varphi(t)=-\infty$ and $\lim_{t\to\omega_+-T}\varphi(t)=+\infty$, there exists $s_1\in]\omega_-,\omega_+-T[$ such that $\varphi(s_1)=0$, i.e. $x(s_1+T)=x(s_1)$. Let $w:\mathbb{R}\to\mathbb{R}$ be the T-periodic extension to $\mathbb{R}$ of $x|_{[s_1,s_1+T]}$. Since w is a solution of (1.3) and (1.14) holds, we conclude that $x>w$ and w is the maximum solution of (1.3). Define a function $x_1:]\omega_-,\omega_+-T[\to\mathbb{R}$ by setting $x_1(t)=x(t)$ on $]\omega_-,s_1]$ and $x_1(t)=x(t+T)$ on $]s_1,\omega_+-T[$. The function x_1 is a solution of (1.1) satisfying $x_1>w$ on $]\omega_-,\omega_+-T[$, $x_1(s_1)=w(s_1)$ and $\lim_{t\to\omega_-}x_1(t)=$

$\lim_{t\to\omega_+-T} x_1(t) = +\infty$. Proceeding recursively we can find $s_n \in]\omega_-, \omega_+ - nT[$ and construct a solution $x_n :]\omega_-, \omega_+ - nT[\to \mathbb{R}$ of (1.1) such that $\omega_+ - nT - \omega_- \leqslant T$, $x_n > w$ on $]\omega_-, \omega_+ - nT[$, $x_n(s_n) = w(s_n)$ and $\lim_{t\to\omega_-} x_n(t) = \lim_{t\to\omega_+-nT} x_n(t) = +\infty$. A contradiction will then be achieved from the following result.

CLAIM. *Assume that there exists the maximum solution w of* (1.3). *Let $x :]\omega_-, \omega_+[\to \mathbb{R}$ be a solution of* (1.1) *such that $\omega_+ - \omega_- \leqslant T$, $x > w$ on $]\omega_-, \omega_+[$ and* $\lim_{t\to\omega_-} x(t) = \lim_{t\to\omega_+} x(t) = +\infty$. *Then $x \gg w$ on $]\omega_-, \omega_+[$.*

Suppose by contradiction that there is $t_0 \in]\omega_-, \omega_+[$ such that $x(t_0) = w(t_0)$ and $x(t) > w(t)$ on $]t_0, \omega_+[$. Denote by $\mathcal{Y}$ the set of all solutions $y : [t_0, \lambda_+(y)[\to \mathbb{R}$ of (1.1), which are nonextendible to the right of $\lambda_+(y)$ and satisfy $y(t_0) = w(t_0)$, $y \geqslant w$ on $[t_0, \lambda_+(y)[$ and $y > w$ on $[t_0, \omega_- + T[\cap [t_0, \lambda_+(y)[$. Note that $\mathcal{Y} \neq \emptyset$, as $x \in \mathcal{Y}$. Set $\lambda_+ = \sup_{y\in\mathcal{Y}} \lambda_+(y)$. Let us show that $\lambda_+ > \omega_- + T$. Otherwise, if $\lambda_+ \leqslant \omega_- + T$, we can pick a solution $z : [\lambda_+ - \delta, \lambda_+ + \delta] \to \mathbb{R}$ of (1.1) such that $z(\lambda_+) > w(\lambda_+)$. Let $y \in \mathcal{Y}$ be such that $\lambda_+(y) > \lambda_+ - \delta$. As $\lim_{t\to\lambda_+(y)} y(t) = +\infty$, we can suppose, possibly for a smaller $\delta > 0$, that $y(t) > z(t)$ on $[\lambda_+ - \delta, \lambda_+(y)[$. Hence z can be continued to a right-nonextendible solution v of (1.1) defined on $[t_0, \lambda_+(v)[$, with $\lambda_+(v) > \lambda_+ + \delta$, and satisfying $v(t_0) = w(t_0)$, $v \geqslant w$ on $[t_0, \lambda_+(v)[$ and $v > w$ on $[t_0, \omega_- + T[\cap [t_0, \lambda_+(v)[$. Since $v \in \mathcal{Y}$ we get a contradiction. Therefore $\lambda_+ > \omega_- + T$. Pick $y \in \mathcal{Y}$ such that $\lambda_+(y) > \omega_- + T$. As $\lim_{t\to(\omega_-+T)^+} x(t - T) = +\infty$, there is $\delta > 0$ such that $w(t) \leqslant y(t) < x(t - T)$ on $]\omega_- + T, \omega_- + T + \delta]$. Hence y can be continued to a solution u of (1.1) defined on $[t_0, t_0 + T]$ and satisfying $w(t) \leqslant u(t) \leqslant x(t - T)$ on $]\omega_- + T, t_0 + T]$. Since u is a solution of (1.3) and $u > w$ on $[t_0, \omega_- + T]$, the maximality of w yields a contradiction. Therefore the claim is proved. □

REMARK 1.10. The following more general conclusion can be achieved. Let $x :]\omega_-, \omega_+[\to \mathbb{R}$ be a nonextendible solution of (1.1) such that $\omega_-, \omega_+ \in \mathbb{R}$, $\omega_+ - \omega_- \leqslant T$ and, for any solution u of (1.3), either $x < u$ or $x > u$. Then two possibilities may occur:

- for all solutions u of (1.3), either $x \ll u$ or $x \gg u$;
- there exist a solution u of (1.3) and $t_0 \in]\omega_-, \omega_+[$ such that $u(t_0) = x(t_0)$ and, e.g., $u < x$. In this case there exists $\bar{t} \in]t_0, t_0 + T[$ such that, for every $\bar{x} > u(\bar{t})$, there is a solution v of (1.3) with $u < v < x$ on $]\omega_-, \omega_+[$ and $v(\bar{t}) = \bar{x}$.

A simple example where the latter possibility occurs follows.

EXAMPLE 1.2. Let us consider Eq. (1.1) taking $f : [-\frac{3}{2}, \frac{3}{2}[\times \mathbb{R} \to \mathbb{R}$ defined by $f(t, x) = 2x^2 \operatorname{sgn}(t)$, if $|x| > 1$, and $f(t, x) = 2\sqrt{|x|} \operatorname{sgn}(t)$, if $|x| \leqslant 1$, and extending f by 3-periodicity with respect to the t-variable. Let $x :]-\frac{3}{2}, \frac{3}{2}[\to \mathbb{R}$ be defined by $x(t) = \frac{1}{3+2t}$ on $]-\frac{3}{2}, -1]$, $x(t) = t^2$ on $]-1, 1]$, $x(t) = \frac{1}{3-2t}$ on $]1, \frac{3}{2}[$. Note that x is a nonextendible solution of (1.1) such that $x > u$ for any solution u of (1.3), 0 is a solution of (1.3), $x(0) = 0$ and, for every $\bar{x} > 0$, there is a solution v of (1.3) with $0 < v < x$ on $] - \frac{3}{2}, \frac{3}{2}[$ and $v(\frac{3}{2}) = \bar{x}$.

The Massera Existence Theorem

When uniqueness of solutions of (1.2) fails, there may exist solutions which are not T-monotone (see Example 1.3). It is useful to notice that existence of a solution that is not T-monotone yields existence of T-periodic solutions. Proposition 1.38 will actually show that infinitely many T-periodic solutions do exist in this case.

COROLLARY 1.11. *Assume* (C). *Let* $x :]t_0, +\infty[\to \mathbb{R}$ *be a solution of* (1.1). *If* x *is not* T*-monotone, then there exists a solution* u *of* (1.3) *such that* $u \not\leqslant x$ *and* $u \not\geqslant x$.

PROOF. Since x is not T-monotone, there are points $t_1, t_2 \in]t_0, +\infty[$ such that $x(t_1) < x(t_1 + T)$ and $x(t_2) > x(t_2 + T)$. Hence we can proceed as in the first step of the proof of Theorem 1.10 to get the conclusion. □

We finally observe that combining Corollary 1.11 and Proposition 1.7 immediately yields the following statement.

PROPOSITION 1.12. *Assume* (C). *Suppose that Eq.* (1.1) *has a bounded solution defined on an unbounded interval. Then problem* (1.3) *has at least one solution.*

Proposition 1.12 provides an extension to the Carathéodory setting of the existence part of the Massera Theorem. Yet, it must be observed that, when uniqueness for (1.1) fails, convergence of a bounded solution x to a T-periodic solution u is not generally guaranteed, as the following example will show. A possible extension of the convergence part of the Massera Theorem to the case where the uniqueness assumption is dropped will be discussed in Section 1.7.

EXAMPLE 1.3. We consider Eq. (1.12) again. As described in Example 1.1, for each $p \in]-2^{-10}, 2^{-10}[$, there exists a 1-periodic solution u_p of (1.12) such that $u_p(-\frac{1}{2}) = p$ and $u_p(0) = 0$. Let $p_1 \neq p_2$ and set $x(t) = u_{p_1}(t)$ on $[0, 1[$ and $x(t) = u_{p_2}(t)$ on $[1, 2[$. Since $x(\frac{1}{2}) = p_1 \neq p_2 = x(\frac{3}{2})$, the extension to $\mathbb{R}$ of x by 2-periodicity is a 2-periodic solution of (1.12) which is not 1-periodic; hence it is a bounded solution which does not converge to any 1-periodic solution of (1.12). As we shall see in Example 1.4, x is a subharmonic solution of order 2 of (1.12), i.e. its minimal period is 2.

1.5. *Structure and dynamics near one-sided isolated periodic solutions*

We investigate in this section the dynamics of all solutions of (1.1) in the vicinity of one-sided isolated T-periodic solutions; namely, solutions of (1.1) which lie between a pair of strictly ordered T-periodic solutions, or above the maximum T-periodic solution, or below the minimum T-periodic solution, provided they exist. T-monotonicity plays an important role in this study. Incidentally we notice that the maximum solution of (1.3) always exists, provided the solution set of (1.3) is bounded from above. Similarly the minimum solution of (1.3) always exists, provided the solution set of (1.3) is bounded from below.

Between two ordered T-periodic solutions

We start by observing that, given a pair of T-periodic solutions, either they are strictly ordered, or there exist infinitely many T-periodic solutions hitting them.

PROPOSITION 1.13. *Assume* (C). *Let* v, w *be solutions of* (1.3), *with* $v \neq w$. *Then either* $v \ll w$, *or* $v \gg w$, *or for every* $t_0 \in [0, T[$ *and every* $x_0 \in [\min\{v(t_0), w(t_0)\}, \max\{v(t_0), w(t_0)\}]$ *there exists a solution* u *of* (1.3) *with* $u(t_0) = x_0$ *and* $\min\{v, w\} \leqslant u \leqslant \max\{v, w\}$.

PROOF. Assume that there exists $\bar{t} \in [0, T[$ such that $v(\bar{t}) = w(\bar{t})$. Let $t_0 \in [0, T]$ and $x_0 \in [\min\{v(t_0), w(t_0)\}, \max\{v(t_0), w(t_0)\}]$. We may assume that $t_0 \leqslant \bar{t}$, otherwise we replace $\bar{t}$ with $\bar{t} + T$. Let us consider the T-periodic functions α and β defined by $\alpha(t_0) = x_0$, $\alpha(t) = \min\{v(t), w(t)\}$ on $]t_0, \bar{t}]$, $\alpha(t) = \max\{v(t), w(t)\}$ on $]\bar{t}, t_0 + T[$, $\beta(t_0) = x_0$, $\beta(t) = \max\{v(t), w(t)\}$ on $]t_0, \bar{t}]$, $\beta(t) = \min\{v(t), w(t)\}$ on $]\bar{t}, t_0 + T[$. Then α and β are respectively a lower solution and an upper solution of (1.3), hence, by Theorem 1.9, there exists a solution u of (1.3) with $\min\{\alpha, \beta\} \leqslant u \leqslant \max\{\alpha, \beta\}$. In particular $u(t_0) = x_0$ and $\min\{v, w\} \leqslant u \leqslant \max\{v, w\}$. □

We consider here the case where a pair of strictly ordered T-periodic solutions is given, the complementary situation will be discussed in Section 1.6. We prove the existence of heteroclinic solutions connecting a pair of strictly ordered T-periodic solutions, when there is no further T-periodic solution in between.

THEOREM 1.14. *Assume* (C). *Let v, w be solutions of* (1.3) *such that $v \ll w$ and there is no solution u of* (1.3) *satisfying $v < u < w$. Then either any nonextendible solution x of* (1.1), *with $v < x < w$, exists on $\mathbb{R}$, is T-increasing and satisfies*

$$\lim_{t \to -\infty} \big(x(t) - v(t)\big) = \lim_{t \to +\infty} \big(x(t) - w(t)\big) = 0,$$

or any nonextendible solution x of (1.1), *with $v < x < w$, exists on $\mathbb{R}$, is T-decreasing and satisfies*

$$\lim_{t \to -\infty} \big(x(t) - w(t)\big) = \lim_{t \to +\infty} \big(x(t) - v(t)\big) = 0.$$

PROOF. Pick any nonextendible solution x of (1.1), with $v < x < w$. Clearly, there is a t_0 such that $v(t_0) < x(t_0) < w(t_0)$. Since x exists on $\mathbb{R}$ and satisfies the assumptions of Theorem 1.10, we have that either x is T-increasing, or x is T-decreasing. Assume the former case occurs, the latter one being treated similarly. Then Proposition 1.7 and Remark 1.7 imply that there exist solutions u_1, u_2 of (1.3) satisfying $u_1 < x < u_2$, as x is not a solution of (1.3), and $\lim_{t \to -\infty}(x(t) - u_1(t)) = \lim_{t \to +\infty}(x(t) - u_2(t)) = 0$. Since $v \leqslant u_1 < u_2 \leqslant w$ and there is no solution u of (1.3) satisfying $v < u < w$, we conclude that $u_1 = v$ and $u_2 = w$.

Finally, let us suppose by contradiction that there exist two solutions x_1, x_2 of (1.1), with $v < x_1 < w$ and $v < x_2 < w$ on $\mathbb{R}$, such that $\lim_{t \to -\infty}(x_1(t) - v(t)) = \lim_{t \to +\infty}(x_1(t) - w(t)) = 0$ and $\lim_{t \to -\infty}(x_2(t) - w(t)) = \lim_{t \to +\infty}(x_2(t) - v(t)) = 0$. Set $x_* = \min\{x_1, x_2\}$ and $x^* = \max\{x_1, x_2\}$. Since either $v < x_* < w$, or $v < x^* < w$, we find a solution x of (1.1), with $v < x < w$ in $\mathbb{R}$, such that either $\lim_{t \to -\infty}(x(t) - v(t)) = \lim_{t \to +\infty}(x(t) - v(t)) = 0$, or $\lim_{t \to -\infty}(x(t) - w(t)) = \lim_{t \to +\infty}(x(t) - w(t)) = 0$, thus contradicting the conclusion previously achieved. □

Maximal and minimal T-periodic solutions

We show now that any bounded set of solutions of (1.3) has a maximal and a minimal element.

LEMMA 1.15. *Assume* (C). *Let* $\mathcal{S} \subseteq W^{1,1}(0,T)$ *be a nonempty set of solutions of* (1.3). *Assume that* $\mathcal{S}$ *is uniformly bounded from above, i.e. there exists a constant* M *such that* $\max_{[0,T]} u(t) \leqslant M$ *for all* $u \in \mathcal{S}$. *Then there exists a maximal solution* w *of* (1.3) *in the* C^0*-closure* $\bar{\mathcal{S}}$ *of* $\mathcal{S}$. *Similarly, if* $\mathcal{S}$ *is uniformly bounded from below, then there exists a minimal solution* v *of* (1.3) *in* $\bar{\mathcal{S}}$.

PROOF. We only prove the former case. We show that $(\bar{\mathcal{S}}, \leqslant)$ is inductively ordered. Let $\mathcal{T} = \{u_i \mid i \in I\}$ be a totally ordered subset of $\bar{\mathcal{S}}$. We prove that $\mathcal{T}$ has an upper bound in $\bar{\mathcal{S}}$. Let us set $u(t) = \sup_{i \in I} u_i(t)$ on $[0,T]$. Let $\mathcal{D} = \{t_m \mid m \in \mathbb{N}\}$ be a countable dense subset of $[0,T]$ and define a sequence in $\mathcal{T}$ as follows: for $n = 1$, take $u_1 \in \mathcal{T}$ such that $u_1(t_1) \geqslant u(t_1) - 1$, for $n = 2$, take $u_2 \in \mathcal{T}$, with $u_2 \geqslant u_1$, such that $u_2(t_2) \geqslant u(t_2) - \frac{1}{2}$, $u_2(t_1) \geqslant u(t_1) - \frac{1}{2}$, and so on. In this way, we construct a sequence $(u_n)_n$ in $\mathcal{T}$, with $u_1 \leqslant u_2 \leqslant \cdots \leqslant u_n \leqslant u_{n+1} \leqslant \cdots$, such that $u_n(t_k) \geqslant u(t_k) - \frac{1}{n}$, for $1 \leqslant k \leqslant n$. It is clear that $(u_n)_n$ converges to u pointwise on $\mathcal{D}$. On the other hand, as $(u_n)_n$ is relatively compact in $C^0([0,T])$, there is a subsequence of $(u_n)_n$ which converges uniformly to some function $\hat{u} \in C^0([0,T])$. Actually, by monotonicity, the whole sequence $(u_n)_n$ converges uniformly to $\hat{u}$. Using the L^1-Carathéodory conditions, we conclude that the convergence takes place in $W^{1,1}(0,T)$, the limit $\hat{u} \in W^{1,1}(0,T)$ and $\hat{u}$ is a solution of (1.3). Moreover, $\hat{u} = u$ on $\mathcal{D}$ and $\hat{u} \leqslant u$ on $[0,T]$. Let us show that $\hat{u} = u$ on $[0,T]$. Indeed, otherwise, one can find a point $t_0 \in [0,T]$ and a function $u_0 \in \mathcal{T}$ such that $\hat{u}(t_0) < u_0(t_0) \leqslant u(t_0)$. The continuity of $\hat{u}$ and u_0 and the density of $\mathcal{D}$ yield a contradiction. This proves that $u \in \bar{\mathcal{S}}$ is an upper bound of $\mathcal{T}$. Since $(\bar{\mathcal{S}}, \leqslant)$ is inductively ordered, Zorn Lemma guarantees the existence of a maximal element $w \in \bar{\mathcal{S}}$. □

By the lattice structure of the solution set of (1.3), we get the following conclusion.

COROLLARY 1.16. *Suppose* (C). *Assume that the set of all solutions of* (1.3) *is uniformly bounded from above. Then there exists the maximum solution of* (1.3). *Assume that the set of all solutions of* (1.3) *is uniformly bounded from below. Then there exists the minimum solution of* (1.3).

Above the maximum, or below the minimum T-periodic solution

We now assume that there exists the maximum solution of (1.3) and describe the qualitative behaviour of all solutions of (1.1) that lie above it. Symmetric conclusions can be established for solutions lying below the minimum T-periodic solution, whenever it exists.

THEOREM 1.17. *Assume* (C). *Suppose that there exists the maximum solution* w *of* (1.3). *Then either any nonextendible solution* x *of* (1.1), *with* $x > w$, *is* T*-increasing or any nonextendible solution* x *of* (1.1), *with* $x > w$, *is* T*-decreasing. Furthermore, for any such* $x :]\omega_-, \omega_+[\to \mathbb{R}$, *either*

$$\omega_-, \omega_+ \in \mathbb{R}, \quad \omega_+ - \omega_- \leqslant T, \quad x \gg w \quad \textit{on }]\omega_-, \omega_+[\quad \textit{and}$$
$$\lim_{t \to \omega_-} x(t) = \lim_{t \to \omega_+} x(t) = +\infty,$$

or

$$\omega_- = -\infty, \quad \lim_{t\to-\infty}\big(x(t) - w(t)\big) = 0 \quad \textit{and} \quad \limsup_{t\to\omega_+} x(t) = +\infty,$$

or

$$\omega_+ = +\infty, \quad \limsup_{t\to\omega_-} x(t) = +\infty \quad \textit{and} \quad \lim_{t\to+\infty}\big(x(t) - w(t)\big) = 0.$$

PROOF. Let $x :]\omega_-, \omega_+[\to \mathbb{R}$ be a nonextendible solution of (1.1) with $x > w$.

Assume first that $\omega_-, \omega_+ \in \mathbb{R}$ and that $\omega_+ - \omega_- \leqslant T$. The Claim stated in the proof of Theorem 1.10 implies that $x \gg w$ on $]\omega_-, \omega_+[$. Then the conclusion follows.

Suppose next that $\omega_+ - \omega_- > T$. By Theorem 1.10 either x is T-increasing, or x is T-decreasing. Assume that the former situation occurs, the latter case being treated similarly. Since x is bounded from below, we have $\omega_- = -\infty$. Let us verify that $\lim_{t\to-\infty}(x(t) - w(t)) = 0$. Remark 1.7 implies that there exists a solution u of (1.3), with $u \leqslant x$, such that $\lim_{t\to-\infty}(x(t) - u(t)) = 0$. Since $u \geqslant w$, the maximality of w yields $u = w$. Of course, if $\omega_+ < +\infty$, then $\lim_{t\to\omega_+} x(t) = +\infty$. Let us show that, if $\omega_+ = +\infty$, then $\limsup_{t\to\omega_+} x(t) = +\infty$. Indeed, otherwise, x would be bounded and Proposition 1.7 would imply the existence of a solution u of (1.3), with $u \geqslant x$, such that $\lim_{t\to+\infty}(x(t) - u(t)) = 0$. Since $u > w$, the maximality of w would yield a contradiction.

Finally, let us suppose, by contradiction, that there exist $x_1 :]-\infty, \omega_+[\to \mathbb{R}$, $x_2 :]\omega_-, +\infty[\to \mathbb{R}$ both solutions of (1.1), with $x_1 > w$ and $x_2 > w$, such that x_1 is T-increasing and x_2 is T-decreasing. Then $\lim_{t\to-\infty}(x_1(t) - w(t)) = 0$ and $\lim_{t\to+\infty}(x_2(t) - w(t)) = 0$. Possibly replacing $x_1(t)$ with $x_1(t - kT)$ for some $k \in \mathbb{N}$, we may assume $\omega_+ > \omega_-$. Set $x_* = \min\{x_1, x_2\}$ and $x^* = \max\{x_1, x_2\}$. Then either $w < x_*$ or $w \not\ll x^*$; in both cases we contradict the conclusions previously achieved. □

1.6. *Stability and detours*

In this section we study the stability properties of T-periodic solutions of (1.1) with the aid of lower and upper solutions. We start by recalling the classical notion of one-sided Lyapunov stability. Yet, in the present context, where uniqueness of the solutions of (1.2) is not guaranteed, such a definition does not generally seem the most appropriate to be considered; indeed, some weaker concept might fit better in order to detect certain residual forms of stability. As an alternative notion to Lyapunov stability we use order stability; this is common in the frame of monotone dynamical systems [93,58,60] and appears suited to our approach based on lower and upper solutions. Lyapunov stability implies order stability, but not vice versa; however, these concepts are equivalent if uniqueness, either in the past or in the future, holds for solutions of (1.2). We further give a characterization of order stability in terms of the asymptotic behaviour of solutions of (1.1) by introducing a third kind of stability, which is named weak stability. This notion was first defined in [130] in the context of multivalued dynamical systems (see also [50]) and is related to that of weak positive invariance considered in [105,152]. We show that weak stability and order stability are always equivalent. Using these concepts, we give a precise description of the stability properties of a T-periodic solution in terms of the existence of a lower or an upper solution close to it. Hence, when a pair of lower and upper solutions is given, we can discuss the stability or instability of the minimum and the maximum T-periodic solutions v and w wedged between them, thus getting a completion of Theorem 1.9. Afterwards, we turn to investigate the behaviour of the solutions lying between v and w. If the lower and the upper solutions are ordered in the standard way, i.e. the lower solution is below the upper solution, we find between v and w a totally ordered continuum of weakly stable T-periodic solutions. Whereas, in the complementary case we see that Lyapunov instability always occurs and, in the absence of uniqueness for (1.2), solutions of (1.1) having very complicated, even chaotic-like, dynamics may exist. In particular, we can find homoclinics, subharmonics of any order and almost periodic solutions, which are not periodic of any fixed period.

One-sided Lyapunov stability

DEFINITION 1.14.
- A solution u of (1.3) is said *Lyapunov stable* (briefly, *$\mathcal{L}$-stable*) *from below* if, for every $\varepsilon > 0$, there is $\delta > 0$ such that, for every $t_0 \in [0, T[$ and every x_0 with $u(t_0) - \delta < x_0 < u(t_0)$, every right-nonextendible solution x of (1.2), with $x \leqslant u$, exists on $[t_0, +\infty[$ and satisfies

$$u(t) - \varepsilon < x(t) \leqslant u(t) \quad \text{on } [t_0, +\infty[. \tag{1.15}$$

- If, further,

$$\lim_{t \to +\infty} \big(x(t) - u(t)\big) = 0, \tag{1.16}$$

  u is said *$\mathcal{L}$-asymptotically stable from below*.
- *$\mathcal{L}$-stability* and *$\mathcal{L}$-asymptotic stability from above* are defined similarly.
- A solution u of (1.3) is said *$\mathcal{L}$-stable* if, for every $\varepsilon > 0$, there is $\delta > 0$ such that, for every $t_0 \in [0, T[$ and every x_0 with $|x_0 - u(t_0)| < \delta$, every right-nonextendible solution x of (1.2) exists on $[t_0, +\infty[$ and satisfies

$$\big|x(t) - u(t)\big| < \varepsilon \quad \text{on } [t_0, +\infty[. \tag{1.17}$$

  If, further, (1.16) holds, u is said *$\mathcal{L}$-asymptotically stable*.
- A solution u of (1.3) is said *$\mathcal{L}$-unstable from below* if it is not $\mathcal{L}$-stable from below. *$\mathcal{L}$-instability from above* and *$\mathcal{L}$-instability* are defined similarly.

REMARK 1.11. The notion of $\mathcal{L}$-stability from below given in Definition 1.14 does not require any condition on solutions $x : [t_0, \omega_+[\to \mathbb{R}$ of (1.2), satisfying $u(t_0) - \delta < x_0 < u(t_0)$, but not $x \leqslant u$. A similar remark holds for the $\mathcal{L}$-stability from above.

DEFINITION 1.15. A solution u of (1.3) is said *isolated from below* in $C^0([0,T])$ if there exists $\varepsilon > 0$ such that no solution z of (1.3) exists satisfying $u - \varepsilon < z < u$. Similarly, a solution u of (1.3) is said *isolated from above* in $C^0([0,T])$ if there exists $\varepsilon > 0$ such that no solution z of (1.3) exists satisfying $u < z < u + \varepsilon$.

PROPOSITION 1.18. *Assume* (C). *A solution u of* (1.3) *is $\mathcal{L}$-asymptotically stable from below* (*respectively $\mathcal{L}$-asymptotically stable from above*) *if and only if it is $\mathcal{L}$-stable from below and isolated from below* (*respectively $\mathcal{L}$-stable from above and isolated from above*) *in $C^0([0,T])$.*

PROOF. Assume u is $\mathcal{L}$-asymptotically stable from below, then u is clearly $\mathcal{L}$-stable and isolated from below in $C^0([0,T])$ as a solution of (1.3). Suppose now that u is $\mathcal{L}$-stable from below and isolated from below. Let $\varepsilon > 0$ be such that no solution z of (1.3) exists satisfying $u - \varepsilon < z < u$. By the $\mathcal{L}$-stability from below of u, there is $\delta > 0$ such that every right-nonextendible solution x of (1.2), with $u(t_0) - \delta < x_0 < u(t_0)$ and $x \leqslant u$, exists on $[t_0, +\infty[$ and satisfies $u - \varepsilon/2 < x < u$ on $[t_0, +\infty[$. We claim that any such solution x satisfies $x(t_0) < x(t_0 + T)$. Indeed, otherwise $x|_{[t_0, t_0+T[}$ is a proper upper solution of (1.3). Hence, by Proposition 1.8 there exists a solution v of (1.3) such that $\lim_{t\to+\infty}(x(t) - v(t)) = 0$ and $u - \varepsilon < v < x < u$ on $[t_0, t_0 + T]$, thus contradicting the assumption that u is isolated from below. Hence $x|_{[t_0, t_0+T[}$ is a proper lower solution of (1.3) and, by Proposition 1.8, $\lim_{t\to+\infty}(u(t) - x(t)) = 0$. A similar argument proves the statement about $\mathcal{L}$-stability from above. □

One-sided order stability

DEFINITION 1.16.

- A solution u of (1.3) is said *order stable* (briefly, *$\mathcal{O}$-stable*) (respectively *properly $\mathcal{O}$-stable*, respectively *strictly $\mathcal{O}$-stable*) *from below* if there exists a sequence $(\alpha_n)_n$ of regular lower solutions (respectively proper regular lower solutions, respectively strict regular lower solutions) of (1.3) such that $\alpha_n < u$ for every n and $\alpha_n \to u$ uniformly on $[0,T]$.
- A solution u of (1.3) is said *$\mathcal{O}$-stable* (respectively *properly $\mathcal{O}$-stable*, respectively *strictly $\mathcal{O}$-stable*) *from above* if there exists a sequence $(\beta_n)_n$ of regular upper solutions (respectively proper regular upper solutions, respectively strict regular upper solutions) of (1.3) such that $\beta_n > u$ for every n and $\beta_n \to u$ uniformly on $[0,T]$.
- A solution u of (1.3) is said *$\mathcal{O}$-stable* (respectively *properly $\mathcal{O}$-stable*, respectively *strictly $\mathcal{O}$-stable*) if it is $\mathcal{O}$-stable (respectively properly $\mathcal{O}$-stable, respectively strictly $\mathcal{O}$-stable) from above and from below.

REMARK 1.12. The requirement for the lower and upper solutions to be regular in Definition 1.16 is not essential. Indeed, we see, for instance, that between a proper lower solution α and a solution u of (1.3), with $\alpha < u$, it is always possible to fit in a proper regular lower solution $\bar{\alpha}$. This follows from Proposition 1.20. A similar observation holds for a proper upper solution β.

In order to prove Proposition 1.20 we first need the following result concerning a property of Carathéodory functions. A more general version of it will be proved in Proposition 2.3.

PROPOSITION 1.19. *Assume* (C). *Then, for each* $\rho > 0$, *there exists a* L^1-*Carathéodory function* $h : [0, T] \times [-\rho, \rho] \times [-\rho, \rho] \to \mathbb{R}$ *such that*

(i) *for a.e.* $t \in [0, T]$ *and every* $x \in [-\rho, \rho]$, $h(t, \cdot, x) : [-\rho, \rho] \to \mathbb{R}$ *is strictly increasing*;

(ii) *for a.e.* $t \in [0, T]$ *and every* $y \in [-\rho, \rho]$, $h(t, y, \cdot) : [-\rho, \rho] \to \mathbb{R}$ *is strictly decreasing*;

(iii) *for a.e.* $t \in [0, T]$ *and every* $(x, y) \in [-\rho, \rho] \times [-\rho, \rho]$, $h(t, x, y) = -h(t, y, x)$;

(iv) *for a.e.* $t \in [0, T]$ *and every* $(x, y) \in [-\rho, \rho] \times [-\rho, \rho]$, *with* $x < y$,

$$\left| f(t, y) - f(t, x) \right| < h(t, y, x).$$

PROPOSITION 1.20. *Assume* (C). *Suppose that* u *is a solution of* (1.3). *If* α *is a proper lower solution of* (1.3), *with* $\alpha < u$, *then there exists a proper regular lower solution* $\bar{\alpha}$ *of* (1.3), *satisfying* $\lim_{t \to 0^+} \bar{\alpha}(t) = \bar{\alpha}(0) = \lim_{t \to T^-} \bar{\alpha}(t)$ *and* $\alpha < \bar{\alpha} < u$. *Similarly, if* β *is a proper upper solution of* (1.3), *with* $\beta > u$, *then there exists a proper regular upper solution* $\bar{\beta}$ *of* (1.3), *satisfying* $\lim_{t \to 0^+} \bar{\beta}(t) = \bar{\beta}(0) = \lim_{t \to T^-} \bar{\beta}(t)$ *and* $u < \bar{\beta} < \beta$.

PROOF. Let h be the function associated with f by Proposition 1.19 and corresponding to $\rho = \max\{\|\alpha\|_\infty, \|u\|_\infty\}$. Consider the periodic problem

$$x' = f(t, \alpha) - h(t, x, \alpha), \quad x(0) = x(T).$$

Since α is a proper lower solution and u is a proper upper solution, this problem has a solution $\bar{\alpha}$, satisfying $\alpha < \bar{\alpha} < u$. The properties of h imply that $\bar{\alpha}$ is a proper lower solution of (1.3). □

Lyapunov stability and order stability

We prove now that $\mathcal{L}$-stability implies $\mathcal{O}$-stability. We start with a result, which establishes the $\mathcal{O}$-stability of an isolated solution of (1.3) in the presence of a lower or an upper solution.

PROPOSITION 1.21. *Assume* (C). *Suppose that* z *is a solution of* (1.3).

- *If* α *is a proper lower solution of* (1.3) *such that* $\alpha < z$ *and there is no solution* u *of* (1.3) *satisfying* $\alpha < u < z$, *then there exists a sequence* $(\alpha_n)_n$ *of proper regular lower solutions of* (1.3), *with* $\alpha < \alpha_n < \alpha_{n+1} < z$ *for each* n, *which converges in* $W^{1,1}(0, T)$ *to* z.
- *If* β *is a proper upper solution of* (1.3) *such that* $\beta > z$ *and there is no solution* u *of* (1.3) *satisfying* $z < u < \beta$, *then there exists a sequence* $(\beta_n)_n$ *of proper regular upper solutions of* (1.3), *with* $z < \beta_{n+1} < \beta_n < \beta$ *for each* n, *which converges in* $W^{1,1}(0, T)$ *to* z.

PROOF. We only prove the statement concerning the lower solution. Repeating recursively the argument in the proof of Proposition 1.20, we get a sequence $(\alpha_n)_n$ of proper regular lower solutions of (1.3) such that, for each n, $\alpha < \alpha_n < \alpha_{n+1} < z$ and α_{n+1} is a solution of

$$x' = f(t, \alpha_n) - h(t, x, \alpha_n), \quad x(0) = x(T).$$

Since $(\alpha_n)_n$ is a bounded sequence, we see, arguing as in the proof of Proposition 1.7, that $(\alpha_n)_n$ converges in $W^{1,1}(0, T)$ to a function $\hat{\alpha}$ such that $\hat{\alpha} \leqslant z$ and $\hat{\alpha}(0) = \hat{\alpha}(T)$. Moreover, as $\lim_{n\to+\infty} h(t, \alpha_{n+1}(t), \alpha_n(t)) = 0$ a.e. on $[0, T]$, we see that $\hat{\alpha}$ is a solution of (1.3) and, hence, $\hat{\alpha} = z$. □

PROPOSITION 1.22. *Assume* (C). *If a solution u of* (1.3) *is $\mathcal{L}$-stable from below* (*respectively $\mathcal{L}$-stable from above*, *respectively $\mathcal{L}$-stable*), *then it is $\mathcal{O}$-stable from below* (*respectively $\mathcal{O}$-stable from above*, *respectively $\mathcal{O}$-stable*).

PROOF. Let u be a solution of (1.3) which is $\mathcal{L}$-stable from below. If u is not isolated from below in $C^0([0, T])$ as a solution of (1.3), there is a sequence $(\alpha_n)_n$ of (regular lower) solutions of (1.3) such that $\alpha_n < u$ for every n and $\alpha_n \to u$ uniformly on $[0, T]$, that is, u is $\mathcal{O}$-stable from below. Let us suppose that u is isolated from below in $C^0([0, T])$. Then, arguing as in the proof of Proposition 1.18, we can construct a proper lower solution α of (1.3) such that $\alpha < u$ and there is no solution w of (1.3) with $\alpha < w < u$. By Proposition 1.21, we conclude that u is $\mathcal{O}$-stable from below. □

REMARK 1.13. The converse of Proposition 1.22 does not generally hold. Indeed, there may exist solutions of (1.3) which are $\mathcal{O}$-stable, but $\mathcal{L}$-unstable. A simple example is given by (1.3), with $f(x) = \operatorname{sgn}(x)\sqrt{|x \sin(\frac{1}{x})|}$ if $x \neq 0$ and $f(x) = 0$ if $x = 0$. Here, the equilibrium 0 is obviously $\mathcal{O}$-stable, but it is $\mathcal{L}$-unstable both from below and from above.

Lyapunov stability and order stability in case of uniqueness

We show that $\mathcal{L}$-stability and $\mathcal{O}$-stability are equivalent concepts if we assume uniqueness in the future or in the past for solutions of (1.2). We need two preliminary results: the former concerns the validity of a comparison principle, the latter establishes a form of continuous dependence.

LEMMA 1.23. *Suppose* (C). *Assume that uniqueness in the past holds for solutions of* (1.2). *Let α be a lower solution and β be an upper solution of* (1.1) *on* $[a, b]$ *and suppose that $\alpha(a) < \beta(a)$. Then $\alpha \ll \beta$ on* $[a, b]$.

PROOF. We first show that $\alpha \leqslant \beta$. Notice that α and β are a lower solution and an upper solution for any initial value problem $x' = f(t, x)$, $x(a) = x_0$ with $\alpha(a) \leqslant x_0 \leqslant \beta(a)$. Hence, if we assume, by contradiction, that $\alpha \not\leqslant \beta$, then Proposition 1.1 yields the existence of $t_0 \in]a, b]$ such that $\alpha(t_0) = \beta(t_0)$. Theorem 1.4 implies then that Eq. (1.1) has solutions x_1, x_2, with $x_1(a) = \alpha(a)$ and $x_2(a) = \beta(a)$, satisfying $\alpha \leqslant x_1 < x_2 \leqslant \beta$ on $[a, t_0]$ and hence $x_1(t_0) = x_2(t_0)$; thus contradicting uniqueness in the past for solutions of (1.2). We just proved that $\alpha < \beta$. Finally, by arguing as above, uniqueness in the past for solutions of (1.2) implies that $\alpha \ll \beta$ on $[a, b]$. □

REMARK 1.14. Assume that uniqueness in the past holds for solutions of (1.2). If α is a lower solution and u is a solution of (1.3), with $\alpha < u$, then $\alpha \ll u$. This implies that, if u is $\mathcal{O}$-stable from below, i.e. if there exists a sequence $(\alpha_n)_n$ of regular lower solutions of (1.3) such that $\alpha_n < u$ for every n and $\alpha_n \to u$ uniformly on $[0, T]$, then there is a subsequence $(\alpha_{n_k})_k$ of $(\alpha_n)_n$ such that $\alpha_{n_k} \ll \alpha_{n_{k+1}} \ll u$ for every k. Further we can see that proper $\mathcal{O}$-stability from below is equivalent to strict $\mathcal{O}$-stability from below.

LEMMA 1.24. *Suppose* (C). *Assume that uniqueness in the future holds for solutions of* (1.2). *Let z be a solution of* (1.1) *on* $[a, b]$. *Then for every $\varepsilon > 0$ there is $\delta > 0$ such that, for every $t_0 \in [a, b[$ and every x_0, with $z(t_0) - \delta < x_0 < z(t_0)$, any right-nonextendible solution x of* (1.2) *exists* (*at least*) *on* $[t_0, b]$ *and satisfies* $z - \varepsilon < x < z$ *on* $[t_0, b]$.

PROOF. We start with the following basically known statement (cf. [121]), which we prove for the sake of completeness.

CLAIM. *For every $\varepsilon > 0$ there is $\delta > 0$ such that every right-nonextendible solution x of* (1.1), *with $z(a) - \delta < x(a) < z(a)$, exists and satisfies $z(t) - \varepsilon < x(t) \leqslant z(t)$ on* $[a, b]$.

Assume by contradiction that there exists $\varepsilon > 0$ and a sequence $(x_n)_n$ of solutions of (1.1) on $[a, b_n]$, with $b_n < b$, such that $x_n(a) \to z(a)$, $z(t) - \varepsilon < x_n(t) < z(t)$ on $[a, b_n[$ and $x_n(b_n) = z(b_n) - \varepsilon$. We can assume that $(x_n(a))_n$ is strictly increasing and hence, by uniqueness in the future, $x_n < x_{n+1}$ on $[a, b_n]$. Therefore, $(b_n)_n$ is increasing, so that $b_n \to \bar{b} \in]a, b]$. Let x be a solution of (1.1), with $x(\bar{b}) = z(\bar{b}) - 2\varepsilon$, defined on $[\bar{b} - \eta, \bar{b}]$, for some $\eta > 0$. By uniqueness in the future, we have $x \leqslant x_n \leqslant z$ on $[\bar{b} - \eta, \bar{b}] \cap [a, b_n]$, for all large n. Hence any such a solution x_n can be continued to $\bar{b}$ and satisfies $z - \varepsilon \leqslant x_n \leqslant z$ on $[a, b_n[$ and $x \leqslant x_n \leqslant z$ on $[b_n, \bar{b}]$. Further, since the sequence $(x_n)_n$ is bounded, by the L^1-Carathéodory conditions, it is also equicontinuous on $[a, \bar{b}]$ and therefore, by monotonicity, it converges to some function y uniformly on $[a, \bar{b}]$. Since the convergence takes place in $W^{1,1}(0, T)$ as well, y is a solution of (1.1), which satisfies $y(a) = z(a)$, as $x_n(a) \to z(a)$. Accordingly, by uniqueness in the future, we conclude that $y = z$ on $[a, \bar{b}]$. On the other hand, as $x_n(b_n) = z(b_n) - \varepsilon \to z(\bar{b}) - \varepsilon$, and, by equicontinuity, $x_n(b_n) \to y(\bar{b})$, we get $y(\bar{b}) = z(\bar{b}) - \varepsilon$, which is a contradiction. Thus, the proof of the claim is finished.

Conclusion. Fix $\varepsilon > 0$ and, according to the Claim, for every $t_0 \in [a, b[$ let $\delta(t_0)$ be the supremum of all $\delta > 0$ such that every right-nonextendible solution x of (1.1), with $z(t_0) - \delta < x(t_0) < z(t_0)$, exists and satisfies $z(t) - \varepsilon < x(t) \leqslant z(t)$ on $[t_0, b]$. This lemma will be proved if we show that $\inf_{t_0 \in [a,b[} \delta(t_0) > 0$. Assume by contradiction that $\inf_{t_0 \in [a,b[} \delta(t_0) = 0$ and let $(t_n)_n$ be a sequence such that $\delta(t_n) \to 0$. We can assume that $(t_n)_n$ is increasing and converges to some $\bar{t} \in]a, b]$. If $(t_n)_n$ is decreasing, the proof is similar. Let $\bar{\delta} > 0$ be the constant associated with $\bar{t}$ by the Claim. We can find a solution $\bar{x}$ of (1.1), with $z(\bar{t}) - \bar{\delta} < \bar{x}(\bar{t}) < z(\bar{t})$, which exists on $[\bar{t} - \eta, b]$, for some $\eta > 0$, and satisfies $z - \varepsilon < \bar{x} < z$ on $[\bar{t} - \eta, b]$ and $\bar{x} \ll z$ on $[\bar{t} - \eta, \bar{t}]$. Then every right-nonextendible solution x of (1.1), with $\bar{x}(\bar{t} - \eta) < x(\bar{t} - \eta) < z(\bar{t} - \eta)$, exists and satisfies $\bar{x} < x < z$ and, hence, $z - \varepsilon < x < z$

on $[\bar{t}-\eta, b]$. This yields a contradiction as we have $\delta(t) \geqslant \min_{[\bar{t}-\eta,\bar{t}]}(z(t)-\bar{x}(t)) > 0$, for every $t \in [\bar{t}-\eta, \bar{t}]$. □

PROPOSITION 1.25. *Assume* (C). *Suppose that uniqueness in the past, or in the future, holds for solutions of* (1.2). *Then a solution u of* (1.3) *is $\mathcal{L}$-stable from below* (*respectively $\mathcal{L}$-stable from above, respectively $\mathcal{L}$-stable*) *if and only if it is $\mathcal{O}$-stable from below* (*respectively $\mathcal{O}$-stable from above, respectively $\mathcal{O}$-stable*).

PROOF. Let u be a solution of (1.3), which is $\mathcal{O}$-stable from below, and let $(\alpha_n)_n$ be a sequence of regular lower solutions of (1.3) such that $\alpha_n < u$ for every n and $\alpha_n \to u$ uniformly on $[0, T]$.

Case 1: *uniqueness in the past holds for* (1.2). By Lemma 1.23 and Remark 1.14, we can suppose that $\alpha_n \ll u$ for each n. Fix $\varepsilon > 0$ and pick n such that $\|\alpha_n - u\|_\infty < \varepsilon$. Set $\delta = \inf_{[0,T]}(u-\alpha_n) > 0$. Then for every $t_0 \in [0, T[$ and x_0 such that $\alpha_n(t_0) \leqslant u(t_0) - \delta < x_0 < u(t_0)$, any right-nonextendible solution x of (1.2) satisfies, according to Lemma 1.23, $\alpha_n \ll x \ll u$ on $[t_0, b]$ for any given $b > t_0$. Hence x exists and satisfies $u - \varepsilon < x < u$ on $[t_0, +\infty[$. This means that u is $\mathcal{L}$-stable from below.

Case 2: *uniqueness in the future holds for* (1.2). The conclusion is an immediate consequence of the following statement.

CLAIM. *Let α be a regular lower solution of* (1.3), *with $\alpha < u$, and set $\eta = \|\alpha - u\|_\infty$. Then there exists $\delta > 0$ such that for every $t_0 \in [0, T[$ every right-nonextendible solution x of* (1.2), *with $u(t_0) - \delta < x_0 < u(t_0)$, exists and satisfies $u - \eta < x < u$ on $[t_0, +\infty[$.*

If $\alpha \ll u$, then the result follows setting $\delta = \inf_{[0,T]}(u-\alpha) > 0$ and using Corollary 1.5. Therefore, suppose $\inf_{[0,T]}(u-\alpha) = 0$. Hence there exist points $t_1 \in [0, T]$ and $t_2 \in]t_1, t_1 + T[$ such that $\alpha(t_1) = u(t_1) - \eta$, $\lim_{t \to t_2^-} \alpha(t) = u(t_2)$ and $u(t) - \eta \leqslant \alpha(t) < u(t)$ on $[t_1, t_2[$. By Lemma 1.24, there is $\delta \in]0, \eta[$ such that, for every $t_0 \in [0, T[$, any right-nonextendible solution x of (1.2), with $u(t_0) - \delta < x_0 < u(t_0)$, exists and satisfies $u(t) - \eta < x(t) \leqslant u(t)$ on $[t_0, t_1 + T]$. Since $\alpha(t_1 + T) = u(t_1 + T) - \eta \leqslant x(t_1 + T) \leqslant u(t_1 + T)$ and α is a lower solution, x exists and satisfies $\alpha(t) \leqslant x(t) \leqslant u(t)$ on $[t_1 + T, t_2 + T[$ and $x(t) = u(t)$ on $[t_2 + T, +\infty[$. □

One-sided weak stability

The following definition of stability is a possible weakening of $\mathcal{L}$-stability suited to deal with cases where uniqueness of the solutions of (1.2) is not guaranteed (see [50]).

DEFINITION 1.17.

- We say that a solution u of (1.3) is *weakly stable* (briefly, *$\mathcal{W}$-stable*) *from below* if, for every $\varepsilon > 0$, there is a uniformly continuous function $\delta : [0, T[\to \mathbb{R}$, with $\delta > 0$, such that, for every $t_0 \in [0, T[$ and every x_0, with $u(t_0) - \delta(t_0) \leqslant x_0 \leqslant u(t_0)$, there exists a solution $x : [t_0, +\infty[\to \mathbb{R}$ of (1.2) which satisfies (1.15).
- If, further, (1.16) holds, we say that u is *$\mathcal{W}$-asymptotically stable from below*.
- *$\mathcal{W}$-stability* and *$\mathcal{W}$-asymptotic stability from above* are defined in a similar way.
- We say that a solution u of (1.3) is *$\mathcal{W}$-stable* if, for every $\varepsilon > 0$, there is a uniformly continuous function $\delta : [0, T[\to \mathbb{R}$, with $\delta > 0$, such that, for every $t_0 \in [0, T[$ and

every x_0, with $|x_0 - u(t_0)| \leqslant \delta(t_0)$, there exists a solution $x : [t_0, +\infty[\to \mathbb{R}$ of (1.2) which satisfies (1.17). If, further, (1.16) holds, we say that u is *$\mathcal{W}$-asymptotically stable*.

Order stability and weak stability

The notion of $\mathcal{W}$-stability is in any case equivalent to that of $\mathcal{O}$-stability.

PROPOSITION 1.26. *Assume* (C). *Let u be a solution of* (1.3). *Then u is $\mathcal{W}$-stable from below* (*respectively $\mathcal{W}$-stable from above*, *respectively $\mathcal{W}$-stable*) *if and only if it is $\mathcal{O}$-stable from below* (*respectively $\mathcal{O}$-stable from above*, *respectively $\mathcal{O}$-stable*).

PROOF. We only discuss stability from below. The proof is divided into two steps.

Step 1. Let u be a solution of (1.3), which is $\mathcal{O}$-stable from below, and let $(\alpha_n)_n$ be a sequence of regular lower solutions of (1.3) such that $\alpha_n < u$ for every n and $\alpha_n \to u$ uniformly on $[0, T]$. Fix $\varepsilon > 0$ and take n such that $\|\alpha_n - u\|_\infty < \varepsilon$. Then set $\delta = u - \alpha_n$ on $[0, T[$. Corollary 1.5 implies that, for every $t_0 \in [0, T]$ and every x_0, with $u(t_0) - \delta(t_0) \leqslant x_0 \leqslant u(t_0)$, there is a solution x which exists and satisfies $u - \varepsilon < x \leqslant u$ on $[t_0, +\infty[$.

Step 2. Let u be a solution of (1.3), which is $\mathcal{W}$-stable from below. If u is not isolated from below as a solution of (1.3), then it is obviously $\mathcal{O}$-stable. Accordingly, suppose that u is isolated from below. Let $\varepsilon > 0$ be such that there is no solution z of (1.3), satisfying $z < u$ and $\|z - u\|_\infty \leqslant \varepsilon$, and let δ be the function associated with ε. Since $\delta > 0$, there exists $t_0 \in [0, T[$ such that $\delta(t_0) > 0$. Fix a strictly increasing sequence $(x_0^{(n)})_n$ such that, for every n, $u(t_0) - \delta(t_0) < x_0^{(n)} < u(t_0)$ and $x_0^{(n)} \to u(t_0)$. Denote by $(x_n)_n$ the corresponding sequence of solutions of (1.2), satisfying, for every n, $x_n(t_0) = x_0^{(n)}$, $u - \varepsilon < x_n \leqslant u$ and $x_n < x_{n+1}$ on $[t_0, +\infty[$. Fix n and observe that $x_n(t_0 + T) > x_n(t_0)$. Indeed, otherwise $x_n|_{[t_0, t_0+T[}$ is a proper upper solution of (1.3) and hence, by Proposition 1.8, there exists a solution z of (1.3) such that $z < x_n$ on $[t_0, +\infty[$, $x_n(t) - z(t) \to 0$, as $t \to +\infty$, and hence $\|z - u\|_\infty \leqslant \varepsilon$, which is a contradiction. Therefore, denoting by α_n the T-periodic extension of $x_n|_{[t_0, t_0+T[}$ to $\mathbb{R}$, we conclude that $(\alpha_n)_n$ is a sequence of proper lower solutions of (1.3), satisfying, for every n, $\alpha_n < \alpha_{n+1} < u$ and $\alpha_n \to u$ uniformly on $[0, T]$. According to Proposition 1.20, we can construct a further sequence $(\bar{\alpha}_n)_n$ of proper regular lower solutions of (1.3) such that, for every n, $\bar{\alpha}_n < u$ and $\bar{\alpha}_n \to u$ uniformly on $[0, T]$. This implies that u is $\mathcal{O}$-stable from below. □

Stability via lower and upper solutions

We now come to the core of this section. In the following results we use lower and upper solutions to give a precise description of the stability properties of the T-periodic solutions of (1.1). In the next theorem we prove the stability of a one-sided isolated solution z of (1.3) in the presence of a lower solution below z, or of an upper solution above z.

THEOREM 1.27. *Assume* (C). *Let z be a solution of* (1.3).

(i) *If α is a proper lower solution of* (1.3) *such that $\alpha < z$ and there is no solution u of* (1.3) *satisfying $\alpha < u < z$, then z is $\mathcal{W}$-asymptotically stable from below.*

(ii) *If β is a proper upper solution of* (1.3) *such that $\beta > z$ and there is no solution u of* (1.3) *satisfying $z < u < \beta$, then z is $\mathcal{W}$-asymptotically stable from above.*

PROOF. We prove only the former statement. Since α is a proper lower solution, we know, by Proposition 1.21, that z is (properly) $\mathcal{O}$-stable from below, hence, by Proposition 1.26, z is $\mathcal{W}$-stable from below. The asymptotic $\mathcal{W}$-stability is then a consequence of Corollary 1.5 and Proposition 1.8. □

Conversely, Theorem 1.30 will show that instability occurs in all the other cases; it is based on the following two propositions.

PROPOSITION 1.28. *Assume* (C). *Let z be a solution of* (1.3).

(i) *If α is a proper lower solution of* (1.3) *such that $\alpha > z$ and there is no solution u of* (1.3) *satisfying $\alpha > u > z$, then z is $\mathcal{L}$-unstable from above.*

(ii) *If β is a proper upper solution of* (1.3) *such that $\beta < z$ and there is no solution u of* (1.3) *satisfying $z > u > \beta$, then z is $\mathcal{L}$-unstable from below.*

PROOF. We prove only the former statement. Let $t_0 \in [0, T]$ be such that $\alpha(t_0) > z(t_0)$. By Remark 1.9 there exists a solution $\widetilde{\alpha} :]-\infty, t_0] \to \mathbb{R}$ of (1.2), with $x_0 = \alpha(t_0)$, satisfying $z \leqslant \widetilde{\alpha} \leqslant \alpha$ on $]-\infty, t_0]$ and such that

$$\lim_{t\to-\infty} \big(\widetilde{\alpha}(t) - z(t)\big) = 0. \tag{1.18}$$

Set $\varepsilon = \alpha(t_0) - z(t_0) > 0$ and take any $\delta > 0$. By (1.18), there exists a positive integer n such that $z(t_0 - nT) \leqslant \widetilde{\alpha}(t_0 - nT) < z(t_0 - nT) + \delta$. Define, on $]-\infty, t_0 + nT]$, $x(t) = \widetilde{\alpha}(t - nT)$. Then x is a solution of (1.1) with $z(t_0) \leqslant x(t_0) < z(t_0) + \delta$ but $x(t_0 + nT) - z(t_0 + nT) = \varepsilon$. Hence we easily conclude that z is $\mathcal{L}$-unstable from above. □

PROPOSITION 1.29. *Assume* (C). *Let z be a solution of* (1.3).

(i) *If there exist a lower solution α of* (1.3) *and points t_1, t_2 such that $z(t_1) = \alpha(t_1)$ and $z(t_2) < \alpha(t_2)$, then z is $\mathcal{L}$-unstable from above.*

(ii) *If there exist an upper solution β of* (1.3) *and points t_1, t_2 such that $z(t_1) = \beta(t_1)$ and $z(t_2) > \beta(t_2)$, then z is $\mathcal{L}$-unstable from below.*

PROOF. We prove only the former statement. Assume without loss of generality that $t_1 < t_2$. By Proposition 1.8, we know that either every right-nonextendible solution $x : [t_1, \omega_+[\to \mathbb{R}$ of (1.2), with $t_0 = t_1$, $x_0 = \alpha(t_1) = z(t_1)$ and $x \geqslant \alpha$, is such that

$$\limsup_{t\to\omega_+} x(t) = +\infty,$$

or there exist the minimum solution $\widetilde{\alpha} : [t_1, +\infty[\to \mathbb{R}$ in $[\alpha, +\infty[$ of (1.2), with $t_0 = t_1$, $x_0 = \alpha(t_1) = z(t_1)$, and the minimum solution v in $[\alpha, +\infty[$ of (1.3), satisfying $v \geqslant \widetilde{\alpha}$ and

$$\lim_{t\to+\infty} \big(\widetilde{\alpha}(t) - v(t)\big) = 0.$$

Since there exists t_2 such that $\alpha(t_2) > z(t_2)$, we have $v \neq z$ and hence

$$\limsup_{t\to+\infty}\bigl(\widetilde{\alpha}(t) - z(t)\bigr) > 0.$$

In both cases we easily conclude that z is $\mathcal{L}$-unstable from above. □

THEOREM 1.30. *Assume* (C). *Let z be a solution of* (1.3).

(i) *If there exists a lower solution α of* (1.3) *such that $z \not\geqslant \alpha$ and, in case $z < \alpha$, there is no solution u of* (1.3) *satisfying $z < u < \alpha$, then z is $\mathcal{L}$-unstable from above.*

(ii) *If there exists an upper solution β of* (1.3) *such that $z \not\leqslant \beta$ and, in case $z > \beta$, there is no solution u of* (1.3) *satisfying $\beta < u < z$, then z is $\mathcal{L}$-unstable from below.*

PROOF. We prove only the former statement. Two cases may occur: either $\alpha > z$ or, by Proposition 1.6, there exist points t_1, t_2 such that $z(t_1) = \alpha(t_1)$ and $z(t_2) < \alpha(t_2)$. In the former case, the conclusion follows from Proposition 1.28, whereas in the latter it follows from Proposition 1.29. □

A direct consequence of the previous results is the following completion of Theorem 1.9, for what concerns the stability properties of solutions of (1.3) in the presence of a pair of lower and upper solutions.

THEOREM 1.31. *Assume* (C). *Suppose that α is a lower solution and β is an upper solution of* (1.3). *Denote by v and w the minimum and the maximum solutions in* $[\min\{\alpha,\beta\}, \max\{\alpha,\beta\}]$ *of* (1.3).

(i) *If $\alpha < \beta$, then v is $\mathcal{W}$-asymptotically stable from below, provided α is a proper lower solution, and w is $\mathcal{W}$-asymptotically stable from above, provided β is a proper upper solution.*

(ii) *If $\alpha > \beta$, then v is $\mathcal{L}$-unstable from below, provided β is a proper upper solution, and w is $\mathcal{L}$-unstable from above, provided α is a proper lower solution.*

(iii) *If $\alpha \not\leqslant \beta$ and $\beta \not\leqslant \alpha$, then every solution u of* (1.3) *satisfying $v \leqslant u \leqslant w$ is $\mathcal{L}$-unstable from below and from above.*

PROOF. In case (i) the conclusion follows from Theorem 1.27. In case (ii) the conclusion follows from Proposition 1.28. In case (iii), if u is a solution of (1.3) satisfying $\min\{\alpha,\beta\} \leqslant v \leqslant u \leqslant w \leqslant \max\{\alpha,\beta\}$, then according to Proposition 1.6, there exist points t_1 and t_2 such that $\alpha(t_1) = \beta(t_1) = u(t_1)$ and either $\beta(t_2) < u(t_2)$, or $u(t_2) < \alpha(t_2)$. The instability of u then follows from Proposition 1.29. □

REMARK 1.15. In case (i), from Proposition 1.8 we further deduce the following invariance and attractivity result. For every t_0, every solution $x : [t_0, +\infty[\to \mathbb{R}$ of (1.1) is such that: if $\alpha \leqslant x \leqslant \beta$ then $\widetilde{\alpha} \leqslant x \leqslant \widetilde{\beta}$, if $\alpha \leqslant x \leqslant v$ then $\lim_{t\to+\infty}(v(t) - x(t)) = 0$, if $w \leqslant x \leqslant \beta$ then $\lim_{t\to+\infty}(x(t) - w(t)) = 0$.

From Propositions 1.18, 1.25, 1.26 and Remark 1.14 we immediately get the following result.

COROLLARY 1.32. *Assume* (C). *Suppose that α is a proper lower solution, β is a proper upper solution of* (1.3) *and $\alpha < \beta$. Denote by v and w the minimum and the maximum solutions in* $[\alpha, \beta]$ *of* (1.3). *Suppose further that either uniqueness in the past for solutions of* (1.2) *holds, or uniqueness in the future for solutions of* (1.2) *holds and α and β are strict. Then v is $\mathcal{L}$-asymptotically stable from below and w is $\mathcal{L}$-asymptotically stable from above.*

Continua of stable or unstable T-periodic solutions

Theorem 1.31 describes the stability properties of the minimum and the maximum solutions v and w of (1.3) wedged between a given pair of lower and upper solutions. Now, we turn to investigate the behaviour of the solutions between v and w. We start by specifying statement (i) in Theorem 1.31. Namely, we prove the existence of a totally ordered continuum of (two-sided) $\mathcal{W}$-stable T-periodic solutions lying between a pair of lower and upper solutions ordered in the standard way, i.e. the lower solution is below the upper solution.

THEOREM 1.33. *Assume* (C). *Suppose that α is a proper lower solution and β is a proper upper solution of* (1.3), *satisfying $\alpha < \beta$, and denote by v and w respectively the minimum and the maximum solution of* (1.3) *in* $[\alpha, \beta]$. *Then there exists a totally ordered continuum $\mathcal{K}$ in $C^0([0, T])$, such that every $u \in \mathcal{K}$ is an $\mathcal{O}$-stable solution of* (1.3) *satisfying $v \leqslant u \leqslant w$; further, $u_1 = \min \mathcal{K}$ is properly $\mathcal{O}$-stable from below and $u_2 = \max \mathcal{K}$ is properly $\mathcal{O}$-stable from above.*

PROOF. Let us denote by $\mathcal{S}_1$ the set of all solutions u of (1.3), with $\alpha < u < \beta$, which are properly $\mathcal{O}$-stable from below. Since, by Proposition 1.21, the minimum solution v is properly $\mathcal{O}$-stable from below, $\mathcal{S}_1$ is non-empty. Notice also that, since $\mathcal{S}_1$ is bounded in $C^0([0, T])$, the L^1-Carathéodory conditions imply that it is also equicontinuous and hence relatively compact in $C^0([0, T])$.

Let us show that the set $(\mathcal{S}_1, \leqslant)$ is inductively ordered. Let $\mathcal{T}$ be a totally ordered subset of $\mathcal{S}_1$. Since $\mathcal{T}$ is uniformly bounded from above, by Lemma 1.15 there exists a maximal solution $\hat{w}$ of (1.3) in the C^0-closure $\bar{\mathcal{T}}$ of $\mathcal{T}$. Actually $\hat{w} \in \mathcal{T}$. Indeed, let $(w_n)_n$ be an increasing sequence in $\mathcal{T}$ converging to $\hat{w}$ in $C^0([0, T])$; since each w_n is properly $\mathcal{O}$-stable from below, by a diagonal argument we conclude that $\hat{w}$ as well is properly $\mathcal{O}$-stable from below. Hence any totally ordered subset of $\mathcal{S}_1$ has an upper bound in $\mathcal{S}_1$. By Zorn Lemma, there exists a maximal element $u_1 \in \mathcal{S}_1$.

Let us denote by $\mathcal{S}_2$ the set of all solutions u of (1.3), with $u_1 \leqslant u < \beta$, which are properly $\mathcal{O}$-stable from above. Since the maximum solution w is properly $\mathcal{O}$-stable from above, $\mathcal{S}_2$ is non-empty. Arguing as above, we see that there exists a minimal solution $u_2 \in \mathcal{S}_2$.

If $u_1 = u_2$, the conclusion is achieved. Therefore, let us suppose that $u_1 < u_2$ and let us denote by $\mathcal{S}_3$ the set of all solutions u of (1.3), with $u_1 \leqslant u \leqslant u_2$. Notice that $\mathcal{S}_3$ is compact in $C^0([0, T])$.

Let us observe that there is no proper lower solution and no proper upper solution of (1.3) between u_1 and u_2. Indeed, if we assume that there exists, for instance, a proper lower solution α^*, with $u_1 < \alpha^* < u_2$, and we denote by z the minimum solution of (1.3), with

$\alpha^* < z \leqslant u_2$, Proposition 1.21 implies that z is properly $\mathcal{O}$-stable from below, thus contradicting the maximality of u_1.

Next, we prove that if $z_1, z_2 \in \mathcal{S}_3$, with $z_1 < z_2$, then there exists a solution z_3 of (1.3) such that $z_1 < z_3 < z_2$. Indeed, if we assume that there is no solution z of (1.3), with $z_1 < z < z_2$, then any nonextendible solution x of (1.1), with $z_1 < x < z_2$, is either a proper lower solution or a proper upper solution, thus contradicting our preceding conclusion. Hence $\mathcal{S}_3$ is dense-in-itself with respect to the order.

Now, let us fix a solution $u_0 \in \mathcal{S}_3$ and denote by $\mathcal{S}(u_0)$ a maximal totally ordered subset of $\mathcal{S}_3$ with $u_0 \in \mathcal{S}(u_0)$. As $u_1, u_2 \in \mathcal{S}(u_0)$, this set is nondegenerate. By Lemma 1.3, $\mathcal{S}(u_0)$ is homeomorphic to a compact interval of $\mathbb{R}$, so it is connected.

Finally, it is clear that every $u \in \mathcal{S}(u_0)$ is $\mathcal{O}$-stable and $u_1 = \min \mathcal{S}(u_0)$ and $u_2 = \max \mathcal{S}(u_0)$ are properly $\mathcal{O}$-stable from below and above, respectively. Therefore, we can set $\mathcal{K} = \mathcal{S}(u_0)$. □

REMARK 1.16. The set $\mathcal{S}_3$, defined in the proof of Theorem 1.33, is obviously connected too. Indeed, for each $u_0 \in \mathcal{S}_3$, the set $\mathcal{S}(u_0)$ is connected. Since $u_1, u_2 \in \bigcap_{u_0 \in \mathcal{S}_3} \mathcal{S}(u_0)$, the set $\mathcal{S}_3 = \bigcup_{u_0 \in \mathcal{S}_3} \mathcal{S}(u_0)$ is connected.

REMARK 1.17. We notice that if α is a lower solution and β is an upper solution of (1.3), with $\alpha > \beta$, then one cannot generally guarantee the existence in between of a solution of (1.3) which is $\mathcal{L}$-unstable both from below and from above. A simple example is given by (1.3), with $f(x) = x|\sin(1/x)|$ for $x \neq 0$ and $f(x) = 0$ for $x = 0$. Here, $\beta = -1$ is a strict upper solution and $\alpha = 1$ is a strict lower solution; in between there is a countable set of equilibria, none of which is simultaneously $\mathcal{L}$-unstable from below and from above. Of course, according to Theorem 1.31, there are equilibria which are $\mathcal{L}$-unstable either from below or from above.

Finally, we consider the case where α and β are not comparable. A counterpart of Theorem 1.33, which specifies statement (iii) in Theorem 1.31, is based on the following preliminary result.

LEMMA 1.34. *Assume* (C). *Let* v, w *be solutions of* (1.3), *such that* $v \neq w$ *and* $v(t_0) = w(t_0)$ *for some* t_0. *Then the set*

$$\mathcal{K} = \big\{u : [t_0, t_0 + T] \to \mathbb{R} \mid u \text{ is a solution of } (1.1) \text{ with } \min\{v, w\} \leqslant u \leqslant \max\{v, w\}\big\}$$

is a nondegenerate continuum in $C^0([t_0, t_0 + T])$ *and every* $u \in \mathcal{K}$, *extended by* T*-periodicity onto* $\mathbb{R}$, *is a* $\mathcal{L}$*-unstable solution of* (1.3).

PROOF. Every solution u of (1.1), with $\min\{v, w\} \leqslant u \leqslant \max\{v, w\}$ on $[t_0, t_0 + T]$, extended by T-periodicity onto $\mathbb{R}$, is a solution of (1.3). Since $\min\{v, w\}$ and $\max\{v, w\}$ are solutions of (1.2), with $x_0 = v(t_0) = w(t_0)$, Theorem 1.4 guarantees that the set $\mathcal{K}$ is a continuum in $C^0([t_0, t_0 + T])$. The instability finally follows from Proposition 1.29. □

THEOREM 1.35. *Assume* (C). *Suppose that* α *is a lower solution and* β *is an upper solution of* (1.3), *with* $\alpha \not\leqslant \beta$ *and* $\alpha \not\geqslant \beta$. *Then the set*

$$\mathcal{K} = \{u \mid u \text{ is a solution of } (1.3) \text{ with } \min\{\alpha, \beta\} \leqslant u \leqslant \max\{\alpha, \beta\}\}$$

is a continuum in $C^0([0, T])$ *and every* $u \in \mathcal{K}$ *is* $\mathcal{L}$*-unstable.*

Complicated dynamics

In the frame of Lemma 1.34, the existence of a nondegenerate continuum entails fairly complicated dynamics for the solutions of (1.1) between $\min\{v, w\}$ and $\max\{v, w\}$. In particular, there exist homoclinics, subharmonics of any order and almost periodic solutions, which are not periodic of any fixed period; this last assertion will now be proved.

THEOREM 1.36. *Assume* (C). *Let* v, w *be solutions of* (1.3) *such that* $v \neq w$ *and* $v(t_0) = w(t_0)$ *for some* t_0. *Then there exists a* (*Čech–Lebesgue*) *infinite dimensional set* $\mathcal{X} \subseteq L^\infty(\mathbb{R})$ *such that every* $x \in \mathcal{X}$ *is a* (*Bohr*) *almost periodic solution of* (1.1), *with* $\min\{v, w\} < x < \max\{v, w\}$, *which is not periodic of any fixed period.*

PROOF. Possibly replacing v with $\min\{v, w\}$ and w with $\max\{v, w\}$, we can assume $v < w$ and $v(t_0) = w(t_0)$ for some t_0. It is not even restrictive to suppose $t_0 = 0$. Let $\mathcal{K} \subseteq C^0([0, T])$ be the set of all solutions x of (1.3) with $v \leqslant x \leqslant w$. By Lemma 1.34, $\mathcal{K}$ is a nondegenerate continuum and it is dense-in-itself with respect to the order (see also Proposition 1.13).

Step 1. *Existence of almost periodic solutions.* Without loss of generality we may assume $T = 1$. Fix a sequence $(u_n)_n$ of solutions of (1.3) such that $v \leqslant u_n \leqslant w$ and

$$0 < \|u_{n+\nu} - u_n\|_\infty < 2^{-n} \tag{1.19}$$

for every $n \in \mathbb{N}$ and $\nu \in \mathbb{N}^+$. Hence $(u_n)_n$ uniformly converges to a solution u_∞ of (1.3), with $v \leqslant u_\infty \leqslant w$. Let us define a function $x : \mathbb{R} \to \mathbb{R}$ by setting $x(t) = u_\infty(t)$ on $[0, 1[$ and, for each $n \in \mathbb{N}$ and $k \in \mathbb{Z}$, $x(t) = u_n(t)$ on $[(2k+1)2^n, (2k+1)2^n + 1[$. The function x is well-defined, since every $m \in \mathbb{Z} \setminus \{0\}$ can be represented in a unique way as $m = (2k+1)2^n$ for some $k \in \mathbb{Z}$ and $n \in \mathbb{N}$. Moreover, since $u_n(0) = u_\infty(0)$ for each n, x is a solution of (1.1).

We want to prove that x is almost periodic and it is not periodic of any fixed period. Let us recall that a continuous function $x : \mathbb{R} \to \mathbb{R}$ is almost periodic if, for every $\varepsilon > 0$, there exists $l > 0$ such that, for every t_0, there is $\tau \in [t_0, t_0 + l]$ with $|x(t+\tau) - x(t)| < \varepsilon$ on $\mathbb{R}$.

Fix $\varepsilon > 0$ and pick $n \in \mathbb{N}^+$ such that

$$2^{-n} < \varepsilon \tag{1.20}$$

and

$$\|u_j - u_\infty\|_\infty < \varepsilon \tag{1.21}$$

for all $j \geqslant n$. Set $l = 2^{n+2}$ and take $t_0 \in \mathbb{R}$. Let $k \in \mathbb{Z}$ be such that $(2k-1)2^n < t_0 \leqslant (2k+1)2^n$. Accordingly, set $\tau = (2k+1)2^n$ and note that $\tau \in [t_0, t_0 + 2^{n+1}[\subseteq [t_0, t_0 + l]$. We want to prove that, for every t, $|x(t+\tau) - x(t)| < \varepsilon$.

Let $t \in [0, 1[$ and hence $t + \tau \in [(2k+1)2^n, (2k+1)2^n + 1[$. We have, by (1.21),

$$\left|x(t+\tau) - x(t)\right| = \left|u_n(t) - u_\infty(t)\right| < \varepsilon.$$

Let $t \in [(2j+1)2^p, (2j+1)2^p + 1[$, for some $j \in \mathbb{Z}$ and $p \in \mathbb{N}$, and hence $t + \tau \in [(2j+1)2^p + (2k+1)2^n, (2j+1)2^p + (2k+1)2^n + 1[$. In order to estimate $x(t+\tau) - x(t)$, we distinguish three cases.

• Let $p < n$. Then there exists $r \in \mathbb{N}$ such that $p + 1 + r = n$ and $(2j+1)2^p + (2k+1)2^n = (2j+1)2^p + (2k+1)2^{p+1+r} = (2(j + (2k+1)2^r) + 1)2^p$. Hence we have $x(t+\tau) = u_p(t) = x(t)$.

• Let $p > n$. Then there exists $r \in \mathbb{N}$ such that $n + 1 + r = p$ and $(2j+1)2^p + (2k+1)2^n = (2j+1)2^{n+r+1} + (2k+1)2^n = (2(k + (2j+1)2^r) + 1)2^n$. Hence, by (1.19) and (1.20), we have $x(t+\tau) = u_n(t)$ and

$$\left|x(t+\tau) - x(t)\right| = \left|u_n(t) - u_p(t)\right| < 2^{-n} < \varepsilon.$$

• Let $p = n$. Then we have $(2j+1)2^p + (2k+1)2^n = (j+k+1)2^{p+1}$. If $j+k+1 = 0$, then $t + \tau \in [0, 1[$ and, by (1.20) and (1.21),

$$\left|x(t+\tau) - x(t)\right| = \left|u_\infty(t) - u_p(t)\right| = \left|u_\infty(t) - u_n(t)\right| < \varepsilon.$$

If $j + k + 1 \neq 0$, then there exist $r \in \mathbb{N}$ and $h \in \mathbb{Z}$ such that $j + k + 1 = (2h+1)2^r$; hence $(j+k+1)2^{p+1} = (2h+1)2^{p+r+1}$. Accordingly, by (1.19) and (1.20), we get

$$\left|x(t+\tau) - x(t)\right| = \left|u_{p+r+1}(t) - u_p(t)\right| = \left|u_{n+r+1}(t) - u_n(t)\right| < 2^{-n} < \varepsilon.$$

Finally, we prove that x is not T-periodic for any $T > 0$. We first recall that, for each $n \in \mathbb{N}$, $x = u_n$ on $[2^n, 2^n + 1]$ and $x = u_{n+1}$ on $[2^{n+1}, 2^{n+1} + 1]$. Since $u_n \neq u_{n+1}$, x is not T-periodic for any $T \in \mathbb{Q}^+$. Assume by contradiction that x is T-periodic for some $T \in \mathbb{R}^+ \setminus \mathbb{Q}$. For each $n \in \mathbb{Z}$, we have $x(n) = x(0)$, by construction, and $x(n + mT) = x(0)$ for every $m \in \mathbb{Z}$, by T-periodicity. Using the density in $\mathbb{R}$ of the set $\{n + mT \mid n, m \in \mathbb{Z}\}$ and the continuity of x, we conclude that x is constant, which is a contradiction.

Step 2. *Existence of an infinite dimensional set of almost periodic solutions.* Fix a sequence $(z_n)_n$ of solutions of (1.3) satisfying $v < z_{n+1} < z_n < w$ and $\|z_n - z_{n+\nu}\|_\infty < 2^{-n}$ for every $n, \nu \in \mathbb{N}$. For each n, let $\mathcal{S}_n$ be the set of all solutions u of (1.3) such that $z_{2n+1} \leqslant u \leqslant z_{2n}$. By Lemma 1.34, $\mathcal{S}_n \subseteq C^0([0, T])$ is compact and dense-in-itself with respect to the order. Hence Lemma 1.3 implies that there exists a continuum $\mathcal{T}_n \subseteq \mathcal{S}_n$ which is homeomorphic to $[0, 1]$. For any sequence $(u_n)_n \in \prod_{n=0}^{+\infty} \mathcal{T}_n$ denote by $\Phi((u_n)_n)$ the almost periodic solution x of (1.1) constructed in Step 1 and set $\mathcal{X} = \Phi\big(\prod_{n=0}^{+\infty} \mathcal{T}_n\big)$. Fix a sequence $(\hat{u}_n)_n \in \prod_{n=0}^{+\infty} \mathcal{T}_n$ and let N be any positive integer. The map $\Phi_N : \prod_{n=0}^{N-1} \mathcal{T}_n \to L^\infty(\mathbb{R})$ defined by setting $\Phi_N(u_0, u_1, \ldots, u_{N-1}) = \Phi(u_0, u_1, \ldots, u_{N-1}, \hat{u}_N, \hat{u}_{N+1}, \ldots)$ is one-to-one and continuous, once $\prod_{n=0}^{N-1} \mathcal{T}_n$ is endowed with the product topology. Actually, since

$\prod_{n=0}^{N-1} \mathcal{T}_n$ is compact, Φ_N is a homeomorphism between $\prod_{n=0}^{N-1} \mathcal{T}_n$ and $\Phi_N(\prod_{n=0}^{N-1} \mathcal{T}_n) = \mathcal{X}_N$. Since $\mathcal{X}_N \subseteq \mathcal{X}$ and $\dim \mathcal{X}_N = N$, we conclude that $\dim \mathcal{X} = +\infty$ (see [47, Chapter 7]). □

EXAMPLE 1.4. We consider once more equation (1.12). Let $\alpha : [-\frac{1}{2}, \frac{1}{2}[\to \mathbb{R}$ be defined by $\alpha(t) = (\frac{\sqrt{2}-1}{4} t)^2$ on $[-\frac{1}{2}, 0]$, and $\alpha(t) = (\frac{1}{4} t)^2$ on $]0, \frac{1}{2}[$. Then α is a solution of (1.12) and a proper lower solution of the associated 1-periodic problem. Note that $\beta = 0$ is a proper upper solution of the 1-periodic problem. By Theorem 1.9, there exists a 1-periodic solution w of (1.12) such that $0 < w < \alpha$ and $w(0) = 0$. Observe that, by symmetry, if x is a solution of (1.12), then $y(t) = -x(-t)$ is also a solution of (1.12). Hence the function $v(t) = -w(-t)$ is a 1-periodic solution of (1.12) such that $v < 0$ and $v(0) = 0$. Applying Lemma 1.34 we see that the set $\mathcal{K}$ of all solutions $u : [0, 1] \to \mathbb{R}$ of (1.12), with $v \leqslant u \leqslant w$, is a nondegenerate continuum in $C^0([0, 1])$ and every $u \in \mathcal{K}$ is a $\mathcal{L}$-unstable 1-periodic solution of (1.12). In addition, we can see that $v(t) < 0 < w(t)$ on $]0, 1[$ and the continuum $\mathcal{K}$ is totally ordered. Indeed, by a result in [29, Corollary 3.5], uniqueness for the Cauchy problem associated with Eq. (1.12) is guaranteed for any t_0, x_0 with $(t_0, x_0) \neq (0, 0)$. Hence, if $u_1, u_2 \in \mathcal{K}$, we have $u_1(t) = u_2(t)$ if and only if $t \in \mathbb{Z}$; the 1-periodicity of u_1 and u_2 yields either $u_1 < u_2$ or $u_2 < u_1$. By Theorem 1.36 there exists an infinite dimensional set $\mathcal{X} \subseteq L^\infty(\mathbb{R})$ such that every $x \in \mathcal{X}$ is an almost periodic solution of (1.12), with $v < x < w$, and x is not periodic of any fixed period. Furthermore, there exist periodic solutions x of (1.12), with $v < x < w$, that are subharmonics of any order. To verify this let $p_1, p_2, \dots, p_n$ be two by two distinct numbers such that $v(\frac{1}{2}) \leqslant p_i \leqslant w(\frac{1}{2})$ for $i = 1, \dots, n$. Let $u_i \in \mathcal{K}$ be such that $u_i(\frac{1}{2}) = p_i$. Define $x(t) = u_i(t)$ on $[i-1, i[$, for $i = 1, \dots, n$, and extend x by n-periodicity to $\mathbb{R}$. Then x is an n-periodic solution of (1.12). Clearly, x is not k-periodic for any positive integer $k < n$. Suppose by contradiction that x is T-periodic, for some non-integer $T < n$. Then $x(T) = x(0) = 0$ and $x'(T) = x'(0) = 0$, while $x'(T) = \sqrt{|x(T)|} + h(T) = h(T) < 0$. Finally, we observe that homoclinic solutions of (1.12) can be trivially constructed as well.

REMARK 1.18. The solution set of (1.3) is the union of a family of mutually ordered connected components in $C^0([0, T])$. Yet, unlike the case where the property of uniqueness holds for (1.2) and hence nondegenerate continua are necessarily one-dimensional, such components may have arbitrarily large dimension (see [114]). We have seen that the dynamics of the trajectories of (1.1), which lie in the plane region filled by the graphs of the T-periodic solutions in a given connected component, may be quite complicated. Therefore any attempt to get a global qualitative portrait of solutions of (1.1) appears at least awkward, if solutions having graph in such regions are included in the analysis. Instead, it seems more meaningful to study only the dynamics of the solutions whose graph lies outside. In the light of Theorem 1.10, the qualitative behaviour of these solutions closely resembles the case where uniqueness holds. Then one could be naturally lead to look at the connected components of the set of solutions of (1.3) as single objects and to discuss the stability of each of them, by checking the stability from below of its minimum element and the stability from above of its maximum element, whenever they exist. Of course the consideration of a multivalued Poincaré operator would play a role here.

1.7. *Asymptotic behaviour of bounded solutions*

We already noticed in Section 1.4 that the existence of a solution u of (1.3) is always guaranteed in the presence of a bounded solution x of (1.1) defined on an unbounded interval. In this section we make this statement more precise, providing a complete extension of the Massera Convergence Theorem to the case where the uniqueness assumption for solutions of (1.2) is dropped. Namely, we see that either x converges to u, or the set of all solutions u of (1.3), whose range contains ω-limit points of x, forms a nondegenerate closed connected set in $C^0([0,T])$.

Lack of T-monotonicity and continua of T-periodic solutions
We start with a technical lemma saying that if a solution x of (1.1) and a solution v of (1.3) attain the same values at two points $a < b$, then there is a nondegenerate closed connected set of solutions u of (1.3) whose graphs completely fill the graph of x on $[a,b]$.

LEMMA 1.37. *Assume* (C). *Let v be a solution of* (1.3) *and $x:[a,b]\to\mathbb{R}$ a solution of* (1.1). *Suppose that $x(a)=v(a)$, $x(b)=v(b)$ and $x(t)>v(t)$ on $]a,b[$. Then the set $\mathcal{C}$ of all solutions $u\geqslant v$ of* (1.3) *such that there exists $t_0\in[a,b]$ with $u(t_0)=x(t_0)$ is a nondegenerate closed connected subset of $C^0([0,T])$. Furthermore, for each $t_0\in[a,b]$ there exists $u\in\mathcal{C}$ with $u(t_0)=x(t_0)$.*

PROOF. The set $\mathcal{C}$ is clearly closed and non-empty.

Step 1. Let us prove that $\mathcal{C}$ is dense-in-itself with respect to the order. Indeed, let $u_1,u_2\in\mathcal{C}$ be such that $u_1<u_2$. If $u_1(t_0)=u_2(t_0)$ for some t_0, then Lemma 1.34 yields the existence of a solution u of (1.3) with $u_1<u<u_2$. Clearly, $u\in\mathcal{C}$. If $u_1\ll u_2$ and no solution u of (1.3) exists with $u_1<u<u_2$, we set $\hat{x}=x$ on $[a,b]$ and $\hat{x}=v$ on $\mathbb{R}\setminus[a,b]$, $y=\min\{\max\{\hat{x},u_1\},u_2\}$. Then y is a nonextendible solution of (1.1) such that $u_1<y<u_2$ and $\lim_{t\to-\infty}(y(t)-u_1(t))=\lim_{t\to+\infty}(y(t)-u_1(t))=0$, contradicting the conclusions of Theorem 1.14. Hence there is a solution u of (1.3) such that $u_1<u<u_2$. Clearly, $u\in\mathcal{C}$.

Step 2. Let us prove that $\mathcal{C}$ is not a singleton. We distinguish two cases.

• Let $b-a\leqslant T$. Consider the function $w:[a,a+T]\to\mathbb{R}$ defined by $w(t)=x(t)$ on $[a,b]$ and $w(t)=v(t)$ on $[b,a+T]$, and extend w by T-periodicity onto $\mathbb{R}$. Then both v and w belong to $\mathcal{C}$.

• Let $b-a>T$. We show that v is not isolated from above in the set of all solutions of (1.3). Indeed, if v were the maximum solution of (1.3), then the function $\hat{x}:\mathbb{R}\to\mathbb{R}$ defined by $\hat{x}(t)=x(t)$ on $[a,b]$ and $\hat{x}(t)=v(t)$ on $\mathbb{R}\setminus[a,b]$ would be a solution of (1.1) contradicting the conclusions of Theorem 1.17. Whereas, if w were a solution of (1.3) with $v\ll w$ and such that there is no solution u of (1.3) with $v<u<w$, then the function $\hat{x}:\mathbb{R}\to\mathbb{R}$ defined by $\hat{x}(t)=\min\{x(t),w(t)\}$ on $[a,b]$ and $\hat{x}(t)=v(t)$ on $\mathbb{R}\setminus[a,b]$ would be a solution of (1.1) contradicting the conclusions of Lemma 1.14. Hence there exists a sequence $(u_n)_n$ of solutions of (1.3), with $u_n>v$ for every n, which converges uniformly to v. Then $u_n\in\mathcal{C}$ for all large n.

Step 3. Let us prove that $\mathcal{C}$ is connected. Pick $w\in\mathcal{C}$, with $w>v$. Set $\mathcal{K}=\{u\mid u$ is a solution of (1.3) with $v\leqslant u\leqslant w\}$. Notice that $\mathcal{K}\subseteq\mathcal{C}$, $\mathcal{K}$ is compact and, by Step 1, dense-in-itself with respect to the order. Let $\mathcal{T}\subseteq\mathcal{K}$ be a maximal totally ordered subset of $\mathcal{K}$ and notice that both v and w belong to $\mathcal{T}$. By Lemma 1.3, $\mathcal{T}$ is homeomorphic to a compact interval of $\mathbb{R}$. This shows that any pair $w_1, w_2 \in \mathcal{C}$ can be connected by a continuum (in fact an arc) containing v and contained in $\mathcal{C}$.

Step 4. Let us prove that for each $t_0 \in [a,b]$ there exists $u \in \mathcal{C}$ with $u(t_0) = x(t_0)$. We distinguish two cases again.

• If $b - a \leqslant T$, the conclusion follows arguing as in Step 2.

• If $b - a > T$, we prove that for each $t_0 \in [a,b]$ there exists $u \in \mathcal{C}$ with $u(t_0) \geqslant x(t_0)$. As $\mathcal{C}$ is connected the conclusion will follow. Accordingly, let us suppose by contradiction there is a point $t_0 \in [a,b]$ such that $u(t_0) < x(t_0)$ for all $u \in \mathcal{C}$. Assume first that $\mathcal{C}$ is bounded. Then $\mathcal{C}$ is a continuum. Let $\mathcal{T}$ be a nondegenerate maximal totally ordered subset of $\mathcal{C}$. Since $\mathcal{C}$ is dense-in-itself, by Lemma 1.3, $\mathcal{T}$ is homeomorphic to a compact interval. Set $w = \max \mathcal{T}$. There are points a_1, b_1 such that $a \leqslant a_1 < t_0 < b_1 \leqslant b$, $x(a_1) = w(a_1)$, $x(b_1) = w(b_1)$ and $x(t) > w(t)$ on $]a_1, b_1[$. By Step 2 there exist a solution u of (1.3), with $u > w$, and a point $t_1 \in [a_1, b_1]$ such that $u(t_1) = x(t_1)$. Hence we conclude that $u \in \mathcal{C}$ and $u > w$, thus contradicting the maximality of w. Assume now that $\mathcal{C}$ is unbounded from above and let $u \in \mathcal{C}$ be such that $\max u > \max x$. Since $u(t_0) < x(t_0)$, there are points a_1, b_1, with $a_1 < t_0 < b_1 < a_1 + T$, such that $x(a_1) = u(a_1)$, $x(b_1) = u(b_1)$ and $x(t) > u(t)$ on $]a_1, b_1[$. Since $b_1 - a_1 < T$, a contradiction is obtained, arguing as in Step 2. □

Next, we specify Corollary 1.11 by showing that existence of solutions of (1.1) which are not T-monotone yields existence of nondegenerate continua of solutions of (1.3).

PROPOSITION 1.38. *Assume* (C). *Let* $x :]\omega_-, \omega_+[\to \mathbb{R}$ *be a nonextendible solution of* (1.1) *and suppose that* x *is not* T*-monotone. Then there exists a nondegenerate continuum* $\mathcal{K}$ *of solutions of* (1.3) *such that, for every* $u \in \mathcal{K}$, *there is* $t_0 \in]\omega_-, \omega_+[$ *with* $u(t_0) = x(t_0)$.

PROOF. We proceed in two steps.

Step 1. Assume either $\omega_- = -\infty$, or $\omega_+ = +\infty$. We know from the first two steps in the proof of Theorem 1.10 that there are points t_1, t_2, p_1, p_2, with $p_1 < p_2$, such that, for every $p \in]p_1, p_2[$, there are a lower solution α and an upper solution β of (1.3) such that $\alpha(t_1) = \beta(t_1) = x(t_1)$ and $\alpha(t_2) = \beta(t_2) = p$. Then Theorem 1.9 and Lemma 1.34 yield the existence of a nondegenerate continuum $\mathcal{K}$ satisfying the required conditions.

Step 2. Assume $\omega_+, \omega_- \in \mathbb{R}$, with $\omega_+ - \omega_- > T$. We distinguish two cases.

• Suppose that

$$\lim_{t \to \omega_-} x(t) = \lim_{t \to \omega_+} x(t) = -\infty;$$

the case where $\lim_{t \to \omega_-} x(t) = \lim_{t \to \omega_+} x(t) = +\infty$ being treated similarly. Since x is not T-monotone, Theorem 1.10 implies the existence of a solution v of (1.3) such that $x(t_0) > v(t_0)$ for some $t_0 \in]\omega_-, \omega_+[$. Hence we can find $a, b \in]\omega_-, \omega_+[$, with $a < b$, such that $x(a) = v(a)$, $x(b) = v(b)$ and $x(t) > v(t)$ on $]a, b[$. Lemma 1.37 yields the existence of a nondegenerate continuum $\mathcal{K}$ having the desired properties.

• Suppose that

$$\lim_{t \to \omega_-} x(t) = -\infty \quad \text{and} \quad \lim_{t \to \omega_+} x(t) = +\infty;$$

the case where $\lim_{t\to\omega_-} x(t) = +\infty$ and $\lim_{t\to\omega_+} x(t) = -\infty$ being treated similarly. Set $\varphi(t) = x(t+T) - x(t)$ on $]\omega_-, \omega_+ - T[$. We have $\lim_{t\to\omega_-} \varphi(t) = \lim_{t\to\omega_+ - T} \varphi(t) = +\infty$ and, as x is not T-monotone, $\varphi(t_0) = x(t_0 + T) - x(t_0) < 0$ for some $t_0 \in]\omega_-, \omega_+ - T[$. Define

$$t_1 = \min\bigl\{t \in]\omega_-, \omega_+ - T[\mid \varphi(t) = 0\bigr\},$$

$$t_2 = \max\bigl\{t \in]\omega_-, \omega_+ - T[\mid \varphi(t) = 0\bigr\}$$

and set

$$u_1 = x|_{[t_1, t_1+T]}, \quad u_2 = x|_{[t_2, t_2+T]}.$$

The T-periodic extensions to $\mathbb{R}$ of u_1, u_2 are solutions of (1.3).

If $u_1 = u_2$, we set $v = u_1 = u_2$. Observe that either $x(t_0) > v(t_0)$, or $x(t_0 + T) < v(t_0 + T)$. Suppose that $x(t_0) > v(t_0)$; the other case being treated similarly. Since $t_0 \in]t_1, t_2[$, we have that $\hat{x} = \max\{v, x|_{[t_1,t_2]}\}$ is a solution of (1.1) satisfying $\hat{x} > v$ on $[t_1, t_2]$, $\hat{x}(t_1) = v(t_1)$ and $\hat{x}(t_2) = v(t_2)$. Lemma 1.37 yields the existence of a nondegenerate continuum $\mathcal{K}$ satisfying the required conditions.

If $u_1 \neq u_2$ and $u_1(\bar{t}) = u_2(\bar{t})$ for some $\bar{t}$, then Lemma 1.34 implies the existence of a nondegenerate continuum $\mathcal{K}$ having the desired properties.

If $u_1 \ll u_2$, then $t_2 - t_1 > T$. The function $\hat{x} : \mathbb{R} \to \mathbb{R}$ defined by $\hat{x}(t) = u_1(t)$ on $]-\infty, t_1[$, $\hat{x}(t) = x(t)$ on $[t_1, t_2]$ and $\hat{x}(t) = u_2(t)$ on $]t_2, +\infty[$ is a solution of (1.1) satisfying $\hat{x}(t_1 + nT) = u_2(t_1 + nT) = u_2(t_1) > u_1(t_1) = \hat{x}(t_1)$, for all large $n \in \mathbb{N}$. Accordingly, $\hat{x}$ is not T-decreasing. On the other hand, $\hat{x}$ is not T-increasing. Otherwise $\varphi(t) \geqslant 0$ on $]\omega_-, \omega_+ - T[$ and hence x should be T-increasing as well, a contradiction. Therefore, $\hat{x}$ is not T-monotone and, by Step 1, we can find a nondegenerate continuum $\mathcal{K}$ satisfying the required conditions.

If $u_1 \gg u_2$, then there exist $a, b \in]\omega_-, \omega_+[$ such that $x(a) = u_2(a)$, $x(b) = u_2(b)$ and $x(t) > u_2(t)$ on $]a, b[$. Lemma 1.37 implies the existence of a nondegenerate continuum $\mathcal{K}$ having the desired properties. □

The Massera Convergence Theorem revisited

We are now in position of proving the announced extension of the Massera Convergence Theorem.

THEOREM 1.39. *Assume* (C). *Suppose that* $x :]t_0, +\infty[\to \mathbb{R}$ *is a bounded solution of* (1.1). *Then either*

- *there exists a solution u of* (1.3) *such that*

$$\lim_{t\to+\infty} \bigl|x(t) - u(t)\bigr| = 0, \tag{1.22}$$

or

- *the set* $\mathcal{C}$ *of all solutions* u *of* (1.3) *such that*

$$\liminf_{t\to+\infty}\left|x(t)-u(t)\right|=0 \tag{1.23}$$

is a nondegenerate closed connected subset of $C^0([0,T])$.

REMARK 1.19. It may happen that there exists a solution u of (1.3) such that x converges to u, nevertheless the set $\mathcal{C}$ is nondegenerate and even unbounded (see Example 1.2).

PROOF. Let us suppose that there is no solution u of (1.3) such that (1.22) holds. Then, by Proposition 1.7, x is not eventually T-monotone. We start by showing that $\mathcal{C}$ is not empty. Next we prove that $\mathcal{C}$ has at least two elements. Finally we show that $\mathcal{C}$ is closed and connected.

Step 1. $\mathcal{C}$ *is nonempty*. Let us denote by $\mathcal{S}_x$ the set of all solutions u of (1.3) for which there are unbounded sequences of real numbers $(a_n)_n$ and $(b_n)_n$ such that $u(a_n)<x(a_n)$ and $u(b_n)>x(b_n)$ for all n. Notice that $\mathcal{S}_x\subseteq\mathcal{C}$.

We claim that $\mathcal{S}_x$ is not empty. Assume by contradiction that $\mathcal{S}_x=\emptyset$. Since x is not eventually T-monotone, by Corollary 1.11 we can construct an increasing unbounded sequence $(a_n)_n$ of real numbers and a sequence $(u_n)_n$ of solutions of (1.3) such that either

$$u_n(a_n)<x(a_n) \quad \text{and} \quad u_n\geqslant x \quad \text{on } [a_{n+1},+\infty[,$$

or

$$u_n(a_n)>x(a_n) \quad \text{and} \quad u_n\leqslant x \quad \text{on } [a_{n+1},+\infty[,$$

for all n. We shall suppose that the former eventuality occurs. A similar argument can be used in the latter case. For each n, we set $\tilde{u}_n=\min\{u_1,u_2,\dots,u_n\}$. Notice that $\tilde{u}_n$ is a solution of (1.3), $\tilde{u}_n(a_n)<x(a_n)$ and $\tilde{u}_n\geqslant x$ on $[a_{n+1},+\infty[$. Since $\tilde{u}_n\geqslant\tilde{u}_{n+1}\geqslant\inf\{x(t)\mid t\in[t_0,+\infty[\}$ on $[0,T]$, the sequence $(\tilde{u}_n)_n$ converges uniformly to a solution u of (1.3), which satisfies $u(a_n)\leqslant\tilde{u}_n(a_n)<x(a_n)$ for all n.

Now, either $u\leqslant x$ on some interval $[b,+\infty[$ and hence $u\leqslant x\leqslant\tilde{u}_n$ on $[a_{n+1},+\infty[$ for all large n, or there is an increasing unbounded sequence of real numbers $(b_n)_n$ such that $u(b_n)>x(b_n)$. In the former case we conclude that (1.22) holds. In the latter case we conclude that $u\in\mathcal{S}_x$. In both cases we get a contradiction.

Step 2. $\mathcal{C}$ *is not a singleton*. Pick $u_0\in\mathcal{C}$. Since

$$0=\liminf_{t\to+\infty}\left|x(t)-u_0(t)\right|<\limsup_{t\to+\infty}\left|x(t)-u_0(t)\right|,$$

we can find a positive number ε and an increasing unbounded sequence of real numbers $(a_n)_n$ such that either $x(a_n)-u_0(a_n)>\varepsilon$ for all n or $u_0(a_n)-x(a_n)>\varepsilon$ for all n. We shall suppose that the former eventuality occurs. A similar argument can be used in the latter case.

Let us show that u_0 cannot be isolated from above in the set of all solutions u of (1.3). We proceed by contradiction. First assume that u_0 is the maximum solution of (1.3).

Let $y = \max\{u_0, x\}$. By Theorem 1.17 we should have either $\lim_{t\to+\infty} y(t) = +\infty$ or $\lim_{t\to+\infty}(y(t) - u_0(t)) = 0$; in both cases we get a contradiction. Next notice that, if there are a solution w of (1.3) and a point $\hat{t}$ such that $u_0 < w$ and $u_0(\hat{t}) = w(\hat{t})$, then, by Lemma 1.34, u_0 is not isolated from above. Finally suppose that there exists a solution w of (1.3) such that $u_0 \ll w$ and no solution u of (1.3) exists with $u_0 < u < w$. Set $y = \min\{\max\{x, u_0\}, w\}$. By Lemma 1.14 we should have either $\lim_{t\to+\infty}(y(t) - u_0(t)) = 0$ or $\lim_{t\to+\infty}(y(t) - w(t)) = 0$; in both cases we get again a contradiction.

Since u_0 is not isolated from above, we can pick a solution u of (1.3) with $u > u_0$ and $\|u - u_0\|_\infty < \varepsilon$. Then we have $u(a_n) < u_0(a_n) + \varepsilon < x(a_n)$ for all n. Since $\liminf_{t\to+\infty} |x(t) - u_0(t)| = 0$, we conclude that $u \in \mathcal{C}$ and therefore $\mathcal{C}$ has at least two elements.

Step 3. *$\mathcal{C}$ is closed and connected.* Since $\mathcal{C}$ is obviously closed, we only need to prove that it is connected. We start showing that $\mathcal{C}$ is dense-in-itself with respect to the order. Notice that if u is a solution of (1.3) and $u_1 < u < u_2$, then $u \in \mathcal{C}$. Assume first that $u_1(\hat{t}) = u_2(\hat{t})$ for some $\hat{t}$. Then, by Lemma 1.34, there exists a nondegenerate continuum of solutions u of (1.3) with $u_1 \leqslant u \leqslant u_2$. Assume next that $u_1 \ll u_2$ and that no solution u of (1.3) exists with $u_1 < u < u_2$. Set $y = \min\{\max\{x, u_1\}, u_2\}$. Then y is a nonextendible solution of (1.1) such that $u_1 < y < u_2$. By Lemma 1.14 we should have either $\lim_{t\to+\infty}(u_2(t) - y(t)) = 0$ or $\lim_{t\to+\infty}(y(t) - u_1(t)) = 0$, contradicting the fact that both u_1 and u_2 belong to $\mathcal{C}$.

Now, let $u_1, u_2 \in \mathcal{C}$ and assume $u_1 < u_2$. Let $\mathcal{T}$ be a nondegenerate maximal totally ordered subset of $\mathcal{C} \cap [u_1, u_2]$. By Lemma 1.3, $\mathcal{T}$ is an arc connecting u_1 and u_2. Accordingly, in order to prove that $\mathcal{C}$ is (arcwise) connected, it is enough to show that for every $u_1, u_2 \in \mathcal{C}$ there exists $u \in \mathcal{C}$ which is order-comparable with both u_1 and u_2. To achieve this aim, we set $v = \min\{u_1, u_2\}$ and $w = \max\{u_1, u_2\}$. We claim that either $v \in \mathcal{C}$ or $w \in \mathcal{C}$. Suppose $v \notin \mathcal{C}$. Then there exists $\varepsilon > 0$ and $\hat{t} \in \mathbb{R}$ such that $x(t) - v(t) > \varepsilon$ for all $t > \hat{t}$. Let us verify that $w \in \mathcal{C}$. Since $u_1 \in \mathcal{C}$, there exists an increasing unbounded sequence of real numbers $(a_n)_n$ with $|x(a_n) - u_1(a_n)| < \varepsilon$ for every n. If $\hat{n}$ is such that $a_{\hat{n}} > \hat{t}$, then $x(a_n) - v(a_n) > \varepsilon > |x(a_n) - u_1(a_n)|$ for all $n > \hat{n}$. Hence we deduce that $v(a_n) \neq u_1(a_n)$; therefore $u_1(a_n) = w(a_n)$ and $|x(a_n) - w(a_n)| < \varepsilon$ for all $n > \hat{n}$. □

The following statement conversely shows that each ω-limit point of a bounded solution defined on an unbounded interval must be in the range of a T-periodic solution.

PROPOSITION 1.40. *Assume* (C). *Suppose that* $x : [t_0, +\infty[\to \mathbb{R}$ *is a bounded solution of* (1.1). *Then, for each* $\hat{x}$, *with* $\liminf_{t\to+\infty} x(t) \leqslant \hat{x} \leqslant \limsup_{t\to+\infty} x(t)$, *and for each sequence* $(t_n)_n$, *with* $\lim_{n\to+\infty} t_n = +\infty$ *and* $\lim_{n\to+\infty} x(t_n) = \hat{x}$, *there exists a solution* u *of* (1.3) *such that* $\liminf_{n\to+\infty} |x(t_n) - u(t_n)| = 0$.

PROOF. If there exists a solution u of (1.3) such that (1.22) holds, the conclusion trivially follows. Hence we may assume that this is not the case. Let $\mathcal{C}$ be the set of all solutions u of (1.3) such that (1.23) holds. By Theorem 1.39, $\mathcal{C}$ is a nondegenerate closed connected set.

Step 1. Suppose that, for each n, $t_n = \hat{t} + k_n T$ with $\hat{t} \in [t_0, t_0 + T]$ and $k_n \in \mathbb{N}^+$. Let us show that there exists a function $u \in \mathcal{C}$ such that $u(\hat{t}) = \hat{x}$. We start by assuming, by contradiction, that $u(\hat{t}) < \hat{x}$ for all $u \in \mathcal{C}$.

Assume that $\mathcal{C}$ is bounded from above. Then, by Lemma 1.15, there exists a maximal element w of $\mathcal{C}$. We first observe that there is no $\bar{t} > t_0$ such that $x > w$ on $[\bar{t}, +\infty[$. Suppose not. Since $0 = \liminf_{t\to+\infty} |x(t) - w(t)| < \limsup_{t\to+\infty} |x(t) - w(t)|$, by Theorem 1.17 there exists a solution u of (1.3) such that $w < u$, and, by Lemma 1.14, w is not isolated from above in the set of all solutions u of (1.3). Hence there exists a sequence $(w_k)_k$ of solutions of (1.3), converging uniformly to w and such that $w_k > w$ for all k. Since $\limsup_{t\to+\infty} |x(t) - w(t)| > 0$, there exists k such that $w_k \in \mathcal{C}$, thus yielding a contradiction to the maximality of w. Nevertheless, since $w(\hat{t}) < \hat{x} = \lim_{n\to+\infty} x(t_n)$, there are $\varepsilon > 0$ and $\bar{n}$ such that $w(\hat{t}) = w(t_n) < \hat{x} - \varepsilon < x(t_n)$ for all $n \geqslant \bar{n}$. Hence there exist a sequence of positive integers $(n_j)_j$, with $n_0 \geqslant \bar{n}$, and sequences of real numbers $(a_j)_j$ and $(b_j)_j$, with $a_j < t_{n_j} < b_j$ such that $w(a_j) = x(a_j)$, $w(b_j) = x(b_j)$ and $x(t) > w(t)$ on $]a_j, b_j[$ for all j. By Lemma 1.37, there exists a solution u_0 of (1.3) such that $u_0 \geqslant w$ and $u_0(t_{n_0}) = \hat{x} - \varepsilon$. Notice that, for all j, we both have $u_0(t_{n_j}) = \hat{x} - \varepsilon < x(t_{n_j})$ and $x(a_j) = w(a_j) \leqslant u_0(a_j)$. Hence $u_0 \in \mathcal{C}$, a contradiction with the maximality of w.

Assume now that $\mathcal{C}$ is unbounded from above. Then there exists $w \in \mathcal{C}$ such that $\max w > \sup x$. Let $a \in]\hat{t} - T, \hat{t}[$ be such that $w(a) = \max w$. Since $w(\hat{t}) < \hat{x} = \lim_{n\to+\infty} x(t_n)$, there are an integer $\bar{n}$ and an increasing sequence $(p_n)_n$ of real numbers such that $\lim_{n\to+\infty} p_n = \hat{x}$ and, for all $n \geqslant \bar{n}$, $w(\hat{t}) = w(t_n) < p_n < \min\{x(t_n), \hat{x}\}$. For each $n \geqslant \bar{n}$, let $w_n : [a + k_n T, a + (k_n + 1)T] \to \mathbb{R}$ be the function defined by $w_n(t) = \max\{w(t), x(t)\}$. Notice that $w_n(a + k_n T) = w(a + k_n T) = w(a)$, $w_n(a + (k_n + 1)T) = w(a + (k_n + 1)T) = w(a)$ and $w_n > w$ on $[a + k_n T, a + (k_n + 1)T]$. Since $w(t_n) < p_n < w_n(t_n)$, by Lemma 1.37, there exists a solution v_n of (1.3) such that $v_n \geqslant w$ and $v_n(t_n) = p_n$. Let $u_n = \min\{v_n, w_n\}$. Then, for each $n \geqslant \bar{n}$, $u_n(a) = w(a)$ and $u_n(\hat{t}) = p_n$. Hence, for all $n \geqslant \bar{n}$, both $u_{\bar{n}}(a + k_n T) = \max w > x(a + k_n T)$ and $u_{\bar{n}}(t_n) = p_{\bar{n}} \leqslant p_n < x(t_n)$, and therefore $u_{\bar{n}} \in \mathcal{C}$. Furthermore, the sequence $(u_n)_n$ is bounded and hence there is a subsequence $(u_{n_j})_j$ converging uniformly to a function $u \in \mathcal{C}$. Since $u_{n_j}(\hat{t}) = p_{n_j}$, we get $u(\hat{t}) = \hat{x}$, a contradiction.

In a similar way we can show that the assumption that $u(\hat{t}) > \hat{x}$ for all $u \in \mathcal{C}$ leads to a contradiction. Hence either there exists $u \in \mathcal{C}$ such that $u(\hat{t}) = \hat{x}$, or there exist $u_1, u_2 \in \mathcal{C}$ with $u_1(\hat{t}) \leqslant \hat{x} \leqslant u_2(\hat{t})$. In the latter case the connectedness of $\mathcal{C}$ yields anyhow the existence of a function $u \in \mathcal{C}$ such that $u(\hat{t}) = \hat{x}$.

Step 2. Let us show that we can always reduce ourselves to the situation considered in Step 1. Namely, for each n let k_n be the integer such that $t_n \in [t_0 + k_n T, t_0 + (k_n + 1)T[$ and set $\hat{t}_n = t_n - k_n T$. Let $\hat{t} \in [t_0, t_0 + T]$ be a cluster point of the sequence $(\hat{t}_n)_n$ and pick a subsequence $(\hat{t}_{n_j})_j$ converging to $\hat{t}$. Set $s_j = \hat{t} + k_{n_j} T$. Notice that $\lim_{j\to+\infty} |t_{n_j} - s_j| = \lim_{j\to+\infty} |\hat{t}_{n_j} - \hat{t}| = 0$. Since the function x is uniformly continuous on $[t_0, +\infty[$, we have $\lim_{j\to+\infty} |x(t_{n_j}) - x(s_j)| = 0$ and therefore $\lim_{j\to+\infty} x(s_j) = \hat{x}$. By Step 1 there exists a solution u of (1.3) such that $\liminf_{j\to+\infty} |x(s_j) - u(s_j)| = 0$. Since $|x(t_{n_j}) - u(t_{n_j})| \leqslant |x(t_{n_j}) - x(s_j)| + |x(s_j) - u(s_j)| + |u(s_j) - u(t_{n_j})|$, $\lim_{j\to+\infty} |x(t_{n_j}) - x(s_j)| = 0$ and $\lim_{j\to+\infty} |u(s_j) - u(t_{n_j})| = \lim_{j\to+\infty} |u(\hat{t}) - u(\hat{t}_{n_j})| = 0$, we conclude that $\liminf_{n\to+\infty} |x(t_n) - u(t_n)| = 0$. □

REMARK 1.20. In the light of Proposition 1.38, Proposition 1.7 and Theorem 1.39, one might guess that, if $x :]t_0, +\infty[\to \mathbb{R}$ is a bounded solution of (1.1), which is not eventually T-monotone, then the set $\mathcal{C}$ of all solutions u of (1.3), such that $\liminf_{t\to+\infty} |u(t) - x(t)| =$

0, is always a nondegenerate connected set. Yet, as the next example shows, this is not actually true.

EXAMPLE 1.5. We construct a function f, which is 1-periodic with respect to the t-variable, such that equation (1.1) has a bounded solution x that is not eventually 1-monotone, but the set $\mathcal{C}$ of all solutions u of (1.3), such that $\liminf_{t\to+\infty}|u(t)-x(t)| = 0$, is a singleton $\mathcal{C}=\{u\}$ and actually $\lim_{t\to+\infty}(u(t)-x(t))=0$. Our construction is based on a modification of Example 1.4. Let $h:\mathbb{R}\to\mathbb{R}$ be the 1-periodic function defined by $h(t)=-\frac{1}{8}|t|$ on $[-\frac{1}{2},\frac{1}{2}[$. Let us define f by setting, for every t, $f(t,x)=\sqrt{|x|}+h(t)$, if $x\leqslant-\frac{1}{8}$, $f(t,x)=\sqrt{|x-2+\frac{1}{2^{n-1}}|}+\frac{1}{2^n}h(t)$, if $n\geqslant 0$ and $x\in[2-\frac{17}{2^{3+n}},2-\frac{15}{2^{3+n}}]$, $f(t,x)=0$, if $x\geqslant 2$. Then we extend f onto $\mathbb{R}^2$ by linear interpolation with respect to x. We can see that Eq. (1.1) has a sequence $(\mathcal{K}_n)_n$ of continua of 1-periodic solutions, which approximate the equilibrium $w\equiv 2$. Each continuum $\mathcal{K}_n$ is a "squeezing" of the continuum $\mathcal{K}$ produced in Example 1.4. Denote by v_n and w_n respectively the minimum and the maximum of $\mathcal{K}_n$. One can verify that for each n, between w_n and v_{n+1}, all solutions are heteroclinics that leave w_n and reach v_{n+1} in a finite time. Indeed, for any $m_n\in\mathbb{Z}$ we can find a solution y_n of (1.1) which pulls ahead of w_n at m_n and reaches the value $2-\frac{15}{2^{3+n}}$ in a finite time. Since f is positive there, y_n enters a region where f is defined linearly. There y_n is still increasing until it reaches the value $2-\frac{17}{2^{4+n}}$, where f is still positive. Then y_n enters a region where all solutions reach v_{n+1} in a finite time.

Let us now outline a possible construction of the solution x of (1.1) having the required properties. Start taking x on [0, 1] to be any $x_1\in\mathcal{K}_1$. Then there is a solution y_1 of (1.1), which pulls ahead of w_1 at $m_1=1$ and reaches v_2 in a finite time. On this interval x is y_1. Afterwards, we take x to be for a while an almost periodic solution lying between v_2 and w_2, careful enough to destroy 1-monotonicity. Then we pull ahead of w_2 at some m_2 and we define x to be a function of the form of y_2. Proceeding in this way we get our desired solution x.

A careful reading of our construction actually shows that x can be defined so that it is not T-monotone for any fixed $T>0$.

2. Second Order Periodic Parabolic PDEs

2.1. *Introduction*

In this part we consider the parabolic boundary value problem

$$\begin{cases}\partial_t u + A(x,t,\partial_x)u = f(x,t,u,\nabla_x u) & \text{in } \Omega\times I,\\ u=0 & \text{on } \partial\Omega\times I.\end{cases} \tag{2.1}$$

Here Ω is a bounded smooth domain in $\mathbb{R}^N$, with $N\geqslant 1$, I is a real interval, $\partial_t+A(x,t,\partial_x)$ is a linear second order uniformly parabolic operator. The coefficients of A are T-periodic in t, $T>0$ being a given period. The function $f:\Omega\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ is T-periodic in t and satisfies the L^p-Carathéodory conditions, for some $p>N+2$, as well as a Nagumo condition, that is, $f(x,t,s,\xi)$ grows at most quadratically with respect to ξ. According to these assumptions, it is natural to consider solutions of (2.1) intended in the strong, i.e. $W_p^{2,1}$-, sense. In this work for simplicity we only deal with homogeneous Dirichlet data in (2.1), however more general boundary conditions, such as Robin and mixed ones, could be considered as well (see [42]).

Nonlinear diffusion equations such as (2.1) typically occur in the mathematical description of various biological, chemical, physical and economical phenomena and they have particular interest in population ecology models where the data periodically depend on time due to seasonal fluctuations (see, e.g., [49,58,155,25,115,103]).

We are mainly interested here in studying existence, localization and qualitative properties of the T-periodic solutions of (2.1), assuming that a lower solution α and an upper solution β of the T-periodic problem associated with (2.1) are given.

If we suppose that the L^p-Carathéodory function f satisfies further regularity assumptions, e.g., f is locally Lipschitz continuous with respect to s and ξ, then the initial value problem associated with (2.1) is well-posed, i.e. it has a unique solution which depends continuously on the given data; moreover, the regularity of f, in conjunction with the parabolic maximum principle, yields the validity of comparison principles of Nagumo-Westphal type (see [151,106]). In this frame the Poincaré operator associated with (2.1), which assigns to every initial datum the value of the solution after one period, is well-defined and strictly order preserving; so that the above mentioned questions can be tackled by the theory of order preserving discrete-time semidynamical systems as in [58,60]. Our perspective in this work is different. Here we assume that f satisfies only the L^p-Carathéodory conditions: in this case evidently such theory does not apply, due to the failure both of uniqueness for the initial value problem and of comparison principles. Nevertheless, we shall see that a careful direct analysis of the partial differential equation (2.1) allows to reinterpret, or to adapt to a more general setting, or to extend the validity of several results which are known within the frame of order preserving dynamical systems, always keeping minimal regularity assumptions.

The study of the existence of periodic solutions of certain special forms of (2.1) seems to have been initiated in the late thirties and continued in the forties by D.H. Karimov [64–67]. In the early fifties, G. Prodi [125–127] introduced lower and upper solutions and a Nagumo-type condition for studying the solvability of the periodic problem associated with (2.1). The stability of periodic solutions was also discussed by G. Prodi in [124], using the concept of Lyapunov stability as extended to parabolic equations by R. Bellmann [17]. In the fifties and the sixties, further progress was obtained, concerning both existence and stability (see, e.g., [107,137,75,23,69,48,12,138,70,78]). These authors approached the problem by various methods and techniques, but still only special cases of (2.1) were considered. A general answer to the question of the existence of a T-periodic solution, when the lower and the upper solutions satisfy

$$\alpha(x,t) \leqslant \beta(x,t) \quad \text{in } \Omega \times]0,T[, \tag{2.2}$$

was eventually given by Ju.S. Kolesov [72,73] in the late sixties. Namely, under additional regularity assumptions on the coefficients and the nonlinearity, he proved the existence of a classical T-periodic solution u of (2.1) such that

$$\alpha(x,t) \leqslant u(x,t) \leqslant \beta(x,t) \quad \text{in } \Omega \times]0,T[. \tag{2.3}$$

This result was improved by H. Amann [6,8] in the seventies. In particular, he proved some basic estimates in connection with the above mentioned Nagumo condition, established the existence of a minimum and of a maximum T-periodic solution between α and β and also allowed Robin boundary conditions in (2.1).

The method of Ju.S. Kolesov and H. Amann consists in finding periodic solutions of (2.1) as fixed points of the Poincaré operator associated with (2.1), whose existence is guaranteed by the smoothness assumptions on f. Further, it was observed apparently for the first time in [8] that the Poincaré operator has the important property of being strongly order preserving with respect to the order induced by the positive cone of $C_0^1(\overline{\Omega})$. In [8] and other related works, this property follows from the validity of a strong comparison principle of the Nagumo–Westphal type for (2.1), which is in turn a consequence of the linear parabolic maximum principle and, again, of the smoothness of f.

These ideas are useful for solving both theoretically and numerically the periodic problem associated with (2.1), but they also revealed to be quite fruitful for investigating the stability properties of the periodic solutions of (2.1). Indeed, strongly influenced by some works of H. Matano [92,93] who successfully integrated monotonicity methods with dynamical systems techniques for the study of autonomous parabolic initial value problems (see also M.W. Hirsch [59] and H.L. Smith [136]), E.N. Dancer and P. Hess [33] were able to provide, remaining in the same frame of [8], an answer to the question of the stability of periodic solutions of (2.1). Namely, they proved that, if α and β are strict lower and upper solutions, for which (2.2) holds, then there exists at least one T-periodic solution u satisfying (2.3) which is stable. Although preliminary results in this direction can be found, e.g., in [71,72,134,92], the novelty of the approach of E.N. Dancer and P. Hess lies in the introduction of a strongly order preserving discrete-time semidynamical system, which allows to detect many qualitative properties of the initial value problems associated with (2.1). In [33] this discrete-time semidynamical system is determined by the solution operator corresponding to the T-periodic problem related to (2.1), although the smoothness assumptions imposed on (2.1) would allow to consider as well the discrete-time semidynamical system induced by the Poincaré operator. In their successive paper [34], as well as in the monograph by P. Hess [58], the approach based on the Poincaré operator was developed in an abstract setting and systematically employed to describe the dynamics of (2.1). Thus, several stability, instability and stabilization results for the T-periodic solutions of (2.1) were obtained. In particular, the existence of heteroclinic orbits, connecting periodic solutions, was established, extending previous results, such as those in [93]. In [32] E.N. Dancer carried on these studies further, investigating the relations between fixed point index, one-sided stability and existence of lower and upper solutions close to a periodic solution, as well as the possibility of weakening the order preserving requirements on the semiflow. Further progress in the theory of discrete-time order preserving semidynamical systems has been obtained in the last ten years; we refer to the very recent works [60,123,61] for updated surveys on this theory, as well as on its applications to parabolic boundary value problems of the form of (2.1).

We notice that all the above cited papers deal with classical solutions of (2.1); whereas, the existence and localization of weak T-periodic solutions in the presence of lower and upper solutions were discussed in [44], even for more general quasilinear equations. In this frame, the existence of a minimum and a maximum solution was established in [26] and their one-sided stability in [46]. To the best of our knowledge, strong solutions have not been explicitly considered in the literature in connection with lower and upper solutions, although they naturally occur in the Carathéodory frame.

So far we have considered situations dealing with pairs of lower and upper solutions satisfying (2.2). Now, we turn to discuss the complementary case where there exist a lower solution α and an upper solution β for which condition (2.2) fails, that is,

$$\alpha(x_0, t_0) > \beta(x_0, t_0) \quad \text{for some } (x_0, t_0) \in \Omega \times]0, T[. \tag{2.4}$$

There are concrete motivations to consider this situation. Indeed, condition (2.2) turns out to be restrictive in some situations. This was remarked by J. Kazdan and F. Warner [68] for elliptic equations, but their observation immediately extends to periodic parabolic problems. Let us denote by λ_1 the principal eigenvalue of the linear T-periodic problem associated with (2.1). Assume for example that the function f in (2.1) is independent of $\nabla_x u$. If f satisfies

$$\operatorname*{ess\,sup}_{\Omega \times]0,T[\times \mathbb{R}} \partial_s f(x, t, s) < \lambda_1, \tag{2.5}$$

then one can construct lower and upper solutions α, β satisfying (2.2). On the other hand, if f satisfies

$$\operatorname*{ess\,inf}_{\Omega \times]0,T[\times \mathbb{R}} \partial_s f(x, t, s) > \lambda_1 \tag{2.6}$$

then, for any pair of lower and upper solutions α, β satisfying (2.2), one easily verifies that $\alpha = \beta$ and they must already be solutions. Hence any result involving such lower and upper solutions is of no use here. Yet, in this situation lower and upper solutions satisfying (2.4) occur quite naturally. Indeed, one can show that condition (2.6) implies the existence of pairs of lower and upper solutions α, β such that

$$\alpha(x, t) > \beta(x, t) \quad \text{in } \Omega \times]0, T[. \tag{2.7}$$

Moreover, the principle of linearized instability suggests that, if a T-periodic solution of (2.1) existed in the presence of lower and upper solutions satisfying (2.4), it should be presumably unstable, unlike what has been established when (2.2) holds. This was conjectured in the frame of elliptic boundary value problems by D.H. Sattinger [134], who was the first to pose these questions in the early seventies. Afterwards, it was pointed out in [7] that the mere existence of a lower solution and an upper solution for which (2.4) holds is generally not sufficient to guarantee the solvability. Adapted to our setting, the example in [7] is of the following type:

$$\begin{cases} \partial_t u - \Delta u = \lambda_m u + \varphi_m & \text{in } \Omega \times]0, T[, \\ u = 0 & \text{on } \partial\Omega \times [0, T], \\ u(\cdot, 0) = u(\cdot, T) & \text{in } \Omega, \end{cases} \tag{2.8}$$

where λ_m is an eigenvalue of $-\Delta$ on $H_0^1(\Omega)$ greater than λ_1 and φ_m is a corresponding nonzero eigenfunction. Of course, problem (2.8) has no solution, although one can construct a lower solution α and an upper solution β satisfying (2.4). Accordingly, one is led to conclude that, to achieve solvability, some further conditions must be assumed, in order to prevent the interference of f with the higher part of the spectrum. This can be expressed in various ways. In [9], where the first important contribution was given in the frame of elliptic equations, H. Amann, A. Ambrosetti and G. Mancini proved that the existence of a lower solution α and of an upper solution β, not necessarily satisfying any ordering condition, implies solvability, provided f is a bounded perturbation of $\lambda_1 s$. Further results, mainly in the direction of weakening this last condition, as well as of obtaining information on the localization of the solution, were later proved, always for elliptic equations, in [54,55,40]. In [41] the existence result of [40] was extended to the periodic problem for (2.1) and, apparently for the first time, the study of the stability properties was faced in this context. Indeed, assuming the existence of strict lower and upper solutions satisfying (2.4) and some growth restrictions on f, it was proved in [41] the existence of a solution u of (2.1), which is unstable and satisfies, for some $(x_1, t_1), (x_2, t_2) \in \Omega \times]0, T[$,

$$u(x_1, t_1) < \alpha(x_1, t_1) \quad \text{and} \quad u(x_2, t_2) > \beta(x_2, t_2). \tag{2.9}$$

We stress that this result is the direct counterpart of the one in [33] when condition (2.2) is replaced by (2.4), because it yields the existence of a solution, information about its localization in terms of α and β and its instability. We further notice that in [41] the analysis was entirely performed in the frame of strong solutions of (2.1) without requiring, unlike [33] and the other quoted results, any further regularity on f in addition to the Carathéodory conditions.

It should be observed that in several cases the principles of linearized stability, or the construction of Lyapunov functionals, can be successfully used to detect the stability of periodic solutions (see, e.g., [37,154,123]). On the other hand, the use of lower and upper solutions for discussing the stability properties of a periodic solution appears appropriate in some critical cases where such methods fail, in particular when the periodic solution under consideration is degenerate, or the nonlinearity f does not satisfy the necessary differentiability requirements, and obviously in all situations where the existence of a Lyapunov functional is not known. This remark also provides a strong motivation for studying the applicability of the lower and upper solution method to detect stability, or instability, and to investigate other qualitative properties, assuming the least of regularity on f. In this work we pursue this program.

The remainder of this introduction is devoted to describe with some detail the plan of this part of our work.

In Section 2.2 we collect some definitions and preliminary statements, having a basic or technical nature, which will be extensively used in the sequel. Most of these are variations or extensions of known ones. Nevertheless, we state them in a precise form, providing even proofs, whenever we are not able to supply adequate references. We introduce in particular a rather general notion of lower and upper solutions for problem (2.1), which appear appropriate in the frame of strong solutions. Namely, a lower solution is defined, locally in time, as the pointwise maximum of a finite number of $W_p^{2,1}$-lower solutions. Similarly an upper solution is defined, locally in time, as the pointwise minimum of a finite number of $W_p^{2,1}$-upper solutions. In this way we allow “angles” in the space variables and “jumps” in the time variable. This level of generality is motivated by the fact that “angles” and “jumps” naturally occur in practical and theoretical situations even in the frame of smooth solutions: typically when considering minima or maxima of solutions, which are not solutions due to the lack of lattice structure, or studying solutions of the initial value problem lying between lower and upper solutions of the periodic problem. After reviewing a parabolic maximum principle for strong solutions, some notions of orderings, order norms and other topics, we further prove in Proposition 2.3 an elementary fact, which, roughly speaking, says that any Carathéodory function satisfies a local regularity condition, which is the natural generalization of the Lipschitz or Hölder continuity and will allow us to compare problem (2.1) with other ones possessing a certain extent of monotonicity. This simple observation has also some interesting connections with the use of a monotone iterative scheme, namely the Chaplyghin method (see [100,13,8]), for the construction of the minimum and the maximum solution of (2.1), lying between a lower and an upper solution satisfying (2.2). However, for brevity, we do not discuss this topic here and we refer to [42].

In Section 2.3 we describe some basic facts concerning the initial value problem associated with (2.1). Namely, we discuss existence, localization and structure of the set of strong solutions of (2.1) in the presence of a pair of possibly discontinuous lower and upper solutions satisfying (2.2). A crucial step of this section is the Nagumo-type result stated in Proposition 2.5 (a different version suited for the periodic problem will be proved later in Proposition 2.10): these statements basically show that suitable $W_p^{2-2/p}$-, or L^∞-, bounds on the initial values force $W_p^{2,1}$-bounds on the solutions of (2.1), provided f satisfies a Nagumo condition. We reproduce in both cases their rather delicate proofs in detail, even if these follow the argument in [6], because some specifications are needed in order to extend their validity to our setting. Then we prove the main results of this section: Theorem 2.6, dealing with compact cylinders, and Corollary 2.7, dealing with noncompact cylinders. They extend to strong solutions and possibly discontinuous lower and upper solutions some existence and localization results for the initial value problem associated with (2.1) which are well known in the frame of classical solutions. We also describe the topological structure of the solution set, proving the Hukuhara–Kneser property. Since our analysis is performed just assuming that f satisfies the Carathéodory conditions, we get an improvement of previous statements obtained in [76], for f Hölder continuous, and in [81], for f continuous. Our proof uses an abstract theorem in [77,153], but differs from the standard ones in the approximation method, as we adapt here an idea introduced in [139] dealing with ordinary differential equations. As a byproduct we get, by a simple proof, a basically known result concerning local existence, continuation and ultimate behaviour of solutions of (2.1).

Section 2.4 is devoted to some existence and localization statements for the periodic problem associated with (2.1), when a pair α, β of possibly discontinuous lower and upper solutions is given. After some preliminaries, mainly devoted to state the above mentioned Nagumo-type result for the periodic problem, we introduce in the frame of (2.1) the notion of T-monotonicity. This property is a simple tool to prove convergence of a bounded solution defined in $\overline{\Omega} \times [t_0, +\infty[$ to a T-periodic solution. We use it to get an extension to the present context of the Monotone Convergence Criterion known in the frame of order preserving discrete-time semidynamical systems (see, e.g., [60, Section 5]). The version of this criterion we give in Proposition 2.13 says that any bounded solution of (2.1) emanating from and lying above a lower solution of the T-periodic problem associated with (2.1), or respectively emanating from and lying below an upper solution, is T-monotone and hence converges to a T-periodic solution. We point out that, in space dimension $N \geqslant 2$, convergence to a T-periodic solution is not the behaviour of all bounded solutions of (2.1), due to the possible occurrence of subharmonic solutions (see, e.g., [141,35] and [123, Section 3]). In our frame additional difficulties come from the lack of regularity of f, so that even generic convergence results fail (see [123, Section 3]). We then give an application of this result to the existence of strong periodic solutions of the linear problem associated with (2.1). Although this topic was recently considered in [84], we show here that a simple approach, based on a monotone iterative scheme, yields a constructive existence proof. This technique is very much in the spirit of this work, since the iteration here introduced is used several times elsewhere in the paper. Additional statements concerning the principal eigenvalue λ_1 of the linear problem associated with (2.1), as well as some Fredholm-type results, are produced in Appendix.

After these premises, which however have an independent interest, we prove the main theorems of this section: the existence and localization of solutions of the periodic problem associated with (2.1) when $\alpha \leqslant \beta$, or when $\alpha \nleqslant \beta$. Theorem 2.15 yields the existence of a minimum and a maximum T-periodic solution of (2.1) between two possibly discontinuous lower and upper solutions α, β satisfying the condition $\alpha \leqslant \beta$. This extends to strong solutions previous results obtained, e.g., in [73,8,6,33,58] for classical solutions. The proof of Theorem 2.15 makes use of a quite standard argument, substantially similar to that used for proving Theorem 2.6. Nevertheless, as its counterpart for the initial value problem, the modification and truncation method we use are new in this frame, since some specific devices are required in order to deal with nonregular lower and upper solutions, as well as with Carathéodory gradient dependent nonlinearities, for handling which some care is required. We also show the existence of a minimum and of a maximum solution using a neat argument based on compactness. For later use, relations with the topological degree are also established, when the lower and upper solutions are strict.

Afterwards we discuss the existence of T-periodic solutions of (2.1) in the presence of generalized lower and upper solutions α, β such that $\alpha \nleqslant \beta$. As we already pointed out, in this frame the sole existence of a pair of lower and upper solutions, even satisfying $\alpha > \beta$, does not generally guarantee the existence of a solution. Here we prove in Theorem 2.17 a result which is related to the classical Amann–Kolesov Three Solutions Theorem [73,5,7] and requires the existence of a further pair of lower and upper solutions α_1, β_1, with $\alpha_1 \leqslant \beta_1$, bracketing α, β. Theorem 2.17 then yields the existence of maximal and minimal T-periodic solutions of (2.1) lying between α_1, β_1 and belonging to the $C^{1,0}$-closure of the set of functions satisfying (2.9). The solvability is established evaluating, by means of the additivity and excision properties, the topological degree of a solution operator associated with (2.1), whose fixed points are precisely the periodic solutions of (2.1). The existence of minimal and maximal solutions is proved using some a-priori bounds in connection with Zorn Lemma. For related results we refer to [41,42], where alternative conditions are used and several applications are indicated.

The aim of Section 2.5 is to extend to our general context the Order Interval Trichotomy, first proved in [34] for order preserving discrete-time semidynamical systems. We stress once more that our results are obtained without requiring any regularity on the function f besides the Carathéodory conditions. Nevertheless with the aid of Proposition 2.3, which plays a crucial role in this context, we can make deformations of our original problem to other problems displaying a certain amount of monotonicity, from which we eventually infer the information needed to prove the existence of some type of connections. Namely, if we fix an ordered pair of T-periodic solutions, say u_1, u_2, then we prove the existence either of a further T-periodic solution between u_1 and u_2, or of a double sequence of lower, or upper, solutions connecting u_1 and u_2. The last assertion expresses a form of stability, or instability, of u_1 and u_2, which is referred to as order stability. Once this version of the Order Interval Trichotomy is established, we apply it to the study of the dynamics near one-sided isolated T-periodic solutions of (2.1). Namely we prove the existence of T-monotone heteroclinic solutions connecting a pair of comparable T-periodic solutions and we study the qualitative behaviour of certain solutions of (2.1) lying above a maximal, or below a minimal, T-periodic solution. Our statements extend previous results in [34,58, 32] (see also [60,123]).

In Section 2.6 we study the stability properties of the T-periodic solutions of (2.1) with the aid of lower and upper solutions. We start by recalling the definition of one-sided Lyapunov–Bellmann stability. Instead of using the more usual C^1-, or L^∞-, norm for measuring the distance between solutions, we use the order norm induced by a fixed function belonging to the interior of the positive cone in $C_0^1(\overline{\Omega})$. This seems natural in order to fit better with the features of (2.1) and with our approach. Since we have to take care of the fact that uniqueness for the initial value problem and validity of comparison principles are not assumed, the notion of Lyapunov stability may be not the most appropriate to be considered here; indeed, some weaker concept might be more suited to detect certain residual forms of stability. As an alternative definition to Lyapunov stability we consider that of order stability; this is commonly used in the frame of order preserving semidynamical systems [93,33,58,60] and appears suited to our approach based on lower and upper solutions. We perform a rather detailed examination of the relationship occurring between the notions of Lyapunov and order stability. In particular we discuss the meaning of these notions of stability, when some versions of the comparison principle hold for solutions of (2.1), implying or not uniqueness for the initial value problem. Explicit conditions on the nonlinearity f, entailing their validity, are also exhibited. We show in particular that Lyapunov stability implies order stability, whereas the converse implication is not generally true; however, these concepts are equivalent if a comparison principle holds. Using these notions, we give a precise description of the stability properties of a T-periodic solution in terms of the existence of a lower or an upper solution close to it. Hence, when a pair of lower and upper solutions α, β, with $\alpha \leqslant \beta$, is given, we discuss the stability of the minimum and the maximum T-periodic solutions v and w lying in between, thus getting a completion of Theorem 2.15. Namely, as a consequence of Proposition 2.13 we prove, in Theorem 2.33, an invariance property of the set of T-periodic solutions which lie between a lower and an upper solution satisfying (2.2) and, in particular, a form of relative attractivity of the minimum and the maximum solutions. Afterwards, we discuss the structure of the set of T-periodic solutions lying between v and w, as well as the dynamics occurring between them. In particular, we prove the existence of a totally ordered continuum of order stable solutions. Assuming the validity of some comparison principle, we obtain refinements and strengthenings of the results we have established in a more general context. In particular we can recover several known results, about stability, asymptotic stability, instability of periodic solutions, using an approach which singles out the essential hypotheses. Finally, we describe with some detail the instability properties of maximal T-periodic solutions which are not above a strict lower solution, or not below a strict upper solution. As a consequence, in the case where $\alpha \not\leqslant \beta$, we get an instability result which completes the conclusions of Theorem 2.17. The last subsection is devoted to a discussion of some relations between stability properties and multiplicity of T-periodic solutions.

2.2. *Preliminaries*

In this section we state some basic facts concerning the parabolic problem

$$\begin{cases} \partial_t u + A(x,t,\partial_x)u = f(x,t,u,\nabla_x u) & \text{in } \Omega \times I, \\ u = 0 & \text{on } \partial\Omega \times I, \end{cases} \tag{2.10}$$

where $I \subseteq \mathbb{R}$ is an interval.

Basic assumptions
We suppose that

(D) $\Omega \subset \mathbb{R}^N$ is a bounded domain, having a boundary $\partial\Omega$ of class C^2, and $T > 0$ is a fixed number.

Hereafter, we set $Q_T = \Omega \times]0,T[$ and $\Sigma_T = \partial\Omega \times [0,T]$. The differential operator has the form

$$A(x,t,\partial_x)u = -\sum_{i,j=1}^{N} \partial_{x_i}\big(a_{ij}(x,t)\partial_{x_j} u\big) + \sum_{i=1}^{N} a_i(x,t)\partial_{x_i} u + a_0(x,t)u.$$

We assume that

(A) $a_{ij} \in C^{1,0}(\overline{Q}_T)$, $a_{ij}(x,t) = a_{ji}(x,t)$ in $\overline{Q}_T$, $a_{ij}(x,0) = a_{ij}(x,T)$ in $\overline{\Omega}$, for $i,j = 1,\ldots,N$, and there exists $\eta > 0$ such that, for all $(x,t) \in \overline{Q}_T$ and $\xi \in \mathbb{R}^N$,

$$\sum_{i,j=1}^{N} a_{ij}(x,t)\xi_i\xi_j \geqslant \eta|\xi|^2;$$

$a_i \in L^\infty(Q_T)$ for $i = 0,\ldots,N$; $\operatorname{ess\,inf}_{Q_T} a_0 > 0$.

We further suppose that

(C) $f : Q_T \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfies the L^p-Carathéodory conditions for some $p > N+2$, i.e. for every $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$, $f(\cdot,\cdot,s,\xi)$ is measurable on Q_T; for a.e. $(x,t) \in Q_T$, $f(x,t,\cdot,\cdot)$ is continuous on $\mathbb{R} \times \mathbb{R}^N$; for each $\rho > 0$, there exists $\gamma \in L^p(Q_T)$ such that $|f(x,t,s,\xi)| \leqslant \gamma(x,t)$, for a.e. $(x,t) \in Q_T$ and every $(s,\xi) \in [-\rho,\rho] \times [-\rho,\rho]^N$;

and

(N) for every $M > 0$, there exist $h \in L^p(Q_T)$, with $p > N+2$, and $K > 0$ such that for a.e. $(x,t) \in Q_T$, every $s \in [-M, M]$ and every $\xi \in \mathbb{R}^N$,

$$\left|f(x,t,s,\xi)\right| \leqslant h(x,t) + K|\xi|^2. \tag{2.11}$$

REMARK 2.1. All functions considered in (A) and (C) are identified with their T-periodic extensions onto $\Omega \times \mathbb{R}$ and $\Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N$, respectively.

Solutions and lower and upper solutions

We notice that, assuming (D) and $p > N+2$, the Sobolev space $W_p^{2,1}(\Omega \times]t_1, t_2[)$, with $t_1, t_2 \in \mathbb{R}$ and $t_1 < t_2$, is (compactly) embedded into $C^{1,0}(\overline{\Omega} \times [t_1, t_2])$ (see [80, Chapter II, Section 3]). Accordingly, the following definitions make sense.

DEFINITION 2.1.

- A *solution of* (2.10) *in* $\overline{\Omega} \times [t_1, t_2]$, with $t_1, t_2 \in \mathbb{R}$ and $t_1 < t_2$, is a function $u \in W_p^{2,1}(\Omega \times]t_1, t_2[)$ such that

$$\begin{aligned} &\partial_t u + A(x,t,\partial_x)u = f(x,t,u,\nabla_x u) && \text{a.e. in } \Omega \times]t_1, t_2[, \\ &u(x,t) = 0 && \text{on } \partial\Omega \times [t_1, t_2]. \end{aligned}$$

- A *solution of* (2.10) *in* $\overline{\Omega} \times J$, where J is a noncompact interval having endpoints t_1, t_2 with $-\infty \leqslant t_1 < t_2 \leqslant +\infty$, is a function u such that, for every compact interval $K \subset J$, $u|_{\overline{\Omega}\times K}$ is a solution of (2.10) in $\overline{\Omega} \times K$.

REMARK 2.2. In the literature this kind of solutions of (2.10) are usually referred to as *strong* solutions.

DEFINITION 2.2. A solution $u : \overline{\Omega} \times [t_0, \omega[\to \mathbb{R}$ of (2.10) is said *nonextendible* (*to the right of* ω), if either $\omega = +\infty$, or $\omega < +\infty$ and there is no solution $\hat{u}$ of (2.10) in $\overline{\Omega} \times [t_0, \omega]$ such that $\hat{u}|_{\overline{\Omega}\times[t_0,\omega[} = u$.

We now introduce two notions of lower and upper solutions for (2.10) of increasing generality, allowing to the utmost “angles” in the space variables and “jumps” in the time variable.

NOTATION 2.3. For a function $u : \overline{\Omega} \times [t_1, t_2] \to \mathbb{R}$, we write $u|_{\overline{\Omega}\times]t_1,t_2[} \in W_p^{2,1}(\Omega \times]t_1, t_2[)$ if there exists $\widetilde{u} \in W_p^{2,1}(\Omega \times]t_1, t_2[) \cap C^{1,0}(\overline{\Omega} \times [t_1, t_2])$ such that $\widetilde{u} = u$ on $\overline{\Omega} \times]t_1, t_2[$.

DEFINITION 2.4.

- A *regular lower solution of* (2.10) *in* $\overline{\Omega} \times [t_1, t_2]$, with $t_1, t_2 \in \mathbb{R}$ and $t_1 < t_2$, is a function $\alpha : \bar{\Omega} \times [t_1, t_2] \to \mathbb{R}$, such that $\alpha|_{\overline{\Omega}\times]t_1,t_2[} \in W_p^{2,1}(\Omega \times]t_1, t_2[)$,

$$\begin{aligned} &\partial_t \alpha + A(x,t,\partial_x)\alpha \leqslant f(x,t,\alpha,\nabla_x \alpha) && \text{a.e. in } \Omega \times]t_1, t_2[, \\ &\alpha(x,t) \leqslant 0 && \text{on } \partial\Omega \times [t_1, t_2], \end{aligned}$$

and, for every $x \in \overline{\Omega}$,

$$\alpha(x, t_1) \geqslant \lim_{t \to t_1^+} \alpha(x, t), \qquad \alpha(x, t_2) \leqslant \lim_{t \to t_2^-} \alpha(x, t).$$

- A *lower solution of* (2.10) *in* $\overline{\Omega} \times [t_1, t_2]$ is a function $\alpha : \overline{\Omega} \times [t_1, t_2] \to \mathbb{R}$ for which there are points $t_1 = \sigma_0 < \sigma_1 < \cdots < \sigma_h < \sigma_{h+1} < \cdots < \sigma_k = t_2$, such that, for each $h \in \{0, \dots, k-1\}$, $\alpha|_{\bar{\Omega} \times [\sigma_h, \sigma_{h+1}]} = \max_{1 \leqslant i \leqslant m} \alpha_i(x, t)$ in $\overline{\Omega} \times [\sigma_h, \sigma_{h+1}]$, where, for each $i = 1, \dots, m$, α_i is a regular lower solution of (2.10) in $\overline{\Omega} \times [\sigma_h, \sigma_{h+1}]$.
- A *lower solution* (respectively a *regular lower solution*) *of* (2.10) *in* $\overline{\Omega} \times J$, where J is a noncompact interval having endpoints t_1, t_2 with $-\infty \leqslant t_1 < t_2 \leqslant +\infty$, is a function α such that, for every compact interval $K \subset J$, $\alpha|_{\overline{\Omega} \times K}$ is a lower solution (respectively a regular lower solution) of (2.10) in $\overline{\Omega} \times K$.
- A *regular upper solution of* (2.10) *in* $\overline{\Omega} \times [t_1, t_2]$, with $t_1, t_2 \in \mathbb{R}$ and $t_1 < t_2$, is a function $\beta : \overline{\Omega} \times [t_1, t_2] \to \mathbb{R}$ such that $\beta|_{\overline{\Omega} \times]t_1, t_2[} \in W_p^{2,1}(\Omega \times]t_1, t_2[)$,

$$\begin{array}{ll} \partial_t \beta + A(x, t, \partial_x)\beta \geqslant f(x, t, \beta, \nabla_x \beta) & \text{a.e. in } \Omega \times]t_1, t_2[, \\ \beta(x, t) \geqslant 0 & \text{on } \partial\Omega \times [t_1, t_2], \end{array}$$

and, for every $x \in \bar{\Omega}$,

$$\beta(x, t_1) \leqslant \lim_{t \to t_1^+} \beta(x, t), \qquad \beta(x, t_2) \geqslant \lim_{t \to t_2^-} \beta(x, t).$$

- An *upper solution of* (2.10) *in* $\overline{\Omega} \times [t_1, t_2]$ is a function $\beta : \overline{\Omega} \times [t_1, t_2] \to \mathbb{R}$ for which there are points $t_1 = \rho_0 < \rho_1 < \cdots < \rho_h < \rho_{h+1} < \cdots < \rho_l = t_2$, such that, for each $h \in \{0, \dots, l-1\}$, $\beta|_{\overline{\Omega} \times [\rho_h, \rho_{h+1}]} = \min_{1 \leqslant j \leqslant n} \beta_j(x, t)$ in $\overline{\Omega} \times [\rho_h, \rho_{h+1}]$, where, for each $j = 1, \dots, n$, β_j is a regular upper solution of (2.10) in $\bar{\Omega} \times [\rho_h, \rho_{h+1}]$.
- An *upper solution* (respectively a *regular upper solution*) *of* (2.10) *in* $\overline{\Omega} \times J$, where J is a noncompact interval having endpoints t_1, t_2 such that $-\infty \leqslant t_1 < t_2 \leqslant +\infty$, is a function β such that, for every compact interval $K \subset J$, $\beta|_{\overline{\Omega} \times K}$ is an upper solution (respectively a regular upper solution) of (2.10) in $\overline{\Omega} \times K$.

REMARK 2.3. We notice that a regular lower solution α is continuous in $\bar{\Omega} \times]t_1, t_2[$, but it may be discontinuous with respect to t at the endpoints of the interval $[t_1, t_2]$. A similar observation holds for a regular upper solution β.

REMARK 2.4. Whenever a lower solution and an upper solution are simultaneously considered, we can assume, without loss of generality, that the sequences $(\sigma_h)_h$ and $(\rho_h)_h$ coincide.

Orderings

DEFINITION 2.5. Assume (D). Given functions $u, v : \bar{\Omega} \times I \to \mathbb{R}$, with $I \subseteq \mathbb{R}$ an interval, we write

- $u \leqslant v$ if $u(x,t) \leqslant v(x,t)$ in $\bar{\Omega} \times I$;
- $u < v$ if $u \leqslant v$ and $u \neq v$;
- $u \ll v$ if there exists a function $w \in C^{1,0}(\overline{\Omega} \times I)$, with $w(x,t) > 0$ for every $(x,t) \in \Omega \times I$, $w(x,t) = 0$ on $\partial\Omega \times I$ and $\frac{\partial w}{\partial \nu}(x,t) < 0$ on $\partial\Omega \times I$, where $\nu = \nu(x) \in \mathbb{R}^N$ is the unit outer normal to Ω at $x \in \partial\Omega$, such that $u + w \leqslant v$.

DEFINITION 2.6. Given functions $v, w : \overline{\Omega} \times I \to \mathbb{R}$, with $I \subseteq \mathbb{R}$ an interval and $v < w$, we define the *order intervals*

$$[v, w] = \{u : \overline{\Omega} \times I \to \mathbb{R} \mid v \leqslant u \leqslant w\},$$

$$[v, +\infty[= \{u : \overline{\Omega} \times I \to \mathbb{R} \mid v \leqslant u\},$$

$$]-\infty, w] = \{u : \overline{\Omega} \times I \to \mathbb{R} \mid u \leqslant w\}.$$

All these notations, with the same meaning, will be also used for functions defined on $\partial\Omega \times I$, or $\overline{\Omega}$, or $\partial\Omega$.

DEFINITION 2.7. Let $\mathcal{S}$ be a given set of solutions of (2.10).

- We say that a solution z of (2.10), with $z \in \mathcal{S}$, is a *maximal solution of* (2.10) *in* $\mathcal{S}$ (respectively a *minimal solution of* (2.10) *in* $\mathcal{S}$) if there is no solution u of (2.10), with $u \in \mathcal{S}$, such that $u > z$ (respectively $u < z$).
- We say that a solution z of (2.10), with $z \in \mathcal{S}$, is the *maximum solution of* (2.10) *in* $\mathcal{S}$ (respectively the *minimum solution of* (2.10) *in* $\mathcal{S}$) if every solution u of (2.10), with $u \in \mathcal{S}$, is such that $u \leqslant z$ (respectively $u \geqslant z$).

The following elementary statement is related to the properties of the positive cone in $C^{1,0}(\bar{\Omega} \times [t_1, t_2])$ and can be proved repeating almost verbatim the argument of [55, Lemma 3.1].

PROPOSITION 2.1. *Assume* (D) *and let* $t_1, t_2 \in \mathbb{R}$, *with* $t_1 \leqslant t_2$, *be given. Let* $u \in C^{1,0}(\bar{\Omega} \times [t_1, t_2])$ *be such that* $u = 0$ *on* $\partial\Omega \times [t_1, t_2]$ *and* $u \gg 0$. *Then the following conclusions hold*:

(i) *for every* $c \in]0, 1[$, *there exists* $\delta > 0$ *such that, for any* $v \in C^{1,0}(\bar{\Omega} \times [t_1, t_2])$ *with* $v \geqslant 0$ *on* $\partial\Omega \times [t_1, t_2]$, *if* $\|u - v\|_{C^{1,0}(\bar{\Omega} \times [t_1, t_2])} < \delta$ *then* $v \geqslant cu$ *in* $\bar{\Omega} \times [t_1, t_2]$;

(ii) *there exists* $\delta > 0$ *such that, for any* $v \in C^{1,0}(\bar{\Omega} \times [t_1, t_2])$ *with* $v \leqslant 0$ *on* $\partial\Omega \times [t_1, t_2]$, *if* $\|v\|_{C^{1,0}(\bar{\Omega} \times [t_1, t_2])} < \delta$ *then* $v \leqslant u$ *in* $\bar{\Omega} \times [t_1, t_2]$.

REMARK 2.5. Conclusion (ii) of Proposition 2.1 is still valid if $u \geqslant 0$ on $\partial\Omega \times [t_1, t_2]$.

Order norm

Following [7] we introduce the notion of order norm.

DEFINITION 2.8. Assume (D) and let $t_1, t_2 \in \mathbb{R}$, with $t_1 \leqslant t_2$, be given. Fix a function $e \in C^1(\bar{\Omega})$ such that $e(x) > 0$ in Ω, $e(x) = 0$ and $\partial_\nu e(x) < 0$ on $\partial\Omega$, where $\nu = \nu(x)$ is the unit outer normal to Ω at $x \in \partial\Omega$. For any function $u : \overline{\Omega} \times [t_1, t_2] \to \mathbb{R}$, satisfying $u = 0$ on $\partial\Omega \times [t_1, t_2]$, we set

$$\|u\|_e = \inf\bigl\{\lambda > 0 \mid |u| \leqslant \lambda e \text{ in } \overline{\Omega} \times [t_1, t_2]\bigr\} \ (\in [0, +\infty]).$$

REMARK 2.6. There are constants $c_1, c_2 > 0$ such that, for every $u \in C^{1,0}(\overline{\Omega} \times [t_1, t_2])$ satisfying $u = 0$ on $\partial\Omega \times [t_1, t_2]$, $c_1 \|u\|_{L^\infty(\Omega \times]t_1, t_2[)} \leqslant \|u\|_e \leqslant c_2 \|u\|_{C^{1,0}(\overline{\Omega} \times [t_1, t_2])}$.

In the remainder of this section we collect some basic results that will be systematically used in the sequel.

Parabolic maximum principle

The following is a version of the parabolic strong maximum principle, expressed both in the interior and in the boundary forms (see [37] and also [45,83,128,145]).

PROPOSITION 2.2. *Assume* (D) *and* (A). *Let $t_1, t_2 \in \mathbb{R}$, with $t_1 < t_2$, be fixed and let $q \in L^\infty(\Omega \times]t_1, t_2[)$ be such that*

$$a_0(x, t) + q(x, t) \geqslant 0 \quad \text{a.e. in } \Omega \times]t_1, t_2[.$$

Assume that $u \in W^{2,1}_p(\Omega \times]t_1, t_2[)$, with $p > N + 2$, satisfy

$$\partial_t u + A(x, t, \partial_x) u + q(x, t) u \geqslant 0 \quad \text{a.e. in } \Omega \times]t_1, t_2[.$$

Moreover, set $m = \min_{\overline{\Omega} \times [t_1, t_2]} u$ and suppose that $m \leqslant 0$. Then the following conclusions hold:

(i) *if m is attained at $(x_0, t_0) \in \Omega \times]t_1, t_2]$, then $u = m$ in $\bar{\Omega} \times [t_1, t_0]$;*

(ii) *if m is attained at $(x_0, t_0) \in \partial\Omega \times]t_1, t_2]$ and u is not constant in $\Omega \times]t_1, t_0]$, then $\partial_\mu u(x_0, t_0) < 0$, where $\mu \in \mathbb{R}^N$ is such that $\mu \cdot \nu > 0$, $\nu = \nu(x_0)$ being the unit outer normal to Ω at $x_0 \in \partial\Omega$.*

REMARK 2.7. In the special case where $m = 0$, Proposition 2.2 remains valid without the assumption $a_0(x, t) + q(x, t) \geqslant 0$ a.e. in $\Omega \times]t_1, t_2[$. Indeed, in that case, as $u \geqslant 0$, we have a.e. in $\Omega \times]t_1, t_2[$

$$\partial_t u + A(x, t, \partial_x) u - a_0(x, t) u + (a_0 + q)^+(x, t) u \geqslant (a_0 + q)^-(x, t) u \geqslant 0$$

and we can apply the previous result.

A generalized Hölder-type continuity

The following result points out a property, shared by all Carathéodory functions, which can be seen as a generalization of the usual Hölder continuity. In spite of its quite elementary character, it will turn out crucial in the sequel.

PROPOSITION 2.3. *Assume that, for some* $p \geqslant 1$, $f : \Omega \times]t_1, t_2[\times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ *satisfies the* L^p*-Carathéodory conditions. Then for each* $\rho > 0$, *there exists a* L^p*-Carathéodory function* $h : \Omega \times]t_1, t_2[\times [-\rho, \rho] \times [-\rho, \rho] \times \mathbb{R}^N \to \mathbb{R}$ *such that*

(i) *for a.e.* $(x, t) \in \Omega \times]t_1, t_2[$ *and every* $(r, \xi) \in [-\rho, \rho] \times \mathbb{R}^N$, $h(x, t, \cdot, r, \xi) : [-\rho, \rho] \to \mathbb{R}$ *is strictly increasing*;

(ii) *for a.e.* $(x, t) \in \Omega \times]t_1, t_2[$ *and every* $(s, \xi) \in [-\rho, \rho] \times \mathbb{R}^N$, $h(x, t, s, \cdot, \xi) : [-\rho, \rho] \to \mathbb{R}$ *is strictly decreasing*;

(iii) *for a.e.* $(x, t) \in \Omega \times]t_1, t_2[$ *and every* $(r, s, \xi) \in [-\rho, \rho] \times [-\rho, \rho] \times \mathbb{R}^N$, $h(x, t, s, r, \xi) = -h(x, t, r, s, \xi)$;

(iv) *for a.e.* $(x, t) \in \Omega \times]t_1, t_2[$ *and every* $(r, s, \xi) \in [-\rho, \rho] \times [-\rho, \rho] \times \mathbb{R}^N$, *with* $r < s$,

$$\bigl|f(x, t, s, \xi) - f(x, t, r, \xi)\bigr| < h(x, t, s, r, \xi);$$

(v) *if, moreover, there exist* $K > 0$ *and* $k \in L^p(\Omega \times]t_1, t_2[)$ *such that for a.e.* $(x, t) \in \Omega \times]t_1, t_2[$, *every* $s \in [-\rho, \rho]$ *and every* $\xi \in \mathbb{R}^N$,

$$\bigl|f(x, t, s, \xi)\bigr| \leqslant k(x, t) + K|\xi|^2,$$

then, for a.e. $(x, t) \in \Omega \times]t_1, t_2[$, *every* $(r, s) \in [-\rho, \rho] \times [-\rho, \rho]$ *and every* $\xi \in \mathbb{R}^N$,

$$\bigl|h(x, t, s, r, \xi)\bigr| \leqslant 2\bigl(k(x, t) + \rho + K|\xi|^2\bigr).$$

PROOF. The function h will be constructed by using the modulus of continuity ω of f. We start by showing that ω is a continuous function.

Step 1. *Let* $g : [a, b] \to \mathbb{R}$ *be a continuous function and let* $\omega : [0, +\infty[\to [0, +\infty[$ *be the modulus of continuity of* g, *defined by setting, for every* $\delta \geqslant 0$,

$$\omega(\delta) = \max_{|x-y| \leqslant \delta} \bigl|g(x) - g(y)\bigr|.$$

Then ω *is a continuous function on* $[0, +\infty[$. Clearly, ω is increasing on $[0, +\infty[$ and is continuous at 0 with $\omega(0) = 0$. Let us prove that ω is continuous at any point $\delta_0 > 0$. Fix such a $\delta_0 > 0$. By monotonicity, we have $\lim_{\delta \to \delta_0^-} \omega(\delta) \leqslant \omega(\delta_0) \leqslant \lim_{\delta \to \delta_0^+} \omega(\delta)$. To show that $\lim_{\delta \to \delta_0^+} \omega(\delta) \leqslant \omega(\delta_0)$, pick a decreasing sequence $(\delta_n)_n$ such that $\delta_n \to \delta_0$. For each n, let $x_n, y_n \in [a, b]$ be such that $|x_n - y_n| \leqslant \delta_n$ and $|g(x_n) - g(y_n)| = \omega(\delta_n)$. Possibly passing to subsequences, we may assume $x_n \to \bar{x}$ and $y_n \to \bar{y}$, so that

$$\omega(\delta_n) = \bigl|g(x_n) - g(y_n)\bigr| \to \bigl|g(\bar{x}) - g(\bar{y})\bigr| \leqslant \omega(\delta_0)$$

and therefore $\lim_{\delta \to \delta_0^+} \omega(\delta) = \lim_{n \to +\infty} \omega(\delta_n) \leqslant \omega(\delta_0)$. To show that $\lim_{\delta \to \delta_0^-} \omega(\delta) \geqslant \omega(\delta_0)$, let $x_0, y_0 \in [a, b]$ be such that $|x_0 - y_0| \leqslant \delta_0$ and $|g(x_0) - g(y_0)| = \omega(\delta_0)$. Pick an increasing sequence $(\delta_n)_n$ such that $\delta_n \to \delta_0$. By convexity, there exist sequences $(x_n)_n$ and $(y_n)_n$ such that $x_n \to x_0$, $y_n \to y_0$ and $|x_n - y_n| \leqslant \delta_n$, for each n. Hence it follows that

$$\omega(\delta_0) = \left|g(x_0) - g(y_0)\right| = \lim_{n\to+\infty}\left|g(x_n) - g(y_n)\right| \leqslant \lim_{n\to+\infty} \omega(\delta_n) = \lim_{\delta\to\delta_0^-} \omega(\delta).$$

Step 2. *Let* $g : [a, b] \times \mathbb{R}^N \to \mathbb{R}$ *be a continuous function and let* $\omega : [0, +\infty[\times \mathbb{R}^N \to [0, +\infty[$ *be the modulus of continuity of* g *with respect to the first variable, defined by setting, for each* $\delta \geqslant 0$ *and* $\xi \in \mathbb{R}^N$,

$$\omega(\delta, \xi) = \max_{|x-y|\leqslant\delta}\left|g(x, \xi) - g(y, \xi)\right|.$$

Then ω *is a continuous function in* $[0, +\infty[\times \mathbb{R}^N$. Assume, by contradiction, that ω is not continuous at $(\delta_0, \xi_0) \in [0, +\infty[\times \mathbb{R}^N$. Then there exist $\varepsilon > 0$ and, for each $\eta > 0$, a point $(\delta_\eta, \xi_\eta) \in [0, +\infty[\times \mathbb{R}^N$ such that $|\delta_0 - \delta_\eta| + |\xi_0 - \xi_\eta| < \eta$ and $|\omega(\delta_0, \xi_0) - \omega(\delta_\eta, \xi_\eta)| > \varepsilon$. By Step 1, there is $\eta_1 > 0$ such that, if $|\delta_0 - \delta| < \eta_1$, then

$$\left|\omega(\delta_0, \xi_0) - \omega(\delta, \xi_0)\right| < \varepsilon/2. \tag{2.12}$$

By the continuity of g, there is $\eta_2 > 0$ such that, if $|\xi_0 - \xi| < \eta_2$, then

$$\left|g(x, \xi) - g(x, \xi_0)\right| < \varepsilon/6$$

for all $x \in [a, b]$. Pick $\eta = \min\{\eta_1, \eta_2\}$ and the corresponding point (δ_η, ξ_η) such that $|\delta_0 - \delta_\eta| + |\xi_0 - \xi_\eta| < \eta$, $|\omega(\delta_0, \xi_0) - \omega(\delta_\eta, \xi_\eta)| > \varepsilon$ and, using (2.12),

$$\left|\omega(\delta_\eta, \xi_0) - \omega(\delta_\eta, \xi_\eta)\right| > \varepsilon/2.$$

Assume that $\omega(\delta_\eta, \xi_0) > \omega(\delta_\eta, \xi_\eta) + \varepsilon/2$, the other case being similar. Take $x_0, y_0 \in [a, b]$ such that $|x_0 - y_0| \leqslant \delta_\eta$ and

$$\omega(\delta_\eta, \xi_0) = \left|g(x_0, \xi_0) - g(y_0, \xi_0)\right|.$$

Take further $x_\eta, y_\eta \in [a, b]$ such that $|x_\eta - y_\eta| \leqslant \delta_\eta$ and

$$\omega(\delta_\eta, \xi_\eta) = \left|g(x_\eta, \xi_\eta) - g(y_\eta, \xi_\eta)\right|.$$

Then we get the contradiction

$$\begin{aligned}
\omega(\delta_\eta, \xi_\eta) &\geqslant \left|g(x_0, \xi_\eta) - g(y_0, \xi_\eta)\right| \\
&\geqslant \left|g(x_0, \xi_0) - g(y_0, \xi_0)\right| - \left|g(x_0, \xi_0) - g(x_0, \xi_\eta)\right| \\
&\quad - \left|g(y_0, \xi_0) - g(y_0, \xi_\eta)\right| \\
&> \omega(\delta_\eta, \xi_0) - \varepsilon/3 > \omega(\delta_\eta, \xi_\eta) + \varepsilon/6.
\end{aligned}$$

Step 3. *Definition of* h. Fix $\rho > 0$ and set, for a.e. $(x,t) \in \Omega \times]t_1,t_2[$, every $\delta \geqslant 0$ and every $\xi \in \mathbb{R}^N$,

$$\omega(x,t,\delta,\xi) = \max\bigl\{\bigl|f(x,t,r,\xi) - f(x,t,s,\xi)\bigr| \mid |r| \leqslant \rho, |s| \leqslant \rho, |r-s| \leqslant \delta\bigr\}.$$

We have, for a.e. $(x,t) \in \Omega \times]t_1,t_2[$, every $r,s \in [-\rho,\rho]$ with $r \leqslant s$ and $\xi \in \mathbb{R}^N$,

$$\bigl|f(x,t,r,\xi) - f(x,t,s,\xi)\bigr| \leqslant \omega(x,t,s-r,\xi).$$

Let us define for a.e. $(x,t) \in \Omega \times]t_1,t_2[$ and every $r,s \in [-\rho,\rho]$, $\xi \in \mathbb{R}^N$,

$$h(x,t,s,r,\xi) = \operatorname{sgn}(s-r)\bigl(\omega\bigl(x,t,|s-r|,\xi\bigr) + |s-r|\bigr).$$

Then the following conclusions hold:

- for a.e. $(x,t) \in \Omega \times]t_1,t_2[$, $h(x,t,\cdot,\cdot,\cdot)$ is continuous;
- for every $(r,s,\xi) \in [-\rho,\rho] \times [-\rho,\rho] \times \mathbb{R}^N$, $h(\cdot,\cdot,s,r,\xi)$ is measurable (this is due to the fact that $\omega(x,t,\delta,\xi) = \sup\{|f(x,t,r,\xi) - f(x,t,s,\xi)| \mid |r-s| \leqslant \delta$, with $r,s \in \mathbb{Q}\}$);
- for each $\sigma > 0$, there exists $\gamma \in L^p(\Omega \times]t_1,t_2[)$ such that, for a.e. $(x,t) \in \Omega \times]t_1,t_2[$ and every $(r,s,\xi) \in [-\rho,\rho] \times [-\rho,\rho] \times [-\sigma,\sigma]^N$, $|h(x,t,s,r,\xi)| \leqslant \gamma(x,t)$;
- for a.e. $(x,t) \in \Omega \times]t_1,t_2[$ and every $(r,\xi) \in [-\rho,\rho] \times \mathbb{R}^N$, $h(x,t,\cdot,r,\xi) : [-\rho,\rho] \to \mathbb{R}$ is strictly increasing;
- for a.e. $(x,t) \in \Omega \times]t_1,t_2[$ and every $(s,\xi) \in [-\rho,\rho] \times \mathbb{R}^N$, $h(x,t,s,\cdot,\xi) : [-\rho,\rho] \to \mathbb{R}$ is strictly decreasing;
- for a.e. $(x,t) \in \Omega \times]t_1,t_2[$ and every $(r,s,\xi) \in [-\rho,\rho] \times [-\rho,\rho] \times \mathbb{R}^N$, $h(x,t,s,r,\xi) = -h(x,t,r,s,\xi)$;
- for a.e. $(x,t) \in \Omega \times]t_1,t_2[$ and every $(r,s,\xi) \in [-\rho,\rho] \times [-\rho,\rho] \times \mathbb{R}^N$, with $r < s$,

  $$\bigl|f(x,t,s,\xi) - f(x,t,r,\xi)\bigr| < h(x,t,s,r,\xi);$$

- if, moreover, there exist $K > 0$ and $k \in L^p(\Omega \times]t_1,t_2[)$ such that for a.e. $(x,t) \in \Omega \times]t_1,t_2[$, every $s \in [-\rho,\rho]$ and every $\xi \in \mathbb{R}^N$,

  $$\bigl|f(x,t,s,\xi)\bigr| \leqslant k(x,t) + K|\xi|^2,$$

  then, for a.e. $(x,t) \in \Omega \times]t_1,t_2[$, every $(r,s) \in [-\rho,\rho] \times [-\rho,\rho]$ and every $\xi \in \mathbb{R}^N$,

  $$\bigl|h(x,t,s,r,\xi)\bigr| \leqslant 2\bigl(k(x,t) + \rho + K|\xi|^2\bigr). \qquad \square$$

2.3. *The initial boundary value problem*

Let us consider the parabolic initial value problem

$$\begin{array}{ll} \partial_t u + A(x,t,\partial_x)u = f(x,t,u,\nabla_x u) & \text{in } \Omega \times I, \\ u = 0 & \text{on } \partial\Omega \times I, \\ u(\cdot,t_0) = u_0 & \text{in } \Omega, \end{array} \tag{2.13}$$

where $I \subseteq \mathbb{R}$ is an interval containing a right neighbourhood of t_0 and $u_0 \in W_p^{2-2/p}(\Omega)$ is such that $u_0 = 0$ on $\partial\Omega$.

In this section we collect some basic facts concerning problem (2.13). Namely, we discuss existence, localization and structure of the set of strong solutions of (2.13) in the presence of a pair of possibly nonregular lower and upper solutions.

Solutions and lower and upper solutions
We notice that, assuming (D) and $p > N + 2$, $W_p^{2-2/p}(\Omega)$ is (compactly) embedded in $C^1(\overline{\Omega})$ (see [1, Chapter VII]). Therefore the following definitions make sense.

DEFINITION 2.9. A *solution of* (2.13) in $\overline{\Omega} \times I$, with $I \subseteq \mathbb{R}$ an interval containing a right neighbourhood of t_0, is a solution u of (2.10) in $\overline{\Omega} \times I$ such that

$$u(x, t_0) = u_0(x) \quad \text{in } \overline{\Omega}.$$

The following notion of lower and upper solutions for the initial value problem (2.13) is used.

DEFINITION 2.10.

- A *lower solution* (respectively a *regular lower solution*) *of* (2.13) *in* $\overline{\Omega} \times I$, with $I \subseteq \mathbb{R}$ an interval containing a right neighbourhood of t_0, is a function α which is a lower solution (respectively a regular lower solution) of (2.10) in $\overline{\Omega} \times I$ and satisfies

$$\alpha(x, t_0) \leqslant u_0(x) \quad \text{in } \Omega.$$

- An *upper solution* (respectively a *regular upper solution*) *of* (2.13) *in* $\overline{\Omega} \times I$, with $I \subseteq \mathbb{R}$ an interval containing a right neighbourhood of t_0, is a function β which is an upper solution (respectively a regular upper solution) of (2.10) in $\overline{\Omega} \times I$ and satisfies

$$\beta(x, t_0) \geqslant u_0(x) \quad \text{in } \Omega.$$

The linear initial value problem
The next statement, which follows from [80, Chapter IV, Section 9 and Chapter VII, Section 10], regards existence, uniqueness and regularity of solutions of the linear initial value problem

$$\begin{array}{ll} \partial_t u + A(x, t, \partial_x)u = h(x, t) & \text{in } \Omega \times]t_0, \tau[, \\ u = 0 & \text{on } \partial\Omega \times [t_0, \tau], \\ u(\cdot, t_0) = u_0 & \text{in } \Omega. \end{array} \tag{2.14}$$

PROPOSITION 2.4. *Assume* (D) *and* (A). *Fix* $p > N + 2$ *and let* $t_0 \in \mathbb{R}$ *and* $\tau \in]t_0, +\infty[$ *be given. Then for every* $h \in L^p(\Omega \times]t_0, \tau[)$ *and every* $u_0 \in W_p^{2-2/p}(\Omega)$ *with* $u_0 = 0$ *on* $\partial\Omega$, *problem* (2.14) *has a unique solution* $u \in W_p^{2,1}(\Omega \times]t_0, \tau[)$. *Moreover, there exists an increasing function* $C(\tau)$, *which is independent of* h *and* u_0, *such that*

$$\|u\|_{W_p^{2,1}(\Omega \times]t_0, \tau[)} \leqslant C(\tau)\big(\|h\|_{L^p(\Omega \times]t_0, \tau[)} + \|u_0\|_{W_p^{2-2/p}(\Omega)}\big).$$

A Nagumo-type result
The following result will be systematically used in the sequel.

PROPOSITION 2.5. *Assume* (D) *and* (A). *Let* $p > N + 2$, $t_0 \in \mathbb{R}$, $\omega \in]t_0, +\infty[$, $h \in L^p(\Omega \times]t_0, \omega[)$, $K > 0$ *and* $R > 0$ *be given. Then there exists a constant* $C > 0$ *such that for each* $u : \bar{\Omega} \times [t_0, \omega[\to \mathbb{R}$, *with* $u \in W_p^{2,1}(\Omega \times]t_0, \tau[)$ *for every* $\tau < \omega$, *satisfying*

$$\begin{array}{ll} \left|\partial_t u + A(x,t,\partial_x)u\right| \leqslant h(x,t) + K|\nabla_x u|^2 & \text{a.e. in } \Omega \times]t_0, \omega[, \\ u = 0 & \text{on } \partial\Omega \times [t_0, \omega[\end{array}$$

and

$$\left\| u(\cdot, t_0) \right\|_{W_p^{2-2/p}(\Omega)} \leqslant R,$$

we have $u \in W_p^{2,1}(\Omega \times]t_0, \omega[)$ *and*

$$\|u\|_{W_p^{2,1}(\Omega \times]t_0, \omega[)} \leqslant C.$$

PROOF. We start with a preliminary result, whose proof closely follows an argument in [6, Lemma 2.1] (see also [149]). We present here a detailed proof in order to verify that the constant γ_0, appearing below, is independent of $\tau \in]\delta, \omega[$, whenever $\delta \in]t_0, \omega[$ is fixed. Hereafter, for any $\sigma \in]t_0, \omega]$, we set $Q_\sigma = \Omega \times]t_0, \sigma[$ and $\Sigma_\sigma = \partial\Omega \times [t_0, \sigma[$.

CLAIM. *Fix any constant* $\delta \in]t_0, \omega[$. *Let a function* $g \in L^\infty(Q_\omega)$ *and* $u_0 \in W_p^{2-2/p}(\Omega)$ *with* $u_0 = 0$ *on* $\partial\Omega$ *be given. Then, for every* $\tau \in]\delta, \omega[$, *the initial value problem*

$$\begin{array}{ll} \partial_t u + A(x,t,\partial_x)u = g(x,t)\left(1 + |\nabla_x u|^2\right) & \text{in } Q_\tau, \\ u = 0 & \text{on } \Sigma_\tau, \\ u(\cdot, t_0) = u_0 & \text{in } \Omega \end{array} \tag{2.15}$$

has a unique solution u, which satisfies

$$\|u\|_{W_p^{2,1}(Q_\tau)} \leqslant \gamma_0\left(\|g\|_{L^\infty(Q_\omega)}, \|u_0\|_{W_p^{2-2/p}(\Omega)}\right),$$

where γ_0 *depends in an increasing way on the indicated quantities, but is independent of* $\tau \in]\delta, \omega[$.

Let us consider, for $\tau \in]t_0, \omega[$ and $\lambda \in [0, 1]$, the initial value problem

$$\begin{array}{ll} \partial_t u + A(x,t,\partial_x)u = g(x,t)\left(\lambda + |\nabla_x u|^2\right) & \text{in } Q_\tau, \\ u = 0 & \text{on } \Sigma_\tau, \\ u(\cdot, t_0) = \lambda u_0 & \text{in } \Omega. \end{array} \tag{2.16}$$

Step 1. *For every* $\tau \in]t_0, \omega[$ *and* $\lambda \in [0, 1]$, (2.16) *has at most one solution*. Indeed, if u, v are solutions of (2.16), then $w = u - v$ satisfies

$$\begin{aligned}
&\partial_t w + A(x,t,\partial_x)w - g\nabla_x(u+v)\cdot\nabla_x w = 0 && \text{in } Q_\tau,\\
&w = 0 && \text{on } \Sigma_\tau,\\
&w(\cdot,t_0) = 0 && \text{in } \Omega
\end{aligned}$$

and hence $w = 0$, as it easily follows from Proposition 2.2.

For each $\tau \in\,]t_0,\omega[$, let us denote by Λ_τ the set of all $\lambda \in [0,1]$ such that (2.16) has a solution.

Step 2. There exists a constant $\varepsilon = \varepsilon(\|g\|_{L^\infty(Q_\omega)}, \|u_0\|_{W_p^{2-2/p}(\Omega)}) > 0$ *such that, for every* $\tau \in\,]t_0,\omega[$ *and* $\lambda_1, \lambda_2 \in \Lambda_\tau$, *with* $|\lambda_1 - \lambda_2| \leqslant \varepsilon$, *the corresponding solutions* u_1, u_2 *of* (2.16) *satisfy*

$$\|u_1 - u_2\|_{W_p^{2,1}(Q_\tau)} \leqslant \gamma_1\big(\|g\|_{L^\infty(Q_\omega)}, \|u_0\|_{W_p^{2-2/p}(\Omega)}, \|u_1\|_{C^{1,0}(\overline{Q}_\tau)}\big), \tag{2.17}$$

where γ_1 *depends in an increasing way on the indicated quantities.* Set $w = u_1 - u_2$ and observe that w satisfies

$$\begin{aligned}
&\partial_t w + A(x,t,\partial_x)w = g\nabla_x(u_1+u_2)\cdot\nabla_x w + (\lambda_1-\lambda_2)g && \text{in } Q_\tau,\\
&w = 0 && \text{on } \Sigma_\tau,\\
&w(\cdot,t_0) = (\lambda_1-\lambda_2)u_0 && \text{in } \Omega.
\end{aligned} \tag{2.18}$$

Let $\eta = \min\{1, \operatorname{ess\,inf}_{\Omega\times\mathbb{R}} a_0\} > 0$ and define

$$M = \eta^{-1}|\lambda_1-\lambda_2|\max\big\{\|g\|_{L^\infty(Q_\tau)}, \|u_0\|_{L^\infty(\Omega)}\big\}.$$

The function $M - w$ satisfies

$$\begin{aligned}
&\partial_t(M-w) + A(x,t,\partial_x)(M-w) - g\nabla_x(u_1+u_2)\cdot\nabla_x(M-w)\\
&\quad = a_0 M - (\lambda_1-\lambda_2)g \geqslant 0 && \text{in } Q_\tau,\\
&M - w \geqslant 0 && \text{on } \Sigma_\tau,\\
&M - w(\cdot,t_0) = M - (\lambda_1-\lambda_2)u_0 \geqslant 0 && \text{in } \Omega.
\end{aligned}$$

Proposition 2.2 then implies that $M - w \geqslant 0$ in Q_τ. Similarly, one proves that $M + w \geqslant 0$ in Q_τ and hence

$$\|w\|_{L^\infty(Q_\tau)} \leqslant \eta^{-1}|\lambda_1-\lambda_2|\max\big\{\|g\|_{L^\infty(Q_\tau)}, \|u_0\|_{L^\infty(\Omega)}\big\}. \tag{2.19}$$

Next, using the inequality

$$\big|\nabla_x(u_1+u_2)\cdot\nabla_x w\big| \leqslant |\nabla_x u_1|^2 + 2|\nabla_x w|^2,$$

we get from (2.18)

$$\big\|\partial_t w + A(x,t,\partial_x)w\big\|_{L^p(Q_\tau)}$$

$$\leqslant 2\|g\|_{L^\infty(Q_\tau)}\big\||\nabla_x w|^2\big\|_{L^p(Q_\tau)}$$
$$+\big(\mathrm{meas}(Q_\tau)\big)^{1/p}\|g\|_{L^\infty(Q_\tau)}\big[\|u_1\|^2_{C^{1,0}(\overline{Q}_\tau)}+1\big]. \tag{2.20}$$

Now, by Hölder inequality and Fubini theorem, we have

$$\begin{aligned}\big\||\nabla_x w|^2\big\|_{L^p(Q_\tau)} &\leqslant \sum_{i=1}^N \big\||\partial_{x_i} w|^2\big\|_{L^p(Q_\tau)}\\ &= \sum_{i=1}^N\bigg(\int_0^\tau\bigg(\int_\Omega\big|\partial_{x_i} w(x,t)\big|^{2p}\,\mathrm{d}x\bigg)\,\mathrm{d}t\bigg)^{1/p}\\ &= \sum_{i=1}^N\bigg(\int_0^\tau\big(\big\|\partial_{x_i} w(\cdot,t)\big\|^2_{L^{2p}(\Omega)}\big)^p\,\mathrm{d}t\bigg)^{1/p}\end{aligned}$$

and hence, by an interpolation inequality [51, Theorem 1.10.1],

$$\begin{aligned}\big\||\nabla_x w|^2\big\|_{L^p(Q_\tau)} &\leqslant \sum_{i=1}^N\bigg(\int_0^\tau\big(\big\|\partial_{x_i} w(\cdot,t)\big\|^2_{L^{2p}(\Omega)}\big)^p\,\mathrm{d}t\bigg)^{1/p}\\ &\leqslant \sum_{i=1}^N\bigg(\int_0^\tau C_1^p\big\|w(\cdot,t)\big\|^p_{W^2_p(\Omega)}\big\|w(\cdot,t)\big\|^p_{L^\infty(\Omega)}\,\mathrm{d}t\bigg)^{1/p}\\ &\leqslant NC_1\|w\|_{L^\infty(Q_\tau)}\bigg(\int_0^\tau\|w(\cdot,t)\|^p_{W^2_p(\Omega)}\,\mathrm{d}t\bigg)^{1/p}\\ &\leqslant NC_1\|w\|_{L^\infty(Q_\tau)}\|w\|_{W^{2,1}_p(Q_\tau)},\end{aligned} \tag{2.21}$$

where C_1 is a constant depending only on Ω and p. Combining (2.21) with (2.19) yields

$$\big\||\nabla_x w|^2\big\|_{L^p(Q_\tau)} \leqslant \frac{NC_1}{\eta}|\lambda_1-\lambda_2|\max\big\{\|g\|_{L^\infty(Q_\tau)},\|u_0\|_{L^\infty(\Omega)}\big\}\|w\|_{W^{2,1}_p(Q_\tau)} \tag{2.22}$$

and hence, inserting (2.22) into (2.20),

$$\begin{aligned}&\big\|\partial_t w+A(x,t,\partial_x)w\big\|_{L^p(Q_\tau)}\\ &\quad\leqslant \frac{2NC_1}{\eta}\|g\|_{L^\infty(Q_\tau)}|\lambda_1-\lambda_2|\max\big\{\|g\|_{L^\infty(Q_\tau)},\|u_0\|_{L^\infty(\Omega)}\big\}\|w\|_{W^{2,1}_p(Q_\tau)}\\ &\qquad+\big(\mathrm{meas}(Q_\tau)\big)^{1/p}\|g\|_{L^\infty(Q_\tau)}\big(\|u_1\|^2_{C^{1,0}(\overline{Q}_\tau)}+1\big).\end{aligned} \tag{2.23}$$

On the other hand, Proposition 2.4 implies that

$$\|w\|_{W_p^{2,1}(Q_\tau)} \leqslant C_2(\tau)\big(\big\|\partial_t w + A(x,t,\partial_x)w\big\|_{L^p(Q_\tau)} + |\lambda_1-\lambda_2|\|u_0\|_{W_p^{2-2/p}(\Omega)}\big), \tag{2.24}$$

where, for every $\tau \in \,]t_0,\omega[$, $C_2(\tau) \leqslant C_2(\omega) < +\infty$ as $\omega < +\infty$. Therefore, if we set

$$\begin{aligned} &C_3\big(\|g\|_{L^\infty(Q_\omega)}, \|u_0\|_{L^\infty(\Omega)}\big)\\ &\quad = C_2(\omega)2\eta^{-1}NC_1\|g\|_{L^\infty(Q_\omega)}\max\big\{\|g\|_{L^\infty(Q_\omega)}, \|u_0\|_{L^\infty(\Omega)}\big\} \end{aligned} \tag{2.25}$$

and

$$\begin{aligned} &C_4\big(\|g\|_{L^\infty(Q_\omega)}, \|u_0\|_{W_p^{2-2/p}(\Omega)}, \|u_1\|_{C^{1,0}(\overline{Q}_\tau)}\big)\\ &\quad = C_2(\omega)\|u_0\|_{W_p^{2-2/p}(\Omega)}\\ &\qquad + C_2(\omega)\big(\mathrm{meas}(Q_\omega)\big)^{1/p}\|g\|_{L^\infty(Q_\omega)}\big(\|u_1\|^2_{C^{1,0}(\overline{Q}_\tau)}+1\big), \end{aligned} \tag{2.26}$$

we get

$$\begin{aligned} \|w\|_{W_p^{2,1}(Q_\tau)} \leqslant{}& C_3\big(\|g\|_{L^\infty(Q_\omega)}, \|u_0\|_{L^\infty(\Omega)}\big)|\lambda_1-\lambda_2|\|w\|_{W_p^{2,1}(Q_\tau)}\\ &+ C_4\big(\|g\|_{L^\infty(Q_\omega)}, \|u_0\|_{W_p^{2-2/p}(\Omega)}, \|u_1\|_{C^{1,0}(\overline{Q}_\tau)}\big). \end{aligned}$$

Accordingly, letting

$$\varepsilon = \varepsilon\big(\|g\|_{L^\infty(Q_\omega)}, \|u_0\|_{L^\infty(\Omega)}\big) = \big[2C_3\big(\|g\|_{L^\infty(Q_\omega)}, \|u_0\|_{L^\infty(\Omega)}\big)\big]^{-1},$$

we conclude that

$$\|w\|_{W_p^{2,1}(Q_\tau)} \leqslant 2C_4\big(\|g\|_{L^\infty(Q_\omega)}, \|u_0\|_{W_p^{2-2/p}(\Omega)}, \|u_1\|_{C^{1,0}(\overline{Q}_\tau)}\big),$$

provided $|\lambda_1-\lambda_2| \leqslant \varepsilon$. Hence (2.17) follows setting

$$\begin{aligned} &\gamma_1\big(\|g\|_{L^\infty(Q_\omega)}, \|u_0\|_{W_p^{2-2/p}(\Omega)}, \|u_1\|_{C^{1,0}(\overline{Q}_\tau)}\big)\\ &\quad = 2C_4\big(\|g\|_{L^\infty(Q_\omega)}, \|u_0\|_{W_p^{2-2/p}(\Omega)}, \|u_1\|_{C^{1,0}(\overline{Q}_\tau)}\big), \end{aligned}$$

where γ_1 depends in an increasing way on the indicated quantities.

Step 3. *For every* $\tau \in \,]t_0,\omega[$ *and* $\lambda \in [0,\varepsilon]$, *the initial value problem* (2.16) *has a solution* u_λ, *which is unique by Step* 1 *and satisfies*

$$\|u_\lambda\|_{W_p^{2,1}(Q_\tau)} \leqslant \gamma_1\big(\|g\|_{L^\infty(Q_\omega)}, \|u_0\|_{W_p^{2-2/p}(\Omega)}, 0\big).$$

For each $\tau \in]t_0, \omega[$, $0 \in \Lambda_\tau$ and for every $\lambda \in [0, \varepsilon]$, any possible solution u_λ of (2.16) satisfies, by letting $u_1 = 0$ and $u_2 = u_\lambda$ in (2.17),

$$\|u_\lambda\|_{C^{1,0}(\overline{Q}_\tau)} < R(\tau) = C_5(\tau)\gamma_1\big(\|g\|_{L^\infty(Q_\omega)}, \|u_0\|_{W_p^{2-2/p}(\Omega)}, 0\big) + 1,$$

where $C_5(\tau)$ is the embedding constant of $W_p^{2,1}(Q_\tau)$ into $C^{1,0}(\overline{Q}_\tau)$. Therefore, the homotopy invariance of the degree implies that

$$\deg\big(I - S_{\tau,\lambda}, B\big(0, R(\tau)\big)\big) = \deg\big(I - S_{\tau,0}, B\big(0, R(\tau)\big)\big),$$

where $S_{\tau,\lambda} : C^{1,0}(\overline{Q}_\tau) \to C^{1,0}(\overline{Q}_\tau)$ is the solution operator associated with (2.16) and $B(0, R(\tau))$ is the open ball in $C^{1,0}(\overline{Q}_\tau)$ of center 0 and radius $R(\tau)$. As for $\mu \in [0, 1]$, $u = \mu S_{\tau,0}u$ has only the trivial solution, we have also

$$\begin{aligned}\deg\big(I - S_{\tau,0}, B\big(0, R(\tau)\big)\big) &= \deg\big(I - \mu S_{\tau,0}, B\big(0, R(\tau)\big)\big)\\ &= \deg\big(I, B\big(0, R(\tau)\big)\big) = 1,\end{aligned}$$

and hence

$$\deg\big(I - S_{\tau,\lambda}, B\big(0, R(\tau)\big)\big) = 1.$$

The existence property of the degree yields the conclusion.

Conclusion. Now, arguing as in Step 3, letting in (2.17) $u_1 = u_\varepsilon$ and $u_2 = u_\lambda$ with $\lambda \in [\varepsilon, \min\{2\varepsilon, 1\}]$, we get $[0, \min\{2\varepsilon, 1\}] \subseteq \Lambda_\tau$ and

$$\|u_{2\varepsilon}\|_{W_p^{2,1}(Q_\tau)} \leqslant 2\gamma_1\big(\|g\|_{L^\infty(Q_\omega)}, \|u_0\|_{W_p^{2-2/p}(\Omega)}, \|u_\varepsilon\|_{C^{1,0}(\overline{Q}_\tau)}\big),$$

where

$$\|u_\varepsilon\|_{C^{1,0}(\overline{Q}_\tau)} \leqslant C_5(\delta, \omega)\gamma_1\big(\|g\|_{L^\infty(Q_\omega)}, \|u_0\|_{W_p^{2-2/p}(\Omega)}, 0\big),$$

$C_5(\delta, \omega)$ denoting a constant such that $C_5(\delta, \omega) > C_5(\tau)$ for every $\tau \in [\delta, \omega]$ (cf. [80, Lemma II.3.3]). Finally, iterating this process a finite number of times, the assertion of the claim follows.

We are now in position to conclude the proof of this proposition. Let $u : \overline{\Omega} \times [t_0, \omega[\to \mathbb{R}$ be a function such that $u \in W_p^{2,1}(Q_\tau)$ for every $\tau \in]t_0, \omega[$, and

$$\begin{array}{ll} \big|\partial_t u + A(x, t, \partial_x)u\big| \leqslant h(x, t) + K|\nabla_x u|^2 & \text{a.e. in } Q_\omega,\\ u = 0 & \text{on } \Sigma_\omega,\\ u(\cdot, t_0) = u_0 & \text{in } \Omega, \end{array}$$

where $u_0 \in W_p^{2-2/p}(\Omega)$ is such that $u_0 = 0$ on $\partial\Omega$ and

$$\|u_0\|_{W_p^{2-2/p}(\Omega)} \leqslant R.$$

Define $k = \frac{\partial_t u + A(x,t,\partial_x)u}{1+h+K|\nabla_x u|^2} \in L^\infty(Q_\omega)$. Let $v \in W_p^{2,1}(Q_\omega)$ be the solution of

$$\begin{array}{ll} \partial_t v + A(x,t,\partial_x)v = k(x,t)h(x,t) & \text{in } Q_\omega, \\ v = 0 & \text{on } \Sigma_\omega, \\ v(\cdot, t_0) = 0 & \text{in } \Omega, \end{array}$$

which satisfies, by Proposition 2.4,

$$\|v\|_{W_p^{2,1}(Q_\omega)} \leqslant C_2(\omega)\|h\|_{L^p(Q_\omega)} \tag{2.27}$$

and hence, by the embedding of $W_p^{2,1}(Q_\omega)$ into $C^{1,0}(\overline{Q}_\omega)$,

$$\|v\|_{C^{1,0}(\overline{Q}_\omega)} \leqslant C_6\|h\|_{L^p(Q_\omega)}. \tag{2.28}$$

Set $w = u - v$ and $g = k\frac{1+K|\nabla_x u|^2}{1+|\nabla_x w|^2}$. The function w satisfies

$$\partial_t w + A(x,t,\partial_x)w = k(x,t)\big(1 + K|\nabla_x u|^2\big) = g(x,t)\big(1 + |\nabla_x w|^2\big)$$

a.e. in Q_ω and therefore, for every $\tau \in]0, \omega[$, it is a solution of

$$\begin{array}{ll} \partial_t w + A(x,t,\partial_x)w = g(x,t)\big(1 + |\nabla_x w|^2\big) & \text{in } Q_\tau, \\ w = 0 & \text{on } \Sigma_\tau, \\ w(\cdot, t_0) = u_0 & \text{in } \Omega. \end{array}$$

Since, by (2.28),

$$\|\nabla_x u\|^2_{L^\infty(Q_\omega)} \leqslant 2\|\nabla_x w\|^2_{L^\infty(Q_\omega)} + 2\big(C_6\|h\|_{L^p(Q_\omega)}\big)^2,$$

it follows that $g \in L^\infty(Q_\omega)$ and

$$\|g\|_{L^\infty(Q_\omega)} \leqslant \max\{1, 2K\} + 2K\big(C_6\|h\|_{L^p(Q_\omega)}\big)^2.$$

Therefore, if we fix $\delta \in]t_0, \omega[$, the above claim implies that, for every $\tau \in]\delta, \omega[$,

$$\|w\|_{W_p^{2,1}(Q_\tau)} \leqslant \gamma_0\Big(\max\{1, 2K\} + 2K\big(C_6\|h\|_{L^p(Q_\omega)}\big)^2, R\Big),$$

which, in turn, yields

$$\|w\|_{C^{1,0}(\overline{Q}_\tau)} \leqslant C_5(\delta, \omega)\gamma_0\Big(\max\{1, 2K\} + 2K\big(C_6\|h\|_{L^p(Q_\omega)}\big)^2, R\Big)$$

and, in particular,

$$\|\nabla_x w\|_{L^\infty(Q_\omega)} \leqslant C_5(\delta, \omega)\gamma_0\Big(\max\{1, 2K\} + 2K\big(C_6\|h\|_{L^p(Q_\omega)}\big)^2, R\Big).$$

Accordingly, denoting by $z \in W^{2,1}_p(Q_\omega)$ the unique solution of

$$\begin{array}{ll} \partial_t z + A(x,t,\partial_x) z = g(x,t)\big(1 + |\nabla_x w|^2\big) & \text{in } Q_\omega, \\ z = 0 & \text{on } \Sigma_\omega, \\ z(\cdot, t_0) = u_0 & \text{in } \Omega, \end{array}$$

we have $w = z$ in $\overline{\Omega} \times [t_0, \omega[$ and hence $w \in W^{2,1}_p(Q_\omega)$. Finally, using (2.27), we conclude that $u \in W^{2,1}_p(Q_\omega)$ and

$$\begin{aligned} \|u\|_{W^{2,1}_p(Q_\omega)} &\leqslant \gamma_0\Big(\max\{1, 2K\} + 2K\big(C_6 \|h\|_{L^p(Q_\omega)}\big)^2, R\Big) \\ &\quad + C_2(\omega)\|h\|_{L^p(Q_\omega)}. \end{aligned}$$ □

Existence of solutions and the Hukuhara–Kneser property

In this section we extend to strong solutions and nonregular lower and upper solutions some existence and localization results for the initial value problem (2.13) which are well known in the frame of classical solutions. We also describe the topological structure of the solution set, proving the Hukuhara–Kneser property. Since our analysis is performed just assuming that f satisfies the Carathéodory conditions, we get an improvement of results obtained in [76], for f Hölder continuous, and in [81], for f continuous. Our proof, which differs from the standard ones in the approximation method, adapts an idea introduced in [139] dealing with ordinary differential equations.

We first deal with the initial value problem on a compact interval.

THEOREM 2.6. *Assume* (D), (A), (C), (N) *and let* $u_0 \in W^{2-2/p}_p(\Omega)$ *be such that* $u_0 = 0$ *on* $\partial\Omega$. *Let* $t_0, t_1 \in \mathbb{R}$ *be such that* $t_0 < t_1$ *and suppose that* α *is a lower solution and* β *is an upper solution of* (2.13) *in* $\overline{\Omega} \times [t_0, t_1]$, *satisfying* $\alpha \leqslant \beta$. *Then there exist the minimum solution* v *and the maximum solution* w *in* $[\alpha, \beta]$ *of problem* (2.13) *in* $\overline{\Omega} \times [t_0, t_1]$. *Further, the set*

$$\mathcal{K} = \big\{u : \overline{\Omega} \times [t_0, t_1] \to \mathbb{R} \mid u \text{ is a solution of (2.13) with } \alpha \leqslant u \leqslant \beta \\ \text{in } \overline{\Omega} \times [t_0, t_1]\big\}$$

is a continuum, i.e. a compact and connected set, in $C^{1,0}(\overline{\Omega} \times [t_0, t_1])$.

PROOF. The proof is divided into three parts.

Part 1. *Existence of a solution* u *of* (2.13) *in* $\overline{\Omega} \times [t_0, t_1]$, *with* $\alpha \leqslant u \leqslant \beta$. We can assume that the sequences $(\sigma_h)_{0 \leqslant h \leqslant k}$ and $(\rho_h)_{0 \leqslant h \leqslant l}$ coincide. We also notice that, by our definition of lower and upper solutions and the condition $\alpha \leqslant \beta$, there exists a constant $M > 0$ such that $-M \leqslant \alpha(x,t) \leqslant \beta(x,t) \leqslant M$ in $\overline{\Omega} \times [t_0, t_1]$. Hence, by condition (N) and Proposition 2.5, there exists $R > 0$ such that, for every function f satisfying (2.11) and every solution u of (2.13) in $\overline{\Omega} \times [t_0, t_1]$ with $\alpha \leqslant u \leqslant \beta$, we have

$$\|u\|_{C^{1,0}(\overline{\Omega} \times [t_0, t_1])} < R. \tag{2.29}$$

For $h \in \{0, \dots, k-1\}$, let $\alpha = \max_{1 \leqslant i \leqslant m_h} \alpha_i^{(h)}$ and $\beta = \min_{1 \leqslant j \leqslant n_h} \beta_j^{(h)}$ in $\overline{\Omega} \times]\sigma_h, \sigma_{h+1}[$, with $\alpha_i^{(h)}, \beta_j^{(h)} \in W_p^{2,1}(\Omega \times]\sigma_h, \sigma_{h+1}[) \cap C^{1,0}(\overline{\Omega} \times [\sigma_h, \sigma_{h+1}])$. Take $\overline{R}$ such that

$$\overline{R} > R + \max_{0 \leqslant h \leqslant k-1} \Big\{ \max_{1 \leqslant i \leqslant m_h} \big\| \alpha_i^{(h)} \big\|_{C^{1,0}(\overline{\Omega} \times [\sigma_h, \sigma_{h+1}])}, \max_{1 \leqslant j \leqslant n_h} \big\| \beta_j^{(h)} \big\|_{C^{1,0}(\overline{\Omega} \times [\sigma_h, \sigma_{h+1}])} \Big\}$$

and set

$$\bar{f}(x,t,s,\xi) = \begin{cases} f(x,t,s,\xi) & \text{if } |\xi| \leqslant \overline{R}, \\ f\big(x,t,s,\overline{R}\frac{\xi}{|\xi|}\big) & \text{if } |\xi| > \overline{R}, \end{cases}$$

for a.e. $(x,t) \in \Omega \times \mathbb{R}$ and every $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$. For any given $h \in \{0, \dots, k-1\}$, we define the functions

$$g_i^{(h)}(x,t,s,\xi) = \begin{cases} \bar{f}\big(x,t,\alpha_i^{(h)}(x,t),\xi\big) + \omega_{1i}^{(h)}\big(x,t,\alpha_i^{(h)}(x,t) - s\big) \\ \qquad\qquad \text{if } s < \alpha_i^{(h)}(x,t), \\ \bar{f}(x,t,s,\xi) \quad \text{if } s \geqslant \alpha_i^{(h)}(x,t), \end{cases}$$

$$h_j^{(h)}(x,t,s,\xi) = \begin{cases} \bar{f}\big(x,t,\beta_j^{(h)}(x,t),\xi\big) - \omega_{2j}^{(h)}\big(x,t,s - \beta_j^{(h)}(x,t)\big) \\ \qquad\qquad \text{if } s > \beta_j^{(h)}(x,t), \\ \bar{f}(x,t,s,\xi) \quad \text{if } s \leqslant \beta_j^{(h)}(x,t), \end{cases}$$

where

$$\begin{aligned} \omega_{1i}^{(h)}(x,t,\delta) = \max_{|\xi| \leqslant \delta} \big| &\bar{f}\big(x,t,\alpha_i^{(h)}(x,t), \nabla_x \alpha_i^{(h)}(x,t) + \xi\big) \\ &- \bar{f}\big(x,t,\alpha_i^{(h)}(x,t), \nabla_x \alpha_i^{(h)}(x,t)\big)\big|, \\ \omega_{2j}^{(h)}(x,t,\delta) = \max_{|\xi| \leqslant \delta} \big| &\bar{f}\big(x,t,\beta_j^{(h)}(x,t), \nabla_x \beta_j^{(h)}(x,t) + \xi\big) \\ &- \bar{f}\big(x,t,\beta_j^{(h)}(x,t), \nabla_x \beta_j^{(h)}(x,t)\big)\big|, \end{aligned}$$

for $i \in \{1, \dots, m_h\}$ and $j \in \{1, \dots, n_h\}$, and

$$F^{(h)}(x,t,s,\xi) = \begin{cases} \max_{1 \leqslant i \leqslant m_h} g_i^{(h)}(x,t,s,\xi) & \text{if } s \leqslant \alpha(x,t), \\ \bar{f}(x,t,s,\xi) & \text{if } \alpha(x,t) < s < \beta(x,t), \\ \min_{1 \leqslant j \leqslant n_h} h_j^{(h)}(x,t,s,\xi) & \text{if } s \geqslant \beta(x,t), \end{cases} \tag{2.30}$$

for a.e. $(x,t) \in \Omega \times]\sigma_h, \sigma_{h+1}[$ and every $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$. Then we consider the modified problem

$$\begin{array}{ll} \partial_t u + A(x,t,\partial_x)u = F(x,t,u,\nabla_x u) & \text{in } \Omega \times]t_0,t_1[, \\ u = 0 & \text{on } \partial\Omega \times [t_0,t_1], \\ u(\cdot,t_0) = u_0 & \text{in } \Omega, \end{array} \tag{2.31}$$

where $F : \Omega \times]t_0,t_1[\times \mathbb{R} \times \mathbb{R}^N$ is a L^p-Carathéodory function such that, for each $h \in \{0,\dots,k-1\}$,

$$F(x,t,s,\xi) = F^{(h)}(x,t,s,\xi),$$

for a.e. $(x,t) \in \Omega \times]\sigma_h, \sigma_{h+1}[$ and every $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$. Notice that there exists $\gamma \in L^p(\Omega \times]t_0,t_1[)$ such that

$$\big|F(x,t,s,\xi)\big| \leqslant \gamma(x,t) \tag{2.32}$$

for a.e. $(x,t) \in \Omega \times]t_0,t_1[$ and every $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$.

Step 1. *Every solution u of* (2.31) *satisfies $\alpha \leqslant u \leqslant \beta$*. Assume there is a solution u of (2.31) such that $\inf_{\Omega\times]t_0,t_1[}(u-\alpha) < 0$. By our definition of a lower solution of (2.31), we can find $h \in \{0,\dots,k-1\}$ such that $\inf_{\Omega\times]\sigma_h,\sigma_{h+1}[}(u-\alpha) < 0$ and $u(x,\sigma_h) \geqslant \lim_{t\to\sigma_h^+}\alpha(x,t)$ in Ω. Since $\alpha = \max_{1\leqslant i\leqslant m_h}\alpha_i^{(h)}$, there is $i \in \{1,\dots,m_h\}$ such that, setting $v = u - \alpha_i^{(h)} \in W_p^{2,1}(\Omega \times]\sigma_h,\sigma_{h+1}[)$, we have $\min_{\overline{\Omega}\times[\sigma_h,\sigma_{h+1}]} v < 0$, $v(x,\sigma_h) \geqslant 0$ in Ω and $v(x,t) \geqslant 0$ on $\partial\Omega \times [\sigma_h,\sigma_{h+1}]$. Hence there exists $(\bar{x},\bar{t}) \in \Omega \times]\sigma_h,\sigma_{h+1}]$ such that $\min_{\overline{\Omega}\times[\sigma_h,\sigma_{h+1}]} v = v(\bar{x},\bar{t})$. As $\bar{x} \in \Omega$, we have $\nabla_x v(\bar{x},\bar{t}) = 0$ and there is an open ball $B \subseteq \Omega$, with $\bar{x} \in B$, and a point $t_1 \in]\sigma_h,\bar{t}[$ such that, a.e. in $B \times]t_1,\bar{t}]$, $|\nabla_x v(x,t)| \leqslant |v(x,t)|$, $v(x,t) < 0$ and

$$\begin{aligned} &\partial_t v + A(x,t,\partial_x)v \\ &\quad \geqslant F^{(h)}\big(x,t,u(x,t),\nabla_x u(x,t)\big) - f\big(x,t,\alpha_i^{(h)}(x,t),\nabla_x\alpha_i^{(h)}(x,t)\big) \\ &\quad \geqslant \bar{f}\big(x,t,\alpha_i^{(h)}(x,t),\nabla_x u(x,t)\big) + \omega_{1i}^{(h)}\big(x,t,\alpha_i^{(h)}(x,t) - u(x,t)\big) \\ &\qquad - \bar{f}\big(x,t,\alpha_i^{(h)}(x,t),\nabla_x\alpha_i^{(h)}(x,t)\big) \\ &\quad \geqslant -\omega_{1i}^{(h)}\big(x,t,\big|\nabla_x v(x,t)\big|\big) + \omega_{1i}^{(h)}\big(x,t,\big|v(x,t)\big|\big) \\ &\quad \geqslant 0, \end{aligned}$$

because $\omega_{1i}^{(h)}(x,t,\cdot)$ is increasing. Hence Proposition 2.2 implies that $v(x,t) = v(\bar{x},\bar{t}) = \min_{\overline{\Omega}\times[\sigma_h,\sigma_{h+1}]} v$ in $B \times]t_1,\bar{t}]$ and, as $\operatorname{ess\,inf} a_0 > 0$,

$$\partial_t v + A(x,t,\partial_x)v = a_0 v < 0,$$

a.e. in $B \times]t_1,\bar{t}]$, thus contradicting the previous inequality. Therefore, we conclude that $u \geqslant \alpha$. Similarly, one proves that $u \leqslant \beta$.

Step 2. *Every solution u of* (2.31) *is a solution of* (2.13). In Step 1, we proved that every solution u of (2.31) satisfies $\alpha \leqslant u \leqslant \beta$ and hence it is a solution of

$$\begin{array}{ll} \partial_t u + A(x,t,\partial_x)u = \bar{f}(x,t,u,\nabla_x u) & \text{in } \Omega \times]t_0,t_1[, \\ u = 0 & \text{on } \partial\Omega \times [t_0,t_1], \\ u(\cdot,t_0) = u_0 & \text{in } \Omega. \end{array}$$

As $\bar{f}$ satisfies (2.11), we have $\|u\|_{C^{1,0}(\overline{\Omega}\times[t_0,t_1])} < \overline{R}$ and hence u is a solution of (2.13).

Step 3. *Problem* (2.31) *has at least one solution*. Let us consider the solution operator $S : C^{1,0}(\overline{\Omega} \times [t_0,t_1]) \to C^{1,0}(\overline{\Omega} \times [t_0,t_1])$ associated with (2.31), which sends any function $u \in C^{1,0}(\overline{\Omega} \times [t_0,t_1])$ onto the unique solution $v \in W_p^{2,1}(\overline{\Omega} \times [t_0,t_1])$ of

$$\begin{array}{ll} \partial_t v + A(x,t,\partial_x)v = F(x,t,u,\nabla_x u) & \text{in } \Omega \times]t_0,t_1[, \\ v = 0 & \text{on } \partial\Omega \times [t_0,t_1], \\ v(\cdot,t_0) = u_0 & \text{in } \Omega. \end{array}$$

The operator S is continuous, has a relatively compact range and its fixed points are the solutions of (2.31). In particular, there exists a constant $\widetilde{R} > 0$ such that

$$\|Su\|_{C^{1,0}(\overline{\Omega}\times[t_0,t_1])} < \widetilde{R}$$

for every $u \in C^{1,0}(\overline{\Omega} \times [t_0,t_1])$. Then standard results of degree theory imply that

$$\deg\big(I - S, B\big(0,\widetilde{R}\big)\big) = 1,$$

where I is the identity operator in $C^{1,0}(\overline{\Omega} \times [t_0,t_1])$ and $B(0,\widetilde{R})$ is the open ball of center 0 and radius $\widetilde{R}$ in $C^{1,0}(\overline{\Omega} \times [t_0,t_1])$. Therefore S has a fixed point. By the conclusions of Steps 1 and 2, we get the existence of a solution u of (2.13) in $\overline{\Omega} \times [t_0,t_1]$ satisfying $\alpha \leqslant u \leqslant \beta$.

Part 2. *Existence of extremal solutions*. We know, from Part 1, that the solutions u of (2.13) in $\overline{\Omega} \times [t_0,t_1]$, with $\alpha \leqslant u \leqslant \beta$, are precisely the fixed points of the solution operator S associated with (2.31), i.e.

$$\mathcal{K} = \big\{u \in C^{1,0}\big(\overline{\Omega} \times [t_0,t_1]\big) \mid u = Su\big\},$$

and $\mathcal{K}$ is a nonempty compact subset of $C^{1,0}(\overline{\Omega} \times [t_0,t_1])$. Next, for each $u \in \mathcal{K}$, define the closed set $\mathcal{C}_u = \{z \in \mathcal{K} \mid z \leqslant u\}$. The family $\{\mathcal{C}_u \mid u \in \mathcal{K}\}$ has the finite intersection property, as it follows from Part 1 observing that if $u_1, u_2 \in \mathcal{K}$, then $\min\{u_1,u_2\}$ is an upper solution of (2.31) with $\alpha \leqslant \min\{u_1,u_2\}$. By the compactness of $\mathcal{K}$ there exists $v \in \bigcap_{u\in\mathcal{K}} \mathcal{C}_u$; clearly, v is the minimum solution in $[\alpha,\beta]$ of (2.13) in $\overline{\Omega} \times [t_0,t_1]$. The maximum solution w can be found in a similar way.

Part 3. *$\mathcal{K}$ is a continuum.* We know from Part 2 that $\mathcal{K}$ is compact in $C^{1,0}(\bar{\Omega} \times [t_0, t_1])$; we apply [77, Theorem 48.2] or [153, Theorem 13.E] to show that $\mathcal{K}$ is connected too. We already proved in Part 1 that there is $R > 0$ such that

$$\deg\bigl(I - S, B(0, R)\bigr) = 1,$$

where $S : C^{1,0}(\overline{\Omega} \times [t_0, t_1]) \to C^{1,0}(\overline{\Omega} \times [t_0, t_1])$ is the solution operator associated with (2.31). Next, for each $n \geqslant 1$, set $\tau_n = \frac{t_1 - t_0}{n}$ and denote by $S_n : C^{1,0}(\overline{\Omega} \times [t_0, t_1]) \to C^{1,0}(\overline{\Omega} \times [t_0, t_1])$ the operator which sends any function $v \in C^{1,0}(\overline{\Omega} \times [t_0, t_1])$ onto the function $w \in C^{1,0}(\overline{\Omega} \times [t_0, t_1])$ defined by

$$w(x, t) = \begin{cases} u_0^*(x, t) & \text{in } \overline{\Omega} \times [t_0, t_0 + \tau_n], \\ u(x, t) & \text{in } \overline{\Omega} \times]t_0 + \tau_n, t_1], \end{cases}$$

where u_0^* is the solution of

$$\begin{aligned} &\partial_t u + A(x, t, \partial_x) u = 0 && \text{in } \Omega \times]t_0, t_1[, \\ &u = 0 && \text{on } \partial\Omega \times [t_0, t_1], \\ &u(\cdot, t_0) = u_0 && \text{in } \Omega \end{aligned}$$

and u is the solution of

$$\begin{aligned} &\partial_t u + A(x, t, \partial_x) u \\ &\quad = F\bigl(x, t - \tau_n, v(x, t - \tau_n), \nabla_x v(x, t - \tau_n)\bigr) && \text{in } \Omega \times]t_0 + \tau_n, t_1[, \\ &u = 0 && \text{on } \partial\Omega \times [t_0 + \tau_n, t_1], \\ &u(\cdot, t_0 + \tau_n) = u_0^*(\cdot, t_0 + \tau_n) && \text{in } \Omega. \end{aligned} \tag{2.33}$$

Each operator S_n is continuous and has a relatively compact range. Further, $I - S_n$ is one-to-one. Indeed, let $v_1, v_2 \in C^{1,0}(\overline{\Omega} \times [t_0, t_1])$ be such that $(I - S_n)v_1 = (I - S_n)v_2$. This means that $v_1(x, t) = v_2(x, t)$ in $\overline{\Omega} \times [t_0, t_0 + \tau_n]$ and $v_1(x, t) - u_1(x, t) = v_2(x, t) - u_2(x, t)$ in $\bar{\Omega} \times [t_0 + \tau_n, t_1]$, where, for $i = 1, 2$, u_i is the solution of (2.33) corresponding to $v = v_i$. Hence we get by recursion that $u_1(x, t) = u_2(x, t)$ and $v_1(x, t) = v_2(x, t)$ in $\bar{\Omega} \times [t_0 + (k-1)\tau_n, t_0 + k\tau_n]$, for $k = 2, \ldots, n$, and then $u_1(x, t) = u_2(x, t)$ and $v_1(x, t) = v_2(x, t)$ in $\overline{\Omega} \times [t_0 + \tau_n, t_1]$. This yields $v_1 = v_2$.

At last we prove that the sequence $(S_n)_n$ converges to S uniformly in $C^{1,0}(\overline{\Omega} \times [t_0, t_1])$. Assume by contradiction this is false, i.e. there exist a constant $\varepsilon > 0$ and a sequence $(v_k)_k$ in $C^{1,0}(\overline{\Omega} \times [t_0, t_1])$ such that

$$\|S_{n_k} v_k - S v_k\|_{C^{1,0}(\overline{\Omega} \times [t_0, t_1])} \geqslant \varepsilon. \tag{2.34}$$

For each k, define

$$g_k(x, t) = F\bigl(x, t, v_k(x, t), \nabla_x v_k(x, t)\bigr) \quad \text{a.e. in } \Omega \times]t_0, t_1[,$$

and

$$\ell_k(x,t) = \begin{cases} 0 & \text{a.e. in } \Omega \times]t_0, t_0 + \tau_{n_k}[, \\ F\big(x, t - \tau_{n_k}, v_k(x, t - \tau_{n_k}), \nabla_x v_k(x, t - \tau_{n_k})\big) \\ \qquad \text{a.e. in } \Omega \times]t_0 + \tau_{n_k}, t_1[. \end{cases}$$

Then $Sv_k = w_k$ is the solution of

$$\begin{aligned} &\partial_t u + A(x,t,\partial_x)u = g_k(x,t) && \text{in } \Omega \times]t_0, t_1[, \\ &u = 0 && \text{on } \partial\Omega \times [t_0, t_1], \\ &u(\cdot, t_0) = u_0 && \text{in } \Omega \end{aligned}$$

and $S_{n_k} v_k = z_k$ is the solution of

$$\begin{aligned} &\partial_t u + A(x,t,\partial_x)u = \ell_k(x,t) && \text{in } \Omega \times]t_0, t_1[, \\ &u = 0 && \text{on } \partial\Omega \times [t_0, t_1], \\ &u(\cdot, t_0) = u_0 && \text{in } \Omega. \end{aligned}$$

By condition (2.32), we can suppose, possibly passing to subsequences, that $(g_k)_k$ converges weakly in $L^p(\Omega \times]t_0, t_1[)$ to g and $(w_k)_k$ converges weakly in $W_p^{2,1}(\Omega \times]t_0, t_1[)$ to the solution w of

$$\begin{aligned} &\partial_t u + A(x,t,\partial_x)u = g(x,t) && \text{in } \Omega \times]t_0, t_1[, \\ &u = 0 && \text{on } \partial\Omega \times [t_0, t_1], \\ &u(\cdot, t_0) = u_0 && \text{in } \Omega. \end{aligned}$$

Similarly, we can suppose that $(\ell_k)_k$ converges weakly in $L^p(\Omega \times]t_0, t_1[)$ to ℓ and $(z_k)_k$ converges weakly in $W_p^{2,1}(\Omega \times]t_0, t_1[)$ to the solution z of

$$\begin{aligned} &\partial_t u + A(x,t,\partial_x)u = \ell(x,t) && \text{in } \Omega \times]t_0, t_1[, \\ &u = 0 && \text{on } \partial\Omega \times [t_0, t_1], \\ &u(\cdot, t_0) = u_0 && \text{in } \Omega. \end{aligned}$$

We shall prove that $g = \ell$. Hence the sequences $(w_k)_k$ and $(z_k)_k$ will have the same limit in $C^{1,0}(\overline{\Omega} \times [t_0, t_1])$, thus contradicting (2.34). For any $k \geqslant 1$ and $\psi \in L^{\frac{p}{p-1}}(\Omega \times]t_0, t_1[)$, let us compute

$$\begin{aligned} &\int_{\Omega \times]t_0, t_1[} \ell_k \psi \\ &\quad = \int_\Omega \left(\int_{t_0 + \tau_{n_k}}^{t_1} F\big(x, t - \tau_{n_k}, v_k(x, t - \tau_{n_k}), \nabla_x v_k(x, t - \tau_{n_k})\big) \psi(x,t)\,\mathrm{d}t \right) \mathrm{d}x \\ &\quad = \int_\Omega \left(\int_{t_0}^{t_1 - \tau_{n_k}} F\big(x, s, v_k(x,s), \nabla_x v_k(x,s)\big) \psi(x, s + \tau_{n_k})\,\mathrm{d}s \right) \mathrm{d}x \\ &\quad = \int_{\Omega \times]t_0, t_1[} g_k \psi_k, \end{aligned}$$

where we set

$$\psi_k(x,t) = \begin{cases} \psi(x, t+\tau_{n_k}) & \text{a.e. in } \Omega \times]t_0, t_1 - \tau_{n_k}[, \\ 0 & \text{a.e. in } \Omega \times [t_1 - \tau_{n_k}, t_1[. \end{cases}$$

Since $(g_k)_k$ is bounded in $L^p(\Omega \times]t_0, t_1[)$ and

$$\int_{\Omega\times]t_0,t_1[} \ell_k \psi = \int_{\Omega\times]t_0,t_1[} g_k \psi + \int_{\Omega\times]t_0,t_1[} g_k(\psi_k - \psi),$$

for proving that $g = \ell$ it is enough to show that $\|\psi_k - \psi\|_{L^{\frac{p}{p-1}}(\Omega\times]t_0,t_1[)} \to 0$, as $k \to +\infty$. Let us denote by $\widetilde{\psi}$ the zero extension of ψ outside $\Omega \times]t_0, t_1[$ and define

$$\widetilde{\psi}_k(x,t) = \widetilde{\psi}(x, t+\tau_{n_k}) \quad \text{a.e. in } \mathbb{R}^{N+1}.$$

By [22, Lemma IV.4] $(\widetilde{\psi}_k)_k$ converges to $\widetilde{\psi}$ in $L^{\frac{p}{p-1}}(\mathbb{R}^{N+1})$. Hence we obtain

$$\begin{aligned}
&\|\psi_k - \psi\|^{\frac{p}{p-1}}_{L^{\frac{p}{p-1}}(\Omega\times]t_0,t_1[)} \\
&\quad = \int_\Omega \left(\int_{t_0}^{t_1-\tau_{n_k}} \left|\psi_k(x,t) - \psi(x,t)\right|^{\frac{p}{p-1}} \mathrm{d}t \right) \mathrm{d}x \\
&\qquad + \int_\Omega \left(\int_{t_1-\tau_{n_k}}^{t_1} \left|\psi(x,t)\right|^{\frac{p}{p-1}} \mathrm{d}t \right) \mathrm{d}x \\
&\quad = \int_\Omega \left(\int_{\mathbb{R}} \left|\widetilde{\psi}_k(x,t) - \widetilde{\psi}(x,t)\right|^{\frac{p}{p-1}} \mathrm{d}t \right) \mathrm{d}x \\
&\qquad - \int_\Omega \left(\int_{t_0-\tau_{n_k}}^{t_0} \left|\widetilde{\psi}_k(x,t)\right|^{\frac{p}{p-1}} \mathrm{d}t \right) \mathrm{d}x \\
&\quad = \|\widetilde{\psi}_k - \widetilde{\psi}\|^{\frac{p}{p-1}}_{L^{\frac{p}{p-1}}(\mathbb{R}^{N+1})} - \int_\Omega \left(\int_{t_0}^{\tau_{n_k}+t_0} \left|\widetilde{\psi}(x,t)\right|^{\frac{p}{p-1}} \mathrm{d}t \right) \mathrm{d}x \to 0,
\end{aligned}$$

as $k \to +\infty$. This concludes the proof. □

REMARK 2.8. Condition (N) can be obviously replaced in Theorem 2.6 by

(N′) there exist $h \in L^p(\Omega \times]t_0, t_1[)$, with $p > N+2$, and $K > 0$ such that, for a.e. $(x,t) \in \Omega \times]t_0, t_1[$, every $s \in [\alpha(x,t), \beta(x,t)]$ and every $\xi \in \mathbb{R}^N$,

$$\left|f(x,t,s,\xi)\right| \leqslant h(x,t) + K|\xi|^2.$$

According to Proposition 2.5, these Nagumo-type conditions prevent the formation of singularities of the space gradient of locally bounded solutions of (2.13).

Now we extend Theorem 2.6 to noncompact intervals.

COROLLARY 2.7. *Assume* (D), (A), (C), (N) *and let* $u_0 \in W_p^{2-2/p}(\Omega)$ *be such that* $u_0 = 0$ *on* $\partial\Omega$. *Let* $t_0 \in \mathbb{R}$ *and* $\tau \in \,]t_0, +\infty]$. *Suppose that* α *is a lower solution and* β *is an upper solution of* (2.13) *in* $\overline{\Omega} \times [t_0, \tau[$ *satisfying* $\alpha \leqslant \beta$. *Then there exist the minimum solution* v *and the maximum solution* w *in* $[\alpha, \beta]$ *of* (2.13) *in* $\overline{\Omega} \times [t_0, \tau[$. *Further, the set*

$$\mathcal{K} = \{u : \overline{\Omega} \times [t_0, \tau[\, \to \mathbb{R} \mid u \text{ is a solution of } (2.13) \text{ with } \alpha \leqslant u \leqslant \beta\}$$

is a continuum in $C^{1,0}(\overline{\Omega} \times [t_0, \tau[)$, *endowed with the topology of* $C^{1,0}$*-convergence on compact subsets of* $\overline{\Omega} \times [t_0, \tau[$.

PROOF. Let $(\tau_n)_n$, with $\tau_0 = t_0$, be a strictly increasing sequence converging to τ.

Step 1. *Existence of extremal solutions*. In order to prove the existence of the minimum solution v in $[\alpha, \beta]$ of (2.13) in $\overline{\Omega} \times [t_0, \tau[$, we apply recursively Theorem 2.6 to (2.13) in $\overline{\Omega} \times [t_0, \tau_n]$ for each $n \geqslant 1$. Let us denote by v_n the minimum solution v in $[\alpha, \beta]$ of (2.13) in $\bar{\Omega} \times [t_0, \tau_n]$. By the minimality of v_n, we have $v_{n+1}|_{\overline{\Omega} \times [t_0, \tau_n]} \geqslant v_n$. On the other hand, $v_n(\cdot, \tau_n) \in W_p^{2-2/p}(\Omega)$, $v_n(\cdot, \tau_n) = 0$ on $\partial\Omega$ and the functions $\alpha|_{\overline{\Omega} \times [\tau_n, \tau_{n+1}]}$ and $v_{n+1}|_{\bar{\Omega} \times [\tau_n, \tau_{n+1}]}$ are, respectively, a lower and an upper solution of the initial value problem

$$\begin{array}{ll} \partial_t u + A(x, t, \partial_x)u = f(x, t, u, \nabla_x u) & \text{in } \Omega \times \,]\tau_n, \tau_{n+1}[, \\ u = 0 & \text{on } \partial\Omega \times [\tau_n, \tau_{n+1}], \\ u(\cdot, \tau_n) = v_n(\cdot, \tau_n) & \text{in } \Omega. \end{array}$$

Hence this problem has a solution v_n^* satisfying $\alpha|_{\bar{\Omega} \times [\tau_n, \tau_{n+1}]} \leqslant v_n^* \leqslant v_{n+1}|_{\bar{\Omega} \times [\tau_n, \tau_{n+1}]}$. Then we define $\hat{v}_{n+1} : \bar{\Omega} \times [t_0, \tau_{n+1}] \to \mathbb{R}$ by setting

$$\hat{v}_{n+1}(x, t) = \begin{cases} v_n(x, t) & \text{in } \overline{\Omega} \times [t_0, \tau_n[, \\ v_n^*(x, t) & \text{in } \overline{\Omega} \times [\tau_n, \tau_{n+1}]. \end{cases}$$

Let us prove that $\hat{v}_{n+1} \in W_p^{2,1}(\Omega \times \,]t_0, \tau_{n+1}[)$. Let $w \in W_p^{2,1}(\Omega \times \,]t_0, \tau_{n+1}[)$ be the unique solution of the linear initial value problem

$$\begin{array}{ll} \partial_t u + A(x, t, \partial_x)u = f(x, t, \hat{v}_{n+1}, \nabla_x \hat{v}_{n+1}) & \text{in } \Omega \times \,]t_0, \tau_{n+1}[, \\ u = 0 & \text{on } \partial\Omega \times [t_0, \tau_{n+1}], \\ u(\cdot, t_0) = u_0 & \text{in } \Omega. \end{array}$$

Since both v_n and $w|_{\overline{\Omega} \times [t_0, \tau_n]}$ are solutions of the linear initial value problem

$$\begin{array}{ll} \partial_t u + A(x, t, \partial_x)u = f(x, t, v_n, \nabla_x v_n) & \text{in } \Omega \times \,]t_0, \tau_n[, \\ u = 0 & \text{on } \partial\Omega \times [t_0, \tau_n], \\ u(\cdot, t_0) = u_0 & \text{in } \Omega, \end{array}$$

by uniqueness, we get $v_n = w$ in $\overline{\Omega} \times [t_0, \tau_n]$. Further, as $w(\cdot, \tau_n) = v_n(\cdot, \tau_n)$ in Ω, v_n^* and $w|_{\overline{\Omega}\times[\tau_n,\tau_{n+1}]}$ are both solutions of the linear initial value problem

$$\begin{array}{ll} \partial_t u + A(x,t,\partial_x)u = f(x,t,v_n^*, \nabla_x v_n^*) & \text{in } \Omega \times]\tau_n, \tau_{n+1}[, \\ u = 0 & \text{on } \partial\Omega \times [\tau_n, \tau_{n+1}], \\ u(\cdot, \tau_n) = v_n(\cdot, \tau_n) & \text{in } \Omega. \end{array}$$

Hence, by uniqueness, we have $v_n^* = w$ in $\overline{\Omega} \times [\tau_n, \tau_{n+1}]$ and we conclude that $\hat{v}_{n+1} = w$ in $\overline{\Omega} \times [t_0, \tau_{n+1}]$. Therefore $\hat{v}_{n+1}$ is a solution of (2.13) in $\overline{\Omega} \times [t_0, \tau_{n+1}]$ satisfying $\alpha \leqslant \hat{v}_{n+1} \leqslant v_{n+1}$. By the minimality of v_{n+1}, we conclude that $\hat{v}_{n+1} = v_{n+1}$ and hence $v_{n+1}|_{\overline{\Omega}\times[t_0,\tau_n]} = v_n$. Then we define $v : \overline{\Omega} \times [t_0, \tau[\to \mathbb{R}$ by setting $v(x,t) = v_n(x,t)$ in $\overline{\Omega} \times [t_0, \tau_n]$. We have that v is the minimum solution in $[\alpha, \beta]$ of (2.13) in $\overline{\Omega} \times [t_0, \tau[$, because, if u were a solution of (2.13) in $\overline{\Omega} \times [t_0, \tau[$ with $u \geqslant \alpha$ and $u \ngeqslant v$, then it should follow $u|_{\overline{\Omega}\times[t_0,\tau_n]} \ngeqslant v_n$ for some n, thus contradicting the minimality of v_n. Similarly, we prove the existence of the maximum solution w in $[\alpha, \beta]$ of (2.13) in $\overline{\Omega} \times [t_0, \tau[$.

Step 2. *$\mathcal{K}$ is a continuum.* We denote by $\mathcal{K}_n$ the set of all solutions $u : \overline{\Omega} \times [t_0, \tau_n] \to \mathbb{R}$ of (2.13) such that $\alpha \leqslant u \leqslant \beta$ on $\overline{\Omega} \times [t_0, \tau_n]$. By Theorem 2.6, $\mathcal{K}_n$ is a continuum in $C^{1,0}(\overline{\Omega} \times [t_0, \tau_n])$. For every $m < n$, let also $\pi_m^n : \mathcal{K}_n \to \mathcal{K}_m$ be the restriction map on $\overline{\Omega} \times [t_0, \tau_m]$, i.e. $\pi_m^n(u) = u|_{\overline{\Omega}\times[t_0,\tau_m]}$ for all $u \in \mathcal{K}_n$. For each $u \in \mathcal{K}$, let us set now $u_n = u|_{\overline{\Omega}\times[t_0,\tau_n]}$ and define a function $\chi : \mathcal{K} \to \prod_{n=1}^{+\infty} \mathcal{K}_n$, by $\chi(u) = (u_n)_n$. Observe that χ is a homeomorphism of $\mathcal{K}$ into $\prod_{n=1}^{+\infty} \mathcal{K}_n$, when $\prod_{n=1}^{+\infty} \mathcal{K}_n$ is endowed with the Tychonoff product topology, and its range $\chi(\mathcal{K})$ is the set of all sequences $(u_n)_n \in \prod_{n=1}^{+\infty} \mathcal{K}_n$ such that, for all $m < n$, $\pi_m^n(u_n) = u_m$, i.e. $\chi(\mathcal{K})$ is the inverse limit of the sequence $(\mathcal{K}_n)_n$ with bonding maps π_m^n (see [47]). As the inverse limit of a sequence of continua is a continuum [47, Theorem 6.1.20] we deduce that $\chi(\mathcal{K})$ is a continuum as well. Since $\mathcal{K}$ is homeomorphic to $\chi(\mathcal{K})$, the conclusion is achieved. □

Continuation and ultimate behaviour of local solutions

The following two results concern existence, continuation and ultimate behaviour of local solutions of problem (2.13); although they are basically known, we provide here simple proofs based on Proposition 2.5 and Theorem 2.6.

LEMMA 2.8. *Assume* (D), (A), (C), (N) *and let $u_0 \in W_p^{2-2/p}(\Omega)$ be such that $u_0 = 0$ on $\partial\Omega$. Fix $t_0 \in \mathbb{R}$ and $t_1 \in]t_0, +\infty]$. If α is a lower solution of* (2.13) *in $\Omega \times]t_0, t_1[$, then there exist $\omega \in]t_0, t_1]$ and a solution v of* (2.13) *in $\overline{\Omega} \times [t_0, \omega[$, with $v \geqslant \alpha$, such that every upper solution β of* (2.13) *in $\overline{\Omega} \times [t_0, \sigma]$, with $\beta \geqslant \alpha$, satisfies $\sigma \leqslant \omega$ and $\beta \geqslant v$ in $\overline{\Omega} \times [t_0, \sigma]$. Similarly, if β is an upper solution of* (2.13) *in $\Omega \times]t_0, t_1[$, then there exist $\omega \in]t_0, t_1]$ and a solution w of* (2.13) *in $\overline{\Omega} \times [t_0, \omega[$, with $w \leqslant \beta$, such that every lower solution α of* (2.13) *in $\overline{\Omega} \times [t_0, \sigma]$, with $\alpha \leqslant \beta$, satisfies $\sigma \leqslant \omega$ and $\alpha \leqslant w$ in $\overline{\Omega} \times [t_0, \sigma]$.*

PROOF. The proof is carried out through two steps.

Step 1. *Existence of a local solution* $u \geqslant \alpha$. Let $\alpha = \max_{1\leqslant i\leqslant m} \alpha_i$ in $\overline{\Omega} \times [t_0, \sigma_1]$, where $\sigma_1 > t_0$ comes from the definition of a lower solution. Define, for each $i \in \{1, \dots, m\}$,

$$g_i(x,t,s,\xi) = \begin{cases} f\big(x,t,\alpha_i(x,t),\xi\big) + \omega_i\big(x,t,\alpha_i(x,t)-s\big) & \text{if } s < \alpha_i(x,t), \\ f(x,t,s,\xi) & \text{if } s \geqslant \alpha_i(x,t), \end{cases}$$

where

$$\begin{aligned} &\omega_i(x,t,\delta) \\ &\quad = \max_{|\xi|\leqslant\delta} \big| f\big(x,t,\alpha_i(x,t), \nabla_x\alpha_i(x,t)+\xi\big) - f\big(x,t,\alpha_i(x,t), \nabla_x\alpha_i(x,t)\big)\big| \end{aligned}$$

for a.e. $(x,t) \in \Omega \times]t_0, \sigma_1[$ and every $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$. Fix a constant

$$R > \max\big\{\|\alpha\|_{L^\infty(\Omega\times]t_0,\sigma_1[)}, \|u_0\|_{L^\infty(\Omega)}\big\}$$

and consider the modified problem

$$\begin{array}{ll} \partial_t u + A(x,t,\partial_x)u = F(x,t,u,\nabla_x u) & \text{in } \Omega \times]t_0, \sigma_1[, \\ u = 0 & \text{on } \partial\Omega \times [t_0, \sigma_1], \\ u(\cdot, t_0) = u_0 & \text{in } \Omega, \end{array} \tag{2.35}$$

where

$$F(x,t,s,\xi) = \begin{cases} \max\limits_{1\leqslant i\leqslant m} g_i(x,t,s,\xi) & \text{if } s \leqslant \alpha(x,t), \\ f(x,t,s,\xi) & \text{if } \alpha(x,t) < s < R, \\ \big(R+1-|s|\big) f(x,t,s,\xi) & \text{if } R \leqslant s \leqslant R+1, \\ 0 & \text{if } s > R+1, \end{cases}$$

for a.e. $(x,t) \in \Omega \times]t_0, \sigma_1[$ and every $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$. Problem (2.35) admits any large positive constant as an upper solution. Therefore, by Theorem 2.6, it has at least one solution, which gives rise to a local solution of (2.13) defined in $\overline{\Omega} \times [t_0, \tau]$, for some $\tau > t_0$.

Step 2. *Existence of* ω *and* v. Let ω be the supremum of all $\sigma \in]t_0, t_1[$ such that problem (2.13) has a solution u in $\overline{\Omega} \times [t_0, \sigma]$ with $u \geqslant \alpha$. Let $(\sigma_n)_n$ be a strictly increasing sequence converging to ω and $(u_n)_n$ be the corresponding sequence of solutions of (2.13) defined in $\overline{\Omega} \times [t_0, \sigma_n]$ with $u_n \geqslant \alpha$. By Theorem 2.6, for each n, there exists the minimum solution v_n in $[\alpha, u_n]$ of (2.13) in $\overline{\Omega} \times [t_0, \sigma_n]$. As in the proof of Corollary 2.7, we see that the function $v : \overline{\Omega} \times [t_0, \omega[\to \mathbb{R}$, defined by setting $v(x,t) = v_n(x,t)$ in $\overline{\Omega} \times [t_0, \sigma_n]$, is the minimum solution in $[\alpha, +\infty[$ of (2.13) in $\overline{\Omega} \times [t_0, \omega[$. Finally, we notice that every upper solution β of (2.13) in $\overline{\Omega} \times [t_0, \sigma]$, with $\beta \geqslant \alpha$, is such that $\sigma \leqslant \omega$ and $\beta \geqslant v$ in $\overline{\Omega} \times [t_0, \sigma]$. Indeed, there is a solution z of (2.13) in $\overline{\Omega} \times [t_0, \sigma]$, satisfying $\alpha \leqslant z \leqslant \beta$ and hence, by construction of v, $z \geqslant v$ in $\overline{\Omega} \times [t_0, \sigma]$. □

PROPOSITION 2.9. *Assume* (D), (A), (C), (N) *and let* $u_0 \in W_p^{2-2/p}(\Omega)$ *be such that* $u_0 = 0$ *on* $\partial\Omega$. *Then problem* (2.13) *has at least one solution* $u : \overline{\Omega} \times [t_0, \tau] \to \mathbb{R}$, *for some* $\tau > t_0$, *and every such a solution can be continued to a nonextendible solution. Further every nonextendible solution* $u:\overline{\Omega}\times[t_0,\omega[\,\to\mathbb{R}$, *with* $\omega<+\infty$, *satisfies*

$$\limsup_{t\to\omega}\big\|u(\cdot,t)\big\|_{L^\infty(\Omega)}=+\infty.$$

PROOF. We proceed through three steps.

Step 1. *Existence of local solutions*. We fix a constant $R>\|u_0\|_{L^\infty(\Omega)}$ and we set, for a.e. $(x,t)\in\Omega\times\mathbb{R}$ and every $(s,\xi)\in\mathbb{R}\times\mathbb{R}^N$,

$$F(x,t,s,\xi)=\begin{cases} f(x,t,s,\xi) & \text{if } |s|<R,\\ (R+1-|s|)f(x,t,s,\xi) & \text{if } R\leqslant|s|\leqslant R+1,\\ 0 & \text{if } |s|>R+1.\end{cases}$$

The corresponding initial value problem

$$\begin{array}{ll} \partial_t u+A(x,t,\partial_x)u=F(x,t,u,\nabla_x u) & \text{in } \Omega\times\,]t_0,t_0+1[,\\ u=0 & \text{on } \partial\Omega\times[t_0,t_0+1],\\ u(\cdot,t_0)=u_0 & \text{in } \Omega,\end{array}$$

admits any large negative constant as a lower solution and any large positive constant as an upper solution. Therefore, by Theorem 2.6, it has a solution, which gives rise to a local solution of (2.13) defined on $\overline{\Omega}\times[t_0,\tau]$, for some $\tau>0$.

Step 2. *Existence of nonextendible solutions*. Let u be a solution of (2.13) in $\overline{\Omega}\times[t_0,\tau]$, with $\tau\in\,]t_0,+\infty[$. Consider the set $\mathcal{V}$ of all functions $v:\overline{\Omega}\times[t_0,\rho[\,\to\mathbb{R}$, for some $\rho>\tau$, such that v is a solution of (2.13) in $\overline{\Omega}\times[t_0,\sigma]$, for every $\sigma<\rho$, and $v|_{\overline{\Omega}\times[t_0,\tau]}=u$. This set $\mathcal{V}$ is non-empty, since $u(\cdot,\tau)\in W_p^{2-2/p}(\Omega)$ and $u(\cdot,\tau)=0$ on $\partial\Omega$ and therefore u can be locally continued to the right of τ. If $v_1,v_2\in\mathcal{V}$, we set $v_1\prec v_2$ whenever $\rho_1\leqslant\rho_2$ and $v_2|_{\overline{\Omega}\times[t_0,\rho_1]}=v_1$. It is easily verified that $\mathcal{V}$ is inductively ordered. Hence Zorn Lemma yields the existence of maximal elements in $\mathcal{V}$, i.e. the existence of nonextendible solutions of (2.13).

Step 3. *Behaviour of nonextendible solutions*. Let $u:\overline{\Omega}\times[t_0,\omega[\,\to\mathbb{R}$, with $\omega<+\infty$, be a nonextendible solution of (2.13) and assume that

$$\limsup_{t\to\omega}\big\|u(\cdot,t)\big\|_{L^\infty(\Omega)}<+\infty,$$

i.e. there is a constant $M>0$ such that $|u(x,t)|\leqslant M$ in $\overline{\Omega}\times[t_0,\omega[$. By the Nagumo condition (N), there exist $h\in L^p(\Omega\times\,]t_0,\omega[)$ and $K>0$ such that

$$\big|f(x,t,u,\nabla_x u)\big|\leqslant h(x,t)+K|\nabla_x u|^2$$

for a.e. $(x,t)\in\Omega\times\,]t_0,\omega[$. Proposition 2.5 implies that $u\in W_p^{2,1}(\Omega\times\,]t_0,\omega[)$. Hence, in particular, $u(\cdot,\omega)\in W_p^{2-2/p}(\Omega)$ and $u(\cdot,\omega)=0$ on $\partial\Omega$. Accordingly, u can be continued to the right of ω, thus contradicting the nonextendibility of u. This allows to conclude that

$$\limsup_{t\to\omega}\big\|u(\cdot,t)\big\|_{L^\infty(\Omega)}=+\infty.$$

□

REMARK 2.9. Without the Nagumo condition (N), the conclusions of Proposition 2.9 remain true, but now

$$\limsup_{t\to\omega} \| u(\cdot,t)\|_{C^1(\overline{\Omega})} = +\infty.$$

To prove the first step in that case, we define F as in the above proof and

$$\overline{F}(x,t,s,\xi) = \max\{-m(x,t), \min\{F(x,t,s,\xi), m(x,t)\}\}$$

where $m \in L^p(\Omega \times]t_0, t_0+1[)$ satisfies $m(x,t) \geqslant |f(x,t,u,\xi)| + 1$, for a.e. $(x,t) \in \Omega \times]t_0, t_0+1[$ and every $(u,\xi) \in \mathbb{R} \times \mathbb{R}^N$ with $|u_0(x) - u| \leqslant 1$ and $|\nabla_x u_0(x) - \xi| \leqslant 1$. We then consider the modified problem

$$\begin{array}{ll} \partial_t u + A(x,t,\partial_x)u = \overline{F}(x,t,u,\nabla_x u) & \text{in } \Omega \times]t_0, t_0+1[, \\ u = 0 & \text{on } \partial\Omega \times [t_0, t_0+1], \\ u(\cdot, t_0) = u_0 & \text{in } \Omega \end{array}$$

and obtain the existence of a local solution as above. The remainder of the proof is similar.

2.4. *The periodic boundary value problem*

Let us consider the periodic problem

$$\begin{array}{ll} \partial_t u + A(x,t,\partial_x)u = f(x,t,u,\nabla_x u) & \text{in } Q_T, \\ u = 0 & \text{on } \Sigma_T, \\ u(\cdot,0) = u(\cdot,T) & \text{in } \Omega, \end{array} \tag{2.36}$$

where, we recall, Q_T and Σ_T stand for $\Omega \times]0,T[$ and $\partial\Omega \times [0,T]$ respectively. When dealing with problem (2.36) it is convenient to introduce the following space of functions, satisfying the Dirichlet-periodic boundary conditions,

$$C_B^{1,0}(\overline{Q}_T) = \{u \in C^{1,0}(\overline{Q}_T) \mid u = 0 \text{ on } \Sigma_T \text{ and } u(\cdot,0) = u(\cdot,T) \text{ in } \Omega\},$$

endowed with the $C^{1,0}$-norm. We notice that, for $u \in C_B^{1,0}(\overline{Q}_T)$, we have $u \gg 0$ if and only if $u(x,t) > 0$ in $\Omega \times [0,T]$ and $\partial_\nu u(x,t) < 0$ on $\partial\Omega \times [0,T]$, where $\nu = \nu(x)$ is the unit outer normal to Ω at $x \in \partial\Omega$.

In this section we discuss existence and localization of solutions of the periodic problem (2.36) in the presence of a pair α, β of possibly discontinuous lower and upper solutions. We further introduce the notion of T-monotonicity and we use it to extend to the present context the Monotone Convergence Criterion known in the frame of order preserving discrete-time semidynamical systems (see, e.g., [60, Section 5]). We then give a simple application of this result to the study of a linear periodic parabolic problem. After these preliminaries we prove the main theorems of this section: existence and localization of solutions of (2.36) when $\alpha \leqslant \beta$, or when $\alpha \not\leqslant \beta$. It is worthy to notice again that for problem

(2.36) the mere existence of a pair of lower and upper solutions, even satisfying $\alpha > \beta$, does not generally guarantee the existence of a solution.

Solutions and lower and upper solutions

DEFINITION 2.11. A *solution of* (2.36) is a solution u of (2.10) in $\overline{\Omega} \times \mathbb{R}$ which is T-periodic with respect to the second variable, i.e. $u(x, t+T) = u(x, t)$ in $\overline{\Omega} \times \mathbb{R}$.

DEFINITION 2.12.

- A *lower solution of* (2.36) is a function $\alpha : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$, T-periodic with respect to the second variable, which is a lower solution of (2.10) in $\overline{\Omega} \times \mathbb{R}$.
- A *regular lower solution of* (2.36) is a lower solution α of (2.36) such that $\alpha|_{\overline{\Omega}\times[0,T]}$ is a regular lower solution of (2.10) in $\overline{\Omega} \times [0, T]$.
- An *upper solution of* (2.36) is a function $\beta : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$, T-periodic with respect to the second variable, which is an upper solution of (2.10) in $\overline{\Omega} \times \mathbb{R}$.
- A *regular upper solution of* (2.36) is an upper solution β of (2.36) such that $\beta|_{\overline{\Omega}\times[0,T]}$ is a regular upper solution of (2.10) in $\overline{\Omega} \times [0, T]$.
- A lower solution of (2.36) (respectively an upper solution of (2.36)) is *proper* if it is not a solution of (2.36).
- A proper lower solution α of (2.36) is *strict* if every solution u of (2.36), with $u > \alpha$, is such that $u \gg \alpha$. Similarly, a proper upper solution β of (2.36) is *strict* if every solution u of (2.36), with $u < \beta$, is such $u \ll \beta$.

REMARK 2.10. We notice that even regular lower and upper solutions of (2.36) may be discontinuous with respect to t at the endpoints of the interval $[0, T]$.

REMARK 2.11. Sometimes we speak of solutions of (2.36) with reference to functions defined on $\overline{\Omega} \times [t_0, t_0 + T]$, and of lower and upper solutions of (2.36) with reference to functions defined on $\overline{\Omega} \times [t_0, t_0 + T[$ for some $t_0 \in \mathbb{R}$. In these cases it is understood that their T-periodic extensions to $\overline{\Omega} \times \mathbb{R}$ have to be considered.

REMARK 2.12. If α is a lower solution of (2.36) and $u \in C^{1,0}_B(\overline{Q}_T)$ is such that $u > \alpha$, then there exist an open ball $B \subseteq \Omega$ and points $t_1, t_2 \in [0, T]$, with $t_1 < t_2$, such that $u(x, t) > \alpha(x, t)$ in $B \times]t_1, t_2[$. If α is a lower solution of (2.36) and $u \in C^{1,0}_B(\overline{Q}_T)$ is such that $u \not\geq \alpha$, then there exist an open ball $B \subseteq \Omega$ and points $t_1, t_2 \in [0, T]$, with $t_1 < t_2$, such that $u(x, t) < \alpha(x, t)$ in $B \times]t_1, t_2[$. Similar conclusions hold for an upper solution β and a function $u \in C^{1,0}_B(\overline{Q}_T)$, such that $u < \beta$, or $u \not\leq \beta$.

A further Nagumo-type result

In the frame of problem (2.36) we need a Nagumo-type result that is a variant of Proposition 2.5, where the $W^{2-2/p}_p$-bound on the initial condition is replaced by a L^∞-bound on the solution itself.

PROPOSITION 2.10. *Assume* (D) *and* (A). *Let* $p > N+2$, $h \in L^p(\Omega \times]-T,T[)$, $K > 0$ *and* $R > 0$ *be given. Then there exists a constant* $C > 0$ *such that, for each* $u \in W_p^{2,1}(\Omega \times]-T,T[)$ *satisfying*

$$\begin{array}{ll} \left|\partial_t u + A(x,t,\partial_x)u\right| \leqslant h(x,t) + K|\nabla_x u|^2 & \textit{a.e. in } \Omega \times]-T,T[, \\ u = 0 & \textit{on } \partial\Omega \times [-T,T], \end{array}$$

and

$$\|u\|_{L^\infty(\Omega\times]-T,T[)} \leqslant R,$$

we have

$$\|u\|_{W_p^{2,1}(\Omega\times]0,T[)} \leqslant C.$$

PROOF. As in the proof of Proposition 2.5, it is enough to prove the following result.

CLAIM. *Let* $g \in L^\infty(\Omega \times]-T,T[)$ *and* $R > 0$ *be given. Then there is a constant* $C > 0$ *such that every* $u \in W_p^{2,1}(\Omega \times]-T,T[)$, *for which*

$$\begin{array}{ll} \partial_t u + A(x,t,\partial_x)u = g(x,t)\left(1 + |\nabla_x u|^2\right) & \textit{in } \Omega \times]-T,T[, \\ u = 0 & \textit{on } \partial\Omega \times [-T,T], \end{array} \tag{2.37}$$

and

$$\|u\|_{L^\infty(\Omega\times]-T,T[)} \leqslant R \tag{2.38}$$

hold, satisfies

$$\|u\|_{W_p^{2,1}(\Omega\times]-T,T[)} \leqslant C.$$

Fix a function u satisfying (2.37) and (2.38), and consider, for each $\lambda \in [0,1]$, the initial value problem

$$\begin{array}{ll} \partial_t u_\lambda + A(x,t,\partial_x)u_\lambda = g(x,t)\left(\lambda + |\nabla_x u_\lambda|^2\right) & \text{in } \Omega \times]-T,T[, \\ u_\lambda = 0 & \text{on } \partial\Omega \times [-T,T], \\ u_\lambda(\cdot,-T) = \lambda u(\cdot,-T) & \text{in } \Omega. \end{array} \tag{2.39}$$

As in the proof of Proposition 2.5, we see that (2.39) has at most one solution. Let $\eta = \operatorname{ess\,inf}_{\Omega\times]-T,T[} a_0 > 0$ and $M = \max\{\eta^{-1}\|g\|_{L^\infty(\Omega\times]-T,T[)}, R\}$. Since $-M$ is a lower solution and M is an upper solution of (2.39), Theorem 2.6 implies that, for every $\lambda \in [0,1]$, (2.39) has a unique solution u_λ.

Moreover, as in Proposition 2.5, we can prove that, for every $\lambda_1, \lambda_2 \in [0,1]$, the corresponding solutions u_{λ_1}, u_{λ_2} of (2.39) satisfy

$$\|u_{\lambda_1} - u_{\lambda_2}\|_{L^\infty(\Omega\times]-T,T[)} \leqslant C_2|\lambda_1 - \lambda_2|. \tag{2.40}$$

Next, we set in $\Omega \times [-T, T]$,

$$v_\lambda(x,t) = \chi(t) u_\lambda(x,t),$$

where $\chi \in C^\infty(\mathbb{R}, [0,1])$ is such that

$$\chi(t) = \begin{cases} 0 & \text{if } t \leqslant -T/2, \\ 1 & \text{if } t \geqslant 0. \end{cases}$$

Each v_λ satisfies

$$\begin{array}{ll} \partial_t v_\lambda + A(x,t,\partial_x) v_\lambda = \chi(t) g(x,t)\big(\lambda + |\nabla_x u_\lambda|^2\big) + \chi'(t) u_\lambda & \text{in } \Omega \times]-T, T[, \\ v_\lambda = 0 & \text{on } \partial\Omega \times [-T, T], \\ v_\lambda(\cdot, -T) = 0 & \text{in } \Omega. \end{array} \tag{2.41}$$

For each $\lambda_1, \lambda_2 \in [0,1]$, set $w = v_{\lambda_1} - v_{\lambda_2} = \chi(u_{\lambda_1} - u_{\lambda_2})$, which satisfies

$$\begin{array}{ll} \partial_t w + A(x,t,\partial_x) w = \chi g(\lambda_1 - \lambda_2) + g \nabla_x (u_{\lambda_1} + u_{\lambda_2}) \cdot \nabla_x w + \chi'(u_{\lambda_1} - u_{\lambda_2}) & \\ \qquad\qquad \text{in } \Omega \times]-T, T[, & \\ w = 0 \qquad \text{on } \partial\Omega \times [-T, T], & \\ w(\cdot, -T) = 0 \qquad \text{in } \Omega. & \end{array} \tag{2.42}$$

and, by (2.40),

$$\|w\|_{L^\infty(\Omega \times]-T,T[)} \leqslant C_2 |\lambda_1 - \lambda_2|.$$

Since

$$\begin{aligned} \big|\nabla_x (u_{\lambda_1} + u_{\lambda_2}) \cdot \nabla_x w\big| &= \chi \big|\nabla_x (u_{\lambda_1} + u_{\lambda_2}) \cdot \nabla_x (u_{\lambda_1} - u_{\lambda_2})\big| \\ &\leqslant \chi |\nabla_x u_{\lambda_1}|^2 + 2\chi \big|\nabla_x (u_{\lambda_1} - u_{\lambda_2})\big|^2, \end{aligned}$$

we get

$$\begin{aligned} &\big\|\partial_t w + A(x,t,\partial_x) w\big\|_{L^p(\Omega \times]-T,T[)} \\ &\quad \leqslant C_4 \big(\big\|\chi |\nabla_x u_{\lambda_1}|^2\big\|_{L^p(\Omega \times]-T,T[)} + \big\|\chi |\nabla_x (u_{\lambda_1} - u_{\lambda_2})|^2\big\|_{L^p(\Omega \times]-T,T[)}\big) + C_3. \end{aligned} \tag{2.43}$$

As in the proof of Proposition 2.5, we also have, by (2.40),

$$
\begin{aligned}
&\big\|\chi\big|\nabla_x(u_{\lambda_1}-u_{\lambda_2})\big|^2\big\|_{L^p(\Omega\times]-T,T[)}\\
&\quad\leqslant C_5\|u_{\lambda_1}-u_{\lambda_2}\|_{L^\infty(\Omega\times]-T,T[)}\big\|\chi(u_{\lambda_1}-u_{\lambda_2})\big\|_{W_p^{2,1}(\Omega\times]-T,T[)}\\
&\quad\leqslant C_2C_5|\lambda_1-\lambda_2|\|w\|_{W_p^{2,1}(\Omega\times]-T,T[)},
\end{aligned}
\tag{2.44}
$$

and in particular,

$$
\big\|\chi|\nabla_x u_{\lambda_1}|^2\big\|_{L^p(\Omega\times]-T,T[)}\leqslant C_2C_5\lambda_1\|v_{\lambda_1}\|_{W_p^{2,1}(\Omega\times]-T,T[)}.
\tag{2.45}
$$

Using (2.43), (2.44), (2.45) and Proposition 2.4, we get

$$
\begin{aligned}
&\|w\|_{W_p^{2,1}(\Omega\times]-T,T[)}\\
&\quad\leqslant C_6+C_7\|v_{\lambda_1}\|_{W_p^{2,1}(\Omega\times]-T,T[)}+C_8|\lambda_1-\lambda_2|\|w\|_{W_p^{2,1}(\Omega\times]-T,T[)}.
\end{aligned}
$$

This also yields

$$
\|w\|_{W_p^{2,1}(\Omega\times]-T,T[)}\leqslant 2C_6+2C_7\|v_{\lambda_1}\|_{W_p^{2,1}(\Omega\times]-T,T[)}
$$

provided that $|\lambda_1-\lambda_2|\leqslant\varepsilon=(2C_8)^{-1}$. Hence, in particular, we have, for each $\lambda\in[0,\varepsilon]$,

$$
\|v_\lambda\|_{W_p^{2,1}(\Omega\times]-T,T[)}\leqslant 2C_6.
$$

Finally, iterating this process a finite number of times, we conclude that there is a constant C_9 such that

$$
\|v_1\|_{W_p^{2,1}(\Omega\times]-T,T[)}\leqslant C_9
$$

and hence

$$
\|u\|_{W_p^{2,1}(\Omega\times]0,T[)}\leqslant C_9. \qquad\square
$$

T-monotonicity and asymptotic behaviour of bounded solutions

DEFINITION 2.13. A function $u:\overline{\Omega}\times I\to\mathbb{R}$, where $I\subseteq\mathbb{R}$ is an interval, is said *T-increasing* (respectively *T-decreasing*) if $u(x,t)\leqslant u(x,t+T)$ (respectively $u(x,t+T)\geqslant u(x,t)$) for every $x\in\overline{\Omega}$ and each $t\in I$ such that $t+T\in I$. A function is said *T-monotone* if it is either T-increasing or T-decreasing.

REMARK 2.13. Let u be a solution of (2.10) on $\overline{\Omega}\times[t_0,+\infty[$. If u is T-increasing then the sequence $(\alpha_n)_n$, defined by setting, for each $n\geqslant t_0/T$, $\alpha_n(x,t)=u(x,t+nT)$ in $\overline{\Omega}\times[0,T[$, gives rise to an increasing sequence of regular lower solutions of (2.36). Similarly, any T-decreasing solution of (2.10) in $\overline{\Omega}\times[t_0,+\infty[$, gives rise to a decreasing sequence $(\beta_n)_n$ of regular upper solutions of (2.36).

The T-monotonicity property is a basic tool to prove convergence of a bounded solution u defined on $\overline{\Omega} \times [t_0, +\infty[$ to a T-periodic solution v. We start with a simple preliminary result where the T-monotonicity of u is replaced by a weaker request: the monotonicity of the sequence $(u(\cdot, nT))_n$, with $n \geqslant t_0/T$. However, in this case only the existence of a T-periodic solution v lying in the ω-limit set of u is guaranteed, not the convergence of u to v.

LEMMA 2.11. *Assume* (D), (A), (C) *and* (N). *Let u be a solution of* (2.10) *in* $\overline{\Omega} \times [t_0, +\infty[$, *which is bounded in* $\overline{\Omega} \times [t_0, +\infty[$ *and such that either* $u(\cdot, nT) \leqslant u(\cdot, (n+1)T)$ *in* $\overline{\Omega}$ *for every* $n \geqslant t_0/T$, *or* $u(\cdot, nT) \geqslant u(\cdot, (n+1)T)$ *in* $\overline{\Omega}$ *for every* $n \geqslant t_0/T$. *Then problem* (2.36) *has at least one solution.*

PROOF. Assume that $u(\cdot, nT) \leqslant u(\cdot, (n+1)T)$ in $\overline{\Omega}$ for every $n \geqslant t_0/T$. Define a sequence $(u_n)_n$ by setting $u_n(x, t) = u(x, t + nT)$ in $\overline{Q}_T$. By Proposition 2.10, we know that the sequence $(u_n)_n$ is bounded in $W_p^{2,1}(Q_T)$. Hence there exists a subsequence $(u_{n_k})_k$ which converges to a solution v of (2.10) in $\overline{Q}_T$ weakly in $W_p^{2,1}(Q_T)$ and strongly in $C^{1,0}(\overline{Q}_T)$. Moreover, as the sequences $(u_n(\cdot, 0))_n$ and $(u_n(\cdot, T))_n$ satisfy $u_n(\cdot, 0) \leqslant u_n(\cdot, T) = u_{n+1}(\cdot, 0)$, we have $u_{n_k}(\cdot, 0) \leqslant u_{n_k}(\cdot, T) \leqslant u_{n_{k+1}}(\cdot, 0)$ and, passing to the limit, $v(\cdot, 0) \leqslant v(\cdot, T) \leqslant v(\cdot, 0)$. Therefore v is a solution of (2.36). □

PROPOSITION 2.12. *Assume* (D), (A), (C) *and* (N). *Let u be a solution of* (2.10) *in* $\overline{\Omega} \times [t_0, +\infty[$, *which is bounded in* $\bar{\Omega} \times [t_0, +\infty[$ *and T-monotone. Then there exists a solution v of* (2.36) *such that*

$$\lim_{t \to +\infty} \left\| u(\cdot, t) - v(\cdot, t) \right\|_{C^1(\overline{\Omega})} = 0.$$

PROOF. Assume that u is T-increasing. Define a sequence $(u_n)_n$ by setting, for every $n \geqslant t_0/T$, $u_n(x, t) = u(x, t + nT)$ in $\overline{Q}_T$. From the proof of Lemma 2.11, we know that a subsequence of $(u_n)_n$ converges to a solution v of (2.36) weakly in $W_p^{2,1}(Q_T)$ and strongly in $C^{1,0}(\overline{Q}_T)$. The monotonicity of $(u_n)_n$ then implies that the whole sequence $(u_n)_n$ converges to v. It remains to prove that

$$\lim_{t \to +\infty} \left\| u(\cdot, t) - v(\cdot, t) \right\|_{C^1(\overline{\Omega})} = 0.$$

Indeed, since $(u_n)_n$ converges in $C^{1,0}(\overline{Q}_T)$ to v, given $\varepsilon > 0$ there is $\bar{n}$ such that, for any $n \geqslant \bar{n}$,

$$\left\| u_n(\cdot, t) - v(\cdot, t) \right\|_{C^1(\bar{\Omega})} < \varepsilon, \quad \text{for all } t \in [0, T].$$

Hence, if we take $t \geqslant \bar{n}T$, with $t \in [nT, (n+1)T[$ for some $n \geqslant \bar{n}$, we obtain

$$\left\| u(\cdot, t) - v(\cdot, t) \right\|_{C^1(\overline{\Omega})} = \left\| u_n(\cdot, t - nT) - v(\cdot, t - nT) \right\|_{C^1(\overline{\Omega})} < \varepsilon,$$

by the periodicity of v. □

A Monotone Convergence Criterion

We now prove that any bounded solution of (2.10) emanating from and lying above a lower solution of (2.36), or respectively emanating from and lying below an upper solution of (2.36), is T-monotone and hence converges to a T-periodic solution. We point out that, in space dimension $N \geqslant 2$, convergence to a T-periodic solution is not the behaviour of all bounded solutions of (2.10), due to the possible occurrence of (linearly stable) subharmonic solutions (see, e.g., [141,35] and [123, Section 3]). In our frame additional difficulties come from the lack of regularity of f, so that even generic convergence results, possibly to subharmonic solutions of (2.10), are not available (see [123, Section 3]). For a discussion of the regular case in space dimension $N = 1$ we refer to [56].

PROPOSITION 2.13. *Assume* (D), (A), (C) *and* (N).

(i) *Let α be a lower solution of* (2.36) *such that, for some t_0, $\alpha(\cdot, t_0) \in W_p^{2-2/p}(\Omega)$ and $\alpha(\cdot, t_0) = 0$ on $\partial\Omega$. Then there exist $\omega \in]t_0, +\infty]$ and a T-increasing solution $\widetilde{\alpha} : \overline{\Omega} \times [t_0, \omega[\to \mathbb{R}$ of* (2.13), *with $u_0 = \alpha(\cdot, t_0)$, satisfying $\widetilde{\alpha} \geqslant \alpha$ in $\overline{\Omega} \times [t_0, \omega[$. Further, every solution $u : \overline{\Omega} \times [t_0, \sigma[\to \mathbb{R}$ of* (2.10), *with $u \geqslant \alpha$, is such that $\sigma \leqslant \omega$ and $u \geqslant \widetilde{\alpha}$ in $\overline{\Omega} \times [t_0, \sigma[$. Finally, we have that either every nonextendible solution $u : \overline{\Omega} \times [t_0, \sigma[\to \mathbb{R}$ of* (2.10), *with $u \geqslant \alpha$, is such that*

$$\limsup_{t \to \sigma} \Bigl(\max_{\overline{\Omega}} u(\cdot, t) \Bigr) = +\infty$$

and (2.36) *has no solution in $[\alpha, +\infty[$, or there exists the minimum solution v in $[\alpha, +\infty[$ of* (2.36); *in the latter case $\omega = +\infty$, $v \geqslant \widetilde{\alpha}$ and*

$$\lim_{t \to +\infty} \bigl\| \widetilde{\alpha}(\cdot, t) - v(\cdot, t) \bigr\|_{C^1(\overline{\Omega})} = 0.$$

(ii) *Let β be an upper solution of* (2.36) *such that, for some t_0, $\beta(\cdot, t_0) \in W_p^{2-2/p}(\Omega)$ and $\beta(\cdot, t_0) = 0$ on $\partial\Omega$. Then there exist $\omega \in]t_0, +\infty]$ and a T-decreasing solution $\widetilde{\beta} : \overline{\Omega} \times [t_0, \omega[\to \mathbb{R}$ of* (2.13), *with $u_0 = \beta(\cdot, t_0)$, satisfying $\widetilde{\beta} \leqslant \beta$ in $\overline{\Omega} \times [t_0, \omega[$. Further, every solution $u : \overline{\Omega} \times [t_0, \sigma[\to \mathbb{R}$ of* (2.10), *with $u \leqslant \beta$, is such that $\sigma \leqslant \omega$ and $u \leqslant \widetilde{\beta}$ in $\overline{\Omega} \times [t_0, \sigma[$. Finally, we have that either every nonextendible solution $u : \overline{\Omega} \times [t_0, \sigma[\to \mathbb{R}$ of* (2.10), *with $u \leqslant \beta$, is such that*

$$\liminf_{t \to \sigma} \Bigl(\min_{\overline{\Omega}} u(\cdot, t) \Bigr) = -\infty$$

and (2.36) *has no solution in $]-\infty, \beta]$, or there exists the maximum solution w in $]-\infty, \beta]$ of* (2.36); *in the latter case $\omega = +\infty$, $w \leqslant \widetilde{\beta}$ and*

$$\lim_{t \to +\infty} \bigl\| \widetilde{\beta}(\cdot, t) - w(\cdot, t) \bigr\|_{C^1(\overline{\Omega})} = 0.$$

PROOF. We only prove statement (i), as the proof of (ii) is similar.

The existence of $\widetilde{\alpha}$ follows from Lemma 2.8. Namely, there exist $\omega \in]t_0, +\infty]$ and a solution $\widetilde{\alpha} : \overline{\Omega} \times [t_0, \omega[\to \mathbb{R}$ of (2.13), with $u_0 = \alpha(\cdot, t_0)$, satisfying $\widetilde{\alpha} \geqslant \alpha$ in $\overline{\Omega} \times [t_0, \omega[$;

further, every solution $u : \overline{\Omega} \times [t_0, \sigma] \to \mathbb{R}$ of (2.10), with $u \geqslant \alpha$, is such that $\sigma \leqslant \omega$ and $u \geqslant \widetilde{\alpha}$ in $\overline{\Omega} \times [t_0, \sigma]$.

Let us prove that $\widetilde{\alpha}$ is T-increasing. We can assume that $\omega > t_0 + T$ as otherwise there is nothing to prove. Notice first that $\widetilde{\alpha}(\cdot, t_0 + T) \geqslant \alpha(\cdot, t_0 + T) = \alpha(\cdot, t_0) = \widetilde{\alpha}(\cdot, t_0)$ in $\overline{\Omega}$. Assume, by contradiction, there exist $x_1 \in \Omega$ and $t_1 \geqslant t_0$ such that

$$\widetilde{\alpha}(x_1, t_1 + T) < \widetilde{\alpha}(x_1, t_1).$$

If $\omega < +\infty$, we set $\beta(x,t) = \min\{\widetilde{\alpha}(x, t+T), \widetilde{\alpha}(x,t)\}$ in $\overline{\Omega} \times [t_0, \omega - T[$. Then β is an upper solution of (2.13), with $u_0 = \alpha(\cdot, t_0)$, in $\overline{\Omega} \times [t_0, \omega - T[$, satisfying $\beta \geqslant \alpha$. Hence, by Theorem 2.6, there exists a solution w of (2.13), with $u_0 = \alpha(\cdot, t_0)$, in $\overline{\Omega} \times [t_0, \omega - T[$, satisfying $\alpha \leqslant w \leqslant \beta$. If $\omega = +\infty$, we set $\beta(x,t) = \min\{\widetilde{\alpha}(x, t+T), \widetilde{\alpha}(x,t)\}$ in $\overline{\Omega} \times [t_0, +\infty[$. Then β is an upper solution of (2.13), with $u_0 = \alpha(\cdot, t_0)$, in $\overline{\Omega} \times [t_0, +\infty[$, satisfying $\beta \geqslant \alpha$. Hence, by Corollary 2.7, there exists a solution w of (2.13), with $u_0 = \alpha(\cdot, t_0)$, in $\overline{\Omega} \times [t_0, +\infty[$, satisfying $\alpha \leqslant w \leqslant \beta$. In both cases, since $\beta < \widetilde{\alpha}$, we get $w < \widetilde{\alpha}$, thus contradicting the definition of $\widetilde{\alpha}$.

By Proposition 2.9, either every nonextendible solution $u : \overline{\Omega} \times [0, \sigma[\to \mathbb{R}$ of (2.10), with $u \geqslant \alpha$, is such that

$$\limsup_{t \to \sigma} \Big(\max_{\overline{\Omega}} u(\cdot, t) \Big) = +\infty$$

and in particular (2.36) has no solution in $[\alpha, +\infty[$, or $\omega = +\infty$ and $\widetilde{\alpha}$ is bounded in $\overline{\Omega} \times [t_0, +\infty[$. In the latter case, as $\tilde{\alpha}$ is T-increasing, by Proposition 2.12, there exists a solution $v \geqslant \widetilde{\alpha}$ of (2.36) such that

$$\lim_{t \to +\infty} \big\| \widetilde{\alpha}(\cdot, t) - v(\cdot, t) \big\|_{C^1(\overline{\Omega})} = 0.$$

Let us prove that v is the minimum solution of (2.36) in $[\alpha, +\infty[$. Otherwise, if $\bar{v}$ were a solution of (2.36) with $\bar{v} \geqslant \alpha$ and $\bar{v} \ngeqslant v$, then $\min\{v, \bar{v}\}$ should be an upper solution of (2.13), with $u_0 = \alpha(\cdot, t_0)$, in $\overline{\Omega} \times [t_0, +\infty[$, such that $\min\{v, \bar{v}\} \geqslant \alpha$ and $\min\{v, \bar{v}\} \ngeqslant \widetilde{\alpha}$. By Corollary 2.7, there should exist a solution z of (2.13), with $u_0 = \alpha(\cdot, t_0)$, in $\overline{\Omega} \times [t_0, +\infty[$, such that $\alpha \leqslant z \leqslant \min\{v, \bar{v}\}$, thus contradicting the definition of $\widetilde{\alpha}$. □

A linear periodic problem

As a simple application of Proposition 2.13, we get a result that concerns existence, uniqueness and regularity of the solution of the linear periodic problem

$$\begin{aligned} &\partial_t u + A(x, t, \partial_x) u + \sigma u = h(x,t) && \text{in } Q_T, \\ &u = 0 && \text{on } \Sigma_T, \\ &u(\cdot, 0) = u(\cdot, T) && \text{in } \Omega, \end{aligned} \tag{2.46}$$

where $\sigma \geqslant 0$ is fixed and $h \in L^p(Q_T)$ for some $p > N + 2$. Although proofs can be found in [84, Lemma 4.1 and Corollary 5.6], we use here an alternative approach which is more in the spirit of this work and will yield, as a byproduct, the global asymptotic stability of the solution (see Proposition 2.35). We associate with problem (2.46) the linear operator

$$L : W^{2,1}_{p,B}(Q_T) \to L^p(Q_T), \tag{2.47}$$

defined by

$$Lu = \partial_t u - \sum_{i,j=1}^{N} \partial_{x_i}\big(a_{ij}(x,t)\partial_{x_j} u\big) + \sum_{i=1}^{N} a_i(x,t)\partial_{x_i} u + a_0(x,t)u, \tag{2.48}$$

where

$$\begin{aligned} W^{2,1}_{p,B}(Q_T) &= W^{2,1}_p(Q_T) \cap C^{1,0}_B\big(\overline{Q_T}\big) \\ &= \big\{u \in W^{2,1}_p(Q_T) \mid u = 0 \text{ on } \Sigma_T \text{ and } u(\cdot,0) = u(\cdot,T) \text{ in } \Omega\big\}. \end{aligned}$$

Notice that $W^{2,1}_{p,B}(Q_T)$ is a Banach subspace of $W^{2,1}_p(Q_T)$ and L is a bounded operator.

PROPOSITION 2.14. *Assume* (D) *and* (A). *Let $p > N + 2$ be fixed. Then, for every $\sigma \geqslant 0$ and for any given $h \in L^p(Q_T)$, problem* (2.46) *has a unique solution $u \in W^{2,1}_p(Q_T)$. Moreover, there exists a constant C, independent of h, such that*

$$\|u\|_{W^{2,1}_p(Q_T)} \leqslant C\|h\|_{L^p(Q_T)}.$$

PROOF. The proof consists of four steps.

Step 1. *Problem* (2.46) *has at most one solution*. The conclusion immediately follows showing that any function $u \in W^{2,1}_p(Q_T)$, satisfying

$$\begin{array}{ll} \partial_t u + A(x,t,\partial_x)u + \sigma u \geqslant 0 & \text{in } Q_T, \\ u \geqslant 0 & \text{on } \Sigma_T, \\ u(\cdot,0) = u(\cdot,T) & \text{in } \Omega, \end{array}$$

is such that $u \geqslant 0$. Indeed, assume by contradiction that $\min_{\overline{Q_T}} u < 0$. Since $u(\cdot,0) = u(\cdot,T)$ in Ω and $u \geqslant 0$ on Σ_T, there exists $(x_0,t_0) \in \Omega \times]0,T]$ such that $u(x_0,t_0) = \min_{\overline{Q_T}} u$. Then Proposition 2.2 implies that u is constant in $\bar{\Omega} \times [0,t_0]$. This yields a contradiction, as then

$$\partial_t u + A(x,t,\partial_x)u + \sigma u = (a_0 + \sigma)u < 0, \quad \text{a.e. in } \Omega \times [0,t_0].$$

Step 2. *Problem* (2.46) *has a solution for any given $h \geqslant 0$*. The function $\alpha = 0$ is obviously a lower solution of (2.46) with $\alpha(\cdot,0) \in W^{2-2/p}_p(\Omega)$ and $\alpha(\cdot,0) = 0$ on $\partial\Omega$. Let us build an upper solution of (2.46). Let z be the unique solution of

$$\begin{array}{ll} \partial_t u + A(x,t,\partial_x)u + \sigma u = h(x,t) & \text{in } Q_T, \\ u = 0 & \text{on } \Sigma_T, \\ u(\cdot,0) = 0 & \text{in } \Omega. \end{array}$$

Define a function $\beta \in W_p^{2,1}(Q_T)$, by setting

$$\beta(x,t) = z(x,t) + at + b \quad \text{in } \overline{\Omega} \times [0, T[,$$

where $a \leqslant 0$ and $b \geqslant 0$ are constants chosen in such a way that

$$aT + \|z(\cdot, T)\|_{L^\infty(\Omega)} \leqslant 0 \quad \text{and} \quad a + \Big(\sigma + \operatorname*{ess\,inf}_{Q_T} a_0\Big)(aT + b) \geqslant 0.$$

Accordingly, β satisfies

$$\begin{aligned} &\partial_t \beta + A(x,t,\partial_x)\beta + \sigma\beta \geqslant h(x,t) && \text{in } Q_T, \\ &\beta \geqslant 0 && \text{on } \Sigma_T, \\ &\beta(\cdot, 0) \geqslant \lim_{t \to T^-} \beta(\cdot, t) && \text{in } \Omega \end{aligned}$$

and hence β is an upper solution of (2.46). We apply Corollary 2.7 and Proposition 2.13 to deduce the existence of a T-increasing solution $\widetilde{\alpha} : \overline{\Omega} \times [0, +\infty[\to \mathbb{R}$ of

$$\begin{aligned} &\partial_t u + A(x,t,\partial_x)u + \sigma u = h(x,t) && \text{in } \Omega \times]0, +\infty[, \\ &u = 0 && \text{on } \partial\Omega \times [0, +\infty[, \\ &u(\cdot, 0) = 0 && \text{in } \Omega \end{aligned}$$

and of a solution v of (2.46) in $[0, \beta]$, which by Step 1 is the unique solution of (2.46). Further, we have

$$\lim_{t \to +\infty} \|\widetilde{\alpha}(\cdot, t) - v(\cdot, t)\|_{C^1(\bar{\Omega})} = 0.$$

Step 3. *Problem* (2.46) *has a solution for any given* h. Let us define the functions $h^+ = \max\{h, 0\}$, $h^- = h^+ - h$ and consider the periodic problems

$$\begin{aligned} &\partial_t u + A(x,t,\partial_x)u + \sigma u = h^+(x,t) && \text{in } Q_T, \\ &u = 0 && \text{on } \Sigma_T, \\ &u(\cdot, 0) = u(\cdot, T) && \text{in } \Omega \end{aligned}$$

and

$$\begin{aligned} &\partial_t u + A(x,t,\partial_x)u + \sigma u = h^-(x,t) && \text{in } Q_T, \\ &u = 0 && \text{on } \Sigma_T, \\ &u(\cdot, 0) = u(\cdot, T) && \text{in } \Omega. \end{aligned}$$

Denote by u_+ and u_- the corresponding solutions, constructed in Step 2. By linearity, we conclude that $u = u_+ - u_-$ is the solution of (2.46).

Step 4. *Continuous dependence*. Since the linear operator $L + \sigma I : W^{2,1}_{p,B}(Q_T) \to L^p(Q_T)$ is continuous, one-to-one and onto and its domain $W^{2,1}_{p,B}(Q_T)$ is a Banach space, the Open Mapping Theorem yields the estimate

$$\|u\|_{W^{2,1}_p(Q_T)} \leqslant C\|h\|_{L^p(Q_T)}. \qquad \square$$

Existence in case $\alpha \leqslant \beta$

The following result yields the existence of extremal solutions of (2.36) in the presence of lower and upper solutions α, β satisfying the condition $\alpha \leqslant \beta$. This extends to strong solutions previous results obtained, e.g., in [73,8,6,33,58] for classical solutions. For later use we also evaluate the degree of the fixed point operator associated with (2.36), when the lower and the upper solutions are strict. To this end we denote by $S : C^{1,0}_B(\overline{Q}_T) \to C^{1,0}_B(\overline{Q}_T)$ the operator, which according to Proposition 2.14 sends any function $u \in C^{1,0}_B(\overline{Q}_T)$ onto the unique solution $v \in W^{2,1}_p(Q_T)$ of

$$\begin{array}{ll} \partial_t v + A(x,t,\partial_x)v = f(x,t,u,\nabla_x u) & \text{in } Q_T, \\ v = 0 & \text{on } \Sigma_T, \\ v(\cdot,0) = v(\cdot,T) & \text{in } \Omega. \end{array} \tag{2.49}$$

THEOREM 2.15. *Assume* (D), (A), (C) *and* (N). *Suppose that* α *is a lower solution and* β *is an upper solution of* (2.36) *satisfying*

$$\alpha \leqslant \beta.$$

Then there exist the minimum solution v *and the maximum solution* w *of* (2.36) *in* $[\alpha, \beta]$. *Further, if* α *and* β *are strict, then the operator* S *defined by* (2.49) *has no fixed point on the boundary of*

$$\mathcal{U} = \{u \in C^{1,0}_B(\overline{Q}_T) \mid \alpha \ll u \ll \beta\},$$

the set of fixed points of S *in* $\mathcal{U}$ *is bounded in* $C^{1,0}_B(\overline{Q}_T)$ *and*

$$\deg\big(I - S, \mathcal{U} \cap B(0,R)\big) = 1,$$

where I *is the identity operator in* $C^{1,0}_B(\overline{Q}_T)$ *and* $B(0,R)$ *is the open ball of center* 0 *and radius* R *in* $C^{1,0}_B(\overline{Q}_T)$, *with* R *so large that all fixed points of* S *in* $\mathcal{U}$ *belong to* $B(0,R)$.

PROOF. The proof is divided into three parts.

Part 1. *Existence of a solution* u *of* (2.36) *with* $\alpha \leqslant u \leqslant \beta$. According to Remark 2.4, we can assume that the sequences $(\sigma_h)_{0\leqslant h\leqslant k}$ and $(\rho_h)_{0\leqslant h\leqslant l}$ coincide. We also notice that, by our definition of lower and upper solutions of (2.36) and the condition $\alpha \leqslant \beta$, there exists $M > 0$ such that $-M \leqslant \alpha(x,t) \leqslant \beta(x,t) \leqslant M$ in $\overline{Q}_T$. Hence, by condition (N) and

Proposition 2.10, there exists $R > 0$ such that, for every function f satisfying (2.11) and every solution u of (2.36) with $\alpha \leqslant u \leqslant \beta$, we have

$$\|u\|_{C^{1,0}(\overline{Q}_T)} < R. \tag{2.50}$$

Exactly as in the first part of the proof of Theorem 2.6, we define a function $F : Q_T \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ with $\Omega \times]t_0, t_1[$ replaced by $\Omega \times]0, T[$. Then we consider the modified problem

$$\begin{array}{ll} \partial_t u + A(x,t,\partial_x)u = F(x,t,u,\nabla_x u) & \text{in } Q_T, \\ u = 0 & \text{on } \Sigma_T, \\ u(\cdot,0) = u(\cdot,T) & \text{in } \Omega. \end{array} \tag{2.51}$$

Step 1. *Every solution u of* (2.51) *satisfies* $\alpha \leqslant u \leqslant \beta$. This can be proved as in the corresponding step of the proof of Theorem 2.6. We just observe that, if u is a solution of (2.51) such that $\inf_{Q_T}(u-\alpha) < 0$, then by our definition of a lower solution of (2.51) either there exists $h \in \{0, \dots, k-1\}$ such that $\inf_{\Omega \times]\sigma_h, \sigma_{h+1}[}(u-\alpha) < 0$ and, for all $x \in \Omega$, $\lim_{t \to \sigma_h^+}(u-\alpha)(x,t) > \inf_{\Omega \times]\sigma_h, \sigma_{h+1}[}(u-\alpha)$, or $\inf_{\Omega \times]\sigma_h, \sigma_{h+1}[}(u-\alpha) = \inf_{\Omega \times]\sigma_{h-1}, \sigma_h[}(u-\alpha)$ for every $h \in \{1, \dots, k-1\}$. In any case we can find $h \in \{0, \dots, k-1\}$ and $i \in \{1, \dots, m_h\}$ such that, setting $v = u - \alpha_i^{(h)} \in W_p^{2,1}(\Omega \times]\sigma_h, \sigma_{h+1}[)$, we have $v(x,t) \geqslant 0$ on $\partial\Omega \times [\sigma_h, \sigma_{h+1}]$ and there exists $(\bar{x}, \bar{t}) \in \Omega \times]\sigma_h, \sigma_{h+1}]$ such that $\min_{\overline{\Omega} \times [\sigma_h, \sigma_{h+1}]} v = v(\bar{x}, \bar{t}) < 0$.

Step 2. *Every solution u of* (2.51) is *a solution of* (2.36). Again this can be proved as in the corresponding step of the proof of Theorem 2.6.

Step 3. *Problem* (2.51) *has at least one solution*. Let us consider the solution operator $\overline{S} : C_B^{1,0}(\overline{Q}_T) \to C_B^{1,0}(\overline{Q}_T)$ associated with (2.51), which sends any function $u \in C_B^{1,0}(\overline{Q}_T)$ onto the unique solution $v \in W_p^{2,1}(Q_T)$ of

$$\begin{array}{ll} \partial_t v + A(x,t,\partial_x)v = F(x,t,u,\nabla_x u) & \text{in } Q_T, \\ v = 0 & \text{on } \Sigma_T, \\ v(\cdot,0) = v(\cdot,T) & \text{in } \Omega. \end{array}$$

The operator $\overline{S}$ is continuous, has a relatively compact range and its fixed points are the solutions of (2.51). Hence there exists a constant $\overline{R} > 0$, that we can suppose larger than R, such that

$$\|\overline{S}u\|_{C^{1,0}(\overline{Q}_T)} < \overline{R},$$

for every $u \in C_B^{1,0}(\overline{Q}_T)$, and

$$\deg\big(I - \overline{S}, B\big(0, \overline{R}\big)\big) = 1, \tag{2.52}$$

where I is the identity operator in $C_B^{1,0}(\overline{Q}_T)$ and $B(0, \overline{R})$ is the open ball of center 0 and radius $\overline{R}$ in $C_B^{1,0}(\overline{Q}_T)$. Therefore $\overline{S}$ has a fixed point. By the conclusions of Steps 1 and 2, we get the existence of a solution u of (2.13) satisfying $\alpha \leqslant u \leqslant \beta$.

Part 2. *Extremal solutions*. Again this can be proved as in the corresponding part of the proof of Theorem 2.6.

Part 3. *Degree computation*. Now, let us assume that α and β are strict lower and upper solutions respectively. Since there exists a solution u of (2.36), which satisfies $\alpha \leqslant u \leqslant \beta$, and every such a solution satisfies $\alpha \ll u \ll \beta$, it follows that $\alpha \ll \beta$. Hence $\mathcal{U}$ is a non-empty open set in $C_B^{1,0}(\overline{Q}_T)$ and there is no fixed point either of S or of $\overline{S}$ on its boundary $\partial\mathcal{U}$. Moreover, by (2.50), the sets of fixed points of S and $\overline{S}$ coincide on $\mathcal{U} \cap B(0, R)$ and we have

$$\deg\bigl(I - S, \mathcal{U} \cap B(0, R)\bigr) = \deg\bigl(I - \overline{S}, \mathcal{U} \cap B(0, R)\bigr).$$

Furthermore, by the excision property of the degree, we get from (2.50) and (2.52)

$$\deg\bigl(I - \overline{S}, B(0, R)\bigr) = 1.$$

Finally, since all fixed points of $\overline{S}$ are in $\mathcal{U} \cap B(0, R)$, we conclude

$$\begin{aligned}\deg\bigl(I - S, \mathcal{U} \cap B(0, R)\bigr) &= \deg\bigl(I - \overline{S}, \mathcal{U} \cap B(0, R)\bigr)\\ &= \deg\bigl(I - \overline{S}, B(0, R)\bigr) = 1.\end{aligned}$$ □

REMARK 2.14. We assumed condition (N) in Theorem 2.15 in order to unify the presentation with the rest of this work. However, here it would be enough to assume the existence of $K > 0$ and $h \in L^p(Q_T)$, with $p > N + 2$, such that, for a.e. $(x, t) \in Q_T$, every $s \in [\alpha(x, t), \beta(x, t)]$ and every $\xi \in \mathbb{R}^N$,

$$\bigl|f(x, t, s, \xi)\bigr| \leqslant h(x, t) + K|\xi|^2.$$

Existence in case $\alpha \nleqslant \beta$

We now discuss the existence of solutions of (2.36) in the presence of lower and upper solutions α, β such that $\alpha \nleqslant \beta$. As we already pointed out, in the frame of problem (2.36) the sole existence of a pair of lower and upper solutions, even satisfying $\alpha > \beta$, does not generally guarantee the existence of a solution. At the light of the example presented in the introduction of this part (cf. (2.8)), we prove a result which is related to the classical Amann–Kolesov Three Solutions Theorem [73,5,7] and requires the existence of a further pair of lower and upper solutions α_1, β_1 such that $\alpha_1 \leqslant \beta_1$ and $\alpha, \beta \in [\alpha_1, \beta_1]$. Theorem 2.17 yields the existence of maximal and minimal solutions of (2.36) in $[\alpha_1, \beta_1]$ belonging to the $C^{1,0}$-closure of the set

$$\mathcal{V} = \bigl\{u \in C_B^{1,0}\bigl(\overline{Q}_T\bigr) \mid u \ngeqslant \alpha \text{ and } u \nleqslant \beta\bigr\}. \tag{2.53}$$

The proof of Theorem 2.17 makes use of the following statement concerning the existence of extremal solutions of (2.36).

LEMMA 2.16. *Assume* (D), (A), (C) *and* (N). *Let $\mathcal{S}$ be a nonempty set of solutions of* (2.36). *Suppose that $\mathcal{S}$ is uniformly bounded from above, i.e. there is a constant M such that* $\max_{Q_T} u \leqslant M$ *for all $u \in \mathcal{S}$. Then there exists a maximal solution w of* (2.36) *in the $C^{1,0}$-closure $\bar{\mathcal{S}}$ of $\mathcal{S}$. Similarly, if $\mathcal{S}$ is uniformly bounded from below, then there exists a minimal solution v of* (2.36) *in the $C^{1,0}$-closure $\bar{\mathcal{S}}$ of $\mathcal{S}$.*

PROOF. We can repeat step by step the proof of Lemma 1.15, with just the following specifications. The set $\mathcal{D} = \{(x_m, t_m) \mid m \in \mathbb{N}\}$ is a countable dense subset of Q_T. The sequence $(u_n)_n$, constructed as in Lemma 1.15, is bounded in $L^\infty(Q_T)$ and, by Proposition 2.10, it is bounded in $W_p^{2,1}(Q_T)$. Hence there is a subsequence converging weakly in $W_p^{2,1}(Q_T)$ to some function $\hat{u} \in W_p^{2,1}(Q_T)$, which is a solution of (2.36). The conclusion then follows as in Lemma 1.15. □

THEOREM 2.17. *Assume* (D), (A), (C) *and* (N). *Suppose that α is a lower solution and β is an upper solution of* (2.36) *satisfying*

$$\alpha \nleqslant \beta.$$

Moreover, assume that there exist a lower solution α_1 and an upper solution β_1 of (2.36) *such that $\alpha_1 \leqslant \beta_1$ and $\alpha, \beta \in [\alpha_1, \beta_1]$. Then problem* (2.36) *has at least one minimal solution v and at least one maximal solution w in $\overline{\mathcal{V}} \cap [\alpha_1, \beta_1]$ with $\mathcal{V}$ defined by* (2.53). *Further, if α, β and α_1, β_1 are strict, then the solution operator S corresponding to* (2.49) *has no fixed point on the boundary of $\mathcal{V} \cap [\alpha_1, \beta_1]$, the set of fixed points of S in $\mathcal{V} \cap [\alpha_1, \beta_1]$ is bounded in $C_B^{1,0}(\overline{Q}_T)$ and*

$$\deg\big(I - S, \mathcal{V} \cap \mathcal{U} \cap B(0, R)\big) = -1,$$

where I is the identity operator in $C_B^{1,0}(\overline{Q}_T)$, $\mathcal{U} = \{u \in C_B^{1,0}(\overline{Q}_T) \mid \alpha_1 \ll u \ll \beta_1\}$ and $B(0, R)$ is the open ball of center 0 *and radius R in $C_B^{1,0}(\overline{Q}_T)$, with R so large that all fixed points of S in $\mathcal{V} \cap \mathcal{U}$ belongs to $B(0, R)$.*

PROOF. In the course of this proof we denote the given lower and upper solutions α, β by α_0, β_0, respectively. Let us consider problem (2.51), where the function F is defined as in the first part of the proof of Theorem 2.15, with α and β replaced by α_1 and β_1 respectively. In case α_1 or β_1 are not strict for (2.36), and hence for (2.51), we further replace α_1 with $\alpha_1 - 1$ and β_1 with $\beta_1 + 1$. Hence we can suppose that α_1 and β_1 are strict lower and upper solutions of (2.51) such that $\alpha_1 \ll \alpha_0 \ll \beta_1$ and $\alpha_1 \ll \beta_0 \ll \beta_1$. Let us set, for $(i, j) \in \{0, 1\}^2 \setminus \{(0, 0)\}$,

$$\mathcal{U}_{(i,j)} = \big\{u \in C_B^{1,0}\big(\overline{Q}_T\big) \mid \alpha_i \ll u \ll \beta_j\big\}.$$

In particular, we have $\mathcal{U}_{(1,1)} = \mathcal{U}$. Notice that $\mathcal{U}_{(i,j)}$ are nonempty open sets in $C_B^{1,0}(\overline{Q}_T)$, since $\alpha_i \ll \beta_j$ for $(i, j) \in \{0, 1\}^2 \setminus \{(0, 0)\}$. Moreover, we have that

$$\mathcal{V} \cap \mathcal{U}_{(1,1)} = \mathcal{U}_{(1,1)} \setminus \overline{(\mathcal{U}_{(0,1)} \cup \mathcal{U}_{(1,0)})}.$$

Let S be the solution operator corresponding to (2.49) and observe that, by Theorem 2.15, all fixed points $u \in \overline{\mathcal{V} \cap \mathcal{U}_{(1,1)}}$ of S satisfy

$$\|u\|_{C^{1,0}(\overline{Q}_T)} < R,$$

for some constant $R > 0$. Then, either there is a solution $u \in \partial\mathcal{V}$ of (2.36) and the existence part of the theorem is proved, or there is no solution $u \in \partial\mathcal{V}$ of (2.36), which means that α_0 and β_0 are strict. In that case, by the additivity-excision property of the degree and Theorem 2.15, we get

$$\begin{aligned}
1 &= \deg\big(I - S, \mathcal{U}_{(1,1)} \cap B(0, R)\big) \\
&= \deg\big(I - S, \mathcal{U}_{(0,1)} \cap B(0, R)\big) + \deg\big(I - S, \mathcal{U}_{(1,0)} \cap B(0, R)\big) \\
&\quad + \deg\big(I - S, \mathcal{V} \cap \mathcal{U}_{(1,1)} \cap B(0, R)\big) \\
&= 2 + \deg\big(I - S, \mathcal{V} \cap \mathcal{U}_{(1,1)} \cap B(0, R)\big)
\end{aligned}$$

and hence

$$\deg\big(I - S, \mathcal{V} \cap \mathcal{U}_{(1,1)} \cap B(0, R)\big) = -1,$$

where $B(0, R)$ is the open ball of center 0 and radius R in $C_B^{1,0}(\overline{Q}_T)$. This argument yields, in any case, the existence of a solution $u \in \overline{\mathcal{V}} \cap [\alpha_1, \beta_1]$ of (2.36).

Finally, the existence of minimal and maximal solutions follows from Lemma 2.16 with $\mathcal{S} = \mathcal{V} \cap [\alpha_1, \beta_1]$. □

2.5. *The Order Interval Trichotomy and applications*

The aim of this section is to extend to our general context the Order Interval Trichotomy, first proved in [34] for order preserving discrete-time semidynamical systems. We stress once more that our results are obtained without requiring any regularity on the function f besides the Carathéodory conditions. Therefore no continuous or discrete semiflow can be naturally associated with (2.10). Moreover, even if a semiflow were defined by (2.13), there is no apparent reason for which it should be order preserving. Nevertheless with the aid of Proposition 2.3, which plays a crucial role in this context, we can deform our original problem to other problems possessing a certain amount of monotonicity, from which we infer the information needed to prove the existence of some kind of connections.

Once a version of the Order Interval Trichotomy is established, we apply it to the study of the dynamics near one-sided isolated solutions of (2.36), namely we prove the existence of heteroclinic solutions connecting a pair of comparable solutions of (2.36) and we study the qualitative behaviour of solutions of (2.10) lying above a maximal, or below a minimal, solution of (2.36).

Connecting T-periodic solutions by lower or upper solutions
The basic result in this context is the following statement.

PROPOSITION 2.18. *Assume* (D), (A), (C) *and* (N). *Suppose that* u_1, u_2 *are solutions of* (2.36) *such that* $u_1 < u_2$ *and there is no solution* u *of* (2.36) *with* $u_1 < u < u_2$. *Then either*

- *there exists a sequence* $(\alpha_n)_n$ *of proper regular lower solutions of* (2.36) *such that, for each* n, $\alpha_n|_{\overline{Q}_T} \in W^{2,1}_{p,B}(Q_T)$ *and* $u_1 < \alpha_n < u_2$, *which converges weakly in* $W^{2,1}_p(Q_T)$ *and strongly in* $C^{1,0}(\overline{Q}_T)$ *to* u_1 *as* $n \to +\infty$, *or*
- *there exists a sequence* $(\beta_n)_n$ *of proper regular upper solutions of* (2.36) *such that, for each* n, $\beta_n|_{\overline{Q}_T} \in W^{2,1}_{p,B}(Q_T)$ *and* $u_1 < \beta_n < u_2$, *which converges weakly in* $W^{2,1}_p(Q_T)$ *and strongly in* $C^{1,0}(\overline{Q}_T)$ *to* u_2 *as* $n \to +\infty$.

PROOF. Let us set, for every $(x,t) \in \overline{Q}_T$ and $s \in \mathbb{R}$,

$$\gamma(x,t,s) = \max\bigl\{u_1(x,t), \min\bigl\{s, u_2(x,t)\bigr\}\bigr\}.$$

Clearly, $\gamma : \overline{Q}_T \times \mathbb{R} \to \mathbb{R}$ is continuous and, for each $(x,t) \in \overline{Q}_T$, $\gamma(x,t,\cdot) : \mathbb{R} \to \mathbb{R}$ is increasing. Moreover, for $i = 1, 2$, let us set, for a.e. $(x,t) \in Q_T$ and every $\varepsilon > 0$,

$$\begin{aligned}&\omega_i(x,t,\varepsilon)\\ &\quad= \max_{|\xi| \leqslant \varepsilon}\bigl| f\bigl(x,t,u_i(x,t), \nabla_x u_i(x,t) + \xi\bigr) - f\bigl(x,t,u_i(x,t), \nabla_x u_i(x,t)\bigr)\bigr|.\end{aligned}$$

Then define, for a.e. $(x,t) \in Q_T$, every $s \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$,

$$\omega(x,t,s) = \begin{cases} \omega_1\bigl(x,t,u_1(x,t) - s\bigr) & \text{if } s < u_1(x,t),\\ 0 & \text{if } u_1(x,t) \leqslant s \leqslant u_2(x,t),\\ -\omega_2\bigl(x,t,s - u_2(x,t)\bigr) & \text{if } s > u_2(x,t)\end{cases}$$

and

$$\bar f(x,t,s,\xi) = f\bigl(x,t,\gamma(x,t,s),\xi\bigr) + \omega(x,t,s).$$

Clearly, $\bar f : Q_T \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfies the Carathéodory condition (C) and the Nagumo condition (N). Let us consider the modified problem

$$\begin{aligned} &\partial_t u + A(x,t,\partial_x)u = \bar f(x,t,u,\nabla_x u) && \text{in } Q_T,\\ &u = 0 && \text{on } \Sigma_T,\\ &u(\cdot,0) = u(\cdot,T) && \text{in } \Omega.\end{aligned} \tag{2.54}$$

Further, take $\rho = \max\{\|u_1\|_{L^\infty(Q_T)}, \|u_2\|_{L^\infty(Q_T)}\}$ and let h be the function associated with f and ρ whose existence is guaranteed by Proposition 2.3. Consider, for each $\mu \in [0,1]$, the following problems

$$\begin{aligned} &\partial_t u + A(x,t,\partial_x)u = \mu \bar f(x,t,u,\nabla_x u) + (1-\mu)\bigl[\bar f(x,t,u_1,\nabla_x u)\\ &\qquad + h(x,t,u_1,\gamma(x,t,u),\nabla_x u) + \omega(x,t,u)\bigr] && \text{in } Q_T,\\ &u = 0 && \text{on } \Sigma_T,\\ &u(\cdot,0) = u(\cdot,T) && \text{in } \Omega\end{aligned} \tag{2.55}$$

and

$$\begin{aligned}
&\partial_t u + A(x,t,\partial_x)u = \mu \bar f(x,t,u,\nabla_x u) + (1-\mu)\big[\bar f(x,t,u_2,\nabla_x u) \\
&\quad + h(x,t,u_2,\gamma(x,t,u),\nabla_x u) + \omega(x,t,u)\big] \quad \text{in } Q_T, \\
&u = 0 \quad \text{on } \Sigma_T, \\
&u(\cdot,0) = u(\cdot,T) \quad \text{in } \Omega.
\end{aligned} \tag{2.56}$$

It is clear that, if $\mu = 1$, (2.55) and (2.56) reduce to (2.54). Note that the right hand side of the equations in (2.55) and (2.56) satisfy the Carathéodory condition (C) and the Nagumo condition (N), uniformly with respect to $\mu \in [0,1]$. Proposition 2.10 implies that there is a constant $C > 0$ such that, if u is a solution of (2.55), or (2.56), for any $\mu \in [0,1]$, satisfying $u_1 \leqslant u \leqslant u_2$, then

$$\|u\|_{W_p^{2,1}(Q_T)} \leqslant C. \tag{2.57}$$

CLAIM 1. *For any $\mu \in [0,1]$, every solution u of* (2.55), *or* (2.56), *satisfies $u_1 \leqslant u \leqslant u_2$ and*, *hence*, (2.57). *In particular*, *u_1 and u_2 are the only solutions of* (2.54).

Let u be a solution of (2.55) and prove that $u \geqslant u_1$. The argument is similar to the proof of Theorem 2.6 observing that, if $v = u - u_1$ is such that $v(x,t) < 0$ and $|\nabla_x v(x,t)| \leqslant |v(x,t)|$ on $B \times]t_1,t_0]$, for some ball $B \subseteq \Omega$, then on this set

$$\begin{aligned}
\partial_t v + A(x,t,\partial_x)v &= \mu\big[\bar f(x,t,u,\nabla_x u) - f(x,t,u_1,\nabla_x u_1)\big] \\
&\quad + (1-\mu)\big[\bar f(x,t,u_1,\nabla_x u) - f(x,t,u_1,\nabla_x u_1) \\
&\quad + \omega(x,t,u) + h(x,t,u_1,\gamma(x,t,u),\nabla_x u)\big] \\
&= f(x,t,u_1,\nabla_x u) + \omega_1\big(x,t,\big|v(x,t)\big|\big) - f(x,t,u_1,\nabla_x u_1) \\
&\geqslant f(x,t,u_1,\nabla_x u) - f(x,t,u_1,\nabla_x u_1 + \nabla_x v) \\
&\quad + f(x,t,u_1,\nabla_x u_1) - f(x,t,u_1,\nabla_x u_1) \\
&= 0.
\end{aligned}$$

Now, let us prove that $u \leqslant u_2$. Again, we argue as in the proof of Theorem 2.6 observing that, if $v = u_2 - u$ is such that $v(x,t) < 0$ and $|\nabla_x v(x,t)| \leqslant |v(x,t)|$ in $B \times]t_1,t_0]$, for some ball $B \subseteq \Omega$, then on this set

$$\begin{aligned}
&\partial_t v + A(x,t,\partial_x)v \\
&\quad = \mu\big[f(x,t,u_2,\nabla_x u_2) - \bar f(x,t,u,\nabla_x u)\big] + (1-\mu)\big[f(x,t,u_2,\nabla_x u_2) \\
&\qquad - \bar f(x,t,u_1,\nabla_x u) - \omega(x,t,u) - h(x,t,u_1,\gamma(x,t,u),\nabla_x u)\big] \\
&\quad = \mu\big[f(x,t,u_2,\nabla_x u_2) - f(x,t,u_2,\nabla_x u) + \omega_2\big(x,t,\big|v(x,t)\big|\big)\big] \\
&\qquad + (1-\mu)\big[f(x,t,u_2,\nabla_x u_2) - f(x,t,u_1,\nabla_x u) + \omega_2\big(x,t,\big|v(x,t)\big|\big) \\
&\qquad - h(x,t,u_1,u_2,\nabla_x u)\big]
\end{aligned}$$

$$
\begin{aligned}
&\geqslant \mu\big[f(x,t,u_2,\nabla_x u_2) - f(x,t,u_2,\nabla_x u) - f(x,t,u_2,\nabla_x u_2) \\
&\quad + f(x,t,u_2,\nabla_x u_2 - \nabla_x v)\big] \\
&\quad + (1-\mu)\big[f(x,t,u_2,\nabla_x u_2) - f(x,t,u_1,\nabla_x u) \\
&\quad - f(x,t,u_2,\nabla_x u_2) + f(x,t,u_2,\nabla_x u_2 - \nabla_x v) + h(x,t,u_2,u_1,\nabla_x u)\big] \\
&\geqslant (1-\mu)\big[-f(x,t,u_1,\nabla_x u) + f(x,t,u_2,\nabla_x u) + h(x,t,u_2,u_1,\nabla_x u)\big] \\
&\geqslant 0.
\end{aligned}
$$

In a completely similar way, it can be seen that the same conclusion holds for (2.56). Hence Claim 1 is proved.

CLAIM 2. *For any* $\mu \in [0,1]$, *every solution of* (2.55) *is a lower solution of* (2.36) *and every solution of* (2.56) *is an upper solution of* (2.36).

We verify only the former statement, since the latter can be proved similarly. Indeed, if u is a solution of (2.55), we have, as $u_1 \leqslant u \leqslant u_2$,

$$
\begin{aligned}
&\partial_t u + A(x,t,\partial_x)u \\
&\quad = \mu \bar f(x,t,u,\nabla_x u) \\
&\qquad + (1-\mu)\big[\bar f(x,t,u_1,\nabla_x u) + h(x,t,u_1,\gamma(x,t,u),\nabla_x u) + \omega(x,t,u)\big] \\
&\quad = \mu f(x,t,u,\nabla_x u) + (1-\mu)\big[f(x,t,u_1,\nabla_x u) + h(x,t,u_1,u,\nabla_x u)\big] \\
&\quad \leqslant \mu f(x,t,u,\nabla_x u) + (1-\mu) f(x,t,u,\nabla_x u) = f(x,t,u,\nabla_x u).
\end{aligned}
$$

Thus, Claim 2 is proved.

Now, let us associate with problems (2.55) and (2.56) the corresponding solution operators $S_{1,\mu}, S_{2,\mu} : C_B^{1,0}(\overline{Q}_T) \to C_B^{1,0}(\overline{Q}_T)$, with $\mu \in [0,1]$, whose fixed points are precisely the solutions of (2.55) and (2.56), respectively. Note that $S_{1,1} = S_{2,1}$ is the solution operator corresponding to (2.54).

CLAIM 3. *For every* $\delta > 0$, $u_1 - \delta$ *and* $u_1 + \delta$ *are, respectively, a lower solution and an upper solution of* (2.55), *with* $\mu = 0$.

As $\operatorname{ess\,inf}_{Q_T} a_0 > 0$, this can be deduced from the fact that

$$
\begin{aligned}
&\bar f\big(x,t,u_1,\nabla_x(u_1-\delta)\big) + h\big(x,t,u_1,\gamma\big(x,t,(u_1-\delta)\big),\nabla_x(u_1-\delta)\big) \\
&\quad = f(x,t,u_1,\nabla_x u_1), \\
&\bar f\big(x,t,u_1,\nabla_x(u_1+\delta)\big) + h\big(x,t,u_1,\gamma\big(x,t,(u_1+\delta)\big),\nabla_x(u_1+\delta)\big) \\
&\quad = f(x,t,u_1,\nabla_x u_1) + h\big(x,t,u_1,\gamma\big(x,t,(u_1+\delta)\big),\nabla_x u_1\big) \\
&\quad \leqslant f(x,t,u_1,\nabla_x u_1)
\end{aligned}
$$

and

$$
\omega(x,t,u_1-\delta) \geqslant 0 \geqslant \omega(x,t,u_1+\delta).
$$

Thus, Claim 3 is proved.

Moreover, $u_1 - \delta$ is a strict lower solution. Indeed, if u is a solution of (2.55) with $\mu = 0$, satisfying $u \geqslant u_1 - \delta$, then $u \geqslant u_1$ and hence $u \gg u_1 - \delta$. On the other hand, for what concerns $u_1 + \delta$, either there is $\delta_0 > 0$ such that, for every $\delta \in]0, \delta_0[$, $u_1 + \delta$ is a strict upper solution, or for every $\delta_0 > 0$ there are $\delta \in]0, \delta_0[$ and a solution u_δ of (2.55) with $\mu = 0$, such that $u_1 \leqslant u_\delta \leqslant \min\{u_2, u_1 + \delta\}$ and $u_\delta(x_0, t_0) = u_1(x_0, t_0) + \delta$ for some $(x_0, t_0) \in \overline{Q}_T$. Hence, in the latter case, we have $\|u_1 - u_\delta\|_{L^\infty(Q_T)} = \delta$ and therefore $u_\delta \to u_1$ in $L^\infty(Q_T)$, as $\delta \to 0$. By (2.57) there is a constant $C > 0$ such that $\|u_\delta\|_{W_p^{2,1}(Q_T)} \leqslant C$, for every $\delta \in]0, \delta_0[$, and therefore $u_\delta \to u_1$ in $C^{1,0}(\overline{Q}_T)$, as $\delta \to 0$. Recalling that each u_δ is a lower solution of (2.36), the conclusion of this Proposition is achieved.

Therefore, let us suppose that there is $\delta_0 > 0$ such that, for every $\delta \in]0, \delta_0[$, $u_1 + \delta$ is a strict upper solution of (2.55). Accordingly Theorem 2.15 guarantees that, for some $R > 0$ and every $\delta \in]0, \delta_0[$,

$$\deg\big(I - S_{1,0}, \mathcal{U}_1^\delta \cap B(0, R)\big) = 1, \tag{2.58}$$

where $\mathcal{U}_1^\delta = \{u \in C_B^{1,0}(\overline{Q}_T) \mid u_1 - \delta \ll u \ll u_1 + \delta\}$.

In a similar way one proves the following result.

CLAIM 4. *For every* $\delta > 0$, $u_2 - \delta$ *and* $u_2 + \delta$ *are, respectively, a lower solution and an upper solution of* (2.56), *with* $\mu = 0$.

Moreover, $u_2 + \delta$ is a strict upper solution. On the other hand, for what concerns $u_2 - \delta$, as in Claim 3, we can suppose that there is $\delta_0 > 0$ such that, for every $\delta \in]0, \delta_0[$, $u_2 - \delta$ is a strict lower solution of (2.56) as otherwise the conclusion is achieved. Accordingly Theorem 2.15 guarantees that, for some $R > 0$ and for every $\delta \in]0, \delta_0[$,

$$\deg\big(I - S_{2,0}, \mathcal{U}_2^\delta \cap B(0, R)\big) = 1, \tag{2.59}$$

where $\mathcal{U}_2^\delta = \{u \in C_B^{1,0}(\overline{Q}_T) \mid u_2 - \delta \ll u \ll u_2 + \delta\}$.

CLAIM 5. *The functions* $u_1 - 1$ *and* $u_2 + 1$ *are, respectively, a strict lower and a strict upper solution of* (2.54), *i.e. of* (2.55) *and* (2.56) *with* $\mu = 1$.

We have

$$\begin{aligned}
&\partial_t(u_1 - 1) + A(x, t, \partial_x)(u_1 - 1) = \partial_t u_1 + A(x, t, \partial_x)u_1 - a_0 \\
&\quad = f(x, t, u_1, \nabla_x u_1) - a_0 \\
&\quad \leqslant f\big(x, t, \gamma\big(x, t, (u_1 - 1)\big), \nabla_x(u_1 - 1)\big) + \omega_1(x, t, 1) - a_0 \\
&\quad \leqslant \bar{f}\big(x, t, u_1 - 1, \nabla_x(u_1 - 1)\big)
\end{aligned}$$

and

$$\begin{aligned}
&\partial_t(u_2 + 1) + A(x, t, \partial_x)(u_2 + 1) = \partial_t u_2 + A(x, t, \partial_x)u_2 + a_0 \\
&\quad = f(x, t, u_2, \nabla_x u_2) + a_0 \\
&\quad \geqslant f\big(x, t, \gamma\big(x, t, (u_2 + 1)\big), \nabla_x(u_2 + 1)\big) - \omega_2(x, t, 1) + a_0 \\
&\quad \geqslant \bar{f}\big(x, t, u_2 + 1, \nabla_x(u_2 + 1)\big).
\end{aligned}$$

Moreover, $u_1 - 1$ and $u_2 + 1$ are strict, since u_1 and u_2 are the only solutions of (2.54). Thus, Claim 5 is proved.

Accordingly Theorem 2.15 guarantees that, for some $R > 0$,

$$\deg\bigl(I - S_{1,1}, \mathcal{U} \cap B(0,R)\bigr) = \deg\bigl(I - S_{2,1}, \mathcal{U} \cap B(0,R)\bigr) = 1, \tag{2.60}$$

where $\mathcal{U} = \{u \in C_B^{1,0}(\overline{Q}_T) \mid u_1 - 1 \ll u \ll u_2 + 1\}$. Using again the fact that u_1 and u_2 are the only solutions of (2.55) with $\mu = 1$, we conclude, by the excision and additivity properties of the degree, that

$$\begin{aligned}\deg\bigl(I - S_{1,1}, \mathcal{U} \cap B(0,R)\bigr) &= \deg\bigl(I - S_{1,1}, \bigl(\mathcal{U}_1^\delta \cup \mathcal{U}_2^\delta\bigr) \cap B(0,R)\bigr)\\ &= \deg\bigl(I - S_{1,1}, \mathcal{U}_1^\delta \cap B(0,R)\bigr)\\ &\quad + \deg\bigl(I - S_{1,1}, \mathcal{U}_2^\delta \cap B(0,R)\bigr)\end{aligned} \tag{2.61}$$

for every $\delta \in \,]0, \min\{1, \delta_0, \|u_1 - u_2\|_\infty\}[$.

Now, let us assume that for every $\delta_0 > 0$ there exists $\delta \in \,]0, \delta_0[$ such that, for every $\mu \in [0,1]$, (2.55) has no solution on $\partial\mathcal{U}_1^\delta$ and (2.56) has no solution on $\partial\mathcal{U}_2^\delta$. The homotopy property of the degree then implies, by (2.58) and (2.59),

$$\deg\bigl(I - S_{1,1}, \mathcal{U}_1^\delta \cap B(0,R)\bigr) = \deg\bigl(I - S_{1,0}, \mathcal{U}_1^\delta \cap B(0,R)\bigr) = 1$$

and

$$\deg\bigl(I - S_{2,1}, \mathcal{U}_2^\delta \cap B(0,R)\bigr) = \deg\bigl(I - S_{2,0}, \mathcal{U}_2^\delta \cap B(0,R)\bigr) = 1. \tag{2.62}$$

Finally, combining relations (2.60)–(2.62) and using the fact that $S_{1,1} = S_{2,1}$, we obtain,

$$\begin{aligned}2 &= \deg\bigl(I - S_{1,0}, \mathcal{U}_1^\delta \cap B(0,R)\bigr) + \deg\bigl(I - S_{2,0}, \mathcal{U}_2^\delta \cap B(0,R)\bigr)\\ &= \deg\bigl(I - S_{1,1}, \mathcal{U}_1^\delta \cap B(0,R)\bigr) + \deg\bigl(I - S_{2,1}, \mathcal{U}_2^\delta \cap B(0,R)\bigr)\\ &= \deg\bigl(I - S_{1,1}, \mathcal{U}_1^\delta \cap B(0,R)\bigr) + \deg\bigl(I - S_{1,1}, \mathcal{U}_2^\delta \cap B(0,R)\bigr)\\ &= \deg\bigl(I - S_{1,1}, \mathcal{U} \cap B(0,R)\bigr) = 1,\end{aligned}$$

which is a contradiction. Hence we can conclude that there is $\delta_0 > 0$ such that for every $\delta \in \,]0, \delta_0[$ either there is a solution u_δ of (2.55), for some $\mu \in [0,1]$, with $u_\delta \in \partial\mathcal{U}_1^\delta$, and hence $u_1 \leqslant u_\delta \leqslant \min\{u_2, u_1 + \delta\}$ and $\|u_1 - u_\delta\|_{L^\infty(Q_T)} = \delta$, or there is a solution u_δ of (2.56), for some $\mu \in [0,1]$, with $u_\delta \in \partial\mathcal{U}_2^\delta$, and hence $\max\{u_1, u_2 - \delta\} \leqslant u_\delta \leqslant u_2$ and $\|u_2 - u_\delta\|_{L^\infty(Q_T)} = \delta$. Condition (2.57) and recalling that solutions u of (2.55), with $u \neq u_1$, are proper lower solutions of (2.36) and solutions u of (2.56), with $u \neq u_2$, are proper upper solutions of (2.36) yield the conclusion. □

REMARK 2.15. Condition (N) can be obviously replaced in Proposition 2.18 by assuming that there exist $h \in L^p(Q_T)$ and $K > 0$ such that for a.e. $(x,t) \in Q_T$, every $s \in [u_1(x,t), u_2(x,t)]$ and every $\xi \in \mathbb{R}^N$,

$$\left|f(x,t,s,\xi)\right| \leqslant h(x,t) + K|\xi|^2.$$

REMARK 2.16. From the proof of Proposition 2.18 it follows that, if u_1 is isolated in $C^{1,0}(\overline{Q}_T)$ as a fixed point of $I - S_{1,1}$ and $\operatorname{ind}(I - S_{1,1}, u_1) \neq 1$, then there exists a sequence $(\alpha_n)_n$ of proper lower solutions of (2.36) such that, for each n, $\alpha_n|_{\overline{Q}_T} \in W^{2,1}_{p,B}(Q_T)$ and $u_1 < \alpha_n < u_2$, which converges weakly in $W^{2,1}_p(Q_T)$ and strongly in $C^{1,0}(\overline{Q}_T)$ to u_1 as $n \to +\infty$. A similar result holds for u_2.

In order to prove the existence of connecting sequences of lower and upper solutions, the following two results are needed.

LEMMA 2.19. *Assume* (D), (A), (C) *and* (N) *and let z be a solution of* (2.36).

(i) *If α is a proper lower solution of* (2.36) *such that $\alpha < z$, then there exists a proper regular lower solution $\bar{\alpha}$ of* (2.36), *satisfying $\bar{\alpha}|_{\overline{Q}_T} \in W^{2,1}_{p,B}(Q_T)$ and $\alpha < \bar{\alpha} < z$.*

(ii) *If β is a proper upper solution of* (2.36) *such that $\beta > z$, then there exists a proper regular upper solution $\bar{\beta}$ of* (2.36), *satisfying $\bar{\beta}|_{\overline{Q}_T} \in W^{2,1}_{p,B}(Q_T)$ and $z < \bar{\beta} < \beta$.*

PROOF. We only prove the former statement. Let h be the function associated with f by Proposition 2.3 and corresponding to $\rho = \max\{\|\alpha\|_{L^\infty(Q_T)}, \|z\|_{L^\infty(Q_T)}\}$. Consider the periodic problem

$$\begin{array}{ll} \partial_t u + A(x,t,\partial_x)u = f(x,t,\alpha,\nabla_x u) - h(x,t,u,\alpha,\nabla_x u) & \text{in } Q_T, \\ u = 0 & \text{on } \Sigma_T, \\ u(\cdot,0) = u(\cdot,T) & \text{in } \Omega. \end{array} \tag{2.63}$$

The right-hand side of the equation satisfies the Carathéodory condition (C) and the Nagumo condition (N). Since α is a proper lower solution and z is a proper upper solution of (2.63), with $\alpha < z$, by Theorem 2.15, this problem has a solution $\bar{\alpha}$, satisfying $\alpha < \bar{\alpha} < z$. The properties of h imply that $\bar{\alpha}$ is a proper lower solution of (2.36). □

PROPOSITION 2.20. *Assume* (D), (A), (C) *and* (N) *and let z be a solution of* (2.36).

(i) *Let α be a proper lower solution of* (2.36) *such that $\alpha < z$ and there is no solution u of* (2.36) *with $\alpha < u < z$. Then there exists a sequence $(\alpha_n)_n$ of proper regular lower solutions of* (2.36), *such that, for each $n \geqslant 1$, $\alpha_n|_{\overline{Q}_T} \in W^{2,1}_{p,B}(Q_T)$ and $\alpha < \alpha_n < \alpha_{n+1} < z$, which converges weakly in $W^{2,1}_p(Q_T)$ and strongly in $C^{1,0}(\overline{Q}_T)$ to z.*

(ii) *Let β be a proper upper solution of* (2.36) *such that $z < \beta$ and there is no solution u of* (2.36) *with $z < u < \beta$. Then there exists a sequence $(\beta_n)_n$ of proper regular upper solutions of* (2.36), *such that, for each $n \geqslant 1$, $\beta_n|_{\overline{Q}_T} \in W^{2,1}_{p,B}(Q_T)$ and $z < \beta_{n+1} < \beta_n < \beta$, which converges weakly in $W^{2,1}_p(Q_T)$ and strongly in $C^{1,0}(\overline{Q}_T)$ to z.*

PROOF. We only prove the former statement. Repeating recursively the argument of the proof of Lemma 2.19, we get a sequence of proper regular lower solutions $(\alpha_n)_n$ such that $\alpha_0 = \alpha$ and, for each $n \geqslant 1$, $\alpha_n|_{\overline{Q}_T} \in W^{2,1}_{p,B}(Q_T)$, $\alpha < \alpha_{n-1} < \alpha_n < z$ and α_n is a solution of

$$\begin{array}{ll} \partial_t u + A(x,t,\partial_x)u = f(x,t,\alpha_{n-1},\nabla_x u) - h(x,t,u,\alpha_{n-1},\nabla_x u) & \text{in } Q_T, \\ u = 0 & \text{on } \Sigma_T, \\ u(\cdot,0) = u(\cdot,T) & \text{in } \Omega. \end{array} \tag{2.64}$$

Since the right-hand side of the equation in (2.64) satisfies conditions (C) and (N), as in the proof of Proposition 2.12, we see that the sequence $(\alpha_n)_n$ converges weakly in $W^{2,1}_p(Q_T)$ and strongly in $C^{1,0}(\overline{Q}_T)$ to a solution of (2.36), which must be z. □

As a direct consequence of Proposition 2.18 and Proposition 2.20 we are finally able to state our general version of the Order Interval Trichotomy, which extends a result in [34] (see also [58,60]).

THEOREM 2.21. *Assume* (D), (A), (C) *and* (N). *Suppose that* u_1, u_2 *are solutions of* (2.36) *such that* $u_1 < u_2$. *Then either*

- *there is a solution* u *of* (2.36) *with* $u_1 < u < u_2$,

or

- *there exists a double sequence* $(\alpha_m)_{m\in\mathbb{Z}}$ *of proper regular lower solutions of* (2.36), *such that, for each* m, $\alpha_m|_{\overline{Q}_T} \in W^{2,1}_{p,B}(Q_T)$ *and* $u_1 < \alpha_m < u_2$, *which converges weakly in* $W^{2,1}_p(Q_T)$ *and strongly in* $C^{1,0}(\overline{Q}_T)$ *to* u_2 *as* $m \to -\infty$ *and to* u_1 *as* $m \to +\infty$,

or

- *there exists a double sequence* $(\beta_m)_{m\in\mathbb{Z}}$ *of proper regular upper solutions of* (2.36), *such that, for each* m, $\beta_m|_{\overline{Q}_T} \in W^{2,1}_{p,B}(Q_T)$ *and* $u_1 < \beta_m < u_2$, *which converges weakly in* $W^{2,1}_p(Q_T)$ *and strongly in* $C^{1,0}(\overline{Q}_T)$ *to* u_1 *as* $m \to -\infty$ *and to* u_2 *as* $m \to +\infty$.

REMARK 2.17. In the frame of Theorem 2.21, if $u_1 \ll u_2$ and there is no solution u of (2.36) with $u_1 < u < u_2$, then the remaining two alternatives are mutually exclusive.

Heteroclinic solutions

We now interpret Theorem 2.21 in terms of the dynamics of solutions of (2.10) by showing that T-monotone solutions connecting two ordered T-periodic solutions do exist. Our statement extends previous results in [34,58,60]. We point out that the proof of the existence of heteroclinic solutions is more involved here than in the case where uniqueness for the initial value problem and validity of comparison principles are assumed.

THEOREM 2.22. *Assume* (D), (A), (C) *and* (N). *Suppose that* u_1, u_2 *are solutions of* (2.36) *such that there is no solution* u *of* (2.36) *with* $u_1 < u < u_2$. *Then there exists a* T*-monotone solution* $z : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ *of* (2.10) *such that either*

$$\lim_{t \to -\infty} \| z(\cdot, t) - u_1(\cdot, t) \|_{C^1(\overline{\Omega})} = 0 \quad \textit{and} \quad \lim_{t \to +\infty} \| z(\cdot, t) - u_2(\cdot, t) \|_{C^1(\overline{\Omega})} = 0, \tag{2.65}$$

or

$$\lim_{t \to -\infty} \| z(\cdot, t) - u_2(\cdot, t) \|_{C^1(\overline{\Omega})} = 0 \quad \textit{and} \quad \lim_{t \to +\infty} \| z(\cdot, t) - u_1(\cdot, t) \|_{C^1(\bar{\Omega})} = 0. \tag{2.66}$$

PROOF. According to Proposition 2.18, we assume, for example, the existence of a sequence $(\alpha_n)_n$ of proper regular lower solutions of (2.36) such that, for each n, $\alpha_n|_{\overline{Q}_T} \in W^{2,1}_{p,B}(Q_T)$ and $u_1 < \alpha_n < u_2$, which converges weakly in $W^{2,1}_p(Q_T)$ and strongly in $C^{1,0}(\overline{Q}_T)$ to u_1 as $n \to +\infty$.

For each n, we also consider the solution $\widetilde{\alpha}_n : \overline{\Omega} \times [0, +\infty[\to \mathbb{R}$ of (2.10), with $\widetilde{\alpha}_n(\cdot, 0) = \alpha_n(\cdot, 0)$, $u_1 < \widetilde{\alpha}_n < u_2$ in $\overline{\Omega} \times [0, +\infty[$ and $\|\widetilde{\alpha}_n(\cdot, t) - u_2(\cdot, t)\|_{C^1(\overline{\Omega})} \to 0$ as $t \to +\infty$, whose existence is guaranteed by Proposition 2.13.

We are going to select a subsequence of $(\widetilde{\alpha}_n)_n$, we shall still denote for simplicity by $(\widetilde{\alpha}_n)_n$, as follows. Let us define $\delta = \frac{1}{2}\|u_1 - u_2\|_{L^\infty(Q_T)}$. We first choose $\widetilde{\alpha}_1$ such that there is $m_1 \geqslant 1$ for which $\|\widetilde{\alpha}_1(\cdot, kT) - u_1(\cdot, 0)\|_{L^\infty(\Omega)} < \delta$, if $k \in \{0, \ldots, m_1 - 1\}$, and $\|\widetilde{\alpha}_1(\cdot, m_1 T) - u_1(\cdot, 0)\|_{L^\infty(\Omega)} \geqslant \delta$. Next, we pick $\widetilde{\alpha}_2$ such that there is $m_2 \geqslant m_1 + 1$ for which $\|\widetilde{\alpha}_2(\cdot, kT) - u_1(\cdot, 0)\|_{L^\infty(\Omega)} < \delta$, if $k \in \{0, \ldots, m_2 - 1\}$, and $\|\widetilde{\alpha}_2(\cdot, m_2 T) - u_1(\cdot, 0)\|_{L^\infty(\Omega)} \geqslant \delta$. If we cannot find such a function $\widetilde{\alpha}_2$, then there exist $\ell \in \{1, \ldots, m_1\}$ and a subsequence $(\widetilde{\alpha}_{n_k})_k$ such that $\|\widetilde{\alpha}_{n_k}(\cdot, \ell T) - u_1(\cdot, 0)\|_{L^\infty(\Omega)} \geqslant \delta$. Since $(\widetilde{\alpha}_{n_k})_k$ is bounded in $L^\infty(\Omega \times]0, \ell T[)$, $(\widetilde{\alpha}_{n_k}(\cdot, 0))_k$ is bounded in $W^{2-2/p}_p(\Omega)$ and $\|\widetilde{\alpha}_{n_k}(\cdot, 0) - u_1(\cdot, 0)\|_{L^\infty(\Omega)} \to 0$, by Proposition 2.5 there is a subsequence of $(\widetilde{\alpha}_{n_k})_k$ converging weakly in $W^{2,1}_p(\Omega \times]0, \ell T[)$ to a solution v of

$$\begin{aligned} &\partial_t u + A(x, t, \partial_x) u = f(x, t, u, \nabla_x u) && \text{in } \Omega \times]0, \ell T[, \\ &u = 0 && \text{on } \partial\Omega \times [0, \ell T], \end{aligned}$$

satisfying $v(\cdot, 0) = u_1(\cdot, 0)$ and $\|v(\cdot, \ell T) - u_1(\cdot, 0)\|_{L^\infty(\Omega)} \geqslant \delta$. Let $i \in \{0, \ldots, \ell - 1\}$ be such that $v(\cdot, (i+1)T) > v(\cdot, iT) = u_1(\cdot, 0)$. Then the function

$$\alpha(x, t) = v(x, t + iT) \quad \text{in } \overline{\Omega} \times [0, T[,$$

is a proper lower solution of (2.36), with $u_1 < \alpha < u_2$ and $\alpha(\cdot, 0) = u_1(\cdot, 0)$. Proposition 2.13 yields the existence of a T-increasing solution $\widetilde{\alpha}$ of (2.13), with $t_0 = 0$ and $u_0 = u_1(\cdot, 0)$, such that

$$\lim_{t \to +\infty} \| \widetilde{\alpha}(\cdot, t) - u_2(\cdot, t) \|_{C^1(\bar{\Omega})} \to 0.$$

Accordingly there exists a T-increasing solution $z:\overline{\Omega}\times\mathbb{R}\to\mathbb{R}$ of (2.10) satisfying (2.65). Then, proceeding by induction, once $\widetilde{\alpha}_{n-1}$ has been defined, either we construct just as above, a T-increasing solution $z:\overline{\Omega}\times\mathbb{R}\to\mathbb{R}$ of (2.10) satisfying (2.65), or we find $\widetilde{\alpha}_n$ for which there is $m_n \geqslant m_{n-1}+1$ such that $\|\widetilde{\alpha}_n(\cdot,kT)-u_1(\cdot,0)\|_{L^\infty(\Omega)}<\delta$, if $k\in\{0,\dots,m_n-1\}$, and $\|\widetilde{\alpha}_n(\cdot,m_nT)-u_1(\cdot,0)\|_{L^\infty(\Omega)}\geqslant\delta$. Suppose that the latter eventuality always occurs and set, for each n,

$$v_n(x,t)=\widetilde{\alpha}_n(x,t+m_nT)\quad \text{in } \overline{\Omega}\times[-m_nT,0]$$

and, for $k\in\{1,\dots,m_n\}$,

$$v_n^{(k)}(x,t)=v_n(x,t)\quad \text{in } \overline{\Omega}\times\big[-kT,-(k-1)T\big].$$

Fix $k\geqslant 1$ and observe that the sequence $(v_n^{(k)})_n$, with n such that $m_n\geqslant k+1$, is bounded in $L^\infty(\Omega\times\,]-(k+1)T,-(k-1)T[)$. Hence, by Proposition 2.10, there is a subsequence $(v_{n_j^{(k)}}^{(k)})_j$ converging weakly in $W_p^{2,1}(\Omega\times\,]-kT,-(k-1)T[)$ to a solution w_k of

$$\begin{array}{ll}\partial_t u+A(x,t,\partial_x)u=f(x,t,u,\nabla_x u) & \text{in } \Omega\times\big]-kT,-(k-1)T\big[,\\ u=0 & \text{on } \partial\Omega\times\big[-kT,-(k-1)T\big].\end{array}$$

Since we can choose the sequence $(n_j^{(k+1)})_j$ to be a subsequence of $(n_j^{(k)})_j$, the sequence $(w_k)_k$ is such that, for each $k\geqslant 1$, $w_{k+1}(\cdot,-kT)=w_k(\cdot,-kT)$ and $\|w_1(\cdot,0)-u_1(\cdot,0)\|_{L^\infty(\Omega)}\geqslant\delta$. Moreover, as each $\widetilde{\alpha}_n$ is T-increasing, we have $w_k(\cdot,-kT)\leqslant w_k(\cdot,-(k-1)T)$. Setting for each $k\geqslant 1$

$$z(x,t)=w_k(x,t)\quad \text{in } \overline{\Omega}\times\big[-kT,-(k-1)T\big],$$

we get a solution of

$$\begin{array}{ll}\partial_t u+A(x,t,\partial_x)u=f(x,t,u,\nabla_x u) & \text{in } \Omega\times\,]-\infty,0[,\\ u=0 & \text{on } \partial\Omega\times\,]-\infty,0],\end{array}$$

which satisfies

$$z(\cdot,-kT)\leqslant z\big(\cdot,-(k-1)T\big), \tag{2.67}$$

$\|z(\cdot,-kT)-u_1(\cdot,0)\|_{L^\infty(\Omega)}\leqslant\delta$, for any $k\geqslant 1$, and $\|z(\cdot,0)-u_1(\cdot,0)\|_{L^\infty(\Omega)}\geqslant\delta$.

Let us define, for each $k\geqslant 1$,

$$z_k(x,t)=z\big(x,t-(k-1)T\big)\quad \text{in } \overline{\Omega}\times[-T,0].$$

As z is bounded in $\overline{\Omega}\times\,]-\infty,0]$ and satisfies (2.67), arguing as in the proof of Lemma 2.11, we prove that the sequence $(z_k)_k$ has a subsequence $(z_{k_j})_j$ converging weakly in $W^{2,1}_p(\Omega\times\,]-T,0[)$ and strongly in $C^{1,0}(\overline{\Omega}\times[-T,0])$ to a solution ζ of

$$\begin{array}{ll} \partial_t u + A(x,t,\partial_x)u = f(x,t,u,\nabla_x u) & \text{in } \Omega\times\,]-T,0[,\\ u=0 & \text{on } \partial\Omega\times[-T,0],\\ u(\cdot,0)=u(\cdot,-T) & \text{in } \Omega, \end{array}$$

that is a solution of (2.36). As $\|\zeta(\cdot,0)-u_1(\cdot,0)\|_{L^\infty(\Omega)}\leqslant\delta$ and $u_1\leqslant\zeta\leqslant u_2$, we have $\zeta=u_1$. Since this argument applies to show that any subsequence of $(z_k)_k$ has a subsequence which converges weakly in $W^{2,1}_p(\Omega\times\,]-T,0[)$ to u_1, we can conclude that

$$\lim_{t\to-\infty}\bigl\|z(\cdot,t)-u_1(\cdot,t)\bigr\|_{C^1(\overline{\Omega})}=0.$$

At last, as $z(\cdot,-T)\leqslant z(\cdot,0)$, $z(\cdot,0)>u_1(\cdot,0)$ and $\|z(\cdot,-T)-u_1(\cdot,-T)\|_{L^\infty(\Omega)}\leqslant\delta$, we see that $z|_{\overline{\Omega}\times[-T,0[}$ gives rise to a proper lower solution of (2.36) and, by Proposition 2.13, it can be extended to a T-increasing solution of (2.10) defined in $\overline{\Omega}\times\mathbb{R}$ and satisfying (2.65). □

REMARK 2.18. In the frame of Theorem 2.22, if $u_1\ll u_2$, then the two alternatives are mutually exclusive. Further, if (2.65) holds, then for every $t_0\in\mathbb{R}$ and every $u_0\in W^{2-2/p}_p(\Omega)$, with $u_0=0$ on $\partial\Omega$ and $u_1(\cdot,t_0)\ll u_0\leqslant u_2(\cdot,t_0)$ in Ω, there is a solution $u:\overline{\Omega}\times[t_0,+\infty[\,\to\mathbb{R}$ of (2.13) such that

$$\lim_{t\to+\infty}\bigl\|u(\cdot,t)-u_2(\cdot,t)\bigr\|_e=0.$$

Similar conclusions are valid if (2.66) holds.

Above a maximal, or below a minimal, T-periodic solution

We now apply Theorem 2.22 to discuss the behaviour of solutions of (2.10) lying above a maximal solution of (2.36). Of course, symmetric conclusions can be established for solutions lying below a minimal solution of (2.36).

THEOREM 2.23. *Assume* (D), (A), (C) *and* (N). *Suppose that u_2 is a maximal solution of* (2.36). *Then either there exist $\omega_+\in\,]-\infty,+\infty]$ and a T-increasing solution $z:\overline{\Omega}\times\,]-\infty,\omega_+[\,\to\mathbb{R}$ of* (2.10) *such that $z>u_2$ and*

$$\lim_{t\to-\infty}\bigl\|z(\cdot,t)-u_2(\cdot,t)\bigr\|_{C^1(\overline{\Omega})}=0 \quad\textit{and}\quad \limsup_{t\to\omega_+}\Bigl(\max_{\overline{\Omega}} z(\cdot,t)\Bigr)=+\infty, \tag{2.68}$$

or, for any given $t_0 \in \mathbb{R}$, *there exists a* T*-decreasing solution* $z : \overline{\Omega} \times [t_0, +\infty[\to \mathbb{R}$ *of* (2.10) *such that* $z > u_2$ *and*

$$\lim_{t \to +\infty} \left\| z(\cdot, t) - u_2(\cdot, t) \right\|_{C^1(\overline{\Omega})} = 0. \tag{2.69}$$

PROOF. Take $R > 1 + \max_{\overline{Q}_T} u_2$ and define a function $f_R : Q_T \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ by setting, for a.e. $(x, t) \in Q_T$ and every $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$,

$$f_R(x, t, s, \xi) = \begin{cases} f\big(x, t, u_2(x, t), \xi\big) + \omega\big(x, t, u_2(x, t) - s\big) & \text{if } s < u_2(x, t), \\ f(x, t, s, \xi) & \text{if } u_2(x, t) \leqslant s \leqslant R, \\ (R + 1 - s) f(x, t, s, \xi) & \text{if } R < s \leqslant R + 1, \\ 0 & \text{if } s > R + 1, \end{cases}$$

where

$$\omega(x, t, \delta) = \max_{|\xi| \leqslant \delta} \big| f\big(x, t, u_2(x, t), \nabla_x u_2(x, t) + \xi\big) \\ - f\big(x, t, u_2(x, t), \nabla_x u_2(x, t)\big)\big|.$$

Of course, f_R satisfies the conditions (C) and (N). Consider the modified problem

$$\begin{array}{ll} \partial_t u + A(x, t, \partial_x) u = f_R(x, t, u, \nabla_x u) & \text{in } Q_T, \\ u = 0 & \text{on } \Sigma_T, \\ u(\cdot, 0) = u(\cdot, T) & \text{in } \Omega. \end{array} \tag{2.70}$$

Arguing as in the proof of Theorem 2.15, we see that every solution u of (2.70) is such that $u \geqslant u_2$. Notice that $\beta = R + 2$ is a proper regular upper solution of (2.70). Let us show that it is strict. Otherwise, there exists a solution u of (2.70) satisfying $u < \beta$ but not $u \ll \beta$. Let $v = \beta - u$. Since $v(x, t) > 0$ on $\partial\Omega \times [0, T]$ and $v(x, 0) = v(x, T)$ in Ω, there exists $(x_0, t_0) \in \Omega \times]0, T]$ such that $v(x_0, t_0) = \min_{\overline{Q}_T} v = 0$. Hence we can find an open ball $B \subseteq \Omega$, with $x_0 \in B$, and a point $t_1 \in]0, t_0[$ such that $0 \leqslant v(x, t) \leqslant 1$ in $B \times]t_1, t_0]$ and $v(\bar{x}, \bar{t}) > 0$ for some $(\bar{x}, \bar{t}) \in \bar{B} \times [t_1, t_0]$. Using the definition of f_R and β, we get

$$\partial_t v + A(x, t, \partial_x) v = a_0(R + 2) \geqslant 0,$$

a.e. in $B \times]t_1, t_0[$. Then Proposition 2.2 implies that v is constant in $\bar{B} \times [t_1, t_0]$, which is a contradiction. Assume now that there is no solution u of (2.70) with $u_2 < u \leqslant \beta$. By Lemma 2.19, there exists a proper regular upper solution $\bar{\beta}$ of (2.70) such that $\bar{\beta}|_{\overline{Q}_T} \in W^{2,1}_{p,B}(Q_T)$ and $u_2 < \bar{\beta} < \beta$. Hence, Proposition 2.13 implies that, for any given $t_0 \in \mathbb{R}$, there exists a T-decreasing solution $z : \overline{\Omega} \times [t_0, +\infty[\to \mathbb{R}$ of (2.10), with $z > u_2$, satisfying (2.69).

Next, assume that there exists a solution u of (2.70) with $u > u_2$. Then, as (2.36) has no solution u with $u > u_2$, any possible solution u of (2.70), but u_2, must satisfy

$$\max_{\overline{Q}_T} (u - u_2) \geqslant 1. \tag{2.71}$$

Denote by $\mathcal{S}$ the set of all solutions of (2.70) satisfying (2.71). By Lemma 2.16, there is a minimal solution $v \in \mathcal{S}$ of (2.70). By Theorem 2.22, there exists a T-monotone solution ζ of

$$\begin{array}{ll} \partial_t u + A(x,t,\partial_x)u = f_R(x,t,u,\nabla_x u) & \text{in } \Omega \times \mathbb{R}, \\ u = 0 & \text{on } \partial\Omega \times \mathbb{R}, \end{array}$$

such that either

$$\lim_{t\to-\infty} \left\| \zeta(\cdot,t) - u_2(\cdot,t) \right\|_{C^1(\bar{\Omega})} = 0 \quad \text{and} \quad \lim_{t\to+\infty} \left\| \zeta(\cdot,t) - v(\cdot,t) \right\|_{C^1(\overline{\Omega})} = 0,$$

or

$$\lim_{t\to-\infty} \left\| \zeta(\cdot,t) - v(\cdot,t) \right\|_{C^1(\overline{\Omega})} = 0 \quad \text{and} \quad \lim_{t\to+\infty} \left\| \zeta(\cdot,t) - u_2(\cdot,t) \right\|_{C^1(\bar{\Omega})} = 0.$$

In the former case, there exists $t_0 \in \mathbb{R}$ such that ζ is a solution of

$$\begin{array}{ll} \partial_t u + A(x,t,\partial_x)u = f(x,t,u,\nabla_x u) & \text{in } \Omega \times]-\infty, t_0], \\ u = 0 & \text{on } \partial\Omega \times]-\infty, t_0]. \end{array}$$

Proposition 2.13 implies that ζ can be continued to a T-increasing solution z of (2.10), which is defined in $\overline{\Omega} \times]-\infty, \omega_+[$ for some $\omega_+ \in]-\infty, +\infty]$ and satisfies (2.68). In the latter case, there exists $\omega_- \in \mathbb{R}$ such that ζ is a solution of

$$\begin{array}{ll} \partial_t u + A(x,t,\partial_x)u = f(x,t,u,\nabla_x u) & \text{in } \Omega \times]\omega_-, +\infty[, \\ u = 0 & \text{on } \partial\Omega \times]\omega_-, +\infty[. \end{array}$$

Then, for any given $t_0 \in \mathbb{R}$, we take $k_0 \in \mathbb{N}$ such that $\omega_- - k_0 T < t_0$ and we set $z(x,t) = \zeta(x, t + k_0 T)$ in $\overline{\Omega} \times [t_0, +\infty[$. Clearly, z is a T-decreasing solution of (2.10) satisfying (2.69). □

2.6. *Stability matters*

In this section we study the stability properties of the T-periodic solutions of (2.10) with the aid of lower and upper solutions. It seems natural to use in this context the order norm $\|\cdot\|_e$ introduced in Definition 2.8. We start by recalling the notion of one-sided Lyapunov–Bellmann stability. Since we have to take care of the fact that uniqueness for the initial value problem and validity of comparison principles are not assumed, such a definition does not generally seem the most appropriate to be considered here; indeed, some weaker concept might fit better in order to detect certain residual forms of stability. As an alternative notion to Lyapunov stability we use here order stability; this is common in the frame of order preserving semidynamical systems [93,33,58,60] and appears suited to our approach based on lower and upper solutions. We remark that Lyapunov stability implies order stability, whereas the converse implication is not generally true; however, these concepts are equivalent if a comparison principle holds for solutions of (2.10). Using these notions, we give a precise description of the stability properties of a T-periodic solution in terms of the existence of a lower or an upper solution close to it. Hence, when a pair of lower and upper solutions α, β, with $\alpha \leqslant \beta$, is given, we discuss the stability of the minimum and the maximum T-periodic solutions v and w lying in between, thus getting a completion of Theorem 2.15. Afterwards, we turn to investigate the behaviour of the solutions between v and w, finding in some cases a totally ordered continuum of order stable T-periodic solutions. Finally, in the complementary case where $\alpha \not\leqslant \beta$, we see that Lyapunov instability occurs, thus completing Theorem 2.17.

One-sided Lyapunov stability
The following definitions are in the spirit of [17].

DEFINITION 2.14.

- A solution z of (2.36) is said *Lyapunov stable* (briefly, *$\mathcal{L}$-stable*) *from below* if, for every $\varepsilon > 0$, there is $\delta > 0$ such that, for every $t_0 \in [0, T[$ and for every $u_0 \in W_p^{2-2/p}(\Omega)$, with $u_0 = 0$ on $\partial\Omega$, $u_0 < z(\cdot, t_0)$ and $\|u_0 - z(\cdot, t_0)\|_e < \delta$, every nonextendible solution u of (2.13), with $u \leqslant z$, exists in $\overline{\Omega} \times [t_0, +\infty[$ and satisfies

$$\left\|u(\cdot, t) - z(\cdot, t)\right\|_e < \varepsilon \quad \text{in } [t_0, +\infty[. \tag{2.72}$$

- If, further,

$$\lim_{t\to+\infty} \left\|u(\cdot, t) - z(\cdot, t)\right\|_e = 0, \tag{2.73}$$

  z is said *$\mathcal{L}$-asymptotically stable from below*.
- *$\mathcal{L}$-stability from above* and *$\mathcal{L}$-asymptotic stability from above* are defined similarly.
- A solution z of (2.36) is said *$\mathcal{L}$-stable* if, for every $\varepsilon > 0$, there is $\delta > 0$ such that, for every $t_0 \in [0, T[$ and for every $u_0 \in W_p^{2-2/p}(\Omega)$ with $\|u_0 - z(\cdot, t_0)\|_e < \delta$, every nonextendible solution u of (2.13) exists in $\overline{\Omega} \times [t_0, +\infty[$ and satisfies (2.72).
  If, further, (2.73) holds, z is said *$\mathcal{L}$-asymptotically stable*.
- A solution z of (2.36) is said *$\mathcal{L}$-unstable from below* if it is not $\mathcal{L}$-stable from below. *$\mathcal{L}$-instability from above* and *$\mathcal{L}$-instability* are defined similarly.

REMARK 2.19. The notion of $\mathcal{L}$-stability from below given in Definition 2.14 does not require any condition on solutions $u : \overline{\Omega} \times [t_0, \omega[\to \mathbb{R}$ of (2.13), satisfying $u_0 < z(\cdot, t_0)$, $\|u(\cdot, t_0) - z(\cdot, t_0)\|_e < \delta$, but not $u \leqslant z$. A similar remark holds for the $\mathcal{L}$-stability from above.

One-sided order stability
The following definitions are in the spirit of [93,33].

DEFINITION 2.15.

- A solution z of (2.36) is said *order stable* (briefly, *$\mathcal{O}$-stable*) (respectively *properly $\mathcal{O}$-stable*, respectively *strictly $\mathcal{O}$-stable*) *from below* if there exists a sequence $(\alpha_n)_n$

of regular lower solutions (respectively proper regular lower solutions, respectively strict regular lower solutions) of (2.36) such that, for every n, $\alpha_n < z$ and $\alpha_n = 0$ on Σ_T, and $\|\alpha_n - z\|_e \to 0$ as $n \to +\infty$.

- A solution z of (2.36) is said *$\mathcal{O}$-stable* (respectively *properly $\mathcal{O}$-stable*, respectively *strictly $\mathcal{O}$-stable*) *from above* if there exists a sequence $(\beta_n)_n$ of regular upper solutions (respectively proper regular upper solutions, respectively strict regular upper solutions) of (2.36) such that, for every n, $\beta_n > z$ and $\beta_n = 0$ on Σ_T, and $\|\beta_n - z\|_e \to 0$ as $n \to +\infty$.
- A solution z of (2.36) is said *$\mathcal{O}$-stable* (respectively *properly $\mathcal{O}$-stable*, respectively *strictly $\mathcal{O}$-stable*) if it is $\mathcal{O}$-stable (respectively properly $\mathcal{O}$-stable, respectively strictly $\mathcal{O}$-stable) from below and from above.
- A solution z of (2.36) is said *$\mathcal{O}$-unstable from below* (respectively *properly $\mathcal{O}$-unstable from below*, respectively *strictly $\mathcal{O}$-unstable from below*) if there exists a sequence $(\beta_n)_n$ of regular upper solutions (respectively proper regular upper solutions, respectively strict regular upper solutions) of (2.36) such that, for every n, $\beta_n < z$ and $\beta_n = 0$ on Σ_T, and $\|\beta_n - z\|_e \to 0$ as $n \to +\infty$.
- A solution z of (2.36) is said *$\mathcal{O}$-unstable from above* (respectively *properly $\mathcal{O}$-unstable from above*, respectively *strictly $\mathcal{O}$-unstable from above*) if there exists a sequence $(\alpha_n)_n$ of regular lower solutions (respectively proper regular lower solutions, respectively strict regular lower solutions) of (2.36) such that, for every n, $\alpha_n > z$ and $\alpha_n = 0$ on Σ_T, and $\|\alpha_n - z\|_e \to 0$ as $n \to +\infty$.

REMARK 2.20. The notion of strict $\mathcal{O}$-stability corresponds to the strong stability introduced by H. Matano in [93] and, as pointed out there, it represents a form of structural stability.

REMARK 2.21. In the frame of Theorem 2.15, if α is a proper lower solution, then the minimum solution v of (2.36) is, by virtue of Proposition 2.20, properly $\mathcal{O}$-stable from below and, if β is a proper upper solution, the maximum solution w of (2.36) is properly $\mathcal{O}$-stable from above. If α is a strict lower solution, v is also not $\mathcal{O}$-unstable from below and, if β is a strict upper solution, w is also not $\mathcal{O}$-unstable from above.

REMARK 2.22. In the definition of $\mathcal{O}$-stability the request that the lower and the upper solutions are regular is not essential at the light of Lemma 2.19.

Lyapunov stability and order stability

We show here that $\mathcal{L}$-stability implies $\mathcal{O}$-stability.

DEFINITION 2.16. A solution u of (2.36) is said *isolated from above* (respectively *isolated from below*) *in* $C^{1,0}(\overline{Q}_T)$ if there is no sequence $(u_n)_n$ of solutions of (2.36) such that $u_n > u$ (respectively $u_n < u$) for every n and $u_n \to u$ in $C^{1,0}(\overline{Q}_T)$.

REMARK 2.23. We point out that, under conditions (D), (A), (C) and (N), it is equivalent to say that a solution z of (2.36) is isolated (from above or from below) in $C^{1,0}(\overline{Q}_T)$, or in $L^\infty(Q_T)$, or in $W_p^{2,1}(Q_T)$. This is a direct consequence of Proposition 2.10.

PROPOSITION 2.24. *Assume* (D), (A), (C) *and* (N). *If a solution* z *of* (2.36) *is* $\mathcal{L}$*-stable from above* (*respectively* $\mathcal{L}$*-stable from below*, *respectively* $\mathcal{L}$*-stable*), *then it is* $\mathcal{O}$*-stable from above* (*respectively* $\mathcal{O}$*-stable from below*, *respectively* $\mathcal{O}$*-stable*). *Moreover, if* z *is isolated from above* (*respectively from below*) *in* $C^{1,0}(\overline{Q}_T)$, *then it is properly* $\mathcal{O}$*-stable from above* (*respectively from below*).

PROOF. Let z be a solution of (2.36) which is $\mathcal{L}$-stable from above and suppose that z is isolated from above in $C^{1,0}(\overline{Q}_T)$, because otherwise there is a sequence of solutions $(u_n)_n$, with $u_n > z$, converging to z in $C^{1,0}(\overline{Q}_T)$ and hence z is $\mathcal{O}$-stable from above. Let $\rho > 0$ be such that every solution u of (2.36), with $u > z$, satisfies $\|u - z\|_{C^{1,0}(\overline{Q}_T)} \geqslant \rho$. We then argue as in the proof of Theorem 2.23. We take $R > \rho + \max_{\bar{\Omega}} z$ and we define a function $f_R : Q_T \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ as there, replacing u_2 with z. Consider the modified problem (2.70) and notice that $\beta = R + 2$ is a strict regular upper solution of (2.70).

Assume there is no solution u of (2.70) with $z < u \leqslant \beta$. Then Proposition 2.20 implies that z is properly $\mathcal{O}$-stable from above.

Next, assume that there exists a solution u of (2.70) with $u > z$. Since every solution u of (2.70) is such that $u \geqslant z$ and (2.36) has no solution u, with $\|u - z\|_{C^{1,0}(\overline{Q}_T)} < \rho$, any possible solution u of (2.70), but z, must satisfy

$$\|u - z\|_{C^{1,0}(\overline{Q}_T)} \geqslant \rho. \tag{2.74}$$

Denote by $\mathcal{S}$ the set of all solutions of (2.70) satisfying (2.74). By Lemma 2.16, there is a minimal solution $v \in \mathcal{S}$ of (2.70), with $v > z$. By Proposition 2.18, z is either properly $\mathcal{O}$-stable from above, or properly $\mathcal{O}$-unstable from above. In the latter case, as by Proposition 2.18 $\alpha_n|_{\overline{Q}_T} \in W^{2,1}_{p,B}(Q_T)$ for each n, Proposition 2.13 imply the existence of a sequence $(\widetilde{\alpha}_n)_n$ of solutions of (2.10) such that $\|\widetilde{\alpha}_n(\cdot, 0) - z(\cdot, 0)\|_e \to 0$ as $n \to +\infty$, but $\limsup_{t \to +\infty} \|\widetilde{\alpha}_n(\cdot, t) - z(\cdot, t)\|_e \geqslant \frac{1}{2}\|v - z\|_e$, thus contradicting the $\mathcal{L}$-stability from above of z. □

Comparison principles

In order to discuss further relations between $\mathcal{O}$-stability and $\mathcal{L}$-stability, we introduce the following versions of the comparison principle for parabolic equations.

DEFINITION 2.17.

- We say that the *first comparison principle* holds for the equation

$$\partial_t u + A(x, t, \partial_x)u = f(x, t, u, \nabla_x u) \tag{2.75}$$

if for every $t_1, t_2 \in \mathbb{R}$, with $t_1 < t_2$, and for every $u_1, u_2 \in W^{2,1}_p(\Omega \times]t_1, t_2[)$, with $p > N + 2$, satisfying

$$\begin{aligned} &\partial_t u_1 + A(x, t, \partial_x)u_1 - f(x, t, u_1, \nabla_x u_1) \\ &\qquad \leqslant \partial_t u_2 + A(x, t, \partial_x)u_2 - f(x, t, u_2, \nabla_x u_2) && \text{in } \Omega \times]t_1, t_2[, \\ &u_1 \leqslant u_2 && \text{on } \partial\Omega \times [t_1, t_2], \\ &u_1(\cdot, t_1) \leqslant u_2(\cdot, t_1), && \text{in } \Omega, \end{aligned} \tag{2.76}$$

we have

$$u_1 \leqslant u_2 \quad \text{in } \overline{\Omega} \times [t_1, t_2].$$

- We say that the *second comparison principle* holds for (2.75) if for every $t_1, t_2 \in \mathbb{R}$, with $t_1 < t_2$, and for every $u_1, u_2 \in W_p^{2,1}(\Omega \times]t_1, t_2[)$, with $p > N + 2$, satisfying

$$\begin{aligned} &\partial_t u_1 + A(x,t,\partial_x)u_1 - f(x,t,u_1,\nabla_x u_1) \\ &\quad \leqslant \partial_t u_2 + A(x,t,\partial_x)u_2 - f(x,t,u_2,\nabla_x u_2) \quad \text{in } \Omega \times]t_1, t_2[, \\ &u_1 \leqslant u_2 \quad \text{in } \overline{\Omega} \times [t_1, t_2], \\ &u_1(\cdot, t_1) < u_2(\cdot, t_1), \quad \text{in } \Omega, \end{aligned} \tag{2.77}$$

we have

$$u_1 \ll u_2 \quad \text{in } \overline{\Omega} \times]t_1, t_2].$$

The first or the second comparison principle holds provided that f satisfies certain local one-sided Lipschitz conditions.

PROPOSITION 2.25. *Assume* (D) *and* (A). *Moreover, suppose that*

(L^+) *for every $\rho > 0$ there is a constant $L \geqslant 0$ such that, for a.e. $(x,t) \in Q_T$, every r, $s \in [-\rho, \rho]$, with $r < s$, and every $\xi, \eta \in [-\rho, \rho]^N$,*

$$f(x,t,s,\xi) - f(x,t,r,\eta) \leqslant L\big(s - r + |\xi - \eta|\big).$$

Then the first comparison principle holds.

PROOF. Let t_1, t_2, with $t_1 < t_2$, be given and let $u_1, u_2 \in W_p^{2,1}(\Omega \times]t_1, t_2[)$ satisfy (2.76). Let L be the constant associated with $\rho = \max\{\|u_1\|_{C^{1,0}(\overline{\Omega}\times[t_1,t_2])}, \|u_2\|_{C^{1,0}(\overline{\Omega}\times[t_1,t_2])}\}$ by condition (L^+). Set $v(x,t) = e^{-Lt}(u_2(x,t) - u_1(x,t))$ in $\overline{\Omega} \times [t_1, t_2]$ and suppose that $\min_{\overline{\Omega}\times[t_1,t_2]} v < 0$. Since $v(x,t) \geqslant 0$ on $\partial\Omega \times [t_1, t_2]$ and $v(\cdot, t_1) \geqslant 0$ in Ω, there exists $(x_0, t_0) \in \Omega \times]t_1, t_2]$ such that $v(x_0, t_0) = \min_{\overline{\Omega}\times[t_1,t_2]} v$. Hence there exist an open ball $B \subseteq \Omega$, with $x_0 \in B$, and a point $t^* \in]t_1, t_0]$ such that, for a.e. $(x,t) \in B \times]t^*, t_0]$, $v(x,t) < 0$ and

$$\begin{aligned} &\partial_t v + A(x,t,\partial_x)v \\ &\quad = -Le^{-Lt}(u_2 - u_1) + e^{-Lt}\partial_t(u_2 - u_1) + e^{-Lt}A(x,t,\partial_x)(u_2 - u_1) \\ &\quad \geqslant -Le^{-Lt}(u_2 - u_1) + e^{-Lt}\big(f(x,t,u_2,\nabla_x u_2) - f(x,t,u_1,\nabla_x u_1)\big) \\ &\quad \geqslant -Le^{-Lt}(u_2 - u_1) + e^{-Lt}L(u_2 - u_1) - e^{-Lt}L|\nabla_x u_2 - \nabla_x u_1| \\ &\quad = -\sum_{n=1}^{N} \tilde{a}_i(x,t)\partial_{x_i} v, \end{aligned}$$

where, for each $i = 1, \dots, N$, $\tilde{a}_i \in L^\infty(\Omega \times]t_1, t_2[)$ is defined by

$$\tilde{a}_i(x,t) = \begin{cases} L \dfrac{\partial_{x_i} v(x,t)}{|\nabla_x v(x,t)|} & \text{if } \nabla_x v(x,t) \neq 0, \\ 0 & \text{if } \nabla_x v(x,t) = 0. \end{cases}$$

Proposition 2.2 then implies that $v(x,t) = v(x_0, t_0) < 0$ in $B \times]t^*, t_0]$ and therefore, as $\operatorname{ess\,inf}_{Q_T} a_0 > 0$,

$$\partial_t v + A(x,t,\partial_x)v + \sum_{n=1}^{N} \tilde{a}_i(x,t)\partial_{x_i} v = a_0 v < 0$$

a.e. in $B \times]t_1, t_0]$, thus contradicting the previous inequality. Accordingly, we conclude that $\min_{\overline{\Omega} \times [t_1,t_2]} v \geqslant 0$ and hence $u_1 \leqslant u_2$ in $\overline{\Omega} \times [t_1, t_2]$. □

PROPOSITION 2.26. *Assume* (D) *and* (A). *Moreover, suppose that*

(L$^-$) *for every $\rho > 0$ there is a constant $L \geqslant 0$ such that, for a.e. $(x,t) \in Q_T$, every r, $s \in [-\rho, \rho]$, with $r < s$, and every $\xi, \eta \in [-\rho, \rho]^N$,*

$$f(x,t,s,\xi) - f(x,t,r,\eta) \geqslant -L\big(s - r + |\xi - \eta|\big).$$

Then the second comparison principle holds.

PROOF. Let t_1, t_2, with $t_1 < t_2$, be given and let $u_1, u_2 \in W_p^{2,1}(\Omega \times]t_1, t_2[)$ satisfy (2.77). Let L be the constant associated with $\rho = \max\{\|u_1\|_{C^{1,0}(\overline{\Omega} \times [t_1,t_2])}, \|u_2\|_{C^{1,0}(\overline{\Omega} \times [t_1,t_2])}\}$ by condition (L$^-$). Set $v(x,t) = u_2(x,t) - u_1(x,t)$ in $\overline{\Omega} \times [t_1, t_2]$ and suppose that either there is $(x_0, t_0) \in \Omega \times]t_1, t_2]$ such that $v(x_0, t_0) = 0$, or there is $(x_0, t_0) \in \partial\Omega \times]t_1, t_2]$ such that $v(x_0, t_0) = 0$ and $\partial_\nu v(x_0, t_0) = 0$, where ν is the outer normal to Ω at x_0. Since $v \geqslant 0$ in $\overline{\Omega} \times [t_1, t_2]$, this implies $v(x_0, t_0) = \min_{\overline{\Omega} \times [t_1,t_2]} v$. Moreover we have, for a.e. $(x,t) \in \Omega \times]t_1, t_0]$,

$$\begin{aligned} \partial_t v + A(x,t,\partial_x)v &\geqslant f(x,t,u_2,\nabla_x u_2) - f(x,t,u_1,\nabla_x u_1) \\ &\geqslant -Lv - L|\nabla_x v| = -Lv - \sum_{n=1}^{N} \tilde{a}_i(x,t)\partial_{x_i} v, \end{aligned}$$

where, for each $i = 1, \dots, N$, $\tilde{a}_i \in L^\infty(\Omega \times]t_1, t_2[)$ is defined by

$$\tilde{a}_i(x,t) = \begin{cases} L \dfrac{\partial_{x_i} v(x,t)}{|\nabla_x v(x,t)|} & \text{if } \nabla_x v(x,t) \neq 0, \\ 0 & \text{if } \nabla_x v(x,t) = 0. \end{cases}$$

If $x_0 \in \Omega$, Proposition 2.2 implies that $v(x,t) = v(x_0, t_0) = 0$ in $\bar{\Omega} \times [t_1, t_0]$, thus contradicting the assumption $v(\cdot, t_1) > 0$ in Ω. If $x_0 \in \partial\Omega$ and $v(x,t) > 0$ in $\Omega \times]t_1, t_0]$, Proposition 2.2 implies that $\partial_\nu v(x_0, t_0) < 0$, which is again a contradiction. Hence we conclude that $u_1 \ll u_2$ in $\overline{\Omega} \times]t_1, t_2]$. □

PROPOSITION 2.27. *Assume* (D) *and* (A). *If the second comparison principle holds, then for every* $t_1, t_2 \in \mathbb{R}$, *with* $t_1 < t_2$, *and for every* $u_1, u_2 \in W_p^{2,1}(\Omega \times]t_1, t_2[)$, *with* $p > N + 2$, *satisfying*

$$\begin{aligned} &\partial_t u_1 + A(x,t,\partial_x)u_1 - f(x,t,u_1,\nabla_x u_1) \\ &\quad \leqslant \partial_t u_2 + A(x,t,\partial_x)u_2 - f(x,t,u_2,\nabla_x u_2) && \text{in } \Omega \times]t_1, t_2[, \\ &u_1 \leqslant u_2 && \text{on } \partial\Omega \times [t_1, t_2], \\ &u_1(\cdot, t_1) \ll u_2(\cdot, t_1) && \text{in } \Omega, \end{aligned} \tag{2.78}$$

we have

$$u_1 \ll u_2 \quad \text{in } \overline{\Omega} \times [t_1, t_2].$$

PROOF. Let t_1, t_2 be such that $t_1 < t_2$ and let $u_1, u_2 \in W_p^{2,1}(\Omega \times]t_1, t_2[)$ satisfy (2.78). By continuity, we have $t^* = \sup\{t \in]t_1, t_2] \mid u_1(\cdot, t) \ll u_2(\cdot, t) \text{ in } \Omega\} > t_1$. Suppose that $t^* < t_2$. As $u_1(\cdot, t) \leqslant u_2(\cdot, t)$ in $\Omega \times]t_1, t^*]$ and $u_1(\cdot, t_1) < u_2(\cdot, t_1)$ in Ω, the second comparison principle implies, in particular, that $u_1(\cdot, t^*) \ll u_2(\cdot, t^*)$ in Ω, which is a contradiction. Hence we conclude that $u_1 \ll u_2$ in $\overline{\Omega} \times [t_1, t_2]$. □

REMARK 2.24. The first comparison principle implies uniqueness in the future for the initial value problem (2.13). On the contrary, this is not true for the second comparison principle: examples in this direction can be found, for instance, in [16].

Lyapunov and order stability in case of validity of comparison principles

In this section we discuss some relations between the notion of $\mathcal{O}$-stability and that of $\mathcal{L}$-stability when the first or the second comparison principle holds. We start with the following observation.

PROPOSITION 2.28. *Assume* (D), (A), (C) *and* (N). *Further, suppose that the first or the second comparison principle holds. If a solution* z *of* (2.36) *is* $\mathcal{L}$*-stable* (*respectively* $\mathcal{L}$*-asymptotically stable*) *from above and from below, then it is* $\mathcal{L}$*-stable* (*respectively* $\mathcal{L}$*-asymptotically stable*).

PROOF. Let z be a solution of (2.36) which is $\mathcal{L}$-stable from above and from below. Then, for every $\varepsilon > 0$, there is $\delta > 0$ such that, for every $t_0 \in [0, T[$ and for every $v_0, w_0 \in W_p^{2-2/p}(\Omega)$, with $v_0 = w_0 = 0$ on $\partial\Omega$ and $z(\cdot, t_0) - \delta e < v_0 < z(\cdot, t_0) < w_0 < z(\cdot, t_0) + \delta e$, there exist, by Lemma 2.8 and the $\mathcal{L}$-stability from above and from below, solutions $v, w : \overline{\Omega} \times [t_0, +\infty[\to \mathbb{R}$ of (2.13), with $u_0 = v_0$ and $u_0 = w_0$ respectively, such that $z(\cdot, t) - \varepsilon e < v(\cdot, t) < z(\cdot, t) < w(\cdot, t) < z(\cdot, t) + \varepsilon e$ in $[t_0, +\infty[$. Pick now $u_0 \in W_p^{2-2/p}(\Omega)$, with $u_0 = 0$ on $\partial\Omega$ and $z(\cdot, t_0) - \delta e \ll u_0 \ll z(\cdot, t_0) + \delta e$, and let $u : \overline{\Omega} \times [t_0, \omega[\to \mathbb{R}$ be a nonextendible solution of (2.13) which exists by Proposition 2.9. Take $v_0, w_0 \in W_p^{2-2/p}(\Omega)$ such that $v_0 = w_0 = 0$ on $\partial\Omega$ and $z(\cdot, t_0) - \delta e < v_0 \ll u_0 \ll w_0 < z(\cdot, t_0) + \delta e$ and let v and w be the corresponding solutions as in the first part of the proof. If the first comparison principle holds, then, by Corollary 2.7 and Remark 2.24,

$\omega = +\infty$, $v < u < w$ and u satisfies (2.72). If the second comparison principle holds, then, by Proposition 2.27, $v \ll u \ll w$, u exists in $\overline{\Omega} \times [t_0, +\infty[$ and satisfies (2.72). □

PROPOSITION 2.29. *Assume* (D), (A), (C) *and* (N). *Further, suppose that the first comparison principle holds. If a solution z of* (2.36) *is strictly $\mathcal{O}$-stable from below* (*respectively strictly $\mathcal{O}$-stable from above*, *respectively strictly $\mathcal{O}$-stable*), *then it is $\mathcal{L}$-stable from below* (*respectively $\mathcal{L}$-stable from above*, *respectively $\mathcal{L}$-stable*).

PROOF. Since z is strictly $\mathcal{O}$-stable from below, there exists a sequence $(\alpha_n)_n$ of strict regular lower solutions of (2.36) such that, for each n, $\alpha_n \ll z$, $\alpha_n = 0$ on Σ_T and $\|\alpha_n - z\|_e \to 0$ as $n \to +\infty$. Fix $\varepsilon > 0$ and take n such that $\|\alpha_n - z\|_e < \varepsilon$. Since $\alpha_n \ll z$, by Proposition 2.1, there exists a constant $\delta > 0$ such that $\delta e \leqslant z(\cdot, t_0) - \alpha_n(\cdot, t_0)$ for every $t_0 \in [0, T]$. Let $u_0 \in W_p^{2-2/p}(\Omega)$, with $u_0 = 0$ on $\partial\Omega$, be such that $z(\cdot, t_0) - \delta e < u_0 \leqslant z(\cdot, t_0)$. Since $\alpha_n(\cdot, t_0) \leqslant z(\cdot, t_0) - \delta e \leqslant u_0 \leqslant z(\cdot, t_0)$, $\alpha_n|_{\bar{\Omega} \times [t_0, +\infty[}$ is a lower solution and $z|_{\overline{\Omega} \times [t_0, +\infty[}$ is an upper solution of (2.13), Corollary 2.7 implies that there exists a solution $u : \overline{\Omega} \times [t_0, +\infty[\to \mathbb{R}$ of (2.13), which is unique by Remark 2.24, satisfying $\alpha_n \leqslant u \leqslant z$ in $\overline{\Omega} \times [t_0, +\infty[$ and hence $\|u(\cdot, t) - z(\cdot, t)\|_e < \varepsilon$ in $[t_0, +\infty[$. □

LEMMA 2.30. *Assume* (D), (A), (C) *and* (N). *Further, suppose that the second comparison principle holds. If α is a lower solution of* (2.36) *and z is a solution of* (2.36), *with $\alpha < z$, then $\alpha \ll z$.*

PROOF. According to the definition of lower solution, let $0 = \sigma_0 < \sigma_1 < \cdots < \sigma_k = T$ and for each $h \in \{0, \ldots, k-1\}$, $\alpha = \max_{1 \leqslant i \leqslant m_h} \alpha_i^{(h)}$ in $\overline{\Omega} \times]\sigma_h, \sigma_{h+1}[$, with $\alpha_i^{(h)} \in W_p^{2,1}(\Omega \times]\sigma_h, \sigma_{h+1}[)$ for every $i \in \{1, \ldots, m_h\}$. Since $\alpha < z$, our definition of a lower solution implies that there is a point $t_1 \in]\sigma_h, \sigma_{h+1}[$, for some $h \in \{0, \ldots, k-1\}$, such that $\alpha(\cdot, t_1) < z(\cdot, t_1)$ and hence $\alpha_i^{(h)}(\cdot, t_1) < z(\cdot, t_1)$ for each $i \in \{1, \ldots, m_h\}$. The second comparison principle implies that $\alpha_i^{(h)} \ll z$ in $\overline{\Omega} \times]t_1, \sigma_{h+1}]$ for each $i \in \{1, \ldots, m_h\}$ and therefore $\alpha(\cdot, \sigma_{h+1}) \leqslant \lim_{t \to \sigma_{h+1}^-} \alpha(\cdot, t) = \lim_{t \to \sigma_{h+1}^-} \max_{1 \leqslant i \leqslant m_h} \alpha_i^{(h)}(\cdot, t) \ll z(\cdot, \sigma_{h+1})$ in Ω. Applying recursively the second comparison principle and using the T-periodicity we conclude that $\alpha \ll z$ in $\overline{\Omega} \times [0, T]$. □

REMARK 2.25. Suppose that the second comparison principle holds. Lemma 2.30 implies that a lower solution of (2.36) is strict if and only if it is proper and hence, in particular, that proper $\mathcal{O}$-stability from below is equivalent to strict $\mathcal{O}$-stability from below. Similar conclusions obviously hold for upper solutions, for proper $\mathcal{O}$-stability from above and for proper $\mathcal{O}$-stability.

PROPOSITION 2.31. *Assume* (D), (A), (C) *and* (N). *Further, suppose that the second comparison principle holds. A solution z of* (2.36) *is $\mathcal{O}$-stable from below* (*respectively $\mathcal{O}$-stable from above*, *respectively $\mathcal{O}$-stable*) *if and only if it is $\mathcal{L}$-stable from below* (*respectively $\mathcal{L}$-stable from above*, *respectively $\mathcal{L}$-stable*).

PROOF. Let z be a solution of (2.36), which is $\mathcal{O}$-stable from below, and let $(\alpha_n)_n$ be a sequence of regular lower solutions of (2.36) such that, for each n, $\alpha_n < z$, $\alpha_n = 0$ on Σ_T and $\|\alpha_n - z\|_e \to 0$ as $n \to +\infty$. Fix $\varepsilon > 0$ and take n such that $\|\alpha_n - z\|_e < \varepsilon$. Since, by Lemma 2.30, $\alpha_n \ll z$, Proposition 2.1 implies the existence of a constant $\delta > 0$ such that $\delta e \leqslant z(\cdot, t_0) - \alpha_n(\cdot, t_0)$ for every $t_0 \in [0, T]$. Let $u_0 \in W_p^{2-2/p}(\Omega)$, with $u_0 = 0$ on $\partial\Omega$, be such that $u_0 \leqslant z(\cdot, t_0)$ and $\|u_0 - z(\cdot, t_0)\|_e < \delta$. Since $\alpha_n(\cdot, t_0) \leqslant z(\cdot, t_0) - \delta e \ll u_0 \leqslant z(\cdot, t_0)$, $\alpha_n|_{\bar{\Omega}\times[t_0,+\infty[}$ is a lower solution and $z|_{\overline{\Omega}\times[t_0,+\infty[}$ is an upper solution of (2.13), Corollary 2.7 implies that there exists a solution $u : \overline{\Omega} \times [t_0, +\infty[\to \mathbb{R}$ of (2.13), satisfying $\alpha_n \leqslant u \leqslant z$ in $\bar{\Omega} \times [t_0, +\infty[$ and hence $\|u(\cdot, t) - z(\cdot, t)\|_e < \varepsilon$ in $[t_0, +\infty[$. Moreover, Proposition 2.27 implies that any nonextendible solution $u : \overline{\Omega} \times [t_0, \omega[\to \mathbb{R}$ of (2.13), with $\alpha_n(\cdot, t_0) \ll u_0 \leqslant z(\cdot, t_0)$ and $u \leqslant z$, satisfies $\alpha_n(\cdot, t) \ll u(\cdot, t)$ and hence $\|u(\cdot, t) - z(\cdot, t)\|_e < \varepsilon$ in $\overline{\Omega} \times [t_0, \omega[$. Proposition 2.9 finally implies that $\omega = +\infty$. To conclude we use Proposition 2.24 and Proposition 2.28. $\square$

Stability via lower and upper solutions

In this section we use lower and upper solutions α, β, with $\alpha \leqslant \beta$, to describe the stability properties of the T-periodic solutions of (2.36) lying in between. We start with an immediate consequence of Proposition 2.13, which shows a certain form of stability of a one-sided isolated solution z of (2.36) in the presence of a lower solution $\alpha < z$, or of an upper solution $\beta > z$.

PROPOSITION 2.32. *Assume* (D), (A), (C) *and* (N). *Moreover, let z be a solution of* (2.36).

(i) *Let α be a proper lower solution of* (2.36) *such that, for some t_0, $\alpha(\cdot, t_0) \in W_p^{2-2/p}(\Omega)$ and $\alpha(\cdot, t_0) = 0$ on $\partial\Omega$. Assume $\alpha < z$ and there is no solution u of* (2.36) *with $\alpha < u < z$. Then there exists the minimum solution $\widetilde{\alpha} : \overline{\Omega} \times [t_0, +\infty[\to \mathbb{R}$ in $[\alpha, z]$ of* (2.13), *with $u_0 = \alpha(\cdot, t_0)$. Further, $\widetilde{\alpha}$ is T-increasing and satisfies*

$$\lim_{t\to+\infty} \left\|\widetilde{\alpha}(\cdot, t) - z(\cdot, t)\right\|_{C^1(\overline{\Omega})} = 0.$$

(ii) *Let β be a proper upper solution of* (2.36) *such that, for some t_0, $\beta(\cdot, t_0) \in W_p^{2-2/p}(\Omega)$ and $\beta(\cdot, t_0) = 0$ on $\partial\Omega$. Assume $z < \beta$ and there is no solution u of* (2.36) *with $z < u < \beta$. Then there exists the maximum solution $\widetilde{\beta} : \overline{\Omega} \times [t_0, +\infty[\to \mathbb{R}$ in $[z, \beta]$ of* (2.13), *with $u_0 = \beta(\cdot, t_0)$. Further, $\widetilde{\beta}$ is T-decreasing and satisfies*

$$\lim_{t\to+\infty} \left\|\widetilde{\beta}(\cdot, t) - z(\cdot, t)\right\|_{C^1(\overline{\Omega})} = 0.$$

From this proposition we get an attractivity and invariance result, which completes Theorem 2.15. It regards the set of solutions of problem (2.36) lying between α and β. This is a classical topic which has been extensively investigated in the last fifty years (see [124,71,72,33,34,58,37,60]). Unlike all these works, here we require no regularity on the function f besides the Carathéodory conditions. So that our assumptions do not guarantee either uniqueness for the initial value problem or validity of comparison principles, which are the basic tools for applying the theory of order preserving discrete-time semidynamical systems.

THEOREM 2.33. *Assume* (D), (A), (C) *and* (N). *Suppose that* α *is a proper lower solution and* β *is a proper upper solution of* (2.36), *such that, for some* t_0, $\alpha(\cdot, t_0)$, $\beta(\cdot, t_0) \in W_p^{2-2/p}(\Omega)$ *and* $\alpha(\cdot, t_0) = 0$, $\beta(\cdot, t_0) = 0$ *on* $\partial\Omega$. *Finally, assume*

$$\alpha < \beta$$

and denote by v *the minimum solution and by* w *the maximum solution of* (2.36) *in* $[\alpha, \beta]$. *Then the following conclusions hold*:

(i) *there exist a* T*-increasing solution* $\widetilde{\alpha} : \overline{\Omega} \times [t_0, +\infty[\to \mathbb{R}$ *of* (2.13), *with* $u_0 = \alpha(\cdot, t_0)$, *and a* T*-decreasing solution* $\widetilde{\beta} : \overline{\Omega} \times [t_0, +\infty[\to \mathbb{R}$ *of* (2.13), *with* $u_0 = \beta(\cdot, t_0)$, *such that*

$$\alpha < \widetilde{\alpha} \leqslant v \leqslant w \leqslant \widetilde{\beta} < \beta$$

and

$$\lim_{t \to +\infty} \left\| \widetilde{\alpha}(\cdot, t) - v(\cdot, t) \right\|_{C^1(\bar{\Omega})} = 0 = \lim_{t \to +\infty} \left\| \widetilde{\beta}(\cdot, t) - w(\cdot, t) \right\|_{C^1(\bar{\Omega})};$$

(ii) *if* $u_0 \in W_p^{2-2/p}(\Omega)$ *is such that* $u_0 = 0$ *on* $\partial\Omega$ *and* $\alpha(\cdot, t_0) \leqslant u_0 \leqslant v(\cdot, t_0)$, *then there exists a solution* $u : \overline{\Omega} \times [t_0, +\infty[\to \mathbb{R}$ *of* (2.13), *satisfying* $\widetilde{\alpha} \leqslant u \leqslant v$ *and hence*

$$\lim_{t \to +\infty} \left\| u(\cdot, t) - v(\cdot, t) \right\|_e = 0;$$

similarly, if $u_0 \in W_p^{2-2/p}(\Omega)$ *is such that* $u_0 = 0$ *on* $\partial\Omega$ *and* $w(\cdot, t_0) \leqslant u_0 \leqslant \beta(\cdot, t_0)$, *then there exists a solution* $u : \overline{\Omega} \times [t_0, +\infty[\to \mathbb{R}$ *of* (2.13), *satisfying* $w \leqslant u \leqslant \widetilde{\beta}$ *and hence*

$$\lim_{t \to +\infty} \left\| u(\cdot, t) - w(\cdot, t) \right\|_e = 0;$$

(iii) *any solution* u *of* (2.10), *such that* $\alpha \leqslant u \leqslant \beta$ *in* $\overline{\Omega} \times [t_0, +\infty[$ *satisfies* $\widetilde{\alpha} \leqslant u \leqslant \widetilde{\beta}$ *in* $\overline{\Omega} \times [t_0, +\infty[$.

PROOF. We first notice that conclusion (i) directly follows from Proposition 2.32. Next, we take $u_0 \in W_p^{2-2/p}(\Omega)$ such that $u_0 = 0$ on $\partial\Omega$ and $\alpha(\cdot, t_0) \leqslant u_0 \leqslant v(\cdot, t_0)$. Since $\widetilde{\alpha}$ and v are, respectively, a lower and an upper solution in $\overline{\Omega} \times [t_0, +\infty[$ of (2.13), Corollary 2.7 implies the existence of a solution $u : \overline{\Omega} \times [t_0, +\infty[\to \mathbb{R}$ of (2.13), satisfying $\widetilde{\alpha} \leqslant u \leqslant v$ and hence

$$\lim_{t \to +\infty} \left\| u(\cdot, t) - v(\cdot, t) \right\|_e = 0.$$

Similarly, if $u_0 \in W_p^{2-2/p}(\Omega)$ is such that $u_0 = 0$ on $\partial\Omega$ and $w(\cdot, t_0) \leqslant u_0 \leqslant \beta(\cdot, t_0)$, then there is a solution $u : \overline{\Omega} \times [t_0, +\infty[\to \mathbb{R}$ of (2.13), satisfying $w \leqslant u \leqslant \widetilde{\beta}$ and hence

$$\lim_{t \to +\infty} \| u(\cdot, t) - w(\cdot, t) \|_e = 0.$$

This proves conclusion (ii). Finally, if $u : \overline{\Omega} \times [t_0, +\infty[\to \mathbb{R}$ is any solution of (2.10), satisfying $\alpha \leqslant u \leqslant \beta$, then by the minimality properties of $\widetilde{\alpha}$ described in Proposition 2.13 we conclude that $u \geqslant \widetilde{\alpha}$. In a similar way we prove that $u \leqslant \widetilde{\beta}$. This yields conclusion (iii). □

REMARK 2.26. We assumed in Theorem 2.33 the existence of a common value t_0 such that $\alpha(\cdot, t_0)$, $\beta(\cdot, t_0) \in W_p^{2-2/p}(\Omega)$ and $\alpha(\cdot, t_0) = 0$, $\beta(\cdot, t_0) = 0$ on $\partial\Omega$. Of course, this request is not essential: the above conditions could be satisfied at different values t_0', t_0'', respectively.

The property expressed in Theorem 2.33 is a form of relative attractivity, which is somehow related to the notion of relative stability introduced in [17] and to that of weak positive invariance defined in [15].

Assuming the validity of the second comparison principle, the following stronger stability result holds.

COROLLARY 2.34. *Assume* (D), (A), (C), (N) *and suppose that the second comparison principle holds. Assume that* α *is a proper lower solution and* β *is a proper upper solution of* (2.36), *such that, for some* t_0, $\alpha(\cdot, t_0)$, $\beta(\cdot, t_0) \in W_p^{2-2/p}(\Omega)$ *and* $\alpha(\cdot, t_0) = 0$, $\beta(\cdot, t_0) = 0$ *on* $\partial\Omega$. *Further, suppose that* $\alpha < \beta$ *and denote by* v *and* w *the minimum and the maximum solutions of* (2.36) *in* $[\alpha, \beta]$. *Then* v *is* $\mathcal{L}$*-asymptotically stable from below and* w *is* $\mathcal{L}$*-asymptotically stable from above.*

PROOF. By Remark 2.21, we know that v is properly $\mathcal{O}$-stable from below and hence, by Proposition 2.31, it is $\mathcal{L}$-stable from below. Notice that, by Lemma 2.30, $\alpha \ll v$. Let $\widetilde{\alpha}$ be the solution of (2.13), with $u_0 = \alpha(\cdot, t_0)$, whose existence is guaranteed by Theorem 2.33, satisfying, by Proposition 2.27, $\alpha < \widetilde{\alpha} \ll v$ in $\overline{\Omega} \times [t_0, +\infty[$ and

$$\lim_{t \to +\infty} \| \widetilde{\alpha}(\cdot, t) - v(\cdot, t) \|_{C^1(\overline{\Omega})} = 0.$$

Then, for every $u_0 \in W_p^{2-2/p}(\Omega)$ such that $u_0 = 0$ on $\partial\Omega$ and $\widetilde{\alpha}(\cdot, t_0) \ll u_0 < v(\cdot, t_0)$, any nonextendible solution u of (2.13) with $u \leqslant v$ satisfies, by Proposition 2.27, $\widetilde{\alpha} \ll u \leqslant v$ in $\overline{\Omega} \times [t_0, +\infty[$ and hence

$$\lim_{t \to +\infty} \| u(\cdot, t) - v(\cdot, t) \|_e = 0.$$

This implies that v is $\mathcal{L}$-asymptotically stable from below. By a similar argument we get the corresponding conclusion for w. □

REMARK 2.27. Under the assumptions of Corollary 2.34, we have that, for every $u_0 \in W_p^{2-2/p}(\Omega)$ such that $u_0 = 0$ on $\partial\Omega$ and $\widetilde{\alpha}(\cdot, t_0) \ll u_0 \ll \widetilde{\beta}(\cdot, t_0)$, any nonextendible solution $u : \overline{\Omega} \times [t_0, \omega[\to \mathbb{R}$ of (2.13) satisfies, by Proposition 2.27 and Proposition 2.9, $\widetilde{\alpha} \ll u \ll \widetilde{\beta}$ in $\overline{\Omega} \times [t_0, \omega[$ with $\omega = +\infty$.

We conclude this section stating some illustrative results where the existence of the pair of lower and upper solutions α, β is replaced by assumptions relating the behaviour of the function f to the principal eigenvalue λ_1 (see Appendix 2.7). Alternative conditions are discussed in [41,42]. We first apply Corollary 2.34 to show the global $\mathcal{L}$-asymptotic stability of the T-periodic solution of a linear problem.

PROPOSITION 2.35. *Assume* (D) *and* (A). *Let $q \in L^\infty(Q_T)$ satisfy $q(x,t) \leqslant \lambda_1$ a.e. in Q_T and $q(x,t) < \lambda_1$ on a set of positive measure. Let $h \in L^p(Q_T)$, with $p > N+2$. Then the unique solution z of*

$$\begin{aligned} &\partial_t u + A(x,t,\partial_x)u = q(x,t)u + h(x,t) && \textit{in } Q_T, \\ &u = 0 && \textit{on } \Sigma_T, \\ &u(\cdot,0) = u(\cdot,T) && \textit{in } \Omega, \end{aligned} \tag{2.79}$$

is globally $\mathcal{L}$-asymptotically stable.

PROOF. Let $\varphi_1 \gg 0$ be the principal eigenfunction (see Proposition 2.49). For any $\sigma > 0$ the functions $\alpha = z - \sigma\varphi_1$ and $\beta = z + \sigma\varphi_1$ are respectively a strict lower and a strict upper solution of problem (2.79), with $\alpha \ll z \ll \beta$. Hence, by Corollary 2.34, z is $\mathcal{L}$-asymptotically stable from below and from above. Now, take $t_0 \in [0, T[$ and $u_0 \in W_p^{2-2/p}(\Omega)$ such that $u_0 = 0$ on $\partial\Omega$. Let $\sigma > 0$ be so large that $\alpha(\cdot, t_0) \leqslant u_0 \leqslant \beta(\cdot, t_0)$ in Ω. By Remark 2.27 the solution u of

$$\begin{aligned} &\partial_t u + A(x,t,\partial_x)u = q(x,t)u + h(x,t) && \text{in } \Omega \times [t_0, +\infty[, \\ &u = 0 && \text{on } \partial\Omega \times [t_0, +\infty[, \\ &u(\cdot, t_0) = u_0 && \text{in } \Omega, \end{aligned}$$

satisfies $\widetilde{\alpha} \ll u \ll \widetilde{\beta}$ in $\overline{\Omega} \times [t_0, +\infty[$, where $\widetilde{\alpha}$, $\widetilde{\beta}$ are the functions whose existence is guaranteed by Theorem 2.33, and hence $\lim_{t\to+\infty} \|u(\cdot,t) - z(\cdot,t)\|_e = 0$. □

For the nonlinear problem (2.36) we state the following stabilization result (see [41,42]). Related, less general, statements can be found in [3,34].

PROPOSITION 2.36. *Assume* (D), (A), (C) *and* (N). *Suppose that there exist functions $a, b \in L^p(Q_T)$, with $p > N+2$, and $\gamma, \Gamma \in L^\infty(Q_T)$, such that, for a.e. $(x,t) \in Q_T$ and all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$ with $s \neq 0$,*

$$\gamma(x,t)|s| + a(x,t) \leqslant f(x,t,s,\xi)\,\mathrm{sgn}(s) \leqslant \Gamma(x,t)|s| + b(x,t),$$

with $\Gamma(x,t) \leqslant \lambda_1$ *a.e. in* Q_T *and* $\Gamma(x,t) < \lambda_1$ *on a subset of positive measure. Then there exist the minimum solution* v *and the maximum solution* w *of* (2.36). *Moreover, for each* $t_0 \in [0, T[$ *and every* $u_0 \in W_p^{2-2/p}(\Omega)$, *with* $u_0 = 0$ *on* $\partial\Omega$, *the following conclusions hold*:

- *if* $u_0 \leqslant v(\cdot, t_0)$, *then there exists a solution* $u : \bar{\Omega} \times [t_0, +\infty[\to \mathbb{R}$ *of* (2.13) *such that* $u \leqslant v$ *and*

$$\lim_{t\to+\infty} \left\| u(\cdot,t) - v(\cdot,t) \right\|_e = 0;$$

- *if* $u_0 \geqslant w(\cdot, t_0)$, *then there exists a solution* $u : \bar{\Omega} \times [t_0, +\infty[\to \mathbb{R}$ *of* (2.13) *such that* $u \geqslant w$ *and*

$$\lim_{t\to+\infty} \left\| u(\cdot,t) - w(\cdot,t) \right\|_e = 0.$$

PROOF. The proof is carried out through some steps.

CLAIM 1. *There exists* $R > 0$ *such that all solutions* u *of* (2.36) *satisfy* $\|u\|_{C^{1,0}(\overline{Q}_T)} < R$.

Without loss of generality, we can suppose $a(x,t) \leqslant 0 \leqslant b(x,t)$ and $\gamma(x,t) \leqslant \Gamma(x,t)$ a.e. in Q_T. Let $q : Q_T \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be a L^p-Carathéodory function, with $p > N+2$, such that, for a.e. $(x,t) \in Q_T$ and every $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$,

$$\gamma(x,t) \leqslant q(x,t,s,\xi) \leqslant \Gamma(x,t)$$

and

$$q(x,t,s,\xi) = \max\left\{\gamma(x,t), \min\left\{s^{-1} f(x,t,s,\xi), \Gamma(x,t)\right\}\right\}, \quad \text{if } |s| \geqslant 1.$$

Define

$$h(x,t,s,\xi) = f(x,t,s,\xi) - q(x,t,s,\xi)s$$

and observe that there exists a function $c \in L^p(Q_T)$, with $p > N+2$, such that, for a.e. $(x,t) \in Q_T$ and every $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$,

$$\left|h(x,t,s,\xi)\right| \leqslant c(x,t).$$

Assume by contradiction that there exists a sequence $(u_n)_n$ of solutions of (2.36) such that $\|u_n\|_{C^{1,0}(\overline{Q}_T)} \to +\infty$. For each n, set $v_n = u_n / \|u_n\|_{C^{1,0}(\overline{Q}_T)}$. Clearly, v_n satisfies

$$\begin{aligned}
&\partial_t v_n + A(x,t,\partial_x) v_n = q(x,t,u_n,\nabla u_n) v_n + \frac{h(x,t,u_n,\nabla u_n)}{\|u_n\|_{C^{1,0}(\overline{Q}_T)}} && \text{in } Q_T,\\
&v_n = 0 && \text{on } \Sigma_T,\\
&v_n(\cdot,0) = v_n(\cdot,T) && \text{in } \Omega.
\end{aligned}$$

We have that $q(\cdot,\cdot,u_n,\nabla u_n) \to q$ weakly in $L^p(Q_T)$, where $q \in L^\infty(Q_T)$ satisfies $\gamma(x,t) \leqslant q(x,t) \leqslant \Gamma(x,t)$ a.e. in Q_T, and $h_n(\cdot,\cdot,u_n,\nabla u_n)/\|u_n\|_{C^{1,0}(\overline{Q}_T)} \to 0$ strongly in $L^p(Q_T)$. Moreover, $(v_n)_n$ is bounded in $W_p^{2,1}(Q_T)$ and therefore, possibly passing to a subsequence, it converges weakly in $W_p^{2,1}(Q_T)$ and strongly in $C^{1,0}(\overline{Q}_T)$ to some function v with $\|v\|_{C^{1,0}(\overline{Q}_T)} = 1$. Hence v is a nontrivial solution of

$$\begin{array}{ll} \partial_t v + A(x,t,\partial_x)v = q(x,t)v & \text{in } Q_T, \\ v = 0 & \text{on } \Sigma_T, \\ v(\cdot,0) = v(\cdot,T) & \text{in } \Omega, \end{array}$$

thus contradicting Proposition 2.51. This proves the claim.

CLAIM 2. *There exist a strict regular lower solution α and a strict regular upper solution β of* (2.36) *such that every solution u of* (2.36) *satisfies $\alpha \ll u \ll \beta$.*

Denote by $w_\pm$ the solutions of

$$\begin{array}{ll} \partial_t u + A(x,t,\partial_x)u = \Gamma(x,t)u \pm b(x,t) & \text{in } Q_T, \\ u = 0 & \text{on } \Sigma_T, \\ u(\cdot,0) = u(\cdot,T) & \text{in } \Omega, \end{array}$$

which exist by Proposition 2.51. Proposition 2.49 and Proposition 2.1 then imply the existence of a constant $k > 0$ such that $w_- - k\varphi_1 \ll u \ll w_+ + k\varphi_1$, for all $u \in C^{1,0}(\overline{Q}_T)$ with $\|u\|_{C^{1,0}(\overline{Q}_T)} < R$. It is now easy to verify that $\alpha = w_- - k\varphi_1$ and $\beta = w_+ + k\varphi_1$ are, respectively, strict regular lower and upper solutions of (2.36). The conclusion then follows using Claim 1.

From the above claims, we deduce by Theorem 2.15 the existence of the minimum solution v and the maximum solution w in $[\alpha,\beta]$; as every solution u of (2.36) satisfies $\alpha \leqslant u \leqslant \beta$, v and w are the minimum and the maximum solution. Possibly modifying the choice of k in the definition of α and β, we can assume that, if u_0 is such that $u_0 \leqslant v(\cdot,t_0)$ (respectively $w(\cdot,t_0) \leqslant u_0$), then $\alpha(\cdot,t_0) \leqslant u_0 \leqslant v(\cdot,t_0)$ (respectively $w(\cdot,t_0) \leqslant u_0 \leqslant \beta(\cdot,t_0)$). The result then follows from Theorem 2.33. □

Continua of stable T-periodic solutions

We now prove a result which generalizes [33, Theorem 1] (see also [34,58]) and provides the existence of $\mathcal{O}$-stable solutions of (2.36) in the presence of lower and upper solutions α, β, with $\alpha \leqslant \beta$. It also provides information about the topological structure of the set of $\mathcal{O}$-stable solutions lying between α and β.

In the proof we shall use a property of compact subsets of $C^{1,0}(\overline{Q}_T)$, which are dense-in-itself with respect to the order, analogous to that one we have described in Lemma 1.3 for compact subsets of $C^0([t_0,t_1])$.

DEFINITION 2.18. A subset $\mathcal{S}$ of $C^{1,0}(\overline{Q}_T)$ is said *dense-in-itself with respect to the order* if for any $u_1, u_2 \in \mathcal{S}$, with $u_1 < u_2$, there exists $u_3 \in \mathcal{S}$ with $u_1 < u_3 < u_2$.

The next result can be proved exactly in the same way as Lemma 1.3.

LEMMA 2.37. *Let $\mathcal{S} \subset C^{1,0}(\overline{Q}_T)$ be a compact set which is dense-in-itself with respect to the order. Let $\mathcal{T} \subseteq \mathcal{S}$ be a maximal nondegenerate totally ordered subset of $\mathcal{S}$. Then $\mathcal{T}$ is homeomorphic to a nondegenerate compact interval of $\mathbb{R}$.*

THEOREM 2.38. *Assume* (D), (A), (C) *and* (N). *Suppose that α is a proper lower solution and β is a proper upper solution of* (2.36) *satisfying $\alpha < \beta$. Denote, respectively, by v and w the minimum and the maximum solution of* (2.36) *in $[\alpha, \beta]$. Then there exists a totally ordered continuum $\mathcal{K}$ in $C^{1,0}(\overline{Q}_T)$ such that every $u \in \mathcal{K}$ is an $\mathcal{O}$-stable solution of* (2.36) *satisfying $v \leqslant u \leqslant w$; moreover, $u_1 = \min \mathcal{K}$ is properly $\mathcal{O}$-stable from below and $u_2 = \max \mathcal{K}$ is properly $\mathcal{O}$-stable from above.*

PROOF. The proof closely follows up the proof of Theorem 1.33. We denote by $\mathcal{S}_1$ the set of all solutions u of (2.36), with $\alpha \leqslant u \leqslant \beta$, which are properly $\mathcal{O}$-stable from below. Since, by Remark 2.21, the minimum solution v is properly $\mathcal{O}$-stable from below, $\mathcal{S}_1$ is not empty. Notice also that, since $\mathcal{S}_1$ is bounded in $L^\infty(Q_T)$, the Nagumo condition (N) implies, by Proposition 2.10, that this set is bounded in $W_p^{2,1}(Q_T)$ and therefore it is relatively compact in $C^{1,0}(\overline{Q}_T)$. Arguing as in the proof of Theorem 1.33, with the aid here of Lemma 2.16 instead of Lemma 1.15, we prove the existence of a maximal solution u_1 of (2.36) in $\mathcal{S}_1$. Proceeding further as in that proof, we denote by $\mathcal{S}_2$ the set of all solutions u of (2.36), with $u_1 \leqslant u \leqslant \beta$, which are properly $\mathcal{O}$-stable from above. By Remark 2.21, $w \in \mathcal{S}_2$. Arguing as above, we get a minimal solution u_2 of (2.36) in $\mathcal{S}_2$.

If $u_1 = u_2$, the conclusion is achieved. Otherwise we denote by $\mathcal{S}_3$ the set of all solutions u of (2.36), with $u_1 \leqslant u \leqslant u_2$. Notice that $\mathcal{S}_3$ is compact in $C^{1,0}(\overline{Q}_T)$. To show that $\mathcal{S}_3$ is dense-in-itself with respect to the order we argue as in Theorem 1.33, with the aid here of Remark 2.21, to show that no proper lower or upper solution of (2.36) may exist between u_1 and u_2, and of Proposition 2.18. Finally, we fix a solution $u_0 \in \mathcal{S}_3$ and we denote by $\mathcal{S}(u_0)$ a maximal totally ordered subset of $\mathcal{S}_3$, with $u_0 \in \mathcal{S}(u_0)$. Then we apply Lemma 2.37 to prove that $\mathcal{S}(u_0)$ is compact and connected. Clearly, every $u \in \mathcal{S}(u_0)$ is $\mathcal{O}$-stable and $u_1 = \min \mathcal{S}(u_0)$ and $u_2 = \max \mathcal{S}(u_0)$ are properly $\mathcal{O}$-stable from below and above, respectively. □

Assuming the validity of the second comparison principle stronger conclusions can be achieved.

COROLLARY 2.39. *Assume* (D), (A), (C), (N) *and suppose that the second comparison principle holds. Let α be a proper lower solution and β be a proper upper solution of* (2.36) *satisfying $\alpha < \beta$. Denote, respectively, by v and w the minimum and the maximum solution of* (2.36) *in $[\alpha, \beta]$. Then there exists a totally ordered continuum $\mathcal{K}$ in $C^{1,0}(\overline{Q}_T)$ such that every $u \in \mathcal{K}$ is a $\mathcal{L}$-stable solution of* (2.36) *satisfying $v \leqslant u \leqslant w$. Moreover, if $u_1 = \min \mathcal{K}$ is isolated from below in $C^{1,0}(\overline{Q}_T)$, then u_1 is $\mathcal{L}$-asymptotically stable from below and if $u_2 = \max \mathcal{K}$ is isolated from above in $C^{1,0}(\overline{Q}_T)$, then u_2 is $\mathcal{L}$-asymptotically stable from above. In particular, if $u_1 = u_2 = u^*$ is isolated in $C^{1,0}(\overline{Q}_T)$, then u^* is $\mathcal{L}$-asymptotically stable.*

PROOF. The former conclusion follows from Theorem 2.38 and Proposition 2.31. The latter conclusion, concerning the $\mathcal{L}$-asymptotic stability from below of u_1, is a consequence of Corollary 2.34. Indeed, since u_1 is isolated from below in $C^{1,0}(\overline{Q}_T)$ and, by Theorem 2.38, is properly $\mathcal{O}$-stable from below, there exists a proper lower solution α^*, with $\alpha^* < u_1$, such that there is no solution u of (2.36) with $\alpha^* < u < u_1$, i.e. u_1 is the minimum solution of (2.36) in $[\alpha^*, \beta]$. Hence u_1 is $\mathcal{L}$-asymptotically stable from below. The conclusion about u_2 follows from a similar argument. The $\mathcal{L}$-asymptotic stability of u^* is a consequence of Proposition 2.28. □

Instability via lower and upper solutions

We now show that the presence of lower and upper solutions α, β, with $\alpha \nleqslant \beta$, yields the existence of unstable T-periodic solutions of (2.10). We stress that all the results in this section are obtained without requiring the validity of any comparison principle.

PROPOSITION 2.40. *Assume* (D), (A), (C) *and* (N). *Let z be a solution of* (2.36). *If z is $\mathcal{O}$-unstable from above and isolated from above in $C^{1,0}(\overline{Q}_T)$, then it is $\mathcal{L}$-unstable from above. If z is $\mathcal{O}$-unstable from below and isolated from below in $C^{1,0}(\overline{Q}_T)$, then it is $\mathcal{L}$-unstable from below.*

PROOF. According to Remark 2.23, we pick $\varepsilon > 0$ such that any solution v of (2.36), with $v > z$, satisfies $\|z - v\|_{L^\infty(Q_T)} \geqslant \varepsilon$. Further, let $(\alpha_n)_n$ be a sequence of regular lower solutions of (2.36) such that $\alpha_n > z$, $\alpha_n = 0$ on Σ_T and $\|\alpha_n - z\|_e \to 0$, as $n \to +\infty$. Fix $t_0 \in [0, T[$. Then, for every $\delta > 0$, there exists n such that, setting $u_0 = \alpha_n(\cdot, t_0)$, we have $u_0 \in W_p^{2-2/p}(\Omega)$, $u_0 = 0$ on $\partial\Omega$, $u_0 \geqslant z(\cdot, t_0)$ and $\|u_0 - z(\cdot, t_0)\|_e < \delta$. Moreover, problem (2.13) has, by Proposition 2.13, a nonextendible solution $u : \overline{\Omega} \times [t_0, \omega[\to \mathbb{R}$, satisfying $u > z$ in $\overline{\Omega} \times [t_0, \omega[$ and either $\lim_{t\to\omega} \|u(\cdot, t)\|_{L^\infty(\Omega)} = +\infty$, or $\lim_{t\to+\infty} \|u(\cdot, t) - v(\cdot, t)\|_{L^\infty(\Omega)} = 0$, for some solution v of (2.36), with $v > z$ and $\|z - v\|_{L^\infty(Q_T)} \geqslant \varepsilon$. Anyhow we conclude that z is $\mathcal{L}$-unstable from above. □

The following result is a counterpart, for what concerns instability, of Proposition 2.32.

PROPOSITION 2.41. *Assume* (D), (A), (C) *and* (N). *Moreover, let z be a solution of* (2.36).

(i) *If α is a strict lower solution of* (2.36) *and β is an upper solution of* (2.36) *such that $\alpha \leqslant \beta$, $\alpha \nleqslant z$, $z < \beta$ and there is no solution u of* (2.36) *satisfying $z < u \leqslant \beta$ and $u \ngeqslant \alpha$, then z is $\mathcal{L}$-unstable from above.*

(ii) *If β is a strict upper solution of* (2.36) *and α is a lower solution of* (2.36) *such that $\alpha \leqslant \beta$, $\beta \ngeqslant z$, $\alpha < z$ and there is no solution u of* (2.36) *satisfying $\alpha \leqslant u < z$ and $u \nleqslant \beta$, then z is $\mathcal{L}$-unstable from below.*

PROOF. We prove only the former statement; the proof of the latter being similar. Let v be the minimum solution of (2.36) in $[\max\{\alpha, z\}, \beta]$ given by Theorem 2.15. Since α is a strict lower solution, we have $v \gg \alpha$ and hence $v > \max\{\alpha, z\}$. Let us observe that there is no solution u of (2.36) such that $z < u < v$. Indeed, if u were such a solution, by the minimality of v, it should satisfy $u \ngeqslant \max\{\alpha, z\}$ and hence $u \ngeqslant \alpha$. This contradicts the assumptions on z. Then Proposition 2.18 implies that either there exists a sequence $(\alpha_n)_n$

of proper regular lower solutions of (2.36) such that, for each n, $\alpha_n|_{\overline{Q}_T} \in W^{2,1}_{p,B}(Q_T)$ and $z < \alpha_n < v$, which converges in $C^{1,0}(\overline{Q}_T)$ to z as $n \to +\infty$, or there exists a sequence $(\beta_n)_n$ of proper regular upper solutions of (2.36) such that, for each n, $\beta_n|_{\overline{Q}_T} \in W^{2,1}_{p,B}(Q_T)$ and $z < \beta_n < v$, which converges in $C^{1,0}(\overline{Q}_T)$ to v as $n \to +\infty$. Let us show that the latter alternative cannot occur. Indeed, otherwise, as $v \gg \alpha$, we could find an upper solution $\hat{\beta}$ of (2.36), with $\max\{\alpha, z\} \leqslant \hat{\beta} < v$. Hence there should exist a solution u of (2.36), with $\max\{\alpha, z\} \leqslant u \leqslant \hat{\beta}$ and therefore $z < u < v$, as $z \not\geqslant \alpha$. This yields a contradiction with a preceding conclusion. Therefore, the former alternative necessarily occurs, i.e. z is properly $\mathcal{O}$-unstable from above. Arguing as in Proposition 2.40, we get the conclusion. □

An immediate consequence of these statements is the following instability result, in the presence of a lower solution α and an upper solution β satisfying the condition $\alpha \not\leqslant \beta$. It yields a completion of Theorem 2.17; as there, we write

$$\mathcal{V} = \{u \in C^{1,0}_B(\overline{Q}_T) \mid u \not\geqslant \alpha \text{ and } u \not\leqslant \beta\}.$$

THEOREM 2.42. *Assume* (D), (A), (C) *and* (N). *Suppose that α is a strict lower solution and β is a strict upper solution of* (2.36) *satisfying*

$$\alpha \not\leqslant \beta.$$

Further, assume that there exist a lower solution α_1 and an upper solution β_1 of (2.36) *such that $\alpha_1 \leqslant \beta_1$ and $\alpha, \beta \in [\alpha_1, \beta_1]$. Then any minimal solution v and any maximal solution w of* (2.36) *in $\overline{\mathcal{V}} \cap [\alpha_1, \beta_1]$ is, respectively, $\mathcal{L}$-unstable from below and $\mathcal{L}$-unstable from above.*

PROOF. We first notice that, as α is a strict lower solution and β is a strict upper solution, $v, w \in \mathcal{V}$. We then apply the former statement of Proposition 2.41, with β and z replaced respectively by β_1 and w, to show that w is $\mathcal{L}$-unstable from above. Whereas, we apply the latter statement of Proposition 2.41, with α and z replaced respectively by α_1 and v, to show that v is $\mathcal{L}$-unstable from below. □

We conclude this section stating some illustrative results where the existence of the lower and upper solutions α, α_1, β, β_1 is replaced by assumptions relating the behaviour of the function f to the principal eigenvalue λ_1. Alternative conditions can be found in [41,42]. We first discuss the $\mathcal{L}$-instability of the T-periodic solution of a linear problem.

PROPOSITION 2.43. *Assume* (D) *and* (A). *Let $q \in L^\infty(Q_T)$ satisfy $\lambda_1 \leqslant q(x,t) \leqslant \mu$ a.e. in Q_T, with $\mu > \lambda_1$ defined in Proposition* 2.52, *and $\lambda_1 < q(x,t)$ on a subset of positive measure. Let $h \in L^p(Q_T)$, with $p > N+2$. Then the unique solution z of problem* (2.79) *is $\mathcal{L}$-unstable from below and from above.*

PROOF. Let $\varphi_1 \gg 0$ be the principal eigenfunction. For any $\sigma > 0$ the functions $\alpha = z + \sigma\varphi_1$ and $\beta = z - \sigma\varphi_1$ are respectively a strict lower and a strict upper solution of problem (2.79), with $\alpha \gg z \gg \beta$. Hence z is $\mathcal{O}$-unstable from above and from below. Proposition 2.40 yields the conclusion. □

A nonlinear counterpart of Proposition 2.43 for problem (2.36) is the following instability result (see [41,42]).

PROPOSITION 2.44. *Assume* (D), (A), (C) *and* (N). *Suppose that there exist functions $a, b \in L^p(Q_T)$, with $p > N+2$, and $\gamma \in L^\infty(Q_T)$ such that, for a.e. $(x,t) \in Q_T$ and all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$ with $s \neq 0$,*

$$\gamma(x,t)|s| + a(x,t) \leqslant f(x,t,s,\xi)\,\mathrm{sgn}(s) \leqslant \mu|s| + b(x,t),$$

where $\gamma(x,t) \geqslant \lambda_1$ a.e. in Q_T, $\gamma(x,t) > \lambda_1$ on a subset of positive measure and μ is defined in Proposition 2.52. *Then there exist a minimal solution v and a maximal solution w of* (2.36) *which are*, *respectively*, *$\mathcal{L}$-unstable from below and $\mathcal{L}$-unstable from above.*

PROOF. The proof is carried out through several steps.

CLAIM 1. *There exists a minimal solution v and a maximal solution w of* (2.36).

Arguing as in Claim 1 of Proposition 2.36 and using Proposition 2.52 instead of Proposition 2.51, we prove the existence of a constant $M > 0$ such that all solutions u of (2.36) satisfy $\|u\|_{C^{1,0}(\overline{Q}_T)} < M$. Hence we conclude by Lemma 2.16.

CLAIM 2. *There exist a strict regular lower solution α and a strict regular upper solution β of* (2.36) *such that every solution u of* (2.36) *satisfies $\beta \ll u \ll \alpha$.*

Using Proposition 2.52, Proposition 2.49 and Proposition 2.1, we define, as in Claim 2 in the proof of Proposition 2.36, $\alpha = w_+ + k\varphi_1$ and $\beta = w_- - k\varphi_1$, where $w_\pm$ are the solutions of

$$\begin{aligned}
&\partial_t u + A(x,t,\partial_x)u = \gamma(x,t)u \pm a(x,t) && \text{in } Q_T,\\
&u = 0 && \text{on } \Sigma_T,\\
&u(\cdot,0) = u(\cdot,T) && \text{in } \Omega,
\end{aligned}$$

and $k > 0$ is so large that $\beta \ll u \ll \alpha$, for all $u \in C^{1,0}(\overline{Q}_T)$ with $\|u\|_{C^{1,0}(\overline{Q}_T)} < M$. It is now easy to verify that α and β are, respectively, strict regular lower and upper solutions of (2.36).

For each $r > 1$, consider the modified problem

$$\begin{aligned}
&\partial_t u + A(x,t,\partial_x)u = f_r(x,t,u,\nabla_x u) && \text{in } Q_T,\\
&u = 0 && \text{on } \Sigma_T,\\
&u(\cdot,0) = u(\cdot,T) && \text{in } \Omega,
\end{aligned} \tag{2.80}$$

where $f_r : Q_T \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is defined, for a.e. $(x,t) \in Q_T$ and every $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$, by

$$f_r(x,t,s,\xi) = \begin{cases} f(x,t,s,\xi) & \text{if } |s| < r, \\ \big(|s| - r\big)\big(\lambda_1 - \frac{1}{r}\big)s & \\ \quad + \big(r + 1 - |s|\big) f(x,t,s,\xi) & \text{if } r \leqslant |s| \leqslant r+1, \\ \big(\lambda_1 - \frac{1}{r}\big)s & \text{if } |s| > r+1. \end{cases}$$

Of course, f_r satisfies condition (C). Let us set

$$\mathcal{V} = \big\{ u \in C_B^{1,0}\big(\overline{Q}_T\big) \mid u \not\geqslant \alpha \text{ and } u \not\leqslant \beta \big\}.$$

CLAIM 3. *There exists $K > 0$ such that every solution $u \in \overline{\mathcal{V}}$ of* (2.80) *with $r > K$ satisfies $\|u\|_{C^{1,0}(\overline{Q}_T)} < K$.*

Assume by contradiction that, for every n, there exist $r_n > n$ and $u_n \in \overline{\mathcal{V}}$, solution of (2.80) for $r = r_n$, such that $\|u_n\|_{C^{1,0}(\overline{Q}_T)} \geqslant n$. As in Claim 1 of Proposition 2.36, we can write, for each $r > 1$,

$$f_r(x,t,s,\xi) = q_r(x,t,s,\xi)s + h_r(x,t,s,\xi)$$

for a.e. $(x,t) \in Q_T$ and every $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$, where q_r and h_r are L^p-Carathéodory functions satisfying

$$\lambda_1 - \frac{1}{r} \leqslant q_r(x,t,s,\xi) \leqslant \mu \quad \text{and} \quad \big|h_r(x,t,s,\xi)\big| \leqslant c(x,t),$$

for some $c \in L^p(Q_T)$ independent of r. Setting $v_n = u_n / \|u_n\|_{C^{1,0}(\overline{Q}_T)}$, we have that, possibly passing to a subsequence, $(v_n)_n$ converges weakly in $W_p^{2,1}(Q_T)$ and strongly in $C^{1,0}(\overline{Q}_T)$ to a solution v of

$$\begin{aligned} &\partial_t v + A(x,t,\partial_x)v = q(x,t)v && \text{in } Q_T, \\ &v = 0 && \text{on } \Sigma_T, \\ &v(\cdot,0) = v(\cdot,T) && \text{in } \Omega, \end{aligned}$$

where $\|v\|_{C^{1,0}(\overline{Q}_T)} = 1$ and $q \in L^\infty(Q_T)$ satisfies $\lambda_1 \leqslant q(x,t) \leqslant \mu$ a.e. in Q_T. Proposition 2.52 implies that $q = \lambda_1$ a.e. in Q_T and $v = a\varphi_1$ for some $a \neq 0$. Hence, by Proposition 2.1, we conclude that, if $a > 0$, there exists $d > 0$ such that, for all n large enough,

$$u_n \geqslant d\|u_n\|_{C^{1,0}(\overline{Q}_T)} a\varphi_1 \gg \alpha$$

and, if $a < 0$, there exists $d > 0$ such that, for all n large enough,

$$u_n \leqslant d\|u_n\|_{C^{1,0}(\overline{Q}_T)} a\varphi_1 \ll \beta,$$

thus contradicting anyhow the assumption $u_n \in \overline{\mathcal{V}}$. This proves the claim.

Let us set $R = 1 + \max\{K, M, \|\alpha\|_{L^\infty(Q_T)}, \|\beta\|_{L^\infty(Q_T)}\}$. We first notice that α and β are strict lower and upper solutions of (2.80), with $r = R$. Indeed, otherwise there exists a solution $u \in \overline{\mathcal{V}}$ of (2.80), with $r = R$, which, by Claim 3, satisfies $\|u\|_{C^{1,0}(\overline{Q}_T)} < K$. Hence u is a solution of (2.36) too, contradicting the fact that α and β are strict lower and upper solutions of (2.36).

CLAIM 4. *Problem* (2.80), *with $r = R$, admits a lower solution α_1 and an upper solution β_1 such that $\alpha_1 \leqslant \beta_1$ and $\alpha, \beta \in [\alpha_1, \beta_1]$.*

We show how to build the upper solution, as the lower solution can be constructed similarly. Let z be the solution of

$$\begin{array}{ll} \partial_t u + A(x,t,\partial_x)u = \left(\lambda_1 - \frac{1}{R}\right)u + (a_0 - \lambda_1)(R+2) & \text{in } Q_T, \\ u = 0 & \text{on } \Sigma_T, \\ u(\cdot,0) = u(\cdot,T) & \text{in } \Omega. \end{array}$$

Choose, by Proposition 2.1, a constant $a > 0$ so large that $a\varphi_1 - z \geqslant 0$. We then easily verify that

$$\beta_1 = a\varphi_1 - z + R + 2$$

is an upper solution of (2.80), with $r = R$. This proves the claim.

We finally notice that the minimal and the maximal solution v and w of (2.36), we obtained in Claim 1, are minimal and maximal solutions in $\overline{\mathcal{V}} \cap [\alpha_1, \beta_1]$ of (2.80), with $r = R$. Theorem 2.42 implies that they are, respectively, $\mathcal{L}$-unstable from below and $\mathcal{L}$-unstable from above as solutions of (2.80), with $r = R$, and hence, by our choice of R, of (2.36). □

Stability and multiplicity of periodic solutions

We discuss in this section some relations between the stability properties and the multiplicity of solutions of (2.36). Propositions 2.45 and 2.47, which are direct consequences of Theorem 2.21 and Theorem 2.38, yield some results expressed in terms of $\mathcal{O}$-stability. Afterwards Propositions 2.46 and 2.48 provide an interpretation of these statements in terms of $\mathcal{L}$-stability.

PROPOSITION 2.45. *Assume* (D), (A), (C) *and* (N). *Let u_1, u_2 be solutions of* (2.36) *such that $u_1 < u_2$, u_1 is not properly $\mathcal{O}$-stable from above and u_2 is not properly $\mathcal{O}$-stable from below. Then there exists a solution u_3 of* (2.36) *with $u_1 < u_3 < u_2$.*

If moreover u_1 is not $\mathcal{O}$-stable from above, then it is properly $\mathcal{O}$-unstable from above and there exists a solution u_3 of (2.36), *with $u_1 < u_3 < u_2$, which is properly $\mathcal{O}$-stable from below. Similarly, if moreover u_2 is not $\mathcal{O}$-stable from below then it is properly $\mathcal{O}$-unstable from below and there exists a solution u_3 of* (2.36) *with $u_1 < u_3 < u_2$ which is properly $\mathcal{O}$-stable from above.*

Finally, if moreover both u_1 is not $\mathcal{O}$-stable from above and u_2 is not $\mathcal{O}$-stable from below, then there exists a solution u_3 of (2.36), *with $u_1 < u_3 < u_2$, which is $\mathcal{O}$-stable.*

PROOF. If u_1 is not properly $\mathcal{O}$-stable from above and u_2 is not properly $\mathcal{O}$-stable from below, then Theorem 2.21 implies that there exists a solution u_3 of (2.36), with $u_1 < u_3 < u_2$.

If u_1 is not $\mathcal{O}$-stable from above, then there exists $\varepsilon > 0$ such that every solution u of (2.36), with $u_1 < u < u_2$, satisfies $\|u - u_1\|_{C^{1,0}(\overline{Q}_T)} \geqslant \varepsilon$. By Lemma 2.16, there is a solution v of (2.36) such that $u_1 < v < u_2$, $\|v - u_1\|_{C^{1,0}(\overline{Q}_T)} \geqslant \varepsilon$ and there is no solution u of (2.36) with $u_1 < u < v$. Theorem 2.21 then implies that u_1 is properly $\mathcal{O}$-unstable from above and v is properly $\mathcal{O}$-stable from below.

Similarly, if u_2 is not $\mathcal{O}$-stable from below, then there exists a solution w of (2.36), with $u_1 < w < u_2$, such that there is no solution u of (2.36) with $w < u < u_2$. Theorem 2.21 implies that u_2 is properly $\mathcal{O}$-unstable from below and w is properly $\mathcal{O}$-stable from above.

Finally, if u_1 is not $\mathcal{O}$-stable from above and u_2 is not $\mathcal{O}$-stable from below, then there exist solutions v and w of (2.36), with $u_1 < v \leqslant w < u_2$, such that there is no solution u of (2.36) with either $u_1 < u < v$ or $w < u < u_2$. Further, v is properly $\mathcal{O}$-stable from below and w is properly $\mathcal{O}$-stable from above. Theorem 2.38 then implies that there exists a solution u_3 of (2.36), with $v \leqslant u_3 \leqslant w$, which is $\mathcal{O}$-stable. □

PROPOSITION 2.46. *Assume* (D), (A), (C) *and* (N). *Further, suppose that the second comparison principle holds. Let u_1, u_2 be solutions of* (2.36) *such that $u_1 < u_2$, u_1 is $\mathcal{L}$-unstable from above and u_2 is $\mathcal{L}$-unstable from below. Then there exists a solution u_3 of* (2.36) *with $u_1 \ll u_3 \ll u_2$ which is $\mathcal{L}$-stable.*

PROOF. Proposition 2.31 implies that u_1 is not $\mathcal{O}$-stable from above and u_2 is not $\mathcal{O}$-stable from below. Hence Proposition 2.45 yields the existence of a solution u_3 of (2.36) with $u_1 < u_3 < u_2$ which is $\mathcal{O}$-stable and therefore, by Proposition 2.31 again, it is $\mathcal{L}$-stable. □

REMARK 2.28. With reference to Proposition 2.46, we have that $u_1 \ll u_2$, u_1 is isolated from above in $C^{1,0}(\overline{Q}_T)$ and u_2 is isolated from below in $C^{1,0}(\overline{Q}_T)$, by Proposition 2.31. Hence, by Lemma 2.16, there exist solutions v and w of (2.36), with $u_1 \ll v \leqslant w \ll u_2$, such that there is no solution u satisfying either $u_1 < u < v$ or $w < u < u_2$. By Proposition 2.45 and Corollary 2.34, v is $\mathcal{L}$-asymptotically stable from below and w is $\mathcal{L}$-asymptotically stable from above.

The following results are dual versions of Proposition 2.45 and Proposition 2.46.

PROPOSITION 2.47. *Assume* (D), (A), (C) *and* (N). *Let u_1, u_2 be solutions of* (2.36) *such that $u_1 < u_2$, u_1 is not properly $\mathcal{O}$-unstable from above and u_2 is not properly $\mathcal{O}$-unstable from below. Then there exists a solution u_3 of* (2.36) *with $u_1 < u_3 < u_2$.*

If moreover u_1 is not $\mathcal{O}$-unstable from above, then it is properly $\mathcal{O}$-stable from above and there exists a solution u_3 of (2.36), *with $u_1 < u_3 < u_2$, which is properly $\mathcal{O}$-unstable from below. Similarly, if moreover u_2 is not $\mathcal{O}$-unstable from below, then it is properly $\mathcal{O}$-stable from below and there exists a solution u_3 of* (2.36), *with $u_1 < u_3 < u_2$, which is properly $\mathcal{O}$-unstable from above.*

PROOF. If u_1 is not properly $\mathcal{O}$-unstable from above and u_2 is not properly $\mathcal{O}$-unstable from below, then Theorem 2.21 implies that there exists a solution u_3 of (2.36) with $u_1 < u_3 < u_2$.

If u_1 is not $\mathcal{O}$-unstable from above, then there exists $\varepsilon > 0$ such that every solution u of (2.36) with $u_1 < u < u_2$ satisfies $\|u - u_1\|_{C^{1,0}(\overline{Q}_T)} \geqslant \varepsilon$. By Lemma 2.16, there exists a solution v of (2.36) such that $u_1 < v < u_2$, $\|v - u_1\|_{C^{1,0}(\overline{Q}_T)} \geqslant \varepsilon$ and there is no solution u of (2.36) with $u_1 < u < v$. Moreover, Theorem 2.21 implies that u_1 is properly $\mathcal{O}$-stable from above and v is properly $\mathcal{O}$-unstable from below.

Similarly, if u_2 is not $\mathcal{O}$-unstable from below, then there exists a solution w of (2.36) with $u_1 < w < u_2$ and there is no solution u of (2.36) such that $w < u < u_2$. Theorem 2.21 implies then that u_2 is properly $\mathcal{O}$-stable from below and w is properly $\mathcal{O}$-unstable from above. □

PROPOSITION 2.48. *Assume* (D), (A), (C) *and* (N). *Let u_1, u_2 be solutions of* (2.36) *such that $u_1 < u_2$, u_1 is $\mathcal{L}$-stable from above and u_2 is $\mathcal{L}$-stable from below. Then there exists a solution u_3 of* (2.36) *with $u_1 < u_3 < u_2$.*

If moreover u_1 is isolated from above in $C^{1,0}(\overline{Q}_T)$, then there exists a solution u_3 of (2.36), *with $u_1 < u_3 < u_2$, which is $\mathcal{L}$-unstable from below. Similarly, if moreover u_2 is isolated from below in $C^{1,0}(\overline{Q}_T)$, then there exists a solution u_3 of* (2.36), *with $u_1 < u_3 < u_2$, which is $\mathcal{L}$-unstable from above.*

PROOF. If we assume that there is no solution u_3 of (2.36), with $u_1 < u_3 < u_2$, then by Theorem 2.21 either u_1 is properly $\mathcal{O}$-unstable from above or u_2 is properly $\mathcal{O}$-unstable from below. In the former case there is a sequence $(\alpha_n)_n$ of proper regular lower solutions of (2.36) such that, for every n, $u_1 < \alpha_n < u_2$, $\alpha_n = 0$ on Σ_T and $\|\alpha_n - u_1\|_e \to 0$, as $n \to +\infty$. Fix $t_0 \in]0, T[$. Then, for every $\delta > 0$, there exists n such that, setting $u_0 = \alpha_n(\cdot, t_0)$, we have $u_0 \in W_p^{2-2/p}(\Omega)$, $u_0 = 0$ on $\partial\Omega$, $u_0 \geqslant u_1(\cdot, t_0)$ and $\|u_0 - u_1(\cdot, t_0)\|_e < \delta$. By Proposition 2.13, problem (2.13) has a nonextendible solution u, satisfying $u > u_1$ in $\overline{\Omega} \times [t_0, +\infty[$ and $\lim_{t\to+\infty} \|u(\cdot, t) - u_2(\cdot, t)\|_e = 0$. Therefore, we get $\limsup_{t\to+\infty} \|u(\cdot, t) - u_1(\cdot, t)\|_e \geqslant \frac{1}{2}\|u_1 - u_2\|_e$, thus contradicting the $\mathcal{L}$-stability from above of u_1. Similarly, we verify that u_2 cannot be properly $\mathcal{O}$-unstable from below. Accordingly, there exists a solution u_3 of (2.36), with $u_1 < u_3 < u_2$.

Suppose further that u_1 is isolated from above in $C^{1,0}(\overline{Q}_T)$, i.e. there is $\delta > 0$ such that every solution u of (2.36), with $u > u_1$, satisfies $\|u - u_1\|_{C^{1,0}(\overline{Q}_T)} \geqslant \delta$, then by Lemma 2.16, there exists a solution u_3 of (2.36), satisfying $u_3 \in [u_1, u_2]$ and $\|u_3 - u_1\|_{C^{1,0}(\overline{Q}_T)} \geqslant \delta$, such that there is no solution u, with $u_1 < u < u_3$. Arguing as in the previous step, we see that u_3 is properly $\mathcal{O}$-unstable from below. Hence, arguing as in Proposition 2.40, we conclude that u_3 is $\mathcal{L}$-unstable from below. Similarly, if we suppose further that u_2 is isolated from below, we derive the existence of a solution u_3 of (2.36) which is $\mathcal{L}$-unstable from above. □

REMARK 2.29. We stress again that the conclusions of Proposition 2.48 are obtained without requiring the validity of any comparison principle. On the other hand, if we assume in Proposition 2.48 that the second comparison principle holds, then u_1 is $\mathcal{L}$-asymptotically stable from above, provided it is isolated from above, and u_2 is $\mathcal{L}$-asymptotically stable from below, provided it is isolated from below. The $\mathcal{L}$-asymptotic stability from above of u_1 easily follows from Corollary 2.34. Indeed, any solution u_3 of (2.36), satisfying $u_1 \ll u_3 \leqslant u_2$ and such that there is no solution u with $u_1 < u < u_3$, is properly $\mathcal{O}$-unstable from below and hence there exist proper regular upper solutions β with $u_1 \ll \beta \ll u_3$. The analogous conclusion about u_2 is obtained in a similar way.

2.7. *Appendix: The principal eigenvalue*

This appendix is devoted to the definition and the discussion of some properties of the principal eigenvalue of the periodic problem

$$\begin{array}{ll} \partial_t u + A(x,t,\partial_x)u = \lambda u & \text{in } Q_T, \\ u = 0 & \text{on } \Sigma_T, \\ u(\cdot,0) = u(\cdot,T) & \text{in } \Omega. \end{array} \tag{2.81}$$

Our first result is an extension of a similar statement proved in [82] (see also [27,18]) under stronger regularity conditions.

PROPOSITION 2.49. *Assume* (D) *and* (A). *Let $p > N+2$ be fixed. Then there exist a number $\lambda_1 > 0$ and a function $\varphi_1 \in W^{2,1}_{p,B}(Q_T)$ such that*

$$L\varphi_1 = \lambda_1 \varphi_1,$$

where L is defined by (2.47) *and* (2.48). *Moreover, the following conclusions hold*:

(i) $\varphi_1 \gg 0$;

(ii) *if $L\psi = \lambda_1\psi$ for some $\psi \in W^{2,1}_{p,B}(Q_T)$, then $\psi = k\varphi_1$ for some $k \in \mathbb{R}$;*

(iii) *if $L\psi = \lambda\psi$ for some $\psi \in W^{2,1}_{p,B}(Q_T)$, with $\psi > 0$, then $\lambda = \lambda_1$;*

(iv) *for every $\lambda < \lambda_1$ and $h \in L^p(Q_T)$, the problem $Lu - \lambda u = h$ has a unique solution.*

PROOF. Let us introduce the operator $S : C^{1,0}_B(\overline{Q}_T) \to C^{1,0}_B(\overline{Q}_T)$ which sends any $h \in C^{1,0}_B(\overline{Q}_T)$ onto the unique solution $u \in C^{1,0}_B(\overline{Q}_T)$ of (2.46), with $\sigma = 0$. S is compact and strongly positive with respect to the order induced in $C^{1,0}_B(\overline{Q}_T)$ by the cone $K = \{u \in C^{1,0}_B(\overline{Q}_T) \mid u \geqslant 0\}$. Observe that $\mathring{K} = \{u \in C^{1,0}_B(\overline{Q}_T) \mid u \gg 0\}$. The strong form of Krein–Rutman Theorem (see, e.g., [153, Theorem 7C]) yields the existence of $\varphi_1 \in W^{2,1}_{p,B}(Q_T)$ satisfying $L\varphi_1 = \lambda_1\varphi_1$, with λ_1 the reciprocal of the spectral radius $r(S)$ of S, as well as conditions (i)–(iii). Finally, condition (iv) is a consequence of Proposition 2.14, if $\lambda \leqslant 0$. For $\lambda \in]0, \lambda_1[$, let us consider the extension $\tilde{S}$ of S to $L^p(Q_T)$, i.e. the operator $\tilde{S} : L^p(Q_T) \to L^p(Q_T)$ which sends any $h \in L^p(Q_T)$ onto the unique solution $u \in L^p(Q_T)$ of (2.46), with $\sigma = 0$. $\tilde{S}$ is compact and positive with respect to the order induced in $L^p(Q_T)$ by the cone $\tilde{K} = \{u \in L^p(Q_T) \mid u \geqslant 0\}$. By the weak form of Krein–Rutman Theorem (see, e.g., [153, Proposition 7.26]), the spectral radius $r(\tilde{S})$ of $\tilde{S}$ is an eigenvalue of $\tilde{S}$, with eigenfunction $\tilde{\varphi}_1 \in L^p(Q_T)$ satisfying $\tilde{\varphi}_1(x,t) > 0$ a.e. in Q_T. Since, by condition (iii), $r(\tilde{S}) = r(S)$, the result follows from the Fredholm Alternative (see, e.g., [144, Theorem 1.L]) applied to the operator $\tilde{S}$. □

PROPOSITION 2.50. *Assume* (D) *and* (A). *Let* $q_1, q_2 \in L^\infty(Q_T)$ *be such that* $q_1(x,t) \leqslant q_2(x,t)$ *a.e. in* Q_T *and let* u_1, u_2 *be nontrivial solutions of* $Lu_1 = q_1 u_1$ *and* $Lu_2 = q_2 u_2$, *respectively. If* $u_2 \geqslant 0$, *then* $q_1 = q_2$ *and* $u_1 = cu_2$, *for some* $c \in \mathbb{R}$.

PROOF. Since $Lu_2 - q_2 u_2 \geqslant 0$ and $u_2 > 0$, we deduce from Remark 2.7 that $u_2 \gg 0$. If we set $c = \min\{d \in \mathbb{R} \mid du_2 \geqslant u_1\}$ and $v = cu_2 - u_1$, as $v \geqslant 0$, we get $Lv - q_1 v \geqslant 0$ and hence either $v \gg 0$ or $v = 0$. The minimality of c actually yields $v = cu_2 - u_1 = 0$. This finally implies $0 = Lv - q_1 v = (q_2 - q_1)cu_2$ and therefore $q_1 = q_2$. □

PROPOSITION 2.51. *Assume* (D) *and* (A). *Let* $q \in L^\infty(Q_T)$ *satisfy* $q(x,t) \leqslant \lambda_1$ *a.e. in* Q_T *and* $q(x,t) < \lambda_1$ *on a subset of positive measure. Then the problem* (2.79) *has a unique solution, for any given* $h \in L^p(Q_T)$ *with* $p > N+2$.

PROOF. Setting $q_1 = q$ and $q_2 = \lambda_1$, Proposition 2.50 implies that the homogeneous equation $Lu = qu$ has only the trivial solution. The Fredholm Alternative applied to the operator $T = q\tilde{S} : L^p(Q_T) \to L^p(Q_T)$, where $\tilde{S}$ has been defined in the proof of Proposition 2.49, yields the conclusion. □

PROPOSITION 2.52. *Assume* (D) *and* (A). *Then there exists* $\mu > \lambda_1$ *such that, if* $q \in L^\infty(Q_T)$ *satisfies* $\lambda_1 \leqslant q(x,t) \leqslant \mu$ *a.e. in* Q_T *and* $\lambda_1 < q(x,t)$ *on a subset of positive measure, problem* (2.79) *has a unique solution, for any given* $h \in L^p(Q_T)$ *with* $p > N+2$.

PROOF. Let us show that there exists $\mu > \lambda_1$ such that, if $q \in L^\infty(Q_T)$ satisfies $\lambda_1 \leqslant q(x,t) \leqslant \mu$ a.e. in Q_T and $\lambda_1 < q(x,t)$ on a set of positive measure, the problem

$$\begin{array}{ll} \partial_t u + A(x,t,\partial_x)u = q(x,t)u & \text{in } Q_T, \\ u = 0 & \text{on } \Sigma_T, \\ u(\cdot,0) = u(\cdot,T) & \text{in } \Omega, \end{array}$$

has only the trivial solution. Indeed, assume by contradiction that, for every n, there exist $q_n \in L^\infty(Q_T)$, satisfying $\lambda_1 \leqslant q_n(x,t) \leqslant \lambda_1 + 1/n$ a.e. in Q_T and $\lambda_1 < q_n(x,t)$ on a set of positive measure, and a solution u_n of

$$\begin{array}{ll} \partial_t u_n + A(x,t,\partial_x)u_n = q_n(x,t)u_n & \text{in } Q_T, \\ u_n = 0 & \text{on } \Sigma_T, \\ u_n(\cdot,0) = u_n(\cdot,T) & \text{in } \Omega, \end{array}$$

with $\|u_n\|_{C^{1,0}(\overline{Q}_T)} = 1$. Proposition 2.50 implies that each u_n changes sign in Q_T. Possibly passing to a subsequence, $(u_n)_n$ converges weakly in $W_p^{2,1}(Q_T)$ and strongly in $C^{1,0}(\overline{Q}_T)$ to a nontrivial solution u of

$$\begin{array}{ll} \partial_t u + A(x,t,\partial_x)u = \lambda_1 u & \text{in } Q_T, \\ u = 0 & \text{on } \Sigma_T, \\ u(\cdot,0) = u(\cdot,T) & \text{in } \Omega, \end{array}$$

which, therefore, satisfies either $u \gg 0$, or $u \ll 0$. Hence Proposition 2.1 yields, for n large enough, either $u_n \gg 0$, or $u_n \ll 0$, which is a contradiction. The Fredholm Alternative applied to the operator $T = q\tilde{S} : L^p(Q_T) \to L^p(Q_T)$, where $\tilde{S}$ has been defined in the proof of Proposition 2.49, yields the conclusion. □

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CHAPTER 4

Bifurcation Theory of Limit Cycles of Planar Systems

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Contents

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In this chapter we present the bifurcation theory of limit cycles of planar systems with relatively simple dynamics. The theory studies the changes of orbital behavior in the phase space, especially the number of limit cycles as we vary the parameters of the system. This theory has been considered by many mathematicians starting with Poincaré who first introduced the notion of limit cycles. A fundamental step towards modern bifurcation theory in differential equations occurred with the definition of structural stability and the classification of structurally stable systems in the plane in 1937 developed by Andronov, Leontovich and Pontryagin. More precisely, Andronov and Pontryagin introduced the notion of a rough system and presented the necessary and sufficient conditions of roughness for systems on the plane. Almost at the same time, Andronov and Leontovich carried out a systematic classification of all principal bifurcations of limit cycles on the plane for the simplest nonrough systems. A further development of the theory had taken yet another direction, namely by selecting bifurcation sets of codimension one for primary bifurcations, and of arbitrary codimension in the general case for degenerate bifurcations. In the two-dimensional case, as was proved in Andronov et al. [2], rough systems compose an open and dense set in the space of all systems on a plane. The nonrough systems fill the boundaries between different regions of structural stability in this space.

In the following sections we concentrate on an in-depth study of limit cycles with general methods of both local and global bifurcations in high codimensional case. Many results are closely related to the second part of Hilbert's 16th problem which concerns with the number and location of limit cycles of a planar polynomial vector field of degree n posed in 1901 by Hilbert [73].

1. Limit cycle and its perturbations

1.1. *Basic notations*

Consider a planar system defined on a region $G \subset \mathbb{R}^2$ of the form

$$\dot{x} = f(x), \tag{1.1}$$

where $f : G \to \mathbb{R}^2$ is a C^r function, $r \geqslant 1$. Then for any point $x_0 \in G$ (1.1) has a unique solution $\varphi(t, x_0)$ satisfying $\varphi(0, x_0) = x_0$. Let $\varphi^t(x_0) = \varphi(t, x_0)$. The family of the transformations $\varphi^t : G \to \mathbb{R}^2$ satisfy the following properties:

(i) $\varphi^0 = \mathrm{Id}$;

(ii) $\varphi^{t+s} = \varphi^t \circ \varphi^s$.

The function φ is called the flow generated by (1.1) or by the vector field f. Let $I(x_0)$ denote the maximal interval of definition of $\varphi(t, x_0)$ in t. If $x_0 \in G$ is such that $\varphi(t, x_0)$ is constant for all $t \in I(x_0)$, then $f(x_0) = 0$. In this case, x_0 is called *a singular point* of (1.1). A point that is not singular is called *a regular point*.

For any regular point $x_0 \in G$, the solution $\varphi(t, x_0)$ defines two planar curves as follows:

$$\gamma^+(x_0) = \{\varphi(t, x_0) \colon t \in I(x_0), t \geqslant 0\}, \quad \gamma^-(x_0) = \{\varphi(t, x_0) \colon t \in I(x_0), t \leqslant 0\},$$

which are called respectively the *positive* and the *negative orbit* of (1.1) through x_0. The union $\gamma(x_0) = \gamma^+(x_0) \cup \gamma^-(x_0)$ is called the *orbit* of (1.1) through x_0. The theorem about the existence and uniqueness of solutions ensures that there is one and only one orbit through any point in G. A *periodic orbit* of (1.1) is an orbit which is a closed curve. The minimal positive number satisfying $\varphi(T, x_0) = x_0$ is said to be the *period* of the periodic orbit $\gamma(x_0)$. Obviously, $\gamma(x_0)$ is a periodic orbit of period T if and only if the corresponding representation $\varphi(t, x_0)$ is a periodic solution of the same period.

DEFINITION 1.1. A periodic orbit of (1.1) is called a *limit cycle* if it is the only periodic orbit in a neighborhood of it. In other words, a limit cycle is an isolated periodic orbit in the set of all periodic orbits.

Now let (1.1) have a limit cycle L: $x = u(t), 0 \leqslant t \leqslant T$. Since (1.1) is autonomous, for any given point $p \in L$ we may suppose $p = u(0)$, and hence, $u(t) = \varphi(t, p)$. Further, for definiteness, let L be oriented clockwise. Introduce a unit vector

$$Z_0 = \frac{1}{|f(p)|}\big(-f_2(p), f_1(p)\big)^{\mathrm{T}}.$$

Then there exists a cross section l of (1.1) which passes through p and is parallel to Z_0. Clearly, a point $x_0 \in l$ near p can be written as $x_0 = p + aZ_0$, $a = (x_0 - p)^{\mathrm{T}} Z_0 \in \mathbb{R}$.

LEMMA 1.1. *There exist a constant* $\varepsilon > 0$ *and* C^r *functions* P *and* $\tau : (-\varepsilon, \varepsilon) \to \mathbb{R}$ *with* $P(0) = 0$ *and* $\tau(0) = T$ *such that*

$$\varphi\big(\tau(a), p + aZ_0\big) = p + P(a)Z_0 \in l, \quad |a| < \varepsilon. \tag{1.2}$$

PROOF. Define $Q(t, a) = [f(p)]^{\mathrm{T}}(\varphi(t, p + aZ_0) - p)$. We have

$$Q(T, 0) = 0, \qquad Q_t(T, 0) = \big|f(p)\big|^2 > 0.$$

Note that Q is C^r for (t, a) near $(T, 0)$. The implicit function theorem implies that a C^r function $\tau(a) = T + \mathrm{O}(a)$ exists satisfying

$$Q\big(\tau(a), a\big) = 0 \quad \text{or} \quad \big[f(p)\big]^{\mathrm{T}}\big(\varphi\big(\tau(a), p + aZ_0\big) - p\big) = 0.$$

It follows that the vector $\varphi(\tau(a), p + aZ_0) - p$ is parallel to Z_0. Hence, it can be rewritten as $\varphi(\tau(a), p + aZ_0) - p = P(a)Z_0$, where $P(a) = Z_0^{\mathrm{T}}(\varphi(\tau(a), p + aZ_0) - p)$. It is obvious that $P \in C^r$ for $|a|$ small with $P(0) = 0$. This ends the proof. □

The above proof tells us that the function τ is the time of the first return to l. By Definition 1.1, the periodic orbit L is a limit cycle if and only if $P(a) \neq a$ for $|a| > 0$ sufficiently small.

DEFINITION 1.2. The function $P : (-\varepsilon, \varepsilon) \to \mathbb{R}$ defined by (1.2) is called a *Poincaré map* or return map of (1.1) at $p \in l$.

For convenience, we sometimes use the notation $P : l \to l$.

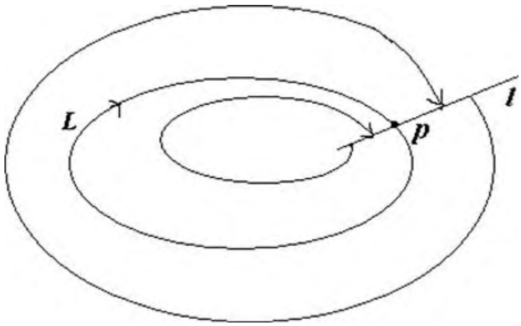

Fig. 1. Behavior of a stable limit cycle.

DEFINITION 1.3. The limit cycle L is said to be *outer stable* (*outer unstable*) if for $a > 0$ sufficiently small,

$$a\big(P(a) - a\big) < 0\ (> 0).$$

The limit cycle L is said to be *inner stable* (*inner unstable*) if the inequality above holds for $-a > 0$ sufficiently small. A limit cycle is called *stable* if it is both inner and outer stable. A limit cycle is called *unstable* if it is not stable.

For example, if L is stable, then the orbits near it behave like the phase portrait as shown in Fig. 1.

Let $P^k(a)$ denote the kth iterate of a under P. It is evident that $\{P^k(a)\}$ is monotonic in k and $P^k(a) > 0$ (< 0) for $a > 0$ (< 0). Thus, it is easy to see that L is outer stable if and only if $P^k(a) \to 0$ as $k \to \infty$ for all $a > 0$ sufficiently small. Similar conclusions hold for outer unstable, inner stable and inner unstable cases.

REMARK 1.1. If the limit cycle L is oriented anti-clockwise we can define its stability in a similar manner by using the Poincaré map P defined by (1.2). For instance, it is said to be inner stable (inner unstable) if $a(P(a) - a) < 0$ (> 0) for $a > 0$ sufficiently small.

DEFINITION 1.4. The limit cycle L is said to be *hyperbolic or of multiplicity one* if $P'(0) \neq 1$. It is said to have *multiplicity* k, $2 \leqslant k \leqslant r$, if $P'(0) = 1$, $P^{(j)}(0) = 0$, $j = 2, \ldots, k-1$, $P^{(k)}(0) \neq 0$.

By Definition 1.3, one can see that L is stable (unstable) if $|P'(0)| < 1$ (> 1).

1.2. *Multiplicity, stability and their property*

Next, we give formulas for $P'(0)$ and $P''(0)$. For the purpose, let

$$v(\theta) = \frac{u'(\theta)}{|u'(\theta)|} = \big(v_1(\theta), v_2(\theta)\big)^{\mathrm{T}}, \qquad Z(\theta) = \big(-v_2(\theta), v_1(\theta)\big)^{\mathrm{T}},$$

and introduce a transformation of coordinates of the form

$$x = u(\theta) + Z(\theta)b, \quad 0 \leqslant \theta \leqslant T,\ |b| < \varepsilon. \tag{1.3}$$

LEMMA 1.2. *The transformation* (1.3) *carries* (1.1) *into the system*

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = 1 + g_1(\theta, b), \qquad \frac{\mathrm{d}b}{\mathrm{d}t} = A(\theta)b + g_2(\theta, b), \tag{1.4}$$

where

$$\begin{aligned}
A(\theta) &= Z^{\mathrm{T}}(\theta) f_x\big(u(\theta)\big)Z(\theta) = \operatorname{tr} f_x\big(u(\theta)\big) - \frac{\mathrm{d}}{\mathrm{d}\theta}\ln\big|f\big(u(\theta)\big)\big|,\\
g_1(\theta, b) &= h(\theta, b)\big[f\big(u(\theta) + Z(\theta)b\big) - f\big(u(\theta)\big)\big] - h(\theta, b)Z'(\theta)b,\\
g_2(\theta, b) &= Z^{\mathrm{T}}(\theta)\big[f\big(u(\theta) + Z(\theta)b\big) - f\big(u(\theta)\big) - f_x\big(u(\theta)\big)Z(\theta)b\big],\\
h(\theta, b) &= \big(\big|f\big(u(\theta)\big)\big| + v^{\mathrm{T}}(\theta)Z'(\theta)b\big)^{-1}v^{\mathrm{T}}(\theta),
\end{aligned}$$

and $\operatorname{tr} f_x(u(\theta))$ *denotes the trace of the matrix* $f_x(u(\theta))$, *which is called the divergence of the vector field* f *evaluated at* $u(\theta)$.

PROOF. By (1.3) and (1.1) we have

$$\big(u' + Z'b\big)\frac{\mathrm{d}\theta}{\mathrm{d}t} + Z\frac{\mathrm{d}b}{\mathrm{d}t} = f(u + Zb). \tag{1.5}$$

In order to obtain (1.4) we need to solve $\frac{\mathrm{d}\theta}{\mathrm{d}t}$ and $\frac{\mathrm{d}b}{\mathrm{d}t}$ from (1.5). First, multiplying (1.5) by v^{T} from the left-hand side and using

$$v^{\mathrm{T}}Z = 0, \qquad v^{\mathrm{T}}f(u) = v^{\mathrm{T}}u' = \big|u'\big| = \big|f(u)\big|,$$

we can obtain

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \big[\big|f(u)\big| + v^{\mathrm{T}}Z'b\big]^{-1}v^{\mathrm{T}}f(u + Zb) = h(\theta, b)f(u + Zb).$$

Note that

$$h(\theta, b)f(u) = h(\theta, b)\big[f(u) + Z'b\big] - h(\theta, b)Z'b = 1 - h(\theta, b)Z'b.$$

It follows that

$$h(\theta, b)f(u + Zb) = h(\theta, b)\big[f(u + Zb) - f(u)\big] - h(\theta, b)Z'b + 1.$$

Then the first equation in (1.4) follows.

Now multiplying (1.5) by Z^{T} from the left and using

$$Z^{\mathrm{T}}Z = 1, \qquad Z^{\mathrm{T}} f(u) = 0, \qquad Z^{\mathrm{T}}Z' = \frac{1}{2}\big(|v|^2\big)' = 0,$$

we obtain

$$\frac{\mathrm{d}b}{\mathrm{d}t} = Z^{\mathrm{T}}\big[f(u+Zb) - f(u) - f_x(u)Zb\big] + Z^{\mathrm{T}} f_x(u)Zb.$$

It is direct to prove that

$$Z^{\mathrm{T}} f_x(u)Z = \operatorname{tr} f_x(u) - \frac{\mathrm{d}}{\mathrm{d}\theta}\ln\big|f(u)\big|.$$

Then the second equation of (1.4) follows. This finishes the proof. □

Set

$$B(\theta) = \big[f_x(u+Zb)\big]'_b\big|_{b=0}, \quad C(\theta) = v^{\mathrm{T}}\big[f_x(u)Z - Z'(\theta)\big], \tag{1.6}$$

and

$$R(\theta, b) = \frac{A(\theta)b + g_2(\theta, b)}{1 + g_1(\theta, b)}.$$

Then by Lemma 1.2, we can write

$$\begin{aligned} R(\theta, b) &= A(\theta)b + \frac{1}{2}\left[Z^{\mathrm{T}}BZ - \frac{2AC}{|f(u)|}\right]b^2 + \mathrm{O}\big(b^3\big) \\ &= A(\theta)b + \frac{1}{2}A_1(\theta)b^2 + \mathrm{O}\big(b^3\big). \end{aligned} \tag{1.7}$$

For $|b|$ small we have from (1.4)

$$\frac{\mathrm{d}b}{\mathrm{d}\theta} = R(\theta, b) \tag{1.8}$$

which is a T-periodic equation. From Lemma 1.2 we know that the function R is C^{r-1} in (θ, b) and C^r in b. Let $b(\theta, a)$ denote the solution of (1.8) with $b(0, a) = a$. We have:

LEMMA 1.3. $P(a) = b(T, a)$.

PROOF. Consider the equation

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = 1 + g_1\big(\theta, b(\theta, a)\big).$$

It has a unique solution $\theta = \theta(t, a)$ satisfying $\theta(0, a) = 0$. From (1.7) it implies $b(\theta, 0) = 0$. This yields $\theta(T, 0) = T$, $\frac{\partial\theta}{\partial t}(T, 0) = 1$. Hence, by the implicit function theorem a unique function $\tilde{\tau}(a) = T + \mathrm{O}(a)$ exists such that $\theta(\tilde{\tau}, a) = T$.

For $x_0 = u(0) + Z(0)a$, we have by (1.3)

$$\varphi(t, x_0) = u\big(\theta(t, a)\big) + Z\big(\theta(t, a)\big)b\big(\theta(t, a), a\big).$$

In particular,

$$\varphi(\tilde{\tau}, x_0) = u(T) + Z(T)b(T, a) = u(0) + Z(0)b(T, a).$$

Thus, it follows from Lemma 1.1 that $\tau = \tilde{\tau}$ and $P(a) = b(T, a)$.

The proof is completed. □

For $|a|$ small we can write

$$b(\theta, a) = b_1(\theta)a + b_2(\theta)a^2 + \mathrm{O}\big(a^3\big),$$

where $b_1(0) = 1, b_2(0) = 0$. By (1.7) and (1.8) one can obtain $b_1' = Ab_1, b_2' = Ab_2 + \frac{1}{2}A_1 b_1^2$, which give

$$b_1(\theta) = \exp\int_0^\theta A(s)\,\mathrm{d}s, \qquad b_2(\theta) = b_1(\theta)\int_0^\theta \frac{1}{2}A_1(s)b_1(s)\,\mathrm{d}s.$$

Then by Lemma 1.3 we have

$$P'(0) = b_1(T) = \exp\int_0^T A(s)\,\mathrm{d}s = \exp\int_0^T \operatorname{tr} f_x\big(u(t)\big)\,\mathrm{d}t,$$
$$P''(0) = 2b_2(T) = b_1(T)\int_0^T A_1(s)b_1(s)\,\mathrm{d}s.$$

Thus, noting (1.7) we obtain the following theorem.

THEOREM 1.1. *Suppose P is a Poincaré map of* (1.1) *at $p \in L$. Then*

(i) $P'(0) = \exp\oint_L \operatorname{div} f\,\mathrm{d}t, \quad \operatorname{div} f = \operatorname{tr} f_x,$

(ii) $P''(0) = P'(0)\int_0^T \mathrm{e}^{\int_0^t A(s)\,\mathrm{d}s}\left[Z^T(t)B(t)Z(t) - \dfrac{2A(t)C(t)}{|f(u(t))|}\right]\mathrm{d}t.$

Hence, L is stable (*unstable*) *if $I(L) = \oint_L \operatorname{div} f\,\mathrm{d}t < 0$ (> 0).*

We remark that Theorem 1.1 remains true in the case of counter clockwise orientation of L.

EXAMPLE 1.1. Consider the quadratic system

$$\dot{x} = -y(1 + cx) - \left(x^2 + y^2 - 1\right),$$
$$\dot{y} = x(1 + cx), \quad 0 < c < 1.$$

This system has the circle L: $x^2 + y^2 = 1$ as its limit cycle. We claim that the cycle is unstable.

In fact, we have

$$I(L) = \oint_L (-2x - cy)\,\mathrm{d}t = \oint_L \left(\frac{c\,\mathrm{d}x}{1 + cx} - \frac{2\,\mathrm{d}y}{1 + cx}\right)$$
$$= \iint_{x^2+y^2 \leqslant 1} \frac{\mathrm{d}x\,\mathrm{d}y}{(1 + cx)^2} > 0.$$

EXAMPLE 1.2. The system

$$\dot{x} = -y - x\left(x^2 + y^2 - 1\right)^2,$$
$$\dot{y} = x - y\left(x^2 + y^2 - 1\right)^2$$

has a unique limit cycle given by L: $(x, y) = (\cos t, \sin t), 0 \leqslant t \leqslant 2\pi$. For the system, it is easy to see that $v(\theta) = (-\sin\theta, \cos\theta)^{\mathrm{T}}$, $Z(\theta) = (-\cos\theta, -\sin\theta)^{\mathrm{T}}$. By Lemma 1.2 and (1.6) we then have

$$A(\theta) = 0, \quad B(\theta) = \begin{pmatrix} 8\cos^2\theta & 8\sin\theta\cos\theta \\ 8\sin\theta\cos\theta & 8\sin^2\theta \end{pmatrix}.$$

Thus from Theorem 1.1 it follows $P'(0) = 1$, $P''(0) = 16\pi$. This shows that L is a limit cycle of multiplicity 2.

From (1.6) and formulas for $P'(0)$ and $P''(0)$ in Theorem 1.1 the derivatives $P'(0)$ and $P''(0)$ are independent of the choice of the cross section l. This fact suggests that the stability and the multiplicity of a limit cycle should have the same property. Below we will prove this in detail even if the cross section l is taken as a C^r smooth curve.

To do this, let L be a limit cycle of (1.1) as before and let l_1 be a C^r curve which has an intersection point $p_1 \in L$ with L and is not tangent to L at p_1. Then it can be represented as

$$l_1\colon\ x = p_1 + q(a), \quad q(0) = 0,\ \det\left(f(p_1), q'(0)\right) > 0,$$

where $q : (-\varepsilon, \varepsilon) \to \mathbb{R}$ is C^r for a constant $\varepsilon > 0$. The condition $\det(f(p_1), q'(0)) > 0$ means that the point $p_1 + q(a)$ is outside L if and only if $a > 0$.

In the same way as Lemma 1.1 we can prove that there exist two C^r functions

$$P_1, \tau_1 : (-\varepsilon, \varepsilon) \to \mathbb{R}, \quad P_1(0) = 0, \quad \tau_1(0) = T$$

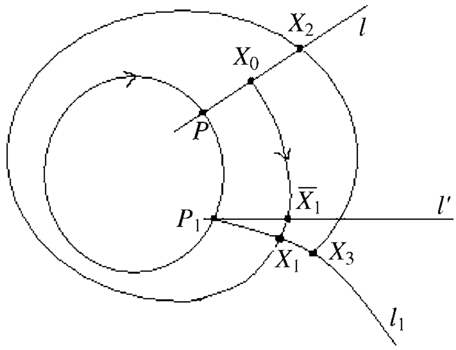

Fig. 2. Two Poincaré maps.

such that

$$\varphi\big(\tau_1(a), p_1 + q(a)\big) = p_1 + q\big(P_1(a)\big) \in l_1. \tag{1.9}$$

This yields another Poincaré map $P_1 : (-\varepsilon, \varepsilon) \to \mathbb{R}$.

LEMMA 1.4. *Let P and P_1 be two Poincaré maps defined by* (1.2) *and* (1.9), *respectively. Then there exists a C^r function $h_1 : (-\varepsilon, \varepsilon) \to \mathbb{R}$ with $h_1(0) = 0$, $h_1'(0) > 0$ such that $h_1 \circ P = P_1 \circ h_1$.*

PROOF. Since $p = u(0)$ we can suppose $p_1 = u(t_1)$ for some $t_1 \in [0, T)$. Similar to Lemma 1.1 again, there exist two C^r functions h_1 and τ^*, both from $(-\varepsilon, \varepsilon)$ to $\mathbb{R}$, with $h_1(0) = 0$ and $\tau^*(0) = t_1$ such that

$$\varphi\big(\tau^*(a), p + aZ_0\big) = p_1 + q\big(h_1(a)\big) \in l_1. \tag{1.10}$$

See Fig. 2. Let $x_0 = p + aZ_0$, $x_1 = \varphi(\tau^*(a), x_0)$, $x_2 = \varphi(\tau(a), x_0)$. By (1.10) and (1.2) we have $x_1 = p_1 + q(a_1)$, $a_1 = h_1(a)$ and $x_2 = p + P(a)Z_0$. Hence, by (1.9) and (1.10) we have

$$\varphi\big(\tau_1(a_1), x_1\big) = p_1 + q\big(P_1(a_1)\big),$$
$$\varphi\big(\tau^*\big(P(a)\big), x_2\big) = p_1 + q(a_2), \quad a_2 = h_1\big(P(a)\big).$$

On the other hand, by the flow property of φ we have

$$x_3 = \varphi\big(\tau_1(a_1), x_1\big) = \varphi\big(\tau_1(a_1) + \tau^*(a), x_0\big) = \varphi\big(\tau^*\big(P(a)\big) + \tau(a), x_0\big)$$
$$= \varphi\big(\tau^*\big(P(a)\big), x_2\big),$$

which, together with the above, follows that $q(P_1(a_1)) = q(a_2)$ or $a_2 = P_1(a_1)$. Hence $h_1 \circ P = P_1 \circ h_1$.

It needs only to prove $h_1'(0) > 0$. Let $a \geqslant 0$. Introduce one more cross section below

$$l'\colon\ x = u(t_1) + Z(t_1)a, \quad 0 \leqslant a \leqslant \varepsilon.$$

Let $\tilde{\tau}_1(a) = t_1 + \mathrm{O}(a)$ be such that $\theta(\tilde{\tau}_1, a) = t_1$. By (1.3) we have

$$\overline{x}_1 = \varphi(\tilde{\tau}_1, x_0) = u(t_1) + Z(t_1)b(t_1, a) \in l'.$$

Then $b(t_1, a) = |p_1\overline{x}_1|$. By the proof of Lemma 1.3,

$$\frac{\partial b}{\partial a}(t_1, 0) = \exp\int_0^{t_1} A(s)\,\mathrm{d}s > 0.$$

Consider the triangle formed by points p_1, x_1 and $\overline{x}_1$. There exists a point x^* on the orbital arc $\widehat{x_1\overline{x}_1}$ such that $f(x^*)$ is parallel to the side $x_1\overline{x}_1$. Since the arc $\widehat{x_1\overline{x}_1}$ approaches p_1 as $a \to 0$ we have $x^* \to p_1$, $f(x^*) \to f(p_1)$ as $a \to 0$. Hence, if we let α_1 denote the angle between sides $p_1\overline{x}_1$ and $\overline{x}_1x_1$, and α_2 the angle between sides p_1x_1 and $\overline{x}_1x_1$, then we have $\alpha_1 \to \frac{\pi}{2}, \alpha_2 \to \alpha_0$ as $a \to 0$, where $\alpha_0 \in (0, \frac{\pi}{2}]$ is the angle between the vectors $f(p_1)$ and $q'(0)$. That is, α_0 is the angle between L and l_1 at p_1. By the Sine theorem, it follows

$$\frac{|p_1\overline{x}_1|}{\sin\alpha_2} = \frac{|p_1x_1|}{\sin\alpha_1}, \quad \text{or}$$

$$\bigl|q\bigl(h_1(a)\bigr)\bigr| = \frac{\sin\alpha_1}{\sin\alpha_2}b(t_1, a) = \frac{a}{\sin\alpha_0}\exp\int_0^{t_1} A(s)\,\mathrm{d}s\bigl(1 + \mathrm{O}(a)\bigr).$$

On the other hand, $q(h_1(a)) = q'(0)h_1'(0)a + \mathrm{O}(a^2)$ which gives

$$\bigl|q\bigl(h_1(a)\bigr)\bigr| = \bigl|q'(0)\bigr| \cdot \bigl|h_1'(0)\bigr|a + \mathrm{O}\bigl(a^2\bigr), \quad a > 0.$$

Hence, we obtain

$$\bigl|h_1'(0)\bigr| = \frac{1}{|q'(0)|\sin\alpha_0}\exp\int_0^{t_1} A(s)\,\mathrm{d}s \neq 0.$$

Noting that $h_1(a) > 0$ for $a > 0$ we have $h_1'(0) > 0$. The proof is completed. □

COROLLARY 1.1. *The stability and the multiplicity of the limit cycle L are independent of the choice of cross sections.*

PROOF. By Lemma 1.4 we have

$$h_1'(\overline{a})\bigl[P(a) - a\bigr] = P_1\bigl(h_1(a)\bigr) - h_1(a), \tag{1.11}$$

where $\overline{a}$ lies between a and $P(a)$. By (1.11) and Definition 1.3, the stability of L under P is the same as that of it under P_1. Also, if the limit cycle L has multiplicity k under P, then $P(a) - a = c_k a^k + O(a^{k+1})$ for some $c_k \neq 0$. It follows from (1.11) that

$$P_1(a) - a = \overline{c}_k a^k + O\big(a^{k+1}\big), \quad \overline{c}_k = \frac{c_k}{[h_1'(0)]^{k-1}}.$$

Therefore, L has the same multiplicity under P_1. This ends the proof. □

In the following we discuss the relation of equivalence between two planar systems. Consider (1.1) and another system of the form

$$\dot{y} = g(y), \tag{1.12}$$

where $g : D \to \mathbb{R}^2$ is C^r $(r \geqslant 1)$ on a region $D \subset \mathbb{R}^2$.

DEFINITION 1.5. Let $U \subset G$, $V \subset D$ be two regions. Two planar systems defined by vector fields $f|_U$ and $g|_V$ are said to be C^k $(1 \leqslant k \leqslant r)$ *equivalent* if there exists a C^k diffeomorphism $h : U \to V$ which takes orbits of (1.1) on U to orbits of (1.12) on V preserving their orientation.

Let $\psi(t, y)$ denote the flow generated by (1.12). Then under the condition of Definition 1.5 a C^k function $s(t, x)$ exists with $s(0, x) = 0$, $\frac{\partial s}{\partial t} > 0$ such that

$$h\big(\varphi(t, x)\big) = \psi\big(s(t, x), h(x)\big), \tag{1.13}$$

as long as $\varphi(t, x) \in U$.

If the functions f and g have the following relationship:

$$g\big(h(x)\big) = \mathrm{D}h(x) f(x),$$

then the system (1.12) is obtained from (1.1) by making the coordinate transformation $y = h(x)$. In this case, (1.13) becomes $h \circ \varphi^t = \psi^t \circ h$. If f and g satisfy $g(x) = K(x) f(x)$ where $K : \mathbb{R}^2 \to \mathbb{R}$ is a C^r positive function, then orbits of (1.1) and (1.12) are identical, and the flows φ and ψ satisfy

$$K\big(\varphi(t, x)\big) \frac{\mathrm{d}\varphi}{\mathrm{d}t} = g\big(\varphi(t, x)\big), \qquad \frac{\mathrm{d}\psi}{\mathrm{d}s} = g\big(\psi(s, x)\big).$$

Hence, we are to sink a function $s(t, x)$ with $s(0, x) = 0$ and satisfying

$$\varphi(t, x) = \psi\big(s(t, x), x\big).$$

Differentiating the equality in t gives

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{\mathrm{d}\psi}{\mathrm{d}s} \frac{\mathrm{d}s}{\mathrm{d}t}.$$

Substituting the equality above into the previous equations yields

$$\frac{\mathrm{d}\psi}{\mathrm{d}s} = K\big(\varphi(t,x)\big)\frac{\mathrm{d}\varphi}{\mathrm{d}t} = K\big(\varphi(t,x)\big)\frac{\mathrm{d}\psi}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t}.$$

Thus, the function s should satisfy $\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{1}{K(\varphi(t,x))}$, which has the solution

$$s(t,x) = \int_0^t \frac{\mathrm{d}t}{K(\varphi(t,x))}.$$

With this choice of $s(t,x)$ above one can prove $\varphi(t,x) = \psi(s(t,x),x)$ by the uniqueness of initial solutions.

For C^k equivalent systems we have the following lemma.

LEMMA 1.5. *Suppose* (1.1) *and* (1.12) *are C^k equivalent under a C^k diffeomorphism $h: U \to V$, $1 \leqslant k \leqslant r$. Let $L \subset U$ and $L_1 = h(L) \subset V$ be limit cycles of* (1.1) *and* (1.12) *respectively. Then*:

(i) *The cycles L and L_1 have the same multiplicity.*

(ii) *The inner* (*outer*) *stability of L is the same as the inner* (*outer*) *stability of L_1 if $h((Int.L) \cap U) \subset Int.L_1$, and the inner* (*outer*) *stability of L is the same as the outer* (*inner*) *stability of L_1 if $h((Int.L) \cap U) \subset Ext.L_1$.*

PROOF. For the limit cycle L, take a cross section l at $p \in L$ as before. Then the curve $l_1 = h(l)$ is a cross section of L_1 at $p_1 = h(p)$. Let

$$Z_1 = \frac{1}{|g(p_1)|}\big(-g_2(p_1), g_1(p_1)\big)^{\mathrm{T}}, \quad Z_1' = h_x(p)Z_0.$$

Note that $h(p + aZ_0) = p_1 + Z_1'a + \mathrm{O}(a^2)$. It is easy to see that the vector Z_1' is tangent to l_1 at p_1. According to the orientation of L_1 and the direction of Z_1' there are four cases to consider as follows:

- Case 1. $Z_1 \cdot Z_1' > 0$ with L_1 clockwise oriented;
- Case 2. $Z_1 \cdot Z_1' < 0$ with L_1 clockwise oriented;
- Case 3. $Z_1 \cdot Z_1' > 0$ with L_1 counter clockwise oriented;
- Case 4. $Z_1 \cdot Z_1' < 0$ with L_1 counter clockwise oriented.

See Fig. 3.

By (1.13) the periods T of L and T_1 of L_1 have the relation $T_1 = s(T, p)$. Let P be the Poincaré map near L defined by (1.2). The cross section l_1 has a representation

$$y = h(p + aZ_0) = p_1 + q(a),$$

where $q(a) = h(p + aZ_0) - h(p) = Z_1'a + \mathrm{O}(a^2)$. Introduce a function q_1 as follows:

$$q_1(a) = \begin{cases} q(a) & \text{for case 1 or 3,} \\ q(-a) & \text{for case 2 or 4.} \end{cases}$$

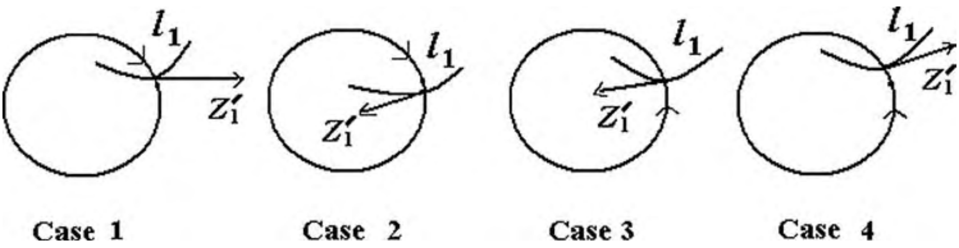

Fig. 3. Four possible cases for L_1 and Z_1'.

Then l_1 can be rewritten as l_1: $y = p_1 + q_1(a)$, $|a| \ll 1$ with q_1 satisfying $Z_1' \cdot q_1'(0) > 0$. On l_1 we can define a Poincaré map P_1 near L_1 by

$$\psi\big(\tau_1(a), p_1 + q_1(a)\big) = p_1 + q_1\big(P_1(a)\big) \in l_1, \tag{1.14}$$

where $\tau_1(a) = T_1 + \mathrm{O}(a)$ is the time of the first return to l_1.

Since $h(p + aZ_0) = p_1 + q(a) = p_1 + q_1((-1)^{i-1}a)$ for case i, we have

$$h\big(p + P(a)Z_0\big) = p_1 + q\big(P(a)\big) = p_1 + q_1\big((-1)^{i-1}P(a)\big) \in l_1$$

and hence by (1.2)

$$h\big(\varphi\big(\tau(a), p + aZ_0\big)\big) = p_1 + q_1\big((-1)^{i-1}P(a)\big) \in l_1. \tag{1.15}$$

On the other hand, by (1.13) we have

$$\begin{aligned} h\big(\varphi\big(\tau(a), p + aZ_0\big)\big) &= \psi\big(s\big(\tau(a), p + aZ_0\big), h(p + aZ_0)\big) \\ &= \psi\big(\tau^*(a), p_1 + q_1\big((-1)^{i-1}a\big)\big) \in l_1, \end{aligned} \tag{1.16}$$

for case i, where $\tau^*(a) = s(\tau(a), p + aZ_0) = T_1 + \mathrm{O}(a)$.

Hence, comparing (1.14) with (1.16) we obtain

$$\tau^*(a) = \tau_1\big((-1)^{i-1}a\big), \qquad h\big(\varphi\big(\tau(a), p + aZ_0\big)\big) = p_1 + q_1\big(P_1\big((-1)^{i-1}a\big)\big).$$

Therefore, it follows from (1.15) that $(-1)^{i-1}P(a) = P_1((-1)^{i-1}a)$ for case i. Thus for case i we have

$$a\big(P(a) - a\big) = (-1)^{i-1}a\big[P_1\big((-1)^{i-1}a\big) - (-1)^{i-1}a\big].$$

Hence, similar to Corollary 1.1, one can prove easily that L and L_1 have the same multiplicity. Furthermore, they have the same stability for cases 1 and 4. However, for cases 2 and 3, the inner (outer) stability of L is the same as the outer (inner) stability of L_1.

Then noting that $h((Int.L) \cap U) \subset Int.L_1$ for cases 1 and 4, and $h((Int.L) \cap U) \subset Ext.L_1$ for cases 2 and 3, the proof follows from Corollary 1.1. □

EXAMPLE 1.3. Consider the system

$$\dot{x} = y, \qquad \dot{y} = -x + y\big(x^2 + y^2 - 1\big).$$

By Theorem 1.1, the system has a unique limit cycle L: $x^2 + y^2 = 1$, and it is hyperbolic and unstable.

Define four functions

$$h_1(x, y) = (x, y)^{\mathrm{T}}, \quad h_2(x, y) = \left(\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2}\right)^{\mathrm{T}},$$
$$h_3(x, y) = \left(\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2}\right)^{\mathrm{T}}, \quad h_4(x, y) = (y, x)^{\mathrm{T}},$$

and a region $U = \{(x, y)\colon x^2 + y^2 > \frac{1}{2}\}$.

Let

$$\dot{u} = g_{i1}(u, v), \qquad \dot{v} = g_{i2}(u, v)$$

denote the system obtained from the previous cubic system by making the coordinate transformation $(u, v)^{\mathrm{T}} = h_i(x, y)$, $(x, y) \in U$. Note that the unit circle is invariant under each h_i. It is evident that the case i in Fig. 3 occurs with $L_1 = L$ for each $i \in \{1, 2, 3, 4\}$.

This example shows that each case in Fig. 3 can happen. In particular, the orientation of a limit cycle may be changed under a coordinate transformation.

1.3. *Perturbations of a limit cycle*

In the rest, we return to the perturbations of a limit cycle.

Consider the following system

$$\dot{x} = f(x) + F(x, \mu) \tag{1.17}$$

where $\mu \in \mathbb{R}^m$ is a vector parameter with $m \geqslant 1$, and $F : G \times \mathbb{R}^m \to \mathbb{R}^2$ is a C^r function with $F(x, 0) = 0$. Thus, system (1.1) is the unperturbed system of (1.17). As before, let (1.1) have a limit cycle L: $x = u(t)$, $0 \leqslant t \leqslant T$. Then completely similar to Lemma 1.2 we have the following lemma.

LEMMA 1.6. *The transformation* (1.3) *carries* (1.17) *into the system*

$$\begin{aligned} \dot{\theta} &= 1 + g_1(\theta, b) + h(\theta, b)F\big(u(\theta) + Z(\theta)b, \mu\big), \\ \dot{b} &= A(\theta)b + g_2(\theta, b) + Z^{\mathrm{T}}(\theta)F\big(u(\theta) + Z(\theta)b, \mu\big), \end{aligned} \tag{1.18}$$

where the functions A, g_1, g_2 and h are the same as those given in Lemma 1.2.

By (1.18) we have

$$\frac{\mathrm{d}b}{\mathrm{d}\theta} = R(\theta, b, \mu), \tag{1.19}$$

where $R(\theta, b, \mu) = A(\theta)b + Z^{\mathrm{T}}(\theta)F_\mu(u(\theta), 0)\mu + \mathrm{O}(|b, \mu|^2)$.

Let $b(0, a, \mu)$ denote the solution of (1.19) with $b(0, a, \mu) = a$. We can write

$$b(\theta, a, \mu) = b_1(\theta)a + b_0(\theta)\mu + \mathrm{O}\big(|a, \mu|^2\big),$$

where $b_1(0) = 1$, $b_0(0) = 0$. Inserting the above into (1.19) yields

$$b_1' = Ab_1, \quad b_0' = Ab_0 + Z^{\mathrm{T}}F_\mu\big(u(s), 0\big).$$

It follows that $b_1(\theta) = \exp\int_0^\theta A(s)\,\mathrm{d}s$ as before, and

$$b_0(\theta) = b_1(\theta)\int_0^{\mathrm{T}} b_1^{-1}(s)Z^{\mathrm{T}}(s)F_\mu\big(u(s), 0\big)\,\mathrm{d}s.$$

Therefore,

$$b(T, a, \mu) = \exp\int_0^{\mathrm{T}} A(s)\,\mathrm{d}s\left[a + \int_0^{\mathrm{T}} b_1^{-1}(s)Z^{\mathrm{T}}(s)F_\mu\big(u(s), 0\big)\,\mathrm{d}s\,\mu\right] + \mathrm{O}\big(|a, \mu|^2\big). \tag{1.20}$$

On the other hand, using (1.2) we can define a Poincaré map $P(a, \mu)$ similarly. By Lemma 1.3 it follows

$$P(a, \mu) = b(T, a, \mu). \tag{1.21}$$

Obviously, for $|\mu|$ small (1.17) has a limit cycle near L_1 if and only if P has a fixed point in a near $a = 0$.

The simplest case is that L is hyperbolic. In this case, L will persist under perturbations. In other words, we have:

THEOREM 1.2. *Let L be hyperbolic. Then there exist $\varepsilon > 0$ and a neighborhood U of L such that* (1.17) *has a unique limit cycle in U for $|\mu| < \varepsilon$. Moreover, the limit cycle is also hyperbolic and has the same stability as L.*

PROOF. Since L is hyperbolic we have $P_a'(0, 0) \neq 1$, or equivalently $I(L) = \oint_L \operatorname{div} f\,\mathrm{d}t \neq 0$. By (1.20) and (1.21),

$$P(a, \mu) - a = \big(\mathrm{e}^{I(L)} - a\big)a + N_0\mu + \mathrm{O}\big(|a, \mu|^2\big), \tag{1.22}$$

where $N_0 = \mathrm{e}^{I(L)}\int_0^{\mathrm{T}} \exp(-\int_0^t A(s)\,\mathrm{d}s)Z^{\mathrm{T}}(s)F_\mu(u(s), 0)\,\mathrm{d}s$.

For (a, μ) near zero applying the implicit function theorem to the equation $P(a, \mu) - a = 0$ we find a unique fixed point $a = a^*(\mu) = \mathrm{O}(\mu)$ of P. This means that for $|\mu|$ small (1.17) has a unique limit cycle, denoted by L_μ, near L. The cycle has a representation $\varphi(t, p + a^*(\mu)Z_0, \mu)$ where $\varphi(t, x, \mu)$ denotes the flow of (1.17) with $\varphi(0, x, \mu) = x$. Moreover, by (1.22) we have $P'_a(a^*(\mu), \mu) = \mathrm{e}^{I(L)} + \mathrm{O}(|a^*(\mu), \mu|)$, which gives

$$\big(\mathrm{e}^{I(L)} - 1\big)\big[P'_a\big(a^*(\mu), \mu\big) - 1\big] > 0$$

for all $|\mu|$ small. Then the conclusion follows from Theorem 1.1. This ends the proof. □

Further, for the nonhyperbolic case we have

THEOREM 1.3. *Let L be a nonhyperbolic limit cycle of the C^r system* (1.1) *with $r \geqslant 2$.*

(i) *If L has multiplicity* 2, *then there exist $\varepsilon > 0$, a neighborhood U of L and a C^r function $\Delta(\mu) = \mathrm{O}(\mu)$ such that* (1.17) *has no limit cycles* (*respectively a limit cycle of multiplicity* 2, *two hyperbolic limit cycles*) *in U as $\Delta(\mu) < 0$* (*respectively* $= 0$, > 0) *for $|\mu| < \varepsilon$.*

(ii) *If L has multiplicity k with $3 \leqslant k \leqslant r$, then there exist $\varepsilon > 0$ and a neighborhood U of L such that* (1.17) *has at most k limit cycles in U for $|\mu| < \varepsilon$. Moreover,* (1.17) *has at least a limit cycle in U for $|\mu| < \varepsilon$ if k is odd.*

PROOF. Since $r \geqslant 2$, $I(L) = 0$ and L has multiplicity 2 we can rewrite (1.22) as

$$P(a, \mu) - a = Q_0(\mu) + Q_1(\mu)a + Q_2(\mu)a^2\big(1 + \mathrm{o}(1)\big),$$

where $Q_0(0) = Q_1(0) = 0$, $Q_2(0) \neq 0$.

Let $G(a, \mu) = P(a, \mu) - a$, which is called a bifurcation function of (1.17). By the implicit function theorem a unique C^r function $q(\mu) = \mathrm{O}(\mu)$ exists such that $G_a(q(\mu), \mu) = 0$ for $|\mu|$ small. Then Taylor's formula yields

$$G(a, \mu) = G\big(q(\mu), \mu\big) + \frac{1}{2}G_{aa}\big(q(\mu), \mu\big)\big(a - q(\mu)\big)^2\big(1 + \mathrm{o}(1)\big).$$

Now it is clear that the conclusion (i) follows by taking $\Delta(\mu) = -G(q(\mu), \mu)Q_2(0)$.

To prove the second conclusion let us assume (1.17) has $k + 1$ limit cycles in an arbitrary neighborhood of L for some sufficiently small $\mu \neq 0$. Then the function G has $k + 1$ zero in a for this small μ. From Rolle's theorem, $\frac{\partial G}{\partial a}$ has k zeros in a. Using the same theorem repeatedly we see that $\frac{\partial^k G}{\partial a^k}$ has a zero $a_0(\mu)$ which can be arbitrarily small as μ goes to zero.

On the other hand, we have

$$\frac{\partial^k G}{\partial a^k} = \frac{\partial^k P}{\partial a^k}, \qquad P(a, 0) - a = q_k a^k + \mathrm{o}\big(a^k\big), \quad q_k \neq 0,$$

which implies $\frac{\partial^k G}{\partial a^k} = k!q_k + \mathrm{o}(1) \neq 0$ for (a, μ) near zero. This is a contradiction. Hence, (1.17) has at most k limit cycles near L for $|\mu|$ small.

Finally, if k is odd, we have then $q_k a[P(a,0)-a]>0$ for $|a|=\varepsilon$, where $\varepsilon>0$ is a small constant. Thus, there exists a constant $\delta>0$ such that $q_k a[P(a,\mu)-a]>0$ for $|a|=\varepsilon, |\mu|\leqslant\delta$. Therefore, a function $a^*(\mu)$ exists with $|a^*(\mu)|<\varepsilon$ such that $P(a^*(\mu),\mu)-a^*(\mu)=0, |\mu|<\delta$. The proof is completed. □

EXAMPLE 1.4. Consider

$$\begin{aligned}\dot{x}&=-y+x\big(x^2+y^2-1\big)^2-x\big[\mu_1\big(x^2+y^2\big)+\mu_2\big],\\ \dot{y}&=x+y\big(x^2+y^2-1\big)^2-y\big[\mu_1\big(x^2+y^2\big)+\mu_2\big].\end{aligned}\tag{1.23}$$

For $\mu_1=\mu_2=0$, (1.23) has a unique limit cycle

$$L:\quad x=\cos t,\quad y=\sin t,\quad 0\leqslant t\leqslant 2\pi.$$

Similar to Example 1.2, the limit cycle L has multiplicity 2 with $P_a''(0,0)=-16\pi$. We make a transformation of the form $(x,y)=(1-b)(\cos\theta,\sin\theta)$, $0\leqslant\theta\leqslant 2\pi$, so that (1.23) becomes

$$\dot{\theta}=1,\qquad \dot{b}=-(1-b)\big[h^2-\mu_1 h-(\mu_1+\mu_2)\big],$$

where $h=(1-b)^2-1$. Hence we have

$$\frac{\mathrm{d}b}{\mathrm{d}\theta}=-(1-b)\big[h^2-\mu_1 h-(\mu_1+\mu_2)\big].$$

Obviously, the solution $b(\theta,a,\mu_1,\mu_2)$ of the above equation with $b(0,a,\mu_1,\mu_2)=a$ is 2π-periodic near $b=0$ if and only if the initial data a satisfies

$$G^*(a,\mu_1,\mu_2)\equiv a^2-\mu_1 a-(\mu_1+\mu_2)=0.$$

Let $\Delta(\mu_1,\mu_2)=\mu_1^2+4(\mu_1+\mu_2)$. Then for (μ_1,μ_2) near $(0,0)$, (1.23) has no limit cycles (respectively a unique multiple 2 limit cycle, two hyperbolic limit cycles) if $\Delta(\mu_1,\mu_2)<0$ (respectively $=0$, >0). The equation $\Delta(\mu_1,\mu_2)=0$ defines a curve $\mu_2=-\mu_1-\frac{1}{4}\mu_1^2$ on the (μ_1,μ_2)-plane, which is called a bifurcation curve of saddle-node type. See Fig. 4.

Turn back to system (1.17). Note that

$$A(\theta)=\operatorname{tr} f_x(u)-\frac{\mathrm{d}}{\mathrm{d}\theta}\ln\big|f(u)\big|,\qquad Z(\theta)=\frac{1}{|f(u)|}\big(-f_2(u),f_1(u)\big).$$

The constant N_0 in (1.22) can be written as $N_0=\frac{1}{|f(u(0))|}\mathrm{e}^{I(L)}M$, where

$$M=\int_0^{\mathrm{T}}\mathrm{e}^{-\int_0^t \operatorname{tr} f_x(u(s))\,\mathrm{d}s}f\big(u(t)\big)\wedge F_\mu\big(u(t),0\big)\,\mathrm{d}t,\tag{1.24}$$

with $(a_1,a_2)\wedge(b_1,b_2)=a_1b_2-a_2b_1$, $\mu\in\mathbb{R}$.

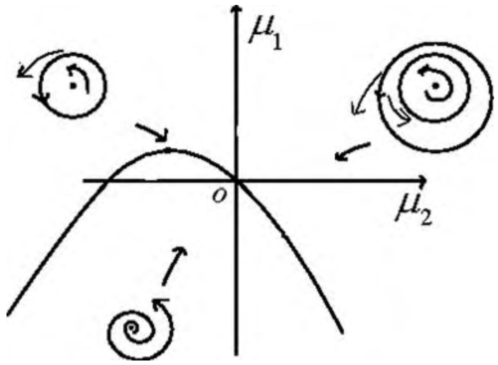

Fig. 4. Bifurcation diagram (saddle-node type) for (1.23).

Thus, if $P(a,0) - a = q_k a^k + \mathrm{o}(a^k)$, we can rewrite (1.22) as

$$G(a,\mu) = P(a,\mu) - a = q_k a^k + \frac{\mathrm{e}^{I(L)}}{|f(u(0))|} M\mu + \mathrm{o}\big(|\mu, a^k|\big). \tag{1.25}$$

Using (1.25) we can prove

THEOREM 1.4. *Let* (1.17) *be analytic with* $\mu \in \mathbb{R}$. *If* $M \neq 0$, *then exist* $\varepsilon > 0$ *and a neighborhood* U *of* L *such that for* $|\mu| < \varepsilon$ *and in the region* U, (1.17) *has*

(i) *a unique limit cycle with multiplicity one if* L *is of odd multiplicity*;
(ii) *two limit cycles with each having multiplicity one for* μ *lying one side of* $\mu = 0$ *and no limit cycles for* μ *lying the opposite side if* L *is of even multiplicity*;
(iii) *no limit cycles if* L *is nonisolated.*

PROOF. By Theorem 1.2 we may suppose $I(L) = 0$. By (1.25), a unique function of the form

$$\mu = -\frac{q_k}{M}\big|f\big(u(0)\big)\big|a^k + \mathrm{O}\big(a^{k+1}\big) = \mu^*(a) \tag{1.26}$$

exists such that $G(a, \mu^*(a)) = 0$. If k is odd with $q_k \neq 0$ the function μ^* has a unique inverse

$$a = \left(\frac{-M\mu}{q_k|f(u(0))|}\right)^{1/k}\big(1 + \mathrm{o}\big(\mu^{1/k}\big)\big) = a^*(\mu),$$

which satisfies $\frac{\partial G}{\partial a}(a^*(\mu), \mu) \neq 0$.

Then the conclusion (i) follows. The conclusion (ii) follows just similarly.

In the case that L is a nonisolated periodic orbit, we have $G(a,0) \equiv 0$ for $|a|$ small. This implies that $\mu^*(a) \equiv 0$ since $G(a,\mu) = 0$ if and only if $\mu = \mu^*(a)$. Hence, for all $|\mu| > 0$ we have $G(a,\mu) \neq 0$. This means that (1.17) has no limit cycles near L for $|\mu| > 0$ small. This ends the proof. □

EXAMPLE 1.5. Consider (1.23) again with (μ_1, μ_2) varying on a straight line. That is, let $\mu_1 = \mu$, $\mu_2 = c\mu$, where c is a constant. Then (1.23) becomes

$$\begin{aligned} \dot{x} &= -y + x\left(x^2 + y^2 - 1\right)^2 - \mu x\left(x^2 + y^2 + c\right), \\ \dot{y} &= x + y\left(x^2 + y^2 - 1\right)^2 - \mu y\left(x^2 + y^2 + c\right). \end{aligned} \tag{1.27}$$

By (1.24) we have $M = (1 + c)2\pi$. Note that $q_2 = \frac{1}{2} P_a''(0, 0) = -8\pi$. The formula (1.26) becomes $\mu^*(a) = \frac{4}{1+c} a^2 + O(a^3)$, which has inverse functions

$$a = a_j(\mu) = \left[\frac{1+c}{4}\mu\right]^{1/2} \left[(-1)^j + O\left(|\mu|^{1/2}\right)\right], \quad j = 1, 2.$$

Thus, when $1 + c \neq 0$, (1.27) has no limit cycles (2 limit cycles) for $(1 + c)\mu < 0$ (> 0). When $(1 + c) = 0$, then (1.27) has always two limit cycles given by $x^2 + y^2 = 1$ and $x^2 + y^2 = 1 - \mu$ for all $|\mu| > 0$ small. This shows that the condition $M \neq 0$ in Theorem 1.4 is somehow necessary.

One can give the bifurcation diagram of (1.27) and compare it with Fig. 4.

REMARK 1.2. A typical result on qualitative theory of differential equations is the so-called Poincaré–Bendixson theorem which says that the positive limit set of a bounded positive semi-orbit is a connected curve consisting of singular points and orbits connecting them. One can find a proof of the theorem in Hale [35], Chicone [14] and Zhang et al. [127]. An important corollary of the theorem is that for an analytic system a positively invariant set with no singular points contains at least one limit cycle. This corollary has many applications to various planar systems to study the existence of a limit cycle. On the other hand, there are also many results on the nonexistence of a limit cycle or the existence of two or more limit cycles. The reader can consult Zhang et al. [127], Ye [116], Ye et al. [117] and Luo et al. [101] for a general theory of limit cycles. For a given planar system with or without parameters it is usually very significant and also difficult to find the number of limit cycles of it. Thousands of papers have been published on this aspect. For recent works, one can see [1,6,8,11,15,18,19,22–26,28–34,97–99].

Results in Theorems 1.1–1.4 are fundamental which can be found in Andronov [2], Chow and Hale [16] and Han [51]. The conclusions in Lemma 1.5 seem very natural, but not obvious. The proof presented here is new.

2. Focus values and Hopf bifurcation

2.1. *Poincaré map and focus value*

In this section we consider local behavior of a planar C^∞ systems near an elementary focus.

After removing the focus to the origin, the system can be written as

$$\begin{aligned}\dot{x} &= ax + by + F(x, y) = f(x, y),\\ \dot{y} &= cx + dy + G(x, y) = g(x, y),\end{aligned} \tag{2.1}$$

where $c \neq 0$, F and $G : \mathbb{R}^2 \to \mathbb{R}$ are C^∞ functions with

$$F(0,0) = G(0,0) = 0, \qquad \frac{\partial(F,G)}{\partial(x,y)}(0,0) = 0,$$

and the eigenpolynomial $h(\lambda) = \lambda^2 - (a+d)\lambda + ad - bc$ has a pair of conjugate complex zero $\alpha \pm \mathrm{i}\beta \neq 0$.

Let

$$C = \begin{pmatrix} 1 & \dfrac{\alpha - a}{\beta} \\ 0 & -\dfrac{c}{\beta} \end{pmatrix}, \qquad C^{-1} = \begin{pmatrix} 1 & \dfrac{\alpha - a}{c} \\ 0 & -\dfrac{\beta}{c} \end{pmatrix}.$$

By using $h(\alpha \pm \mathrm{i}\beta) = 0$, or

$$\alpha^2 - \beta^2 - (a+d)\alpha + ad - bc = 0, \qquad \big(2\alpha - (a+d)\big)\beta = 0,$$

it is easy to see that

$$\begin{aligned} &C^{-1}\begin{pmatrix} a & b \\ c & d \end{pmatrix} C = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix},\\ &\alpha = \frac{1}{2}(a+d), \quad |\beta| = \frac{1}{2}\sqrt{-4bc - (a-d)^2}. \end{aligned} \tag{2.2}$$

Then the linear transformation $(x, y)^{\mathrm{T}} = C(u, v)^{\mathrm{T}}$ carries (2.1) into the form

$$\dot{u} = \alpha u + \beta v + \widetilde{F}(u, v), \qquad \dot{v} = -\beta u + \alpha v + \widetilde{G}(u, v),$$

where $\widetilde{F}, \widetilde{G} = \mathrm{O}(|u, v|^2)$. We call the above system a first order normal form of (2.1).

Let us define a Poincaré map of (2.1) near the origin. For any given $\theta_0 \in [0, 2\pi)$, let

$$Z_0 = (\cos\theta_0, \sin\theta_0)^{\mathrm{T}}, \qquad Z_0^{\perp} = (\sin\theta_0, -\cos\theta_0)^{\mathrm{T}}.$$

The unit vector Z_0 determines a cross section l below

$$l: \quad (x, y)^{\mathrm{T}} = rZ_0, \quad 0 < r < r_0$$

for $r_0 > 0$ small. For $(x_0, y_0)^{\mathrm{T}} = rZ_0 \in l$, let $(x(t, x_0, y_0), y(t, x_0, y_0))$ be the solution of (2.1) with $x(0, x_0, y_0) = x_0$, $y(0, x_0, y_0) = y_0$. Then the formula of constant variation follows:

$$\begin{pmatrix} x(t, x_0, y_0) \\ y(t, x_0, y_0) \end{pmatrix} = \left[\exp\begin{pmatrix} at & bt \\ ct & dt \end{pmatrix}\right]\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \mathrm{O}\big(|x_0, y_0|^2\big). \tag{2.3}$$

Thus,

$$\begin{pmatrix} x(t, x_0, y_0) \\ y(t, x_0, y_0) \end{pmatrix} - \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \left[\exp\begin{pmatrix} at & bt \\ ct & dt \end{pmatrix} - I\right] Z_0 r + \mathrm{O}\big(r^2\big) \equiv rV(t, r).$$

By (2.2) we have

$$V(t, r) = C\left[\exp\begin{pmatrix} \alpha t & \beta t \\ -\beta t & \alpha t \end{pmatrix} - I\right] C^{-1} Z_0 + \mathrm{O}(r).$$

Let $H(t, r) = [V(t, r)]^{\mathrm{T}} Z_0^{\perp}$. Define

$$V(t, 0) = \lim_{r\to 0} V(t, r) = C\left[\exp\begin{pmatrix} \alpha t & \beta t \\ -\beta t & \alpha t \end{pmatrix} - I\right] C^{-1} Z_0.$$

Also, if we allow the variable r to be negative in the definition of l then H is well defined for $|r| < r_0$. It is easy to see that

$$H(T, 0) = \big(\mathrm{e}^{\alpha T} - 1\big) Z_0 Z_0^{\perp} = 0, \qquad H_t(T, 0) = \mathrm{e}^{\alpha T} \beta \neq 0,$$

where $T = \frac{2\pi}{|\beta|}$. Hence, the implicit function theorem yields that a unique C^{∞} function $\tau(r) = T + \mathrm{O}(r)$ exists such that $H(\tau(r), r) = 0$ for $|r| < r_0$. That is to say, for $0 < r < r_0$, $\tau(r)$ is the time of the first return to l by a circle. Therefore, we can write

$$\begin{pmatrix} x(\tau, x_0, y_0) \\ y(\tau, x_0, y_0) \end{pmatrix} = P(r) Z_0 \in l, \quad 0 < r < r_0, \tag{2.4}$$

where $P \in C^{\infty}$ for $|r| < r_0$ with $rP(r) > 0$ for $r \neq 0$.

DEFINITION 2.1. Let (2.1) satisfy (2.2). The function P defined by (2.4) is called a *Poincaré map* of (2.1) near the origin. The map is often written as $P: l \to l$.

DEFINITION 2.2. If there exists $r_0' > 0$ such that $P(r) - r < 0 (> 0)$ for $0 < r < r_0'$, we say the origin is a *stable* (*unstable*) *focus*. If $P(r) - r = 0$ for $0 < r < r_0'$, we say the origin is a *center*.

REMARK 2.1. Similar to Corollary 1.1 we can prove that the stability of the origin is independent of the choice of θ_0 which is the angle defining the section l.

In fact, for any given $\theta_0, \theta_1 \in [0, 2\pi)$ with $\theta_0 \neq \theta_1$, let the corresponding cross sections and Poincaré maps be l_0, l_1 and $P_0 : l_0 \to l_0$ and $P_1 : l_1 \to l_1$, respectively. Similarly, there exist unique C^∞ functions $\tau^*(r) = t_1 + \mathrm{O}(r)$ and $h_1(r) = \mathrm{O}(r)$ such that

$$\begin{pmatrix} x(\tau^*, rZ_0) \\ y(\tau^*, rZ_0) \end{pmatrix} = h_1(r) Z_1, \quad t_1 \in (0, T).$$

Differentiating the equation above in r and using formula of constant variation (2.3) we obtain

$$\left[\exp \begin{pmatrix} at_1 & bt_1 \\ ct_1 & dt_1 \end{pmatrix}\right] Z_0 = h_1'(0) Z_1,$$

which yields $h_1'(0) \neq 0$. It is clear that $h_1(r) > 0$ for $r > 0$. Thus, $h_1'(0) > 0$. Then in the same way as in Lemma 1.4 we have

$$h_1 \circ P_0 = P_1 \circ h_1,$$

which gives the conclusion in Remark 2.1 easily.

LEMMA 2.1. *For any given $\theta_0 \in [0, 2\pi)$ the origin is a stable* (*unstable*) *focus of* (2.1) *if and only if*

$$r\big[P(r) - r\big] < 0 \ (> 0) \quad \textit{for } 0 < |r| < r_0.$$

Hence, *if for some $k \geqslant 1$ it holds that*

$$P(r) - r = 2\pi v_k r^k + \mathrm{O}\big(r^{k+1}\big), \quad v_k \neq 0, \tag{2.5}$$

then k is odd, *and the origin is stable* (*unstable*) *if $v_k < 0$ (> 0)*.

PROOF. Let P_0 denote the function P defined by (2.4). For any $\theta \in [0, 2\pi)$ there exists a unique $\theta_1 \in [0, 2\pi)$ such that

$$\theta_1 = \theta_0 + \pi \quad \text{if } 0 \leqslant \theta_0 < \pi; \qquad \theta_0 = \theta_1 + \pi \quad \text{if } \pi \leqslant \theta_0 < 2\pi.$$

Let P_1 denote the Poincaré map associated to θ_1. We claim that

$$P_0(r) = -P_1(-r), \quad |r| < r_0. \tag{2.6}$$

For definiteness, assume $0 \leqslant \theta_0 < \pi$ and $0 < r < r_0$. Let $Z_1 = (\cos\theta_1, \sin\theta_1)$. Then $Z_1 = -Z_0$ and $(x_0, y_0)^{\mathrm{T}} = rZ_0 = (-r)Z_1$. Hence, by (2.4) we have

$$\begin{pmatrix} x(\tau, x_0, y_0) \\ y(\tau, x_0, y_0) \end{pmatrix} = P_0(r) Z_0 = P_1(-r) Z_1,$$

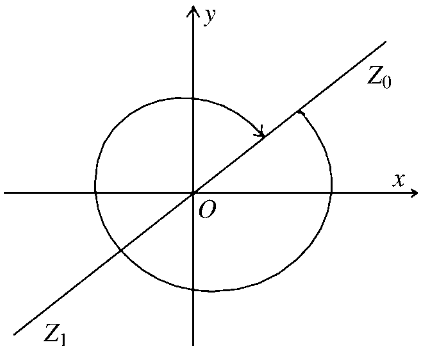

Fig. 5. The Poincaré maps P_0 and P_1 $(b > 0)$.

which gives (2.6). See Fig. 5.

By Definition 2.2, the origin is stable if and only if

$$P_0(r) - r < 0 \quad \text{for } 0 < r < r_0. \tag{2.7}$$

By Remark 2.1, the above inequality is equivalent to

$$P_1(r) - r < 0 \quad \text{for } 0 < r < r_0.$$

Note that by (2.6)

$$P_1(r) - r = -\big[P_0(-r) - (-r)\big].$$

Hence, (2.7) is equivalent to

$$P_0(r) - r > 0 \quad \text{for } 0 < -r < r_0.$$

Then combining (2.7) and the above together we see that (2.7) holds if and only if

$$r\big[P_0(r) - r\big] < 0 \quad \text{for } 0 < |r| < r_0.$$

Thus, if (2.5) holds, the number k must be odd and the sign of v_k determines the stability of the origin. This ends the proof. □

DEFINITION 2.3. Let (2.5) hold with $k = 2m + 1$, $m \geqslant 0$. We call the origin to be *a focus of order m*, and v_k the *mth Lyapunov constant* or *focus value*. The focus at the origin is called *rough or hyperbolic* (*weak or fine*) as $m = 0$ $(m \geqslant 1)$.

REMARK 2.2. Similar to Remark 2.1 and Corollary 1.1, one can prove that the order and the first nonzero Lyapunov constant of the origin are independent of the choice of θ_0 and l.

For the sake of convenience, we will take $\theta_0 = 0$ below. In this case we have $(x_0, y_0) = (r, 0)$, and hence from (2.4) and (2.2)–(2.3) we obtain

$$\begin{aligned} P(r) &= \big(x(\tau, x_0, y_0), y(\tau, x_0, y_0)\big) Z_0 \\ &= \left[C\mathrm{e}^{\alpha\tau} \begin{pmatrix} \cos\beta\tau & \sin\beta\tau \\ -\sin\beta\tau & \cos\beta\tau \end{pmatrix} C^{-1} \begin{pmatrix} r \\ 0 \end{pmatrix} + \mathrm{O}\big(r^2\big) \right]^{\mathrm{T}} Z_0, \end{aligned}$$

which yields $P(r) = r\mathrm{e}^{\alpha T} + \mathrm{O}(r^2)$ since $\tau(0) = T$. Comparing with (2.5) gives that

$$v_1 = \frac{1}{2\pi}\big[\mathrm{e}^{\alpha T} - 1\big] = \frac{1}{2\pi}\big[\mathrm{e}^{\frac{a+d}{2}T} - 1\big].$$

Thus, the origin is stable (unstable) if

$$\operatorname{div}(f, g)|_0 = f_x(0, 0) + g_y(0, 0) < 0 \ (> 0).$$

Next, we suppose $\alpha = 0$ and give a computation formula for v_3. For the purpose it is convenient to use the first order normal form of (2.1). Without loss of generality we may suppose (2.1) has been of the form. In other words, we may assume (2.2) holds with C being the identity. Introducing the polar coordinate to (2.1) we obtain

$$\dot r = \cos\theta F + \sin\theta G, \qquad \dot\theta = -b + (\cos\theta G - \sin\theta F)/r \tag{2.8}$$

and

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = \frac{\cos\theta F + \sin\theta F}{-b + (\cos\theta G - \sin\theta F)/r} \equiv R(\theta, r). \tag{2.9}$$

Let $r(\theta, r_0)$ be the solution of (2.9) with $r(0, r_0) = r_0$. We have

LEMMA 2.2. *Let $a = d = 0, b = -c \neq 0$. Then*

(i) $P(r_0) = r(2\pi, r_0)$ *if* $b < 0$, *and* $P(r(2\pi, r_0)) = r_0$ *if* $b > 0$,

(ii) $v_3 = \frac{1}{2\pi}\operatorname{sgn}(-b)\int_0^{2\pi} R_3(\theta)\,\mathrm{d}\theta$, *where* $R_3(\theta) = \frac{1}{3!}\frac{\partial^3 R}{\partial r^3}(\theta, 0)$.

PROOF. The conclusion (i) can be proved in a similar manner to Lemma 1.3. Let

$$R(\theta, r) = R_2(\theta) r^2 + R_3(\theta) r^3 + \cdots \tag{2.10}$$

and

$$r(\theta, r_0) = r_1(\theta) r_0 + r_2(\theta) r_0^2 + r_3(\theta) r_0^3 + \cdots,$$

where $r_1(0) = 1, r_2(0) = r_3(0) = 0$. Inserting the above solution and (2.10) into (2.9), and then comparing the like powers of r_0 we obtain

$$r_1'(\theta) = 0, \qquad r_2'(\theta) = R_2 r_1^2, \qquad r_3'(\theta) = R_3 r_1^2 + 2R_2 r_1 r_2,$$

or

$$r_1(\theta) = 1, \qquad r_2(\theta) = \int_0^\theta R_2(\theta)\,\mathrm{d}\theta,$$
$$r_3(\theta) = \int_0^\theta \big[R_3(\theta) + 2R_2(\theta)r_2(\theta)\big]\,\mathrm{d}\theta.$$

Hence, if we let

$$\widetilde{P}(r_0) = r(2\pi, r_0) = r_0 + 2\pi\,\tilde{v}_2 r_0^2 + 2\pi\,\tilde{v}_3 r_0^3 + \mathrm{O}\big(r_0^4\big),$$

then

$$\begin{aligned}
\tilde{v}_2 &= \frac{1}{2\pi}\int_0^{2\pi} R_2(\theta)\,\mathrm{d}\theta,\\
\tilde{v}_3 &= \frac{1}{2\pi}\int_0^{2\pi} \big[R_3(\theta) + 2R_2(\theta)r_2(\theta)\big]\,\mathrm{d}\theta\\
&= \frac{1}{2\pi}\left[\int_0^{2\pi} R_3(\theta)\,\mathrm{d}\theta + \int_0^{2\pi} 2r_2(\theta)r_2'(\theta)\,\mathrm{d}\theta\right]\\
&= \frac{1}{2\pi}\int_0^{2\pi} R_3(\theta)\,\mathrm{d}\theta + 2\pi(\tilde{v}_2)^2.
\end{aligned}$$

If $b < 0$, we have $P(r_0) = \widetilde{P}(r_0)$, and by Lemma 2.1, $\tilde{v}_2 = 0$. Thus

$$v_3 = \tilde{v}_3 = \frac{1}{2\pi}\int_0^{2\pi} R_3(\theta)\,\mathrm{d}\theta.$$

If $b > 0$, we have $P(r_0) = \widetilde{P}^{-1}(r_0)$. Note that if $\widetilde{P}(r_0) = r_0 + 2\pi\,\tilde{v}_k r_0^k + \mathrm{O}(r_0^{k+1})$ then

$$\widetilde{P}^{-1}(r_0) = r_0 - 2\pi\,\tilde{v}_k r_0^k + \mathrm{O}\big(r_0^{k+1}\big).$$

Hence, using Lemma 2.1 we have $\tilde{v}_2 = 0$ and

$$v_3 = -\tilde{v}_3 = -\frac{1}{2\pi}\int_0^{2\pi} R_3(\theta)\,\mathrm{d}\theta.$$

This ends the proof. □

Based on the above lemma we can prove:

THEOREM 2.1. *Let $a = d = 0$, $b = -c \neq 0$, and*

$$F(x,y) = \sum_{i+j=2}^{3} a_{ij} x^i y^j + \mathrm{o}\big(|x,y|^3\big),$$

$$G(x,y) = \sum_{i+j=2}^{3} b_{ij} x^i y^j + \mathrm{o}\big(|x,y|^3\big).$$

Then

$$\begin{aligned} v_3 &= \frac{1}{8|b|}\bigg\{3(a_{30}+b_{03}) + a_{12} + b_{21} - \frac{1}{b}\big[a_{11}(a_{20}+a_{02}) - b_{11}(b_{20}+b_{02}) \\ &\quad + 2(a_{02}b_{02} - a_{20}b_{20})\big]\bigg\} \\ &= \frac{1}{16|b|}\bigg\{F_{xxx} + F_{xyy} + G_{yyy} - \frac{1}{b}\big[F_{xy}(F_{xx}+F_{yy}) - G_{xy}(G_{xx}+G_{yy}) \\ &\quad - F_{xx}G_{xx} + F_{yy}G_{yy}\big]\bigg\}\bigg|_{(0,0)}. \end{aligned}$$

PROOF. By (2.9) and (2.10),

$$R_2(\theta) = -b^{-1}P_2(\theta), \qquad R_3(\theta) = -b^{-1}\big[P_3(\theta) + b^{-1}P_2(\theta)S_2(\theta)\big],$$

where

$$\begin{aligned} P_k(\theta) &= \cos\theta F_k(\cos\theta,\sin\theta) + \sin\theta G_k(\cos\theta,\sin\theta), \\ S_k(\theta) &= \cos\theta G_k(\cos\theta,\sin\theta) - \sin\theta F_k(\cos\theta,\sin\theta), \\ F_k(x,y) &= \sum_{i+j=k} a_{ij}x^i y^j, \qquad G_k(x,y) = \sum_{i+j=k} b_{ij}x^i y^j. \end{aligned}$$

It follows directly that

$$\begin{aligned} P_3(\theta) &= (a_{12}+b_{21})\sin^2\theta\cos^2\theta + a_{30}\cos^4\theta + b_{03}\sin^4\theta + K_0(\theta), \\ P_2(\theta)S_2(\theta) &= -a_{02}b_{02}\sin^6\theta + a_{20}b_{20}\cos^6\theta - N_1\sin^4\theta\cos^2\theta \\ &\quad - N_2\sin^2\theta\cos^4\theta + K_1(\theta), \end{aligned}$$

where K_0 and K_1 are 2π-periodic functions with zero mean value over the interval $[0, 2\pi]$, and

$$\begin{aligned} N_1 &= 2a_{02}a_{11} - 2b_{02}b_{11} + a_{02}b_{20} + a_{11}b_{11} + a_{20}b_{02} - a_{02}b_{02}, \\ N_2 &= 2a_{20}a_{11} - 2b_{20}b_{11} - a_{02}b_{20} - a_{11}b_{11} + a_{20}b_{20} - a_{20}b_{02}. \end{aligned}$$

Then we have

$$\frac{1}{2\pi}\int_0^{2\pi} R_3(\theta)\,\mathrm{d}\theta = -\frac{\pi}{2\pi b}\bigg[(a_{30}+b_{03})\frac{3}{4}+(a_{12}+b_{21})\frac{1}{4} \\ -\frac{1}{b}\bigg[(a_{02}b_{02}-a_{20}b_{20})\frac{5}{8}+(N_1+N_2)\frac{1}{8}\bigg]\bigg].$$

Thus, the conclusion follows from Lemma 2.2(ii). □

Along the above line one can find formulas for computing v_5, v_7, etc. See [31] for more detail.

2.2. *Normal form and Lyapunov technique*

Next, we introduce another method (called normal form method) to find focus values.

LEMMA 2.3. *Let* $a=d=0$, $b=-c\neq 0$. *Then for any integer* $m\geqslant 1$ *there exists a polynomial change of variables of the form*

$$(x,y)^{\mathrm{T}} = (u,v)^{\mathrm{T}} + \mathrm{O}\big(|u,v|^2\big) = Q(u,v)$$

which transforms the system (2.1) *into*

$$\begin{aligned} \dot u &= bv + \sum_{j=1}^{m}(a_j u + b_j v)\big(u^2+v^2\big)^j + \mathrm{O}\big(|u,v|^{2m+2}\big),\\ \dot v &= -bu + \sum_{j=1}^{m}(-b_j u + a_j v)\big(u^2+v^2\big)^j + \mathrm{O}\big(|u,v|^{2m+2}\big). \end{aligned} \tag{2.11}$$

PROOF. For convenience we introduce complex variables $z=x+\mathrm{i}y$, $\bar z = x-\mathrm{i}y$, or

$$\begin{pmatrix} z\\ \bar z\end{pmatrix} = \begin{pmatrix} 1 & \mathrm{i}\\ 1 & -\mathrm{i}\end{pmatrix}\begin{pmatrix} x\\ y\end{pmatrix} \tag{2.12}$$

so that (2.1) becomes

$$\begin{aligned} \dot z &= f\left(\frac{z+\bar z}{2},\frac{z-\bar z}{2\mathrm{i}}\right) + \mathrm{i}g\left(\frac{z+\bar z}{2},\frac{z-\bar z}{2\mathrm{i}}\right) \equiv h(z,\bar z),\\ \dot{\bar z} &= f\left(\frac{z+\bar z}{2},\frac{z-\bar z}{2\mathrm{i}}\right) - \mathrm{i}g\left(\frac{z+\bar z}{2},\frac{z-\bar z}{2\mathrm{i}}\right) \equiv \bar h(z,\bar z). \end{aligned} \tag{2.13}$$

It is easy to verify that $\bar h$ is the conjugate of h. Hence, we may neglect the second equation in (2.13), since it can be obtained from the first one by taking complex conjugation.

By (2.3) we can write

$$h(z,\bar z) = -\mathrm{i}bz + \sum_{2\leqslant j+k\leqslant 2m+1} A_{jk} z^j \bar z^k + \mathrm{O}\big(|z|^{2m+2}\big). \tag{2.14}$$

We would like to make a change of variables of the form

$$z = \omega + \sum_{2\leqslant j+k\leqslant 2m+1} C_{jk}\omega^j \overline{\omega}^k = \omega + p(\omega,\overline{\omega}), \tag{2.15}$$

and expect that the resulting equation which has the form in general

$$\dot\omega = -\mathrm{i}b\omega + \sum_{2\leqslant j+k\leqslant 2m+1} B_{jk}\omega^j \overline{\omega}^k + \mathrm{O}\big(|\omega|^{2m+2}\big) \tag{2.16}$$

is as simple as possible.

For the purpose, differentiating both sides of (2.15) in t and using (2.13), (2.14) and (2.16) we obtain

$$\begin{aligned}
&\left[1 + \sum_{j+k=2}^{2m+1} C_{jk} j\omega^{j-1}\overline{\omega}^k\right]\left[-\mathrm{i}b\omega + \sum_{j+k=2}^{2m+1} B_{jk}\omega^j\overline{\omega}^k\right] \\
&\quad + \sum_{j+k=2}^{2m+1} C_{jk} k\omega^j \overline{\omega}^{k-1}\left[\mathrm{i}b\overline{\omega} + \sum_{j+k=2}^{2m+1} \overline{B}_{jk}\omega^k\overline{\omega}^j\right] \\
&\ = -\mathrm{i}b\left[\omega + \sum_{2\leqslant j+k\leqslant 2m+1} C_{jk}\omega^j\overline{\omega}^k\right] \\
&\quad + \sum_{j+k=2}^{2m+1} A_{jk}\big(\omega + p(\omega,\overline{\omega})\big)^j \big(\overline{\omega} + \bar p(\omega,\overline{\omega})\big)^k + \mathrm{O}\big(|\omega|^{2m+1}\big).
\end{aligned}$$

By considering terms of $\omega^j\overline{\omega}^k$ for $j+k=2$ we obtain

$$-\mathrm{i}bC_{jk} j\omega^j\overline{\omega}^k + B_{jk}\omega^j\overline{\omega}^k + \mathrm{i}bC_{jk}k\omega^j\overline{\omega}^k = -\mathrm{i}bC_{jk}\omega^j\overline{\omega}^k + A_{jk}\omega^j\overline{\omega}^k,$$

which yields

$$B_{jk} = A_{jk} + \mathrm{i}bC_{jk}(j-k-1). \tag{2.17}$$

Hence, to nullify the coefficients B_{jk} we can choose

$$C_{jk} = \frac{A_{jk}}{\mathrm{i}b(1+k-j)}$$

if

$$j \neq k+1, \tag{2.18}$$

which always holds for $j+k=2$. Thus it appears that $B_{jk}=0$ for the chosen C_{jk} for $j+k=2$. In other words, all quadratic terms will be nullified in the new equation (2.16) so far if we choose the coefficient of quadratic terms in (2.15) proper.

By equating the coefficients of $\omega^j\overline{\omega}^k$ for $j+k=3$, we obtain

$$B_{jk} = A_{jk} + \mathrm{i}bC_{jk}(j-k-1) + Z_{jk}, \tag{2.19}$$

where Z_{jk} depends on $C_{j'k'}$ with $j'+k'=2$. Thus, in this case, for j and k satisfying (2.18) we can also choose C_{jk} by

$$C_{jk} = \frac{1}{\mathrm{i}b(1+k-j)}[A_{jk}+Z_{jk}]$$

so that $B_{jk}=0$ for the new equation. The only monomial left up to now is $B_{21}\omega^2\overline{\omega}$ which is called a resonant term, and by (2.19) the coefficient C_{21} can be chosen freely, say $C_{21}=0$.

For higher values of $j+k$, the expression (2.19) remains valid, where Z_{jk} depends only on $C_{j'k'}$ with $2 \leqslant j'+k' < j+k$. Hence, by continuing the way above, an appropriate change of variables can be found that eliminates all those monomials $B_{jk}\omega^j\overline{\omega}^k$ for which (2.18) are satisfied. The only monomials which survive, called resonant terms, have the form $B_{k+1,k}\omega^{k+1}\overline{\omega}^k$. Therefore, eventually, Eq. (2.16) takes the form

$$\dot{\omega} = -\mathrm{i}b\omega + \sum_{k=1}^{m} B_{k+1,k}\omega^{k+1}\overline{\omega}^k + \mathrm{O}\big(|\omega|^{2m+2}\big) = R(\omega,\overline{\omega}).$$

More precisely, there exists a change of variables

$$z = \omega + p(\omega,\overline{\omega}), \qquad \overline{z} = \overline{\omega} + \bar{p}(\omega,\overline{\omega})$$

which carries (2.13) into the form of

$$\dot{\omega} = R(\omega,\overline{\omega}), \qquad \dot{\overline{\omega}} = \overline{R}(\omega,\overline{\omega}).$$

Substituting $\omega = u+\mathrm{i}v$, $\overline{\omega} = u-\mathrm{i}v$ into the above gives the desired system (2.11) with

$$a_j = \operatorname{Re} B_{j+1,j}, \quad b_j = -\operatorname{Im} B_{j+1,j}.$$

The coordinate change from (2.1) to (2.11) is given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & \mathrm{i} \\ 1 & -\mathrm{i} \end{pmatrix}^{-1} \begin{pmatrix} \omega + p(\omega,\overline{\omega}) \\ \overline{\omega} + \bar{p}(\omega,\overline{\omega}) \end{pmatrix}, \qquad \begin{pmatrix} \omega \\ \overline{\omega} \end{pmatrix} = \begin{pmatrix} 1 & \mathrm{i} \\ 1 & -\mathrm{i} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

where (x, y) and (u, v) are both real variables, and

$$\begin{pmatrix} 1 & \mathrm{i} \\ 1 & -\mathrm{i} \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{\mathrm{i}}{2} & \frac{\mathrm{i}}{2} \end{pmatrix}.$$

The proof is completed. □

We call the system (2.11) the normal form of (2.1) of order $2m + 1$. It is easy to see that in polar coordinates the normal form equation takes the form

$$\begin{aligned} \dot{r} &= a_1 r^3 + \cdots + a_m r^{2m+1} + \mathrm{O}\big(r^{2m+2}\big), \\ \dot{\theta} &= -\big(b + b_1 r^2 + \cdots + b_m r^{2m}\big) + \mathrm{O}\big(r^{2m+2}\big). \end{aligned} \tag{2.20}$$

For a relationship between focus values appeared in (2.5) and the constants a_j in (2.20) we have

LEMMA 2.4. *Let* $a = d = 0$, $b = -c \neq 0$. *Then* (2.5) *holds with* $k = 2m + 1 \geqslant 3$ *if and only if* $a_j = 0$, $j = 1, \ldots, m - 1$, $a_m \neq 0$. *Moreover,* $v_{2m+1} = a_m/|b|$ *as* $a_j = 0$, $j = 1, \ldots, m - 1$.

PROOF. Let P and P^* denote respectively the Poincaré maps of (2.1) and (2.11) both associated to $\theta_0 = 0$. For $r_0 > 0$ small, let

$$\begin{aligned} &l\colon\ (x, y) = (r, 0), \quad 0 < r < r_0, \\ &l_1 = Q^{-1}(l) = \big\{\big(q_1(r), q_2(r)\big), 0 < r < r_0\big\}, \end{aligned}$$

where Q^{-1} is the inverse of the transformation Q appeared in Lemma 2.3, and $q_1(r) = r + \mathrm{O}(r^2)$, $q_2(r) = \mathrm{O}(r^2)$. Let $P_1 : l_1 \to l_1$ denote the Poincaré map of (2.11) defined by a similar way to (1.14). Then by the proof of Lemma 1.5 with the case 1 we have

$$P_1 = P. \tag{2.21}$$

Let $l' = \{(u, v)\colon\ u = r, v = 0, 0 < r < r_0\}$ which is tangent to l_1 at the origin. Denote by $(u(t, r),\ v(t, r))$ the solution of (2.11) with $(u(0, r),\ v(0, r)) = (r, 0)$. It is easy to see from (2.11) that

$$u(t, r) = r\cos bt + \mathrm{O}\big(r^2\big), \qquad v(t, r) = -r\sin bt + \mathrm{O}\big(r^2\big).$$

Note that l_1 can be represented as

$$l_1 = \big\{\big(r', c\big(r'\big)\big),\ 0 < r' < q_1(r_0)\big\}$$

where c is a C^∞ function satisfying $c(0) = c'(0) = 0$. Consider the function

$$K(t, r) = c\big(u(t, r)\big) - v(t, r) = rK_1(t, r),$$

where $K_1(t,r) = \sin bt + \mathrm{O}(r)$. By the implicit function theorem, there exists a C^∞ function $\tau^*(r) = \mathrm{O}(r)$ such that $K_1(\tau^*(r), r) \equiv 0$. Then similar to (1.10), there exists a unique function $h_1(r)$ such that

$$\big(u\big(\tau^*(r), r\big), v\big(\tau^*(r), r\big)\big) = \big(q_1\big(h_1(r)\big), q_2\big(h_1(r)\big)\big) \in l_1. \tag{2.22}$$

Moreover, the function satisfies

$$h_1 \circ P^* = P_1 \circ h_1. \tag{2.23}$$

By (2.22) it follows that

$$q_1(h_1) = u(\tau^*, r) = r \cos b\tau^* + \mathrm{O}\big(r^2\big) = r + \mathrm{O}\big(r^2\big),$$

which gives that

$$h_1 = q_1^{-1}\big(r + \mathrm{O}\big(r^2\big)\big) = r + \mathrm{O}\big(r^2\big).$$

Therefore, by (2.21) and (2.23) we obtain

$$h_1 \circ P^* = P \circ h_1, \quad h_1(r) = r + \mathrm{O}\big(r^2\big). \tag{2.24}$$

Then, it is easy to verify that (2.5) holds if and only if

$$P^* - r = 2\pi v_k r^k + \mathrm{O}\big(r^{k+1}\big), \quad v_k \neq 0. \tag{2.25}$$

On the other hand, let $a_j = 0$, $j = 1, \dots, n-1$, $a_n \neq 0$. We then have from (2.20)

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = -\frac{a_n}{b} r^{2n+1} + \mathrm{O}\big(r^{2n+2}\big)$$

which has the solution

$$r(\theta, r_0) = r_0 - \frac{a_n}{b}\theta r_0^{2n+1} + \mathrm{O}\big(r_0^{2n+2}\big).$$

Thus,

$$r(2\pi, r_0) = r_0 - \frac{2\pi a_n}{b} r_0^{2n+1} + \mathrm{O}\big(r_0^{2n+2}\big).$$

Hence, by Lemma 2.2 we have

$$P^*(r) = r + \frac{2\pi a_n}{|b|} r^{2n+1} + \mathrm{O}\big(r^{2n+2}\big).$$

Obviously, for $n \geqslant 1$ the above holds if and only if (2.25) holds with $k = 2m+1 = 2n+1$. This ends the proof. □

We remark that some algorithms have been established for calculating the coefficients $a_1, \dots, a_m$ in (2.20). An efficient one can be found in Yu [119].

We finally give a third method to determine the stability and the order of a focus originated by Lyapunov.

LEMMA 2.5. *For any given integer $N > 1$ and expansions of f and g of the form*

$$f = by + f_1 + \mathrm{O}\big(|x, y|^{2N+2}\big), \qquad g = -bx + g_1 + \mathrm{O}\big(|x, y|^{2N+2}\big),$$

where

$$f_1(x, y) = \sum_{2\leqslant i+j\leqslant 2N+1} a_{ij} x^i y^j, \quad g_1(x, y) = \sum_{2\leqslant i+j\leqslant 2N+1} b_{ij} x^i y^j,$$

there exist constants $L_2, \dots, L_{N+1}$ and a polynomial

$$V(x, y) = \sum_{k=2}^{2N+2} V_k(x, y),$$

where

$$V_2(x, y) = x^2 + y^2, \qquad V_k(x, y) = \sum_{i+j=k} c_{ij} x^i y^j, \quad 3 \leqslant k \leqslant 2N + 2$$

such that

$$V_x f + V_y g = \sum_{k=2}^{N+1} L_k \big(x^2 + y^2\big)^k + \mathrm{O}\big(|x, y|^{2N+3}\big). \tag{2.26}$$

Moreover, for $2 \leqslant k \leqslant N + 1$, L_{k+1} depends only on a_{ij} and b_{ij} with $i + j \leqslant 2k + 1$.

PROOF. We want to find a polynomial V with the supposed form and constants $L_2, \dots, L_{N+1}$ satisfying (2.26). Note that for the given form of f, g and V we have

$$\begin{aligned} V_x f + V_y g &= b(yV_x - xV_y) + (V_x f_1 + V_y g_1) + \mathrm{O}\big(|x, y|^{2N+3}\big) \\ &= b \sum_{k=3}^{2N+2} (yV_{kx} - xV_{ky}) - \sum_{k=3}^{2N+2} G_k + \mathrm{O}\big(|x, y|^{2N+3}\big), \end{aligned}$$

where

$$\sum_{k=3}^{2N+2} G_k = -(V_x f_1 + V_y g_1) + \mathrm{O}\big(|x, y|^{2N+3}\big) \tag{2.27}$$

with G_k being a homogeneous polynomial of degree k depending only on the coefficients a_{ij}, b_{ij} and c_{ij} with $2 \leqslant i + j < k$.

Then neglecting terms of degree more than $2N + 2$ Eq. (2.26) is equivalent to

$$b \sum_{k=3}^{2N+2} (yV_{kx} - xV_{ky}) = \sum_{k=3}^{2N+2} G_k + \sum_{k=2}^{N+1} L_k \big(x^2 + y^2\big)^k.$$

In order to find V_k and L_k satisfying this equation it suffices to solve the following equation for $3 \leqslant k \leqslant 2N + 2$:

$$b(yV_{kx} - xV_{ky}) = G_k \qquad \text{for } k \text{ odd,} \tag{2.28}$$

$$b(yV_{kx} - xV_{ky}) = G_k + L_{k/2}\big(x^2 + y^2\big)^{k/2} \quad \text{for } k \text{ even.} \tag{2.29}$$

The above equations can be rewritten as

$$b\frac{\mathrm{d}}{\mathrm{d}\theta} V_k(\cos\theta, \sin\theta) = G_k(\cos\theta, \sin\theta) \qquad \text{for } k \text{ odd,} \tag{2.30}$$

$$b\frac{\mathrm{d}}{\mathrm{d}\theta} V_k(\cos\theta, \sin\theta) = L_{k/2} + G_k(\cos\theta, \sin\theta) \quad \text{for } k \text{ even} \tag{2.31}$$

since

$$b(yV_{kx} - xV_{ky})(r\cos\theta, r\sin\theta) = b\,r^k \frac{\mathrm{d}}{\mathrm{d}\theta} V_k(\cos\theta, \sin\theta).$$

Let us solve (2.30) and (2.31) by induction in k. First, for $k = 3$, we have

$$\int_0^{2\pi} G_3(\cos\theta, \sin\theta)\,\mathrm{d}\theta = 0.$$

Hence there is a function $\widetilde{G}_3$ of the form

$$\widetilde{G}_3(\theta) = \sum_{i+j=3} \tilde{g}_{ij} \cos^i\theta \sin^j\theta$$

such that $\widetilde{G}_3'(\theta) = G_3(\cos\theta, \sin\theta)$. Denote $\widetilde{G}_3$ by

$$\widetilde{G}_3(\theta) = \int G_3(\cos\theta, \sin\theta)\,\mathrm{d}\theta.$$

We can write

$$\int G_3(\cos\theta, \sin\theta)\,\mathrm{d}\theta = \sum_{i+j=3} \tilde{g}_{ij} \cos^i\theta \sin^j\theta.$$

Then by (2.30) we have a solution for V_3 below

$$V_3(x, y) = \frac{1}{b} \sum_{i+j=3} \tilde{g}_{ij} x^i y^j.$$

That is, we have $c_{ij} = \frac{1}{b}\tilde{g}_{ij}$ for $i + j = 3$.

For $k = 4$, G_4 will be known when V_3 is definite. Thus, we can choose L_2 to be such that

$$L_2 + \frac{1}{2\pi} \int_0^{2\pi} G_4(\cos\theta, \sin\theta)\, \mathrm{d}\theta = 0.$$

It then follows that

$$\int \left[L_2 + G_4(\cos\theta, \sin\theta)\right] \mathrm{d}\theta = \sum_{i+j=4} \tilde{g}_{ij} \cos^i\theta \sin^j\theta$$

as before. By (2.31), V_4 will be determined by taking $c_{ij} = \frac{1}{b}\tilde{g}_{ij}$ for $i + j = 4$.

For higher values of k, V_k can be determined in the same procedure. This ends the proof. □

The following lemma says that the first nonzero constants of $L_2, \ldots, L_{N+1}$ will determine the stability and the order of the focus of (2.11) at the origin if $a = 0$.

LEMMA 2.6. *Let* $a = d = 0$, $b = -c \neq 0$. *Then* (2.5) *holds with* $k = 2m + 1 \geqslant 3$ *if and only if*

$$L_j = 0, \quad j = 2, \ldots, m, \qquad L_{m+1} \neq 0.$$

Moreover, $v_{2m+1} = \frac{L_{m+1}}{2|b|}$ *as* $L_j = 0$, $j = 2, \ldots, m$.

PROOF. Let $r > 0$ be small and L denote the orbit arc of (2.1) from $(r, 0)$ to $(P(r), 0)$. Note that for any $\mu \in (0, 1)$,

$$r + \mu\left[P(r) - r\right] = r + \mathrm{O}\left(r^3\right).$$

The mean value theorem implies that

$$\begin{aligned} V\left(P(r), 0\right) - V(r, 0) &= V_x\left(r + \mu\left[P(r) - r\right], 0\right)\left(P(r) - r\right) \\ &= 2r\left[P(r) - r\right]\left(1 + \mathrm{O}\left(r^2\right)\right), \end{aligned}$$

where V is the function in Lemma 2.5.

On the other hand, by the formula of constant variation we have $x^2 + y^2 = r^2(1 + \mathrm{O}(r))$ along L. Thus, by (2.26)

$$V\big(P(r),0\big)-V(r,0)=\int_L \mathrm{d}V=\int_L (V_x f+V_y g)\,\mathrm{d}t$$
$$=\int_0^{\tau(r)}\left[\sum_{k=2}^{N+1} L_k r^{2k}\big(1+\mathrm{O}(r)\big)+\mathrm{O}\big(r^{2N+3}\big)\right]\mathrm{d}t$$
$$=\frac{2\pi}{|b|}\sum_{k=2}^{N+1} L_k r^{2k}\big(1+\mathrm{O}(r)\big)+\mathrm{O}\big(r^{2N+3}\big),$$

where $\tau(r)=\frac{2\pi}{|b|}+\mathrm{O}(r)$ is the time along L. It follows that

$$P(r)-r=\frac{\pi}{|b|}\sum_{k=2}^{N+1} L_k r^{2k-1}\big(1+\mathrm{O}(r)\big)+\mathrm{O}\big(r^{2N+2}\big).$$

Then the conclusion follows easily and the proof is completed. □

The proof of Lemma 2.6 presents an algorithm to compute constants $L_2, L_3, \ldots$. It includes the following three steps in a loop:

(i) Find G_{2m+1} by (2.27),
(ii) Find V_{2m+1} by (2.30),
(iii) Find G_{2m+2}, L_{m+1} and V_{2m+2} by (2.31).

To begin with $m=1$ we can get L_2 by executing the 3 steps. We can get L_3 further by doing the same procedure for $m=2$. See Chicone [14] for more detail.

By (2.3) and (2.5) we can prove the following corollary in a similar way to Lemma 2.6.

COROLLARY 2.1. *Let $a=d=0$, $b=-c\neq 0$. Suppose there exists a function $\widetilde{V}(x,y)=x^2+y^2+\mathrm{O}(|x,y|^3)$ such that*

$$\dot{\widetilde{V}}=\widetilde{V}_x f+\widetilde{V}_y g=H_{2k}(x,y)+\mathrm{O}\big(|x,y|^{2k+1}\big),\quad k\geqslant 2,$$

where H_{2k} is a homogeneous polynomial of order $2k$ satisfying

$$\widetilde{L}_k=\frac{|b|}{2\pi}\int_0^{2\pi/|b|} H_{2k}(\cos bt,-\sin bt)\,\mathrm{d}t<0\ (>0).$$

Then the origin is a stable (unstable) focus of order $k-1$ of Eq. (2.1).

Now let us recall some facts obtained so far in this section. We will outline them for analytic systems. Suppose f and g in (2.1) are analytic functions near the origin and (2.3) holds with $a+d=0$. Then by Lemma 2.1, the proof of Lemma 2.6 and the definition of the Poincaré map in (2.4) we have

$$P(r)-r=\frac{\pi}{|b|}\sum_{k=2}^{N+1} L_k r^{2k-1}\big(1+\mathrm{O}(r)\big)+\mathrm{O}\big(r^{2N+2}\big)=2\pi\sum_{k=3}^{\infty} v_k r^k,\tag{2.32}$$

where

$$v_{2m} = \mathrm{O}\big(|v_3, v_5, \ldots, v_{2m-1}|\big), \quad m \geqslant 2.$$

The implicit function theorem ensures the convergence of the series in the right-hand side of (2.32).

If $a = d = 0,\ b = -c \neq 0$, by Lemma 2.3, there exists a formal change of variables given by the formal series

$$(x, y)^{\mathrm{T}} = (u, v)^{\mathrm{T}} + \sum_{i+j \geqslant 2} q_{ij} u^i v^j = Q(u, v)$$

which carries (2.1) formally into

$$\begin{aligned} \dot{u} &= bv + \sum_{j \geqslant 1} (a_j u + b_j v)\big(u^2 + v^2\big)^j, \\ \dot{v} &= -bu + \sum_{j \geqslant 1} (-b_j u + a_j v)\big(u^2 + v^2\big)^j. \end{aligned} \tag{2.33}$$

By Lemma 2.5, there exist a formal series

$$V(x, y) = x^2 + y^2 + \sum_{i+j \geqslant 3} c_{ij} x^i y^j$$

and constants $L_2, L_3, \ldots,$ such that

$$V_x f + V_y g = \sum_{k \geqslant 2} L_k \big(x^2 + y^2\big)^k. \tag{2.34}$$

Lemmas 2.4 and 2.6 show that the following three statements are equivalent to each other:

(i) $v_{2j+1} = 0,\ j = 1, \ldots, m-1,\ v_{2m+1} \neq 0$;

(ii) $a_j = 0,\ j = 1, \ldots, m-1,\ a_m \neq 0$;

(iii) $L_j = 0,\ j = 1, \ldots, m,\ L_{m+1} \neq 0$.

Moreover, when one of the conditions (i)–(iii) holds, we have

$$v_{2m+1} = \frac{a_m}{|b|} = \frac{L_{m+1}}{2|b|}.$$

Based on this relationship, we call either $v_{2m+1},\ a_m$ or L_{m+1} the mth Lyapunov quantity or focus value.

Note that the series in (2.32) is always convergent for $|r|$ small if (2.1) is analytic. We therefore obtain

THEOREM 2.2. *Suppose* (2.1) *is an analytic system satisfying* $a = d = 0,\ b = -c \neq 0$. *Then* (2.1) *has a center at the origin if and only if one of the following holds*:

(a) $v_{2k+1} = 0$ *for all* $k \geqslant 1$;
(b) $a_k = 0$ *for all* $k \geqslant 1$;
(c) $L_k = 0$ *for all* $k \geqslant 2$.

Lyapunov proved that the formal series for $V(x, y)$ is convergent near the origin if $L_k = 0$ for all $k \geqslant 2$. Hence, by (2.34) in this case the function V represents a first integral whose level sets are orbits of the system (2.1).

Bryuno [9] proved that the formal series for $Q(u, v)$ is convergent near the origin if and only if $a_k = 0$ for all $k \geqslant 1$. In this case, system (2.33) becomes

$$\dot{u} = v\left[b + \sum_{j\geqslant 1} b_j\left(u^2 + v^2\right)^j\right],$$
$$\dot{v} = -u\left[b + \sum_{j\geqslant 1} b_j\left(u^2 + v^2\right)^j\right]$$

which has a first integral of the form $u^2 + v^2$. This implies that the original system (2.1) has a first integral of the form $W_1^2(x, y) + W_2^2(x, y)$, where $W(x, y) = (W_1(x, y), W_2(x, y))$ is the inverse of the coordinate change $(x, y)^{\mathrm{T}} = Q(u, v)$.

Thus, by Theorem 2.2 and the next theorem we know that under the condition of Theorem 2.2 system (2.1) has a center at the origin if and only if it has an analytic first integral of the form $x^2 + y^2 + \mathrm{O}(|x, y|^3)$.

The following theorem gives a sufficient condition for (2.1) to have a center at the origin.

THEOREM 2.3. *Consider the* C^∞ *system* (2.1). *Let* (2.2) *hold. If one of the conditions below is satisfied*:
(i) $f(-x, y) = f(x, y), g(-x, y) = -g(x, y)$;
(ii) *there exists a* C^∞ *function* $H(x, y)$ *for* (x, y) *near the origin satisfying* $H(0, 0) = 0$, $H(x, y) \neq 0$ *for* $0 < x^2 + y^2 \ll 1$ *such that* $H_x f + H_y g = 0$,

then (2.1) *has a center at the origin.*

PROOF. Let (i) hold first. Without loss of generality we assume that the orbits of (2.1) near the origin are oriented clockwise. For $y_0 > 0$ small let $(x(t), y(t))$ be the solution of (2.1) with $(x(0), y(0)) = (0, y_0)$. Set

$$L_1 = \left\{\left(x(t), y(t)\right) \mid 0 \leqslant t \leqslant t_1\right\}, \qquad L_2 = \left\{\left(x(t), y(t)\right) \mid t_1 \leqslant t \leqslant t_2\right\},$$
$$L_1' = \left\{\left(-x(-t), y(-t)\right) \mid -t_1 \leqslant t \leqslant 0\right\},$$

where

$$t_1 = \min\left\{t > 0 \mid x(t) = 0, y(t) < 0\right\}, \qquad t_2 = \min\left\{t > 0 \mid x(t) = 0, y(t) > 0\right\}.$$

By our assumption, L_1' is an orbit arc of (2.1) starting at $(0, y(t_1))$. This implies that $L_2 = L_1'$ and hence $y(t_2) = y_0$. In other words, $(x(t), y(t))$ is a periodic solution.

Now let (ii) hold. For definiteness, we suppose $H(x, y) > 0$ for $0 < x^2 + y^2 \ll 1$. Then the function $z = H(x, y)$ takes a minimal value at $(x, y) = (0, 0)$. This implies that for sufficiently small $h > 0$, the equation $H(x, y) = h$ defines a closed curve near the origin. Our assumption ensures that the curve is a periodic orbit which approaches the origin as $h \to 0$. The proof is completed. □

EXAMPLE 2.1. Consider a C^∞ system of the form

$$\dot{x} = y - xh(x, y), \qquad \dot{y} = -x - yh(x, y),$$

where

$$h(x, y) = \begin{cases} 0, & \text{for } (x, y) = (0, 0), \\ \mathrm{e}^{-\frac{1}{x^2+y^2}}, & \text{otherwise.} \end{cases}$$

Let $V(x, y) = x^2 + y^2$. Then for the system, (2.34) becomes

$$V_x f + V_y g = -2\big(x^2 + y^2\big)h(x, y) < 0$$

for $x^2 + y^2 > 0$. By the proof of Lemma 2.6, the origin is a stable focus. However, in this case we have $L_k = 0$ for all $k \geqslant 2$.

This example shows that Theorem 2.2 is no longer true for C^∞ systems.

EXAMPLE 2.2. Consider the following cubic Liénard system:

$$\begin{aligned} \dot{x} &= y - \big(a_3x^3 + a_2x^2 + a_1x\big), \\ \dot{y} &= -x, \end{aligned} \tag{2.35}$$

where a_1, a_2 and a_3 are real constants. The divergence of (2.35) takes value $-a_1$ at the origin. Hence, the origin is a stable (unstable) focus if $a_1 > 0$ (< 0). Further, by Theorem 2.1 we have $V_3 = -\frac{3}{8}a_3$ as $a_1 = 0$. Thus, in this case, the origin is stable (unstable) if $a_3 > 0$ (< 0).

Let $a_1 = a_3 = 0$. Theorem 2.3(i) implies that (2.35) has a center at the origin now.

EXAMPLE 2.3. For a given C^∞ function $H : \mathbb{R}^2 \to \mathbb{R}$, it induces a planar system of the form

$$\dot{x} = H_y, \qquad \dot{y} = -H_x$$

which is called a Hamiltonian system with the Hamiltonian function H. The level sets of the function give orbits of the system. By Theorem 2.3(ii), if there is a point $(x_0, y_0) \in \mathbb{R}^2$

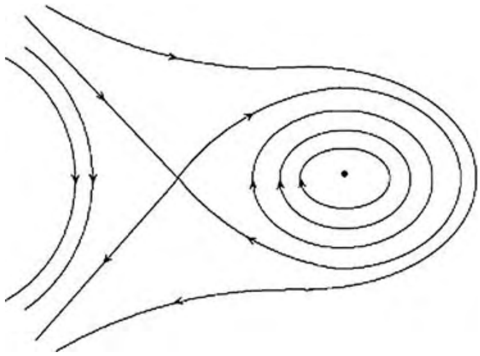

Fig. 6. The phase portrait of (2.36) with $g(x) = x^2 - x$.

satisfying

$$H_x(x_0, y_0) = H_y(x_0, y_0) = 0,$$
$$-\left[H_{xy}(x_0, y_0)\right]^2 + H_{xx}(x_0, y_0)H_{yy}(x_0, y_0) > 0$$

then the above Hamiltonian system has a center at the point.

For example, let

$$H(x, y) = \frac{1}{2}y^2 + G(x).$$

Then the induced system is

$$\dot{x} = y, \qquad \dot{y} = -g(x), \tag{2.36}$$

where $g(x) = G'(x)$. In mechanics, the function H is called the total energy of the system (2.36), while the term $\frac{1}{2}y^2$ is called the kinetic energy and the function G is the potential energy. It is evident that (2.36) has a center at point $(x_0, 0)$ if $g(x_0) = 0,\ g'(x_0) > 0$. A singular point $(x_0, 0)$ satisfying $g'(x_0) < 0$ is a saddle point. Taking $g(x) = x^2 - x$, (2.34) becomes

$$\dot{x} = y, \qquad \dot{y} = x - x^2$$

which has two singular points: center (1,0) and saddle (0,0). Note that $H(0, 0) = 0$. There is a nontrivial orbit γ defined by the equation $H(x, y) = 0$. The orbit approaches the saddle point both positively and negatively. An orbit with this property is called a homoclinic orbit. The phase portrait of the above quadratic Hamiltonian system has been drawn in Fig. 6.

2.3. *Hopf bifurcation*

In the next part, we consider a planar C^∞ system with a vector parameter of the form

$$\dot{x} = f(x, y, \mu), \qquad \dot{y} = g(x, y, \mu), \tag{2.37}$$

where $\mu \in \mathbb{R}^m$, $m \geqslant 1$. Suppose (2.37) has an elementary focus for $\mu = 0$. Without loss of generality we can assume that for all small $|\mu|$ (2.37) has a focus at the origin. Then we can write near the origin

$$\begin{aligned} f(x, y, \mu) &= a(\mu)x + b(\mu)y + \mathrm{O}\big(|x, y|^2\big), \\ g(x, y, \mu) &= c(\mu)x + d(\mu)y + \mathrm{O}\big(|x, y|^2\big), \end{aligned} \tag{2.38}$$

where $a(0) = d(0)$, $b(0) = -c(0) \neq 0$.

Let $P(r, \mu)$ denote the Poincaré map of (2.37) near the origin. By Lemma 2.1, similar to (2.32) for any integer $N > 0$, P has the following expansion:

$$P(r, \mu) = r + 2\pi \sum_{k=1}^{N} v_k(\mu) r^k + \mathrm{O}\big(r^{N+1}\big), \tag{2.39}$$

where

$$v_{2m} = \mathrm{O}\big(|v_1, v_3, \ldots, v_{2m-1}|\big), \quad 1 \leqslant m \leqslant \frac{N}{2}. \tag{2.40}$$

Sometimes, for convenience, we call $v_{2m+1}(\mu)$ in (2.39) the mth Lyapunov constant of (2.37) at the origin, $m \geqslant 1$. Introduce

$$d(r, \mu) = P(r, \mu) - r,$$

which is called a succession function or bifurcation function of (2.37). This function is also called the displacement function.

By using an analogous formula to (2.6) it is easy to obtain

LEMMA 2.7. *For $|\mu|$ small, Eq. (2.37) has a limit cycle near the origin if and only if there exist $r_1(\mu) > 0$, $r_2(\mu) < 0$ near $r = 0$ such that*

$$d\big(r_j(\mu), \mu\big) = 0, \quad j = 1, 2.$$

By (2.38) and the discussion after Remark 2.2 we know that $v_1(\mu)$ has the same sign as $a(\mu) + d(\mu)$. In fact, $v_1(\mu) = \frac{1}{2\pi}[\mathrm{e}^{2\pi\alpha/|\beta|} - 1]$ with $\alpha = \frac{1}{2}(a + d)$, $|\beta| = \frac{1}{2}\sqrt{-4bc - (a - d)^2}$. Hence, if $a(0) + d(0) \neq 0$ then $v_1(0) \neq 0$ and $d(r, \mu)$ has no positive zero near $r = 0$ for all $|\mu|$ small. Thus, if $a(0) + d(0) \neq 0$, there is no limit cycle near the origin for all $|\mu|$ small.

Let $a(0)+d(0)=0$, and let v_{30} denote the first order focus value of (2.37) for $\mu=0$ at the origin. Then by (2.39) and (2.40) we have

$$d(r,\mu)=2\pi r\big[v_1(\mu)+v_2(\mu)r+v_3(\mu)r^2+\mathrm{O}\big(r^3\big)\big],$$

where $v_1(0)=v_2(0)=0,\ v_3(0)=v_{30}$. Similar to Theorem 1.3(i) we can prove that if $v_{30}\neq 0,\ d(r,\mu)$ has two zeros in r with one positive and the other negative for $v_1(\mu)v_{30}<0$ and has no nontrivial zero for $v_1(\mu)v_{30}>0$. Thus, by Lemma 2.7 and noting

$$v_1=\frac{a(\mu)+d(\mu)}{2|b(0)|}\big(1+\mathrm{O}(\mu)\big),$$

we have proved the following

THEOREM 2.4. *If for $\mu=0$ (2.37) has a first order focus at the origin, then it has at most one limit cycle near the origin for all $|\mu|$ small. Moreover, the limit cycle exists if and only if $(a(\mu)+d(\mu))v_{30}<0$.*

In general, similar to Theorem 1.3(ii), we have:

THEOREM 2.5. *If for $\mu=0$ (2.37) has a kth order focus at the origin ($k\geqslant 2$), then it has at most k limit cycles near the origin for $|\mu|$ small. Moreover, k limit cycles can appear by suitable perturbations.*

EXAMPLE 2.4. Consider a cubic Liénard system

$$\begin{aligned}\dot{x}&=y-\big(x^5+\mu_1x^3+\mu_2x\big),\\ \dot{y}&=-x,\end{aligned}\tag{2.41}$$

where μ_1 and μ_2 are small parameters.

First, by (2.39) and (2.40), we have

$$\begin{aligned}v_1&=-\frac{1}{2}\mu_2+\mathrm{O}\big(\mu_2^2\big), &\quad v_2&=\mathrm{O}(\mu_2),\\ v_3&=-\frac{3}{8}\mu_1+\mathrm{O}(\mu_2), &\quad v_4&=\mathrm{O}\big(|v_1,v_3|\big).\end{aligned}$$

It is easy to check that for

$$V(x,y)=x^2+y^2-\frac{5}{8}xy^5-\frac{5}{3}x^3y^3-\frac{11}{8}x^5y$$

it holds along the orbits of (2.41)

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{\mu_1=\mu_2=0}=-\frac{5}{8}\big(x^2+y^2\big)^3+\mathrm{O}\big(|x,y|^{10}\big).$$

Hence, by Lemma 2.6, it follows that

$$v_5 = -\frac{5}{16} + \mathrm{O}\big(|\mu_1, \mu_2|\big).$$

This shows that for $\mu_1 = \mu_2 = 0$ the origin is a stable focus of order 2 for (2.41).

We claim that there is a function

$$\delta(\mu_1) = \frac{9}{40}\mu_1^2 + \mathrm{O}\big(|\mu_1|^{5/2}\big)$$

such that for $|\mu_1| + |\mu_2|$ small Eq. (2.41) has

(i) no limit cycle if either $\mu_1 \geqslant 0$ and $\mu_2 \geqslant 0$ or $\mu_1 < 0$ and $\mu_2 > \delta(\mu_1)$;

(ii) a unique simple limit cycle if either $\mu_1 \geqslant 0$ and $\mu_2 < 0$ or $\mu_1 < 0$ and $\mu_2 \leqslant 0$;

(iii) a unique double limit cycle if $\mu_1 < 0$ and $\mu_2 = \delta(\mu_1)$;

(iv) two simple limit cycles if $\mu_1 < 0$ and $\mu_2 < \delta(\mu_1)$.

In fact, by $v_2 = \mathrm{O}(v_1)$, $v_3 = -\frac{3}{8}\mu_1 + \mathrm{O}(v_1)$ and $v_4 = \mathrm{O}(|v_1| + |\mu_1|)$ we can write

$$\begin{aligned} d(r,\mu) &= 2\pi r P_1(r,u)\left[v_1 - \frac{3}{8}\mu_1 r^2 P_2(r,\mu) + v_5^* r^4 P_3(r,\mu)\right] \\ &= 2\pi r P_1(r,u) d_1(r,\mu), \end{aligned}$$

where $P_i = 1 + \mathrm{O}(r) \in C^\infty$, $i = 1,2,3$, $v_5^* = -\frac{5}{16} + \mathrm{O}(|\mu_1|)$. The function d_1 can be written further as

$$d_1(r,\mu) = v_1 - \frac{3}{8}\mu_1\rho^2 + v_5^*\rho^4 P_3^*(\rho,\mu) \equiv d_2(\rho,\mu),$$

where $\rho = r\sqrt{P_2(r,\mu)} = r + \mathrm{O}(r^2) \in C^\omega$, $P_3^* = 1 + \mathrm{O}(\rho) \in C^\omega$.

Clearly, for $\mu_1 \geqslant 0$ we have $\frac{\partial d_2}{\partial \rho} < 0$ for $0 < \rho \ll 1$. This implies that for $\mu_1 \geqslant 0$, d_1 has a unique zero if and only if $v_1 > 0$. Note that Eq. (2.41) has a double limit cycle near the origin if and only if the function $d_2(\rho,\mu)$ has a double zero in ρ near $\rho = 0$. Let us consider the equations below

$$d_2(\rho,\mu) = 0, \qquad \frac{\partial d_2}{\partial \rho}(\rho,\mu) = 0.$$

We can solve form the above equations

$$\begin{aligned} \rho &= \sqrt{-\frac{3}{5}\mu_1}\big(1 + \delta_1\big(\sqrt{-\mu_1}\big)\big), \\ v_1 &= \frac{9}{256 v_5^*}\mu_1^2\big(1 + \delta_2\big(\sqrt{-\mu_1}\big)\big) \end{aligned}$$

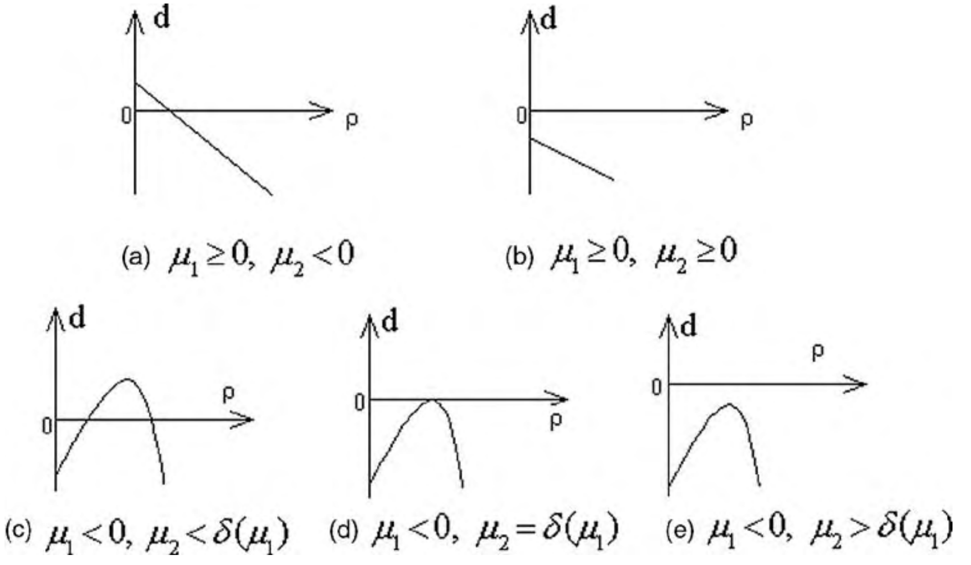

Fig. 7. Curves determined by $d_2(\rho, \mu)$.

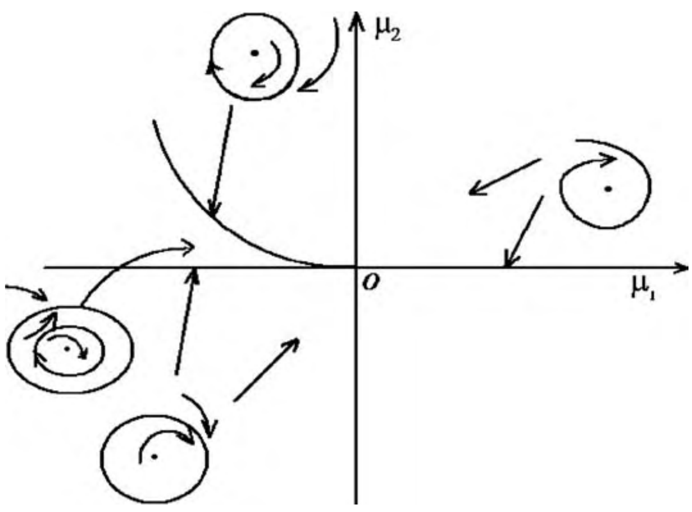

Fig. 8. The bifurcation diagram of (2.41).

for $\mu_1 < 0$, where $\delta_i(u) = \mathrm{O}(u) \in C^\omega,\ i = 1, 2$. The second equation above determines a unique function

$$\mu_2 = \delta(\mu_1) = \frac{9}{40}\mu_1^2 + \mathrm{O}\big(\mu_1^{5/2}\big).$$

Now it is easy to draw curves determined by the function $d = d_2(\rho, \mu)$ on (d, ρ) plane, see Fig. 7.

Then the claim follows easily. For the bifurcation diagram of (2.41) see Fig. 8.

On the (μ_1, μ_2) plane in Fig. 8, the line $\mu_2 = 0$ (near the origin) is Hopf bifurcation curve, and the curve $\mu_2 = \delta(\mu_1)$ is the double limit cycle bifurcation curve, or in other words, the saddle-node bifurcation curve for limit cycles.

If (2.37) has a center at the origin for $\mu = 0$, we can also study the bifurcation of limit cycles for $|\mu|$ small. We give an example to illustrate this phenomenon.

EXAMPLE 2.5. Consider the cubic system

$$\begin{aligned}\dot{x} &= y + x^2 + \varepsilon\bigl(x^3 - \varepsilon x\bigr),\\ \dot{y} &= -\bigl(x - x^3\bigr).\end{aligned}$$

We prove that the system has a unique limit cycle near the origin for $0 < \varepsilon \ll 1$.

For the purpose, let us consider

$$\begin{aligned}\dot{x} &= y + x^2 + \varepsilon\bigl(x^3 - \delta x\bigr),\\ \dot{y} &= -\bigl(x - x^3\bigr),\end{aligned} \tag{2.42}$$

and prove that the system has a unique limit cycle near the origin for $0 < |\varepsilon| < \varepsilon_0,\ 0 < \delta < \varepsilon_0$ for a small constant $\varepsilon_0 > 0$.

First, by Theorem 2.3(i), (2.42) has a center at the origin for $\varepsilon = 0$. Hence, the succession function $d(r, \varepsilon, \delta)$ of (2.42) can be written as

$$\begin{aligned}d(r, \varepsilon, \delta) &= 2\pi\varepsilon r\bigl[v_1^*(\varepsilon, \delta) + v_2^*(\varepsilon, \delta)r + v_3^*(\varepsilon, \delta)r^2 + \mathrm{O}\bigl(r^3\bigr)\bigr]\\ &= 2\pi\varepsilon r d^*(r, \varepsilon, \delta),\end{aligned}$$

where

$$\varepsilon v_1^* = v_1 = \frac{1}{2\pi}\bigl[\mathrm{e}^{-\varepsilon\delta\pi} - 1\bigr] = -\frac{\varepsilon\delta}{2}\bigl(1 + \mathrm{O}(\varepsilon\delta)\bigr),$$

$$v_2^* = \mathrm{O}\bigl(v_1^*\bigr), \quad \varepsilon v_3^* = v_3 = \frac{3}{8}\varepsilon + \mathrm{O}(v_1).$$

Therefore,

$$d^* = -\frac{\delta}{2}\bigl(1 + \mathrm{O}(\varepsilon\delta)\bigr) + \mathrm{O}(\delta)r + \left(\frac{3}{8} + O(\delta)\right)r^2 + \mathrm{O}\bigl(r^3\bigr)$$

which has a positive zero $r = \frac{2}{\sqrt{3}}\sqrt{\delta}(1 + \mathrm{O}(\varepsilon\delta))$ near $r = 0$ if $0 < \delta \ll 1$. Then the conclusion for (2.42) follows directly.

2.4. *Degenerate Hopf bifurcation*

We now turn to the degenerate Hopf bifurcation near a center and establish a general theory by using the first order Melnikov function.

Consider a C^∞ planar system of the form

$$\begin{aligned} \dot{x} &= f(x, y) + \varepsilon p(x, y, \varepsilon, \delta), \\ \dot{y} &= g(x, y) + \varepsilon q(x, y, \varepsilon, \delta), \end{aligned} \tag{2.43}$$

where $\varepsilon \in \mathbb{R}$, $\delta \in D \subset \mathbb{R}^m$ $(m \geqslant 1)$ with D compact, and

$$\left.\begin{aligned} &\big(f(x, y), g(x, y)\big) = \mu(H_y, -H_x), \quad \mu = \pm 1, \\ &H(x, y) = K\big(x^2 + y^2\big) + \mathrm{O}\big(|x, y|^3\big), \quad K > 0, \\ &p(0, 0, \varepsilon, \delta) = q(0, 0, \varepsilon, \delta) = 0. \end{aligned}\right\} \tag{2.44}$$

Then for $0 < h \ll 1$, the equation $H(x, y) = h$ defines a periodic orbit L_h of (2.43) ($\varepsilon = 0$) which intersects the positive x-axis at $A(h) = (a(h), 0)$. Let $B(h, \varepsilon, \delta)$ denote the first intersection point of the positive orbit of (2.43) starting at $A(h)$ with the positive x-axis. Then by (2.44) we have

$$H(B) - H(A) = \int_{AB} \mathrm{d}H = \varepsilon\big[M(h, \delta) + \mathrm{O}(\varepsilon)\big], \tag{2.45}$$

where

$$\begin{aligned} M(h, \delta) &= \frac{1}{\mu} \oint_{L_h} (fq - gp)|_{\varepsilon=0}\, \mathrm{d}t = \frac{1}{\mu} \oint_{L_h} (q\, \mathrm{d}x - p\, \mathrm{d}y)|_{\varepsilon=0} \\ &= \frac{1}{|\mu|} \iint_{H \leqslant h} (p_x + q_y)|_{\varepsilon=0}\, \mathrm{d}x\, \mathrm{d}y. \end{aligned} \tag{2.46}$$

We call M the first order Melnikov function. We will see that the function plays an important role in the study of the number of limit cycles.

Let $u(t, c)$ be the solution of (2.43) ($\varepsilon = 0$) with initial value $(c, 0)$. Then we have by (2.44)

$$\begin{aligned} &H\big(u(t, c)\big) = H(c, 0) = r^2(c), \quad t \in \mathbb{R}, \\ &r(c) = c\sqrt{K + S(c)}, \quad S(c) = \mathrm{O}(c) \in C^\infty. \end{aligned}$$

Denote by $c = c(r) = \frac{r}{\sqrt{K}} + \mathrm{O}(r^2)$ the inverse of $r = r(c)$. Then

$$H\big(v(t, r)\big) = r^2, \qquad v\big(T(r), r\big) = \big(c(r), 0\big),$$

where $v(t,r)=u(t,c(r))$, and $T(r)$ is the period of the function v in t. Clearly, the functions $v(t,r)$ and $T(r)$ are both C^∞ with $T(0)=\frac{\pi}{K|\mu|}>0$. Introduce a C^∞ function as follows:

$$G(\theta,r)=v\left(\frac{T(r)}{2\pi}\theta,r\right).$$

Obviously, $G=\mathrm{O}(r)$ is 2π-periodic in θ, and

$$H\big(G(\theta,r)\big)=r^2,\quad \theta\in\mathbb{R}. \tag{2.47}$$

LEMMA 2.8. *The change of variables*

$$(x,y)^{\mathrm{T}}=G(\theta,r) \tag{2.48}$$

transforms the system (2.43) *into the* C^∞ *system*

$$\begin{aligned}\dot\theta&=\frac{2\pi}{T(r)}\left[1-\frac{\varepsilon}{2\mu r}G_r\wedge\big(p(G,\varepsilon,\delta),q(G,\varepsilon,\delta)\big)\right],\\ \dot r&=\frac{\varepsilon}{2r}DH(G)\cdot\big(p(G,\varepsilon,\delta),q(G,\varepsilon,\delta)\big)^{\mathrm{T}}.\end{aligned} \tag{2.49}$$

PROOF. Differentiating (2.48) in t yields

$$G_\theta\dot\theta+G_r\dot r=\big(f(G)+\varepsilon p(G,\varepsilon,\delta),g(G)+\varepsilon q(G,\varepsilon,\delta)\big)^{\mathrm{T}}. \tag{2.50}$$

By (2.47) and the definition of G, we have

$$\mathrm{D}H(G)G_\theta=0,\qquad \mathrm{D}H(G)G_r=2r,\qquad G_\theta=\frac{T(r)}{2\pi}\big(f(G),g(G)\big).$$

Multiplying (2.50) by $\mathrm{D}H(G)$ from the left-hand side gives

$$2r\dot r=\varepsilon\mathrm{D}H(G)\cdot\big(p(G,\varepsilon,\delta),q(G,\varepsilon,\delta)\big),$$

which gives the second equation in (2.49).

Further, noting that

$$G_r\wedge G_\theta=\frac{T(r)}{2\pi}G_r\wedge\big(f(G),g(G)\big)=-\frac{\mu T(r)}{2\pi}\mathrm{D}H(G)G_r=-\frac{\mu T(r)}{\pi}r,$$

it follows from (2.50) that

$$(G_r\wedge G_\theta)\dot\theta=G_r\wedge\big(f(G),g(G)\big)+\varepsilon G_r\wedge(p,q),$$

or

$$-\frac{\mu T(r)}{\pi}r\dot\theta=-2\mu r+\varepsilon G_r\wedge(p,q)$$

which gives the first equation in (2.49). The proof is completed. □

By (2.49) we obtain

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = \varepsilon R(\theta, r, \varepsilon, \delta), \tag{2.51}$$

where R is C^∞ and 2π-periodic in θ with $R = \mathrm{O}(r)$ and

$$R(\theta, r, \theta, \delta) = \frac{T(r)}{4\pi r}\mathrm{D}H(G) \cdot \big(p(G, 0, \delta), q(G, 0, \delta)\big)^{\mathrm{T}}. \tag{2.52}$$

Let $r(\theta, \rho, \varepsilon, \delta)$ denote the solution of (2.51) with $r(0, \rho, \varepsilon, \delta) = \rho$, and $(x(t, c, \varepsilon, \delta), y(t, c, \varepsilon, \delta))$ the solution of (2.43) with $(x(0, c, \varepsilon, \delta), y(0, c, \varepsilon, \delta)) = (c, 0)$. Then the Poincaré map $P(c, \varepsilon, \delta)$ of (2.43) is given by

$$P(c, \varepsilon, \delta) = x(\tau, c, \varepsilon, \delta) = P(c)$$

where

$$\tau = \min\big\{t > 0 \mid cx(t, c, \varepsilon, \delta) > 0,\ y(t, c, \varepsilon, \delta) = 0\big\}.$$

By the definition of G we have $G(0, r) = (c(r), 0)$ which yields

$$G\big(0, P_1(\rho)\big) = \big(c\big(P_1(\rho)\big), 0\big),$$

where

$$P_1(\rho) = r(2\pi, \rho, \varepsilon, \delta) = P_1(\rho, \varepsilon, \delta)$$

which is called the Poincaré map of (2.51).

On the other hand, by (2.48) for $c = c(\rho)$ we have

$$\big(P(c, \varepsilon, \delta), 0\big) = G\big(2\pi, r(2\pi, \rho, \varepsilon, \delta)\big) = G\big(0, P_1(\rho)\big).$$

Hence

$$\big(P\big(c(\rho), \varepsilon, \delta\big), 0\big) = \big(c\big(P_1(\rho)\big), 0\big)\ ,$$

or

$$P \circ c = c \circ P_1. \tag{2.53}$$

THEOREM 2.6. *Let* (2.43) *satisfy* (2.44). *Then*:

(i) *The function M is C^∞ at $h = 0$. It is analytic at $h = 0$ if* (2.43) *is analytic*.

(ii) *If there exist a compact subset D_0 of D and an integer $k \geqslant 0$ such that for $0 < h \ll 1$*

$$M(h,\delta) = B_k(\delta)h^{k+1} + \mathrm{O}\big(h^{k+2}\big), \quad B_k(\delta) \neq 0,\ \delta \in D_0, \tag{2.54}$$

then there exist $\varepsilon_0 > 0$, an open set $U(D_0) \supset D_0$ and a neighborhood V of the origin such that (2.43) *has at most k limit cycles in V for $0 < |\varepsilon| < \varepsilon_0$, $\delta \in U(D_0)$.*

PROOF. It is easy to see that the Poincaré map P_1 of (2.51) can be written as

$$P_1(r,\varepsilon,\delta) = r + \varepsilon r F(r,\varepsilon,\delta), \tag{2.55}$$

where

$$rF(r,0,\delta) = \int_0^{2\pi} R(\theta,r,0,\delta)\,\mathrm{d}\theta \equiv R_0(r,\delta).$$

By (2.44) and (2.52) we have

$$\begin{aligned} R_0(r,\delta) &= \frac{1}{2r\mu}\oint_{H=r^2}(f,g)\wedge(p,q)|_{\varepsilon=0}\,\mathrm{d}t \\ &= \frac{1}{2r\mu}\oint_{H=r^2}(q\,\mathrm{d}x - p\,\mathrm{d}y)|_{\varepsilon=0} \end{aligned} \tag{2.56}$$

which immediately follows $R_0(-r,\delta) = -R_0(r,\delta)$. Hence, the function $M^*(r)$ defined by

$$M^*(r) = 2rR_0(r,\delta)$$

is even. Since $M^* \in C^\infty$, for any integer $j > 1$, we have

$$M^*(r) = \sum_{i=1}^{j} A_i r^{2i} + N(r),$$

where $N \in C^\infty$ is even and $N^{(i)}(0) = 0$, $i = 0,1,\dots,2j$.

Let $\overline{N}(h) = N(\sqrt{h})$. We claim that

$$\overline{N}^{(i)}(0) = 0, \quad i = 0,1,\dots,j.$$

In fact, we can prove

$$\overline{N}^{(i)}(h) = h^{j-i}\widetilde{N}_i\big(\sqrt{h}\big), \quad i = 0,1,\dots,j, \tag{2.57}$$

by induction in i, where $\widetilde{N}_i(r)$ is C^∞ in r and $\widetilde{N}_i(0) = 0$.

First, it is not hard to see that

$$N^{(i)}(r) = r^{2j-i}N_i(r), \quad N_i \in C^\infty, \qquad N_i(0) = 0, \quad i = 0,1,\dots,2j.$$

Hence, (2.57) holds for $i=0$. That is,

$$\overline{N}(h)=h^j N_0(\sqrt{h})=h^j \widetilde{N}_0(\sqrt{h}).$$

It then follows that

$$(\overline{N}(h))'=h^{j-1}\left[jN_0(\sqrt{h})+\frac{1}{2}\sqrt{h}N_0'(\sqrt{h})\right].$$

Let

$$\widetilde{N}_1(r)=jN_0(r)+\frac{1}{2}rN_0'(r).$$

Then $\widetilde{N}_1 \in C^\infty$ and $\widetilde{N}_1(0)=0$. Thus, (2.57) holds for $i=1$. Let (2.57) hold for $i=k$. Then we have

$$\overline{N}^{(k+1)}(h)=\left[h^{j-k}\widetilde{N}_k(\sqrt{h})\right]'=h^{j-k-1}\left[(j-k)\widetilde{N}_k(\sqrt{h})+\frac{1}{2}\sqrt{h}\widetilde{N}_k'(\sqrt{h})\right].$$

Set

$$\widetilde{N}_{k+1}(r)=(j-k)\widetilde{N}_k(r)+\frac{1}{2}r\widetilde{N}_k'(r).$$

It follows that (2.57) holds for $i=k+1$. Hence, (2.57) has been proved.

By (2.57) it is immediate that $\overline{N}\in C^j$ for $0\leqslant h\ll 1$. Let

$$\overline{M}(h)=\sum_{i=1}^{j}A_i h^i+\overline{N}(h).$$

Then $\overline{M}\in C^j$. Therefore $\overline{M}\in C^\infty$ since j is arbitrarily large.

Note that by (2.46) and (2.56) we have

$$\overline{M}(h)=M^*(\sqrt{h})=2\sqrt{h}R_0(\sqrt{h},\delta)=M(h,\delta).$$

Thus, $M\in C^\infty$ in h at $h=0$, and if (2.54) holds, then

$$R_0(r,\delta)=\frac{1}{2}B_k(\delta)r^{2k+1}+\mathrm{O}(r^{2k+3}).$$

By (2.53) and (2.55) and Lemma 2.7, Eq. (2.43) has a limit cycle near the origin if and only if F has two zeros correspondingly with one positive and the other negative. Hence, by Rolle theorem, (2.54) ensures that at most k limit cycles can appear near the origin for $|\varepsilon|$ small and δ in a neighborhood of D_0.

Finally, if (2.43) is analytic, then (2.51) is analytic. Hence the functions R_0 and M^* are also analytic. This implies that the series

$$M^*(r) = \sum_{i \geqslant 1} A_i r^{2j}$$

is convergent for $|r|$ small. Therefore M is analytic since $M(h, \delta) = M^*(\sqrt{h})$. The proof is completed. □

By Theorem 2.6, for any $k \geqslant 1$ we have the following expansion for M:

$$M(h, \delta) = h\big[b_0(\delta) + b_1(\delta)h + \cdots + b_k(\delta)h^k + \mathrm{O}\big(h^{k+1}\big)\big], \quad 0 < h \ll 1. \tag{2.58}$$

We can use the coefficients $b_0, b_1, \ldots, b_k$ to study the Hopf bifurcation of limit cycles.

COROLLARY 2.2. *Suppose that there exist $k \geqslant 1$, $\delta_0 \in D$ such that $b_k(\delta_0) \neq 0$ and*

$$b_j(\delta_0) = 0, \quad j = 0, 1, \ldots, k-1, \qquad \det \frac{\partial(b_0, \ldots, b_{k-1})}{\partial(\delta_1, \ldots, \delta_k)}(\delta_0) \neq 0, \tag{2.59}$$

where $\delta = (\delta_1, \ldots, \delta_m)$, $m \geqslant k$. Then for any $\varepsilon_0 > 0$ and any neighborhood V of the origin there exist $0 < \varepsilon < \varepsilon_0$ and $|\delta - \delta_0| < \varepsilon_0$ such that (2.43) *has precisely k limit cycles in V.*

PROOF. Fix $\delta_j = \delta_{j0}$ for $j = k+1, \ldots, m$. By (2.59) the change of parameters

$$b_j = b_j(\delta), \quad j = 0, \ldots, k-1$$

has the inverse $\delta_j = \delta_j(b_0, \ldots, b_{k-1})$, $j = 1, \ldots, k$. Then (2.58) becomes

$$M(h, \delta) = h\big[b_0 + b_1 h + \cdots + b_{k-1}h^{k-1} + b_k h^k + \mathrm{O}\big(h^{k+1}\big)\big]$$

where $b_k = b_k(\delta_0) \neq 0$ as $b_0 = \cdots = b_{k-1} = 0$. By changing the sign of $b_{k-1}, b_{k-2}, \ldots, b_0$ in turn such that

$$b_{j-1}b_j < 0, \quad j = k, k-1, \ldots, 1, \qquad 0 < |b_0| \ll |b_1| \ll \cdots \ll |b_{k-1}| \ll 1,$$

we can find k simple positive zeros $h_1, h_2, \ldots, h_k$ with $0 < h_k < h_{k-1} < \cdots < h_1 \ll 1$. Let $r_j = \sqrt{h_j}$, $j = 1, \ldots, k$. Then $r_1, \ldots, r_k$ are simple positive zeros of $R_0(r, \delta)$. By (2.55) and the implicit function theorem, the function F has k zeros $r_j + \mathrm{O}(\varepsilon)$, $j = 1, \ldots, k$. This ends the proof. □

The following is evident from Theorem 2.6(ii).

COROLLARY 2.3. *If for any* $\varepsilon_0 > 0$ *and any neighborhood* V *of the origin* (2.43) *has* k *limit cycles in* V *for some* (ε, δ) *satisfying* $0 < |\varepsilon| < \varepsilon_0$, $|\delta - \delta_0| < \varepsilon_0$, *then* $M(h, \delta_0) = O(h^{k+1})$.

In the multiple parameter case it often occurs that $M(h, \delta_0) \equiv 0$, $M(h, \delta) \not\equiv 0$ for $\delta \neq \delta_0$. In this case (2.54) fails to hold, and we have further the following result.

THEOREM 2.7. *Let* (2.44) *hold. Suppose*

(i) *the coefficients* $b_0, \ldots, b_{k-1}$ *in* (2.58) *satisfy* (2.59);

(ii) *there exists a* k*-dimensional vector function* $\varphi(\varepsilon, \delta_{k+1}, \ldots, \delta_m)$ *such that* (2.43) *has a center at the origin when* $(\delta_1, \ldots, \delta_k) = \varphi(\varepsilon, \delta_{k+1}, \ldots, \delta_m)$ *for* $|\varepsilon| + |\delta - \delta_0|$ *small.*

Then there exist $\varepsilon_0 > 0$ *and a neighborhood* V *of the origin such that* (2.43) *has at most* $k - 1$ *limit cycles in* V *for* $0 < |\varepsilon| < \varepsilon_0$, $|\delta - \delta_0| < \varepsilon_0$. *Moreover,* $k - 1$ *limit cycles can appear in an arbitrary neighborhood of the origin for some* (ε, δ) *sufficiently near* $(0, \delta_0)$.

PROOF. By (2.58) and noting that $M(h, \delta) = 2rR_0(r, \delta)$, $r = \sqrt{h}$, we have

$$R_0(r, \delta) = r\left[\sum_{j=0}^{k} b_j(\delta) r^{2j} + O\big(r^{2k+2}\big)\right].$$

Then the function F in (2.55) has the following expansion:

$$F(r, \varepsilon, \delta) = \sum_{j=0}^{2k-1} c_j(\varepsilon, \delta) r^j + r^{2k} Q(r, \varepsilon, \delta), \tag{2.60}$$

where $Q \in C^\infty$, and

$$c_{2j}(0, \delta) = b_j(\delta), \quad c_{2j+1}(0, \delta) = 0, \quad j = 0, \ldots, k - 1.$$

By (2.59), the equations

$$b_j = c_{2j}(\varepsilon, \delta), \quad j = 0, \ldots, k - 1$$

have the solution

$$(\delta_1, \ldots, \delta_k) = \bar{\varphi}(\varepsilon, b_0, \ldots, b_{k-1}, \delta_{k+1}, \ldots, \delta_m). \tag{2.61}$$

By condition (ii) we have $F(r, \varepsilon, \delta) = 0$ and hence $c_{2j}(\varepsilon, \delta) = 0$, $j = 0, \ldots, k - 1$ as long as $(\delta_1, \ldots, \delta_k) = \varphi(\varepsilon, \delta_{k+1}, \ldots, \delta_m)$. The uniqueness of the solution $\bar{\varphi}$ implies that

$$\bar{\varphi} = \varphi \quad \text{if and only if} \quad b_0 = \cdots = b_{k-1} = 0. \tag{2.62}$$

Inserting (2.61) into (2.60) we obtain

$$F = \sum_{j=0}^{k-1} \big[b_j r^{2j} + c_{2j+1}(\varepsilon, \bar{\varphi}, \delta_{k+1}, \ldots, \delta_m) r^{2j+1}\big] + r^{2k} Q(r, \varepsilon, \bar{\varphi}, \delta_{k+1}, \ldots, \delta_m)$$
$$\equiv \widetilde{F}\big(r, \varepsilon, \tilde{\delta}\big), \quad \tilde{\delta} = (b_0, \ldots, b_{k-1}, \delta_{k+1}, \ldots, \delta_m). \tag{2.63}$$

It follows from (2.61) and (2.62) that $\widetilde{F} = 0$ as $b_j = 0,\ j = 0, \ldots, k-1$. Hence, by the mean value theorem we can write

$$c_{2j+1}(\varepsilon, \bar{\varphi}, \delta_{k+1}, \ldots, \delta_m) = \varepsilon \sum_{i=0}^{k-1} b_i A_{ij}\big(\varepsilon, \tilde{\delta}\big),$$
$$Q(r, \varepsilon, \bar{\varphi}, \delta_{k+1}, \ldots, \delta_m) = \sum_{i=0}^{k-1} b_i Q_i\big(r, \varepsilon, \tilde{\delta}\big),$$

where $Q_i \in C^\infty$, $i = 0, \ldots, k-1$. Note that $\varepsilon r^2 F = r(P_1 - r)$ keeps sign for $0 < |r| \ll 1$ by Lemma 2.1 and (2.53). By (2.63), it follows that

$$c_{2j+1}(\varepsilon, \bar{\varphi}, \delta_{k+1}, \ldots, \delta_m) = 0, \quad \text{if } b_0 = \cdots = b_j = 0,\ j = 0, \ldots, k-1.$$

This yields that

$$A_{ij} = 0, \quad j+1 \leqslant i \leqslant k-1,\ j = 0, \ldots, k-2.$$

Therefore, the function $\widetilde{F}$ in (2.63) can be written as

$$\widetilde{F}\big(r, \varepsilon, \tilde{\delta}\big) = \sum_{j=0}^{k-1} b_j r^{2j} P_j\big(r, \varepsilon, \tilde{\delta}\big), \tag{2.64}$$

where

$$P_j = 1 + \varepsilon \sum_{i=j}^{k-1} A_{ji}\big(\varepsilon, \tilde{\delta}\big)\, r^{2(i-j)+1} + r^{2(k-j)} Q_j, \quad 0 \leqslant j \leqslant k-1.$$

We particularly have

$$\widetilde{F}\big(r, \varepsilon, \tilde{\delta}\big) = \sum_{j=0}^{k-1} b_j r^{2j} + \mathrm{O}\big(|r|^{2k} + |\varepsilon|\big).$$

By using this form we can prove, in a similar manner to Corollary 2.2, that $k-1$ limit cycles can appear in an arbitrary neighborhood of the origin for some (ε, δ) sufficiently close to $(0, \delta_0)$.

It needs only to prove that $\widetilde{F}$ has at most $k-1$ positive zeros near $r=0$ for $|b_0|+\cdots+|b_{k-1}|>0$ small. We do this by induction in k. For convenience, denote by F_{k-1} the right-hand side function of (2.64).

First, for $k=1$, we have

$$F_0 = b_0 P_0 = b_0\big(1+\mathrm{O}\big(|\varepsilon|+|r|\big)\big) \neq 0 \quad \text{for } |b_0|>0.$$

Suppose for $k=n$ the function

$$F_{n-1} = \sum_{j=0}^{n-1} b_j r^{2j} P_j$$

has at most $n-1$ positive zeros in r near $r=0$ for $|\varepsilon|$ and $|b_0|+\cdots+|b_{n-1}|>0$ small, where $P_j = 1+\mathrm{O}(|\varepsilon|+|r|) \in C^\infty$, $j=0,\ldots,n-1$. Consider the function

$$F_n = \sum_{j=0}^{n} b_j r^{2j} P_j,$$

where $P_j = 1+\mathrm{O}(|\varepsilon|+|r|) \in C^\infty$. We have

$$F_n = P_0 \widetilde{F}_n, \quad \widetilde{F}_n = \sum_{j=0}^{n} b_j r^{2j} \widetilde{P}_j,$$

where $\widetilde{P}_0 = 1$, $\widetilde{P}_j = P_j/P_0 = 1+\mathrm{O}(|\varepsilon|+|r|) \in C^\infty$, $j=1,\ldots,n$. Then

$$\frac{\mathrm{d}\widetilde{F}_n}{\mathrm{d}r} = \sum_{j=1}^{n} 2j b_j r^{2j-1}\left(\widetilde{P}_j + \frac{r}{2}\frac{\mathrm{d}\widetilde{P}_j}{\mathrm{d}r}\right) = r\sum_{j=0}^{n-1} \bar{b}_j r^{2j} \overline{P}_j = r\overline{F}_{n-1},$$

where

$$\bar{b}_j = 2(j+1)b_{j+1}, \quad \overline{P}_j = \widetilde{P}_{j+1} + \frac{r}{2}\frac{\mathrm{d}\widetilde{P}_{j+1}}{\mathrm{d}r} = 1+\mathrm{O}\big(|\varepsilon|+|r|\big) \in C^\infty.$$

By the induction assumption, the function $\overline{F}_{n-1}$ has at most $n-1$ positive zeros in r near $r=0$ for $|\varepsilon|$ and $|\bar{b}_0|+\cdots+|\bar{b}_{n-1}|>0$ small. Hence, by Rolle's theorem it follows that $\widetilde{F}_n$ has at most n positive zeros near $r=0$ in r for $|\varepsilon|$ and $|b_0|+\cdots+|b_n|>0$ small. This finishes the proof. □

Observe that in both Theorems 2.6 and 2.7 the parameter δ is required to vary near δ_0. The results are local in this sense. In many cases the functions p and q in (2.43) depend on δ linearly. This enables us to obtain a global result based on Theorems 2.6 and 2.7 as follows.

THEOREM 2.8. *Let* (2.44) *hold, and let the functions* p *and* q *in* (2.43) *be linear in* δ. *Suppose further that for an integer* $k \geqslant 1$

(i) $\operatorname{rank}\dfrac{\partial(b_0,\dots,b_{k-1})}{\partial(\delta_1,\dots,\delta_m)} = k,\ m \geqslant k$;

(ii) *Eq.* (2.43) *has a center at the origin as* $b_j(\delta) = 0,\ j = 0, 1, \dots, k-1$.

Then for any given $N > 0$, *there exist* $\varepsilon_0 > 0$ *and a neighborhood* V *of the origin such that Eq.* (2.43) *has at most* $k-1$ *limit cycles in* V *for* $0 < |\varepsilon| < \varepsilon_0$, $|\delta| \leqslant N$. *Moreover,* $k-1$ *limit cycles can appear in an arbitrary neighborhood of the origin for some* (ε, δ).

PROOF. Since p and q are linear in δ, each coefficient b_j in (2.58) is also linear in δ. Hence, by condition (i) we can suppose

$$\det \frac{\partial(b_0,\dots,b_{k-1})}{\partial(\delta_1,\dots,\delta_k)} \neq 0.$$

It follows that the equations $b_j(\delta) = 0,\ j = 0,\dots,k-1$ have a solution $(\delta_1,\dots,\delta_k) = \varphi(\delta_{k+1},\dots,\delta_m)$.

For $\delta_0^* = (\varphi(0), 0) \in \mathbb{R}^m$, by Theorem 2.7, Eq. (2.43) can have $k-1$ limit cycles near the origin for some (ε,δ) near $(0,\delta_0^*)$. Then we need to prove that $k-1$ is the maximal number of limit cycles. If it is not the case, then there exist $N > 0$ and sequences $\varepsilon_n \to 0$, $\delta^{(n)} \in \mathbb{R}^m$ with $|\delta^{(n)}| \leqslant N$ such that for $(\varepsilon,\delta) = (\varepsilon_n, \delta^{(n)})$ Eq. (2.43) has k limit cycles which approach the origin as $n \to \infty$. We may suppose $\delta^{(n)} \to \delta_0$ for some $\delta_0 \in \mathbb{R}^m$ as $n \to \infty$. First, by Theorem 2.6 we must have $b_j(\delta_0) = 0,\ j = 0,\dots,k-1$. By our assumption (ii), Eq. (2.43) has a center at the origin for $(\delta_1,\dots,\delta_k) = \varphi(\delta_{k+1},\dots,\delta_m)$. Hence, it follows from Theorem 2.7 that (2.43) has at most $k-1$ limit cycles near the origin for all (ε,δ) sufficiently close to $(0,\delta_0)$. This is a contradiction since Eq. (2.43) has k limit cycles approaching the origin as $(\varepsilon,\delta) = (\varepsilon_n,\delta^{(n)}) \to (0,\delta_0)$. This ends the proof. □

Theorems 2.5 and 2.6 tell us that the coefficients in the expansion of M act like focus values. The following lemma gives a relation between the two groups of the values.

LEMMA 2.9. *Let* (2.44) *and* (2.58) *hold. Then*

$$b_0 = 4\pi v_1^*, \qquad b_j = \frac{4\pi}{K^j}\left[v_{2j+1}^* + \mathrm{O}\left(\left|v_1^*, v_3^*, \dots, v_{2j-1}^*\right|\right)\right], \quad j = 1,\dots,k-1,$$

where

$$v_{2j+1}^* = \left.\frac{\partial v_{2j+1}}{\partial \varepsilon}\right|_{\varepsilon=0}, \quad j = 0,\dots,k-1,$$

and v_{2j+1} *is the* j*th Lyapunov constant of* (2.43) *at the origin.*

PROOF. Let $P(r,\varepsilon,\delta)$ denote the Poincaré map of (2.43). Then the point B in (2.45) is given by $(P(a,\varepsilon,\delta),0)$, where $a = \sqrt{\frac{h}{K}}(1 + \mathrm{O}(\sqrt{h}))$. Note that $P - a = \mathrm{O}(|\varepsilon a|)$. By the mean value theorem we have

$$H(B) - H(A) = H_x\big(a\big(1 + O(\varepsilon)\big), 0\big)(P - a)$$
$$= 2Ka\big[1 + O\big(|\varepsilon| + |a|\big)\big](P - a).$$

By (2.39) and (2.40) we can write formally

$$P(a, \varepsilon, \delta) - a = 2\pi \sum_{i \geqslant 0} v_{2i+1} a^{2i+1} P_i,$$

where $P_i = 1 + O(a)$, $i \geqslant 0$. Hence

$$H(B) - H(A) = 4\pi h \sum_{i \geqslant 0} \frac{v_{2i+1}}{K^i} h^i \overline{P}_i,$$

where $\overline{P}_i = 1 + O(|\varepsilon| + \sqrt{h})$, $i \geqslant 0$. Inserting the above into (2.45) and noting $v_{2i+1} = \varepsilon v_{2i+1}^* + O(\varepsilon^2)$ we obtain

$$4\pi h \sum_{i \geqslant 0} \frac{v_{2i+1}^*}{K^i} h^i \big(1 + O\big(\sqrt{h}\big)\big) = M(h, \delta).$$

Since M is C^∞ at $h = 0$ it follows from the above that

$$M(h, \delta) = 4\pi h\left[v_1^* + \frac{1}{K}\big(v_3^* + O\big(v_1^*\big)\big)h + \frac{1}{K^2}\big(v_5^* + O\big(v_1^*, v_3^*\big)\big)h^2 + \cdots\right].$$

Thus the conclusion follows by comparing with (2.58). This completes the proof. □

EXAMPLE 2.6. Consider

$$\begin{aligned} &\dot{x} = y - \big(\mu_0 x^5 + \mu_1 x^3 + \mu_2 x\big), \\ &\dot{y} = -x. \end{aligned} \tag{2.65}$$

We claim that there exist $\varepsilon_0 > 0$ and a neighborhood V of the origin such that (2.65) has at most 2 limit cycles in V for $|\mu_0| + |\mu_1| + |\mu_2| < \varepsilon_0$, and 2 limit cycles can appear.

In fact, by the discussion to (2.41) we know that

$$v_1 = -\frac{1}{2}\mu_2 + O\big(\mu_2^2\big), \qquad v_3 = -\frac{3}{8}\mu_1 + O(\mu_2),$$
$$v_5 = -\frac{5}{16}\mu_0 + O\big(|\mu_1| + |\mu_2|\big).$$

Let $\mu_i = \varepsilon\delta_i$, $\varepsilon = \sqrt{\mu_0^2 + \mu_1^2 + \mu_2^2}$, $\delta_0^2 + \delta_1^2 + \delta_2^2 = 1$. Then by Lemma 2.9 (taking $\mu = 1$ and $K = 1/2$) we have

$$b_0 = -2\pi\delta_2, \qquad b_1 = -3\pi\delta_1 + O(\delta_2), \qquad b_2 = -5\pi\delta_0 + O\big(|\delta_1| + |\delta_2|\big).$$

For the sake of convenience, we take $\delta_0, \delta_1, \delta_2$ as free parameters with $|\delta_i| \leqslant 1$. Then the conditions of Theorem 2.8 are satisfied with $m = k = 3$. Therefore, the claim follows.

Note that for $\varepsilon = 0$ the origin is a linear center of (2.65). Hence, in this case the function M can be obtained directly by using (2.46).

DEFINITION 2.4. We say that Eq. (2.43) has Hopf cyclicity $k - 1$ at the origin if the conclusion of Theorem 2.8 holds.

Thus, both systems (2.41) and (2.65) have Hopf cyclicity 2 at the origin.

EXAMPLE 2.7. Consider a Liénard system of the form

$$\dot{x} = y - \varepsilon \sum_{i=1}^{2n+1} a_i x^i, \qquad \dot{y} = -x \tag{2.66}$$

where $n \geqslant 1$, $|a_i| \leqslant 1$, and $\varepsilon > 0$ is small. We claim that the system has Hopf cyclicity n at the origin.

In fact, we have $H(x, y) = \frac{1}{2}(x^2 + y^2)$ with $\mu = 1$, $K = \frac{1}{2}$. The curve L_h given by $H(x, y) = h$ has the representation $(x, y) = \sqrt{2h}(\cos t, -\sin t)$. Hence, by (2.46) we have

$$M(h) = -\sum_{j=0}^{n} 2^{j+1} N_j a_{2j+1} h^{j+1},$$

where

$$N_j = \int_0^{2\pi} \cos^{2(j+1)} t \, \mathrm{d}t > 0.$$

Set $\delta = (a_1, a_2, \ldots, a_{2n+1})$, $b_j = -2^{j+1} N_j a_{2j+1}$, $j = 0, \ldots, n$ and $k = n + 1$. Note that by Theorem 2.3(i) Eq. (2.66) has a center at the origin as $b_j = 0$, $j = 0, \ldots, n$. Thus, the claim follows from Theorem 2.8.

In the proof of Corollary 2.2 we have given a way to find limit cycles near the origin. In the case of (2.66), $M(h)$ can have n zeros $h_1 > h_2 > \cdots > h_n > 0$ for some $(a_1, a_3, \ldots, a_{2n+1})$. Then the corresponding function F in (2.55) has n zeros $\sqrt{h_j} + \mathrm{O}(\varepsilon)$, $j = 1, \ldots, n$. It follows that the n limit cycles of (2.66) approach the curves $\frac{1}{2}(x^2 + y^2) = h_j$, $j = 1, \ldots, n$, respectively, as $\varepsilon \to 0$.

In the following we give another way to obtain limit cycles near the origin. For convenience, take $a_{2j} = 0$, $j = 1, \ldots, n$.

First, from Lemma 2.9, we have

$$v_{2j+1} = \varepsilon v_{2j+1}^* + \mathrm{O}(\varepsilon^2), \quad j = 0, \ldots, n,$$

where

$$v_1^* = \frac{1}{4\pi} b_0 = -\frac{N_0}{2\pi} a_1,$$

$$v_{2j+1}^* = \frac{1}{2^{j+2}\pi} b_j + \mathrm{O}\big(|b_0, \ldots, b_{j-1}|\big) = -\frac{N_j}{2\pi} a_{2j+1} + \mathrm{O}\big(|a_1, a_3, \ldots, a_{2j-1}|\big).$$

On the other hand, for $\widetilde{V} = x^2 + y^2$ we have along the orbit of (2.66)

$$\dot{\widetilde{V}} = -2\varepsilon \sum_{j=0}^{n} a_{2j+1} x^{2j+2}.$$

By Corollary 2.1, it holds that

$$\widetilde{L}_{k+1} = -\frac{\varepsilon}{\pi} a_{2k+1} N_k$$

if $a_{2j+1} = 0$ for $j = 0, \ldots, k-1$. It then follows that

$$v_{2j+1} = -\varepsilon \left[\frac{N_j}{2\pi} a_{2j+1} + \mathrm{O}\big(|a_1, a_3, \ldots, a_{2j-1}|\big) \right] \big(1 + \mathrm{O}(\varepsilon)\big), \quad j = 0, \ldots, n.$$

For $\varepsilon > 0$, let us vary $a_1, a_3, \ldots, a_{2n+1}$ such that

$$a_{2j-1} a_{2j+1} < 0, \qquad 0 < |a_{2j-1}| \ll |a_{2j+1}| \ll \varepsilon, \quad j = 1, \ldots, n$$

which yields

$$v_{2j-1} v_{2j+1} < 0, \qquad 0 < |v_{2j-1}| \ll |v_{2j+1}| \ll \varepsilon^2, \quad j = 1, \ldots, n.$$

Thus, by (2.32), the corresponding function $P(r) - r$ will have n zeros which give n limit cycles of (2.66). Geometrically, the n limit cycles are obtained by changing the stability of the origin n times. Moreover, unlike above they approach the origin as $\varepsilon \to 0$.

REMARK 2.3. There are different ways to prove Lemma 2.3 for the normal form system (2.11). Here the proof is given by following [111]. The conclusions in Lemmas 2.4 and 2.6 are well known. However, the definite relationship between constants a_m, v_{2m+1} and L_m given in the lemmas appears for the first time here. For more discussions on the level set of a Hamiltonian system, the reader can consult [36]. On the bifurcation of limit cycles, Theorems 2.4 and 2.5 are basic tools, while Theorems 2.6–2.8 were recently obtained by [47]. More discussions similar to Lemma 2.9 can be found in [80]. Under condition (2.44), it was proved in [64] that the right-hand side function in (2.45) is C^∞ in $\sqrt{h}$ at $h = 0$, but not C^2 in h at $h = 0$ generally.

As we know, in 1901, Hilbert [73] posed 23 mathematical problems of which the second part of the 16th one is to find the maximal number and relative position of limit cycles of planar polynomial systems. Many works have been done on the study of the problem, especially for quadratic and cubic systems, see [3,4,8,10–13,15,18–34,38–51,54–72,74–110, 113–126,128,129]. A detailed introduction and related literatures can be found in Li [85], Schlomiuk [109] and Ilyashenko [78]. As was showed in Examples 2.4 and 2.7, a typical way to find limit cycles in Hopf bifurcation is to compute focal values and change the stability of the focus by using the values. Certain nice results have been obtained for quadratic and cubic systems. Bautin [5] proved that a focus of a quadratic system has order at most three and that for this system a focus or center can generate at most three limit cycles under perturbations of its coefficients. Then Chen and Wang [12] (by bifurcation method) and Shi [110] (by using Poincaré–Bendixson theorem) separately found a quadratic system having four limit cycles. More and more mathematicians believe that quadratic system have at most four limit cycles. However, up to now the problem is still open. Li and Li [89], Li and Huang [88], Li and Liu [91–93], and Liu, Yang and Jiang [96] found different cubic systems having eleven limit cycles by using Melnikov function method. James and Lloyd [79] gave a cubic system having eight limit cycles in a neighborhood of a focus. Recently, Han, Lin and Yu [60] and Yu and Han [120,121] obtained sufficient conditions for a cubic system to have 10 or 12 limit cycles, respectively (all limit cycles having small amplitude). It seems that the maximal number of limit cycle for cubic system is 12.

For Hopf bifurcation in higher dimension, the reader can see Hale [35], Chow and Hale [16], Chow, Li and Wang [17] and Han [51] etc. For bifurcation of periodic solutions of delay-differential equations, see [7,37,52,53] et al.

3. Perturbations of Hamiltonian systems

3.1. *General theory*

In this section we will study a C^∞ system of the form

$$\dot{x} = H_y + \varepsilon p(x, y, \varepsilon, \delta), \qquad \dot{y} = -H_x + \varepsilon q(x, y, \varepsilon, \delta), \tag{3.1}$$

where $H(x, y), p(x, y, \varepsilon, \delta), q(x, y, \varepsilon, \delta)$ are C^∞ functions, $\varepsilon \geqslant 0$ is small and $\delta \in D \subset \mathbb{R}^m$ is a vector parameter with D compact. For $\varepsilon = 0$ (3.1) becomes

$$\dot{x} = H_y, \qquad \dot{y} = -H_x \tag{3.2}$$

which is Hamiltonian. Hence, Eq. (3.1) is called a near-Hamiltonian system.

For Eq. (3.2) we suppose there exist a family of periodic orbits given by

$$L_h\colon\ H(x, y) = h, \quad h \in (\alpha, \beta)$$

such that L_h approaches an elementary center point, denoted by L_α, as $h \to \alpha$, and an invariant curve, denoted by L_β, as $h \to \beta$. Without loss of generality, we can assume that each L_h is oriented clockwise.

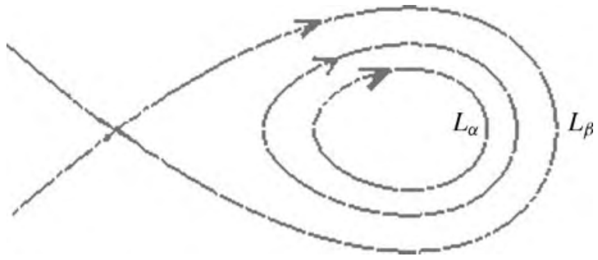

Fig. 9. The phase portrait of (3.2) with a L_β homoclinic loop.

If L_β is bounded, it usually is a homoclinic loop consisting of a saddle and a connection or a heteroclinic loop consisting of at least two saddles and connections between them. In the homoclinic case, the phase portrait of the family $\{L_h\colon \alpha \leqslant h \leqslant \beta\}$ is given in Fig. 9.

Introduce an open set G as follows:

$$G = \bigcup_{\alpha<h<\beta} L_h.$$

Our main purpose is to study the number of limit cycles of Eq. (3.1) in a neighborhood of the closure $\overline{G}$ of G for $\varepsilon > 0$ small and $\delta \in D$.

Note that if Eq. (3.1) has a limit cycle $L(\varepsilon, \delta)$ for $\varepsilon > 0$ small and $\delta \in D_0 \subset D$, then the limit of the cycle as $\varepsilon \to 0$ is either the center L_α, a periodic orbit L_h with $h \in (\alpha, \beta)$ or the boundary L_β. That is,

$$\lim_{\varepsilon\to 0} L(\varepsilon, \delta) = L_h, \qquad h \in [\alpha, \beta].$$

In this case, we say that the limit cycle $L(\varepsilon, \delta)$ is generated from L_h. Thus, in order to study the number of limit cycles, we first need to study the number of limit cycles generated from each L_h.

For the purpose, similar to Definition 2.4 we first introduce a notation below.

DEFINITION 3.1. We say that Eq. (3.1) has *cyclicity* k at a given L_h, $h \in [\alpha, \beta]$, if there exist $\varepsilon_0 > 0$ and a neighborhood V of L_h such that Eq. (3.1) has at most k limit cycles in V for $0 < \varepsilon < \varepsilon_0$, $\delta \in D$ and if k limit cycles can appear in an arbitrary neighborhood of L_h for some (ε, δ) with $\varepsilon > 0$ sufficiently small. More specifically, k is said to be *Hopf* (*Poincaré or homoclinic*) *cyclicity* when $h = \alpha$ ($h \in (\alpha, \beta)$ or $h = \beta$ with L_β being a homoclinic loop).

In the last section we gave some method to find Hopf cyclicity. This section concerns with global bifurcations of limit cycles and presents further methods to find Hopf, Poincaré and homoclinic cyclicity.

Take $h = h_0 \in (\alpha, \beta)$ and $A(h_0) \in L_{h_0}$. Let l be a cross section of Eq. (3.2) passing through $A(h_0)$. Then for h near h_0 the periodic orbit L_h has a unique intersection point with l, denoted by $A(h)$. That is, $A(h) = L_h \cap l$. Consider the positive orbit $\gamma(h, \varepsilon, \delta)$

of Eq. (3.1) starting at $A(h)$. Let $B(h, \varepsilon, \delta)$ denote the first intersection point of the orbit with l. Then similar to (2.45) we have

$$H(B) - H(A) = \varepsilon\big[M(h, \delta) + \mathrm{O}(\varepsilon)\big] = \varepsilon F(h, \varepsilon, \delta), \tag{3.3}$$

where

$$\begin{aligned} M(h, \delta) &= \oint_{L_h} (H_y q + H_x p)|_{\varepsilon=0}\, \mathrm{d}t \\ &= \oint_{L_h} (q\, \mathrm{d}x - p\, \mathrm{d}y)|_{\varepsilon=0} = \iint_{H \leqslant h} (p_x + q_y)|_{\varepsilon=0}\, \mathrm{d}x\, \mathrm{d}y. \end{aligned} \tag{3.4}$$

The function $F(h, \varepsilon, \delta)$ in (3.3) is called a bifurcation function of Eq. (3.1). It has the following property.

LEMMA 3.1. *For $\varepsilon > 0$ small and $\delta \in D$, Eq.* (3.1) *has a limit cycle near L_{h_0}, $h_0 \in (\alpha, \beta)$, if and only if the equation $F(h, \varepsilon, \delta) = 0$ has a zero in h near h_0.*

PROOF. Since the orbit $\gamma(h, \varepsilon, \delta)$ starting at $A(h)$ is closed if and only if $A = B$, we need only to prove that $A = B$ if and only if $H(A) = H(B)$ by (3.3). It is easy to see that $B = A + \mathrm{O}(\varepsilon)$. Hence, by Taylor formula for $\varepsilon > 0$ small we have

$$H(B) - H(A) = \big(H_x(A), H_y(A)\big) \cdot (B - A) + \mathrm{O}\big(|B - A|^2\big).$$

Note that the cross section l can be taken to be parallel to the gradient $(H_x(A), H_y(A))$. It follows that

$$H(B) - H(A) = \Big[\pm\sqrt{H_x^2(A) + H_y^2(A)} + \mathrm{O}\big(|B - A|\big)\Big] \cdot |B - A|,$$

which gives the desired conclusion. The proof is completed. □

As in the situation of Hopf bifurcation, the Melnikov function $M(h, \delta)$ can also be used to determine the cyclicity at a periodic orbit. First, by (3.3) we have

THEOREM 3.1. *Let $h_0 \in (\alpha, \beta)$, $\delta_0 \in D$.*

(i) *There is no limit cycle near L_{h_0} for $\varepsilon + |\delta - \delta_0|$ small if $M(h_0, \delta_0) \neq 0$.*

(ii) *There is exactly one* (*at least one*, *respectively*) *limit cycle $L(\varepsilon, \delta)$ for $\varepsilon + |\delta - \delta_0|$ small which approaches L_{h_0} as $(\varepsilon, \delta) \to (0, \delta_0)$ if $M(h_0, \delta_0) = 0$, $M_h(h_0, \delta_0) \neq 0$* (*$h_0$ is a zero of $M(h, \delta_0)$ with odd multiplicity*, *respectively*).

(iii) *The cyclicity of Eq.* (3.1) *at L_{h_0} is at most k if for any $\delta \in D$ there exists $0 \leqslant j \leqslant k$ such that*

$$M_h^{(j)}(h_0, \delta) \neq 0.$$

PROOF. By (3.3) and Lemma 3.1, the first conclusion is clear. For the second one, the conclusion follows from the implicit function theorem if h_0 is a simple zero of $M(h, \delta_0)$. Let h_0 be a multiple zero of $M(h, \delta_0)$ with odd multiplicity. Then for $\varepsilon_0 > 0$ small we have

$$M(h_0 - \varepsilon_0, \delta_0) \cdot M(h_0 + \varepsilon_0, \delta_0) < 0.$$

Hence, by (3.3) we have

$$F(h_0 - \varepsilon_0, \varepsilon, \delta) \cdot F(h_0 + \varepsilon_0, \varepsilon, \delta) < 0$$

for $0 < \varepsilon < \varepsilon_0$, $|\delta - \delta_0| < \varepsilon_0$ as long as ε_0 is sufficiently small. Thus the function $F(h, \varepsilon, \delta)$ has a zero $h^* \in (h_0 - \varepsilon_0, h_0 + \varepsilon_0)$.

For the third conclusion, if Eq. (3.1) has cyclicity at least $k + 1$ at L_{h_0}, then there exist some $\varepsilon_n \to 0$, $\delta^{(n)} \in D$ such that for $(\varepsilon, \delta) = (\varepsilon_n, \delta^{(n)})$ Eq. (3.1) has at least $k + 1$ limit cycles which approach L_{h_0} as $n \to \infty$. We can suppose $\delta^{(n)} \to \delta_0$ as $n \to \infty$. On the other hand, by our assumption $M(h, \delta_0)$ has h_0 as a zero of multiplicity at most k. By (3.3) and Rolle's theorem for $\varepsilon + |\delta - \delta_0|$ small the function $F(h, \varepsilon, \delta)$ has at most k zeros in h near h_0, which follows that at most k limit cycles exist near L_{h_0} for $\varepsilon + |\delta - \delta_0|$ small. Thus a contradiction appears if $(\varepsilon, \delta) = (\varepsilon_n, \delta^{(n)})$ with n sufficiently large.

The proof is completed. □

By the above proof we have immediately

COROLLARY 3.1. *Suppose for some $\delta_0 \in D$, $M(h, \delta_0)$ has k zeros in $h \in (\alpha, \beta)$ with each having odd multiplicity. Then for $\varepsilon + |\delta - \delta_0|$ small Eq.* (3.1) *has k limit cycles in a compact subset of the open set G.*

COROLLARY 3.2. *If there exist $h_0 \in (\alpha, \beta), \delta_0 \in D$ such that for an arbitrary neighborhood of L_{h_0}, Eq.* (3.1) *has k limit cycles in the neighborhood for some (ε, δ) with $\varepsilon + |\delta - \delta_0|$ sufficiently small, then $M(h, \delta_0) = \mathrm{O}(|h - h_0|^k)$.*

Let $L(\varepsilon, \delta)$ be the limit cycle appeared in Theorem 3.1. To determine its stability we need to consider the sigh of the integral

$$\varepsilon \oint_{L(\varepsilon,\delta)} (p_x + q_y)\,\mathrm{d}t = \varepsilon \left[\oint_{L_{h_0}} (p_x + q_y)|_{\varepsilon=0}\,\mathrm{d}t + \mathrm{O}(\varepsilon) \right].$$

Obviously, if

$$\sigma(h_0, \delta_0) = \oint_{L_{h_0}} (p_x + q_y)|_{\varepsilon=0,\delta=\delta_0}\,\mathrm{d}t \neq 0$$

then the stability will be determined easily. The following lemma gives a relation between $\sigma(h, \delta)$ and $M_h(h, \delta)$.

LEMMA 3.2. *Suppose L_h is oriented clockwise. Then*

$$M_h(h,\delta) = \pm \oint_{L_h} (p_x + q_y)|_{\varepsilon=0}\,\mathrm{d}t = \pm\sigma(h,\delta),$$

where "+" (respectively "−") is taken when L_h expands (respectively shrinks) with h increasing.

PROOF. Fix $h_0 \in (\alpha,\beta)$. For definiteness, suppose that L_h expands with h increasing. Then for $h > h_0$, applying Green's formula we have

$$M(h,\delta) - M(h_0,\delta) = \iint_{\Delta(h)} (p_x + q_y)|_{\varepsilon=0}\,\mathrm{d}x\,\mathrm{d}y, \tag{3.5}$$

where $\Delta(h)$ denotes the annulus bounded by L_h and L_{h_0}. Let $u(t,h)$ denote a representation of L_h satisfying

$$H\big(u(t,h)\big) = h, \quad 0 \leqslant t \leqslant T(h),\ h \in (\alpha,\beta).$$

Here $T(h)$ denotes the period of L_h. Consider the integral transformation of variables given by

$$(x,y) = u(t,r), \quad 0 \leqslant t \leqslant T(r),\ h_0 < r < h.$$

Note that

$$\mathrm{D}H(u)\cdot \mathrm{D}_r u = \det\frac{\partial u(t,r)}{\partial(t,r)} = 1.$$

We obtain from (3.5)

$$M(h,\delta) - M(h_0,\delta) = \int_{h_0}^{h} \mathrm{d}r \int_0^{T(r)} (p_x + q_y)\big(u(t,r),0,\delta\big)\,\mathrm{d}t.$$

Then differentiating the above in h yields

$$M_h(h,\delta) = \int_0^{T(h)} (p_x + q_y)\big(u(t,h),0,\delta\big)\,\mathrm{d}t.$$

This ends the proof. □

EXAMPLE 3.1. Consider van der Pol equation

$$\ddot{x} + \varepsilon\big(x^2 - 1\big)\dot{x} + x = 0.$$

The equation is equivalent to

$$\dot{x} = y, \qquad \dot{y} = -x - \varepsilon\big(x^2 - 1\big)y. \tag{3.6}$$

For $\varepsilon = 0$, Eq. (3.6) has periodic orbits L_h: $\frac{1}{2}(x^2 + y^2) = h$, $h > 0$. By (3.4) we have

$$M(h) = \iint_{x^2+y^2\leqslant 2h} \big(1 - x^2\big)\,\mathrm{d}x\,\mathrm{d}y = \pi h(2 - h).$$

The function M has a unique positive zero $h = 2$. By Lemma 3.2 we have

$$\sigma_0 = \oint_{L_2} \big(1 - x^2\big)\,\mathrm{d}t = M'(2) = -2\pi.$$

Hence, by Theorem 3.1, Eq. (3.6) has a unique limit cycle $L(\varepsilon)$ for $\varepsilon > 0$ which is stable, simple and approaches the circle $x^2 + y^2 = 4$ as $\varepsilon \to 0$.

In many cases (polynomial systems for example), the first order Melnikov function has the form

$$M(h, \delta) = \sum_{i=1}^{k} b_i(\delta)\, I_i(h), \quad k \geqslant 2.$$

Let

$$I_1'(\alpha) \neq 0, \qquad I_1(h) \neq 0, \quad \alpha < h < \beta. \tag{3.7}$$

Then we can write

$$M(h, \delta) = I_1(h) \sum_{i=1}^{k} b_i\, J_i(h) = I_1(h) N(h, b), \tag{3.8}$$

where $b = (b_1, \ldots, b_n)$, $b_i = b_i(\delta)$, $J_i(h) = I_i(h)/I_1(h)$, $i = 1, 2, \ldots, k$.

Introduce the Wronskian of the functions $J_1, J_2, \ldots, J_k$

$$W(h) = \begin{vmatrix} J_1(h) & J_2(h) & \ldots & J_k(h) \\ J_1'(h) & J_2'(h) & \ldots & J_k'(h) \\ \ldots & \ldots & \ldots & \ldots \\ J_1^{(k-1)}(h) & J_2^{(k-1)}(h) & \ldots & J_k^{(k-1)}(h) \end{vmatrix}. \tag{3.9}$$

Clearly, by a property of determinant we have

$$W'(h)=\begin{vmatrix} J_1(h) & J_2(h) & \dots & J_k(h)\\ J_1'(h) & J_2'(h) & \dots & J_k'(h)\\ \cdots & \cdots & \cdots & \cdots\\ J_1^{(k-2)}(h) & J_2^{(k-2)}(h) & \dots & J_k^{(k-2)}(h)\\ J_1^{(k)}(h) & J_2^{(k)}(h) & \dots & J_k^{(k)}(h)\end{vmatrix}. \tag{3.10}$$

We have

THEOREM 3.2. *Suppose that* (3.7) *and* (3.8) *hold. Let* $|b|>0$ *for all* $\delta\in D$ *and* $h_0\in[\alpha,\beta)$.

(i) *If* $W(h_0)\neq 0$, *then Eq.* (3.1) *has cyclicity at most* $k-1$ *at* L_{h_0}.

(ii) *If* $W(h_0)=0$, $W'(h_0)\neq 0$, *then Eq.* (3.1) *has cyclicity at most* k *at* L_{h_0}.

PROOF. For the first conclusion, let us suppose Eq. (3.1) has cyclicity at least k at L_{h_0}. Then there exist $\varepsilon_n\to\infty$, $\delta_n\in D$ such that for $(\varepsilon,\delta)=(\varepsilon_n,\delta_n)$ Eq. (3.1) has k limit cycles $L_j^{(n)}$ with $L_j^{(n)}\to L_{h_0}$ as $n\to\infty$, $j=1,2,\dots,k$. Without loss of generality, we can suppose

$$\delta_n\to\delta_0,\quad b(\delta_n)\to b_0=(b_{10},\dots,b_{k0})\quad \text{as } n\to\infty.$$

Near h_0 we have the following expansion for $N(h,b)$:

$$N(h,b)=\bar b_1+\bar b_2(h-h_0)+\dots+\bar b_k(h-h_0)^{k-1}+\bar b_{k+1}(h-h_0)^k+\cdots, \tag{3.11}$$

where, by (3.8),

$$\bar b_{j+1}=\left[\frac{M(h,\delta)}{I_1(h)}\right]^{(j)}_{h=h_0}=\frac{1}{j!}\left[b_1J_1^{(j)}(h_0)+\dots+b_kJ_k^{(j)}(h_0)\right],$$
$$j=0,1,\dots,k-1,k. \tag{3.12}$$

It follows that

$$\det\frac{\partial(\bar b_1,\dots,\bar b_k)}{\partial(b_1,\dots,b_k)}=\frac{W(h_0)}{1!2!\cdots(k-1)!}\neq 0. \tag{3.13}$$

Note that $|b_0|>0$ by our assumption. By (3.12) and (3.13) we obtain

$$\left(\bar b_1,\dots,\bar b_k\right)\big|_{\delta=\delta_0}=\left(\bar b_{10},\dots,\bar b_{k0}\right)\neq 0.$$

Thus, an integer l satisfying $1\leqslant l\leqslant k$ exists such that

$$\bar b_{j0}=0,\quad j=1,\dots,l-1,\qquad \bar b_{l0}\neq 0.$$

Hence, we have by (3.11)

$$N(h, b_0) = \bar{b}_{l0}(h - h_0)^{l-1} + \mathrm{O}\big(|h - h_0|^l\big), \quad \bar{b}_{l0} \neq 0. \tag{3.14}$$

Therefore, by (3.7) and Corollary 2.3 (if $h_0 = \alpha$) or Corollary 3.2 (if $h_0 > \alpha$), there exist $\varepsilon_0 > 0$ and a neighborhood U of L_{h_0} such that for $0 < \varepsilon < \varepsilon_0$, $|\delta - \delta_0| < \varepsilon_0$ Eq. (3.1) has at most $l - 1$ limit cycles in U. This contradicts that Eq. (3.1) has k limit cycles approaching L_{h_0} for $(\varepsilon, \delta) = (\varepsilon_n, \delta_n)$ and $n \to \infty$. This finishes the proof of conclusion (i).

Let $W(h_0) = 0$, $W'(h_0) \neq 0$. If the conclusion (ii) is not true, then, as before, there exists a sequence $(\varepsilon_n, \delta_n)$ approaching $(0, \delta_0)$ such that for $(\varepsilon, \delta) = (\varepsilon_n, \delta_n)$, Eq. (3.1) has $k + 1$ limit cycles approaching L_{h_0} as $n \to \infty$. In this case, formulas (3.11) and (3.12) remain true with

$$\det \frac{\partial(\bar{b}_1, \ldots, \bar{b}_{k-1}, \bar{b}_{k+1})}{\partial(b_1, \ldots, b_k)} = \frac{W'(h_0)}{1!2! \cdots (k-2)!k!} \neq 0, \tag{3.15}$$

and hence

$$\big(\bar{b}_1, \ldots, \bar{b}_{k-1}, \bar{b}_{k+1}\big)\big|_{\delta=\delta_0} = \big(\bar{b}_{10}, \ldots, \bar{b}_{k-1,0}, \bar{b}_{k+1,0}\big) \neq 0.$$

It follows that there exists $1 \leqslant l \leqslant k + 1$ and $l \neq k$ such that (3.14) holds. Thus, a contradiction appears in the same way as above. This ends the proof. □

It often happens that $b = 0$ for some $\delta \in D$. In this case, the condition of Theorem 3.2 fails and the following one can apply further.

THEOREM 3.3. *Suppose the following conditions are satisfied.*

(a) *The vector b is linear in δ with rank $\frac{\partial b}{\partial \delta} = k$ and $b(\delta_0) = 0$ for some $\delta_0 \in D$.*

(b) *Eq. (3.1) is analytic on the closure $\overline{G}$ and has a center near L_α when $b = 0$.*

(c) *There exists $h_0 \in [\alpha, \beta)$ such that $|W(h_0)| + |W'(h_0)| \neq 0$.*

Then

(i) *When $W(h_0) \neq 0$, Eq. (3.1) has cyclicity $k - 1$ at L_{h_0}.*

(ii) *When $W(h_0) = 0$, $W'(h_0) \neq 0$, Eq. (3.1) has cyclicity k (respectively $k - 1$ or k) at L_{h_0} if $h_0 > \alpha$ (respectively $h_0 = \alpha$).*

(iii) *For each $h \in [\alpha, \beta)$, Eq. (3.1) has cyclicity at least $k - 1$ at L_h.*

PROOF. Let $W(h_0) \neq 0$ first. By Definition 3.1, it suffices to prove the following two points:

(1) There are at most $k - 1$ limit cycles near L_{h_0} for $\varepsilon > 0$ small and $\delta \in D$.

(2) There can appear $k - 1$ limit cycles in any neighborhood of L_{h_0} for some arbitrarily small $\varepsilon + |\delta - \delta_0|$.

We proceed the proof by contradiction. If the conclusion (1) is not true, then there exists a sequence $(\varepsilon_n, \delta_n) \to (0, \delta^*)$ with $\delta^* \in D$ such that for $(\varepsilon, \delta) = (\varepsilon_n, \delta_n)$ Eq. (3.1) has k

limit cycles which approach L_{h_0} as $n \to \infty$. The proof of Theorem 3.2 implies $b(\delta^*) = 0$. By condition (a) we may assume

$$\det \frac{\partial(b_1, \ldots, b_k)}{\partial(\delta_1, \ldots, \delta_k)} \neq 0.$$

Then for δ near δ^* the linear equation $b = b(\delta)$ has a unique set of solutions $\delta_j = \tilde{\delta}_j(b, \delta_{k+1}, \ldots, \delta_m)$, $j = 1, \ldots, k$. By (3.12) we have

$$\det \frac{\partial(\bar{b}_1, \ldots, \bar{b}_k)}{\partial(\delta_1, \ldots, \delta_k)} = \det \frac{\partial(\bar{b}_1, \ldots, \bar{b}_k)}{\partial(b_1, \ldots, b_k)} \det \frac{\partial(b_1, \ldots, b_k)}{\partial(\delta_1, \ldots, \delta_k)} \neq 0.$$

Further, by condition (b) Eq. (3.1) has a center near L_α when $(\bar{b}_1, \ldots, \bar{b}_k) = 0$. Thus, in the case of $h_0 = \alpha$, by (3.7), (3.8), (3.11) and Theorem 2.7 we know that Eq. (3.1) has Hopf cyclicity $k-1$ at L_α for $\varepsilon + |\delta - \delta^*|$ small. This contradicts to the existence of k limit cycles near L_α for $(\varepsilon, \delta) = (\varepsilon_n, \delta_n)$. The conclusion (1) above is proved for $h_0 = \alpha$. Since $b(\delta_0) = 0$, using δ_0 instead of δ^*, the above discussion implies that $k-1$ limit cycles can appear in any neighborhood of L_α for arbitrarily small $\varepsilon + |\delta - \delta_0|$. Then the conclusion (2) follows for $h_0 = \alpha$.

For the case of $h_0 > \alpha$, by (3.3), (3.7), (3.8) and (3.11) we have

$$\begin{aligned} F(h, \varepsilon, \delta) &= I_1(h)\left[\sum_{j=1}^{k} b_j J_j(h) + \mathrm{O}(\varepsilon)\right] \\ &= I_1(h)\left[\sum_{j=1}^{k} \tilde{b}_j (h - h_0)^{j-1} + \mathrm{O}\big(|h - h_0|^k\big)\right], \end{aligned} \tag{3.16}$$

where $\tilde{b}_j = \bar{b}_j + \mathrm{O}(\varepsilon)$, $j = 1, \ldots, k$.

Also, by (3.13) and condition (b) Eq. (3.1) has a center near L_α for $(\bar{b}_1, \ldots, \bar{b}_k) = 0$ and hence for h near h_0,

$$F(h, \varepsilon, \delta) = 0 \quad \text{if } \big(\bar{b}_1, \ldots, \bar{b}_k\big) = 0.$$

Thus (3.16) can be rewritten as

$$F(h, \varepsilon, \delta) = I_1(h) \sum_{j=1}^{k} \tilde{b}_j (h - h_0)^{j-1}\big[1 + P_j(h, \varepsilon, \delta)\big], \tag{3.17}$$

where $\tilde{b}_j = \bar{b}_j + \mathrm{O}(\varepsilon|\bar{b}_1, \ldots, \bar{b}_k|)$, $P_j(h, \varepsilon, \delta) = \mathrm{O}(|h - h_0|^{k-j+1})$, $j = 1, \ldots, k$. Then using the form of (3.17) and similar to the proof of Theorem 2.7 we can prove that F has at most $k-1$ zeros near $h = h_0$ for $\varepsilon + |\delta - \delta^*|$ small. A contradiction occurs too as before. Also, as before, by using (3.17) F must have $k-1$ zeros in h near $h = h_0$ for $(\tilde{b}_1, \ldots, \tilde{b}_k)$

satisfying $0 < |\tilde{b}_j| \ll |\tilde{b}_{j+1}| \ll |\tilde{b}_k|$, $\tilde{b}_j \tilde{b}_{j+1} < 0$, $j = 1, \ldots, k-1$. Hence, the proof of the first conclusion of the theorem is completed.

To prove the second one, suppose $W(h_0) = 0$, $W'(h_0) \neq 0$ with $h_0 > \alpha$. Then by (3.9) and (3.10) there exist constants $\alpha_0, \ldots, \alpha_{k-2}$ such that

$$\big(J_1^{(k-1)}(h_0), \ldots, J_k^{(k-1)}(h_0)\big) = \sum_{j=0}^{k-2} \alpha_j \big(J_1^{(j)}(h_0), \ldots, J_k^{(j)}(h_0)\big).$$

Hence, by (3.12) we have

$$\bar{b}_k = \sum_{j=1}^{k-1} \bar{\alpha}_j \bar{b}_j, \tag{3.18}$$

for some constants $\bar{\alpha}_1, \ldots, \bar{\alpha}_{k-1}$. Therefore, by (3.15), (3.18), similar to (3.17) we obtain

$$\begin{aligned} F(h, \varepsilon, \delta) = I_1(h)\Bigg[&\sum_{j=1, j\neq k}^{k+1} \tilde{b}_j (h-h_0)^{j-1}\big(1 + \mathrm{O}\big(|h-h_0|^{k-j+2}\big)\big) \\ &+ \Bigg(\sum_{j=1}^{k-1} \bar{\alpha}_j \bar{b}_j + \mathrm{O}\big(\varepsilon\big|\bar{b}_1, \ldots, \bar{b}_{k-1}, \bar{b}_{k+1}\big|\big)\Bigg) \cdot (h-h_0)^{k-1}\Bigg], \end{aligned}$$

where

$$\tilde{b}_j = \bar{b}_j + \mathrm{O}\big(\varepsilon\big|\bar{b}_1, \ldots, \bar{b}_{k-1}, \bar{b}_{k+1}\big|\big), \quad j = 1, \ldots, k-1, k+1.$$

Note that

$$\bar{b}_j = \tilde{b}_j + \mathrm{O}\big(\varepsilon\big|\tilde{b}_1, \ldots, \tilde{b}_{k-1}, \tilde{b}_{k+1}\big|\big), \quad j = 1, \ldots, k-1, k+1.$$

We have further

$$F = I_1(h) \sum_{j=1}^{k+1} \tilde{b}_j (h-h_0)^{j-1}\big(1 + \widetilde{P}_j\big), \tag{3.19}$$

where

$$\begin{aligned} &\widetilde{P}_j = \big(\bar{\alpha}_j + \mathrm{O}(\varepsilon)\big)(h-h_0)^{k-j}, \quad j = 1, \ldots, k-1, \\ &\widetilde{P}_k = 0, \qquad \widetilde{P}_{k+1} = \mathrm{O}\big(|h-h_0|\big), \qquad \tilde{b}_k = \mathrm{O}\big(\varepsilon \tilde{b}_{k+1}\big). \end{aligned}$$

Using (3.19), similar to Theorem 2.7, one can prove that Eq. (3.1) has at most k limit cycles near L_{h_0} for ε small and $\delta \in D$.

Furthermore, we can prove that k limit cycles can appear in any neighborhood of L_{h_0}. In fact, let us take

$$h - h_0 = \lambda\varepsilon, \qquad \tilde{b}_j = B_j, \varepsilon^{k+1-j}, \quad j = 1, \ldots, k-1, k+1,$$

where B_j are such constants that the polynomial in λ

$$g(\lambda) = B_1 + B_2\lambda + \cdots + B_{k-1}\lambda^{k-2} + B_{k+1}\lambda^k$$

has k simple zeros $\lambda_j \neq 0$, $j = 1, \ldots, k$. Then (3.19) becomes

$$F = \varepsilon^k I_1(h)\bigl[g(\lambda) + \mathrm{O}(\varepsilon)\bigr],$$

which has k simple zeros $\tilde{\lambda}_j = \lambda_j + \mathrm{O}(\varepsilon)$ in λ, $j = 1, \ldots, k$. Thus F has k simple zeros $h_j = h_0 + \tilde{\lambda}_j\varepsilon$ in h, $j = 1, \ldots, k$.

For $h_0 = \alpha$, we may assume $\alpha = 0$. Then noting (3.18), similar to (2.64) we have

$$F = \bar{b}_1\bigl(1 + \mathrm{O}\bigl(|\varepsilon, r|\bigr)\bigr) + \bar{b}_2 r^2\bigl(1 + \mathrm{O}\bigl(|\varepsilon, r|\bigr)\bigr) + \cdots + \bar{b}_{k-1} r^{2(k-2)}$$
$$\times \bigl(1 + \mathrm{O}\bigl(|\varepsilon, r|\bigr)\bigr) + \bar{b}_{k+1} r^{2(k-1)}\mathrm{O}(\varepsilon) + \bar{b}_{k+1} r^{2k}\bigl(1 + \mathrm{O}\bigl(|\varepsilon, r|\bigr)\bigr).$$

It is easy to see that F has at most k zeros in $r > 0$ and $k-1$ zeros can appear.

For conclusion (iii), let $h \in [\alpha, \beta)$ and U be any neighborhood of L_h. Since W is analytic on $[\alpha, \beta)$ there exists $\bar{h} > h$ such that $W(\bar{h}) \neq 0$ and $L_{\bar{h}} \subset U$. By the conclusion (i) Eq. (3.1) has $k-1$ limit cycles in U for some (ε, δ). Since U is arbitrary it follows that L_h has cyclicity at least $k-1$. The proof is completed. □

By (3.17) and (3.19) one can prove easily

COROLLARY 3.3. *Suppose the conditions* (a)–(c) *hold with* $h_0 > \alpha$. *If*

(i) $W(h_0) \neq 0$, $1 \leqslant l \leqslant k-1$, *or*

(ii) $W(h_0) = 0$, $W'(h_0) \neq 0$, $1 \leqslant l \leqslant k$, $l \neq k-1$,

then there exists a function $\delta = \delta(\varepsilon)$ *with* $\delta(0) \in D$ *such that for* $\delta = \delta(\varepsilon)$ *and* $\varepsilon > 0$ *small Eq.* (3.1) *has a limit cycle of multiplicity* l *which approaches* L_{h_0} *as* $\varepsilon \to 0$.

REMARK 3.1. Under conditions of Theorem 3.3, the function W is analytic and has only isolated zeros on $[\alpha, \beta)$. Thus for any $\bar{h} \in (\alpha, \beta)$, there exist $\alpha \leqslant h_1 < h_2 < \cdots < h_l < \bar{h}$, $l \geqslant 0$ such that Eq. (3.1) has cyclicity $k-1$ at L_h for $h \in [\alpha, \bar{h}) - \{h_1, \ldots, h_l\}$.

EXAMPLE 3.2. Consider the Liénard system (2.66) discussed in Example 2.7

$$\dot{x} = y - \varepsilon \sum_{i=1}^{2n+1} a_i x^i, \qquad \dot{y} = -x.$$

As before, suppose that $\varepsilon > 0$ is small and $|a_i| \leqslant 1$ for $i = 1, \ldots, 2n+1$ with $n \geqslant 1$.

We claim that for each $h \geqslant 0$ the above system has cyclicity n at the circle $x^2 + y^2 = 2h$.

In fact, by the discussion in Example 2.7 we have

$$M(h)=\sum_{j=0}^{n} b_{j+1}h^{j+1}, \quad b_{j+1}=-2^{j+1}N_j a_{2j+1},\ N_j>0,\ j=0,\dots,n.$$

Comparing with (3.8) we may take

$$I_1(h)=h, \qquad J_j(h)=h^{j-1}, \quad j=1,\dots,n+1.$$

Hence, by (3.9) we have $W(h)=1$ for all $h\geqslant 0$. Thus the claim follows from Theorem 3.3 and Theorem 2.3(i).

EXAMPLE 3.3. For the system

$$\dot{x}=y-\varepsilon\sum_{i=0}^{n} a_{2i+1}x^{2i+1}, \qquad \dot{y}=-x(1-x),$$

we have

$$M(h)=\sum_{j=0}^{n} b_{j+1}h^{j+1}\big(1+\mathrm{O}(h)\big)$$

for $0<h\ll 1$. It follows $W(0)=1$. Thus, by Remark 3.1 there may exist finitely many values $0<h_1<\cdots<h_l$ in the interval $(0,\frac{1}{6})$ such that for any $h\in[0,\frac{1}{6})-\{h_1,\dots,h_l\}$ (respectively $h\in\{h_1,\dots,h_l\}$) the above system has cyclicity n (respectively at least n) at L_h.

DEFINITION 3.2. Let $U\in\mathbb{R}^2$ be a bounded set and N_0 a positive integer. We call N_0 *the cyclicity of* U for Eq. (3.1) with $\delta\in D$ and $\varepsilon>0$ small if the following are satisfied:

(i) For any given compact set $V\in U$ Eq. (3.1) has at most N_0 limit cycles in V for all $\delta\in D$ and $0<\varepsilon<\varepsilon_0$, where $\varepsilon_0=\varepsilon_0(V)>0$.

(ii) Eq. (3.1) has N_0 limit cycles for some (ε,δ) with $\delta\in D$ whose limits as $\varepsilon\to 0$ are in U.

The cyclicity of the set $G=\bigcup_{h\in(\alpha,\beta)} L_h$ for Eq. (3.1) is also called the cyclicity of the period annulus $\{L_h\colon \alpha<h<\beta\}$.

The following theorem gives a sufficient condition for finding the cyclicity of the open sets G and $G\cup L_\alpha$.

THEOREM 3.4. *Suppose* (3.7) *and* (3.8) *hold. Let the conditions* (a) *and* (b) *of Theorem* 3.3 *are satisfied. Let further, for each* $b\neq 0$ *the function* $N(h,b)$ *has at most* N_0 *zeros* (*taking multiplicity into account*) *in* $h\in[\alpha,\beta)$ (*respectively* $h\in(\alpha,\beta)$) *and for some* $b\neq 0$ *it has* N_0 *simple zeros in* $h\in(\alpha,\beta)$. *Then the cyclicity of the set* $G\cup L_\alpha$ (*respectively* G) *for Eq.* (3.1) *is* N_0 *as* G *is bounded.*

PROOF. For the sake of simplicity, we suppose that the singular point of Eq. (3.1) near L_α is at the origin with $\alpha = 0$. Then we can take a cross section l which is on the positive x-axis with an endpoint at the origin. In this case we have

$$A(h) = L_h \cap l = \big(a(h), 0\big), \quad B(h, \varepsilon, \delta) = \big(b^*(h, \varepsilon, \delta), 0\big)$$

for $h > 0$ small. By (2.4), the function b^* is analytic in a. Thus, $b^* = \mathrm{O}(a)$ and $H(B) - H(A) = \mathrm{O}(a^2) = \mathrm{O}(h)$. Then it follows from (3.3) that $F(h, \varepsilon, \delta) = \mathrm{O}(h)$. As before, suppose

$$\det \frac{\partial(b_1, \dots, b_k)}{\partial(\delta_1, \dots, \delta_k)} \neq 0.$$

Then we can solve $\delta_i = \tilde{\delta}_i(b, \delta_{k+1}, \dots, \delta_m)$, $i = 1, \dots, k$ from $b = b(\delta)$. Hence, for $h \in [0, \beta)$ we have

$$F(h, \varepsilon, \delta) = I_1(h)\big[N(h, b) + \mathrm{O}(\varepsilon)\big] \equiv F^*(h, \varepsilon, b, \delta_{k+1}, \dots, \delta_m). \tag{3.20}$$

By our assumption and Corollary 3.1 Eq. (3.1) can have N_0 limit cycles for some (ε, δ). What we need to do is to prove that for any given constants $0 < \lambda < \beta$ (respectively $0 < \mu < \lambda < \beta$) there exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$, $\delta \in D$ the function F has at most N_0 zeros in $h \in (0, \lambda]$ (respectively $[\mu, \lambda]$). If the conclusion is not true, then there exists a sequence $\{(\varepsilon_j, \delta_j)\}$ with $\varepsilon_j \to 0, \delta_j \to \delta_0 \in D$ as $j \to \infty$ such that $F(h, \varepsilon_j, \delta_j)$ has $N_0 + 1$ zeros h_{ij}, $i = 1, \dots, N_0 + 1$ in $h \in (0, \lambda]$ (respectively $[\mu, \lambda]$). We can suppose $h_{ij} \to h_{i0} \in [0, \lambda]$ (respectively $[\mu, \lambda]$) as $j \to \infty$. Then by (3.20)

$$N(h_{i0}, b_0) = 0, \quad i = 1, \dots, N_0 + 1,$$

where $b_0 = b(\delta_0)$. Hence, by Corollaries 2.3 and 3.2 the function $N(h, b_0)$ has at least $N_0 + 1$ zeros (multiplicity taken into account) in $[0, \lambda]$ (respectively $[\mu, \lambda]$). This implies $b_0 = 0$.

On the other hand, we have $F^*|_{b=0} = 0$ on $[0, \lambda]$. This yields

$$\begin{aligned} F^* &= \sum_{i=1}^{k} b_i F_i(h, \varepsilon, b, \delta_{k+1}, \dots, \delta_m) \\ &= I_1(h)\big[N(h, b) + \mathrm{O}\big(\varepsilon|b|\big)\big] \\ &= |b| I_1(h)\big[N(h, c) + \mathrm{O}(\varepsilon)\big] \\ &\equiv |b| I_1(h) G(h, \varepsilon, c, \delta_{k+1}, \dots, \delta_m), \end{aligned} \tag{3.21}$$

where $c = \frac{b}{|b|} = (c_1, \dots, c_k)$. Let

$$c^{(j)} = \frac{b(\delta_j)}{|b(\delta_j)|}, \quad j \geqslant 1.$$

Then $b(\delta_j) \to b(\delta_0) = 0$, $j \to \infty$. We can assume $c^{(j)} \to c_0$ with $|c_0| = 1$. Note that

$$F(h_{ij}, \varepsilon_j, \delta_j) = 0, \quad j \geqslant 1,\ i = 1, \dots, N_0 + 1.$$

It follows from (3.20) and (3.21) that $N(h_{i0}, c_0) = 0$, $i = 1, \dots, N_0 + 1$, which contradicts to our assumption. The proof is completed. □

REMARK 3.2. Let the conditions of Theorem 3.4 be satisfied. If G is unbounded, then there exists a compact set $V_0 \subset G \cup L_\alpha$ (respectively $\subset G$) such that for any compact set V satisfying $V_0 \subset V \subset G \cup L_\alpha$ (respectively $V_0 \subset V \subset G$) Eq. (3.1) has cyclicity N_0 on V.

EXAMPLE 3.4. Consider Eq. (2.66) again. Since

$$N(h, b) = \sum_{j=0}^{n} b_{j+1} h^j,$$

in this case, by the discussion in Example 3.2 for any compact set V containing the origin its cyclicity is n.

An open problem for Eq. (3.1) is: What is the maximal number of limit cycles on the plane? The above example suggests that the answer be n. The most difficult part is to study the number of limit cycles which disappear into infinity as $\varepsilon \to 0$.

3.2. *Existence of 2 and 3 limit cycles*

By (3.8), the function N can be written in the form

$$N(h, b) = b_1 - P(h, b_2, \dots, b_k). \tag{3.22}$$

The function P here is called a detection function and its graph on the (h, b_1) plane a detection curve. For a given system, an interesting problem is to find a point $(b_{10}, b_{20}, \dots, b_{k0})$ such that the line $b_1 = b_{10}$ and the curve $b_1 = P(h, b_{20}, \dots, b_{k0})$ have as many intersection points as possible.

The simplest case is that P is monotonic in $h \in (\alpha, \beta)$. Some sufficient conditions for some special systems on the monotonicity of P were obtained by Li and Zhang [84], and Han [49]. Obviously, if $P(\alpha, b_2, \dots, b_k) \neq P(\beta, b_2, \dots, b_k)$ for some $(b_2, \dots, b_k)$ then Eq. (3.1) can have a limit cycle.

Next, we give some conditions for Eq. (3.1) to have 2 or 3 limit cycles. We will suppose $\beta < \infty$ and L_β is a homoclinic loop with a hyperbolic saddle S on it. First, we prove

LEMMA 3.3. *Let $\phi(x, y)$ be a C^∞ function with $\phi(S) = 0$. Then along the orbit L_h of Eq.* (3.2) *the limit* $\lim_{h\to\beta} \oint_{L_h} \phi(x, y)\,\mathrm{d}t = \oint_{L_\beta} \phi(x, y)\,\mathrm{d}t$ *exists finitely.*

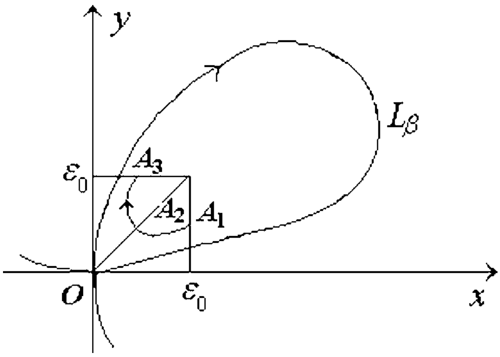

Fig. 10.

PROOF. Without loss of generality, we may suppose the saddle S is at the origin and the Eq. (3.2) has the form

$$\dot{x} = \lambda x + f(x, y), \quad \dot{y} = -\lambda y + g(x, y) \tag{3.23}$$

where f, $g = \mathrm{O}(|x, y|^2)$, which implies that $H(x, y) = \lambda xy + \mathrm{O}(|x, y|^3)$, $\lambda \neq 0$ with $\beta = 0$. For definiteness, assume $\lambda < 0$ and L_h locates in the first quadrant. See Fig. 10.

For $\varepsilon_0 > 0$ small take points $A_1 \in L_h \cap \{x = \varepsilon_0\}$, $A_2 \in L_h \cap \{x = y\}$, $A_3 \in L_h \cap \{y = \varepsilon_0\}$. Then the coordinate (a, a) of A_2 satisfies $a = a(h) = \sqrt{\frac{h}{\lambda}} + \mathrm{O}(h)$. Let $y = xu$. Then

$$H(x, y) = x^2\big[\lambda u + x\phi_0(x, u)\big],$$

where, by the integral mean value theorem, $\phi_0 \in C^\infty$. The orbit arc $\widehat{A_1A_2}$ satisfies the equation

$$H(x, y) = h, \quad a(h) \leqslant x \leqslant \varepsilon_0,$$

which is equivalent to

$$V(x, u, v) = 0, \quad a(h) \leqslant x \leqslant \varepsilon_0, \tag{3.24}$$

where $V(x, u, v) = \lambda u + x\phi_0(x, u) - v$, $v = \frac{h}{x^2} \in [\frac{h}{\varepsilon_0^2}, \frac{h}{a^2}]$. Note that

$$V\left(0, \frac{v}{\lambda}, v\right) = 0, \qquad V_u\left(0, \frac{v}{\lambda}, v\right) = \lambda.$$

Taking v as a parameter we can solve uniquely from (3.24)

$$u = u(x, v) = \frac{v}{\lambda} + \mathrm{O}(x) \quad \text{with } u_v = \frac{1}{\lambda} + \mathrm{O}(x).$$

It is clear that along $\widehat{A_1A_2}$

$$\frac{\phi(x,y)}{\lambda x+f(x,y)}=\frac{\phi(x,xu(x,v))}{\lambda x+f(x,xu(x,v))}=\frac{\phi(S)}{\lambda x}+R_0(x)+R_1(x,v)v, \tag{3.25}$$

where $R_0, R_1\in C^\infty$. Hence, by (3.23) and $\phi(S)=0$

$$\int_{\widehat{A_1A_2}}\phi(x,y)\,\mathrm{d}t=\int_{\varepsilon_0}^a(R_0+R_1v)\,\mathrm{d}x=\int_{\varepsilon_0}^a R_0\,\mathrm{d}x+\int_{\varepsilon_0}^a R_1v\,\mathrm{d}x.$$

Since

$$\left|\int_{\varepsilon_0}^a v\,\mathrm{d}x\right|=\left|h\int_{\varepsilon_0}^a\frac{\mathrm{d}x}{x^2}\right|=\left|\frac{h}{a}-\frac{h}{\varepsilon_0}\right|=\mathrm{O}\big(|h|^{1/2}\big),$$

it follows that

$$\lim_{h\to 0}\int_{\widehat{A_1A_2}}\phi(x,y)\,\mathrm{d}t=\int_{\varepsilon_0}^0 R_0\,\mathrm{d}x=\int_{\widehat{A_{10}A_{20}}}\phi(x,y)\,\mathrm{d}t\in\mathbb{R},$$

where $A_{i0}=\lim_{h\to 0}A_i$, $i=1,2,3$.

Similarly,

$$\lim_{h\to 0}\int_{\widehat{A_2A_3}}\phi(x,y)\,\mathrm{d}t=\int_{\widehat{A_{20}A_{30}}}\phi(x,y)\,\mathrm{d}t\in\mathbb{R}.$$

Also, it is obvious that

$$\lim_{h\to 0}\int_{\widehat{A_3A_1}}\phi(x,y)\,\mathrm{d}t=\int_{\widehat{A_{30}A_{10}}}\phi(x,y)\,\mathrm{d}t\in\mathbb{R}.$$

Thus

$$\begin{aligned}\lim_{h\to 0}\oint_{L_h}\phi(x,y)\,\mathrm{d}t&=\lim_{h\to 0}\int_{\widehat{A_1A_2}\cup\widehat{A_2A_3}\cup\widehat{A_3A_1}}\phi(x,y)\,\mathrm{d}t\\&=\int_{\widehat{A_{10}A_{20}}}\phi\,\mathrm{d}t+\int_{\widehat{A_{20}A_{30}}}\phi\,\mathrm{d}t+\int_{\widehat{A_{30}A_{10}}}\phi\,\mathrm{d}t\\&=\oint_{L_\beta}\phi(x,y)\,\mathrm{d}t\in\mathbb{R}.\end{aligned}$$

This finishes the proof. □

By Lemmas 3.2 and 3.3 we have immediately

COROLLARY 3.4. *Let L_h be oriented clockwise and expand with h increasing. Then for any C^∞ functions $\bar{p}(x,y)$ and $\bar{q}(x,y)$ the Abelian*

$$I(h) = \oint_{L_h} \bar{q}\,\mathrm{d}x - \bar{p}\,\mathrm{d}y$$

has the derivative

$$I'(h) = \oint_{L_h} (\bar{p}_x + \bar{q}_y)\,\mathrm{d}t,$$

and

$$I'(\alpha) = (\bar{p}_x + \bar{q}_y)(L_\alpha)T_\alpha, \qquad I'(h) = c_0 T_h + c_1 + c_2(h),$$

where T_h denotes the period of L_h, $T_\alpha = \lim_{h\to\alpha} T_h$, $c_0 = (\bar{p}_x + \bar{q}_y)(S)$, $c_1 = \oint_{L_\beta}[\bar{p}_x + \bar{q}_y - c_0]\,\mathrm{d}t$, $\lim_{h\to\beta} c_2(h) = 0$. Further, by (3.25) *it is easy to see that*

$$\lim_{h\to\beta} \frac{T_h}{\ln|h-\beta|} = p_0 < 0. \tag{3.26}$$

The following lemma gives formulas for computing the value of the function P and its derivative at $h = \alpha$.

LEMMA 3.4. *Let* (3.7), (3.8) *and* (3.22) *hold. Suppose*

$$(p_x + q_y)(L_\alpha, 0, \delta) = \sum_{j=1}^{k} d_{j0} b_j, \quad d_{10} \neq 0, \tag{3.27}$$

and

$$v_3^* = \sum_{j=2}^{k} v_{3j} b_j \quad \text{as } b_1 = -\frac{1}{d_{10}} \sum_{j=2}^{k} d_{j0} b_j, \tag{3.28}$$

where $v_3^* = \frac{\partial v_3}{\partial \varepsilon}|_{\varepsilon=0}$ and v_3 is the first focus value of Eq. (3.1) *at the focus near L_α obtained by using Theorem* 2.1, *and d_{j0} $(1 \leqslant j \leqslant k)$ and v_{3j} $(2 \leqslant j \leqslant k)$ are constants. Then*

$$P(\alpha, b_2, \ldots, b_k) = -\frac{1}{d_{10}} \sum_{j=2}^{k} d_{j0} b_j,$$

$$P'_h(\alpha, b_2, \ldots, b_k) = -\frac{4\pi}{K b_{10}} \sum_{j=2}^{k} v_{3j} b_j,$$

where $K > 0$, and $b_{10} = I_1'(\alpha)$ which can be obtained by Corollary 3.4.

PROOF. Let

$$I_j(h) = b_{j0}(h-\alpha) + b_{j1}(h-\alpha)^2 + \mathrm{O}\big(|h-\alpha|^3\big), \quad j = 1, \dots, k.$$

Then

$$M(h,\delta) = b_0^*(h-\alpha) + b_1^*(h-\alpha)^2 + \mathrm{O}\big(|h-\alpha|^3\big),$$

where $b_i^* = \sum_{j=1}^k b_{ji} b_j$, $i = 0, 1$. By Lemma 2.9 and Corollary 3.4 we have

$$b_0^* = 0 \quad \text{if and only if} \quad (p_x + q_y)(L_\alpha, 0, \delta) = 0,$$

and

$$v_3^* = \frac{K}{4\pi} b_1^* \quad \text{when } b_0^* = 0.$$

Therefore, by (3.27) $b_0^* = 0$ implies

$$b_1 = -\frac{1}{b_{10}} \sum_{j=2}^k b_{j0} b_j = -\frac{1}{d_{10}} \sum_{j=2}^k d_{j0} b_j,$$

and hence

$$v_3^* = \frac{K}{4\pi}\left[b_{11}b_1 + \sum_{j=2}^k b_{j1} b_j\right] = \frac{K}{4\pi b_{10}} \sum_{j=2}^k (b_{10}b_{j1} - b_{11}b_{j0}) b_j,$$

when $b_0^* = 0$. Hence, by (3.22) and (3.8)

$$\begin{aligned}
P(h, b_2, \dots, b_k) &= -\sum_{j=2}^k \frac{b_j I_j(h)}{I_1(h)} \\
&= -\sum_{j=2}^k \frac{b_j[b_{j0} + b_{j1}(h-\alpha) + \mathrm{O}(|h-\alpha|^2)]}{b_{10} + b_{11}(h-\alpha) + \mathrm{O}(|h-\alpha|^2)} \\
&= -\frac{1}{b_{10}^2} \sum_{j=2}^k b_j\big[b_{10}b_{j0} + (b_{10}b_{j1} - b_{11}b_{j0})(h-\alpha) \\
&\quad + \mathrm{O}\big(|h-\alpha|^2\big)\big] \\
&= -\frac{1}{b_{10}} \sum_{j=2}^k b_{j0} b_j - \frac{4\pi}{K b_{10}} v_3^*(h-\alpha) + \mathrm{O}\big(|h-\alpha|^2\big)
\end{aligned}$$

$$= -\frac{1}{d_{10}} \sum_{j=2}^{k} d_{j0} b_j - \frac{4\pi}{K b_{10}} \sum_{j=2}^{k} v_{3j} b_j (h - \alpha) + \mathrm{O}\big(|h - \alpha|^2\big).$$

Then the conclusion follows from (3.28) and the above easily. The proof is ended. □

Now we can give a condition for the existence of 2 limit cycles.

THEOREM 3.5. *Let L_h be oriented clockwise and expand with h increasing. Suppose* (3.7), (3.8), (3.22), (3,27) *and* (3.28) *are satisfied. If there exists $\delta_0 \in D$ such that*

$$\text{(i)}\qquad (p_x + q_y)(S, 0, \delta_0) = \sum_{j=1}^{k} d_j b_j(\delta_0),$$

$$\sigma(\delta_0) = \sum_{j=2}^{k} \left(d_j - \frac{\mathrm{d}_1 I_j(\beta)}{I_1(\beta)} \right) b_j(\delta_0) \neq 0;$$

$$\text{(ii)}\qquad \sigma(\delta_0) \sum_{j=2}^{k} v_{3j} b_j(\delta_0) < 0;$$

$$\text{(iii)}\qquad b_1'(\delta_0) \neq 0 \quad \textit{and}$$

$$b_1(\delta_0) = \begin{cases} \min\big\{P\big(\alpha, b_2(\delta_0), \ldots, b_k(\delta_0)\big), P\big(\beta, b_2(\delta_0), \ldots, b_k(\delta_0)\big)\big\} \\ \quad \textit{as } \sigma(\delta_0) I_1(\beta) < 0, \\ \max\big\{P\big(\alpha, b_2(\delta_0), \ldots, b_k(\delta_0)\big), P\big(\beta, b_2(\delta_0), \ldots, b_k(\delta_0)\big)\big\} \\ \quad \textit{as } \sigma(\delta_0) I_1(\beta) > 0 \end{cases}$$

then Eq. (3.1) *has at least two limit cycles for some* (ε, δ) *near* $(0, \delta_0)$.

PROOF. By (3.8) and (3.22) we have $M = I_1(b_1 - P)$. Hence $M' = I_1'(b_1 - P) - I_1 P'$ and

$$\frac{P'}{T_h} = \frac{I_1'}{T_h} \cdot \frac{b_1 - P}{I_1} - \frac{M'}{T_h} \cdot \frac{1}{I_1}.$$

Take $b_1 = P(\beta, b_2, \ldots, b_k)$ and apply Corollary 3.4 to function I_1 and M so that

$$\begin{aligned} \lim_{h \to \beta} \frac{P'}{T_h} &= -\frac{1}{I_1(\beta)} \lim_{h \to \beta} \frac{M'}{T_h} \bigg|_{b_1 = P(\beta, b_2, \ldots, b_k)} \\ &= -\frac{1}{I_1(\beta)} (p_x + q_y)(S, 0, \delta)|_{b_1 = P(\beta, b_2, \ldots, b_k)} \\ &= -\frac{1}{I_1(\beta)} \sigma(\delta). \end{aligned} \tag{3.29}$$

Then noting $I_1'(\alpha)I_1(\beta) > 0$, by Lemma 3.4 we know that for h near β the product $P_h'(\alpha, b_2, \dots, b_k)P_h'(h, b_2, \dots, b_k)$ has the same sign as $\sigma(\delta)\sum_{j=2}^{k} v_{3j}b_j$. Then it follows from (ii) that for $\delta = \delta_0$ the function $P(h, b_2, \dots, b_k)$ has a minimum or maximum in the interval (α, β). Therefore, by (3.29) and (iii) for some δ near δ_0 the function $b_1 - P(h, b_2, \dots, b_k)$ has two zeros with odd multiplicity in the interval (α, β). Then the conclusion follows from Theorem 3.1. This ends the proof. □

EXAMPLE 3.5. Consider the system

$$\dot{x} = y(1-y) - \varepsilon\big(x^3 - \delta x\big), \qquad \dot{y} = -x, \tag{3.30}$$

where $\varepsilon > 0$ is small and $\delta \in \mathbb{R}$ is bounded. For $\varepsilon = 0$, Eq. (3.30) has a family of periodic orbits giving by

$$L_h\colon\ \frac{1}{2}\big(x^2 + y^2\big) - \frac{1}{3}y^3 = h, \quad 0 < h < \frac{1}{6},\ y < 1.$$

By (3.4) we have

$$M(h,\delta) = \delta I_1(h) - 3I_2(h) = I_1(h)\big(\delta - P(h)\big)$$

where

$$P(h) = \frac{3I_2(h)}{I_1(h)},$$

$$I_j(h) = -\frac{1}{2j-1}\oint_{L_h} x^{2j-1}\,\mathrm{d}y = \iint_{Int.L_h} x^{2j-2}\,\mathrm{d}x\,\mathrm{d}y, \quad j = 1, 2.$$

By Corollary 3.4, $I_1(h) = 2\pi h + \mathrm{O}(h^2)$, $I_2(h) = \mathrm{O}(h^2)$. By Theorem 2.1, we have $v_3^* = -\frac{3}{8}$ as $\delta = 0$. Let

$$\sigma = \frac{3I_2(\frac{1}{6})}{I_1(\frac{1}{6})} = \delta_0.$$

Since $L_{1/6}$ can be represented as $x^2 = \frac{2}{3}(y-1)^2(y+\frac{1}{2})$, $-\frac{1}{2} \leqslant y \leqslant 1$, it is easy to see that

$$I_1\left(\frac{1}{6}\right) = 2\int_{-1/2}^{1}(1-y)\sqrt{1 - \frac{2}{3}(1-y)}\,\mathrm{d}y = \frac{6}{5},$$

$$I_2\left(\frac{1}{6}\right) = \frac{2}{3}\int_{-1/2}^{1}(1-y)^3\left(1 - \frac{2}{3}(1-y)\right)^{3/2}\mathrm{d}y = \frac{108}{385}.$$

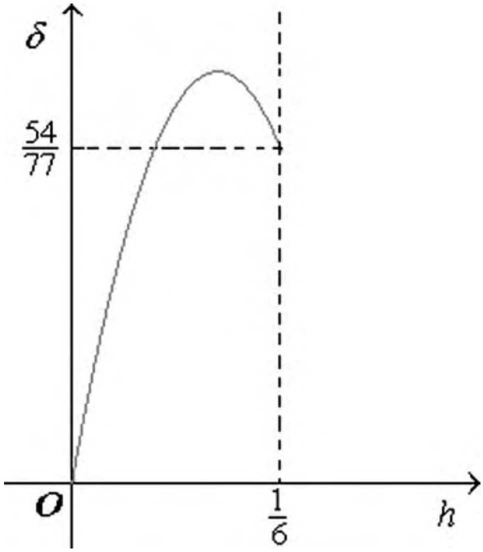

Fig. 11. The graph of $P(h)$ for Eq. (3.27).

Thus, $\delta_0 = \frac{54}{77}$. By Theorem 3.5 for Eq. (3.30) the system has two limit cycles for (ε, δ) near $(0, \frac{54}{77})$ with $\delta > \frac{54}{77}$. The graph of the function $\delta = P(h)$ on the (h, δ) plan is as shown in Fig. 11.

For the existence of three limit cycles, we have:

THEOREM 3.6. *Let L_h be oriented clockwise and expand with h increasing. Suppose* (3.7), (3.8), (3.22), (3.27) *and* (3.28) *are satisfied. Assume further*

(i) *there exists $\delta_0 \in D$ such that*

$$(p_x + q_y)(S, 0, \delta_0) \sum_{j=2}^{k} v_{3j} b_j(\delta_0) > 0, \quad b_1'(\delta_0) \neq 0,$$

$$b_1(\delta_0) = P\big(\alpha, b_2(\delta_0), \dots, b_k(\delta_0)\big) = P\big(\beta, b_2(\delta_0), \dots, b_k(\delta_0)\big);$$

(ii) *there exists δ^* near δ_0 such that*

$$I_1(\beta)(p_x + q_y)(S, 0, \delta_0) \\ \times \big[P\big(\alpha, b_1(\delta^*), \dots, b_k(\delta^*)\big) - P\big(\beta, b_1(\delta^*), \dots, b_k(\delta^*)\big)\big] > 0.$$

Then for $\varepsilon > 0$ small and $\delta = \delta^*$ Eq. (3.1) *has* 3 *limits cycles.*

We will give an example to show the way to prove the above theorem.

EXAMPLE 3.6. Consider a system of the form

$$\begin{cases} \dot{x} = y(1-y) - \varepsilon\big(x^3 - \delta_1 x + \delta_2 xy\big), \\ \dot{y} = -x. \end{cases} \tag{3.31}$$

We claim that for $\varepsilon > 0$, $\delta_1 < 0$, $\delta_2 < -54/11$ and $\varepsilon + |\delta_1| + |\delta_2 + 54/11|$ small Eq. (3.31) has 3 limit cycles.

For this system we have

$$M(h) = b_1 I_1(h) + b_2 I_2(h) + b_3 I_3(h),$$

where $b_1 = \delta_1$, $b_2 = -3$, $b_3 = \delta_2$ and $I_1(h)$, $I_2(h)$ are the same as in Example 3.5, and $I_3(h) = \oint_{L_h} xy\,\mathrm{d}y$ with $I_3(\frac{1}{6}) = -\frac{6}{35}$. Let

$$P(h, b_2, b_3) = -\frac{1}{I_1(h)}\big[b_2 I_2(h) + b_3 I_3(h)\big].$$

Also, let $b_{10} = \delta_{10} = 0$, $b_{20} = -3$, $b_{30} = \delta_{20} = 3I_2(\frac{1}{6})/I_3(\frac{1}{6}) = -\frac{54}{11}$. Then

$$b_{10} = P(0, b_{20}, b_{30}) = P\left(\frac{1}{6}, b_{20}, b_{30}\right) = 0.$$

Further, by Lemma 3.4 and (3.29) we have

$$P_h'(0, b_{20}, b_{30}) = \frac{1}{2}(3 + \delta_{20}) < 0,$$

$$\lim_{h \to \frac{1}{6}} \frac{P_h'(h, b_{20}, b_{30})}{T_h} = \frac{\delta_{20}}{I_1(\frac{1}{6})} < 0,$$

$$P\left(\frac{1}{6}, b_2, b_3\right) = -\frac{I_3(\frac{1}{6})}{I_1(\frac{1}{6})}(\delta_2 - \delta_{20}).$$

Denote by Γ^0 and Γ^* respectively the graph of the function $P(h, b_2, b_3)$ on the (b_1, h) plane for $\delta_2 = \delta_{20}$ and $\delta_2 = \delta_2^*$, where $\delta_2^* < \delta_{20}$ with $|\delta_2^* - \delta_{20}|$ small. See Fig. 12.

It is clear that for $\delta_1^* < 0$ and $|\delta_1^*|$ sufficiently small, the line $b_1 = \delta_1^*$ and Γ^*: $b_1 = P(h, -3, \delta_2^*)$ have at least 3 intersection points. Then the conclusion follows easily.

3.3. *Near-Hamiltonian polynomial systems*

We now give an important application of Theorem 3.3 to polynomial systems.

Consider a near-Hamiltonian polynomial system of the form

$$\begin{aligned} \dot{x} &= H_y(x, y, \mu) + \varepsilon p(x, y, a), \\ \dot{y} &= -H_x(x, y, \mu) + \varepsilon q(x, y, a), \end{aligned} \tag{3.32}$$

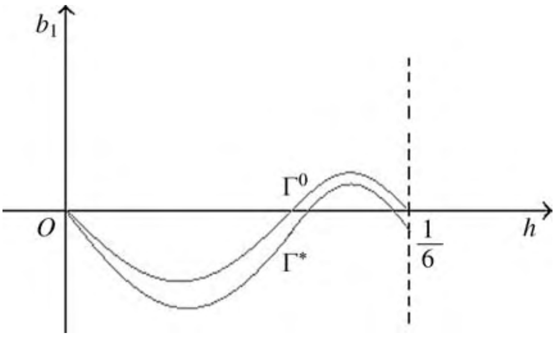

Fig. 12. Curves Γ^0 and Γ^*.

where

$$H(x, y, \mu) = \frac{1}{2}\left(x^2 + y^2\right) + \sum_{i+j=3}^{m} h_{ij} x^i y^j, \quad m \geqslant 3,$$

$$p(x, y, a) = \sum_{i+j=1}^{n} a_{ij} x^i y^j, \qquad q(x, y, a) = \sum_{i+j=1}^{n} b_{ij} x^i y^j,$$

and $\mu = \{h_{ij}\} \in \mathbb{R}^r$, $r = \frac{1}{2}(m+1)(m+2) - 6$, $a = \{a_{ij}, b_{ij}\} \in \mathbb{R}^{n^2+3n}$, $n \geqslant 2$. For each μ, there exists $\beta = \beta(\mu) \in (0, +\infty]$ such that for $h \in (0, \beta)$ the equation $H(x, y, \mu) = h$ defines a smooth closed curve L_h which surrounds a unique singular point (the origin) of the system (3.32) ($\varepsilon = 0$). By (3.4) we can write

$$M(h, a) = \sum_{i+j=0}^{n-1} c_{ij} I_{ij}(h) = \sum_{j=1}^{k} b_j I_j(h),$$

where

$$I_{ij} = \iint_{H \leqslant h} x^i y^j \, \mathrm{d}x \, \mathrm{d}y,$$

$$\{I_1, \ldots, I_k\} = \{I_{ij},\ 0 \leqslant i + j \leqslant n - 1\}, \quad I_1 = I_{00},$$

$$\{b_1, \ldots, b_k\} = \{c_{ij},\ 0 \leqslant i + j \leqslant n - 1\}, \quad k = \frac{1}{2} n(n+1).$$

Denote by $W(h, \mu)$ the Wronskian defined by (3.9). Then the equation $W(0, \mu) = 0$ defines an $(r-1)$-dimensional surface $\Sigma_{r-1}^{(1)}$ in $\mathbb{R}^r$ and the equations $W(h, \mu) = W_h'(h, \mu) = 0$ with $0 \leqslant h < \beta(\mu)$ define another $(r-1)$-dimensional surface $\Sigma_{r-1}^{(2)}$. Let

$$B_r = \mathbb{R}^r - \Sigma_{r-1}^{(1)} - \Sigma_{r-1}^{(2)}.$$

Then B_r is an open set in $\mathbb{R}^r$ with boundary $\Sigma_{r-1}^{(1)} \cup \Sigma_{r-1}^{(2)}$. For each $\mu \in B_r$, we have

$$W(0, \mu) \neq 0, \qquad \left|W(h, \mu)\right| + \left|W_h'(h, \mu)\right| \neq 0, \quad h \in [0, \beta).$$

Note that for each $\mu \in B_r$ and $\bar{h} \in (0, \beta)$, the analytic function $W(h, \mu)$ has only finitely many zeros $h_1, \ldots, h_l$ on $(0, \bar{h}]$. Then by Theorem 3.3 we have immediately:

THEOREM 3.7. *There exists an open set B_r in $\mathbb{R}^r$ whose boundary consists of $r - 1$ dimensional surfaces such that for each $\mu \in B_r$, and $\bar{h} \in (0, \beta)$ there may exist constants $0 < h_1 < \cdots < h_l$ with $l \geqslant 0$ such that*

(i) *for $h \in [0, \bar{h}] - \{h_1, \ldots, h_l\}$, Eq.* (3.32) *has cyclicity $\frac{1}{2}n(n + 1) - 1$ at L_h;*

(ii) *for $h \in \{h_1, \ldots, h_l\}$, Eq.* (3.32) *has cyclicity $\frac{1}{2}n(n + 1)$ at L_h.*

In particular, for each $\mu \in B_r$, Eq. (3.32) *has Hopf cyclicity $\frac{1}{2}n(n + 1) - 1$ at the origin.*

Roughly speaking, for almost all $\mu \in \mathbb{R}^r$, Eq. (3.32) has cyclicity $\frac{1}{2}n(n + 1) - 1$ or $\frac{1}{2}n(n + 1)$ at each L_h with $h \in [0, \beta)$.

REMARK 3.3. It is easy to see that in some cases, Theorem 3.7 is still valid if μ is a vector parameter of dimension r with $r < \frac{1}{2}(m + 1)(m + 2) - 6$.

As an application, let us consider quadratic systems. By Ye et al. [117], a quadratic system having a focus or center can be changed into the form

$$\begin{aligned} \dot{x} &= -y + \delta x + lx^2 + mxy + ny^2, \\ \dot{y} &= x(1 + ax + by). \end{aligned}$$

The system is Hamiltonian if and only if $\delta = m = b + 2l = 0$. Then taking $\delta = \varepsilon\delta_1$, $m = \varepsilon m_1$, $b = -2l_0 + \varepsilon b_1$, $l = l_0 + \varepsilon l_1$, $a = a_0 + \varepsilon a_1$, it becomes

$$\begin{aligned} \dot{x} &= H_y + \varepsilon\big(\delta_1 x + l_1 x^2 + m_1 xy + n_1 y^2\big), \\ \dot{y} &= -H_x + \varepsilon\big(a_1 x^2 + b_1 xy\big), \end{aligned} \tag{3.33}$$

where

$$H(x, y) = -\frac{1}{2}\big(x^2 + y^2\big) + l_0 x^2 y + \frac{1}{3}n_0 y^3 - \frac{1}{3}a_0 x^3.$$

For (3.33), we have

$$M(h) = \delta_1 I_1(h) + \bar{b}_1 I_2(h) + m_1 I_3(h), \quad \bar{b}_1 = b_1 + 2l_1,$$

where

$$\begin{aligned} I_1(h) &= \iint_{-H \leqslant h} \mathrm{d}x\, \mathrm{d}y, \quad I_2(h) = \iint_{-H \leqslant h} x\, \mathrm{d}x\, \mathrm{d}y, \\ I_3(h) &= \iint_{-H \leqslant h} y\, \mathrm{d}x\, \mathrm{d}y, \quad h \in [0, \beta). \end{aligned}$$

By [117], up to a positive constant, the first four focus values of the origin can be taken as

$$W_0 = \varepsilon V_0, \qquad W_i = \varepsilon V_i + \mathrm{O}(\varepsilon^2), \quad i = 1, 2, 3,$$

where

$$V_0 = \delta_1, \quad V_1 = m_1(n_0 + l_0) - a_0\bar{b}_1,$$
$$V_2 = 5m_1 a_0^2\big[(l_0 + n_0)^2(n_0 - 2l_0) - a_0^2 n_0\big].$$

Thus,

$$\det \frac{\partial(V_0, V_1, V_2)}{\partial(\delta_1, \bar{b}_1, m_1)} = -5a_0^3\big[(l_0 + n_0)^2(n_0 - 2l_0) - a_0^2 n_0\big].$$

Hence, by Lemma 2.9 and (3.12), it is easy to see that

$$W(0, a_0, l_0, n_0) = N_0 a_0^3\big[(l_0 + n_0)^2(n_0 - 2l_0) - a_0^2 n_0\big], \quad N_0 \neq 0.$$

Then from the discussion before Theorem 3.7 it follows that for any $(a_0, l_0, n_0) \in \mathbb{R}^3$ satisfying

$$a_0\big[(l_0 + n_0)^2(n_0 - 2l_0) - a_0^2 n_0\big] \neq 0,$$

the Hopf cyclicity of (3.33) at the origin is 2.

3.4. *Homoclinic bifurcation*

In the rest of the chapter, we introduce a way to find limit cycles in a neighborhood of the homoclinic loop L_β.

As a preliminary, we first discuss the stability of an isolated homoclinic loop. Consider a C^∞ planar system of the form

$$\dot{x} = f(x, y), \qquad \dot{y} = g(x, y). \tag{3.34}$$

Suppose that (3.34) has a homoclinic loop L with a hyperbolic saddle. Assume that the saddle is at the origin. Then on the stability of L we have

LEMMA 3.5. *Let*

$$c_1 = (f_x + g_y)(0), \qquad c_2 = \oint_L (f_x + g_y - c_1)\,\mathrm{d}t. \tag{3.35}$$

Then L is stable (*unstable*) *if* $c_1 < 0$, *or* $c_1 = 0$, $c_2 < 0$ ($c_1 > 0$, *or* $c_1 = 0$, $c_2 > 0$).

For a proof of the lemma, see Han [47,51], Han and Chen [57], and Feng and Qian [27].

Further, let $c_1 = c_2 = 0$. We can assume the functions f and g have the following form:

$$f(x, y) = \lambda x + O\big(|x, y|^2\big), \quad g(x, y) = -\lambda y + O\big(|x, y|^2\big), \quad \lambda > 0.$$

In this case, let

$$c_3 = \frac{1}{2\lambda}\left[f_{xxy} + g_{xyy} - \frac{1}{\lambda}(f_{xx}f_{xy} - g_{xy}g_{yy})\right]\bigg|_{x=y=0}, \tag{3.36}$$

which is called the first saddle value of (3.34) at the origin.

If f and g satisfy

$$f(x, y) = \lambda y + O\big(|x, y|^2\big), \qquad g(x, y) = \lambda x + O\big(|x, y|^2\big),$$

then, instead of (3.36), let

$$\begin{aligned} c_3 = -\frac{1}{2\lambda}\bigg[& f_{xxx} - f_{xyy} + g_{xxy} - g_{yyy} + \frac{1}{\lambda}\big(f_{xy}(f_{yy} - f_{xx}) \\ & + g_{xy}(g_{yy} - g_{xx}) - f_{xx}g_{xx} + f_{yy}g_{yy}\big)\bigg]\bigg|_{x=y=0}. \end{aligned} \tag{3.37}$$

Then we have the following lemma obtained by Han, Hu and Liu [58].

LEMMA 3.6. *Let $c_1 = c_2 = 0$, and $c_3 \neq 0$. Then L is stable if and only if one of the following occurs*:

(i) $c_3 < 0$, *a Poincaré map is well-defined near inside L and L is oriented anti-clockwise*;

(ii) $c_3 < 0$, *a Poincaré map is well-defined near outside L and L is oriented clockwise*;

(iii) $c_3 > 0$, *a Poincaré map is well-defined near outside L and L is oriented anti-clockwise*;

(iv) $c_3 > 0$, *a Poincaré map is well-defined near inside L and L is oriented clockwise.*

We remark that the formula (3.37) can be obtained from (3.36) by introducing a liner transformation $x = \frac{1}{\sqrt{2}}(u - v)$ and $y = \frac{1}{\sqrt{2}}(u + v)$ where u and v are new variables.

Going back to (3.1) we introduce the following four functions:

$$\begin{aligned} d_0(\delta) &= \oint_{L_\beta} (q\,\mathrm{d}x - p\,\mathrm{d}y)\bigg|_{\varepsilon=0} = M(\beta, \delta), \\ d_1(\delta) &= (p_x + q_y)(S, 0, \delta), \\ d_2(\delta) &= \oint_{L_\beta} \big[p_x + q_y - d_1(\delta)\big]\,\mathrm{d}t, \\ d_3(\delta) &= \frac{\partial c_3}{\partial \varepsilon}(\varepsilon, \delta)\bigg|_{\varepsilon=0}, \end{aligned} \tag{3.38}$$

where c_3 is obtained by (3.36) or (3.37).

The following theorem was obtained by Han [43].

THEOREM 3.8. (i) *Let there exist* $\delta_0 = (\delta_{10}, \dots, \delta_{m0}) \in D$, $m \geqslant 2$ *such that*

$$d_0(\delta_0) = d_1(\delta_0) = 0, \qquad d_2(\delta_0) \neq 0, \qquad \det \frac{\partial(d_0, d_1)}{\partial(\delta_1, \delta_2)}(\delta_0) \neq 0.$$

Then for any $\varepsilon_0 > 0$ *and neighborhood* U *of* δ_0 *there exists an open subset* $V_\varepsilon \subset U$ *for* $0 < \varepsilon < \varepsilon_0$ *such that Eq.* (3.1) *has* 2 *limit cycles near* L_β *for* $a \in V_\varepsilon$.

(ii) *Let there exist* $\delta_0 = (\delta_{10}, \dots, \delta_{m0}) \in D$, $m \geqslant 3$, *such that*

$$d_i(\delta_0) = 0, \quad i = 0, 1, 2, \qquad d_3(\delta_0) \neq 0, \qquad \det \frac{\partial(d_0, d_1, d_2)}{\partial(\delta_1, \delta_2, \delta_3)}(\delta_0) \neq 0.$$

Then for any $\varepsilon_0 > 0$ *and neighborhood* U *of* δ_0 *there exists an open subset* $V_\varepsilon \subset U$ *for* $0 < \varepsilon < \varepsilon_0$ *such that Eq.* (3.1) *has* 3 *limit cycles near* L_β *for* $a \in V_\varepsilon$.

We briefly outline the proof of the first conclusion. For definiteness, suppose L_β is oriented clockwise and the Poincaré map is well defined inside it as before. For $\varepsilon > 0$ small a unique saddle S_ε near S and two separatrices L_ε^u and L_ε^s near L_β exist. The directed distance between L_ε^s and L_ε^u on a cross section is given by (see [14,51,112])

$$d(\varepsilon, \delta) = \varepsilon N\big[d_0(\delta) + \mathrm{O}(\varepsilon)\big], \quad N > 0.$$

By the assumption, we can suppose $d'_{00} = \frac{\partial d_0}{\partial \delta_1}(\delta_0) \neq 0$. The implicit function theorem implies that a unique function $\delta_1 = \varphi_1(\varepsilon, \delta_2, \dots, \delta_m) = \varphi_{10}(\delta_2, \dots, \delta_m) + \mathrm{O}(\varepsilon)$ exists such that for $\varepsilon > 0$ and $|\delta - \delta_0|$ small $d(\varepsilon, \delta) \geqslant 0$ if and only if $d'_{00}[\delta_1 - \varphi_1] \geqslant 0$. Hence, a homoclinic loop L_ε^* appears near L_β if $\delta_1 = \varphi_1$.

Let $\delta_1 = \varphi_1$ and define

$$c_1(\varepsilon, \delta_2, \dots, \delta_m) = \varepsilon(p_x + q_y)(S_\varepsilon, 0, \delta) = \varepsilon\big[c_{10}(\delta_2, \dots, \delta_m) + \mathrm{O}(\varepsilon)\big],$$

where

$$c_{10} = (p_x + q_y)(S, 0, \delta)|_{\delta_1 = \varphi_{10}}.$$

Then our assumption implies that

$$c_{10}(\delta_{20}, \dots, \delta_{m0}) = 0, \qquad d'_{10} = \frac{\partial c_{10}}{\partial \delta_2}(\delta_{20}, \dots, \delta_{m0}) \neq 0.$$

Thus a unique function $\delta_2 = \varphi_2(\varepsilon, \delta_3, \dots, \delta_m) = \varphi_{20}(\delta_3, \dots, \delta_m) + \mathrm{O}(\varepsilon)$ exists such that

$$c_1(\varepsilon, \delta_2, \dots, \delta_m) \geqslant 0 \quad \text{if and only if} \quad d'_{10}[\delta_2 - \varphi_2] \geqslant 0.$$

Let $\delta_1 = \varphi_1, \delta_2 = \varphi_2$ and define

$$c_2(\varepsilon, \delta_3, \ldots, \delta_m) = \varepsilon \oint_{L_\varepsilon^*} (p_x + q_y)\,\mathrm{d}t = \varepsilon\big[c_{20}(\delta_3, \ldots, \delta_m) + \mathrm{O}(\varepsilon)\big],$$

where

$$c_{20}(\delta_3, \ldots, \delta_m) = \oint_{L_\beta} (p_x + q_y)|_{\delta_1=\varphi_{10},\, \delta_2=\varphi_{20}}\,\mathrm{d}t.$$

It is easy to see that

$$c_{20}(\delta_{30}, \ldots, \delta_{m0}) = d_2(\delta_0) \neq 0.$$

Without loss of generality, suppose $d_2(\delta_0) > 0$. Then for $\varepsilon > 0$, $\delta_1 = \varphi_1, \delta_2 = \varphi_2$ and $\varepsilon + |\delta - \delta_0|$ small L_ε^* is unstable by Lemma 3.6. Fix $\varepsilon > 0$ and δ_j near δ_{j0} for $j = 3, \ldots, m$, and vary δ_1 and δ_2 such that

$$\delta_1 = \varphi_1, \qquad 0 < |\delta_2 - \varphi_2| \ll 1, \qquad c_1(\varepsilon, \delta_2, \ldots, \delta_m) < 0.$$

Then, L_ε^* has changed its stability from unstable into stable and therefore an unstable limit cycle has appeared near it at the same time. Next, noting that we have assumed that L_β is oriented clockwise and the Poincaré map is well-defined inside it, we then change δ_1 such that $0 < |\delta_1 - \varphi_1| \ll |\delta_2 - \varphi_2|$, $d(\varepsilon, \delta) < 0$. Clearly, L_ε^* has broken and a stable limit cycle has appeared. Therefore, 2 limit cycles can appear near L_β.

We can summarize the method used above into 3 steps:

1. For $\varepsilon > 0$ fixed, a homoclinic loop L_ε^* appears for δ on a codimension 1 surface in $\mathbb{R}^m$ when $d(\varepsilon, \delta) = 0$.
2. Keep L_ε^* to appear and study its stability and then change the stability of L_ε^* in turn to produce limit cycles.
3. Find a final limit cycle by making the homoclinic loop broken.

This method was first used by Han [43] and then developed by Han and Chen [57] and Han, Hu and Liu [58] to study the number of limit cycles near a double homoclinic loop.

General theorems like Theorem 3.8 on double homoclinic bifurcations can be found in Han and Zhang [71]. Interesting applications of the method to quadratic and cubic systems et al. for the existence limit cycles are given in [65–68,70,72,122–126].

EXAMPLE 3.7. Consider

$$\begin{aligned} \dot{x} &= y - \varepsilon\big(a_1 x + a_2 x^2 + x^4\big), \\ \dot{y} &= x - x^2. \end{aligned} \tag{3.39}$$

For $\varepsilon = 0$, (3.39) has a first integral of the form

$$H(x, y) = \frac{1}{2}\big(y^2 - x^2\big) + \frac{1}{3}x^3,$$

which gives a family of periodic orbits

$$L_h\colon\ H(x,y)=h,\quad -\frac{1}{6}<h<0.$$

The limit of L_h as $h\to 0$ is a homoclinic loop L_0. We can prove that (3.39) has 2 limit cycles near L_0 for some (a_1,a_2).

In fact, for (3.39) we have

$$d_0(a_1,a_2)=-a_1I_{00}-2a_2I_{01}-4I_{03},$$

where

$$I_{0j}=\oint_{L_0}x^jy\,\mathrm{d}x=2\int_0^{3/2}x^{j+1}\sqrt{1-\frac{2}{3}x}\,\mathrm{d}x,\quad j=0,1,3.$$

It easy to get that

$$I_{00}=\frac{6}{5},\qquad I_{01}=\frac{36}{35},\qquad I_{03}=\frac{72}{77}I_{00}.$$

Thus,

$$d_0(a_1,a_2)=-I_{00}\left[a_1+\frac{12}{7}a_2+\frac{288}{77}\right].$$

By (3.38), we have further

$$\begin{aligned}
d_1(a_1,a_2)&=-a_1,\\
d_2(a_1,a_2)&=-\oint_{L_0}\left(2a_2x+4x^3\right)\mathrm{d}t\\
&=-2\int_0^{3/2}\frac{2a_2x+4x^3}{\sqrt{1-\frac{2}{3}x}}\,\mathrm{d}x=-6\left[2a_2+\frac{24}{5}\right].
\end{aligned}$$

Hence, if $a_1=0, a_2=-\frac{24}{11}$, then

$$d_0=d_1=0,\qquad d_2\neq 0.$$

It follows from Theorem 2.8 that Eq. (3.39) has 2 limit cycles for $\varepsilon>0$ small and some (a_1,a_2) near $(0,-\frac{24}{11})$.

From Han [51] we know Eq. (3.39) has at most 2 limit cycles on the plane for all $\varepsilon>0$ small and (a_1,a_2) bounded.

REMARK 3.4. Study of the number of limit cycles for near-Hamiltonian systems has been a significant and important part of bifurcation theory for decades. Most results can be divided into two aspects. One is to give a lower bound of the number for a given system with parameters. In other words, this aspect mainly concentrates on finding limit cycles as many as possible by choosing suitable parameters. The other is to study the maximal number of limit cycles and give an upper bound of it. In this section we mainly concern with the theory and methods in the first aspect. Theorem 3.1 is well known as Poincaré–Pontryagin–Andronov theorem. The formula in Lemma 3.2 was first obtained by Han [39]. Theorems 3.2, 3.3 and 3.7 are just recently obtained by Han, Chen and Sun [56]. Theorem 3.4 is from Han [51]. Results in Lemma 3.3 were given by Luo, Han and Zhu [100]. Here we present a new and simple proof. Conclusions of Lemma 3.4 were first established in Han and Ye [69]. The homoclinic bifurcation under the condition of Lemma 3.6 was studied in Zhu [130] in detail. Theorems 3.5 and 3.6 are new and obtained by the author. There have been many interesting results in the second aspect on the estimate of an upper bound for the number of limit cycles planar systems. For the theory and methods on this aspect, the reader can consult [13,15–18,23–26,42–44,49,74,76,83,107,109,115–118,127, 128]. For the bifurcation of periodic solutions of higher dimensional systems, see [14,16, 35,37,39,41,51–53,111,112,131].

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CHAPTER 5

Functional Differential Equations with State-Dependent Delays: Theory and Applications

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1. Introduction

Studies of single differential equations with state-dependent delayed or advanced arguments go back at least to Poisson [181], but as an object of a broader mathematical activity the area is rather young.

Most work during the past 50 years is devoted to equations with state-dependent delays, which arise as models in applications. A prominent example is the two-body problem of electrodynamics, which remained a mathematical terra incognita until R.D. Driver's work began to appear in the sixties of the last century.

The present survey reports about the more recent work on equations with state-dependent delays, with emphasis on particular models and on the emerging theory from the dynamical systems point of view. Several new results are presented. It will also become obvious that challenging problems remain to be solved.

State-dependent delays were addressed earlier in survey papers on the larger area of functional differential equations, notably by Halanay and Yorke [94] and Myshkis [165]. It is tempting to borrow as a motto from Halanay and Yorke [94] their nice statement

> *This ... proved once more, if necessary, that the delay existing at the present time between the moment results are obtained and the moment of publication (as well as the great number of publications which are very difficult to follow) makes it necessary to present from time to time such reports on yet unpublished results and unsolved problems.*

The simplest example of a differential equation with constant delay is the linear equation

$$y'(t) = ay(t-h)$$

with the fixed delay $h > 0$ and a parameter $a \in \mathbb{R}$. Analogues with state-dependent delay like

$$x'(t) = ax\big(t - r\big(x(t)\big)\big),$$

with a bounded delay map $r : \mathbb{R} \to [0, h]$, are already nonlinear in general. Both equations can be written in the same general form

$$x'(t) = f(x_t) \tag{1.0.1}$$

of a delay differential equation. Here $f : U \to \mathbb{R}^n$ is defined on a subset U of the set $(\mathbb{R}^n)^{[-h,0]}$ of all functions $\phi : [-h, 0] \to \mathbb{R}^n$. The *solution segment* $x_t : [-h, 0] \to \mathbb{R}^n$ is given by

$$x_t(s) = x(t+s), \quad -h \leqslant s \leqslant 0.$$

In case of the examples above, we have

$$n = 1, \quad U = \mathbb{R}^{[-h,0]}, \quad f(\phi) = a\phi(-h) \quad \text{and} \quad f(\phi) = a\phi\big(-r\big(\phi(0)\big)\big),$$

respectively.

The maps $f : U \to \mathbb{R}^n$ describing equations with state-dependent delay have in general less smoothness properties than those representing equations with constant delay, and the theory of Retarded Functional Differential Equations (RFDEs) which has been developed since the fifties of the last century (see, e.g., [54,98]) is not applicable to equations with state-dependent delay. This concerns already basic questions of existence, uniqueness and smooth dependence on initial data for the initial value problem (IVP)

$$x'(t) = f(x_t), \quad x_{t_0} = \phi \tag{1.0.2}$$

which for data $\phi \in U$ and for $t_0 \in \mathbb{R}$ is associated with Eq. (1.0.1). When in the sequel we speak of a solution $x : [t_0 - h, T) \to \mathbb{R}^n$, $t_0 < T \leqslant \infty$, to the IVP (1.0.2) it is always understood that x at least satisfies

$$x_t \in U \quad \text{for all } t \in [0, T), \quad x_{t_0} = \phi,$$

and that x is differentiable on (t_0, T) with

$$x'(t) = f(x_t) \quad \text{for all } t \in (0, T).$$

In Section 3 solutions of well-posed IVPs will even be continuously differentiable.

The following Section 2 describes examples of differential equations with state-dependent delays which arise in physics, automatic control, neural networks, infectious diseases, population growth, and cell production. Some of these models differ considerably from others, and most of them do not look simple. Typically the delay is not given explicitly as a function of what seems to be the natural state variable; the delay may be defined implicitly by a functional, integral or differential equation and should often be considered as part of the state variables.

Modelling systems with state-dependent delays seems to require extra care, perhaps because there is not much experience with this phenomenon. We tried to avoid models for which as yet no consistent motivation can be given.

The models in Section 2 indicate the types of equations to which the subsequent sections are confined. Not covered are, for example, nonautonomous systems, delays of the form $r = r(t) = a\,t + b\,x(t)$ or $r(t) = at + bx''(t)$ like in [186–188], and constructions of explicit solutions as in [104]. Also we do not say much about state-dependent delays in control theory. Early work in this area is found in [80–82,160].

Section 3 presents a framework for the study of the IVP (1.0.2). We analyze how state-dependent delays prevent the IVP (1.0.2) from being well-posed on open subsets U of familiar Banach spaces, and see that under mild smoothness hypotheses the IVP (1.0.2) *is* well-posed for data only in a submanifold of finite codimension. This *solution manifold* is given by the equation considered and generalizes the familiar domain of the generator of the semigroup given by a linear autonomous RFDE (as in [54,98]). On the solution manifold the IVP (1.0.2) with $t_0 = 0$ defines a semiflow of continuously differentiable solution operators. This resolves the problem of linearization for equations with state-dependent delay, which had been pointed out earlier by Cooke and Huang [48]. The widely known heuristic technique of *freezing the delay at equilibrium and then linearizing the resulting*

RFDE, often skilfully applied, can now be understood in terms of the true linearization. It is clarified why familiar characteristic equations for linear autonomous RFDEs can be used to analyze local dynamics generated by differential equations with state-dependent delay.

At stationary points the continuously differentiable solution operators have local center, stable, and unstable manifolds. It is shown that these stable and unstable manifolds of maps yield local stable and unstable manifolds also for the semiflow. In particular there is a convenient Principle of Linearized Stability. Center manifolds for the semiflow can not be immediately obtained as just described; they are constructed in Section 4.

Examples to which the results of Section 3 apply are given in [206,208]; for the proof in [208] that hyperbolic stable periodic orbits exist the smoothness results of Section 3 are indispensable. Most other work about which we report was accomplished before the basic theory of Section 3 was developed; some further work was done in parallel. We do not attempt to present these other results in the framework of Section 3. In fact, for many models it remains to be studied whether they fit into this framework or not.

In Section 4 a new result is proved, namely existence of Lipschitz continuous local center manifolds for the semiflow found in Section 3, at stationary points. The more technical proof that these center manifolds actually are continuously differentiable will appear elsewhere. An important open problem is to obtain more smoothness, as it was established for local unstable manifolds [129].

Section 5 is about local Hopf bifurcation, i.e., about the appearance of small periodic orbits close to a stationary point when a parameter in the underlying differential equation is varied and passes a critical value. We state a Hopf bifurcation theorem recently obtained by M. Eichmann [65], which seems to be the first such result for differential equations with state-dependent delays.

Section 6 presents results about differentiability of solutions with respect to parameters and initial data, for a certain class of nonautonomous differential equations with state-dependent delay. The framework is different from the one developed in Section 3. The IVP is considered for Lipschitz continuous initial data, and a quasi-normed space derived from Sobolev spaces turns out to be useful for studying differentiability under relaxed smoothness assumptions, which may be convenient for applications.

Section 7 deals with periodic orbits. The search for periodic solutions has been an important topic in the study of nonlinear autonomous delay differential equations since the sixties of the last century. By now, several methods have been developed in this area. The most general results on existence and global bifurcation employ fixed point theorems and the fixed point index. Others are based on the study of 2-dimensional invariant sets, or on Poincaré–Bendixson type analysis of plane curves which are obtained from evaluations like $x_t \mapsto (x(t), x(t-1))$ along certain solutions. There are local and global Hopf bifurcation theorems for RFDEs, the Fuller index counting periodic orbits is used, and certain symmetric periodic solutions can be obtained from associated ordinary differential systems. Not all approaches mentioned here have been tried for equations with state-dependent delays, which cause complications. We describe results which use the topological concept of ejectivity, and an approach which yields stable periodic orbits. Let us add here that topological tools (fixed point theorems, degree, coincidence degree) have also been employed to prove existence of periodic solutions to nonautonomous, *periodic* differential equations with state-dependent delay [43,138,139,221,225]; related work on nonautonomous equations is found in [50,55].

The topic of Section 8 is limiting behaviour with respect to the independent variable and with respect to parameters. Section 8.1 presents a study of a two-dimensional attractor with periodic orbits, which extends a part of a result for equations with constant delay. Section 8.2 reports about results of Mallet-Paret and Nussbaum on the precise asymptotic shape of periodic solutions in a singular perturbation problem. These results are genuine for equations with state-dependent delay, and involve work on unusual eigenvalue problems for so-called *max-plus operators*. Section 8.3 describes an approach to periodic solutions in case of small delay. Section 8.4 contains an application of the monotone dynamical systems theory which yields a generic convergence result, and Section 8.5 comments on further results about stability and oscillation properties.

Section 9 deals with numerical methods. The study of numerical approximation of solutions to differential equations with state-dependent delay goes back at least to the mid-sixties of the last century, and since then it has been an intensively investigated area. The section begins with a brief summary of continuous Runge–Kutta methods for ordinary differential equations and reports about modifications and extensions which are necessary in case of state-dependent delays.

Let us mention here a few out of many open questions, in addition to those addressed in the subsequent sections. Do equations with state-dependent delay generate semiflows with better smoothness properties than obtained in Section 3, on suitable invariant sets? The results from [129] on unstable manifolds point in this direction. A suspicion is that periodic solutions of equations with state-dependent delay may have stronger stability properties than their counterparts in related equations with constant delay. Can this be made precise and established, for suitable classes of equations? Complicated motion, like chaos, has not yet been rigorously shown to exist for equations with state-dependent delay. Very little is known about the general two-body problem of electrodynamics with two charged particles in a 3-dimension configuration space. Vanishing state-dependent delays, like in a collision in the two-body problem, are also limiting cases of advanced arguments; this indicates that a better understanding of more general differential equations with both delayed and advanced state-dependent arguments may be needed.

It is convenient to end this introduction with notation, for function spaces which occur frequently in the sequel. The Banach spaces of continuous, Lipschitz continuous, and continuously differentiable maps $\phi : [-h, 0] \to \mathbb{R}^n$ are denoted by

$$C = C\big([-h, 0]; \mathbb{R}^n\big), \quad C^{0,1} = C^{0,1}\big([-h, 0]; \mathbb{R}^n\big), \quad \text{and}$$
$$C^1 = C^1\big([-h, 0]; \mathbb{R}^n\big),$$

respectively. The norms on these spaces are given by

$$\|\phi\|_C = \max_{-h\leqslant t\leqslant 0}\big|\phi(t)\big|, \quad \|\phi\|_{C^{0,1}} = \|\phi\|_C + \sup_{t\neq s}\frac{|\phi(t)-\phi(s)|}{|t-s|},$$
$$\|\phi\|_{C^1} = \|\phi\|_C + \big\|\phi'\big\|_C,$$

respectively.

If X and Y are real or complex Banach spaces then $L(X, Y)$ denotes the Banach space of continuous linear mappings $T : X \to Y$, with the norm given by $\|T\| = \sup_{\|x\| \leqslant 1} \|Tx\|$.

2. Models and applications

A remark in [206] says that *state-dependent delays arise in various circumstances, but it seems not obvious how to single out a tractable class of equations which contains a large set of examples which are well motivated*. The difficulty of singling out a tractable class of equations to include many interesting models may prove to be an extremely valuable source to stimulate new mathematical techniques and theories. In this section we describe differential equations with state-dependent delay that arise from electrodynamics, automatic and remote control, machine cutting, neural networks, population biology, mathematical epidemiology and economics.

2.1. *A two-body problem of classical electrodynamics*

In Driver [58] (see also [57,59]), a mathematical model for a two-body problem of classical electrodynamics incorporating retarded interaction is proposed and analyzed. He considers the motion for two charged particles moving along the x-axis and substituted the expressions for the field of a moving charge, calculated from the Liénard–Wiechert potential, into the Lorentz–Abraham force law. Radiation reaction is omitted, but time delays are incorporated due to the finite speed of propagation, c, of electrical effects. As a result, the model is a system of delay differential equations involving time delays, which depend on the unknown trajectories. From this model and after some analysis, he obtains a system of six delay-differential equations for the evolution of the states, the velocities and the time delays.

To describe his model, we denote by $x_i(t)$ $(i = 1, 2)$ the positions of the two point charges on the axis in a given inertial system at time t, the time of an observer in that system. Let $v_i(t) = x_i'(t)$ $(i = 1, 2)$ be the velocities of the charges. As mentioned above, we omit radiation reaction but allow an external electric field, $E_{\text{ext}}(t, x)$, in the x-direction, that is assumed to be continuous over some open set D in the (t, x)-plane. Then the equation of motion of charge i is

$$\frac{m_i v_i'(t)}{[1 - v_i^2(t)/c^2]^{3/2}} = q_i E_j\big(t, x_i(t)\big) + q_i E_{\text{ext}}\big(t, x_i(t)\big), \quad i, j \in \{1, 2\},\ j \neq i, \tag{2.1.1}$$

where m_i is the rest mass and q_i is the magnitude of charge i, c is the speed of light, and $E_j(t, x)$ is the electric field at (t, x) due to other charge $j \neq i$. The magnetic field of charge j is not involved in this one-dimensional case.

The field at time t and at the point $x_i(t)$ produced by charge j is assumed to be that computed from the Liénard–Wiechert potentials. The expression for this field involves a time lag, $t - \tau_{ji}$, representing the instant at which a light signal would have to leave charge j in order to arrive at $x_i(t)$ at the instant t. Therefore, the delay $\tau_{ji}(t)$ must be a solution of the functional equation

$$\tau_{ji}(t) = \left|x_i(t) - x_j\big(t - \tau_{ji}(t)\big)\right|/c. \tag{2.1.2}$$

Clearly, $\tau_{ji}(t)$ cannot be written explicitly.

Because of the occurrence of time delays in the model equation (2.1.1), one needs to specify initial trajectories of the two charges over some appropriate interval $[\alpha, t_0]$. Consider now those initial trajectories and their extensions $(x_1(t), x_2(t))$ defined on some interval $[\alpha, \beta)$, where $\beta > t_0$, such that

(a) each $x_i'(t)$ is continuous and $|x_i'(t)| < c$ for all $t \in [\alpha, \beta)$;
(b) $x_2(t) > x_1(t)$ and $(t, x_i(t)) \in D$ for all $t \in [t_0, \beta)$;
(c) the two functional equations $\tau_{ji}^0 = |x_i(t_0) - x_j(t_0 - \tau_{ji}^0)|/c$ have solutions τ_{ji}^0, $i \neq j$, $i, j \in \{1, 2\}$.

Then Driver proves that $(x_1(t), x_2(t))$ is a solution of (2.1.1)–(2.1.2) if and only if it satisfies the following system of six delay differential equations for $t \in (t_0, \beta)$:

$$\begin{cases} x_i'(t) = v_i(t), \\ \tau_{ji}'(t) = \dfrac{(-1)^i v_i(t) - (-1)^i v_j(t - \tau_{ji}(t))}{c - (-1)^i v_j(t - \tau_{ji}(t))}, \\ \dfrac{v_i'(t)}{[1 - v_i^2(t)/c^2]^{3/2}} = \dfrac{(-1)^i a_i c}{\tau_{ji}^2(t)} \cdot \dfrac{c + (-1)^i v_j(t - \tau_{ji}(t))}{c - (-1)^i v_j(t - \tau_{ji}(t))} + q_i E_{\text{ext}}\big(t, x_i(t)\big)/m_i, \end{cases} \tag{2.1.3}$$

where $\tau_{ji}(t_0) = \tau_{ji}^0$, $a_i = q_1 q_2/(4\pi\epsilon_0 m_i c^3)$ (a constant, and in particular, ϵ_0 is the dielectric constant of free space), and $(i, j) = (1, 2)$ or $(2, 1)$.

It is shown in Driver [58] that if given initial trajectories satisfy condition (a) for $\alpha \leqslant t \leqslant t_0$, condition (b) at t_0, and condition (c), and if $E_{\text{ext}}(t, x)$ is Lipschitz continuous with respect to x in each compact subset of D and if the initial velocity of each particle is Lipschitz continuous, then a unique solution does exist. This solution can be continued as long as the charges do not collide ($\lim x_1(t) = \lim x_2(t)$ as t approaches the right endpoint of the maximal interval for existence) and neither $(t, x_1(t))$ nor $(t, x_2(t))$ approaches the boundary D.

We remark here that in Driver and Norris [64], the above Lipschitz continuity for the initial velocities is relaxed to the integrability of the initial velocity on $[\alpha, t_0]$. In Driver [61], one special case was given where the positions and velocities of the particles *at some instant* will determine the state of the system. More precisely, in this example of electrodynamic equations of motion, instantaneous values of positions and velocities of the particles will determine their trajectories, if the solutions are defined for all future time. This property was frequently conjectured, asserted, or implicitly assumed, as in Newtonian mechanics and as indicated by the long list of related references in Driver [61], but this property should not be expected for general electrodynamic equations.

In the case where $E_{\mathrm{ext}}(t,x)=0$ for all $(t,x)\in\mathbb{R}^2$ and if $q_1q_2>0$ (two point charges of like sign), then $\lim_{t\to\infty}[x_2(t)-x_1(t)]=\infty$ and $|v_i(t)|\leqslant\bar{c}<c$ for all $t\geqslant\alpha$. This is a quite interesting result as it indicates that the delay $\tau_{ji}(t)$ may become unbounded, as such, one obtains a system of functional differential equation with unbounded state-dependent delays.

It is noted that if three-dimensional motions are considered, then one obtains a functional differential system of neutral type where the delays are dependent on the states, and the change rate of v_i at the current time also depends on its historical value $v_j'(t-\tau_{ji})$. More precisely, if we introduce a unit vector

$$u_i=\frac{x_i-x_j(t-\tau_{ji})}{c\tau_{ji}}$$

and a scalar quantity

$$\gamma_{ij}=1-\frac{1}{c}v_j(t-\tau_{ji})\cdot u_i$$

as Driver [62] does, where $\cdot$ indicates the dot or scalar product in $\mathbb{R}^3$ (note, of course, x_1, x_2 are now vectors in $\mathbb{R}^3$), then the Lorentz force law yields

$$v_i'(t)=\frac{q_i(1-|v_i|^2/c^2)^{1/2}}{m_i}\big[E_j+(v_i/c\cdot E_j)(u_i-v_i/c)-(v_i/c\cdot u_i)E_j\big], \tag{2.1.4}$$

where E_j is the retarded (vector-valued) electric field arriving at x_i at the instant t from particle j. This field, in $\mathbb{R}^3$, can be found from the Liénard–Weichert potentials as

$$\begin{aligned}E_j={}&\frac{kcq_j}{\tau_{ji}^2\gamma_{ij}^3}\big[u_i-v_i(t-\tau_{ji})/c\big]\big[1-|v_j|^2(t-\tau_{ji})\big]\\&+\frac{kq_j}{\tau_{ji}\gamma_{ij}^3}u_i\times\big(\big[u_i-v_j(t-\tau_{ji})/c\big]\times v_j'(t-\tau_{ji})\big),\end{aligned} \tag{2.1.5}$$

where $k>0$ is a constant depending on the units, and $\times$ indicates the vector cross product in $\mathbb{R}^3$. The dynamical adaptation for τ_{ji} is given by

$$\tau_{ji}'(t)=\frac{u_i\cdot[v_i-v_j(t-\tau_{ji})]}{c\gamma_{ij}}. \tag{2.1.6}$$

In the above discussions, the motion of each particle is influenced by the electromagnetic fields of the others, and due to the finite speed of the propagation of these fields, the model equations describing the motion of charged particles via action at a distance will involve time delays which depends on the state of the whole system. In Driver [63] and in Hoag and Driver [111], it is noted that if one considers that the basic laws of physics are symmetric with respect to time reversal, then the existence of these delays implies that there should also be advanced terms in the equations, and thus one is led to a system of functional differential equations with mixed arguments (Hoag and Driver [111]), and of neutral type (Driver [63]).

In summary and in conclusion, despite the fact that much of the work by Driver and his collaborators on electrodynamics was published nearly 40 years ago, many interesting questions related to the fundamental issues of electrodynamics remain unsolved mathematically and Driver's models remain as a source of inspiration for the theoretical development and a testing tool for new results.

2.2. *Position control*

State-dependent delays arise naturally in automatic position control if in the feedback loop running times of signals between the object of study and a reference point are taken into account.

In [205], the following simple and idealized situation is considered: An object moves along a line and regulates its position relative to an obstacle by means of signals which are reflected by the obstacle. Let $x(t)$ denote the position of the object at time $t \in \mathbb{R}$. The aim of control is that the object should not collide with the obstacle located at $-w < 0$, and that the object should be close to a preferred position at distance w from the obstacle, i.e., at $0 \in \mathbb{R}$. The difficulty for the control is that measurement of the position via signal running times takes time during which the object is moving. More precisely, assume that signals travel from the object to the obstacle at a speed $c > 0$ and are reflected. The object senses the reflected signals and measures the signal running time $s(t)$ between the emission and detection at time t:

$$cs = \left|x\big(t - s(t)\big) + w\right| + \left|x(t) + w\right|.$$

Then it uses s to compute a distance d from the obstacle according to

$$d = \frac{c}{2}s.$$

This seems to be reasonable since it gives the true distance at time t at least if at times $t - s(t)$ and t the object is in the same position. We must however emphasize that in general d is only a *computed* length and not the true distance, and that we consider a situation where there is no direct, immediate access to the true position.

Depending on the computed distance $d - w$ from the preferred the object adjusts its speed in size and direction, with a reaction time lag $r > 0$ which is assumed to be constant. This negative feedback mechanism is then described by the differential equation

$$x'(t) = v\big(d(t - r) - w\big)$$

where $v : \mathbb{R} \to \mathbb{R}$ represents negative feedback with respect to the preferred position $0 \in \mathbb{R}$ in the sense that

$$\delta v(\delta) < 0 \quad \text{for all } \delta \neq 0$$

holds. Therefore, we are led to the system

$$x'(t) = v\left(\frac{c}{2}s(t-r) - w\right), \tag{2.2.1}$$

$$cs(t) = \left|x\big(t - s(t)\big) + w\right| + \left|x(t) + w\right| \tag{2.2.2}$$

for positive parameters c, w, r and a negative feedback nonlinearity v. Notice that for motion $x : \mathbb{R} \to \mathbb{R}$ with speed $|x'|$ bounded by a constant $b < c$, that is, slower than the signal speed, the equation

$$s = \frac{1}{c}\big(\left|x(t-s) + w\right| + \left|x(t) + w\right|\big)$$

equivalent to (2.2.2) has a unique solution $s = \sigma(x|_{(-\infty,t]})$ because for given x and t the right-hand side of the equation defines a contraction $[0, \infty) \to [0, \infty)$. Then one can take the right-hand side of Eq. (2.2.2) with $t - r - \sigma(x|_{(-\infty,t-r]})$ and $t - r$ instead of $t - s$ and t, respectively, and replace $cs(t - r)$ in Eq. (2.2.1), which yields a single delay differential equation with state-dependent delay. Essentially the same reasoning shows that for Lipschitz continuous solutions $x : [-h, t_e) \to \mathbb{R}$, $h > 0$ and $t_e > 0$, which have Lipschitz constants strictly less than c and satisfy suitable boundedness conditions, the system (2.2.1)–(2.2.2) can be rewritten as a single delay differential equation of the form (1.0.1).

A closely related model, with an explicit expression for a fraction of the signal running time instead of Eq. (2.2.2) for the total signal running time, was mentioned earlier by Nussbaum [177]. A similar model was also proposed by Messer [161].

It is perhaps more realistic to replace the first order differential equation (2.2.1) with the constant reaction lag $r > 0$ by Newton's law

$$x'' = A$$

with an instantaneous restoring force A which depends on the computed distance d. Also friction might be taken into account. Such a model was studied in [208], by means of the fundamental theory presented in the next section.

We describe the main results from [205,208], namely existence of stable periodic orbits, in Section 7.3.

In [32,33] Büger and Martin study a case of velocity control which involves signal running times. They consider an object travelling along a line which, ideally, should have a certain prescribed constant velocity v_0 throughout its whole journey. The object regulates its velocity $v = x'$ by adjusting its acceleration $a = v'$ according to a negative feedback relation

$$a \cdot (v - v_0) < 0 \quad \text{for all } v \in \mathbb{R} \setminus \{v_0\},$$

that is, the object speeds up if $v < v_0$ and slows down when $v > v_0$. This strategy is remotely controlled by a base located at $x = 0$ to which the object transfers its current velocity v with a certain transmission velocity $c_1 > 0$, and the negative feedback information is then transmitted back from the base to the object with a transmission velocity $c > 0$. The total signal running time of the signals emitted from the object in the past and transmitted back from the base to the object as to arrive at time t is given by

$$s = s_1 + s_2$$

where

$$s_1 = \frac{1}{c_1}\bigl|x(t-s)\bigr| \quad \text{and} \quad s_2 = \frac{1}{c}\bigl|x(t)\bigr|.$$

If an additional constant reaction time $r > 0$ of the base is taken into account then the total transmission delay is

$$r + s(t)$$

where now

$$s(t) = \frac{1}{c_1}\bigl|x\bigl(t - r - s(t)\bigr)\bigr| + \frac{1}{c}\bigl|x(t)\bigr|.$$

The preceding equation and the second order differential equation

$$x''(t) = A\bigl(x'\bigl(t - r - s(t)\bigr) - v_0\bigr)$$

with a negative feedback nonlinearity $A : \mathbb{R} \to \mathbb{R}$ constitute the model. Büger and Martin investigated a simplification which neglects the running time of the signal from the object to the base, in which case the model is reduced to the single equation

$$x''(t) = A\left(x'\left(t - r - \frac{1}{c}\bigl|x(t)\bigr|\right) - v_0\right) \tag{2.2.3}$$

with an explicitly given state-dependent delay. It is shown in [32,33] that close to segments of constant velocity solutions $t \mapsto v_0 t + c$ there exist segments of solutions of Eq. (2.2.3) for which $t - r - \frac{1}{c}|x(t)|$ remains bounded from above by some T. In this case the object reacts only to velocities achieved before $t = T$, which may not be adequate for larger velocities reached later. The phenomenon occurs for solutions whose speed $|v| = |x'|$ approaches the signal speed c or grows beyond. Büger and Martin call it the *escaping disaster*. In [33] they design another control mechanism which overcomes this kind of instability of the constant velocity solutions.

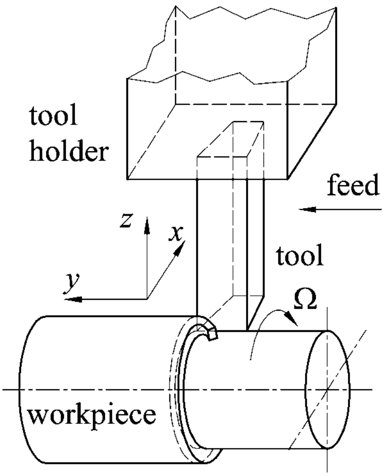

Fig. 1. A cutting process.

2.3. *Mechanical models*

State-dependent models have been proposed and investigated in mechanics, as well. Johnson [123] studied a steel rolling mill control system in 1972 where state-dependent delays have already been encountered. Nevertheless, state-dependent delay models have not frequently used in mechanics, since the required mathematical methods, like linearization techniques, were only recently developed (see Sections 3.4 and 3.6).

Insperger, Stépán and Turi [117] proposed a two degree of freedom model for turning process. This models a machine tool where a workpiece is rotating, the tool cuts the surface that was formed in the previous cut, see Fig. 1. The chip thickness is determined by the current and a previous position of the tool and the workpiece. In standard models the time delay between two succeeding cuts is considered to be equal to the period of the workpiece rotation. More realistic models given in the machine tool literature include the feed motion and the consequent trochoidal path of the cutter tooth. In this case the time delay between the succeeding cuts is not constant, it changes periodically in time. Time periodic delays also arise in the model of varying spindle speed machining. If the regeneration process is modeled more accurately, then the vibration of the tool is also included in the model, and this results a model with state-dependent delay. The system can be modeled as a two degree of freedom oscillator that is excited by the cutting force, so the governing equations are

$$m\ddot{x}(t) + c_x\dot{x}(t) + k_x x(t) = F_x, \tag{2.3.1}$$

$$m\ddot{y}(t) + c_y\dot{y}(t) + k_y y(t) = -F_y, \tag{2.3.2}$$

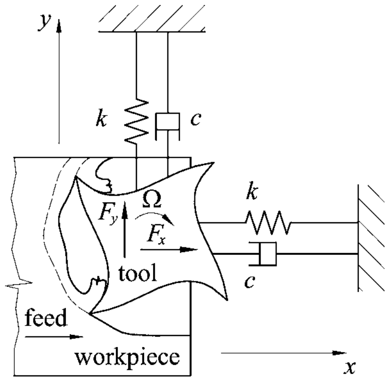

Fig. 2. A milling process.

where the associated model parameters are mass m, damping c_x, c_y and stiffness k_x, k_y. The x and y components of the cutting force are, respectively,

$$F_x = K_x w h^q, \quad F_y = K_y w h^q,$$

where K_x and K_y are the cutting coefficients in the x and y components, w is the depth of cut, h is the chip thickness, and the exponent $0 < q < 1$ is constant. The time delay between the present and the previous cuts is determined by the equation

$$R\Omega\tau(x_t) = 2R\pi + x(t) - x\big(t - \tau(x_t)\big), \tag{2.3.3}$$

where Ω is the spindle speed given in [rad/s] and R is the radius of the workpiece. This is an implicit equation for the time delay, and τ depends on the solution, as well. The chip thickness satisfies

$$h = v\tau(x_t) + y(t) - y\big(t - \tau(x_t)\big),$$

where v is the feed speed. Therefore the model equations are

$$m\ddot{x}(t) + c_x\dot{x}(t) + k_x x(t) = K_x w\big(v\tau(x_t) + y(t) - y\big(t - \tau(x_t)\big)\big)^q,$$

$$m\ddot{y}(t) + c_y\dot{y}(t) + k_y y(t) = -K_y w\big(v\tau(x_t) + y(t) - y\big(t - \tau(x_t)\big)\big)^q,$$

where the delay function is defined by (2.3.3). In [117] the linearized stability of an equilibrium solution was also studied using the method of [101] and [108]. It was shown that the linearized equation is almost equal to the standard constant delay machine tool vibration equation, the difference is a term with a small coefficient.

A related problem was studied in [116], where a two degree freedom model of milling process is considered. In this machine a tool with equally spaced teeth rotating with constant spindle speed, and cuts the surface that was formed in the previous cut, see Fig. 2. In this case the system again can be described by (2.3.1)–(2.3.2), but now the damping and stiffness parameters are equal in both x and y directions: $c_x = c_y = c$, $k_x = k_y = k$, and it was shown that the model equations have the form

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = \sum_{j=1}^{N} \alpha_{x,j}(t)\Big(R\big(1 - \cos(\Omega\tau_j - \vartheta)\big) + \big(v\tau_j + x(t-\tau_j) - x(t)\big)\sin\varphi_j(t) + \big(y(t-\tau_j) - y(t)\big)\cos\varphi_j(t)\Big)^q, \tag{2.3.4}$$

$$m\ddot{y}(t) + c\dot{y}(t) + ky(t) = \sum_{j=1}^{N} \alpha_{y,j}(t)\Big(R\big(1 - \cos(\Omega\tau_j - \vartheta)\big) + \big(v\tau_j + x(t-\tau_j) - x(t)\big)\sin\varphi_j(t) + \big(y(t-\tau_j) - y(t)\big)\cos\varphi_j(t)\Big)^q. \tag{2.3.5}$$

Here N is the number of teeth,

$$\alpha_{x,j}(t) = wg\big(\varphi_j(t)\big)\big(K_t\cos\big(\varphi_j(t)\big) + K_n\sin\big(\varphi_j(t)\big)\big),$$
$$\alpha_{y,j}(t) = wg\big(\varphi_j(t)\big)\big(K_n\cos\big(\varphi_j(t)\big) - K_t\sin\big(\varphi_j(t)\big)\big),$$

K_t and K_n are tangential and normal cutting coefficients, g is a screen function, it is equal to 1 if the jth tooth is active, and it is 0 if not, $\phi_j(t) = -\Omega t + (j-1)\vartheta$, Ω is the spindle speed, $\vartheta = 2\pi/N$ is the pitch angle, and the time delays $\tau_j = \tau_j(t, x_t, y_t)$ are defined by the implicit relations

$$\big(v\tau_j + x(t-\tau_j) - x(t)\big)\cos\phi_j(t) - \big(y(t-\tau_j) - y(t)\big)\sin\phi_j(t) = R\sin(\Omega\tau_j - \vartheta) \tag{2.3.6}$$

for $j = 1, \ldots, N$. It is easy to check that the functions on the right-hand sides of (2.3.4) and (2.3.5) are periodic in time with period $\bar{\tau} = 2\pi/(N\Omega)$, and the time delay functions τ_j are periodic in time with period $T = N\bar{\tau}$. Moreover, $\tau_j(t + \bar{\tau}, x_t, y_t) = \tau_{j-1}(t, x_t, y_t)$ gives the connection between time delays associated to two succeeding cuts. Note that if the vibration of the tool is not included in the delay model, i.e., $x(t) = 0$ and $y(t) = 0$, then Eq. (2.3.6) is simplified to

$$v\tau_j\cos\phi_j(t) = R\sin(\Omega\tau_j - \vartheta),$$

which yields that the delay depends only on time. This case of time periodic delay was investigated earlier in the literature. If the feed is negligible relatively to the diameter, i.e., $v\tau_j \ll R$, then if we substitute $v\tau_j = 0$ into the previous equation we get $\sin(\Omega\tau_j - \vartheta) = 0$.

This gives the constant delays $\tau_j = \bar{\tau} = \vartheta/\Omega$ that was usually used in standard milling models. Linearized stability of model (2.3.4)–(2.3.5) was also studied in [116] using the results of [101] and was compared to the stability of the standard time-periodic time delay models.

2.4. *Delay adaptation in neural networks and distributed systems*

A synaptic connection between two neurons is referred as to a delay line if the signal transmission is delayed. In neural circuits, time delays arise because interneural distances and axonal conduction times are finite [5,141–143,147,163,219]. In several sensory systems, delay lines are essential for coordinating activities. Examples include the auditory system of barn owls, echo location in bats, and the lateral line system of weakly electric fish. See [41] and the survey article [40]. Delay lines are also important for managing distributed systems, for such systems a fundamental problem concerns how the flow of information from distinct, independent components can be best regulated to optimize a prescribed performance of the network. For example, in parallel computing machines the asynchronous output of independent processors must be integrated to yield well-defined results.

Several possible biophysical mechanisms can be envisioned by which adjustable delays could be achieved, and recent development in the physiology of synapses and dendrites suggests that not only synaptic weights, but also synaptic delays vary [1,215] and changing synaptic delays have significant impact on the neural signal processing.

Time delays have important influence on learning algorithms. As noted in [19], not only delays affect the learning of other parameters such as gains, time constants or synaptic weights but also *delays themselves may be part of the adjustable parameters of a neural system so as to increase the range of its dynamics*. There exist numerous examples of finely tuned delays and delay lines, and *certainly many delays are subject to variations, for instance during the growth of an organism*.

Much of the existing work related to delay adaptation in neural networks have been concentrating on the fine tuning of a selected set of parameters in architectures already endowed of a certain degree of structure. In these applications, the delays are arranged in orderly arrays of delay lines, these delay lines are essentially part of a feedforward network for which the learning task is much more simple and the delays adjust on a slow time scale. In [199], the successive parts of a spoken word are delayed differentially in order to arrive simultaneously onto a unit assigned to the recognition of that particular word. Baldi and Atiya [19] viewed this as a "time warping" technique for optimal matching in the sense that for a given input $I(t)$, the output of the ith delay line is given by the convolution

$$O_i(t) = \int_0^\infty K(i,s)I(t-s)\,\mathrm{d}s,$$

where $K(i,s)$ is a Gaussian delay kernel

$$K(i,s) = \frac{1}{\sqrt{2\pi}\sigma}\mathrm{e}^{-(s-iT)^2/(2\sigma^2)}$$

and T is a parameter used to discretize the possible delays. In this application, the adjustment of delays on a slow time scale may take place across several speakers during the basic development of a speech recognizer and thus the adjustment is independent of the dynamics of the network. However, it is also desirable that delays adjust on a fast scale to adapt in real time to one particular speaker during normal functioning. The above work of Baldi and Atiya, as well as those in [51,166], develops models for delay adaptation with the help of a global teacher signal. It is shown in [51] that a network with adaptable delays can achieve smaller errors than a network with fixed delays, if they both have the same number of neurons and connections. Other rules and algorithms have been developed by which excitatory postsynaptic potentials from different synapses can be gradually pulled into coincidence, see, for example, [113].

Examples of self-organized delay adaptation can be found in [73]. For example, time delays in the optic nerve are equalized [192], signals in visual callosal axons arrive simultaneously at all axonal endings [115], and neurons in vitro can inhibit the formation of a myelin sheet by firing at a low frequency [193].

Two different mechanisms, delay shift and delay selection, are investigated in [73,74] for the self-organized adaptation of transmission delays in the nervous system.

To formulate the model for a network consisting of a large number of presynaptic neurons and one postsynaptic neuron which receives its input lines, we consider the idealized situation where there is a continuous set of input connections described by two functions, $\rho(\tau,t)$ and $\omega(\tau,t)$, for the delays and weights, respectively: $\rho(\tau,t)\,\mathrm{d}\tau$ gives the fraction of connections with delays in $[\tau,\tau+\mathrm{d}\tau]$, and $\omega(\tau,t)$ is the average weight of connections with delay τ. We assume that delays in the system adapt on a developmental time scale, and thus the model formulated below does not involve the internal dynamics of the network. In the continuous description, the input density $J(\tau,t)$ provided by the synapses at delay τ after presentation of a pattern has the simple form

$$J(\tau,t)=\rho(\tau,t)\omega(\tau,t).$$

The dynamics of the input are governed by two simultaneous equations: a balance equation for the input density

$$\frac{\partial}{\partial t}J(\tau,t)=-\frac{\partial}{\partial\tau}\big[J(\tau,t)v(\tau,t)\big]+Q(\tau,t), \tag{2.4.1}$$

and a continuity equation for $\rho(\tau,t)$, indicating the conservation of the number of neural connections

$$\frac{\partial}{\partial}\rho(\tau,t)=-\frac{\partial}{\partial\tau}\big[\rho(\tau,t)v(\tau,t)\big]. \tag{2.4.2}$$

The drift velocity, $v(\tau,t)$, and the source term, $Q(\tau,t)$, are defined according to the Hebbian principles. In particular, in the case of delay shifts, the weights are not modified and the source term vanishes. Therefore, the dynamics is completely governed by (2.4.2) where the velocity, $v = \mathrm{d}\tau/\mathrm{d}t$, of the delays realizes the Hebbian adaptation

$$v(\tau, t) = \gamma_\tau \int_{-\infty}^{+\infty} W_\tau(\tau - s) P(s, t)\, \mathrm{d}s, \tag{2.4.3}$$

and γ_τ denotes the learning rate, $W_\tau(x)$ denotes a learning window for delay adaptation. $W_\tau(x)$ should be positive when the presynaptic contribution precedes the postsynaptic spike, and negative in the other case, and this rule will adjust the delays such that their effects will align in time at the soma [72]. The distribution of spike times, $P(s, t)$, of a neuron depends on the input and its statistics, and this is assumed to be of the form $P(s, t) = \beta J(s, t)$ in the work of [73,74] and it is justified if the input is sufficiently high and if there is some random background activity. Note that delay shift mechanism assumes that the transmission delays themselves are altered. This mechanism is possible because transmission velocities in the nervous system can be altered, for example, by changing the length and thickness of dendrites and axons, the extent of myelination of axons, or the density and type of ion channels. See [74,75].

In the case of delay selection, the drift velocity of the delays vanishes and the total input of the postsynaptic neuron is not conserved. Equations (2.4.1) and (2.4.2) result in

$$\rho(\tau, t) \frac{\partial}{\partial t} \omega(\tau, t) = Q(\tau, t). \tag{2.4.4}$$

Again, the source term is derived from the Hebbian rule by

$$Q(\tau, t) = \gamma_\omega \omega(\tau, t) \rho(\tau, t) \int_{-\infty}^{+\infty} W_\omega(\tau - s) P(s, t)\, \mathrm{d}s,$$

with γ_ω denoting the corresponding learning rate, and W_ω representing a learning window that is maximal just before the time of spiking—leading to a selection of delay lines for which the effects align at soma.

In the aforementioned delay-adaptation models, it is assumed that delays in the system adapt on a developmental time scale, and thus the temporal development of ρ and ω is determined by an average over an ensemble of presynaptic input patterns. In the recent work [114], the self-organized adaptation of transmission delays is incorporated into the projective adaptive resonance theory developed in [38,39], and this self-organized adaptation of delay is driven by the dissimilarity between input patterns and stored patterns in a neural network designed for pattern recognition from data sets in high dimensional spaces. This adaptation can be regarded as a consequence of the Hebbian learning law, and the dynamic adaptation can be modeled by a nonlinear differential equation and hence a system of delay differential equations with adaptive delay is used.

We now describe the model for such a network that consists of two layers of neurons. Denote the nodes in F_1 layer (Comparison/Input Processing layer) by P_i, $i \in \Lambda_p := \{1, \ldots, m\}$; nodes in F_2 layer (Clustering layer) by C_j, $j \in \Lambda_c := \{m+1, \ldots, m+n\}$; the activation of F_1 node P_i by x_i, the activation of F_2 node C_j by y_j; the bottom-up weight from P_i to C_j by z_{ij}, the top-down weight from C_j to P_i by w_{ji}.

The STM (short-term memory) equations for neurons in F_1 layer are given by

$$\epsilon_p \frac{\mathrm{d}x_i(t)}{\mathrm{d}t} = -x_i(t) + I_i, \quad t \geqslant -1,\ i \in \Lambda_p, \tag{2.4.5}$$

where $0 < \epsilon_p \ll 1$, I_i is the constant input imposed on P_i.

The change of the STM for a F_2 neuron depends on the internal decay, the excitation from self-feedback, the inhibition from other F_2 neurons and the excitation by the bottom-up filter inputs from F_1 neurons. Namely, we have the STM equations for the committed neurons in F_2 layer:

$$\begin{aligned}\epsilon_c \frac{\mathrm{d}y_j(t)}{\mathrm{d}t} &= -y_j(t) + \big[1 - Ay_j(t)\big]\big[f_c\big(y_j(t)\big) + T_j(t)\big] \\ &\quad - \big[B + Cy_j(t)\big] \sum_{k\in\Lambda_c\setminus\{j\}} f_c\big(y_k(t)\big), \quad t \geqslant 0,\ j \in \Lambda_c,\end{aligned} \tag{2.4.6}$$

where $0 < \epsilon_c \ll 1$, $f_c : \mathbb{R} \to \mathbb{R}$ is a signal function, A, B, and C are non-negative constants, and the bottom-up filter input T_j is given by

$$T_j(t) = D \sum_{i\in\Lambda_p} z_{ij}(t) f_p\big(x_i\big(t - \tau_{ij}(t)\big)\big)\mathrm{e}^{-\alpha\tau_{ij}(t)}, \quad t \geqslant 0, \tag{2.4.7}$$

where D is a scaling constant, $f_p : \mathbb{R} \to \mathbb{R}$ is the signal function of the input layer. It is assumed here that the signal transmissions between two layers are not instantaneous and the signal decays exponentially at a rate $\alpha > 0$. This assumption that signal strength decays if the transmission is delayed can be replaced by the mechanism of *delay selection* by replacing (2.4.7) by

$$\begin{aligned}T_j(t) &= D_f \sum_{i\in\Lambda_p,\tau_{ij}(t)=0} z_{ij}(t) f_p\big(x_i(t)\big) \\ &\quad + D_d \sum_{i\in\Lambda_p,\tau_{ij}(t)>0} z_{ij}(t) f_p\big(x_i\big(t - \tau_{ij}(t)\big)\big)\end{aligned}$$

with two different weight factors $0 < D_d \ll D_f$.

The term τ_{ij} is the signal transmission delay between the cluster neuron C_j and the input neuron P_i. It is assumed that this delay is driven by the dissimilarity in the sense that the signal processing from the input neuron P_i to the cluster neuron C_j is faster when the output from P_i is similar to the corresponding component of w_{ji} of the feature vector $w_j = (w_{ji})_{i\in\Lambda_p}$ of the cluster neuron C_j. Therefore, we have

$$\beta \frac{\mathrm{d}\tau_{ij}(t)}{\mathrm{d}t} = -\tau_{ij}(t) + E\big[1 - h_{ij}(t)\big], \quad t \geqslant 0,\ i \in \Lambda_p,\ j \in \Lambda_c, \tag{2.4.8}$$

where $\beta > 0$, $E \in (0, 1)$ are constants and

$$h_{ij}(t) = S\big(d\big(f_p\big(x_i(t), w_{ji}(t)\big)\big), z_{ij}(t)\big)$$

is the similarity measure between the output signal $f_p(x_i(t))$ and the corresponding component $w_{ji}(t)$ of the feature vector of the cluster neuron C_j, with respect to the significance factor of the bottom-up synaptic weight $z_{ij}(t)$, here d is the usual absolute value function and $S:\mathbb{R}^+ \times [0,1] \to [0,1]$ is a given function, non-increasing with respect to the first argument and nondecreasing with respect to the second argument. Moreover, $S(0,1) = 1$ (the similarity measure is 1 with complete similarity and maximal synaptic bottom-up weight) and $S(+\infty, z) = S(x, 0) = 0$ for all $z \in [0,1]$ and $x \in \mathbb{R}^+$ (the similarity measure is 0 with complete dissimilarity or minimal bottom-up synaptic weight). Therefore, if $\tau_{ij}(0) = 0$ then from (2.4.8) it follows that $0 \leqslant \tau_{ij}(t) \leqslant E$ for all $t \in \mathbb{R}^+$, and moreover, if $h_{ij}(t) = 1$ on an interval $[0, b)$ for a given $b > 0$ then $\tau_{ij}(t) = 0$ for all $t \in [0, b)$.

The equation governing the change of the weights follows from the usual synaptic conservation rule and only connections to activated neurons are modified. The top-down weights are modified so that the template will point to the direction of the delayed and exponentially decayed outputs from F_1 layer. Therefore, we have

$$\gamma \frac{\mathrm{d}w_{ji}(t)}{\mathrm{d}t} = f_c\bigl(y_j(t)\bigr)\bigl[-w_{ji}(t) + f_p\bigl(x_i\bigl(t - \tau_{ij}(t)\bigr)\mathrm{e}^{-\alpha\tau_{ij}(t)}\bigr)\bigr] \tag{2.4.9}$$

for $t \geqslant 0$, $i \in \Lambda_p$, $j \in \Lambda_c$, where $\gamma > 0$ is a given constant.

The bottom-up weights are changed according to the competitive learning law and Weber law that says that LTM size should vary inversely with input pattern scale. Thus the LTM equations for committed neurons C_j in F_2 layer are

$$\begin{aligned} \delta \frac{\mathrm{d}z_{ij}(t)}{\mathrm{d}t} = {} & f_c\bigl(y_j(t)\bigr)\Bigl[\bigl(1 - z_{ij}(t)\bigr)h_{ij}(t)L - z_{ij}(t)\bigl(1 - h_{ij}(t)\bigr) \\ & - z_{ij}(t) \sum_{k \in \Lambda_p \setminus \{i\}} h_{kj}(t)\Bigr], \quad t \geqslant 0,\ i \in \Lambda_p,\ j \in \Lambda_c, \end{aligned} \tag{2.4.10}$$

where $0 < \delta \ll \gamma = \mathrm{O}(1)$ and $L > 0$ is a given constant.

Equations (2.4.5)–(2.4.10) give a system of functional differential equations where the dynamics of the delay $\tau_{ij}(t)$ is adaptive and is described by the nonlinear equation (2.4.8). The dynamics is investigated in [114] in the case where the signal functions f_p and f_c are step functions, though much remains to be done in the general case.

2.5. *Disease transmission and threshold phenomena*

Delay differential equations with state-dependent delay arise naturally from the modeling of infection disease transmission, the modeling of immune response systems and the modeling of respiration, where the delay is due to the time required to accumulate an appropriate dosage of infection or antigen concentration.

Following [201], we consider a particular infectious disease in an isolated population that is divided into several disjoint classes (compartments) of individuals: the susceptible class (those individuals who are not infective but are capable of contracting the disease and become infective), the exposed class (those who are exposed but not yet infective), and the infective class (those individuals who are capable of transmitting the disease to others), and the removed class (those who have had the disease and are dead, or have recovered and are permanently immune, or are isolated until recovery and permanent immunity occur). Let $S(t)$, $E(t)$, $I(t)$ and $R(t)$ be the size of each class at time t, and assume the following:

(i) the rate of exposure of susceptibles to infectives at time t is given by $-rS(t)I(t)$;

(ii) an individual who becomes infective at time t recovers from the infection (is thus removed from the population) at time $t+\sigma$, where σ is a positive constant;

(iii) an individual who is first exposed at time τ becomes infective at time t if

$$\int_\tau^t \left[\rho_1(x)+\rho_2(x)I(x)\right]\mathrm{d}x = m,$$

where ρ_1, ρ_2 are nonnegative functions and m is a given positive constant;

(iv) the population size remains to be a constant N.

The motivation for assumption (iii), the basis for a threshold model, is that human body can often control a small exposure to an infection, that is, there is a tolerance level below which the body's immune system can combat exposure to infection. When too large an exposure results, the individual contracts the disease. The amount of exposure received depends on the duration of the exposure and the amount of infectivity around the individual, that is assumed to be proportional to the number of infective individuals in the population. Thus, during the time interval $[t, t+h]$ an exposure of $\int_t^{t+h} \rho_2(x)I(x)\,\mathrm{d}x$ is accumulated where ρ_2 is a proportionality function which is a measure of the amount of infection communicated per infective (virulence). When the total exposure reaches the threshold m, the individual moves from class (E) to class (I). The term ρ_1 is the rate of accumulation of exposure independent of the number of infectives (such as constant input of virus from the external environment). In what follows, we consider the simple case where $\rho_1 = 0$, $\rho_2 = \rho$.

We assume the initial distribution of the infectives is given by a monotone function $I_0 : [-\sigma, 0] \to [0, \infty)$ such that $I_0(-\sigma) = 0$, and $I_0(0) > 0$ infective individuals are inserted in the population at $t = 0$. It is convenient to extend I_0 to the whole real line

$$I_0^{\mathrm{e}}(t) = \begin{cases} 0, & |t| > \sigma; \\ I_0(t), & -\sigma < t \leqslant 0; \\ I_0(0) - I_0(t-\sigma), & 0 \leqslant t \leqslant \sigma, \end{cases}$$

so that the extension $I_0^{\mathrm{e}}(t)$ describes the number of initial infectives who are still present as infectives at time $t \in [-\sigma, \infty)$.

From (i) and (ii), it follows that the number of new infectives introduced into the population at time t is given by $-\int_{t-\sigma}^t \frac{\mathrm{d}}{\mathrm{d}x} S(\tau(x))H(x)\,\mathrm{d}x$, where $H(x) = 0$, $x < 0$ and $H(x) = 1$, $x > 0$. Therefore,

$$I(t) = I_0^{\mathrm{e}}(t) - \int_{t-\sigma}^t \frac{\mathrm{d}}{\mathrm{d}x} S\bigl(\tau(x)\bigr)H(x)\,\mathrm{d}x, \quad t \geqslant 0.$$

For the infection to spread, some of the initial susceptible population must become infective before time σ. Thus, we assume there is $t_0 < \sigma$ so that the following "admissibility" condition,

$$\int_0^{t_0} \rho(x) I_0^{\mathrm{e}}(x)\,\mathrm{d}s = m,$$

must be met.

Then the model equations become

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} S(t) = -rS(t)I(t), \\ I(t) = I_0^{\mathrm{e}}(t) - \displaystyle\int_{t-\sigma}^{t} \frac{\mathrm{d}}{\mathrm{d}x} S\big(\tau(x)\big) H(x)\,\mathrm{d}x, \\ E(t) = S\big(\tau(t)\big) - S(t), \\ R(t) = N - S(t) - I(t) - E(t), \\ \displaystyle\int_{\tau(t)}^{t} \rho(x) I(x)\,\mathrm{d}x = m. \end{cases}$$

If we further adopt the convention that $\tau(t) = 0$ for $t \leqslant t_0$, then we obtain a state-dependent delay differential equation for $S(t)$:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} S(t) = -rS(t)I(t), \\ I(t) = I_0(t) + S\big(\tau(t-\sigma)\big) - S\big(\tau(t)\big), \\ \displaystyle\int_{\tau(t)}^{t} \rho(x) I(x)\,\mathrm{d}x = m, \quad t > t_0, \\ \tau(t) = 0, \quad t \leqslant t_0. \end{cases}$$

Other related models involving threshold conditions that determine the state-dependence of delay can be found in Gatica and Waltman [87–89], Hoppensteadt and Waltman [112], Smith [189], and Waltman [201]. Relatively complete references can be found in the work of Kuang and Smith [136,137], where the prototype equation takes the form

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} x(t) = -\nu x(t) - \mathrm{e}^{-\eta\tau} f\big(x(t-\tau)\big), \\ \displaystyle\int_{t-\tau}^{t} k\big(x(t), x(s)\big)\,\mathrm{d}s = m, \end{cases} \tag{2.5.1}$$

with nonnegative constants ν, η and m, and a positive function k, as well as a nonlinearity $f : \mathbb{R} \to \mathbb{R}$. Again, in the contents of epidemiological modeling, $x(t)$ may represent the proportion of a population which is infective at time t and the second equation in system (2.5.1) may reflect that an individual who is first exposed to the disease at time $t - \tau$ becomes infectious at time t if, during the interval from $t - \tau$ to t, a threshold level of exposure is accumulated where per unit time exposure depends on the infective fraction $x(s)$ via $k(x(t), x(s))$.

2.6. *Population models and size-dependent interaction*

Recent efforts in modeling state-dependent phenomena in population dynamics involve structured models, and state-dependent delay normally arises from a certain threshold condition. For example, in the work [170], Nisbet and Gurney considered insect populations which have several life stages (instars). After constructing a mathematical model consisting of an equation for the mass density function of the population, and under the homogeneity assumption for the population at each life stage, they reduced the model to a system of delay differential equations for the size of the population in each life stage. The threshold delays then appear due to the assumption that the insect must spend an amount of time in the larval stage sufficient to accumulate a threshold amount of food. See also [3].

This idea was adopted in the work of Arino, Hbid and Bravo de la Parra [8] for the growth of a population of fish where they introduce an additional stage between the eggs and the mobile larvae, the so-called (S1) larval stage. The state variable for this stage is $n_1(a,t)$, and the passage through (S1) is described with the help of another variable $q_1(a,t)$, the amount of food eaten up to time t by an individual entered in (S1) a units of time earlier. The introduction of this variable makes it possible to formulate the condition for any individual to have eaten a certain amount of food Q_1 (threshold) during the whole duration (bounded above by T_1) an individual can spend in (S1). The variation of ingested food is governed by the standard structured model (see [162]) subject to zero boundary and initial conditions that can be solved by integration along the characteristics to give

$$q_1(a,t) = \int_{t-a}^{t} \frac{K_1}{N_1(\sigma) + C_1}\,\mathrm{d}\sigma, \qquad t > a,$$

where K_1 is the quantity of food flowing into the species habitat per unit of volume, per unit of time, C_1 is the food(converted into a number of individuals) taken per unit of volume by consumers other than (S1) stage, and $N_1(t) = \int_0^{T_1} n_1(a,t)\,\mathrm{d}a$ is the population in stage (S1) which is susceptible to enter the next stage at time t per unit of volume. Hence an individual moves out of the (S1) stage exactly at time t if it entered in the (S1) stage $a_1(t)$ units of time earlier, where $a_1(t)$ is given by the threshold condition

$$\int_{t-a_1(t)}^{t} \frac{K_1}{N_1(\sigma) + C_1}\,\mathrm{d}\sigma = Q_1,$$

from which it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t} a_1(t) = -\frac{N_1(t - a_1(t)) - N_1(t)}{N_1(t) + C_1}. \tag{2.6.1}$$

The introduction of $a_1(t)$ through the above threshold condition ties the change of individual states to the dynamics of the population at the population level. From the definition of $a_1(t)$, one naturally has $n_1(a,t)=0$ for all $a>a_1(t)$. The dynamics for $n_1(a,t)$ with $0<a<a_1(t)$ is given by

$$\begin{cases} \dfrac{\partial}{\partial a}n_1(a,t)+\dfrac{\partial}{\partial t}n_1(a,t)=-f(a)n_1(a,t), & 0<a<a_1(t),\ t>0,\\ n_1(a,0)=0, \qquad n_1(0,t)=B(t), \end{cases}$$

where the function f is related to the individual resistance to fluctuation of food capacities, and $B(t)$ is the density of eggs laid per unit of volume at time t. In the special case where the eggs of a given year are determined directly in terms of passive larvae that survived some years earlier, we have

$$B(t)=kN_1(t-r)$$

for some positive constants r and k. This yields

$$\begin{cases} N_1(t)=k\displaystyle\int_{t-a_1(t)}^{t}\exp\left(-\int_0^{t-a}f(\sigma)\,\mathrm{d}\sigma\right)N_1(a-r)\,\mathrm{d}a,\\ \displaystyle\int_{t-a_1(t)}^{t}\frac{K_1}{N_1(\sigma)+C_1}\,\mathrm{d}\sigma=Q_1, \end{cases} \tag{2.6.2}$$

or, equivalently,

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}N_1(t)=kN_1(t-r)-\big(1-a_1'(t)\big)\exp\left(-\displaystyle\int_0^{a_1(t)}f(\sigma)\,\mathrm{d}\sigma\right)kN_1(t-r)\\ \qquad\qquad -k\displaystyle\int_{t-a_1(t)}^{t}f(t-a)\exp\left(-\int_0^{t-a}f(\sigma)\,\mathrm{d}\sigma\right)N_1(t-a)\,\mathrm{d}a,\\ \displaystyle\int_{t-a_1(t)}^{t}\frac{K_1}{N_1(\sigma)+C_1}\,\mathrm{d}\sigma=Q_1. \end{cases} \tag{2.6.3}$$

Note that in the above system of FDEs, there are two components of the delay: a constant delay r and a state-dependent delay.

As the second equation can be written as (2.6.1) by differentiation, it is natural that Arino, Hadeler and Hbid [7] and Magal and Arino [148] considered the system with adaptive delays

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}x(t)=-f\big(x\big(t-\tau(t)\big)\big),\\ \dfrac{\mathrm{d}}{\mathrm{d}t}\tau(t)=h\big(x(t),\tau(t)\big). \end{cases}$$

The existence of periodic solutions for the above system with adaptive delays is considered, and their results will be described in Section 7.2.

In the context of population dynamics, the delay arises frequently as the maturation time from birth to adulthood and this time is in some cases a function of the total population [11]. In [52], the consequence of size-dependent competition among the individuals is investigated by using a system of delay differential equations with state-dependent delays for a population consisting of only two distinct size classes, juveniles and adults, under the assumption that individuals are born at a size $s = s_b$ and remain juvenile as long as $s < s_m$, and individuals mature on reaching the maturation size threshold $s = s_m$. Therefore, if the density of juvenile and adult individuals at time t are denoted by $J(t)$ and $A(t)$, respectively, and if juveniles and adults feed on a shared resource, denoted by $F(t)$, then

$$s_m - s_b = \int_{t-\tau(t)}^{t} \epsilon_g a F(x)\,\mathrm{d}x, \tag{2.6.4}$$

where it is assumed that juvenile individuals feed at a rate aF and use all ingested food for growth in size with conversion efficiency ϵ_g, and $\tau(t)$ is the juvenile delay at time t. Assuming further that adult individuals feed at a rate qaF and use all ingested food for reproduction with conversion efficiency ϵ_b (here q is the ratio between the adult and juvenile feeding rate), and that the (instantaneous) mortality is inversely proportional to food intake with proportionality constant μ, de Roos and Persson obtain the following set of equations for the dynamics of $(J(t), A(t), F(t))$:

$$\begin{cases} \dfrac{\mathrm{d}F(t)}{\mathrm{d}t} = D - aFA - qaFA, \\ \dfrac{\mathrm{d}J(t)}{\mathrm{d}t} = R(t) - R\big(t-\tau(t)\big)P(t)\dfrac{F(t)}{F(t-\tau(t)))} - \dfrac{\mu}{aF(t)}J(t), \\ \dfrac{\mathrm{d}A(t)}{\mathrm{d}t} = R\big(t-\tau(t)\big)P(t)\dfrac{F(t)}{F(t-\tau(t))} - \dfrac{\mu}{qaF(t)}A(t), \end{cases} \tag{2.6.5}$$

where

$$R(t) = \epsilon_b q a F(t) A(t)$$

is the total population birth rate at time t and

$$P(t) = \exp\left(-\int_{t-\tau(t)}^{t} \frac{\mu}{aF(x)}\,\mathrm{d}x\right)$$

denotes the probability that an individual which should mature at time t has survived its juvenile period. The ratio $F(t)/F(t-\tau(t))$ counts for the change in maturation rate due to a change in the juvenile delay $\tau(t)$, and this can be derived by considering the model formulation in terms of a hyperbolic partial differential equation for the size distribution of the consumer population $n(t, s)$. The corresponding boundary conditions reflect the fact that the flow rate at time t across the boundary of the size domain $s = s_b$ into the juvenile class equals to total population birth rate $R(t)$ at that time, and that the flow rate at time t across the boundary $s = s_m$ into the adult class equals $\epsilon_g aF(t)n(t, s_m)$.

Note that differentiation of (2.6.4) then yields an equation to govern the evolution of the delay

$$\frac{d\tau}{dt} = 1 - \frac{F(t)}{F(t-\tau)}.$$

Similarly, in [23], it is showed that size dependent birth processes where the lifespan of individuals is a function of the current population size lead to a certain integral equation,

$$x(t) = \int_{t-L(x(t))}^{t} b\big(x(s)\big)\,ds, \tag{2.6.6}$$

differentiation then leads to the delay differential equation

$$\frac{d}{dt}x(t) = \frac{b(x(t)) - b(x(t-L(x(t))))}{1 - L'(x(t))b(x(t-L(x(t))))}. \tag{2.6.7}$$

One should emphasize that, despite a close correspondence between solutions of (2.6.6) and (2.6.7), caution must be employed in using one equation to investigate the other. For example, any constant is a solution of (2.6.7), whereas equation (2.6.6) only admits constant solutions whose values satisfy $x = b(x)L(x)$.

We note that early development of state-dependent delay models in economics and population biology was motivated by phenomenological considerations. For example, in [25, 146] the dynamics of price adjustment in a single commodity market that involves time delays due to production lags and storage policies is considered. The importance of the incorporation of a variable production delay is pointed out as certain commodities, once produced, may be stored for a variable period of time until market prices are deemed advantageous by the producer. In the field of population dynamics, motivated by the observation in Gambell [85] that for antarctic whale and seal populations, the length of time to maturity is a function of the amount of food (mostly krill) available, Aiello, Freedman and Wu [2] propose a stage-structured model of population growth, where the time to maturity is itself state dependent and the special form is suggested by the work [6] that describes how the duration of larval development of flies is a nonlinear increasing function of larval density. This model is further analyzed in [224]. An alternative version, which is designed to address the drawback in the Aiello–Freedman–Wu model that the maturation time for any newborn depends on the existing population size at the same time, is proposed in [76] based on the assumption that the maturation time depends on the size of the population which existed at the time of birth. In particular, if $r(t)$ is the date of birth of individuals who become mature at time $t \geqslant 0$, then the age length up to maturity at time t will be given by the function $\tau(z(r(t)))$. Consequently, $r(t)$ can be solved implicitly by

$$r(t) = t - \tau\big(z\big(r(t)\big)\big), \quad t \geqslant 0.$$

This modification in defining the density-delay dependent term results in the following system for the population with two stages: immature and mature, whose densities are denoted by $z(t) - x_m(t)$ and $x_m(t)$:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} z(t) = -\gamma z(t) + (\alpha + \gamma) x_m(t) - f\bigl(x_m(t)\bigr) x_m(t), \\ \dfrac{\mathrm{d}}{\mathrm{d}t} x_m(t) = \alpha x_m\bigl(r(t)\bigr) r'(t) \exp\bigl[-\gamma \tau\bigl(z\bigl(r(t)\bigr)\bigr)\bigr] - f\bigl(x_m(t)\bigr) x_m(t), \end{cases}$$

where $\gamma > 0$ and $f(x_m)$ are the mortality rates during the immature and mature stages. A detailed derivation of the above model and a careful analysis of conditions which assure existence, uniqueness, positiveness and boundedness of solutions can be found in [76], and the additional analysis such as the existence of steady-state solutions and how the delay affects stability can be found in [9,11].

Not much progress has been made for population dynamics involving both spatial dispersal and state-dependent delay, which would naturally involve certain types of partial functional differential equations. In [184], a new class of non-local partial functional differential equations is proposed for the evolution of a single species population that involves delayed feedback, where the delay such as the time length for reproduction, is selective and the selection depends on the status of the system. The abstract model in the work [184] is the following non-local partial differential equation with state-dependent selective delay:

$$\frac{\partial}{\partial t} u(t, x) + A u(t, x) + d u(t, x) = \bigl(F(u_t)\bigr)(x), \quad x \in \Omega, \tag{2.6.8}$$

where

$$\bigl(F(u_t)\bigr)(x) := \int_{-r}^{0} \left\{ \int_{\Omega} b\bigl(u(t+\theta, y)\bigr) f(x-y)\, \mathrm{d}y \right\} \xi\bigl(\theta, \|u(t)\|\bigr)\, \mathrm{d}\theta,$$

A is a densely-defined self-adjoint positive linear operator with domain $D(A) \subset L^2(\Omega)$ and with compact resolvent, Ω is a smooth bounded domain in $\mathbb{R}^n$, $f : (\Omega - \Omega) \to \mathbb{R}$ is a certain bounded function, $b : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz map and satisfies $|b(w)| \leqslant C_1|w| + C_2$ with $C_1 \geqslant 0$ and $C_2 \geqslant 0$, d is a positive constant. The function $u(\cdot, \cdot) : [-r, +\infty) \times \Omega \to \mathbb{R}$ is given such that for any t the function $u(t) \equiv u(t, \cdot) \in L^2(\Omega)$, $\|\cdot\|$ is the $L^2(\Omega)$-norm. The function $\xi : [-r, 0] \times \mathbb{R} \to \mathbb{R}$ represents the state-selective delay, for example, if

$$\xi(\theta, s) = \mathrm{e}^{-\beta(g(s)-\theta)^2}, \quad \theta \in [-r, 0],\ s \in \mathbb{R},$$

then the function g gives the coordinate of the maximum of ξ. Thus the system selects the maximal historical impact on the current change rate according to the system's current state.

2.7. *Cell biology and hematological disorders*

The monograph [162] contains a brief discussion of a modified structured model for the control of the bone marrow stem cell population which supplies the circulating red blood cell population. For this case, the maturing stem cell population is structured by a maturation variable and the rate of maturation is assumed to depend only on the total mature red blood cell population. Since a threshold level of maturation is required in order for an immature cell to enter the population, a threshold delay differential equation arises naturally.

Age-structured models have also been used in the study of hematological disorders, and these models can be reduced via the method of characteristics to systems of threshold type differential delay equations, see [24,26,149,185]. Here we describe the work [149].

We first note that the precursor cells begin from a pool of burst-forming units of the erythroid line (BFU) that have differentiated into a self-sustaining population which eventually leads to the production of mature erythrocytes. At some point, the BFUs further differentiate and start down a proliferative path that can ultimately produce erythrocytes. Early in this proliferative phase of development the hormone Epo, alone with other hormones, affects the number of BFU that become erythrocytes. Increase in the concentration of Epo may increase the number of BFU recruited to mature into erythrocytes. Alternatively, there may be a relatively constant supply of committed BFU, but only the cells tagged with sufficient Epo survive the rapidly proliferating colony forming units (CFU) phase to complete maturation.

Let $p(t, \mu)$ denote the population of precursor cells at time t with age μ, and let $V(E)$ be the velocity of maturation, which may depend on the hormone concentration, E. The maturity level μ for erythropoiesis can represent the accumulation of hemoglobin in the precursor cells. If $S_0(E)$ is the number of cells recruited into the proliferating precursor population, then the entry of new precursor cells into the age-structured model satisfies the boundary condition

$$V(E)p(t,0) = S_0(E). \tag{2.7.1}$$

Let μ_F be the maximum age for a cell reaching maturity, then the dynamics of the precursor cells is governed by the age-structured model

$$\left(\frac{\partial}{\partial t} + V(E)\frac{\partial}{\partial \mu}\right)p(t,\mu) = \big[\beta(\mu,E) - H(\mu)\big]p(t,\mu), \quad t > 0,\ 0 < \mu < \mu_F, \tag{2.7.2}$$

where $\beta(\mu, E)$ is the net birth rate for proliferating precursor cells, and $H(\mu)$ is the disappearance rate given by

$$H(\mu) = \frac{h(\mu - \bar{\mu})}{\int_{\mu}^{\mu_F} h(s - \bar{\mu})\,\mathrm{d}s}$$

and $h(\mu - \bar{\mu})$ is the distribution of maturity levels of the cells when released into the circulating blood, with $\bar{\mu}$ being the mean age of mature precursor cells and the normalization $\int_0^{\mu_F} h(\mu - \bar{\mu})\,\mathrm{d}\mu = 1$.

Let $m(t, \nu)$ be the population of mature non-proliferating cells at time t and age ν, and assume the mature cells age at a constant rate W. The boundary condition for cells entering the maturation population is given by

$$Wm(t, 0) = V(E)\int_0^{\mu_F} h(\mu - \bar{\mu})p(t, \mu)\,\mathrm{d}\mu. \tag{2.7.3}$$

The complete feedback involves the Epo level E, its growth is regulated by the total population of mature cells. Therefore, we need to determine the maximum age $\nu_F(t)$ of erythrocytes, which varies in t as the destruction of erythrocytes occurs by active removal of the oldest cells. We assume a constant erythrocyte removal rate Q, which can be justified by either assuming a constant supply of markers or a constant number of phagocytes that become satiated in their destruction of the oldest erythrocytes, then consideration of the moving boundary condition at $\nu = \nu_F(t)$ yields

$$\bigl(W - \nu_F'(t)\bigr)m\bigl(t, \nu_F(t)\bigr) = Q, \tag{2.7.4}$$

or equivalently,

$$\frac{\mathrm{d}}{\mathrm{d}t}\nu_F(t) = W - \frac{Q}{m(t, \nu_F(t))}. \tag{2.7.5}$$

To obtain this unusual moving boundary condition, we notice that when erythrocytes age, their cell membrane breaks down and marcophages destroy the least pliable cells. On the other hand, since we assume the macrophages are in constant supply and are saturated in their consumption of erythrocytes, the age of destruction of erythrocytes then varies. In particular, during the time interval $[t, t + \Delta t]$, one can use the Mean Value Theorem to find $\chi, \eta \in [t, t + \Delta t]$ so that

$$Q\Delta t + \bigl[\nu_F(t + \Delta t) - \nu_F(t)\bigr]m\bigl(\eta, \nu_F(\eta)\bigr) = W\Delta t m\bigl(\chi, \nu_F(\chi)\bigr)$$

from which (2.7.4) follows.

The dynamics of matured cells before its maximum age is governed by the age-structured model

$$\left(\frac{\partial}{\partial t} + W\frac{\partial}{\partial \nu}\right)m(t, \nu) = -W\gamma(\nu)m(t, \nu), \quad t > 0,\ 0 < \nu < \nu_F(t), \tag{2.7.6}$$

where $\gamma(\nu)$ is the age-dependent death rate of mature cells.

To complete the feedback cycle, we note that the Epo level E is governed by a negative feedback of the total population of mature cells

$$M(t) = \int_0^{\nu_F(t)} m(t, \nu)\,\mathrm{d}\nu, \tag{2.7.7}$$

via

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) = F\big(M(t)\big) - kE(t), \tag{2.7.8}$$

where k is the decay constant for the hormone and $F(M)$ is a monotonically decreasing function of M.

In the simple case, that we consider in the remaining part of this subsection, where $V(E) = W = 1$ and $\beta(\mu, E) = \beta(\mu)$ is independent of E, if the initial condition

$$p(0, \mu) = \phi(\mu), \qquad m(0, \nu) = \psi(\nu)$$

is given, then the method of characteristics yields that

$$p(t, \mu) = \begin{cases} \phi(\mu - t)\exp\left[\int_0^t F(\mu - t + w)\,\mathrm{d}w\right], & t < \mu; \\ S_0\big(E(t-\mu)\big)\exp\left[\int_{t-\mu}^t F(w - t + \mu)\,\mathrm{d}w\right], & t > \mu, \end{cases} \tag{2.7.9}$$

with $F(\mu) = \beta(\mu) - H(\mu)$, and

$$m(t, \nu) = \begin{cases} \psi(\nu - t)\exp\left[-\int_0^t (\nu + \sigma - t)\,\mathrm{d}\sigma\right], & t < \nu; \\ \int_0^{\mu_F} h(\mu - \bar{\mu})p(t - \nu, \mu)\,\mathrm{d}\mu \exp\left[-\int_0^\nu \gamma(\sigma)\,\mathrm{d}\sigma\right], & t > \nu. \end{cases} \tag{2.7.10}$$

If t is sufficiently large, then we get

$$M(t) = \int_0^{\nu_F(t)} \int_0^{\mu_F} h(\mu - \bar{\mu})p(t - \nu, \mu)\,\mathrm{d}\mu \exp\left[-\int_0^\nu \gamma(\sigma)\,\mathrm{d}\sigma\right]\mathrm{d}\nu. \tag{2.7.11}$$

Thus, (2.7.5), (2.7.8) and (2.7.11), with p given in (2.7.9) form a complete system of integro-differential equations with the delay ν_F being adaptive.

3. A framework for the initial value problem

3.1. *Preliminaries*

For differential delay equations with state-dependent delays it is less obvious than in case of time-invariant delays on which state space IVPs are well-posed. For initial data in the familiar space $C = C([-h, 0]; \mathbb{R}^n)$ solutions to equations with state-dependent delay are in general not unique in cases where for similar equations with constant delay the IVP is well-posed. Winston [216] gave the following example: The functions defined by

$$x(t) = t + 1 \quad \text{and} \quad x(t) = t + 1 - t^{3/2}$$

for small $t > 0$ both are solutions of the equation

$$x'(t) = -x\bigl(t - \bigl|x(t)\bigr|\bigr)$$

with initial values

$$x(t) = \begin{cases} -1, & \text{if } t \leqslant -1; \\ \frac{3}{2}(t+1)^{1/3} - 1, & \text{if } -1 < t \leqslant -\frac{7}{8}; \\ \frac{10}{7}t + 1, & \text{if } -\frac{7}{8} < t \leqslant 0. \end{cases}$$

Early results on existence, uniqueness, and continuous dependence for solutions of IVPs with state-dependent delays are due to Driver [57–62], who studied cases of the two-body problem of electrodynamics; see also work by Driver and Norris [64], Travis [198], and Hoag and Driver [111]. The latter investigated equations with both delayed and advanced state-dependent shifted arguments. Winston [216,218] studied uniqueness for special scalar equations, among others. Well-posed initial value problems are the basis for work on periodic solutions, notably by Nussbaum [174], Alt [4], Mallet-Paret and Nussbaum [151, 152,155], Kuang and Smith [136,137,190], Mallet-Paret, Nussbaum and Paraskevopoulos [156], Arino, Hadeler and Hbid [7], Krisztin and Arino [132], Magal and Arino [148], and in [205]. Further existence and uniqueness results are due to Gatica and Waltman [88,89] and Jackiewicz [119,121], and to Ito and Kappel [118] in an approach to more general IVPs. The delay differential and integral equations addressed in these results belong to special classes where the state-dependent delay appears explicitly or is defined implicitly by an additional equation. Typically, the IVP is uniquely solved for initial and other data which satisfy suitable Lipschitz conditions. Manitius [160] and Brokate and Colonius [29] deal with differentiability of operators given by the right-hand side of differential equations with state-dependent delay, in the context of control theory. Section 6 below reports about work of Hartung and Turi [106] who proved differentiability of solutions with respect to parameters—including Lipschitz continuous initial data—in Sobolev spaces and related quasi-normed spaces. Louihi, Hbid, and Arino [144] consider a class of equations with state-dependent delay in the framework of nonlinear semigroup theory. Their results are used in an approach of Ouifki and Hbid [178] to periodic solutions which is described in Section 8.3 below.

Let us now see where the difficulty with uniqueness arises if a given differential equation with state-dependent delay is written in the general form (1.0.1). Our discussion of the uniqueness problem will naturally lead us to a manifold on which Eq. (1.0.1) defines a continuous semiflow with continuously differentiable solution operators. We shall obtain continuously differentiable local stable and unstable manifolds of the semiflow at stationary points, also center manifolds for the solution operators, and a convenient Principle of Linearized Stability, among others.

We begin with an equation

$$x'(t) = g\big(x\big(t - r(x_t)\big)\big), \tag{3.1.1}$$

with a given map $g:\mathbb{R}^n \to \mathbb{R}^n$ and a given delay functional $r:U \to [0,h]$, for $h > 0$, $n \in \mathbb{N}$, and $U \subset (\mathbb{R}^n)^{[-h,0]}$. Equation (3.1.1) has the form (1.0.1) for

$$f = g \circ ev \circ \big(id \times (-r)\big)$$

where

$$ev: \big(\mathbb{R}^n\big)^{[-h,0]} \times [-h,0] \to \mathbb{R}^n$$

is the evaluation map defined by

$$ev(\phi, s) = \phi(s).$$

Notice that the restriction of ev to $C \times [-h,0]$ is not locally Lipschitz continuous: Lipschitz continuity would imply Lipschitz continuity of elements $\phi \in C$. Therefore in general f is not locally Lipschitz continuous on open subsets of C, and the familiar results on existence, uniqueness, and dependence of solutions on initial data and parameters for RFDEs from, say, [54,98] fail.

The difficulty just mentioned disappears if C is replaced by the smaller Banach space $C^1 = C^1([-h,0];\mathbb{R}^n)$ since the restricted evaluation map

$$Ev: C^1 \times [-h,0] \ni (\phi, s) \mapsto \phi(s) \in \mathbb{R}^n$$

is continuously differentiable, with

$$\mathrm{D}_1 Ev(\phi,s)\chi = Ev(\chi,s) \quad \text{and} \quad \mathrm{D}_2 Ev(\phi,s)1 = \phi'(s).$$

So, for $g:\mathbb{R}^n \to \mathbb{R}^n$ and $r:U \to [0,h]$, $U \subset C^1$ open, both continuously differentiable, the resulting functional

$$f = g \circ Ev \circ \big(id \times (-r)\big)$$

is continuously differentiable from U to $\mathbb{R}^n$, with

$$\begin{aligned}\mathrm{D}f(\phi)\chi &= \mathrm{D}g\big(\phi\big(-r(\phi)\big)\big)\big[\mathrm{D}_1 Ev\big(\phi, -r(\phi)\big)\chi - \mathrm{D}_2 Ev\big(\phi, -r(\phi)\big)\mathrm{D}r(\phi)\chi\big] \\ &= \mathrm{D}g\big(\phi\big(-r(\phi)\big)\big)\big[\chi\big(-r(\phi)\big) - \mathrm{D}r(\phi)\chi\phi'\big(-r(\phi)\big)\big]\end{aligned}$$

for $\phi \in U$ and $\chi \in C^1$.

However, yet another obstacle is in the way. Suppose $U \subset C^1$ is open, $f:U \to \mathbb{R}^n$ is continuously differentiable, and the IVP

$$x'(t) = f(x_t) \quad \text{for } t > 0, \quad x_0 = \phi, \tag{3.1.2}$$

is well-posed for $\phi \in U$. A solution $x:[-h, t_e) \to \mathbb{R}^n$, $0 < t_e \leqslant \infty$, has segments $x_t \in U \subset C^1$, $0 \leqslant t < t_e$. Therefore x is continuously differentiable, and the curve $[0, t_e) \ni t \mapsto x_t \in C^1$ is continuous. Continuity at $t = 0$ yields

$$\phi'(0) = x'(0) = f(x_0) = f(\phi),$$

a necessary condition on initial data which may or may not be satisfied.

In any case, the last equation suggests to consider the IVP (3.1.2) for initial data only in the closed subset

$$X_f = \{\phi \in U:\ \phi'(0) = f(\phi)\}$$

of $U \subset C^1$.

From now on, a solution is understood to be a continuously differentiable function $x:[-h, t_*) \to \mathbb{R}^n$, $0 < t_* \leqslant \infty$, which satisfies $x_t \in U$ for $0 \leqslant t < t_*$, $x_0 = \phi$, and $x'(t) = f(x_t)$ for $0 < t < t_*$.

Incidentally, notice that X_f is a nonlinear analogue of the domain

$$\{\phi \in C^1:\ \phi'(0) = L\phi\}$$

of the generator of the semigroup given by the solutions to the linear IVP

$$y'(t) = Ly_t, \quad y_0 = \phi \in C,$$

for $L: C \to \mathbb{R}^n$ linear continuous [54,98].

3.2. *The semiflow on the solution manifold*

There is a mild smoothness condition (S) on the functional $f: U \to \mathbb{R}^n$, $U \subset C^1$ open, which is often satisfied if f represents an equation with state-dependent delay, and which implies all the desired results, namely

(S1) f is continuously differentiable,

(S2) each derivative $Df(\phi)$, $\phi \in U$, extends to a linear map $D_e f(\phi): C \to \mathbb{R}^n$, and

(S3) the map

$$U \times C \ni (\phi, \chi) \mapsto D_e f(\phi)\chi \in \mathbb{R}^n$$

is continuous.

Let us see what condition (S) means for the example

$$f = g \circ Ev \circ \big(id \times (-r)\big)$$

above, with $r: U \to [0, h]$, $U \subset C^1$ open, and $g: \mathbb{R}^n \to \mathbb{R}^n$ continuously differentiable. Suppose the delay functional r satisfies condition (S) (with $n = 1$). If then $D_e f$ is defined by the formula above for $\mathrm{D}f$, with $\mathrm{D}r$ replaced by $\mathrm{D}_e r$, one sees that condition (S) holds for f. Notice that r satisfies (S) provided it is the restriction of a continuously differentiable map $V \to [0, h]$, $V \subset C$ open, to $U = V \cap C^1$.

THEOREM 3.2.1. *Suppose $U \subset C^1$ is open, $f : U \to \mathbb{R}^n$ has property* (S), *and $X_f \neq \emptyset$. Then X_f is a continuously differentiable submanifold of U with codimension n, and each $\phi \in X_f$ uniquely defines a noncontinuable solution $x^\phi : [-h, t_+(\phi)) \to \mathbb{R}^n$ of the IVP* (3.1.2). *All segments x_t^ϕ, $0 \leqslant t < t_+(\phi)$ and $\phi \in X_f$, belong to X_f, and the relations*

$$F(t,\phi) = x_t^\phi, \quad \phi \in X_f,\ 0 \leqslant t < t_+(\phi)$$

define a domain $\Omega \subset \mathbb{R} \times X_f$ and a continuous semiflow $F : \Omega \to X_f$. Each map

$$F(t,\cdot) : \{\phi \in X_f\colon\ (t,\phi) \in \Omega\} \to X_f$$

is continuously differentiable, and for all $(t,\phi) \in \Omega$ and $\chi \in T_\phi X_f$ we have

$$\mathrm{D}_2 F(t,\phi)\chi = v_t^{\phi,\chi}$$

with the solution $v^{\phi,\chi} : [-h, t_+(\phi)) \to \mathbb{R}^n$ of the linear IVP

$$v'(t) = \mathrm{D}f\big(F(t,\phi)\big)v_t, \quad v_0 = \chi. \tag{3.2.1}$$

At each (t,ϕ) with $\phi \in X_f$ and $h < t < t_+(\phi)$, the partial derivative $\mathrm{D}_1 F(t,\phi)$ exists, and

$$\mathrm{D}_1 F(t,\phi)1 = \big(x_t^\phi\big)'.$$

The restriction of F to the submanifold $\{(t,\phi) \in \Omega\colon\ h < t\}$ of $\mathbb{R} \times X_f$ is continuously differentiable.

Notice that the tangent spaces of the manifold X_f are given by

$$T_\phi X_f = \{\chi \in C^1\colon\ \chi'(0) = \mathrm{D}f(\phi)\chi\}.$$

Theorem 3.2.1 is proved in [206,207]. The construction of the semiflow is also sketched in [209].

The first part (S1) of the hypothesis and continuity of each extension $\mathrm{D}_e f(\phi)$ suffice for X_f to be a continuously differentiable submanifold. The proof of this is simple: For

$$p : C^1 \ni \phi \mapsto \phi'(0) \in \mathbb{R}^n,$$

$$X_f = (p - f)^{-1}(0),$$

and the Implicit Function Theorem yields local representations of X_f as graphs over the kernels of the linear maps $\mathrm{D}(p - f)(\phi) = p - \mathrm{D}f(\phi))$, $\phi \in X_f$, provided these linear maps are surjective. In case $n = 1$ surjectivity follows since using the continuity of $\mathrm{D}_e f(\phi)$ at $0 \in C$ one finds $\chi \in C^1 \subset C$ with $\chi'(0) = 1$ and $\mathrm{D}f(\phi)\chi = \mathrm{D}_e f(\phi)\chi < 1 = p\chi$, which gives $(p - \mathrm{D}f(\phi))C^1 = \mathbb{R}$. In case $n > 1$ similar arguments yield a basis of $\mathbb{R}^n$ in $(p - \mathrm{D}f(\phi))C^1$.

Condition (S3) implies that the map $D_e f$ is locally bounded. From this one derives easily the following local Lipschitz property:

(L) For every $\phi \in U$ there are a neighbourhood V and $L \geqslant 0$ with $|f(\psi) - f(\chi)| \leqslant L\|\psi - \chi\|_C$ for all $\psi \in V$, $\chi \in V$,

see Corollary 1 in [207]. Notice that the last norm is the norm on C and not the larger norm on the smaller space C^1. So (L) is not a consequence of (S1). Property (L) (together with (S1), (S2) and continuity of each $\mathrm{D}_e f(\phi)$) yields existence and continuity of the semiflow as well as the properties of the maps $F(t, \cdot)$ stated in Theorem 3.2.1. Only for the two last statements of Theorem 3.2.1, on smoothness, the full hypothesis (S) is needed.

It is worth noting that continuity of the map

$$U \ni \phi \mapsto \mathrm{D}_e f(\phi) \in L\big(C, \mathbb{R}^n\big),$$

which seems only slightly stronger than property (S3) above, does in general *not* hold for functionals f which represent differential equations with state-dependent delay.

A key issue in the proof of Theorem 3.2.1 is how the local Lipschitz property (L) is used for the construction of local solutions to the IVP. Let us briefly explain this.

In order to solve the integrated version

$$x(t) = \phi(0) + \int_0^t f(x_s)\,\mathrm{d}s, \quad x_0 = \phi \in X_f,$$

of the IVP for $0 \leqslant t \leqslant T$ by a continuously differentiable map

$$x : [-h, T] \to \mathbb{R}^n,$$

which is continuously differentiable with respect to ϕ, the desired solution is first written as the sum of a linear, continuously differentiable continuation $\hat{\phi}$ of ϕ and of a function u which is zero on $[-h, 0]$. The fixed point problem for u is

$$u(t) = \int_0^t \big(f\big(u_s + \hat{\phi}_s\big) - f(\phi)\big)\,\mathrm{d}s, \quad 0 \leqslant t \leqslant T.$$

The advantage of this formulation is that for the operator $u \mapsto A(\phi, u)$ given by the right hand side dependence on ϕ is more explicit than in the equation for x. For $A(\phi, \cdot)$ to become a contraction with respect to the norm $\|u\|_{T,1} = \max_{0 \leqslant s \leqslant T} |u(s)| + \max_{0 \leqslant s \leqslant T} |u'(s)|$ we use the following estimate for $v = A(\phi, u)$, $\overline{v} = A(\phi, \overline{u})$, and $0 \leqslant t \leqslant T$:

$$\big|v'(t) - \overline{v}'(t)\big| = \big|f\big(u_t + \hat{\phi}_t\big) - f\big(\overline{u}_t + \hat{\phi}_t\big)\big| \leqslant L\|u_t - \overline{u}_t\|_C$$

(due to the local Lipschitz property, for T and u and $\overline{u}$ small)

$$\leqslant L \max_{0\leqslant s\leqslant T} \big|u(s) - \overline{u}(s)\big| = L \max_{0\leqslant s\leqslant T} \left| \int_0^s \big(u'(r) - \overline{u}'(r)\big)\,\mathrm{d}r \right| \leqslant LT \|u - \overline{u}\|_{T,1}.$$

The proof that the semiflow F is continuously differentiable on the manifold given by $t > h$ is based on growth estimates and smoothness properties of solutions to the IVP

$$v'(t) = \mathrm{D}_e f\big(F(t,\phi)\big)v_t, \quad v_0 = \chi \tag{3.2.2}$$

for initial data $\chi \in C$.

The previous result is optimal in the sense that typically the semiflow F has no partial derivatives with respect to the first variable for $0 \leqslant t < h$.

The framework presented up to here is instrumental in the proof in [208] that for a certain differential system with state-dependent delay, which models position control by echo, hyperbolic stable periodic orbits exist.

With regard to results for solutions $x : [-h, t_+) \to \mathbb{R}^n$ to equations of the form (3.1.1) and generalizations thereof in the sense that x is continuous but differentiable only for $0 < t < t_+$, notice that for $h \leqslant t < t_+$ we have

$$x_t \in X_f$$

(with $f = g \circ ev \circ (id \times (-r))$ in case of Eq. (3.1.1)), so all dynamical properties like structure and stability of invariant sets (stationary points, periodic orbits, unstable manifolds, global attractors, ...) are determined by the semiflow F on X_f—provided the mild smoothness condition (S) holds.

Some notions of the approach to well-posedness and smoothness which we described here are related to ideas from earlier work. The Lipschitz property (L) from the proof of Theorem 3.2.1 was used before in [205] and is analogous to the notion of being *locally almost Lipschitzian* from [156]. The condition that (S1) holds and each $\mathrm{D}_e f(\phi)$ is continuous was introduced in [156] as *almost Fréchet differentiability*. It also is a special case of a smoothness condition used in [129]. Sets analogous to X_f were considered in [134] as state space for neutral functional differential equations, and by Louihi, Hbid and Arino [144]. In [144] the IVP for a certain class of equations of the form (3.1.1) defines a semigroup of locally Lipschitz continuous solution operators on the space $C^{0,1}$. The positively invariant subset $E \subset C^{0,1}$ of data $\phi \in C^1$ with $\phi'(0) = g(\phi(-r(\phi)))$ is identified as the domain of strong continuity of the semigroup. In the present notation, $E = X_f$ for $f = g \circ Ev \circ (id \times (-r))$. It is stated in [144] that E is a Lipschitz manifold.

3.3. *Compactness*

Recall that a map from a subset of a Banach space into a Banach space is called compact if images of bounded sets have compact closure. A simple compactness result for the semiflow of Theorem 3.2.1 is the following. Notice in which way the hypotheses strengthen property (S).

PROPOSITION 3.3.1. *Suppose $f:U \to \mathbb{R}^n$, $U \subset C^1$ open, is bounded, satisfies condition* (S), *and*

(Lb) *for every bounded set $B \subset U$ there exists $L_B \geqslant 0$ with $|f(\phi) - f(\psi)| \leqslant L_B \|\phi - \psi\|_C$ for all ϕ, ψ in B.*

Then all maps $F(t,\cdot)$, $t \geqslant h$, are compact.

PROOF. Let $t \geqslant h$ and let a bounded set $B \subset \{\phi \in X_f\colon (t,\phi) \in \Omega\}$ be given. In order that $\overline{F(t,B)} \subset C^1$ be compact it is enough to show that every sequence in $F(t,B)$ has a convergent subsequence, which follows by means of the Ascoli–Arzelà Theorem provided both sets $M = \{x_t^\phi \in C^1\colon \phi \in B\}$ and $M' = \{(x_t^\phi)' \in C\colon \phi \in B\}$ are bounded with respect to the norm $\|\cdot\|_C$ and equicontinuous. The boundedness of B and f in combination with Eq. (1.0.1) shows that

$$b = \sup_{\phi \in B, -h \leqslant s \leqslant t} \left|\left(x^\phi\right)'(s)\right| < \infty.$$

Then the boundedness of B and integration yield

$$\sup_{\phi \in B, -h \leqslant s \leqslant t} \left|x^\phi(s)\right| < \infty.$$

In particular, M' and M are bounded with respect to $\|\cdot\|_C$. The boundedness of M' yields the equicontinuity of M. We also have that the set

$$Y = \left\{F(s,\phi)\colon \phi \in B, 0 \leqslant s \leqslant t\right\} \subset X_f \subset U$$

is bounded with respect to $\|\cdot\|_{C^1}$. Now equicontinuity of M' follows from the estimate

$$\begin{aligned}
\left|\left(x_t^\phi\right)'(s) - \left(x_t^\phi\right)'(u)\right| &= \left|\left(x^\phi\right)'(t+s) - \left(x^\phi\right)'(t+u)\right| \\
&= \left|f\left(F(t+s,\phi)\right) - f\left(F(t+u,\phi)\right)\right| \\
&\leqslant L_Y \left\|F(t+s,\phi) - F(t+u,\phi)\right\|_C \\
&= L_Y \max_{-h \leqslant v \leqslant 0} \left|x^\phi(t+s+v) - x^\phi(t+u+v)\right| \\
&= L_Y \max_{-h \leqslant v \leqslant 0} \left|\int_{t+u+v}^{t+s+v} (x^\phi)'(w)\,\mathrm{d}w\right| \leqslant L_Y b|s-u|
\end{aligned}$$

for all $\phi \in B$ and all s, u in $[-h, 0]$. □

In case of the example $f = g \circ Ev \circ (id \times (-r))$ with $r:U \to [0,h]$, $U \subset C^1$ open, and $g:\mathbb{R}^n \to \mathbb{R}^n$ continuously differentiable the hypotheses of Proposition 3.3.1 are satisfied if in addition U is convex, r has property (S), g is bounded, and $\mathrm{D}_e r$ and $\mathrm{D}g$ map bounded sets onto bounded sets: The boundedness of f is obvious. The formula which computes $\mathrm{D}_e f$ from $\mathrm{D}g$ and $\mathrm{D}_e r$ shows that $\mathrm{D}_e f$ maps bounded sets onto bounded subsets. For a given bounded set $B \subset U$ there is a larger convex bounded subset $B^* \subset U$, and for ϕ, ψ in B^* we have

$$\begin{aligned}\big|f(\phi) - f(\psi)\big| &= \left|\int_0^1 \mathrm{D}f\big(\psi + t(\phi - \psi)\big)(\phi - \psi)\,\mathrm{d}t\right| \\ &= \left|\int_0^1 \mathrm{D}_e f\big(\psi + t(\phi - \psi)\big)(\phi - \psi)\,\mathrm{d}t\right| \\ &\leqslant \int_0^1 \big|\mathrm{D}_e f\big(\psi + t(\phi - \psi)\big)(\phi - \psi)\big|\,\mathrm{d}t \\ &\leqslant \sup_{\chi \in B^*} \big\|\mathrm{D}_e f(\chi)\big\| \|\phi - \psi\|_C,\end{aligned}$$

which yields property (Lb).

For another result on compactness, for solution operators on the space $C^{0,1}$, see [144].

3.4. *Linearization at equilibria*

Theorem 3.2.1 reveals in particular how to linearize semiflows defined by differential equations with state-dependent delay, an issue which had been mysterious before. Consider $f : U \to \mathbb{R}^n$, $U \subset C^1$ open, with property (S). Let a stationary point $\phi_0 \in X_f$ of the semiflow F from Theorem 3.2.1 be given. The linearization of F at ϕ_0 is the semigroup T of the linear continuous operators $T(t) = \mathrm{D}_2 F(t, \phi_0)$, $t \geqslant 0$, on the Banach space $T_{\phi_0} X_f$ with the norm $\|\cdot\|_{C^1}$. T is strongly continuous since the solutions $v^{\phi_0,\chi} : [-h, \infty) \to \mathbb{R}^n$ of the IVP (3.2.1) with $\chi \in T_{\phi_0} X_f$ are continuously differentiable.

Before the present approach was available a heuristic technique had been developed in order to circumvent the linearization problem in studies of local stability and instability properties of equilibria. This technique associates to the given nonlinear equation an auxiliary linear equation in the following way: First the delay is frozen at equilibrium, then the resulting nonlinear equation with constant delay is linearized. Of course, this makes sense only for equations where the delay appears explicitly, like, e.g., in Eq. (3.1.1). Let us use Eq. (3.1.1) to show that the auxiliary equation found by the heuristic technique is

$$v'(t) = \mathrm{D}_e f(\phi_0) v_t$$

in our framework. Suppose for simplicity that $n = 1$, that $g : \mathbb{R} \to \mathbb{R}$ satisfies $g(0) = 0$ and that g and the delay functional $r : U \to [0, h]$, $U \subset C^1$ open, are continuously differentiable. Freezing the delay in Eq. (3.1.1) at $\phi_0 = 0$ and linearizing the resulting RFDE

$$x'(t) = g\big(x\big(t - r(0)\big)\big)$$

at the zero solution yields

$$v'(t) = g'(0) v\big(t - r(0)\big).$$

On the other hand, for the functional $f_{g,r} = g \circ Ev \circ (id \times (-r))$ and $\chi \in C^1$ the computation of derivatives above yields

$$\mathrm{D} f_{g,r}(0)\chi = \mathrm{D}g(0)\chi\big(-r(0)\big) = g'(0)\chi\big(-r(0)\big).$$

Obviously the right-hand side of this equation defines a continuous linear extension $\mathrm{D}_e f_{g,r}(0) : C \to \mathbb{R}$ of $\mathrm{D} f_{g,r}(0)$, which verifies our previous statement. In other words, the auxiliary equation found by the heuristic method yields the true linear variational equation by restriction to the tangent space $T_0 X_f$.

In the sequel we clarify the relation between the spectral properties of the linearization T of the semiflow F of Theorem 3.2.1 at a stationary point $\phi_0 \in X_f$ and the strongly continuous semigroup T_e on the space C which is defined by the solutions of the IVP (3.2.2). Recall that the generator $G_e : dom \to C$ of T_e is given by $dom = \{\phi \in C^1\colon\ \phi'(0) = D_e f(\phi_0)\phi\}$ and $G_e\phi = \phi'$.

We have $T_{\phi_0} X_f = dom$, $T(t)\phi = T_e(t)\phi$ for $t \geqslant 0$ and $\phi \in dom$, and the norm $\|\cdot\|_{C^1}$ coincides with the graph norm $\|\cdot\|_e = \|\cdot\|_C + \|G_e \cdot\|_C$ of the operator G_e.

It is a simple general fact that for a strongly continuous semigroup S of linear continuous operators on a Banach space B, with generator $A : D_A \to B$, the induced linear operators

$$D_A \ni x \mapsto S(t)x \in D_A$$

are continuous with respect to the graph norm $\|\cdot\|_A = \|\cdot\| + \|A \cdot\|$ and form a strongly continuous semigroup on the Banach space $(D_A, \|\cdot\|_A)$, with the generator A_d defined on the domain

$$D_d = \{x \in D_A\colon\ Ax \in D_A\}$$

of A^2 and given by $A_d x = Ax$. Proofs are immediate from $S(t)D_A \subset D_A$ and

$$S(t)Ax = AS(t)x \quad \text{on } D_A$$

for all $t \geqslant 0$.

Our semigroups and their generators are precisely in the relation just described, so we have

$$\begin{aligned} D = (D_d =) &\big\{\chi \in C^1\colon\ \chi'(0) = \mathrm{D}f(\phi_0)\chi,\ \chi' \in C^1,\ \chi''(0) = \mathrm{D}f(\phi_0)\chi'\big\} \\ &= \big\{\chi \in C^2\colon\ \chi'(0) = \mathrm{D}f(\phi_0)\chi,\ \chi''(0) = \mathrm{D}f(\phi_0)\chi'\big\} \end{aligned}$$

and $G\chi = \chi'$ on D.

We return to the general case. Below, ρ and σ denote resolvent sets and spectra, respectively.

PROPOSITION 3.4.1. *Suppose B is a Banach space over $\mathbb{C}$, S is a strongly continuous semigroup on B with generator $A : D_A \to B$, and A_d is the generator of the semigroup induced on the Banach space $(D_A, \|\cdot\|_A)$. Then the following holds.*

(i) $\rho(A) \subset \rho(A_d) \subset \rho(A) \cup \{\lambda \in \mathbb{C}\colon\ A - \lambda I$ *injective and* $(A - \lambda I)D_A \neq B\}$, *and* $\sigma(A_d) \subset \sigma(A)$. *For all* $\lambda \in \rho(A)$, $(A_d - \lambda I)^{-1}x = (A - \lambda I)^{-1}x$ *on* D_A.

(ii) *Let* γ *be a simple closed curve in* $\rho(A)$ *and let* $\sigma_{\gamma,d} = int(\gamma) \cap \sigma(A_d)$, $\sigma_\gamma = int(\gamma) \cap \sigma(A)$. *Then the spectral projections* P_d, P *and generalized eigenspaces* $\mathcal{G}_d = P_d D_A$, $\mathcal{G} = PB$ *associated with the sets* $\sigma_{\gamma,d} \subset \sigma(A_d)$ *and* $\sigma_\gamma \subset \sigma(A)$, *respectively, satisfy*

$$P_d x = Px \quad on\ D_A, \quad \mathcal{G}_d \subset \mathcal{G}, \quad and \quad \overline{\mathcal{G}_d}^B = \mathcal{G}.$$

PROOF. 1. Proof of (i).

1.1. Let $\lambda \in \rho(A)$ be given. The continuity of the inverse with respect to the norm on B and the definition of the graph norm combined show that for all $y \in D_A$ we have

$$\begin{aligned}\left\|(A-\lambda I)^{-1}y\right\|_A &= \left\|(A-\lambda I)^{-1}y\right\| + \left\|A(A-\lambda I)^{-1}y\right\| \\ &\leqslant \left\|(A-\lambda I)^{-1}y\right\| + \left\|(A-\lambda I)(A-\lambda I)^{-1}y\right\| \\ &\quad + \left\|\lambda(A-\lambda I)^{-1}y\right\| \\ &\leqslant \bigl((1+|\lambda|)\left\|(A-\lambda I)^{-1}\right\| + 1\bigr)\|y\|.\end{aligned}$$

It follows that $(A-\lambda I)^{-1}$ defines a linear continuous map R from $(D_A, \|\cdot\|_A)$ into itself. The range of R contains the domain D_d of the generator A_d since for any $x \in D_A$ with $Ax \in D_A$ we have that $y = (A_d - \lambda I)x = (A - \lambda I)x \in D_A$, hence $x = (A - \lambda I)^{-1}y \in (A-\lambda I)^{-1}D_A = RD_A$.

Conversely, for $x = Ry = (A-\lambda I)^{-1}y$ with $y \in D_A$ we have $x \in D_A$ and $(A-\lambda I)x = y \in D_A$, hence $Ax = \lambda x + y \in D_A$, or $x \in D_d$.

So R maps D_A injectively onto the domain D_d of A_d. We have $R(A_d - \lambda I)y = R(A - \lambda I)y = y$ on D_d, $(A_d - \lambda I)Rx = (A-\lambda I)Rx = x$ on D_A, and it follows that $\lambda \in \rho(A_d)$ and $(A_d - \lambda I)^{-1}y = (A-\lambda I)^{-1}y$ on D_A.

The shown inclusion of resolvent sets yields the asserted result for spectra.

1.2. Proof of the remaining inclusion. Let $\lambda \in \rho(A_d)$ be given. $A - \lambda I$ is injective since $(A-\lambda I)x = 0$ implies $Ax \in D_A$, or $x \in D_d$, and therefore $(A_d - \lambda I)x = (A-\lambda I)x = 0$, which in turn gives $x = 0$, by $\lambda \in \rho(A_d)$.

Consider the injective continuous map

$$A_c : \bigl(D_A, \|\cdot\|_A\bigr) \ni x \mapsto (A - \lambda I)x \in B.$$

In case $(A-\lambda I)D_A = B$ the open mapping theorem shows that the inverse of A_c is continuous, which implies that

$$B \ni y \mapsto (A-\lambda I)^{-1}y \in B$$

is continuous. So in this case, $\lambda \in \rho(A)$, and the assertion becomes obvious.

2. Proof of (ii). The representation of $P_d x$ and Px, $x \in D_A$, as contour integrals in $(D_A, \|\cdot\|_A)$ and B, respectively, along the curve γ with integrands given by

$$(A_d - \lambda I)^{-1} x = (A - \lambda I)^{-1} x, \quad \lambda \in |\gamma|,$$

shows that $P_d x = Px$ on D_A. Consequently,

$$\mathcal{G}_d = P_d D_A \subset P D_A \subset PB = \mathcal{G},$$

and finally

$$\overline{\mathcal{G}_d}^B = \overline{P_d D_A}^B = \overline{P D_A}^B = PB$$

(since $\overline{D_A} = B$ and since P has closed range)

$$= \mathcal{G}. \qquad \square$$

It is easy to see that for a given semiflow S on a Banach space B over $\mathbb{R}$ and its generator $A : D_A \to B$, the complexification $A_{\mathbb{C}}$ of A coincides with the generator of the complexified semigroup $S_{\mathbb{C}} : t \mapsto S(t)_{\mathbb{C}}$, and that the semigroup induced by $S_{\mathbb{C}}$ on $(D_A)_{\mathbb{C}}$ is generated by the complexification of the generator $A_d : D_d \to (D_A, \|\cdot\|_A)$.

Returning to our case of semigroups T_e on C, T on $T_{\phi_0} X_f$ and their generators G_e, G, respectively, we recall that the embedding $(dom, \|\cdot\|_e) \to C$ is compact, by the Ascoli–Arzelà theorem. This yields that all resolvents of $(G_e)_{\mathbb{C}}$, which define continuous maps from $C_{\mathbb{C}}$ onto the complexification of the Banach space $(dom, \|\cdot\|_e)$, are compact. Therefore $\sigma(G_e) := \sigma((G_e)_{\mathbb{C}})$ is discrete and consists of eigenvalues with finite-dimensional generalized eigenspaces. Using Proposition 3.4.1 and the remarks following it we infer

$$\rho(G) = \rho(G_e), \qquad \sigma(G) = \sigma(G_e),$$

and for the spectral projections and generalized eigenspaces of $G_{\mathbb{C}}$ and $(G_e)_{\mathbb{C}}$ which are associated with $\lambda \in \sigma(G) = \sigma(G_e)$,

$$P(\lambda)\chi = P_e(\lambda)\chi \quad \text{on } T_{\phi_0} X_f = dom,$$

$$\mathcal{G}(\lambda) = \mathcal{G}_e(\lambda).$$

Recall that to the right of any line parallel to the imaginary axis there are at most a finite number of eigenvalues of G_e. Let C_u and C_c denote the unstable and center spaces of G_e, i.e., the finite-dimensional realified generalized eigenspaces given by the eigenvalues with positive real part and on the imaginary axis, respectively. The stable space C_s of G_e is the realified generalized eigenspace given by the spectrum with negative real part. More precisely, C_s is the realification of the space $(id - P_{\geqslant 0}) C_{\mathbb{C}}$ where $P_{\geqslant 0} : C_{\mathbb{C}} \to \mathbb{C}_{\mathbb{C}}$ denotes the spectral projection associated with the finite spectral set of all $\lambda \in \sigma(G_e)$ with $Re\lambda \geqslant 0$. We have

$$C_u \oplus C_c \subset D \subset dom = T_{\phi_0} X_f$$

while the infinite-dimensional, complementary space C_s is not contained in *dom*. Using the previous relations between spectral projections and generalized eigenspaces of the generators G and G_e it follows easily that the unstable and center spaces of G coincide with C_u and C_c, respectively, and that the stable space of G is

$$C_s \cap dom.$$

For each map $F(t,\cdot)$, $t > 0$, we have

$$\sigma\big(\mathrm{D}F(t,\cdot)(\phi_0)\big) = \sigma\big(T(t)\big) \subset \{0\} \cup \big\{\mathrm{e}^{zt}\colon\ z \in \sigma(G)\big\}, \tag{3.4.1}$$

and the linear unstable, center, and stable spaces for the derivative $\mathrm{D}F(t,\cdot)(\phi_0) = \mathrm{D}_2 F(t,\phi_0) = T(t)$, which are defined by the eigenvalues outside, on, and inside the unit circle, are

$$C_u, C_c, \text{ and } C_s \cap T_{\phi_0} X_f, \text{ respectively.}$$

3.5. *Local invariant manifolds at stationary points*

Consider $f: U \to \mathbb{R}^n$, $U \subset C^1$ open, with property (S) and the associated semiflow $F: \Omega \to X_f$ of Theorem 3.2.1. It is convenient to set

$$\Omega_t = \big\{\phi \in X_f\colon\ (t,\phi) \in \Omega\big\} \quad \text{and} \quad F_t = F(t,\cdot),$$

for each $t \geqslant 0$. A stationary point $\phi_0 \in X_f$ of the semiflow F is a fixed point for each map $F_t: \Omega_t \to X_f$, $t > 0$. Results about local invariant manifolds for continuously differentiable maps on open subsets of Banach spaces [97,44,171,133] yield continuously differentiable local stable, center, and unstable manifolds

$$W_s \subset X_f, \quad W_c \subset X_f, \quad W_u \subset X_f$$

of F_t $t > 0$, at the fixed point ϕ_0, with tangent spaces at ϕ_0 given by

$$T_{\phi_0} W_s = C_s \cap T_{\phi_0} X_f, \qquad T_{\phi_0} W_c = C_c, \qquad T_{\phi_0} W_u = C_u.$$

In the sequel we recall details of this and use them to show that local stable manifolds W_s of any map F_a, $a > 0$, provide local stable manifolds of the semiflow F at the given stationary point. The last fact is familiar for flows of continuously differentiable vector fields in finite dimensions, but here we are concerned with semiflows on a Banach manifold of infinite dimension, and some care should be taken which properties of the underlying delay differential equation (1.0.1) ensure the result.

For local unstable manifolds an analogous result holds, and the proof is similar. Existence of smooth local center manifolds for the semiflow is more difficult, and the relationship between center manifolds of its time-t-maps and center manifolds for the semiflow is more subtle. This will be discussed in the next section.

So let a stationary point $\phi_0 \in X_f$ of the semiflow F be given. Recall from the preceding subsection the linear stable, center, and unstable spaces

$$C_s \cap T_{\phi_0} X_f, C_c, C_u$$

of the generator G of the semigroup T on $T_{\phi_0} X_f$. Choose $\beta > 0$ so that

$$-\beta > \max\{Re\, z\colon\ z \in \sigma(G), Re\, z < 0\}.$$

Let $a > 0$. In order to introduce the local stable manifold of the map $F_a : \Omega_a \to X_f$ at the fixed point ϕ_0 we use a manifold chart of X_f at ϕ_0. Let $Y = T_{\phi_0} X_f$. There is a subspace $E \subset C^1$ of dimension n which is a complement of Y in C^1. Let $P : C^1 \to C^1$ denote the projection along E onto Y. Then the equation $K(\phi) = P(\phi - \phi_0)$ defines a manifold chart on an open neighbourhood V of ϕ_0 in $\Omega_a \subset X_f$, with $Y_0 = K(V)$ an open neighbourhood of $0 = K(\phi_0)$ in the Banach space Y (with the norm given by $\|\cdot\|_{C^1}$). The inverse of K is given by a continuously differentiable map $R : Y_0 \to C^1$. Both derivatives $\mathrm{D}K(\phi_0)$ and $\mathrm{D}R(0)$ are given by the identity on Y. We may assume that there is a Lipschitz constant $L_R \geqslant 0$ so that

$$\|R(\chi) - R(\psi)\|_{C^1} \leqslant L_R \|\chi - \psi\|_{C^1} \quad \text{for all } \chi \in Y_0, \psi \in Y_0.$$

Choose an open neighbourhood $Y_1 \subset Y_0$ of 0 in Y with $F_a(R(Y_1)) \subset V$. In local coordinates the map F_a is represented by the continuously differentiable map

$$H : Y_1 \ni \chi \mapsto K\big(F_a\big(R(\chi)\big)\big) \in Y.$$

Obviously, $H(0) = 0$, $\mathrm{D}H(0) = \mathrm{D}F_a(\phi_0) = T(a)$, and $H(Y_1) \subset Y_0$. Using the last statement of the preceding subsection we infer that the linear stable, center, and unstable spaces of H at its fixed point $0 \in Y_1$ are $C_s \cap Y$, C_c, and C_u, respectively. Set $\lambda = \mathrm{e}^{-a\beta}$. Then (3.4.1) gives

$$\max\{|\zeta|\colon\ \zeta \in \sigma\big(\mathrm{D}H(0)\big), |\zeta| < 1\} < \lambda < 1.$$

The Stable Manifold Theorem (see Theorem I.2 in [133]) yields the following result.

PROPOSITION 3.5.1. *There exist* $\alpha \in (0, \lambda)$, *convex open neighbourhoods* $C_{s,2}$ *of* 0 *in* $C_s \cap Y$ *and* $C_{cu,2}$ *of* 0 *in* $C_c \oplus C_u$ *with* $N = C_{s,2} + C_{cu,2} \subset Y_2$, *a continuously differentiable map* $w : C_{s,2} \to C_{cu,2}$ *with* $w(0) = 0$ *and* $\mathrm{D}w(0) = 0$, *and an equivalent norm* $\|\cdot\|_H$ *on* Y *such that the following holds.*

(i) *The graph* $W = \{\chi + w(\chi)\colon\ \chi \in C_{s,2}\}$ *is equal to the set of all initial points* $\psi = \psi_0$ *of trajectories* $(\psi_j)_0^\infty$ *of* H *which satisfy* $\lambda^{-j}\psi_j \in N$ *for all* $j \in \mathbb{N}_0$ *and* $\lambda^{-j}\psi_j \to 0$ *as* $j \to \infty$.

(ii) $H(W) \subset W$.

(iii) $\|H(\phi) - H(\psi)\|_H \leqslant \alpha\|\phi - \psi\|_H$ *for all* $\psi \in W$, $\phi \in W$.

(iv) *For every trajectory* $(\psi_j)_0^\infty$ *of* H *with* $\lambda^{-j}\psi_j \in N$ *for all* $j \in \mathbb{N}_0$,

$$\psi_0 \in W.$$

Here, trajectories are defined by the equations $\psi_{j+1} = H(\psi_j)$ for all integers $j \geqslant 0$. The local stable manifold of F_a at ϕ_0 is the continuously differentiable submanifold

$$W_s = R(W)$$

of X_f. Obviously, $W_s \subset V$, $\phi_0 \in W_s$, and $T_{\phi_0}W_s = C_s \cap Y = C_s \cap T_{\phi_0}X_f$.

COROLLARY 3.5.2.

(i) $F_a(W_s) \subset W_s$, *and each neighbourhood of* ϕ_0 *in* W_s *contains a neighbourhood* $W_{s,1}$ *of* ϕ_0 *in* W_s *with* $F_a(W_{s,1}) \subset W_{s,1}$.

(ii) *There exists* $c_s \geqslant 0$ *so that for every trajectory* $(\psi_j)_0^\infty$ *of* F_a *in* W_s *and for all integers* $j \geqslant 0$,

$$\|\psi_j - \phi_0\|_{C^1} \leqslant c_s\alpha^j\|\psi_0 - \phi_0\|_{C^1}.$$

PROOF. 1. The first inclusion in assertion (i) follows from

$$K\big(F_a(W_s)\big) = K\big(F_a\big(R(W)\big)\big) = H(W) \subset W = K\big(R(W)\big)$$

by application of R. Proof of the second part of (i): For $\epsilon > 0$, set

$$Y_{H,\epsilon} = \big\{\psi \in Y\colon\ \|\psi\|_H < \epsilon\big\}.$$

Any given neighbourhood of ϕ_0 in $V \subset X_f$ contains $V_\epsilon = R(Y_{H,\epsilon})$ for some $\epsilon > 0$, and $R(W \cap Y_{H,\epsilon}) = R(W) \cap R(Y_{H,\epsilon}) = W_s \cap V_\epsilon$. Part (iii) of Proposition 3.5.1 yields $F_a(W_s \cap V_\epsilon) = R(K(F_a(R(W \cap Y_{H,\epsilon})))) = R(H(W \cap Y_{H,\epsilon})) \subset R(W \cap Y_{H,\epsilon}) = W_s \cap V_\epsilon$.

2. Proof of assertion (ii). There are positive constants $c_1 \leqslant c_2$ with

$$c_1\|\chi\|_{C^1} \leqslant \|\chi\|_H \leqslant c_2\|\chi\|_{C^1} \quad \text{for all } \chi \in Y.$$

Let a trajectory $(\psi_j)_0^\infty$ of F_a in W_s be given. The points $\chi_j = K(\psi_j) \in W$ form a trajectory of H since

$$\chi_{j+1} = K(\psi_{j+1}) = K\big(F_a(\psi_j)\big) = K\big(F_a\big(R(\chi_j)\big)\big) = H(\chi_j)$$

for each integer $j \geqslant 0$. Hence

$$\begin{aligned}\|\psi_j - \phi_0\|_{C^1} &= \left\| R(\chi_j) - R(0) \right\|_{C^1} \\ &\leqslant L_R \|\chi_j\|_{C^1} \\ &\leqslant L_R c_1^{-1} \|\chi_j\|_H \\ &\leqslant L_R c_1^{-1} \alpha^j \|\chi_0\|_H \\ &\leqslant L_R \frac{c_2}{c_1} \alpha^j \|\chi_0\|_{C^1} \\ &= L_R \frac{c_2}{c_1} \alpha^j \|P\|_{C^1} \|\psi_0 - \phi_0\|_{C^1}.\end{aligned}$$

□

We want to show that the semiflow F maps a piece W^s of W_s close to ϕ_0 into W_s, in the sense that $F([0,\infty) \times W^s) \subset W_s$. The proof requires a quantitative version of continuous dependence on initial conditions. Notice that the proof of the latter employs the Lipschitz property (L) of the functional f.

PROPOSITION 3.5.3. *There exist an open neighbourhood $X_{f,a}$ of ϕ_0 in X_f and a constant $c_a \geqslant 0$ so that $[0,a] \times X_{f,a} \subset \Omega$ and*

$$\left\| F(t,\phi) - \phi_0 \right\|_{C^1} \leqslant c_a \|\phi - \phi_0\|_{C^1} \quad \textit{for all } (t,\phi) \in [0,a] \times X_{f,a}.$$

PROOF. 1. Using continuity of the semiflow and compactness of the interval $[0,a]$ we find an open neighbourhood $X_{f,a}$ of the stationary point ϕ_0 in X_f so that $[0,a] \times X_{f,a} \subset \Omega$ and $F([0,a] \times X_{f,a})$ is contained in a neighbourhood of ϕ_0 in the domain U of f on which the Lipschitz estimate (L) holds.

2. Let $\xi = \phi_0(0)$, and let $\phi \in X_{f,a}$ be given. Set $x = x^\phi$. For $0 \leqslant t \leqslant a$, we have

$$\begin{aligned}\left| x(t) - \xi \right| &= \left| x(0) - \xi + \int_0^t x'(s)\,\mathrm{d}s \right| = \left| x(0) - \xi + \int_0^t f(x_s)\,\mathrm{d}s \right| \\ &\leqslant \|x_0 - \phi_0\|_C + L \int_0^t \|x_s - \phi_0\|_C\,\mathrm{d}s,\end{aligned}$$

as $f(\phi_0) = 0$. For any $t \in [0,a]$ there exists $t_0 \in [t-h,t]$ with $\|x_t - \phi_0\|_C = |x(t_0) - \xi|$. In case $t_0 < 0$ we obtain

$$\|x_t - \phi_0\|_C \leqslant \|x_0 - \phi_0\|_C$$

while in case $t_0 \geqslant 0$,

$$\begin{aligned}\|x_t - \phi_0\|_C &\leqslant \|x_0 - \phi_0\|_C + L \int_0^{t_0} \|x_s - \phi_0\|_C\,\mathrm{d}s \\ &\leqslant \|x_0 - \phi_0\|_C + L \int_0^t \|x_s - \phi_0\|_C\,\mathrm{d}s.\end{aligned}$$

In both cases,

$$\|x_t - \phi_0\|_C \leqslant \|x_0 - \phi_0\|_C + L\int_0^t \|x_s - \phi_0\|_C \, \mathrm{d}s.$$

Gronwall's lemma yields

$$\left\|F(t,\phi) - \phi_0\right\|_C \leqslant \|\phi - \phi_0\|_C \, \mathrm{e}^{Lt} \quad \text{on } [0,a] \times X_{f,a}.$$

Consequently, for all $(t,\phi) \in [0,a] \times X_{f,a}$,

$$\left|\left(x^\phi\right)'(t)\right| = \left|f\left(x_t^\phi\right) - f(\phi_0)\right| \leqslant L\left\|x_t^\phi - \phi_0\right\|_C \leqslant L\,\mathrm{e}^{Lt}\|\phi - \phi_0\|_C.$$

It follows that

$$\left\|\left(x_t^\phi\right)' - (\phi_0)'\right\|_C \leqslant L\,\mathrm{e}^{Lt}\|\phi - \phi_0\|_{C^1},$$

and finally

$$\left\|F(t,\phi) - \phi_0\right\|_{C^1} \leqslant (L+1)\mathrm{e}^{La}\|\phi - \phi_0\|_{C^1}$$

on $[0,a] \times X_{f,a}$. □

Now we can prove the desired invariance property of W_s with respect to the semiflow F, and an exponential estimate. Set

$$\gamma = -\frac{\log\alpha}{a};$$

then $0 < \beta < \gamma$.

PROPOSITION 3.5.4. *There exists an open neighbourhood W^s of ϕ_0 in W_s so that $[0,\infty) \times W^s \subset \Omega$, $F([0,\infty) \times W^s) \subset W_s$, and there is a constant $c_w \geqslant 0$ so that*

$$\left\|F(t,\psi) - \phi_0\right\|_{C^1} \leqslant c_w\,\mathrm{e}^{-\gamma t}\|\psi - \phi_0\|_{C^1} \quad \textit{for all } \psi \in W^s,\ t \geqslant 0. \tag{3.5.1}$$

PROOF. 1. Set $V_N = R(N)$. Choose c_s according to Corollary 3.5.2(ii) and $X_{f,a}$ and c_a according to Proposition 3.5.3. It follows that there is an open neighbourhood W^s of ϕ_0 in $W_s \cap X_{f,a} \subset V_N \cap X_{f,a}$ so that

$$F_a\left(W^s\right) \subset W^s, \tag{3.5.2}$$

$$F\left([0,a] \times W^s\right) \subset V_N, \tag{3.5.3}$$

and

$$\left\{\chi \in Y\colon\ \|\chi\|_{C^1} \leqslant \|P\|_{C^1}\, c_a\, c_s \sup_{\eta \in W^s} \|\eta - \phi_0\|_{C^1}\right\} \subset N. \tag{3.5.4}$$

Using (3.5.2) and (3.5.3) and properties of the semiflow we get $[0,\infty)\times W^s \subset \Omega$ and $F([0,\infty)\times W^s) \subset V_N$.

2. Proof of $F([0,\infty)\times W^s)\subset W_s$. Let $t \geqslant 0$ and $\psi \in W^s$ be given. The assertion $\rho = F(t,\psi) \in W_s$ is equivalent to

$$K(\rho) \in K(W_s) = W. \tag{3.5.5}$$

By the remarks in part 1 the point ρ defines a trajectory $(\rho_j)_0^\infty$ of F_a in V_N, with $\rho_0 = \rho$, and the point $\psi \in W^s$ defines a trajectory $(\psi_j)_0^\infty$ of F_a in $W^s \subset V_N$, with $\psi_0 = \psi$. The points $\chi_j = K(\rho_j)$ form a trajectory of H in N since

$$\chi_{j+1} = K(\rho_{j+1}) = K\big(F_a(\rho_j)\big) = K\big(F_a\big(R(\chi_j)\big)\big) = H(\chi_j)$$

for all integers $j \geqslant 0$. Proposition 3.5.1(iv) shows that (3.5.5) follows from

$$\lambda^{-j}\chi_j \in N \quad \text{for all integers } j \geqslant 0. \tag{3.5.6}$$

Proof of (3.5.6). Let $j \in \mathbb{N}_0$, $k \in \mathbb{N}_0$, $ka \leqslant t < (k+1)a$. Then

$$\begin{aligned}
\|\chi_j\|_{C^1} &= \big\|K(\rho_j)\big\|_{C^1} \\
&= \big\|P(\rho_j - \phi_0)\big\|_{C^1} \\
&\leqslant \|P\|_{C^1}\|\rho_j - \phi_0\|_{C^1} \\
&= \|P\|_{C^1}\big\|F(ja,\rho) - \phi_0\big\|_{C^1} \\
&= \|P\|_{C^1}\big\|F\big(t - ka, F\big((j+k)a,\psi\big)\big) - \phi_0\big\|_{C^1} \\
&\leqslant \|P\|_{C^1} c_a \|\psi_{j+k} - \phi_0\|_{C^1} \\
&\leqslant \|P\|_{C^1} c_a c_s \alpha^{j+k}\|\psi - \phi_0\|_{C^1},
\end{aligned}$$

hence

$$\big\|\lambda^{-j}\chi_j\big\|_{C^1} \leqslant \|P\|_{C^1} c_a c_s \cdot 1 \cdot \sup_{\eta\in W^s}\|\eta - \phi_0\|_{C^1},$$

which yields

$$\lambda^{-j}\chi_j \in N,$$

according to (3.5.4).

3. Proof of (3.5.1). Let $\psi \in W^s$, $t \geqslant 0$, $j \in \mathbb{N}_0$, $ja \leqslant t < (j+1)a$. Then

$$\begin{aligned}
\big\|F(t,\psi) - \phi_0\big\|_{C^1} &= \big\|F\big(t - ja, F(ja,\psi)\big) - \phi_0\big\|_{C^1} \\
&\leqslant c_a\big\|F(ja,\psi) - \phi_0\big\|_{C^1}
\end{aligned}$$

$$
\begin{aligned}
&\leqslant c_a c_s \alpha^j \|\psi - \phi_0\|_{C^1} \\
&= c_a c_s \mathrm{e}^{j \log \alpha} \|\psi - \phi_0\|_{C^1} \\
&= c_a c_s \mathrm{e}^{(t \log \alpha)/a} \mathrm{e}^{(j-t/a) \log \alpha} \|\psi - \phi_0\|_{C^1} \\
&\leqslant c_a c_s \mathrm{e}^{(t \log \alpha)/a} \alpha^{-1} \|\psi - \phi_0\|_{C^1}. \qquad \square
\end{aligned}
$$

The C^1-submanifold W^s of X_f is the local stable manifold of F at ϕ_0. It is locally positively invariant under F, with tangent space

$$T_{\phi_0} W^s = C_s \cap T_{\phi_0} X_f,$$

and it has the following uniqueness property: There exists a constant $c > 0$ so that all initial data $\psi \in X_f$ with $[0, \infty) \times \{\psi\} \subset \Omega$ and

$$\mathrm{e}^{\beta t} \left\| F(t, \psi) - \phi_0 \right\|_{C^1} < c \quad \text{for all } t \geqslant 0$$

belong to W^s. This property is easily established by means of estimates as in the preceding proofs.

We said already that analogously to the approach to local *stable* manifolds just presented one obtains continuously differentiable local *unstable* manifolds for the semiflow from local unstable manifolds of the maps F_a, $a > 0$.

Local *unstable* manifolds for certain classes of differential equations with state-dependent delay were also obtained in earlier work, by Krishnan [126,127] and more generally in [129]. The proofs in [126,127] and [129] proceed without knowledge of a semiflow and use the heuristic approach with the auxiliary linear equation mentioned in Section 3.4. It is remarkable that in [129] higher order differentiability of local unstable manifolds is achieved.

Related to work on local invariant manifolds is a result of Arino and Sanchez [10] about saddle point behaviour of solutions close to a stationary point which is hyperbolic, i.e., there is no spectrum of the generator G_e on the imaginary axis. Also in [10] no semiflow is available, and the auxiliary linear equation is used.

3.6. *The principle of linearized stability*

For $f : U \to \mathbb{R}^n$, $U \subset C^1$ open, with property (S) and the associated semiflow F of Theorem 3.2.1 the results of Section 3.4 and Proposition 3.5.4 yield the following Principle of Linearized Stability.

THEOREM 3.6.1. *If all eigenvalues of G_e have negative real part then ϕ_0 is exponentially asymptotically stable for the semiflow F.*

For applications, recall that the eigenvalues of G_e and their multiplicities are given by the familiar transcendental characteristic equation which is obtained from the Ansatz $v(t) = \mathrm{e}^{\lambda t} c$ for a solution to the equation

$$v'(t) = \mathrm{D}_e f(\phi_0) v_t.$$

Earlier Principles of Linearized Stability, for certain classes of differential equations with state-dependent delay, are due to Cooke and Huang [48] and to Hartung and Turi [105, 108]. The proofs employ the heuristic approach described in Section 3.4. Related is work by Hartung [101] on exponential stability of periodic solutions to nonautonomous, periodic differential equations with state-dependent delay, and a result by Győri and Hartung [92] who derive exponential stability of the zero solution of the nonautonomous equation

$$x'(t) = a(t) x\bigl(t - r\bigl(t, x(t)\bigr)\bigr)$$

from exponential stability of the zero solution to the linear nonautonomous RFDE

$$y'(t) = a(t) y\bigl(t - r(t, 0)\bigr).$$

4. Center manifolds

4.1. *Preliminaries*

Assume $f : U \to \mathbb{R}^n$, $U \subset C^1$ open, with property (S) of Section 3, and define X_f as in Section 3. By Theorem 3.2.1, in case $X_f \neq \emptyset$ the solutions of the equation

$$x'(t) = f(x_t) \tag{1.0.1}$$

with initial function $x_0 = \phi \in X_f$ define a semiflow $F : \Omega \to X_f$. Assume that $0 \in U$ and 0 is a stationary point of F. Define $L = \mathrm{D}f(0)$, $L_e = \mathrm{D}_e f(0)$ and

$$r : U \ni \phi \mapsto f(\phi) - L\phi \in \mathbb{R}^n.$$

Clearly, r also satisfies (S), and $r(0) = 0$, $\mathrm{D}r(0) = 0$. Then (1.0.1) is equivalent to

$$x'(t) = Lx_t + r(x_t). \tag{4.1.1}$$

The solutions of the IVP

$$y'(t) = L_e y_t, \quad y_0 = \phi \in C$$

define the strongly continuous semigroup $(T_e(t))_{t \geqslant 0}$ on C with generator $G_e : \mathrm{dom}(G_e) \to C$, $\mathrm{dom}(G_e) = \{\phi \in C^1\colon\ \phi'(0) = L_e(\phi)\}$, $G_e\phi = \phi'$. The realified generalized eigenspaces of G_e given by the eigenvalues with negative, zero and positive real part are the stable C_s, center C_c and unstable C_u spaces, respectively. We have the decomposition

$$C = C_s \oplus C_c \oplus C_u,$$

C_s is infinite dimensional, C_c and C_u are finite dimensional, $C_c \subset \operatorname{dom}(G_e)$, $C_u \subset \operatorname{dom}(G_e)$. The set $C_s^1 = C_s \cap C^1$ is a closed subset of C^1. Hence we get the decomposition

$$C^1 = C_s^1 \oplus C_c \oplus C_u \tag{4.1.2}$$

of C^1.

The derivatives $D_2F(t,0)$, $t \geqslant 0$, form the strongly continuous semigroup $T(t)$, $t \geqslant 0$, on $T_0X_f = \operatorname{dom}(G_e)$ with generator G. The stable, center and unstable subspaces of G are $C_s \cap \operatorname{dom}(G_e) = C_s^1 \cap \operatorname{dom}(G_e)$, C_c and C_u, respectively, and

$$T_0X_f = \operatorname{dom}(G_e) = \big(C_s^1 \cap \operatorname{dom}(G_e)\big) \oplus C_c \oplus C_u.$$

In the sequel we assume

$$\dim C_c \geqslant 1.$$

The main result of this section guarantees the existence of a Lipschitz smooth (local) center manifold of F at the stationary point 0.

THEOREM 4.1.1. *There exist open neighbourhoods $C_{c,0}$ of* 0 *in C_c and $C^1_{su,0}$ in $C_s^1 \oplus C_u$ with $N = C_{c,0} + C^1_{su,0} \subset U$, a Lipschitz continuous map $w_c : C_{c,0} \to C^1_{su,0}$ such that $w_c(0) = 0$ and for the graph*

$$W_c = \big\{\phi + w_c(\phi)\colon \phi \in C_{c,0}\big\}$$

of w_c the following holds.

(i) *$W_c \subset X_f$, and W_c is a $\dim C_c$-dimensional Lipschitz smooth submanifold of X_f.*

(ii) *If $x : \mathbb{R} \to \mathbb{R}^n$ is a continuously differentiable solution of* (1.0.1) *on $\mathbb{R}$ with $x_t \in N$ for all $t \in \mathbb{R}$, then $x_t \in W_c$ for all $t \in \mathbb{R}$.*

(iii) *W_c is locally positively invariant with respect to the semiflow F, i.e., if $\phi \in W_c$ and $\alpha > 0$ such that $F(t,\phi)$ is defined for all $t \in [0,\alpha)$, and $F(t,\phi) \in N$ for all $t \in [0,\alpha)$, then $F(t,\phi) \in W_c$ for all $t \in [0,\alpha)$.*

The proof will be given in Section 4.3. We follow the approach of [54] by applying the Lyapunov–Perron method. The variation-of-constants formula is from [54] which requires dual spaces and adjoint operators. Other forms of the variation-of-constants formula, obtained, e.g., via integrated semigroups or extrapolation theory, could also be used. The existence of a global center manifold for a modified version of Eq. (4.1.1) is formulated as a fixed point problem with a parameter. However, as the right-hand side of (4.1.1) has smoothness properties only in the space C^1, the space, where we look for fixed points, should contain smoother functions than the corresponding space in [54]. This is why the proof is not a straightforward application of the technique of [54]. The C^1-smoothness of W_c also holds under the hypotheses of Theorem 4.1.1. A proof can be found in [131].

4.2. *The linear inhomogeneous equation*

Let $|\cdot|$ be a norm in $\mathbb{R}^n$. The spaces C, C^1 and their norms are defined as in Section 1. Let $L^\infty(-h,0;\mathbb{R}^n)$ denote the Banach space of measurable and essentially bounded functions from $(-h,0)$ into $\mathbb{R}^n$ equipped with the essential least upper bound norm $\|\cdot\|_\infty$.

We denote dual spaces and adjoint operators by an asterisk $*$ in the sequel. The elements $\phi^\odot$ of C^* for which the curve

$$[0,\infty) \ni t \mapsto T_e^*(t)\phi^\odot \in C^*$$

is continuous form a closed subspace $C^\odot$ (of C^*) which is positively invariant under $T_e^*(t)$, $t \geqslant 0$. The operators

$$T_e^\odot(t): C^\odot \ni \phi^\odot \mapsto T_e^*(t)\phi^\odot \in C^\odot, \quad t \geqslant 0,$$

constitute a strongly continuous semigroup on $C^\odot$. Similarly, we can introduce the dual space $C^{\odot *}$ and the semigroup of adjoint operators $T_e^{\odot *}(t)$, $t \geqslant 0$, which is strongly continuous on $C^{\odot\odot}$. There is an isometric isomorphism between $\mathbb{R}^n \times L^\infty(-h,0;\mathbb{R}^n)$ equipped with the norm $\|(\alpha,\phi)\| = \max\{|\alpha|, \|\phi\|_\infty\}$ and $C^{\odot *}$. We will identify $C^{\odot *}$ with $\mathbb{R}^n \times L^\infty(-h,0;\mathbb{R}^n)$ and omit the isomorphism. The original state space is sun-reflexive in the sense that, for the norm-preserving linear map $j: C \to C^{\odot *}$ given by $j(\phi) = (\phi(0),\phi)$, we have $j(C) = C^{\odot\odot}$. We also omit the embedding operator j and identify C and $C^{\odot\odot}$. All of these results as well as the decomposition of $C^{\odot *}$ and the variation-of-constants formula can be found in [54].

Let $Y^{\odot *}$ denote the subspace $\mathbb{R}^n \times \{0\}$ of $C^{\odot *}$. For the kth unit vector e_k in $\mathbb{R}^n$ set $r_k^{\odot *} = (e_k, 0) \in Y^{\odot *}$. Let $l: \mathbb{R}^n \to Y^{\odot *}$ be the linear map given by $l(e_k) = r_k^{\odot *}$, $k \in \{1,2,\ldots,n\}$. Then l has an inverse l^{-1}, and $\|l\| = \|l^{-1}\| = 1$.

Let $G_e^{\odot *}$ denote the generator of $T_e^{\odot *}$. For the spectra $\sigma(G_e)$ and $\sigma(G_e^{\odot *})$ we have $\sigma(G_e) = \sigma(G_e^{\odot *})$. Recall that we assumed

$$\sigma(G_e) \cap \mathrm{i}\mathbb{R} \neq \emptyset.$$

Then $C^{\odot *}$ can be decomposed as

$$C^{\odot *} = C_s^{\odot *} \oplus C_c \oplus C_u, \tag{4.2.1}$$

where $C_s^{\odot *}$, C_c, C_u are closed subspaces of $C^{\odot *}$, C_c and C_u are contained in C^1, $1 \leqslant \dim C_c < \infty$, $\dim C_u < \infty$. The subspaces $C_s^{\odot *}$, C_c and C_u are invariant under $T_e^{\odot *}(t)$,

$t \geqslant 0$, and $T_e(t)$ can be extended to a one-parameter group on both C_c and C_u. There exist real numbers $K \geqslant 1$, $a < 0$, $b > 0$ and $\epsilon > 0$ with $\epsilon < \min\{-a, b\}$ such that

$$\begin{aligned} \|T_e(t)\phi\| &\leqslant K\mathrm{e}^{bt}\|\phi\|, && t \leqslant 0,\ \phi \in C_u, \\ \|T_e(t)\phi\| &\leqslant K\mathrm{e}^{\epsilon|t|}\|\phi\|, && t \in \mathbb{R},\ \phi \in C_c, \\ \|T_e^{\odot *}(t)\phi\| &\leqslant K\mathrm{e}^{at}\|\phi\|, && t \geqslant 0,\ \phi \in C_s^{\odot *}. \end{aligned} \tag{4.2.2}$$

Using the identification of C and $C^{\odot\odot}$, we obtain $C_s^1 = C^1 \cap C_s^{\odot *}$. The decompositions (4.1.2) and (4.2.1) define the projection operators P_s, P_c, P_u and $P_s^{\odot *}$, $P_c^{\odot *}$, $P_u^{\odot *}$ with ranges C_s^1, C_c, C_u and $C_s^{\odot *}, C_c, C_u$, respectively.

We need a variation-of-constants formula for solutions of

$$x'(t) = L_e x_t + q(t) \tag{4.2.3}$$

with a continuous function $q : \mathbb{R} \to \mathbb{R}^n$.

If c, d are reals with $c \leqslant d$, and $w : [c, d] \to C^{\odot *}$ is continuous, then the weak-star integral

$$\int_c^d T_e^{\odot *}(d-\tau)w(\tau)\,\mathrm{d}\tau \in C^{\odot *}$$

is defined by

$$\left(\int_c^d T_e^{\odot *}(d-\tau)w(\tau)\,\mathrm{d}\tau\right)\left(\phi^{\odot}\right) = \int_c^d T_e^{\odot *}(d-\tau)w(\tau)\left(\phi^{\odot}\right)\mathrm{d}\tau$$

for all $\phi^{\odot} \in C^{\odot}$.

If $I \subset \mathbb{R}$ is an interval, $q : I \to \mathbb{R}^n$ is continuous and $x : I + [-h, 0] \to \mathbb{R}^n$ is a solution of (4.2.3) on I, then the curve $u : I \ni t \mapsto x_t \in C$ satisfies the integral equation

$$u(t) = T_e(t-s)u(s) + \int_s^t T_e^{\odot *}(t-\tau)Q(\tau)\,\mathrm{d}\tau, \quad t, s \in I,\ s \leqslant t, \tag{4.2.4}$$

with $Q(t) = l(q(t))$, $t \in I$. Moreover, if $Q : I \to Y^{\odot *}$ is continuous, and $u : I \to C$ satisfies (4.2.4), then there is a continuous function $x : I + [-h, 0] \to \mathbb{R}^n$ such that $x_t = u(t)$ for all $t \in I$, and x satisfies (4.2.3) with $q(t) = l^{-1}(Q(t))$, $t \in I$. So, there is a one-to-one correspondence between the solutions of (4.2.3) and (4.2.4).

For a Banach space B with norm $\|\cdot\|$ and a real $\eta \geqslant 0$, we define the Banach space

$$C_\eta(\mathbb{R}, B) = \left\{b \in C(\mathbb{R}, B)\colon \sup_{t\in\mathbb{R}} \mathrm{e}^{-\eta|t|}\|b(t)\| < \infty\right\}$$

with norm

$$\|b\|_{C_\eta^0(\mathbb{R},B)} = \sup_{t\in\mathbb{R}} \mathrm{e}^{-\eta|t|}\|b(t)\|.$$

For $\eta \geqslant 0$, we introduce the notation

$$Y_\eta = C_\eta\big(\mathbb{R}, Y^{\odot *}\big), \quad C_\eta^0 = C_\eta(\mathbb{R}, C), \quad C_\eta^1 = C_\eta\big(\mathbb{R}, C^1\big).$$

We need the following smoothing property of Eq. (4.2.4).

PROPOSITION 4.2.1. *Let $\eta \geqslant 0$, $Q \in Y_\eta$, $u \in C_\eta^0$, and assume that u satisfies*

$$u(t) = T_e(t-s)u(s) + \int_s^t T_e^{\odot *}(t-\tau)Q(\tau)\,\mathrm{d}\tau, \quad -\infty < s \leqslant t < \infty.$$

Then $u \in C_\eta^1$ and

$$\|u\|_{C_\eta^1} \leqslant \big(1 + \mathrm{e}^{\eta h}\|L_e\|\big)\|u\|_{C_\eta^0} + \mathrm{e}^{\eta h}\|Q\|_{Y_\eta}.$$

PROOF. Define $q : \mathbb{R} \to \mathbb{R}^n$ by $q(t) = l^{-1}(Q(t))$, $t \in \mathbb{R}$. Then $q \in C_\eta(\mathbb{R}, \mathbb{R}^n)$, and

$$\|q\|_{C_\eta(\mathbb{R},\mathbb{R}^n)} = \|Q\|_{Y_\eta}.$$

The function $x : \mathbb{R} \to \mathbb{R}^n$, given by $x(t) = u(t)(0)$, satisfies $x_t = u(t)$, $t \in \mathbb{R}$, and Eq. (4.2.3) holds for all $t \in \mathbb{R}$. Then x is C^1-smooth, $x_t \in C^1$ for all $t \in \mathbb{R}$, and the mapping $\mathbb{R} \ni t \mapsto x_t \in C^1$ is continuous. Moreover, for all $t \in \mathbb{R}$,

$$\begin{aligned}
\big|x'(t)\big| &\leqslant \|L_e\|\|x_t\|_C + \big|q(t)\big| \\
&= \|L_e\|\big\|u(t)\big\|_C + \big\|Q(t)\big\|_{Y^{\odot *}} \\
&\leqslant \mathrm{e}^{\eta|t|}\big(\|L_e\|\|u\|_{C_\eta^0} + \|Q\|_{Y_\eta}\big).
\end{aligned}$$

Hence

$$\begin{aligned}
\sup_{t\in\mathbb{R}} \mathrm{e}^{-\eta|t|}\big\|x_t'\big\|_C &= \sup_{t\in\mathbb{R}} \mathrm{e}^{-\eta|t|} \sup_{-h\leqslant s\leqslant 0} \big|x'(t+s)\big| \\
&\leqslant \big(\|L_e\|\|u\|_{C_\eta^0} + \|Q\|_{Y_\eta}\big) \sup_{t\in\mathbb{R}} \mathrm{e}^{-\eta|t|} \sup_{-h\leqslant s\leqslant 0} \mathrm{e}^{\eta|t+s|} \\
&\leqslant \mathrm{e}^{\eta h}\big(\|L_e\|\|u\|_{C_\eta^0} + \|Q\|_{Y_\eta}\big).
\end{aligned}$$

Therefore, $u \in C_\eta^1$, and

$$\begin{aligned}
\|u\|_{C_\eta^1} &= \sup_{t\in\mathbb{R}} \mathrm{e}^{-\eta|t|}\|x_t\|_{C^1} = \sup_{t\in\mathbb{R}} \mathrm{e}^{-\eta|t|}\big(\|x_t\|_C + \big\|x_t'\big\|_C\big) \\
&\leqslant \|u\|_{C_\eta^0} + \mathrm{e}^{\eta h}\|L_e\|\|u\|_{C_\eta^0} + \mathrm{e}^{\eta h}\|Q\|_{Y_\eta}.
\end{aligned}$$

□

For a given $Q:\mathbb{R}\to Y^{\odot *}$ we (formally) define

$$(\mathcal{K}Q)(t)=\int_0^t T_e^{\odot *}(t-\tau)P_c^{\odot *}Q(\tau)\,\mathrm{d}\tau+\int_\infty^t T_e^{\odot *}(t-\tau)P_u^{\odot *}Q(\tau)\,\mathrm{d}\tau$$
$$+\int_{-\infty}^t T_e^{\odot *}(t-\tau)P_s^{\odot *}Q(\tau)\,\mathrm{d}\tau.$$

PROPOSITION 4.2.2. *Assume* $\eta\in(\epsilon,\min\{-a,b\})$. *Then the mapping*

$$\mathcal{K}_\eta : Y_\eta \ni Q\mapsto \mathcal{K}Q\in C^1_\eta$$

is linear bounded with norm

$$\|\mathcal{K}_\eta\|\leqslant K\bigl(1+\mathrm{e}^{\eta h}\|L_e\|\bigr)\left(\frac{1}{\eta-\epsilon}+\frac{1}{-a-\eta}+\frac{1}{b-\eta}\right)+\mathrm{e}^{\eta h}.$$

If $Q\in Y_\eta$ *then* $u=\mathcal{K}Q$ *is the unique solution of*

$$u(t)=T_e(t-s)u(s)+\int_s^t T_e^{\odot *}(t-\tau)Q(\tau)\,\mathrm{d}\tau,\quad -\infty<s\leqslant t<\infty, \tag{4.2.5}$$

in C^1_η *with* $P_c^{\odot *}u(0)=0$.

PROOF. Lemma IX.3.2 of [54] shows that $\mathcal{K}$ as a mapping from Y_η into C^0_η is linear bounded such that its norm is bounded by

$$K\left(\frac{1}{\eta-\epsilon}+\frac{1}{-a-\eta}+\frac{1}{b-\eta}\right),$$

moreover $u=\mathcal{K}Q$ with $Q\in Y_\eta$ is the unique solution of (4.2.5) with $P_c^{\odot *}u(0)=0$. Hence Proposition 4.2.1 yields the boundedness of $\mathcal{K}_\eta$ with the stated bound for the norm. □

4.3. *Construction of a center manifold*

Now we prove Theorem 4.1.1.

As $\dim C_c<\infty$, there is a norm $|\cdot|_c$ on C_c which is C^∞-smooth on $C_c\setminus\{0\}$. Then

$$|\phi|_1=\max\bigl\{|P_c\phi|_c,\ \bigl\|(\mathrm{id}_{C^1}-P_c)\phi\bigr\|_{C^1}\bigr\},\quad \phi\in C^1,$$

defines the new norm $|\cdot|_1$ on C^1 which is equivalent to $\|\cdot\|_{C^1}$.

Let $\rho:\mathbb{R}\to\mathbb{R}$ be a C^∞-smooth function so that $\rho(t)=1$ for $t\leqslant 1$, $\rho(t)=0$ for $t\geqslant 2$, and $\rho(t)\in(0,1)$ for $t\in(1,2)$.

Define

$$\hat{r}(\phi) = \begin{cases} r(\phi), & \text{if } \phi \in U; \\ 0, & \text{if } \phi \notin U. \end{cases}$$

For any $\delta > 0$, let

$$r_\delta(\phi) = \hat{r}(\phi)\rho\left(\frac{|P_c\phi|_c}{\delta}\right)\rho\left(\frac{|(\mathrm{id}_{C^1} - P_c)\phi|_1}{\delta}\right), \quad \phi \in C^1.$$

For $\gamma > 0$ set $B_\gamma(C^1) = \{\phi \in C^1\colon\ |\phi|_1 < \gamma\}$.

Choose $\delta_0 > 0$ so that

$$B_{2\delta_0}\big(C^1\big) \subset U,$$

and $r|_{B_{2\delta_0}(C^1)}$, $\mathrm{D}r|_{B_{2\delta_0}(C^1)}$ are bounded. Then, for any $\delta \in (0, \delta_0)$

$$r_\delta|_{\{\phi\in C^1\colon\ |(\mathrm{id}_{C^1} - P_c)\phi|_1 < \delta\}}(\phi) = \hat{r}(\phi)\rho\left(\frac{|P_c\phi|_c}{\delta}\right), \quad \phi \in C^1,$$

and $r_\delta|_{\{\phi\in C^1\colon\ |(\mathrm{id}_{C^1} - P_c)\phi|_1 < \delta\}}$ is a bounded and C^1-smooth function with bounded derivative.

There exist $\delta_1 \in (0, \delta_0)$ and a nondecreasing function $\mu : [0, \delta_1] \to [0, 1]$ such that μ is continuous at 0, $\mu(0) = 0$, and for all $\delta \in (0, \delta_1]$ and for all $\phi, \psi \in C^1$

$$\begin{aligned} &\big|r_\delta(\phi)\big| \leqslant \delta\mu(\delta), \\ &\big|r_\delta(\phi) - r_\delta(\psi)\big| \leqslant \mu(\delta)\|\phi - \psi\|_{C^1}. \end{aligned} \tag{4.3.1}$$

For a proof of completely analogous estimates see, e.g., Proposition II.2 in [133].

For $\delta \in (0, \delta_1]$ we consider the modified equations

$$x'(t) = Lx_t + r_\delta(x_t), \quad t \in \mathbb{R}, \tag{4.3.2}$$

and

$$u(t) = T_e(t - s) + \int_s^t T_e^{\odot *}(t - \tau) l\big(r_\delta\big(u(\tau)\big)\big)\,\mathrm{d}\tau, \quad -\infty < s \leqslant t < \infty. \tag{4.3.3}$$

These equations are equivalent in the following sense: If $x : \mathbb{R} \to \mathbb{R}^n$ is C^1-smooth and is a solution of Eq. (4.3.2), then $u : \mathbb{R} \ni t \mapsto x_t \in C^1$ is a solution of Eq. (4.3.3), and conversely, a continuous $u : \mathbb{R} \to C^1$ satisfying (4.3.3) defines a C^1-smooth solution of (4.3.2) by $x(t) = u(t)(0)$, $t \in \mathbb{R}$.

Now we fix the reals $\eta \in (\epsilon, \min\{-a, b\})$ and $\delta \in (0, \delta_1)$ such that

$$\|\mathcal{K}_\eta\| \mu(\delta) < \frac{1}{2}. \tag{4.3.4}$$

Let the substitution operator

$$R : \left(C^1\right)^{\mathbb{R}} \to \left(Y^{\odot *}\right)^{\mathbb{R}}$$

of the map $C^1 \ni \phi \mapsto l(r_\delta(\phi)) \in Y^{\odot *}$ be given by

$$R(u)(t) = l\big(r_\delta\big(u(t)\big)\big).$$

Inequalities (4.3.1) and $\|l\| = 1$ yield that

$$R\big(C^1_\eta\big) \subset Y_\eta,$$

and for the induced map

$$R_{\delta\eta} : C^1_\eta \to Y_\eta,$$

the inequalities

$$\big\| R_{\delta\eta}(u) \big\|_{Y_\eta} \leqslant \delta \mu(\delta), \quad u \in C^1_\eta,$$

and

$$\big\| R_{\delta\eta}(u) - R_{\delta\eta}(v) \big\|_{Y_\eta} \leqslant \mu(\delta) \|u - v\|_{C^1_\eta}, \quad u, v \in C^1_\eta, \tag{4.3.5}$$

hold.

Let the mapping $S : C_c \to C^1_\eta$ be given by $(S\phi)(t) = T_e(t)\phi$, $\phi \in C_c$, $t \in \mathbb{R}$. For all $\phi \in C_c$ we have $\|T_e(t)\phi\|_{C^1} = \|T_e(t)\phi\|_C + \|\frac{\mathrm{d}}{\mathrm{d}t}(T_e(t)\phi)\|_C$ and $\frac{\mathrm{d}}{\mathrm{d}t}(T_e(t)\phi) = T_e(t)G_e\phi = T_e(t)\phi'$. Therefore, by applying the second inequality in (4.2.2) and $\eta > \epsilon$, we find

$$\begin{aligned} \|S\phi\|_{C^1_\eta} &= \sup_{t \in \mathbb{R}} \mathrm{e}^{-\eta|t|} \big(\big\| T_e(t)\phi \big\|_C + \big\| T_e(t)\phi' \big\|_C \big) \\ &\leqslant K \big(\|\phi\|_C + \big\|\phi'\big\|_C \big) \\ &\leqslant K \|\phi\|_{C^1}. \end{aligned} \tag{4.3.6}$$

Define the mapping

$$\mathcal{G} : C^1_\eta \times C_c \to C^1_\eta$$

by

$$\mathcal{G}(u, \phi) = S\phi + \mathcal{K}_\eta \circ R_{\delta\eta}(u), \quad u \in C^1_\eta,\ \phi \in C_c.$$

For all $u, v \in C^1_\eta$ and $\phi \in C_c$, (4.3.4) and (4.3.5) yield

$$\begin{aligned}\left\|\mathcal{G}(u,\phi)-\mathcal{G}(v,\phi)\right\|_{C^1_\eta} &\leqslant \|\mathcal{K}_\eta\|\left\|R_{\delta\eta}(u)-R_{\delta\eta}(v)\right\|_{Y_\eta}\\ &\leqslant \|\mathcal{K}_\eta\|\mu(\delta)\|u-v\|_{C^1_\eta}\\ &\leqslant \frac{1}{2}\|u-v\|_{C^1_\eta}.\end{aligned}$$

If $\gamma > 0$ and $\phi \in C_c$ with $\|\phi\|_{C^1} \leqslant \gamma/(2K)$, and $u \in \overline{B_\gamma(C^1_\eta)}$, then, by using (4.3.4), (4.3.5) and (4.3.6),

$$\begin{aligned}\left\|\mathcal{G}(u,\phi)\right\|_{C^1_\eta} &\leqslant K\|\phi\|_{C^1}+\|\mathcal{K}_\eta\|\mu(\delta)\|u\|_{C^1_\eta}\\ &\leqslant \frac{\gamma}{2}+\frac{\gamma}{2}=\gamma.\end{aligned}$$

Therefore, $\mathcal{G}(\cdot,\phi)$ maps $\overline{B_\gamma(C^1_\eta)}$ into itself provided $\gamma \geqslant 2K\|\phi\|_{C^1}$. In addition, $\mathcal{G}(\cdot,\phi)$ is Lipschitz continuous with Lipschitz constant $1/2$.

Consequently, there is a map

$$u^*: C_c \to C^1_\eta$$

such that, for each $\phi \in C_c$, $u = u^*(\phi)$ is the unique solution in C^1_η of the equation

$$u = \mathcal{G}(u,\phi).$$

The mapping u^* is globally Lipschitz continuous since

$$\left\|u^*(\phi)-u^*(\psi)\right\|_{C^1_\eta} \leqslant K\|\phi-\psi\|_{C^1}+\frac{1}{2}\left\|u^*(\phi)-u^*(\psi)\right\|_{C^1_\eta},$$

yielding

$$\left\|u^*(\phi)-u^*(\psi)\right\|_{C^1_\eta} \leqslant 2K\|\phi-\psi\|_{C^1}$$

for all $\phi, \psi \in C_c$.

The set

$$W = \left\{u^*(\phi)(0):\ \phi \in C_c\right\}$$

is called the global center manifold of Eq. (4.3.2) at the stationary point 0.

Setting

$$w: C_c \ni \phi \mapsto (\mathrm{id}_{C^1} - P_c)u^*(\phi)(0) \in C^1_s \oplus C_u,$$

we get the graph representation

$$W = \{\phi + w(\phi)\colon\ \phi \in C_c\}$$

for W.

For all $\phi \in C_c$ we have

$$\begin{aligned}
|w(\phi)|_1 &= \|w(\phi)\|_{C^1} = \|(\mathrm{id}_{C^1} - P_c)u^*(\phi)(0)\|_{C^1} \\
&= \|\mathcal{K}_\eta(R_{\delta\eta}(u^*(\phi)))(0)\|_{C^1} \leqslant \|\mathcal{K}_\eta(R_{\delta\eta}(u^*(\phi)))\|_{C^1_\eta} \\
&\leqslant \|\mathcal{K}_\eta\| \|R_{\delta\eta}(u^*(\phi))\|_{Y_\eta} \\
&\leqslant \|\mathcal{K}_\eta\| \delta\mu(\delta) < \delta.
\end{aligned}$$

An important consequence is that

$$W \subset \{\phi \in C^1\colon\ |(\mathrm{id}_{C^1} - P_c)\phi|_1 < \delta\},$$

that is, W is contained in the δ-neighbourhood of C_c, where r_δ is C^1-smooth with bounded derivative. This fact is essential in the proof of the C^1-smoothness of the center manifold.

Setting

$$\begin{aligned}
&C_{c,0} = \{\phi \in C_c\colon\ |\phi|_1 < \delta\}, \\
&C^1_{su,0} = \{\phi \in C^1_s \oplus C_u\colon\ |\phi|_1 < \delta\}, \\
&N = C_{c,0} + C^1_{su,0} = \{\phi \in C^1\colon\ |\phi|_1 < \delta\}, \\
&w_c = w|_{C_{c,0}}, \\
&W_c = \{\phi + w_c(\phi)\colon\ \phi \in C_{c,0}\},
\end{aligned}$$

we obtain that $w_c(C_{c,0}) \subset C^1_{su,0}$ and w_c is Lipschitz continuous. As $\mathcal{G}(0,0) = 0$, it follows that $u^*(0) = 0$, and consequently $w_c(0) = 0$.

Let $v \in C^1_\eta$ be a solution of Eq. (4.3.3). Define $z : \mathbb{R} \to C^1$ by

$$z(t) = v(t) - T_e(t)P_c v(0), \quad t \in \mathbb{R}.$$

Obviously, $v \in C^1_\eta$ implies $z \in C^1_\eta$. Moreover,

$$z(t) = T_e(t-s)z(s) + \int_s^t T_e^{\odot *}(t-\tau)l(r_\delta(v(\tau)))\,\mathrm{d}\tau, \quad -\infty < s \leqslant t < \infty.$$

As $P_c z(0) = 0$ and $v \in C^1_\eta$, Proposition 4.2.2 yields

$$z = \mathcal{K}_\eta(R_{\delta\eta}(v)).$$

Therefore

$$v = S\big(P_c v(0)\big) + \mathcal{K}_\eta\big(R_{\delta\eta}(v)\big),$$

and

$$v(0) = u^*\big(P_c v(0)\big) \in W^c.$$

For any $t \in \mathbb{R}$ and $\hat{v} : \mathbb{R} \ni s \mapsto v(t+s) \in C^1$, it is clear that $\hat{v} \in C^1_\eta$, and $\hat{v}$ is also a solution of Eq. (4.3.3). Therefore, $v(t) = \hat{v}(0) \in W$ follows for all $t \in \mathbb{R}$. Consequently, for each $v \in C^1_\eta$ satisfying Eq. (4.3.3), $v(t) \in W$ holds for all $t \in \mathbb{R}$.

If $x : \mathbb{R} \to \mathbb{R}^n$ is a solution of Eq. (1.0.1) with $x_t \in N$ for all $t \in \mathbb{R}$, then (4.3.2) also holds, and $u(t) = x_t$, $t \in \mathbb{R}$, satisfies Eq. (4.3.3) since $r|_N = r_\delta|_N$, and $u \in C^1_\eta$. Thus, $x_t \in W$, $t \in \mathbb{R}$. This proves (ii) of Theorem 4.1.1.

In order to show (iii) in Theorem 4.1.1, let $\phi \in W_c$. Then $u^*(P_c\phi) \in C^1_\eta$, and $u^*(P_c\phi)(t) \in W$ for all $t \in \mathbb{R}$. Let $\beta \in (0, \infty]$ be maximal so that

$$u^*(P_c\phi)(t) \in N \quad \text{for all } t \in [0, \beta),$$

that is,

$$u^*(P_c\phi)(t) \in W_c \quad \text{for all } t \in [0, \beta).$$

Then there exists a C^1-smooth function $y : [-h, \beta) \to \mathbb{R}^n$ so that $y_t = u^*(P_c\phi)(t)$, $t \in [0, \beta)$, and

$$\begin{cases} y'(t) = Ly_t + r(y_t), & 0 < t < \beta, \\ y_0 = \phi. \end{cases}$$

If $x^\phi : [-h, \alpha)$ is also a solution of the above IVP with $x^\phi_t \in N$, $t \in [0, \alpha)$, then the result on unique continuation of solutions in Section 3 yields $\alpha \leqslant \beta$ and $x^\phi(t) = y(t)$, $t \in [-h, \alpha)$.

For any $\phi \in W_c$, the function $x : \mathbb{R} \to \mathbb{R}^n$ defined by $x_t = u^*(P_c\phi)(t)$, $t \in \mathbb{R}$, is continuously differentiable and satisfies Eq. (4.3.2). Consequently, $x'(t) = Lx_t + r_\delta(x_t)$, $t \in \mathbb{R}$. In particular, $\phi'(0) = L\phi + r_\delta(\phi)$. Using $\phi \in W_c \subset N$ and $r_\delta|_N = r|_N$, $\phi \in X_f$ follows. Thus, $W_c \subset X_f$.

Recall from Section 3.5 that there is an n-dimensional subspace $E \subset C^1$ which is a complement of $Y = T_0 X_f$ in C^1. If $e_1, \dots, e_n$ is a basis for E, then using the decomposition $C^1 = C^1_s \oplus C_c \oplus C_u$ of C^1, for each $i \in \{1, \dots, n\}$ we have

$$e_i = s_i + c_i + u_i$$

for some $s_i \in C^1_s$, $c_i \in C_c$, $u_i \in C_u$. As $C_c \oplus C_u \subset Y$, we have $s_i \notin Y$. Then the subspace $\hat{E}$ spanned by the vectors

$$\hat{e}_i = e_i - c_i - u_i, \quad i \in \{1, \dots, n\},$$

is also an n-dimensional complementary subspace of Y in C^1, and in addition $\hat{E} \subset C_s^1$. Therefore, without loss of generality, we may assume $E \subset C_s^1$. Then

$$C_s^1 = E \oplus \left(C_s^1 \cap Y\right),$$

$$Y = \left(C_s^1 \cap Y\right) \oplus C_c \oplus C_u,$$

and

$$C^1 = E \oplus \left(C_s^1 \cap Y\right) \oplus C_c \oplus C_u = E \oplus Y.$$

Let $P : C^1 \to C^1$ denote the projection along E onto Y. There is an open neighbourhood V of 0 in X_f so that $P : V \to Y$ is a manifold chart of X_f. Set $Y_0 = P(V)$. The inverse of $P : V \to Y_0$ is C^1-smooth. If $\delta > 0$ is sufficiently small, then $W_c \subset V$ and $PW_c \subset Y_0$. In order to complete the proof of (i) in Theorem 4.1.1, it is enough to show that PW_c is a $\dim C_c$-dimensional Lipschitz submanifold of Y. Indeed,

$$PW_c = \left\{P\left(\phi + w_c(\phi)\right)\colon \phi \in C_{c,0}\right\} = \left\{\phi + Pw_c(\phi)\colon \phi \in C_{c,0}\right\}.$$

As $w_c(\phi) \in C_s^1 \oplus C_u$, we have

$$Pw_c(\phi) \in \left(C_s^1 \cap Y\right) \oplus C_u.$$

Thus, PW_c is the graph of the Lipschitz continuous map

$$\left\{\phi \in C_c\colon |\phi|_1 < \delta\right\} \ni \chi \mapsto Pw_c(\chi) \in \left(C_s^1 \cap Y\right) \oplus C_u.$$

This completes the proof of Theorem 4.1.1.

4.4. *Discussion*

It is also true that the local center manifold W_c given in Theorem 4.1.1 is a C^1-submanifold of X_f. The proof will appear in [131]. It is based on the fact that for the global center manifold W of the modified Eq. (4.3.1)

$$W \subset \left\{\phi \in C^1\colon \left|(\mathrm{id}_{C^1} - P_c)\phi\right|_1 < \delta\right\},$$

and r_δ is C^1-smooth on the subset $\{\phi \in C^1\colon |(\mathrm{id}_{C^1} - P_c)\phi|_1 < \delta\}$ of C^1 with bounded derivative. The techniques of [54] or [133] can be modified to our situation in order to show that the map $w : C_c \to C_s^1 \oplus C_u$ is C^1-smooth.

Another way to obtain C^1-smooth center manifolds is to consider, for some $a > 0$, the time-a map

$$F_a : \Omega_a \to X_f,$$

and construct a local center manifold of F_a at its fixed point 0. However, as a 2-dimensional ordinary differential equation example shows in [130], the obtained center manifold of F_a is not necessarily a locally invariant center manifold of the semiflow F. There is a standard technique to overcome this difficulty (see, e.g., [130]). The idea is that the modification of the map F_a should be done through the modification of the semiflow F. This requires a modification and extension of F from a small neighbourhood of $[0, a] \times \{0\}$ in Ω to a certain global semiflow. This is a nontrivial task. It is an open problem to work out the complete proof by using this approach.

Local bifurcation results for functional differential equations with state-dependent delay through the center manifold reduction would require C^k-smooth local center manifolds also with $k > 1$. As far as we know such results are not available at the moment.

A first step towards the proof of a C^k-smooth center manifold could be a C^k-smooth version of the results of Section 3. Then a C^k-smooth time-a map could be the basis to construct a C^k-smooth center manifold as suggested above for $k = 1$.

Another possible way is the extension of the approach explained in this section. Notice that it does not require the existence of a smooth semiflow. We remark that this idea worked for a construction of C^k-smooth unstable manifolds under natural conditions on f which are satisfied by equations with state-dependent delay [129].

5. Hopf bifurcation

Hopf bifurcation is the phenomenon that under certain conditions small periodic orbits appear close to a stationary point when in the underlying differential equation a parameter is varied and passes a critical value. In this section we state a Hopf bifurcation theorem for differential equations with state-dependent delay which has recently been proved by M. Eichmann [65]. The equation considered is a parametrized version of Eq. (1.0.1), namely

$$x'(t) = g(\alpha, x_t). \tag{5.0.1}$$

The map $g : J \times U \to \mathbb{R}^n$ in Eq. (5.0.1) is defined on the product of an open interval $J \subset \mathbb{R}$ and an open subset $U \subset C^1 = C^1([-h, 0], \mathbb{R}^n)$, with $h > 0$ and $n \in \mathbb{N}$. Let $C = C([-h, 0], \mathbb{R}^n)$ and let C^2 denote the Banach space of twice continuously differentiable functions $\phi : [-h, 0] \to \mathbb{R}^n$, with the norm given by $\|\phi\|_{C^2} = \|\phi\|_C + \|\phi'\|_C + \|\phi''\|_C$. The set

$$U^* = U \cap C^2$$

is an open subset of C^2. The following hypotheses on smoothness are assumed.

(H1) The mapping $g : J \times U \to \mathbb{R}^n$ is continuously differentiable.

(H2) For each $(\alpha, \phi) \in J \times U$ the partial derivative $\mathrm{D}_2 g(\alpha, \phi)$ of g with respect to ϕ extends to a continuous linear map

$$\mathrm{D}_{2,e} g(\alpha, \phi) : C \to \mathbb{R}^n.$$

(H3) The mapping

$$J \times U \times C \ni (\alpha, \phi, \chi) \mapsto \mathrm{D}_{2,e} g(\alpha, \phi)\chi \in \mathbb{R}^n$$

is continuous.

(H4) The restriction $g^* = g|J \times U^*$ is twice continuously differentiable.

(H5) For each $(\alpha, \phi) \in J \times U^*$ the second order partial derivative $D_2^2 g^*(\alpha, \phi) : C^2 \times C^2 \to \mathbb{R}^n$ of g^* with respect to ϕ has a continuous bilinear extension

$$\mathrm{D}_{2,e}^2 g^*(\alpha, \phi) : C^1 \times C^1 \to \mathbb{R}^n.$$

(H6) The mappings

$$J \times U^* \times C^1 \times C^1 \ni (\alpha, \phi, \chi_1, \chi_2) \mapsto \mathrm{D}_{2,e}^2 g^*(\alpha, \phi)(\chi_1, \chi_2) \in \mathbb{R}^n$$

and

$$J \times U^* \times C^1 \ni (\alpha, \phi, \chi) \mapsto \mathrm{D}_{2,e}^2 g^*(\alpha, \phi)(\chi, \cdot) \in L\big(C^2, \mathbb{R}^n\big)$$

are continuous.

The hypotheses (H1)–(H3) imply that each map $g(\alpha, \cdot) : U \to \mathbb{R}^n$, $\alpha \in J$, satisfies the hypotheses (S1)–(S3) of Theorem 3.2.1. Notice that condition (H3) is weaker than continuity of the map

$$J \times U \ni (\alpha, \phi) \mapsto \mathrm{D}_{2,e} g(\alpha, \phi) \in L\big(C, \mathbb{R}^n\big).$$

In (H4), differentiability refers to the norm given by $\|(\alpha, \phi)\| = |\alpha| + \|\phi|_{C^2}$ on $\mathbb{R} \times C^2$. Notice that condition (H6) is weaker than continuity of the map

$$J \times U^* \ni (\alpha, \phi) \mapsto \mathrm{D}_{2,e}^2 g^*(\alpha, \phi) \in L^2\big(C^1, \mathbb{R}^n\big),$$

where $L^2(C^1, \mathbb{R}^n)$ denotes the Banach space of continuous bilinear maps $C^1 \times C^1 \to \mathbb{R}^n$, with the appropriate norm.

Suppose $\phi^* \in U^* \subset C^2$ satisfies

$$g^*\big(\alpha, \phi^*\big) = 0 \quad \text{for all } \alpha \in J,$$

so that ϕ^* is a stationary point for all $\alpha \in J$.

For $\alpha \in J$, set $L(\alpha) = \mathrm{D}_2 g(\alpha, \phi^*)$, and let $A(\alpha)$ denote the generator of the strongly continuous semigroup on C given by the IVP

$$y'(t) = \mathrm{D}_{2,e} g(\alpha, \phi^*) y_t, \quad y_0 = \chi \in C.$$

The spectral assumptions for the Hopf bifurcation theorem from [65] are the following.

(L1) There is a continuously differentiable map $\lambda : I \to \mathbb{C}$, $I \subset J$ an open interval, such that each $\lambda(\alpha)$, $\alpha \in I$, is a simple eigenvalue of $A(\alpha)$.

(L2) For some $\alpha_0 \in I$, $Re\,\lambda(\alpha_0) = 0$ and $\omega_0 = Im\,\lambda(\alpha_0) > 0$ and

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}(Re\,\lambda)(\alpha_0) \neq 0.$$

(L3) For every integer $k \in \mathbb{Z} \setminus \{-1, 1\}$, $\mathrm{i}k\omega_0$ is not an eigenvalue of $A(\alpha_0)$.

The following local Hopf bifurcation theorem is obtained in [65]:

THEOREM 5.0.1. *Suppose* (H1)–(H6) *and* (L1)–(L3) *hold. Then there are an interval $M \subset \mathbb{R}$ with $0 \in M$ and continuously differentiable mappings $u^* : M \to C^1$, $\omega^* : M \to \mathbb{R}$ and $\alpha^* : M \to I$ with*

$$u^*(0) = \phi^*, \quad \alpha^*(0) = \alpha_0, \quad \omega^*(0) = \omega_0$$

such that for each $a \in M$ there is a periodic solution $x : \mathbb{R} \to \mathbb{R}^n$ of the equation

$$x'(t) = g\big(\alpha^*(a), x_t\big)$$

with $x_0 = u^*(a)$ and with period $\frac{\omega^*(a)}{2\pi}$.

To our knowledge, Theorem 5.0.1 is the first Hopf bifurcation result for differential equations with state-dependent delay. A related earlier result is due to H.L. Smith [189] who proved bifurcation of periodic solutions from a stationary point for a system of integral equations with state-dependent delay, by reduction to a Hopf bifurcation theorem of Hale and de Oliveira [95] for equations with time-invariant but parameter-dependent delay.

6. Differentiability of solutions with respect to parameters

6.1. *Preliminaries*

This section deals with nonautonomous parametrized state-dependent delay systems of the form

$$x'(t) = g\big(t, x(t), x\big(t - \tau(t, x_t, \sigma)\big), \theta\big), \quad t \in [0, T], \tag{6.1.1}$$

with initial condition

$$x(t) = \phi(t), \quad t \in [-h, 0]. \tag{6.1.2}$$

Here σ and θ are parameters in the delay function τ and in g belonging to normed linear spaces Σ and Θ, respectively. In the sequel we consider also the initial function ϕ in the IVP (6.1.1)–(6.1.2) as parameters, and denote the corresponding solution by $x(\cdot; \phi, \sigma, \theta)$, and its segment function at t by $x(\cdot; \phi, \sigma, \theta)_t$.

Suppose, e.g., a system of the form

$$y'(s) = \tilde{g}\big(s, y(s), y\big(s - \tilde{\tau}(s, y_t)\big)\big), \quad s \in [t_0, t_0 + T]$$

is given. Then introducing $t = s - t_0$ and $x(t) = y(t + t_0)$ we can transform the equation into the form (6.1.1) with $g(t, \psi, u, \theta) = \tilde{g}(t + \theta, \psi, u)$ with $\theta = t_0$, and $\tau(s, \psi, \sigma) = \tilde{\tau}(s + \sigma, \psi)$ with $\sigma = t_0$. In this case $\Sigma = \Theta = \mathbb{R}$. Of course, (6.1.1) contains more general cases as well, e.g., σ and θ can be coefficient functions in the delay function τ and g, respectively. In this case Σ and Θ will be infinite dimensional function spaces.

As we have seen in Section 3, in general the IVP (6.1.1)–(6.1.2) has a unique solution only if the initial function ϕ is Lipschitz continuous, or equivalently, if ϕ belongs to the Banach space $W^{1,\infty}$ of absolutely continuous functions $\phi : [-h, 0] \to \mathbb{R}^n$ with essentially bounded derivatives, with the norm defined by

$$|\phi|_{W^{1,\infty}} = \max\big\{|\phi|_C, \operatorname{ess\,sup}\big\{\big|\phi'(s)\big| : s \in [-h, 0]\big\}\big\}.$$

We define the parameter space for the IVP (6.1.1)–(6.1.2) as

$$\Gamma = W^{1,\infty} \times \Sigma \times \Theta,$$

and the norm on Γ by

$$|\gamma|_\Gamma = \big|(\phi, \sigma, \theta)\big|_\Gamma = |\phi|_{W^{1,\infty}} + |\sigma|_\Sigma + |\theta|_\Theta.$$

We assume throughout this section that $\Omega_1 \subset \mathbb{R}^n$, $\Omega_2 \subset \mathbb{R}^n$, $\Omega_3 \subset \Theta$, $\Omega_4 \subset C$ and $\Omega_5 \subset \Sigma$ are open subsets of the respective spaces, $T > 0$ is finite or $T = \infty$ (in the latter case $[0, T]$ means $[0, \infty)$), and

(D1) (i) $g : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \Theta \supset [0, T] \times \Omega_1 \times \Omega_2 \times \Omega_3 \to \mathbb{R}^n$ is continuous; and

(ii) g is locally Lipschitz continuous with respect to its second, third and fourth variables in the following sense: For every $\alpha \in (0, T]$, for every compact subsets $M_i \subset \Omega_i$ $(i = 1, 2)$ of $\mathbb{R}^n$, and for every closed and bounded subset $M_3 \subset \Omega_3$ of Θ there exists $L_1 = L_1(\alpha, M_1, M_2, M_3)$ such that $|f(t, v, w, \theta) - f(t, \bar{v}, \bar{w}, \bar{\theta})| \leqslant L_1(|v - \bar{v}| + |w - \bar{w}| + |\theta - \bar{\theta}|_\Theta)$, for $t \in [0, \alpha]$, $v, \bar{v} \in M_1$, $w, \bar{w} \in M_2$, and $\theta, \bar{\theta} \in M_3$;

(D2) (i) $\tau : \mathbb{R} \times C \times \Sigma \supset [0, T] \times \Omega_4 \times \Omega_5 \to \mathbb{R}$ is continuous, $0 \leqslant \tau(t, \psi, \sigma) \leqslant h$ for $t \in [0, T]$, $\psi \in \Omega_4$, and $\sigma \in \Omega_5$; and

(ii) $\tau(t, \psi, \sigma)$ is locally Lipschitz-continuous in ψ and σ in the following sense: For every $\alpha \in (0, T]$, for every compact subset $M_4 \subset \Omega_4$ of C and for every closed, bounded subset $M_5 \subset \Omega_5$ of Σ there exists a constant $L_2 = L_2(\alpha, M_4, M_5)$ such that $|\tau(t, \psi, \sigma) - \tau(t, \bar{\psi}, \bar{\sigma})| \leqslant L_2(|\psi - \bar{\psi}|_C + |\sigma - \bar{\sigma}|_\Sigma)$ for $t \in [0, \alpha]$, $\psi, \bar{\psi} \in M_4$, and $\sigma, \bar{\sigma} \in M_5$.

We, of course, assume that a parameter $\bar{\gamma} = (\bar{\phi}, \bar{\sigma}, \bar{\theta}) \in \Gamma$ satisfies the compatibility condition

$$\bar{\phi}(0) \in \Omega_1, \quad \bar{\phi}\big(-\tau\big(0, \bar{\phi}, \bar{\sigma}\big)\big) \in \Omega_2, \quad \bar{\theta} \in \Omega_3, \quad \bar{\phi} \in \Omega_4, \quad \text{and} \quad \bar{\sigma} \in \Omega_5. \tag{6.1.3}$$

It is known [57] that the IVP (6.1.1)–(6.1.2) has a unique solution for each parameter $(\bar{\phi}, \bar{\sigma}, \bar{\theta}) \in \Gamma$, moreover, the solution is Lipschitz continuous with respect to the parameters [99]. We denote the open ball with radius δ centered at $\bar{\gamma}$ in Γ by $G_\Gamma(\bar{\gamma}, \delta)$, i.e., $G_\Gamma(\bar{\gamma}, \delta) = \{\gamma \in \Gamma\colon |\gamma - \bar{\gamma}|_\Gamma < \delta\}$. The next result is proved, e.g., in [99].

THEOREM 6.1.1. *Suppose* (D1)(i)–(ii) *and* (D2)(i)–(ii). *For any* $\bar{\gamma} = (\bar{\phi}, \bar{\sigma}, \bar{\theta}) \in \Gamma$ *satisfying* (6.1.3) *there exist* $\alpha > 0$, $\delta > 0$ *and* $L = L(\alpha, \bar{\gamma}, \delta)$ *such that the IVP* (6.1.1)–(6.1.2) *has a unique solution on* $[-h, \alpha]$ *for any* $\gamma \in G_\Gamma(\bar{\gamma}, \delta)$, *and*

$$\big|x(\cdot; \gamma)_t - x(\cdot; \bar{\gamma})_t\big|_{W^{1,\infty}} \leqslant L|\gamma - \bar{\gamma}|_\Gamma \quad \textit{for } t \in [0, \alpha],\ \gamma \in G_\Gamma(\bar{\gamma}, \delta).$$

6.2. *Pointwise differentiability with respect to parameters*

In this subsection we study differentiability of the function $\gamma \mapsto x(t; \gamma)$ where t is fixed. In addition to (D1)(i)–(ii) and (D2)(i)–(ii) we need

- (D1) (iii) $g : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \Theta \supset [0, T] \times \Omega_1 \times \Omega_2 \times \Omega_3 \to \mathbb{R}^n$ is continuously differentiable with respect to its second, third and fourth variables;
- (D2) (iii) $\tau : \mathbb{R} \times C \times \Sigma \supset [0, T] \times \Omega_4 \times \Omega_5 \to [0, \infty)$ is continuously differentiable with respect to its second and third variables.

As it was shown in Section 3.2, solutions corresponding to a parameter value from the parameter set

$$\Pi = \big\{(\phi, \sigma, \theta) \in \Gamma\colon \phi'(0) = g\big(0, \phi(0), \phi\big(-\tau(0, \phi, \sigma)\big), \theta\big),\ \phi \in C^1\big\}$$

are continuously differentiable on $[-h, \alpha]$. The key point of the proof of the following result is the same as that of Theorem 3.2.1, namely, the differentiability of the evaluation map $Ev : C^1 \times [-h, 0] \to \mathbb{R}^n$.

THEOREM 6.2.1. *Suppose* (D1)(i)–(iii) *and* (D2)(i)–(iii), $(\bar{\phi}, \bar{\sigma}, \bar{\theta}) \in \Pi$, *and let* $\delta > 0$ *and* $\alpha > 0$ *be such that the IVP* (6.1.1)–(6.1.2) *has a unique solution on* $[-h, \alpha]$ *for any* $\gamma \in G_\Gamma(\bar{\gamma}, \delta)$. *Then for any* $t \in [0, \alpha]$ *the function*

$$x(t; \cdot) : \Gamma \supset G_\Gamma(\bar{\gamma}, \delta) \to \mathbb{R}^n$$

is differentiable at $\bar{\gamma}$, *and its derivative is given by*

$$\mathrm{D}_2 x(t; \bar{\gamma})u = z(t; \bar{\gamma}, u), \quad u \in \Gamma,$$

where $z(\cdot; \bar{\gamma}, u)$ *is the solution of the time-dependent delay system*

$$z'(t; \bar{\gamma}, u) = \mathrm{D}_2 g\big(t, \bar{x}(t), \bar{x}\big(t - \bar{r}(t)\big), \bar{\theta}\big) z(t; \bar{\gamma}, u)$$

$$
\begin{aligned}
&\quad + D_3 g\big(t, \bar{x}(t), \bar{x}\big(t-\bar{r}(t)\big), \bar{\theta}\big)\Big(-\bar{x}'\big(t-\bar{r}(t)\big)D_2\tau(t, \bar{x}_t, \bar{\sigma}) \\
&\quad \times z(\cdot; \bar{\gamma}, u)_t + z\big(t-\bar{r}(t); \bar{\gamma}, u\big) - \bar{x}'\big(t-\bar{r}(t)\big)D_3\tau\big(t, \bar{x}_t, \bar{\sigma}\big)u^{\sigma}\Big) \\
&\quad + D_4 g\big(t, \bar{x}(t), \bar{x}\big(t-\bar{r}(t)\big), \bar{\theta}\big)u^{\theta}, \quad t \in [0, \alpha], \\
z(t; \bar{\gamma}, u) &= u^{\phi}(t), \quad t \in [-r, 0],
\end{aligned}
$$

and where $\bar{x}(t) = x(t; \bar{\gamma})$, $\bar{r}(t) = \tau(t, \bar{x}_t, \bar{\sigma})$, *and* $u = (u^{\phi}, u^{\sigma}, u^{\theta}) \in \Gamma$.

We refer to [99] for the proof of Theorem 6.2.1. Here we just make some remarks on the choice of the norm on Γ. It is clear that at some point in the proof of Theorem 6.2.1 it is necessary to be able to differentiate the composite function $F(t, \psi) = \psi(-\tau(t, \psi))$. Suppose $\psi \in C^1$ is fixed, and consider

$$
\begin{aligned}
F(t, \psi + u) - F(t, \psi) &= \psi\big(-\tau\big(t, \psi + u\big)\big) + u\big(-\tau(t, \psi + u)\big) - \psi\big(-\tau(t, \psi)\big) \\
&= \psi'\big(-\tau(t, \psi)\big)\big(-\tau(t, \psi + u) + \tau(t, \psi)\big) \\
&\quad + u\big(-\tau(t, \psi)\big) + \omega(t, \psi, u) + u\big(-\tau(t, \psi + u)\big) \\
&\quad - u\big(-\tau(t, \psi)\big),
\end{aligned}
$$

where

$$
\begin{aligned}
\omega(t, \psi, u) &= \psi\big(-\tau(t, \psi + u)\big) - \psi\big(-\tau(t, \psi)\big) \\
&\quad - \psi'\big(-\tau(t, \psi)\big)\big(-\tau(t, \psi + u) + \tau(t, \psi)\big).
\end{aligned}
$$

Therefore we expect that

$$
D_2 F(t, \psi)u = -\psi'\big(-\tau(t, \psi)\big)D_2\tau(t, \psi)u + u\big(-\tau(t, \psi)\big)
$$

using an appropriate norm on the domain. The assumptions $\psi \in C^1$, (D2)(iii) and the chain rule combined imply immediately that $|\omega(t, \psi, u)|/|u|_C \to 0$ as $|u|_C \to 0$. But to control the term $u(-\tau(t, \psi + u)) - u(-\tau(t, \psi))$ the C-norm is not suitable, we need the stronger $W^{1,\infty}$-norm: The Mean Value Theorem yields

$$
\frac{|u(-\tau(t, \psi + u)) - u(-\tau(t, \psi))|}{|u|_{W^{1,\infty}}} \leqslant \big|\tau(t, \psi + u) - \tau(t, \psi)\big|,
$$

and the right-hand side of the preceding inequality tends to 0 as $|u|_{W^{1,\infty}} \to 0$, due to the continuity of τ. Therefore $D_2 F$ defined above is, in fact, the derivative of the function $F(t, \cdot) : W^{1,\infty} \to \mathbb{R}^n$ at $\psi \in C^1$ for any t.

6.3. *Differentiability with respect to parameters in norm*

Since the condition $\bar{\gamma} \in \Pi$ in the previous subsection may be inconvenient for certain applications, we explore different spaces to study differentiability in it for the case when the initial function and the solution segments are only $W^{1,\infty}$ functions.

First we introduce some notation and definitions. $W_\alpha^{1,p}$ $(1 \leqslant p < \infty)$ denotes the Banach space of absolutely continuous functions $\psi : [-h, \alpha] \to \mathbb{R}^n$ of finite norm

$$|\psi|_{W_\alpha^{1,p}} = \left(\int_{-h}^{\alpha} \left|\psi(s)\right|^p + \left|\psi'(s)\right|^p \, \mathrm{d}s \right)^{1/p}.$$

Similarly, $W_\alpha^{1,\infty}$ denotes the Banach space $W_\alpha^{1,\infty}([-h, \alpha], \mathbb{R}^n)$.

Consider a linear space Y and let $|\cdot|$ and $\|\cdot\|$ be two norms defined on Y. We say that $(Y, |\cdot|)$ is a quasi-Banach space with respect to the norm $\|\cdot\|$ if for any $r > 0$ the set $\{y \in Y\colon \|y\| \leqslant r\}$ is complete in the norm $|\cdot|$. See [96].

In addition to (D1)(i)–(iii) and (D2)(i)–(iii) we use in this subsection the following condition

(D2) (iv) $\tau : \mathbb{R} \times C \times \Sigma \supset [0, T] \times \Omega_4 \times \Omega_5 \to [0, \infty)$ is continuously differentiable with respect to its first variable; and

(v) $D_1\tau$, $D_2\tau$ and $D_3\tau$ are locally Lipschitz continuous in the following sense: For every $\alpha \in (0, T]$, for every compact subset $M_4 \subset \Omega_4$ of C and for every closed, bounded subset $M_5 \subset \Omega_5$ of Σ there exists a constant $L_3 = L_3(\alpha, M_4, M_5)$ such that

$$\left|D_1\tau(t, \psi, \sigma) - D_1\tau\left(t, \bar{\psi}, \bar{\sigma}\right)\right| \leqslant L_3\left(\left|\psi - \bar{\psi}\right|_C + |\sigma - \bar{\sigma}|_\Sigma\right),$$

$$\left\|D_2\tau(t, \psi, \sigma) - D_2\tau\left(t, \bar{\psi}, \bar{\sigma}\right)\right\|_{\mathcal{L}(C,\mathbb{R})} \leqslant L_3\left(\left|\psi - \bar{\psi}\right|_C + |\sigma - \bar{\sigma}|_\Sigma\right),$$

$$\left\|D_3\tau(t, \psi, \sigma) - D_3\tau\left(t, \bar{\psi}, \bar{\sigma}\right)\right\|_{\mathcal{L}(\Sigma,\mathbb{R})} \leqslant L_3\left(\left|\psi - \bar{\psi}\right|_C + |\sigma - \bar{\sigma}|_\Sigma\right)$$

for $t \in [0, \alpha]$, $\psi, \bar{\psi} \in M_4$, and $\sigma, \bar{\sigma} \in M_5$.

Hale and Ladeira [96] investigated differentiability of solutions to the constant delay equation

$$x'(t) = f\left(x(t), x(t - \tau)\right)$$

with respect to the delay, τ. They showed by means of an extension of the Uniform Contraction Principle to quasi-Banach spaces that the map

$$[0, h] \to W_\alpha^{1,1}, \quad \tau \mapsto x(\cdot; \tau)$$

is differentiable. Note that in their proof the integral norm of $W_\alpha^{1,1}$ played a crucial role; it can be replaced by a more general $W_\alpha^{1,p}$-norm, but not by the stronger $W_\alpha^{1,\infty}$-norm. This result suggests that the set $W_\alpha^{1,\infty}$ equipped with the norm $|\cdot|_{W_\alpha^{1,p}}$ could possibly be used as the state space for solutions. It might be a reasonable choice since (see, e.g., [99]) the parameter map $(\phi, \theta, \sigma) \mapsto x(\cdot; \phi, \theta, \sigma)_t$ is Lipschitz continuous in both the $|\cdot|_{W^{1,\infty}}$ and $|\cdot|_{W^{1,p}}$-norms while the time map $t \mapsto x(\cdot; \phi, \theta, \sigma)_t$ is continuous only in the $|\cdot|_{W^{1,p}}$, but not in the $|\cdot|_{W^{1,\infty}}$-norm. This indicates that the set $W^{1,\infty}$ equipped with the $|\cdot|_{W^{1,p}}$-norm (which is not a Banach space, it is only a quasi-Banach space with respect to the $|\cdot|_{W^{1,\infty}}$-norm) could be considered as a “natural” state space for state-dependent delay equations.

We follow the usual procedure to study differentiability with respect to parameters: Introducing $y(t) = x(t) - \tilde{\phi}(t)$ where $\tilde{\phi}$ is the extension of ϕ to $[-h, \alpha]$ by $\tilde{\phi}(t) = \phi(0)$ for $0 < t \leqslant \alpha$, we rewrite the IVP (6.1.1)–(6.1.2) as a fixed point equation $S(y, \phi, \theta, \sigma) = y$, with the operator S given by

$$S(y, \phi, \theta, \sigma)(t) = \begin{cases} 0, & t \in [-h, 0], \\ \displaystyle\int_0^t g\big(u, y(u) + \tilde{\phi}(u), \Lambda\big(u, y_u + \tilde{\phi}_u, \sigma\big), \theta\big)\,\mathrm{d}u, & t \in [0, \alpha], \end{cases}$$

and with $\Lambda(t, \psi, \sigma) = \psi(-\tau(t, \psi, \sigma))$. In order to apply the Uniform Contraction Principle in this setting we need continuous differentiability of S with respect to y, ϕ, θ and σ in the $W^{1,p}_\alpha$ norm. It turns out that instead of the pointwise differentiability of Λ with respect to ψ and σ studied in the previous subsection it is enough to have the differentiability of the composite function $t \mapsto \Lambda(t, x_t, \sigma)$ with respect to x and σ in a norm of “L^p-type”, for $x \in W^{1,\infty}_\alpha$.

Brokate and Colonius [29] studied equations of the form

$$x'(t) = f\big(t, x\big(r\big(t, x(t)\big)\big)\big), \quad t \in [a, b]$$

and investigated differentiability of the composition operator

$$A : W^{1,\infty}\big([a, b]; \mathbb{R}\big) \supset \bar{X} \to L^p\big([a, b]; \mathbb{R}\big), \quad A(x)(t) = x\big(r\big(t, x(t)\big)\big).$$

They assumed that r is twice continuously differentiable satisfying $a \leqslant r(t, v) \leqslant b$ for all $t \in [a, b]$ and $v \in \mathbb{R}$, and considered as domain of A the set

$$\bar{X} = \Big\{ x \in W^{1,\infty}\big([a, b]; \mathbb{R}\big) \colon \text{there exists } \epsilon > 0 \text{ s.t. } \frac{\mathrm{d}}{\mathrm{d}t}\big(r\big(t, x(t)\big)\big) \geqslant \epsilon \\ \text{for a.e. } t \in [a, b] \Big\}.$$

It was shown in [29] that under these assumptions A is continuously differentiable with the derivative given by

$$\big(\mathrm{D}A(x)u\big)(t) = x'\big(r\big(t, x(t)\big)\big)\mathrm{D}_2 r\big(t, x(t)\big)u(t) + u\big(r\big(t, x(t)\big)\big)$$

for $u \in W^{1,\infty}([a, b], \mathbb{R})$.

Both the strong $W^{1,\infty}$ norm on the domain and the weak L^p norm on the range, together with the choice of the domain seemed to be necessary to obtain the results in [29]. Note that Manitius in [160] used a similar domain and norm when he studied linearization for a class of state-dependent delay systems.

Clearly, to apply the Uniform Contraction Principle to the operator S we have to use the same norm on the domain and range of S. It turns out that the following "product norm" preserves the essential properties of the different norms used in [96] and [29]: Let $x \in W_\alpha^{1,\infty}$, and decompose x as $x = y + \tilde{\phi}$, (where $\phi(t) = x(t)$ for $t \in [-r, 0]$, and $\tilde{\phi}$ is the extension of ϕ to $[-r, \alpha]$ by $\tilde{\phi}(t) = \phi(0)$), and define the norm of x by

$$|x|_{X_\alpha^{1,p}} = \left(\int_0^\alpha \left|y'(u)\right|^p \mathrm{d}u\right)^{1/p} + |\phi|_{W^{1,\infty}},$$

and consider the normed linear space $X_\alpha^{1,p} \equiv (W_\alpha^{1,\infty}, |\cdot|_{X_\alpha^{1,p}})$. The norm $|\cdot|_{X_\alpha^{1,p}}$ is weaker than the $|\cdot|_{W_\alpha^{1,\infty}}$ norm, but stronger than the $|\cdot|_{W_\alpha^{1,p}}$ norm (see [106]).

This norm is "strong enough" that the methods of [29], with minor modifications, provide differentiability of the composition map

$$B : X_\alpha^{1,p} \times \Sigma \supset U_1 \times U_2 \to L^p\big([0,\alpha]; \mathbb{R}^n\big), \quad B(x,\sigma)(t) = \Lambda(t, x_t, \sigma),$$

on a suitable domain $U_1 \times U_2$. On the other hand, $|\cdot|_{X_\alpha^{1,p}}$ is "weak enough" that using the differentiability of the operator B above we can obtain differentiability of the operator $S : X_\alpha^{1,p} \times W^{1,\infty} \times \Theta \times \Sigma \supset V_1 \times V_2 \times V_3 \times V_4 \to X_\alpha^{1,p}$ with respect to y, ϕ, θ and σ. Moreover it is possible to use a modification of the Uniform Contraction Principle to get differentiability of the fixed point (the solution of the IVP) with respect to the parameters ϕ, θ and σ in the $|\cdot|_{X_\alpha^{1,p}}$ norm. Since this product norm is stronger than the $|\cdot|_{W_\alpha^{1,p}}$ norm, the result implies the differentiability of solutions in the latter norm as well. For more details and the proof of the next result we refer to [106].

THEOREM 6.3.1. *Suppose* (D1)(i)–(iii) *and* (D2)(i)–(v), *and let* $\bar{\delta} > 0$ *and* $\alpha > 0$ *be such that the IVP* (6.1.1)–(6.1.2) *has a unique solution on* $[-h, \alpha]$ *for any* $\gamma \in G_\Gamma(\bar{\gamma}, \bar{\delta})$, *and suppose there exists* $\epsilon > 0$ *such that the solution* $\bar{x} = x(\cdot; \bar{\gamma})$ *satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(t - \tau(t, \bar{x}_t, \bar{\sigma})\big) \geqslant \epsilon \quad \textit{a.e. } t \in [0, \alpha].$$

Then there exists $\delta > 0$ *such that the functions*

$$\Gamma \supset G_\Gamma(\bar{\gamma}, \delta) \to X_\alpha^{1,p}, \quad \gamma \mapsto x(\cdot; \gamma)$$

and

$$\Gamma \supset G_\Gamma(\bar{\gamma}, \delta) \to W_\alpha^{1,p}, \quad \gamma \mapsto x(\cdot; \gamma)$$

are continuously differentiable.

An application of differentiability of solutions with respect to parameters was given in [100], where estimation of unknown parameters in state-dependent delay equations was studied. The goal of the work was to find a parameter value which minimizes a least square cost function $P(\gamma) = \sum_{k=1}^{N}(x(t_i;\gamma) - y_i)^2$, where y_i $(i = 1, \ldots, N)$ are measurements of the solution at time points t_i $(i = 1, \ldots, N)$. The so-called method of quasilinearization was adopted and numerically tested for state-dependent delay equations for cases where the parameters were infinite dimensional, e.g., the initial function. This algorithm is based on Newton's method and uses the derivative of P, hence also $\mathrm{D}_2x(t;\gamma)$. The convergence of the estimation method was observed also for those cases where $\mathrm{D}_2x(t;\gamma)$ did not exist in the pointwise sense of Theorem 6.2.1 but only in a norm, as stated in Theorem 6.3.1. For example, when the initial function is approximated by piecewise linear splines, then Theorem 6.2.1 is not applicable to solutions corresponding to such parameters since they belong to $W^{1,\infty}$ but not to C^1.

7. Periodic solutions via fixed point theory

Over the last 40 years the most general results on existence of periodic solutions to autonomous delay differential equations, with constant and also with state-dependent delay, have been obtained using topological fixed point theorems and the fixed point index. The first step in applying these tools is the construction of a *return map*: For initial data in a suitably chosen set K one follows the solution segments until they return to K. Fixed points of the return map define periodic solutions. The search for a suitable set K requires a priori knowledge about the desired periodic solutions and about their role in the global dynamics generated by the delay equation. Hypotheses of fixed point theorems must also be satisfied. This indicates that in general the search for K may be a nontrivial task. The finer (more restrictive) a structure a domain K of a return map has, the more qualitative information about the periodic solution is provided.

It is not uncommon that domains of return maps or their closures contain a known fixed point which also is a stationary point of the semiflow. Therefore, to obtain a *nonconstant* periodic solution, one needs to find another fixed point. While impossible in case the trivial fixed point is globally attracting, there is hope in case the trivial fixed point is unstable. A weak topological notion of instability, which proved very useful, is Browder's concept of *ejectivity* [30]. A fixed point x of a map $f : M \to N$, N a topological space and $M \subset N$, is called ejective if there exists a neighbourhood V of x in M so that for each $y \in V \setminus x$ there is $j \in \mathbb{N}$ with $f^j(y) \notin V$. Ejectivity was deeply explored and first applied by Nussbaum (see, e.g., [174,175]), who also proved the first result on existence of periodic solutions for differential equations with state-dependent delay [174]. In the next subsection, Section 7.1, we describe a very general result of Mallet-Paret, Nussbaum, and Paraskevopoulos [156] on existence of periodic solutions and its proof, which is based on ejectivity.

The Section 7.2 deals with a model from Section 2.6 where the delay is governed by a differential equation which involves the state. This example was studied by Arino, Hadeler and Hbid [7] and Magal and Arino [148] and shows some of the specific difficulties caused by state-dependent delays in the search for periodic solutions.

Ejectivity is used also in the existence proofs in [4,136,137,151].

Of course, ejectivity does not adequately reflect the unstable behaviour of solutions to decent differential equations. Close to unstable manifolds of equilibria, or in cones around such unstable manifolds, solutions to delay and other differential equations behave much more regular than expressed by ejectivity. Accordingly one can prove existence of periodic solutions and global bifurcation from stationary points also without recourse to ejectivity. Schauder's fixed point theorem and simple calculations of the fixed point index suffice if only unstable solution behaviour close to equilibria is exploited to a larger extent. This was done in [202,203] and in Chapter XV of [54] for a class of RFDEs with constant delay. For equations with state-dependent delay, a proof of existence of periodic solutions along these lines has not yet been carried out.

Beyond existence and outside the scope of purely topological tools, uniqueness and stability of periodic orbits are of interest. In Section 7.3 we present results from [205, 208] where return maps are contractions or have a locally attracting fixed point, with the associated periodic orbit nontrivial, stable and hyperbolic. The approach applies to single equations and systems with state-dependent delay where the nonlinearities are given by functions which do not vary much on long intervals.

7.1. *A general result by continuation*

In [156] Mallet-Paret, Nussbaum and Paraskevopoulos prove existence of periodic solutions for a rather general class of scalar RFDEs which include equations with state-dependent delay. They consider Eq. (1.0.1) with $f : C \to \mathbb{R}$ continuous, $C = C([-h, 0], \mathbb{R})$, and assume that there are $\tau_0 \in (0, h]$ and a locally Lipschitz continuous function $g : \mathbb{R} \to \mathbb{R}$ satisfying the negative feedback condition

$$\xi g(\xi) < 0 \quad \text{for all } \xi \neq 0$$

so that

$$f(\phi) = g\big(\phi(-\tau_0)\big)$$

for ϕ in the closed hyperplane H given by $\phi(0) = 0$. In particular, $f(0) = 0$, τ_0 is the delay on H, and the constant function zero is a solution to Eq. (1.0.1). In Section 3 we mentioned the property (alL) from [156] of being *almost locally Lipschitzian*, to which condition (L) in Section 3 is closely related. Property (alL) requires local Lipschitz estimates for f which involve the norm on C but only arguments of f which are Lipschitz continuous. For f with property (alL) and at most linear growth Lipschitz continuous initial data $\phi \in C$ uniquely determine solutions $x : [-h, \infty) \to \mathbb{R}$ of Eq. (1.0.1) with $x_0 = \phi$. Also bounds for solutions and continuous dependence on initial data are established. Under a more restrictive negative feedback condition, now for the functional f and involving also data $\phi \in C \setminus H$, it is shown that segments x_t, $t \geqslant 0$, of solutions which start from $x_0 = \phi$ in a closed bounded convex set

$$G^+ \subset \big\{\phi \in H\colon \phi(t) \geqslant 0 \text{ for all } t \in [-h, 0]\big\}$$

return to G^+ at a certain well-defined zero $z_2 = z_2(x_0) > 0$ provided the solution has at least one sign change on $(0, \infty)$ and a zero thereafter. On a subset $U^+ \subset G^+$ which is open with respect to the topology on G^+ induced by C the previous result yields a continuous return map $\Gamma_0 : U^+ \to G^+$. Actually, more is achieved here for later use: $\Gamma_0 = \Gamma(\cdot, 0)$, for a homotopy $\Gamma : U^+ \times [0, 1] \to G^+$ of modified return maps $\Gamma(\cdot, \alpha)$ associated with the members of a one-parameter family of RFDEs; the equation at $\alpha = 1$ has the property that the values of initial data on $[-h, -\tau_0)$ have no influence on the solution.

The extremal point $0 \in G^+$ of G^+ does not belong to U^+, and each fixed point of the return map Γ_0 defines a non-constant periodic solution of Eq. (1.0.1). A fixed point exists if the fixed point index $i_{G^+}(\Gamma_0, U^+)$ of Γ_0 is defined and non-zero.

The remaining steps towards

$$i_{G^+}\bigl(\Gamma_0, U^+\bigr) \neq 0 \tag{7.1.1}$$

require some sort of linearization of the RFDE at the zero solution, in order to describe and exploit conditions for instability of the zero solution. Here the hypothesis (aFd) that f is *almost Fréchet differentiable* at 0 comes into play. It requires that the restriction of f to the space $C^{0,1} = C^{0,1}([-h, 0], \mathbb{R})$ of Lipschitz continuous data has a derivative at $0 \in C^1 \subset C^{0,1}$. The properties (alL) and (aFd) combined imply that this derivative $\mathrm{D}(f|_{C^{0,1}})(0)$ extends to a continuous linear map $\mathrm{D}_e(f|_{C^{0,1}})(0) : C \to \mathbb{R}$, like in conditions (S2) and (S3) from Section 3. A look back at the hypotheses on f shows that the recipe *freeze the delay at equilibrium, then linearize* mentioned in Section 3 is not sufficient to compute this extended derivative. The desired linear equation has the form

$$\begin{aligned} v'(t) &= \mathrm{D}_e(f|_{C^{0,1}})(0)v_t \\ &= -\beta\, v(t) - \gamma v(t - \tau_0) \end{aligned} \tag{7.1.2}$$

with constants $\beta \geqslant 0, \gamma \geqslant 0$, and is considered for initial data in the Banach space $C_{\tau_0} = C([-\tau_0, 0], \mathbb{R})$. On the cone

$$K = \bigl\{\phi \in C_{\tau_0}\colon\ \phi(t) \geqslant 0 \text{ for all } t \in [-\tau_0, 0],\ \phi(0) = 0\bigr\}$$

there is a return map $S_{\beta,\gamma} : K \to K$ similar to Γ_0, given by the solutions to Eq. (7.2.2), with $S_{\beta,\gamma}(0) = 0$. In case the zero solution of Eq. (7.2.2) is unstable the trivial fixed point $0 \in K$ has index zero;

$$i_K(S_{\beta,\gamma}, 0) = 0. \tag{7.1.3}$$

To see this one can use ejectivity as in [174,175], or follow [176], or proceed as in [203,54].

The major steps in the proof of (7.2.1) are a reduction to

$$i_{G^+}\bigl(\Gamma_1, U^+\bigr) \neq 0$$

by homotopy invariance, and the deduction of the preceding inequality from (7.2.3). In this last step all basic properties normalization, additivity, homotopy invariance, and commutativity of the fixed point index are instrumental.

The skillful proof in [156] overcomes more obstacles than this brief outline indicates. The main theorem of [156] yields a wide variety of results on existence of periodic solutions for explicitly given RFDEs, in particular also for equations with multiple state-dependent delays of the form

$$x'(t) = G\big(x(t), x\big(t - r_1\big(x(t)\big)\big), \ldots, x\big(t - r_m\big(x(t)\big)\big)\big).$$

7.2. *Periodic solutions when delay is described by a differential equation*

As mentioned above, Nussbaum [174] used the ejective fixed point theorem to prove the existence of periodic solutions to the equation

$$x'(t) = -\alpha x\big(t - 1 - \big|x(t)\big|\big)\big(1 - x^2(t)\big).$$

Alt [4] and Kuang and Smith [136,137] obtained periodic solutions for equations where the delay is given by a threshold condition.

While the major steps towards existence of so-called slowly oscillating periodic solutions remain essentially the same as for delay differential equations with constant delays, the technical details become more involved when delays are state-dependent. This refers both to the construction of the domain of a return map and to the verification that a trivial fixed point is ejective. To illustrate this we describe in the sequel work of Arino, Hadeler and Hbid [7] and of Magal and Arino [148] for the system

$$\begin{cases} x'(t) = -f\big(x\big(t - r(t)\big)\big), \\ r'(t) = q\big(x(t), r(t)\big). \end{cases} \tag{7.2.1}$$

Here the variation of the delay is determined by an ordinary differential equation. Standing assumptions are that the functions $f:\mathbb{R} \to [-M, M]$, $M > 0$, and $q:\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuously differentiable, with

$$xf(x) > 0 \quad \text{for all } x \neq 0 \text{ and } f \text{ nondecreasing,}$$

and that for some fixed reals $r_2 > r_1 > 0$ and for all $x \in \mathbb{R}$,

$$q(x, r_2) < 0 < q(x, r_1).$$

Let $C = C([-r_2, 0], \mathbb{R})$. The hypotheses ensure that the IVP given by the system (7.2.1) and initial data $(\phi, r_0) \in C \times [r_1, r_2]$ with ϕ Lipschitz continuous has a unique solution $(x, r) = (x^{\phi,r_0}, r^{\phi,r_0})$, with x defined on $[-r_2, \infty)$ and r defined on $[0, \infty)$. The solution component x is Lipschitz continuous with $|x'(t)| \leqslant M$ for all $t > 0$, and the hypothesis on q yields $r_1 < r(t) < r_2$ for all $t > 0$.

Incidentally, note that the system (7.2.1) can be reformulated as a special case of (1.0.1) with $n = 2$, $h = r_2$, $U = (\mathbb{R} \times [0, h])^{[-h,0]}$, and the corresponding functional being given by $(-f(\phi(-r_0)), q(\phi(0), r_0))$ for $(\phi, r_0) \in U$.

It also follows that the function $q(0,\cdot)$ has zeros in $[r_1, r_2]$. The additional hypothesis

$$\frac{\partial q}{\partial r}(0,r) < 0 \quad \text{on } [r_1, r_2]$$

implies that there is exactly one zero r_0^* of $q(0,\cdot)$ in $[r_1, r_2]$, and that the constant solution $r^*: t \mapsto r_0^*$ of the autonomous equation

$$r' = q(0,r)$$

is asymptotically stable. Below we shall see that this fact causes complications in view of ejectivity of a fixed point corresponding to the constant solution of system (7.2.1) given by $x(t) = 0$ and r^*.

In order to obtain that $t \mapsto t - r(t)$ is strictly increasing it is furthermore assumed that

$$q(x,r) < 1 \quad \text{on } \mathbb{R} \times [r_1, r_2].$$

The notion of a slowly oscillating solution, which is familiar from work on equations with constant delay $h > 0$ like, e.g.,

$$x'(t) = -f\big(x(t-h)\big)$$

and means that zeros are isolated and spaced at distances larger than the delay h, is modified for the x-components of solutions to the system (7.2.1) according to the following definition: A function $x : [t_0, \infty) \to \mathbb{R}$, $t_0 \in \mathbb{R}$, is called slowly oscillating if its zeros form a disjoint union of closed intervals Z whose left endpoints have no accumulation point and are spaced at distances not less than r_2, with

$$\lim_{t \nearrow a} \operatorname{sign}\big(x(t)\big) = -\lim_{s \searrow b} \operatorname{sign}\big(x(s)\big)$$

at each compact interval $Z = [a,b]$ with $t_0 < a$. Notice that the definition allows nonconstant functions which are zero on some unbounded interval $[t_1, \infty)$, $t_1 > t_0$, a phenomenon which occurs among the first components of solutions to the system (7.2.1).

The first step towards a return map is to show that certain initial data define solutions with slowly oscillating first component. In [7] it is shown that the first components of solutions with initial value component ϕ in the cone

$$\Gamma = \big\{\phi \in C([-r_2, 0]; \mathbb{R}) \colon \text{ there exists } \theta \in [-r_2, -r_1] \text{ so that } \phi(\theta) = 0 \\ \text{and } \phi(s) < 0 \text{ for } s < \theta;\ 0 \leqslant \phi(\theta) \leqslant \phi(0) \text{ for } \theta \leqslant s \leqslant 0\big\}$$

return to Γ at a sequence of times whose distances are not less than r_2. More precisely, if (x, r) is a solution with initial value (ϕ, r_0) and $\pm\phi \in \Gamma$ then there are two sequences, possibly finite, of points t_i^* and zeros t_i of x in $[0, \infty)$ such that

$$t_0^* \geqslant 0, \quad t_i^* + (r_2 - r_1) \leqslant t_{i+1} \leqslant t_{i+1}^* - r_1$$

and $\pm(-1)^{i+1}x$ is nondecreasing on the interval $[t_i^*, t_{i+1}^*]$. In particular, $\pm(-1)^i x_{t_i^*} \in \Gamma$ for each index $i > 0$. To reach the conclusion, it is required that the delay variation satisfies the smallness condition $(r_2 - r_1)|f(x)| < |x|$ for $x \neq 0$.

Uniqueness with respect to initial data is needed, and the domain for a return map should be convex. Therefore Γ must be modified. The smaller set

$$\Gamma_1 = \{\phi \in \Gamma\colon\ \phi \text{ is Lipschitzian and nondecreasing}\}$$

is convex, and closed in $C^{0,1}([-r_2, 0], \mathbb{R})$. Let $E = \Gamma_1 \times [r_1, r_2]$. For each integer $j \geqslant 1$, it is now natural to introduce operators P_j and P_j^+ on the space E by

$$P_j(\phi, r_0) = \bigl(x_{t_j^*}, r\bigl(t_j^*\bigr)\bigr),$$

$$P_j^+(\phi, r_0) = \bigl((-1)^j x_{t_j^*}, r\bigl(t_j^*\bigr)\bigr).$$

Obviously, the existence of $P_j(\phi, r_0)$ is subject to the condition that $t_1^*, t_2^*, \ldots, t_{j-1}^*$ exist. If this condition is satisfied but the solution starting from $(x_{t_{j-1}^*}, r(t_{j-1}^*))$ does not cross zero, then it is proved in [7] that $x(t) \to 0$ and $r(t) \to r^*$ as $t \to \infty$. In this case, one defines $P_j(\phi, r_0) = (0, r^*)$.

P_1 sends bounded sets of E into bounded sets of the product space $C([-r_2, 0], \mathbb{R}) \times [r_1, r_2]$, and P_1^+ is a compact and continuous operator from E into itself.

It is important to know when a first positive zero $t_1 = t_1(\phi, r_0)$ exists: This property is ensured for every $(\phi, r_0) \in (\Gamma_1 \cup (-\Gamma_1)) \times [r_1, r_2]$ if there exist $M > 0$ and $R_0 > 0$ with $M(r_2 - r_1) \leqslant R_0$, $|f(x)| \leqslant M$ for $x \in \mathbb{R}$ and $(2r_1 - r_2)|f(x)| \geqslant |x|$ for $|x| \leqslant R_0$.

Unfortunately, the fixed point $(0, r^*)$ of any return map defined in a set containing $\{0\} \times [r_1, r_2]$ is not ejective. This follows immediately from positive invariance of the set $\{0\} \times [r_1, r_2]$ and asymptotic stability of the solution r^* to the equation $r' = q(0, r)$.

In order to obtain ejectivity smaller convex subsets

$$E_K = \bigl\{(\phi, r_0) \in E\colon\ \bigl|r_0 - r_0^*\bigr| \leqslant K\|\phi\|_C,\ \|\phi\|_C = \bigl|\phi(0)\bigr|\bigr\}$$

for $K > 0$ are introduced which contain from the obstacle for ejectivity $\{0\} \times [r_1, r_2]$ only the point $(0, r_0^*)$. For $R > 0$ and $K > 0$ let

$$E_{R,K} = \bigl\{(\phi, r_0) \in E\colon\ \|\phi\|_C \leqslant R,\ \bigl|r_0 - r_0^*\bigr| \leqslant K\|\phi\|_C\bigr\},$$

and define

$$\Gamma_2 = \Bigl\{\phi \in \Gamma_1\colon\ \phi(0) = \sup_{-r_2 \leqslant s \leqslant 0} \bigl|\phi(s)\bigr|\Bigr\}.$$

In order for P_1^+ to map $(\Gamma_2 \times [r_1, r_2]) \cap E_{R,K}$ into E_K, a restriction on r_1 is needed. In [7] it is shown that for each $R > 0$ there exist $\tilde{r}_1 > 0$ and $K > 0$ such that for each $r_1 > \tilde{r}_1$

the operator P_1^+ maps $(\Gamma_2 \times [r_1, r_2]) \cap E_{R,K}$ into E_K. The key to the proof of the above statement is that for each $R > 0$ there exists $C(r_1) \in (0, 1)$ such that

$$\left\| x_{t_1^*}^{\phi, r_0} \right\|_C \geqslant C(r_1) \|\phi\|_C \tag{7.2.2}$$

for each $(\phi, r_0) \in \Gamma_2 \times [r_1, r_2]$ with $0 < \|\phi\|_C \leqslant R$. Estimates of this type exclude super-exponential decay of slowly oscillating solutions and play an important role in work on the global dynamics of equations with constant delay.

The map $(\phi, r_0) \mapsto -x_{t_1^*}^{\phi, r_0}$ sends $E = \Gamma_1 \times [r_1, r_2]$ into Γ_2. The fact that $t_1 = t_1(\phi, r_0)$ is a zero of $x = x^{\phi, r_0}$ and integration of the first equation in (7.2.1) yield the estimate

$$\left\| x_{t_1^*}^{\phi, r_0} \right\|_C \leqslant M r_2,$$

for each $(\phi, r_0) \in E$. With this preparation, we can define $R = M r_2$ and introduce the set

$$X = \left\{ \phi \in \Gamma_2 \colon \|\phi\|_C \leqslant R, \ \operatorname*{ess\,sup}_{-r_2 \leqslant s \leqslant 0} \left|\phi'(s)\right| \leqslant M \right\}.$$

Choose $K > 0$ and $\tilde{r}_1 > 0$ as above, assume $r_1 > \tilde{r}_1$, and set

$$Y = \bigl(X \times [r_1, r_2]\bigr) \cap E_{R,K}.$$

Y is closed and convex, and the iterate P_1^2 defines a map $P : Y \to Y$.

It remains to show that $(0, r_0^*)$ is an ejective fixed point of P. Ejectivity of $(0, r_0^*)$ is a consequence of instability of the constant solution $t \mapsto (0, r_0^*)$ to the system (7.2.1), which in turn follows from instability for the linearized system. In [7] the technique *freeze the delay at equilibrium, then linearize* mentioned in Section 3.4 is applied to the first equation in (7.2.1) and yields

$$y'(t) = -y\bigl(t - r_0^*\bigr), \tag{7.2.3}$$

which is unstable for $r_0^* > \frac{\pi}{2}$ when f normalized so that $f'(0) = 1$. More precisely, the eigenvalues of the generator of the semigroup generated by Eq. (7.2.2) on the space

$$C\bigl(\bigl[-r_0^*, 0\bigr], \mathbb{R}\bigr)$$

with largest real part are a complex conjugate pair $u \pm i v$ in the open right halfplane, and the associated realified generalized eigenspace U is 2-dimensional and consists of segments of solutions

$$\mathbb{R} \ni t \mapsto \mathrm{e}^{ut}\bigl(a \cos(vt) + b \sin(vt)\bigr)$$

to Eq. (7.2.3). For $(a, b) \neq (0, 0)$ these solutions are slowly oscillating, due to $|v| < \frac{\pi}{r_0^*}$. The problem is now to transfer such unstable solution behaviour to the slowly oscillating solution components x of solutions (x, r) to the nonlinear system (7.2.1) which start from small initial data $(\phi, \rho) \in Y$ in the other state space $C([-r_2, 0], \mathbb{R}) \times [r_1, r_2]$. The proof in [7] proceeds by contradiction. Ejectivity means that there exists $\epsilon > 0$ so that for each $(\phi, r_0) \in Y$ with $0 < \|\phi\|_C \leqslant \epsilon$ there is an integer $j > 0$ such that $(x, r) = (x^{\phi, r_0}, r^{\phi, r_0})$ and $t_j^* = t_j^*(\phi, r_0)$ satisfy $\|x_{t_j^*}\|_C \geqslant \epsilon$ or $|r(t_j^*) - r_0^*| \geqslant \epsilon$. It can be shown that in case the previous statement on ejectivity is not true, then there exists a constant $d > 0$ so that for all $(\phi, r_0) \in Y$ with $\|\phi\|_C \leqslant \epsilon$ and for all $t \geqslant 0$ we have $\|x_t^{\phi, r_0}\|_C \leqslant d\epsilon$ and $|r(t) - r_0^*| \leqslant d\epsilon$.

The next step is to show that the spectral projection Π_U onto the realified generalized eigenspace U associated with $u \pm v$ in $C([-r_0^*, 0], \mathbb{R})$ satisfies

$$\gamma := \inf_{\|\phi\|=1, \phi \in \tilde{\Gamma}_2} \|\Pi_U \phi\| > 0, \tag{7.2.4}$$

under a further restriction about the variation of the delay

$$r_2 - r_1 < \delta \tag{7.2.5}$$

for some constant $\delta > 0$ (explicitly given in [7]) that depends on r_0^* only. Here and in what follows, $\tilde{\Gamma}_2$ is defined in a similar fashion as Γ_2, except we replace Γ_1 and Γ by $\tilde{\Gamma}_1$ and $\tilde{\Gamma}$ respectively, where the domain of ϕ is $[-r_0^*, 0]$ rather than $[-r_2, 0]$.

Let $\sup_{j \in \mathbb{N}} \|x_{t_j^*} \phi, r_0\|_C = \tilde{\epsilon} \leqslant \epsilon$. We can choose an integer $j_0 > 0$ so that $\|x_{t_{j_0}^*} \phi, r_0\|_C \geqslant C\tilde{\epsilon}$ and thus, $|y(t_{j_0}^*)| \geqslant C\tilde{\epsilon}\gamma$, where $y(t) = \Pi_U x_t$, C and γ are given in (7.2.2) and (7.2.4). Note that the first equation of the system (7.2.1) can be written as $x'(t) = -x(t - r^*) + o(\tilde{\epsilon})$ for large t, by using the linearization and the fact that $|x'(t)| \leqslant M$ and $|r(t) - r_0^*| \leqslant d\tilde{\epsilon}$ for all $t \geqslant 0$.

Therefore,

$$y'(t) = A_U y(t) + \mathrm{o}(\tilde{\epsilon})$$

with $A_U = \left(\begin{smallmatrix} \alpha & -\beta \\ \beta & \alpha \end{smallmatrix}\right)$. It then follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\big|y(t)\big| = \alpha\big|y(t)\big| + \mathrm{o}(\tilde{\epsilon}).$$

Therefore,

$$\big|y(t)\big| = \mathrm{e}^{\alpha(t-\sigma)}\left[\big|y(\sigma)\big| + \int_\sigma^t \mathrm{e}^{\alpha(\sigma - s)}\mathrm{o}(\tilde{\epsilon})\,\mathrm{d}s\right],$$

which implies

$$\big|y(t)\big| \geqslant \mathrm{e}^{\alpha(t - t_{j_0}^*)}\left[C\tilde{\epsilon}\gamma - \frac{\mathrm{o}(\tilde{\epsilon})}{\alpha}\right] \geqslant \mathrm{e}^{\alpha(t - t_{j_0}^*)}\frac{C\gamma}{2}\tilde{\epsilon}$$

if $\tilde{\epsilon}$ is small. This leads to a contradiction since $y(t) = \Pi_U x_t$ should be bounded by a constant multiple of $\tilde{\epsilon}$ for all $t \geqslant 0$.

In summary, under a few technical conditions including the negative feedback condition on the state variable x and the delay r, the smallness of the delay variation and an instability condition for an associated linear equation, Arino, Hadeler and Hbid obtain in [7] the existence of periodic solution with slowly oscillating first component.

In [148] Magal and Arino obtain such periodic solutions under weaker conditions on the delay variation—(7.2.5) is no longer needed—by means of a different argument which uses a modification of ejectivity and employs other cones of initial data. Magal and Arino consider a cone that was already used by Kuang and Smith in [136,137], namely,

$$E_{\mathrm{KS}} = \big\{(\phi, r_0) \in C^{0,1}\big([-r_2, 0], \mathbb{R}\big) \times [r_1, r_2]\colon \phi(-r_0) = 0 \\ \text{and } \phi \text{ nonincreasing on } [-r_0, 0]\big\}.$$

Using arguments similar to those in [7] one can show that for $\epsilon = \pm 1$ and $(\epsilon\phi, r_0) \in E_{KS}$ there are reals $t_i^* = t_i^*(\phi, r_0)$ and zeros $t_i = t_i(\phi, r_0)$ of $x = x^{\phi, r_0}$ such that $t_0 = -r_0$, $t_0^* = 0$, $t_i^* \leqslant t_{i+1}$, $t_i = t_i^* - r(t_i^*)$ for integers $i \geqslant 0$, and $\epsilon(-1)^{i+1}x(t)$ is nonincreasing on $[t_i^*, t_{i+1}^*]$, $x(t_i) = 0$, and $x(t_i^*) \neq 0$ if $\phi(0) \neq 0$. So, $(\epsilon(-1)^{i+1}x_{t_i^*}, r(t_i^*)) \in E_{\mathrm{KS}}$. Let $X_0 = C^1([-r_2, 0], \mathbb{R}) \times [r_1, r_2]$, introduce

$$E_0 = \big\{(\phi, r_0) \in X_0\colon \phi'(s) \geqslant 0 \text{ for } s \in [-r_0, 0], \phi(-r_0) = 0, \phi'(0) = 0\big\},$$

and define the return maps P_j and P_j^+ on E_0 exactly as in [7]. For each integer $j > 0$ one finds $P_{2j}(E_0) \subset E_0$.

The fact that for initial data $(\phi, r_0) \in C^1([-r_2, 0], \mathbb{R}) \times [r_1, r_2]$ the map $t \mapsto t - r^{\phi, r_0}(t)$ is increasing implies that on $[0, \infty)$ the solution $(x^{\phi, r_0}, r^{\phi, r_0})$ does not depend on the restriction of ϕ to $[-r_2, -r_0)$. The preceding observation suggests to consider initial data $(\tilde{\phi}, r_0)$ with $\tilde{\phi}$ defined only on $[-r_0, 0]$, and secondly, to modify the fixed point problem by transforming the initial delay r_0 to 1. Let

$$X_1 = C^1\big([-1, 0], \mathbb{R}\big) \times [r_1, r_2].$$

Magal and Arino introduce maps

$$L : X_0 \to X_1 \quad \text{and} \quad Q : X_1 \to X_0$$

by

$$L(\phi, r_0) = (\psi, r_0), \quad \psi(s) = \phi(s\, r_0)$$

and

$$Q(\psi, r_0) = (\phi, r_0), \quad \phi(s) = \psi(s/r_0) \quad \text{on } [-r_0, 0],$$

$$\phi(s) = \frac{\psi'(-1)}{r_0}(s - r_0) \quad \text{on } [-r_2, -r_0).$$

Let E_1 denote the analogue of E_0 where X_0 and r_0 are replaced by X_1 and 1, respectively. E_1 is closed and convex, and $Q(E_1) \subset E_0$, $L(E_0) \subset E_1$. The maps $F_{2j} = L \circ P_{2j} \circ (Q|E_1)$ send E_1 into E_1. We have $F_2(0, r_0^*) = (0, r_0^*)$, and fixed points (ϕ, r_0) of F_2 with $\phi \neq 0$ yield periodic solutions of the system (7.2.1) with slowly oscillating first component. However, the second component of $F_2 : E_1 \to C^1([-1,0], \mathbb{R}) \times [r_1, r_2]$ is not continuous at points $(0, r_0)$ with $r_0 \neq r_0^*$, which requires a further modification. The map $\tilde{F}_2 : E_1 \to E_1$ resulting from this is continuous and compact (with respect to the topology on $C^1([-1,0], \mathbb{R})$) and retains the property that nontrivial fixed points define periodic solutions to (7.2.1) with slowly oscillating first component. It is shown in [148] that

(i) $\tilde{F}_2(0, r_0^*) = (0, r_0^*)$,
(ii) $\tilde{F}_2(\{0\} \times [r_1, r_2]) \subset \{0\} \times [r_1, r_2]$, and
(iii) for every $\epsilon > 0$ there exist $c > 0$ and $\gamma \in [0, 1)$ with

$$\left|\left(\tilde{F}_1^{2j}\right)_2(\phi, r_0) - r_0^*\right| \leqslant \gamma \left|r_0 - r_0^*\right|$$

for all $(\phi, r_0) \in E_1$ with

$$\|\phi\|_{C^1([-1,0],\mathbb{R})} + \left|r_0 - r_0^*\right| \leqslant \epsilon$$

and

$$\|\phi\|_{C^1([-1,0],\mathbb{R})} \leqslant c\left|r_0 - r_0^*\right|,$$

and for all integers $j \geqslant 1$.

Here the index 2 denotes the second component of the map $\tilde{F}_2^{2j}$. Properties (ii) and (iii) combined exclude that the fixed point $(0, r^*)$ of $\tilde{F}_2$ is ejective. Now the modification of ejectivity comes into play. Let X be a Banach space, $A \subset Y \subset X$, and assume that $g : Y \to Y$ has a fixed point $x_0 \in \partial_Y A$. Then x_0 is called semi-ejective on $Y \setminus A$ if there is a neighbourhood V of x_0 in Y so that for each $y \in V \setminus A$ there is an integer $m \geqslant 1$ with

$$g^m(y) \in Y \setminus V.$$

Arguments similar to those in [7] which exploit the instability of Eq. (7.2.3) yield in [148] that the fixed point $(0, r_0^*)$ is semi-ejective on $E_1 \setminus (\{0\} \times [r_1, r_2])$. Finally, an extension of the ejective fixed point theorem in [148] to the case of semi-ejective fixed points guarantees that $\tilde{F}_2$ has a fixed point (ϕ, r_0) with $\phi \neq 0$, which defines the desired periodic solution.

7.3. *Attracting periodic orbits*

In [205] the system (2.2.1)–(2.2.2)

$$x'(t) = v\left(\frac{c}{2}s(t - r) - w\right),$$

$$cs = \left|x(t-s)+w\right| + \left|x(t)+w\right|,$$

with positive parameters c, r, w, is studied. The function $v : \mathbb{R} \to \mathbb{R}$ is assumed to satisfy a negative feedback condition

$$\delta v(\delta) < 0 \quad \text{for all } \delta \neq 0.$$

For continuous and bounded v the system defines a continuous semiflow on a set O which is open in a compact set $M \subset C = C([-h,0],\mathbb{R})$, with $h = \frac{4w}{c} + r$. The set M consists of Lipschitz continuous functions. The set O contains a closed subset K of initial data which define solutions whose segments x_t return to K, after an excursion into the ambient space O. This yields a continuous return map, on a domain without stationary points. If v is Lipschitz continuous and close to constants in $(-\infty, -\beta]$ and $[\beta, \infty)$, respectively, with $\beta > 0$ sufficiently small, then one can estimate the Lipschitz constant of the return map in terms of Lipschitz constants for v and for its restrictions $v|_{(-\infty,-\beta]}$ and $v|_{[\beta,\infty)}$; under suitable further assumptions the return map becomes a contraction. The unique fixed point of the contracting return map belongs to a periodic orbit of the system (2.2.1)–(2.2.2) which is stable and exponentially attracting with asymptotic phase.

The observation which led to the method used in [205] and in earlier work on equations with constant delays is the following: If the function $g : \mathbb{R} \to \mathbb{R}$ in the equation

$$y'(t) = g\big(y(t-1)\big) \tag{7.3.1}$$

is constant on some interval I and if y remains long enough in I, say, for $t_0 - 1 \leqslant t \leqslant t_0$, then for $t \geqslant t_0$ the solution y depends only on $y(t_0)$ and $g(I)$. This can be used to design simple-looking nonlinearities g, representing negative feedback, for which periodic solutions of Eq. (7.3.1) can be computed explicitly, see, e.g., Chapter XV in [54]. Moreover, solutions which start from initial data close to the periodic orbit eventually merge into it. This is an extremely strong kind of orbital stability, giving hope that also for nonlinearities which are only close to constants on some intervals attracting periodic orbits may exist. If instead of the scalar equation (7.3.1) more generally systems are considered then suitable nonlinearities which are constant on nontrivial intervals yield low-dimensional subsets of the state space which are positively invariant under the semiflow and absorb flowlines from a neighbourhood.

In [208] the system

$$u' = v, \tag{7.3.2}$$

$$v' = -r\,v + A(p), \tag{7.3.3}$$

$$p = \frac{c}{2}s - w, \tag{7.3.4}$$

$$cs = u(t-s) + u(t) + 2w \tag{7.3.5}$$

is studied. In contrast to the system (2.2.1)–(2.2.2) the model (7.3.2)–(7.3.5) for position control by echo is now based on Newton's law. Instead of the constant time lag $r > 0$ in Eq. (2.2.1) there is now a friction term $-rv$ in Eq. (7.3.3).

For suitable positive values of the parameters w, c, r and for certain functions $A : \mathbb{R} \to \mathbb{R}$ which represent negative feedback and are constant on $(-\infty, -\beta]$ and on $[\beta, \infty)$, with $\beta > 0$ small, the existence of a hyperbolic stable periodic orbit is established. The proof begins with a reformulation of the system as an equation of the form (1.0.1), with a continuously differentiable functional f defined on an open subset U of the space $C^1 = C^1([-h, 0], \mathbb{R}^2)$, for suitable $h > 0$. Several steps of the proof rely on the smoothness properties of the semiflow F on the solution manifold $X_f \subset U$ which are provided by Theorem 3.2.1. In X_f a thin, infinite-dimensional set I of initial data ϕ is found to which the flowlines $F(\cdot, \phi)$ return, after a journey through the ambient part of the manifold. The associated return map is not necessarily compact, which precludes an immediate application of Schauder's theorem in order to find a fixed point—not to speak of an attracting fixed point. But the return map is semiconjugate to an interval map which is differentiable. Estimates of derivatives of the interval map yield a unique, attracting fixed point of the latter, which can be lifted to the return map. The proof that the resulting periodic orbit of the system (7.3.2)–(7.3.5) is hyperbolic and stable involves a continuously differentiable Poincaré return map, on a hyperplane transversal to the periodic orbit, in addition to the previous return map on the thin set $I \subset X_f$, and a discussion of derivatives of iterates.

8. Attractors, singular perturbation, small delay, generic convergence, stability and oscillation

This section deals with limiting behaviour. Section 8.1 is concerned with long term dynamics and reports about the structure of a global attractor [132]. Section 8.2 describes work of Mallet-Paret and Nussbaum [151–155] about the asymptotic shape of periodic solutions when a parameter becomes large, Section 8.3 sketches an approach of Ouifki and Hbid [178] to existence of periodic solutions when delays are small, Section 8.4 reports about work of Bartha [21] on generic convergence of solutions, and Section 8.5 comments on further results about stability and oscillatory solution behaviour.

8.1. *An attracting disk*

The paper [132] studies the equation

$$x'(t) = -\mu x(t) + f\big(x\big(t - r\big(x(t)\big)\big)\big) \tag{8.1.1}$$

with $\mu > 0$, $f \in C^2(\mathbb{R}, \mathbb{R})$, $f(0) = 0$, $f'(u) < 0$ for all $u \in \mathbb{R}$, $r \in C^1(\mathbb{R}, \mathbb{R})$, $r(0) = 1$, and $\sup_{u \in \mathbb{R}} f(u) < \infty$ provided $r(u) \geqslant 0$ for all $u \in \mathbb{R}$. The case $\mu = 0$ can also be handled with a slight modification. Then the delayed logistic equation (or Wright's equation)

$$y'(t) = -\alpha y\big(t - r\big(y(t)\big)\big)\big[1 + y(t)\big]$$

with state-dependent delay and solutions satisfying $y(t) > -1$ is a particular case. Indeed, after the transformation $x = \log(1+y)$ we obtain

$$x'(t) = -\alpha\left[e^{x(t-r(e^{x(t)}-1))} - 1\right].$$

The aim is to describe the asymptotic behaviour of the slowly oscillatory solutions of Eq. (8.1.1). Here a solution x of (8.1.1) is called slowly oscillatory if $|z'-z| > r(0) = 1$ for every pair of zeros z', z of x. The results are in part analogous to those of Walther [204] for the constant delay case $r \equiv 1$.

Let I_r denote the largest subinterval of $\mathbb{R}$ with $0 \in I_r$ and $r(u) \geqslant 0$ for all $u \in I_r$. First it is shown that for every element ϕ of the space $BC((-\infty,0], I_r)$ of bounded continuous functions from $(-\infty,0]$ into I_r, there is a solution $x:\mathbb{R}\to\mathbb{R}$ of Eq. (8.1.1) through ϕ, i.e., x is continuous on $\mathbb{R}$, continuously differentiable on $(0,\infty)$, (8.1.1) holds for all $t>0$, and $x|_{(-\infty,0]} = \phi$. If ϕ is Lipschitz continuous then x is unique.

In the next step four positive constants A, B, R, K are constructed such that

$$r\big([-B,A]\big) \subset (0,R],$$

$$\max_{(u,v)\in[-B,A]\times[-B,A]} \big|-\mu u + f(v)\big| \leqslant K,$$

moreover, for any solution $x:\mathbb{R}\to\mathbb{R}$ of (8.1.1) with $x|_{(-\infty,0]} \in BC((-\infty,0], I_r)$ there exists $T \geqslant 0$ such that

$$x(t) \in [-B,A] \quad \text{for all } t \geqslant T.$$

Therefore, from the point of view of the asymptotic ($t\to\infty$) behaviour, only those solutions are interesting which have values in $[-B,A]$.

Let C_R denote the space $C([-R,0],\mathbb{R})$ equipped with the supremum norm. The set

$$L_K = \left\{\phi \in C_R\colon \phi\big([-R,0]\big) \subset [-B,A],\ \sup_{-R\leqslant s<t\leqslant 0} \frac{|\phi(t)-\phi(s)|}{t-s} \leqslant K\right\}$$

is a compact convex subset of C_R. For every $\phi \in L_K$, Eq. (8.1.1) has a unique solution $x^\phi : [-R,\infty) \to \mathbb{R}$ with $x|_{[-R,0]} = \phi$ and $x(t) \in [-B,A]$ for all $t \geqslant 0$. Then the relations

$$F(t,\phi) = x_t^\phi, \quad t \geqslant 0, \qquad x_t^\phi(s) = x^\phi(t+s), \quad -R \leqslant s \leqslant 0,$$

define a continuous semiflow F on L_K. In the sequel, only those solutions of (8.1.1) are considered whose segments are in L_K.

Define the compact subset

$$S = \big\{\phi \in L_K\colon \operatorname{sch}\big(\phi,[t-1,t]\big) \leqslant 1 \text{ for all } t \in [-R+1,0]\big\}$$

of L_K, where $\operatorname{sch}(\phi,[t-1,t])$ denotes the number of sign changes of ϕ on the interval $[t-1,t]$. If x is a slowly oscillatory solution of (8.1.1), then all segments x_t belong to S.

The set S is positively invariant under the semiflow F. The restriction of F to $[0, \infty) \times S$ defines the continuous semiflow F_S on the compact metric space S. The global attractor $\mathcal{A}$ of F_S has the following properties:

(i) $\mathcal{A}$ is a compact connected subset of $S \subset L_K$.

(ii) For each $\phi \in \mathcal{A}$ there is a unique solution of (8.1.1) through ϕ on $\mathbb{R}$, which is also denoted by x^ϕ. The map $F_\mathcal{A} : \mathbb{R} \times \mathcal{A} \ni (t, \phi) \mapsto x_t^\phi \in \mathcal{A}$ is a continuous flow.

(iii) $\mathcal{A}$ is the union of $0 \in C_R$ and the segments x_t of the globally defined slowly oscillating solutions $x : \mathbb{R} \to [-B, A]$ of Eq. (8.1.1).

The first main result is that a Poincaré–Bendixson type theorem holds on $\mathcal{A}$: The α- and ω-limit sets of phase curves in $\mathcal{A}$ are either $\{0\}$ or periodic orbits given by slowly oscillating periodic solutions. The second main result is that in case $\mathcal{A} \neq \{0\}$, the set $\mathcal{A}$ is homeomorphic to the 2-dimensional closed unit disk so that the unit circle corresponds to a periodic orbit given by a slowly oscillating periodic solution.

Below we list some of the technical tools used in the proofs.

There is an additional assumption on the delay function r: either

$$\left|r'(u)\right| < \frac{1}{K}, \quad u \in [-B, A],$$

or

$$r \in C^2\big([-B, A], \mathbb{R}\big), \text{ and there is } a \in (0, 1) \text{ with } r''(u) \leqslant a\mu\big[r'(u)\big]^2, \\ u \in [-B, A].$$

This assumption and the fact that the dependence of the delay on the state is of the simple form $r(x(t))$ seem to be crucial in several parts of the proof.

An important consequence of the above hypothesis on r is that the function

$$t \mapsto t - r\big(x(t)\big)$$

is strictly increasing for solutions of (8.1.1). Another important fact is that for a suitable weighted difference $v : \mathbb{R} \to \mathbb{R}$ of two solutions x and y on $\mathbb{R}$, an equation of the form

$$v'(t) = \alpha(t) v\big(t - r\big(x(t)\big)\big) \tag{8.1.2}$$

holds on $\mathbb{R}$ with a negative, bounded and continuous α. The backward uniqueness of solutions is a corollary.

A modified version of the well-known discrete Lyapunov functional of Mallet-Paret and Sell [150,157,158] is also introduced. Instead of on intervals with fixed length, the sign changes are counted for a solution x on intervals of the form $[t - r(x(t)), t]$. The properties are completely analogous to those of the constant delay case. In particular, this functional can be used to exclude the existence of solutions decaying to zero at ∞ or $-\infty$ faster than any exponential. By applying the discrete Lyapunov functional to a weighted difference of two solutions satisfying Eq. (8.1.2), the number of sign changes of the difference can be controlled.

A return map P on the compact convex set

$$U = \{\phi \in L_K\colon\ \phi(0) = 0,\ \phi(s) \geqslant 0 \text{ for all } s \in [-1, 0]\}$$

is defined by $P(\phi) = x^{\phi}_{z_2}$, where at z_2 the second sign change of x^{ϕ} in $(0, \infty)$ occurs, and $P(\phi) = 0$ if there is at most one sign change in $(0, \infty)$. P is not necessarily continuous on U. However, $P|_{\mathcal{A}\cap U}$ is continuous. In addition, P restricted to $\{\phi \in \mathcal{A} \cap U\colon\ P(\phi) \neq 0\}$ is a homeomorphism onto $\mathcal{A} \cap U \setminus \{0\}$. It is also an essential step that $\mathcal{A} \cap U$ is connected. The elements of $\mathcal{A} \cap U \setminus \{0\}$ are exactly those segments x_s of globally defined slowly oscillating solutions $x : \mathbb{R} \to [-B, A]$ for which $x(s) = 0$ and $x'(s) < 0$.

An asymptotic expansion for slowly oscillating solutions converging to zero as $t \to -\infty$ is also proved. It relates solutions of Eq. (8.1.1) to solutions of the associated linear equation

$$y'(t) = -\mu y(t) + f'(0)y(t-1).$$

This result is used to verify that for any two elements ϕ, ψ of $\mathcal{A}$, the difference of the solutions $x^{\phi} - x^{\psi}$ has at most one sign change in all intervals $[t - r(x^{\phi}(t)), t]$ and $[t - r(x^{\psi}(t)), t]$, $t \in \mathbb{R}$. This fact is important in the proof of the injectivity of the map

$$\Pi : \mathcal{A} \ni \phi \mapsto \begin{pmatrix} \phi(0) \\ \phi(-r(\phi(0))) \end{pmatrix} \in \mathbb{R}^2.$$

The paper [22] considers Eq. (8.1.1) in the positive feedback case, i.e., $f' > 0$, and proves certain results which are analogous to the constant delay case in [133].

We remark that in the constant delay case, it is also known that the attractor of the slowly oscillating solutions is a C^1-smooth submanifold of the phase space [210]. Another remarkable result is that the domain of attraction is an open dense subset of the phase space [159]. Whether these remain true for the state-dependent delay case are open problems.

8.2. *Limiting profiles for a singular perturbation problem*

In their series of papers [151,152,155] Mallet-Paret and Nussbaum determine the asymptotic shape, or *limiting profile*, of periodic solutions to equations of the form

$$\epsilon x'(t) = f\big(x(t), x\big(t - r\big(x(t)\big)\big)\big), \tag{8.2.1}$$

for $\epsilon \to 0$. A limiting profile is a subset Ω of the plane $\mathbb{R}^2$ which arises as limit of a sequence of solutions

$$x^k = \{\big(t, x^k(t)\big) \in \mathbb{R}^2\colon\ t \in \mathbb{R}\}, \quad k \in \mathbb{N},$$

to Eq. (8.2.1) with $\epsilon = \epsilon_k$, in case $\lim_{k\to\infty} \epsilon_k = 0$. Convergence of a sequence of closed subsets of the plane means that intersections with given compact subsets converge in the Hausdorff metric.

The standing hypotheses in [155] are the following. f is a Lipschitz continuous real-valued map defined on a square $I \times I$, $I = [-D, C]$ with $C > 0$ and $D > 0$, and $r : I \to [0, \infty)$ is Lipschitz continuous with

$$r(0) = 1 \quad \text{and} \quad r(\xi) > 0 \quad \text{for } -D < \xi < C.$$

The zero set $f^{-1}(0) \subset I \times I$ is a strictly decreasing continuous function $g : I \to I$ with $g(0) = 0$, and f is positive below its zero set and negative above it. Moreover,

$$\left|g^2(\xi)\right| < |\xi| \quad \text{on } (-D, C) \setminus \{0\}.$$

In case $r(C) > 0$ it is assumed that $g(C) = -D$ while in case $r(-D) > 0$, $g(-D) = C$. Finally, f is differentiable at $(0, 0)$ with $\mathrm{D}_2 f(0, 0)1 < D_1 f(0, 0)1$.

Notice that the distribution of the signs of f in $I \times I$ generalizes the negative feedback inequality for functions of a single variable.

The periodic solutions considered are slowly oscillating in the sense that their zeros are spaced at distances larger than $r(0) = 1$, which is the delay at equilibrium. Their minimal periods are given by 3 consecutive zeros, and they are sine-like in the sense that the period interval between 3 successive zeros consists of 3 adjacent subintervals on each of which the periodic solution is monotone. Existence of sine-like slowly oscillating periodic solutions for sufficiently small $\epsilon > 0$ is proved in [152].

Limiting profiles exist, due to a result from [152] that for every sequence of parameters $\epsilon_k > 0$ with $\lim_{k\to\infty} \epsilon_k = 0$ and for every sequence of sine-like slowly oscillating periodic solutions $x^k : \mathbb{R} \to I$ of Eq. (8.2.1) with $\epsilon = \epsilon_k$ there is a subsequence (k_j) for which the graphs x^{k_j} converge.

It may happen that the limiting profile is simply the abscissa $\mathbb{R} \times \{0\}$. Theorem 5.1 in [152] provides sufficient conditions on r which exclude this case, like for example $r'(0) \neq 0$.

The first step towards the description of nontrivial limiting profiles is an appropriate interpretation of the formal limit of Eq. (8.2.1) for $\epsilon \to 0$, which reads

$$0 = f\bigl(x(t), x\bigl(t - r\bigl(x(t)\bigr)\bigr)\bigr) \tag{8.2.2}$$

and can be considered as a difference equation in implicit form for functions on the real line. But this is too narrow, as limiting profiles may contain vertical line segments. Notice that for any point $(t, x(t))$ on a solution $x : \mathbb{R} \to I$ of Eq. (8.2.2) there is another point $(s, x(s))$ on x with

$$s = t - r\bigl(x(t)\bigr) \leqslant t \quad \text{and} \quad x(s) = g\bigl(x(t)\bigr)$$

since Eq. (8.2.2) is solved for the second argument by g. This suggests to consider the *backdating map* $\Phi : \mathbb{R} \times I \to \mathbb{R} \times I$ given by

$$\Phi(\tau, \xi) = \bigl(\tau - r(\xi), g(\xi)\bigr)$$

and its trajectories (τ_n, ξ_n), which satisfy the system

$$0 = f(\xi_n, \xi_{n-1}),$$
$$\tau_{n-1} = \tau_n - r(\xi_n).$$

Properties of the backdating map are the key to the description of limiting profiles.

Theorem 1.3 in [155] establishes that the minimal periods $p^k > 2$ of sine-like slowly oscillating periodic solutions $x^k : \mathbb{R} \to I$, $k \in \mathbb{N}$, of Eq. (8.2.1) with $\epsilon = \epsilon_k$, $\lim_{k\to\infty} \epsilon_k = 0$, are bounded.

Each nontrivial limiting profile $\Omega \subset \mathbb{R} \times I$ is periodic, i.e.,

$$\Omega = \Omega + (p, 0)$$

with the (existing) limit $p \geqslant 2$ of the minimal periods of the approximating sequence of periodic solutions, and the intersection of Ω with a suitable vertical strip of width p can be written as the union of a horizontal line segment below the abscissa, an ascending part, a horizontal line segment above the abscissa, and a descending part. The horizontal parts may be singletons, and the ascending and descending parts may contain both horizontal and vertical line segments.

The first main result, Theorem A in [155], describes a nontrivial limiting profile Ω in the following way, using the (existing) limits $\mu > 0$ of the maxima and $-\nu < 0$ of the minima of the approximating periodic solutions: There is a sequence of continuous functions $\psi_n : [-\nu, \mu] \setminus \{0\} \to \mathbb{R}$, $n \in \mathbb{Z}$, with right and left limits at 0, so that for each integer n the function $(-1)^n \psi_n$ is increasing, $\psi_n \leqslant \psi_{n+1}$, $\psi_{n+2} = \psi_n + p$, and

$$\Omega = \left(\bigcup_n \psi_n^* \cup \left(A_n \times \{0\}\right)\right) \cup \left(\bigcup_n B_n \times \{\lambda_n\}\right)$$

where

$$\psi_n^* = \left\{\left(\psi_n(\xi), \xi\right) \colon 0 \neq \xi \in [-\nu, \mu]\right\},$$

$$A_n = \begin{cases} \left[\psi_n(0-), \psi_n(0+)\right] & \text{if } n \text{ is even,} \\ \left[\psi_n(0+), \psi_n(0-)\right] & \text{if } n \text{ is odd,} \end{cases}$$

and

$$B_n = \left[\psi_n(\lambda_n), \psi_{n+1}(\lambda_n)\right]$$

with

$$\lambda_n = \begin{cases} \mu & \text{if } n \text{ is even,} \\ -\nu & \text{if } n \text{ is odd.} \end{cases}$$

Moreover, there exist $\delta_0 > 0$ and $\delta_1 > 0$ so that for $0 \neq \xi \in [-\nu, \delta_0]$,

$$\psi_{2m}(\xi) = \max_{-\nu \leqslant s \leqslant \xi} \big(r(s) + \psi_{2m-1}\big(g(s)\big)\big) \tag{8.2.3}$$

while for $0 \neq \xi \in [-\delta_1, \mu]$,

$$\psi_{2m+1}(\xi) = \max_{\xi \leqslant s \leqslant \mu} \big(r(s) + \psi_{2m}\big(g(s)\big)\big). \tag{8.2.4}$$

The nonlocal *max-plus operators* given by the right-hand sides of Eqs. (8.2.3)–(8.2.4) bear analogies with linear Fredholm integral operators. To see this, replace addition in (8.2.3)–(8.2.4) by multiplication and maximization by integration. In [153,154] Mallet-Paret and Nussbaum study max-plus operators and associated eigenvalue problems; the theory is applied in [155].

Theorem B in [155] deals with monotone delay functions r and provides more detailed information about limiting profiles, in terms of f, g, r and $h = r + r \circ g$. Here the functions ψ_n for n odd are solutions to an eigenvalue problem

$$p + \psi_n(\xi) = \max_{\xi \leqslant s \leqslant \mu} \big(h(s) + \psi_n\big(g^2(s)\big)\big)$$

for a max-plus operator, with the period p as an *additive eigenvalue*. The functions ψ_n for n even are computed from ψ_n for n odd.

Theorem C in [155] establishes uniqueness of limiting profiles, under further conditions on the data f and r.

A simple-looking example for which there is a unique limiting profile is the equation

$$\epsilon x'(t) = -x(t) - kx\big(t - 1 - cx(t)\big)$$

with $k > 1$ and $c > 0$. In this case, $p = 1 + k$ and

$$\Omega \cap \big((-1, k) \times \mathbb{R}\big) = \left\{\left(\tau, \frac{1}{c}\tau\right):\ -1 < \tau < k\right\},$$

$$\Omega \cap \big(\{k\} \times \mathbb{R}\big) = \left\{(k, \xi):\ -\frac{1}{c} \leqslant \xi \leqslant \frac{k}{c}\right\}.$$

So Ω is uniquely determined and looks like sawteeth.

8.3. *Small delay*

In [178] Ouifki and Hbid obtain existence of periodic solutions for a system of the form

$$x'(t) = g\big(x\big(t - r(x_t)\big)\big) \tag{8.3.1}$$

with a map $g:\mathbb{R}^2 \to \mathbb{R}^2$ which satisfies $g(0)=0$, is smooth of class C^4, and has Jacobian $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ at $0 \in \mathbb{R}^2$. Also the delay functional $r:C \to [0,h] \subset \mathbb{R}$, $C = C([-h,0];\mathbb{R}^2)$, is assumed to be smooth of class C^4, and several smallness conditions are imposed. The approach in [178] is based on a decomposition of the functional $f:C \to \mathbb{R}^2$ corresponding to the right hand side of Eq. (8.3.1) into a smooth map $f_r:C \to \mathbb{R}^2$ and a remainder term, both depending on r. The decomposition holds for arguments in a closed subset E_1 of the space $C^{0,1} = C^{0,1}([-h,0];\mathbb{R}^2)$. The set E_1 is contained in the analogue $E \subset C^1 = C^1([-h,0];\mathbb{R}^2) \subset C^{0,1}$ of the solution manifold X_f from Section 3; in [144] it was shown that a nonlinear semigroup on $C^{0,1}$ generated by equations like (8.3.1) becomes strongly continuous if restricted to E. Smallness assumptions on r yield that the set E_1 is positively invariant. Each truncated equation

$$y'(t) = f_r(y_t) \tag{8.3.2}$$

defines a semiflow on the space C, as f_r is sufficiently smooth. The delay $\mu = r(0)$ at equilibrium is then considered as a parameter; Eq. (8.3.2) is rewritten as

$$y'(t) = f_{\tilde{r}}(\mu, z_t), \tag{8.3.3}$$

with $\tilde{r}(\phi) = r(\phi) - r(0)$. Under assumptions which guarantee certain stability properties of the stationary point 0 of Eq. (8.3.3) with $\mu = 0$ and $\tilde{r}$ small, a combination of center manifold theory with a Hopf bifurcation theorem yields attracting periodic orbits $o(\mu,\tilde{r})$ of the truncated equation (8.3.2) for small $\mu > 0$ and small $\tilde{r}$. Upon that a return map is constructed following solutions of the original equation (8.3.1) which start from initial data in E_1 close to a chosen point on $o(\mu,\tilde{r})$. This requires further smallness properties of $\tilde{r}$; closeness refers to the topology of $C^{0,1}$. With respect to this topology the return map is continuous and compact, Schauder's theorem is applied, and resulting fixed points define periodic solutions of Eq. (8.3.1).

8.4. *Generic convergence*

In [21] Bartha considers the scalar equation

$$x'(t) = -\mu x(t) + f\big(x\big(t - r\big(x(t)\big)\big)\big) \tag{8.4.1}$$

assuming that $\mu > 0$, $f \in C^1(\mathbb{R},\mathbb{R})$, $f(0) = 0$, $f'(u) > 0$ for all $u \in \mathbb{R}$, $r \in C^1(\mathbb{R},\mathbb{R})$, and there is $A > 0$ with

$$\big|f(u)\big| < \mu|u| \quad \text{for all } u \in \mathbb{R} \setminus (-A, A).$$

In addition, it is also required that

$$r(u) > 0 \quad \text{for all } u \in [-A, A].$$

Setting $R = \max_{u\in[-A,A]} |r(u)|$, the metric space X is defined as the space of Lipschitz continuous functions $\phi : [-R, 0] \to [-A, A]$ equipped with the metric

$$d(\phi, \psi) = \max_{s\in[-R,0]} \left|\phi(s) - \psi(s)\right|.$$

Then, for every $\phi \in X$, Eq. (8.4.1) has a unique solution $x^\phi : [-R, \infty) \to [-A, A]$ with $x^\phi|_{[-R,0]} = \phi$, and the relations

$$F(t, \phi) = x_t^\phi, \quad t \geqslant 0, \quad x_t^\phi(s) = x^\phi(t + s), \quad -R \leqslant s \leqslant 0,$$

define a continuous semiflow on X.

Using the standard ordering

$$\phi \leqslant \psi \quad \text{iff} \quad \phi(s) \leqslant \psi(s), \quad -R \leqslant s \leqslant 0,$$

it is relatively straightforward to show that the semiflow F is monotone, i.e., $F(t, \phi) \leqslant F(t, \psi)$ whenever $\phi \in X$, $\psi \in X$, $\phi \leqslant \psi$ and $t \geqslant 0$. However, the strongly order preserving property (SOP), which is a crucial hypothesis in the generic convergence theorem of Smith and Thieme in [191], does not hold in general for F. Recall that SOP of F means the monotonicity of F, and that in case $\phi \in X$, $\psi \in X$, $\phi \leqslant \psi$, $\phi \neq \psi$ there exist $t_0 > 0$ and open subsets U, V of X with $\phi \in U$ and $\psi \in V$ such that $F(t_0, U) \leqslant F(t_0, V)$. Here, for subsets S, T of X we write $S \leqslant T$ if $\phi \leqslant \psi$ holds for all $\phi \in S$ and $\psi \in T$. The main reason of the failure of the SOP property for F is that for different elements ϕ, ψ of X with $\phi \leqslant \psi$, $x^\phi(t) = x^\psi(t)$ may happen for all $t \geqslant 0$.

In [21] the SOP property is replaced by the weaker mildly order preserving property (MOP). Introduce

$$\phi <_F \psi$$

for elements ϕ, ψ of X if $\phi \leqslant \psi$, $\phi \neq \psi$, and $F(t, \phi) \neq F(t, \psi)$ for all $t \geqslant 0$. Then F is said to be MOP if it is monotone, and for every ϕ, ψ in X with $\phi <_F \psi$, there exist $t_0 > 0$ and open subsets U, V of X with $\phi \in U$ and $\psi \in V$ such that $F(t_0, U) \leqslant F(t_0, V)$.

[21] proves that F has the MOP property. An important step toward this result is that for two globally defined solutions $x : \mathbb{R} \to [-A, A]$ and $y : \mathbb{R} \to [-A, A]$ with $x_0 = y_0$, it is true that

$$x(t) = y(t) \quad \text{for all } t \in \mathbb{R}.$$

The abstract generic convergence result of Smith and Thieme from [191] is modified in [21] so that SOP is replaced by MOP. Then the main result of [21] is that there is an open dense subset Y of X such that

$$\lim_{t\to\infty} x^\phi(t) \text{ exists}$$

for all $\phi \in Y$.

8.5. *Stability and oscillation*

Several results in the literature which deal with limiting behaviour of solutions to nonautonomous differential equations with non-constant delay are also valid for equations with state-dependent delays, see, e.g., [128,222]. Most of these papers concentrate on the behaviour of given solutions, not on questions of existence, uniqueness, and continuous dependence.

Here we list a few papers where the presence of the state-dependent delay is emphasized since it causes new technical difficulties.

Kuang [135] considers the scalar nonautonomous state-dependent delay differential equation

$$x'(t) = -g\big(t, x(t)\big) - \mathrm{e}^{-\eta\tau(x_t)} f\big(t, x\big(t - \tau(x_t)\big)\big).$$

Sharp conditions for the boundedness of solutions, global and uniform stability of the trivial solution are presented.

Cooke and Huang [47] study the scalar equation

$$x'(t) = x(t)\left(a - bx(t) - \sum_{i=1}^{L} b_i x(t - r_i) - cx\big(t - \tau(x_t)\big)\right),$$

where a, b_i, c are positive constants, τ is a functional of the history of $x(\cdot)$ over all times before t. They obtain results on convergence of positive solutions, periodic and oscillatory behaviour which extend work of G. Seifert for the constant delay case.

Győri and Hartung [93] consider the linear delay differential systems

$$x'(t) = A_i(t)x\big(t - \sigma_i(t)\big) \tag{8.5.1}$$

with continuous coefficient functions $A_i : [0, \infty) \to \mathbb{R}^{n\times n}$ and continuous delay functions $\sigma_i : [0, \infty) \to \mathbb{R}$, $i \in \{1, 2\}$, such that, for some $r > 0$, $0 \leqslant \sigma_i(t) \leqslant t + r$, $t \geqslant 0$, and $\liminf_{t\to\infty}[t - \sigma_i(t)] > 0$. Assuming that the zero solution of Eq. (8.5.1) with $i = 1$ is exponentially stable, explicit neighbourhoods of A_1 and σ_1 are constructed so that if A_2 and σ_2 belong to the corresponding neighbourhoods of A_1 and σ_1, then the zero solution of Eq. (8.5.1) with $i = 2$ is also exponentially stable. As an application, among others, sufficient conditions are given to guarantee the exponential stability of the zero solution of the scalar equation

$$x'(t) = a(t)x\big(t - \tau(t, x_t)\big)$$

with delay functional τ defined by the threshold relation

$$\int_{t-\tau(t,x_t)}^{t} f(t, s, x_t)\,\mathrm{d}s = m$$

provided that the zero solution of

$$x'(t) = a(t)x\big(t - \tau(t, 0)\big)$$

is exponentially stable.

Cao, Fan and Gard [37] study the two-stage population model of Aiello, Freedman and Wu [2] with density-dependent delay. They show that no Hopf bifurcation can occur in the sense that the characteristic equation, associated with linearization at any strictly positive equilibrium, never has imaginary roots. Instability can arise together with the creation of multiple equilibria. The attractivity regions of the equilibrium points are also estimated.

Bélair [24] considers an age-structured model, and reduces it to a system of delay differential equations with state-dependent delay. Assuming that a center manifold reduction is valid, a supercritical Hopf bifurcation is established.

Rai and Robertson [182,183] study stage-structured population models with delay where the delay is a function of the total population density, and they prove positivity, boundedness and stability of the solutions.

Bartha [20] addresses the stability and convergence of solutions for a class of neutral functional differential equations with state-dependent delay. The equation is transformed into a retarded differential equation with infinite delay. The state-dependent delay causes that the transformation depends on each particular solution. For the retarded equation with infinite delay a result of Krisztin [128] can be applied to get sharp sufficient conditions for the stability of the zero solution. The second part of [20] contains attractivity results.

Pinto [180] gives conditions assuring asymptotic expansions of the form

$$y(t) = \exp\biggl(\int_{t_0}^{t} a(s)\,\mathrm{d}s\biggr)\biggl[\xi + \mathrm{O}\biggl(\int_{t}^{\infty} \lambda(s)\,\mathrm{d}s\biggr)\biggr] \quad \text{as } t \to \infty$$

for the solutions of the scalar state-dependent differential equation

$$y'(t) = a(t)y\big(t - r\big(t, y(t)\big)\big)$$

with certain $\xi \in \mathbb{R}$, $\lambda \in L_1[0, \infty)$ constructed by means of the coefficient function a and the delay function r. These results are extended to systems in [83].

Asymptotic solution behaviour for various classes of autonomous equations with state-dependent delays is investigated also in [45,46,53,104].

Gatica and Rivero [86] obtain sufficient conditions for the oscillation of all nontrivial solutions of a scalar equation with a state-dependent delay given by a threshold condition.

Domoshnitsky, Drakhlin and Litsyn [56] study the equation

$$x'(t) + \sum_{i=1}^{m} A_i(t)x\big(t - (H_i x)(t)\big) = f(t)$$

in $\mathbb{R}^n$ with measurable and essentially bounded functions A_i and f, and measurable delay functional H_i. Sufficient conditions are obtained for the boundedness, oscillation and nonoscillation of the solutions by using an associated linear equation.

Additional stability and oscillation results can be found in [36,46,84,92,172,173,194, 195,200,217,220,223].

9. Numerical methods

9.1. *Preliminaries*

The study of numerical approximation for state-dependent delay equations goes back at least to the mid sixties of the last century [28,77], and since then it is an intensively investigated research area [12–14,17,18,27,31,35,49,68–70,78,79,90,125,140,145,168,169,196, 197,211].

In this section we concentrate on numerical methods for state-dependent delay equations (SD-DDEs) of the form

$$x'(t) = f\big(t, x(t), x\big(t - \tau\big(t, x(t)\big)\big)\big), \quad t \in [t_0, t_N] \tag{9.1.1}$$

with an associated initial condition

$$x(t) = \phi(t), \quad t \in [t_0 - h, t_0]. \tag{9.1.2}$$

For simplicity we assume that $f : [t_0, t_N] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and Lipschitz continuous in its second and third variables, $\tau : [t_0, t_N] \times \mathbb{R} \to [0, h]$ is continuous, and $\phi \in C^{0,1}([t_0, t_N]; \mathbb{R})$, therefore the IVP (9.1.1)–(9.1.2) has a unique solution. The results we present can usually be easily extended to the system or multiple delays case.

We mention that there are a large number of publications dealing with other types of state-dependent differential equations including neutral SD-DDEs of the form

$$x'(t) = f\big(t, x(t), x\big(t - \tau\big(t, x(t)\big)\big), x'\big(t - \sigma\big(t, x(t)\big)\big)\big)$$

(see, e.g., [16,27,34,42,69,119,122,169,179]), the so-called "implicit" neutral SD-DDEs of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(x(t) - g\big(t, x\big(t - \sigma\big(t, x(t)\big)\big)\big)\big) = f\big(t, x(t), x\big(t - \tau\big(t, x(t)\big)\big)\big)$$

[102,140], Volterra differential equations with state-dependent delays [13,17,31,35,120, 196], and differential-algebraic equations with state-dependent delays [109].

9.2. *Continuous Runge–Kutta methods for ODEs*

Before we discuss approximation of state-dependent equations, first we recall some basic notations and definitions for numerical approximation of ODEs. Consider the IVP

$$x'(t) = g\big(t, x(t)\big), \quad t \in [t_0, t_N], \tag{9.2.1}$$

$$x(t_0) = x_0, \tag{9.2.2}$$

and suppose mesh points $\Delta = \{t_0 < t_1 < \cdots < t_N\}$ are given. *Discrete* one- or multistep methods associated to the IVP (9.2.1)–(9.2.2) produce a sequence $y_0, \ldots, y_N$ to approximate the solution x at the mesh points Δ. In this paper for simplicity we restrict our discussion to a popular class of one-step methods, the Runge–Kutta (RK) methods. A discrete RK method is defined by $y_0 = x_0$, and

$$y_{n+1} = y_n + h_n \sum_{i=1}^{s} b_i k_{n,i}, \quad n = 0, 1, \ldots, N-1, \tag{9.2.3}$$

where the stage values $k_{n,1}, \ldots, k_{n,s}$ are determined by the system of algebraic equations

$$k_{n,i} = g\left(t_n + h_n c_i, y_n + h_n \sum_{j=1}^{s} a_{ij} k_{n,j}\right), \quad i = 1, 2, \ldots, s, \tag{9.2.4}$$

and $h_n = t_{n+1} - t_n$. s is called the number of *stages*, the b_i's are the *weights*, the c_i's are the *abscissae* of the method satisfying $c_i \in [0, 1]$. The coefficients a_{ij} are collected in a matrix $\mathbf{A}$, the weights and abscissae in the vectors $\mathbf{b}$ and $\mathbf{c}$, and the parameters are usually listed in the Butcher tableau $\begin{array}{c|c} \mathbf{c} & \mathbf{A} \\ \hline & \mathbf{b}^{\mathrm{T}} \end{array}$. If the matrix $\mathbf{A}$ is lower triangular with zero diagonal entries then the RK method is explicit, otherwise (9.2.3) and (9.2.4) implicitly define the sequence $y_0, \ldots, y_N$.

In the mid eighties of the last century and at the beginning of the nineties the interest in the study and application of *continuous extensions* of numerical methods has been increased, since if a *dense output* is required by an ODE solver, e.g., when plotting the graph of the numerical solution, then a discrete solver is not efficient enough. Continuous methods are especially important for the approximation of delay equations where the evaluation of the approximate solution is needed in between mesh points.

One possible way to derive a continuous extension of the RK method (called CRK method) (9.2.3)–(9.2.4) is the following: Define

$$u(t_n + \theta h_n) = y_n + h_n \sum_{i=1}^{s} w_i(\theta) k_{n,i}, \quad \theta \in [0, 1], \; n = 0, 1, \ldots, N-1, \tag{9.2.5}$$

where w_i are polynomials of degree less than or equal to δ satisfying

$$w_i(0) = 0, \quad w_i(1) = b_i, \quad i = 1, \ldots, s.$$

This formula is called *interpolant of the first class* of the discrete RK method, δ is the *degree of interpolant*. The Butcher tableau of a CRK method has the form $\begin{array}{c|c} \mathbf{c} & \mathbf{A} \\ \hline & \mathbf{w}^{\mathrm{T}}(\theta) \end{array}$.

There are several methods to define the polynomial interpolation w_i in (9.2.5). A typical way is to use a cubic Hermite interpolation, or when a higher order interpolant is required, a so-called fully Hermite interpolation [71,110]. Another approach is to consider (9.2.4)–(9.2.5) as a discrete RK method with coefficients a_{ij}/θ, weights $w_i(\theta)/\theta$ and step size θh_n, and apply order conditions known for the discrete RK method, see, e.g., [27]. We mention that there are more general extensions of discrete RK methods than that of the form (9.2.5) (see, e.g., [27,110]), but we do not go into details here.

Let z be the solution of the local problem

$$z'(t) = g\big(t, z(t)\big), \quad t \in [t_n, t_{n+1}], \tag{9.2.6}$$

$$z(t_n) = z^*. \tag{9.2.7}$$

We say that the discrete RK method (9.2.3)–(9.2.4) has *order* p if $p \geqslant 1$ is the largest integer such that for all C^p-functions g in (9.2.1) and for all meshes Δ we have

$$\big|z(t_n) - y_n\big| = \mathrm{O}\big(h_n^{p+1}\big)$$

uniformly with respect to z^* in any bounded subset of $\mathbb{R}$ and $n = 0, \ldots, N-1$, where z is the solution of (9.2.6)–(9.2.7). Similarly, the CRK method (9.2.4)–(9.2.5) has *uniform order* q if $q \geqslant 1$ is the largest integer such that for all C^q-functions g in (9.2.1) and for all meshes Δ we have

$$\max\big\{\big|z(t) - u(t)\big|\colon\ t_n \leqslant t \leqslant t_{n+1}\big\} = \mathrm{O}\big(h_n^{q+1}\big)$$

uniformly with respect to z^* in any bounded subset of $\mathbb{R}$ and $n = 0, \ldots, N-1$, where z is the solution of (9.2.6)–(9.2.7).

We recall the following result from [27]:

THEOREM 9.2.1. *If the discrete RK method* (9.2.3)–(9.2.4) *has order* p *and if* g *is a* C^p-*function*, *then the method is convergent of global order* p *on any bounded interval* $[t_0, t_N]$, *i.e.*,

$$\max\big\{\big|x(t_n) - y_n\big|\colon\ n = 1, \ldots, N\big\} = \mathrm{O}\big(h^p\big),$$

where $h = \max\{h_0, \ldots, h_{N-1}\}$.

Moreover, if the CRK method (9.2.4)–(9.2.5) *has uniform order* q, *then it has uniform global order* $q' = \min\{p, q+1\}$, *i.e.*,

$$\max\big\{\big|x(t) - u(t)\big|\colon\ t_0 \leqslant t \leqslant t_N\big\} = \mathrm{O}\big(h^{q'}\big).$$

It can be checked (see [27]) that, in order to get the uniform order q, the interpolant must be of degree $\delta \geqslant q$, and if a discrete RK method has a continuous extension $u(t)$ of uniform order q with degree $\delta > q$, then it has a continuous extension $\tilde{u}(t)$ of uniform order q with degree $\delta = q$, as well.

As an example we present the Butcher tableau of a continuous extension of the classical four-stage discrete RK method which has uniform order 3:

$$\begin{array}{c|cccc} 0 & 0 & & & \\ \frac{1}{2} & \frac{1}{2} & 0 & & \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & \\ 1 & 0 & 0 & 1 & 0 \\ \hline & w_1(\theta) & w_2(\theta) & w_3(\theta) & w_4(\theta) \end{array} \quad \text{where} \quad \begin{aligned} w_1(\theta) &= \tfrac{2}{3}\theta^3 - \tfrac{3}{2}\theta^2 + \theta, \\ w_2(\theta) &= -\tfrac{2}{3}\theta^3 + \theta^2, \\ w_3(\theta) &= -\tfrac{2}{3}\theta^3 + \theta^2, \\ w_4(\theta) &= \tfrac{2}{3}\theta^3 - \tfrac{1}{2}\theta^2. \end{aligned}$$

One-step collocation methods can be considered as CRK methods: Pick distinct abscissae $c_1, \dots, c_s \in [0, 1]$, and define

$$\begin{aligned} \ell_i(v) &= \prod_{k=1,\ k\neq i}^{s} \frac{v - c_k}{c_i - c_k}, \quad && i = 1, \dots, s, \\ a_{ij} &= \int_0^{c_i} \ell_j(v)\,\mathrm{d}v, && i, j = 1, \dots, s, \\ w_i(\theta) &= \int_0^{\theta} \ell_i(v)\,\mathrm{d}v, && i = 1, \dots, s. \end{aligned}$$

Then the corresponding CRK method (9.2.4)–(9.2.5) defines a polynomial u of degree $\leqslant s$ satisfying the collocation equations

$$u'(t_n + h_n c_i) = g\big(t_n + h_n c_i, u(t_n + h_n c_i)\big), \quad i = 1, \dots, s, \quad u(t_n) = y_n.$$

It is known (see, e.g., [27]) that the uniform global order of this CRK method is $q' = s$ or $s + 1$.

Nowadays an efficient differential equation solver uses a higher order continuous method, or usually a pair of higher order methods (to estimate local errors in the step size selection). For other examples of CRK methods we refer to [27,49,70,71].

9.3. *The standard approach to approximation of SD-DDEs*

A typical approach (called "the standard approach" in [27]) to obtain a numerical approximation to the solution x of the state-dependent IVP (9.1.1)–(9.1.2) is the numerical analogue of the method of steps well-known for computing exact solutions of constant delay equations. In this subsection we describe this approach, but for simplicity, we formulate it using the class of one-step CRK methods. Clearly, it can be adopted using many other types of ODE discretization techniques.

Pick mesh points $\Delta = \{t_0 < t_1 < \cdots < t_N\}$, let $y_0 = \phi(t_0)$, and consider the sequence of local IVPs for $n = 0, 1, \dots, N - 1$:

$$z_n'(t) = f\Big(t, z_n(t), y\big(t - \tau\big(t, z_n(t)\big)\big)\Big), \quad t \in [t_n, t_{n+1}], \tag{9.3.1}$$

$$z_n(t_n) = y_n, \tag{9.3.2}$$

where

$$y(s) = \begin{cases} \phi(s), & s \in [t_0 - h, t_0], \\ u(s), & s \in [t_0, t_n], \\ z_n(s), & s \in [t_n, t_{n+1}]. \end{cases}$$

Here u denotes the function $u : [t_0, t_n] \to \mathbb{R}$ whose restriction to $[t_i, t_{i+1}]$ $(i = 0, \dots, n-1)$ is the numerical solution, i.e., a CRK interpolant of the solution of the ith IVP. Then u is already defined in the previous steps. Now we solve this IVP on $[t_n, t_{n+1}]$ using the CRK method of the form (9.2.5) corresponding to $g(t, x) = f(t, x, u(t - \tau(t, x)))$, i.e., where $k_{n,i}$'s are defined by

$$\begin{cases} t_{n,i} = t_n + h_n c_i, \\ y_{n,i} = y_n + h_n \sum_{j=1}^{s} a_{ij} k_{n,j}, \\ k_{n,i} = f\big(t_{n,i}, y_{n,i}, y\big(t_{n,i} - \tau(t_{n,i}, y_{n,i})\big)\big). \end{cases} \tag{9.3.3}$$

Then we extend u from $[t_0, t_n]$ to $[t_0, t_{n+1}]$ by this CRK interpolant, define $y_{n+1} = u(t_{n+1})$, and continue with the next local IVP in the same manner.

If τ is bounded below by a positive constant $\bar{\tau}$, and the step size is chosen so that $h_n \leqslant \bar{\tau}$, then (9.3.1) is an ODE, since y in (9.3.1) never takes an argument from $[t_n, t_{n+1}]$, so it is explicitly defined. Therefore if the discrete RK method is explicit, i.e., $\mathbf{A}$ is lower triangular with zeros in the diagonal, then its continuous extension (9.2.5)–(9.3.3) is also explicit.

The numerical difficulty arises in the *vanishing delay case*, i.e., when τ can be arbitrary small. Then h_n can be such that $t - \tau(t, z(t)) > t_n$ for some $t \in [t_n, t_{n+1}]$, so the evaluation of y in (9.3.3) depends also on the unknown interpolant u on $[t_n, t_{n+1}]$. (This is called *overlapping*.) In this case the method is implicit, even if the original discrete RK method was explicit. Therefore the existence of the numerical approximation, i.e., the solvability of the algebraic equations (9.2.4)–(9.3.3) for the stage values $k_{n,1}, \dots, k_{n,s}$ is not trivial. It can be shown [27] that the above problem has a positive answer: Suppose the functions f, τ and ϕ are Lipschitz continuous, then in the overlapping case there always exist a sufficiently small h_n and a suitable degree of interpolants so that the implicit relations (9.2.4)–(9.3.3) have a unique solution for the stage values $k_{n,1}, \dots, k_{n,s}$; therefore u has a unique extension from $[t_0, t_n]$ to $[t_n, t_{n+1}]$. The proof of this result shows that this extension can be determined as a limit of a fixed point iteration, and therefore, it is common to estimate it by iteration using a predictor–corrector method. Next we show a possible algorithm to compute the nth step of the approximation in the case when the original discrete RK method is explicit, i.e., $a_{ij} = 0$ for $i \leqslant j$ and $c_1 = 0$. In this algorithm we first (Step 1) predict a starting value of the stage values using the last computed approximate solution value in the overlapping case instead of interpolating values. If overlapping occurs, then in an iteration (Step 2) we correct the stage values. It can be done using a fixed number of steps (m in the algorithm), or by testing the numerical convergence in a loop. Finally (Step 3) we update the interpolant.

Step 1: Prediction—computation of initial stage values

$$
\begin{aligned}
& t_{n,1} = t_n \\
& z_{n,1}^{(0)} = y_n \\
& k_{n,1}^{(0)} = f\big(t_n, z_{n,1}^{(0)}, u\big(t_n - \tau\big(t_n, z_{n,1}^{(0)}\big)\big)\big) \\
& \text{for } i = 2, \ldots, s \text{ do} \\
& \quad t_{n,i} = t_n + h_n c_i \\
& \quad z_{n,i}^{(0)} = y_n + h_n \sum_{j=1}^{i-1} a_{ij} k_{n,j}^{(0)} \\
& \quad d_{n,i}^{(0)} = t_{n,i} - \tau\big(t_{n,i}, z_{n,i}^{(0)}\big) \\
& \quad \text{if } d_{n,i}^{(0)} \leqslant t_n \text{ then} \\
& \qquad k_{n,i}^{(0)} = f\big(t_{n,i}, z_{n,i}^{(0)}, u\big(d_{n,i}^{(0)}\big)\big) \\
& \quad \text{else} \\
& \qquad k_{n,i}^{(0)} = f\big(t_{n,i}, z_{n,i}^{(0)}, z_{n,i}^{(0)}\big) \\
& \quad \text{end if} \\
& \text{end for}
\end{aligned}
$$

Step 2: Correction by iteration is needed if $d_{n,i}^{(0)} > t_n$ was for any $i \geqslant 2$ in Step 1

$$
\begin{aligned}
& \text{for } r = 1, \ldots, m \text{ do} \\
& \quad \text{for } i = 1, \ldots, s \text{ do} \\
& \qquad z_{n,i}^{(r)} = y_n + h_n \sum_{j=1}^{i-1} a_{ij} k_{n,j}^{(r-1)} \\
& \qquad d_{n,i}^{(r)} = t_{n,i} - \tau\big(t_{n,i}, z_{n,i}^{(r-1)}\big) \\
& \qquad \text{if } d_{n,i}^{(r)} \leqslant t_n \text{ then} \\
& \qquad\quad k_{n,i}^{(r)} = f\big(t_{n,i}, z_{n,i}^{(r)}, u\big(d_{n,i}^{(r)}\big)\big) \\
& \qquad \text{else} \\
& \qquad\quad \theta_i = \frac{d_{n,i}^{(r)} - t_n}{h_n}
\end{aligned}
$$

$$\hat{u}_i = y_n + h_n \sum_{j=1}^{i-1} w_j(\theta_i) k_{n,j}^{(r)} + h_n \sum_{j=i}^{s} w_j(\theta_i) k_{n,j}^{(r-1)}$$

$$k_{n,i}^{(r)} = f\big(t_{n,i}, z_{n,i}^{(r)}, \hat{u}_i\big)$$

end if

end for

end for

Step 3: Computation of the extension of u to $[t_n, t_{n+1}]$

$$u(t_n + \theta h_n) = y_n + h_n \sum_{i=1}^{s} w_i(\theta) k_{n,i}^{(m)}, \quad \theta \in [0, 1].$$

Most modern differential equation solvers use non-uniform mesh size, therefore the selection of the step size in each integration step (the so-called *primary step size selection*) is an important practical issue in such softwares. Concerning this topic we refer to [27,110, 125] for more details.

The next result says that in order a method be of order p it is necessary that the solution be at least C^p on each interval $[t_n, t_{n+1}]$. We say that the function x has *discontinuity of order* p at ξ if $x^{(p-1)}$ exists and is continuous at ξ, and $x^{(p)}$ has jump discontinuity at ξ, i.e., $x^{(p)}(\xi+) \neq x^{(p)}(\xi-)$.

For the proof of the next result see [27]:

THEOREM 9.3.1. *Suppose f, τ and ϕ are C^p functions. Moreover,*

(1) *the mesh $\Delta = \{t_0 < t_1 < \cdots < t_N\}$ includes all discontinuity points $\xi_1 < \xi_2 < \cdots < \xi_m$ of the solution of order $\leqslant p$;*

(2) *the discrete RK method* (9.2.1)–(9.2.3) *is consistent of order p and the CRK method* (9.2.4)–(9.2.5) *is consistent of uniform order q.*

Then the method (9.2.4)–(9.3.3) *for solving the IVP* (9.1.1)–(9.1.2) *is convergent of uniform global order $q' = \min\{p, q+1\}$, i.e.*

$$\max\big\{\big|x(t) - u(t)\big|\colon\ t_0 \leqslant t \leqslant t_N\big\} = \mathrm{O}\big(h^{q'}\big),$$

where $h = \max\{h_0, \ldots, h_{N-1}\}$.

We note that there are many papers which follow the basic method of steps described in this subsection, i.e., approximate the solution of an SD-DDE by that of an ODE, but defined by different discrete or continuous one- or multistep ODE solvers or with different definitions of the associated ODE [15,49,68,70,77,91,124,125,169,179].

Rewriting a constant delay equation as an equivalent abstract Cauchy problem and using Trotter–Kato-type approximations is another popular approach especially in control applications. This technique was extended to SD-DDEs in [118].

9.4. *Tracking of derivative discontinuities*

Theorem 9.3.1 indicates that a high order method must locate all the discontinuity points of the solutions up to order p, and add them to the mesh. For constant or time-dependent delay equations this is a relatively simple task, but in the state-dependent delay case it leads to significant difficulties. Indeed, the location of the discontinuity points can not be computed *a priori* because they depend on the solution, and, on the other hand, only approximate locations can be computed.

We assume throughout this subsection that f, τ and ϕ are all C^p-functions. One can show that in case $\phi'(t_0-) \neq f(t_0, \phi(t_0), \phi(-\tau(t_0)))$ the corresponding solution x of the IVP (9.1.1)–(9.1.2) has discontinuity of order 1 at $t = t_0$, but for $t > t_0$ the solution is $C1$-smooth. If $\xi > t_0$ is such that $\xi - \tau(\xi, x(\xi)) = t_0$, it is easy to check that $x''(\xi-) \neq x''(\xi+)$, so x has a discontinuity of order 2 at $t = \xi$. Similarly, if we set $\xi_0 = t_0$ and define a (finite or infinite) sequence by the relation $\xi_{i+1} - \tau(\xi_{i+1}, x(\xi_{i+1})) = \xi_i$ for $i = 0, 1, \ldots,$ the sequence $\xi_0, \xi_1, \ldots$ consists of discontinuity points of increasing order: ξ_i has order $i + 1$. If a solution x is such that the time lag function $t \mapsto t - \tau(t, x(t))$ is strictly monotone increasing (which is satisfied in many applications), the above sequence will contain all discontinuity points of the solution.

On the other hand, if the above time lag function is not strictly monotone increasing, then the equation $\xi - \tau(\xi, x(\xi)) = \xi_i$ may have many solutions ξ. It can be checked (see [27,78,79,168]) that at a solution ξ of the above equation the function x has a discontinuity of order $\geqslant 1$, if and only if the graph of $t \mapsto t - \tau(t, x(t))$ crosses the level ξ_i, i.e., the root ξ has odd multiplicity. Therefore in this case the discontinuity points can be naturally stored in a tree: $\xi_{0,1} = t_0$ is the root of the tree, level 1 of the tree contains the solutions $\xi_{1,1}, \ldots, \xi_{1,\ell_1}$ of

$$\xi - \tau\bigl(\xi, x(\xi)\bigr) = \xi_{i,j} \tag{9.4.1}$$

with odd multiplicity for $i = 0$ and $j = 1$. Then for $i = 1$ and any $j = 1, \ldots, \ell_1$ the solutions $\xi_{2,1}, \ldots, \xi_{2,\ell_2}$ of (9.4.1), if exist, are placed in the second level of the tree, as the descendents of the respective $\xi_{1,j}$, etc.

It was shown in [168] that there are only finitely many computationally important points (i.e., of order less or equal to p) in the discontinuity propagation tree, so they can be ordered and relabeled as an increasing sequence $t_0 = \xi_0 < \xi_1 < \xi_2 < \cdots < \xi_\ell \leqslant t_N$. In the sequel we use this notation.

Let ξ_j be a descendent of ξ_i in the discontinuity tree, i.e.,

$$\xi_j - \tau\bigl(\xi_j, x(\xi_j)\bigr) = \xi_i. \tag{9.4.2}$$

It is easy to see that as discontinuity propagates from ξ_i to ξ_j, smoothing occurs. More precisely, we recall the next result from [168].

THEOREM 9.4.1. *Suppose ξ_i and ξ_j satisfying* (9.4.2) *are discontinuity points of order k_i and k_j, respectively, and ξ_j has odd multiplicity m_j. Then $m_j k_i + 1 \leqslant k_j$.*

Since the exact discontinuity points $\xi_1, \dots, \xi_\ell$ depend on the solution, suitable approximations are required. Let u denote the continuous interpolant (in the previous subsection it was a CRK interpolant) of the numerical approximation of x using certain mesh and approximate values $y_0 = \phi(t_0), y_1, \dots, y_N$. Let $\tilde{\xi}_0 = t_0$, and define the approximate discontinuity points as solutions of

$$\xi - \tau\big(\xi, u(\xi)\big) = \tilde{\xi}_i \tag{9.4.3}$$

where u is the approximate solution of the IVP satisfying $|x(\tilde{\xi}_j) - u(\tilde{\xi}_j)| = \mathrm{O}(h^p)$, and $h = \max\{h_0, \dots, h_{N-1}\}$. The following result is cited from [78]:

THEOREM 9.4.2. *Let u be a pth order approximation of x at $\tilde{\xi}_j$, i.e.,* $|x(\tilde{\xi}_j) - u(\tilde{\xi}_j)| = \mathrm{O}(h^p)$, $h = \max\{h_0, \dots, h_{N-1}\}$, *$m_j$ be the multiplicity of ξ_j in* (9.4.2), *and* $|\xi_i - \tilde{\xi}_i| = \mathrm{O}(h^{r_i})$. *Then* $|\xi_j - \tilde{\xi}_j| = \mathrm{O}(h^{r_j})$ *where* $r_j = \min\{p, r_i\}/m_j$.

Feldstein and Neves [78] suggested the following *secondary step size control* to select the step size of the numerical integration method so that the approximate discontinuity points are collected to the mesh to keep the global order of the method high: Suppose at the nth step $y_0, \dots, y_n$ are defined, and the approximate discontinuity points found so far are $t_0 = \tilde{\xi}_0 < \tilde{\xi}_1 < \tilde{\xi}_2 < \cdots < \tilde{\xi}_{\ell_n} \leqslant t_N$.

Step 1: Predict the next approximate value of the integration method by $y_{n+1} = u(t_n + h_n)$ using the continuous method u of order p and the step size h_n selected by the primary step size control method of u.

Step 2: For $i = 1, \dots, \ell_n$ find the first i such that

$$\big(t_n - \tau(t_n, y_n) - \tilde{\xi}_i\big)\big(t_{n+1} - \tau(t_{n+1}, y_{n+1}) - \tilde{\xi}_i\big) < 0,$$

i.e., (9.4.3) corresponding to the right-hand side $\tilde{\xi}_i$ has a solution $\tilde{\xi}$. If such i exists then proceed with Step 3, otherwise we accept h_n and y_{n+1} and finish this algorithm, i.e, go to the next iterate of computing y_{n+2}.

Step 3: Using a root-finding method (e.g., bisection) combined with the definition of u find an approximate solution $\tilde{\xi}$ with the above property (if more solutions are found, the least one is used).

Step 4: Include $\tilde{\xi}$ to the new mesh $\hat{t}_0 = t_0, \dots, \hat{t}_n = t_n, \hat{t}_{n+1} = \tilde{\xi}, \hat{t}_{n+2} = t_{n+1}$, redefine the approximate solution values $\hat{y}_0 = y_0, \dots, \hat{y}_n = y_n, \hat{y}_{n+1} = u(\tilde{\xi})$. Restart the numerical integration from $\hat{t}_{n+1} = \tilde{\xi}$ to $\hat{t}_{n+2}$. Let $\hat{u}$ be the new interpolant on $[\hat{t}_{n+1}, t_{n+2}]$, and define $\hat{y}_{n+2} = \hat{u}(t_{n+2})$.

Step 5: Continue with the next iterate.

See [78,79] for more details. Concerning the global order of the above method combined with a pth order continuous numerical approximation method u Feldstein and Neves [78] showed:

THEOREM 9.4.3. *Suppose the order of discontinuity of ξ_j is k_j* ($j = 1, \dots, \ell$), *$\tilde{\xi}_j$ is the corresponding approximate discontinuity point, i.e., the solution of* (9.4.3), $|\xi_j - \tilde{\xi}_j| =$

$\mathrm{O}(h^{r_j})$, $\hat{\xi}_j$ *is a numerical approximation of* $\tilde{\xi}_j$ *with order* $|\hat{\xi}_j - \tilde{\xi}_j| = \mathrm{O}(h^{s_j})$, *and the uniform order of u is* p. *Then*

$$\max\bigl\{\bigl|x(t) - u(t)\bigr|: t \in [t_0, t_N]\bigr\} = \mathrm{O}\bigl(h^s\bigr) \quad \textit{where } s = \min_{j=1,\dots,\ell}\{p, k_j r_j, k_j s_j\}.$$

Moreover, if $s_j \geqslant p/k_j$ *for* $j = 1, \dots, \ell$, *then*

$$\max\bigl\{\bigl|x(t) - u(t)\bigr|: t \in [t_0, t_N]\bigr\} = \mathrm{O}\bigl(h^p\bigr).$$

In this subsection we assumed that the parameters of the IVP, f, τ and ϕ are all C^p-functions. If they are only piecewise C^p-functions, i.e., there are points where any of the tree parameters has smaller smoothness, then starting from such a point we can build a discontinuity propagation tree similar to what we described in this subsection. Such points are called *secondary discontinuity points*. Similarly, multiple delays can also be handled (see, e.g., [27]).

We note that the extension of the notion of the discontinuity tree from the scalar case to the system case is far from being obvious, and was investigated by Willé and Baker [212, 213]. They associated a so called dependency network of oriented graphs to the discontinuity points, see also [27]. For more discussions on numerical problems related to tracking discontinuity points we refer to [211,214].

Tracking discontinuities can be computationally expensive, especially when the number of discontinuity points is large. Another typical approach to handle the loss of numerical accuracy due to the presence of derivative discontinuities is the method of *defect control* developed by Enright and Hayashi [67,69] (see also [16,79]). In this method the size of the defect, i.e., $\max\{|u'(t) - f(t, u(t), u(t - \tau(t, u(t))))|: t \in [t_n, t_{n+1}]\}$ is monitored and its size is controlled at each step of the integration.

9.5. *Concluding remarks*

There are several software packages available for solving SD-DDEs. Without completeness we list some of them: ARCHI (Paul [179]), DDE-STRIDEL (Butcher [34]), DDVERK (Enright and Hayashi [68]), DMRODE (Neves [167]), DKLAG6 (Corwin, Sarafyan and Thompson [49]), RADAR5 (Guglielmi and Hairer [90]), SNDDELM (Jackiewicz and Lo [122]), SYSDEL (Karouri and Vaillancourt [125]). We refer to [27,79,197] for further discussion and comparison of available solvers for state-dependent delay equations.

In this section we discussed some problems related to the design and analysis of numerical approximation schemes for state-dependent delay equations. Other important qualitative issues, like stability of numerical methods are not discussed here, we mention [12, 27,140] for studies in this directions. We also mention that there are many topics beyond the scope of this survey, e.g., numerical bifurcation analysis (see [66,145]), boundary value problems (see [18]) or parameter estimation (see [14,100,103,107,164]) for state-dependent delay equations.

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CHAPTER 6

Global Solution Branches and Exact Multiplicity of Solutions for Two Point Boundary Value Problems

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Dedicated to the memory of my father Lev Korman

Contents

HANDBOOK OF DIFFERENTIAL EQUATIONS
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1. Introduction

We consider solutions of the two point boundary value problems

$$u''(x) + \lambda f\big(x, u(x)\big) = 0 \quad \text{for } a < x < b, \qquad u(a) = u(b) = 0, \tag{1.1}$$

depending on a parameter λ. We wish to know how many *exactly* solutions does problem (1.1) have, and how these solutions change with λ. What is the role of the parameter λ? Of course, it could be absorbed into the nonlinearity f. However, as is often the case, it is helpful to have something "extra" in the statement of the problem. Consider for example the problem

$$u''(x) + 4e^{\frac{5u(x)}{5+u(x)}} = 0 \quad \text{for } 0 < x < 1, \qquad u(0) = u(1) = 0. \tag{1.2}$$

Problems of this type come up in combustion theory, referred to as "perturbed Gelfand problem", see, e.g., J. Bebernes and D. Eberly [7]. It will follow from a result we present below that this problem has *exactly* three positive solutions. It appears next to impossible to establish this result directly. We introduce a parameter λ, and consider

$$u''(x) + \lambda e^{\frac{5u(x)}{5+u(x)}} = 0 \quad \text{for } 0 < x < 1, \qquad u(0) = u(1) = 0. \tag{1.3}$$

We now study *curves of solutions*, $u = u(x, \lambda)$. The advantage of this approach is that some parts of the solution curve are easy to understand, and it also becomes clear what are the tougher parts of the solution curve that we need to study—the *turning points*. For example, it is easy to understand the "small" solutions of (1.3), by applying the implicit function theorem (in Banach spaces) in the neighborhood of the trivial solution $\lambda = 0$, $u = 0$. We then continue this curve of solutions for increasing $\lambda > 0$ until a critical solution is reached, i.e. the implicit function theorem is no longer applicable. We show that at the critical solution the Crandall–Rabinowitz Theorem 1.2 (see below) applies. It implies that either the solution curve continues forward in λ through the critical solution, or it just bends back (no secondary bifurcations or other eccentric behaviour is possible). We then show that the global solution curve makes exactly two turns, and the value of $\lambda = 4$ from (1.2) lies between the turns, thus establishing the existence of three solutions. The bifurcation approach, just described, has been developed in the recent years by Y. Li, T. Ouyang, J. Shi and the present author. It applies also to the semilinear elliptic problems for balls in $\mathbb{R}^n$, however in the present paper we restrict to the ODE case (1.1).

The most detailed results are obtained when one considers positive solutions of autonomous problems, i.e. when $f = f(u)$. Since in that case both the length and the position of the interval (a, b) are irrelevant, and since positive solutions are symmetric with respect to the midpoint of the interval, it is convenient to pose the problem on the interval $(-1, 1)$, i.e. we consider

$$u''(x) + \lambda f\big(u(x)\big) = 0 \quad \text{for } -1 < x < 1, \qquad u(-1) = u(1) = 0. \tag{1.4}$$

It turns out that convexity properties of $f(u)$ are important for determining the direction of the turn for solution curves. Accordingly, in the simplest case $f''(u) > 0$ and $f(u) > 0$, for $u > 0$, we can give an exhaustive analysis of the problem. (In case $f''(u) < 0$ and $f(u) > 0$, for $u > 0$, it is easy to prove uniqueness of solutions.) The next case in order of complexity is when $f(u)$ changes concavity exactly once. The prominent case is when $f(u)$ is modelled on a cubic with simple roots:

$$u'' + \lambda(u-a)(u-b)(c-u) = 0 \quad \text{for } -1 < x < 1, \qquad u(-1) = u(1) = 0. \tag{1.5}$$

We assume that $0 \leqslant a < b < c$, since the analysis is easier if some root(s) is negative. We wish to describe how many *exactly* positive solutions does problem (1.5) have for various λ.

This problem was studied in a 1981 paper by J. Smoller and A. Wasserman [57]. They succeeded in solving the problem for $a = 0$, while their proof for $a > 0$ case contained an error. This error was discovered by S.-H. Wang [59], who was able to solve the problem under some restriction on $a > 0$. Both papers used the phase-plane analysis. Then P. Korman, Y. Li and T. Ouyang [30] used bifurcation theory to attack the problem, but again some restrictions were necessary (all of the above mentioned papers covered more general $f(u)$, behaving like cubic). Very recently, P. Korman, Y. Li and T. Ouyang [33], building on their previous work, have given a computer assisted proof for general cubic. It turns out that the set of all positive solutions consists of two curves, with the lower curve monotone in λ, and the upper curve having exactly one turn. The computations in P. Korman, Y. Li and T. Ouyang [30] also showed that the approach in J. Smoller and A. Wasserman [57] could not possibly cover the general cubic. (That approach required a certain integral to be positive, in order to derive a differential inequality for a time map. However, that integral changes sign for some cubics.) In the next section we state the optimal result, and describe the approach taken in [30] and [33].

Another prominent class of problems where $f(u)$ changes concavity exactly once is

$$u'' + \lambda e^{\frac{au}{u+a}} = 0 \quad \text{for } -1 < x < 1, \qquad u(-1) = u(1) = 0, \tag{1.6}$$

from combustion theory. Here a is a second parameter. In case $a = 5$, we have the problem (1.3), discussed above. If $a \leqslant 4$, the problem is easy. In that case the solution curve is monotone, and it continues for all $\lambda > 0$ without any turns. Following some earlier results of K.J. Brown, M.M.A. Ibrahim and R. Shivaji [9] and others (see [60] and [29] for the earlier references), S.-H. Wang [60] has proved existence of a constant a_0, so that for $a > a_0$ the solution curve of (2.21) is exactly S-shaped, i.e. it starts at $\lambda = 0$, $u = 0$, it makes exactly two turns, and then it continues for all $\lambda > 0$ without any more turns. S.-H. Wang [60] gave an approximation of the constant $a_0 \simeq 4.4967$. That paper, as well as all previous ones, used a time map approach. P. Korman and Y. Li [29] have applied the bifurcation approach to the problem. Since bifurcation approach is more general, this opened a way to do other problems. In fact, Y. Du and Y. Lou [14] have used a similar approach, with several additional tricks of their own, to prove that for a ball in two dimensions a similar result holds for sufficiently large a.

P. Korman and Y. Li [29] had also improved the value of the constant to $a_0 \simeq 4.35$, i.e. for $a > a_0$ the solution sets are S-shaped curves. But what about $4 < a < a_0$? S.-H. Wang [60] has conjectured existence of a critical number $\bar{a}$, so that for $a \leqslant \bar{a}$ the solution curve is monotone, while for $a > \bar{a}$ the solution curve is exactly S-shaped (the number a_0, mentioned above, is just an upper bound for $\bar{a}$). Recently, P. Korman, Y. Li and T. Ouyang [33] has given a computer assisted proof of the S.-H. Wang's conjecture. Numerical calculations show that $\bar{a} \simeq 4.07$.

Other topics we discuss using the bifurcation approach involve pitchfork bifurcation and symmetry breaking, sign changing solutions, and the Neumann problem. We also present a recent formula from P. Korman, Y. Li and T. Ouyang [33], which allows one to compute all possible values of $\alpha = u(0)$, at which solution of (1.4), with the maximal value equal to α, is singular.

The case when $f = f(x, u)$ is much harder than the autonomous case. In particular, the time map method does not apply. Bifurcation approach works, but it becomes much more complicated. For example, solutions of the corresponding linearized problem need not be of one sign (an implicit example of that is provided by the Theorem 1.10 in W.-M. Ni and R.D. Nussbaum [46]). In the papers P. Korman and T. Ouyang a class of $f(x, u)$ has been identified, for which the theory of positive solutions is very similar to that for the autonomous case, see, e.g., [34–36]. Further results in this direction have been given in P. Korman, Y. Li and T. Ouyang [30], and P. Korman and J. Shi [40]. Namely, assume that $f \in C^2$ satisfies

$$f(-x, u) = f(x, u) \quad \text{for all } -1 < x < 1 \text{ and } u > 0,$$
$$f_x(x, u) \leqslant 0 \quad \text{for all } 0 < x < 1 \text{ and } u > 0.$$

Under the above conditions any positive solution of (1.1) is an even function, with $u'(x) < 0$ for all $x \in (0, 1]$, see B. Gidas, W.-M. Ni and L. Nirenberg [15]. We show that any solution of the corresponding linearized problem is of one sign, and then outline a number of exact multiplicity results.

Without symmetry assumption on $f(x, u)$ things are even more hard. In Section 4 we present extensions of the previous results in P. Korman and T. Ouyang [38]. The notion of Schwarzian derivative from Complex Analysis turns out to play a role here.

The bifurcation approach is effective for other problems, in addition to the two point problems that we discuss in the present paper. Most notably, similar results were developed for PDE's on a ball or annulus in $\mathbb{R}^n$, see, e.g., P. Korman, Y. Li and T. Ouyang [31] or T. Ouyang and J. Shi [50]. It was also used for systems of equations in P. Korman [22], for fourth order equations in P. Korman [26], and for periodic problems in P. Korman and T. Ouyang [37].

In Section 5 we give a brief review of time map method. Let $u = u(t)$ be solution of the initial value problem,

$$u'' + f(u) = 0, \qquad u(0) = 0, \quad u'(0) = p.$$

Using ballistic analogy, we can interpret this as "shooting" from the ground level, at an angle $p > 0$. Let $T/2$ denote the time it takes for the projectile to reach its maximum amplitude. By symmetry of positive solutions, $T = T(p)$ is then the time when the projectile falls back to the ground, the *time map*. The function $u(t)$ then satisfies the two point Dirichlet problem

$$u'' + f(u) = 0 \quad \text{for } 0 < t < T, \qquad u(0) = u(T) = 0,$$

which by rescaling is equivalent to (1.4). There are two completely different formulas for the same time map $T = T(p)$. The first one is obtained by direct integration, see, e.g., W.S. Loud [43] for an early reference, while the second one was derived by R. Schaaf [53] through a change of variables, which converts the problem into a harmonic oscillator. Both formulas for the time map are nontrivial to use. The first one involves improper integrals, while the second one is highly implicit. ("Name your poison", so to say.) However, both approaches are well developed by now, see the book by R. Schaaf [53], and the papers of S.-H. Wang and his coworkers, of I. Addou, and many other papers, including J. Cheng [10,11], and K.J. Brown et al. [9]. We give an exposition of the second approach, and connect it to the notion of *generalized averages* from P. Korman and Y. Li [28].

In the final Section 6 we discuss numerical computation of solutions of (1.4). Again, the autonomous case is much easier. We describe two efficient ways to compute the solutions, and explain why finite differences (or finite elements) are not appropriate for autonomous problems.

The basic tool for continuation of solutions is the implicit function theorem in Banach spaces. We present it here in the formulation of M.G. Crandall and P.H. Rabinowitz [12], see also L. Nirenberg [45], and A. Ambrosetti and G. Prodi [5].

THEOREM 1.1. *Let X, Λ and Z be Banach spaces, and $f(x,\lambda)$ a continuous mapping of an open set $U \subset X \times \Lambda \to Z$. Assume that f has a Frechet derivative with respect to x, $f_x(x,\lambda)$ which is continuous on U. Assume that*

$$f(x_0, \lambda_0) = 0 \quad \textit{for some } (x_0, \lambda_0) \in U.$$

If $f_x(x_0,\lambda_0)$ is an isomorphism (i.e. $1:1$ and onto) of X onto Z, then there is a ball $B_r(\lambda_0) = \{\lambda\colon \|\lambda - \lambda_0\| < r\}$ and a unique continuous map $x(\lambda): B_r(\lambda_0) \to X$, such that

$$f\bigl(x(\lambda), \lambda\bigr) \equiv 0, \quad x(\lambda_0) = x_0.$$

If f is of class C^p, so is $x(\lambda)$, $p \geqslant 1$.

In the conditions of the above theorem, we refer to (x_0, λ_0) as a regular solution, otherwise we call a solution singular. What happens at a singular solution? (I.e. when $f_x(x_0, \lambda_0)$ is not an isomorphism.) In general, practically anything imaginable may happen, as one can see even for functions of two variables. However, in a lucky case solution will continue through a critical point, either by making a simple turn there, or maybe it even continues forward in λ (the critical point is then like a point of inflection). M.G. Crandall and P.H. Rabinowitz [13] have given conditions for that to occur. The following result is one of our principal tools.

THEOREM 1.2. [13] *Let X and Y be Banach spaces. Let $(\bar{\lambda}, \bar{x}) \in \mathbb{R} \times X$ and let F be a continuously differentiable mapping of an open neighborhood of $(\bar{\lambda}, \bar{x})$ into Y. Let the null-space $N(F_x(\bar{\lambda}, \bar{x})) = \operatorname{span} x_0$ be one-dimensional and* $\operatorname{codim} R(F_x(\bar{\lambda}, \bar{x})) = 1$. *Let $F_\lambda(\bar{\lambda}, \bar{x}) \notin R(F_x(\bar{\lambda}, \bar{x}))$. If Z is a complement of $\operatorname{span} x_0$ in X, then the solutions of $F(\lambda, x) = F(\bar{\lambda}, \bar{x})$ near $(\bar{\lambda}, \bar{x})$ form a curve $(\lambda(s), x(s)) = (\bar{\lambda} + \tau(s), \bar{x} + s x_0 + z(s))$, where $s \to (\tau(s), z(s)) \in \mathbb{R} \times Z$ is a continuously differentiable function near $s = 0$ and* $\tau(0) = \tau'(0) = 0$, $z(0) = z'(0) = 0$.

Except for a brief discussion on p-Laplace equations, we consider only the classical solutions throughout this paper. We shall denote the derivatives of $u(x)$ by either $u'(x)$ or u_x, and mix both notations sometimes to make our discussion more transparent.

Most of the results in the present paper are based on our joint papers with Y. Li, T. Ouyang and J. Shi. Working with these talented colleagues has been a wonderful experience for me, and I wish to thank them for this opportunity. I also wish to thank Professors A. Canada, P. Drabek and A. Fonda for inviting me to write this review paper.

2. Bifurcation theory approach

2.1. *Some general properties of solutions of autonomous problems*

We will consider positive, negative and sign-changing solutions of the Dirichlet problem (for $u = u(x)$)

$$u'' + \lambda f(u) = 0 \quad \text{for } -1 < x < 1, \qquad u(-1) = u(1) = 0, \tag{2.1}$$

depending on a parameter λ. We assume throughout this section that $f(u) \in C^2(\bar{\mathbb{R}}_+)$. We choose to consider the problem on the interval $(-1, 1)$ for convenience (which is related to the symmetry of solutions). By shifting and scaling, we can replace the interval $(-1, 1)$ by any other interval (a, b).

LEMMA 2.1. *Let $\xi \in (-1, 1)$ be any critical point of $u(x)$, i.e. $u'(\xi) = 0$. Then $u(x)$ is symmetric with respect to ξ.*

PROOF. Let $v(x) \equiv u(2\xi - x)$. Then $v(x)$ satisfies the same equation (2.1), and, moreover, $v(\xi) = u(\xi)$ and $v'(\xi) = u'(\xi) = 0$. By uniqueness of initial value problems, $u(x) \equiv v(x)$, and the proof follows. □

LEMMA 2.2. *Solution of* (2.1) *cannot have points of positive minimum, and of negative maximum.*

PROOF. Let us rule out the case of positive minimums, with the other case being similar. Assume on the contrary that there are points of positive minimums, and let ξ be the largest such point. Since $u(\xi) > 0$ and $u(1) = 0$, we can find a point $\eta \in (\xi, 1)$, so that $u(\xi) = u(\eta)$. Observe that $u'(\eta) < 0$. Indeed, if we had $u'(\eta) = 0$, then by the preceding lemma,

η would have to be a point of minimum, contradicting the maximality of ξ. We know that the energy $E(x) = \frac{1}{2}u'(x)^2 + \lambda F(u(x))$ is constant, but by above, $E(\eta) > E(\xi)$, a contradiction. □

We now consider positive solutions of (2.1). It follows from the lemmas above, that any positive solution is an even function, with $u'(x) > 0$ on $(-1, 0)$, and $u'(x) < 0$ on $(0, 1)$. (Of course, by the classical theorem of B. Gidas, W.-M. Ni and L. Nirenberg [15] this result holds for balls in $\mathbb{R}^n$ for any $n \geqslant 1$.) Hence $\alpha \equiv u(0)$ is the maximal value of solution. We show next that it is impossible for two solutions of (2.1) to share the same α.

LEMMA 2.3. *The value of $u(0) = \alpha$ uniquely identifies the solution pair $(\lambda, u(x))$ (i.e. there is at most one λ, with at most one solution $u(x)$, so that $u(0) = \alpha$).*

PROOF. Assume on the contrary that we have two solution pairs $(\lambda, u(x))$ and $(\mu, v(x))$, with $u(0) = v(0) = \alpha$. Clearly, $\lambda \neq \mu$, since otherwise we have a contradiction with uniqueness of initial value problems. (Recall that $u'(0) = v'(0) = 0$.) Then $u(\frac{1}{\sqrt{\lambda}}x)$ and $v(\frac{1}{\sqrt{\mu}}x)$ are both solutions of the same initial value problem

$$u'' + f(u) = 0, \qquad u(0) = \alpha, \quad u'(0) = 0,$$

and hence $u(\frac{1}{\sqrt{\lambda}}x) \equiv v(\frac{1}{\sqrt{\mu}}x)$, but that is impossible, since the first function vanishes at $x = \sqrt{\lambda}$, while the second one at $x = \sqrt{\mu}$. □

Bifurcation theory approach revolves around the study of the linearized equation for (2.1)

$$w'' + \lambda f'\big(u(x)\big)w = 0 \quad \text{for } -1 < x < 1, \qquad w(-1) = w(1) = 0, \tag{2.2}$$

where $u(x)$ is a solution of (2.1). If this problem has a nontrivial solution, we call $u(x)$ a singular solution of (2.1). We say that the solution $u(x)$ is nonsingular, if $w(x) \equiv 0$ is the only solution of (2.2). The following lemma is easy to prove in the autonomous case.

LEMMA 2.4. *Let $u(x)$ be a positive solution of* (2.1), *with*

$$u'(1) < 0. \tag{2.3}$$

If the problem (2.2) *admits a nontrivial solution*, *then it does not change sign*, *i.e. we may assume that $w(x) > 0$ on $(-1, 1)$.*

PROOF. The function $u'(x)$ also satisfies the linear equation (2.2). By the condition (2.3), $u'(x)$ is not a multiple of $w(x)$. Hence its roots are interlaced with those of $w(x)$. If $w(x)$ had a root ξ inside say $(-1, 0)$, then $u'(x)$ would have to vanish on $(-1, \xi)$, which is impossible by the remarks following Lemma 2.2. □

Condition (2.3) will hold for any positive solution, provided that

$$f(0) \geqslant 0, \tag{2.4}$$

see, e.g., p. 107 in M. Renardi and R.C. Rogers [52]. If $f(0) < 0$ it is possible to have $u'(1) = 0$. We shall encounter such a situation later, in connection with symmetry-breaking bifurcation. What we see here is a manifestation of the "divide" between the problems when (2.4) holds, and the case of $f(0) < 0$.

LEMMA 2.5. *If problem* (2.2) *admits nontrivial solutions*, *then the solution set is one dimensional*. *If moreover* $u(x)$ *is a positive solution*, *satisfying* (2.3), *then* $w(x)$ *is an even function*.

PROOF. By uniqueness of initial value problem the value of $w'(1)$ uniquely determines $w(x)$, and hence the null space is one dimensional. Turning to the second claim, if $u(x)$ is positive, then it is even. Hence $w(-x)$ also solves (2.2). Since the null space is one dimensional, $w(-x) = cw(x)$ for some constant c. Evaluating this relation at $x = 0$, we conclude that $c = 1$ (since $w(0) > 0$ by the previous lemma), which is the desired symmetry. □

The following lemma gives a simple condition for positive solutions of (2.1) to be nonsingular.

LEMMA 2.6. *Assume that either*

$$f'(u) > \frac{f(u)}{u} \quad \textit{for all } u > 0,$$

or the opposite inequality holds. *Then the linearized problem* (2.2) *has only the trivial solution*.

PROOF. If we rewrite Eq. (2.1) in the form

$$u'' + \lambda \frac{f(u)}{u} u = 0,$$

and use the Sturm comparison theorem, we conclude that the positive solution $u(x)$ oscillates faster than $w(x)$, and hence it must vanish on $(-1, 1)$, which is impossible. □

Another very simple condition is the following.

LEMMA 2.7. *Assume that*

$$f'(u) < 0 \quad \textit{for all } u > 0.$$

Then the linearized problem (2.2) *has only the trivial solution*.

PROOF. Multiplying Eq. (2.2) by w, and integrating, we conclude that problem (2.2) can have only the trivial solution. □

We shall need the following lemma, which "connects" the solutions of (2.1) and (2.2). It will allow us to verify the crucial condition of the Crandall–Rabinowitz Theorem 1.1 for both positive and sign changing solutions.

LEMMA 2.8. *If problem* (2.2) *admits a nontrivial solution*, *then*

$$\int_{-1}^{1} f(u) w \, \mathrm{d}x = \frac{1}{\lambda} u'(1) w'(1). \tag{2.5}$$

PROOF. The quantity $u''(x)w(x) - u'(x)w'(x)$ is a constant, and hence

$$u''(x)w(x) - u'(x)w'(x) = -u'(1)w'(1).$$

Integrating over $(-1, 1)$ (by parts), we conclude the lemma. □

Very often one is looking for positive solutions of (2.1). A possible reason for this emphasis, is that only positive solutions have a chance to be stable, a property significant for applications. Let us recall the notion of stability. For any solution $u(x)$ of (2.1) let $(\mu, w(x))$ denote the principal eigenpair of the corresponding linearized equation, i.e. $w(x) > 0$ satisfies

$$w'' + \lambda f'(u)w + \mu w = 0 \quad \text{for } -1 < x < 1, \qquad w(-1) = w(1) = 0. \tag{2.6}$$

The solution $u(x)$ of (2.1) is called unstable if $\mu < 0$, it is called stable if $\mu > 0$, and neutrally stable in case $M = 0$. (This is so-called linear stability. It means, roughly, that solutions of the corresponding heat equation, with the initial data near $u(x)$ will tend to $u(x)$, as $t \to \infty$, see the book by D. Henry [16].)

PROPOSITION 1. *Let* $u(x)$ *be a solution of* (2.1) *that changes sign on* $(-1, 1)$. *Then* $u(x)$ *is unstable*.

PROOF. Let $(\mu, w(x))$ denote the principal eigenpair of (2.6). Assume that on the contrary $\mu \geqslant 0$. Since $u(x)$ changes sign, we can find $-1 < x_1 < x_2 < 1$, such that $u'(x_1) = u'(x_2) = 0$ and say $u'(x) < 0$ on (x_1, x_2) (the other case is similar). Observe that $u''(x_1) < 0$ and $u''(x_2) > 0$ ($u'(x)$ satisfies a linear equation, it cannot vanish together with its derivative). Denoting $p(x) = u''(x)w(x) - u'(x)w'(x)$, we have

$$p'(x) = \mu u'(x)w(x) \leqslant 0 \quad \text{for } x \in (x_1, x_2).$$

We see that $p(x)$ is nonincreasing on (x_1, x_2). But, $p(x_1) = u''(x_1)w(x_1) < 0$ and $p(x_2) = u''(x_2)w(x_2) > 0$, a contradiction. □

This result was also proved by R. Schaaf [53]. A similar result for balls in $\mathbb{R}^n$ can be found in C.S. Lin and W.-M. Ni [42].

LEMMA 2.9. *Any two positive solutions of* (2.1) *do not intersect inside* $(-1,1)$ (*i.e. they are strictly ordered on* $(-1,1)$).

PROOF. Let $u(x)$ and $v(x)$ be two intersecting solutions. Since both of them are even functions, they intersect on the half-interval $(0,1)$ as well. Let $0<\xi<\eta<1$ be two consecutive intersection points. If $v(x)>u(x)$ on (ξ,η), then $|u'(\xi)|>|v'(\xi)|$, while $|u'(\eta)|<|v'(\eta)|$. The energy $E(x)=\frac{1}{2}u'(x)^2+\lambda F(u(x))$ is constant for any solution $u(x)$. But at ξ, $u(x)$ has higher energy than $v(x)$, and at η the order is reversed, a contradiction. □

The following result from P. Korman [21] gives a detailed description of the solution shape for large λ. (In [21] we proved this result for balls in $\mathbb{R}^n$.) If for some reason solutions cannot be of that shape, it follows that there are no positive solutions of (2.1) for large λ. Recall that root α of $f(u)$ is called *stable* if $f(\alpha)=0$ and $f'(\alpha)<0$.

THEOREM 2.1. *Let* $u(x,\lambda)$ *be a positive solution of* (2.1), *that exists for all* $\lambda>\bar{\lambda}$, *for some* $\bar{\lambda}>0$. *Assume that either* $\lim_{u\to\infty}\frac{f(u)}{u}=\infty$, *or there is* $u_0>0$, *so that* $f(u)\leqslant 0$ *for* $u\geqslant u_0$. *Then the interval* $(-1,1)$ *can be decomposed into a union of open intervals, whose total length* $=2$, *so that on each such subinterval* $u(x,\lambda)$ *tends to a stable root of* $f(u)$, *as* $\lambda\to\infty$.

EXAMPLE. Assume that $f(u)<0$ for $0<u<\bar{u}$, with some $\bar{u}>0$, $f(u)>0$ for $u>\bar{u}$, and $\lim_{u\to\infty}\frac{f(u)}{u}=\infty$, e.g., $f(u)=u^p-1$, with $p>1$. Then the problem

$$u''+\lambda\bigl(u^p-1\bigr)=0,\quad -1<x<1,\qquad u(-1)=u(1)=0$$

has no positive solution for λ large enough. Indeed, since $f(u)$ has no stable roots, solution cannot exhibit the behaviour described in the above theorem, and hence the solution cannot exist for all large λ.

The bifurcation approach applies also to the quasilinear problems of the type

$$\bigl(\varphi(u')\bigr)'+\lambda f(u)=0\quad\text{for } -1<x<1,\qquad u(-1)=u(1)=0. \tag{2.7}$$

The prominent example is that of p-Laplacian $\varphi(t)=t|t|^{p-2}$, with $p\geqslant 2$. Motivated by this example, we assume that $\varphi(t)\in C^2(\mathbb{R}\setminus\{0\})$ satisfies

$$\varphi'(t)>0\quad\text{for all } t\neq 0. \tag{2.8}$$

We consider weak solutions of (2.7), which are of class $C[-1,1]\cap C^1(-1,1)\cap C^2((-1,1)\setminus\{0\})$. (Since the problem is degenerate elliptic, the value of $u''(0)$ might be infinite.)

Assuming condition (2.8), we will show that the conclusions of Lemmas 2.1, 2.2, 2.4 and 2.5 hold verbatim for problem (2.7). In case $\varphi(t) = t|t|^{p-2}$, Lemmas 2.3 and 2.8 hold too. The proofs are basically the same, but there are some difficulties due to degeneracy. For example, in the proof of Lemma 2.1 it is not apparent that $u(x)$ is symmetric with respect to ξ in case $\xi = 0$. (If $\xi \neq 0$, the proof is as before.) We therefore combine the first two lemmas to assert that the solution has the same shape as before.

LEMMA 2.10. *Assuming condition* (2.8), *any positive solution of* (2.7) *is an even function*, *with* $u'(x) > 0$ *on* $(-1, 0)$, *and* $u'(x) < 0$ *on* $(0, 1)$.

PROOF. We need to adjust the definition of energy. Define $\Phi(z) = \int_0^z t\varphi'(t)\,\mathrm{d}t$. Then the energy $\Phi(u'(x)) + \lambda F(u(x))$ is constant (in case of p-Laplacian, $\frac{p}{p-1}|u'(x)|^p + \lambda F(u(x)) = constant$). Using the energy, we conclude as before that $u(x)$ cannot have any points of minimum inside the interval $(-1, 1)$. Also, since the energy is constant, it follows that

$$u'(-1) = -u'(1). \tag{2.9}$$

To prove the symmetry, we consider $v(x) \equiv u(-x)$. The function $v(x)$ satisfies the same problem (2.7). By (2.9) it has the same initial conditions at $x = 1$ as $u(x)$. Hence $u(x) \equiv v(x)$. And finally, observe that an even function with no interior minimums has the desired shape. □

It is easy to see that Lemma 2.3 holds in case of p-Laplacian. Since we need homogeneity for rescaling, we cannot assert it for the general problem (2.7). Next we consider the linearized problem for (2.7)

$$\begin{aligned} &\big(\varphi'(u')w'\big)' + \lambda f'(u)w = 0 \quad \text{for } -1 < x < 1, \\ &w(-1) = w(1) = 0. \end{aligned} \tag{2.10}$$

LEMMA 2.11. *Assume that condition* (2.3) *holds*. *If problem* (2.10) *admits a nontrivial solution*, *then it does not change sign*, *i.e. we may assume that* $w(x) > 0$ *on* $(-1, 1)$. *Moreover, in the case of p-Laplacian*, *the following generalization of the formula* (2.5) *holds*:

$$\int_{-1}^{1} f(u)w\,\mathrm{d}x = \frac{2}{p\lambda}\varphi'\big(u'(1)\big)u'(1)w'(1). \tag{2.11}$$

PROOF. The proof of the first statement is exactly the same as before. Turning to the other one, we differentiate Eq. (2.10)

$$\big(\varphi'(u')u_x'\big)' + \lambda f'(u)u_x = 0. \tag{2.12}$$

Combining problems (2.10) and (2.12), we have

$$\varphi'\big(u'(x)\big)\big(u''(x)w(x) - u'(x)w'(x)\big) = -\varphi'\big(u'(1)\big)u'(1)w'(1). \tag{2.13}$$

We now integrate over $(-1, 1)$. In the case of p-Laplacian, $\varphi(t) = t|t|^{p-2}$, $\varphi'(t) = (p-1)|t|^{p-2}$, and $t\varphi'(t) = (p-1)\varphi(t)$. Hence the second term on the left is equal to $-(p-1)\int_{-1}^{1} \varphi(u'(x))w'(x)\,\mathrm{d}x$. Integrating this term by parts, we can combine it with the first term of the resulting equation. Finally, we observe that $\varphi'(u'(x))u''(x) = -\lambda f(u(x))$. □

2.2. *Convex nonlinearities*

For convex nonlinearities one can give an exhaustive description of the bifurcation diagrams for problem (2.1), since we are able to show that the solution curve cannot turn more than once. Namely, we assume that $f(u) \in C^2(\overline{\mathbb{R}}_+)$ satisfies

$$f(0) > 0, \quad \text{and} \quad f(u) > 0 \quad \text{for } u > 0, \tag{2.14}$$

$$f''(u) > 0 \quad \text{for } u > 0, \tag{2.15}$$

$$f(u) \geqslant au - b \quad \text{for } u > 0 \text{ and some constants } a > 0 \text{ and } b \geqslant 0. \tag{2.16}$$

THEOREM 2.2. *Problem* (2.1), *under the above conditions, has at most two positive solutions for any λ. Moreover, all positive solutions lie on a unique curve in the* $(\lambda, u(0))$ *plane. This curve begins at the point* $(\lambda = 0, u(0) = 0)$, *and either it tends to infinity at some* $\lambda_0 > 0$, *or else it bends back at some* $\lambda_0 > 0$, *and then continues without any more turns, and tends to infinity at some* $\bar{\lambda}$, $0 \leqslant \bar{\lambda} < \lambda_0$.

PROOF. When $\lambda = 0$ we have a trivial solution $u = 0$. It follows by the implicit function theorem that for small $\lambda > 0$ there is a continuous in λ curve of solutions, passing through $(0, 0)$. We claim that this solution curve cannot be continued indefinitely for all $\lambda > 0$. Assume on the contrary that solutions can be continued as $\lambda \to \infty$. Write problem (2.1) in the corresponding integral form,

$$u(x) = \lambda \int_{-1}^{1} G(x, \xi) f\bigl(u(\xi)\bigr)\,\mathrm{d}\xi, \tag{2.17}$$

where $G(x, \xi)$ is the corresponding Green's function. It is well known that $G(x, \xi) > 0$ for all $0 < x, \xi < 1$. Since by our assumptions $f(u)$ is bounded from below by a positive constant, it follows that $u(x)$ will become uniformly large, as $\lambda \to \infty$. Let $\phi_1(x)$ be the principal eigenvalue of $-u''$ on the interval $(-1, 1)$ subject to zero boundary conditions, and λ_1 the corresponding principal eigenvalue (here $\phi_1(x) = \cos\frac{\pi}{2}x$, and $\lambda_1 = \frac{\pi^2}{4}$). Multiplying Eq. (2.1) by $\phi_1(x)$ and integrating, we have

$$\begin{aligned}\lambda_1 \int_{-1}^{1} u\phi_1\,\mathrm{d}x &= -\int_{-1}^{1} u''\phi_1\,\mathrm{d}x = \lambda \int_{-1}^{1} f(u)\phi_1\,\mathrm{d}x \\ &\geqslant \lambda a \int_{-1}^{1} u\phi_1\,\mathrm{d}x - \lambda b \int_{-1}^{1} \phi_1\,\mathrm{d}x.\end{aligned}$$

But this leads to a contradiction, as $\lambda \to \infty$, since $\int_{-1}^{1} u\phi_1 \, dx \to \infty$. ($u(x)$ is a convex function, tending to infinity.)

Let λ_0 denote the supremum of λ, for which the solution curve continues to the right. It is possible that solutions become unbounded as $\lambda \to \lambda_0$ (this is one of the possibilities discussed in the statement of the theorem). So assume that the solutions stay bounded, as $\lambda \to \lambda_0$. Passing to the limit in the integral form of the equation, see (2.17), we conclude the existence of a bounded solution $u_0(x)$, which our solution curve enters at $\lambda = \lambda_0$. Clearly the pair $(\lambda_0, u_0(x))$ is a singular solution of (2.1) (since it cannot be continued to the right in λ). We show next that the Crandall–Rabinowitz Theorem 1.2 applies at $(\lambda_0, u_0(x))$.

We begin by recasting the equation in operator form $F(\lambda, u) = 0$, where the map $F(\lambda, u) : \mathbb{R}_+ \times C^2(-1, 1) \to C(-1, 1)$ is defined by $F(\lambda, u) = u''(x) + \lambda f(u(x))$. Observe that $F_u(\lambda, u)w$ is given by the left-hand side of the linearized equation (2.2). Since the point (λ_0, u_0) is singular, it follows that the linearized equation (2.2) has a nontrivial solution $w(x)$, which is positive by Lemma 2.4. By Lemma 2.5 it follows that the null-space $N(F_u(\lambda_0, u_0)) = span\{w(x)\}$ is one-dimensional, and then $codim R(F_u(\lambda_0, u_0)) = 1$, since $F_u(\lambda_0, u_0)$ is a Fredholm operator of index zero. To apply the Crandall–Rabinowitz Theorem 1.2, it remains to show that $F_\lambda(\lambda_0, u_0) \notin R(F_u(\lambda_0, u_0))$. Assuming the contrary would imply existence of a nontrivial $v(x)$, a solution of

$$v'' + \lambda_0 f_u(\lambda_0, u_0)v = f(\lambda_0, u_0) \quad \text{for } x \in (-1, 1), \qquad v(-1) = v(1) = 0.$$

By the Fredholm alternative (or just multiplying this equation by w, Eq. (2.2) by v, subtracting and integrating)

$$\int_{-1}^{1} f(\lambda_0, u_0) w(x) \, dx = 0,$$

which contradicts Lemma 2.8. (Since $f(0) > 0$, we have $u'(1) < 0$, and also $w'(1) < 0$ by uniqueness of initial value problems. Hence by Lemma 2.8, the above integral is positive.) Hence the Crandall–Rabinowitz Theorem 1.2 applies at $(\lambda_0, u_0(x))$.

Next we compute the direction of bifurcation at the point $(\lambda_0, u_0(x))$. According to the Crandall–Rabinowitz Theorem 1.2, the solution set near the point $(\lambda_0, u_0(x))$ is a curve $\lambda = \lambda(s)$, $u = u(s)$, with $\lambda(0) = \lambda_0$ and $u(0) = u_0(x)$. Observe that $\lambda'(0) = 0$, and $u_s(0) = w(x)$, according to the Crandall–Rabinowitz theorem. If we can show that $\lambda''(0) < 0$, it would follow that the solution curve turns to the left at $(\lambda_0, u_0(x))$, since the function $\lambda(s)$ has a maximum at $s = 0$. To express $\lambda''(s)$, we differentiate Eq. (2.1) twice in s, obtaining

$$u_{ss}'' + \lambda f_u u_{ss} + \lambda f_{uu} u_s^2 + 2\lambda' f_u u_s + \lambda'' f = 0, \qquad u_{ss}(-1) = u_{ss}(1) = 0.$$

Letting $s = 0$, we have by the above remarks

$$u_{ss}'' + \lambda_0 f_u u_{ss} + \lambda_0 f_{uu} w^2 + \lambda''(0) f = 0, \qquad u_{ss}(-1) = u_{ss}(1) = 0. \tag{2.18}$$

Multiplying this equation by w, Eq. (2.2) by u_{ss}, subtracting and integrating

$$\lambda''(0) = -\lambda_0 \frac{\int_{-1}^{1} f_{uu}(\lambda_0, u_0(x)) w^3(x)\,\mathrm{d}x}{\int_{-1}^{1} f(\lambda_0, u_0) w(x)\,\mathrm{d}x} < 0, \tag{2.19}$$

with the last inequality due to convexity of $f(u)$ and Lemma 2.8.

The above analysis is valid not only at the point $(\lambda_0, u_0(x))$, but also at any other critical point. Hence, locally near any critical point, the solution set consists of a parabola-like curve, facing to the left in the $(\lambda, u(0))$ plane. Hence, after bending back at the point $(\lambda_0, u_0(x))$, our solution curve continues for decreasing λ, without ever encountering critical points. (At any critical point, we could not possibly have a parabola-like curve, described above, since our curve has arrived from the right.) Hence, the solution curve continues globally, without any turns, and it has to go to infinity at some $\bar{\lambda} \geqslant 0$. We then have one of the solution curves, described in the theorem, and the maximum value of solutions on this curve, $u(0)$ varies from zero to infinity. Hence all possible maximum values are "taken", and so by Lemma 2.3 this curve exhausts the solution set. □

REMARKS.

1. If, moreover, we have

$$\lim_{u \to \infty} \frac{f(u)}{u} = \infty, \tag{2.20}$$

then the curve cannot go to infinity at a finite λ. Hence, it will bend back at some λ_0, and go to infinity at $\lambda = 0$. Indeed, since $f(u) > 0$, the solutions $u(x, \lambda)$ are concave in x. So that if $u(x, \lambda)$ gets large near some $\lambda = \lambda_1$, it would have to get uniformly large on some interval, say on $(-1/2, 1/2)$. Writing our equation in the form $u'' + \lambda \frac{f(u)}{u} u = 0$, and using the Sturm comparison theorem, we conclude that the positive solution $u(x)$ has to vanish on $(-1/2, 1/2)$, which is impossible.
2. One can show that the solutions on the lower branch are increasing in λ, for all $x \in (-1, 1)$, see P. Korman and T. Ouyang, [34,35]. (On the upper branch this is no longer true, but the maximal value $u(0)$ is increasing as we trace the branch, i.e. it is decreasing in λ.)
3. Our assumptions did not require for $f(u)$ to be increasing.
4. All three possibilities, mentioned in the theorem, can actually occur, see Fig. 1 for the results of numerical computations. Observe that in the second and third cases $f(u)$ is asymptotically linear, and bifurcation from infinity happens.

2.3. *S-shaped solution curves*

We saw in the previous section that solution curves are relatively simple for convex $f(u)$. If $f(u)$ changes concavity, then the solution curve may admit more than one turn. One

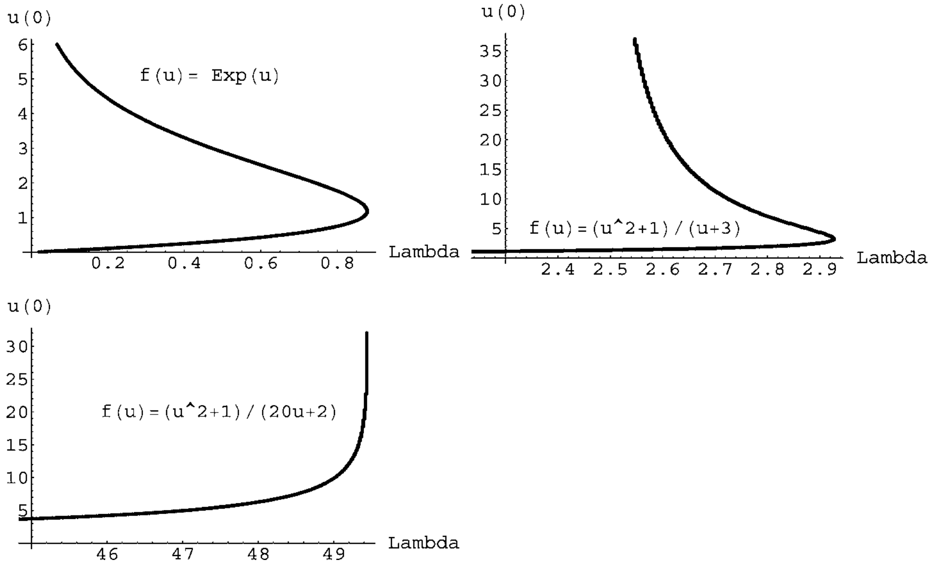

Fig. 1. Three types of solution curves for convex $f(u)$.

prominent nonlinearity, with change in concavity, is connected to combustion theory, see the nice book of J. Bebernes and D. Eberly [7]. Namely, we consider the problem

$$u'' + \lambda e^{\frac{au}{u+a}} = 0 \quad \text{for } -1 < x < 1, \qquad u(-1) = u(1) = 0, \tag{2.21}$$

where a is a constant (this problem is referred to as "the perturbed Gelfand problem" in [7]). In case $a \leqslant 4$, the problem is easy. In that case $uf'(u) - f(u) < 0$ for all $u > 0$, and hence all positive solutions are nonsingular. This means that the solution curve is monotone, i.e. it continues for all $\lambda > 0$ without any turns. S.-H. Wang [59] has proved existence of a constant a_0, so that for $a > a_0$ the solution curve of (2.21) is exactly S-shaped, i.e. it starts at $\lambda = 0$, $u = 0$, it makes exactly two turns, and then it continues for all $\lambda > 0$ without any more turns. S.-H. Wang [59] gave an approximation of the constant $a_0 \simeq 4.4967$. That paper, as well as all previous ones, used a time map approach. P. Korman and Y. Li [28] have applied the bifurcation approach to the problem to show the exactness of the S-shaped curves, and they also improved the value of the constant to $a_0 \simeq 4.35$, i.e. for $a > a_0$ the solution sets are S-shaped curves. S.-H. Wang [59] has conjectured existence of critical number $\bar{a}$, so that for $a \leqslant \bar{a}$ the solution curve is monotone, while for $a > \bar{a}$ the solution curve is exactly S-shaped. Recently, P. Korman, Y. Li and T. Ouyang [33] has given a computer assisted proof of the S.-H. Wang's conjecture. Numerical calculations show that $\bar{a} \simeq 4.07$.

We are going to discuss the S-shaped curves, mostly following P. Korman and Y. Li [29]. However, in that paper time maps were still used at one point. Subsequently, in P. Korman and J. Shi [40] an argument not using time maps was given. Next we present this result, dealing with instability of solutions (it also turned out to be of independent interest), after we recall the notion of stability. For any solution $u(x)$ of (2.1) let $(\mu, w(x))$ denote the principal eigenpair of the corresponding linearized equation, i.e. $w(x) > 0$ satisfies

$$w'' + \lambda f'(u)w + \mu w = 0 \quad \text{for } -1 < x < 1, \qquad w(-1) = w(1) = 0. \tag{2.22}$$

The solution $u(x)$ of (2.1) is called unstable if $\mu < 0$, otherwise it is stable.

Let $F(u) = \int_0^u f(t)\,\mathrm{d}t$, $h(u) = 2F(u) - uf(u)$. The instability result from P. Korman and J. Shi [40] is

THEOREM 2.3. *Assume that* $f \in C^1[0, \infty)$, $f(0) > 0$, *and for some* $\alpha > \beta > 0$ *we have*:

$$h'(u) \geqslant 0 \quad \textit{for } 0 < u < \beta, \qquad h'(u) \leqslant 0 \quad \textit{for } \beta < u < \alpha, \tag{2.23}$$

$$h(\alpha) \leqslant 0. \tag{2.24}$$

Then the solution of (2.1) *with* $u(0) = \alpha$ *is unstable*, *if it exists*.

PROOF. We have $h(0) = 0$, $h'(u) = f(u) - uf'(u)$, $h'(0) = f(0) > 0$. It follows from our conditions that $h(u)$ is unimodular on $[0, \alpha]$, and it takes its positive maximum at $u = \beta$. Define $x_0 \in (0, 1)$ by $u(x_0) = \beta$. We then conclude

$$\begin{aligned} &f\big(u(x)\big) - u(x)f'\big(u(x)\big) \leqslant 0 \quad \text{on } (0, x_0),\\ &f\big(u(x)\big) - u(x)f'\big(u(x)\big) \geqslant 0 \quad \text{on } (x_0, 1). \end{aligned} \tag{2.25}$$

We also remark that by the condition (2.24),

$$\int_0^1 \big[f(u) - uf'(u)\big]u'(x)\,\mathrm{d}x = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}x} h\big(u(x)\big)\,\mathrm{d}x = -h(\alpha) \geqslant 0. \tag{2.26}$$

Assume now that $u(x)$ is stable, i.e. $\mu \geqslant 0$ in (2.22). Without loss of generality, we assume that $w > 0$ in $(-1, 1)$. By the maximum principle, $u'(1) < 0$, so near $x = 1$ we have $-u'(x) > w(x)$. Since $-u'(0) = 0$, while $w(0) > 0$, the functions $w(x)$ and $-u'(x)$ change their order at least once on $(0, 1)$. We claim that the functions $w(x)$ and $-u'(x)$ change their order exactly once on $(0, 1)$. (We ignore the points where these functions merely “touch”.) Observe that $-u'(x)$ satisfies

$$(-u')'' + \lambda f'(u)(-u') = 0 \quad \text{on } (0, 1), \tag{2.27}$$

while $w(x)$ (and any of its positive multiples) is a supersolution of the same equation. Let $x_3 \in (0, 1)$ be the largest point where $w(x)$ and $-u'(x)$ change the order. Assuming the claim to be false, let x_2, with $0 < x_2 < x_3$, be the next point where the order changes. We have $w > -u'$ on (x_2, x_3), and the opposite inequality to the left of x_2. Since $w(0) > -u'(0)$, there is another point $x_1 < x_2$, where the order is changed. We can now find a constant $\gamma > 1$, and a point $x_0 \in (x_1, x_2)$ so that a $\gamma w(x)$, a supersolution of (2.27), touches at x_0 from above a solution $-u'(x)$ of the same equation, a contradiction.

Since the point of changing of order is unique, by scaling of $w(x)$ we can achieve

$$\begin{aligned} -u'(x) &\leqslant w(x) \quad \text{on } (0, x_0), \\ -u'(x) &\geqslant w(x) \quad \text{on } (x_0, 1). \end{aligned} \tag{2.28}$$

Using (2.25), (2.28), and also (2.26), we have

$$\int_0^1 \bigl[f(u) - uf'(u)\bigr] w(x)\,\mathrm{d}x < \int_0^1 \bigl[f(u) - uf'(u)\bigr]\bigl(-u'(x)\bigr)\,\mathrm{d}x \leqslant 0, \tag{2.29}$$

since the integrand on the left is pointwise smaller than the one on the right. On the other hand, multiplying Eq. (2.22) by u, Eq. (2.1) by w, subtracting and integrating over (0, 1), we have

$$\int_0^1 \bigl[f(u) - uf'(u)\bigr] w(x)\,\mathrm{d}x = \frac{\mu}{\lambda}\int_0^1 uw\,\mathrm{d}x \geqslant 0,$$

which contradicts (2.29). So $\mu < 0$. □

We will consider a class of nonlinearities, including $f(u) = \mathrm{e}^{\frac{au}{u+a}}$, so let us list our assumptions. We assume that $f(u) \in C^2[0, \bar{u}]$ for some $0 < \bar{u} \leqslant \infty$, and that it satisfies

$$f(u) > 0 \quad \text{for all } 0 \leqslant u < \bar{u}. \tag{2.30}$$

We assume $f(u)$ to be convex-concave, i.e. there an $\alpha \in (0, \bar{u})$, such that

$$f''(u) > 0 \quad \text{for } u \in (0, \alpha), \qquad f''(u) < 0 \quad \text{for } u \in (\alpha, \bar{u}). \tag{2.31}$$

We define a function $I(u) = f^2(u) - 2F(u)f'(u)$, where as before $F(u) = \int_0^u f(t)\,\mathrm{d}t$. Assume there is a $\beta > \alpha$, such that

$$I(\beta) = f^2(\beta) - 2F(\beta)f'(\beta) \geqslant 0. \tag{2.32}$$

The following lemma has originated from P. Korman, Y. Li and T. Ouyang [30].

LEMMA 2.12. *Assume that $f(u)$ satisfies conditions* (2.30)–(2.32). *Let (λ, u) be any critical point of* (2.1), *such that $u(0) \geqslant \beta$, and let $w(x)$ be the corresponding solution of the linearized problem* (2.2). *Then*

$$\int_0^1 f''\bigl(u(x)\bigr) u_x(x) w^2(x)\,\mathrm{d}x > 0. \tag{2.33}$$

PROOF. We shall derive a convenient expression for the integral in (2.33). Differentiate (2.2)

$$w_x'' + \lambda f'(u) w_x + \lambda f''(u) u_x w = 0. \tag{2.34}$$

Multiplying Eq. (2.34) by w, Eq. (2.2) by w_x, integrating and subtracting, we express

$$\lambda \int_0^1 f''(u) u_x w^2 \, dx = w'^2(1) - \lambda w^2(0) f'\big(u(0)\big). \tag{2.35}$$

By differentiation, we verify that $u''(x)w(x) - u'(x)w'(x)$ is constant for all x, and hence

$$u''(x)w(x) - u'(x)w'(x) = -\lambda w(0) f\big(u(0)\big) \quad \text{for all } x \in [-1, 1]. \tag{2.36}$$

Evaluating this expression at $x = 1$, we obtain

$$w'(1) = \frac{\lambda w(0) f(u(0))}{u'(1)}. \tag{2.37}$$

Multiplying (2.1) by u', and integrating over $(0, 1)$, we have

$$u'^2(1) = 2\lambda F\big(u(0)\big). \tag{2.38}$$

Using (2.38) and (2.37) in (2.35), we finally express

$$\lambda \int_0^1 f''(u) u_x w^2 \, dx = \frac{w^2(0)}{2F(\rho)} I(\rho), \tag{2.39}$$

where we denote $\rho = u(0)$. By our assumption, $I(\beta) \geqslant 0$. Since

$$I'(\rho) = -2F(\rho) f''(\rho) > 0 \quad \text{for } \rho \geqslant \beta,$$

we conclude that $I(\rho) > I(\beta) \geqslant 0$, and the lemma follows. □

The following lemma contains the crucial trick, which has originated from P. Korman, Y. Li and T. Ouyang [30]. It says that for convex-concave problems only turns to the right are possible in the (λ, α) plane, once the maximum value of the solution, $u(0)$, has reached a certain level.

LEMMA 2.13. *In the conditions of the preceding Lemma* 2.12, *assume again that* $u(0) \geqslant \beta$, *and* $w(x)$ *the corresponding solution of the linearized problem* (2.2). *Then*

$$\int_0^1 f''\big(u(x)\big) w^3(x) \, dx < 0. \tag{2.40}$$

PROOF. Let $(\lambda, u(x))$ be a critical point of (2.1). Since $u(0) \geqslant \beta > \alpha$, it follows that the function $f''(u(x))$ changes sign exactly once on $(0, 1)$, say at x_0. Then we have

$$f''\big(u(x)\big) < 0 \quad \text{for } x \in (0, x_0), \qquad f''\big(u(x)\big) > 0 \quad \text{for } x \in (x_0, 1). \tag{2.41}$$

We have proved in Theorem 2.3 that the functions $w(x)$ and $-u'(x)$ intersect exactly once on $(0, 1)$. By scaling $w(x)$ we may assume that they intersect at x_0. ($w(x)$ is a solution of a linear problem, and hence it is defined up to a constant multiple.) In view of Lemma 2.12, we then have

$$\int_0^1 f''\big(u(x)\big)w^3(x)\,dx < \int_0^1 f''\big(u(x)\big)w^2(-u_x)\,dx < 0,$$

since by (2.41) the integrand on the right is pointwise greater than the one the left. □

The same approach can be used to prove the following more general theorem, which was implicit in P. Korman, Y. Li and T. Ouyang [30], see also T. Ouyang and J. Shi [50].

THEOREM 2.4 [30].
(i) *Assume that* $f(0) \geqslant 0$, $f''(u) < 0$ *for* $0 < u < u_0$, $f''(u) > 0$ *for* $u > u_0$. *Then only turns to the left are possible on the solution curve.*

(ii) *Assume that* $f(0) \leqslant 0$, $f''(u) > 0$ *for* $0 < u < u_0$, $f''(u) < 0$ *for* $u > u_0$. *Then only turns to the right are possible on the solution curve.*

(Of course, in both cases we conclude existence of at most two positive solutions, with the maximum values lying in the first positive hump of $f(u)$.)

We are now ready for the main result of this section, see P. Korman and Y. Li [29].

THEOREM 2.5. *Assume that* $f(u)$ *satisfies conditions* (2.30) *and* (2.31), *and moreover,*

$$\lim_{u\to\infty} \frac{f(u)}{u} = 0. \tag{2.42}$$

With $h(u) \equiv 2F(u) - uf(u)$, *assume that*

$$h(\alpha) < 0. \tag{2.43}$$

Then the solution set of (2.1) *consists of one curve, which is exactly S-shaped, i.e. it starts at* $\lambda = 0$, $u = 0$, *it makes exactly two turns, and then it continues for all* $\lambda < \infty$, *without any more turns.*

PROOF. By the implicit function theorem there is a curve of positive solutions of (2.1), starting at $\lambda = 0$, $u = 0$. As in Theorem 2.2, this curve continues for increasing λ, until a possible singular solution (λ_0, u_0) is reached, at which point the Crandall–Rabinowitz Theorem 1.2 applies. By (2.19) it follows that only turns to the left are possible if $u(0) < \alpha$, since $f(u)$ is convex for $u \in (0, \alpha)$. Until the first critical point (λ_0, u_0) is reached, the solutions are stable. Indeed, the solution curve starts at $(\lambda = 0, u = 0)$, which is a stable solution (the principal eigenvalue of the corresponding linearized problem $= \frac{\pi^2}{2}$), while any change of stability requires a passage through a singular point. By Theorem 2.3 when

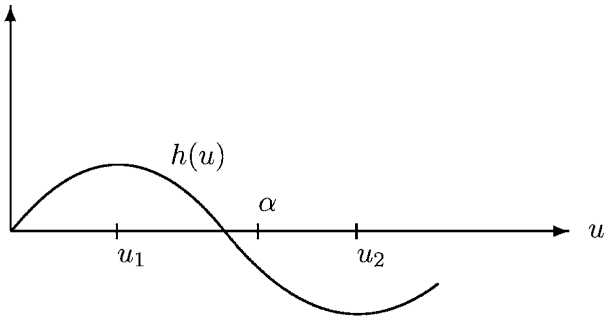

Fig. 2. The function $h(u)$.

$u(0) = \alpha$ the solution is unstable. Hence a singular solution was reached before that, and since only turns to the left are possible when $u(x) < \alpha$, it follows that *exactly one* turn has occurred, and at $u(0) = \alpha$ the solution curve travels to the left.

We now show that the solution curve keeps traveling to the left, until $u(0)$ increases to the level when only turns to the right are possible. For that we take a close look at the function $h(u) = 2F(u) - uf(u)$. Since

$$h'(u) = f(u) - uf'(u), \qquad h''(u) = -uf''(u),$$

it follows that the function $h'(u)$ is decreasing on $(0, \alpha)$ and increasing on (α, ∞). We have $h'(0) = f(0) > 0$, and so $h'(u)$ can have at most two roots. We claim that it has exactly two roots, u_1 and u_2 with $h'(u)$ being positive on $(0, u_1) \cup (u_2, \infty)$, and negative on (u_1, u_2). Indeed, existence of the first root is clear, since $h(0) = 0$ and $h(\alpha) < 0$. As for the second root u_2, if it did not exist, we would have

$$uf'(u) > f(u) \quad \text{for all } u > \alpha. \tag{2.44}$$

Integrating (2.44),

$$f(u) > \frac{f(\alpha)}{\alpha} u \quad \text{for all } u > \alpha,$$

contradicting the assumption (2.42). So that the function $h(u)$ starts with $h(0) = 0$, it is increasing on $(0, u_1)$, decreasing on (u_1, u_2), with absolute minimum at u_2, and then it increases on the interval (u_2, ∞) (see Fig. 2). By Theorem 2.3 the solution curve keeps traveling to the left, while $u(0) \in (\alpha, u_2)$.

We claim that for $u(0) > u_2$ Lemma 2.13 applies. For that we need to check that for $\beta = u_2$ the condition (2.32) holds. Indeed, since $h(u_2) < 0$, we have $f(u_2)u_2 > 2F(u_2)$. Hence

$$I(u_2) = f^2(u_2) - 2F(u_2)f'(u_2) > f^2(u_2) - u_2 f(u_2) f'(u_2) = 0,$$

and the claim follows (observe that $f'(u_2) = \frac{f(u_2)}{u_2} > 0$). By Lemma 2.13, only turns to the right are possible when $u(0) > u_2$.

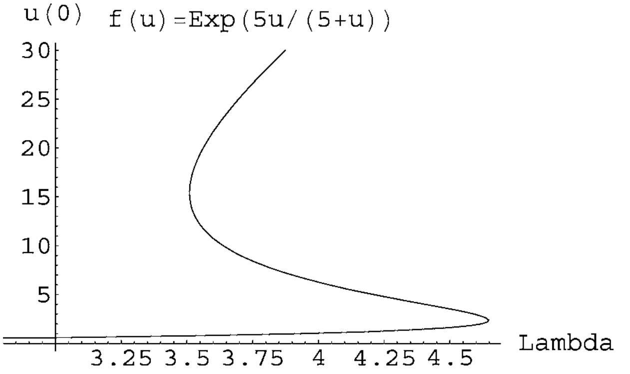

Fig. 3. An S-shaped solution curve.

Let us now put it all together. We have a curve of solutions, which starts at $(\lambda = 0, u = 0)$. As we travel on this curve, $u(0)$ is always increasing. By the time we reach $u(0) = \alpha$ level, the solution curve has made exactly one turn to the left. When $\alpha < u(0) < u_2$ the solution curve travels to the left. When $u(0) > u_2$, the solution curve cannot travel to the left indefinitely, since it is easy to see that solutions are bounded for bounded λ. Hence, the curve must turn to the right. Since for $u(0) > u_2$ only turns to the right are possible, exactly one such turn occurs. It follows that the solution curve is exactly S-shaped. □

In Fig. 3 we give an example of an S-shaped solution curve. Notice that *Mathematica* has drawn the vertical axis around $\lambda = 3$. Also observe that an actual S-shaped solution curve is way different from what most people would draw by hand.

2.4. *Cubic-like nonlinearities*

We again consider the problem

$$u'' + \lambda f(u) = 0, \quad x \in (-1, 1), \qquad u(-1) = u(1) = 0, \tag{2.45}$$

where $f(u)$ behaves like a cubic with three distinct roots, with a model example $f(u) = (u - a)(u - b)(c - u)$. Namely, we assume that the function $f(u) \in C^2(\overline{\mathbb{R}}_+)$ has three nonnegative roots at $0 \leqslant a < b < c$, and

$$\begin{aligned} &f(u) > 0 \quad \text{on } [0, a) \cup (b, c), \qquad f(u) < 0 \quad \text{on } (a, b) \cup (c, \infty), \\ &\int_a^c f(u)\,\mathrm{d}u > 0. \end{aligned} \tag{2.46}$$

Moreover, we assume there is an $\alpha > b$, so that

$$f''(u) > 0 \quad \text{for } 0 \leqslant u < \alpha, \qquad f''(u) < 0 \quad \text{for } u > \alpha. \tag{2.47}$$

This problem was originally studied using time-maps, see J. Smoller and A. Wasserman [57] and S.-H. Wang [59,60]. In P. Korman, Y. Li and T. Ouyang [30] the bifurcation approach was applied. The case of $a = 0$ turned out to be easier for both time-maps and bifurcation approaches, while in case $a > 0$ some restriction on a (a bound from above) was necessary for both approaches.

We shall do the case $a = 0$ first, after two simple lemmas. (I.e. $f(u)$ is modeled on $f(u) = u(u-b)(c-u)$.) Let $\beta \in (b, c)$ be the unique point satisfying

$$f'(\beta) = \frac{f(\beta)}{\beta}. \tag{2.48}$$

(I.e. the point where the straight line through the origin is tangent to the graph of $y = f(u)$.) Clearly, $\beta > \alpha$. The following lemma shows that no turns of the solution curve are possible until the maximum value of the solution reaches a certain level.

LEMMA 2.14. *Assume that $f(u) \in C^2$ satisfies conditions* (2.46) *and* (2.47). *If $u(x)$ is a critical solution of* (2.45) *then*

$$u(0) > \beta. \tag{2.49}$$

PROOF. We claim that

$$f'(u) > \frac{f(u)}{u} \quad \text{for } 0 < u < \beta. \tag{2.50}$$

Indeed, denote $p(u) = uf'(u) - f(u)$. Then $p(0) = p(\beta) = 0$, and $p'(u) = uf''(u)$, which implies that $p(u)$ is increasing on $(0, \alpha)$ and decreasing on (α, β). Then (2.50) follows, and hence by Lemma 2.6 the linearized equation has only the trivial solution, in case $u(0) \leqslant \beta$. □

LEMMA 2.15. *Assume that $f(u) \in C^2$ satisfies conditions* (2.46) *and* (2.47), *with $a = 0$. Assume $u(x)$ is a critical solution of* (2.45), *and let $w(x)$ be solution of the corresponding linearized problem* (2.2). *Then*

$$\int_0^1 f''(u) w^3 \, dx < 0. \tag{2.51}$$

PROOF. We begin by showing that

$$\int_0^1 f''(u) u_x^2 w \, dx = 0. \tag{2.52}$$

We have (using Eqs. (2.45) and (2.2))

$$(u''w' - u'w'')' = \lambda f''(u) u_x^2 w.$$

Integrating over $(0,1)$, and using that $w''(1) = -\lambda f'(u(1))w(1) = 0$ and $u''(1) = -\lambda f(u(1)) = -\lambda f(0) = 0$, we conclude (2.52) (it is here that we use that $f(0) = 0$, i.e. $a = 0$).

We now proceed similarly to Lemma 2.13. Similarly to that lemma, we show that the functions $w(x)$ and $-u'(x)$ intersect exactly once on $(0,1)$. Observe that by Lemma 2.14, we have $u(0) > \beta > \alpha$ at any critical solution $u(x)$. Let $\xi \in (0,1)$ be the point where $u(\xi) = \alpha$. By scaling $w(x)$ we may assume that $w(x)$ and $-u'(x)$ intersect at ξ. Then on the interval $(0,\xi)$, where $f''(u(x))$ is negative, we have $u_x^2 < w^2$, while on the interval $(\xi,1)$, where $f''(u(x))$ is positive, we have $u_x^2 > w^2$. We then have, in view of (2.52),

$$\int_0^1 f''(u)w^3\,\mathrm{d}x < \int_0^1 f''(u)u_x^2 w\,\mathrm{d}x = 0,$$

since the integral on the left is pointwise smaller than the one on the right. □

The following theorem is from P. Korman, Y. Li and T. Ouyang [30].

THEOREM 2.6. *Assume that* $f(u) \in C^2$ *satisfies conditions* (2.46) *and* (2.47), *with* $a = 0$. *Then there is a critical* λ_0 *such that for* $\lambda < \lambda_0$ *problem* (2.45) *has no positive solutions*, *it has exactly one positive solution at* $\lambda = \lambda_0$, *and exactly two positive solutions for* $\lambda > \lambda_0$. *Moreover, all solutions lie on a single solution curve*, *which for* $\lambda > \lambda_0$ *has two branches* $0 < u^-(x,\lambda) < u^+(x,\lambda)$, *with* $u^+(x,\lambda)$ *strictly monotone increasing in* λ, *and* $\lim_{\lambda\to\infty} u^+(x,\lambda) = c$. *On the lower branch*, $u^-(0,\lambda)$ *is monotone decreasing*, $\lim_{\lambda\to\infty} u^-(x,\lambda) = 0$ *for all* $x \neq 0$, *while* $u^-(0,\lambda) > b$ *for all* λ. *We also have* $\lim_{\lambda\to\infty} u^-(0,\lambda) = \theta$, *where* θ *is defined by the relation* $\int_0^\theta f(u)\,\mathrm{d}u = 0$.

PROOF. If $\lambda c > \frac{\pi^2}{4}$, then existence of positive solutions follows by monotone iterations. Indeed, $\phi = c$ is a supersolution of (2.45), while $\psi = \epsilon \cos\frac{\pi}{2}x$ is a subsolution of the same problem, if ϵ is chosen sufficiently small ($\lambda = \frac{\pi^2}{4}$ and $\phi_1 = \cos\frac{\pi}{2}x$ give, of course, the principal eigenpair of the Laplacian on $(-1,1)$). We now continue the positive solution (any one) for decreasing λ. At regular points we use the implicit function theorem for continuation, while the singular point(s) will be discussed below. We cannot continue this curve indefinitely for decreasing λ, since it has no place to go. Indeed, solutions are bounded by c, and so the right hand side of Eq. (2.45) goes to zero, and hence $u(x) \to 0$ as $\lambda \to 0$. But that is impossible, since $f(u)$ is negative near $u = 0$, while at the point of maximum $u''(0) \leqslant 0$. Hence at some critical $\lambda = \lambda_0$ and $u = u_0$ the solution curve cannot be continued further for decreasing λ.

As before, we show that the Crandall–Rabinowitz Theorem 1.2 applies at (λ_0, u_0). According to Lemma 2.15 a turn to the right must occur at this, and any other critical point. Hence, exactly one turn happens, and the solution curve has exactly two branches.

The properties of the solution branches are easy to prove, see [30]. □

In Fig. 4 we give an example for the above theorem. Notice that *Mathematica* has chosen the point $(6,2)$ as the point where the axes intersect.

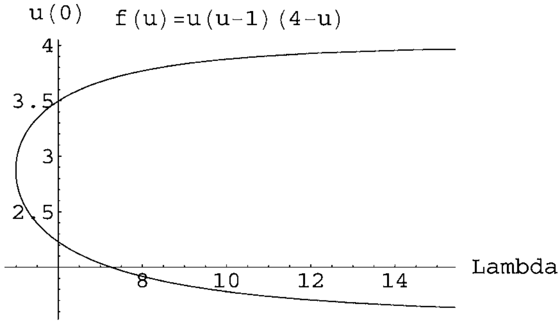

Fig. 4. A parabola-like solution curve, when $a = 0$.

Next we turn to the case when $a > 0$. I.e. we assume that $f(u)$ satisfies conditions (2.46) and (2.47), with the cubic $f(u) = (u - a)(u - b)(c - u)$ being our model example. Similarly to the above, we denote by $\beta \in (b, c)$ the unique point satisfying

$$f'(\beta) = \frac{f(\beta)}{\beta - a}. \tag{2.53}$$

(I.e. the point where the straight line through the point $(a, 0)$ is tangent to the graph of $y = f(u)$.) Clearly, $\beta > \alpha$. The proof of the following lemma is similar to that of Lemma 2.14, and so we omit it (see [33]).

LEMMA 2.16. *Assume that $f(u) \in C^2$ satisfies conditions* (2.46) *and* (2.47). *If $u(x)$ is a critical solution of* (2.45) *then*

$$u(0) > \beta. \tag{2.54}$$

We define a constant $\tau \in (b, c)$ by $f'(\tau) = 0$, i.e. τ is the second root of $f'(u)$. We also recall the function $I(u) = f^2(u) - 2F(u)f'(u)$, defined previously.

LEMMA 2.17. *Assume that $f(u) \in C^2$ satisfies conditions* (2.46) *and* (2.47). *Assume that either*

$$\int_a^\tau f(u)\,\mathrm{d}u \leqslant 0, \tag{2.55}$$

or

$$I(\beta) \geqslant 0. \tag{2.56}$$

If $u(x)$) is a critical solution of (2.45), *and $w(x)$ is a solution of the corresponding linearized problem*, *then*

$$\int_0^1 f''\big(u(x)\big)u'(x)w^2(x)\,\mathrm{d}x > 0. \tag{2.57}$$

PROOF. For any solution of (2.45) we have $\int_a^{u(0)} f(u)\,\mathrm{d}u > 0$ (just multiply the equation by u' and integrate between $x = 0$ and the point $x = \xi$, such that $u(\xi) = a$). So that if (2.55) holds, then $u(0) > \tau$, i.e. $f'(u(0)) < 0$. Then (2.57) follows from formula (2.35) for the integral (2.57). In case condition (2.56) holds, the proof proceeds the same way as in Lemma 2.12. □

REMARK. We can replace condition (2.55) by requiring that $u(0) > \tau$.

The following result was essentially proved in P. Korman, Y. Li and T. Ouyang [33].

THEOREM 2.7. *Assume that* $f(u) \in C^2$ *satisfies conditions* (2.46) *and* (2.47). *Assume either the condition* (2.56) *is satisfied, or else assume that any solution of problem* (2.45), *with* $u(0) \in (\beta, \tau)$ *is noncritical. Then there exists a critical* λ_0, *such that problem* (2.45) *has exactly one positive solution for* $0 < \lambda < \lambda_0$, *exactly two positive solutions at* $\lambda = \lambda_0$, *and exactly three positive solutions for* $\lambda_0 < \lambda < \infty$. *Moreover, all solutions lie on two smooth solution curves. One of the curves, referred to as the lower curve, starts at* ($\lambda = 0, u = 0$), *it is increasing in* λ, *and* $\lim_{\lambda\to\infty} u(x, \lambda) = a$ *for* $x \in (-1, 1)$. *The upper curve is a parabola-like curve with exactly one turn to the right.*

PROOF. The properties of the lower curve are easy to prove. According to the implicit function theorem there is a curve of positive solutions, starting at $\lambda = 0$ and $u = 0$. Since $f'(u) < 0$ when $u < a$, it follows by Lemma 2.7 that solutions are nondegenerate, and hence they can be continued for all $\lambda > 0$. It is easy to see that the solutions on this curve are increasing in λ, and $\lim_{\lambda\to\infty} u(x, \lambda) = a$ for all $x \in (-1, 1)$, see [30].

Turning to the upper curve, recall that critical solutions are possible only if $u(0) > \beta$. If condition (2.56) holds then we have (2.57). The same way as in Lemma 2.12 we show that at any critical point

$$\int_0^1 f''\big(u(x)\big) w^3(x)\,\mathrm{d}x < 0, \tag{2.58}$$

which means that only turns to the right are possible on the upper curve. Similarly to Theorem 2.6 for the $a = 0$ case, we show existence of solutions on the upper curve, and that the upper curve has to turn. Hence, exactly one turn occurs on the upper curve, and its other properties are proved similarly to the $a = 0$ case. In the other case, when (2.55) holds, we know that no critical points are possible, until $u(0) > \tau$. But then again (2.57) holds, which implies (2.58), and we proceed the same way as in the first case. □

The above result shows that either one gets "lucky" at the level $u(0) = \beta$, i.e. condition (2.56) holds, and the above Theorem 2.7 applies, or else the interval (β, τ) is "dangerous", i.e. we need to rule out the possibility of any turns when $\beta < u(0) < \tau$ (since we cannot tell their direction). For that computer assisted proofs are feasible. In fact, in a recent paper P. Korman, Y. Li and T. Ouyang [33] have given three independent computer assisted proofs in case of a cubic. We describe their result next. Let $f(u) = (u - a)(u - b)(c - u)$,

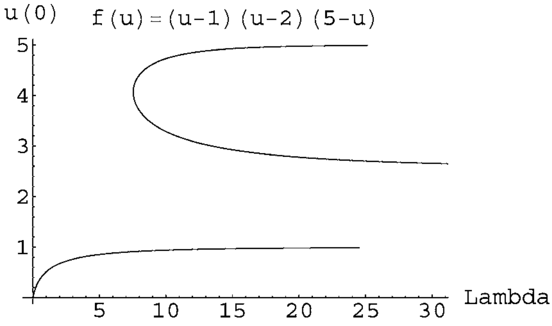

Fig. 5. A two-piece solution curve for a cubic, with $a > 0$.

with $0 < a < b < c$. For problem (2.45) to have a positive solution it is necessary that $\int_a^c f(u)\,du > 0$, i.e.

$$b < \frac{a+c}{2}. \tag{2.59}$$

It was shown in [33] that under the necessary condition (2.59) the above Theorem 2.7 applies, providing an exact multiplicity result for the general cubic. In the next section we present a new tool, used in [33] to give one of the computer assisted proofs.

2.5. *Computing the location of bifurcation*

Assume that for the problem

$$u''(x) + f\big(u(x)\big) = 0, \quad x \in (-1, 1), \qquad u(-1) = u(1) = 0 \tag{2.60}$$

bifurcation occurs at $u(0) = \alpha$, i.e. the corresponding linearized problem

$$w''(x) + f'\big(u(x)\big)w(x) = 0, \quad x \in (-1, 1), \qquad w(-1) = w(1) = 0 \tag{2.61}$$

admits a nontrivial solution. The following result of P. Korman, Y. Li and T. Ouyang [33] provides a way to determine all possible α's at which bifurcation may occur, i.e. the corresponding solution of (2.60) is singular.

THEOREM 2.8. *A positive solution of problem* (2.60) *with the maximal value* $\alpha = u(0)$ *is singular if and only if*

$$G(\alpha) \equiv F(\alpha)^{1/2} \int_0^\alpha \frac{f(\alpha) - f(\tau)}{[F(\alpha) - F(\tau)]^{3/2}}\, d\tau - 2 = 0. \tag{2.62}$$

PROOF. We need to show that the problem (2.61) has a nontrivial solution. By direct verification the function $w(x) = -u'(x)\int_x^1 \frac{1}{u'^2(t)}\,\mathrm{d}t$ satisfies Eq. (2.61). Also $w(1) = 0$. If we also have

$$w'(0) = 0, \tag{2.63}$$

then since $u(x)$ is an even function, the function $w(x)$ is also even (by uniqueness for initial value problems), and hence $w(-1) = 0$, which gives us a nontrivial solution of (2.61). Conversely, every nontrivial solution of (2.61) is an even function, and hence (2.63) is satisfied.

Using Eq. (2.60), we compute

$$w'(x) = f\big(u(x)\big)\int_x^1 \frac{1}{u'^2(t)}\,\mathrm{d}t + \frac{1}{u'(x)}.$$

Since the energy $\frac{u'^2}{2}(x) + F(u(x))$ is constant,

$$\frac{u'^2}{2}(x) + F\big(u(x)\big) = F\big(u(0)\big) = F(\alpha).$$

On the interval (0, 1) we express

$$u'(x) = -\sqrt{2}\sqrt{F(\alpha) - F(u(x))}. \tag{2.64}$$

We use this formula in the integral $\int_x^1 \frac{1}{u'(t)^2}\,\mathrm{d}t$, and then we make a change of variables $t \to s$, by letting $s = u(t)$. We have

$$\begin{aligned} 2^{3/2}\int_x^1 \frac{1}{u'^2(t)}\,\mathrm{d}t &= -\int_x^1 \frac{u'(t)\,\mathrm{d}t}{[F(\alpha) - F(u(t))]^{3/2}} \\ &= -\int_{u(x)}^0 \frac{1}{[F(\alpha) - F(s)]^{3/2}}\,\mathrm{d}s. \end{aligned} \tag{2.65}$$

Using formulas (2.64) and (2.65), we express

$$2^{3/2}w'(x) = \int_0^{u(x)} \frac{f(u(x))}{[F(\alpha) - F(\tau)]^{3/2}}\,\mathrm{d}\tau - \frac{2}{[F(\alpha) - F(u(x))]^{1/2}}. \tag{2.66}$$

If we try to set here $x = 0$, then both terms on the right are infinite. Instead, we observe that

$$\begin{aligned} -\frac{2}{[F(\alpha) - F(u)]^{1/2}} &= -\int_0^u \frac{\mathrm{d}}{\mathrm{d}\tau}\frac{2}{[F(\alpha) - F(\tau)]^{1/2}}\,\mathrm{d}\tau - \frac{2}{F(\alpha)^{1/2}} \\ &= -\int_0^u \frac{f(\tau)}{[F(\alpha) - F(\tau)]^{3/2}}\,\mathrm{d}\tau - \frac{2}{F(\alpha)^{1/2}}. \end{aligned} \tag{2.67}$$

Using (2.67) in (2.66), we obtain

$$2^{3/2}w'(x) = \int_0^{u(x)} \frac{f(u(x)) - f(\tau)}{[F(\alpha) - F(\tau)]^{3/2}}\,\mathrm{d}\tau - \frac{2}{F(\alpha)^{1/2}}. \tag{2.68}$$

The integral on the right is now nonsingular, as we let $x \to 0$. At $x = 0$ we see that (2.63) is equivalent to (2.62). □

In case of a cubic $f(u) = (u - a)(u - b)(c - u)$, P. Korman, Y. Li and T. Ouyang [33] have used formula (2.62) to give a computer assisted proof that there are no turning points in the dangerous region, $u(0) \in (\beta, \tau)$, thus establishing the exact multiplicity result (Theorem 2.7) from the previous section.

2.6. *Computing the direction of bifurcation*

We have seen that at a critical solution $u(x)$ of (2.60) the integral $I = \int_0^1 f''(u)w^3\,\mathrm{d}x$ governs the direction of bifurcation. Also, it is known that in case $I \neq 0$ a critical solution $u(x)$ is *nondegenerate*, i.e. it persists when the equation is perturbed slightly (i.e. the turning points persist under perturbations), see, e.g., [32]. The following result from P. Korman, Y. Li and T. Ouyang [33] allows one to compute the integral I as a function of $\alpha = u(0)$.

THEOREM 2.9. *At any critical solution $u(x)$ of* (2.60), *with* $u(0) = \alpha$,

$$I = c\int_0^{\alpha} f''(u)\left(\int_u^{\alpha} f(s)\,\mathrm{d}s\right)\left(\int_0^{u} \frac{\mathrm{d}s}{(\int_s^{\alpha} f(t)\,\mathrm{d}t)^{3/2}}\right)^3 \mathrm{d}u, \tag{2.69}$$

where $c = \frac{1}{4\sqrt{2}}u'^3(1)w'^3(1) > 0$.

This formula is rather involved, but using *Mathematica* it can be evaluated numerically. In a future paper, with Y. Li and T. Ouyang, we use this result to handle equations modeled on polynomials of arbitrary power.

2.7. *Pitchfork bifurcation and symmetry breaking*

So far for the problem

$$u'' + \lambda f(u) = 0 \quad \text{for } -1 < x < 1, \qquad u(-1) = u(1) = 0 \tag{2.70}$$

we have considered the cases when $f(0) \geqslant 0$. As we have observed earlier, this condition implies that $|u_x(\pm 1, \lambda)| \neq 0$ for any positive solution $u(x, \lambda)$. Since we also know that

$u_x(x, \lambda) < 0$ for all $x \in (0, 1)$, there is no way for a positive solution to become sign-changing, as we vary λ (no interior roots, or zero slope at the boundary are possible). The situation changes drastically in case

$$f(0) < 0. \tag{2.71}$$

A solution may develop a zero slope at the boundary, and become sign-changing. In fact, a pitchfork bifurcation usually happens. In addition to the sign-changing symmetric solution, two symmetry-breaking solution emerge, see P. Korman [20].

Let us consider the problem

$$u''(x) + u^{2k}(x) - \lambda = 0 \quad \text{for } -1 < x < 1, \qquad u(-1) = u(1) = 0, \tag{2.72}$$

which we will relate to the problems of type (2.70) shortly. Here $k \geqslant 1$ is an integer. For $k = 1$ this problem was exhaustively analyzed in J.C. Scovel's Ph.D. thesis [54], and in H.P. McKean and J.C. Scovel [44]. They used explicit integration via elliptic functions, which means that their method does not work for $k > 1$. It turned out that the solution set of (2.72) for $k = 1$ consists of infinitely many identically looking curves. Each curve is a parabola like curve, with pitchfork bifurcation on one of the branches. (I.e. there is exactly one turn, and exactly one point of pitchfork bifurcation on each curve, see Fig. 6.) V. Anuradha and R. Shivaji [6] have studied a related problem. Using the quadrature technique, they showed existence of infinitely many points of bifurcation. In [20] P. Korman had used bifurcation theory to approach problem (2.72), and in particular the case of $f(u) = u^{2k}$, with $k > 1$. We were able to generalize some, but not all, of the results of H.P. McKean and J.C. Scovel [44].

It is well known that at $\lambda = 0$ there exists a unique positive solution of (2.72). This solution is known to be nondegenerate, so that we can continue it for small $\lambda > 0$. Setting $u(x) = \mu v(x)$, with μ determined by the relation $\mu^{2k} = \lambda$, we convert problem (2.72) into (a particular case of the problem (2.70))

$$\begin{aligned} &v''(x) + \lambda\bigl(v^{2k}(x) - 1\bigr) = 0, \quad \text{for } -1 < x < 1,\\ &v(-1) = v(1) = 0, \end{aligned} \tag{2.73}$$

where λ is a new parameter (equal to μ^{2k-1}). With the parameter now in front of the nonlinearity, Lemma 2.8 applies, and hence we can always continue both positive and sign-changing solutions of (2.73) (and also of (2.72). Observe that the curve of positive solutions does not turn for $\lambda > 0$ (for $g(v) \equiv v^{2k} - 1$, we have $vg'(v) > g(v)$ for all $v > 0$). By Theorem 2.1 this curve of positive solutions cannot be continued for all $\lambda > 0$ (the function $g(v) = v^{2k} - 1$ has no "stable" roots, i.e. roots where derivative is negative). By the Sturm's comparison theorem, it is easy to see that positive solutions cannot become unbounded at a finite λ. Hence, solutions on this curve must eventually stop being positive, and the only way this can happen is that $u'(\pm 1) = 0$ at some λ_0 (in view of the symmetry of positive solutions).

We now outline the pitchfork bifurcation analysis for the general problem (2.70), and more details can be found in P. Korman [20]. So suppose problem (2.70) has a curve of positive solutions $u(x,\lambda)$, so that for $\lambda < \lambda_0$ we have $u_x(1,\lambda) < 0$, while at $\lambda = \lambda_0$ we have $u_x(1,\lambda_0) = 0$. The function $u_x(x,\lambda_0)$ is then a solution of the corresponding linearized problem ($u_0 = u(x,\lambda_0)$)

$$w'' + \lambda_0 f'(u_0)w = 0 \quad \text{for } -1 < x < 1, \qquad w(-1) = w(1) = 0. \tag{2.74}$$

The null-space of the linearized problem is one-dimensional (by Lemma 2.5), and it is spanned by the odd function $u_x(x,\lambda_0)$. If we now restrict to the space of even functions, the null-space will be empty, and hence by the implicit function theorem the solution curve $u(x,\lambda)$ continues for $\lambda > \lambda_0$, as sign-changing symmetric (even) solutions. We can compute the tangential direction for this curve at $\lambda = \lambda_0$:

$$u_\lambda(x,\lambda_0) = xu_x(x,\lambda_0). \tag{2.75}$$

Indeed, the function $u_\lambda(x,\lambda_0) - xu_x(x,\lambda_0)$ is an even function, solving the linearized problem (2.74). Hence it must be zero, justifying (2.75). If we let $v = u - u(x,\lambda)$, where $u(x,\lambda)$ is the curve of sign-changing symmetric solutions, then for $\lambda > \lambda_0$ we have a trivial solution $v = 0$. We showed in [20] that the conditions of the Crandall–Rabinowitz theorem on bifurcation from the trivial solutions are satisfied at $\lambda = \lambda_0$, giving rise to a parabola-like curve of symmetry breaking solutions (see also M. Ramaswamy [51], which we used in [20]). Their tangential direction is given by $u_x(x,\lambda_0)$.

One of the reasons we were not able to fully recover the beautiful results of McKean and Scovel [44], is that we could not tell the direction of the pitchfork bifurcation: which way the symmetry breaking solutions bifurcate, toward $\lambda > \lambda_0$ or $\lambda < \lambda_0$? Recently X. Hou, P. Korman and Y. Li [17] has given a computer assisted way (again computer assisted!) to settle this question. Here is their result, which says that the pitchfork opens forward.

THEOREM 2.10. *Consider problem* (2.72), *with* $1 \leqslant k \leqslant 720$. *Let* λ_0 *be the point of pitchfork bifurcation*. (*The value of* λ_0 *was explicitly computed in* [17].) *Then there is a negative* $\bar{\lambda} = \bar{\lambda}(k) < 0$, *so that problem* (2.72) *has exactly two positive solutions for* $\bar{\lambda} < \lambda < 0$, *it has exactly one positive and one negative solution on* $(0,\lambda_0)$. *Moreover, there is a* $\lambda_1(k) > \lambda_0$, *so that problem* (2.72) *has four solutions on* (λ_0,λ_1), *one negative* (*and symmetric*), *one sign-changing and symmetric* (*with* $u(0) > 0$), *and two asymmetric solutions*.

In Fig. 6 we present a picture of pitchfork bifurcation from [17] (produced by X. Hou). We draw $u'(-1)$ as a function of λ. In that figure solid lines denote positive and negative solutions, the dashed line denotes sign-changing symmetric solutions, and the doted lines stand for the symmetry breaking solutions.

2.8. *Sign-changing solutions*

We consider sign-changing solutions of the two point problem

$$u'' + f(u) = 0 \quad \text{for } x \in (0,1), \qquad u(0) = u(1) = 0. \tag{2.76}$$

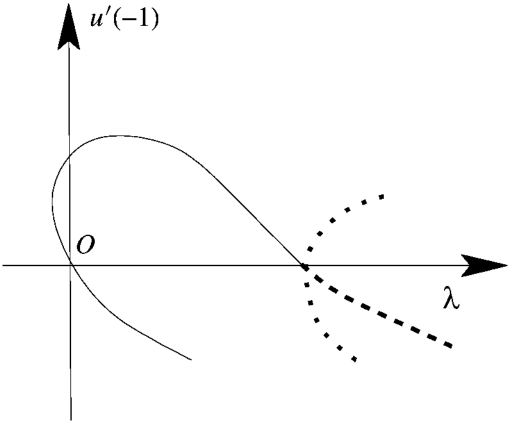

Fig. 6. Pitchfork bifurcation.

Notice that we pose the problem over the interval (0, 1), since sign-changing solutions need not be symmetric. Also we do not have λ in front of $f(u)$ (one can think that λ is absorbed into $f(u)$). Corresponding linearized problem is

$$w'' + f'(u)w = 0 \quad \text{for } x \in (0, 1), \qquad w(0) = w(1) = 0. \tag{2.77}$$

The following result is from P. Korman and T. Ouyang [39].

THEOREM 2.11. *Let* $f \in C^2(\mathbb{R})$, *and assume that either one of the following two inequalities holds*:

$$\frac{f(u)}{u} - f'(u) > 0\ (< 0) \quad \textit{for almost every } u \in \mathbb{R}. \tag{2.78}$$

Then any solution of problem (2.76), *satisfying* $u'(0) \neq 0$, *is nonsingular* (*i.e.* (2.77) *admits only the trivial solution*).

PROOF. Assume on the contrary that problem (2.77) admits a nontrivial solution $w(x)$.

Step 1. We show that the number of roots of u and w inside (0, 1) differs by one. Assume for definiteness that $f'(u) > \frac{f(u)}{u}$ for almost every $u \in \mathbb{R}$. If we regard (2.76) as a linear equation $u'' + \frac{f(u)}{u}u = 0$, then by the Sturm comparison theorem the function $w(x)$ has a root between any two roots of $u(x)$. Since both functions vanish at the endpoints, $x = 0$ and $x = 1$, it follows that w has one more interior root than u.

Step 2. We will show that u and w have the same number of interior roots. This will result in a contradiction, proving the theorem. We denote by n_u the number of interior roots of u, and use the same notation for other functions. The functions w and u' satisfy the same linear equation, and hence their roots are interlaced. Since w vanishes at the endpoints and u' does not, it follows that $n_{u'} = n_w + 1$. Since $n_{u'} = n_u + 1$, it follows that $n_u = n_w$. □

Corresponding to the solution $u(x)$ of (2.76) we may consider an eigenvalue problem

$$\varphi'' + f'(u)\varphi + \mu\varphi = 0 \quad \text{for } x \in (0,1), \qquad \varphi(0) = \varphi(1) = 0. \tag{2.79}$$

The eigenvalues of (2.79) form a sequence $\mu_1 < \mu_2 < \cdots < \mu_n < \cdots$, tending to infinity. The number of negative eigenvalues is called the *Morse index* of $u(x)$. The following theorem is from P. Korman and T. Ouyang [39].

THEOREM 2.12. *Let* $u(x)$ *any solution of problem* (2.76), *with* k *interior roots and satisfying* $u'(0) \neq 0$. *Then the Morse index of* $u(x)$ *is either* k *or* $k+1$. *Moreover, the Morse index equals* k *if the first inequality in* (2.78) *holds, and it equals* $k+1$ *if the second inequality in* (2.78) *holds.*

Similar results for balls in $\mathbb{R}^n$ have been given in J. Shi and J. Wang [56].

2.9. *The Neumann problem*

Consider the Neumann problem

$$u'' + \lambda f(u) = 0 \quad \text{for } 0 < x < 1, \qquad u'(0) = u'(1) = 0. \tag{2.80}$$

We are interested in the solution branches bifurcating off constant solutions. By translation we may assume the constant solution to be zero, i.e. we assume that

$$f(0) = 0, \tag{2.81}$$

and that $f(u) \in C^2(a^-, a^+)$ for some $-\infty \leqslant a^- < 0 < a^+ \leqslant \infty$. We assume that

$$uf(u) > 0 \quad \text{on } \left(a^-, a^+\right). \tag{2.82}$$

We consider solutions of (2.80) such that $u(x) \in (a^-, a^+)$. It suffices to consider only the *increasing* solutions of (2.80), i.e. $u'(x) > 0$ on $(0,1)$, since other solutions can be produced from them by reflection, pasting and scaling. Clearly we have $u(0) < 0 < u(1)$, since $x = 0$ and $x = 1$ are points of minimum and maximum, respectively.

The corresponding linearized problem is

$$w'' + \lambda f'(u)w = 0 \quad \text{for } 0 < x < 1, \qquad w'(0) = w'(1) = 0. \tag{2.83}$$

If this problem has only the trivial solution, then the solution branches bifurcating from zero do not turn. A simple condition for this to happen goes back to Z. Opial [47], see also R. Schaaf [53]. Namely, we assume that either one of the following two inequalities holds:

$$\frac{f(u)}{u} - f'(u) > 0 \ (< 0) \quad \text{for every } u \in \left(a^-, a^+\right) \setminus \{0\}. \tag{2.84}$$

It is an elementary exercise to show that (2.84) will follow if either one of the following two inequalities holds:

$$uf''(u) > 0 \ (< 0) \quad \text{for every } u \in \left(a^-, a^+\right) \setminus \{0\}. \tag{2.85}$$

The following result is of course known, see [47] and [53], although previously it was stated in different terms (involving monotonicity of time maps), and proved by different methods.

THEOREM 2.13. *Assume that conditions* (2.81), (2.82) *and* (2.84) (*or* (2.85)) *hold*, *and* $u(x) \in (a^-, a^+)$ *for all* $x \in (0, 1)$. *Then the linearized problem* (2.83) *admits only the trivial solution*.

PROOF. Assume on the contrary that $w(x)$ is a nontrivial solution of (2.83). Observe that $u'(x)$ satisfies the same equation (2.83), $u'(x) > 0$ for $x \in (0, 1)$ and $u'(0) = u'(1) = 0$. It follows by the Sturm comparison theorem that $w(x)$ has exactly one root on (0, 1); we call it η, i.e. $w(\eta) = 0$. We may assume (by scaling) that $w(0) < 0$ and $w(1) > 0$, and hence $w'(\eta) > 0$. Let ξ denote the unique root of the increasing solution $u(x)$, i.e. $u(\xi) = 0$ and $u'(\xi) > 0$. Writing Eq. (2.80) in the form $u'' + \lambda \frac{f(u)}{u} u = 0$, and combining it with (2.83), we have

$$(u'w - uw')' + \lambda \left[\frac{f(u)}{u} - f'(u) \right] uw = 0. \tag{2.86}$$

We now consider two cases.

Case 1. $\xi \leqslant \eta$. Assume that the first inequality holds in (2.84), i.e. the quantity in the square bracket in (2.86) is positive. We integrate (2.86) over the interval $(\eta, 1)$, where both $u(x)$ and $w(x)$ are positive

$$u(\eta)w'(\eta) + \lambda \int_\eta^1 \left[\frac{f(u)}{u} - f'(u) \right] uw \, dx = 0.$$

We have a contradiction, since both terms on the left are positive.

If the second inequality holds in (2.84), i.e. the quantity in the square bracket in (2.86) is negative, we integrate (2.86) over the interval $(0, \xi)$, where both $u(x)$ and $w(x)$ are negative

$$u(\xi)w'(\xi) + \lambda \int_0^\xi \left[\frac{f(u)}{u} - f'(u) \right] uw \, dx = 0.$$

Again, we have a contradiction, since both terms on the left are negative.

Case 2. $\xi \geqslant \eta$. If the first inequality holds in (2.84), we integrate over $(0, \eta)$, where both $u(x)$ and $w(x)$ are negative

$$-u(\eta)w'(\eta) + \lambda \int_0^\eta \left[\frac{f(u)}{u} - f'(u)\right] uw \, \mathrm{d}x = 0.$$

Both terms on the left are positive, a contradiction. If the second inequality holds in (2.84), we integrate over $(\xi, 1)$, where both $u(x)$ and $w(x)$ are positive

$$-u'(\xi)w(\xi) + \lambda \int_\xi^1 \left[\frac{f(u)}{u} - f'(u)\right] uw \, \mathrm{d}x = 0.$$

Both terms on the left are negative, again we have a contradiction. □

Beyond this simple theorem, we know of only two results on the Neumann problem. The first one is due to R. Schaaf [53]. It dealt with monotonicity of time maps, here we rephrase it in terms of nondegeneracy of solutions.

THEOREM 2.14 [53]. *Assume that the function $f(u)$ is either an $A - B$ or C function on the interval (a^-, a^+). Then the linearized problem* (2.83) *admits only the trivial solution.*

The other one is from P. Korman [23].

THEOREM 2.15. *Assume that $f(u)$ satisfies $f'(u) > 0$ and $f'''(u) < 0$ on the interval $(0, a^+)$, and $f''(u) > 0$ on (a^-, a^+). Then the linearized problem* (2.83) *admits only the trivial solution.*

The last result is not very satisfactory. Its only advantage is that no third order assumptions on $f(u)$ are made on $(a^-, 0)$, while on $(0, a^+)$ such functions are of class $A - B$. R. Schaaf's result is better.

According to the condition (2.85), we can handle the cases when $f(u)$ changes concavity at its root $u = 0$. But what if it keeps the same concavity? We wish to pose the following problem.

PROBLEM. Assume that $f(0) = 0$, and

$$f''(u) > 0 \quad \text{for } u \in \left(a^-, a^+\right).$$

Is it true that any increasing solution of the Neumann problem (2.80), with values in (a^-, a^+), is nondegenerate (i.e. (2.83) has only the trivial solution)?

Using bifurcation approach, we can also treat some nonautonomous problems. For example,

$$u'' + \lambda b(x) f(u) = 0 \quad \text{for } 0 < x < 1, \qquad u'(0) = u'(1) = 0, \tag{2.87}$$

with the given function $b(x)$ being positive and continuous. The corresponding linearized problem is now

$$w'' + \lambda b(x) f'(u)w = 0 \quad \text{for } 0 < x < 1, \qquad w'(0) = w'(1) = 0. \tag{2.88}$$

Formula (2.86) still holds here (with an extra factor of $b(x)$ in front of the square bracket), and hence the arguments of the above theorem can be used unchanged. In particular, we conclude as above that any nontrivial solution of (2.88) cannot vanish exactly once. Unlike the autonomous problem, we cannot yet conclude that $w(x)$ is zero, since we cannot automatically exclude the possibilities that $w(x)$ has no roots, or at least two roots. We need to introduce another condition:

$$f'(u) \geqslant 0 \quad \text{for every } u \in \left(a^-, a^+\right). \tag{2.89}$$

THEOREM 2.16. *Assume that $b(x)$ is positive and continuous on* $[0, 1]$, *and $f(u) \in C^1[a^-, a^+]$ satisfies conditions* (2.81), (2.82), (2.89), *and the first inequality holds in* (2.84). *Let $u(x)$ be an increasing solution of the Neumann problem* (2.80), *satisfying $u(x) \in (a^-, a^+)$ for all $x \in (0, 1)$. Then the linearized problem* (2.88) *admits only the trivial solution.*

PROOF. As we mentioned above, the arguments used in proof of Theorem 2.13 apply here as well. In case the first inequality holds in (2.84), we have proved in Theorem 2.13 that $w(x)$ cannot vanish on either side of ξ, the root of $u(x)$. Hence $w(x)$ keeps the same sign over $(0, 1)$. But then integrating the linearized equation (2.88),

$$\int_0^1 b(x) f'(u) w \, \mathrm{d}x = 0,$$

which is a contradiction, since the integrand is of one sign. □

As an example, the function $f(u) = u - u^3$ satisfies the conditions of this theorem on the interval $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$.

2.10. *Similarity of the solution branches*

We saw in the previous two sections that under the same condition (2.84) we could prove nondegeneracy for both sign-changing solutions, and for the Neumann problem. It turns out that a curve of solutions with an odd number of sign changes is always similar to curves of solutions of Neumann problems. (I.e. both curves have the same number of critical points, with the same direction of turns.) Let us fix the notation, before we state the result. We consider the problem

$$u'' + \lambda f(u) = 0 \quad \text{on } (0, 1), \tag{2.90}$$

subject to either Dirichlet

$$u(0) = u(1) = 0, \tag{2.91}$$

or Neumann

$$u'(0) = u'(1) = 0 \tag{2.92}$$

boundary conditions. We shall consider the solutions as the positive parameter λ varies, and refer to the solution curves as either Dirichlet or Neumann branches, depending on the boundary conditions used. Recall that by Lemma 2.1 any solution of Eq. (2.90) is symmetric with respect to any of its critical points. This implies, in particular, that either minimum or maximum occurs at any critical point. It follows that any solution of Neumann problem is determined by its values on any subinterval $I \subset (0, 1)$, whose end-points are two consecutive critical points of $u(x)$. We can then obtain the solution on the entire interval $(0, 1)$ through reflections and translations. We refer to I, and any other interval uniquely determining the solution through reflections and translations, as a *determining interval*. The interval I, joining two consecutive critical points of $u(x)$, is also a determining interval for the Dirichlet problem. Another determining interval for Dirichlet problem is (ξ, η), where $0 \leqslant \xi < \theta < \eta \leqslant 1$ are three consecutive roots of $u(x)$. This interval contains both positive and negative humps (and all positive (negative) humps are translations of one another).

As we vary λ the number of roots on Dirichlet branches, as well as the number of monotonicity changes on Neumann branches, remain constant. Indeed, by Lemma 2.2 solutions of (2.90) cannot have points of positive maximum and negative minimum, and there is no other mechanism by which extra roots (or monotonicity changes) may be created.

The natural way to distinguish the Dirichlet branches is by the number of interior roots, and the Neumann branches can be identified by the number of changes of monotonicity (both properties remain constant on the solution curves). Any solution of the Dirichlet problem with at least one interior root contains a solution of the Neumann problem on a subinterval of $(0, 1)$. Indeed, just consider the solution between two consecutive critical points. In order for solutions of the Neumann problem to contain in turn a solution of the Dirichlet problem, we need to impose some conditions on $f(u)$. Namely, we assume that

$$f(0) = 0, \tag{2.93}$$

and there exist two constants $-\infty \leqslant m < 0 < M \leqslant \infty$ so that

(f1) $f(u) > 0$ for $u \in (0, M)$,

(f2) $f(u) < 0$ for $u \in (m, 0)$.

LEMMA 2.18. *Under conditions* (2.93), (f1) *and* (f2) *any solution of the Neumann problem for* (2.90), *satisfying*

$$m < u(x) < M \quad \textit{for all } x \tag{2.94}$$

has a root between any two critical points.

PROOF. Follows immediately, by multiplying the equation (2.90) by u', and integrating between any two consecutive critical points. □

DEFINITION. We call two solution branches of (2.90) to be *similar* if for any solution on the either branch there is a determining interval so that by stretching of x, or by reflection $x \to 2a - x$, for some $a \in (0, 1)$, we obtain a solution from the other branch on a (different) determining interval. Clearly, if solution branches are similar then the corresponding solution curves in (λ, u) "plane" have the same shape.

The following result was proved in this form by P. Korman [24], although it can also be found in R. Schaaf [51].

THEOREM 2.17. *All Neumann branches of* (2.90) *are similar, and if* $f(u)$ *satisfies conditions* (2.93), (f1) *and* (f2), *while all solutions satisfy* (2.94), *then the Neumann branches are similar to the Dirichlet ones with an odd number of interior roots* (*and these Dirichlet branches are also all similar*).

PROOF. We begin with Neumann branches. If a Neumann solution changes monotonicity twice, then its increasing part is a reflection of its decreasing part with respect to $x = \frac{1}{2}$. If a Neumann solution changes monotonicity n times, then all critical points occur at i/n, $i = 1, \dots, n-1$, and the graphs of solution on all intervals where it is increasing (decreasing) are translations of one another. Since an interval connecting any two critical points is a determining interval, the equivalence of the Neumann branches follows (via rescaling). □

If a Dirichlet solution has $2k - 1$ interior roots, it has k identical positive humps and k identical negative humps. Assume for definiteness that solution starts with a negative hump, followed by a positive one, and so on. If ξ is the first point of (negative) minimum of $u(x)$, then the first interior root occurs at 2ξ. If $2\xi + \eta$ is the point of the first (positive) maximum, then the second interior root occurs at $2\xi + 2\eta$. The last critical point, a positive maximum, occurs at $1 - \eta$. Observe that $k(2\xi + 2\eta) = 1$, i.e. $\xi + \eta = \frac{1}{2k}$. So while both ξ and η vary with λ, $u(x)$ solves the Neumann problem on the interval $(\xi, 1 - \eta)$, and this interval has a *fixed* length of

$$1 - \eta - \xi = \frac{2k - 1}{2k}.$$

So that any Dirichlet solution curve "carries" inside it a solution of a Neumann problem on a fixed interval (which can be made to be (0, 1) by rescaling), and hence the Dirichlet branch cannot have any more complexity (like extra turns) than any Neumann branch.

Conversely, consider the Neumann problem with $2k + 1$ changes of monotonicity. Assume for definiteness that $u(0) < 0$. Then $u(1) > 0$. Assume that $\xi = \xi(\lambda)$ is the smallest interior root, and $1 - \eta$ is the largest one, $\eta = \eta(\lambda)$. On the interval (0, 1) we then have $2k + 1$ negative half-humps, each of width ξ, and $2k + 1$ positive ones, each of width η. So that $\xi + \eta = \frac{1}{2k+1}$. On the interval $(\xi, 1 - \eta)$ we have a solution of the Dirichlet problem with $2k - 1$ interior roots, and the length of this interval is

$$1 - \eta - \xi = \frac{2k}{2k + 1},$$

which does not vary with λ. So that any Neumann branch "carries" inside it a solution of a Dirichlet problem on a fixed interval, and hence the Neumann branch cannot have any more complexity than the corresponding Dirichlet branch with an odd number of interior zeroes.

Finally, the Dirichlet branches with odd number of interior zeroes are all similar, since any two such branches are similar to a pair of Neumann branches, but Neumann branches are all similar.

The Dirichlet branches with even number of interior zeroes may behave differently, as the following example due to R. Schaaf [53] shows.

EXAMPLE [53]. For the problem

$$u'' + \lambda\big(\mathrm{e}^u - 1\big) = 0 \quad \text{on } (0, 1), \qquad u(0) = u(1) = 0$$

the branch bifurcating from the principal eigenvalue does not turn, while all other branches have exactly one turn.

3. A class of symmetric nonlinearities

For the autonomous equation (2.1) both phase-plane analysis and bifurcation theory apply. If we allow explicit dependence of the nonlinearity on x, i.e. consider

$$u'' + \lambda f(x, u) = 0 \quad \text{for } -1 < x < 1, \qquad u(-1) = u(1) = 0, \tag{3.1}$$

then the problem becomes much more complicated. For example, solutions of the corresponding linearized problem need not be of one sign. In the papers P. Korman and T. Ouyang a class of $f(x, u)$ has been identified, for which the theory of positive solutions is very similar to that for the autonomous case, see, e.g., [34–36]. Further results in this direction have been given in P. Korman, Y. Li and T. Ouyang [30], and P. Korman and J. Shi [40]. Namely, we assume that $f \in C^2$ satisfies

$$f(-x, u) = f(x, u) \quad \text{for all } -1 < x < 1 \text{ and } u > 0, \tag{3.2}$$

$$f_x(x, u) \leqslant 0 \quad \text{for all } 0 < x < 1 \text{ and } u > 0. \tag{3.3}$$

Under the above conditions any positive solution of (3.1) is an even function, with $u'(x) < 0$ for all $x \in (0, 1]$, see B. Gidas, W.-M. Ni and L. Nirenberg [15]. (For the one-dimensional problem (3.1) a different proof of the symmetry of solutions is given in P. Korman [18]. It is a little simpler than the moving plane method of [15], and it allows to relax somewhat the condition (3.3).) As before the linearized problem

$$w'' + \lambda f_u(x, u)w = 0 \quad \text{for } -1 < x < 1, \qquad w(-1) = w(1) = 0 \tag{3.4}$$

will be important for the multiplicity results.

LEMMA 3.1 [34]. *Under conditions* (3.2) *and* (3.3) *any nontrivial solution of* (3.4) *is of one sign. Moreover*, $w(x)$ *is an even function*, *and it spans the null set of* (3.4).

PROOF. Assume that $w(x)$ has a root on $[0, 1)$ (the case when $w(x)$ vanishes on $(-1, 0]$ is similar). We may assume (taking $-w$ if necessary) that there is a subinterval (x_1, x_2), $0 \leqslant x_1 < x_2 \leqslant 1$, so that $w(x) > 0$ on (x_1, x_2), and $w(x_1) = w(x_2) = 0$. Integrating the relation $[u'w' - u''w]' = \lambda f_x w$ over (x_1, x_2),

$$u'(x_2)w'(x_2) - u'(x_1)w'(x_1) = \lambda \int_{x_1}^{x_2} f_x w \, \mathrm{d}x.$$

We have a contradiction, since the quantity on the left is positive, while the one on the right is nonpositive.

The null set of (3.4) is one dimensional, since it can be parameterized by $w'(1)$. To prove that $w(x)$ is even, observe that $w(-x)$ is also a solution of (3.1), and hence $w(-x) = cw(x)$ for some constant c (since the null set of (3.4) is one dimensional). Evaluating this at $x = 0$, we conclude that $c = 1$ (since $w(0) > 0$), and the claim follows. □

The next lemma shows that the Crandall–Rabinowitz Theorem 1.2 applies at any critical solution.

LEMMA 3.2. *Under conditions* (3.2) *and* (3.3) *let* $u(x)$ *be a critical solution of* (3.1), *and* $w(x)$ *a solution of the corresponding linearized problem. Then we have*

$$\int_0^1 f(x, u)w \, \mathrm{d}x > \frac{1}{2\lambda} u'(1)w'(1) > 0. \tag{3.5}$$

PROOF. By the preceding lemma we may assume that $w(x) > 0$. We then have

$$(u''w - u'w')' = -\lambda f_x w > 0 \quad \text{for } x > 0.$$

So that the function $u''w - u'w'$ is increasing on $(0, 1)$, and then

$$u''w - u'w' < -u'(1)w'(1) \quad \text{for } x > 0.$$

Integrating this over $(0, 1)$, and expressing u'' from Eq. (3.1), we conclude (3.5). □

The following result from P. Korman and J. Shi [40] is an extension of Lemma 2.3. Unlike the autonomous case, several conditions are now needed.

THEOREM 3.1 See [40]. *In addition to* (3.2) *and* (3.3) *assume that*

$$f(x, u) > 0 \quad \textit{for all } -1 < x < 1 \textit{ and } u > 0. \tag{3.6}$$

Then the set of positive solutions of (3.1) *can be parameterized by their maximum values* $u(0)$. (*I.e.* $u(0)$ *uniquely determines the pair* $(\lambda, u(x))$.)

PROOF. Assume on the contrary $v(x)$ is another solution of (3.1), corresponding to some parameter $\mu \geqslant \lambda$, but $u(0) = v(0)$. The case of $\mu = \lambda$ is not possible in view of uniqueness of initial value problems, so assume that $\mu > \lambda$. Then $v(x)$ is a supersolution of (3.1), i.e.

$$v'' + \lambda f(x, v) < 0 \quad \text{for } -1 < x < 1, \qquad v(-1) = v(1) = 0. \tag{3.7}$$

Since $v''(0) < u''(0)$, it follows that $v(x) < u(x)$ for $x > 0$ small. Let $0 < \xi \leqslant 1$ be the first point where the graphs of $u(x)$ and $v(x)$ intersect (i.e. $v(x) < u(x)$ on $(0, \xi)$). We now multiply Eq. (3.1) by u', and integrate over $(0, \xi)$. Denoting by $x_2(u)$ the inverse function of $u(x)$ on $(0, \xi)$, we have

$$\frac{1}{2}u'^2(\xi) + \int_{u(0)}^{u(\xi)} f\big(x_2(u), u\big)\,\mathrm{d}u = 0. \tag{3.8}$$

Similarly denoting by $x_1(u)$ the inverse function of $v(x)$ on $(0, \xi)$, we have from (3.7)

$$\frac{1}{2}v'^2(\xi) + \int_{u(0)}^{u(\xi)} f\big(x_1(u), u\big)\,\mathrm{d}u > 0. \tag{3.9}$$

Subtracting (3.9) from (3.8), noticing that $x_2(u) > x_1(u)$ for all $u \in (u(\xi), u(0))$, and using condition (3.3), we have

$$\frac{1}{2}\big[u'^2(\xi) - v'^2(\xi)\big] + \int_{u(\xi)}^{u(0)} \big[f\big(x_1(u), u\big) - f\big(x_2(u), u\big)\big]\,\mathrm{d}u < 0. \tag{3.10}$$

Since both terms on the left are positive, we obtain a contradiction. □

Next we consider positive solutions of the boundary value problem

$$u'' + \lambda b(x) f(u) = 0 \quad \text{for } -1 < x < 1, \qquad u(-1) = u(1) = 0. \tag{3.11}$$

We assume that $b(x) \in C^1[-1, 1]$ satisfies $b(x) > 0$ for $x \in [-1, 1]$, and $b(x) = b(-x)$, $b'(x) < 0$ for $x \in (0, 1)$. We also assume that $f(u) > 0$, so that this problem belongs to the class discussed above. For any solutions $u(x)$ let $(\mu, w(x))$ denote the principal eigenpair of the corresponding linearized equation, i.e. we assume that $w(x) > 0$ satisfies

$$\begin{aligned} &w'' + \lambda b(x) f'(u) w + \mu w = 0 \quad \text{for } -1 < x < 1, \\ &w(-1) = w(1) = 0. \end{aligned} \tag{3.12}$$

The following theorem is taken from P. Korman and J. Shi [40].

THEOREM 3.2. *Assume $f \in C^2[0, \infty)$, $f(u) > 0$, $f'(u) > 0$ and $f''(u) > 0$ for all $u > 0$, and for some $\alpha > 0$ condition* (2.43) *is satisfied. Then the solution of* (3.11) *with $u(0) = \alpha$ is unstable if it exists.*

PROOF. In the proof of Theorem 2.3, (2.25) and (2.26) are still true. Assume now that $u(x)$ is stable, i.e. $\mu \geqslant 0$ in (3.12). Then $w(x)$ is a positive solution of the problem

$$w'' + g(x, w) = 0 \quad \text{for } -1 < x < 1, \qquad w(-1) = w(1) = 0, \tag{3.13}$$

with $g(x, w) = \lambda b(x) f'(u(x))w + \mu w$. Since $g(x, w)$ is even in x, and

$$g_x = \lambda b'(x) f'(u)w + \lambda b(x) f''(u)u'w < 0 \quad \text{on } (0, 1),$$

the theorem of B. Gidas, W.-M. Ni and L. Nirenberg [15] applies to (3.13). It follows that $w(x)$ is an even function with $w'(x) < 0$ on $(0, 1)$. Recall that $w(x)$ is determined up to a constant multiple. Since $w(x)$ is decreasing, while $-u'(x)$ is increasing on $(0, 1)$, by scaling $w(x)$ we can achieve (2.28). Using (2.25), (2.28), and also (2.26), we have (2.29).

Since $b(x) > 0$, $b'(x) < 0$ in $(0, 1)$ using (2.25) and (2.29), we have

$$\begin{aligned}
&\int_0^1 b(x)\bigl[f(u) - uf'(u)\bigr]w(x)\,\mathrm{d}x \\
&\quad = \int_0^{x_0} b(x)\bigl[f(u) - uf'(u)\bigr]w(x)\,\mathrm{d}x \\
&\qquad + \int_{x_0}^1 b(x)\bigl[f(u) - uf'(u)\bigr]w(x)\,\mathrm{d}x \\
&\quad < \int_0^{x_0} b(x_0)\bigl[f(u) - uf'(u)\bigr]w(x)\,\mathrm{d}x \\
&\qquad + \int_{x_0}^1 b(x_0)\bigl[f(u) - uf'(u)\bigr]w(x)\,\mathrm{d}x \\
&\quad = b(x_0)\int_0^1 \bigl[f(u) - uf'(u)\bigr]w(x)\,\mathrm{d}x \leqslant 0.
\end{aligned} \tag{3.14}$$

On the other hand, multiplying Eq. (3.12) by u, Eq. (3.1) by w, subtracting and integrating over $(0, 1)$, we have

$$\int_0^1 b(x)\bigl[f(u) - uf'(u)\bigr]w(x)\,\mathrm{d}x = \frac{\mu}{\lambda}\int_0^1 uw\,\mathrm{d}x \geqslant 0. \tag{3.15}$$

We reach a contradiction by combining (3.14) and (3.15). □

As an application we have the following exact multiplicity result from P. Korman and J. Shi [40]. It extends the corresponding result in [34] by not restricting the behavior of $f(u)$ at infinity. Theorem 3.1 above allows us to conclude the uniqueness of the solution curve.

THEOREM 3.3. *We assume that* $b(x) \in C^1[-1, 1]$ *satisfies* $b(x) > 0$ *for* $x \in [-1, 1]$, *and* $b(x) = b(-x)$, $b'(x) < 0$ *for* $x \in (0, 1)$. *Assume* $f \in C^2[0, \infty)$, $f(u) > 0$, $f'(u) > 0$ *and*

$f''(u) > 0$ *for all* $u > 0$, *while* $h(\alpha) \leqslant 0$ *for some* $\alpha > 0$. *Then there exist two constants* $0 \leqslant \bar{\lambda} < \lambda_0$, *so that problem* (3.11) *has no solution for* $\lambda > \lambda_0$, *exactly two solutions for* $\bar{\lambda} < \lambda < \lambda_0$, *and in case* $\bar{\lambda} > 0$ *it has exactly one solution for* $0 < \lambda < \bar{\lambda}$. *Moreover, all solutions lie on a unique smooth solution curve. If we moreover assume that* $\lim_{u\to\infty} \frac{f(u)}{u} = \infty$, *then* $\bar{\lambda} = 0$.

EXAMPLE. Theorem 3.3 applies (with $\bar{\lambda} = 0$) to an example from combustion theory

$$u'' + \lambda b(x)\mathrm{e}^u = 0 \quad \text{for } -1 < x < 1, \qquad u(-1) = u(1) = 0,$$

where $b(x)$ satisfies the above conditions.

P. Korman and T. Ouyang [34] have considered a class of *indefinite* problems

$$\begin{aligned} &u''(x) + \lambda u(x) + h(x)u^p(x) = 0, \quad -1 < x < 1,\\ &u(-1) = u(1) = 0. \end{aligned} \tag{3.16}$$

Here $p > 1$, and λ a real parameter. The given function $h(x)$ is assumed to be even, and it is allowed to change sign on $(-1, 1)$. By using bifurcation analysis, as above, as well as earlier work of T. Ouyang [48,49], it was possible to give an exhaustive description of the set of positive solutions of (3.16).

We denote by $\phi_1 = \cos\frac{\pi}{2}x$, the principal eigenfunction of $-u''$ on $(-1, 1)$, corresponding to the principal eigenvalue $\lambda_1 = \frac{\pi^2}{4}$. We assume that $h(x) \in C^1(-1, 1) \cap C^0[-1, 1]$ is an even function, and moreover,

$$h(0) > 0, \quad \text{and} \quad h'(x) < 0 \quad \text{for } x \in (0, 1), \tag{3.17}$$

$$\int_{-1}^{1} h(x)\phi_1^{p+1}(x)\,\mathrm{d}x < 0. \tag{3.18}$$

(Notice that the last assumption implies that $h(x)$ changes sign.)

THEOREM 3.4 [34]. *Assume that conditions* (3.17) *and* (3.18) *hold for problem* (3.16). *Then there is a critical* λ_0, $\lambda_0 > \lambda_1$, *so that for* $-\infty < \lambda \leqslant \lambda_1$ *problem* (3.16) *has exactly one positive solution*, *it has exactly two positive solutions for* $\lambda_1 < \lambda < \lambda_0$, *exactly one at* $\lambda = \lambda_0$, *and no positive solutions for* $\lambda > \lambda_0$. *Moreover, all positive solutions lie on a unique continuous in* λ *curve*, *which bifurcates from zero at* $\lambda = \lambda_1$ (*to the right*), *it continues without any turns to* $\lambda = \lambda_0$, *at which it turns to the left*, *and then continues without any more turns for all* $-\infty < \lambda \leqslant \lambda_0$. *We also have* $\max_x u(x) \to \infty$ *as* $\lambda \to -\infty$.

As far as we know, this is still the only known exact multiplicity result for indefinite problems.

Let us mention next the cubic problems

$$\begin{aligned} &u'' + \lambda\big(u - a(x)\big)\big(u - b(x)\big)\big(c(x) - u\big) = 0 \quad \text{for } -1 < x < 1,\\ &u(-1) = u(1) = 0, \end{aligned} \tag{3.19}$$

with given even functions $0 \leqslant a(x) \leqslant b(x) \leqslant c(x)$. As we discussed above, for constant a, b and c, the exact multiplicity question has been settled only recently by P. Korman, Y. Li and T. Ouyang [33], via a computer assisted proof. One can expect that under some conditions the same global picture holds for variable coefficients. This was established for several special cases. P. Korman, Y. Li and T. Ouyang [30] have given an exact multiplicity result in case $a = b = 0$. P. Korman and T. Ouyang [36] had done the same in case $a = 0$, and P. Korman and T. Ouyang [38] had given an exact multiplicity result in case when $a > 0$ is a constant.

Next we indicate an extension. Consider a problem with a variable diffusion coefficient

$$\big(a(x)u'\big)' + \lambda f(u) = 0 \quad \text{for } -1 < x < 1, \qquad u(-1) = u(1) = 0. \tag{3.20}$$

We assume that a given function $a(x) \in C^1[-1, 1]$ is even, and it satisfies

$$a(x) > 0 \quad \text{and} \quad xa'(x) \leqslant 0, \quad \text{for } x \in [-1, 1]. \tag{3.21}$$

We perform a change of variables $x \to s$, given by

$$s = \int_0^x \frac{\mathrm{d}t}{a(t)}.$$

If we denote by $s_0 = \int_0^1 \frac{\mathrm{d}t}{a(t)}$, then this transformation gives a one-to-one map of the interval $(-1, 1)$ onto $(-s_0, s_0)$. Moreover, $s > 0$ (< 0) iff $x > 0$ (< 0). The problems (3.20) transforms into

$$\begin{aligned} &u_{ss} + \lambda a\big(x(s)\big) f(u) = 0 \quad \text{for } -s_0 < x < s_0, \\ &u(-s_0) = u(s_0) = 0. \end{aligned} \tag{3.22}$$

Observe that the function $s = s(x)$ is odd, and hence its inverse $x = x(s)$ is also odd, and then $a(x(s))$ is even. In view of (3.21)

$$\frac{\mathrm{d}}{\mathrm{d}s} a\big(x(s)\big) = a'\big(x(s)\big) a\big(x(s)\big) \leqslant 0 \ (\geqslant 0) \quad \text{if } s > 0 \ (< 0).$$

If we now assume that $f(u) > 0$ for $u > 0$, then the problem (3.22) satisfies conditions (3.2) and (3.3). Hence, we can translate our results, in particular Theorem 3.3, to problem (3.20).

4. General nonlinearities

Without the symmetry assumptions on $f(x, u)$ the problem is much harder. We restrict to a subclass of such problems, i.e. we now consider positive solutions of the boundary value problem

$$u'' + \lambda \alpha(x) f(u) = 0 \quad \text{for } a < x < b, \qquad u(a) = u(b) = 0, \tag{4.1}$$

on an arbitrary interval (a, b). We assume that $f(u)$ and $\alpha(x)$ are positive functions of class C^2, i.e.

$$f(u) > 0 \quad \text{for } u > 0, \qquad \alpha(x) > 0 \quad \text{for } x \in [a, b]. \tag{4.2}$$

As before, it will be crucial for bifurcation analysis to prove positivity for the corresponding linearized problem

$$w'' + \lambda\alpha(x) f'(u)w = 0 \quad \text{for } a < x < b, \qquad w(a) = w(b) = 0. \tag{4.3}$$

The following result was proved in P. Korman and T. Ouyang [38], although our exposition here is a little different.

LEMMA 4.1. *In addition to conditions* (4.2), *assume that*

$$\frac{3}{2}\frac{\alpha'^2}{\alpha} - \alpha'' < 0 \quad \textit{for all } x \in (a, b). \tag{4.4}$$

If the linearized problem (4.3) *admits a nontrivial solution*, *then we may assume that* $w(x) > 0$ *on* (a, b).

PROOF. Let $z(x) = g(x)u'(x)$, with $g(x)$ to be chosen shortly. Then $z(x)$ satisfies the equation

$$z'' + \lambda\alpha(x) f'(u)z = g''(x)u'(x) - \lambda\big(2g'(x)\alpha(x) + \alpha'(x)g(x)\big) f.$$

We now chose $g(x) = \alpha(x)^{-1/2}$. Then $2g'(x)\alpha(x) + \alpha'(x)g(x) = 0$, while

$$g''(x) < 0 \quad \text{for } a < x < b, \tag{4.5}$$

in view of the condition (4.4).

Notice that any positive solution of (4.1) is a concave function, and hence it has only one critical point, the point of global maximum. Let x_0 be the point of maximum of $u(x)$. We have

$$z'' + \lambda\alpha(x) f'(u)z = g''(x)u'(x), \tag{4.6}$$

with the right-hand side negative on (a, x_0) and positive on (x_0, b). This will make it impossible for $w(x)$ to vanish inside (a, b). Indeed, if we assume that $w(x)$ vanishes on say (x_0, b), we could find two consecutive roots of $w(x)$, $x_0 \leqslant x_1 < x_2 \leqslant b$ so that $w(x_1) = w(x_2) = 0$, while $w(x) > 0$ on (x_1, x_2). We now multiply the equation (4.6) by $w(x)$, Eq. (4.3) by $z(x)$, subtract and integrate over (x_1, x_2), obtaining

$$-g(x_2)u'(x_2)w'(x_2) + g(x_1)u'(x_1)w'(x_1) = \int_{x_1}^{x_2} g''(x)u'(x)w(x)\,\mathrm{d}x.$$

We have a contradiction, since the quantity on the left is negative, and the integral on the right is positive. □

REMARKS.

1. Recall the Schwarzian derivative from Complex Analysis and Dynamical Systems

$$(Sf)(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2.$$

   If one denotes $A(x) = \int \alpha(x)\,\mathrm{d}x$, then our condition (4.4) says that the Schwarzian derivative of $A(x)$ is positive.
2. Semilinear equations on an annulus in $\mathbb{R}^n$, $n > 2$, can be reduced by a standard change of variables to the problem (4.1), with $\alpha(x) = x^{-2k}$ and $k = 1 + \frac{1}{n-2}$, see, e.g., [19]. One sees that our condition (4.4) just misses this kind of functions. In [19] positivity of $w(x)$ was proved under an extra assumption that the annulus is “thin”.

We shall present a new result on positivity of $w(x)$, after we prove a simple lemma.

LEMMA 4.2. *Assuming conditions* (4.2), *let* x_0 *be the unique point of maximum of the positive solution of* (4.1). *Assume that*

$$\alpha'(x) < 0 \quad \textit{on } (x_0, b). \tag{4.7}$$

If the corresponding linearized problem (4.3) *admits a nontrivial solution* $w(x)$, *then this solution cannot vanish inside* (x_0, b).

PROOF. Assuming the contrary, let γ be the largest root of $w(x)$ on (x_0, b), and assume that $w(x) > 0$ on (γ, b). (The number of roots of $w(x)$ inside (a, b) is at most finite, as follows by the Sturm’s comparison theorem, since both functions $f'(u(x))$ and $\alpha(x)$ are bounded on $[a, b]$, and λ is fixed. Hence, there is a largest root γ.) Differentiate Eq. (4.1)

$$u_x'' + \lambda\alpha(x) f'(u)u_u + \lambda\alpha'(x) f(u) = 0. \tag{4.8}$$

Multiplying Eq. (4.8) by $w(x)$, Eq. (4.3) by $u'(x)$, subtracting and integrating, we have

$$-u'(b)w'(b) + u'(\gamma)w'(\gamma) + \lambda \int_\gamma^b \alpha'(x) f\bigl(u(x)\bigr) w(x)\,\mathrm{d}x = 0.$$

This results in a contradiction, since all terms on the left are negative. □

REMARK. If $\alpha'(x) > 0$ on (a, x_0), then a similar proof shows that $w(x)$ cannot vanish inside (a, x_0).

LEMMA 4.3. *Assume conditions* (4.2) *hold, and in addition assume that*

$$\alpha'(x) < 0 \quad \textit{on } (a, b), \tag{4.9}$$

and

$$2\alpha(x) + x\alpha'(x) > 0 \quad \text{on } (a,b). \tag{4.10}$$

If the linearized problem (4.3) *admits a nontrivial solution, then we may assume that* $w(x) > 0$ *on* (a,b).

PROOF. Let x_0 be the unique point of maximum of the solution $u(x)$. By the previous Lemma 4.2 it follows that $w(x)$ cannot vanish on (x_0,b). Assuming that $w(x)$ vanishes on $(a,x_0]$, let $\gamma \in (a,x_0]$ be the first root of $w(x)$, and we may assume that $w(x) > 0$ on (a,γ). We consider the function $\zeta(x) = x[u'(x)w'(x) + \lambda\alpha(x)f(u(x))w(x)] - u'(x)w(x)$, introduced by M. Tang [58]. One computes

$$\zeta'(x) = \lambda\big[2\alpha(x) + x\alpha'(x)\big]f(u)w. \tag{4.11}$$

Integrating over (a,γ),

$$\gamma u'(\gamma)w'(\gamma) - au'(a)w'(a) = \lambda\int_a^\gamma \big[2\alpha(x) + x\alpha'(x)\big]f(u)w\,\mathrm{d}x.$$

We have a contradiction, since the quantity on the left is negative, while the integral on the right is positive. □

REMARK. Our condition (4.10) again just misses the case of an annulus.

Positivity of $w(x)$ can be used to prove uniqueness and exact multiplicity results. For example, we can prove the following theorem.

THEOREM 4.1. *For problem* (4.1) *assume that conditions* (4.2) *hold, and that either condition* (4.4) *holds, or conditions* (4.9) *and* (4.10) *hold. In addition assume that* $f''(u) > 0$ *for all* $u > 0$, *and* $\lim_{u\to\infty}\frac{f(u)}{u} = \infty$. *Then there is a critical* $\lambda_0 > 0$, *so that problem* (4.1) *has exactly two positive solutions for* $0 < \lambda < \lambda_0$, *exactly one positive solution at* $\lambda = \lambda_0$, *and no positive solutions for* $\lambda > \lambda_0$. *Moreover, all solutions lie on a unique smooth solution curve, which starts at* $(\lambda = 0, u = 0)$, *bends back at* $\lambda = \lambda_0$, *and tends to infinity as* $\lambda \to 0$.

PROOF. The proof is similar to that of Theorem 2.2, except for proving the uniqueness of the solution curve (since the maximum value of the solution no longer identifies that solution). However, if another solution curve existed, one of its ends would have to go through the point $(\lambda = 0, u = 0)$, contradicting the uniqueness of solutions near regular points, which follows by the implicit function theorem. (Since $w > 0$ at the turning point, one of the branches is increasing in λ, i.e. it is decreasing for decreasing λ. By the arguments of [34], or [35], the monotonicity is preserved along the branch, and hence this branch must go into the origin, as $\lambda \to 0$.) □

EXAMPLE. The theorem applies to the problem

$$u'' + \lambda\alpha(x)\mathrm{e}^u = 0, \quad a < x < b, \qquad u(a) = u(b) = 0,$$

if $\alpha(x) > 0$ satisfies either condition (4.4), or conditions (4.9) and (4.10).

5. Time maps

5.1. *There are several different formulas for the time map*

Let $u = u(t)$ be solution of the initial value problem,

$$u'' + f(u) = 0, \qquad u(0) = 0, \quad u'(0) = p.$$

Using ballistic analogy, we can interpret this as "shooting" from the ground level, at an angle $p > 0$. Let $T/2$ denote the time it takes for the projectile to reach its maximum amplitude α, $\alpha = \alpha(p)$. By symmetry of positive solutions, $T = T(p)$ is then the time when the projectile falls back to the ground, the *time map*. Since the energy is constant (as before, $F(u) = \int_0^u f(t)\,\mathrm{d}t$)

$$\frac{1}{2}\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)^2 + F\big(u(t)\big) = F(\alpha) = \frac{1}{2}p^2.$$

Solving this for $\frac{\mathrm{d}t}{\mathrm{d}u}$, and integrating

$$T/2 = \frac{1}{\sqrt{2}}\int_0^\alpha \frac{\mathrm{d}u}{\sqrt{F(\alpha) - F(u)}}, \tag{5.1}$$

which lets us compute $T = T(\alpha)$ (or $T = T(p)$, since $\alpha = \alpha(p)$). This formula has been used extensively for a long time, see, e.g., W.S. Loud [43], T. Laetsch [41], K.J. Brown et al. [9], J. Smoller and A. Wasserman [57], S.-H. Wang [59–61], I. Addou [1,2], I. Addou and S.-H. Wang [3], S.-H. Wang and T.S. Yeh [63], and J. Cheng [10,11]. It is not easy to use this formula. The integral is improper at $u = \alpha$, so that one needs a regularizing substitution before differentiating in α. One regularizing substitution is $u = \alpha\sin\theta$, which gives

$$T/2 = \frac{\alpha}{\sqrt{2}}\int_0^{\pi/2} \frac{\cos\theta\,\mathrm{d}\theta}{\sqrt{F(\alpha) - F(\alpha\sin\theta)}}. \tag{5.2}$$

The integrand is now bounded. Formula (5.2) can be used for numerical computations, as well as for proving theorems. For more information we refer the reader to the above mentioned papers, particularly to the recent papers of S.-H. Wang and his coworkers.

5.2. *Time map formula through global linearization*

We are interested in positive solutions of the two point problem for $u = u(t)$

$$u'' + f(u) = 0, \quad 0 < t < T, \qquad u(0) = u(T) = 0. \tag{5.3}$$

We do not consider the end point T to be fixed, but rather depending on $p = u'(0)$ (or on the maximum value of the solution α, $\alpha = u(T/2)$). To obtain the formula for $T = T(p)$, we begin by transforming (5.3) into the system form

$$\begin{aligned} u' &= y, \\ y' &= -f(u), \end{aligned} \tag{5.4}$$

together with the initial conditions

$$u(0) = 0, \quad y(0) = p. \tag{5.5}$$

Let $F(u) = \int_0^u f(t)\,\mathrm{d}t$. In the linear case when $f(u) = u$, we have $F(u) = \frac{1}{2}u^2$. We now define the function $g(x)$, for $x \geqslant 0$, by

$$F\bigl(g(x)\bigr) = \frac{1}{2}x^2. \tag{5.6}$$

In other words, $g(x) = F^{-1}(\frac{1}{2}x^2)$, and the inverse function F^{-1} is defined, provided we assume throughout this section that $f(u) \in C^2(0,a) \cap C[0,a]$ for some $0 < a \leqslant \infty$, and

$$f(u) > 0 \quad \text{for } u \in (0,a). \tag{5.7}$$

We assume also that

$$\text{either } \ f(0) > 0, \quad \text{or} \quad f(0) = 0 \text{ and } f'(0) > 0. \tag{5.8}$$

Differentiate (5.6)

$$f\bigl(g(x)\bigr)g'(x) = x. \tag{5.9}$$

In (5.4) we let $u = g(x)$, then multiply the second equation by $g'(x)$, and use (5.9)

$$\begin{aligned} g'(x)x' &= y, \\ g'(x)y' &= -f\bigl(g(x)\bigr)g'(x) = -x. \end{aligned} \tag{5.10}$$

We now change the independent variable in (5.10), $t \to \theta$, by solving

$$\frac{\mathrm{d}t}{\mathrm{d}\theta} = g'\bigl(x(t)\bigr), \qquad t(0) = 0. \tag{5.11}$$

Then the system (5.4) is linearized, and problem (5.4), (5.5) transforms into

$$\begin{aligned} &\frac{\mathrm{d}x}{\mathrm{d}\theta} = y, \\ &\frac{\mathrm{d}y}{\mathrm{d}\theta} = -x, \\ &x(0) = 0, \quad y(0) = p. \end{aligned} \tag{5.12}$$

Solution of (5.12) is

$$x = p\sin\theta, \quad y = p\cos\theta.$$

Using this in (5.11) and integrating, we have the formula for the time map

$$T = \int_0^\pi g'(p\sin\theta)\,\mathrm{d}\theta. \tag{5.13}$$

This formula was derived by R. Schaaf [53], and was used by her to obtain a number of uniqueness and multiplicity results.

Separating variables in (5.11) and integrating

$$\int_0^T \frac{f(g(x(t)))}{x(t)}\,\mathrm{d}t = \pi, \tag{5.14}$$

where we have used (5.9) to express g'. From the definition of $g(x)$ we have

$$x = \sqrt{2F(g(x))} = \sqrt{2F(u)},$$

and hence we can rewrite (5.14) as

$$\int_0^T \frac{f(u(t))}{\sqrt{F(u(t))}}\,\mathrm{d}t = \sqrt{2}\pi. \tag{5.15}$$

This formula was derived in a different way by P. Korman and Y. Li [28], where the quantity on the left was referred to as “generalized average” of the solution of (5.3). The reason why this term was chosen is that in case $f(u) = u^3$, this formula gives the average value of the solution: $\int_0^T u(t)\,\mathrm{d}t = \frac{\pi}{\sqrt{2}}$.

REMARKS.

1. We needed the positivity of $f(u)$ so that the inverse function F^{-1} is defined, however there is no need to distinguish between $f(0) = 0$ and $f(0) > 0$ cases for both formulas (5.13) and (5.15). In case $f(0) > 0$ the integral in (5.15) (and in (5.14)) is improper at both end points, however since $u'(0) \neq 0$ and $u'(T) \neq 0$, the integral converges. (For small t, $u(t) \sim u'(0)t$, $F(u(t)) \sim f(0)u(t) \sim f(0)u'(0)t$.)

2. Similarly, in the derivation of the time map formula (5.13), we run into an improper integral in case $f(0) > 0$. Indeed, when solving for $\theta = \theta(t)$ in (5.11), we have $\theta = \int_0^t \frac{\mathrm{d}s}{g'(x(s))}$, which is an improper integral at $s = 0$. However, as we have just seen, it is a convergent integral. Hence the time map formula in (5.13) is valid in both cases $f(0) = 0$, and $f(0) > 0$.
3. Let us collect the properties of the function $g(x)$. We have $g(0) = 0$, $g'(x) = \frac{\sqrt{2F(g(x))}}{f(g(x))} > 0$ for $x > 0$. We also have $g'(0) = 0$ in case $f(0) > 0$, and, by L'Hopital's rule as was observed in R. Schaaf [53], $g'(0) = \frac{1}{\sqrt{f'(0)}}$ in case $f(0) = 0$. Observe that $g'(0)$ is defined, thanks to condition (5.8).

It is sometimes more convenient to express $T = T(\alpha)$, where α is the maximum value of the solution, $\alpha = u(T/2)$. Since the energy $\frac{1}{2}u'^2(x) + F(u(x))$ is constant, it follows that $p = \sqrt{2F(\alpha)}$, and hence

$$T/2 = \int_{\pi/2}^{\pi} g'(p \sin\theta)\,\mathrm{d}\theta = \int_0^{\pi/2} g'\Big(\sqrt{2F(\alpha)}\cos\theta\Big)\,\mathrm{d}\theta. \tag{5.16}$$

$T/2$ is, of course, the time it takes the solution to travel from its maximum to zero.

EXAMPLE. Consider the problem ($u = u(x)$)

$$u'' + \lambda \mathrm{e}^u = 0, \quad x \in (0,1), \qquad u(0) = u(1) = 0. \tag{5.17}$$

By a change of independent variable we convert it to

$$u'' + \mathrm{e}^u = 0, \quad x \in (0,1), \qquad u(0) = u(T) = 0, \tag{5.18}$$

where $T = \sqrt{\lambda}$. Here $f(u) = \mathrm{e}^u$, $F(u) = \mathrm{e}^u - 1$, and $g(x) = \ln(\frac{1}{2}x^2 + 1)$. The integral in (5.16) is then relatively simple, and in fact *Mathematica* gives

$$\sqrt{\lambda}/2 = \sqrt{\frac{2}{\mathrm{e}^\alpha}}\,\mathrm{ArcTanh}\left[\sqrt{\frac{\mathrm{e}^\alpha - 1}{\mathrm{e}^\alpha}}\right]. \tag{5.19}$$

Plotting this formula (with λ along the horizontal axis and α along the vertical one), we obtain the same bifurcation diagram as obtained by standard integration. Computation this way is considerably faster than by integration, and we also observe that here $f(0) > 0$. Formula (5.19) can also be obtained by explicit integration of problem (5.17).

We can proceed similarly for the general case

$$u'' + \lambda f(u) = 0, \quad x \in (0,1), \qquad u(0) = u(1) = 0. \tag{5.20}$$

We do not have a simple formula for $g(x)$ anymore, however from (5.16) (see also (5.9)) we obtain (as before, $T=\sqrt{\lambda}$)

$$\sqrt{\lambda}/2=\int_0^{\pi/2}\frac{\sqrt{2F(\alpha)}\cos\theta}{f(F^{-1}(F(\alpha)\cos^2\theta))}\,\mathrm{d}\theta. \tag{5.21}$$

This formula will provide probably one of the most efficient ways to compute the bifurcation diagrams, once the evaluation of the inverse function F^{-1} is numerically implemented.

Assume now there is an $a>0$, so that $f(a)=0$, and $f(u)>0$ for $u>a$, while no assumptions on the sign of $f(u)$ are made when $u\in(0,a)$. If we denote by $T_1/2$ the time it takes the solution to travel from its maximum to $u(x)=a$, then

$$T_1/2=\int_{\theta_0}^{\pi/2} g'\Big(\sqrt{2F(\alpha)}\cos\theta\Big)\,\mathrm{d}\theta, \quad \text{where } \theta_0=\sin^{-1}\sqrt{\frac{F(a)}{F(\alpha)}}. \tag{5.22}$$

The following theorem we proved in [27].

THEOREM 5.1. *Assume that for some* $0<a<b\leqslant\infty$ *we have*

$$f(u)>0 \quad \textit{for } a<u<b, \tag{5.23}$$

$$f'(u)\int_a^u f(t)\,\mathrm{d}t-\frac{1}{2}f^2(u)>0 \quad \textit{for } a<u<b. \tag{5.24}$$

(*Observe that we implicitly assume that* $f(a)=0$.) *Then the problem*

$$u''+f(u)=0,\quad x\in(0,1),\qquad u(0)=u(1)=0$$

has at most one positive solution, *with* $a<\alpha=u(1/2)<b$.

What is remarkable here is than no assumptions whatsoever are made on $f(u)$ when $u\in(0,a)$. We used generalized averages to prove this result, but it should be possible to obtain it from formula (5.22) too. In fact, there is a similar result in R. Schaaf's book [53] (a little less general than the above theorem). Observe that (5.24) will follow if $f(a)=0$ and $f''(u)>0$ for $a<u<b$. A more general result for p-Laplacian case has been given recently by J. Cheng [10].

We now show how the time map formula gives rise to uniqueness and multiplicity results for the Dirichlet problem. Compute

$$\frac{\mathrm{d}T}{\mathrm{d}p}=\int_0^{\pi} g''(p\sin\theta)\sin\theta\,\mathrm{d}\theta, \tag{5.25}$$

where $g''(x)$ is computed from (5.9) (written in the form $g'(x)=\frac{\sqrt{2F(u)}}{f(u)}$, $u=g(x)$)

$$g''(x)=\frac{f^2-2Ff'}{f^3}(u), \quad \text{with } u=g(x). \tag{5.26}$$

If the time map $T(p)$ is monotone, then clearly the positive solution of the Dirichlet problem (5.3) for any *fixed* T is unique. Hence, we have uniqueness of solutions if either

$$I(u) \equiv f'(u)F(u) - \frac{1}{2}f^2(u) > 0 \quad \text{for almost all } u > 0, \tag{5.27}$$

or the opposite inequality holds. This condition was derived by R. Schaaf [53], and it also follows from the generalized averages in [28].

We observe next that this condition does not add anything to the standard uniqueness condition

$$uf'(u) - f(u) \quad \text{does not change sign for } u > 0. \tag{5.28}$$

I.e. (5.28) holds whenever (5.27) does, and so condition (5.28) is both simpler and more general.

Indeed, we begin by observing

$$\frac{\mathrm{d}^2}{\mathrm{d}u^2}\left(\sqrt{F(u)}\right) = \frac{I(u)}{2F^{3/2}(u)} \equiv J(u),$$

where $J(u)$ has the same sign as $I(u)$. Integrating between some $a > 0$ and $u > 0$,

$$\frac{\mathrm{d}}{\mathrm{d}u}\left(\sqrt{F(u)}\right) = \int_a^u J(\xi)\,\mathrm{d}\xi + c > 0, \tag{5.29}$$

where $c = \frac{f(a)}{2\sqrt{F(a)}} > 0$. Integrating (5.29),

$$\sqrt{F(u)} = cu + c_1 + \int_a^u (u-\xi)J(\xi)\,\mathrm{d}\xi, \tag{5.30}$$

where $c_1 = -\int_0^a \xi J(\xi)\,\mathrm{d}\xi$. From (5.30) we find $F(u)$, and then $f(u)$ and $f'(u)$ by differentiation. We then have

$$uf'(u) - f(u) = 2\sqrt{F(u)}J(u)u + 2\int_0^u \xi J(\xi)\,\mathrm{d}\xi \left(\int_a^u J(\xi)\,\mathrm{d}\xi + c\right).$$

In view of (5.29), the quantity in the bracket is positive, and it follows that if $J(u)$ is positive (negative), so is $uf'(u) - f(u)$.

We now consider the problem

$$u'' + \lambda f(u) = 0, \quad x \in (0,1), \qquad u(0) = u(1) = 0, \tag{5.31}$$

depending on a positive parameter λ. As before, we can convert it to problem (5.3), with $T = \sqrt{\lambda}$. If we can show that $T''(p) > 0$ (or $T''(p) < 0$) for all $p > 0$, it will follow that for any λ there is at most two p's with $T(p) = \sqrt{\lambda}$, i.e. at most two solutions of (5.31). Since

$$T''(p) = \int_0^\pi g'''(p\sin\theta)\sin^2\theta\,\mathrm{d}\theta,$$

it suffices to show that the function $g'''(u)$ keeps the same sign. By formula (1-1-15) in R. Schaaf [53]

$$g'''(u) = -g'(x)\frac{3f'(u)(f^2(u) - 2F(u)f'(u)) + 2F(u)f(u)f''(u)}{f^4(u)},$$
$$\text{with } u = g(x),$$

which led her to the following condition: if

$$3f'(u)\big(f^2(u) - 2F(u)f'(u)\big) + 2F(u)f(u)f''(u) > 0 \text{ (or } < 0),$$
$$\text{for all } u > 0 \tag{5.32}$$

then problem (5.31) has at most two positive solutions. Since condition (5.32) is not easy to verify, R. Schaaf [53] went on to develop her $A - B$ and C conditions, which are sufficient for (5.32) to hold.

Condition (5.32) says that $g'(u)$ is either convex or concave. Working with the generalized averages, P. Korman and Y. Li [28] have shown that the same result is true if $\frac{1}{g'(u)}$ is convex (also in the case $\frac{1}{g'(u)}$ concave, but this possibility is included in the case when $g'(u)$ is concave). This led them to the following condition: if

$$\frac{1}{2}f''(u)F^2(u) + \frac{3}{8}f^3(u) - \frac{3}{4}f(u)f'(u)F(u) > 0 \quad \text{for all } u > 0, \tag{5.33}$$

then problem (5.31) has at most two positive solutions. Observe that this condition is different from (5.32). Conditions (5.32) and (5.33) work in both cases $f(0) = 0$ and $f(0) > 0$. Also, computer algebra can help in verifying these conditions.

EXAMPLE. The function $f(u) = 2 + \mathrm{e}^{-u}\sin u$ satisfies (5.33). This function changes concavity infinitely many times. A straightforward computation, using *Mathematica*, shows that for this function the left-hand side of (5.33) is positive, tending to 10.125 as $u \to \infty$. Hence, problem (5.31) with this $f(u)$ has at most two positive solutions for any $\lambda > 0$.

REMARKS.

1. The time map formula can be also developed for the p-Laplacian case. Actually, even more general case is developed in Section 2.5 of R. Schaaf's book [53].
2. Finally, we mention why we constantly stress that all results about the time map hold in both cases $f(0) = 0$ and $f(0) > 0$. The important book by R. Schaaf [53] treats the $f(0) = 0$ case in Chapter 1, while the case $f(0) > 0$ (and also the case $f(0) < 0$) is postponed to Chapter 3. Some readers might form an incorrect impression that the book covers only the $f(0) = 0$ case (as in the MathSciences Review of that book).

5.3. *Variational formula for the time map*

In addition to the two formulas for the time map, discussed above, a curious variational formula has been discovered by R. Benguria and M.C. Depassier, see [8], which has also references to their earlier papers. If $u(t)$ is a solution of

$$u'' + \lambda f(u) = 0, \quad 0 < t < 1, \qquad u(0) = \alpha, \quad u'(0) = u(1) = 0, \tag{5.34}$$

it is shown by R. Benguria and M.C. Depassier that

$$\lambda = \max_{g \in D} \frac{1}{2} \frac{(\int_0^\alpha g'(y)^{1/3}\,\mathrm{d}y)^3}{\int_0^\alpha f(y) g(y)\,\mathrm{d}y},$$

where $D = \{g | g \in C^1(0,\alpha), g' > 0, g(0) = 0\}$. By rescaling, this formula is of course equivalent to a time map formula. It was used in [8] to obtain lower and upper bounds for time maps.

5.4. *A nonlocal problem*

Using the generalized inverses, we now give a complete description of the solution set of a nonlocal problem. We begin with a simple observation. It is well known that for any $L > 0$ the problem (here $u = u(x)$)

$$u'' + u^3 = 0, \quad 0 < x < L, \qquad u(0) = u(L) = 0$$

has a unique positive solution, and a unique negative solution. If we now take a positive solution on the interval $(0, L/k)$, followed by the negative solution on $(L/k, 2L/k)$, and so on, then we obtain a solution with $k - 1$ sign changes, for any positive integer k.

We now consider a nonlocal problem, where instead of a second boundary condition we prescribe the average value of the solution on some fixed interval $(0, L)$

$$\begin{aligned} &u'' + u^3 = 0, \quad 0 < x < L,\\ &u(0) = 0,\\ &\int_0^L u(s)\,\mathrm{d}s = \alpha, \end{aligned} \tag{5.35}$$

where α is a prescribed constant. We are interested in both positive, negative and sign-changing solutions, i.e. we shall talk of solutions with k sign changes, where $k \geqslant 0$. Without loss of generality we may assume $\alpha \geqslant 0$ (otherwise, consider $v = -u$). If $\alpha = 0$, it is clear that there exists exactly two solution of (5.35) with k sign changes, for any odd $k \geqslant 1$. Indeed, a solution of Eq. (5.35) with $u(0) = u(L) = 0$ having an odd number of roots inside $(0, L)$, and its negative, provide the desired solutions of (5.35). So that we may assume $\alpha > 0$.

THEOREM 5.2 [27]. *For any* $0 < \alpha < \frac{\pi}{\sqrt{2}}$ *there exists exactly one solution of* (5.35) *with k sign changes, for any* $k \geqslant 0$. *For* $\alpha = \frac{\pi}{\sqrt{2}}$ *there exists exactly one solution with k sign changes, for any even* $k \geqslant 0$, *and no solutions if k is odd. For any* $\alpha > \frac{\pi}{\sqrt{2}}$ *problem* (5.35) *has no solutions.*

PROOF. The problem "scales right". Setting $x = bt$, and $u = \frac{1}{b}v$, we see that $v = v(t)$ satisfies

$$\begin{aligned} &v'' + v^3 = 0, \quad 0 < t < \frac{L}{b}, \\ &v(0) = 0, \\ &\int_0^{L/b} v(s)\,\mathrm{d}s = \alpha. \end{aligned} \tag{5.36}$$

Comparing with (5.35), we see that only the length of the interval has changed. Hence we have a one-to-one map between the solution sets on any two intervals. So consider a solution $U(x)$ of the equation $u'' + u^3 = 0$, with $u(0) = 0$, which has k sign changes, whose roots are $x = 1, 2, \ldots$, and such that $U(x) > 0$ on $(0, 1)$, $U(x) < 0$ on $(1, 2)$, and so on. According to formula (5.15), the integral of $U(x)$ over any of its positive humps is equal to $\frac{\pi}{\sqrt{2}}$, while the integral of $U(x)$ over any of its negative humps is $-\frac{\pi}{\sqrt{2}}$. Imagine cutting this solution with a sliding vertical line $x = \xi$. By continuity, for any $\alpha \in (0, \frac{\pi}{\sqrt{2}}]$ we can find a unique $\xi \in (0, 1]$ so that $U(x)$ is positive solution of (5.35) on the interval $(0, \xi)$. We then map this solution to the original interval $(0, L)$ by the above transformation. Similarly, for any $\alpha \in (0, \frac{\pi}{\sqrt{2}})$ we can find a unique $\xi \in (1, 2)$ so that we have a solution of (5.35) on the interval $(0, \xi)$, with exactly one sign change. We then map $U(x)$ to the original interval, as before. Similarly we construct solutions with arbitrarily many sign changes.

By (5.15), no solution is possible in case $\alpha > \frac{\pi}{\sqrt{2}}$. □

6. Numerical computation of solutions

Good analytical understanding of a problem goes hand in hand with efficient numerical calculation of its solution. We know that for positive solutions the maximum value $u(0) = \alpha$ uniquely determines the solution pair $(\lambda, u(x))$ of the problem

$$u'' + \lambda f(u) = 0 \quad \text{for } -1 < x < 1, \qquad u(-1) = u(1) = 0, \tag{6.1}$$

see Lemma 2.3 above. We also know that the parameter λ in (6.1) can be "scaled out", i.e. $v(x) \equiv u(\frac{1}{\sqrt{\lambda}}x)$ solves the equation $v'' + f(v) = 0$, while $v(0) = u(0) = \alpha$, and $v'(0) = u'(0) = 0$. The root of $v(x)$ is $r = \sqrt{\lambda}$. We therefore solve the initial value problem

$$v'' + f(v) = 0, \qquad v(0) = \alpha, \quad v'(0) = 0, \tag{6.2}$$

```
Clear ["Global'*"]
f[u_] = Exp [5u/(5+u)];
t = {};
Δα = 0.2; α0 = 0; nsteps = 150;
Do[up = α0 + j * Δα;
  If[(NIntegrate[f[t], {t, 0, up}] > 0 && f[up] > 0),
    α[j] = α0 + j * Δα;
    Int[θ_] := NIntegrate[f[u], {u, α[j] Sin[θ], α[j]}];
    λ[j] = 2 α[j]^2 (NIntegrate[Cos[θ]/Sqrt[Int[θ]], {θ, 0, Pi/2}])^2;
    t = Append[t, {λ[j], α[j]}]
  ]
  , {j, 1, nsteps}
]
ListPlot[t , PlotJoined → True,
  AxesLabel → {"Lambda", "u(0)"}, PlotLabel → "f(u)=Exp(5u/(5+u))"]
```

and find its first positive root r. Then $\lambda = r^2$ by the above remarks. This way for each α we can find the corresponding λ. After we choose sufficiently many α_n and compute the corresponding λ_n, we can plot the pairs (λ_n, α_n), obtaining a bifurcation diagram in (λ, α) plane. We stress that the resulting two-dimensional bifurcation curve gives a faithful representation of the solution set of (6.1), since the value $u(0) = \alpha$ uniquely determines the solution pair $(\lambda, u(x))$. The program for solving (6.1) is essentially one short loop, involving the **NDSolve** command in *Mathematica*. It can be found at the author's web-page: http://math.uc.edu/~kormanp/.

An equally good way to do numerical computations is by direct integration. For problem (6.2) we have $r = T/2$, where as before r is the first positive root, and T is the time map. I.e. $\lambda = T^2/4$. Using formula (5.2) for the time map, we have

$$\lambda = \frac{1}{2}\alpha^2\left(\int_0^{\pi/2} \frac{\cos\theta}{\sqrt{\int_{\alpha\sin\theta}^{\alpha} f(u)\,\mathrm{d}u}}\,\mathrm{d}\theta\right)^2. \tag{6.3}$$

The *Mathematica* program based on (6.3) is so short and simple, that we include its listing here. It solves problem (6.1) for $f(u) = \mathrm{e}^{5u/(5+u)}$, and produces an S-shaped bifurcation curve, in agreement with our results. (Our program is solving the Dirichlet problem on the interval (0, 1), rather than (−1, 1), which accounts for the extra factor of 4.)

We see absolutely no need to ever use finite differences (or finite elements) for problem (6.1). If we divide the interval (0, 1) into n pieces, with step $h = 1/n$ and subdivision points $x_i = ih$, and denote by u_i the numerical approximation of $u(x_i)$, the finite difference approximation of (6.1) is

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + \lambda f(u_i) = 0, \quad 1 \leqslant i \leqslant n-1, \qquad u_0 = u_n = 0. \tag{6.4}$$

This is a system of *nonlinear* algebraic equations, more complicated in every way than the original problem (6.1). In particular, this system often has more solutions than the corresponding differential equation (6.1). The existence of the extra solutions (not corresponding to the solutions of (6.1)) has been recognized for a while, and a term *spurious* *solutions* has been used. For example in case $f(u) = \mathrm{e}^u$ the solution curve of (6.1) has exactly one turn (as we proved before), while the solution curve of (6.4) has three turns, see P. Korman [25]. Increasing the number of subdivision points n does not remove the two spurious turns, it just moves them closer to $\lambda = 0$. Actually the spurious turns are avoided in the opposite direction, when $n \leqslant 6$. We found this hard to prove, even when $n = 2$. When studying problem (6.4), we can no longer rely on the familiar tools from differential equations. Even in the case $f(u) = u^k$ the analysis of problem (6.4) is very involved, see E.L. Allgower [4].

For the general problem (1.1) (with $f = f(x,u)$) we suggest using the predictor-corrector method. If solution $u(x,\lambda)$ is known, one approximates

$$u(x,\lambda + \Delta\lambda) \simeq u(x,\lambda) + u_\lambda(x,\lambda)\Delta\lambda, \tag{6.5}$$

and then a very accurate approximation of $u(x,\lambda+\Delta\lambda)$ can be usually obtained in around 4 steps of Newton's iteration, with the initial guess given by (6.5). This way we can continue the solution in λ. To find $u_\lambda(x,\lambda)$ one solves a linear problem

$$\begin{aligned} &u_\lambda'' + f_u(x,u)u_\lambda + f(x,u) = 0 \quad \text{for } -1 < x < 1, \\ &u_\lambda(-1) = u_\lambda(1) = 0. \end{aligned} \tag{6.6}$$

To solve (6.6) one uses finite differences. (There are no spurious solutions for linear problems!) The resulting tri-diagonal system is easily solved by Gaussian elimination.

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CHAPTER 7

Singularities and Laplacians in Boundary Value Problems for Nonlinear Ordinary Differential Equations

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Contents

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Abstract

In this text we investigate solvability of various nonlinear singular boundary value problems for ordinary differential equations on the compact interval $[0, T]$. The nonlinearities in differential equations may be singular both in the time and space variables. Location of all singular points in $[0, T]$ need not be known.

The work is divided into 6 sections. Sections 1 and 2 are devoted to singular higher order boundary value problems. The remaining ones deal with the second order case. Motivated by various applications in physics we admit here the left-hand sides of the equations under consideration containing the ϕ-Laplacian or p-Laplacian operator. The special attention is paid to Dirichlet and periodic problems.

Usually, the main ideas of the proofs of the results mentioned are described. More detailed proofs are included in the cases where no proofs are available in literature or where the details are needed later.

0. Notation

Let $J \subset \mathbb{R}$, $k \in \mathbb{N}$, $p \in (1, \infty)$. Then we will write:

- $L_\infty(J)$ for the set of functions essentially bounded and (Lebesgue) measurable on J; the corresponding norm is $\|u\|_\infty = \sup\operatorname{ess}\{|u(t)|\colon t \in J\}$.
- $L_1(J)$ for the set of functions (Lebesgue) integrable on J; the corresponding norm is $\|u\|_1 = \int_J |u(t)|\,\mathrm{d}t$.
- $L_p(J)$ for the set of functions whose pth powers of modulus are integrable on J; the corresponding norm is $\|u\|_p = (\int_J |u(t)|^p\,\mathrm{d}t)^{1/p}$.
- $C(J)$ and $C^k(J)$ for the sets of functions continuous on J and having continuous kth derivatives on J, respectively.
- $AC(J)$ and $AC^k(J)$ for the sets of functions absolutely continuous on J and having absolutely continuous kth derivatives on J, respectively.
- $AC_{loc}(J)$ and $AC^k_{loc}(J)$ for the sets of functions absolutely continuous on each compact interval $I \subset J$ and having absolutely continuous kth derivatives on each compact interval $I \subset J$, respectively.
- If $J = [a, b]$, we will simply write $C[a, b]$ instead of $C([a, b])$ and similarly for other types of intervals and other functional sets defined above.

Further, we use the following notation:

- If $u \in L_\infty[a, b]$ is continuous on $[a, b]$, then $\max\{|u(t)|\colon t \in [a, b]\} = \sup\operatorname{ess}\{|u(t)|\colon t \in [a, b]\}$. Therefore the norm in $C[a, b]$ will be denoted by $\|u\|_\infty = \max\{|u(t)|\colon t \in [a, b]\}$ and the norm in $C^k[a, b]$ by $\|u\|_{C^k} = \sum_{i=0}^k \|u^{(i)}\|_\infty$.
- Let $n \in \mathbb{N}$ and $\mathcal{M} \subset \mathbb{R}^n$. Then $\overline{\mathcal{M}}$ will denote the closure of $\mathcal{M}$, $\partial\mathcal{M}$ the boundary of $\mathcal{M}$ and meas $(\mathcal{M})$ the Lebesgue measure of $\mathcal{M}$.
- $\deg(\mathcal{I} - \mathcal{F}, \Omega)$ stands for the Leray–Schauder degree of $\mathcal{I} - \mathcal{F}$ with respect to Ω, where $\mathcal{I}$ denotes the identity operator.

We say that a function f satisfies *the Carathéodory conditions* on the set $[a, b] \times \mathcal{M}$ if:

$$f(\cdot, x_0, \dots, x_{n-1})\colon [a, b] \to \mathbb{R} \text{ is measurable} \quad \text{for all } (x_0, x_1, \dots, x_{n-1}) \in \mathcal{M}; \tag{0.1}$$

$$f(t, \cdot, \dots, \cdot)\colon \mathcal{M} \to \mathbb{R} \text{ is continuous for a.e. } t \in [a, b]; \tag{0.2}$$

$$\begin{cases} \text{for each compact set } \mathcal{K} \subset \mathcal{M} \text{ there is a function } m_\mathcal{K} \in L_1[a, b] \text{ such that} \\ \quad \left|f(t, x_0, \dots, x_{n-1})\right| \leqslant m_\mathcal{K}(t) \quad \text{for a.e. } t \in [a, b] \\ \text{and all } (x_0, x_1, \dots, x_{n-1}) \in \mathcal{K}. \end{cases} \tag{0.3}$$

In this case we will write $f \in Car([a, b] \times \mathcal{M})$. If $J \subset [a, b]$ and $J \neq \overline{J}$, then $f \in Car(J \times \mathcal{M})$ will mean that $f \in Car(I \times \mathcal{M})$ for each compact interval $I \subset J$.

1. Principles of solvability of singular higher order BVPs

1.1. *Regular and singular BVPs*

For $n \in \mathbb{N}$, $[0,T] \subset \mathbb{R}$, $i \in \{0,1,\dots,n-1\}$ and a closed set $\mathcal{B} \subset C^i[0,T]$ consider the boundary value problem

$$u^{(n)} = f\big(t, u, \dots, u^{(n-1)}\big), \tag{1.1}$$

$$u \in \mathcal{B}. \tag{1.2}$$

In what follows, we will investigate the solvability of problem (1.1), (1.2) on the set $[0,T] \times \mathcal{A}$, where $\mathcal{A}$ is a closed subset of $\mathbb{R}^n$ or $\mathcal{A} = \mathbb{R}^n$. The classical existence results are based on the assumption $f \in Car([0,T] \times \mathcal{A})$. In this case we will say that problem (1.1), (1.2) is *regular on* $[0,T] \times \mathcal{A}$. If $f \notin Car([0,T] \times \mathcal{A})$ we will say that problem (1.1), (1.2) is *singular on* $[0,T] \times \mathcal{A}$.

Motivated by the following applications we will mainly address singular problems.

EXAMPLE 1. In certain problems in fluid dynamics and boundary layer theory (see, e.g., Callegari, Friedman and Nachman [43–45]) the second order differential equation

$$u'' + \frac{\psi(t)}{u^\lambda} = 0 \tag{1.3}$$

arose. Here $\lambda \in (0,\infty)$ and $\psi \in C(0,1)$, $\psi \notin L_1[0,1]$. Equation (1.3) is known as the generalized Emden–Fowler equation. Its solvability with the Dirichlet boundary conditions

$$u(0) = u(1) = 0 \tag{1.4}$$

was investigated by Taliaferro [141] in 1979 and then by many other authors. Problem (1.3), (1.4) has been studied on the set $[0,1] \times [0,\infty)$ because positive solutions have been sought. We can see that $f(t,x) = \psi(t)x^{-\lambda}$ does not fulfill conditions (0.2) and (0.3) with $[a,b] = [0,1]$ and $\mathcal{M} = [0,\infty)$. Hence problem (1.3), (1.4) is singular on $[0,1] \times [0,\infty)$.

EXAMPLE 2. Consider the fourth order degenerate parabolic equation

$$U_t + \big(|U|^\mu U_{yyy}\big)_y = 0,$$

which arises in droplets and thin viscous flows models (see, e.g., [32,33]). The source-type solutions of this equation have the form

$$U(y,t) = t^{-b} u\big(yt^{-b}\big), \quad b = \frac{1}{\mu + 4},$$

which leads to the study of the third order ordinary differential equation on $[-1,1]$,

$$u''' = btu^{1-\mu}.$$

We see that $f(t,x) = btx^{1-\mu}$ is singular on $[-1,1] \times [0,\infty)$ if $\mu > 1$.

EXAMPLE 3. Similarly to Example 2, the sixth order degenerate equation

$$U_t - \big(|U|^\mu U_{yyyyy}\big)_y = 0$$

which arises in semiconductor models (Bernis [30,31]) leads to the fifth order ordinary differential equation

$$-u^{(5)} = \frac{t}{u^\lambda}$$

which is singular for $\lambda > 0$.

A solvability decision for singular boundary value problems requires an exact definition of a solution to such problems. Here, we will work with the same definition of a solution both for regular problems and for singular ones.

DEFINITION 1.1. A function $u \in AC^{n-1}[0,T] \cap \mathcal{B}$ is said to be a *solution of problem* (1.1), (1.2), if it satisfies the equality $u^{(n)}(t) = f(t, u(t), \dots, u^{(n-1)}(t))$ a.e. on $[0,T]$. If we investigate problem (1.1), (1.2) on $\mathcal{A} \neq \mathbb{R}^n$, we moreover require $(u(t), \dots, u^{(n-1)}(t)) \in \mathcal{A}$ for $t \in [0,T]$.

In literature, *an alternative approach to solvability* of singular problems can be found. In this approach, solutions are defined as functions whose $(n-1)$st derivatives can have discontinuities at some points in $[0,T]$. Here we will call them the w-solutions. According to Kiguradze [92] or Agarwal and O'Regan [3] we define them as follows. In contrast to our starting setting, to define w-solutions we assume that $i \in \{0, 1, \dots, n-2\}$ and $\mathcal{B}$ is a closed subset in $C^i[0,T]$.

DEFINITION 1.2. We say that u is *a w-solution of problem* (1.1), (1.2) if there exists a finite number of points $t_\nu \in [0,T]$, $\nu = 1, 2, \dots, r$, such that if we denote $J = [0,T] \setminus \{t_\nu\}_{\nu=1}^r$, then $u \in C^{n-2}[0,T] \cap AC^{n-1}_{loc}(J) \cap \mathcal{B}$ satisfies $u^{(n)}(t) = f(t, u(t), \dots, u^{(n-1)}(t))$ a.e. on $[0,T]$. If $\mathcal{A} \neq \mathbb{R}^n$ we require $(u(t), \dots, u^{(n-1)}(t)) \in \mathcal{A}$ for $t \in J$.

Clearly each solution is a w-solution and each w-solution which belongs to $AC^{n-1}[0,T]$ is a solution.

In the study of singular problem (1.1), (1.2) we will focus our attention on two types of singularities of the function f:

Let $J \subset [0,T]$. We say that $f : J \times \mathcal{A} \to \mathbb{R}$ has singularities in its *time variable* t, if $J \neq \overline{J} = [0,T]$ and

$$f \in Car(J \times \mathcal{A}) \quad \text{and} \quad f \notin Car\big([0,T] \times \mathcal{A}\big). \tag{1.5}$$

Let $\mathcal{D} \subset \mathcal{A}$. We say that $f : [0, T] \times \mathcal{D} \to \mathbb{R}$ has singularities in its *space variables* $x_0, x_1, \dots, x_{n-1}$, if $\mathcal{D} \neq \overline{\mathcal{D}} = \mathcal{A}$ and

$$f \in Car\big([0, T] \times \mathcal{D}\big) \quad \text{and} \quad f \notin Car\big([0, T] \times \mathcal{A}\big). \tag{1.6}$$

We will study particular cases of (1.5) and (1.6) which will be described in Section 1.2 and Section 1.3, respectively.

1.2. *Singularities in time variable*

According to (0.3) and (1.5) a function f has a singularity in its *time variable* t, if f is not integrable on $[0, T]$. Let us define it more precisely. Let $k \in \mathbb{N}$, $t_i \in [0, T]$, $i = 1, \dots, k$, $J = [0, T] \setminus \{t_1, t_2, \dots, t_k\}$ and let $f \in Car(J \times \mathcal{A})$. Assume that for each $i \in \{1, \dots, k\}$ there exists $(x_0, \dots, x_{n-1}) \in \mathcal{A}$ such that

$$\int_{t_i}^{t_i+\varepsilon} \big|f(t, x_0, \dots, x_{n-1})\big| \, dt = \infty \quad \text{or} \quad \int_{t_i-\varepsilon}^{t_i} \big|f(t, x_0, \dots, x_{n-1})\big| \, dt = \infty \tag{1.7}$$

for any sufficiently small $\varepsilon > 0$. Then f does not fulfill (0.3) with $\mathcal{M} = \mathcal{A}$ and, according to (1.5), function f has singularities in its time variable t, namely at the values $t_1, \dots, t_k$. We will call these values the *singular points* of f.

EXAMPLE. Let $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2, \dots, k$, be continuous. Then the function

$$f(t, x_0, \dots, x_{n-1}) = \sum_{i=1}^{k} \frac{1}{t - t_i} f_i(x_0, \dots, x_{n-1}),$$

has singular points $t_1, t_2, \dots, t_k$.

1.3. *Singularities in space variables*

By virtue of (0.2) and (1.6) we see that if f has a singularity in some of its *space variables* then f is not continuous in this variable on a region where f is studied. Motivated by Eq. (1.3) we will consider the following case. Let $\mathcal{A}_i \subset \mathbb{R}$ be a closed interval and let $c_i \in \mathcal{A}_i$, $\mathcal{D}_i = \mathcal{A}_i \setminus \{c_i\}$, $i = 0, 1, \dots, n-1$. Assume that there exists $j \in \{0, 1, \dots, n-1\}$ such that

$$\begin{cases} \limsup\limits_{x_j \to c_j, x_j \in \mathcal{D}_j} \big|f(t, x_0, \dots, x_j, \dots, x_{n-1})\big| = \infty \\ \text{for a.e. } t \in [0, T] \text{ and for some } x_i \in \mathcal{D}_i, \ i = 0, 1, \dots, n-1, \ i \neq j. \end{cases} \tag{1.8}$$

If we put $\mathcal{A} = \mathcal{A}_0 \times \cdots \times \mathcal{A}_{n-1}$, we see that f does not fulfill (0.2) with $\mathcal{M} = \mathcal{A}$ and, according to (1.6), the function f has a singularity in its space variable x_j, namely at the value c_j.

Let u be a solution of (1.1), (1.2) and let a point $t_u \in [0, T]$ be such that $u^{(j)}(t_u) = c_j$ for some $j \in \{0, \ldots, n-1\}$. Then t_u is called *a singular point corresponding to the solution u*.

Now, let u be a w-solution of (1.1), (1.2). Assume that a point $t_u \in [0, T]$ is such that $u^{(n-1)}(t_u)$ does not exist or there is a $j \in \{0, \ldots, n-1\}$ such that $u^{(j)}(t_u) = c_j$. Then t_u is called *a singular point corresponding to the w-solution u*.

EXAMPLE. Let $h_1, h_2, h_3 \in L_1[0, T]$, $h_2 \neq 0$, $h_3 \neq 0$ a.e. on $[0, T]$. Consider the Dirichlet problem

$$u'' + h_1(t) + \frac{h_2(t)}{u} + \frac{h_3(t)}{u'} = 0, \quad u(0) = u(T) = 0. \tag{1.9}$$

Let u be a solution of (1.9). Then 0 and T are singular points corresponding to u. Moreover, there exists at least one point $t_u \in (0, T)$ satisfying $u'(t_u) = 0$, which means that t_u is also a singular point corresponding to u. Note that (in contrast to the points 0 and T) we do not know the location of t_u in $(0, T)$.

In accordance with this example, we will distinguish two types of singular points corresponding to solutions or to w-solutions: *singular points of type I*, where we know their location in $[0, T]$ and *singular points of type II* whose location is not known.

1.4. *Existence principles for BVPs with time singularities*

Singular problems are usually investigated by means of auxiliary regular problems. To establish the existence of a solution of a singular problem we introduce a sequence of approximating regular problems which are solvable. Then we pass to the limit for the sequence of approximate solutions to get a solution of the original singular problem. Here we provide existence principles which contain the main rules for a construction of such sequences to get either w-solutions or solutions.

Consider problem (1.1), (1.2) on $[0, T] \times \mathcal{A}$. For the sake of simplicity assume that f has only one time singularity at $t = t_0$, $t_0 \in [0, T]$. It means that

$$\left\{\begin{array}{l} J = [0, T] \setminus \{t_0\}, \ \ f \in Car(J \times \mathcal{A}) \text{ satisfies at least one of the conditions} \\ \text{(i)} \quad \displaystyle\int_{t_0-\varepsilon}^{t_0} \big|f(t, x_0, \ldots, x_{n-1})\big| \, \mathrm{d}t = \infty, \quad t_0 \in (0, T], \\ \text{(ii)} \quad \displaystyle\int_{t_0}^{t_0+\varepsilon} \big|f(t, x_0, \ldots, x_{n-1})\big| \, \mathrm{d}t = \infty, \quad t_0 \in [0, T), \\ \text{for some } (x_0, x_1, \ldots, x_{n-1}) \in \mathcal{A} \text{ and each sufficiently small } \varepsilon > 0. \end{array}\right. \tag{1.10}$$

Let us have a sequence of regular problems

$$u^{(n)} = f_k\big(t, u, \dots, u^{(n-1)}\big), \quad u \in \mathcal{B}, \tag{1.11}$$

where $f_k \in Car([0,T] \times \mathbb{R}^n)$, $k \in \mathbb{N}$.

THEOREM 1.3 (First existence principle for w-solutions of (1.1), (1.2)). *Let* (1.10) *hold and let* $\mathcal{B}$ *be a closed subset in* $C^{n-2}[0,T]$. *Assume that the conditions*

$$\begin{cases} \text{for each } k \in \mathbb{N} \text{ and each } (x_0, \dots, x_{n-1}) \in \mathcal{A}, \\ f_k(t, x_0, \dots, x_{n-1}) = f(t, x_0, \dots, x_{n-1}) \quad \text{a.e. on } [0,T] \setminus \Delta_k, \\ \text{where } \Delta_k = \big(t_0 - \frac{1}{k}, t_0 + \frac{1}{k}\big) \cap [0,T], \end{cases} \tag{1.12}$$

and

$$\begin{cases} \text{there exists a bounded set } \Omega \subset C^{n-1}[0,T] \text{ such that} \\ \text{for each } k \in \mathbb{N} \text{ the regular problem (1.11) has a solution } u_k \in \Omega \\ \text{and } \big(u_k(t), \dots, u_k^{(n-1)}(t)\big) \in \mathcal{A} \text{ for } t \in [0,T] \end{cases} \tag{1.13}$$

are fulfilled. Then

$$\begin{cases} \text{there exist a function } u \in C^{n-2}[0,T] \text{ and a subsequence} \\ \{u_{k_\ell}\} \subset \{u_k\} \text{ such that } \lim\limits_{\ell \to \infty} \|u_{k_\ell} - u\|_{C^{n-2}} = 0; \end{cases} \tag{1.14}$$

$$\begin{cases} \lim\limits_{\ell \to \infty} u_{k_\ell}^{(n-1)}(t) = u^{(n-1)}(t) \text{ locally uniformly on } J \\ \text{and } \big(u(t), \dots, u^{(n-1)}(t)\big) \in \mathcal{A} \text{ for } t \in J; \end{cases} \tag{1.15}$$

$$\text{the function } u \in AC_{loc}^{n-1}(J) \text{ is a w-solution of problem (1.1), (1.2).} \tag{1.16}$$

SKETCH OF THE PROOF. *Step* 1. *Convergence of the sequence of approximate solutions.*

Condition (1.13) implies that the sequences $\{u_k^{(i)}\}$, $0 \leqslant i \leqslant n-2$, are bounded and equicontinuous on $[0,T]$. By the Arzelà–Ascoli theorem the assertion (1.14) is true and $u \in \mathcal{B} \subset C^{n-2}[0,T]$. Since $\{u_k^{(n-1)}\}$ is bounded on $[0,T]$, we get, due to (1.11) and (1.12), that for each $t \in [0, t_0)$ the sequence $\{u_k^{(n-1)}\}$ is equicontinuous on $[0,t]$ and so the same holds on $[t,T]$ if $t \in (t_0, T]$. The Arzelà–Ascoli theorem and the diagonalization principle yield (1.15).

Step 2. *Properties of the limit* u.

By virtue of (1.12), (1.14) and (1.15) we have

$$\lim_{k_\ell \to \infty} f_{k_\ell}\big(t, u_{k_\ell}(t), \dots, u_{k_\ell}^{(n-1)}(t)\big) = f\big(t, u(t), \dots, u^{(n-1)}(t)\big) \quad \text{a.e. on } [0,T]. \tag{1.17}$$

Hence, using the Lebesgue convergence theorem, we can deduce that if $t_0 \neq 0$ the limit u solves the equation

$$u^{(n-1)}(t) = u^{(n-1)}(0) + \int_0^t f\big(s, u(s), \dots, u^{(n-1)}(s)\big)\,\mathrm{d}s \quad \text{for } t \in [0, t_0) \tag{1.18}$$

and if $t_0 \neq T$ the limit u solves the equation

$$u^{(n-1)}(t) = u^{(n-1)}(T) - \int_t^T f\big(s, u(s), \dots, u^{(n-1)}(s)\big)\,\mathrm{d}s \quad \text{for } t \in (t_0, T], \tag{1.19}$$

which immediately yields (1.16). □

For the existence of a solution $u \in AC^{n-1}[0, T]$ of problem (1.1), (1.2) we will impose additional conditions on f on some neighbourhood of t_0.

THEOREM 1.4 (First existence principle for solutions of (1.1), (1.2)). *Let all assumptions of Theorem* 1.3 *be fulfilled. Further, assume that*

$$\begin{cases} \textit{there exist } \psi \in L_1[0, T],\ \eta > 0 \textit{ and } \lambda_1, \lambda_2 \in \{-1, 1\} \textit{ such that} \\ \quad \lambda_1 f_{k_\ell}\big(t, u_{k_\ell}(t), \dots, u_{k_\ell}^{(n-1)}(t)\big) \geqslant \psi(t) \\ \textit{for all } \ell \in \mathbb{N} \textit{ and for a.e. } t \in [t_0 - \eta, t_0) \cap [0, T] \textit{ provided } (1.10)(\mathrm{i}) \textit{ holds}, \\ \quad \lambda_2 f_{k_\ell}\big(t, u_{k_\ell}(t), \dots, u_{k_\ell}^{(n-1)}(t)\big) \geqslant \psi(t) \\ \textit{for all } \ell \in \mathbb{N} \textit{ and for a.e. } t \in (t_0, t_0 + \eta] \cap [0, T] \textit{ provided } (1.10)(\mathrm{ii}) \textit{ is true}. \end{cases} \tag{1.20}$$

Then the assertions (1.14) *and* (1.15) *are valid and* $u \in AC^{n-1}[0, T]$ *is a solution of problem* (1.1), (1.2).

SKETCH OF THE PROOF. *Step* 1. As in the proof of Theorem 1.3 we get that (1.14)–(1.16) hold.

Step 2. Since u is a w-solution of problem (1.1), (1.2), it remains to prove that $u \in AC^{n-1}[0, T]$. Assume that condition (1.10)(i) holds. Since

$$u_{k_\ell}^{(n-1)}(t) - u_{k_\ell}^{(n-1)}(t_0 - \eta) = \int_{t_0-\eta}^t f_{k_\ell}\big(s, u_{k_\ell}(s), \dots, u_{k_\ell}^{(n-1)}(s)\big)\,\mathrm{d}s \tag{1.21}$$

for $t \in (0, t_0)$, we get due to (1.13) that there is a $c \in (0, \infty)$ such that

$$\lambda_1 \int_{t_0-\eta}^{t_0} f_{k_\ell}\big(s, u_{k_\ell}(s), \dots, u_{k_\ell}^{(n-1)}(s)\big)\,\mathrm{d}s \leqslant c \tag{1.22}$$

for each $\ell \in \mathbb{N}$. By the Fatou lemma, having in mind conditions (1.17), (1.20) and (1.22), we deduce that $f(t, u(t), \dots, u^{(n-1)}(t)) \in L_1[0, t_0]$. Similarly, if condition (1.10)(ii) holds, we deduce that $f(t, u(t), \dots, u^{(n-1)}(t)) \in L_1[t_0, T]$. Therefore $f(t, u(t), \dots, u^{(n-1)}(t)) \in L_1[0, T]$ and due to (1.18) and (1.19) we have that $u \in AC^{n-1}[0, T]$ is a solution of problem (1.1), (1.2). □

In the sequel we will need the following definition:

DEFINITION 1.5. Let $[a, b] \subset \mathbb{R}$ and $\{g_k\} \subset L_1[a, b]$. We say that the sequence $\{g_k\}$ is *uniformly integrable on* $[a, b]$ if

$$\begin{cases} \text{for each } \varepsilon > 0 \text{ there exists } \delta > 0 \text{ such that} \\ \displaystyle\sum_{j=1}^{\infty}(b_j - a_j) < \delta \quad \Longrightarrow \quad \sum_{j=1}^{\infty}\int_{a_j}^{b_j}\big|g_k(t)\big|\,\mathrm{d}t < \varepsilon \\ \text{for each } k \in \mathbb{N} \text{ and each sequence of intervals } \{(a_j, b_j)\} \text{ in } [a, b]. \end{cases} \tag{1.23}$$

Note that condition (1.23) is satisfied for example if there exists $\psi \in L_1[a, b]$ such that $|g_k(t)| \leqslant \psi(t)$ for a.e. $t \in [a, b]$ and all $k \in \mathbb{N}$.

THEOREM 1.6 (Second existence principle for solutions of (1.1), (1.2)). *Let all assumptions of Theorem* 1.3 *be fulfilled and assume in addition that* $\mathcal{B}$ *is a closed subset in* $C^{n-1}[0, T]$ *and that*

$$\begin{cases} \textit{there exists } \eta > 0 \textit{ such that the sequence } \big\{f_k\big(t, u_k(t), \dots, u_k^{(n-1)}(t)\big)\big\} \\ \textit{is uniformly integrable on } [t_0 - \eta, t_0 + \eta] \cap [0, T]. \end{cases} \tag{1.24}$$

Then

$$\begin{cases} \textit{there exist a function } u \in \overline{\Omega} \textit{ and a subsequence } \{u_{k_\ell}\} \subset \{u_k\} \textit{ such that} \\ \displaystyle\lim_{\ell\to\infty} \|u_{k_\ell} - u\|_{C^{n-1}} = 0 \textit{ and } \big(u(t), \dots, u^{(n-1)}(t)\big) \in \mathcal{A} \quad \textit{for } t \in [0, T] \end{cases} \tag{1.25}$$

and $u \in AC^{n-1}[0, T]$ *is a solution of problem* (1.1), (1.2).

SKETCH OF THE PROOF. *Step* 1. By (1.13) we get that the sequences $\{u_k^{(i)}\}$, $0 \leqslant i \leqslant n-2$, are bounded in $C[0,T]$ and equicontinuous on $[0,T]$ and $\{u_k^{(n-1)}\}$ is bounded in $C[0,T]$. Using (1.24) one can show that $\{u_k^{(n-1)}\}$ is also equicontinuous on $[0,T]$. The Arzelà–Ascoli theorem yields (1.25) and $u \in \mathcal{B} \subset C^{n-1}[0,T]$.

Step 2. As in Step 2 of the proof of Theorem 1.3 we get that u is a w-solution of problem (1.1), (1.2).

Step 3. It remains to prove that $u \in AC^{n-1}[0,T]$. Since $u \in AC_{loc}^{n-1}(J)$, it is sufficient to prove

$$u^{(n-1)} \in AC\big([t_0-\eta, t_0+\eta] \cap [0,T]\big). \tag{1.26}$$

Assume that (1.10)(i) holds and $[t_0-\eta, t_0] \subset [0,T]$. By virtue of (1.17) and (1.24), applying Vitali's convergence theorem we obtain $f(t, u(t), \dots, u^{(n-1)}(t)) \in L_1[t_0-\eta, t_0]$. If (1.10)(ii) holds, we can assume $[t_0, t_0+\eta] \subset [0,T]$ and deduce similarly that

$$f\big(t, u(t), \dots, u^{(n-1)}(t)\big) \in L_1[t_0, t_0+\eta].$$

Hence, we get (1.26). □

1.5. *Existence principles for BVPs with space singularities*

Similarly to Section 1.4 we will establish sufficient properties for an approximate sequence of regular problems and of their solutions to pass to a limit and yield a solution of the original singular problem (1.1), (1.2). Let $\mathcal{A}_i \subset \mathbb{R}$, $i = 0, \dots, n-1$, be closed intervals and let $\mathcal{A} = \mathcal{A}_0 \times \dots \times \mathcal{A}_{n-1}$. Consider problem (1.1), (1.2) on $[0,T] \times \mathcal{A}$ and assume that f has only one singularity at each x_i, namely at the values $c_i \in \mathcal{A}_i$, $i = 0, \dots, n-1$. Denoting $\mathcal{D} = \mathcal{D}_0 \times \dots \times \mathcal{D}_{n-1}$, $\mathcal{D}_i = \mathcal{A}_i \setminus \{c_i\}$, $i = 0, \dots, n-1$, we will assume that

$$f \in Car\big([0,T] \times \mathcal{D}\big) \text{ satisfies (1.8)} \quad \text{for } j = 0, \dots, n-1. \tag{1.27}$$

Consider a sequence of regular problems (1.11) where $f_k \in Car([0,T] \times \mathbb{R}^n)$, $k \in \mathbb{N}$. We will use the approach used by Rachůnková and Staněk in [121] and [122].

THEOREM 1.7 (Second existence principle for w-solutions of (1.1), (1.2)). *Let* (1.13), (1.27) *hold and let* $\mathcal{B}$ *be a closed subset in* $C^{n-2}[0,T]$. *Assume that*

$$\begin{cases} \text{for each } k \in \mathbb{N}, \text{ for a.e. } t \in [0,T] \text{ and each } (x_0, \dots, x_{n-1}) \in \mathcal{D} \\ \text{we have} \\ \quad f_k(t, x_0, \dots, x_{n-1}) = f(t, x_0, \dots, x_{n-1}) \\ \quad \text{if } |x_i - c_i| \geqslant \frac{1}{k},\ 0 \leqslant i \leqslant n-1. \end{cases} \tag{1.28}$$

Then the assertion (1.14) *is valid.*

If, moreover, the set

$$\Sigma = \big\{ s \in [0,T]\colon\ u^{(i)}(s) = c_i \text{ for some } i \in \{0, \dots, n-1\} \\ \text{or } u^{(n-1)}(s) \text{ does not exist} \big\}$$

is finite, then the assertion (1.15) *is valid for* $J = [0, T] \setminus \Sigma$. *If, in addition,*

$$\begin{cases} \text{the sequence } \{ f_{k_\ell}(t, u_{k_\ell}(t), \dots, u_{k_\ell}^{(n-1)}(t)) \} \\ \text{is uniformly integrable on each interval } [a, b] \subset J \end{cases} \tag{1.29}$$

then $u \in AC_{loc}^{n-1}(J)$ *is a w-solution of problem* (1.1), (1.2).

SKETCH OF THE PROOF. *Step* 1. *Convergence of the sequence of approximate solutions.*
As in Step 1 of the proof of Theorem 1.3 we get (1.14) and $u \in \mathcal{B} \subset C^{n-2}[0, T]$. Assume that Σ is finite and choose an arbitrary $[a, b] \subset J$. According to (1.27) and (1.28) we can prove that the sequence $\{u_{k_\ell}^{(n-1)}\}$ is equicontinuous on $[a, b]$. Using the Arzelà–Ascoli theorem and the diagonalization principle we deduce that the subsequence $\{u_{k_\ell}\}$ can be chosen so that it fulfills (1.15).

Step 2. *Convergence of the sequence of regular right-hand sides.*
Consider sets

$$\mathcal{V}_1 = \{ t \in [0, T]:\ f(t, \cdot, \dots, \cdot) : \mathcal{D} \to \mathbb{R} \text{ is not continuous} \},$$
$$\mathcal{V}_2 = \{ t \in [0, T]:\ \text{the equality in (1.28) is not valid} \}.$$

We can see that $\operatorname{meas}(\mathcal{V}_1) = \operatorname{meas}(\mathcal{V}_2) = 0$. Denote $\mathcal{U} = \Sigma \cup \mathcal{V}_1 \cup \mathcal{V}_2$ and choose an arbitrary $t \in [0, T] \setminus \mathcal{U}$. By (1.14) and (1.15) there exists $\ell_0 \in \mathbb{N}$ such that for each $\ell \in \mathbb{N}$, $\ell \geqslant \ell_0$,

$$|u^{(i)}(t) - c_i| > \frac{1}{k_\ell}, \quad |u_{k_\ell}^{(i)}(t) - c_i| \geqslant \frac{1}{k_\ell} \quad \text{for } i \in \{0, \dots, n-1\}.$$

According to (1.28) we have

$$f_{k_\ell}(t, u_{k_\ell}(t), \dots, u_{k_\ell}^{(n-1)}(t)) = f(t, u_{k_\ell}(t), \dots, u_{k_\ell}^{(n-1)}(t))$$

and, by (1.14), (1.15),

$$\lim_{\ell \to \infty} f_{k_\ell}(t, u_{k_\ell}(t), \dots, u_{k_\ell}^{(n-1)}(t)) = f(t, u(t), \dots, u^{(n-1)}(t)). \tag{1.30}$$

Since $\operatorname{meas}(\mathcal{U}) = 0$, (1.30) holds for a.e. $t \in [0, T]$.

Step 3. *Existence of a w-solution.*
Choose an arbitrary interval $[a, b] \subset J$. By (1.29) and (1.30) we can use Vitali's convergence theorem [23] to show that $f(t, u(t), \dots, u^{(n-1)}(t)) \in L_1[a, b]$ and if we pass to the limit in the sequence

$$u_{k_\ell}^{(n-1)}(t) = u_{k_\ell}^{(n-1)}(a) + \int_a^t f_{k_\ell}(s, u_{k_\ell}(s), \dots, u_{k_\ell}^{(n-1)}(s))\,\mathrm{d}s, \quad t \in [a, b],$$

we get

$$u^{(n-1)}(t) = u^{(n-1)}(a) + \int_a^t f\big(s, u(s), \ldots, u^{(n-1)}(s)\big)\,\mathrm{d}s, \quad t \in [a, b].$$

Since $[a, b] \subset J$ is an arbitrary interval, we conclude that $u \in AC^{n-1}_{loc}(J)$ satisfies Eq. (1.1) for a.e. $t \in [0, T]$. □

THEOREM 1.8 (Third existence principle for solutions of (1.1), (1.2)). *Let* (1.13), (1.27), (1.28) *hold and let* $\mathcal{B}$ *be a closed subset of* $C^{n-1}[0, T]$. *Further, assume that the sequence*

$$\big\{f_k\big(t, u_k(t), \ldots, u_k^{(n-1)}(t)\big)\big\} \quad \textit{is uniformly integrable on } [0, T]. \tag{1.31}$$

Then the assertion (1.25) *is valid. If, moreover, the functions* $u^{(i)} - c_i$, $0 \leqslant i \leqslant n-1$, *have at most a finite number of zeros in* $[0, T]$, *then* $u \in AC^{n-1}[0, T]$ *is a solution of* (1.1), (1.2).

SKETCH OF THE PROOF. *Step* 1. As in Step 1 in the proof of Theorem 1.6 we get that (1.25) is valid and $u \in \mathcal{B} \subset C^{n-1}[0, T]$.

Step 2. As in Step 2 in the proof of Theorem 1.7 we get that (1.30) is valid.

Step 3. We can argue as in Step 3 in the proof of Theorem 1.7 if we take $[0, T]$ instead of $[a, b]$ and (1.31) instead of (1.29). □

2. Existence results for singular two-point higher order BVPs

In this section we are interested in problems for higher order differential equations having singularities in their space variables only (see Section 1.3). We consider the focal, conjugate, (n, p), Sturm–Liouville and Lidstone boundary conditions which appear most frequently in literature. Boundary conditions considered are two-point, linear and homogeneous.

Existence results for the above singular problems are proved by regularization and sequential techniques which consist in the construction of a proper sequence of auxiliary regular problems and in limit processes (see Section 1.5). To prove solvability of the auxiliary regular problems we use the Nonlinear Fredholm Alternative (see, e.g., [95, Theorem 4] or [146, p. 25]) which we formulate in the form convenient for our problems. In particular, we consider the differential equation

$$u^{(n)} + \sum_{i=0}^{n-1} a_i(t) u^{(i)} = g\big(t, u, \ldots, u^{(n-1)}\big) \tag{2.1}$$

and the corresponding linear homogeneous differential equation

$$u^{(n)} + \sum_{i=0}^{n-1} a_i(t) u^{(i)} = 0 \tag{2.2}$$

where $a_i \in L_1[0,T]$, $0 \leqslant i \leqslant n-1$, $g \in Car([0,T]\times\mathbb{R}^n)$. Further, we introduce boundary conditions

$$\mathcal{L}_j(u) = r_j, \quad 1 \leqslant j \leqslant n, \tag{2.3}$$

and the corresponding homogeneous boundary conditions

$$\mathcal{L}_j(u) = 0, \quad 1 \leqslant j \leqslant n, \tag{2.4}$$

where $\mathcal{L}_j : C^{n-1}[0,T] \to \mathbb{R}$ are linear and continuous functionals and $r_j \in \mathbb{R}$, $1 \leqslant j \leqslant n$.

THEOREM 2.1 (Nonlinear Fredholm Alternative). *Let the linear homogeneous problem* (2.2), (2.4) *have only the trivial solution and let there exist a function* $\psi \in L_1[0,T]$ *such that*

$$\bigl|g(t, x_0, \dots, x_{n-1})\bigr| \leqslant \psi(t) \quad \text{for a.e. } t \in [0,T] \text{ and all } x_0, \dots, x_{n-1} \in \mathbb{R}.$$

Then the nonlinear problem (2.1), (2.3) *has a solution* $u \in AC^{n-1}[0,T]$.

2.1. *Focal conditions*

We discuss the singular $(p, n-p)$ right focal problem

$$(-1)^{n-p} u^{(n)} = f\bigl(t, u, \dots, u^{(n-1)}\bigr), \tag{2.5}$$

$$u^{(i)}(0) = 0, \quad 0 \leqslant i \leqslant p-1, \qquad u^{(j)}(T) = 0, \;\; p \leqslant j \leqslant n-1, \tag{2.6}$$

where $n \geqslant 2$, $p \in \mathbb{N}$ is fixed, $1 \leqslant p \leqslant n-1$, $f \in Car([0,T]\times\mathcal{D})$ with

$$\mathcal{D} = \begin{cases} \underbrace{\mathbb{R}_+^{p+1} \times \mathbb{R}_- \times \mathbb{R}_+ \times \mathbb{R}_- \times \cdots \times \mathbb{R}_+}_{n} & \text{if } n-p \text{ is odd,} \\ \underbrace{\mathbb{R}_+^{p+1} \times \mathbb{R}_- \times \mathbb{R}_+ \times \mathbb{R}_- \times \cdots \times \mathbb{R}_-}_{n} & \text{if } n-p \text{ is even} \end{cases}$$

and f may be singular at the value 0 of all its space variables. Here $\mathbb{R}_- = (-\infty, 0)$ and $\mathbb{R}_+ = (0, \infty)$. Notice that if $f > 0$ then the singular points corresponding to the solutions of problem (2.5), (2.6) are only of type I. The Green function of problem $u^{(n)} = 0$, (2.6) is presented in [18] and [19].

The existence result for the singular problem (2.5), (2.6) is given in the following theorem.

THEOREM 2.2 [120, Theorem 4.3]. *Let* $f \in Car([0,T]\times\mathcal{D})$ *and let there exist positive constants* ε *and* r *such that*

$$\varepsilon(T-t)^r \leqslant f(t, x_0, \dots, x_{n-1})$$

for a.e. $t \in [0,T]$ *and each* $(x_0,\dots,x_{n-1}) \in \mathcal{D}$.

Also assume that for a.e. $t \in [0,T]$ *and each* $(x_0,\dots,x_{n-1}) \in \mathcal{D}$ *we have*

$$f(t,x_0,\dots,x_{n-1}) \leqslant \varphi(t) + \sum_{i=0}^{n-1} \omega_i\bigl(|x_i|\bigr) + \sum_{i=0}^{n-1} h_i(t)|x_i|^{\alpha_i},$$

where $\alpha_i \in (0,1)$, $\varphi, h_i \in L_1[0,T]$ *are nonnegative*, $\omega_i : \mathbb{R}_+ \to \mathbb{R}_+$ *are nonincreasing*, $0 \leqslant i \leqslant n-1$, *and*

$$\int_0^T \omega_i\bigl(t^{r+n-i}\bigr)\,\mathrm{d}t < \infty \quad \textit{for } 0 \leqslant i \leqslant n-1.$$

Then there exists a solution $u \in AC^{n-1}[0,T]$ *of problem* (2.5), (2.6) *with*

$$\begin{cases} u^{(i)} > 0 \quad \textit{on } (0,T] & \textit{for } 0 \leqslant i \leqslant p-1, \\ (-1)^{j-p}u^{(j)} > 0 \quad \textit{on } [0,T) & \textit{for } p \leqslant j \leqslant n-1. \end{cases} \tag{2.7}$$

SKETCH OF PROOF. *Step* 1. *Construction of a sequence of regular differential equations related to Eq.* (2.5).

Put

$$\varphi^*(t) = \varphi(t) + \sum_{i=0}^{n-1} \omega_i(1) + \sum_{i=0}^{n-1} h_i(t) \quad \text{for a.e. } t \in [0,T].$$

Then $\varphi^* \in L_1[0,T]$ and there exists $r^* > 0$ such that the estimate $\|u^{(n-1)}\|_\infty < r^*$ is valid for any function $u \in AC^{n-1}[0,T]$ satisfying (2.6), $(-1)^{n-p}u^{(n)}(t) \geqslant \varepsilon(T-t)^r$ and

$$(-1)^{n-p}u^{(n)}(t) \leqslant \varphi^*(t) + \sum_{i=0}^{n-1} \omega_i\bigl(\bigl|u^{(i)}(t)\bigr|\bigr) + \sum_{i=0}^{n-1} h_i(t)\bigl|u^{(i)}(t)\bigr|^{\alpha_i}$$

for a.e. $t \in [0,T]$. Now for $m \in \mathbb{N}$, $0 \leqslant i \leqslant n-1$ and $x \in \mathbb{R}$, put $\varrho_i = 1 + r^* T^{n-i-1}$ and

$$\sigma_i\left(\tfrac{1}{m}, x\right) = \begin{cases} \frac{1}{m}\operatorname{sign} x & \text{for } |x| < \frac{1}{m}, \\ x & \text{for } \frac{1}{m} \leqslant |x| \leqslant \varrho_i, \\ \varrho_i \operatorname{sign} x & \text{for } \varrho_i < |x|. \end{cases}$$

Extend f onto $[0,T] \times (\mathbb{R}\setminus\{0\})^n$ as an even function in each of its space variables x_i, $0 \leqslant i \leqslant n-1$, and for a.e. $t \in [0,T]$ and each $(x_0,\dots,x_{n-1}) \in \mathbb{R}^n$, define auxiliary functions

$$f_m(t,x_0,\dots,x_{n-1}) = f\left(t, \sigma_0\left(\frac{1}{m}, x_0\right), \dots, \sigma_{n-1}\left(\frac{1}{m}, x_{n-1}\right)\right), \quad m \in \mathbb{N}.$$

In this way we get the family of regular differential equations

$$(-1)^{n-p}u^{(n)} = f_m\big(t, u, \ldots, u^{(n-1)}\big) \tag{2.8}$$

depending on $m \in \mathbb{N}$.

Step 2. *Properties of solutions to problems* (2.8)*,* (2.6).

By Theorem 2.1 we show that for any $m \in \mathbb{N}$, problem (2.8), (2.6) has a solution $u_m \in AC^{n-1}[0, T]$ satisfying (for $t \in [0, T]$)

$$\begin{cases} u_m^{(i)}(t) \geqslant ct^{r+n-i} & \text{if } 0 \leqslant i \leqslant p-1, \\ (-1)^{i-p}u_m^{(i)}(t) \geqslant c(T-t)^{r+n-i} & \text{if } p \leqslant i \leqslant n-1, \end{cases} \tag{2.9}$$

where c is a positive constant and $\|u_m^{(n-1)}\|_\infty < r^*$. Moreover, the sequence $\{u_m^{(n-1)}\}$ is equicontinuous on $[0, T]$. By virtue of the Arzelà–Ascoli theorem, a convergent subsequence $\{u_{k_m}\}$ exists and let $\lim_{m\to\infty} u_{k_m} = u$. Then $u \in C^{n-1}[0, T]$, u satisfies (2.6) and, because of (2.9),

$$\begin{aligned} &u^{(i)}(t) \geqslant ct^{r+n-i} && \text{for } t \in [0, T] \text{ and } 1 \leqslant i \leqslant p-1, \\ &(-1)^{i-p}u^{(i)}(t) \geqslant c(T-t)^{r+n-i} && \text{for } t \in [0, T] \text{ and } p \leqslant i \leqslant n-1. \end{aligned}$$

Also,

$$\left| f_{k_m}\big(t, u_{k_m}(t), \ldots, u_{k_m}^{(n-1)}(t)\big)\right| \leqslant \varrho(t) \quad \text{for a.e. } t \in [0, T] \text{ and all } m \in \mathbb{N},$$

where $\varrho \in L_1[0, T]$ and

$$\lim_{m\to\infty} f_{k_m}\big(t, u_{k_m}(t), \ldots, u_{k_m}^{(n-1)}(t)\big) = f\big(t, u(t), \ldots, u^{(n-1)}(t)\big) \quad \text{for a.e. } t \in [0, T].$$

Now, letting $m \to \infty$ in

$$u_{k_m}^{(n-1)}(t) = u_{k_m}^{(n-1)}(0) + (-1)^{n-p}\int_0^t f_{k_m}\big(s, u_{k_m}(s), \ldots, u_{k_m}^{(n-1)}(s)\big)\,\mathrm{d}s$$

we conclude

$$u^{(n-1)}(t) = u^{(n-1)}(0) + (-1)^{n-p}\int_0^t f\big(s, u(s), \ldots, u^{(n-1)}(s)\big)\,\mathrm{d}s, \quad t \in [0, T].$$

Hence $u \in AC^{n-1}[0, T]$ and u is a solution of problem (2.5), (2.6). □

EXAMPLE. Let $\gamma \in (0,1)$, $\alpha_i \in [0,1)$, $c_i \in (0,\infty)$, $\beta_i \in (0, \frac{1}{n+\gamma-i})$ and let $h_i \in L_1[0,T]$ be nonnegative for $0 \leqslant i \leqslant n-1$. Then, by Theorem 2.2, there exists a solution $u \in AC^{n-1}[0,T]$ of the differential equation

$$(-1)^{n-p} u^{(n)} = \left(\frac{T-t}{t}\right)^{\gamma} + \sum_{i=0}^{n-1} \frac{c_i}{|u^{(i)}|^{\beta_i}} + \sum_{i=0}^{n-1} h_i(t) \left|u^{(i)}\right|^{\alpha_i}$$

satisfying the $(p, n-p)$ right focal boundary conditions (2.6) and (2.7).

REMARK 2.3. Substituting $t = T - s$ in (2.5), (2.6) and using Theorem 2.2 we can also give results for the existence of solutions to singular differential equations satisfying $(n-p, p)$ left focal boundary conditions

$$u^{(i)}(0) = 0, \quad p \leqslant i \leqslant n-1, \qquad u^{(j)}(T) = 0, \quad 0 \leqslant j \leqslant p-1$$

(see [120, Theorem 4.4]).

The singular problem (2.5), (2.6) was also considered on the interval [0, 1] by Agarwal, O'Regan and Lakshmikantham in [12] where f is assumed to be continuous and independent of space variables $x_p, \dots, x_{n-1}$ and may be singular at $x_i = 0$, $0 \leqslant i \leqslant p-1$. They examined the problem

$$\begin{cases} (-1)^{n-p} u^{(n)} = \varphi(t) h\big(t, u, \dots, u^{(p-1)}\big), \\ u^{(i)}(0) = 0, \quad 0 \leqslant i \leqslant p-1, \qquad u^{(j)}(1) = 0, \quad p \leqslant j \leqslant n-1, \end{cases} \tag{2.10}$$

under the assumptions

$$\varphi \in C^0(0,1) \quad \text{with } \varphi > 0 \text{ on } (0,1) \text{ and } \varphi \in L_1[0,1], \tag{2.11}$$

$$h : [0,1] \times \mathbb{R}_+^p \to \mathbb{R}_+ \text{ is continuous,} \tag{2.12}$$

$$\begin{cases} h(t, x_0, \dots, x_{p-1}) \leqslant \displaystyle\sum_{i=0}^{p-1} g_i(x_i) + r\big(\max\{x_0, \dots, x_{p-1}\}\big) \quad \text{on } [0,1] \times \mathbb{R}_+^p \\ \text{with } g_i > 0 \text{ continuous and nonincreasing on } \mathbb{R}_+ \text{ for each } i = 0, \dots, p-1 \\ \text{and } r \geqslant 0 \text{ continuous and nondecreasing on } [0,\infty), \end{cases} \tag{2.13}$$

$$\begin{cases} h(t, x_0, \dots, x_{p-1}) \geqslant \displaystyle\sum_{i=0}^{p-1} h_i(x_i) \quad \text{on } [0,1] \times \mathbb{R}_+^p \\ \text{with } h_i > 0 \text{ continuous and nonincreasing on } \mathbb{R}_+ \text{ for each } i = 0, \dots, p-1, \end{cases} \tag{2.14}$$

$$\begin{cases} \int_0^1 \varphi(t) g_i\big(k_i t^{p-i}\big)\,\mathrm{d}t < \infty \quad \text{for each } i = 0, \dots, p-1, \\ \text{where } k_i > 0 \ (i = 0, \dots, p-1) \text{ are constants} \end{cases} \tag{2.15}$$

and

$$\begin{cases} \text{if } z > 0 \text{ satisfies } z \leqslant a_0 + b_0 r(z) \text{ for constants } a_0 \geqslant 0 \text{ and } b_0 \geqslant 0, \\ \text{then there exists a constant } K \text{ (which may depend only on } a_0 \text{ and } b_0) \\ \text{such that } z \leqslant K. \end{cases} \tag{2.16}$$

The next result was proved by sequential technique and a nonlinear alternative of Leray–Schauder type [77, Theorem 2.3].

THEOREM 2.4 [12, Theorem 2.1]. *Suppose* (2.11)–(2.16) *hold. Then problem* (2.10) *has a solution* $u \in C^{n-1}[0,1] \cap C^n(0,1)$ *with* $u^{(i)} > 0$ *on* $(0,1]$ *for* $0 \leqslant i \leqslant p-1$.

EXAMPLE. Consider the problem

$$\begin{cases} (-1)^{n-p} u^{(n)} = \displaystyle\sum_{i=0}^{p-1} \left(\frac{1}{(u^{(i)})^{\beta_i}} + \mu_i \big(u^{(i)}\big)^{\alpha_i} + \tau_i \right), \\ u^{(i)}(0) = 0, \quad 0 \leqslant i \leqslant p-1, \qquad u^{(j)}(1) = 0, \quad p \leqslant j \leqslant n-1 \end{cases} \tag{2.17}$$

with $\beta_i \in (0,\infty)$, $\mu_i, \tau_i \in [0,\infty)$, $\alpha_i \in [0,1)$ for $0 \leqslant i \leqslant p-1$. In addition, assume $\beta_i(p-i) < 1$ for $i = 0, \dots, p-1$. Theorem 2.4 guarantees that problem (2.17) has a solution $u \in C^{n-1}[0,1] \cap C^n(0,1)$ with $u^{(i)} > 0$ on $(0,1]$ for $0 \leqslant i \leqslant p-1$.

2.2. *Conjugate conditions*

Let $1 \leqslant p \leqslant n-1$ be a fixed natural number. Consider the $(p, n-p)$ conjugate problem

$$(-1)^p u^{(n)} = f\big(t, u, \dots, u^{(n-1)}\big), \tag{2.18}$$

$$u^{(i)}(0) = 0, \quad 0 \leqslant i \leqslant n-p-1, \qquad u^{(j)}(T) = 0, \quad 0 \leqslant j \leqslant p-1 \tag{2.19}$$

where $n \geqslant 2$, $f \in Car([0,T] \times \mathcal{D})$, $\mathcal{D} = (0,\infty) \times (\mathbb{R} \setminus \{0\})^{n-1}$ and f may be singular at the value 0 of all its space variables.

Replacing t by $T - t$, if necessary, we may assume that $p \in \{1, \dots, \frac{n}{2}\}$ for n even and $p \in \{1, \dots, \frac{n+1}{2}\}$ for n odd. We observe that the larger p is chosen, the more complicated structure of the set of all singular points of a solution to (2.18), (2.19) and its derivatives is obtained. We note that solutions of problem (2.18), (2.19) have singular points of type I at $t = 0$ and/or $t = T$ and also singular points of type II. Since the singular problem (2.18), (2.19) for $n = 2$ is the Dirichlet problem discussed in Section 4, we assume that $n > 2$.

THEOREM 2.5 ([121, Theorems 2.1 and 2.7] and [123]). *Let $n > 2$ and $1 \leqslant p \leqslant n-1$ be fixed natural numbers. Suppose that the following conditions are satisfied*:

$$\begin{cases} f \in Car\big([0,T]\times\mathcal{D}\big) \text{ and there exists } c>0 \text{ such that} \\ \quad c \leqslant f(t,x_0,\dots,x_{n-1}) \\ \text{for a.e. } t\in[0,T] \text{ and all } (x_0,\dots,x_{n-1})\in\mathcal{D}, \end{cases} \tag{2.20}$$

$$\begin{cases} h \in Car\big([0,T]\times[0,\infty)\big) \text{ is nondecreasing in its second variable and} \\ \quad \limsup\limits_{z\to\infty} \dfrac{1}{z}\displaystyle\int_0^T h(t,z)\,\mathrm{d}t < \left(1+\sum_{i=0}^{n-2}\frac{T^{n-i-1}}{(n-i-2)!}\right)^{-1}, \end{cases} \tag{2.21}$$

$$\begin{cases} \omega_i : (0,\infty)\to(0,\infty) \text{ are nonincreasing and} \\ \quad \displaystyle\int_0^T \omega_i\big(t^{n-i}\big)\,\mathrm{d}t < \infty \quad \text{for } 0\leqslant i\leqslant n-1, \end{cases} \tag{2.22}$$

$$\begin{cases} f(t,x_0,\dots,x_{n-1}) \leqslant h\left(t,\displaystyle\sum_{i=0}^{n-1}|x_i|\right)+\sum_{i=0}^{n-1}\omega_i\big(|x_i|\big) \\ \text{for a.e. } t\in[0,T] \text{ and all } (x_0,\dots,x_{n-1})\in\mathcal{D}. \end{cases} \tag{2.23}$$

Then the $(p, n-p)$ conjugate problem (2.18), (2.19) *has a solution $u \in AC^{n-1}[0,T]$ and $u > 0$ on $(0,T)$.*

SKETCH OF PROOF. *Step* 1. *Uniform integrability.*
Put

$$\mathcal{B} = \big\{u \in AC^{n-1}[0,T]\text{: } u \text{ satisfies (2.19) and } (-1)^p u^{(n)}(t) \geqslant c \\ \text{for a.e. } t\in[0,T]\big\},$$

where $c > 0$ is taken from (2.20). Conditions (2.20) and (2.22) guarantee that there exists a positive constant A such that

$$\int_0^T \omega_i\big(\big|u^{(i)}(t)\big|\big)\,\mathrm{d}t \leqslant A \quad \text{for each } u\in\mathcal{B} \text{ and } 0\leqslant i\leqslant n-1$$

and that the set of functions $\{\omega_i(|u^{(i)}(t)|)\text{: } u\in\mathcal{B},\ 0\leqslant i\leqslant n-1\}$ is uniformly integrable on $[0,T]$. Also, $u > 0$ on $(0,T)$ for each $u \in \mathcal{B}$.

Step 2. *Estimates of functions belonging to $\mathcal{B}$.*

By virtue of (2.21), there exists $r^* > 1$ such that the estimate $\|u\|_{C^{n-1}} < r^*$ holds for each function $u \in \mathcal{B}$ satisfying

$$u^{(n)}(t) \leqslant h\left(t, n+\sum_{i=0}^{n-1}\big|u^{(i)}(t)\big|\right)+\sum_{i=0}^{n-1}\big[\omega_i\big(\big|u^{(i)}(t)\big|\big)+\omega_i(1)\big] \\ \text{for a.e. } t\in[0,T].$$

Step 3. *Construction of regular problems to* (2.18)*,* (2.19) *and properties of their solutions.*

For $m \in \mathbb{N}$, let $h_m \in Car([0,T] \times ([0,\infty) \times \mathbb{R}^{n-1}))$ be such that $h_m(t,x_0,\ldots,x_{n-1}) = f(t,x_0,\ldots,x_{n-1})$ for a.e. $t \in [0,T]$ and any $x_0 \geqslant \frac{1}{m}$, $|x_j| \geqslant \frac{1}{m}$, $1 \leqslant j \leqslant n-1$. Put

$$f_m(t,x_0,x_1,\ldots,x_{n-1}) = h_m\bigl(t,\sigma_0(x_0),\sigma(x_1),\ldots,\sigma(x_{n-1})\bigr)$$

for a.e. $t \in [0,T]$ and each $(x_0,x_1,\ldots,x_{n-1}) \in \mathbb{R}^n$, where

$$\sigma_0(x) = \begin{cases} |x| & \text{if } |x| \leqslant r^*, \\ r^* & \text{if } |x| > r^*, \end{cases} \qquad \sigma(x) = \begin{cases} x & \text{if } |x| \leqslant r^*, \\ r^* \operatorname{sign} x & \text{if } |x| > r^*. \end{cases}$$

Now, the sequence of regular differential equations

$$(-1)^p u^{(n)} = f_m\bigl(t,u,\ldots,u^{(n-1)}\bigr) \tag{2.24}$$

is considered. It follows from Theorem 2.1 that for each $m \in \mathbb{N}$ there exists a solution u_m of problem (2.24), (2.19) and $\|u_m\|_{C^{n-1}} < r^*$. Moreover, the sequence of functions $\{f_m(t,u_m(t),\ldots,u_m^{(n-1)}(t))\}$ is uniformly integrable on $[0,T]$. By the Arzelà–Ascoli theorem there exists a subsequence $\{u_{k_m}\}$ converging in $C^{n-1}[0,T]$, $\lim_{m\to\infty} u_{k_m} = u$. Then $u \in C^{n-1}[0,T]$ satisfies (2.19) and for each $i \in \{1,\ldots,n-1\}$, the function $u^{(n-i)}$ has a finite number of zeros $0 \leqslant a_{i1} < \cdots < a_{i,p_i} \leqslant T$ and satisfies

$$\bigl|u^{(n-i)}(t)\bigr| \geqslant \frac{c}{i!}(t-a_{ik})^i \quad \text{for } t \in [a_{ik},a_{i,k+1}]$$

or

$$\bigl|u^{(n-i)}(t)\bigr| \geqslant \frac{c}{i!}(a_{i,k+1}-t)^i \quad \text{for } t \in [a_{ik},a_{i,k+1}]$$

(see [123]). Therefore $u \in AC^{n-1}[0,T]$ and u is a solution of problem (2.18), (2.19) due to Theorem 1.8. From Step 1 and (2.20) it follows that $u > 0$ on $(0,T)$. □

EXAMPLE. Let $p \in \{1,\ldots,n-1\}$. Let $\alpha_i \in (0,1)$, $\beta_i \in (0,\frac{1}{n-i})$ and $b_i \in L_1[0,T]$, $c_i \in L_\infty[0,T]$ be nonnegative for $0 \leqslant i \leqslant n-1$. Also, let $\varphi \in L_1[0,T]$ and $\varphi(t) \geqslant c$ for a.e. $t \in [0,T]$ with $c > 0$. Then the differential equation

$$(-1)^p u^{(n)} = \varphi(t) + \sum_{i=0}^{n-1}\left(b_i(t)\bigl|u^{(i)}\bigr|^{\alpha_i} + \frac{c_i(t)}{|u^{(i)}|^{\beta_i}}\right)$$

has a solution $u \in AC^{n-1}[0,T]$ satisfying (2.19) and $u > 0$ on $(0,T)$.

2.3. (n, p) *boundary conditions*

Here we are concerned with the singular (n, p) problem

$$-u^{(n)} = f\big(t, u, \dots, u^{(n-1)}\big), \tag{2.25}$$

$$\begin{cases} u^{(i)}(0) = 0, & 0 \leqslant i \leqslant n-2, \\ u^{(p)}(T) = 0, & p \text{ fixed}, 0 \leqslant p \leqslant n-1, \end{cases} \tag{2.26}$$

where $n \geqslant 2$, $f \in Car([0, T] \times \mathcal{D})$, $\mathcal{D} = (0, \infty) \times (\mathbb{R} \setminus \{0\})^{n-2} \times \mathbb{R}$ and f may be singular at the value 0 of its space variables $x_0, \dots, x_{n-2}$. Notice that the $(n, 0)$ problem is simultaneously also the $(1, n-1)$ conjugate problem. For $f > 0$, solutions of problem (2.25), (2.26) have singular points of type I at $t = 0$, $t = T$ and also singular points of type II.

THEOREM 2.6 [14, Theorem 4.2]. *Suppose*

$$\begin{cases} f \in Car([0, T] \times \mathcal{D}) \text{ and there exist positive } \psi \in L_1[0, T] \text{ and} \\ K \in (0, \infty) \text{ such that } \psi(t) \leqslant f(t, x_0, \dots, x_{n-1}) \quad \text{for a.e. } t \in [0, T] \\ \text{and each } (x_0, \dots, x_{n-1}) \in (0, K] \times (\mathbb{R} \setminus \{0\})^{n-2} \times \mathbb{R}, \end{cases} \tag{2.27}$$

$$\begin{cases} h_j \in L_1[0, T] \text{ is nonnegative}, \omega_i : (0, \infty) \to (0, \infty) \text{ is nonincreasing}, \\ \displaystyle\sum_{k=0}^{n-1} \frac{1}{(n-k-1)!} \int_0^T h_k(t) t^{n-k-1}\,\mathrm{d}t < 1, \quad \int_0^T \omega_i\big(t^{n-i-1}\big)\,\mathrm{d}t < \infty \\ \text{for } 0 \leqslant j \leqslant n-1 \text{ and } 0 \leqslant i \leqslant n-2, \end{cases} \tag{2.28}$$

$$\begin{cases} \text{for a.e. } t \in [0, T] \text{ and each } (x_0, \dots, x_{n-1}) \in \mathcal{D} \text{ we have} \\ \displaystyle f(t, x_0, \dots, x_{n-1}) \leqslant \varphi(t) + \sum_{i=0}^{n-2} \omega_i\big(|x_i|\big) + \sum_{j=0}^{n-1} h_j(t)|x_j| \\ \text{where } \varphi \in L_1[0, T] \text{ is nonnegative}. \end{cases} \tag{2.29}$$

Then there exists a solution $u \in AC^{n-1}[0, T]$ *of problem* (2.25), (2.26) *with* $u^{(i)} > 0$ *on* $(0, T]$ *for* $0 \leqslant i \leqslant p-1$ (*if* $p \geqslant 1$) *and* $u^{(p)} > 0$ *on* $(0, T)$.

SKETCH OF PROOF. *Step* 1. *A priori bounds for solutions of problem* (2.25), (2.26).

Upper and lower bounds for solutions of problem (2.25), (2.26) and their derivatives are given by means of the Green function of the problem $-u^{(n)} = 0$, (2.26) (see, e.g., [1]).

Step 2. *Construction of auxiliary regular problems and properties of their solutions.*

Using Step 1, a sequence of regular differential equations

$$-u^{(n)} = f_m\big(t, u, \dots, u^{(n-1)}\big), \quad m \in \mathbb{N},\ m \geqslant m_0 \geqslant \frac{1}{K}, \tag{2.30}$$

with $f_m \in Car([0, T] \times \mathbb{R}^n)$ is constructed. By the Leray–Schauder degree theory, we prove that for any $m \geqslant m_0$ problem (2.30), (2.26) has a solution u_m. The sequence

$\{u_m\}_{m=m_0}^{\infty}$ is bounded in $C^{n-1}[0,T]$ and $\{u_m^{(n-1)}\}_{m=m_0}^{\infty}$ is equicontinuous on $[0,T]$. The Arzelà–Ascoli theorem guarantees the existence of a subsequence $\{u_k\}_{k=1}^{\infty}$ of $\{u_m\}_{m=m_0}^{\infty}$ converging in $C^{n-1}[0,T]$ to a function u. Then $u \in C^{n-1}[0,T]$ satisfies (2.26) and $u^{(i)} > 0$ on $(0,T]$ for $0 \leqslant i \leqslant p-1$ (if $p \geqslant 1$) and $u^{(p)} > 0$ on $(0,T)$. Moreover, for each $i \in \{p+1,\dots,n-2\}$, the function $u^{(i)}$ has a unique zero ξ_i in $(0,T)$ $(0 < \xi_{n-2} < \xi_{n-1} < \dots < \xi_{p+1} < T)$ and satisfies

$$u^{(i)}(t) \geqslant \begin{cases} ct^{n-i-1} & \text{for } t \in [0,\xi_{i+1}],\\ c(\xi_i - t) & \text{for } t \in [\xi_{i+1},\xi_i], \end{cases} \qquad u^{(i)}(t) \leqslant c(\xi_i - t) \quad \text{for } t \in [\xi_i, T]$$

where c is a positive constant. Since $\{f_k(t,u_k(t),\dots,u_k^{(n-1)}(t))\}$ is uniformly integrable on $[0,T]$, we can use Theorem 1.8 concluding that $u \in AC^{n-1}[0,T]$ and u is a solution of problem (2.25), (2.26). □

A related existence result for the singular (n,p) problem

$$\begin{cases} -u^{(n)} = \varphi(t)h\big(t,u,\dots,u^{(p-1)}\big),\\ u^{(i)}(0) = 0, \quad 0 \leqslant i \leqslant n-2,\\ u^{(p)}(1) = 0, \quad p \text{ fixed}, 1 \leqslant p \leqslant n-1 \end{cases} \tag{2.31}$$

was presented in [12] with h continuous and positive on $[0,1] \times (0,\infty)^p$ and $\varphi \in C^0(0,1) \cap L_1[0,1]$ positive on $(0,1)$. In this setting solutions of problem (2.31) cannot have singular points of type II. The result is the following.

THEOREM 2.7 [12, Theorem 3.1]. *Suppose that* (2.11)–(2.14) *and* (2.16) *hold and*

$$\begin{cases} \displaystyle\int_0^1 \varphi(t)g_i\big(k_i t^{n-1-i}\big)\,\mathrm{d}t < \infty \quad \textit{for each } i = 0,\dots,p-1,\\ \textit{where } k_i > 0,\ i = 0,\dots,p-1,\ \textit{are constants.} \end{cases}$$

Then problem (2.31) *has a solution* $u \in C^{n-1}[0,1] \cap C^n(0,1)$ *with* $u^{(j)} > 0$ *on* $(0,1]$ *for* $0 \leqslant j \leqslant p-1$.

2.4. *Sturm–Liouville conditions*

We are now concerned with the Sturm–Liouville problem for the nth-order differential equation (2.25), $n \geqslant 3$, and the boundary conditions

$$\begin{cases} u^{(i)}(0) = 0, \quad 0 \leqslant i \leqslant n-3,\\ \alpha u^{(n-2)}(0) - \beta u^{(n-1)}(0) = 0,\\ \gamma u^{(n-2)}(T) + \delta u^{(n-1)}(T) = 0, \end{cases} \tag{2.32}$$

where $\alpha, \gamma > 0$ and $\beta, \delta \geqslant 0$. Notice that the function f in Eq. (2.25) may be singular at the value 0 of its space variables $x_0, \dots, x_{n-1}$. If $f > 0$, solutions of problem (2.25), (2.32) have singular points of type I at the end points of the interval $[0, T]$ and also singular points of type II.

We will impose the following conditions on the function f in (2.25):

$$\begin{cases} f \in Car\big([0,T] \times \mathcal{D}\big) \text{ where } \mathcal{D} = (0,\infty)^{n-1} \times \big(\mathbb{R} \setminus \{0\}\big) \\ \text{and there exist positive constants } \varepsilon \text{ and } r \text{ such that} \\ \quad \varepsilon t^r \leqslant f(t, x_0, \dots, x_{n-1}) \\ \text{for a.e. } t \in [0,T] \text{ and each } (x_0, \dots, x_{n-1}) \in \mathcal{D}; \end{cases} \tag{2.33}$$

$$\begin{cases} \text{for a.e. } t \in [0,T] \text{ and each } (x_0, \dots, x_{n-1}) \in \mathcal{D}, \\ \quad f(t, x_0, \dots, x_{n-1}) \leqslant \varphi(t) + \displaystyle\sum_{i=0}^{n-1} \omega_i\big(|x_i|\big) + \sum_{i=0}^{n-1} h_i(t)|x_i|^{\alpha_i} \\ \text{with } \alpha_i \in (0,1),\ \varphi, h_i \in L_1[0,T] \text{ nonnegative,} \\ \omega_i : (0,\infty) \to (0,\infty) \text{ nonincreasing, } 0 \leqslant i \leqslant n-1, \text{ and} \\ \quad \displaystyle\int_0^T \omega_{n-1}\big(t^{r+1}\big)\,\mathrm{d}t < \infty, \quad \int_0^T \omega_i\big(t^{n-i-1}\big)\,\mathrm{d}t < \infty \quad \text{for } 0 \leqslant i \leqslant n-2; \end{cases} \tag{2.34}$$

$$\begin{cases} \text{for a.e. } t \in [0,T] \text{ and each } (x_0, \dots, x_{n-1}) \in \mathcal{D}, \\ \quad f(t, x_0, \dots, x_{n-1}) \leqslant \varphi(t) + \displaystyle\sum_{i=0, i \neq n-2}^{n-1} \omega_i\big(|x_i|\big) + q(t)\omega_{n-2}\big(|x_{n-2}|\big) \\ \qquad + \displaystyle\sum_{i=0}^{n-1} h_i(t)|x_i|^{\alpha_i} \\ \text{with } \alpha_i \in (0,1),\ \varphi, q, h_i \in L_1[0,T] \text{ nonnegative,} \\ \omega_i : (0,\infty) \to (0,\infty) \text{ nonincreasing, } 0 \leqslant i \leqslant n-1, \text{ and} \\ \quad \displaystyle\int_0^T \omega_{n-1}\big(t^{r+1}\big)\,\mathrm{d}t < \infty, \quad \int_0^T \omega_j\big(t^{n-j-2}\big)\,\mathrm{d}t < \infty \quad \text{for } 0 \leqslant j \leqslant n-3. \end{cases} \tag{2.35}$$

The next two theorems show that our sufficient conditions for the solvability of problem (2.25), (2.32) with $\min\{\beta, \delta\} > 0$ are weaker then those for this problem with $\min\{\beta, \delta\} = 0$.

THEOREM 2.8 [120, Theorem 4.1]. *Let conditions* (2.33) *and* (2.34) *be satisfied and let* $\min\{\beta, \delta\} = 0$. *Then problem* (2.25), (2.32) *has a solution* $u \in AC^{n-1}[0,T]$, $u^{(n-2)} > 0$ *on* $(0,T)$ *and* $u^{(j)} > 0$ *on* $(0,T]$ *for* $0 \leqslant j \leqslant n-3$.

SKETCH OF PROOF. *Step* 1. *A priori bounds for the solution of* (2.25), (2.32).

By (2.33) and by the properties of the Green function to problem

$$-u'' = 0, \quad \alpha u(0) - \beta u'(0) = 0, \quad \gamma u(T) + \delta u'(T) = 0,$$

the existence of a positive constant A is proved such that for any function $u \in AC^{n-1}[0,T]$ satisfying (2.32) and $-u^{(n)}(t) \geqslant \varepsilon t^r$ for a.e. $t \in [0,T]$ we have

$$u^{(n-2)}(t) \geqslant \begin{cases} At & \text{for } t \in [0, \frac{T}{2}], \\ A(T-t) & \text{for } t \in (\frac{T}{2}, T], \end{cases} \tag{2.36}$$

$$u^{(j)}(t) \geqslant \frac{A}{4(n-j-1)!} t^{n-j-1} \quad \text{for } t \in [0,T] \text{ and } 0 \leqslant j \leqslant n-3 \tag{2.37}$$

and

$$u^{(n-1)}(t) \begin{cases} \geqslant \frac{\varepsilon}{r+1}(\xi - t)^{r+1} & \text{for } t \in [0, \xi], \\ < -\frac{\varepsilon}{r+1}(t-\xi)^{r+1} & \text{for } t \in (\xi, T], \end{cases} \tag{2.38}$$

where $\xi \in (0,T)$ (depending on the solution u) is the unique zero of $u^{(n-1)}$. Condition (2.34) guarantees the existence of a positive constant S such that $\|u\|_{C^{n-1}} \leqslant S$ for any solution u to (2.25), (2.32).

Step 2. *Construction of regular differential equations.*

Using Step 1, a sequence of regular differential equations (2.30) is constructed where $f_m \in Car([0,T] \times \mathbb{R}^n)$,

$$f_m(t, x_0, \ldots, x_{n-1}) = f(t, x_0, \ldots, x_{n-1})$$

for a.e. $t \in [0,T]$ and all $(x_0, \ldots, x_{n-1}) \in \mathbb{R}^n$ such that

$$\frac{1}{m} \leqslant x_j \leqslant S+1 \quad \text{if } 0 \leqslant j \leqslant n-2, \qquad \frac{1}{m} \leqslant |x_{n-1}| \leqslant S+1$$

and

$$\sup\bigl\{ f_m(t, x_0, \ldots, x_{n-1}) \colon (x_0, \ldots, x_{n-1}) \in \mathbb{R}^n \bigr\} \in L_1[0,T] \quad \text{for all } m \in \mathbb{N}.$$

Then Theorem 2.1 guarantees that the regular problem (2.30), (2.32) has a solution u_m which satisfies (2.36)–(2.38) (with u_m instead of u).

Step 3. *Properties of solutions to regular problems* (2.30), (2.32).

The sequence $\{u_m\}$ is considered. It is proved that $\{u_m\}$ is bounded in $C^{n-1}[0,T]$ and, by (2.34), the sequence of functions $\{f_m(t, u_m(t), \ldots, u_m^{(n-1)}(t))\}$ is uniformly integrable on $[0,T]$, which implies that $\{u_m^{(n-1)}\}$ is equicontinuous on $[0,T]$. Hence a subsequence

$\{u_{k_m}\}$ converging in $C^{n-1}[0,T]$ exists and let $\lim_{m\to\infty} u_{k_m} = u$. Since u satisfies (2.36)–(2.38), the functions $u^{(i)}$, $0 \leqslant i \leqslant n-1$, have a finite number of zeros in $[0,T]$. Therefore, by Theorem 1.8, $u \in AC^{n-1}[0,T]$ and u is a solution of problem (2.25), (2.32) such that $u^{(j)} > 0$ on $(0,T]$ for $0 \leqslant j \leqslant n-3$ and $u^{(n-2)} > 0$ on $(0,T)$. □

THEOREM 2.9 [120, Theorem 4.2]. *Let conditions* (2.33) *and* (2.35) *be satisfied and let* $\min\{\beta,\delta\} > 0$. *Then there exists a solution* $u \in AC^{n-1}[0,T]$ *of problem* (2.25), (2.32) *such that* $u^{(n-2)} > 0$ *on* $[0,T]$ *and* $u^{(j)} > 0$ *on* $(0,T]$ *for* $0 \leqslant j \leqslant n-3$.

SKETCH OF PROOF. Since $\min\{\beta,\delta\} > 0$, there is a positive constant B such that $u^{(n-2)} \geqslant B$ on $[0,T]$ for any solution u of problem (2.25), (2.32). Further, the inequalities (2.37) with B instead of A and (2.38) hold. Next we argue as in the sketch of proof to Theorem 2.8. □

2.5. *Lidstone conditions*

Let $\mathbb{R}_- = (-\infty,0)$, $\mathbb{R}_+ = (0,\infty)$ and $\mathbb{R}_0 = \mathbb{R}\setminus\{0\}$. Here we will consider the singular problem

$$(-1)^n u^{(2n)} = f\bigl(t,u,\dots,u^{(2n-2)}\bigr), \tag{2.39}$$

$$u^{(2j)}(0) = u^{(2j)}(T) = 0, \quad 0 \leqslant j \leqslant n-1, \tag{2.40}$$

where $n \geqslant 1$ and $f \in Car([0,T]\times\mathcal{D})$ with

$$\mathcal{D} = \begin{cases} \underbrace{\mathbb{R}_+\times\mathbb{R}_0\times\mathbb{R}_-\times\mathbb{R}_0\times\cdots\times\mathbb{R}_+}_{4k-3} & \text{if } n = 2k-1,\\ \underbrace{\mathbb{R}_+\times\mathbb{R}_0\times\mathbb{R}_-\times\mathbb{R}_0\times\cdots\times\mathbb{R}_-}_{4k-1} & \text{if } n = 2k \end{cases}$$

(for $n = 1, 2$ and 3 we have $\mathcal{D} = \mathbb{R}_+$, $\mathcal{D} = \mathbb{R}_+\times\mathbb{R}_0\times\mathbb{R}_-$ and $\mathcal{D} = \mathbb{R}_+\times\mathbb{R}_0\times\mathbb{R}_-\times\mathbb{R}_0\times\mathbb{R}_+$, respectively). The function f may be singular at the value 0 of its space variables $x_0,\dots,x_{2n-2}$. If f is positive on $[0,T]\times\mathcal{D}$, solutions of problem (2.39), (2.40) have singular points of type I at $t=0$ and $t=T$ as well as singular points of type II.

THEOREM 2.10 [14, Theorem 4.1]. *Let the following conditions be satisfied*:

$$\begin{cases} f \in Car([0,T]\times\mathcal{D}) \textit{ and there exists } \varphi \in L_1[0,T] \textit{ such that}\\ \quad 0 < \varphi(t) \leqslant f(t,x_0,\dots,x_{2n-2})\\ \textit{for a.e. } t\in[0,T] \textit{ and each } (x_0,\dots,x_{2n-2}) \in \mathcal{D}; \end{cases} \tag{2.41}$$

$$\begin{cases} \textit{for } 0 \leqslant j \leqslant 2n-2,\ h_j \in L_1[0,T] \textit{ are nonnegative and} \\ \displaystyle\sum_{j=0}^{n-1} \frac{T^{2(n-j)-3}}{6^{n-j-1}} \int_0^T t(T-t)h_{2j}(t)\,\mathrm{d}t \\ \displaystyle\qquad + \sum_{j=0}^{n-2} \frac{T^{2(n-j-2)}}{6^{n-j-2}} \int_0^T t(T-t)h_{2j+1}(t)\,\mathrm{d}t < 1 \\ (\textit{here } \sum_{j=0}^{n-2} = 0 \textit{ if } n=1); \end{cases} \tag{2.42}$$

$$\begin{cases} \omega_j : \mathbb{R}_+ \to \mathbb{R}_+ \textit{ are nonincreasing}, \ \Lambda > 0 \textit{ and} \\ \displaystyle\int_0^T \omega_j(s)\,\mathrm{d}s < \infty, \qquad \omega_j(uv) \leqslant \Lambda\omega_j(u)\omega_j(v) \\ \textit{for } 0 \leqslant j \leqslant 2n-2 \textit{ and } u, v \in \mathbb{R}_+; \end{cases} \tag{2.43}$$

$$\begin{cases} \displaystyle f(t, x_0, \dots, x_{2n-2}) \leqslant \psi(t) + \sum_{j=0}^{2n-2} \omega_j\big(|x_j|\big) + \sum_{j=0}^{2n-2} h_j(t)|x_j| \\ \textit{for a.e. } t \in [0,T] \textit{ and each } (x_0, \dots, x_{2n-2}) \in \mathcal{D}, \\ \textit{where } \psi \in L_1[0,T] \textit{ is nonnegative.} \end{cases} \tag{2.44}$$

Then problem (2.39), (2.40) *has a solution* $u \in AC^{2n-1}[0,T]$ *and*

$$(-1)^j u^{(2j)} > 0 \quad \textit{on } (0,T) \quad \textit{for } 0 \leqslant j \leqslant n-1.$$

SKETCH OF PROOF. *Step* 1. *A priori bounds for solutions of problem* (2.39), (2.40).

Using (2.41) and the properties of the Green functions to problems $u^{(2j)} = 0$, $u^{(2i)}(0) = u^{(2i)}(T) = 0$, $0 \leqslant i \leqslant j-1$, it is proved that for any solution u of problem (2.39), (2.40) and for each j, $0 \leqslant j \leqslant n-1$, the inequality $(-1)^j u^{(2j)} > 0$ holds on $(0,T)$ and the function $(-1)^j u^{(2j+1)}$ is decreasing on $[0,T]$ and vanishes at a unique $\xi_j \in (0,T)$ (depending on u). Moreover,

$$\big|u^{(2j)}(t)\big| \geqslant A \frac{T^{2(n-j)-5}}{30^{n-j-1}} t(T-t), \quad t \in [0,T],\ 0 \leqslant j \leqslant n-1,$$

and (if $n > 1$)

$$\big|u^{(2j+1)}(t)\big| \geqslant A \frac{T^{2(n-j)-7}}{30^{n-j-2}} \left| \int_{\xi_j}^t s(T-s)\,\mathrm{d}s \right|, \quad t \in [0,T],\ 0 \leqslant j \leqslant n-2,$$

where $A = \int_0^T t(T-t)\varphi(t)\,\mathrm{d}t$.

Step 2. *Construction of a sequence of regular problems.*

For each $m \in \mathbb{N}$, define $f_m \in Car([0,T] \times \mathbb{R}^{2n-1})$ satisfying

$$f_m(t, x_0, \dots, x_{2n-2}) = f(t, x_0, \dots, x_{2n-2})$$

for a.e. $t \in [0, T]$ and each $(x_0, \dots, x_{2n-2}) \in \mathcal{D}$, $|x_j| \geqslant \frac{1}{m}$, $0 \leqslant j \leqslant 2n-2$. By virtue of (2.44) and a fixed point theorem of Leray–Schauder type (see, e.g., [50, Corollary 8.1]), for each $m \in \mathbb{N}$ there exists a solution u_m of the regular differential equation $(-1)^n u^{(2n)} = f_m(t, u, \dots, u^{(2n-2)})$ satisfying (2.40). Further, $\|u_m\|_{C^{2n-1}} \leqslant B$ for each $m \in \mathbb{N}$ where B is a positive constant and the sequence $\{f_m(t, u_m(t), \dots, u_m^{(2n-2)}(t))\}$ is uniformly integrable on $[0, T]$ due to (2.43) and (2.44).

Step 3. *Limit processes*.

From Step 2 it follows that $\{u_m\}$ is bounded in $C^{2n-1}[0, T]$. Hence, by the Arzelà–Ascoli theorem and a compactness principle, there exists its subsequence $\{u_{k_m}\}$ which converges in $C^{2n-2}[0, T]$ and $\{u_{k_m}^{(2n-1)}(0)\}$ converges in $\mathbb{R}$. Let $\lim_{m\to\infty} u_{k_m} = u$, $\lim_{m\to\infty} u_{k_m}^{(2n-1)}(0) = C$. Then $u \in C^{2n-2}[0, T]$ satisfies (2.40) and, by Step 1, the functions $u^{(i)}$, $0 \leqslant i \leqslant 2n-1$, have a finite number of zeros on $[0, T]$. Therefore, by Theorem 1.8, $u \in AC^{2n-1}[0, T]$ and u is a solution of (2.39). Moreover, $(-1)^j u^{(2j)} > 0$ on $(0, T)$ for $0 \leqslant j \leqslant n-1$. □

2.6. *Historical and bibliographical notes*

Higher order boundary value problems with space singularities have been mostly studied by Agarwal, Eloe, Henderson, Lakshmikantham, O'Regan, Rachůnková and Staněk.

Positive solutions in the set $C^{n-1}[0, 1] \cap C^n(0, 1)$ were obtained in [7] for the singular $(p, n-p)$ right focal problem $(-1)^{n-p} u^{(n)} = \varphi(t) f(t, u)$, (2.6) on the interval $[0, 1]$ where $f \in C^0([0, 1] \times (0, \infty))$ is positive and may be singular at $u = 0$. In [7] the authors also discussed applications in fluid theory and boundary layer theory.

Singular $(p, n-p)$ conjugate problems were studied in [5], [58] (with $p = 1$) and [59] for the differential equation $(-1)^{n-p} u^{(n)} = f(t, u)$ where $f \in C^0((0, 1) \times (0, \infty))$ and may be singular at $u = 0$. Here positive solutions on $(0, 1)$ belong to the class $C^{n-1}[0, 1] \cap C^n(0, 1)$. Existence results in [58] and [59] are proved by a fixed point theorem for operators that are decreasing with respect to a cone and those in [5] by a nonlinear alternative of Leray–Schauder.

The existence of positive solutions on $(0, 1)$ to singular Sturm–Liouville problems for the differential equation $-u^{(n)} = f(t, u, \dots, u^{(n-2)})$ can be found in [20]. There $f \in C((0, 1) \times (0, \infty)^{n-1})$ is positive and may be singular at the value 0 of all its space variables. The results are proved by a fixed point theorem for mappings that are decreasing with respect to a cone in a Banach space.

Existence results for positive solutions to singular $(p, n-p)$ focal, conjugate and (n, p) problems are given in [8,9] for differential equations with the right-hand side $\varphi(t) f(t, u)$ where $f \in C([0, 1] \times (0, \infty))$ and may be singular at $u = 0$. The paper [8] is the first to establish conditions for the existence of two solutions to singular $(p, n-p)$ focal and (n, p) problems. Further multiple solutions for singular $(p, n-p)$ focal, conjugate and (n, p) problems are established in [9]. The technique presented in [8,9] to guarantee the existence of twin solutions to the singular problems combines (i) a nonlinear alternative of Leray–Schauder, (ii) the Krasnoselskii fixed point theorem in a cone, and (iii) lower type inequalities.

Notice that in all cited papers singular points corresponding to solutions of the singular problems under discussion are only of type I.

3. Principles of solvability of singular second order BVPs with ϕ-Laplacian

In the theory of partial differential equations, the p-Laplace equation

$$\operatorname{div}\big(|\nabla v|^{p-2}\nabla v\big) = h\big(|x|, v\big) \tag{3.1}$$

is considered. Here ∇ is the gradient, $p > 1$ and $|x|$ is the Euclidean norm in $\mathbb{R}^n$ of $x = (x_1, \ldots, x_n)$, $n > 1$. Radially symmetric solutions of (3.1) (i.e., solutions that depend only on the variable $r = |x|$) satisfy the ordinary differential equation

$$r^{1-n}\big(r^{n-1}|v'|^{p-2}v'\big)' = h(r, v), \quad ' = \frac{\mathrm{d}}{\mathrm{d}r}. \tag{3.2}$$

If $p = n$, the change of variables $t = \ln r$ transforms (3.2) into the equation

$$\big(|u'|^{p-2}u'\big)' = \mathrm{e}^{nt} h(\mathrm{e}^t, u), \quad ' = \frac{\mathrm{d}}{\mathrm{d}t}$$

and for $p \neq n$, the change of variables $t = r^{(p-n)/(p-1)}$ transforms (3.2) into the equation

$$\big(|u'|^{p-2}u'\big)' = \left|\frac{p-1}{p-n}\right|^p t^{\frac{p-n}{p(1-n)}} h\big(t^{\frac{p-1}{p-n}}, u\big), \quad ' = \frac{\mathrm{d}}{\mathrm{d}t}.$$

The operator $u \to (|u'|^{p-2}u')'$ is called the (one-dimensional) *p-Laplacian*. Its natural generalization is the *ϕ-Laplacian*

$$u \to \big(\phi(u')\big)',$$
$$\text{where } \phi : \mathbb{R} \to \mathbb{R} \text{ is an increasing homeomorphism and } \phi(\mathbb{R}) = \mathbb{R}. \tag{3.3}$$

Therefore Eq. (3.1) was a motivation for discussing the solutions to the differential equations

$$\big(|u'|^{p-2}u'\big)' = f(t, u, u')$$

and

$$\big(\phi(u')\big)' = f(t, u, u')$$

with the p-Laplacian and the ϕ-Laplacian, respectively.

In the next part of this section, we treat problems for second order differential equations with the ϕ-Laplacian on the left-hand side and with nonlinearities on the right-hand sides which can have singularities in their space variables. Boundary conditions under discussion are generally nonlinear and nonlocal. Using regularization and sequential techniques we present general existence principles for solvability of regular and singular problems.

3.1. *Regularization of singular problems with ϕ-Laplacian*

We discuss singular differential equations of the form

$$\big(\phi(u')\big)' = f(t, u, u') \tag{3.4}$$

with the ϕ-Laplacian. Here $f \in Car([0,T] \times \mathcal{D})$, the set $\mathcal{D} = \mathcal{D}_1 \times \mathcal{D}_2 \subset \mathbb{R}^2$ is not necessarily closed, $\mathcal{D}_1$, $\mathcal{D}_2$ are intervals and f may have singularities in its space variables on the boundary $\partial\mathcal{D}_j$ of $\mathcal{D}_j$ $(j = 1, 2)$. We note that f has a singularity on $\partial\mathcal{D}_j$ in its space variable x_j if there is an $a_j \in \partial\mathcal{D}_j$ such that for a.e. $t \in [0, T]$ and some $x_{3-j} \in \mathcal{D}_{3-j}$,

$$\limsup_{x_j \to a_j,\ x_j \in \mathcal{D}_j} \big|f(t, x_1, x_2)\big| = \infty.$$

Let $\mathcal{A}$ denote the set of functionals $\alpha : C^1[0,T] \to \mathbb{R}$ which are
(a) continuous and
(b) bounded, that is, $\alpha(\Omega)$ is bounded (in $\mathbb{R}$) for any bounded $\Omega \subset C^1[0,T]$.
For $\alpha, \beta \in \mathcal{A}$, consider the (generally nonlinear and nonlocal) boundary conditions

$$\alpha(u) = 0, \quad \beta(u) = 0. \tag{3.5}$$

DEFINITION 3.1. A function $u : [0,T] \to \mathbb{R}$ is said to be a *solution of problem* (3.4), (3.5) if $\phi(u') \in AC[0,T]$, u satisfies the boundary conditions (3.5) and $(\phi(u'(t)))' = f(t, u(t), u'(t))$ holds for a.e. $t \in [0,T]$.

Special cases of the boundary conditions (3.5) are the Dirichlet (Neumann; mixed; periodic and Sturm–Liouville type) boundary conditions which we get setting $\alpha(x) = x(0)$, $\beta(x) = x(T)$ $(\alpha(x) = x'(0)$, $\beta(x) = x'(T)$; $\alpha(x) = x(0)$, $\beta(x) = x'(T)$; $\alpha(x) = x(0) - x(T)$, $\beta(x) = x'(0) - x'(T)$ and $\alpha(x) = a_0 x(0) + a_1 x'(0)$, $\beta(x) = b_0 x(T) + b_1 x'(T))$.

In order to obtain an existence result for problem (3.4), (3.5), we use regularization and sequential techniques. For this purpose consider a sequence of regular differential equations

$$\big(\phi(u')\big)' = f_n(t, u, u') \tag{3.6}$$

where $f_n \in Car([0,T] \times \mathbb{R}^2)$. The function f_n is constructed in such a way that

$$f_n(t, x, y) = f(t, x, y) \quad \text{for a.e. } t \in [0,T] \text{ and all } (x, y) \in \mathcal{Q}_n$$

where $\mathcal{Q}_n \subset \mathcal{D}$ and roughly speaking $\mathcal{Q}_n$ converges to $\mathcal{D}$ as $n \to \infty$.

Let $h \in Car([0,T]\times\mathbb{R}^2)$ and consider the regular differential equation

$$\big(\phi(u')\big)' = h(t,u,u'). \tag{3.7}$$

A function $u:[0,T]\to\mathbb{R}$ is called a *solution of the regular problem* (3.7), (3.5) if $\phi(u') \in AC[0,T]$, u satisfies (3.5) and $(\phi(u'(t)))' = h(t,u(t),u'(t))$ for a.e. $t\in[0,T]$.

The next general existence principle can be used for solving the regular problem (3.7), (3.5).

THEOREM 3.2 (General existence principle for regular problems). *Assume* (3.3), $h \in Car([0,T]\times\mathbb{R}^2)$ *and* $\alpha,\beta \in \mathcal{A}$. *Suppose there exist positive constants* S_0 *and* S_1 *such that*

$$\|u\|_\infty < S_0, \quad \|u'\|_\infty < S_1$$

for all solutions u to the problem

$$\big(\phi(u')\big)' = \lambda h(t,u,u'), \quad \alpha(u)=0, \quad \beta(u)=0 \tag{3.8}$$

and each $\lambda\in[0,1]$. *Also assume there exist positive constants* Λ_0 *and* Λ_1 *such that*

$$|A| < \Lambda_0, \quad |B| < \Lambda_1 \tag{3.9}$$

for all solutions $(A,B)\in\mathbb{R}^2$ *of the system*

$$\begin{cases}\alpha(A+Bt) - \mu\alpha(-A-Bt) = 0,\\ \beta(A+Bt) - \mu\beta(-A-Bt) = 0\end{cases} \tag{3.10}$$

and each $\mu\in[0,1]$. *Then problem* (3.7), (3.5) *has a solution.*

PROOF. Set

$$\Omega = \big\{x\in C^1[0,T]\colon\ \|x\|_\infty < \max\{S_0, \Lambda_0+\Lambda_1 T\},\ \|x'\|_\infty < \max\{S_1,\Lambda_1\}\big\}.$$

Then Ω is an open, bounded and symmetric with respect to $0\in C^1[0,T]$ subset of the Banach space $C^1[0,T]$. Define an operator $\mathcal{P}:[0,1]\times\overline{\Omega}\to C^1[0,T]$ by the formula

$$\mathcal{P}(\lambda,x)(t) = x(0)+\alpha(x) + \int_0^t \phi^{-1}\left(\phi\big(x'(0)+\beta(x)\big) + \lambda\int_0^s h\big(v,x(v),x'(v)\big)\,\mathrm{d}v\right)\mathrm{d}s. \tag{3.11}$$

A standard argument shows that $\mathcal{P}$ is a continuous operator. We claim that $\mathcal{P}([0,1]\times\overline{\Omega})$ is compact in $C^1[0,T]$. Indeed, since $\overline{\Omega}$ is bounded in $C^1[0,T]$, we have

$$\big|\alpha(x)\big| \leqslant r, \quad |\beta(x)| \leqslant r, \quad \big|h\big(t,x(t),x'(t)\big)\big| \leqslant \varrho(t)$$

for a.e. $t \in [0,T]$ and $x \in \overline{\Omega}$, where r is a positive constant and $\varrho \in L_1[0,T]$. Set $K = \max\{S_1, \Lambda_1\} + r$ and $V = \max\{|\phi(-K)|, |\phi(K)|\}$. Then

$$\begin{aligned}\big|\mathcal{P}(\lambda,x)(t)\big| &\leqslant \max\{S_0, \Lambda_0 + \Lambda_1 T\} + r \\ &\quad + T\max\big\{\big|\phi^{-1}\big(-V - \|\varrho\|_1\big)\big|, \big|\phi^{-1}\big(V + \|\varrho\|_1\big)\big|\big\}, \\ \big|\mathcal{P}(\lambda,x)'(t)\big| &\leqslant \max\big\{\big|\phi^{-1}\big(-V - \|\varrho\|_1\big)\big|, \big|\phi^{-1}(V + \|\varrho\|_1)\big|\big\}\end{aligned}$$

and

$$\big|\phi\big(\mathcal{P}(\lambda,x)'(t_2)\big) - \phi\big(\mathcal{P}(\lambda,x)'(t_1)\big)\big| \leqslant \left|\int_{t_1}^{t_2} \varrho(t)\,\mathrm{d}t\right|$$

for $t, t_1, t_2 \in [0,T]$ and $(\lambda,x) \in [0,1]\times\overline{\Omega}$. Hence $\mathcal{P}([0,1]\times\overline{\Omega})$ is bounded in $C^1[0,T]$ and $\{\phi[\mathcal{P}(\lambda,x)']\}$ is equicontinuous on $[0,T]$. The mapping ϕ^{-1} being an increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$, we deduce from

$$\big|\mathcal{P}(\lambda,x)'(t_2) - \mathcal{P}(\lambda,x)'(t_1)\big| = \big|\phi^{-1}\big(\phi\big(\mathcal{P}(\lambda,x)'(t_2)\big)\big) - \phi^{-1}\big(\phi\big(\mathcal{P}(\lambda,x)'(t_1)\big)\big)\big|$$

that $\{\mathcal{P}(\lambda,x)'\}$ is also equicontinuous on $[0,T]$. Now the Arzelà–Ascoli theorem shows that $\mathcal{P}([0,1]\times\overline{\Omega})$ is compact in $C^1[0,T]$. Thus $\mathcal{P}$ is a compact operator.

Suppose that x_0 is a fixed point of the operator $\mathcal{P}(1,\cdot)$. Then

$$\begin{aligned}x_0(t) &= x_0(0) + \alpha(x_0) \\ &\quad + \int_0^t \phi^{-1}\left(\phi\big(x_0'(0) + \beta(x_0)\big) + \int_0^s h\big(v, x_0(v), x_0'(v)\big)\,\mathrm{d}v\right)\mathrm{d}s.\end{aligned}$$

Hence $\alpha(x_0) = 0$, $\beta(x_0) = 0$ and x_0 is a solution of the differential equation (3.7). Therefore x_0 is a solution of problem (3.7), (3.5) and to prove our theorem, it suffices to show that

$$\deg\big(\mathcal{I} - \mathcal{P}(1,\cdot), \Omega\big) \neq 0 \tag{3.12}$$

where "deg" stands for the Leray–Schauder degree and $\mathcal{I}$ is the identity operator on $C^1[0,T]$. To see this let a compact operator $\mathcal{K}: [0,1]\times\overline{\Omega} \to C^1[0,T]$ be given by

$$\mathcal{K}(\mu,x)(t) = x(0) + \alpha(x) - \mu\alpha(-x) + \big[x'(0) + \beta(x) - \mu\beta(-x)\big]t.$$

Then $\mathcal{K}(1,\cdot)$ is odd (i.e. $\mathcal{K}(1,-x) = -\mathcal{K}(1,x)$ for $x \in \overline{\Omega}$) and

$$\mathcal{K}(0,\cdot) = \mathcal{P}(0,\cdot). \tag{3.13}$$

If $\mathcal{K}(\mu_0, x_0) = x_0$ for some $\mu_0 \in [0,1]$ and $x_0 \in \overline{\Omega}$, then

$$\begin{aligned}x_0(t) &= x_0(0) + \alpha(x_0) - \mu_0\alpha(-x_0) + \big[x_0'(0) + \beta(x_0) - \mu_0\beta(-x_0)\big]t, \\ &\quad t \in [0,T].\end{aligned}$$

Thus $x_0(t) = A_0 + B_0 t$ where $A_0 = x_0(0) + \alpha(x_0) - \mu_0\alpha(-x_0)$ and $B_0 = x_0'(0) + \beta(x_0) - \mu_0\beta(-x_0)$, so $\alpha(x_0) - \mu_0\alpha(-x_0) = 0$, $\beta(x_0) - \mu_0\beta(-x_0) = 0$. Hence

$$\alpha(A_0 + B_0 t) - \mu_0\alpha(-A_0 - B_0 t) = 0,$$
$$\beta(A_0 + B_0 t) - \mu_0\beta(-A_0 - B_0 t) = 0.$$

Therefore $|A_0| < \Lambda_0$, $|B_0| < \Lambda_1$ and $\|x_0\|_\infty < \Lambda_0 + \Lambda_1 T$, $\|x_0'\|_\infty < \Lambda_1$, which gives $x_0 \notin \partial\Omega$. Now, by the Borsuk antipodal theorem and a homotopy property,

$$\deg\big(\mathcal{I} - \mathcal{K}(0,\cdot), \Omega\big) = \deg\big(\mathcal{I} - \mathcal{K}(1,\cdot), \Omega\big) \neq 0. \tag{3.14}$$

Finally, assume that $\mathcal{P}(\lambda_*, x_*) = x_*$ for some $\lambda_* \in [0,1]$ and $x_* \in \overline{\Omega}$. Then x_* is a solution of problem (3.8) with $\lambda = \lambda_*$ and, by our assumptions, $\|x_*\|_\infty < S_0$ and $\|x_*'\|_\infty < S_1$. Hence $x_* \notin \partial\Omega$ and the homotopy property yields

$$\deg\big(\mathcal{I} - \mathcal{P}(0,\cdot), \Omega\big) = \deg\big(\mathcal{I} - \mathcal{P}(1,\cdot), \Omega\big).$$

This, with (3.13) and (3.14), implies (3.12). Therefore, problem (3.7), (3.5) has a solution. □

REMARK 3.3. If functionals $\alpha, \beta \in \mathcal{A}$ are linear, then system (3.10) has the form

$$A\alpha(1) + B\alpha(t) = 0,$$
$$A\beta(1) + B\beta(t) = 0.$$

All of its solutions (A, B) are bounded if and only if $\alpha(1)\beta(t) - \alpha(t)\beta(1) \neq 0$ (and then $(A, B) = (0, 0)$). This is satisfied for example for the Dirichlet conditions but not for the periodic ones.

3.2. *General existence principle for singular BVPs with ϕ-Laplacian*

Let us consider the singular problem (3.4), (3.5). By regularization and sequential techniques, we construct an approximating sequence of regular problems (3.6), (3.5) for whose solvability Theorem 3.2 can be used. Existence results for the singular problem (3.4), (3.5) can be proved by the following two general existence principles. The first principle uses the Vitali convergence theorem, the other is based on a combination of the Lebesgue dominated convergence theorem and the Fatou theorem.

THEOREM 3.4 (General existence principle for singular problems I). *Assume* (3.3). *Let there exist a bounded set $\Omega \subset C^1[0, T]$ such that*

(i) *for each $n \in \mathbb{N}$, the regular problem* (3.6), (3.5) *has a solution $u_n \in \Omega$,*
(ii) *the sequence $\{f_n(t, u_n(t), u_n'(t))\}$ is uniformly integrable on* $[0, T]$.

Then

(a) *there exist* $u \in \overline{\Omega}$ *and a subsequence* $\{u_{k_n}\}$ *of* $\{u_n\}$ *such that* $\lim_{n\to\infty} u_{k_n} = u$ *in* $C^1[0,T]$,

(b) *u is a solution of problem* (3.4), (3.5) *if*

$$\lim_{n\to\infty} f_{k_n}\big(t, u_{k_n}(t), u'_{k_n}(t)\big) = f\big(t, u(t), u'(t)\big)$$

for a.e. $t \in [0,T]$.

PROOF. Since Ω is bounded in $C^1[0,T]$ and $\{u_n\} \subset \Omega$, we have

$$\|u_n\|_\infty \leqslant r, \quad \|u'_n\|_\infty \leqslant r, \quad n \in \mathbb{N}, \tag{3.15}$$

where r is a positive constant. Now (ii) guarantees that for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\big|\phi\big(u'_n(t_2)\big) - \phi\big(u'_n(t_1)\big)\big| \leqslant \left|\int_{t_1}^{t_2} \big|f_n\big(t, u_n(t), u'_n(t)\big)\big|\,\mathrm{d}t\right| < \varepsilon$$

for each $t_1, t_2 \in [0,T]$, $|t_1 - t_2| < \delta$ and $n \in \mathbb{N}$. Therefore $\{\phi(u'_n)\}$ is equicontinuous on $[0,T]$, and by virtue of (3.15) and the fact that ϕ is continuous and increasing on $\mathbb{R}$, $\{u'_n\}$ is equicontinuous on $[0,T]$ as well. The Arzelà–Ascoli theorem guarantees the existence of a subsequence $\{u_{k_n}\}$ of $\{u_n\}$ converging in $C^1[0,T]$ to some $u \in \overline{\Omega}$.

Suppose that $\lim_{n\to\infty} f_{k_n}(t, u_{k_n}(t), u'_{k_n}(t)) = f(t, u(t), u'(t))$ for a.e. $t \in [0,T]$. By (ii), $\{f_{k_n}(t, u_{k_n}(t), u'_{k_n}(t))\}$ is uniformly integrable on $[0,T]$. Therefore, by Vitali's convergence theorem, $f(t, u(t), u'(t)) \in L_1[0,T]$ and letting $n \to \infty$ in

$$\phi\big(u'_{k_n}(t)\big) = \phi\big(u'_{k_n}(0)\big) + \int_0^t f_{k_n}\big(s, u_{k_n}(s), u'_{k_n}(s)\big)\,\mathrm{d}s, \quad t \in [0,T],\ n \in \mathbb{N},$$

we arrive at

$$\phi\big(u'(t)\big) = \phi\big(u'(0)\big) + \int_0^t f\big(s, u(s), u'(s)\big)\,\mathrm{d}s, \quad t \in [0,T].$$

Consequently, $\phi(u') \in AC[0,T]$ and u is a solution of (3.4). In addition, since

$$\lim_{n\to\infty} u_{k_n} = u \quad \text{in } C^1[0,T]$$

and α and β are continuous in $C^1[0,T]$, it follows that $\alpha(u) = 0$, $\beta(u) = 0$. Hence u is a solution of problem (3.4), (3.5). □

REMARK 3.5. Let f in (3.4) have singularities only at the value 0 of its space variables and let f_n in (3.6) satisfy $f_n(t,x,y) = f(t,x,y)$ for a.e. $t \in [0,T]$ and all $(x,y) \in \mathcal{D}$, $n \in \mathbb{N}$, $|x| \geqslant \frac{1}{n}$ and $|y| \geqslant \frac{1}{n}$. Then the condition

$$\lim_{n\to\infty} f_{k_n}\big(t, u_{k_n}(t), u'_{k_n}(t)\big) = f\big(t, u(t), u'(t)\big)$$

for a.e. $t \in [0, T]$ is satisfied if u and u' have a finite number of zeros.

THEOREM 3.6 (General existence principle for singular problems II). *Assume* (3.3). *Let f have singularities only at the value* 0 *of its space variables. Let f_n in equation* (3.6) *satisfy*

$$\begin{cases} \text{for a.e. } t \in [0, T] \text{ and each } x, y \in \mathbb{R} \setminus \{0\}, \\ 0 \leqslant f_n(t, x, y) \leqslant p(|x|, |y|) \quad \text{where } p \in C((0, \infty) \times (0, \infty)). \end{cases} \tag{3.16}$$

Suppose that for each $n \in \mathbb{N}$, the regular problem (3.6), (3.5) *has a solution u_n and there exists a subsequence $\{u_{k_n}\}$ of $\{u_n\}$ converging in $C^1[0, T]$ to some u. Then u is a solution of the singular problem* (3.4), (3.5) *if u and u' have a finite number of zeros and*

$$\lim_{n\to\infty} f_{k_n}\big(t, u_{k_n}(t), u'_{k_n}(t)\big) = f\big(t, u(t), u'(t)\big) \quad \text{for a.e. } t \in [0, T]. \tag{3.17}$$

PROOF. Assume that (3.17) is true and $0 \leqslant \xi_1 < \xi_2 < \cdots < \xi_m \leqslant T$ are all the zeros of u and u'. Since $\|u_{k_n}\|_\infty \leqslant L$ and $\|u'_{k_n}\|_\infty \leqslant L$ for each $n \in \mathbb{N}$ where L is a positive constant, it follows from (3.16), (3.17),

$$\phi\big(u'_{k_n}(T)\big) - \phi\big(u'_{k_n}(0)\big) = \int_0^T f_{k_n}\big(t, u_{k_n}(t), u'_{k_n}(t)\big)\,\mathrm{d}t, \quad n \in \mathbb{N},$$

and the Fatou theorem that

$$\int_0^T f\big(t, u(t), u'(t)\big)\,\mathrm{d}t \leqslant \phi(L) - \phi(-L).$$

Hence $f(t, u(t), u'(t)) \in L_1[0, T]$. Set $\xi_0 = 0$ and $\xi_{m+1} = T$. We claim that for all $j \in \{0, 1, \ldots, m\}$, $\xi_j < \xi_{j+1}$, the equality

$$\phi\big(u'(t)\big) = \phi\left(u'\left(\frac{\xi_j + \xi_{j+1}}{2}\right)\right) + \int_{\frac{\xi_j+\xi_{j+1}}{2}}^t f\big(s, u(s), u'(s)\big)\,\mathrm{d}s \tag{3.18}$$

is satisfied for $t \in [\xi_j, \xi_{j+1}]$. Indeed, let $j \in \{0, 1, \ldots, m\}$ and $\xi_j < \xi_{j+1}$. Let us look at the interval $[\xi_j + \delta, \xi_{j+1} - \delta]$ where $\delta \in (0, \frac{\xi_j+\xi_{j+1}}{2})$. We know that $|u| > 0$ and $|u'| > 0$ on (ξ_j, ξ_{j+1}) and therefore $|u(t)| \geqslant \varepsilon$, $|u'(t)| \geqslant \varepsilon$ for $t \in [\xi_j + \delta, \xi_{j+1} - \delta]$ with a positive constant ε. Hence there exists $n_0 \in \mathbb{N}$ such that $|u_{k_n}(t)| \geqslant \frac{\varepsilon}{2}$, $|u'_{k_n}(t)| \geqslant \frac{\varepsilon}{2}$ for $t \in [\xi_j + \delta, \xi_{j+1} - \delta]$ and $n \geqslant n_0$. This yields (see (3.16))

$$0 \leqslant f_{k_n}\big(t, u_{k_n}(t), u'_{k_n}(t)\big) \leqslant \max\left\{p(u, v)\colon u, v \in \left[\frac{\varepsilon}{2}, L\right]\right\}$$

for a.e. $t \in [\xi_j + \delta, \xi_{j+1} - \delta]$ and all $n \geqslant n_0$. Letting $n \to \infty$ in

$$\phi\big(u'_{k_n}(t)\big) = \phi\left(u'_{k_n}\left(\frac{\xi_j + \xi_{j+1}}{2}\right)\right) + \int_{\frac{\xi_j+\xi_{j+1}}{2}}^t f_{k_n}\big(s, u_{k_n}(s), u'_{k_n}(s)\big)\,\mathrm{d}s$$

gives (3.18) for $t \in [\xi_j + \delta, \xi_{j+1} - \delta]$ by the Lebesgue dominated convergence theorem. Since $\delta \in (0, \frac{\xi_j+\xi_{j+1}}{2})$ is arbitrary, (3.18) is true on the interval (ξ_j, ξ_{j+1}) and using the fact that $f(t, u(t), u'(t)) \in L_1[0, T]$, we conclude that (3.18) holds also at $t = \xi_j$ and ξ_{j+1}. From (3.18) it follows that $\phi(u') \in AC[0, T]$ and Eq. (3.4) is satisfied for a.e. $t \in [0, T]$. Finally, $\alpha(u_{k_n}) = 0$ and $\beta(u_{k_n}) = 0$ and the continuity of α and β yields $\alpha(u) = 0$ and $\beta(u) = 0$. Hence u is a solution of problem (3.4), (3.5). □

3.3. *Nonlocal singular BVPs*

We consider differential equations of the type

$$\big(\phi(u')\big)' = f(t, u, u') \tag{3.19}$$

where ϕ is an increasing and odd homeomorphism, $\phi(\mathbb{R}) = \mathbb{R}$, f satisfies the Carathéodory conditions on a subset of $[0, T] \times \mathbb{R}^2$ and f may be singular in its space variables.

We also discuss nonlinear nonlocal boundary conditions

$$u(0) = u(T), \quad \max\big\{u(t)\colon\ 0 \leqslant t \leqslant T\big\} = c,\ c \in \mathbb{R}, \tag{3.20}$$

$$u(0) = u(T) = -\gamma \min\big\{u(t)\colon\ 0 \leqslant t \leqslant T\big\}, \quad \gamma \in (0, \infty), \tag{3.21}$$

$$\min\big\{u(t)\colon\ 0 \leqslant t \leqslant T\big\} = 0, \quad \delta(u') = 0, \quad \delta \in \mathcal{B}, \tag{3.22}$$

where $\mathcal{B}$ denotes the set of functionals $\delta : C[0, T] \to \mathbb{R}$ which are

(a) continuous, $\delta(0) = 0$, and

(b) increasing, that is $x, y \in C[0, T]$, $x < y$ on $(0, T) \Rightarrow \delta(x) < \delta(y)$.

EXAMPLE. Let $n \in \mathbb{N}$ and $0 \leqslant a < b \leqslant T$. Then the functionals $\delta_1(x) = x(\frac{T}{2}) + \max\{x(t)\colon 0 \leqslant t \leqslant T\}$, $\delta_2(x) = \int_a^b x^{2n+1}(t)\,\mathrm{d}t$, $\delta_3(x) = \int_0^T \mathrm{e}^{x(t)}\,\mathrm{d}t - T$ belong to the set $\mathcal{B}$. The functionals $\delta_4(x) = x(0)$ and $\delta_5(x) = x(T)$ satisfy condition (a) of $\mathcal{B}$ but do not satisfy condition (b). Hence $\delta_4, \delta_5 \notin \mathcal{B}$.

The boundary conditions (3.20)–(3.22) are special cases of (3.5) where

$$\alpha(x) = x(0) - x(T), \qquad \beta(x) = \max\big\{x(t)\colon\ 0 \leqslant t \leqslant T\big\} - c \qquad \text{for (3.20)},$$

$$\alpha(x) = x(0) - x(T), \qquad \beta(x) = x(0) + \gamma \min\big\{x(t)\colon\ 0 \leqslant t \leqslant T\big\} \quad \text{for (3.21)},$$

and

$$\alpha(x) = \min\big\{x(t)\colon\ 0 \leqslant t \leqslant T\big\}, \qquad \beta(x) = \delta(x') \quad \text{for (3.22)}.$$

The next theorems give sufficient conditions for solvability of the three nonlocal singular problems given above. Their proofs are based on applying general existence principles presented in Theorems 3.2, 3.4 and 3.6. Notice that if $f < 0$ in Eq. (3.19) then the singular points corresponding to the solutions of problem (3.19), (3.20) are of type II and, if $f > 0$, the solutions of problems (3.19), (3.21) and (3.19), (3.22) have singular points of type II.

THEOREM 3.7 [137, Theorem 2.1]. *Suppose* $f \in Car([0,T] \times \mathbb{R} \times (\mathbb{R} \setminus \{0\})$,

$$\begin{cases} -q(x)\big(\omega_1(|y|) + \omega_2(|y|)\big) \leqslant f(t,x,y) \leqslant -a \\ \text{for a.e. } t \in [0,T] \text{ and each } (x,y) \in \mathbb{R} \times (\mathbb{R} \setminus \{0\}), \\ \text{where } a > 0,\ q \in C(\mathbb{R}) \text{ is positive}, \omega_1 \in C[0,\infty) \text{ is nonnegative}, \\ \omega_2 \in C(0,\infty) \text{ is positive and nonincreasing and} \\ \int_0^1 \omega_2\big(\phi^{-1}(s)\big)\,\mathrm{d}s < \infty \end{cases} \tag{3.23}$$

and

$$\int_{d-1}^{d} \frac{\mathrm{d}s}{H^{-1}(\int_s^d q(v)\,\mathrm{d}v)} < \infty \quad \text{for any } d \in \mathbb{R},$$

where

$$H(x) = \int_0^{\phi(x)} \frac{\phi^{-1}(s)\,\mathrm{d}s}{\omega_1(1+\phi^{-1}(s)) + \omega_2(\phi^{-1}(s))} \quad \text{for } x \in [0,\infty).$$

Let

$$\mathcal{S} = \left\{c \in \mathbb{R}\colon \lim_{x\to-\infty} \int_x^c \frac{\mathrm{d}s}{H^{-1}(\int_s^c q(v)\mathrm{d}v)} > \frac{T}{2}\right\}.$$

Then problem (3.19), (3.20) *has a solution for each* $c \in \mathcal{S}$.

SKETCH OF PROOF. *Step* 1. *Regularization*.

A sequence of auxiliary regular differential equations $(\phi(u'))' = f_n(t,u,u')$ is constructed where $f_n \in Car([0,T] \times \mathbb{R}^2)$ and

$$f_n(t,x,y) = f(t,x,y) \quad \text{for a.e. } t \in [0,T] \text{ and each } x \in \mathbb{R},\ |y| \geqslant \frac{1}{n},\ n \in \mathbb{N}.$$

Step 2. *Existence of solutions of regular problems* (3.6), (3.20).

Let $c \in \mathcal{S}$. By (3.23), the existence of positive constants S_0 and S_1 (independent of n and λ) is proved such that $\|u\|_\infty < S_0$ and $\|u'\|_\infty < S_1$ for any $\lambda \in [0,1]$, $n \in \mathbb{N}$ and each solution u of the differential equation

$$\big(\phi(u')\big)' = \lambda f_n(t,u,u') \tag{3.24}$$

satisfying the conditions (3.20). Put $\alpha(x) = x(0) - x(T)$ and $\beta(x) = \max\{x(t)\colon 0 \leqslant t \leqslant T\} - c$ for $x \in C^1[0,T]$. Then system (3.10) has a unique solution $(A,B) = (c\frac{1-\mu}{1+\mu}, 0)$

for each $\mu \in [0, 1]$ and therefore all solutions of this system are bounded in $\mathbb{R}^2$. Hence Theorem 3.2 guarantees that for each $n \in \mathbb{N}$, problem (3.6), (3.20) has a solution u_n and $\|u_n\|_\infty < S_0$, $\|u_n'\|_\infty < S_1$.

Step 3. *Properties of solutions of regular problems* (3.6), (3.20).

The sequence $\{u_n\}$ is considered. It is proved that

$$\begin{aligned} u_n'(t) &\geqslant \phi^{-1}\big(a(\xi_n - t)\big) \quad \text{for } t \in [0, \xi_n], \\ \big|u_n'(t)\big| &\geqslant \phi^{-1}\big(a(t - \xi_n)\big) \quad \text{for } t \in [\xi_n, T], \end{aligned}$$

where ξ_n is the unique zero of u_n' and $a > 0$ appears in (3.23). Next, it is shown that the sequence $\{f_n(t, u_n(t), u_n'(t))\}$ is uniformly integrable on $[0, T]$. Hence $\{u_n\}$ is bounded in $C^1[0, T]$ and $\{u_n'\}$ is equicontinuous on $[0, T]$ and, by the Arzelà–Ascoli theorem and the compactness principle, we can assume without loss of generality that $\{u_n\}$ converges in $C^1[0, T]$ and $\{\xi_n\}$ converges in $\mathbb{R}$. Let

$$\lim_{n\to\infty} u_n = u \quad \text{and} \quad \lim_{n\to\infty} \xi_n = \xi.$$

Then $u \in C^1[0, T]$ satisfies (3.20),

$$\begin{aligned} u'(t) &\geqslant \phi^{-1}\big(a(\xi - t)\big) \quad \text{for } t \in [0, \xi], \\ \big|u'(t)\big| &\geqslant \phi^{-1}\big(a(t - \xi)\big) \quad \text{for } t \in [\xi, T] \end{aligned}$$

and

$$\lim_{n\to\infty} f_n\big(t, u_n(t), u_n'(t)\big) = f\big(t, u(t), u'(t)\big) \quad \text{for a.e. } t \in [0, T].$$

Theorem 3.4 now guarantees that u is a solution of problem (3.19), (3.20). □

EXAMPLE. Let $p > 2$, $\alpha \in [0, p - 2)$ and $\beta \in (0, p - 1)$. Then for any $c \in \mathbb{R}$ there exists a solution of the differential equation

$$\big(|u'|^{p-2}u'\big)' + \big(2 + \sin(tu) + |u|\big)\left(|u'|^\alpha + \frac{1}{|u'|^\beta}\right) = 0$$

satisfying boundary conditions (3.20).

THEOREM 3.8 [138, Theorem 4.1]. *Let $f \in Car([0,T] \times (\mathbb{R} \setminus \{0\})^2)$. Let*

$$\begin{cases} a \leqslant f(t,x,y) \leqslant \big(h_1(|x|) + h_2(|x|)\big)\big(\omega_1(\phi(|y|)) + \omega_2(\phi(|y|))\big) \\ \text{for a.e. } t \in [0,T] \text{ and all } (x,y) \in (\mathbb{R} \setminus \{0\})^2, \\ \text{where } a > 0,\ h_1, \omega_1 \in C[0,\infty) \text{ are nonnegative and nondecreasing}, \\ h_2, \omega_2 \in C(0,\infty) \text{ are positive and nonincreasing}, \\ \int_0^1 h_2(s)\,\mathrm{d}s < \infty, \quad \int_0^1 \omega_2(s)\,\mathrm{d}s < \infty, \quad \int_0^\infty \frac{\mathrm{d}s}{\omega_2(s)} = \infty, \end{cases} \tag{3.25}$$

and let

$$\liminf_{x \to \infty} \int_0^x \frac{\mathrm{d}s}{K^{-1}(\frac{T}{2}(h_1(x) + h_2(s)))} > \frac{T}{2}, \tag{3.26}$$

where

$$K(x) = \int_0^{\phi(x)} \frac{\mathrm{d}s}{\omega_1(\phi(1) + s) + \omega_2(s)}, \quad x \in [0,\infty). \tag{3.27}$$

Then there exists a solution of problem (3.19), (3.21) *for each* $\gamma > 0$.

SKETCH OF PROOF. *Step* 1. *Regularization.*

A sequence of approximating differential equations

$$\big(\phi(u')\big)' = f_n(t,u,u')$$

is introduced where $f_n \in Car([0,T] \times \mathbb{R}^2)$ and

$$f_n(t,x,y) = f(t,x,y) \quad \text{for a.e. } t \in [0,T] \text{ and each } |x| \geqslant \frac{1}{n},\ |y| \geqslant \frac{1}{n},\ n \in \mathbb{N}.$$

Step 2. *Existence of solutions of regular problems* (3.6), (3.21).

Let $\gamma > 0$ in (3.21). Using (3.25) and (3.26), the existence of a positive constant P (depending on γ) is proved such that $\|u\|_\infty < PT$ and $\|u'\|_\infty < P$ for each solution u of problem (3.24), (3.21) with $\lambda \in [0,1]$ and $n \in \mathbb{N}$. Put $\alpha(x) = x(0) - x(T)$ and

$$\beta(x) = x(0) + \gamma \min\big\{x(t)\colon 0 \leqslant t \leqslant T\big\} \quad \text{for } x \in C^1[0,T].$$

The system (3.10) has a unique solution $(A,B) = (0,0)$ for each $\mu \in [0,1]$. Hence, by Theorem 3.2 for each $n \in \mathbb{N}$, there exists a solution u_n of problem (3.6), (3.21) and $\|u_n\|_\infty < PT$, $\|u_n'\|_\infty < P$.

Step 3. *Properties of solutions of regular problems* (3.6), (3.21).

The sequence $\{u_n\}$ is considered. From (3.25) it follows that u'_n is increasing on $[0,T]$ and has a unique zero $\xi_n \in (0,T)$ and u_n vanishes exactly at two points t_{1n}, t_{2n}, $0 < t_{1n} < \xi_n < t_{2n} < T$, $u_n > 0$ on $[0, t_{1n}) \cup (t_{2n}, T]$ and $u_n < 0$ on (t_{1n}, t_{2n}). Further, u_n satisfies the inequality

$$\left|u_n(t)\right| \geqslant \begin{cases} \dfrac{S|t - t_{1n}|}{\xi_n - t_{1n}} & \text{for } t \in [0, \xi_n], \\[2ex] \dfrac{S|t - t_{2n}|}{t_{2n} - \xi_n} & \text{for } t \in [\xi_n, T], \end{cases}$$

where S is a positive constant and the sequence $\{f_n(t, u_n(t), u'_n(t))\}$ is uniformly integrable on $[0,T]$, which implies that $\{u'_n\}$ is equicontinuous on $[0,T]$. Moreover, there exists a positive constant Δ such that

$$t_{1n} \geqslant \gamma\Delta, \quad \xi_n - t_{1n} > \Delta, \quad t_{2n} - \xi_n > \Delta, \quad T - t_{2n} > \gamma\Delta \quad \text{for } n \in \mathbb{N}.$$

Hence, by the Arzelà–Ascoli theorem, there exists a subsequence $\{u_{k_n}\}$ which converges in $C^1[0,T]$ and let $u = \lim_{n\to\infty} u_{k_n}$. Then u vanishes exactly at two points in $[0,T]$, u' has a unique zero and $\lim_{n\to\infty} f_{k_n}(t, u_{k_n}(t), u'_{k_n}(t)) = f(t, u(t), u'(t))$ for a.e. $t \in [0,T]$. Now Theorem 3.4 guarantees that u is a solution of problem (3.19), (3.21). □

REMARK 3.9. If $\lim_{x\to\infty} h_2(x) < A$ for some $A > 0$ and

$$\liminf_{x\to\infty} \frac{x}{K^{-1}(\frac{T}{2}(h_1(x) + A))} > \frac{T}{2},$$

then condition (3.26) is satisfied.

EXAMPLE. Let $q_j \in L_\infty[0,T]$ be nonnegative $(1 \leqslant j \leqslant 6)$, $q_1(t) \geqslant a > 0$ for a.e. $t \in [0,T]$, $p > 1$, $\beta_1, \beta_2, \beta_3 \in (0, p-1)$, $\alpha_1 \in (0, p-1+\beta_2)$, $\alpha_2, \alpha_3 \in (0,1)$. Then for each $\gamma > 0$, there exists a solution of the differential equation

$$\left(|u'|^{p-2}u'\right)' = q_1(t) + q_2(t)|u|^{\alpha_1} + \frac{q_3(t)}{|u|^{\alpha_2}} + \frac{q_4(t)}{|u|^{\alpha_3}|u'|^{\beta_1}} + q_5(t)|u'|^{\beta_2} + \frac{q_6(t)}{|u'|^{\beta_3}}$$

satisfying boundary conditions (3.21).

THEOREM 3.10. *Suppose $f \in Car([0,T] \times (0,\infty) \times (\mathbb{R} \setminus \{0\}))$ and the following conditions are satisfied*:

$$\begin{cases} \varphi(t) \leqslant f(t,x,y) \leqslant \big(h_1(x)+h_2(x)\big)\big[\omega_1\big(\phi\big(|y|\big)\big)+\omega_2\big(\phi\big(|y|\big)\big)\big] \\ \text{for a.e. } t \in [0,T] \text{ and each } (x,y) \in (0,\infty) \times (\mathbb{R} \setminus \{0\}), \\ \text{where } \varphi \in L_\infty[0,T] \text{ is positive}, \\ h_1, \omega_1 \in C[0,\infty) \text{ are positive and nondecreasing}, \\ h_2, \omega_2 \in C(0,\infty) \text{ are positive and nonincreasing}, \\ \displaystyle\int_0^1 h_2(s)\,\mathrm{d}s < \infty \end{cases} \tag{3.28}$$

and

$$\liminf_{x\to\infty} \frac{V(x)}{H(Tx)} > 1 \tag{3.29}$$

where

$$V(x) = \int_0^{\phi(x)} \frac{\phi^{-1}(s)\,\mathrm{d}s}{\omega_1(s+1)+\omega_2(s)},$$
$$H(x) = \int_0^x \big(h_1(s+1)+h_2(s)\big)\,\mathrm{d}s \quad \text{for } x \in [0,\infty).$$

Then for each $\delta \in \mathcal{B}$, problem (3.19), (3.22) *has a solution*.

SKETCH OF PROOF. *Step* 1. *Regularization*.

A sequence of auxiliary regular differential equations $(\phi(u'))' = f_n(t,u,u')$ is constructed with $f_n \in Car([0,T] \times \mathbb{R}^2)$ satisfying

$$f_n(t,x,y) = f(t,x,y) \quad \text{for a.e. } t \in [0,T] \text{ and each } x \geqslant \frac{1}{n},\ |y| \geqslant \frac{1}{n},\ n \in \mathbb{N}.$$

Step 2. *Existence of solutions of regular problems* (3.6), (3.22).

Fix $\delta \in \mathcal{B}$. From (3.28), (3.29) and from the properties of δ we obtain the existence of positive constants M_0 and M_1 such that $\|u\|_\infty < M_0$, $\|u'\|_\infty < M_1$ for each solution u of problem (3.24), (3.22) with $\lambda \in [0,1]$ and $n \in \mathbb{N}$. Set $\alpha(x) = \min\{x(t)\colon\ 0 \leqslant t \leqslant T\}$ and $\beta(x) = \delta(x')$ for $x \in C^1[0,T]$. Then system (3.10) has a unique solution $(A,B) = (0,0)$ for each $\mu \in [0,1]$. Therefore, by Theorem 3.2, for each $n \in \mathbb{N}$ there exists a solution u_n of problem (3.6), (3.22) and $\|u_n\|_\infty < M_0$, $\|u'\|_\infty < M_1$.

Step 3. *Properties of solutions of regular problems* (3.6), (3.22).

By Step 2, $\{u_n\}$ is bounded in $C^1[0,T]$ and from (3.28) it follows that $\{u_n'\}$ is equicontinuous on $[0,T]$. The assumption (3.28) and the properties of δ show that u_n has a unique zero ξ_n, $\xi_n \in (0,T)$, u_n' is increasing on $[0,T]$, $u_n'(\xi_n) = 0$ and

$$\left|u_n'(t)\right| \geqslant \left|\int_{\xi_n}^t \varphi(s)\,\mathrm{d}s\right|, \qquad u_n(t) \geqslant \int_{\xi_n}^t (t-s)\varphi(s)\,\mathrm{d}s \tag{3.30}$$

for $t \in [0,T]$. According to the Arzelà–Ascoli theorem, there exists a subsequence $\{u_{k_n}\}$ converging in $C^1[0,T]$ to some u and from (3.30) we see that u and u' vanish at a unique point. Since

$$\lim_{n\to\infty} f_{k_n}\big(t, u_{k_n}(t), u_{k_n}'(t)\big) = f\big(t, u(t), u'(t)\big) \quad \text{for a.e. } t \in [0,T],$$

Theorem 3.6 gives that u is a solution of problem (3.19), (3.22). □

REMARK 3.11. Problem (3.19), (3.22) was investigated in [140]. The conditions for the solvability of this problem are stronger there than those in Theorem 3.10. This is due to the fact that [140] uses the Vitali convergence theorem in limit processes whereas Theorem 3.10 is proved by Theorem 3.6.

EXAMPLE. Let $\varphi \in L_\infty[0,T]$ be positive, $p > 1$, $c_j > 0$ $(1 \leqslant j \leqslant 4)$, $\beta \in (0,1)$, $\alpha, \gamma, \delta, \lambda \in (0,\infty)$ and $\alpha + \gamma < p - 1$. Then for each $\delta \in \mathcal{B}$, the differential equation

$$\big(|u'|^{p-2}u'\big)' = \varphi(t)\left(1 + c_1 u^\alpha + \frac{c_2}{u^\beta}\right)\left(1 + c_3|u'|^\gamma + \frac{c_4}{|u'|^\lambda}\right)$$

has a solution u satisfying boundary conditions (3.22).

3.4. *Historical and bibliographical notes*

The general existence principles presented in Theorems 3.2 and 3.4 are special cases of the principles stated by Agarwal, O'Regan and Staněk in [16] for a class of second-order functional differential equations. Some general existence principles for second-order regular differential equations with the ϕ-Laplacian and Dirichlet or mixed boundary data have been established using the nonlinear alternative of Leray–Schauder type by O'Regan [109].

Second-order differential equations with the p-Laplacian and the ϕ-Laplacian occur in the study of the p-Laplace equations [91], general diffusion theory [22,40], non-Newtonian fluid theory [81] and the turbulent flow of a polytropic gas in a porous medium [60,36].

In recent years problems for $p(t)$-Laplacian equations have been studied (e.g., [63,64]). The $p(t)$-Laplacian is defined by $u \to (|u'|^{p(t)-2}u')'$ where $p \in C[0,T]$ and $p > 1$ on $[0,T]$. The $p(t)$-Laplacian is a generalization of the p-Laplacian.

4. Singular Dirichlet BVPs with ϕ-Laplacian

Motivated by various significant applications to non-Newtonian fluid theory, diffusion of flows in porous media, nonlinear elasticity and theory of capillary surfaces (see [22,60, 112], and Section 3.4), several authors have proposed the study of equations $(\phi_p(u'))' + f(t,u,u') = 0$ with the p-Laplacian $(\phi_p(u'))'$, where $p \in (1,\infty)$ and $\phi_p(y) = |y|^{p-2}y$ for $y \in \mathbb{R}$. Usually the p-Laplacian is replaced by its abstract and more general version, which leads to clearer exposition and better understanding of the methods that are employed to derive existence results. Therefore, similarly to Section 3, we will work with a ϕ-Laplacian which satisfies (3.3), i.e. ϕ is an increasing homeomorphism with $\phi(\mathbb{R}) = \mathbb{R}$.

We will consider a singular Dirichlet problem of the form

$$\big(\phi(u')\big)' + f(t,u,u') = 0, \quad u(0) = u(T) = 0 \tag{4.1}$$

and its special cases, in particular, a problem of the form

$$u'' + f(t,u,u') = 0, \quad u(0) = u(T) = 0, \tag{4.2}$$

where $\phi(y) = y$ on $\mathbb{R}$.

We will investigate problems (4.1) and (4.2) on the set $[0,T] \times \mathcal{A}$. In general, the function f depends on a time variable $t \in [0,T]$ and on two space variables x and y, where $(x,y) \in \mathcal{A}$ and $\mathcal{A}$ is a closed subset of $\mathbb{R}^2$ or $\mathcal{A} = \mathbb{R}^2$.

We assume that problems (4.1) and (4.2) are singular, which means, by Section 1, that f does not satisfy the Carathéodory conditions on $[0,T] \times \mathcal{A}$. In what follows, the types of singularities of f will be exactly specified for each problem under consideration.

In accordance with Section 1 we define:

DEFINITION 4.1. A function $u : [0,T] \to \mathbb{R}$ with $\phi(u') \in AC[0,T]$ is *a solution of problem* (4.1) if u satisfies $(\phi(u'(t)))' + f(t,u(t),u'(t)) = 0$ a.e. on $[0,T]$ and fulfills the boundary conditions $u(0) = u(T) = 0$.

A function $u \in C[0,T]$ is *a w-solution of problem* (4.1) if there exists a finite number of singular points $t_\nu \in [0,T]$, $\nu = 1,\dots,r$, such that if we denote $J = [0,T] \setminus \{t_\nu\}_{\nu=1}^r$, then $\phi(u') \in AC_{loc}(J)$, u satisfies $(\phi(u'(t)))' + f(t,u(t),u'(t)) = 0$ a.e. on $[0,T]$ and fulfills the boundary conditions $u(0) = u(T) = 0$.

Note that the condition $\phi(u') \in AC[0,T]$ implies $u \in C^1[0,T]$ and the condition $\phi(u') \in AC_{loc}(J)$ implies $u \in C^1(J)$. We will mention some papers where f is supposed to be continuous on $(0,T) \times \mathbb{R}^2$ and can have only time singularities at $t = 0$ and $t = T$. Then any solution (any w-solution) u of (4.1) moreover satisfies $\phi(u') \in C^1(0,T)$. If we investigate the solvability of problem (4.1) or (4.2) on the set $[0,T] \times \mathcal{A}$ and $\mathcal{A} \neq \mathbb{R}^2$, we impose on its solution u in addition the condition

$$\big(u(t), u'(t)\big) \in \mathcal{A} \quad \text{for } t \in [0,T]. \tag{4.3}$$

If u is a w-solution, then one requires it to satisfy (4.3) for $t \in J$ only.

In some cases (see, e.g., (4.9)) f does not depend on y. Then we work with a set $\mathcal{A}$ which is a closed subset of $\mathbb{R}$ or $\mathcal{A} = \mathbb{R}$ and condition (4.3) has the form $u(t) \in \mathcal{A}$ for $t \in [0, T]$.

REMARK 4.2. We will carry out the investigation of the singular problem (4.1) in the spirit of the existence principles presented in Sections 1 and 3:

- the singular problem is approximated by a sequence of solvable regular problems;
- a sequence $\{u_n\}$ of approximate solutions is generated;
- a convergence of a suitable subsequence $\{u_{k_n}\}$ is investigated;
- the type of this convergence determines the properties of its limit u and, among others, determines whether u is a w-solution or a solution of the original singular problem.

There are more possibilities how to construct an approximating sequence of regular problems. Their choice depends on the type of singularities of the nonlinearity f in (4.1) (time, space), on the type of singular points corresponding to a solution (w-solution) of (4.1) (type I, type II), on the type of results desired (existence of a solution, a positive solution, a w-solution, uniqueness), and so on. A common idea is that approximate functions f_n have no singularities, $f_n \neq f$ on neighbourhoods U_n of singular points of f, $f_n = f$ elsewhere, and $\lim_{n\to\infty} \operatorname{meas}(U_n) = 0$.

Having such a sequence of $\{f_n\}$ we study problems

$$\big(\phi(u')\big)' + f_n(t, u, u') = 0, \quad u(0) = u(T) = \varepsilon_n,\ n \in \mathbb{N}, \tag{4.4}$$

where $\varepsilon_n \in \mathbb{R}$, $\lim_{n\to\infty} \varepsilon_n = 0$. In some proofs, one simply puts $\varepsilon_n = 0$ for $n \in \mathbb{N}$.

Solvability of (4.4) can be investigated by means of various methods which have been developed for regular Dirichlet problems (fixed point theorems, topological degree arguments, the topological transversality method, variational methods, lower and upper functions, the Fredholm nonlinear alternative, etc.). See also Section 3. Using one of the above methods we generate a sequence of approximate solutions $\{u_n\}$ of (4.4). The crucial information which enables us to realize the limit process concerns a priori estimates of the approximate solutions u_n.

4.1. *Method of lower and upper functions*

It is well known that for regular second order boundary value problems *the lower and upper functions method* is a profitable instrument for proofs of their solvability and for a priori estimates of their solutions. See, e.g., [47–49,93,94,114,124,146]. Hence, it seems to be a good idea to extend this method to the singular problem (4.1). In literature there are several definitions of lower and upper functions for regular boundary value problems. (Note that in some papers they are called lower and upper solutions.) Here we will use the following definition which is the same both for regular problems with $f \in Car([0, T] \times \mathbb{R}^2)$ and for singular ones with $f \in Car((0, T) \times \mathbb{R}^2)$ having time singularities at $t = 0$ and $t = T$.

DEFINITION 4.3. A function $\sigma : [0, T] \to \mathbb{R}$ with $\phi(\sigma') \in AC[0, T]$ is called *a lower function of* (4.1) if σ satisfies

$$\big(\phi\big(\sigma'(t)\big)\big)' + f\big(t, \sigma(t), \sigma'(t)\big) \geqslant 0 \quad \text{for a.e. } t \in [0, T] \tag{4.5}$$

and

$$\sigma(0) \leqslant 0, \quad \sigma(T) \leqslant 0. \tag{4.6}$$

If the inequalities in (4.5) and (4.6) are reversed, then σ is called *an upper function of* (4.1).

For the special case (4.2) we admit a more general definition.

DEFINITION 4.4. A function $\sigma \in C[0, T]$ is called *a lower function of* (4.2) if there exists a finite set $\Sigma \subset (0, T)$ such that $\sigma \in AC^1_{loc}([0, T] \setminus \Sigma)$, $\sigma'(\tau+), \sigma'(\tau-) \in \mathbb{R}$ for each $\tau \in \Sigma$,

$$\sigma''(t) + f\big(t, \sigma(t), \sigma'(t)\big) \geqslant 0 \quad \text{for a.e. } t \in [0, T], \tag{4.7}$$

$$\sigma(0) \leqslant 0, \quad \sigma(T) \leqslant 0, \quad \sigma'(\tau-) < \sigma'(\tau+) \quad \text{for each } \tau \in \Sigma. \tag{4.8}$$

If the inequalities in (4.7) and (4.8) are reversed, then σ is called *an upper function of* (4.2).

REMARK 4.5. (i) If, moreover, f is continuous on $(0, T) \times \mathbb{R}^2$, then a lower (upper) function σ of (4.1) is supposed to satisfy $\phi(\sigma') \in C^1(0, T)$ and a lower (upper) function of (4.2) belongs also to $C^2(0, T)$.

(ii) If the boundary conditions in (4.1) or in (4.2) are replaced by inhomogeneous ones, i.e. they have the form

$$u(0) = a, \quad u(T) = b$$

for some $a, b \in \mathbb{R}$, then the corresponding boundary inequalities in (4.6) or in (4.8) are modified to

$$\sigma(0) \leqslant a, \quad \sigma(T) \leqslant b.$$

We present straightforward extensions of the classical lower and upper functions method to a singular problem with the p-Laplacian

$$\big(\phi_p(u')\big)' + f(t, u) = 0, \quad u(0) = a, \quad u(T) = b, \tag{4.9}$$

where

$$\begin{cases} \phi_p(y) = |y|^{p-2}y, \ p > 1, \ a, b \in \mathbb{R}, \ f \in Car\big((0, T) \times \mathbb{R}\big), \\ f \text{ can have time singularities at } t = 0 \text{ and } t = T. \end{cases} \tag{4.10}$$

Recall that f has time singularities at $t = 0$ and $t = T$ if there exist $x, y \in \mathbb{R}$ such that

$$\int_0^{\varepsilon} \big|f(t, x, y)\big| \,\mathrm{d}t = \infty, \qquad \int_{T-\varepsilon}^{T} \big|f(t, x, y)\big| \,\mathrm{d}t = \infty$$

for each sufficiently small $\varepsilon > 0$.

Making use of ideas of the papers [100] by Lomtatidze and Torres and [80] by Habets and Zanolin, one can prove the following result for the special case of (4.9) with $p = 2$.

THEOREM 4.6. *Let $p = 2$ and* (4.10) *hold. Let σ_1 and σ_2 be a lower and an upper function for problem* (4.9) *and $\sigma_1 \leqslant \sigma_2$ on* $[0, T]$. *Assume also that there is a function $h \in L_1(I)$ on each compact interval $I \subset (0, T)$ such that*

$$\big|f(t, x)\big| \leqslant h(t) \quad \textit{for a.e. } t \in (0, T) \textit{ and each } x \in \big[\sigma_1(t), \sigma_2(t)\big],$$

and

$$\int_0^T t(T-t)h(t)\,\mathrm{d}t < \infty.$$

Then problem (4.9) *has a w-solution $u \in C[0, T] \cap AC^1_{loc}(0, T)$ such that*

$$\sigma_1(t) \leqslant u(t) \leqslant \sigma_2(t) \quad \textit{for } t \in [0, T]. \tag{4.11}$$

If for a.e. $t \in (0, T)$ the function $f(t, x)$ is nonincreasing in x, then the w-solution is unique. If $h \in L_1[0, T]$, then u belongs to $AC^1[0, T]$, i.e. u is a solution of (4.9).

Theorem 4.6 can be proved by means of the Schauder fixed point theorem which is applied to the operator $\mathcal{T} : C[0, T] \to C[0, T]$, where

$$(\mathcal{T}u)(t) = a + \frac{t}{T}(b-a) + \int_0^T G(t, s) f^*\big(s, u(s)\big)\,\mathrm{d}s.$$

Here G is the Green function of the problem $-u'' = 0$, $u(0) = u(T) = 0$ and f^* is given by

$$f^*(t, x) = \begin{cases} f\big(t, \sigma_1(t)\big) & \text{if } x < \sigma_1(t), \\ f(t, x) & \text{if } \sigma_1(t) \leqslant x \leqslant \sigma_2(t), \\ f\big(t, \sigma_2(t)\big) & \text{if } x > \sigma_2(t) \end{cases}$$

for a.e. $t \in [0, T]$ and each $x \in \mathbb{R}$.

REMARK 4.7. By virtue of (4.11) we can investigate problem (4.9) on $[0, T] \times \mathcal{A}_t$, where $\mathcal{A}_t = [\sigma_1(t), \sigma_2(t)]$ for $t \in [0, T]$. Therefore, in Theorem 4.6, instead of $f \in Car((0, T) \times \mathbb{R})$ it is sufficient to assume $f \in Car((0, T) \times A_t)$.

Jiang in [86] dealt with problem (4.9) under the assumption (4.10) and, in addition,

$$f \in C\big((0,T)\times\mathbb{R}\big). \tag{4.12}$$

He modified Theorem 4.6 for $p \neq 2$.

THEOREM 4.8. *Let* (4.10) *and* (4.12) *hold. Let* σ_1 *and* σ_2 *be a lower and an upper function for problem* (4.9) *and* $\sigma_1 \leqslant \sigma_2$ *on* $[0,T]$. *Assume also that there is a function* $h \in C(0,T)$ *such that*

$$\big|f(t,x)\big| \leqslant h(t) \quad \textit{for } t \in (0,T),\ x \in \big[\sigma_1(t),\sigma_2(t)\big],$$

and that there exist $\mu, \nu \in [0, p-1)$ *such that*

$$\int_0^T t^{\mu}(T-t)^{\nu} h(t)\,\mathrm{d}t < \infty. \tag{4.13}$$

Then problem (4.9) *has a w-solution* $u \in C[0,T]$ *satisfying* $\phi_p(u') \in C^1(0,T)$ *and* (4.11).

In contrast to $p = 2$, there is no Green function for $p \neq 2$, which makes the proof of Theorem 4.8 more difficult and complicated than that for $p = 2$.

REMARK 4.9. Motivated by physical and technical problems, there is a lot of papers studying problems *with both time and space singularities*. If such a problem has a singularity at $x = 0$, one often searches for solutions (w-solutions) which are positive on $(0,T)$. Although they vanish at 0 and T, they are still called *positive solutions* (*positive w-solutions*) in literature. In this case, problems (4.1) and (4.2) are investigated on the set $[0,T]\times\mathcal{A}$, where $\mathcal{A} = [0,\infty)\times\mathbb{R}$, and f in (4.1) or (4.2) is supposed to satisfy $f \in Car((0,T)\times\mathcal{D})$, where $\mathcal{D} = (0,\infty)\times\mathbb{R}$ (or more specifically f is supposed to be continuous on $(0,T)\times\mathcal{D}$). In this case lower and upper functions have to be positive on $(0,T)$ and consequently a lower function σ_1 has to satisfy $\sigma_1(0) = \sigma_1(T) = 0$.

Having in mind Remarks 4.7 and 4.9 we will search for positive w-solutions of a singular problem

$$u'' + f(t,u) = 0, \quad u(0) = u(T) = 0, \tag{4.14}$$

where

$$\begin{cases} f \in Car\big((0,T)\times(0,\infty)\big) \text{ can have} \\ \text{time singularities at } t = 0,\ t = T \text{ and a space singularity at } x = 0. \end{cases} \tag{4.15}$$

Recall that f has a space singularity at $x = 0$ if

$$\limsup_{x\to 0+}\big|f(t,x)\big| = \infty \quad \text{for a.e. } t \in [0,T].$$

Let us present a simple application of Theorem 4.6 in the spirit of Habets and Zanolin [80].

THEOREM 4.10. *Let a function f be positive and satisfy* (4.15). *Assume that for a.e. $t \in (0, T)$ the function $f(t, x)$ is nonincreasing in x. Suppose that there exists a lower function σ_1 of problem* (4.14) *such that $\sigma_1 > 0$ on $(0, T)$ and*

$$\int_0^T f\big(s, \sigma_1(s)\big)\,\mathrm{d}s < \infty.$$

Then problem (4.14) *has a unique positive solution $u \in AC^1[0, T]$ such that $\sigma_1 \leqslant u$ on $[0, T]$.*

Theorem 4.10 follows from Theorem 4.6 and Remark 4.7 if we put $h(t) = f(t, \sigma_1(t))$ and

$$\sigma_2(t) = \int_0^T G(t, s) f(s, k)\,\mathrm{d}s + k,$$

where $k = \max\{\sigma_1(t)\colon\ t \in [0, T]\}$ and G is the Green function of the problem

$$-u'' = 0, \quad u(0) = u(T) = 0.$$

The next result can be viewed as a corollary of [100, Theorem 1.1], where Lomtatidze and Torres studied an equation including an additional term $g(t, u)u'$.

THEOREM 4.11. *Let* (4.15) *hold. Let σ_1 and σ_2 be a lower and an upper function of problem* (4.14) *and*

$$\sigma_2(0) > 0, \quad \sigma_2(T) > 0, \quad 0 < \sigma_1 \leqslant \sigma_2 \quad \text{on } (0, T).$$

Let, moreover, for every $0 < \eta < \min\{\sigma_2(t)\colon\ t \in [0, T]\}$ there exist $h_\eta \in C(0, T)$ such that

$$\big|f(t, x)\big| \leqslant h_\eta(t) \quad \text{for } t \in (0, T) \text{ and all } x \in \big[\sigma_{1\eta}(t), \sigma_2(t)\big],$$

where $\sigma_{1\eta}(t) = \max\{\eta, \sigma_1(t)\}$ and

$$\int_0^T t(T - t) h_\eta(t)\,\mathrm{d}t < \infty.$$

Then problem (4.14) *has a positive w-solution $u \in C[0, T] \cap AC^1_{loc}(0, T)$ such that*

$$\sigma_1(t) \leqslant u(t) \leqslant \sigma_2(t) \quad \text{for } t \in [0, T].$$

SKETCH OF THE PROOF. *Step* 1. *Construction of auxiliary intervals.*

A decreasing sequence $\{a_n\} \subset (0, T)$ and an increasing sequence $\{b_n\} \subset (0, T)$ are constructed such that, among other, $\lim_{n\to\infty} a_n = 0$, $\lim_{n\to\infty} b_n = T$.

Step 2. *Construction of auxiliary regular problems.*

For $t \in [0, T]$, $x \in \mathbb{R}$, $n \in \mathbb{N}$, functions $\chi_n, \alpha_n, \beta, f_n$ are given by

$$\chi_n(x) = \begin{cases} \sigma_1(a_n) & \text{if } x < \sigma_1(a_n), \\ x & \text{if } x \geqslant \sigma_1(a_n), \end{cases}$$

$$\alpha_n(t) = \begin{cases} \sigma_1(a_n) & \text{if } 0 \leqslant t \leqslant a_n, \\ \sigma_1(t) & \text{if } a_n \leqslant t \leqslant b_n, \\ \sigma_1(b_n) & \text{if } b_n \leqslant t \leqslant T, \end{cases} \qquad \beta(t) = \begin{cases} v_1(t) & \text{if } 0 \leqslant t \leqslant a_1, \\ \sigma_2(t) & \text{if } a_1 \leqslant t \leqslant b_1, \\ v_2(t) & \text{if } b_1 \leqslant t \leqslant T, \end{cases}$$

$$f_n(t, x) = \begin{cases} \frac{1}{2}\big[f\big(t, \chi_n(x)\big) + \big|f\big(t, \chi_n(x)\big)\big|\big] & \text{if } t \in (0, a_n] \cup [b_n, T), \\ f\big(t, \chi_n(x)\big) & \text{if } t \in (a_n, b_n), \end{cases}$$

where v_1 and v_2 are solutions of some auxiliary linear Dirichlet problems.

Step 3. *Convergence of the sequence of approximating solutions.*

Solvability of a sequence of regular problems

$$u'' + f_n(t, u) = 0, \quad u(0) = \sigma_1(a_n), \quad u(T) = \sigma_1(b_n), \quad n \in \mathbb{N}, \tag{4.16}$$

is investigated. The functions α_1 and β are a lower and an upper function of (4.16) with $n = 1$, and hence, by Theorem 4.6, there is a w-solution u_1 of (4.16) with $n = 1$ such that $\alpha_1 \leqslant u_1 \leqslant \beta$ on $[0, T]$. Further, α_2 and u_1 are a lower and an upper function of (4.16) with $n = 2$, and so Theorem 4.6 guarantees the existence of a w-solution u_2 of (4.16) with $n = 2$ such that $\alpha_2 \leqslant u_2 \leqslant u_1$ on $[0, T]$. In this way a sequence of w-solutions is obtained and then a limit process is applied. □

At the end of this subsection we will show another existence assertion in terms of the lower and upper functions for a problem with the p-Laplacian of the form

$$\big(\phi_p(u')\big)' + \psi(t)g(t, u) = 0, \quad u(0) = u(1) = 0, \tag{4.17}$$

where $\phi_p(y) = |y|^{p-2}y$, $p > 1$. Here we assume that

$$\begin{cases} \psi : (0, 1) \to (0, \infty) \text{ is continuous} \\ \text{and can have time singularities at } t = 0 \text{ and } t = 1, \end{cases} \tag{4.18}$$

$$\begin{cases} g : [0, 1] \times (0, \infty) \to \mathbb{R} \text{ is continuous} \\ \text{and can have a space singularity at } x = 0. \end{cases} \tag{4.19}$$

In this setting, using the paper by Agarwal, Lü and O'Regan [2], we offer the following result about the existence of positive w-solutions of (4.17).

THEOREM 4.12. *Let* (4.18) *and* (4.19) *hold. Assume that the following conditions are satisfied*:

$$\begin{cases} \{\rho_n\} \text{ is a nonincreasing sequence of real numbers}, \\ \lim_{n\to\infty} \rho_n = 0 \text{ and } n_0 \in \mathbb{N},\ n_0 \geqslant 3 \text{ being fixed}, \end{cases} \tag{4.20}$$

$$\max\left\{ \int_0^{1/2} \phi_p^{-1}\left(\int_s^{1/2} \psi(t)\,\mathrm{d}t \right) \mathrm{d}s, \int_{1/2}^{1} \phi_p^{-1}\left(\int_{1/2}^{s} \psi(t)\,\mathrm{d}t \right) \mathrm{d}s \right\} = b_0 < \infty \tag{4.21}$$

and

$$\psi(t) g(t, \rho_n) \geqslant 0 \quad \text{for } t \in \left[\frac{1}{2^{n+1}}, 1 \right), \quad n \geqslant n_0.$$

Further assume that σ_1 *and* σ_2 *are a lower and an upper function of problem* (4.17) *with* $\sigma_1 > 0$ *on* $(0, 1)$, $\max\{\rho_{n_0}, \sigma_1(t)\} \leqslant \sigma_2(t)$ *for* $t \in [0, 1]$ *and*

$$\left(\phi_p\left(\sigma_2'(t)\right)\right)' + \psi(t) g\left(\frac{1}{2^{n_0+1}}, \sigma_2(t) \right) \leqslant 0 \quad \text{for } t \in \left(0, \frac{1}{2^{n_0+1}}\right).$$

Then problem (4.17) *has a positive w-solution* $u \in C[0, 1]$ *with* $\phi_p(u') \in C^1(0, 1)$ *and*

$$\sigma_1(t) \leqslant u(t) \leqslant \sigma_2(t) \quad \text{for } t \in [0, 1].$$

In Section 4.3 we will show how lower and upper functions for regular problems can be applied to get not only a w-solution but also a solution of a given singular problem (see Theorem 4.18).

4.2. *Positive nonlinearities*

Many papers studying problem (4.1) or (4.2) with a space singularity at $x = 0$ concern the case that the nonlinearity f is positive. Such problems are referred to as *positone* ones in literature, see [10,11,135]. The positivity of f implies that each solution is concave and hence positive on $(0, T)$, and if, moreover, f has a space singularity at $x = 0$ but not at y, then each solution has only two corresponding singular points $0, T$ which are of type I. This makes the study of such problems easier than of those having sign-changing f or space singularities at y.

First we will discuss mixed singularities at t *and* x. In Section 1.1 we have presented problem (1.3), (1.4) the solvability of which was investigated by Taliaferro [141]. This problem has mixed singularities: the time ones at $t = 0$ and $t = 1$ as well as the space one at $x = 0$. Among many papers generalizing Taliaferro's existence results we choose the paper by Tineo [142] devoted to the existence of positive solutions or w-solutions to a singular problem

$$u'' + f(t, u, u') = 0, \quad u(0) = u(1) = 0, \tag{4.22}$$

where

$$\begin{cases} f : (0, 1) \times (0, \infty) \times \mathbb{R} \to (0, \infty) \text{ is continuous and can have} \\ \text{time singularities at } t = 0,\ t = 1 \text{ and a space singularity at } x = 0. \end{cases} \tag{4.23}$$

THEOREM 4.13 [142, Theorem 0.1]. *Let* (4.23) *hold. Suppose that there are continuous functions* $\varphi : (0, 1) \to (0, \infty)$, $\psi : (0, 1) \to [0, \infty)$ *and* $g : (0, \infty) \to (0, \infty)$ *such that* g *is decreasing and*

$$f(t, x, y) \leqslant \varphi(t)g(x) + \psi(t)|y| \quad \textit{for } t \in (0, 1),\ x \in (0, \infty),\ y \in \mathbb{R},$$

$$\int_0^1 t(1-t)\varphi(t)\,\mathrm{d}t < \infty, \qquad \int_0^1 \psi(t)\,\mathrm{d}t < \infty.$$

Assume further that for each constant $M > 0$ *there exists a continuous function* $\varepsilon_M : (0, 1) \to (0, \infty)$ *such that*

$$\varepsilon_M(t) \leqslant f(t, x, y) \quad \textit{for } t \in (0, 1),\ x \in (0, M],\ y \in \mathbb{R}.$$

Then problem (4.22) *has a positive w-solution* $u \in C[0, 1] \cap C^2(0, 1)$. *If, moreover,*

$$\int_0^1 g\big(kt(1-t)\big)\varphi(t)\,\mathrm{d}t < \infty \quad \textit{for all } k > 0,$$

then u *belongs to* $AC^1[0, 1]$, *which means that* u *is a solution of* (4.22).

The proof of Theorem 4.13 proceeds according to Remark 4.2. Solvability of auxiliary regular problems is obtained by the Leray–Schauder degree argument and the limit process is guaranteed by means of a priori estimates of the approximate solutions.

EXAMPLE. Let α, $\beta \in (0, 2)$, k, $\lambda \in (0, \infty)$, $\varepsilon \in C(0, 1)$, $\varepsilon > 0$ on $(0, 1)$. By Theorem 4.13 the problem

$$u'' + \frac{1}{t^\alpha (1-t)^\beta u^\lambda} + t^k |u'| + \varepsilon(t) = 0, \quad u(0) = u(1) = 0$$

has a positive w-solution $u \in C[0, 1] \cap C^2(0, 1)$. If $\alpha + \lambda$, $\beta + \lambda \in (0, 1)$ then, moreover, $u \in C^1[0, 1]$. Hence u is a solution.

Let us turn back to problem (4.14) with f satisfying (4.15). Wang in [148] considered f which can have at most linear growth in x. He illustrated his result by functions

$$f(t,x) = \frac{1}{t^\alpha x^\beta} \quad \text{with } \alpha \in (1,2),\ \beta > 0$$

or

$$f(t,x) = \delta x \exp\left(\frac{1}{x}\right) \quad \text{with a sufficiently small } \delta > 0.$$

For f which is moreover continuous on $(0,T) \times (0,\infty)$, Agarwal and O'Regan [3,4,11] proved the existence of a positive w-solution of (4.14) with f increasing in x for large x. An example of such f is

$$f(t,x) = \delta\left(\frac{1}{x^\alpha} + x^\beta + 1\right), \quad \alpha, \beta, \delta \in (0,\infty).$$

If f has sublinear growth in x (i.e. $\beta \in (0,1)$), then (4.14) has a positive w-solution for each $\delta > 0$. If f has linear or superlinear growth in x (i.e. $\beta = 1$ or $\beta \in (1,\infty)$), then (4.14) has a positive w-solution for any sufficiently small $\delta > 0$. A formula for an upper bound of δ is also given.

Now let us consider space singularities at x and y. We will present conditions ensuring solvability of problems with singularities in space variables x and y and with singular points both of type I and of type II. The main difficulty in the study of singular points of type II is the fact that their location in $[0,T]$ is not known. This is the reason why in mathematical literature there are only few papers concerning solvability of such problems and no results about w-solutions are known.

The first existence result in this direction was reached by Staněk [135] in 2001. The following theorem can be viewed as a corollary of [135, Theorem 1].

For a fixed $A > 0$ we consider a singular problem

$$u'' + \mu f(t,u,u') = 0, \quad u(0) = u(T) = 0, \tag{4.24}$$

with a positive real parameter μ, where

$$\begin{cases} f \text{ is continuous on } \mathcal{D}_A = [0,T] \times (0,A) \times \left[-\frac{2A}{T}, 0\right) \cup \left(0, \frac{2A}{T}\right] \\ \text{and can have space singularities at } x = 0, x = A, y = 0. \end{cases} \tag{4.25}$$

Sufficient conditions on μ and f for the solvability of (4.24) in the set $[0,T] \times \mathcal{A}$, where $\mathcal{A} = [0,A] \times [-\frac{2A}{T}, \frac{2A}{T}]$, are given in the next theorem.

THEOREM 4.14. *Let* (4.25) *hold. Suppose that there exists* $\delta > 0$ *such that* f *satisfies*

$$\delta \leqslant f(t,x,y) \leqslant g(x)\omega(|y|) \quad \text{on } \mathcal{D}_A,$$

where $\omega \geqslant \delta$ *is continuous on* $(0, \frac{2A}{T}]$ *and* $g \in C(0, A) \cap L_1[0, A]$. *Let*

$$\mu_T = \left(\int_0^{2A/T} \frac{y}{\omega(y)}\,\mathrm{d}y\right)\left(\int_0^A g(x)\,\mathrm{d}x\right)^{-1}.$$

Then for any $\mu \in (0, \mu_T]$ *problem* (4.24) *has a solution* $u_\mu \in AC^1[0, T]$ *satisfying*

$$0 < u_\mu(t) < A \quad \textit{for } t \in (0, T). \tag{4.26}$$

Take notice of the fact that for any solution u of problem (4.24) there is a point $t_u \in (0, T)$ with $u'(t_u) = 0$. Since f has a singularity at $y = 0$ and we do not know the position of $t_u \in (0, T)$, t_u is a singular point of type II.

To prove Theorem 4.14 a two-parameter family of regular problems is constructed and their solvability is established by the topological transversality method. Then, a priori bounds for approximate solutions of regular problems are derived. Using these bounds and the Arzelà–Ascoli theorem, a solution of (4.24) is obtained by a limiting process.

EXAMPLE. Let $A > 0$, $a, b, c, d, \gamma \in [0, \infty)$, $a + b + d > 0$, $\alpha, \beta \in (0, 1)$. Then for a sufficiently small $\mu > 0$ the problem

$$u'' + \mu\left(1 + \frac{a}{u^\alpha (A - u)^\beta} + bu^\gamma\right)\left(1 + cu'^2\right)\left(1 + \frac{d}{u'^2}\right) = 0,$$
$$u(0) = u(T) = 0$$

has a solution u_μ satisfying (4.26). The upper bound μ_T is explicitly expressed in [135].

The next existence result is in the spirit of Staněk [139], where a more general state-dependent functional differential equation was studied. Here we consider problem (4.1) with the ϕ-Laplacian and a function f satisfying

$$\begin{cases} f \in Car\big([0, T] \times \mathcal{D}\big),\ \mathcal{D} = (0, \infty) \times \big(\mathbb{R} \setminus \{0\}\big), \\ \text{and } f \text{ can have space singularities at } x = 0,\ y = 0. \end{cases} \tag{4.27}$$

THEOREM 4.15. *Let* (4.27) *hold and let* ϕ *be odd. Suppose that there exists* $\delta \in (0, \infty)$ *such that* f *satisfies*

$$\delta \leqslant f(t, x, y) \leqslant \big(h_1(x) + h_2(x)\big)\big(\omega_1\big(\phi\big(|y|\big)\big) + \omega_2\big(\phi\big(|y|\big)\big)\big)$$

for a.e. $t \in [0, T]$ *and all* $(x, y) \in \mathcal{D}$, *where* $h_1, \omega_1 \in C[0, \infty)$ *are positive and nondecreasing*, $h_2, \omega_2 \in C(0, \infty) \cap L_1[0, 1]$ *are positive and nonincreasing and*

$$\int_0^\infty \frac{\mathrm{d}s}{\omega_1(s)} = \infty.$$

Let

$$\frac{T}{2} < \liminf_{u\to\infty} \frac{u}{K^{-1}(T(h_1(u)+h_2(u)))},$$

where K^{-1} *denotes the inverse function to* $K:[0,\infty)\to[0,\infty)$,

$$K(u) = \int_0^{\phi(u)} \frac{\mathrm{d}s}{\omega_1(s)+\omega_2(s)}.$$

Then problem (4.1) *has a positive solution* $u \in AC^1[0,T]$.

In the proof of this result the solvability of a sequence of regular problems is obtained by the Leray–Schauder degree theory. Limit processes are guaranteed by Vitali's convergence theorem.

EXAMPLE. Let $c \in L_\infty[0,T]$, $p\in(1,\infty)$, $\beta\in(0,1)$, $\gamma,\eta\in(0,p)$, $\delta\in(0,\infty)$ and $\alpha\in(0,p-\gamma)$. Further, let $c(t)\geqslant\delta$ a.e. on $[0,T]$. Then the problem

$$\begin{aligned}&\big(|u'|^{p-2}u'\big)' + c(t)\left(1+u^\alpha+\frac{1}{u^\beta}\right)\left(1+|u'|^\gamma+\frac{1}{|u'|^\eta}\right)=0,\\ &u(0)=u(T)=0\end{aligned}$$

has a positive solution.

4.3. *Sign-changing nonlinearities*

Results about the solvability of singular Dirichlet problems with sign-changing nonlinearities mostly concern w-solutions. Making use of the arguments of Section 1 we can show a new existence principle giving positive solutions to singular Dirichlet problems of the form

$$u'' + f(t,u,u') = 0, \quad u(0)=u(T)=0, \tag{4.28}$$

where

$$\begin{cases} f\in Car\big((0,T)\times\mathcal{D}\big) \text{ can change its sign, } \mathcal{D}=(0,\infty)\times\mathbb{R},\\ \text{and } f \text{ can have mixed singularities at } t=0,\ t=T,\ x=0.\end{cases} \tag{4.29}$$

For $k\in\mathbb{N}$, $k\geqslant\frac{3}{T}$, $t\in[0,T]$, $x\in\mathbb{R}$, put $\Delta_k=[0,\frac{1}{k})\cup(T-\frac{1}{k},T]$,

$$\gamma_k(t)=\begin{cases}\frac{1}{k} & \text{if } t<\frac{1}{k},\\ t & \text{if } t\in[0,T]\setminus\Delta_k,\\ T-\frac{1}{k} & \text{if } t>T-\frac{1}{k},\end{cases}\qquad \delta_k(x)=\begin{cases}|x| & \text{if } |x|\geqslant\frac{1}{k},\\ \frac{1}{k} & \text{if } |x|<\frac{1}{k}.\end{cases}$$

Then we construct the sequence of regular functions

$$f_k(t,x,y) = f\big(\gamma_k(t), \delta_k(x), y\big) \quad \text{for a.e. } t \in [0,T],\ \text{each } x, y \in \mathbb{R} \tag{4.30}$$

and the sequence of regular problems

$$u'' + f_k(t,u,u') = 0, \quad u(0) = u(T) = \frac{1}{k}, \tag{4.31}$$

where $f_k \in Car([0,T] \times \mathbb{R}^2)$, $k \in \mathbb{N}$, $k \geqslant \frac{3}{T}$.

THEOREM 4.16 (Existence principle for solutions of (4.28)). *Let* (4.29) *hold. Assume that*

$$\begin{cases} \textit{there exists a bounded set } \Omega \subset C^1[0,T] \textit{ such that} \\ \textit{problem } (4.31) \textit{ has a solution } u_k \in \Omega \textit{ for each } k \in \mathbb{N},\ k \geqslant \frac{3}{T}, \end{cases} \tag{4.32}$$

$$\begin{cases} \textit{there exists a function } \varepsilon \in C[0,T],\ \varepsilon(0) = \varepsilon(T) = 0, \\ \textit{such that } u_k(t) \geqslant \varepsilon(t) > 0 \textit{ for } t \in (0,T) \textit{ and each } k \in \mathbb{N},\ k \geqslant \frac{3}{T} \end{cases} \tag{4.33}$$

and

$$\textit{the sequence } \big\{f_k\big(t, u_k(t), u_k'(t)\big)\big\} \textit{ is uniformly integrable on } [0,T]. \tag{4.34}$$

Then

$$\begin{cases} \textit{there exist a function } u \in \overline{\Omega} \textit{ and a subsequence } \{u_{k_n}\} \subset \{u_k\} \\ \textit{such that } \lim\limits_{n\to\infty} \|u_{k_n} - u\|_{C^1} = 0, \end{cases} \tag{4.35}$$

and

$$u \in AC^1[0,T] \textit{ is a positive solution of problem } (4.28). \tag{4.36}$$

PROOF. Since f can have singularities both at t and at x, we cannot obtain Theorem 4.16 as a direct consequence of some of the theorems in Section 1. Nevertheless, we can use the ideas of their proofs and argue as follows.

Step 1. *Convergence of the sequence of approximating solutions.*

By (4.32) we get that $\{u_k\}$ and $\{u_k'\}$ are bounded in $C[0,T]$. The boundedness of $\{u_k'\}$ implies the equicontinuity of $\{u_k\}$ on $[0,T]$. Condition (4.34) yields the equicontinuity of $\{u_k'\}$ on $[0,T]$ and hence the Arzelà–Ascoli theorem gives the assertion (4.35).

Step 2. *Convergence of the sequence of regular right-hand sides.*

The conditions $u_{k_n}(0) = u_{k_n}(T) = \frac{1}{k_n}$ imply $u(0) = u(T) = 0$. By (4.33) we get $u(t) > 0$ on $(0,T)$. Choose $\xi \in (0,T)$ such that $f(\xi,\cdot,\cdot) : \mathcal{D} \to \mathbb{R}$ is continuous. Then, by virtue of

(4.30) and (4.33), we have

$$u_{k_n}(\xi) \geqslant \varepsilon(\xi) > \frac{1}{k_n}, \quad \xi \in [0,T] \setminus \Delta_{k_n},$$
$$f_{k_n}\bigl(\xi, u_{k_n}(\xi), u'_{k_n}(\xi)\bigr) = f\bigl(\xi, u_{k_n}(\xi), u'_{k_n}(\xi)\bigr)$$

for a sufficiently large k_n. Therefore

$$\lim_{n\to\infty} f_{k_n}\bigl(t, u_{k_n}(t), u'_{k_n}(t)\bigr) = f\bigl(t, u(t), u'(t)\bigr) \quad \text{for a.e. } t \in [0,T]. \tag{4.37}$$

Step 3. *Properties of the limit u.*

By (4.34), (4.37) and Vitali's convergence theorem, we get that $f(t, u(t), u'(t))$ belongs to $L_1[0,T]$ and we can pass to the limit in the sequence

$$u'_{k_n}(t) = u'_{k_n}(0) - \int_0^t f_{k_n}\bigl(s, u_{k_n}(s), u'_{k_n}(s)\bigr)\,\mathrm{d}s, \quad t \in [0,T],$$

thus obtaining

$$u'(t) = u'(0) - \int_0^t f\bigl(s, u(s), u'(s)\bigr)\,\mathrm{d}s, \quad t \in [0,T].$$

Hence (4.36) is true. □

REMARK 4.17. Having in mind the absolute continuity of the Lebesgue integral we see that if there exists $\varphi \in L_1[0,T]$ such that

$$\bigl|f_k\bigl(t, u_k(t), u'_k(t)\bigr)\bigr| \leqslant \varphi(t) \quad \text{for a.e. } t \in [0,T] \text{ and each } k \in \mathbb{N},\ k \geqslant \tfrac{3}{T}, \tag{4.38}$$

then condition (4.34) is valid.

In Section 4.1, the classical lower and upper functions method has been extended to singular problems (see Theorems 4.6, 4.8, 4.11 and 4.12). Motivated by Agarwal, O'Regan, Lakshmikantham and Leela [13], we will show another approach which consists in the employment of a sequence of lower and upper functions of approximating regular problems.

THEOREM 4.18. *Let* (4.29) *and* (4.30) *hold. Assume that there exists $k_0 \in \mathbb{N}$ such that for each $k \in \mathbb{N}$, $k \geqslant k_0$, the following conditions are satisfied*:

$$\begin{cases} \alpha_k \text{ and } \beta \text{ are a lower and an upper function of (4.31) and} \\ \alpha'_k, \beta' \in L_\infty[0,T], \quad \frac{1}{k} \leqslant \alpha_k(t) \leqslant \beta(t) \quad \text{for } t \in [0,T], \end{cases} \tag{4.39}$$

$$\begin{cases} \bigl|f_k(t,x,y)\bigr| \leqslant \psi(t)g(x)\omega(|y|) \quad \text{for a.e. } t \in [0,T] \\ \text{and for all } x \in \bigl(0, \|\beta\|_\infty\bigr),\ y \in \mathbb{R}, \end{cases} \tag{4.40}$$

with

$$\begin{cases} \text{positive functions } \psi \in L_1[\frac{1}{k}, T - \frac{1}{k}],\ \omega \in C[0,\infty) \text{ and} \\ \text{a positive nonincreasing function } g \in C(0,\infty). \end{cases} \tag{4.41}$$

Further, assume that there is a function $\alpha \in C[0,T]$ *with* $\alpha(0) = \alpha(T) = 0$, $\alpha > 0$ *on* $(0,T)$, $\alpha_k \geqslant \alpha$ *on* $[0,T]$, *such that*

$$\int_0^T \psi(t) g\big(\alpha(t)\big)\,\mathrm{d}t < \int_0^\infty \frac{\mathrm{d}s}{\omega(s)}. \tag{4.42}$$

Then problem (4.28) *has a positive solution* $u \in AC^1[0,T]$ *satisfying*

$$\alpha(t) \leqslant u(t) \leqslant \beta(t) \quad \textit{for } t \in [0,T].$$

SKETCH OF THE PROOF. Theorem 4.18 can be proved by means of Theorem 4.16 in the following way.

Step 1. *Construction of the sequence of regular problems.*

Condition (4.42) implies that there exists $\rho > 0$ such that

$$\int_0^T \psi(t) g\big(\alpha(t)\big)\,\mathrm{d}t < \int_0^\rho \frac{\mathrm{d}s}{\omega(s)}. \tag{4.43}$$

For $k \in \mathbb{N}$, $k \geqslant k_0$, put $\rho_k = \max\{\rho, \|\alpha_k'\|_\infty, \|\beta'\|_\infty\}$ and consider a sequence of regular problems

$$u'' + \widetilde{f}_k(t,u,u') = 0, \quad u(0) = u(T) = \frac{1}{k}, \tag{4.44}$$

where for a.e. $t \in [0,T]$ and all $(x,y) \in \mathbb{R}^2$ we set

$$\widetilde{f}_k(t,x,y) = \chi_k(y) f_k(t,x,y)$$

and

$$\chi_k(y) = \begin{cases} 1 & \text{if } |y| \leqslant \rho_k, \\ 2 - \frac{|y|}{\rho_k} & \text{if } \rho_k < |y| < 2\rho_k, \\ 0 & \text{if } |y| \geqslant 2\rho_k. \end{cases}$$

Then α_k and β are respectively a lower and an upper function of (4.44), $\widetilde{f}_k$ satisfies (4.40) and, moreover, there exists $\widetilde{h}_k \in L_1[0,T]$ such that

$$\big|\widetilde{f}_k(t,x,y)\big| \leqslant \widetilde{h}_k(t) \quad \text{for a.e. } t \in [0,T],\ \text{all } x \in \big[\alpha_k(t), \beta(t)\big],\ y \in \mathbb{R}.$$

The classical existence result based on the lower and upper functions method for regular problems (see, e.g., [146]) guarantees that for each $k \geqslant k_0$ problem (4.44) has a solution u_k with

$$\alpha \leqslant \alpha_k \leqslant u_k \leqslant \beta \quad \text{on } [0, T]. \tag{4.45}$$

Step 2. *A priori estimates of approximate solutions.*

Using (4.40), (4.41) and (4.43) we deduce that $\|u_k'\|_\infty \leqslant \rho$, which implies that u_k is also a solution of (4.31) for $k \geqslant k_0$. Hence, if we put $\varepsilon(t) = \alpha(t)$ for $t \in [0, T]$ and

$$\Omega = \left\{x \in C^1[0, T]\colon\ \alpha \leqslant x \leqslant \beta \text{ on } [0, T],\ \|x'\|_\infty \leqslant \rho\right\},$$

we get that (4.32), (4.33) are fulfilled.

Step 3. *Uniform integrability of regular right-hand sides.*

By (4.40), (4.41) and (4.45) we get

$$\left|f_k\big(t, u_k(t), u_k'(t)\big)\right| \leqslant \psi(t) g\big(u_k(t)\big)\omega\big(\left|u_k'(t)\right|\big) \leqslant M\psi(t) g\big(\alpha(t)\big) = \varphi(t)$$

for a.e. $t \in [0, T]$, where $M = \max\{\omega(|s|)\colon\ s \in [-\rho, \rho]\}$. By virtue of (4.43), $\varphi \in L_1[0, T]$ and we conclude by Remark 4.17 that condition (4.34) is valid. Therefore the assertion follows from Theorem 4.16 and condition (4.45). □

REMARK 4.19. If f does not depend on y, then (4.42) takes the form

$$\int_0^T \psi(t) g\big(\alpha(t)\big)\, \mathrm{d}t < \infty.$$

In the rest of this section we present a selection of existence results about w-solutions which can be obtained by theorems from Section 4.1. Consider a singular Dirichlet problem

$$u'' + f(t, u) = 0, \quad u(0) = u(T) = 0, \tag{4.46}$$

where

$$\begin{cases} f \in Car\big((0, T) \times (0, \infty)\big) \text{ can change its sign and} \\ \text{can have singularities at } t = 0,\ t = T \text{ and at } x = 0. \end{cases} \tag{4.47}$$

The first result is due to Lomtatidze [99].

THEOREM 4.20. *Let* (4.47) *hold. Assume that* f *is nonincreasing as a function of its second argument and that there is* $\varepsilon > 0$ *for which*

$$f(t, \varepsilon) \geqslant 0 \quad \textit{for a.e. } t \in [0, T],\ \text{meas}\left\{t \in (0, T)\colon\ f(t, \varepsilon) > 0\right\} > 0.$$

Then the condition

$$\int_0^T t(T-t)\big|f(t,\delta)\big|\,\mathrm{d}t < \infty \quad \textit{for any } \delta \in (0,\varepsilon]$$

is necessary and sufficient for problem (4.46) *to have a unique positive w-solution* $u \in C[0,T] \cap AC^1_{loc}(0,T)$.

The proof of Theorem 4.20 is based on the lower and upper functions method via Theorem 4.11. More general results were obtained by Lomtatidze and Torres in [100], where a differential equation having moreover the term $g(t,u)u'$ was investigated.

A very similar result for a simpler case of problem (4.46) with f satisfying (4.49) was proved by Habets and Zanolin in [80]. The analogue of their results was proved by Jiang [86] for a singular Dirichlet problem with the p-Laplacian

$$\big(\phi_p(u')\big)' + f(t,u) = 0, \quad u(0) = u(T) = 0, \tag{4.48}$$

where $\phi(y) = |y|^{p-2}y,\ \ p > 1$ and

$$\begin{cases} f \in C\big((0,T)\times(0,\infty)\big) \text{ can change its sign and} \\ \text{can have singularities at } t=0,\ t=T \text{ and at } x=0. \end{cases} \tag{4.49}$$

THEOREM 4.21 [86, Theorem 3]. *Let* (4.49) *hold. Assume that*

(i) *there exists a constant* $L > 0$ *such that for any compact set* $K \subset (0,T)$ *there is* $\varepsilon = \varepsilon_K > 0$ *such that*

$$f(t,x) > L \quad \textit{for all } t \in K,\ x \in (0,\varepsilon],$$

(ii) *for any* $\delta > 0$ *there are* $h_\delta \in C(0,T)$ *and* $\mu, \nu \in [0, p-1)$ *such that*

$$\big|f(t,x)\big| \leqslant h_\delta(t) \quad \textit{for all } t \in (0,T),\ x \geqslant \delta$$

and

$$\int_0^T t^\mu (T-t)^\nu h_\delta(t)\,\mathrm{d}t < \infty.$$

Then problem (4.48) *has a positive w-solution* $u \in C[0,T]$ *with* $\phi_p(u') \in C^1(0,T)$. *If, moreover, for each* $t \in (0,T)$ *the function* $f(t,x)$ *is nonincreasing in* x, *then* u *is a unique w-solution.*

Another existence result for differential equation where the nonlinearity f can depend on u' is due to Jiang in [87]. He studied the singular Dirichlet problem of the form

$$u'' + f(t,u,u') = 0, \quad u(0) = u(1) = 0, \tag{4.50}$$

where

$$\begin{cases} f \in C((0,1) \times (0,\infty) \times \mathbb{R}) \text{ can change its sign} \\ \text{and can have singularities at } t = 0,\ t = 1 \text{ and } x = 0. \end{cases} \tag{4.51}$$

Motivated by the example

$$f(t,x,y) = \frac{\delta}{t^m(1-t)^n}\left(\frac{1}{x^\alpha} + x^\beta + \sin(8\pi t)\right)\left(1 + t(1-t)|y|^{1/\gamma}\right) \tag{4.52}$$

with real numbers $\alpha > 0$, $\beta \geqslant 0$, $\gamma > 1$, $\delta > 0$, $0 \leqslant m, n < 2$, he proved existence of a positive w-solution of (4.50) for f satisfying (4.51) and having superlinear growth in x for large x and sublinear growth in y for large $|y|$. Particularly, if f in (4.50) has the form (4.52), then an upper bound for δ is found guaranteeing that (4.50) has a positive w-solution. The proof is based on the papers [80] and [4].

Let us turn back to problem (4.17). Using the lower and upper functions method established in Theorem 4.12 we can get sufficient conditions for the existence of positive w-solutions. Specifically, we report the result motivated by Agarwal, Lü and O'Regan [2].

THEOREM 4.22. *Let* (4.18)–(4.21) *hold. Assume that there exist* $n_0 \in \mathbb{N}$, $n_0 \geqslant 3$ *and* $c_0 \in (0,\infty)$ *such that*

$$\psi(t)g(t,x) \geqslant c_0 \quad \text{for } t \in \left[\frac{1}{2^{n+1}}, 1 - \frac{1}{2^{n+1}}\right],\ x \in (0,\rho_n],\ n \geqslant n_0$$

and

$$\left|g(t,x)\right| \leqslant \eta(x) + h(x) \quad \text{for } (t,x) \in [0,1] \times (0,\infty),$$

where $\eta \in AC_{loc}(0,\infty)$ *is positive*, $h \in C[0,\infty)$ *is nonnegative and* $\frac{h}{\eta}$ *is nondecreasing on* $(0,\infty)$. *Further assume that*

$$\eta' < 0 \quad \text{a.e. on } (0,R) \quad \text{and} \quad \frac{\eta'}{\eta^2} \in L_1[0,R] \quad \text{for any } R > 0$$

and that there exists $r \in (0,\infty)$ *such that*

$$b_0 < \left(\phi_p^{-1}\left(1 + \frac{h(r)}{\eta(r)}\right)\right)^{-1} \int_0^r \frac{\mathrm{d}u}{\phi_p^{-1}(\eta(u))}.$$

Then problem (4.17) *has a positive w-solution* $u \in C[0,1]$ *with* $\phi_p(u') \in C^1(0,1)$.

4.4. *Sign-changing solutions and w-solutions*

If we consider a singular differential equation with a space singularity at $x = 0$, a question about the existence of sign-changing solutions of such an equation can arise. A single result in this direction was proved by Rachůnková and Staněk in [118], where an equation of the form $(r(u)u')' = \mu q(t) f(t, u)$ has been studied.

Here we will present this result for a simplified equation

$$u'' + \mu f(t, u) = 0 \tag{4.53}$$

with a positive real parameter μ and with boundary conditions

$$u(0) = u(T) = 0, \quad \max\{u(t)\colon t \in [0, T]\} \min\{u(t)\colon t \in [0, T]\} < 0. \tag{4.54}$$

We assume that

$$\begin{cases} f : [0, T] \times (\mathbb{R} \setminus \{0\}) \to \mathbb{R} \text{ is continuous} \\ \text{and can have a space singularity at } x = 0. \end{cases} \tag{4.55}$$

By a *solution of problem* (4.53), (4.54) we mean a function $u \in C^1[0, T]$ having precisely one zero $t_u \in (0, T)$. Moreover, $u \in C^2((0, T) \setminus \{t_u\})$ fulfills (4.54) and there exists $\mu_u > 0$ such that u satisfies (4.53) for $\mu = \mu_u$ and $t \in (0, T) \setminus \{t_u\}$.

THEOREM 4.23. *Let* (4.55) *hold. Assume that for each* $t \in [0, T]$ *the function* $f(t, x)$ *is nondecreasing with respect to* x *on* $(-\infty, 0)$ *and nonincreasing on* $(0, \infty)$ *and*

$$\varepsilon \leqslant f(t, x) \operatorname{sign} x \leqslant g(x) \quad \text{for } (t, x) \in [0, T] \times (\mathbb{R} \setminus \{0\}),$$

where $\varepsilon \in (0, \infty)$ *and* $g \in C(\mathbb{R} \setminus \{0\}) \cap L_1[-1, 1]$. *Then for each* $A \in (0, \infty)$ *and* $B \in (-\infty, 0)$ *there exist solutions* u *and* v *of problem* (4.53), (4.54) *satisfying*

$$\max\{u(t)\colon t \in [0, T]\} = A \quad \text{and} \quad \min\{v(t)\colon t \in [0, T]\} = B. \tag{4.56}$$

By virtue of Theorem 4.23, any solution u of problem (4.53), (4.54) vanishes at some point $t_u \in (0, T)$. Since f has a singularity at $x = 0$ and we do not know the position of t_u, we see that t_u is a singular point of type II.

The proof of Theorem 4.23 is based on a combination of four main theorems in [118], where a new method of proofs was developed. It is based on "gluing" the positive and negative parts of solutions and smoothing them.

In accordance with the paper [119] by Rachůnková and Staněk we define *a w-solution of problem* (4.53), (4.54) as a function $u \in C[0, T]$ having precisely one zero $t_u \in (0, T)$. Further, $u \in C^2((0, T) \setminus \{t_u\})$ fulfills (4.54), there exist finite limits

$$\lim_{t \to t_u -} u'(t), \quad \lim_{t \to t_u +} u'(t)$$

and there exists $\mu_u > 0$ such that u satisfies (4.53) for $\mu = \mu_u$ and $t \in (0, T) \setminus \{t_u\}$.

In [119], the following existence result for w-solutions is proved.

THEOREM 4.24. *Let all assumptions of Theorem* 4.23 *be satisfied. Then for each* $t_0 \in (0, T)$ *and each* $A \in (0, \infty)$, $B \in (-\infty, 0)$ *problem* (4.53), (4.54) *has just two different w-solutions vanishing at* t_0 *and having their maximum value on* $[0, T]$ *equal to* A, *and just two different w-solutions vanishing at* t_0 *and having their minimum value on* $[0, T]$ *equal to* B.

In the proof of Theorem 4.24, w-solutions are constructed by means of solutions of auxiliary Dirichlet problems on $[0, t_0]$ and $[t_0, T]$.

EXAMPLE. Let $\alpha, \beta \in (0, 1)$, $a \in (0, \infty)$, $b \in (-\infty, 0)$ and

$$f(x) = \begin{cases} \frac{a}{x^\alpha} & \text{for } x > 0, \\ \frac{b}{(-x)^\beta} & \text{for } x < 0. \end{cases}$$

Consider the differential equation

$$u'' + \mu f(u) = 0. \tag{4.57}$$

By Theorem 4.23, for each $A > 0$ and $B < 0$ there exist solutions u and v of problem (4.57), (4.54) satisfying (4.56). Moreover, by Theorem 4.24, for each $t_0 \in (0, T)$ and for each $A > 0$ there exist just two different w-solutions u_1 and u_2 of problem (4.57), (4.54) satisfying

$$\max\{u_1(t)\colon t \in [0, T]\} = \max\{u_2(t)\colon t \in [0, T]\} = A.$$

Further, for each $t_0 \in (0, T)$ and for each $B < 0$ there exist just two different w-solutions v_1 and v_2 of problem (4.57), (4.54) satisfying

$$\min\{v_1(t)\colon t \in [0, T]\} = \min\{v_2(t)\colon t \in [0, T]\} = B.$$

4.5. *Historical and bibliographical notes*

A systematic study of solvability of Dirichlet problems having both time and space singularities was initiated in 1979 by Taliaferro [141], who found necessary and sufficient conditions for the existence of solutions (w-solutions) of problem (1.3), (1.4). A contribution to the more general problem (4.22) was published by Bobisud, O'Regan and Royalty [37] in 1988. In 1989, in contrast to the shooting method used in [141] and the topological transversality method applied in [37], Gatica, Oliker and Waltman [72] proved a fixed point theorem for decreasing maps on cones and applying it they obtained solvability of (4.14). However, in these works the nonlinearity f had to be bounded in its space variables x and y for large x and large $|y|$.

An extension of the above results permitting linear growth of f in its third variable y for large $|y|$ was treated by Baxley [24] in 1991 and by Tineo [142] in 1992. The condition of boundedness of f in its second variable x for large x was overcome by Agarwal and O'Regan in [3] (1996), where the existence of a positive w-solution was proved even for f having superlinear growth for large x. In 1999, the first multiplicity result for Dirichlet problems with time and space singularities was reached. Particularly, Agarwal and O'Regan [6] proved the existence of two different positive w-solutions. All these results rely on the fact that nonlinearities in the equations considered are positive.

In 1987, this assumption was removed by Lomtatidze [99] for problem (4.14). We can also refer to papers by Janus and Myjak [83] for a nonhomogeneous equation (1.3) and by Habets and Zanolin [80] for the continuous case of (4.14) which appeared in 1994. From papers providing more general existence results for problems with sign-changing nonlinearities we mention the recent papers by Jiang [87] (2002), by Agarwal, Staněk [17] (2003) or by Lomtatidze, Torres [100] (2003). These papers deal with problems of the type (4.2).

Existence results for problems of the type (4.1) with the ϕ-Laplacian and sign-changing nonlinearities were presented by Wang and Gao in [149] (1996), where Taliaferro's results were extended. Existence results in the spirit of Habets and Zanolin which are applicable to problems with the p-Laplacian of the form (4.48) were given by Jiang in [86] (2001). In 2003, Agarwal, Lü and O'Regan [2] published the existence result for the problem with the p-Laplacian of the form (4.17).

Further results and references for positive and for sign-changing nonlinearities can be found in the monographs by Kiguradze [92] (1975), by Kiguradze and Shekhter [93] (1987), by O'Regan [110] (1994), by Agarwal and O'Regan [10] (2003) and in [11] (2004). Note that there exists a large group of papers investigating Dirichlet boundary value problems having only time singularities. These results are not discussed here but some of them can be found in the above cited monographs.

In the study of Dirichlet problems with space singularities and singular points both of type I and of type II the first existence result was reached by Staněk [135] in 2001, and the existence of sign-changing solutions was proved by Rachůnková and Staněk [118] in 2003. Numerical algorithms and computation of solutions and w-solutions of singular Dirichlet problems were given by Baxley [25] (1995) and by Baxley and Thompson [28] (2000).

5. Singular periodic BVPs with ϕ-Laplacian

The aim of this section is to present existence results for singular periodic problems of the form

$$\big(\phi(u')\big)' = f(t, u, u'), \tag{5.1}$$

$$u(0) = u(T), \quad u'(0) = u'(T), \tag{5.2}$$

where $0 < T < \infty$, $\phi : \mathbb{R} \to \mathbb{R}$ is an increasing and odd homeomorphism such that $\phi(\mathbb{R}) = \mathbb{R}$ and

$$\begin{cases} f \in Car\big([0,T] \times \big((0,\infty) \times \mathbb{R}\big)\big) \quad \text{and} \\ f \text{ can have a space singularity at } x = 0. \end{cases} \tag{5.3}$$

In accordance with Section 1.3, this means that

$$\limsup_{x \to 0+} \big| f(t,x,y) \big| = \infty \quad \text{for a.e. } t \in [0,T] \text{ and some } y \in \mathbb{R}$$

may happen.

REMARK 5.1. Physicists say that f has an *attractive* singularity at $x = 0$ if

$$\liminf_{x \to 0+} f(t,x,y) = -\infty \quad \text{for a.e. } t \in [0,T] \text{ and some } y \in \mathbb{R}$$

since near the origin the force is directed inward. Alternatively, f is said to have a *repulsive* singularity at $x = 0$ if

$$\limsup_{x \to 0+} f(t,x,y) = \infty \quad \text{for a.e. } t \in [0,T] \text{ and some } y \in \mathbb{R}$$

holds.

In the setting of Section 1.3, problem (5.1), (5.2) is investigated on the set $[0,T] \times \mathcal{A}$, where $\mathcal{A} = [0,\infty) \times \mathbb{R}$. In contrast to the Dirichlet problem (4.1), where each solution vanishes at $t = 0$ and $t = T$ and hence enters the space singularity $x = 0$ of f, all known existence results for the periodic problem (5.1), (5.2) under the assumption (5.3) concern positive solutions which do not touch the space singularity $x = 0$ of the function f.

DEFINITION 5.2. A function $u : [0,T] \to \mathbb{R}$ is a *positive solution* to problem (5.1), (5.2) if $\phi(u') \in AC[0,T]$, $u > 0$ on $[0,T]$, $(\phi(u'(t)))' = f(t,u(t),u'(t))$ for a.e. $t \in [0,T]$ and (5.2) is satisfied.

The restriction to positive solutions causes that the general existence principle in Theorem 1.8 about a limit of a sequence of approximate solutions need not be employed here. On the other hand, the singular problem (5.1), (5.2) will be also investigated through regular approximating periodic problems having differential equations of the form

$$\big(\phi(u')\big)' = h(t,u,u'), \tag{5.4}$$

where $h \in Car([0,T] \times \mathbb{R}^2)$. As usual, by a *solution of the regular problem* (5.4), (5.2) we understand a function u such that $\phi(u') \in AC[0,T]$, (5.2) is true and $(\phi(u'(t)))' = h(t,u(t),u'(t))$ for a.e. $t \in [0,T]$.

Notice that the requirement $\phi(u') \in AC[0,T]$ implies that $u \in C^1[0,T]$.

We will also discuss various special cases of (5.1) including the classical one with $\phi(y) \equiv y$ or those with f not depending on u' or with f depending on u' linearly.

Let us notice that the assumption that ϕ is an odd function is only technical and it is sufficient to assume (3.3) as in Section 3 in most cases. We employ it just to simplify some formulas occurring in this section.

5.1. *Method of lower and upper functions for regular problems*

First, we will consider problem (5.4), (5.2), where $h \in Car([0,T] \times \mathbb{R}^2)$. We bring some results which will be exploited in the investigation of the singular problem (5.1), (5.2). The lower and upper functions method combined with the topological degree argument is an important tool for proofs of solvability of regular periodic problems. Several rather general definitions of lower and upper functions are available (see, e.g., [47,48,62,93,124, 147]). However, for our purposes the following one seems to be optimal.

DEFINITION 5.3. We say that a function $\sigma \in C[0,T]$ is a *lower function* of problem (5.4), (5.2) if there is a finite set $\Sigma \subset (0,T)$ such that $\phi(\sigma') \in AC_{loc}([0,T] \setminus \Sigma)$, $\sigma'(\tau+)$, $\sigma'(\tau-) \in \mathbb{R}$ for each $\tau \in \Sigma$ and

$$\big(\phi\big(\sigma'(t)\big)\big)' \geqslant h\big(t, \sigma(t), \sigma'(t)\big) \quad \text{for a.e. } t \in [0,T], \tag{5.5}$$

$$\sigma(0) = \sigma(T), \quad \sigma'(0) \geqslant \sigma'(T), \tag{5.6}$$

$$\sigma'(\tau+) > \sigma'(\tau-) \quad \text{for all } \tau \in \Sigma. \tag{5.7}$$

If the inequalities in (5.5)–(5.7) are reversed, σ is called an *upper function* of problem (5.4), (5.2).

The role of lower and upper functions is demonstrated by the following "maximum principle":

LEMMA 5.4. *Let σ_1 and σ_2 be a lower and an upper function of* (5.4), (5.2) *and* $\sigma_1 \leqslant \sigma_2$ *on* $[0,T]$. *Then for each $d \in [\sigma_1(0), \sigma_2(0)]$ and each $\widetilde{f} \in Car([0,T] \times \mathbb{R}^2)$ such that*

$$\begin{cases} \widetilde{f}(t,x,y) < h\big(t, \sigma_1(t), \sigma_1'(t)\big) \textit{ for a.e. } t \in [0,T], \textit{ all } x \in \big(-\infty, \sigma_1(t)\big) \\ \qquad \textit{and all } y \in \mathbb{R} \textit{ such that } \big|y - \sigma_1'(t)\big| \leqslant \dfrac{\sigma_1(t) - x}{\sigma_1(t) - x + 1}, \\ \widetilde{f}(t,x,y) > h\big(t, \sigma_2(t), \sigma_2'(t)\big) \textit{ for a.e. } t \in [0,T], \textit{ all } x \in \big(\sigma_2(t), \infty\big) \\ \qquad \textit{and all } y \in \mathbb{R} \textit{ such that } \big|y - \sigma_2'(t)\big| \leqslant \dfrac{x - \sigma_2(t)}{x - \sigma_2(t) + 1}, \end{cases} \tag{5.8}$$

any solution u of the problem

$$\big(\phi(u')\big)' = \widetilde{f}(t,u,u'), \quad u(0) = u(T) = d$$

satisfies $\sigma_1 \leqslant u \leqslant \sigma_2$ *on* $[0, T]$.

PROOF. Denote $v = u - \sigma_1$ and assume that $v(\alpha) = \min\{v(t)\colon\ t \in [0, T]\} < 0$. Since $d \in [\sigma_1(0), \sigma_2(0)]$ and thanks to (5.6) and (5.7), we may assume that $\alpha \in (0, T) \setminus \Sigma$, $v'(\alpha) = 0$, and there is $\beta \in (\alpha, T]$ such that $(\alpha, \beta] \cap \Sigma = \emptyset$ and

$$v(t) < 0 \quad \text{and} \quad \left|v'(t)\right| < \frac{-v(t)}{1 - v(t)} \quad \text{for all } t \in [\alpha, \beta].$$

Using (5.5) (where $\sigma = \sigma_1$) and (5.8), we obtain

$$\left(\phi\left(u'(t)\right) - \phi\left(\sigma_1'(t)\right)\right)' < h\left(t, \sigma_1(t), \sigma_1'(t)\right) - \left(\phi\left(\sigma_1'(t)\right)\right)' \leqslant 0$$
$$\text{for a.e. } t \in [\alpha, \beta].$$

Hence

$$0 > \int_\alpha^t \left(\phi\left(u'(s)\right) - \phi\left(\sigma_1'(s)\right)\right)' \mathrm{d}s = \phi\left(u'(t)\right) - \phi\left(\sigma_1'(t)\right) \quad \text{for all } t \in (\alpha, \beta],$$

which leads to a contradiction with the definition of α, i.e. $u \geqslant \sigma_1$ on $[0, T]$. Similarly we can show that $u \leqslant \sigma_2$ on $[0, T]$. □

Problem (5.4), (5.2) is often transformed to a fixed point problem (see, e.g., [41,101,104, 150]). Here we present one possibility how to find an operator representation of (5.4), (5.2) in the space $C^1[0, T]$. Having in mind that the periodic conditions (5.2) can be equivalently rewritten as

$$u(0) = u(T) = u(0) + u'(0) - u'(T),$$

let us consider the quasilinear Dirichlet problem

$$\left(\phi(x')\right)' = b(t) \quad \text{a.e. on } [0, T], \quad x(0) = x(T) = d \tag{5.9}$$

with $b \in L_1[0, T]$ and $d \in \mathbb{R}$. A function $x \in C^1[0, T]$ is a solution of (5.9) if and only if there is $a \in \mathbb{R}$ such that

$$x(t) = d + \int_0^t \phi^{-1}\left(a + \int_0^s b(\tau)\,\mathrm{d}\tau\right) \mathrm{d}s \quad \text{for } t \in [0, T]$$

and

$$\int_0^T \phi^{-1}\left(a + \int_0^s b(\tau)\,\mathrm{d}\tau\right) \mathrm{d}s = 0. \tag{5.10}$$

Since ϕ is increasing on $\mathbb{R}$ and $\phi(\mathbb{R}) = \mathbb{R}$, Eq. (5.10) has exactly one solution $a = a(b) \in \mathbb{R}$ for each $b \in L_1[0, T]$. So, we can define an operator $\mathcal{K}: L_1[0, T] \to C^1[0, T]$ by

$$\big(\mathcal{K}(b)\big)(t) = \int_0^t \phi^{-1}\left(a(b) + \int_0^s b(\tau)\,\mathrm{d}\tau\right) \mathrm{d}s \quad \text{for } t \in [0, T]. \tag{5.11}$$

Let $\mathcal{N}: C^1[0, T] \to L_1[0, T]$ and $\mathcal{F}: C^1[0, T] \to C^1[0, T]$ have the form $(\mathcal{N}(u))(t) = h(t, u(t), u'(t))$ for a.e. $t \in [0, T]$ and

$$\big(\mathcal{F}(u)\big)(t) = u(0) + u'(0) - u'(T) + \big(\mathcal{K}\big(\mathcal{N}(u)\big)\big)(t) \quad \text{for } t \in [0, T]. \tag{5.12}$$

In view of the definition of $\mathcal{K}$, a function $x \in C^1[0, T]$ is a solution to (5.9) if and only if $x = d + \mathcal{K}(b)$. Therefore, $u \in C^1[0, T]$ is a solution to (5.4), (5.2) if and only if it is a fixed point of $\mathcal{F}$.

An alternative representation of the operator $\mathcal{F}$ can be obtained by inserting $\alpha(u) = u(0) - d$, $\beta(u) = d - u(T)$ and $d = u(0) + u'(0) - u'(T)$ into the operator $\mathcal{P}(1, u)$ defined in the proof of Theorem 3.2. In this way we get

$$\begin{aligned}\big(\mathcal{F}(u)\big)(t) &= u(0) + u'(0) - u'(T) \\ &\quad + \int_0^t \phi^{-1}\left(\phi\big(u'(T) + u(T) - u(0)\big) + \int_0^s \big(\mathcal{N}(u)\big)(\tau)\,\mathrm{d}\tau\right) \mathrm{d}s\end{aligned}$$

for $t \in [0, T]$ and $u \in C^1[0, T]$.

Taking into account [101, Proposition 2.2] or the proof of Theorem 3.2, we can summarize:

LEMMA 5.5. *Let $\mathcal{F}: C^1[0, T] \to C^1[0, T]$ be defined by* (5.12). *Then $\mathcal{F}$ is completely continuous and $u \in C^1[0, T]$ is a solution to* (5.4), (5.2) *if and only if $\mathcal{F}(u) = u$.*

The next lemma describes the relationship between lower and upper functions and the Leray–Schauder topological degree. We will consider the class of auxiliary problems

$$\big(\phi(v')\big)' = \eta(v')h(t, v, v'), \quad v(0) = v(T), \quad v'(0) = v'(T), \tag{5.13}$$

where $\eta: \mathbb{R} \to [0, 1]$ may be an arbitrary continuous function.

LEMMA 5.6. *Let σ_1 and σ_2 be a lower and an upper function of* (5.4), (5.2) *and $\sigma_1 < \sigma_2$ on* $[0, T]$. *Furthermore, let there exist $r^* > 0$ such that*

$$\begin{cases} \|v'\|_\infty < r^* \quad \textit{for each continuous } \eta: \mathbb{R} \to [0, 1] \textit{ and for} \\ \textit{each solution } v \textit{ of } (5.13) \textit{ such that } \sigma_1 \leqslant v \leqslant \sigma_2 \textit{ on } [0, T]. \end{cases} \tag{5.14}$$

Finally, assume that $\mathcal{F}: C^1[0, T] \to C^1[0, T]$ is defined by (5.12) *and, for $\rho > 0$, denote*

$$\Omega_\rho = \big\{u \in C^1[0, T]\colon\ \sigma_1 < u < \sigma_2 \textit{ on } [0, T] \textit{ and } \|u'\|_\infty < \rho\big\}. \tag{5.15}$$

Then

$$\deg(\mathcal{I}-\mathcal{F},\Omega_\rho)=1 \quad \textit{for each } \rho \geqslant r^* \textit{ such that } \mathcal{F}(u)\neq u \textit{ on } \partial\Omega_\rho.$$

PROOF. Put $\Omega=\Omega_{r^*}$, $R^*=r^*+\|\sigma_1'\|_\infty+\|\sigma_2'\|_\infty$,

$$\eta(y)=\begin{cases}1 & \text{if } |y|\leqslant R^*,\\ 2-\frac{|y|}{R^*} & \text{if } R^*<|y|\leqslant 2R^*,\\ 0 & \text{if } |y|>2R^*\end{cases}$$

and assume that $\mathcal{F}(x)\neq x$ for all $x\in\partial\Omega$. Then σ_1 and σ_2 are a lower and an upper function for the modified problem (5.13) and there is a $\psi\in L_1[0,T]$ satisfying

$$\bigl|\eta(y)h(t,x,y)\bigr|\leqslant\psi(t) \quad \text{for a.e. } t\in[0,T] \text{ and all } (x,y)\in[\sigma_1(t),\sigma_2(t)]\times\mathbb{R}.$$

We can construct (see the proof of Theorem 2.1 in [127]) the function $\widetilde{f}\in Car([0,T]\times\mathbb{R}^2)$ so that

$$\begin{aligned}&\widetilde{f}(t,x,y)=\eta(y)h(t,x,y)\\&\quad\text{for a.e. } t\in[0,T] \text{ and all } (x,y)\in[\sigma_1(t),\sigma_2(t)]\times\mathbb{R},\\&\bigl|\widetilde{f}(t,x,y)\bigr|\leqslant\widetilde{\psi}(t)\\&\quad\text{for a.e. } t\in[0,T],\ \text{all } (x,y)\in\mathbb{R}^2 \text{ and some } \widetilde{\psi}\in L_1[0,T]\end{aligned}$$

and $\widetilde{f}$ satisfies the assumptions of Lemma 5.4 with $\eta(y)h(t,x,y)$ in place of $h(t,x,y)$. Define $\widetilde{\mathcal{F}}:C^1[0,T]\to C^1[0,T]$ by

$$\widetilde{\mathcal{F}}(u)=\alpha\bigl(u(0)+u'(0)-u'(T)\bigr)+\mathcal{K}\bigl(\widetilde{\mathcal{N}}(u)\bigr),$$

where

$$\bigl(\widetilde{\mathcal{N}}(u)\bigr)(t)=\widetilde{f}\bigl(t,u(t),u'(t)\bigr) \quad \text{for } u\in C^1[0,T] \text{ and a.e. } t\in[0,T],$$

$$\alpha(x)=\begin{cases}\sigma_1(0) & \text{if } x<\sigma_1(0),\\ x & \text{if } \sigma_1(0)\leqslant x\leqslant\sigma_2(0),\\ \sigma_2(0) & \text{if } x>\sigma_2(0)\end{cases}$$

and let $\mathcal{K}:L_1[0,T]\to C^1[0,T]$ be given in (5.11). By Lemma 5.5, the operator $\widetilde{\mathcal{F}}$ is completely continuous. Moreover, it follows from the definition of the operator $\mathcal{K}$ that the problem

$$\bigl(\phi(u')\bigr)'=\widetilde{f}(t,u,u'), \quad u(0)=u(T)=\alpha\bigl(u(0)+u'(0)-u'(T)\bigr)$$

is equivalent to the operator equation $\widetilde{\mathcal{F}}(u) = u$. We can find $r_0 \in (0, \infty)$ such that for any $\lambda \in [0, 1]$, each fixed point u of the operator $\lambda\widetilde{\mathcal{F}}$ belongs to $\mathcal{B}(r_0) = \{x \in C^1[0, T]\colon \|x\|_\infty + \|x'\|_\infty < r_0\}$. So, $\mathcal{I} - \lambda\widetilde{\mathcal{F}}$ is a homotopy on $\overline{\mathcal{B}}(r_0) \times [0, 1]$ and

$$\deg\bigl(\mathcal{I} - \widetilde{\mathcal{F}}, \mathcal{B}(r_0)\bigr) = \deg\bigl(\mathcal{I}, \mathcal{B}(r_0)\bigr) = 1.$$

Put $\Omega_1 = \{u \in \Omega\colon \sigma_1(0) < u(0) + u'(0) - u'(T) < \sigma_2(0)\}$. Clearly, $\widetilde{\mathcal{F}} = \mathcal{F}$ on $\overline{\Omega}_1$ and $u \in \Omega_1$ whenever $\mathcal{F}(u) = u$ and $u \in \Omega$. Using Lemma 5.4, we can prove that

$$\bigl(\widetilde{\mathcal{F}}(u) = u\bigr) \quad \Longrightarrow \quad u \in \Omega_1$$

which, by the excision property of the degree, yields

$$\deg(\mathcal{I} - \mathcal{F}, \Omega) = \deg(\mathcal{I} - \mathcal{F}, \Omega_1) = \deg(\mathcal{I} - \widetilde{\mathcal{F}}, \Omega_1) = \deg\bigl(\mathcal{I} - \widetilde{\mathcal{F}}, \mathcal{B}(r_0)\bigr) = 1.$$

Finally, according to (5.14) all fixed points u of $\mathcal{F}$ such that $\sigma_1 < u < \sigma_2$ on $[0, T]$ belong to Ω. Thus

$$\deg(\mathcal{I} - \mathcal{F}, \Omega_\rho) = \deg(\mathcal{I} - \mathcal{F}, \Omega) = 1$$

for each $\rho \geqslant r^*$ such that $\mathcal{F}(x) \neq x$ on $\partial\Omega_\rho$. □

Lemma 5.6 offers a possibility to get existence results for problems having a pair of lower and upper functions σ_1 and σ_2 satisfying

$$\sigma_1 \leqslant \sigma_2 \quad \text{on } [0, T]. \tag{5.16}$$

In such a case we say that σ_1 and σ_2 are *well-ordered* and the existence of an a priori estimate r^* with the property (5.14) is usually ensured by conditions of Nagumo type. The most general known version of such conditions is provided by the next lemma which is a modified version of the result by Staněk [133, Lemma 1].

LEMMA 5.7. *Let $\sigma_1, \sigma_2 \in C[0, T]$ satisfy* (5.16) *and assume that*

$$\begin{cases} \psi \in L_1[0, T] \text{ is nonnegative}, \ \varepsilon_1, \varepsilon_2 \in \{-1, 1\}, \\ \omega \in C(\mathbb{R}) \text{ is positive and } \displaystyle\int_{-\infty}^0 \frac{\mathrm{d}s}{\omega(s)} = \int_0^\infty \frac{\mathrm{d}s}{\omega(s)} = \infty. \end{cases} \tag{5.17}$$

Then there is an $r^* > 0$ such that

$$\|v'\|_\infty < r^* \tag{5.18}$$

holds for each $v \in C^1[0,T]$ *such that* $\phi(v') \in AC[0,T]$, $v(0) = v(T)$, $v'(0) = v'(T)$, $\sigma_1 \leqslant v \leqslant \sigma_2$ *on* $[0,T]$ *and, for a.e.* $t \in [0,T]$,

$$\begin{cases} \varepsilon_1\big(\phi\big(v'(t)\big)\big)' \leqslant \big(\psi(t) + v'(t)\big)\omega\big(\phi\big(v'(t)\big)\big) & \text{if } v'(t) > 0, \\ \varepsilon_2\big(\phi\big(v'(t)\big)\big)' \leqslant \big(\psi(t) - v'(t)\big)\omega\big(\phi\big(v'(t)\big)\big) & \text{if } v'(t) < 0. \end{cases}$$
□

Lemma 5.6 provides also a crucial argument for the proof of existence of a solution even in the case that the given problem possesses lower and upper functions σ_1 and σ_2 which do not satisfy (5.16), i.e. if

$$\sigma_1(\tau) > \sigma_2(\tau) \quad \text{for some } \tau \in [0,T]. \tag{5.19}$$

In such a case, the following a priori estimate is available.

LEMMA 5.8. *Let* $\psi \in L_1[0,T]$. *Then there is* $r^* > 0$ *such that* (5.18) *holds for each* $v \in C^1[0,T]$ *fulfilling* $\phi(v') \in AC[0,T]$, $v(0) = v(T)$, $v'(0) = v'(T)$ *and* $(\phi(v'(t)))' > \psi(t)$ (*or* $(\phi(v'(t)))' < \psi(t)$) *for a.e.* $t \in [0,T]$.

PROOF. We will restrict ourselves to the case that $(\phi(v'(t)))' > \psi(t)$ for a.e. $t \in [0,T]$. (The other case can be proved by a similar argument.) By the proof of [129, Lemma 1.1], we can see that $\|w\|_\infty < \|\psi\|_1$ holds for each $w \in AC[0,T]$ such that $w(0) = w(T)$, $w(t_w) = 0$ for some $t_w \in (0,T)$ and $w'(t) > \psi(t)$ for a.e. $t \in [0,T]$. The assertion of the lemma follows by setting $w = \phi(v')$ and

$$r^* = \phi^{-1}\big(\|\psi\|_1\big). \tag{5.20}$$
□

The next lemma provides an existence principle which will be helpful later:

LEMMA 5.9. *Let* σ_1 *and* σ_2 *be a lower and an upper function of* (5.4), (5.2) *and let* (5.19) *be true. Furthermore, let there be* $m \in L_1[0,T]$ *such that*

$$h(t,x,y) > m(t) \ (\text{or } h(t,x,y) < m(t)) \quad \text{for a.e. } t \in [0,T] \text{ and all } x, y \in \mathbb{R}$$

and let $r^* > 0$ *be given by* (5.20), *where* $\psi = |m| + 2$. *Then problem* (5.4), (5.2) *has a solution u satisfying*

$$\|u'\|_\infty < r^* \tag{5.21}$$

and

$$\min\big\{\sigma_1(\tau_u), \sigma_2(\tau_u)\big\} \leqslant u(\tau_u) \leqslant \max\big\{\sigma_1(\tau_u), \sigma_2(\tau_u)\big\} \quad \text{for some } \tau_u \in [0,T]. \tag{5.22}$$

SKETCH OF THE PROOF. We follow the ideas of the proof of Theorem 3.2 in [127]. Assume, e.g., that $h(t,x,y) > m(t)$ for a.e. $t \in [0,T]$ and all $x, y \in \mathbb{R}$.

Step 1. *Construction of an auxiliary problem and the operator representation.*

Define $\psi(t){:} = -(|m(t)|+2)$ for a.e. $t \in [0,T]$, find $r^* > 0$ as in Lemma 5.8 and set $c^* = \|\sigma_1\|_\infty + \|\sigma_2\|_\infty + Tr^*$. Consider the auxiliary problem

$$\big(\phi(u')\big)' = \widetilde{f}(t,u,u'), \quad u(0) = u(T), \quad u'(0) = u'(T), \tag{5.23}$$

where

$$\widetilde{f}(t,x,y) = \begin{cases} -\big(|m(t)|+1\big) & \text{if } x \leqslant -(c^*+1), \\ h(t,x,y) + (x+c^*)\big(|m(t)|+1+h(t,x,y)\big) & \\ & \text{if } -(c^*+1) < x < -c^*, \\ h(t,x,y) & \text{if } -c^* \leqslant x \leqslant c^*, \\ h(t,x,y) + (x-c^*)|m(t)| & \text{if } c^* < x < c^*+1, \\ h(t,x,y) + |m(t)| & \text{if } x \geqslant c^*+1. \end{cases}$$

We have

$$\begin{cases} \widetilde{f}(t,x,y) < 0 & \text{if } x \leqslant -(c^*+1), \\ \widetilde{f}(t,x,y) > 0 & \text{if } x \geqslant c^*+1, \\ \widetilde{f}(t,x,y) = h(t,x,y) & \text{if } x \in [-c^*, c^*], \end{cases} \tag{5.24}$$

$$\widetilde{f}(t,x,y) > \psi(t) \quad \text{for a.e. } t \in [0,T] \text{ and all } x, y \in \mathbb{R} \tag{5.25}$$

and σ_1 and σ_2 are a lower and an upper function of (5.23). Moreover, $\sigma_3(t) \equiv -c^* - 2$ and $\sigma_4(t) \equiv c^* + 2$ form another pair of a lower and an upper function for (5.23) and

$$\sigma_3 < \min\{\sigma_1, \sigma_2\} \leqslant \max\{\sigma_1, \sigma_2\} < \sigma_4 \text{ on } [0,T].$$

Denote $\Omega_0 = \{u \in C^1[0,T]{:}\ \sigma_3 < u < \sigma_4 \text{ on } [0,T],\ \|u'\|_\infty < r^*\}$,

$$\Omega_1 = \big\{u \in \Omega_0{:}\ \sigma_3 < u < \sigma_2 \text{ on } [0,T]\big\},$$
$$\Omega_2 = \big\{u \in \Omega_0{:}\ \sigma_1 < u < \sigma_4 \text{ on } [0,T]\big\}$$

and $\Omega = \Omega_0 \setminus \overline{\Omega_1 \cup \Omega_2}$. By Lemma 5.5, problem (5.23) is equivalent to the operator equation $\widetilde{\mathcal{F}}(u) = u$ in $C^1[0,T]$, where $\widetilde{\mathcal{F}}(u) = u(0) + u'(0) - u'(T) + \mathcal{K}(\widetilde{\mathcal{N}}(u))$, $(\widetilde{\mathcal{N}}(u))(t) = \widetilde{f}(t,u(t),u'(t))$ and $\mathcal{K}{:}\ L_1[0,T] \to C^1[0,T]$ is given by (5.11). Clearly, $\widetilde{\mathcal{F}}(u) = \mathcal{F}(u)$ for $u \in C^1[0,T]$ such that $\|u\|_\infty \leqslant c^*$.

Step 2. *A priori estimates.* We show that

$$\|u'\|_\infty < r^* \quad \text{and} \quad \|u\|_\infty < c^*$$

is true for all $u \in \overline{\Omega}$ such that $\widetilde{\mathcal{F}}(u) = u$.

Step 3. *Existence of a solution to* (5.4), (5.2).

Let $\widetilde{\mathcal{F}}(u) = u$ and $u \in \partial\Omega$. By Step 3, we have $\mathcal{F}(u) = \widetilde{\mathcal{F}}(u) = u$ and u solves (5.4), (5.2). Let $\widetilde{\mathcal{F}}(u) \neq u$ on $\partial\Omega$. Then using Lemma 5.6 we get

$$\deg\big(\mathcal{I} - \widetilde{\mathcal{F}}, \Omega_0\big) = \deg\big(\mathcal{I} - \widetilde{\mathcal{F}}, \Omega_1\big) = \deg\big(\mathcal{I} - \widetilde{\mathcal{F}}, \Omega_2\big) = 1.$$

Furthermore, by (5.19), we have $\Omega_1 \cap \Omega_2 = \emptyset$. Therefore, due to the additive property of the degree,

$$\deg\big(\mathcal{I} - \widetilde{\mathcal{F}}, \Omega\big) = \deg\big(\mathcal{I} - \widetilde{\mathcal{F}}, \Omega_0\big) - \deg\big(\mathcal{I} - \widetilde{\mathcal{F}}, \Omega_1\big) - \deg\big(\mathcal{I} - \widetilde{\mathcal{F}}, \Omega_2\big) = -1$$

which implies that $\widetilde{\mathcal{F}}$ has a fixed point $u \in \Omega$. It follows by Step 3 that $\|u\|_\infty < c^*$ which, by virtue of (5.24), means that u solves (5.4), (5.2).

We can proceed analogously when $h(t, x, y) < m(t)$ for a.e. $t \in [0, T]$ and all $x, y \in \mathbb{R}$. □

5.2. *Method of lower and upper functions for singular problems*

Now, we consider problem (5.1), (5.2) where f satisfies (5.3). We present sufficient conditions in terms of lower and upper functions for the existence of positive solutions to (5.1), (5.2). Lower and upper functions σ_1 and σ_2 are defined in the same way as for the regular problem (5.4), (5.2) (see Definition 5.3). However, since problem (5.1), (5.2) is investigated on $[0, T] \times \mathcal{A}$ where $\mathcal{A} = [0, \infty) \times \mathbb{R}$, only such σ_1 and σ_2 which are positive a.e. on $[0, T]$ make sense.

The first existence result concerns problem (5.1), (5.2) having well-ordered lower and upper functions.

THEOREM 5.10. *Let there exist lower and upper functions σ_1 and σ_2 of problem* (5.1), (5.2) *such that* (5.16) *is true and $\sigma_1 > 0$ on* $[0, T]$. *Furthermore, let for a.e. $t \in [0, T]$ and each $(x, y) \in [\sigma_1(t), \sigma_2(t)] \times \mathbb{R}$ the inequalities*

$$\begin{cases} \varepsilon_1 f(t, x, y) \leqslant \big(\psi(t) + y\big)\omega\big(\phi(y)\big) & \textit{if } y > 0, \\ \varepsilon_2 f(t, x, y) \leqslant \big(\psi(t) - y\big)\omega\big(\phi(y)\big) & \textit{if } y < 0 \end{cases} \tag{5.26}$$

hold with $\varepsilon_1, \varepsilon_2, \omega$ and ψ satisfying (5.17). *Then problem* (5.1), (5.2) *has a positive solution u such that*

$$\sigma_1 \leqslant u \leqslant \sigma_2 \quad \textit{on } [0, T]. \tag{5.27}$$

PROOF. *Step* 1. *The case $\sigma_1 < \sigma_2$.*

Assume that $\sigma_1 < \sigma_2$ on $[0, T]$. Consider the auxiliary problem (5.4), (5.2) with $h(t, x, y) = f(t, \max\{\sigma_1(t), \min\{x, \sigma_2(t)\}, y)$ for a.e. $t \in [0, T]$ and $(x, y) \in \mathbb{R}^2$. Clearly, $h \in Car([0, T] \times \mathbb{R}^2)$ and $h(t, x, y) = f(t, x, y)$ if $x \in [\sigma_1(t), \sigma_2(t)]$. Further, σ_1 and σ_2 are a lower and an upper function of (5.4), (5.2). Choose an arbitrary continuous $\eta : \mathbb{R} \to [0, 1]$

and let v be an arbitrary solution of (5.13) fulfilling $\sigma_1 \leqslant v \leqslant \sigma_2$ on $[0, T]$. Since (5.26) is satisfied with h in place of f, we have for a.e. $t \in [0, T]$

$$\begin{aligned}\varepsilon_1\big(\phi\big(v'(t)\big)\big)' &= \varepsilon_1\eta\big(v'(t)\big)h\big(t, v(t), v'(t)\big)\\ &\leqslant \eta\big(v'(t)\big)\big(\psi(t) + v'(t)\big)\omega\big(\phi\big(v'(t)\big)\big)\\ &\leqslant \big(\psi(t) + v'(t)\big)\omega\big(\phi\big(v'(t)\big)\big) \quad \text{if } v'(t) > 0\end{aligned}$$

and

$$\varepsilon_2\big(\phi\big(v'(t)\big)\big)' \leqslant \big(\psi(t) - v'(t)\big)\omega\big(\phi\big(v'(t)\big)\big) \quad \text{if } v'(t) < 0.$$

Hence we can apply Lemma 5.7 to deduce that (5.14) is satisfied. Let $\mathcal{F}: C^1[0, T] \to C^1[0, T]$ and $\Omega = \Omega_{r^*}$ be defined by (5.12) and (5.15), respectively. Then there are two possibilities: either $\mathcal{F}$ has a fixed point $u \in \partial\Omega$ or $\mathcal{F}(u) \neq u$ on $\partial\Omega$.

(a) Let $\mathcal{F}(u) = u$ for some $u \in \partial\Omega$. In view of Lemma 5.5 and of the definition of h, it follows that u is a solution to (5.1), (5.2) fulfilling (5.27).

(b) If $\mathcal{F}(u) \neq u$ on $\partial\Omega$, then by Lemma 5.6 we have $\deg(\mathcal{I} - \mathcal{F}, \Omega) = 1$, which implies that $\mathcal{F}$ has a fixed point $u \in \Omega$. As in (a), this fixed point is a solution to (5.1), (5.2) fulfilling (5.27).

Step 2. *The case* $\sigma_1 \leqslant \sigma_2$.

For each $k \in \mathbb{N}$, the function $\widetilde{\sigma}_k = \sigma_2 + \frac{1}{k}$ is also an upper function of (5.4), (5.2) and $\sigma_1 < \widetilde{\sigma}_k$ on $[0, T]$. Hence, in the general case, when the strict inequality between σ_1 and σ_2 need not hold, we can use Step 1 to show that for each $k \in \mathbb{N}$ there exists a solution u_k to (5.4), (5.2) such that

$$u_k(t) \in \left[\sigma_1(t), \sigma_2(t) + \tfrac{1}{k}\right] \quad \text{for } t \in [0, T] \quad \text{and} \quad \|u_k'\|_\infty < \rho^*,$$

where $\rho^* > 0$ is the a priori estimate given by Lemma 5.7 with $\sigma_2 + 1$ in place of σ_2. Using the Arzelà–Ascoli theorem and the Lebesgue dominated convergence theorem for the sequence $\{u_k\}$ we get a solution u of (5.1), (5.2) as the C^1-limit of a subsequence of $\{u_k\}$. □

REMARK 5.11. Theorem 5.10 provides the existence of a positive solution to problem (5.1), (5.2) with $f(t, x, y) = -h(x)y + g(t, x)$, if $h \in C[0, \infty)$, $g \in Car([0, T] \times (0, \infty))$ and if the existence of well-ordered and positive lower and upper functions is ensured. Indeed, for a.e. $t \in [0, T]$ and each $(x, y) \in [\sigma_1(t), \sigma_2(t)] \times \mathbb{R}$, we have

$$\big|f(t, x, y)\big| \leqslant \big|h(x)\big||y| + \big|g(t, x)\big| \leqslant K\big(\psi(t) + |y|\big)$$

where $K = 1 + \max\{|h(x)|\colon x \in [\delta, \|\sigma_2\|_\infty]\}$, $\psi(t) = \sup\{|g(t, x)|\colon x \in [\delta, \|\sigma_2\|_\infty]\}$ and $\delta = \min\{\sigma_1(t)\colon t \in [0, T]\}$. (By assumption, we have $\delta > 0$.)

Now, we will consider problem (5.1), (5.2) which has lower and upper functions, but no pair of them is well-ordered. We will deal with the periodic problem for the equation

$$\big(\phi(u')\big)' = g(u) + p(t, u, u'). \tag{5.28}$$

THEOREM 5.12. *Assume* $g \in C(0,\infty)$, $p \in Car([0,T]\times\mathbb{R}^2)$ *and*

$$\lim_{x\to 0+}\int_x^1 g(\xi)\,\mathrm{d}\xi = +\infty. \tag{5.29}$$

Let there exist lower and upper functions σ_1 *and* σ_2 *of problem* (5.28), (5.2) *such that* (5.19) *is true and* $\sigma_2 > 0$ *on* $[0,T]$. *Furthermore, let there exist an* $m \in L_1[0,T]$ *such that*

$$g(x) + p(t,x,y) > m(t) \quad \text{for a.e. } t\in[0,T] \text{ and all } x>0,\ y\in\mathbb{R} \tag{5.30}$$

holds and let r^* *be given by* (5.20) *with* $\psi = |m| + 2$. *Then problem* (5.28), (5.2) *has a positive solution* u *satisfying* (5.21) *and* (5.22).

PROOF. We will use the ideas of [130]. Similarly to [130, Lemma 2.5] we can deduce from (5.29) and (5.30) that σ_1 is positive on $[0,T]$. Thus, $\delta := \min\{\{\sigma_1(t),\sigma_2(t)\}\colon t\in[0,T]\} > 0$. Put $R = \|\sigma_1\|_\infty + \|\sigma_2\|_\infty$ and $B = R + r^*T$. Furthermore, as $p \in Car([0,T]\times\mathbb{R}^2)$, there is $\widetilde{p} \in L_1[0,T]$ such that $|p(t,x,y)| \leqslant \widetilde{p}(t)$ for a.e. $t\in[0,T]$ and all $(x,y)\in[0,B]\times[-r^*,r^*]$. Put

$$K = \|\widetilde{p}\|_1 r^* + \int_\delta^B |g(\xi)|\,\mathrm{d}\xi.$$

By (5.29) there exists $\varepsilon \in (0,\delta)$ such that $g(\varepsilon) > 0$ and

$$\int_\varepsilon^\delta g(\xi)\,\mathrm{d}\xi > K. \tag{5.31}$$

For a.e. $t\in[0,T]$ and $(x,y)\in\mathbb{R}^2$, define

$$h(t,x,y) = \widetilde{g}(x) + p(t,x,y), \qquad \text{where } \widetilde{g}(x) = \begin{cases} g(\varepsilon) & \text{if } x<\varepsilon,\\ g(x) & \text{if } x\geqslant\varepsilon.\end{cases}$$

Then $h \in Car([0,T]\times\mathbb{R}^2)$, σ_1 and σ_2 are lower and upper functions of (5.4), (5.2) and, by (5.30), $h(t,x,y) > m(t)$ for a.e. $t\in[0,T]$ and all $x>0$, $y\in\mathbb{R}$. By Lemma 5.9, problem (5.4), (5.2) has a solution u satisfying (5.21) and $\delta \leqslant u(t_u) \leqslant R$ for some $t_u \in [0,T]$. In particular, $u \leqslant B$ for all $t\in[0,T]$. It remains to show that $u \geqslant \varepsilon$ on $[0,T]$.

Let $t_0, t_1 \in [0,T]$ be such that $u(t_0) = \min\{u(t)\colon t\in[0,T]\}$ and $u(t_1) = \max\{u(t)\colon t\in[0,T]\}$. We have $u'(t_0) = u'(t_1) = 0$ and $u(t_1)\in[\delta,B]$. Put $v(t) = \phi(u'(t))$ for $t\in[0,T]$. Then $u'(t) = \phi^{-1}(v(t))$ on $[0,T]$, $v(t_0) = v(t_1) = \phi(0)$ and

$$\int_{t_0}^{t_1} \bigl(\phi\bigl(u'(s)\bigr)\bigr)' u'(s)\,\mathrm{d}s = \int_{t_0}^{t_1} v'(s)\phi^{-1}\bigl(v(s)\bigr)\,\mathrm{d}s = \int_{v(t_0)}^{v(t_1)} \phi^{-1}(\xi)\,\mathrm{d}\xi = 0.$$

Thus, multiplying both sides of the equality $(\phi(u'(t)))' = h(t, u(t), u'(t))$ by $u'(t)$ and integrating from t_0 to t_1, we get

$$\int_{u(t_0)}^{u(t_1)} \widetilde{g}(\xi)\,\mathrm{d}\xi \leqslant \int_{t_0}^{t_1} \big|p\big(t, u(t), u'(t)\big)\big|\big|u'(t)\big|\,\mathrm{d}t \leqslant \|\widetilde{p}\|_1 r^*.$$

Therefore

$$\begin{aligned} g(\varepsilon)\big(\varepsilon - u(t_0)\big) + \int_{\varepsilon}^{\delta} g(\xi)\,\mathrm{d}\xi &= \int_{u(t_0)}^{\delta} \widetilde{g}(\xi)\,\mathrm{d}\xi \\ \leqslant \int_{u(t_0)}^{u(t_1)} \widetilde{g}(\xi)\,\mathrm{d}\xi + \int_{\delta}^{B} \big|g(\xi)\big|\,\mathrm{d}\xi &\leqslant \|\widetilde{p}\|_1 r^* + \int_{\delta}^{B} \big|g(\xi)\big|\,\mathrm{d}\xi = K. \end{aligned}$$

Since $g(\varepsilon) > 0$, this contradicts (5.31) whenever $u(t_0) = \min\{u(t)\colon t \in [0, T]\} < \varepsilon$. Hence, $u(t) \geqslant \varepsilon$ on $[0, T]$ which means that u is a solution to (5.28), (5.2). □

REMARK 5.13. Let g and p fulfill the assumptions of Theorem 5.12 and $f(t, x, y) = g(x) + p(t, x, y)$. Then the condition (5.29) implies that

$$\limsup_{x \to 0+} g(x) = +\infty, \tag{5.32}$$

which means by Remark 5.1 that f and g have a space repulsive singularity at $x = 0$. Each repulsive singularity having the property (5.29) is called a *strong singularity* of f and the corresponding function g is usually called a *strong repulsive singular force*. On the contrary, if (5.32) holds together with

$$\lim_{x \to 0+} \int_x^1 g(\xi)\,\mathrm{d}\xi \in \mathbb{R}, \tag{5.33}$$

then the singularity of f at $x = 0$ is called a *weak singularity* and g is called a *weak repulsive singular force.*

5.3. *Attractive singular forces*

This section is devoted to singular problem (5.1), (5.2) where f can have an attractive singularity at $x = 0$. (See Remark 5.1.)

In what follows we use the standard notation for mean values of integrable functions: for $y \in L_1[0, T]$, the symbol $\overline{y}$ stands for

$$\overline{y} := \frac{1}{T} \int_0^T y(t)\,\mathrm{d}t.$$

THEOREM 5.14. *Let there exist* $r > 0$, $A > r$ *and* $b \in L_1[0, T]$ *such that* $\overline{b} \geqslant 0$,

$$f(t, r, 0) \leqslant 0 \quad \text{for a.e. } t \in [0, T], \tag{5.34}$$

$$f(t, x, y) \geqslant b(t) \quad \text{for a.e. } t \in [0, T] \text{ and all } x \in [A, B] \text{ and } |y| \leqslant \phi^{-1}\big(\|b\|_1\big), \tag{5.35}$$

where

$$B - A \geqslant 2T\phi^{-1}\big(\|b\|_1\big). \tag{5.36}$$

Furthermore, let for a.e. $t \in [0, T]$ *and each* $(x, y) \in [r, B] \times \mathbb{R}$ *the inequalities* (5.26) *hold with* ε_1, ε_2, ω *and* ψ *satisfying* (5.17).

Then problem (5.1), (5.2) *has a positive solution* u *such that*

$$r \leqslant u \leqslant B \quad \text{on } [0, T]. \tag{5.37}$$

PROOF. For a given $d \in \mathbb{R}$, let x_d be a solution of (5.9). Then

$$\phi\big(x_d'(t)\big) = \phi\big(x_d'(t_0)\big) + \int_{t_0}^{t} b(s)\,\mathrm{d}s \quad \text{for all } t, t_0 \in [0, T].$$

Since $\overline{b} \geqslant 0$, it follows that $x_d'(T) \geqslant x_d'(0)$. Since $x_d(0) = x_d(T)$, there is a $t_d \in (0, T)$ such that $x_d'(t_d) = 0$. Thus

$$\phi\big(x_d'(t)\big) = \int_{t_d}^{t} b(s)\,\mathrm{d}s \quad \text{for } t \in [0, T]$$

and so $\|x_d'\|_\infty \leqslant \phi^{-1}(\|b\|_1)$ for each $d \in \mathbb{R}$ and $\|x_0\|_\infty \leqslant T\phi^{-1}(\|b\|_1)$. Put $\sigma_2 = A + T\phi^{-1}(\|b\|_1) + x_0$. Then

$$A \leqslant \sigma_2 \leqslant A + 2T\phi^{-1}\big(\|b\|_1\big) \leqslant B \quad \text{on } [0, T]. \tag{5.38}$$

Having in mind (5.35) and (5.9), we can see that σ_2 is an upper function of (5.1), (5.2). Furthermore, $\sigma_1 = r$ is a lower function of problem (5.1), (5.2) and $0 < \sigma_1 < \sigma_2$ on $[0, T]$. By Theorem 5.10, problem (5.1), (5.2) has a positive solution u satisfying (5.37). □

Now, let us consider the Liénard periodic problem

$$\big(\phi(u')\big)' + h(u)u' = g(t, u) + e(t), \quad u(0) = u(T), \quad u'(0) = u'(T), \tag{5.39}$$

where g can have an attractive space singularity at $x = 0$.

THEOREM 5.15. *Assume*

$$h \in C[0,\infty),\ e \in L_1[0,T],\ g \in Car\big([0,T]\times(0,\infty)\big), \tag{5.40}$$

$$\textit{there exists } \alpha > 0 \textit{ such that } \liminf_{|y|\to\infty} \frac{|\phi(y)|}{|y|^{\alpha}} > 0, \tag{5.41}$$

$$\begin{cases} \textit{there exists } r > 0 \textit{ such that} \\ g(t,r) + e(t) \leqslant 0 \quad \textit{for a.e. } t \in [0,T], \end{cases} \tag{5.42}$$

$$\begin{cases} \textit{there exist } A > r \textit{ and } g_0 \in L_1[0,T] \textit{ such that} \\ g(t,x) \geqslant g_0(t) \quad \textit{for a.e. } t \in [0,T] \textit{ and all } x \geqslant A \end{cases} \tag{5.43}$$

and

$$\overline{g}_0 + \overline{e} \geqslant 0. \tag{5.44}$$

Then problem (5.39) *has a positive solution u such that $u \geqslant r$ on* $[0,T]$.

SKETCH OF THE PROOF. We follow the ideas of the paper [128]. Define

$$f(t,x,y) = -h(x)y + g(t,x) + e(t)$$
$$\text{for a.e. } t \in [0,T] \text{ and all } x \in (0,\infty),\ y \in \mathbb{R}.$$

Step 1. First, notice that, due to (5.42), $\sigma_1(t) \equiv r$ is a lower function of (5.39).

Step 2. Thanks to (5.41), (5.43) and (5.44), we can construct an upper function σ_2 of (5.39). To this aim, take an arbitrary $C \in \mathbb{R}$ and consider a parameter auxiliary problem

$$\big(\phi(v')\big)' + \lambda h(v+C)v' = \lambda b(t), \quad v(0) = v(T) = 0, \quad \lambda \in [0,1], \tag{5.45}$$

where $b(t) = g_0(t) + e(t)$ for a.e. $t \in [0,T]$. By (5.41), there are $k > 0$ and $y_0 > 0$ such that

$$\big|\phi(y)\big| > \frac{k}{2}|y|^{\alpha} \quad \text{for } |y| \geqslant y_0. \tag{5.46}$$

Multiplying (5.45) by $v(t)$ and integrating over $[0,T]$, we obtain

$$-\int_0^T \phi\big(v'(t)\big)v'(t)\,\mathrm{d}t = \lambda \int_0^T b(t)v(t)\,\mathrm{d}t. \tag{5.47}$$

Using (5.46), (5.47) and the Hölder inequality, we can find $\rho \in (0,\infty)$, independent of $C \in \mathbb{R}$, such that $v \in \mathcal{B}(\rho) = \{x \in C^1[0,T]\colon \|x\|_\infty + \|x'\|_\infty < \rho\}$ holds for each $\lambda \in [0,1]$ and each solution v of (5.45). Thus, choosing a proper operator representation of problem (5.45) and using a standard homotopy and topological degree argument we can show that, for each $C \in \mathbb{R}$, problem (5.45) with $\lambda = 1$ has a solution $v_C \in \mathcal{B}(\rho)$. Now, it is already easy to see that if $C > A + \rho$, then $\sigma_2 = v_C + C$ is an upper function of (5.39). Indeed, we have $\sigma_2(0) = \sigma_2(T) = C$ and, due to (5.44),

$$\phi\big(\sigma_2'(T)\big) - \phi\big(\sigma_2'(0)\big) = T\overline{b} = T[\overline{g}_0 + \overline{e}] \geqslant 0.$$

Moreover, $\sigma_2(t) \geqslant C - \rho > A > r$ on $[0, T]$. Hence, by (5.43), we have

$$\begin{aligned}\big(\phi\big(\sigma_2'(t)\big)\big)' &= -h\big(\sigma_2(t)\big)\sigma_2'(t) + g_0(t) + e(t)\\ &\leqslant -h\big(\sigma_2(t)\big)\sigma_2'(t) + g\big(t, \sigma_2(t)\big) + e(t) = f\big(t, \sigma_2(t), \sigma_2'(t)\big)\\ &\quad \text{for a.e. } t \in [0, T].\end{aligned}$$

Step 3. Finally, similarly as in Remark 5.11, we show that f satisfies (5.26) with $\omega(s) \equiv 1 + \max\{|h(x)|\colon x \in [r, \|\sigma_2\|_\infty]\}$, $\psi(t) = |e(t)| + \sup\{|g(t, x)|\colon x \in [r, \|\sigma_2\|_\infty]\}$ and $\varepsilon_1 = \varepsilon_2 = 1$. Therefore, by Theorem 5.10, problem (5.39) has a positive solution u such that $u \geqslant r$ on $[0, T]$. □

REMARK 5.16. If g does not depend on t, i.e. $g(t, x) \equiv g(x)$ for a.e. $t \in [0, T]$ and all $x \in (0, \infty)$, then the condition (5.42) is satisfied if $\liminf_{x\to 0+}(g(x) + \|e\|_\infty) < 0$ which is true, e.g., if $\liminf_{x\to 0+} g(x) = -\infty$ and $\sup\operatorname{ess}\{e(t)\colon t \in [0, T]\} < \infty$. Similarly, the conditions (5.43) and (5.44) are in such a case satisfied if $\liminf_{x\to\infty}(g(x) + \overline{e}) > 0$. In particular, Theorem 5.15 applies to problem (5.39) if $\phi = \phi_p$, $p > 1$, $\sup\operatorname{ess}\{e(t)\colon t \in [0, T]\} < \infty$, $\overline{e} > 0$, $g(t, x) = -\beta(t)x^{-\lambda}$, where $\beta \in L_1[0, T]$, $\beta \geqslant \varepsilon > 0$ a.e. on $[0, T]$ and $\lambda \geqslant 1$. Notice that the condition (5.41) is satisfied, e.g., by $\phi(y) = (|y|y + y)\ln(1 + \frac{1}{|y|})$ or $\phi(y) = y(\exp(y^2) - 1)$.

5.4. *Repulsive singular forces*

In this section we study the singular problem (5.1), (5.2) with f having a repulsive singularity at $x = 0$. Recall (see Remark 5.1) that this means that the relation

$$\limsup_{x\to 0+} f(t, x, y) = \infty \quad \text{for a.e. } t \in [0, T] \text{ and some } y \in \mathbb{R}$$

is true. In general, for the case of a repulsive singularity, the existence of a pair of associated lower and upper functions having opposite order is typical. This causes that such a case is more difficult and more interesting than that of an attractive singularity. The next assertion deals with equation (5.28) and is a direct corollary of Theorem 5.12.

THEOREM 5.17. *Assume $g \in C(0, \infty)$, $p \in Car([0, T] \times \mathbb{R}^2)$, (5.29) and (5.30) with some $m \in L_1[0, T]$. Furthermore, let there be $r > 0$, $A > r$, $B \geqslant A$ and $b \in L_1[0, T]$ such that $\overline{b} \leqslant 0$, (5.36),*

$$g(r) + p(t, r, 0) \geqslant 0 \quad \textit{for a.e. } t \in [0, T], \tag{5.48}$$

and

$$g(x) + p(t,x,y) \leqslant b(t)$$
$$\text{for a.e. } t \in [0,T] \text{ and all } x \in [A,B] \text{ and } |y| \leqslant \phi^{-1}\big(\|b\|_1\big) \tag{5.49}$$

hold. Then problem (5.28), (5.2) *has a positive solution u such that* $u(t_u) \in [r,B]$ *for some* $t_u \in [0,T]$.

PROOF. By (5.48), $\sigma_2(t) \equiv r$ is an upper function of (5.28), (5.2). Furthermore, let x_0 be a solution of

$$\big(\phi(x')\big)' = b(t), \quad u(0) = u(T) = 0.$$

Using (5.49) and having in mind that $\overline{b} \leqslant 0$, we can show by a reasoning analogous to that applied in the proof of Theorem 5.14 to construct an upper function that the function $\sigma_1 = A + T\phi^{-1}(\|b\|_1) + x_0$ is a lower function of (5.28), (5.2). Using Theorem 5.12 we complete the proof. □

In particular, when restricted to the Duffing equation with the ϕ-Laplacian

$$\big(\phi(u')\big)' = g(u) + e(t), \tag{5.50}$$

Theorem 5.17 has the following corollary.

COROLLARY 5.18. *Let* $e \in L_1[0,T]$ *with* $\operatorname{inf\,ess}\{e(t)\colon t \in [0,T]\} > -\infty$ *and let* $g \in C(0,\infty)$ *have a strong repulsive singularity* (5.29). *Further, let*

$$g_* := \inf\big\{g(x)\colon x \in (0,\infty)\big\} > -\infty \tag{5.51}$$

and let there be $A > 0$ *such that*

$$g(x) + \overline{e} \leqslant 0 \quad \text{for } x \in [A,B], \text{ where } B - A \geqslant 2T\phi^{-1}\big(\|e - \overline{e}\|_1\big).$$

Then problem (5.50), (5.2) *has a positive solution u such that* $u(t_u) \leqslant B$ *for some* $t_u \in [0,T]$.

PROOF. By (5.29) we have (5.32) and consequently there is an $r \in (0,A)$ such that $g(r) + e(t) \geqslant 0$ for a.e. $t \in [0,T]$. The assertion follows from Theorem 5.17 if we put $b(t) = e(t) - \overline{e}$ and $m(t) = g_* + e(t)$ a.e. on $[0,T]$. □

Consider the periodic problem for the Liénard equation

$$\big(\phi_p(u')\big)' + h(u)u' = g(u) + e(t) \tag{5.52}$$

with the p-Laplacian $\phi_p(y) = |y|^{p-2}y$. To this end, the following easy corollary of the continuation type principle due to Manásevich and Mawhin turned out to be essential.

LEMMA 5.19 [101, Theorem 3.1] and [84, Lemma 3]. *Let* $p > 1$, $h \in C[0,\infty)$, $g \in C(0,\infty)$ *and* $e \in L_1[0,T]$. *Furthermore, assume there exist* $r > 0$, $R > r$ *and* $R' > 0$ *such that*

(i) *the inequalities* $r < v < R$ *on* $[0,T]$ *and* $\|v'\|_\infty < R'$ *hold for each* $\lambda \in (0,1]$ *and for each positive solution* v *of the problem*

$$\begin{cases} \big(\phi_p(v')\big)' = \lambda\big(-h(v)v' + g(v) + e(t)\big), \\ v(0) = v(T), \quad v'(0) = v'(T), \end{cases} \tag{5.53}$$

(ii) $(g(x) + \overline{e} = 0) \Longrightarrow r < x < R$,

(iii) $(g(r) + \overline{e})(g(R) + \overline{e}) < 0$.

Then problem (5.52), (5.2) *has at least one solution* u *such that* $r < u < R$ *on* $[0,T]$.

Under the assumptions ensuring that g is bounded below on $(0,\infty)$, the following result was delivered by Jebelean and Mawhin.

THEOREM 5.20 [84, Theorem 2]. *Let* $p > 1$, $h \in C[0,\infty)$, $e \in L_1[0,T]$ *and let* $g \in C(0,\infty)$ *have a strong repulsive singularity* (5.29). *Furthermore, assume*

$$\liminf_{x\to\infty} g(x) > -\infty \tag{5.54}$$

and

$$\liminf_{x\to 0+}\big[g(x) + \overline{e}\big] > 0 > \limsup_{x\to\infty}\big[g(x) + \overline{e}\big]. \tag{5.55}$$

Then problem (5.52), (5.2) *has a positive solution*.

PROOF. We will verify that the assumptions of Lemma 5.19 are satisfied.

Step 1. First, we will show that

$$\begin{cases} \text{there are } R_0 > 0 \text{ and } R_1 > R_0 \text{ such that} \\ v(t_v) \in (R_0, R_1) \text{ for some } t_v \in [0,T] \\ \text{holds for each } \lambda \in (0,1] \text{ and each positive solution } v \text{ of (5.53).} \end{cases} \tag{5.56}$$

To this aim, assume that $\lambda \in (0,1]$ and that v is a positive solution to (5.53). Integrating the differential equation in (5.53) over $[0,T]$ and having in mind the periodicity of v, we get:

$$\int_0^T \big(g\big(v(t)\big) + e(t)\big)\,\mathrm{d}t = 0. \tag{5.57}$$

By the first inequality in (5.55), there is an $R_0 > 0$ such that

$$g(x) + \overline{e} > 0 \quad \text{whenever } x \in (0, R_0). \tag{5.58}$$

If $g(v(t)) + \overline{e} > 0$ were valid on $[0, T]$, we would have

$$\int_0^T \big(g\big(v(t)\big) + e(t)\big)\,\mathrm{d}t = \int_0^T \big(g\big(v(t)\big) + \overline{e}\big)\,\mathrm{d}t > 0.$$

Since this contradicts (5.57), we see that $\max\{v(t)\colon t \in [0, T]\} > R_0$. Similarly, by the second inequality in (5.55), there is an $R_1 > R_0$ such that $g(x) + \overline{e} < 0$ for $x \geqslant R_1$ and $v(t_1) < R_1$ for some $t_1 \in [0, T]$. Therefore (5.56) is true.

Step 2. Now we show that

$$\begin{cases} \text{there is } R > 0 \text{ such that } v < R \text{ on } [0, T] \\ \text{for each } \lambda \in (0, 1] \text{ and each positive solution } v \text{ of (5.53).} \end{cases} \tag{5.59}$$

Notice that, due to (5.54) and (5.58), we have $g_* = \inf\{g(x)\colon x \in (0, \infty)\} > -\infty$. Thus, multiplying (5.53) by v and integrating over $[0, T]$, we get

$$\|v'\|_p^p \leqslant \int_0^T \big(|g_*| + |e(t)|\big) v(t)\,\mathrm{d}t.$$

Furthermore, for R_1 given as in Step 1 and $\frac{1}{p} + \frac{1}{q} = 1$, we deduce that

$$\|v'\|_p^p \leqslant \left(\int_0^T \big(|g_*| + |e(t)|\big)\,\mathrm{d}t\right)\big(R_1 + T^{1/q}\|v'\|_p\big).$$

The right-hand side being a linear function of $\|v'\|_p$, this is possible only if there is $C_1 > 0$, independent of v and λ and such that $\|v'\|_p < C_1$. Therefore

$$v(t) = v(t_1) + \int_{t_1}^t v'(s)\,\mathrm{d}s < R_1 + T^{1/q} C_1$$

for all $\lambda \in (0, 1]$ and all positive solutions v of (5.53), i.e. the assertion (5.59) is true with $R := R_1 + T^{1/q} C_1$.

Step 3. Next we show that

$$\begin{cases} \text{there is } R_2 > 0 \text{ such that } |v'| < \lambda^{\frac{1}{p-1}} R_2 \text{ on } [0, T] \\ \text{for each } \lambda \in (0, 1] \text{ and each positive solution } v \text{ of (5.53).} \end{cases} \tag{5.60}$$

Having in mind that v satisfies the periodic conditions, we can see that there is $t' \in [0, T]$ such that $v'(t') = 0$. Integrating the differential equation in (5.53) over $[t', t]$ and taking into account (5.59), we get

$$|v'(t)|^{p-1} \leqslant \lambda \left(\int_0^R |h(x)|\,\mathrm{d}x + \|e\|_1 + \left| \int_{t'}^t |g(v(s))|\,\mathrm{d}s \right| \right)$$
$$\text{for } t \in [0, T]. \tag{5.61}$$

By (5.58), there is $b > 0$ such that $g(x) \geqslant -b$ for all $x \in (0, R]$. So, by (5.59), $g(v(t)) \geqslant -b$ on $[0, T]$ holds for each possible positive solution v of (5.53). Therefore, $|g(v(t))| \leqslant g(v(t)) + 2b$ for all $t \in [0, T]$ wherefrom, using (5.57), we deduce

$$\left| \int_{t'}^t |g(v(s))|\,\mathrm{d}s \right| \leqslant 2bT + \|e\|_1,$$

which inserted into (5.61) yields (5.60) with

$$R_2^{p-1} = \int_0^R |h(x)|\,\mathrm{d}x + 2(b + \|e\|_1) > 0.$$

Step 4. We show that

$$\begin{cases} \text{there is } r \in (0, R_0) \text{ such that } v > r \text{ on } [0, T] \\ \text{for each } \lambda \in (0, 1] \text{ and each positive solution } v \text{ of (5.53).} \end{cases} \tag{5.62}$$

Put $h_R := \max\{|h(x)|\colon x \in [0, R]\}$, $R^* = \frac{R_2^p}{q} + R_2(h_R R_2 T + \|e\|_1)$ and

$$K^* = R^* + \int_{R_0}^R |g(x)|\,\mathrm{d}x. \tag{5.63}$$

By (5.29), there is $r > 0$ such that

$$\int_r^{R_0} g(x)\,\mathrm{d}x > K^*. \tag{5.64}$$

Put $w(t) = \phi_p(v'(t))$ for $t \in [0, T]$. Then $|w(t)|^q = |v'(t)|^p$ for $t \in [0, T]$,

$$v'(t) = |w(t)|^{q-2} w(t) \quad \text{for } t \in [0, T] \tag{5.65}$$

and

$$w'(t) = \lambda\left(-h(v(t))v'(t) + g(v(t)) + e(t)\right) \quad \text{for a.e. } t \in [0, T]. \tag{5.66}$$

Multiplying (5.65) by $w'(t)$ and (5.66) by $v'(t)$ and subtracting we get

$$\frac{1}{q}\big(\left|v'(t)\right|^p\big)' = \lambda\big(-h\big(v(t)\big)\big(v'(t)\big)^2 + g\big(v(t)\big)v'(t) + e(t)v'(t)\big)$$
$$\text{for a.e. } t \in [0, T]. \tag{5.67}$$

Now, suppose that $\min\{v(t)\colon t \in [0, T]\} < r$. Let us extend v to a T-periodic function on $\mathbb{R}$ and choose $t' \in [0, T]$ and $t^* \in (t', t' + T]$ so that $v(t') = r$, $v(t^*) = \max\{v(t)\colon t \in [0, T]\}$ and $v(t) \geqslant r$ on $[t', t^*]$. Integrating (5.67) from t' to t^*, we get

$$\lambda \int_r^{v(t^*)} g(x)\,\mathrm{d}x$$
$$= -\frac{1}{q}\left|v'(t')\right|^p - \lambda\left(\int_{t'}^{t^*} h\big(v(t)\big)\big(v'(t)\big)^2\,\mathrm{d}t + \int_{t'}^{t^*} e(t)v'(t)\,\mathrm{d}t\right).$$

Consequently, by (5.59) and (5.60),

$$\int_r^{v(t^*)} g(x)\,\mathrm{d}x \leqslant \frac{\lambda^{\frac{p}{p-1}}}{q} R_2^p + h_R \lambda^{\frac{2}{p-1}} R_2^2 T + \|e\|_1 \lambda^{\frac{1}{p-1}} R_2$$
$$\leqslant \frac{R_2^p}{q} + R_2\big(h_R R_2 T + \|e\|_1\big) = R^*,$$

which, by virtue of (5.63), finally gives

$$\int_r^{R_0} g(x)\,\mathrm{d}x = \int_r^{v(t^*)} g(x)\,\mathrm{d}x - \int_{R_0}^{v(t^*)} g(x)\,\mathrm{d}x \leqslant R^* + \int_{R_0}^{R} \left|g(x)\right|\mathrm{d}x = K^*.$$

This being contradictory to (5.64) implies that $v > r$ holds on $[0, T]$, i.e. (5.62) is true.

Step 5. To summarize, there are r, R and R' such that the assumption (i) from Lemma 5.19 is satisfied. Furthermore, since by Step 1 we have

$$g(x) + \overline{e} > 0 \quad \text{if } 0 < x < R_0 \quad \text{and} \quad g(x) + \overline{e} < 0 \quad \text{if } x > R_1$$

and $0 < r < R_0 < R_1 < R$, it is easy to see that also the assumptions (ii) and (iii) of Lemma 5.19 are satisfied. □

Assume that the dissipativity condition

$$h(x) \geqslant h_* > 0 \quad \text{or} \quad h(x) \leqslant -h_* < 0 \quad \text{for all } x \in [0, \infty) \tag{5.68}$$

is fulfilled instead of (5.54) and $e \in L_2[0, T]$. Then the existence of a positive solution to problem (5.52) is ensured by Jebelean and Mawhin.

THEOREM 5.21 [85, Theorem 3]. *Let* $p > 1$, $h \in C[0, \infty)$, $e \in L_2[0, T]$ *and let* $g \in C(0, \infty)$ *have a strong repulsive singularity* (5.29). *Furthermore, assume* (5.55) *and* (5.68). *Then problem* (5.52), (5.2) *has a positive solution.*

PROOF. The proof is analogous to that of Theorem 5.20, just the estimate (5.59) is, thanks to (5.68), obtained more easily. Indeed: let $\lambda \in (0,1]$ and let v be a positive solution of (5.53). Let R_0, R_1 and t_1 be found as in Step 1 of the proof of Theorem 5.20, i.e. R_0 is such that (5.58) is true, $R_1 > R_0$, $g(x) + \overline{e} < 0$ for $x \geqslant R_1$ and $v(t_1) < R_1$. Integrating equality (5.67) over $[0,T]$, we get $h_* \|v'\|_2 \leqslant \|e\|_2$ and, consequently,

$$v(t) = v(t_1) + \int_{t_1}^{t} v'(s)\,\mathrm{d}s < R_1 + \sqrt{T}\,\frac{\|e\|_2}{h_*} + 1 \quad \text{for all } t \in [0,T].$$

Thus, (5.59) is true with $R = R_1 + \sqrt{T}\,\frac{\|e\|_2}{h_*} + 1$. Now, we can repeat Steps 3–5 of the proof of Theorem 5.20. □

Lemma 5.19 enables us to prove also the following result concerning both the non-dissipative case and the case where g need not be bounded below on $(0,\infty)$. Recall that the symbol π_p is defined for $p > 1$ by

$$\pi_p = \frac{2\pi(p-1)^{1/p}}{p\sin(\frac{\pi}{p})}$$

and $(\frac{\pi_p}{T})^p$ is the first eigenvalue of the Dirichlet problem

$$\big(\phi_p(u')\big)' + \lambda\phi_p(u) = 0, \quad u(0) = u(T) = 0$$

(see [57]).

THEOREM 5.22. *Let $p > 1$, $h \in C[0,\infty)$, $e \in L_1[0,T]$ and let $g \in C(0,\infty)$ have a strong repulsive singularity* (5.29). *Furthermore, assume* (5.55) *and*

$$\begin{cases} \textit{there exist } a,\ 0 \leqslant a < \big(\frac{\pi_p}{T}\big)^p, \textit{ and } \gamma \geqslant 0 \textit{ such that} \\ g(x)x \geqslant -\big(ax^p + \gamma\big) \quad \textit{for all } x > 0. \end{cases} \tag{5.69}$$

Then problem (5.52), (5.2) *has a positive solution.*

PROOF. Similarly to the proof of Theorem 5.21, it suffices to verify (5.59). Assume that $\lambda \in (0,1]$, v is a positive solution to (5.53) and let R_1 and t_1 have the same meaning as in Step 1 of the proof of Theorem 5.20. Multiplying (5.53) by $v(t)$ and integrating over $[0,T]$, we get

$$\|v'\|_p^p \leqslant a\|v\|_p^p + \|e\|_1\|v\|_\infty + \gamma T. \tag{5.70}$$

Since $v(t_1) \leqslant R_1$, we have

$$0 < v(t) < R_1 + T^{1/q}\|v'\|_p \quad \text{for } t \in [0,T], \tag{5.71}$$

where $\frac{1}{p}+\frac{1}{q}=1$. Now put

$$y(t)=\begin{cases} v(t+t_1)-v(t_1) & \text{if } 0\leqslant t\leqslant T-t_1,\\ v(t+t_1-T)-v(t_1) & \text{if } T-t_1\leqslant t\leqslant T.\end{cases}$$

We have $y\in C^1[0,T]$, $y(0)=y(T)=0$ and $\|y+v(t_1)\|_p^p=\|v\|_p^p$. Therefore, by the generalized Poincaré-Wirtinger inequality (see, e.g., [153, Lemma 3]),

$$\|y\|_p\leqslant\frac{T}{\pi_p}\|y'\|_p=\frac{T}{\pi_p}\|v'\|_p.$$

Hence, for an arbitrary $\varepsilon>0$, there is a $C_1>0$ such that

$$\|v\|_p^p\leqslant\big(\|y\|_p+v(t_1)T^{1/p}\big)^p\leqslant(1+\varepsilon)\left(\frac{T}{\pi_p}\right)^p\|v'\|_p^p+C_1.$$

Inserting this into (5.70), choosing $\varepsilon\in(0,\frac{1}{a}(\frac{\pi_p}{T})^p-1)$ and having in mind (5.71), we deduce that

$$\alpha\|v'\|_p^p\leqslant T^{1/q}\|e\|_1\|v'\|_p+C_2$$

for some $C_2>0$, where $\alpha=(1-a(1+\varepsilon)(\frac{T}{\pi_p})^p)>0$. However, this is possible only if there is $R_p\in(0,\infty)$, independent of λ and v, such that $\|v'\|_p<R_p$. Therefore $0<v(t)<R_1+T^{1/q}R_p+1$ on $[0,T]$ for all $\lambda\in(0,1]$ and all positive solutions v of (5.53), i.e. the assertion (5.59) is true with $R=R_1+T^{1/q}R_p+1$.

By virtue of (5.55), we can choose $b>0$ so that $\inf\{g(x)\colon x\in(0,R]\}\geqslant -b$. Thus, we can continue by Steps 3–5 of the proof of Theorem 5.20 to verify that the assumptions of Lemma 5.19 are satisfied. □

REMARK 5.23. Theorem 5.22 is a slightly modified scalar version of the result by Liu [98, Theorem 1].

In the undamped case of the Duffing type equation

$$\big(\phi_p(u')\big)'=g(u)+e(t), \tag{5.72}$$

condition (5.69) can be replaced by a related asymptotic condition. It is shown in the next theorem which has been proved for $p=2$ by Rachůnková and Tvrdý in [125, Theorem 3.1].

THEOREM 5.24. *Let* $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $g\in C(0,\infty)$ *and* $e\in L_q[0,T]$. *Furthermore, assume* (5.29),

$$\liminf_{x\to 0+} g(x)>-\infty, \tag{5.73}$$

$$\liminf_{x\to\infty} \frac{g(x)}{x^{p-1}} > -\left(\frac{\pi_p}{T}\right)^p. \tag{5.74}$$

Further, assume that there exist $r > 0$ and $A > r$ such that the conditions

$$g(r) + e(t) \geqslant 0 \quad \text{for a.e. } t \in [0, T] \tag{5.75}$$

and

$$g(x) + \overline{e} \leqslant 0 \quad \text{for } x \in [A, B], \tag{5.76}$$

where

$$B - A \leqslant 2T\|e - \overline{e}\|_1^{q-1}, \tag{5.77}$$

are satisfied. Then problem (5.72), (5.2) *has a positive solution u such that $u(t_u) \in [r, B]$ for some $t_u \in [0, T]$.*

PROOF. *Step* 1. *Lower and upper functions.*

By (5.75), $\sigma_2 \equiv r$ is an upper function of (5.72), (5.2). Let v be a solution of the quasilinear Dirichlet problem (5.9) with $b(t) = e(t) - \overline{e}$ a.e. on $[0, T]$ and $d = 0$ and let $\sigma_1 = A + T\phi_p^{-1}(\|e - \overline{e}\|_1) + v$ on $[0, T]$. Let us recall that $\phi_p^{-1} = \phi_q$. Hence $\phi_p^{-1}(\|e - \overline{e}\|_1) = \|e - \overline{e}\|_1^{q-1}$. Having in mind assumption (5.76), we can see, similarly to the proof of Theorem 5.17 (see also the proof of Theorem 5.14), that σ_1 is a lower function of (5.72), (5.2) and $\sigma_1(t) \in [A, B]$ for $t \in [0, T]$.

Step 2. *Construction of an auxiliary problem having a right-hand side bounded below.*

By (5.29) we have (5.32) and hence there is a sequence $\{\varepsilon_n\} \subset (0, r)$ such that

$$g(\varepsilon_n) > 0 \quad \text{for } n \in \mathbb{N}, \qquad \lim_{n\to\infty} \varepsilon_n = 0 \quad \text{and} \quad \lim_{n\to\infty} g(\varepsilon_n) = \infty. \tag{5.78}$$

For $n \in \mathbb{N}$ and $M \in \mathbb{R}$, $M > r$, define

$$g_{n,M}(x) = \begin{cases} 0 & \text{if } x < 0, \\ \frac{g(\varepsilon_n)}{\varepsilon_n^{p-1}} x^{p-1} & \text{if } x \in [0, \varepsilon_n], \\ g(x) & \text{if } x \in [\varepsilon_n, M], \\ g(M) & \text{if } x > M. \end{cases} \tag{5.79}$$

By (5.74), there are $\eta \in (0, (\frac{\pi_p}{T})^p)$ and $x_0 > 1$ such that

$$\frac{g(x)}{x^{p-1}} \geqslant -\left(\left(\frac{\pi_p}{T}\right)^p - \eta\right) \quad \text{for all } x \geqslant x_0.$$

Put

$$p(x) = \begin{cases} 0 & \text{if } x \leqslant 0, \\ \frac{g(x_0)}{x_0^{p-1}} x^{p-1} & \text{if } x \in (0, x_0), \\ g(x) & \text{if } x \geqslant x_0 \end{cases}$$

and $q_{n,M}(x) = g_{n,M}(x) - p(x)$ for $x \in \mathbb{R}$. By virtue of (5.73), there is $\gamma \geqslant 0$ such that $q_{n,M}(x) \geqslant -\gamma$ for all $x \in \mathbb{R}$, $n \in \mathbb{N}$ and $M > r$. Consequently, each function $\widetilde{g}(x) = g_{n,M}(x)$, $n \in \mathbb{N}$, $M > r$, satisfies the estimate

$$\widetilde{g}(x)x \geqslant -\left(\left(\frac{\pi_p}{T}\right)^p - \eta\right)|x|^p - \gamma|x| \quad \text{for all } x \in \mathbb{R}. \tag{5.80}$$

Step 3. *A priori estimates.*

Now, we will give uniform a priori estimates for solutions of periodic problems associated to the equations

$$\left(\phi_p(u')\right)' = \widetilde{g}(u) + e(t), \tag{5.81}$$

where $\widetilde{g}$ may be an arbitrary function satisfying the estimate (5.80). To this aim, we will prove the following assertion.

CLAIM. *Let* $\gamma \geqslant 0$ *and* $\eta \in (0, (\frac{\pi_p}{T})^p)$. *Then for any* $\delta > 0$, *there are* $R \geqslant \delta$ *and* $R' > 0$ *such that the estimates*

$$u \leqslant R \quad \text{on } [0, T] \quad \text{and} \quad \|u'\|_p \leqslant R' \tag{5.82}$$

hold whenever

$$\begin{cases} \widetilde{g} \in C(0, \infty) \text{ fulfills (5.80) and } u \text{ is a solution of (5.81), (5.2)} \\ \text{such that } \min\{u(t)\colon t \in [0, T]\} \leqslant \delta. \end{cases} \tag{5.83}$$

PROOF OF CLAIM. We will follow ideas from the proof of [130, Lemma 2.4]. Suppose that for each $k \in \mathbb{N}$ there are $g_k \in C(0, \infty)$ and a solution u_k of

$$\left(\phi_p(u')\right)' = g_k(u) + e(t), \quad u(0) = u(T), \quad u'(0) = u'(T) \tag{5.84}$$

such that

$$g_k(x)x \geqslant -\left(\left(\frac{\pi_p}{T}\right)^p - \eta\right)|x|^p - \gamma|x| \tag{5.85}$$

and

$$u_k(t_k) = \delta \quad \text{for some } t_k \in [0, T] \quad \text{and} \quad \max\{u_k(t)\colon t \in [0, T]\} > k. \tag{5.86}$$

In particular, we have

$$\lim_{k\to\infty} \max\{u_k(t)\colon\ t\in[0,T]\} = \infty. \tag{5.87}$$

Let us extend u_k and e to functions T-periodic on $\mathbb{R}$. We have

$$\big(\phi_p\big(u_k'(t)\big)\big)' = g_k\big(u_k(t)\big) + e(t) \quad \text{for a.e. } t\in\mathbb{R}.$$

Multiplying this equality by u_k, integrating from t_k to t_k+T and making use of (5.85), we obtain

$$\begin{aligned}\|u_k'\|_p^p &= -\int_{t_k}^{t_k+T} g_k\big(u_k(s)\big)u_k(s)\,\mathrm{d}s - \int_{t_k}^{t_k+T} e(s)u_k(s)\,\mathrm{d}s\\ &\leqslant \left(\left(\frac{\pi_p}{T}\right)^p - \eta\right)\|u_k\|_p^p + \gamma T^{1/q}\|u_k\|_p + \|e\|_q\|u_k\|_p.\end{aligned}$$

Let us set $v_k = u_k - \delta$. By (5.87) we have $\lim_{k\to\infty}\|v_k\|_\infty = \infty$. Therefore, applying the Hölder inequality we can conclude that

$$\lim_{k\to\infty}\|v_k'\|_p = \infty. \tag{5.88}$$

Furthermore, it is easy to verify that

$$\|v_k'\|_p^p \leqslant \left(\left(\frac{\pi_p}{T}\right)^p - \epsilon\right)\|v_k\|_p^p + a\|v_k\|_p + b \tag{5.89}$$

holds with some $\epsilon\in(0,(\frac{\pi_p}{T})^p)$ and $a,\ b\geqslant 0$ not depending on v_k. This, together with (5.88), gives

$$\lim_{k\to\infty}\|v_k\|_p = \infty. \tag{5.90}$$

Moreover, as $v_k(t_k) = v_k(t_k+T) = 0$, we can apply the generalized Poincaré–Wirtinger inequality (see, e.g., [153, Lemma 3]) to get

$$\|v_k\|_p^p \leqslant \left(\frac{T}{\pi_p}\right)^p \|v_k'\|_p^p \quad \text{for each } k\in\mathbb{N}.$$

Hence the inequality (5.89) can be rewritten as

$$\left(\frac{\pi_p}{T}\right)^p \leqslant \frac{\|v_k'\|_p^p}{\|v_k\|_p^p} \leqslant \left(\frac{\pi_p}{T}\right)^p - \epsilon + \frac{a}{\|v_k\|_p^{p-1}} + \frac{b}{\|v_k\|_p^p}, \tag{5.91}$$

which, in view of (5.90), leads to a contradiction

$$\left(\frac{\pi_p}{T}\right)^p \leqslant \left(\frac{\pi_p}{T}\right)^p - \epsilon.$$

As a consequence, we can conclude that the sequences $\{\|v_k\|_\infty\}$ and $\{\|v_k\|_p\}$ are bounded. By (5.89), this implies that also the sequence $\{\|v_k'\|_p\}$ is bounded. In particular, there are $R \in [\delta, \infty)$ and $R' \in (0, \infty)$ such that $u \leqslant R$ and $\|u'\|_p \leqslant R'$ hold whenever (5.83) is true. This completes the proof of Claim.

Now, let $R > B$ and $R' > 0$ be constants given by Claim for $\delta = B$. Put

$$K = \int_A^R \left|g(x)\right| \mathrm{d}x + \|e\|_q R'.$$

It follows from (5.29) and (5.78) that we can choose $\varepsilon = \varepsilon_{n^*} \in \{\varepsilon_n\}$ such that

$$\int_\varepsilon^A g(x)\,\mathrm{d}x > K \quad \text{and} \quad g(\varepsilon) > 0. \tag{5.92}$$

By (5.73), there is $g_R \in \mathbb{R}$ such that

$$g(x) \geqslant g_R \quad \text{for } x \in (0, R]. \tag{5.93}$$

Define

$$\widetilde{g}(x) = g_{n^*,R}(x) \quad \text{for } x \in \mathbb{R} \tag{5.94}$$

and consider the regular periodic problem for the auxiliary equation

$$\left(\phi_p(u')\right)' = \widetilde{g}(u) + e(t). \tag{5.95}$$

Clearly, σ_1 and σ_2 are lower and upper functions of (5.95), (5.2) and $\widetilde{g}(x) + e(t) \geqslant g_R + e(t)$ for a.e. $t \in [0, T]$ and all $x \in \mathbb{R}$. Thus, by Lemma 5.9, problem (5.95), (5.2) possesses a solution u such that $u(t_u) \in [r, B]$ for some $t_u \in [0, T]$. In particular, $\min\{u(t)\colon t \in [0, T]\} \leqslant B$. Furthermore, by Step 2 it is easy to see that $\widetilde{g}$ satisfies the estimate (5.80). Thus, by Claim and by the definitions of R and R', the estimates (5.82) are true.

Step 4. *Existence of a solution to* (5.72), (5.2).

It remains to show that $u \geqslant \varepsilon$ on $[0, T]$. Let $t_0, t_1 \in [0, T]$ be such that

$$u(t_0) = \min\bigl\{u(t)\colon t \in [0, T]\bigr\} \quad \text{and} \quad u(t_1) = \max\bigl\{u(t)\colon t \in [0, T]\bigr\}.$$

Due to the periodicity of u, we have $u'(t_0) = u'(t_1) = 0$. Multiplying the equality $(\phi_p(u'(t)))' = \widetilde{g}(u(t)) + e(t)$ by $u'(t)$ and integrating, we get

$$\int_{u(t_0)}^{u(t_1)} \widetilde{g}(x)\,\mathrm{d}x \leqslant \|e\|_q R'.$$

Therefore,

$$\int_{u(t_0)}^{A} \widetilde{g}(x)\,\mathrm{d}x \leqslant \int_{A}^{R} \bigl|g(x)\bigr|\,\mathrm{d}x + \|e\|_q R' = K.$$

Let $u(t_0) < \varepsilon$. Then, by (5.92),

$$\begin{aligned}\int_{u(t_0)}^{A} \widetilde{g}(x)\,\mathrm{d}x &= \int_{u(t_0)}^{\varepsilon} \widetilde{g}(x)\,\mathrm{d}x + \int_{\varepsilon}^{A} g(x)\,\mathrm{d}x \\ &= g(\varepsilon)\bigl(\varepsilon - u(t_0)\bigr) + \int_{\varepsilon}^{A} g(x)\,\mathrm{d}x \\ &> \int_{\varepsilon}^{A} g(x)\,\mathrm{d}x > K,\end{aligned}$$

a contradiction. So $u(t) \geqslant \varepsilon$ on $[0, T]$, which together with (5.79), (5.82) and (5.94) yields that u is a solution of (5.72), (5.2). □

EXAMPLES. (i) Let $p > 1$, $h \in C[0, \infty)$, $\beta > 0$, $\alpha \geqslant 1$, $e \in L_1[0, T]$. Then, by Theorem 5.20, the problem

$$\bigl(|u'|^{p-2}u'\bigr)' + h(u)u' = \frac{\beta}{u^\alpha} + e(t), \quad u(0) = u(T), \quad u'(0) = u'(T) \tag{5.96}$$

has a positive solution if $\overline{e} < 0$. Integrating both sides of the differential equation in (5.96) over $[0, T]$ and taking into account the positivity of $g(x) = \beta x^{-\alpha}$ on $(0, \infty)$, we can see that the condition $\overline{e} < 0$ is also necessary for the existence of a positive solution to (5.96).

(ii) Let $p > 1$, $c \neq 0$, $a > 1$, $\beta > 0$, $\alpha \geqslant 1$. Then, by Theorem 5.21, the problem

$$\bigl(|u'|^{p-2}u'\bigr)' + cu' = \frac{\beta}{u^\alpha} - a\exp(u) + e(t), \quad u(0) = u(T), \quad u'(0) = u'(T)$$

has a solution for each $e \in L_2[0, T]$.

(iii) Let $p > 1$, $h \in C[0, \infty)$, $0 < a < (\frac{\pi_p}{T})^p$, $\beta > 0$ and $\alpha \geqslant 1$. Then, by Theorem 5.22, the problem

$$\bigl(|u'|^{p-2}u'\bigr)' + h(u)u' = -au^{p-1} + \frac{\beta}{u^\alpha} + e(t), \quad u(0) = u(T), \quad u'(0) = u'(T)$$

has a positive solution for each $e \in L_1[0, T]$.

For the classical case $p = 2$, the following result due to Omari and Ye is known. Its proof combines the lower and upper functions method, the degree theory and connectedness arguments for some properly chosen truncated equations and a posteriori estimates.

THEOREM 5.25 [108, Theorem 1.2]. *Assume* $e \in L_\infty[0,T]$, (5.29), $\lim_{x\to 0+} g(x) = \infty$ *and*

$$\liminf_{x\to\infty} \frac{g(x)}{x} \geqslant -\left(\frac{\pi}{T}\right)^2 \quad \textit{and} \quad \liminf_{x\to\infty} \frac{2G(x)}{x^2} > -\left(\frac{\pi}{T}\right)^2,$$

where

$$G(x) = \int_x^1 g(\xi)\,\mathrm{d}\xi \quad \textit{for } x \in (0,\infty).$$

Then the problem $u'' + h(u)u' = g(u) + e(t)$, (5.2) *has a solution if and only if it possesses a lower function* $\sigma_1 \in AC^1[0,T]$.

Hitherto we have assumed the strong singularity condition (5.29). The next existence principle enables us to treat also problems with weak repulsive singularities. We shall restrict ourselves to the case that $\phi(y) \equiv y$ and f does not depend on u', i.e. we consider the equation

$$u'' = f(t,u), \tag{5.97}$$

where $f : [0,T] \times (0,\infty) \to \mathbb{R}$.

THEOREM 5.26. *Let* $f \in Car([0,T] \times (0,\infty))$, $r > 0$, $A \geqslant r$ *and let* $\mu \in L_1[0,T]$ *and* $\beta \in L_1[0,T]$ *be such that* $\mu(t) \geqslant 0$ *a.e. on* $[0,T]$, $\overline{\mu} > 0$,

$$\overline{\beta} \leqslant 0 \quad \textit{and} \quad f(t,x) \leqslant \beta(t) \quad \textit{for a.e. } t \in [0,T] \textit{ and all } x \in [A,B] \tag{5.98}$$

and

$$f(t,x) \geqslant -\mu(t)(x-r) \quad \textit{for a.e. } t \in [0,T] \textit{ and all } x \in [r,B], \tag{5.99}$$

where

$$B - A \geqslant \frac{T^2}{2}\overline{m},$$

$$m(t) = \max\bigl\{\sup\bigl\{f(t,x)\colon\ x \in [r,A]\bigr\}, \beta(t), 0\bigr\} \quad \textit{for a.e. } t \in [0,T]$$

and

$$\begin{cases} v \geqslant 0 \textit{ on } [0,T] \textit{ holds for each } v \in AC^1[0,T] \textit{ such that} \\ v''(t) + \mu(t)v(t) \geqslant 0 \quad \textit{for a.e. } t \in [0,T], \\ v(0) = v(T), \quad v'(0) = v'(T). \end{cases} \tag{5.100}$$

Then problem (5.97), (5.2) *has a positive solution* u *such that* $r \leqslant u \leqslant B$ *on* $[0,T]$.

PROOF. The proof follows the ideas of the proof of [129, Theorem 2.5]. First, assume that $\overline{\beta} < 0$.

Step 1. *Existence of a solution u to a certain auxiliary problem.*

Put

$$\widetilde{f}(t,x) = \begin{cases} f(t,r) - \mu(t)(x-r) & \text{if } x \leqslant r, \\ f(t,x) & \text{if } x \in [r,B], \\ f(t,B) & \text{if } x \geqslant B \end{cases} \tag{5.101}$$

and consider the problem

$$u'' = \widetilde{f}(t,u), \quad u(0) = u(T), \quad u'(0) = u'(T). \tag{5.102}$$

We have $\widetilde{f} \in Car([0,T] \times \mathbb{R})$. By (5.98), (5.99) and (5.101), the inequalities

$$\widetilde{f}(t,x) \leqslant \beta(t) \quad \text{if } x \geqslant A \tag{5.103}$$

and

$$\widetilde{f}(t,x) \geqslant -\mu(t) \min\{x-r, B-r\} \tag{5.104}$$

are valid for a.e. $t \in [0,T]$ and all $x \in \mathbb{R}$. In particular, $\widetilde{f}(t,x) \geqslant -\mu(t)(B-r)$. By (5.104), $\sigma_2 \equiv r$ is an upper function of (5.102). Further, let σ_0 be the solution of the Dirichlet problem $v'' = b$, $v(0) = v(T) = 0$, where $b(t) = \beta(t) - \overline{\beta}$ for a.e. $t \in [0,T]$, and let $\sigma_c(t) = c + \sigma_0(t)$ for $t \in [0,T]$ and $c \in \mathbb{R}$. Then $\sigma_c'' = b$ a.e. on $t \in [0,T]$ and $\sigma_c(0) = \sigma_c(T) = c$. Moreover, $\sigma_c'(T) - \sigma_c'(0) = T\overline{b} = 0$. Let us choose $c^* > 0$ so that $\sigma_1 := \sigma_{c^*} \geqslant A$ on $[0,T]$. Due to (5.103), where $\beta < b$ a.e. on $[0,T]$, we can see that σ_1 is a lower function of (5.102). Therefore, by Lemma 5.9, the regular problem (5.102) has a solution u such that $u(t_u) \geqslant r$ for some $t_u \in [0,T]$.

Step 2. *Lower estimate for u.*

We shall show that

$$u \geqslant r \quad \text{on } [0,T]. \tag{5.105}$$

Set $z = u - r$. By virtue of (5.99) and (5.101), we have

$$z''(t) + \mu(t)z(t) = u''(t) + \mu(t)z(t) = \widetilde{f}\big(t,u(t)\big) + \mu(t)\big(u(t) - r\big) \geqslant 0$$

for a.e. $t \in [0,T]$. By (5.100), it follows that $z(t) \geqslant 0$ on $[0,T]$, i.e. (5.105) is true.

Step 3. *Upper estimate for u.*

We shall show that

$$u \leqslant B \quad \text{on } [0,T]. \tag{5.106}$$

By the definition of m and by (5.101) and (5.103) we have

$$\widetilde{f}(t,x) \leqslant m(t) \quad \text{for a.e. } t \in [0,T] \text{ and all } x \geqslant r.$$

Hence, we can use Lemma 5.8 (see (5.20)) to get

$$\|u'\|_\infty \leqslant \|m\|_1 = T\,\overline{m}. \tag{5.107}$$

If $u \geqslant A$ were valid on $[0,T]$, then taking into account the periodicity of u' and (5.103), we would get

$$0 = \int_0^T \widetilde{f}\bigl(t,u(t)\bigr)\,\mathrm{d}t \leqslant \int_0^T \beta(t)\,\mathrm{d}t = T\overline{\beta} < 0,$$

a contradiction. Thus, there is $\tau \in [0,T]$ such that $u(\tau) < A$. Now, assume that $u(s) > A$ for some $s \in [0,T]$ and extend u to the function T-periodic on $\mathbb{R}$. There are s_1, s_2 and $s^* \in \mathbb{R}$ such that $s_1 < s^* < s_2$, $s_2 - s_1 < T$, $u(s_1) = u(s_2) = A$ and $u(s^*) = \max\{u(s)\colon s \in [0,T]\} > A$. In particular, due to (5.107),

$$2\bigl(u(s^*) - A\bigr) = \int_{s_1}^{s^*} u'(s)\,\mathrm{d}s + \int_{s_2}^{s^*} u'(s)\,\mathrm{d}s \leqslant T^2\overline{m},$$

wherefrom the estimate

$$u(t) - A \leqslant \frac{T^2}{2}\overline{m} \leqslant B - A \quad \text{on } [0,T]$$

follows. Consequently, (5.106) is true.

Step 4. *Conclusion: u is a solution to* (5.97)*,* (5.2).

The estimates (5.105) and (5.106) mean that $r \leqslant u \leqslant B$ holds on $[0,T]$. By (5.101), we conclude that u is a solution to (5.97), (5.2).

If $\overline{\beta} = 0$, we can approximate the solution to (5.1), (5.2) by solutions of the problems

$$u'' = \widetilde{f}_n(t,u), \quad u(0) = u(T), \quad u'(0) = u'(T),$$

where

$$\widetilde{f}_n(t,x) = \begin{cases} f(t,r) & \text{if } x \leqslant r, \\ f(t,x) & \text{if } x \in [r,A], \\ f(t,x) - \mu(t)\frac{1}{n}\frac{x-A}{x-A+1} & \text{if } x \in [A,B], \\ f(t,B) - \mu(t)\frac{1}{n}\frac{B-A}{B-A+1} & \text{if } x \geqslant B. \end{cases}$$

□

Recently, using the Krasnoselskii fixed point theorem, Torres proved for the Hill equation

$$u'' + \mu(t)u = g(t,u) \tag{5.108}$$

the existence result which is related to Theorem 5.26.

THEOREM 5.27 [144, Theorem 4.5]. *Let $\mu \in L_1[0,T]$ be such that the problem*

$$v'' + \mu(t)v = 0, \quad v(0) = v(T), \quad v'(0) = v'(T) \tag{5.109}$$

possesses the Green function $G(t,s)$ which is positive on $[0,T]\times[0,T]$. Moreover, assume that there is an $R>0$ such that

$$g(t,x) \geqslant 0 \quad \textit{for all } x \in \left(0, \frac{M}{m}R\right] \quad \textit{and}$$

$$g(t,x) \leqslant \frac{1}{TM}x \quad \textit{for all } x \in \left[R, \frac{M}{m}R\right]$$

for a.e. $t \in [0,T]$, where

$$m = \min\{G(t,s)\colon\ t,s \in [0,T]\} \quad \textit{and} \quad M = \max\{G(t,s)\colon\ t,s \in [0,T]\}.$$

Then problem (5.108), (5.2) *has a positive solution.*

It is easy to check that the function $G(t,s) = \sin(\frac{\pi}{T}|t-s|)$, $t,s \in [0,T]$, is the Green function for $v'' + (\frac{\pi}{T})^2 v = 0$, $v(0) = v(T)$, $v'(0) = v'(T)$ and $G(t,s) \geqslant 0$ on $[0,T] \times [0,T]$. Hence, the statement (5.100) holds if $\mu(t) \equiv \mu_1 = (\frac{\pi}{T})^2$. Notice that $\mu_1 = (\frac{\pi}{T})^2$ is the first eigenvalue of the related Dirichlet problem and it is optimal in the sense that for $\mu(t) = \mu$ a.e. on $[0,T]$ and $\mu \in (\mu_1, 4\mu_1)$ the corresponding Green function of $v'' + \mu v = 0$, $v(0) = v(T)$, $v'(0) = v'(T)$ is not nonnegative on $[0,T]\times[0,T]$.

In particular, when restricted to the Duffing equation

$$u'' = g(u) + e(t), \tag{5.110}$$

Theorem 5.26 has the following corollary.

COROLLARY 5.28 [129, Corollary 3.7]. *Suppose that $g \in C(0,\infty)$, $e \in L_1[0,T]$,*

$$\overline{e} + \limsup_{x\to\infty} g(x) < 0$$

and there is $r>0$ such that

$$e(t) + g(x) + \left(\frac{\pi}{T}\right)^2 x \geqslant \left(\frac{\pi}{T}\right)^2 r \quad \textit{for a.e. } t \in [0,T] \textit{ and all } x > r.$$

Then problem (5.110), (5.2) *has a positive solution* u *such that* $u \geqslant r$ *on* $[0, T]$.

More detailed information on the sign properties of the associated Green functions is provided by the next proposition which is due to Torres [144] (see also [154, Lemma 2.5]). Before formulating it, let us define the function $K : [0, \infty] \to (0, \infty)$ by

$$K(z) = \begin{cases} \dfrac{2\pi}{zT^{1+2/z}}\left(\dfrac{2}{2+z}\right)^{1-2/z}\left(\dfrac{\Gamma(\frac{1}{z})}{\Gamma(\frac{1}{2}+\frac{1}{z})}\right)^2 & \text{if } 1 \leqslant z < \infty, \\ \dfrac{4}{T} & \text{if } z = \infty. \end{cases} \tag{5.111}$$

Let us recall that for a given z, $1 \leqslant z \leqslant \infty$, $K(z)$ is the best Sobolev constant for the inequality $C\|u\|_z^2 \leqslant \|u'\|_2^2$, i.e.

$$K(z) = \inf\left\{\frac{\|u'\|_2^2}{\|u\|_z^2}\colon u \in H_0^1[0, T] \setminus \{0\}\right\},$$

where $H_0^1[0, T] = \{u \in AC[0, T]\colon u' \in L_2[0, T],\ u(0) = u(T) = 0\}$.

PROPOSITION 5.29 [144, Corollary 2.3]. *Let* $1 \leqslant q \leqslant \infty$ *and let* $\mu \in L_q[0, T]$. *Then* (5.100) *is true provided*

$$\mu(t) \geqslant 0 \quad \textit{a.e. on } [0, T], \quad \overline{\mu} > 0 \quad \textit{and} \quad \|\mu\|_q \leqslant K(2q^*), \tag{5.112}$$

where

$$\begin{cases} \frac{1}{q} + \frac{1}{q^*} = 1 & \textit{if } 1 < q < \infty, \\ q^* = \infty & \textit{if } q = 1, \\ q^* = 1 & \textit{if } q = \infty \end{cases} \tag{5.113}$$

and the function K *is defined by* (5.111).

Moreover, if $\|\mu\|_q < K(2q^*)$, *then problem* (5.109) *has the Green function which is positive on* $[0, T] \times [0, T]$.

Notice that if $\mu(t) \equiv \mu \in (0, \infty)$ on $[0, T]$, then we can take $q = \infty$, $q^* = 1$ and so we get $K(2q^*) = K(2) = (\frac{\pi}{T})^2$, which confirms the above mentioned fact that in such a case (5.100) is satisfied if $\mu \in (0, (\frac{\pi}{T})^2]$.

EXAMPLE. Consider the Brillouin beam focusing equation

$$u'' + a(1 + \cos t)u = \frac{1}{u} \tag{5.114}$$

on the interval $[0, 2\pi]$, where $a > 0$ is a parameter. (See [34] for a description of the model.) The problem of existence of a positive 2π-periodic solution to (5.114) has been considered by several authors (see, e.g., [55,144,145,152,154]). Put $A = \frac{1}{\sqrt{a}}$. Then for all $x \geqslant A$ and $t \in [0, 2\pi]$ we have

$$f(t,x) := \frac{1}{x} - a(1+\cos t)x \leqslant \beta(t) := \frac{1}{A} - a(1+\cos t)A$$

and $\overline{\beta} = \frac{1}{A} - aA = 0$. So, the assumption (5.98) is satisfied with $B = \infty$ and $T = 2\pi$. Furthermore, for $r \in (0, r_0]$ and a.e. $t \in [0, T]$ define

$$m_r(t) := \max\bigl\{\sup\bigl\{f(t,x)\colon\ x \in [r, A]\bigr\}, \beta(t), 0\bigr\}.$$

Let $r_0 = \frac{1}{\sqrt{2a}}$. Then $\frac{1}{r} - a(1+\cos t)r \geqslant \frac{1}{r_0} - 2ar_0 = 0$ holds for $r \in (0, r_0]$ and $t \in [0, T]$. Consequently,

$$m_r(t) = \frac{1}{r} - a(1+\cos t)r \quad \text{for a.e. } t \in [0,T] \text{ and all } r \in (0, r_0].$$

For a given $r \in (0, r_0]$, put $B_r = A + \pi\overline{m_r}$. Now, it is easy to check that it is possible to find $r \in (0, r_0]$ such that the assumption (5.99) is satisfied with $B = B_r$ and $\mu(t) = a(1+\cos t)$ whenever $a < \frac{1}{2\pi} \approx 0.15915$. Finally, notice that by virtue of Proposition 5.29, the assumption (5.100) is satisfied if

$$a \leqslant K_{\max} := \max\left\{\frac{K(2q^*)}{\|1+\cos t\|_q}\colon\ 1 \leqslant q \leqslant \infty\right\} \approx 0.16488.$$

(The maximum is attained at $q \approx 2.1941$, see [145, Corollary 4.8].) By Theorem 5.26, we can conclude that the equation (5.114) has a positive 2π-periodic solution for $a < \frac{1}{2\pi}$. To compare, notice that for $q = \infty$ and $T = 2\pi$, we get $q^* = 1$ and

$$\frac{K(2q^*)}{\|1+\cos t\|_\infty} = \frac{1}{8} = 0.125.$$

Finally, let us note that using more sophisticated and involved techniques, Zhang proved (see [154, Theorem 4.5]) that for $a < K_{\max}$, $b > 0$, $\lambda \geqslant 1$, $e \in C[0, T]$ and $h \in C[0, \infty)$ the problem

$$u'' + h(u)u' + a(1+\cos t)u = \frac{b}{u^\lambda} + e(t), \quad u(0) = u(2\pi), \quad u'(0) = u'(2\pi)$$

has a positive solution.

The hitherto mentioned conditions for the existence of a positive solution of problem (5.97), (5.2) concern the case when $f(t,x)$ asymptotically behaves like $-kx$ with $k \leqslant \mu_1$, $\mu_1 = (\frac{\pi}{T})^2$ being the first eigenvalue of the related Dirichlet problem. The next theorem deals with the case corresponding to k lying between two adjacent higher eigenvalues.

Let us denote by $\{\mu_k\}_{k=1}^{\infty}$ the sequence of eigenvalues of the related linear Dirichlet problem $u'' + \mu x = 0$, $u(0) = u(T) = 0$, that is,

$$\mu_k = \left(\frac{\pi k}{T}\right)^2, \quad k \in \mathbb{N}. \tag{5.115}$$

Furthermore, we set $\mu_0 = 0$.

THEOREM 5.30 [52, Theorem 1.1]. *Assume that* $f : [0, T] \times (0, \infty) \to \mathbb{R}$ *is continuous and there are positive constants* c, c', δ *and* $\nu \geqslant 1$ *such that*

$$\frac{c'}{x^\nu} \leqslant f(t, x) \leqslant \frac{c}{x^\nu} \quad \textit{for all } x \in (0, \delta). \tag{5.116}$$

Moreover, let there exist a nonnegative integer k *such that*

$$-\mu_{k+1} < \liminf_{x\to\infty} \frac{f(t, x)}{x} \leqslant \limsup_{x\to\infty} \frac{f(t, x)}{x} < -\mu_k \quad \textit{uniformly in } t \in [0, T]. \tag{5.117}$$

Then problem (5.97), (5.2) *has a positive solution.*

SKETCH OF THE PROOF. For a given $e \in C[0, T]$ denote by $\mathcal{R}(e)$ the unique solution of the problem

$$u'' + u = e(t), \quad u(0) = u(T), \quad u'(0) = u'(T).$$

It is known that $\mathcal{R}$ defines a compact linear operator on the space $C_T[0, T]$ of continuous T-periodic functions endowed with the sup norm $\|\cdot\|_\infty$. Problem (5.97), (5.2) is thus equivalent to finding a positive solution $u \in C_T[0, T]$ of the fixed point problem $u = \mathcal{T}(u)$, where

$$\mathcal{T}(u) = \mathcal{R}\big(u + f(\cdot, u)\big) \quad \text{for } u \in C_T[0, T].$$

For $0 < \varepsilon < M$ define $\Omega_{\varepsilon,M} = \{u \in C_T[0, T]:\ \varepsilon < u < M \text{ on } [0, T]\}$. Then $\mathcal{T} : \overline{\Omega}_{\varepsilon,M} \to C_T[0, T]$ is a completely continuous operator.

The proof of the theorem consists in showing that there are ε, M such that $\deg(\mathcal{I} - \mathcal{T}, \Omega_{\varepsilon,M}) \neq 0$:

Let k be from (5.117) and choose an arbitrary $\gamma \in (\mu_k, \mu_{k+1})$. Further, for $\lambda \in [0, 1]$ and $u \in C_T[0, T]$, define

$$\widetilde{\mathcal{T}}(u) = \mathcal{R}\left(\gamma u - \frac{1}{u^\nu} + u\right) \quad \text{and} \quad \mathcal{T}_\lambda(u) = \lambda \widetilde{\mathcal{T}}(u) + (1 - \lambda)\mathcal{T}(u).$$

Notice that

$$\mathcal{T}_\lambda(u) = \mathcal{R}\left(\lambda\left(\gamma u - \frac{1}{u^\nu}\right) + (1-\lambda) f(\cdot, u) + u\right).$$

Furthermore, $\mathcal{T}_0 = \mathcal{T}$ and $\mathcal{T}_1 = \widetilde{\mathcal{T}}$. It can be shown that there are ε, M such that $0 < \varepsilon < M$ and $u \neq \mathcal{T}_\lambda(u)$ for all $u \in \partial\Omega_{\varepsilon,M}$. By the homotopy property of the degree, it follows that $\deg(\mathcal{I} - \mathcal{T}, \Omega_{\varepsilon,M}) = \deg(\mathcal{I} - \widetilde{\mathcal{T}}, \Omega_{\varepsilon,M})$.

Define

$$\widetilde{\mathcal{S}}(u) = \mathcal{R}\left(\gamma u - \frac{\lambda}{u^3} + u\right) \quad \text{and} \quad \mathcal{S}_\lambda(u) = \lambda \widetilde{T}(u) + (1-\lambda)\widetilde{\mathcal{S}}(u)$$

for $u \in C_T[0, T]$ and $\lambda \in [0, 1]$. We have

$$\mathcal{S}_\lambda(u) = \mathcal{R}\left(\gamma u + u - (1-\lambda)\frac{1}{u^\nu} - \lambda\frac{1}{u^3}\right),$$

$\mathcal{S}_0 = \widetilde{\mathcal{S}}$ and $\mathcal{S}_1 = \widetilde{\mathcal{T}}$. Now, we prove that $u \neq \mathcal{S}_\lambda(u)$ on $\partial\Omega_{\varepsilon,M}$ for each $\lambda \in [0, 1]$ and for some suitable ε and M. Similarly to Step 1, this yields that $\deg(\mathcal{I} - \widetilde{\mathcal{T}}, \Omega_{\varepsilon,M}) = \deg(\mathcal{I} - \widetilde{\mathcal{S}}, \Omega_{\varepsilon,M})$.

The proof is completed by proving that $\deg(\mathcal{I} - \widetilde{\mathcal{S}}, \Omega_{\varepsilon,M}) \neq 0$. □

REMARK 5.31. Consider the problem

$$u'' + ku = \frac{\beta}{u^\lambda} + e(t), \quad u(0) = u(T), \quad u'(0) = u'(T) \tag{5.118}$$

with $\lambda > 0$, $\beta > 0$ and $k \geqslant 0$. Denote

$$g(x) = \frac{\beta}{x^\lambda} - kx \quad \text{for } x > 0.$$

If $e \in L_1[0, T]$, $k = 0$ and $\lambda \geqslant 1$, i.e. the function g has a strong singularity at $x = 0$, then by [96, Theorem 3.12] problem (5.118) has a positive solution if and only if $\overline{e} < 0$ and, in the case $\lambda \in (0, 1)$, this condition need not ensure the existence of a positive solution to (5.118) (cf. [96, Theorem 4.1]). Further, if $e \in C[0, T]$ and $\lambda \geqslant 1$, then by Theorem 5.30, problem (5.118) has a positive solution whenever the condition

$$k \neq \left(n\frac{\pi}{T}\right)^2 \quad \text{for all } n \in \mathbb{N}$$

is satisfied. It is worth mentioning that the resonance case of $k = (\frac{\pi}{T})^2$ is covered neither by Theorem 5.30 nor by Theorem 5.25 even for the strong singularity $\lambda \geqslant 1$.

In comparison to these results, it should be pointed out that using Corollary 5.28, we can obtain existence results also for the cases $\lambda \in (0, 1)$ and $k = (\frac{\pi}{T})^2$. In particular, for problem (5.118), with $e \in L_1[0, T]$ we get the existence of a positive solution in the following cases:

$$k = 0, \quad \bar{e} < 0 \quad \text{and} \quad \inf\operatorname{ess}\{e(t)\colon t \in [0, T]\} > -\left(\frac{\pi^2}{T^2\lambda\beta}\right)^{\frac{\lambda}{\lambda+1}}(\lambda + 1)\beta$$

or

$$0 < k < \left(\frac{\pi}{T}\right)^2 \quad \text{and} \quad \inf\operatorname{ess}\{e(t)\colon t \in [0, T]\} > -\left(\frac{\pi^2 - T^2k}{T^2\lambda\beta}\right)^{\frac{\lambda}{\lambda+1}}(\lambda + 1)\beta$$

or

$$k = \left(\frac{\pi}{T}\right)^2 \quad \text{and} \quad \inf\operatorname{ess}\{e(t)\colon t \in [0, T]\} > 0.$$

Notice that for the case $0 < k < (\frac{\pi}{T})^2$, Theorem 5.27 provides a complementary existence condition. For details, see [144, Corollary 4.6].

We close this section by mentioning some results concerning the case when the nonlinearity can have both a space singularity at $x = 0$ and superlinear descent for large x. The first is due to del Pino and Manásevich. It was motivated by [156], where an equation governing the nonlinear vibrations of a radially forced thick-walled and incompressible material was derived. The proof makes use of a version of the Poincaré–Birkhoff theorem due to Ding [56] together with an analysis of some oscillatory properties of solutions to the related initial value problems.

THEOREM 5.32 [51, Theorem 2.1]. *Let $f : \mathbb{R} \times (0, \infty) \to \mathbb{R}$ be continuous, locally Lipschitz in x, T-periodic in t and such that for $s, \beta \in \mathbb{R}$ and $\alpha > 0$, the solution $u(t)$ of the local initial value problem*

$$u'' = f(t, u), \quad u(s) = \alpha, \quad u'(s) = \beta$$

is continuable to the whole real line $\mathbb{R}$ and $u > 0$ on $\mathbb{R}$. Furthermore, assume that

$$0 < \liminf_{x \to 0+} x f(t, x) \leqslant \infty \quad \textit{and} \quad \lim_{x \to \infty} \frac{f(t, x)}{x} = -\infty$$

uniformly in t. Then there is $n_0 \in \mathbb{N}$ such that for every $n \geqslant n_0$ there exist two distinct T-periodic positive solutions u_{n1}, u_{n2} of (5.97), (5.2) *such that both $u_{n1} - 1$ and $u_{n2} - 1$ have exactly $2n$ zeros in $[0, T)$. In particular, problem* (5.97), (5.2) *possesses infinitely many T-periodic positive solutions.*

The other result is due to Ge and Mawhin. It deals with the equation

$$u'' = g(u) + p(t, u, u'). \tag{5.119}$$

Its proof was obtained by use of some continuation theorems valid in absence of a priori bounds and given by Capietto, Mawhin and Zanolin, e.g., in [46] and [103].

THEOREM 5.33 [73, Theorem 1]. *Let $g \in C(0, \infty)$ and $p \in Car([0, T] \times \mathbb{R}^2)$. Furthermore, assume that there are constants α, $\beta \geqslant 1$, $M \geqslant 0$, $L \geqslant 0$ such that*

$$\lim_{x \to 0+} g(x)x^\alpha = \infty, \qquad \lim_{x \to \infty} \frac{g(x)}{x^\beta} = -\infty, \tag{5.120}$$

$$\left|p(t, x, y)\right| \leqslant \begin{cases} M\left(|x|^{\frac{1-\alpha}{2}} + |y|\right) + L & \text{if } 0 < x < 1, \\ M\left(|x|^{\frac{1+\beta}{2}} + |y|\right) + L & \text{if } x \geqslant 1. \end{cases} \tag{5.121}$$

Then problem (5.119), (5.2) *has a positive solution.*

5.5. *Historical and bibliographical notes*

In 1958 Bevc, Palmer and Süsskind [34] searched for positive 2π-periodic solutions of the Brillouin electron beam focusing system (5.114) which is a singular perturbation of the Mathieu equation. Before, in 1950, Pinney [113] considered the so-called Ermakov–Pinney equation $r'' + a(t)r = \frac{K}{r^3}$, where $a(t)$ is T-periodic and $K > 0$. Another example of the singular problem mentioned in literature are the parametric resonances of certain nonlinear Schrödinger systems (see [71]). As mentioned by Mawhin and Jebelean in their exhaustive historical introduction to the paper [84], second order nonlinear differential equations or systems with singularities appear naturally in the description of particles submitted to Newtonian type forces or to forces caused by compressed gases. Their mathematical study started in the sixties by Forbat and Huaux [69], Huaux [82], Derwidué [54] and Faure [65], who considered positive solutions of equations describing, e.g., the motion of a piston in a cylinder closed at one extremity and submitted to a T-periodic exterior force, to the restoring force of a perfect gas and to a viscosity friction. The equations under their study may be after suitable substitutions transformed to

$$u'' + cu' = \frac{\beta}{u} + e(t),$$

where $c \neq 0$ and $\beta \in \mathbb{R}$ can be either positive or negative. Equations of this form are usually called Forbat's equations and their Liénard type generalizations like

$$u'' + h(u)u' = g(t, u) + e(t) \tag{5.122}$$

are sometimes also referred to as the generalized Forbat's equations. It is worth mentioning that, while Forbat and others relied on the dissipativeness properties, Faure made use of the Leray–Schauder topological method.

Later, in the seventies, techniques of critical point theory were applied for the first time by Gordon [76], who also introduced the *strong force condition* of the type (5.29).

In 1988, Gaete and Manásevich [70], using variational methods, proved the existence of at least two different positive T-periodic solutions of the equation $u'' = p(t)u^2 - u + u^{-5}$ which governs the radial oscillations of an elastic spherical membrane made up of a neo-Hookean material, and subjected to an internal pressure $p:\mathbb{R} \to (0,\infty)$ continuous, T-periodic and non-constant.

In 1987, keeping in mind the model equation $u'' = \beta u^{-\lambda} + e(t)$ with $\lambda > 0$, $\beta \neq 0$ and $e \in L_1[0,T]$, Lazer and Solimini [96] employed topological arguments and the lower and upper functions method to investigate the existence of positive solutions to the Duffing equation $u'' = g(u) + e(t)$. The restoring force g was allowed to have an attractive singularity or a strong repulsive singularity at origin. Starting with this paper, the interest in periodic singular problems considerably increased. The results by Lazer and Solimini have been generalized or extended, e.g., by Habets and Sanchez [78] (1990), Mawhin [102] (1991), del Pino, Manásevich and Montero [52] (1992), del Pino and Manásevich [51] (1993), Fonda [66] (1993), Omari and Ye [108] (1995), Zhang [152] (1996) and [154] (1998) and Ge and Mawhin [73] (1998). Some of these papers (e.g., [78,108,152] or [154]) cover also the Liénard equation (5.122) with $g(t,x)$ having at $x = 0$ an essentially autonomous singularity. However, all of them, when dealing with the repulsive singularity, supposed that the strong force condition of the type (5.29) is satisfied. Furthermore, except for [51,52] and [73], they dealt with restoring forces $g(t,x)$ behaving at ∞ like $-kx$ with $0 < k < \mu_1$, $\mu_1 = (\frac{\pi}{T})^2$ being the first Dirichlet eigenvalue of $x'' + \mu x = 0$. The paper [52] was concerned with the cases corresponding to k lying between two adjacent higher eigenvalues, while the papers [51] and [73] dealt with the superlinear case. Recently, Yan and Zhang [151] (2003) proved an existence result assuming that the nonlinearity grows semilinearly as $x \to \infty$ and fulfill a certain higher-order non-resonance condition in terms of the periodic and antiperiodic eigenvalues. Let us mention also that Martinez-Amores and Torres [105] considered in 1996 stability of periodic solutions of problems with singularities of attractive type. Furthermore, in 1998, Torres [143] delivered results on the existence of bounded solutions to singular equations of repulsive type.

In 2001, Rachůnková, Tvrdý and Vrkoč [129], motivated by results on the existence of positive solutions to regular periodic problems by Nkashama and Santanilla [107] (1990) and Sanchez [131] (1992), made use of the lower and upper functions method to deliver related results in the form applicable also to singular problems. Unlike the above mentioned papers, their results concern also the resonance case $k = \mu_1$ and do not need any strong force condition. Later, in 2002, further step was done by Bonheure, Fonda and Smets [39] who made use of the properties of forced isochronous oscillators. Their results are also valid in the resonance case $k = \mu_1$ with a weak singularity. It turned out that in the resonance case $k = \mu_1$ problem (5.118) with $\lambda \geqslant 0$ has a solution whenever there is a $\delta > 0$ such that

$$\min_{t\in[0,T]} \int_t^{t+T} e(t) \sin\left(\pi \frac{s-t}{T}\right) \mathrm{d}s \geqslant \delta.$$

Analogous results were derived also by Bonheure and De Coster [38] in 2003 by means of the lower and upper functions method. Simultaneously, Torres [144] noticed that having a thorough analysis of the sign properties of the related Green's functions, solvability of the periodic problem with a weak singularity can be ensured also by the Krasnoselskii fixed point theorem. His results turned out to be complementary to those already known when $0 < k < \mu_1$.

For related multiplicity results we refer to the papers by Fonda, Manásevich and Zanolin [68] (1993), Rachůnková [115] and [116] (2000) and Rachůnková, Tvrdý and Vrkoč [130] (2003). In particular, in [115] an extension of the results of Gaete and Manásevich from [70] applicable to the equation modelling radial oscillations of an elastic spherical membrane can be found.

The regular periodic problem with ϕ- or p-Laplacian on the left-hand side was considered by several authors. For example, del Pino, Manásevich and Murúa [53] (1992) and Yan [150] (2003) proved the existence or multiplicity of periodic solutions of the equation $(\phi_p(u'))' = f(t,u)$ under non resonance conditions imposed on f. In 1998, general existence principles for the regular vector problem, based on the homotopy to the averaged nonlinearity, were presented by Manásevich and Mawhin [101] (1998) (see also Mawhin [104]). Multiplicity results were given by Liu in [97] (1998) and by Jiang, Chu and Zhang in [88] (2005). The resonance case was considered by Fabry and Fayyad in [61] (1992).

The first steps to establish the lower/upper functions method for problems with a ϕ-Laplacian operator on the left-hand side were done by Cabada and Pouso in [41] (1997) and by Jiang and Wang in [89] (1997), the latter paper dealing with the p-Laplacian. They assumed the existence of a pair of well-ordered lower and upper functions and both-sided Nagumo conditions. These results were extended by Staněk [133] (2001) to the case when a functional right-hand side fulfills one-sided growth conditions of Nagumo type. The paper by Cabada, Habets and Pouso [42] (1999) was the first to present the lower/upper functions method for periodic problems with a ϕ-Laplacian operator under the assumption that lower/upper functions are in the reverse order, see also [42] (2000). If $\phi = \phi_p$ the authors got the solvability for $1 < p \leqslant 2$ only. The general existence principle valid also when lower/upper functions are non-ordered was presented by Rachůnková and Tvrdý in [127] (2005) and for the case when impulses are admitted also in [126] (2005).

The singular periodic problem for the Liénard equation $(\phi_p(u'))' + h(u)u' = g(u) + e(t)$ with p-Laplacian on the left-hand side was treated by Liu [98] (2002) and Jebelean and Mawhin [84] (2002) and [85] (2004). Their main tool was the existence principle due to Manásevich and Mawhin from [101] (1998). Furthermore, in [84], the significance of the lower/upper functions method was shown.

The only existence results for problem (5.1), (5.2) with f having a singularity for $u' = 0$ were delivered by Staněk in [134] (2001) and [136] (2002).

Extensions to vector systems of the second order were not the subject of this text. We can only refer, e.g., to the papers by Habets and Sanchez [79] (1990), Solimini [132] (1990), Fonda [67] (1995) and Zhang [155] (1999) for the classical case and by Manásevich and Mawhin [101] (1998) and Liu [98] (2002) for systems with p-Laplacian operators on their left-hand sides.

6. Other types of two-point second order BVPs

In Sections 4 and 5, under the assumption that ϕ satisfies (3.3), we have investigated the nonlinear second order differential equation of the form

$$\big(\phi(u')\big)' + f(t, u, u') = 0, \tag{6.1}$$

subjected to Dirichlet and periodic boundary conditions, respectively. In this section we will study solvability of Eq. (6.1) with some other types of two-point boundary conditions on the interval $[0, T] \subset \mathbb{R}$.

We will focus our attention on problems with *space* or with *mixed* (*time and space*) singularities. According to Section 1, *solutions* and *w-solutions* of the problems are defined in the same way as for the Dirichlet problem (see Section 4.1) just replacing the Dirichlet conditions by the boundary conditions under consideration. We can define *lower and upper functions* of the second order boundary value problem in the same way as in Definition 4.3 replacing inequalities (4.6) with inequalities corresponding to the boundary conditions in question.

In the sequel we consider two-point linear boundary conditions arising in the study of physical, chemical or engineering problems and having the form

$$\begin{cases} a_0 u(0) - b_0 u'(0) = 0, \quad a_1 u(T) + b_1 u'(T) = 0, \\ a_i, b_i \in \mathbb{R}, \quad a_i^2 + b_i^2 > 0, \quad i = 0, 1. \end{cases} \tag{6.2}$$

Conditions (6.2) include conditions of the Dirichlet type (with $b_0 = b_1 = 0$), of the Neumann type (with $a_0 = a_1 = 0$), of the mixed type (with $a_0 = b_1 = 0$ or $b_0 = a_1 = 0$), of the Robin type (with $a_i > 0$, $b_i > 0$, $i = 0, 1$) and of the standard Sturm–Liouville type (with $a_i, b_i \in [0, \infty)$, $i = 0, 1$). We will also mention problems involving inhomogeneous form of the above boundary conditions, i.e.

$$\begin{cases} a_0 u(0) - b_0 u'(0) = A, \quad a_1 u(T) + b_1 u'(T) = B, \\ a_i, b_i \in \mathbb{R}, \quad a_i^2 + b_i^2 > 0, \quad i = 0, 1, \ A, B \in \mathbb{R}. \end{cases} \tag{6.3}$$

However, there is no restriction in assuming just the homogeneous conditions since a change from $u(t)$ to $y(t) = u(t) - q(t)$, where q is a polynomial satisfying (6.3), will reduce (6.3) to (6.2).

Consider a class of nonlinear singular boundary value problems whose importance is derived, in part, from the fact that they arise when searching for *positive, radially symmetric solutions* to the nonlinear elliptic partial differential equation

$$\Delta u + g(r, u) = 0 \quad \text{on } \Omega, \quad u\,|_{\Gamma} = 0,$$

where Δ is the Laplace operator, Ω is the open unit disk in $\mathbb{R}^n$ (centered at the origin), Γ is its boundary, and r is the radial distance from the origin. Radially symmetric solutions to this problem are solutions of the ordinary differential equation with the mixed boundary conditions

$$u'' + \frac{n-1}{t}u' + g(t,u) = 0, \quad u'(0) = 0, \quad u(1) = 0.$$

See, e.g., [29] or [74]. Particularly, Gatica, Oliker and Waltman [72] investigated the singular problem

$$u'' + \frac{n-1}{t}u' + \psi(t)\frac{1}{u^\alpha} = 0, \quad u'(0) = 0, \quad u(1) = 0, \tag{6.4}$$

where

$$\begin{cases} n \geqslant 2,\ \alpha \in (0,1),\ \psi \in C[0,1) \text{ is non-negative,} \\ \psi \text{ can have a time singularity at } t = 1. \end{cases} \tag{6.5}$$

THEOREM 6.1 [72, Theorem 4.1]. *Let* (6.5) *hold. Assume that*

$$0 < \int_0^1 (1-t)^{-\alpha}\psi(t)\,\mathrm{d}t < \infty.$$

Then problem (6.4) *has a solution that is positive on* $[0,1)$.

The technical arguments in the proof involve concavity of solutions and the use of iterative techniques. The main tool is a fixed point theorem for decreasing mappings on cones.

In the theory of diffusion and reaction a class of differential equations

$$u'' - \eta^2 g_\kappa(u) = 0 \tag{6.6}$$

appears. Here η^2 is the (positive) Thiele modulus, $u \geqslant 0$ is the concentration of one of the reactants and κ is a positive parameter. The functions g_κ are continuous on $[0,\infty)$,

$$\lim_{\kappa \to 0+} g_\kappa(x) = g(x) \quad \text{for } x \in (0,\infty)$$

and g can have a space singularity at $x = 0$. The model functions are

$$g_\kappa(x) = \frac{x}{\kappa + x^{1+\gamma}}, \tag{6.7}$$

where γ is a positive parameter. Aris [21] proposed such equations as descriptions of the steady state for chemicals reacting and diffusing according to the Langmuir–Hinshelwood kinetics. Bobisud [35] studied a class of equations (6.6) on $[-1,1]$ subjected to the inhomogeneous Robin boundary conditions

$$\alpha u(-1) - u'(-1) = A, \quad \alpha u(1) + u'(1) = A, \quad \alpha, A > 0, \tag{6.8}$$

and with functions g_κ behaving qualitatively very much like the model functions in (6.7). He proved that for η^2 in (6.6) sufficiently small the limit problem with $\kappa = 0$ has a positive solution which can be approximated uniformly on $[-1, 1]$ by solutions of (6.6), (6.8) with small κ.

Motivated by problem (6.6), (6.8) as well as by problem (6.4), Baxley and Gersdorff [27] studied a singular equation in which u' can appear nonlinearly,

$$u'' + h(t, u') - \eta^2 g(t, u) = 0, \quad \eta^2 > 0, \tag{6.9}$$

with inhomogeneous Sturm–Liouville boundary conditions

$$u'(0) = 0, \quad \alpha u(T) + \beta u'(T) = A, \quad \alpha, A > 0,\ \beta \geqslant 0, \tag{6.10}$$

where

$$\begin{cases} h \in C\big((0, T] \times [0, \infty)\big) \text{ is non-negative} \\ \text{and can have a time singularity at } t = 0, \end{cases} \tag{6.11}$$

$$\begin{cases} g \in C\big([0, T] \times \big(0, \frac{A}{\alpha}\big]\big) \text{ is positive} \\ \text{and can have a space singularity at } x = 0. \end{cases} \tag{6.12}$$

In contrast to Theorem 6.1 where positive singular nonlinearity $\psi(t)x^{-\alpha}$ appears, the next theorem applies to equations involving a negative singular term $-\eta^2 g(t, x)$.

THEOREM 6.2 [27, Theorem 17]. *Let* (6.11) *and* (6.12) *hold. Assume that*

$$h(t, 0) = 0 \quad \textit{for } t \in (0, T]$$

and that there exists $G \in L[0, \frac{A}{\alpha}]$ *satisfying*

$$g(t, x) \leqslant G(x) \quad \textit{on } [0, T] \times \big(0, \tfrac{A}{\alpha}\big]. \tag{6.13}$$

Then for η^2 *sufficiently small problem* (6.9), (6.10) *has a solution that is positive on* $[0, T]$.

Moreover, if η^2 is sufficiently large, Baxley and Gersdorff guarantee the existence of the so called *dead core* solutions which are defined as functions belonging for some $t_0 \in (0, T)$ to $C^1[0, T] \cap C^2(t_0, T]$, satisfying equation (6.9) on $(t_0, T]$, vanishing on $[0, t_0]$ and fulfilling (6.10). The proof is based on a priori estimates of approximate solutions of auxiliary regular problems and on the Arzelà–Ascoli theorem.

Agarwal, O'Regan and Staněk [15] considered a singular equation with a ϕ-Laplacian generalizing (6.9) and subjected to inhomogeneous mixed conditions

$$\big(\phi(u')\big)' - \mu f(t, u, u') = 0, \quad u'(0) = 0, \quad u(T) = b, \quad b > 0, \tag{6.14}$$

where μ is a real positive parameter and

$$\begin{cases} \phi(0) = 0, \quad f \in Car\big([0,T] \times \big(\mathbb{R} \setminus \{b\}\big) \times \big(\mathbb{R} \setminus \{0\}\big)\big), \\ f \text{ can have space singularities at } x = b \text{ and } y = 0. \end{cases} \tag{6.15}$$

THEOREM 6.3 [15, Theorem 3.1]. *Let* (6.15) *hold. Assume that there exist* $\varepsilon > 0$, $\nu \in (0, T]$ *and a positive non-decreasing function* $\rho \in C[0, T]$ *such that*

$$f\big(t, x, \rho(t)\big) = 0 \quad \textit{for a.e. } t \in [0, T] \textit{ and all } x \in [0, b),$$

$$\varepsilon \leqslant f(t, x, y) \quad \textit{for a.e. } t \in [0, \nu] \textit{ and all } x \in [0, b),\ y \in [0, \nu].$$

Further, let for a.e. $t \in [0, T]$ *and for all* $x \in [0, b)$, $y \in (0, \rho(t)]$

$$0 \leqslant f(t, x, y) \leqslant \big(h_1(x) + h_2(x)\big)\big(\omega_1(y) + \omega_2(y)\big),$$

where $h_1 \in C[0, b]$, $\omega_1 \in C[0, \infty)$ *are non-negative*, $h_2 \in C[0, b)$, $\omega_2 \in C(0, \infty)$ *are positive*, h_1 *and* ω_2 *are non-increasing*, h_2 *and* ω_1 *are non-decreasing and* $\omega_1 + \omega_2$ *is non-increasing on a right neighbourhood of* 0. *Moreover, let*

$$\int_0^b h_2(s)\,\mathrm{d}s < \infty, \quad \int_0^1 \omega_2\big(\phi^{-1}(s)\big)\,\mathrm{d}s < \infty.$$

Finally, let there exist $\mu_* > 0$ *such that*

$$\int_0^b \frac{\mathrm{d}s}{\Omega^{-1}(\mu_* H(s))} = T, \tag{6.16}$$

where

$$H(u) = \int_0^u \big(h_1(s) + h_2(s)\big)\,\mathrm{d}s, \quad \Omega(u) = \int_0^{\phi(u)} \frac{\phi^{-1}(s)\,\mathrm{d}s}{\omega_1(\phi^{-1}(s)) + \omega_2(\phi^{-1}(s))}.$$

Then for each $\mu \in (0, \mu_*)$ *problem* (6.14) *has a solution* u *satisfying*

$$0 < u(t) \leqslant b, \quad 0 \leqslant u'(t) \leqslant \rho(t) \quad \textit{for } t \in [0, T].$$

To prove this existence result the authors used regularization and sequential techniques. First, they defined a family of auxiliary regular differential equations depending on $n \in \mathbb{N}$ and then, using the topological transversality theorem, they obtained a sequence of positive approximate solutions. Applying the Arzelà–Ascoli theorem and the Lebesgue convergence theorem they showed that its limit is a solution of problem (6.14).

REMARK 6.4. Note that if there exists $\mu_0 \in (0,\infty)$ such that

$$\int_0^b \frac{\mathrm{d}s}{\Omega^{-1}(\mu_0 H(s))} \in (T,\infty),$$

then $\mu_* \in (0,\mu_0)$ satisfying (6.16) can be always found.

Comparing problem (6.9), (6.10) and problem (6.14), they seem to be in some sense close, because both of them have negative singular nonlinearities in differential equations and the boundary conditions of (6.14) are contained in (6.10). However, there is a large difference between them. For example, positive solutions of (6.9), (6.10) do not touch the space singularity of f at $x = 0$. On the other hand, each solution u of (6.14) satisfies $u(T) = b$ and hence enters the space singularity of f at $x = b$. Another difference between them consists in the fact that f in (6.14) can have also a space singularity at $y = 0$ and hence Theorem 6.3 can be used in the following example whereas Theorem 6.2 cannot.

EXAMPLE. Let $\alpha \in (0,\infty)$ and $\beta \in (0,1)$. By Theorem 6.3 there exists a positive number μ_* depending on β only such that for any $\mu \in (0,\mu_*)$ the problem

$$u'' - \mu\bigl(1 - |u|^\alpha\bigr)\left(\frac{1}{|u'|^\beta} - 1\right) = 0, \quad u'(0) = 0, \quad u(1) = \frac{1}{2},$$

has a solution u such that $0 < u(t) \leqslant \frac{1}{2}$, $0 \leqslant u'(t) \leqslant 1$ for $t \in [0,1]$. An explicit formula for μ_* can be found in [15].

Assumption (6.13) in Theorem 6.2 means that the space singularity of g at $x = 0$ is a *weak singularity*. See Remark 5.1 for more detail. Note that the assumption (6.13) is not satisfied for the problem

$$u'' + \frac{t^2}{32u^2} - \frac{\lambda^2}{8} = 0, \quad u(0) = 0, \quad 2u'(1) - (1-\nu)u(1) = 0, \tag{6.17}$$

where $\lambda \in (0,\infty)$, $\nu \in (0,1)$, which models the large deflection membrane response of a spherical cap. This problem has been solved numerically by various techniques in engineering literature [75,106,111]. Baxley in [26] proved existence and uniqueness of a solution of this problem, gave qualitative information about the solution, and used this information to suggest an approach to numerical computation. In the proof the maximum principle plays a fundamental role. In contrast to (6.8), (6.11) and (6.14), problem (6.17) has boundary conditions which are not included in the Sturm–Liouville ones, because $\nu < 1$ and so the coefficient $-(1-\nu)$ at $u(1)$ is negative.

Existence results for equations whose nonlinearities have a singularity at $x = 0$ and can be increasing for $x \to \infty$ are proved in [4], where Agarwal and O'Regan obtained the existence of a w-solution $u > 0$ on $(0,1]$ of such equations with mixed boundary conditions. Their theorem can be applied for example to the problem

$$u'' + \left(\frac{1}{u^\alpha} + u^\beta + 1\right)\bigl(1 + (u')^3\bigr) = 0, \quad u'(0) = 0, \quad u(1) = 0, \tag{6.18}$$

with $\alpha \in (0,1)$, $\beta \geqslant 0$. We see that the nonlinearity

$$f(t,x,y) = \left(\frac{1}{x^\alpha} + x^\beta + 1\right)\left(1 + y^3\right) \tag{6.19}$$

has a weak space singularity at $x = 0$ and can be increasing for large x. If $\beta \in (0,1)$ the growth of f is sublinear. For $\beta = 1$ or $\beta > 1$ the growth of f is linear or superlinear, respectively.

In the investigation of singular problems (6.1), (6.2) or (6.1), (6.3), lower and upper functions of the corresponding regular problems can be a fruitful tool. See for example papers by Kannan and O'Regan in [90] or by Agarwal and Staněk in [17]. We will demonstrate the role of lower and upper functions on the following singular problem with mixed boundary conditions

$$u'' + f(t,u,u') = 0, \quad u'(0) = 0, \quad u(T) = 0, \tag{6.20}$$

where

$$\begin{cases} \mathcal{D} = (0,\infty) \times (-\infty, 0), \quad f \in Car\big((0,T) \times \mathcal{D}\big), \\ f \text{ can have time singularities at } t = 0,\ t = T \\ \text{and space singularities at } x = 0,\ y = 0. \end{cases} \tag{6.21}$$

First, consider an auxiliary regular problem

$$u'' + h(t,u,u') = 0, \quad u'(0) = 0, \quad u(T) = 0, \tag{6.22}$$

where $h \in Car([0,T] \times \mathbb{R}^2)$.

DEFINITION 6.5. A function $\sigma \in C[0,T]$ is called *a lower function of* (6.22) if there exists a finite set $\Sigma \subset (0,T)$ such that $\sigma \in AC^1_{loc}([0,T] \setminus \Sigma)$, $\sigma'(\tau+), \sigma'(\tau-) \in \mathbb{R}$ for each $\tau \in \Sigma$,

$$\sigma''(t) + f\big(t, \sigma(t), \sigma'(t)\big) \geqslant 0 \quad \text{for a.e. } t \in [0,T], \tag{6.23}$$

$$\sigma'(0) \geqslant 0, \quad \sigma(T) \leqslant 0, \quad \sigma'(\tau-) < \sigma'(\tau+) \quad \text{for each } \tau \in \Sigma. \tag{6.24}$$

If the inequalities in (6.23) and (6.24) are reversed, then σ is called *an upper function of* (6.22).

In what follows we will need the classical lower and upper functions result for the mixed problem (6.22).

LEMMA 6.6 [93, Lemma 3.7]. *Let σ_1 and σ_2 be a lower and an upper function for problem* (6.22) *such that $\sigma_1 \leqslant \sigma_2$ on $[0,T]$. Assume also that there is a function $\psi \in L_1[0,T]$ such that*

$$\bigl|h(t,x,y)\bigr| \leqslant \psi(t) \quad \textit{for a.e. } t \in [0,T],\ \textit{all } x \in \bigl[\sigma_1(t), \sigma_2(t)\bigr],\ y \in \mathbb{R}. \tag{6.25}$$

Then problem (6.22) *has a solution $u \in AC^1[0,T]$ satisfying*

$$\sigma_1(t) \leqslant u(t) \leqslant \sigma_2(t) \quad \textit{for } t \in [0,T]. \tag{6.26}$$

We will apply Lemma 6.6 to the singular mixed problem (6.20).

THEOREM 6.7 [117, Theorem 3.1]. *Let* (6.21) *hold. Assume that there exist $\varepsilon \in (0,1)$, $\nu \in (0,T)$, $c \in (\nu, \infty)$ such that*

$$f\bigl(t, c(T-t), -c\bigr) = 0 \quad \textit{for a.e. } t \in [0,T], \tag{6.27}$$

$$0 \leqslant f(t,x,y)$$
$$\qquad \textit{for a.e. } t \in [0,T],\ \textit{and all } x \in \bigl(0, c(T-t)\bigr],\ y \in [-c, 0), \tag{6.28}$$

$$\varepsilon \leqslant f(t,x,y)$$
$$\qquad \textit{for a.e. } t \in [0,\nu],\ \textit{and all } x \in \bigl(0, c(T-t)\bigr],\ y \in [-\nu, 0). \tag{6.29}$$

Then problem (6.20) *has a solution $u \in AC^1[0,T]$ satisfying*

$$0 < u(t) \leqslant c(T-t), \quad -c \leqslant u'(t) < 0 \quad \textit{for } t \in (0,T). \tag{6.30}$$

PROOF. Let $k \in \mathbb{N}$, $k \geqslant \frac{3}{T}$.

Step 1. *Approximate solutions.*

For $t \in [\frac{1}{k}, T - \frac{1}{k}]$, $x \in \mathbb{R}$ and $y \in \mathbb{R}$ put

$$\alpha_k(t,x) = \begin{cases} c(T-t) & \text{if } x > c(T-t), \\ x & \text{if } \frac{c}{k} \leqslant x \leqslant c(T-t), \\ \frac{c}{k} & \text{if } x < \frac{c}{k}, \end{cases}$$

$$\beta_k(y) = \begin{cases} -\frac{\varepsilon}{k} & \text{if } y > -\frac{\varepsilon}{k}, \\ y & \text{if } -c \leqslant y \leqslant -\frac{\varepsilon}{k}, \\ -c & \text{if } y < -c, \end{cases} \qquad \gamma(y) = \begin{cases} \varepsilon & \text{if } y \geqslant -\nu, \\ \varepsilon\frac{c+y}{c-\nu} & \text{if } -c < y < -\nu, \\ 0 & \text{if } y \leqslant -c. \end{cases}$$

For a.e. $t \in [0,T]$ and $x, y \in \mathbb{R}$ define

$$f_k(t,x,y) = \begin{cases} \gamma(y) & \text{if } t \in [0, \frac{1}{k}), \\ f\bigl(t, \alpha_k(t,x), \beta_k(y)\bigr) & \text{if } t \in [\frac{1}{k}, T - \frac{1}{k}], \\ 0 & \text{if } t \in (T - \frac{1}{k}, T]. \end{cases}$$

Then $f_k \in Car([0,T] \times \mathbb{R}^2)$ and there is $\psi_k \in L_1[0,T]$ such that

$$\left| f_k(t,x,y) \right| \leqslant \psi_k(t) \quad \text{for a.e. } t \in [0,T], \text{ all } x, y \in \mathbb{R}. \tag{6.31}$$

We have got an auxiliary regular problem

$$u'' + f_k(t,u,u') = 0, \quad u'(0) = 0, \quad u(T) = 0. \tag{6.32}$$

Conditions (6.27) and (6.28) yield

$$f_k\big(t, c(T-t), -c\big) = 0 \quad \text{and} \quad f_k(t,0,0) \geqslant 0 \quad \text{for a.e. } t \in [0,T].$$

Put $\sigma_1(t) = 0$, $\sigma_2(t) = c(T-t)$ for $t \in [0,T]$. Then σ_1 and σ_2 are a lower and an upper function of (6.32). Hence, by Lemma 6.6, problem (6.32) has a solution u_k and

$$0 \leqslant u_k(t) \leqslant c(T-t) \quad \text{on } [0,T]. \tag{6.33}$$

Step 2. *A priori estimates of approximate solutions.*

Since $u_k'(0) = 0$ and $f_k(t,x,y) \geqslant 0$ for a.e. $t \in [0,T]$ and all $x, y \in \mathbb{R}$, we get $u_k'(t) \leqslant 0$ on $[0,T]$. Condition (6.33) and $u_k(T) = 0$ give $u_k(T) - u_k(t) \geqslant -c(T-t)$, which yields $u_k'(T) \geqslant -c$. Since u_k' is nonincreasing on $[0,T]$, we have proved

$$-c \leqslant u_k'(t) \leqslant 0 \quad \text{on } [0,T]. \tag{6.34}$$

Due to $u_k'(0) = 0$, there is $t_k \in (0,T]$ such that

$$-\nu \leqslant u_k'(t) \leqslant 0 \quad \text{for } t \in [0,t_k].$$

If $t_k \geqslant \nu$, we get by (6.29)

$$u_k'(t) \leqslant -\varepsilon t \quad \text{for } t \in [0,\nu]. \tag{6.35}$$

Assume that $t_k < \nu$ and $u_k'(t) < -\nu$ for $t \in (t_k, \nu]$. Then

$$u_k'(t) \leqslant -\varepsilon t \quad \text{for } t \in [0,t_k].$$

Since $-\nu < -\varepsilon t$ for $t \in (t_k, \nu]$, we get (6.35) again. Integrating (6.35) over $[0,\nu]$ and using the concavity of u_k on $[0,T]$ we deduce that

$$\frac{\varepsilon \nu^2}{2T}(T-t) \leqslant u_k(t) \quad \text{on } [0,T]. \tag{6.36}$$

Step 3. *Convergence of a sequence of approximate solutions.*

Consider the sequence $\{u_k\}$. Choose an arbitrary compact interval $J \subset (0,T)$. By virtue of (6.33)–(6.36) there is $k_0 \in \mathbb{N}$ such that for each $k \in \mathbb{N}$, $k \geqslant k_0$,

$$\frac{c}{k_0} \leqslant u_k(t) \leqslant c(T-t), \quad -c \leqslant u_k'(t) \leqslant -\frac{\varepsilon}{k_0} \quad \text{on } J, \tag{6.37}$$

and hence there is $\psi \in L_1(J)$ such that

$$\left| f_k\big(t, u_k(t), u_k'(t)\big)\right| \leqslant \psi(t) \quad \text{for a.e. } t \in J. \tag{6.38}$$

Using conditions (6.33), (6.34), (6.38), the Arzelà–Ascoli theorem and the diagonalization principle, we can choose $u \in C[0,T] \cap C^1(0,T)$ and a subsequence of $\{u_k\}$ which we denote for the sake of simplicity in the same way such that

$$\begin{cases} \lim\limits_{k\to\infty} u_k = u & \text{uniformly on } [0,T], \\ \lim\limits_{k\to\infty} u_k' = u' & \text{locally uniformly on } (0,T). \end{cases} \tag{6.39}$$

Therefore we have $u(T) = 0$.

Step 4. *Convergence of the sequence of approximate problems.*

Choose an arbitrary $\xi \in (0,T)$ such that

$$f(\xi,\cdot,\cdot) \text{ is continuous on } (0,\infty) \times (-\infty,0).$$

By (6.37) there exist a compact interval $J^* \subset (0,T)$ and $k_* \in \mathbb{N}$ such that $\xi \in J^*$ and for each $k \geqslant k_*$

$$u_k(\xi) > \frac{c}{k_*}, \quad u_k'(\xi) < -\frac{\varepsilon}{k_*}, \quad J^* \subset \left[\frac{1}{k}, T - \frac{1}{k}\right].$$

Therefore

$$f_k\big(\xi, u_k(\xi), u_k'(\xi)\big) = f\big(\xi, u_k(\xi), u_k'(\xi)\big)$$

and, due to (6.39),

$$\lim_{k\to\infty} f_k\big(t, u_k(t), u_k'(t)\big) = f\big(t, u(t), u'(t)\big) \quad \text{for a.e. } t \in (0,T). \tag{6.40}$$

Choose an arbitrary $t \in (0,T)$. Then there exists a compact interval $J \subset (0,T)$ such that (6.38) holds for all sufficiently large k. By virtue of (6.32) we get

$$u_k'\left(\frac{T}{2}\right) - u_k'(t) = \int_{T/2}^{t} f_k\big(s, u_k(s), u_k'(s)\big)\,\mathrm{d}s.$$

Letting $k \to \infty$ and using (6.38), (6.39), (6.40) and the Lebesgue convergence theorem on J, we get

$$u'\left(\frac{T}{2}\right) - u'(t) = \int_{T/2}^{t} f\big(s, u(s), u'(s)\big)\,\mathrm{d}s \quad \text{for each } t \in (0, T). \tag{6.41}$$

Therefore $u \in AC^1_{loc}(0, T)$ satisfies

$$u''(t) + f\big(t, u(t), u'(t)\big) = 0 \quad \text{for a.e. } t \in (0, T). \tag{6.42}$$

Further, according to (6.32) and (6.34) we have for each $k \geqslant \frac{3}{T}$

$$\int_0^T f_k\big(s, u_k(s), u_k'(s)\big)\,\mathrm{d}s = -u_k'(T) \leqslant c,$$

which together with (6.28), (6.33), (6.34) and (6.40) yields, by the Fatou lemma, that $f(t, u(t), u'(t)) \in L_1[0, T]$. Therefore, by (6.42), $u \in AC^1[0, T]$. Moreover, for each $k \geqslant \frac{3}{T}$ and $t \in (0, T)$

$$\begin{aligned}\left|u_k'(t)\right| \leqslant{}& \int_0^t \left|f_k\big(s, u_k(s), u_k'(s)\big) - f\big(s, u(s), u'(s)\big)\right|\mathrm{d}s \\ &+ \int_0^t \left|f\big(s, u(s), u'(s)\big)\right|\mathrm{d}s.\end{aligned}$$

Hence by (6.39) and (6.40) for each $\varepsilon > 0$ there exists $\delta > 0$ and for each $t \in (0, \delta)$ there exists $k_0 = k_0(\varepsilon, t) \in \mathbb{N}$ such that

$$\left|u'(t)\right| \leqslant \left|u'(t) - u_{k_0}'(t)\right| + \left|u_{k_0}'(t)\right| < \varepsilon.$$

It means that $u'(0) = \lim_{t\to 0+} u'(t) = 0$. We have proved that u is a solution of problem (6.20). □

EXAMPLE. Let $\alpha > 0$, $\beta \geqslant 0$ be arbitrary numbers. By Theorem 6.7 problem (6.18) has a solution $u \in AC^1[0, 1]$ satisfying

$$0 < u(t) \leqslant 1 - t, \quad -1 \leqslant u'(t) < 0 \quad \text{for } t \in (0, 1).$$

Note that Theorem 6.7 guarantees solvability of problem (6.18) even for the nonlinearity (6.19) having a strong space singularity ($\alpha \geqslant 1$) at $x = 0$.

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