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Textbook of Tensor Calculus and Differential Geometry

Prasun Kumar Nayak

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To

My grandfather
whom I have inherited the most

Contents

Preface

Tensors and differential geometry play an important role in the spheres of mathematics, physics, and engineering due to their inherent viabilities. The aim of this textbook is to give rigorous and thorough analysis and applications of various aspects of tensor algebra and analysis with applications to differential geometry and mechanics, both classical and relativistic. Also, the present book has been designed in a lucid and coherent manner so that the undergraduate and postgraduate students of mathematics and physics of various universities may reap considerable benefit out of it. I have chosen the topics with great care and have tried to present them systematically with various examples.

This book consists of *ten* chapters. Chapter 1 provides an informative introduction concerning the origin and nature of the tensor concept and the scope of the tensor calculus. Then tensor algebra has been developed in an N-dimensional space. In Chapter 2, an N-dimensional Riemannian space has been chosen for the development of Tensor calculus. In Chapter 3, some symbols and their properties are described. Using these symbols, covariant differentiation of tensors is explained in a compact form. Characteristic peculiarity of Riemannian space consists of curvature tensor, which is covered in Chapter 4. In Chapter 5, geometry of space curve is given. The intrinsic property of surfaces is given in Chapter 6. Chapters 7 and 8 consist of surfaces in space and curves on a surface respectively. Chapter 9 deals with the application of tensors to classical mechanics. Differential geometry was used to great advantage by Einstein in his development of relativity, which is explained in Chapter 10.

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Critical evaluation, suggestions and comments for further improvement of the book will be appreciated.

PRASUN KUMAR NAYAK

CHAPTER 1

Tensor Algebra

The concept of a tensor has its origin in the development of differential geometry by Gauss, Riemann and Christoffel. Ricci and Levi-Civita have developed 'tensor calculus' or rather 'tensor analysis', which is generalisation of vector analysis, also known as absolute differential calculus. Tensor analysis is concerned with the study of abstract objects, called *tensors,* whose properties are independent of the reference frames used to describe the objects. If a tensor is defined at every point of a space, we say that we have a tensor field over the space. Tensor calculus is concerned essentially with the study of tensor fields.

1.1 Tensors

Tensor is a natural and logical generalisation of the term vector. A tensor is represented in a particular reference frame by a set of functions, termed as *components,* just as a vector is determined in a given reference frame by a set of components. Whether a given set of functions represents a tensor depends on the law of transformation of these functions from one co-ordinate system to another. The main aim of 'tensor calculus' is the study of those objects of a space endorsed with a co-ordinate system where the components of objects transform according to a law when we change from one co-ordinate system to another. Regarding the concept of a tensor the following points should be noted:

(i) It is an object of a space and depends on the nature of transformation of co-ordinate system and the nature of the law according to which its components in one system are transformed, when referred to another co-ordinate system.

(ii) The components of a tensor may be chosen arbitrarily in any system of co-ordinates. Its components in any other system are uniquely determined by the corresponding law of transformation.

(iii) The components describing a tensor generally change with the change of co-ordinate system, but the concept of a tensor does not change with the change of co-ordinate system.

(iv) The components of a tensor are always supposed to be functions of the co-ordinates of a point. We say, using geometrical language, that they depend on the position of the point and that our tensor algebra is a geometry of position, independent of any notion of measure.

(v) A tensor represents a mathematical object which exists at a point just as a force represents a physical object which exists at a point.

Scalars and vectors are both special cases of more general object, called a tensor of order N whose specification in any co-ordinate system requires 3^N numbers, called the components of tensor. In fact, scalars are tensors of order zero with $3^\circ = 1$ component. Vectors are tensors of order one with $3^1 = 3$ components.

Physical and geometrical facts have the peculiarity that although they may be established by using co-ordinate systems their contents are independent of such systems. This peculiarity is also possessed by a tensor, i.e. although co-ordinate systems are used to describe tensors, their properties are independent of co-ordinate systems. For this reason tensor calculus is an ideal tool for the study of geometrical and physical objects. As a result, tensor calculus has its applications to the branches of theoretical physics.

In the case of tensors, it is not possible (or at least not easy) to make any geometrical pictures, and hence tensors have to be introduced only through their transformations under changes of the co-ordinate systems. In the book vector, authors have developed pictorial ways of representating vectors and one-forms; this can to some extent be extended to tensors of higher type, but the pictures rapidly become very complicated. It is perhaps better to avoid picturing most tensors directly.

A study of tensor calculus requires a certain amount of background material that may seem unimportant in itself, but without which one proceed very far. Included in that prerequisite material is the topic of the present chapter, the summation convention. As the reader proceeds to later chapters he or she will see that it is this convention which makes the results of tensor analysis surveyable.

1.1.1 Space of N Dimensions

An ordered set of N real numbers $x^1, x^2, \ldots, x^N$ is called N-tuple of real numbers and is denoted by $\left(x^1, x^2, \ldots, x^N\right)$. Here the number i in x^i be the index of x, not power of x. The set of all N-tuples of real numbers is said to form an N dimensional arithmetic continuum and each N-tuple is called the point of this continuum. Such a continuum shall be denoted by V_N. The V_N is sometimes called an N dimensional space, because it can be endowed with the structure of an N-dimensional linear space.

For development of algebra of tensor, a co-ordinate system is set up in a certain manner, this implies that the co-ordinates $\left(x^1, x^2, \ldots, x^N\right)$ can be assigned to every point in V_N with respect to a chosen co-ordinate system establishing a one-to-one correspondence between the points of V_N and the set of all co-ordinates like $\left(x^1, x^2, \ldots, x^N\right)$.

If $\left(x^1, x^2, \ldots, x^N\right)$ be the co-ordinate of a point P in V_N we shall say that the co-ordinate of P is x^i and the corresponding co-ordinate system is denoted by $\left(x^i\right)$.

1.1.2 Dummy and Free Index

When in an indexed expression an index occurs once as a lower index and is an upper index so that the summation conversion is applied, then this index is called dummy index (or suffix). For example, in the expression $a_i b^i$ or a_i^i or $a_i^k x^i$, the index i is dummy. In the expression $a_{ij} x^i x^j$, both the indices i and j are dummy.

Dummy or umbral or dextral index can be replaced by another dummy suffix not used in that term. For example, $a_i^k x^i = a_j^k x^j$. Also, two or more than two dummy suffixes can be interchanged. In an indexed expression if an index is not dummy, then it is called free index. For example, $a_{ij} x^i$, the index i is dummy but index j is free.

Note 1.1.1 By a system of order zero, we shall mean a single quantity having no index, such as A.

Note 1.1.2 The upper and lower indices of a system are called its indices of contravariance and covariance, respectively. For example, for the system A^i_{jk}, the index i is the index of contravariance and the indices j, k are indices of covariance. Accordingly, A^{ij} is called a contravariant system, the system A_{ij} is called a covariant system, while the system A^i_j is called mixed system.

Note 1.1.3 The numbers of components of a system of components of a system of kth order in which each of the indices takes values from 1 to N in N^k.

EXAMPLE 1.1.1 *If* $u^i = a^i_p v^p$ *and* $w^i = b^i_q u^q$, *show that* $w^i = b^i_p a^p_q v^q$.

Solution: From $u^i = a^i_p v^p$ we get, $u^q = a^q_p v^p$. Hence,

$$w^i = b^i_q u^q = b^i_q a^q_p v^p = b^i_p a^p_q v^q,$$

where we have to replace the dummy indices q and p by p and q, respectively.

1.1.3 Summation Convention

Let us consider the sum $\sum\limits_{i=1}^{N} \sum\limits_{j=1}^{N} a_{ij} x^i x^j$. In order to avoid such awkward way of expression using the sigmas $(\sum s)$ we shall make use of a convention used by Einstein, in his development of the theory of relativity, which is accordingly called Einstein summation convention. Instead of using the traditional sigma for sums, the strategy is to allow the repeated subscript to become itself the designation for the summation. If in an indexed expression a dummy index occurs, then the expression has to be considered as the sum with respect to that index over the prescribed range. Thus the expression

$$a_1x^1 + a_2x^2 + a_3x^3 + \cdots + a_Nx^N = \sum_{i=1}^{N} a_ix^i$$

will be written in summation convention as a_ix^i in N dimensions. Similarly, according to this convention $\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}x^ix^j$ will be written as $a_{ij}x^ix^j$ in N dimensional space.

Therefore, according to this convention, if an index is repeated in a term, summation over it from 1 to N is implied.

Nonidentities: The following nonidentities should be carefully noted

$$a_{ij}\left(x^i + y^j\right) \neq a_{ij}x^i + a_{ij}y^j; \quad a_{ij}x^iy^j \neq a_{ij}y^ix^j; \quad (a_{ij} + a_{ji})x^iy^j \neq 2a_{ij}x^iy^j.$$

Valid identities: The following identities should be carefully noted:

$$a_{ij}\left(x^j + y^j\right) = a_{ij}x^j + a_{ij}y^j; \quad a_{ij}x^iy^j = a_{ij}y^jx^i$$

$$a_{ij}x^ix^j = a_{ji}x^ix^j; \quad (a_{ij} + a_{ji})x^ix^j = 2a_{ij}x^ix^j; \quad (a_{ij} - a_{ji})x^ix^j = 0.$$

EXAMPLE 1.1.2 *Express the sum* $\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}u^iv^j$ *by using summation convention and hence find all the terms of the sum in which each of the indices takes values from 1 to 3.*

Solution: Since each of the indices i, j occurs twice, once as lower index and again as upper index, the required expression is $a_{ij}u^iv^j$. Since both the indices are dummy,

$$\begin{aligned}
a_{ij}u^iv^j &= \sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij}u^iv^j = \sum_{i=1}^{3}\left(a_{i1}u^iv^1 + a_{i2}u^iv^2 + a_{i3}u^iv^3\right) \\
&= \sum_{i=1}^{3} a_{i1}u^iv^1 + \sum_{i=1}^{3} a_{i2}u^iv^2 + \sum_{i=1}^{3} a_{i3}u^iv^3 \\
&= \left(a_{11}u^1v^1 + a_{21}u^2v^1 + a_{31}u^3v^1\right) + \left(a_{12}u^1v^2 + a_{22}u^2v^2 + a_{32}u^3v^2\right) \\
&\quad + \left(a_{13}u^1v^3 + a_{23}u^2v^3 + a_{33}u^3v^3\right) \\
&= a_{11}u^1v^1 + a_{22}u^2v^2 + a_{33}u^3v^3 + a_{12}u^1v^2 + a_{21}u^2v^1 \\
&\quad + a_{13}u^1v^3 + a_{31}u^3v^1 + a_{23}u^2v^3 + a_{32}u^3v^2.
\end{aligned}$$

EXAMPLE 1.1.3 *If the* a_{ij} *are constants, calculate* $\frac{\partial}{\partial x^k}\left(a_{ij}x^ix^j\right)$.

Solution: Returning to Einstein summation convention, we have

$$\sum_{i,j} a_{ij}x^i x^j = \sum_{i\neq k, j\neq k} a_{ij}x^i x^j + \sum_{i=k, j\neq k} a_{ij}x^i x^j + \sum_{i\neq k, j=k} a_{ij}x^i x^j + \sum_{i=k, j=k} a_{ij}x^i x^j$$
$$= C + \left(\sum_{j\neq k} a_{kj}x^j\right) x^k + \left(\sum_{i\neq k} a_{ik}x^i\right) x^k + a_{kk}(x^k)^2,$$

where C is independent of x^k. Differentiating with respect to x^k,

$$\frac{\partial}{\partial x^k}\left(\sum_{i,j} a_{ij}x^i x^j\right) = 0 + \sum_{j\neq k} a_{kj}x^j + \sum_{i\neq k} a_{ik}x^i + 2a_{kk}x^k$$
$$= \sum_j a_{kj}x^j + \sum_i a_{ik}x^i = a_{ki}x^i + a_{ik}x^i = (a_{ik} + a_{ki})\, x^i,$$

where we are going back to the Einstein summation convention.

Further, if $a_{ij} = a_{ji}$ are constants then

$$\frac{\partial^2}{\partial x^k \partial x^l}\left(a_{ij}x^i x^j\right) = \frac{\partial}{\partial x^k}\left[\frac{\partial}{\partial x^l}\left(a_{ij}x^i x^j\right)\right] = \frac{\partial}{\partial x^k}\left[(a_{lj} + a_{jl})\, x^j\right]$$
$$= \frac{\partial}{\partial x^k}\left[2a_{il}x^i\right] = 2a_{il}\delta^i_k = 2a_{kl}.$$

1.1.4 Kronecker Delta

A particular system of second order, denoted by $\delta^i_j;\ \ i,j = 1,2,\ldots,N$ which is defined as follows:

$$\delta_{ij} = \delta^i_j = \delta^{ij} = \begin{cases} 1\ ; & \text{for } i = j \\ 0\ ; & \text{for } i \neq j. \end{cases} \tag{1.1}$$

Such a system is called a *Kronecker symbol.* It is also called a Kronecker delta.

Property 1.1.1 If the coordinates $x^1, x^2, \ldots, x^N$ are independent, then

$$\frac{\partial x^i}{\partial x^j} = \begin{cases} 1\ ; & \text{for } i = j \\ 0\ ; & \text{for } i \neq j. \end{cases}$$

This implies that, if x^i and x^j belong to the same co-ordinate system and independent, then $\frac{\partial x^i}{\partial x^j} = \delta^i_j$. It is also written as

$$\frac{\partial x^i}{\partial x^j} = \frac{\partial x^i}{\partial x^k}\,\frac{\partial x^k}{\partial x^j} = \delta^i_j$$

where, as per convention, summation over k is implied.

Property 1.1.2 Following Einstein summation convention, we get

$$\delta_i^i = \delta_1^1 + \delta_2^2 + \cdots + \delta_N^N = 1 + 1 + \cdots + 1(N \text{ times}) = N.$$

Property 1.1.3 According to the Einstein summation convention, we get

$$\begin{aligned}\delta_j^1 A^{jk} &= \delta_1^1 A^{1k} + \delta_2^1 A^{2k} + \delta_3^1 A^{3k} + \cdots + \delta_N^1 A^{Nk} \\ &= A^{1k} + 0 + 0 + \cdots + 0 = A^{1k} \\ \delta_j^2 A^{jk} &= A^{2k}, \quad \delta_j^3 A^{jk} = A^{3k}, \ldots, \delta_j^N A^{jk} = A^{Nk}.\end{aligned}$$

Generalising this we obtain $\delta_j^i A^{jk} = A^{ik}$. Also, $\delta_k^j A^{ik} = A^{jk}$, in which in the expression A^{jk} we replace the index j by the i. Similarly, $\delta_j^i A_i^k = A_j^k$. Thus the symbol δ_j^i allows us to replace one index by another, for this reason, the symbol δ_j^i is sometimes called the *substitution operator.*

Property 1.1.4 Using the Einstein summation convention, we get

$$\begin{aligned}\delta_j^i \delta_k^j &= \delta_1^i \delta_k^1 + \delta_2^i \delta_k^2 + \cdots + \delta_i^i \delta_k^i + \cdots + \delta_N^i \delta_k^N \\ &= 0\delta_k^1 + 0\delta_k^2 + \cdots + 1\delta_k^i + \cdots + 0\delta_k^N = \delta_k^i.\end{aligned}$$

Using definition of δ_j^i we get,

$$\delta_j^i \delta_k^j = \frac{\partial x^i}{\partial x^j}\frac{\partial x^j}{\partial x^k} = \frac{\delta x^i}{\delta x^k} = \delta_k^i.$$

In particular, when $k = i$ we get

$$\delta_j^i \delta_i^j = \delta_i^i = \delta_1^1 + \delta_2^2 + \cdots + \delta_N^N = N.$$

EXAMPLE 1.1.4 *If $A^i = g_r^i a_{rs} y_s, y_i = b_{ir} x_r$ and $a_{ir} b_{rj} = \delta_{ij}$, find A^i in terms of the x_r.*

Solution: First write, $y_s = b_{st} x_t$. Then by substitution,

$$A^i = g_r^i a_{rs} b_{st} x_t = g_r^i \delta_{rt} x_t = g_r^i x_r.$$

EXAMPLE 1.1.5 *Evaluate $\delta_r^i \delta_j^r$ and $\delta_j^i \delta_l^j \delta_k^l$ the indices take all values from 1 to N.*

Solution: Using Einstein summation convention, we have,

$$\delta_r^i \delta_j^r = \delta_1^i \delta_j^1 + \delta_2^i \delta_j^2 + \cdots + \delta_N^i \delta_j^N.$$

So, according to summation convention,

$$\begin{aligned}\delta_r^1 \delta_j^r &= \delta_1^1 \delta_j^1 + \delta_2^1 \delta_j^2 + \cdots + \delta_N^1 \delta_j^N \\ &= \delta_j^1 + 0 + \cdots + 0 = \delta_j^1; \text{ for all } j.\end{aligned}$$

Similarly, $\delta_r^2 \delta_j^r = \delta_j^2, \ldots, \delta_r^N \delta_j^r = \delta_j^N$. From these equations it follows that, $\delta_r^i \delta_j^r = \delta_j^i$. Using this result, we get

$$\delta_j^i \delta_l^j \delta_k^l = \frac{\delta x^i}{\delta x^j} \frac{\delta x^j}{\delta x^l} \frac{\delta x^l}{\delta x^k} = \frac{\delta x^i}{\delta x^k} = \delta_k^i.$$

EXAMPLE 1.1.6 *Calculate $\delta_{ij} x_i x_j$ for $N = 3$.*
Solution: For $N = 3$, we have

$$\begin{aligned} \delta_{ij} x_i x_j &= 1 x_1 x_1 + 1 x_2 x_2 + 1 x_3 x_3 \\ &= (x_1)^2 + (x_2)^2 + (x_3)^2. \end{aligned}$$

In general, $\delta_{ij} x_i x_j = x_i x_i$ and $\delta_j^r a_{ir} x_i = a_{ij} x_i$.

1.1.5 Manifolds and Tensors

It is hard to imagine a physical problem which does not involve some sort of continuous space. It might be physical three-dimensional space, four-dimensional space time, phase space for a problem in classical or quantum mechanics, the space of all thermodynamic equilibrium states, or some still more abstract space.

For instance, in dealing with the states of gas determined by the pressure (p), the volume (v), the temperature (T) and the time (t), one may wish to co-ordinate these entities with ordered set of four real numbers (x_1, x_2, x_3, x_4). Here the diagrammatic representation of the states of gas by points in the physical space is already impossible. However, the essential idea in the concept of co-ordinate system is not a pictorial representation but the one-to-one reciprocal correspondence of objects with sets of numbers.

All these spaces have different geometrical properties, but they all share something in common, something which has to do with their being continuous spaces rather than, say, lattices of discrete points. The key to differential geometry's importance to modern physics is that it studies precisely those properties common to all such spaces. The most basic of these properties go into the definition of the differentiable manifold, which is the mathematically precise substitute for the word 'space'.

Definition 1.1.1 Let us denote $\Re^n$ by the set of all n-tuples of real numbers $(x^1, x^2, \ldots, x^n)$. A set of 'points' M is defined to be a manifold if

(i) each point of M has an open neighbourhood and
(ii) has a continuous one-to-one map onto an open set of $\Re^n$, for some n.

Clearly the dimension of M is n. By definition, the map associates with a point P of M an n-tuple $\left[x^1(P), x^2(P), \ldots, x^n(P)\right]$. These numbers $x^1(P), x^2(P), \ldots, x^n(P)$ are called the co-ordinates of P under this map. Then an n dimensional manifold is that it is simply any set which can be given n independent co-ordinates in some neighbourhood of any point, since these co-ordinates actually define the required map to $\Re^n$.

The usefulness of the concept of a manifold really comes from its generality, the fact that it embraces sets which one might not ordinarily regard as spaces. By definition, any set that can be parameterized continuously is a manifold whose dimension is the number of independent parameters. For example,

(i) The set of all rotation of a rigid object in three dimensions is a manifold, since it can be continuously parameterized by the three 'Euler angles'.

(ii) The set of all Lorentz transformation is likewise a three dimensional manifold; the parameters are the three components of the velocity of the boost.

(iii) For N particles the numbers consisting of all their positions (3N numbers) and velocities (3N numbers) define a point in a 6N-dimensional manifold, called phase space.

(iv) Given an equation (algebraic or differential) for a dependent variable y in terms of an independent variable x, one can define the set of all (y, x) to be a manifold; any particular solution is a curve in this manifold. This concept is easily extended to arbitrary numbers of dependent and independent variables.

A particular set of n real numbers $(x_0^1, x_0^2, \dots, x_0^n)$ can be thought to specify a point P_0 in the n-dimensional metric manifold covered by a co-ordinate system (x^i).

Note that, the map is only required to be one-to-one, not to preserve lengths or angles or any other geometrical notion. Indeed, the idea of distance becomes devoid to geometrical sense even in familiar representation of the states of gas (the pressure p and the volume v) by points in the Cartesian pv-plane. It is manifestly absurd to speak of the distance between two states characterised by ordered pairs of numbers (p, v). Thus, length is not defined at this level of geometry, and, we shall encounter physical applications in which we will not want to introduce a notion of distance between two points of our manifolds.

1.2 Transformation of Co-ordinates

We consider a transformation from one co-ordinate system $\left(x^1, x^2, \dots, x^N\right)$ to another co-ordinate system $\left(\overline{x}^1, \overline{x}^2, \dots, \overline{x}^N\right)$ in the same space V_N, related by the N equations

$$T : \overline{x}^i = \phi^i \left(x^1, x^2, \dots, x^N\right); i = 1, 2, \dots, N \tag{1.2}$$

where ϕ^i are single valued, continuous functions of co-ordinates $x^1, x^2, \dots, x^N$ and have continuous partial derivatives up to any desired order and further the determinant

$$J = \begin{vmatrix} \dfrac{\partial \phi^1}{\partial x^1} & \dfrac{\partial \phi^1}{\partial x^2} & \cdots & \dfrac{\partial \phi^1}{\partial x^N} \\ \dfrac{\partial \phi^2}{\partial x^1} & \dfrac{\partial \phi^2}{\partial x^2} & \cdots & \dfrac{\partial \phi^2}{\partial x^N} \\ \vdots & \vdots & \vdots & \vdots \\ \dfrac{\partial \phi^N}{\partial x^1} & \dfrac{\partial \phi^N}{\partial x^2} & \cdots & \dfrac{\partial \phi^N}{\partial x^N} \end{vmatrix} \neq 0. \tag{1.3}$$

This determinant (1.3) is called the Jacobian of transformation (1.2) and is denoted by $\left|\dfrac{\partial \phi^i}{\partial x^i}\right|, \left|\dfrac{\partial \overline{x}^i}{\partial x^i}\right|$ or $\left|\dfrac{\partial \overline{x}}{\partial x}\right|$. A well known theorem from analysis states that T is locally bijective on an open set in $\Re^N$ if and only if $J \neq 0$ at each point on the open set. In virtue of (1.3) the functions ϕ^i are independent and the Eq. (1.2) can be solved for the x^i as functions of $\overline{x}^i$ giving

$$T^{-1} : x^i = \psi^i\left(\overline{x}^1, \overline{x}^2, \ldots, \overline{x}^N\right); \quad i = 1, 2, \ldots, N. \tag{1.4}$$

where the functions $\psi^i(\overline{x}^1, \overline{x}^2, \ldots, \overline{x}^N)$ are single valued.

The relations (1.2) and (1.4) are called formulas of transformation of co-ordinates of V_N. They help to determine the co-ordinates of any point of V_N with respect to one co-ordinate system when the co-ordinates of the same point with respect to another co-ordinate system are known. We shall refer to a class of co-ordinate transformations with this properties as *admissible transformations.* Below are some examples of admissible transformations of co-ordinates.

(i) Consider a system of equations specifying the relation between the spherical polar co-ordinates x^i and the rectangular Cartesian co-ordinates $\overline{x}^i$ in E^3 (three-dimensional Euclidean space),

$$T : \overline{x}^1 = x^1 \sin x^2 \cos x^3; \quad \overline{x}^2 = x^1 \sin x^2 \sin x^3; \quad \overline{x}^3 = x^1 \cos x^2$$

where, $x^1 > 0, 0 < x^2 < \pi, 0 \leq x^3 < 2\pi$. The Jacobian of this transformation is given by

$$J = \begin{vmatrix} \dfrac{\partial \overline{x}^1}{\partial x^1} & \dfrac{\partial \overline{x}^1}{\partial x^2} & \dfrac{\partial \overline{x}^1}{\partial x^3} \\ \dfrac{\partial \overline{x}^2}{\partial x^1} & \dfrac{\partial \overline{x}^2}{\partial x^2} & \dfrac{\partial \overline{x}^2}{\partial x^3} \\ \dfrac{\partial \overline{x}^3}{\partial x^1} & \dfrac{\partial \overline{x}^3}{\partial x^2} & \dfrac{\partial \overline{x}^3}{\partial x^3} \end{vmatrix} = \left(x^1\right)^2 \sin x^2 \neq 0.$$

The inverse transformation is given by

$$T^{-1}: \begin{cases} x^1 = \sqrt{\left(\overline{x}^1\right)^2 + \left(\overline{x}^2\right)^2 + \left(\overline{x}^3\right)^2} \\ x^2 = \cos^{-1} \frac{\overline{x}^3}{\sqrt{\left(\overline{x}^1\right)^2+\left(\overline{x}^2\right)^2+\left(\overline{x}^3\right)^2}} = \tan^{-1} \frac{\sqrt{\left(\overline{x}^1\right)^2+\left(\overline{x}^2\right)^2}}{\overline{x}^3} \\ x^3 = \tan^{-1} \frac{\overline{x}^2}{\overline{x}^1}. \end{cases}$$

(ii) In $\Re^2$, let a curvilinear co-ordinate system $(\overline{x}^i)$ be defined from rectangular co-ordinates (x^i) by the equation

$$T : \overline{x}^1 = x^1 x^2; \;\; \overline{x}^2 = \left(x^2\right)^2$$

so

$$J = \begin{vmatrix} x^2 & x^1 \\ 0 & 2x^2 \end{vmatrix} = 2\left(x^2\right)^2 \neq 0.$$

Thus the curvilinear co-ordinates are admissible for the region $x^2 > 0$ and $x^2 < 0$ (both open sets in a plane).

(iii) The relation between the cylindrical co-ordinates y^i and the rectangular Cartesian co-ordinates x^i is given by

$$x^1 = y^1 \cos y^2; \quad x^2 = y^1 \sin y^2; \quad x^3 = y^3,$$

where, $y^1 \geq 0$, $0 \leq y^2 < 2\pi$, $-\infty < y^3 < \infty$.

(iv) The relation between the parabolic cylindrical co-ordinates y^i and the rectangular Cartesian co-ordinates x^i is given by

$$x^1 = \frac{1}{2}\left[(y^1)^2 - (y^2)^2\right]; \quad x^2 = y^1 y^2; \quad x^3 = y^3,$$

where, $-\infty < y^1 < \infty$, $y^2 \geq 0$, $-\infty < y^3 < \infty$.

(v) The relation between the elliptic cylindrical co-ordinates y^i and the rectangular Cartesian co-ordinates x^i is given by

$$x^1 = a \cosh y^1 \cos y^2; \quad x^2 = a \sinh y^1 \sin y^2; \quad x^3 = y^3,$$

where, $y^1 \geq 0$, $0 \leq y^2 < 2\pi$, $-\infty < y^3 < \infty$.

(vi) The relation between the paraboloidal co-ordinates y^i and the rectangular Cartesian co-ordinates x^i is given by

$$x^1 = y^1 y^2 \cos y^3; \quad x^2 = y^1 y^2 \sin y^3; \quad x^3 = \frac{1}{2}\left[(y^1)^2 - (y^2)^2\right],$$

where, $y^1 \geq 0$, $y^2 \geq 0$, $0 \leq y^3 < 2\pi$.

The components of a tensor change under transformation of co-ordinates but the entity called a tensor does not change under co-ordinate transformation. If, starting from a system of co-ordinates x^i, we allow only transformations which are expressed by function which have derivatives up to any order, and can be solved, these new co-ordinates will be called allowable co-ordinates.

Below are some properties of admissible transformations of co-ordinates.

Property 1.2.1 If a transformation (1.2) of co-ordinates possesses an inverse transformation (1.4) with respective Jacobians J and K, respectively, then $JK = 1$.

Proof: Since $\overline{x}^i$s are independent and x^is are also independent functions of $\overline{x}^i$s, by the formula of partial differentiation and summation convention we can write

$$\frac{\partial \overline{x}^i}{\partial \overline{x}^j} = \frac{\partial \overline{x}^i}{\partial x^k}\frac{\partial x^k}{\partial \overline{x}^j}; \quad k = 1, 2, \ldots, N$$

or

$$\overline{\delta}^i_j = \delta^i_j = \frac{\partial \overline{x}^i}{\partial x^k} \; \frac{\partial x^k}{\partial \overline{x}^j}$$

or

$$\left|\delta^i_j\right| = \left|\frac{\partial \overline{x}^i}{\partial x^k}\right| \; \left|\frac{\partial x^k}{\partial \overline{x}^j}\right|$$

or

$$1 = \left|\frac{\partial \overline{x}^i}{\partial x^k}\right| \; \left|\frac{\partial x^k}{\partial \overline{x}^j}\right|; \text{ as } \left|\delta^i_j\right| = 1$$

or

$$1 = JK; \;\; J = \left|\frac{\partial \overline{x}^i}{\partial x^k}\right| \text{ and } K = \left|\frac{\partial x^k}{\partial \overline{x}^j}\right|.$$

Incidentally it follows from this result that $J \neq 0$.

Property 1.2.2 The Jacobian of the product transformation is equal to the product of the Jacobians of transformations entering in the product.

Proof: Let us consider any two admissible transformations

$$T_1 : \overline{x}^i = \overline{x}^i\left(x^1, x^2, \ldots, x^N\right); \;\; T_2 : \overline{\overline{x}}^i = \overline{\overline{x}}^i\left(\overline{x}^1, \overline{x}^2, \ldots, \overline{x}^N\right),$$

where $i = 1, 2, \ldots, N$. The transformation $T_3 : x^i \to \overline{\overline{x}}^i$ defined by

$$T_3 : \overline{\overline{x}}^i = \overline{\overline{x}}^i\left[\overline{x}^1\left(x^1, x^2, \ldots, x^N\right), \ldots, \overline{x}^N\left(x^1, x^2, \ldots, x^N\right)\right]$$

is called the product of T_2 and T_1 and we write $T_3 = T_2 T_1$ (Figure 1.1). If the Jacobian of T_3 is denoted by J_3, it follows that:

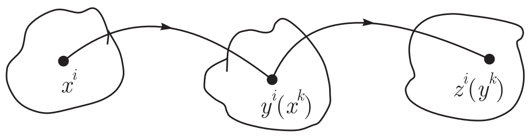

Figure 1.1: The transformation.

$$J_3 = \left| \frac{\partial \overline{\overline{x}}^i}{\partial \overline{x}^k} \; \frac{\partial \overline{x}^k}{\partial x^j} \right| = \left| \frac{\partial \overline{\overline{x}}^i}{\partial \overline{x}^k} \right| \; \left| \frac{\partial \overline{x}^k}{\partial x^j} \right| = J_2 J_1,$$

where J_2 and J_1 are the Jacobians of T_2 and T_1, respectively.

Property 1.2.3 The set of all admissible transformations of co-ordinates forms a group.

Proof: The set of all admissible transformations of co-ordinates forms a group if the following four axioms are satisfied:

(i) The product of two admissible transformations is a transformation belonging to the set of admissible transformations. This property is known as the property of closure.

(ii) The product transformation possesses an inverse, since the transformations appearing in the product have inverses.

(iii) The identity transformation $\left(\overline{x}^i = x^i\right)$ obviously exists.

(iv) The associative law $T_3\left(T_2T_1\right) = \left(T_3T_2\right)T_1$ obviously holds.

These properties are precisely the ones entering in the definition of an abstract group. The fact that admissible transformations forms a group justifies us in choosing as a point of departure any convenient co-ordinate system, as long as it is one of those admitted in the set.

EXAMPLE 1.2.1 *Find Cartesian co-ordinates of a point whose cylindrical co-ordinates are* $\left(4, \frac{\pi}{3}, 2\right)$.

Solution: The relation between the cylindrical co-ordinates y^i and the rectangular Cartesian co-ordinates x^i is given by

$$x^1 = y^1 \cos y^2; \quad x^2 = y^1 \sin y^2; \quad x^3 = y^3.$$

Here, $y^1 = 4,\ y^2 = \frac{\pi}{3}$ and $y^3 = 2$. Therefore,

$$x^1 = 4\cos\frac{\pi}{3} = 4 \cdot \frac{1}{2} = 2; \quad x^2 = 4\sin\frac{\pi}{3} = 4 \cdot \frac{\sqrt{3}}{2} = 2\sqrt{3} \text{ and } x^3 = 2.$$

Thus, the required co-ordinates are $(2, 2\sqrt{3}, 2)$.

EXAMPLE 1.2.2 *Find cylindrical co-ordinates of a point whose spherical co-ordinates are* $\left(4, \frac{\pi}{2}, \frac{\pi}{3}\right)$.

Solution: The relation between the rectangular Cartesian co-ordinates y^i and the spherical co-ordinates x^i is given by

$$y^1 = x^1 \sin x^2 \cos x^3; \quad y^2 = x^1 \sin x^2 \sin x^3; \quad y^3 = x^1 \cos x^2.$$

Here, $x^1 = 4,\ x^2 = \frac{\pi}{2},\ x^3 = \frac{\pi}{3}$. Therefore,

$$y^1 = 4 \sin \frac{\pi}{2} \cos \frac{\pi}{3} = 4 \cdot 1 \cdot \frac{1}{2} = 2$$
$$y^2 = 4 \sin \frac{\pi}{2} \sin \frac{\pi}{3} = 4 \cdot 1 \cdot \frac{\sqrt{3}}{2} = 2\sqrt{3}; \quad y^3 = 4 \cos \frac{\pi}{2} = 0.$$

Thus, Cartesian co-ordinates of the point $\left(4, \frac{\pi}{2}, \frac{\pi}{3}\right)$ are $(2, 2\sqrt{3}, 0)$. Now we are to find the cylindrical co-ordinates of the point, whose Cartesian co-ordinates are $(2, 2\sqrt{3}, 0)$.

The relation between the cylindrical co-ordinates y^i and the rectangular Cartesian co-ordinates x^i is given by $x^1 = y^1 \cos y^2;\ x^2 = y^1 \sin y^2;\ x^3 = y^3$. Here, $x^1 = 2, x^2 = 2\sqrt{3}, x^3 = 0$, therefore,

$$2 = y^1 \cos y^2; \quad 2\sqrt{3} = y^1 \sin y^2$$
$$\Rightarrow 2^2 + (2\sqrt{3})^2 = \left(y^1 \cos y^2\right)^2 + \left(y^1 \sin y^2\right)^2$$
$$\Rightarrow (y^1)^2 = 16 \Rightarrow y^1 = 4.$$

and

$$\tan y^2 = \sqrt{3} \Rightarrow y^2 = \frac{\pi}{3}.$$

Thus, the required co-ordinates are $(4, \frac{\pi}{3}, 0)$.

EXAMPLE 1.2.3 *Find spherical co-ordinates of a point whose cylindrical co-ordinates are* $\left(2\sqrt{2}, \frac{\pi}{4}, 1\right)$.

Solution: The relation between the cylindrical co-ordinates y^i and the rectangular Cartesian co-ordinates x^i is given by $x^1 = y^1 \cos y^2;\ x^2 = y^1 \sin y^2;\ x^3 = y^3$. Here, $y^1 = 2\sqrt{2}, y^2 = \frac{\pi}{4}, y^3 = 1$, therefore,

$$x^1 = 2\sqrt{2} \cos \frac{\pi}{4} = 2\sqrt{2} \cdot \frac{1}{\sqrt{2}} = 2$$
$$x^2 = 2\sqrt{2} \sin \frac{\pi}{4} = 2\sqrt{2} \cdot \frac{1}{\sqrt{2}} = 2 \text{ and } x^3 = 1.$$

Thus, Cartesian co-ordinates of the point $\left(2\sqrt{2}, \frac{\pi}{4}, 1\right)$ are $(2, 2, 1)$. Now, we are to find the spherical co-ordinates of the point, whose rectangular co-ordinates are $(2, 2, 1)$.

The relation between the spherical co-ordinates y^i and the rectangular Cartesian co-ordinates x^i is given by

$$y^1 = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2} = \sqrt{4+4+1} = 3$$

$$y^2 = \cos^{-1} \frac{x^3}{\sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}} = \cos^{-1} \frac{1}{\sqrt{4+4+1}} = \cos^{-1} \frac{1}{3}$$

$$y^3 = \tan^{-1} \frac{x^2}{x^1} = \tan^{-1} 1 = \frac{\pi}{4}.$$

Thus, the required co-ordinates are $\left(3, \cos^{-1} \frac{1}{3}, \frac{\pi}{4}\right)$.

EXAMPLE 1.2.4 *Show that the equation* $x^1 = 4\cos x^2$ *in spherical co-ordinates represents a sphere.*

Solution: The relation between the Cartesian co-ordinates y^i and the spherical co-ordinates is given by

$$y^1 = x^1 \sin x^2 \cos x^3; \quad y^2 = x^1 \sin x^2 \sin x^3; \quad y^3 = x^1 \cos x^2.$$

Then,

$$x^1 = \sqrt{(y^1)^2 + (y^2)^2 + (y^3)^2}$$

and

$$x^2 = \cos^{-1} \left[\frac{y^3}{(y^1)^2 + (y^2)^2 + (y^3)^2} \right]$$

The given equation can therefore be written as

$$\sqrt{(y^1)^2 + (y^2)^2 + (y^3)^2} = 4 \frac{y^3}{\sqrt{(y^1)^2 + (y^2)^2 + (y^3)^2}}$$

or

$$(y^1)^2 + (y^2)^2 + (y^3 - 2)^2 = 4$$

which represents the equation of a sphere.

1.3 e-Systems

In this section we shall define two completely skew-symmetric systems (of functions) $e_{\alpha\beta}$ and $e^{\alpha\beta}$ explicitly. The second order e-system e_{ij} or e^{ij} are defined by

$$\left. \begin{array}{l} e_{11} = 0,\ e_{22} = 0,\ e_{12} = 1,\ e_{21} = -1 \\ e^{11} = 0,\ e^{22} = 0,\ e^{12} = 1,\ e^{21} = -1 \end{array} \right\}. \tag{1.5}$$

The covariant ε tensor of second order is defined by

$$\varepsilon_{11} = 0,\ \varepsilon_{12} = \sqrt{g},\ \varepsilon_{21} = -\sqrt{g},\ \varepsilon_{22} = 0,\ g = |g_{ij}|. \tag{1.6}$$

This tensor is skew-symmetric. Accordingly by tensor law of transformation,

$$\overline{\varepsilon}_{ij} = \frac{\partial u^p}{\partial \overline{u}^i}\frac{\partial u^q}{\partial \overline{u}^j}\varepsilon_{pq}$$

or

$$\overline{\varepsilon}_{ij} = \frac{\partial u^1}{\partial \overline{u}^i}\frac{\partial u^2}{\partial \overline{u}^j}\varepsilon_{12} + \frac{\partial u^2}{\partial \overline{u}^i}\frac{\partial u^1}{\partial \overline{u}^j}\varepsilon_{21};\ \text{as } \varepsilon_{11} = 0, \varepsilon_{22} = 0$$

or

$$\overline{\varepsilon}_{ij} = \sqrt{g}\begin{vmatrix} \dfrac{\partial u^1}{\partial \overline{u}^i} & \dfrac{\partial u^2}{\partial \overline{u}^i} \\ \dfrac{\partial u^1}{\partial \overline{u}^j} & \dfrac{\partial u^2}{\partial \overline{u}^j} \end{vmatrix} = \sqrt{g}\ \frac{\partial(u^1, u^2)}{\partial(\overline{u}^i, \overline{u}^j)}.$$

Therefore, $\overline{\varepsilon}_{11} = \overline{\varepsilon}_{22} = 0$ and

$$\overline{\varepsilon}_{12} = \sqrt{g}\frac{\partial(u^1, u^2)}{\partial(\overline{u}^1, \overline{u}^2)} = \sqrt{g};\ \overline{\varepsilon}_{21} = -\sqrt{g}\frac{\partial(u^1, u^2)}{\partial(\overline{u}^1, \overline{u}^2)} = -\sqrt{g}.$$

Let us consider ε^{ij} as

$$\varepsilon^{ij} = \varepsilon_{rs}g^{ir}g^{js} \tag{1.7}$$

which is called the contravariant ε tensor of second order. Since $\varepsilon_{11} = \varepsilon_{22} = 0$, we find

$$\varepsilon^{ij} = \sqrt{g}\left[g^{i1}g^{j2} + g^{i2}g^{j1}\right]$$

and therefore, $\varepsilon^{11} = \varepsilon^{22} = 0$ and

$$\varepsilon^{12} = \sqrt{g}\left[g^{11}g^{22} - (g^{12})^2\right] = \frac{1}{\sqrt{g}} \text{ and } \varepsilon^{21} = -\frac{1}{\sqrt{g}}.$$

Therefore, the contravariant ε tensor has the components

$$\varepsilon^{11} = 0;\quad \varepsilon^{12} = \frac{1}{\sqrt{g}};\quad \varepsilon^{21} = -\frac{1}{\sqrt{g}};\quad \varepsilon^{22} = 0$$

Similarly, we get ε^{ij} as

$$\varepsilon^{ij} = \frac{1}{\sqrt{g}}e^{ij}. \tag{1.8}$$

The third order e-system e_{ijk} or e^{ijk} known as permutation symbols in three-dimensional space are defined by

$$\left.\begin{aligned} e_{123} = e_{231} = e_{312} = 1;\quad e_{213} = e_{321} = e_{132} = -1 \\ e^{123} = e^{231} = e^{312} = 1;\quad e^{213} = e^{321} = e^{132} = -1 \end{aligned}\right\} \tag{1.9}$$

and the remaining 21 components are zero. Further, let us define

$$\varepsilon_{ijk} = \frac{1}{\sqrt{g}}\, e_{ijk};\ \varepsilon^{ijk} = \sqrt{g}\, e^{ijk};\quad g = |g_{ij}| \tag{1.10}$$

which are, respectively, the covariant and contravariant tensors, are called *permutation tensors* in three-dimensional space. Thus, the *permutation tensor* ϵ_{ijk} is given by

$$\epsilon_{ijk} = \begin{cases} +1; & \text{when } i,j,k \text{ are in even permutation of } 1,2,3 \\ 0; & \text{when any two of the indices } i,j,k \text{ are equal} \\ -1; & \text{when } i,j,k \text{ are in odd permutation of } 1,2,3. \end{cases}$$

These are also called fully antisymmetric tensor of rank 3 with 3! non-vanishing components. We shall now establish some results using e-systems of second and third orders.

Property 1.3.1 e-systems of second order: According to the definition of determinants,

$$\left|a^i_j\right| = \begin{vmatrix} a^1_1 & a^1_2 \\ a^2_1 & a^2_2 \end{vmatrix} = a^1_1 a^2_2 - a^1_2 a^2_1$$

$$= e_{12} a^1_1 a^2_2 + e_{21} a^2_1 a^1_2;\quad \text{as } e_{12} = 1, e_{21} = -1$$

or

$$\left|a^i_j\right| = e_{ij} a^i_1 a^j_2 \quad \text{(by summation convention).} \tag{1.11}$$

Similarly, it can be shown that $\left|a^i_j\right| = e^{ij} a^1_i a^2_j$. Let us now consider the expression $e_{ij} a^i_p a^j_q$ where the indices p and q are free and can be assigned values 1 and 2 at will. We have

$$e_{ij} a^i_1 a^j_2 = e_{12} a^1_1 a^2_2 + e_{21} a^2_1 a^1_2 = e_{12} a^1_1 a^2_2 - e_{12} a^2_1 a^1_2 = e_{12}\left|a^i_j\right|.$$

Similarly, it can be shown that $e_{ij} a^i_2 a^j_1 = e_{21}\left|a^i_j\right|$. From this two results we can write

$$e_{ij} a^i_p a^j_q = e_{pq}\left|a^i_j\right| \Rightarrow \left|a^i_j\right| e_{pq} = e_{ij} a^i_p a^j_q. \tag{1.12}$$

Similarly, it can be shown that

$$\left|a^i_j\right| e^{pq} = e^{ij} a^p_i a^q_j. \tag{1.13}$$

Property 1.3.2 ***e*-system of third order:** If we take,

$$|a^i_j| = \begin{vmatrix} a^1_1 & a^1_2 & a^1_3 \\ a^2_1 & a^2_2 & a^2_3 \\ a^3_1 & a^3_2 & a^3_3 \end{vmatrix},$$

then, by the similar arguments as in property 1.3.1, the following results hold:

$$\left.\begin{aligned} \left|a^i_j\right| &= e_{ijk}a^i_1 a^j_2 a^k_3 \\ \left|a^i_j\right| &= e^{ijk}a^1_i a^2_j a^3_k \\ \left|a^i_j\right| e_{pqr} &= e_{ijk}a^i_p a^j_q a^k_r \\ \left|a^i_j\right| e^{pqr} &= e^{ijk}a^p_i a^q_j a^r_k \end{aligned}\right\}. \tag{1.14}$$

The above definitions of e-systems of second and third order can obviously be extended to define e-systems of nth order $e_{i_1,i_2,\ldots,i_n}$ and $e^{i_1,i_2,\ldots,i_n}$ involving n indices. If

$$|a^i_j| = \begin{vmatrix} a^1_1 & a^1_2 & \cdots & a^1_N \\ a^2_1 & a^2_2 & \cdots & a^2_N \\ \vdots & \vdots & \vdots & \vdots \\ a^N_1 & a^N_2 & \cdots & a^N_N \end{vmatrix}$$

Then results analogous to (1.14) will hold.

Property 1.3.3 The product of e^{ij} and e_{pq} is called the generalised Kronecker delta and is denoted by δ^{ij}_{pq}, i.e.

$$\delta^{ij}_{pq} = e^{ij}e_{pq}. \tag{1.15}$$

Similarly, $\delta^{ijk}_{pqr} = e^{ijk}e_{pqr}$. The product $e^{i_1,i_2,\ldots,i_n}e_{j_1,j_2,\ldots,j_n}$ is called the generalised Kronecker delta is denoted by $\delta^{i_1,i_2,\ldots,i_n}_{j_1,j_2,\ldots,j_n}$. It is to be noted that

$$e^{ijk}e_{pqk} = \delta^{ij}_{pq} \quad \text{and} \quad e^{ijk}e_{pjk} = 2\delta^i_p. \tag{1.16}$$

EXAMPLE 1.3.1 *Evaluate $e_{ij}e^{ik}$, in e-systems of the second order, if $i, j = 1, 2$.*

Solution: According to the Einstein summation convention, we get

$$e_{ij}e^{ik} = e_{1j}e^{1k} + e_{2j}e^{2k}.$$

When, $j = k = 1$ or 2, then

$$\begin{aligned} e_{i1}e^{i1} &= e_{11}e^{11} + e_{21}e^{21} = 1 \cdot 1 + 0 \cdot 0 = 1 \\ e_{i2}e^{i2} &= e_{12}e^{12} + e_{22}e^{22} = 0 \cdot 0 + 1 \cdot 1 = 1. \end{aligned}$$

Thus, when $j = k$, then $e_{ij}e^{ik} = 1$. Again, when $j \neq k$, say $j = 1, k = 2$ or $j = 2, k = 1$, we get

$$\begin{aligned} e_{i1}e^{i2} &= e_{11}e^{12} + e_{21}e^{22} = 1 \cdot 0 + 0 \cdot 1 = 0 \\ e_{i2}e^{i1} &= e_{12}e^{11} + e_{22}e^{21} = 0 \cdot 1 + 1 \cdot 0 = 0. \end{aligned}$$

Thus, when $j \neq k$, then $e_{ij}e^{ik} = 0$. Therefore in general, $e_{ij}e^{ik} = \delta_j^k$. In particular, when $j = k$, we get,

$$e_{ij}e^{ij} = \delta_j^i = \delta_1^1 + \delta_2^2 + \cdots + \delta_N^N = N.$$

EXAMPLE 1.3.2 *Establish the following identity:*

$$e_{rij}e_{rkl} \equiv \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}. \tag{1.17}$$

Solution: The identity implies $n = 3$, so that there are potentially $3^4 = 81$ separate cases to consider. However, this number can be quickly reduced to only four cases as follows:

If either $i = j$ or $k = l$, then both sides vanish. For example, if $i = j$, then the left $e_{rij} = 0$, and on the right

$$\delta_{ik}\delta_{jl} - \delta_{ik}\delta_{jl} = 0.$$

Hence, we need only consider the cases in which both $i \neq j$ and $k \neq l$. Upon writing out the sum on the left, two of the terms drop out, since $i \neq j$,

$$e_{1ij}e_{1kl} + e_{2ij}e_{2kl} + e_{3ij}e_{3kl} = e_{1'2'3'}e_{1'kl}; \quad i = 2', j = 3',$$

where $(1'2'3')$ denotes some permutation of (123). Thus, there are left only two cases, each with two subcases.

Case 1: Let $e_{1'2'3'}e_{1'kl} \neq 0; i = 2', j = 3'$. Here, either $k = 2'$ and $l = 3'$ or $k = 3'$ and $l = 2'$. If the former, then the left member of Eq. (1.17) is +1, while the right member equals

$$\delta_{2'2'}\delta_{3'3'} - \delta_{2'3'}\delta_{3'2'} = 1 - 0 = 1.$$

If the latter, then both members equal to -1, as can be easily verified.

Case 2: Let $e_{1'2'3'}e_{1'kl} = 0; i = 2', j = 3'$. Since, $k \neq l$, either $k = 1'$ and $l = 1'$. If $k = 1'$ then the right member of Eq. (1.17) is

$$\delta_{2'1'}\delta_{3'l} - \delta_{2'l}\delta_{3'1'} = 0 - 0 = 0.$$

If $l = 1'$, we have

$$\delta_{2'k}\delta_{3'1'} - \delta_{2'1'}\delta_{3'k} = 0 - 0 = 0.$$

This completes the examination of all cases, and the identity is established.

1.4 Tensor Notation on Matrices

It is known that if the range of the indices of a system of second order is from 1 to N, then the number of its components is N^2. A system of second order can be of three types, namely, a^i_j, a_{ij} and a^{ij}. By matrices of systems of second order we mean the matrices $\left(a^i_j\right), (a_{ij})$ and $\left(a^{ij}\right)$, i.e.

$$\begin{pmatrix} a^1_1 & a^1_2 & \cdots & a^1_N \\ a^2_1 & a^2_2 & \cdots & a^2_N \\ \vdots & \vdots & \vdots & \vdots \\ a^N_1 & a^N_2 & \cdots & a^N_N \end{pmatrix}, \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{pmatrix} \text{ and } \begin{pmatrix} a^{11} & a^{12} & \cdots & a^{1N} \\ a^{21} & a^{22} & \cdots & a^{2N} \\ \vdots & \vdots & \vdots & \vdots \\ a^{N1} & a^{N2} & \cdots & a^{NN} \end{pmatrix}$$

each of which is an $N \times N$ matrix. The determinants of their matrices are, respectively, denoted by $\left|a^i_j\right|, |a_{ij}|$ and $\left|a^{ij}\right|$. We shall now establish the following results on matrices and determinants of system of second order:

In terms of the Kronecker deltas, the identity matrix of order N is

$$I = (\delta_{ij})_{N\times N} = \left(\delta^i_j\right)_{N\times N} = \left(\delta^{ij}\right)_{N\times N}, \tag{1.18}$$

which has the property $IA = AI = A$ for any square matrix A of order N.

A square matrix $A = (a_{ij})_{N\times N}$ is invertible, if there exists a (unique) matrix $B = (b_{ij})_{N\times N}$, called the *inverse of* A, such that $AB = BA = I$. In terms of components, the criterion reads

$$a_{ir}b_{rj} = b_{ir}a_{rj} = \delta_{ij}; \quad a^i_r b^r_j = b^i_r a^r_j = \delta^i_j; \quad a^{ir}b^{rj} = b^{ir}a^{rj} = \delta^{ij}. \tag{1.19}$$

If $A = (a^i_j)$ and $B = (b^i_j)$ be two matrices conformable for multiplication, then $a^i_j b^j_p = AB$ is the *multiplication of two matrices,* where i and p are not summed on.

Property 1.4.1 If $a^i_j b^j_p = c^i_p$, then $\left(a^i_j\right)\left(b^j_p\right) = \left(c^i_p\right)$ and $\left|a^i_j\right|\left|b^j_p\right| = \left|c^i_p\right|$.

Proof: Since we take a system of second order, so

$$c^i_p = a^i_j b^j_p = a^i_1 b^1_p + a^i_2 b^2_p$$

or

$$\begin{pmatrix} c^1_1 & c^1_2 \\ c^2_1 & c^2_2 \end{pmatrix} = \begin{pmatrix} a^1_1 b^1_1 + a^1_2 b^2_1 & a^1_1 b^1_2 + a^1_2 b^2_2 \\ a^2_1 b^1_1 + a^2_2 b^2_1 & a^2_1 b^1_2 + a^2_2 b^2_2 \end{pmatrix} = \begin{pmatrix} a^1_1 & a^1_2 \\ a^2_1 & a^2_2 \end{pmatrix}\begin{pmatrix} b^1_1 & b^1_2 \\ b^2_1 & b^2_2 \end{pmatrix}$$

or

$$\left(c^i_p\right) = \left(a^i_j\right)\left(b^j_p\right). \tag{1.20}$$

Taking determinants of both sides, we get

$$\left|c^i_p\right| = \left|a^i_j\right|\left|b^j_p\right|; \quad \text{as } |AB| = |A||B|. \tag{1.21}$$

We shall prove these results by taking the range of the indices from 1 to 2. Generalising this to a finite numbers, we get

$$\left|a^i_k b^k_h c^h_i, \ldots, p^r_j\right| = \left|a^i_k\right|\left|b^k_h\right|\left|c^h_i\right| \cdots \left|p^r_j\right|.$$

Thus, the results hold, in general, when the range is from 1 to N.

Property 1.4.2 If $a_{ij} b^{ik} = c^k_j$, then $\left(b^{ik}\right)^T \left(a_{ij}\right) = \left(c^k_j\right)$ and $\left|b^{ik}\right|\left|a_{ij}\right| = \left|c^k_j\right|$, where $\left(b^{ik}\right)^T$ is the transpose of $\left(b^{ik}\right)$.

Proof: Since we take a system of second order, so

$$c^k_j = a_{ij} b^{ik} = a_{1j} b^{1k} + a_{2j} b^{2k}$$

or

$$\begin{pmatrix} c^1_1 & c^1_2 \\ c^2_1 & c^2_2 \end{pmatrix} = \begin{pmatrix} a_{11} b^{11} + a_{21} b^{21} & a_{12} b^{11} + a_{22} b^{21} \\ a_{11} b^{12} + a_{21} b^{22} & a_{12} b^{12} + a_{22} b^{22} \end{pmatrix} = \begin{pmatrix} b^{11} & b^{21} \\ b^{12} & b^{22} \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

or

$$\left(c^k_j\right) = \left(b^{ik}\right)^T \left(a_{ij}\right). \tag{1.22}$$

Taking determinants of both sides, we get

$$\left|c^k_j\right| = \left|b^{ik}\right|^T \left|a_{ij}\right| = \left|b^{ik}\right|\left|a_{ij}\right|; \text{ since } \left|B^T\right| = |B|^T = |B|. \tag{1.23}$$

We shall prove these results by taking the range of the indices from 1 to 2. But the results hold, in general, when the range is from 1 to N.

Property 1.4.3 Let the cofactor of the element a^i_j in the determinant $|a^i_j|$ be denoted by the symbol A^j_i. Then by summation convention,

$$a^i_j A^j_k = a^i_1 A^1_k + a^i_2 A^2_k + \cdots + a^i_N A^N_k = \delta^i_k |a^i_j| = \delta^i_k a, \tag{1.24}$$

and

$$a^i_j A^k_j = a^1_j A^k_1 + a^2_j A^k_2 + \cdots + a^N_j A^k_N = \delta^k_j |a^i_j| = \delta^k_j a. \tag{1.25}$$

where $a = |a^i_j|$. These formulas include the familiar simple Laplace developments of $|a^i_j|$. The first of these then represents the expansion in terms of these elements of the ith row; the second, in terms of the elements of the jth column of $|a^i_j|$.

If the elements of the determinant a is denoted by a_{ij}, we shall write the cofactor of a_{ij} as A_{ij}. Simple Laplace developments corresponding to (1.24) and (1.25) assume the form

$$a_{ij}A_{ij} = a \qquad \text{and} \qquad a_{ij}A_{ik} = a.$$

Property 1.4.4 Let us consider a system of n linear equations as

$$a^i_j x^j = b^i; \quad i, j = 1, 2, \ldots, n \tag{1.26}$$

in n unknown x^i, where $|a^i_j| \neq 0$. Multiplying both sides of equations in (1.26) by A^k_i, and sum with respect to i yields

$$a^i_j A^k_i x^j = b^i A^k_i$$

or,

$$a\delta^k_j x^j = b^i A^k_i; \quad \text{using (1.25)}$$

or,

$$a x^k = b^i A^k_i \Rightarrow x^k = \frac{1}{a} b^i A^k_i. \tag{1.27}$$

This is the Cramer's rule for the solution of the system of n linear equations (1.26).

Property 1.4.5 Consider the determinant $|a^i_j| = a$. Let the elements $a^i_j A^j_k$ be functions of the independent variables $x_1, x_2, \ldots x_N$ then,

$$\frac{\partial a}{\partial x_1} = \begin{vmatrix} \frac{\partial a^1_1}{\partial x_1} & \frac{\partial a^1_2}{\partial x_1} & \cdots & \frac{\partial a^1_N}{\partial x_1} \\ a^2_1 & a^2_2 & \cdots & a^2_N \\ \vdots & \vdots & \cdots & \vdots \\ a^N_1 & a^N_2 & \cdots & a^N_N \end{vmatrix} + \begin{vmatrix} a^1_1 & a^1_2 & \cdots & a^1_N \\ \frac{\partial a^2_1}{\partial x_1} & \frac{\partial a^2_2}{\partial x_1} & \cdots & \frac{\partial a^2_N}{\partial x_1} \\ \vdots & \vdots & \cdots & \vdots \\ a^N_1 & a^N_2 & \cdots & a^N_N \end{vmatrix} + \begin{vmatrix} a^1_1 & a^1_2 & \cdots & a^1_N \\ a^2_1 & a^2_2 & \cdots & a^2_N \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial a^N_1}{\partial x_1} & \frac{\partial a^N_2}{\partial x_1} & \cdots & \frac{\partial a^N_N}{\partial x_1} \end{vmatrix}$$

$$= \left(\frac{\partial a_1^1}{\partial x_1} A_1^1 + \frac{\partial a_2^1}{\partial x_1} A_1^2 + \cdots + \frac{\partial a_N^1}{\partial x_1} A_1^N \right)$$
$$+ \left(\frac{\partial a_1^2}{\partial x_1} A_2^1 + \frac{\partial a_2^2}{\partial x_1} A_2^2 + \cdots + \frac{\partial a_N^2}{\partial x_1} A_2^N \right)$$
$$+ \cdots + \left(\frac{\partial a_1^N}{\partial x_1} A_N^1 + \frac{\partial a_2^N}{\partial x_1} A_N^2 + \cdots + \frac{\partial a_N^N}{\partial x_1} A_N^N \right)$$
$$= \frac{\partial a_i^1}{\partial x_1} A_1^i + \frac{\partial a_i^2}{\partial x_1} A_2^i + \cdots + \frac{\partial a_i^N}{\partial x_1} A_N^i = \frac{\partial a_i^j}{\partial x_1} A_j^i.$$

Therefore, in general, we get

$$\frac{\partial a}{\partial x_p} = \frac{\partial a_i^j}{\partial x_p} A_j^i, \tag{1.28}$$

where A_j^i is the cofactor a_i^j in the determinant $a = |a_j^i|$.

Property 1.4.6 Consider the transformations $z^i = z^i\left(y^k\right)$ and $y^i = y^i\left(x^k\right)$ (Figure 1.1). Let the N functions $z^i\left(y^1, y^2, \ldots, y^N\right)$ be independent on N variables $y^1, y^2, \ldots, y^N$ so that $\left|\frac{\partial z^i}{\partial y^i}\right| \neq 0$. In this case the N equations $z^i = z^i\left(y^k\right)$ are solvable for the z's in terms of y's. Similarly suppose that N functions $y^i\left(x^1, x^2, \ldots, x^N\right)$ are independent functions so that $\left|\frac{\partial y^i}{\partial x^k}\right| \neq 0$. Using the chain rule of differentiation, we get, the relation

$$\frac{\partial z^i}{\partial x^k} = \frac{\partial z^i}{\partial y^1}\frac{\partial y^1}{\partial x^k} + \frac{\partial z^i}{\partial y^2}\frac{\partial y^2}{\partial x^k} + \cdots + \frac{\partial z^i}{\partial y^N}\frac{\partial y^N}{\partial x^k} = \frac{\partial z^i}{\partial y^j}\frac{\partial y^j}{\partial x^k}$$

or

$$\left|\frac{\partial z^i}{\partial x^k}\right| = \left|\frac{\partial z^i}{\partial y^j}\right| \left|\frac{\partial y^j}{\partial x^k}\right| = \left|\frac{\partial z^i}{\partial y^j}\right| \left|\frac{\partial y^i}{\partial x^j}\right|, \tag{1.29}$$

connecting the functional determinants. Consider the particular case in which $z^i = x^i$, then Eq. (1.29) becomes,

$$\left|\frac{\partial x^i}{\partial x^k}\right| = \left|\frac{\partial x^i}{\partial y^j}\right| \left|\frac{\partial y^i}{\partial x^j}\right| \quad \text{or} \quad \left|\delta_k^i\right| = \left|\frac{\partial x^i}{\partial y^j}\right| \left|\frac{\partial y^i}{\partial x^j}\right|$$

or

$$1 = \left|\frac{\partial x^i}{\partial y^j}\right| \left|\frac{\partial y^i}{\partial x^j}\right| \Rightarrow \left|\frac{\partial x^i}{\partial y^j}\right| = \frac{1}{\left|\frac{\partial y^i}{\partial x^j}\right|}.$$

Thus, the Jacobian of direct transformation is the reciprocal of Jacobian of inverse transformation.

EXAMPLE 1.4.1 *Prove that* $\left|a_j^i\right| \left|b_m^k\right| = \left|c_j^i\right|$, *where* $c_m^i = a_p^i b_m^p$.

Solution: Using formula (1.14), we get

$$\begin{aligned} \left|a_j^i\right| \left|b_m^k\right| &= \left|a_j^i\right| e_{kms} b_1^k b_2^m b_3^s = \left[e_{kms} \left|a_j^i\right|\right] b_1^k b_2^m b_3^s \\ &= \left[e_{ijt} a_k^i a_m^j a_s^t\right] b_1^k b_2^m b_3^s; \text{by Eq. (1.14)} \\ &= e_{ijt} \left(a_k^i b_1^k\right) \left(a_m^j b_2^m\right) \left(a_s^t b_3^s\right) \\ &= e_{ijt} c_1^i c_2^j c_3^t = \left|c_j^i\right|. \end{aligned}$$

The above result can be stated as follows:

$$\left|a_j^i\right| \left|b_m^k\right| = \left|a_p^i b_m^p\right|,$$

which is well known result on multiplication of two determinants of third order.

Result 1.4.1 Here we consider second order determinants in various forms

(i) $\left|c_\alpha^\beta\right|$, determinant of mixed form;

(ii) $\left|c_{\alpha\beta}\right|$, determinant of double covariant form;

(iii) $\left|c^{\alpha\beta}\right|$, determinant of double contravariant form.

(i) Now, according to the definition of determinant,

$$\left|c_\alpha^\beta\right| = \begin{vmatrix} c_1^1 & c_2^1 \\ c_1^2 & c_2^2 \end{vmatrix} = c_1^1 c_2^2 - c_2^1 c_1^2 \tag{1.30}$$

$$= e_{12} e_1^1 c_2^2 + e_{21} e_2^1 c_1^2 \tag{1.31}$$

$$= e_{\alpha\beta} c_1^\alpha c_2^\beta = e^{\alpha\beta} c_\alpha^1 c_\beta^2.$$

From this, generalising, we get

$$\left|c_\alpha^\beta\right| e_{\rho\sigma} = e_{\alpha\beta} c_\rho^\alpha c_\sigma^\beta \tag{1.32}$$

and

$$\left|c_\alpha^\beta\right| e^{\rho\sigma} = e^{\alpha\beta} c_\alpha^\rho c_\beta^\sigma. \tag{1.33}$$

(ii) Now, according to the definition of determinant,

$$\begin{aligned} \left|c_{\alpha\beta}\right| &= \begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix} = c_{11} c_{22} - c_{12} c_{21} \\ &= e^{\alpha\beta} c_{1\alpha} c_{2\beta} = e^{\alpha\beta} c_{\alpha 1} c_{\beta 2}. \end{aligned}$$

From this, generalising, we get

$$|c_{\alpha\beta}|\, e_{\rho\sigma} = e^{\alpha\beta} c_{\rho\alpha} c_{\sigma\beta} \tag{1.34}$$

and

$$|c_{\alpha\beta}|\, e_{\rho\sigma} = e^{\alpha\beta} c_{\alpha\rho} c_{\beta\sigma}. \tag{1.35}$$

(iii) In the similar way, we get,

$$\left|c^{\alpha\beta}\right| e^{\rho\sigma} = e_{\alpha\beta} c^{\rho\alpha} c^{\sigma\beta} \tag{1.36}$$

and

$$\left|c^{\alpha\beta}\right| e^{\rho\sigma} = e_{\alpha\beta} c^{\alpha\rho} c^{\beta\sigma}. \tag{1.37}$$

Result 1.4.2 We know, the Jacobian of transformation from $x^i \to \overline{x}^i$ is a determinant in mixed system. Hence,

$$J = \left|\frac{\partial x^\alpha}{\partial \overline{x}^\beta}\right| = e_{\rho\sigma} \frac{\partial x^\rho}{\partial \overline{x}^1} \frac{\partial x^\sigma}{\partial \overline{x}^2}.$$

Generalising,

$$\left|\frac{\partial x^\alpha}{\partial \overline{x}^\beta}\right| e_{\lambda\mu} = e_{\rho\sigma} \frac{\partial x^\rho}{\partial \overline{x}^\lambda} \frac{\partial x^\sigma}{\partial \overline{x}^\mu}$$

or

$$J e_{\lambda\mu} = e_{\rho\sigma} \frac{\partial x^\rho}{\partial \overline{x}^\lambda} \frac{\partial x^\sigma}{\partial \overline{x}^\mu}. \tag{1.38}$$

Tensors are defined by means of their properties of transformation under co-ordinate transformations.

1.5 Contravariant Vector and Tensor

Tensors are defined by means of their properties of transformation under coordinate transformation. Let A^i be a set of N functions of N co-ordinates $x^1, x^2, \ldots, x^N$ in a given co-ordinate system $\left(x^i\right)$. Then the quantities A^i are said to form the components of a contravariant vector, if these components transform according to the following rule on change of co-ordinate system from $\left(x^i\right)$ to another system $\left(\overline{x}^i\right)$:

$$\overline{A}^i = \frac{\partial \overline{x}^i}{\partial x^j} A^j; \quad i = 1, 2, \ldots, N. \tag{1.39}$$

The contravariant vector is also called a *contravariant tensor of rank 1.* Multiplying both sides of Eq. (1.39) by $\frac{\partial x^p}{\partial \overline{x}^i}$, we get

$$\frac{\partial x^p}{\partial \overline{x}^i}\overline{A}^i = \frac{\partial x^p}{\partial \overline{x}^i}\frac{\partial \overline{x}^i}{\partial x^j}A^j = \delta^p_j A^j = A^p$$

or

$$A^p = \frac{\partial x^p}{\partial \overline{x}^i}\overline{A}^i. \tag{1.40}$$

The formula expressing the components A^i in a co-ordinate system $\left(x^i\right)$ in terms of those in another system $\left(\overline{x}^i\right)$.

Let A^{ij} be a set of N^2 functions of N co-ordinates x^i in a given system of co-ordinates $\left(x^i\right)$. The quantities A^{ij} are said to form the components of a contravariant tensor of order two (or of rank 2), if these components transform according to the following rule on change of co-ordinate system from $\left(x^i\right)$ to another system $\left(\overline{x}^i\right)$:

$$\overline{A}^{ij} = \frac{\partial \overline{x}^i}{\partial x^p}\frac{\partial \overline{x}^j}{\partial x^q}A^{pq}; \quad 1 \leq i, j \leq N. \tag{1.41}$$

Multiplying both sides of Eq. (1.41) by $\frac{\partial x^r}{\partial \overline{x}^i}\frac{\partial x^s}{\partial \overline{x}^j}$, we get

$$\begin{aligned}\frac{\partial x^r}{\partial \overline{x}^i}\frac{\partial x^s}{\partial \overline{x}^j}\overline{A}^{ij} &= \frac{\partial x^r}{\partial \overline{x}^i}\frac{\partial x^s}{\partial \overline{x}^j}\frac{\partial \overline{x}^i}{\partial x^p}\frac{\partial \overline{x}^j}{\partial x^q}A^{pq}\\ &= \frac{\partial x^r}{\partial \overline{x}^i}\frac{\partial \overline{x}^i}{\partial x^p}\frac{\partial x^s}{\partial \overline{x}^j}\frac{\partial \overline{x}^j}{\partial x^q}A^{pq}\\ &= \delta^r_p\delta^s_q A^{pq} = A^{rs}\end{aligned}$$

or

$$A^{rs} = \frac{\partial x^r}{\partial \overline{x}^i}\frac{\partial x^s}{\partial \overline{x}^j}\overline{A}^{ij}. \tag{1.42}$$

The tensor A^{ij} is of the type $(2,0)$. Similarly, a contravariant tensor of order n may be defined by considering a system of order n of type $A^{\alpha_1\alpha_2,\ldots,\alpha_n}$.

A tensor of second rank can be written as a square matrix of order N, just as a tensor of the first rank can be treated as an N-component vector. Thus, a contravariant tensor of rank two can be written as

$$A^{ij} = \begin{bmatrix} A^{11} & A^{12} & \cdots & A^{1N}\\ A^{21} & A^{22} & \cdots & A^{2N}\\ \vdots & \vdots & \cdots & \vdots\\ A^{N1} & A^{N2} & \cdots & A^{NN}\end{bmatrix}.$$

However, the converse is not true. The elements of an arbitrary square matrix do not form the components of a second rank tensor.

EXAMPLE 1.5.1 *If the components of a contravariant vector in* $\left(x^i\right)$ *co-ordinate system are* $(3,4)$*, find its components in* $\left(\overline{x}^i\right)$ *co-ordinate system, where*

$$\overline{x}^1 = 7x^1 - 5x^2 \quad \text{and} \quad \overline{x}^2 = -5x^1 + 4x^2.$$

Solution: Here, $A^1 = 3$ and $A^2 = 4$. From the transformation rule, the two sets of functions A^i and $\overline{A}^i$ are connected by relations (1.39). For $i = 1$, we get,

$$\overline{A}^1 = \frac{\partial \overline{x}^1}{\partial x^j}A^j = \frac{\partial \overline{x}^1}{\partial x^1}A^1 + \frac{\partial \overline{x}^1}{\partial x^2}A^2 = 7 \cdot 3 + (-5)\cdot 4 = 1.$$

For $i = 2$, we get

$$\overline{A}^2 = \frac{\partial \overline{x}^2}{\partial x^j}A^j = \frac{\partial \overline{x}^2}{\partial x^1}A^1 + \frac{\partial \overline{x}^2}{\partial x^2}A^2 = (-5)\cdot 3 + 4 \cdot 4 = 1.$$

EXAMPLE 1.5.2 *Let the components of a contravariant vector in* $\left(x^i\right)$ *co-ordinate system are* (x^2, x^1)*, find its components in* $\left(\overline{x}^i\right)$ *co-ordinate system, under the change of co-ordinates*

$$\overline{x}^1 = (x^2)^2 \neq 0 \quad \text{and} \quad \overline{x}^2 = x^1x^2.$$

Solution: Let $A^1 = x^2$ and $A^2 = x^1$, then by definition (1.39) of contravariance,

$$\overline{A}^i = \frac{\partial \overline{x}^i}{\partial x^j}A^j = \frac{\partial \overline{x}^i}{\partial x^1}A^1 + \frac{\partial \overline{x}^i}{\partial x^2}A^2; \quad i = 1,2$$

so

$$\overline{A}^1 = \frac{\partial \overline{x}^1}{\partial x^1}A^1 + \frac{\partial \overline{x}^1}{\partial x^2}A^2 = A^1\cdot 0 + A^2(2x^2) = 2x^1x^2 = 2\overline{x}^2.$$

and

$$\overline{A}^2 = \frac{\partial \overline{x}^2}{\partial x^1}A^1 + \frac{\partial \overline{x}^2}{\partial x^2}A^2 = A^1\cdot x^2 + A^2\cdot x^1 = (x^1)^2 + (x^2)^2 = 2\overline{x}^1 + \frac{(\overline{x}^2)^2}{\overline{x}^1}.$$

EXAMPLE 1.5.3 *Let the components of a contravariant tensor* A^{ij} *of order two in* $\left(x^i\right)$ *co-ordinate system are* $A^{11} = 1, A^{12} = 1, A^{21} = -1$ *and* $A^{22} = 2$*. Find its components* $\overline{A}^{ij}$ *in* $\left(\overline{x}^i\right)$ *co-ordinate system, under the change of co-ordinates* $\overline{x}^1 = (x^1)^2 \neq 0$ *and* $\overline{x}^2 = x^1x^2$*.*

Solution: We have to tackle this problem by using matrices. Writing $J^i_j = J'^j_i \equiv \frac{\partial \overline{x}^i}{\partial x^j}$. We have from (1.41)

$$\overline{A}^{ij} = \frac{\partial \overline{x}^i}{\partial x^p}\frac{\partial \overline{x}^j}{\partial x^q}A^{pq} = J^i_p A^{pq} J'^q_j$$

or

$$\overline{A} = JAJ^T = \begin{pmatrix} 2x^1 & 0 \\ x^2 & x^1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 2x^1 & x^2 \\ 0 & x^1 \end{pmatrix}$$

$$= \begin{pmatrix} 4(x^1)^2 & 2x^1x^2 + 2(x^1)^2 \\ 2x^1x^2 - 2(x^1)^2 & 2(x^1)^2 + (x^2)^2 \end{pmatrix}.$$

In particular, at the point $(1, -2)$,

$$\overline{A}^{11} = 4(1)^2 = 4; \quad \overline{A}^{12} = 2.1(-2) + 2.1^2 = -2$$
$$\overline{A}^{21} = 2.1(-2) - 2.1^2 = -6; \quad \overline{A}^{22} = 2.1^2 + (-2)^2 = 6.$$

EXAMPLE 1.5.4 *If x^i be the co-ordinate of a point in N dimensional space, show that dx^i are components of a contravariant vector.*

Solution: Let $x^1, x^2, \dots, x^N$ are co-ordinates in (x^i) co-ordinate system and $\overline{x}^1, \overline{x}^2, \dots, \overline{x}^N$ are co-ordinates in $(\overline{x}^i)$ co-ordinate system. Now

$$\overline{x}^i = \overline{x}^i\left(x^1, x^2, \dots, x^N\right)$$

or

$$d\overline{x}^i = \frac{\partial \overline{x}^i}{\partial x^1}dx^1 + \frac{\partial \overline{x}^i}{\partial x^2}dx^2 + \cdots + \frac{\partial \overline{x}^i}{\partial x^N}dx^N = \frac{\partial \overline{x}^i}{\partial x^j}dx^j.$$

It is law of transformation of contravariant vector. Therefore, the co-ordinate differentials dx^i are the components of a contravariant tensor of rank one—the infinitesimal displacement vector. Note that the co-ordinate x^i, in spite of the notation, are not the components of a tensor.

EXAMPLE 1.5.5 *Show that fluid velocity and acceleration at any point is a component of contravariant vector.*

Solution: Let the co-ordinates of a point in the fluid be $x^i(t)$ at any time t. Then the velocity v^i at any point in the co-ordinate system (x^i) is given by, $v^i = \frac{dx^i}{dt}$. Here, $\frac{dx^i}{dt}$ are the components of the tangent vector of the point x^i in the (x^i) co-ordinate system. Let the co-ordinates x^i be transformed to new co-ordinates $\overline{x}^j$. In this transformed co-ordinates the velocity $\overline{v}^j$ is given by

$$\overline{v}^j = \frac{d\overline{x}^j}{dt} = \frac{\partial \overline{x}^j}{dx^i}\frac{\partial x^i}{dt} = \frac{\partial \overline{x}^j}{\partial x^i}v^i, \tag{i}$$

using the concept of chain rule of partial differentiation. Note that, the component of the tangent vector in the co-ordinate system $(\overline{x}^i)$ are $\frac{d\overline{x}^i}{dt}$. Thus, we can say that the velocity v^i, i.e. component of tangent vector on the curve in N dimensional space are components of a contravariant vector or contravariant tensor of rank 1.

The co-ordinates x^i is $\frac{dx^i}{dt}$ are the co-ordinates of, say, a particle in motion, while the coefficients $\frac{d\overline{x}^j}{dx^i}$ only denote a relation between two co-ordinate systems, which is independent of time (i), we get

$$\bar{a}^j = \frac{d\overline{v}^j}{dt} = \frac{\partial \overline{x}^j}{\partial x^i}\frac{dv^i}{dt} = \frac{\partial x^j}{\partial x^i}a^i \tag{ii}$$

This shows that, acceleration a^i is also a contravariant vector.

1.6 Covariant Vector and Tensor

Let A_i be a set of N functions of N co-ordinates $x^1, x^2, \ldots, x^N$ in a given co-ordinate system $\left(x^i\right)$. Then the quantities A_i are said to form the components of a covariant vector, if these components transform according to the following rule on change of co-ordinate system from $\left(x^i\right)$ to another system $\left(\overline{x}^i\right)$:

$$\overline{A}_i = \frac{\partial x^j}{\partial \overline{x}^i} A_j; \quad i = 1, 2, \ldots, N. \tag{1.43}$$

The covariant vector is also called a *covariant tensor of rank 1.* Multiplying both sides of Eq. (1.43) by $\frac{\partial \overline{x}^i}{\partial x^p}$, we get

$$\frac{\partial \overline{x}^i}{\partial x^p} \overline{A}_i = \frac{\partial \overline{x}^i}{\partial x^p} \frac{\partial x^j}{\partial \overline{x}^i} A_j = \delta_p^j A_j = A_p$$

or

$$A_p = \frac{\partial \overline{x}^i}{\partial x^p} \overline{A}_i. \tag{1.44}$$

The formula expressing the components A^i in a co-ordinate system $\left(x^i\right)$ in terms of those in another system $\left(\overline{x}^i\right)$.

Let A_{ij} be a set of N^2 functions of N co-ordinates x^i in a given system of co-ordinates $\left(x^i\right)$. Then the quantities A_{ij} are said to form the components of a covariant tensor of order two (or of rank 2), if these components transform according to the following rule on change of co-ordinate system from $\left(x^i\right)$ to another system $\left(\overline{x}^i\right)$:

$$\overline{A}_{ij} = \frac{\partial x^p}{\partial \overline{x}^i} \frac{\partial x^q}{\partial \overline{x}^j} A_{pq}; \quad 1 \leq i, j \leq N. \tag{1.45}$$

Multiplying both sides of Eq. (1.45) by $\frac{\partial \overline{x}^i}{\partial x^r} \frac{\partial \overline{x}^j}{\partial x^s}$, we get

$$\begin{aligned} \frac{\partial \overline{x}^i}{\partial x^r} \frac{\partial \overline{x}^j}{\partial x^s} \overline{A}_{ij} &= \frac{\partial \overline{x}^i}{\partial x^r} \frac{\partial \overline{x}^j}{\partial x^s} \frac{\partial x^p}{\partial \overline{x}^i} \frac{\partial x^q}{\partial \overline{x}^j} A_{pq} \\ &= \frac{\partial \overline{x}^i}{\partial x^r} \frac{\partial x^p}{\partial \overline{x}^i} \frac{\partial \overline{x}^j}{\partial x^s} \frac{\partial x^q}{\partial \overline{x}^j} A_{pq} \\ &= \delta_r^p \delta_s^q A_{pq} = A_{rs} \end{aligned}$$

or

$$A_{rs} = \frac{\partial \overline{x}^i}{\partial x^r} \frac{\partial \overline{x}^j}{\partial x^s} \overline{A}_{ij}. \tag{1.46}$$

The tensor A_{ij} is of the type $(0, 2)$. Similarly, a covariant tensor of order n is of the form $A_{\alpha_1 \alpha_2, \ldots, \alpha_n}$.

Note: In the case of a Cartesian co-ordinate system, the co-ordinate direction x^i coincides with the direction orthogonal to the constant-x^i surface, so that the distinction between the covariant and contravariant tensors vanishes.

EXAMPLE 1.6.1 *Show that* $\frac{\partial \phi}{\partial x^i}$ *is a covariant vector, where* ϕ *is a scalar function.*

Solution: Let $x^1, x^2, \ldots, x^N$ are co-ordinates in (x^i) co-ordinate system and $\overline{x}^1, \overline{x}^2, \ldots, \overline{x}^N$ are co-ordinates in $(\overline{x}^i)$ co-ordinate system. Consider

$$\overline{\phi}\left(\overline{x}^1, \overline{x}^2, \ldots, \overline{x}^N\right) = \phi\left(x^1, x^2, \ldots, x^N\right)$$

or

$$\partial\overline{\phi} = \frac{\partial \phi}{\partial x^1}\partial x^1 + \frac{\partial \phi}{\partial x^2}\partial x^2 + \cdots + \frac{\partial \phi}{\partial x^N}\partial x^N$$

or

$$\frac{\partial\overline{\phi}}{\partial \overline{x}^i} = \frac{\partial \phi}{\partial x^1}\frac{\partial x^1}{\partial \overline{x}^i} + \frac{\partial \phi}{\partial x^2}\frac{\partial x^2}{\partial \overline{x}^i} + \cdots + \frac{\partial \phi}{\partial x^N}\frac{\partial x^N}{\partial \overline{x}^i} = \frac{\partial x^j}{\partial \overline{x}^i}\frac{\partial \phi}{\partial x^j}.$$

It is law of transformation of covariant vector. Therefore, $\frac{\partial \phi}{\partial x^i}$ are components of a covariant vector. This shows that gradient of a scalar field is a covariant vector and is represented in terms of components in the direction orthogonal to the constant co-ordinate surfaces.

EXAMPLE 1.6.2 *Let the components of a covariant vector in* $\left(x^i\right)$ *co-ordinate system are* $(x^2, x^1{+}2x^2)$*, find its components in* $\left(\overline{x}^i\right)$ *co-ordinate system, under the change of co-ordinates* $\overline{x}^1 = (x^2)^2 \neq 0$ *and* $\overline{x}^2 = x^1x^2$*.*

Solution: Let $A^1 = x^2$ and $A^2 = x^1 + 2x^2$, then by definition (1.43) of covariance,

$$\overline{A}_i = \frac{\partial x^j}{\partial \overline{x}^i}A_j = \frac{\partial x^1}{\partial \overline{x}^i}A_1 + \frac{\partial x^2}{\partial \overline{x}^i}A_2; \quad i = 1, 2.$$

so

$$\overline{A}_1 = \frac{\partial x^1}{\partial \overline{x}^1}A_1 + \frac{\partial x^2}{\partial \overline{x}^1}A_2 = A^1\left(\frac{-x^1}{2(x^2)^2}\right) + A^2\frac{1}{2x^2} = 1$$

and

$$\overline{A}_2 = \frac{\partial x^1}{\partial \overline{x}^2}A_1 + \frac{\partial x^2}{\partial \overline{x}^2}A_2 = A^1\left(\frac{1}{x^2}\right) + A^2 \cdot 0 = 1.$$

Hence, the components are $(1, 1)$ at all points in the $(\overline{x}^i)$ system ($\overline{x}^1 = 0$ excluded).

EXAMPLE 1.6.3 *Let the components of a contravariant tensor* A^{ij} *of order two in* $\left(x^i\right)$ *co-ordinate system are* $A_{11} = x^2, A_{12} = 0 = A_{21}$ *and* $A_{22} = x^1$*. Find its components* $\overline{A}_{ij}$ *in* $\left(\overline{x}^i\right)$ *co-ordinate system, under the change of co-ordinates* $\overline{x}^1 = (x^1)^2 \neq 0$ *and* $\overline{x}^2 = x^1x^2$*.*

Solution: In terms of the inverse Jacobian matrix, the covariant transformation law [Eq. (1.45)] is given by

$$\overline{A}_{ij} = \frac{\partial x^r}{\partial \overline{x}^i}A_{rs}\frac{\partial x^s}{\partial \overline{x}^j} = \overline{J}^r_i A_{rs}\overline{J}^s_j = \overline{J'}^i_r A_{rs}\overline{J}^s_j$$

or

$$\overline{A} = \overline{J}^T A \overline{J} = \begin{pmatrix} \dfrac{1}{2x^1} & \dfrac{-x^2}{2(x^1)^2} \\ 0 & \frac{1}{x^1} \end{pmatrix} \begin{pmatrix} x^2 & 0 \\ 0 & x^1 \end{pmatrix} \begin{pmatrix} \dfrac{1}{2x^1} & 0 \\ \dfrac{-x^2}{2(x^1)^2} & \dfrac{1}{x^1} \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{x^1x^2 + (x^2)^2}{4(x^1)^3} & \dfrac{-x^2}{2(x^1)^2} \\ \dfrac{-x^2}{2(x^1)^2} & \dfrac{1}{x^1} \end{pmatrix}.$$

Continuing in the matrix approach from Example 1.5.3, we get

$$A^{ij}A_{ij} = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x^2 & 0 \\ 0 & x^1 \end{pmatrix} = \begin{pmatrix} x^2 & x^1 \\ -x^2 & 2x^1 \end{pmatrix}.$$

Therefore, $t = \text{trace} = x^2 + 2x^1$. Now,

$$\overline{A}^{ij}\overline{A}_{ij} = \begin{pmatrix} 4(x^1)^2 & 2x^1x^2 + 2(x^1)^2 \\ 2x^1x^2 - 2(x^1)^2 & 2(x^1)^2 + (x^2)^2 \end{pmatrix} \begin{pmatrix} \dfrac{x^1x^2 + (x^2)^2}{4(x^1)^3} & \dfrac{-x^2}{2(x^1)^2} \\ \dfrac{-x^2}{2(x^1)^2} & \dfrac{1}{x^1} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 2x^1 \\ -\dfrac{3x^2}{2} & x^2 + 2x^1 \end{pmatrix}.$$

Therefore, $\overline{t} = \text{trace} = x^2 + 2x^1$, so that $A^{ij}A_{ij}$ is an invariant.

EXAMPLE 1.6.4 *Prove that ε_{ij} is a covariant tensor of rank 2.*
Solution: Let us consider a transformation of co-ordinates from $x^i \to \overline{x}^i$. Then,

$$\overline{g}_{\gamma\delta} = g_{\alpha\beta} \frac{\partial x^\alpha}{\partial \overline{x}^\gamma} \frac{\partial x^\beta}{\partial \overline{x}^\delta}$$

or

$$\sqrt{\overline{g}} = \sqrt{g}\ J;\ g = |g_{ij}|,$$

where the transformation is positive. From Eq. (1.38), we get

$$\sqrt{\overline{g}}\ e_{\lambda\mu} = \sqrt{g}\ e_{\rho\sigma} \frac{\partial x^\rho}{\partial \overline{x}^\lambda} \frac{\partial x^\sigma}{\partial \overline{x}^\mu}$$

or

$$\varepsilon_{\lambda\mu} = \varepsilon_{\rho\sigma} \frac{\partial x^\rho}{\partial \overline{x}^\lambda} \frac{\partial x^\sigma}{\partial \overline{x}^\mu};\ \text{from (1.10)}.$$

This shows that ε_{ij} is a covariant tensor of rank 2.

1.7 Mixed Tensor

Let A^i_j be a set of N^2 functions of N co-ordinates $x^1, x^2, \ldots, x^N$ in a given co-ordinate system (x^i). Then the quantities A^i_j are said to form the components of a mixed tensor of order two (or of rank 2), if these components transform according to the following rule on change of co-ordinate system from (x^i) to another system $(\overline{x}^i)$:

$$\overline{A}^i_j = \frac{\partial \overline{x}^i}{\partial x^p}\frac{\partial x^q}{\partial \overline{x}^j} A^p_q. \tag{1.47}$$

Multiplying both sides of (1.47) by $\frac{\partial x^r}{\partial \overline{x}^i}\frac{\partial \overline{x}^j}{\partial x^s}$, we get

$$\begin{aligned}\frac{\partial x^r}{\partial \overline{x}^i}\frac{\partial \overline{x}^j}{\partial x^s}\overline{A}^i_j &= \frac{\partial x^r}{\partial \overline{x}^i}\frac{\partial \overline{x}^j}{\partial x^s}\frac{\partial \overline{x}^i}{\partial x^p}\frac{\partial x^q}{\partial \overline{x}^j} A^p_q \\ &= \frac{\partial x^r}{\partial \overline{x}^i}\frac{\partial \overline{x}^i}{\partial x^p}\frac{\partial \overline{x}^j}{\partial x^s}\frac{\partial x^q}{\partial \overline{x}^j} A^p_q \\ &= \delta^r_p\,\delta^q_s\,A^p_q = A^r_s\end{aligned}$$

or

$$A^r_s = \frac{\partial x^r}{\partial \overline{x}^i}\frac{\partial \overline{x}^j}{\partial x^s}\overline{A}^i_j. \tag{1.48}$$

The formula expressing the components A^i_j in a co-ordinate system (x^i) in terms of those in another system $(\overline{x}^i)$. The tensor A^i_j is of the type $(1,1)$. The upper position of the suffix is reserved to indicate contravariant character, whereas the lower position of the suffix is reserved to indicate covariant character.

Let A^i_{jk} be a set of N^3 functions of N co-ordinates x^i in a given system of co-ordinates (x^i). Then the quantities A^i_{jk} are said to form the components of a mixed tensor of order three (or of rank 3) with first order contravariance and second order covariance, if these components transform according to the following rule on change of co-ordinate system from (x^i) to another system $(\overline{x}^i)$:

$$\overline{A}^i_{jk} = \frac{\partial \overline{x}^i}{\partial x^p}\frac{\partial x^q}{\partial \overline{x}^j}\frac{\partial x^r}{\partial \overline{x}^k} A^p_{qr}. \tag{1.49}$$

Multiplying both sides of (1.49) by $\frac{\partial x^l}{\partial \overline{x}^i}\frac{\partial \overline{x}^j}{\partial x^m}\frac{\partial \overline{x}^k}{\partial x^n}$, we get

$$\begin{aligned}\frac{\partial x^l}{\partial \overline{x}^i}\frac{\partial \overline{x}^j}{\partial x^m}\frac{\partial \overline{x}^k}{\partial x^n}\overline{A}^i_{jk} &= \frac{\partial x^l}{\partial \overline{x}^i}\frac{\partial \overline{x}^j}{\partial x^m}\frac{\partial \overline{x}^k}{\partial x^n}\frac{\partial \overline{x}^i}{\partial x^p}\frac{\partial x^q}{\partial \overline{x}^j}\frac{\partial x^r}{\partial \overline{x}^k} A^p_{qr} \\ &= \frac{\partial x^l}{\partial \overline{x}^i}\frac{\partial \overline{x}^i}{\partial x^p}\frac{\partial \overline{x}^j}{\partial x^m}\frac{\partial x^q}{\partial \overline{x}^j}\frac{\partial \overline{x}^k}{\partial x^n}\frac{\partial x^r}{\partial \overline{x}^k} A^p_{qr} \\ &= \delta^l_p\,\delta^q_m\,\delta^r_n\,A^p_{qr} = A^l_{mn}\end{aligned}$$

or

$$A^l_{mn} = \frac{\partial x^l}{\partial \overline{x}^i} \frac{\partial \overline{x}^j}{\partial x^m} \frac{\partial \overline{x}^k}{\partial x^n} \overline{A}^i_{jk}. \tag{1.50}$$

The formula expressing the components A^i_{jk} in a co-ordinate system $\left(x^i\right)$ in terms of those in another system $\left(\overline{x}^i\right)$. Similarly, a mixed tensor $A^{i_1 i_2,\ldots,i_r}_{j_1 j_2,\ldots,j_s}$ of the type (r, s) of order $r + s$ with rth order of contravariance and sth order of covariance is defined. This tensor has N^{r+s} components.

EXAMPLE 1.7.1 *Show that the Kronecker delta is a mixed tensor of rank 2.*

Solution: Let $x^1, x^2, \ldots, x^N$ are co-ordinates in (x^i) co-ordinate system and $\overline{x}^1, \overline{x}^2, \ldots,$ $\overline{x}^N$ are co-ordinates in $(\overline{x}^i)$ co-ordinate system. Let the component of Kronecker delta in (x^i) system δ^i_j and component of Kronecker delta in $(\overline{x}^i)$ system $\overline{\delta}^i_j$, then by definition,

$$\overline{\delta}^i_j = \frac{\partial \overline{x}^i}{\partial \overline{x}^j} = \frac{\partial \overline{x}^i}{\partial x^k} \frac{\partial x^l}{\partial \overline{x}^j} \frac{\partial x^k}{\partial x^l}$$

or

$$\overline{\delta}^i_j = \frac{\partial \overline{x}^i}{\partial x^k} \frac{\partial x^l}{\partial \overline{x}^j} \delta^k_l.$$

This shows that Kronecker delta δ^i_j is a mixed tensor of rank 2. It is sometimes called the fundamental mixed tensor.

EXAMPLE 1.7.2 *Show that there is no distinction between contravariant and covariant vectors when the transformations are of the type*

$$\overline{x}^i = a^i_m x^m + d^i,$$

where d^i and a^i_m are constants such that $a^i_r a^i_m = \delta^r_m$.

Solution: The given transformation is $\overline{x}^i = a^i_m x^m + d^i$, where d^i and a^i_m are constants such that $a^i_r a^i_m = \delta^r_m$. Differentiating both sides with respect to x^m, we get

$$\frac{\partial \overline{x}^i}{\partial x^m} = a^i_m + 0 = a^i_m.$$

Multiplying both sides of the given transformation by a^i_r, we get

$$\begin{aligned} \overline{x}^i\, a^i_r &= a^i_r\, a^i_m\, x^m + a^i_r\, d^i \\ &= \delta^r_m\, x^m + a^i_r\, d^i = x^r + a^i_r\, d^i \end{aligned}$$

or

$$x^r = \overline{x}^i\ a_r^i - a_r^i\ d^i$$

or

$$x^j = \overline{x}^i\ a_j^i - a_j^i\ d^i.$$

Differentiating with respect to $\overline{x}^i$, we get

$$\frac{\partial x^j}{\partial \overline{x}^i} = a_j^i - 0 = a_j^i \Rightarrow \frac{\partial \overline{x}^i}{\partial x^j} = \frac{\partial x^j}{\partial \overline{x}^i} = a_j^i.$$

Thus, the relation $\overline{A}^i = \frac{\partial \overline{x}^i}{\partial x^j} A^j$ takes the form,

$$\overline{A}^i = \frac{\partial \overline{x}^i}{\partial \overline{x}^j} A^j = a_j^i A^j$$

and the relation $\overline{A}_i = \frac{\partial x^j}{\partial \overline{x}^i} A_j$ takes the form

$$\overline{A}_i = \frac{\partial x^j}{\partial \overline{x}^i} A_j = a_j^i A_j.$$

Thus, we see that both the vectors A^i and A_i transform in an identical manner.

It is important to note the essential difference between a contravariant and a covariant tensor. In the case of contravariant tensor, the tensor is represented by components in the direction of co-ordinate increase, whereas in the case of a covariant tensor, the tensor is represented by components in the directions orthogonal to constant co-ordinate surfaces.

EXAMPLE 1.7.3 *Prove that the transformation of contravariant vectors form a group.*

Solution: Let A^i be a contravariant vector. Let S be the set of transformations of contravariant vectors and T_1, T_2 be two such transformations from the system $\left(x^i\right)$ to the system $\left(\overline{x}^i\right)$ and from $\left(\overline{x}^i\right)$ to $\left(\overline{\overline{x}}^i\right)$ given by

$$T_1 : \overline{A}^i = \frac{\partial \overline{x}^i}{\partial x^p}\ A^p; \quad T_2 : \overline{\overline{A}}^{\,i} = \frac{\partial \overline{\overline{x}}^i}{\partial \overline{x}^r}\ \overline{A}^r.$$

Then the product of transformation $T_2 T_1 : x^i \to \overline{\overline{x}}^i$ is given by

$$T_2 T_1 : \overline{\overline{A}}^{\,i} = \frac{\partial \overline{\overline{x}}^i}{\partial \overline{x}^r} \frac{\partial \overline{x}^r}{\partial x^p}\ A^p = \frac{\partial \overline{\overline{x}}^i}{\partial x^p}\ A^p,$$

from which it follows that $T_2 T_1 \in S$. Let I be the transformation given by

$$I : A^i = \frac{\partial x^p}{\partial x^p}\ \overline{A}^i,$$

then

$$IT_1 = T_1 I = T_1.$$

Hence, I is the identity transformation and from this relation, it follows that $I \in S$. Now, consider, the transformation

$$T_1 : \overline{A}^i = \frac{\partial \overline{x}^i}{\partial x^p} A^p,$$

from which it follows that

$$T_1^* : A^p = \frac{\partial x^p}{\partial \overline{x}^i} \overline{A}^i.$$

Since T_1^* represents the transformation from the system $\left(\overline{x}^i\right)$ to $\left(x^i\right)$, it is the inverse of T_1 and $T_1^* \in S$. Finally, if T_3 represents a transformation from the system $\left(\overline{\overline{x}}^i\right)$ to $\left(\overline{\overline{\overline{x}}}^i\right)$, then

$$T_3(T_2 T_1) = (T_3 T_2)\, T_1.$$

In virtue of these results, it is expressed by saying that transformations of a contravariant vector form a group.

EXAMPLE 1.7.4 *If* $\left(\frac{x^1}{x^2}, \frac{x^2}{x^1}\right)$ *is a covariant vector in cartesian co-ordinates* x^1, x^2*; find its components in polar co-ordinates* $(\overline{x}^1, \overline{x}^2)$.

Solution: The transformation law from the cartesian co-ordinate (x^1, x^2) to polar co-ordinates $(\overline{x}^1, \overline{x}^2)$ are $x^1 = \overline{x}^1 \cos \overline{x}^2; \quad x^2 = \overline{x}^1 \sin \overline{x}^2$.

Let A_i denotes the components of a covariant vector in co-ordinates x^i and $\overline{A}_i$ denotes its components in co-ordinates $\overline{x}^i$, then, we have the transformation law (1.43). In the present case, $A_1 = \frac{x^1}{x^2}; A_2 = \frac{x^2}{x^1}$. Let these quantities are $\overline{A}_1$ and $\overline{A}_2$ in $(\overline{x}^1, \overline{x}^2)$ co-ordinate system. Using the transformation law, we have

$$\begin{aligned} \overline{A}_1 &= \frac{\partial x^j}{\partial \overline{x}^1} A_j = \frac{\partial x^1}{\partial \overline{x}^1} A_1 + \frac{\partial x^2}{\partial \overline{x}^1} A_2 \\ &= \cos \overline{x}^2 \, \frac{\overline{x}^1 \cos \overline{x}^2}{\overline{x}^1 \sin \overline{x}^2} + \sin \overline{x}^2 \, \frac{\overline{x}^1 \sin \overline{x}^2}{\overline{x}^1 \cos \overline{x}^2} = \frac{\cos^3 \overline{x}^2 + \sin^3 \overline{x}^2}{\sin \overline{x}^2 \cos \overline{x}^2}. \end{aligned}$$

Also, using the transformation law, we have

$$\begin{aligned} \overline{A}_2 &= \frac{\partial x^j}{\partial \overline{x}^2} A_j = \frac{\partial x^1}{\partial \overline{x}^2} A_1 + \frac{\partial x^2}{\partial \overline{x}^2} A_2 \\ &= -\overline{x}^1 \sin \overline{x}^2 \, \frac{\overline{x}^1 \cos \overline{x}^2}{\overline{x}^1 \sin \overline{x}^2} + \overline{x}^1 \cos \overline{x}^2 \frac{\overline{x}^1 \sin \overline{x}^2}{\overline{x}^1 \cos \overline{x}^2} \\ &= \overline{x}^1 \left(\sin \overline{x}^2 - \cos \overline{x}^2\right). \end{aligned}$$

EXAMPLE 1.7.5 *If X, Y, Z are the components of a contravariant vector in Cartesian co-ordinates x, y, z in E_3 find the components of the vector in cylindrical co-ordinates.*

Solution: Let $x^1 = x, x^2 = y, x^3 = z$ and $x^1 = r, x^2 = \theta$ and $x^3 = z$, then the relation between Cartesian and cylindrical co-ordinates is given by

$$x^1 = \overline{x}^1 \cos \overline{x}^2; \quad x^2 = \overline{x}^1 \sin \overline{x}^2; \quad x^3 = \overline{x}^3$$

or

$$\overline{x}^1 = \sqrt{(x^1)^2 + (x^2)^2}; \quad \overline{x}^2 = \tan^{-1} \frac{x^2}{x^1}; \quad \text{and } \overline{x}^3 = x^3.$$

Let A^i denotes the components of a contravariant vector in co-ordinates x^i and $\overline{A}^i$ are that in $\overline{x}^i$ co-ordinates. In the present case, $A^1 = X, A^2 = Y$ and $A^3 = Z$. Therefore, from Eq. (1.39), we get

$$\begin{aligned}
\overline{A}^1 &= \frac{\partial \overline{x}^1}{\partial x^j} A^j = \frac{\partial \overline{x}^1}{\partial x^1} A^1 + \frac{\partial \overline{x}^1}{\partial x^2} A^2 + \frac{\partial \overline{x}^1}{\partial x^3} A^3 \\
&= \frac{x^1}{\sqrt{(x^1)^2 + (x^2)^2}} X + \frac{x^2}{\sqrt{(x^1)^2 + (x^2)^2}} Y + 0 \cdot Z \\
&= X \cos \overline{x}^2 + Y \sin \overline{x}^2. \\
\overline{A}^2 &= \frac{\partial \overline{x}^2}{\partial x^j} A^j = \frac{\partial \overline{x}^2}{\partial x^1} A^1 + \frac{\partial \overline{x}^2}{\partial x^2} A^2 + \frac{\partial \overline{x}^2}{\partial x^3} A^3 \\
&= -\frac{x^2}{(x^1)^2 + (x^2)^2} X + \frac{x^1}{(x^1)^2 + (x^2)^2} Y + 0 \cdot Z \\
&= -X \frac{1}{\overline{x}^1} \sin \overline{x}^2 + Y \frac{1}{\overline{x}^1} \cos \overline{x}^2. \\
\overline{A}^3 &= \frac{\partial \overline{x}^3}{\partial x^j} A^j = \frac{\partial \overline{x}^3}{\partial x^1} A^1 + \frac{\partial \overline{x}^3}{\partial x^2} A^2 + \frac{\partial \overline{x}^3}{\partial x^3} A^3 \\
&= 0 \cdot X + 0 \cdot Y + 1 \cdot Z = Z.
\end{aligned}$$

EXAMPLE 1.7.6 *If A_i is a covariant vector, determine whether $\frac{\partial A_i}{\partial x^j}$ is a tensor.*

Solution: Since A_i is a covariant vector, we have the transformation law (1.43). Differentiating both sides with respect to $\overline{x}^k$, we get

$$\begin{aligned}
\frac{\partial \overline{A}_i}{\partial \overline{x}^k} &= \frac{\partial^2 x^j}{\partial \overline{x}^k \partial \overline{x}^j} A_j + \frac{\partial x^j}{\partial \overline{x}^i} \frac{\partial A_j}{\partial \overline{x}^k} \\
&= \frac{\partial^2 x^j}{\partial \overline{x}^k \partial \overline{x}^j} A_j + \frac{\partial x^j}{\partial \overline{x}^i} \frac{\partial A_j}{\partial x^l} \frac{\partial x^l}{\partial \overline{x}^k}.
\end{aligned}$$

From this it follows that $\frac{\partial A_i}{\partial x^j}$ is not a tensor due to the presence of the first term in the right-hand side.

EXAMPLE 1.7.7 *If the relation* $a^i_j v^j = 0$ *holds for arbitrary contravariant vector* v^i*, show that* $a^i_j = 0$.

Solution: The given relation $a^i_j v^j = 0$ can be written as

$$a^i_1 v^1 + a^i_2 v^2 + \cdots + a^i_N v^N = 0.$$

Since v^j is arbitrary contravariant vector, we can choose it at will. First, we take v^j as $(1, 0, \ldots, 0)$, then $v^1 = 1, v^2 = 0, \ldots, v^N = 0$. Hence,

$$a^i_1 \cdot 1 + a^i_2 \cdot 0 + \cdots + a^i_N \cdot 0 = 0 \Rightarrow a^i_1 = 0.$$

Next we take v^j as $(0, 1, 0, \ldots, 0)$, $(0, 0, 1, 0, \ldots, 0)$, $\ldots$, $(0, 0, \ldots, 1)$ in succession. Then we get

$$a^i_2 = 0;\ a^i_3 = 0;\ a^i_N = 0.$$

Hence, it follows that $a^i_j = 0$.

EXAMPLE 1.7.8 *If the relation* $a_{ij} v^i v^j = 0$ *holds for arbitrary contravariant vector* v^i*, show that* $a_{ij} + a_{ji} = 0$.

Solution: Let $P = a_{ij} v^i v^j = 0$. Differentiating with respect to v^k we get

$$\frac{\partial P}{\partial v^k} = a_{ij}\ \frac{\partial v^i}{\partial v^k}\ v^j + a_{ij}\ v^i\ \frac{\partial v^j}{\partial v^k} = 0$$

or

$$\frac{\partial P}{\partial v^k} = a_{ij}\ \delta^i_k\ v^j + a_{ij}\ v^i\ \delta^j_k = 0$$

or

$$\frac{\partial P}{\partial v^k} = a_{kj}\ v^j + a_{ik}\ v^i = 0.$$

Further, differentiating with respect to v^l we get

$$\frac{\partial^2 P}{\partial v^l \partial v^k} = a_{kj} \frac{\partial v^j}{\partial v^l} + a_{ik} \frac{\partial v^i}{\partial v^l} = 0$$

or

$$\frac{\partial^2 P}{\partial v^l \partial v^k} = a_{kj} \delta^j_l + a_{ik} \delta^i_l = 0$$

or

$$\frac{\partial^2 P}{\partial v^l \partial v^k} = a_{kl} + a_{lk} = 0.$$

Replacing the dummy indices k and l by i and j respectively, we get, $a_{ij} + a_{ji} = 0$. If further a_{ij} is symmetric, then $a_{ij} = 0$.

EXAMPLE 1.7.9 *If $B_{ij} = A_{ji}$, a covariant tensor, show that B_{ij} is a tensor of order two.*

Solution: Since A_{ij} is a covariant tensor, we have Eq. (1.45). Now,

$$\begin{aligned}\overline{B}_{ij} = \overline{A}_{ji} &= \frac{\partial x^p}{\partial \overline{x}^j}\,\frac{\partial x^q}{\partial \overline{x}^i}\,A_{pq} = \frac{\partial x^p}{\partial \overline{x}^j}\,\frac{\partial x^q}{\partial \overline{x}^i}\,B_{qp};\quad \text{as } B_{ij} = A_{ji}\\ &= \frac{\partial x^q}{\partial \overline{x}^i}\frac{\partial x^p}{\partial \overline{x}^j}B_{qp}.\end{aligned}$$

From this relation, it follows that B_{ij} is a covariant tensor of order two.

EXAMPLE 1.7.10 *If the relation $a_{ij}u^i u^j = 0$ holds for all vectors u^i such that $u^i\lambda_i = 0$, where λ_i is a given covariant vector, show that*

$$a_{ij} + a_{ji} = \lambda_i v_j + \lambda_j v_i,$$

where v_j is some covariant vector.

Solution: From the given relation $u^i\lambda_i = 0$, we get, $u^i\lambda_i v_j u^j = 0$, where v_j is some covariant vector. Using the given relation $a_{ij}u^i u^j = 0$, we get

$$a_{ij}u^i u^j = u^i\lambda_i v_j u^j$$

or

$$\left(a_{ij} - \lambda_i v_j\right)u^i u^j = 0.$$

Since $u^i u^j$ is arbitrary contravariant vector, $u^i u^j \neq 0$, and so,

$$a_{ij} - \lambda_i v_j = 0 \Rightarrow a_{ij} = \lambda_i v_j.$$

Interchanging the dummy suffixes i, j in the relations $a_{ij}u^i u^j = 0$ and $u^i\lambda_i\, v_j u^j = 0$, we get

$$a_{ji}u^j u^i = 0 \text{ and } u^j\lambda_j v_i u^i = 0$$

or

$$a_{ji}u^j u^i = u^j\lambda_j v_i u^i$$

or

$$\left(a_{ji} - \lambda_j v_i\right)u^i u^j = 0$$

or

$$a_{ji} = \lambda_j v_i;\qquad \text{as } u^i u^j \neq 0.$$

Adding we get

$$a_{ij} + a_{ji} = \lambda_i v_j + \lambda_j v_i.$$

EXAMPLE 1.7.11 *If the equality $a^i_j u_i = \sigma u_j$ holds for any covariant vector u_i such that $u_i v^i = 0$, where v^i is a given contravariant vector, show that*

$$a^i_j = \sigma \delta^i_j + \lambda_j v^i,$$

where λ_i is some covariant vector.

Solution: From the given equality $a^i_j u_i = \sigma u_j$, we get

$$a^i_j u_i = \sigma u_j + \lambda_j u_i v^i; \text{ as } u_i v^i = 0$$

or

$$a^i_j u_i = \sigma \delta^i_j u_i + \lambda_j u_i v^i; \text{as } \delta^i_j u_i = u_j$$

or

$$u_i \left[a^i_j - \sigma \delta^i_j - \lambda_j v^i\right] = 0,$$

where λ_i is some covariant vector. Since u_i is arbitrary, so

$$a^i_j - \sigma \delta^i_j - \lambda_j v^i = 0 \quad \text{or} \quad a^i_j = \sigma \delta^i_j + \lambda_j v^i.$$

EXAMPLE 1.7.12 *If $A = (a_{ij})$ is a symmetric 4×4 matrix such that $a_{ij} x^i x^j = 0$ for all x^i such that $g_{ij} x^i x^j = 0$, prove that there exists a fixed real number λ such that $a_{ij} = \lambda g_{ij}$.*

Solution: Observe that the vector $(1, \pm 1, 0, 0)$ satisfies $g_{ij} x^i x^j = 0$. Hence, substituting these components into the equation $a_{ij} x^1 x^j = 0$ yields,

$$a_{00} \pm a_{01} \pm a_{10} + a_{11} = 0$$

or

$$a_{00} + a_{11} = 0 = a_{01} + a_{10},$$

by symmetry of A. Similarly, using the vectors $(1, 0, \pm 1, 0)$ and $(1, 0, 0, \pm 1)$, we get

$$a_{00} = -a_{11} = -a_{22} = -a_{33} = \lambda; \; c_{ij} = 0; \; i = 0 \text{ or } j = 0.$$

Finally, employing the vectors $(\sqrt{2}, 1, 1, 0)$, $(\sqrt{2}, 1, 0, 1)$ and $(\sqrt{2}, 0, 1, 1)$, we obtain $a_{12} = a_{13} = a_{23} = 0$.

1.8 Invariants

Objects, functions, equations or formulas that are independent of the co-ordinate system used to express them have intrinsic value and are fundamental significance; they are called invariants. Let ϕ be a function of N co-ordinates x^i in a co-ordinate system $\left(x^i\right)$ in V_N and $\overline{\phi}$ be its transform in another co-ordinate system $\left(\overline{x}^i\right)$. Then ϕ is called an *invariant or scalar* of V_N with respect to the transformation

$$x^i = \psi^i\left(\overline{x}^1, \overline{x}^2, \ldots, \overline{x}^N\right); i = 1, 2, \ldots, N,$$

if

$$\overline{\phi}\left(\overline{x}^1, \overline{x}^2, \ldots, \overline{x}^N\right) = \phi\left(x^1, x^2, \ldots, x^N\right)$$

or

$$\overline{\phi}\left(\overline{x}^1, \overline{x}^2, \ldots, \overline{x}^N\right) = \phi\left[\psi^1\left(\overline{x}^1, \ldots, \overline{x}^N\right), \ldots, \psi^N\left(\overline{x}^1, \ldots, \overline{x}^N\right)\right]. \tag{1.51}$$

A scalar is invariant under any co-ordinate transformations. Obvious physical examples of a scalar in Newtonian mechanics are length, mass, energy, volume etc, which are independent of the choice of the co-ordinate system.

Differentiating Eq. (1.51) with respect to $\overline{x}^i$, we get

$$\frac{\partial \overline{\phi}}{\partial \overline{x}^i} = \frac{\partial \overline{\phi}}{\partial x^j}\frac{\partial x^j}{\partial \overline{x}^i} = \frac{\partial x^j}{\partial \overline{x}^i}\frac{\partial \phi}{\partial x^j};\ \text{as } \phi = \overline{\phi}. \tag{1.52}$$

Now, $\frac{\partial \phi}{\partial x^j}$ may be considered as the components of a system of first order of type A_i and Eq. (1.52) shows that these components transform according to a certain rule on transformation of co-ordinates from a system $\left(x^i\right)$ to another system $\left(\overline{x}^i\right)$. The rule indicated by Eq. (1.52) leads to the definition of a covariant vector of V_N.

We shall agree to call an invariant or a scalar a tensor of order zero or of type (0, 0).

EXAMPLE 1.8.1 *Show that the Kronecker delta is an invariant, i.e. it has same components in every co-ordinate system.*

Solution: Let $x^1, x^2, \ldots, x^N$ are co-ordinates in (x^i) co-ordinate system and $\overline{x}^1, \overline{x}^2, \ldots,$ $\overline{x}^N$ are co-ordinates in $(\overline{x}^i)$ co-ordinate system. Let the component of Kronecker delta in (x^i) system δ^i_j and component of Kronecker delta in $(\overline{x}^i)$ system $\overline{\delta}^i_j$, then according to the law of transformation,

$$\begin{aligned}\overline{\delta}^i_j &= \frac{\partial \overline{x}^i}{\partial x^k}\frac{\partial x^l}{\partial \overline{x}^j}\,\delta^k_l = \frac{\partial \overline{x}^i}{\partial x^k}\left(\frac{\partial x^l}{\partial \overline{x}^j}\,\delta^k_l\right)\\ &= \frac{\partial \overline{x}^i}{\partial x^k}\frac{\partial x^k}{\partial \overline{x}^j};\ \text{ as } \frac{\partial x^l}{\partial \overline{x}^j}\,\delta^k_l = \frac{\partial x^k}{\partial \overline{x}^j}\\ &= \frac{\partial \overline{x}^i}{\partial \overline{x}^j} = \delta^i_j;\ \text{ as } \frac{\partial \overline{x}^i}{\partial \overline{x}^j} = \delta^i_j.\end{aligned}$$

This shows that δ^i_j is an invariant.

EXAMPLE 1.8.2 *Show that the determinant of a tensor of type* $(1, 1)$ *is an invariant.*

Solution: Since $\overline{x}^i$s are independent and x^is are also independent functions of the $\overline{x}^i$s, by the formula of partial differentiation and summation convention, we can write

$$\frac{\partial \overline{x}^i}{\partial \overline{x}^j} = \frac{\partial \overline{x}^i}{\partial x^p}\,\frac{\partial x^p}{\partial \overline{x}^j};\ p = 1, 2, \ldots, N$$

or

$$\delta^i_j = \frac{\partial \overline{x}^i}{\partial x^p}\,\frac{\partial x^p}{\partial \overline{x}^j}$$

or

$$\left|\delta^i_j\right| = \left|\frac{\partial \overline{x}^i}{\partial x^p}\right|\,\left|\frac{\partial x^p}{\partial \overline{x}^j}\right|$$

or

$$1 = JJ';\ \text{where, } J = \left|\frac{\partial \overline{x}^i}{\partial x^p}\right|;\ J' = \left|\frac{\partial x^p}{\partial \overline{x}^j}\right|.$$

Let A^i_j be a mixed tensor of type $(1, 1)$, then by tensor law of transformation,

$$\overline{A}^i_j = \frac{\partial \overline{x}^i}{\partial x^p}\frac{\partial x^q}{\partial \overline{x}^j}A^p_q$$

or

$$\left(\overline{A}^i_j\right) = \left(\frac{\partial \overline{x}^i}{\partial x^p}\right)\left(A^p_q\right)\left(\frac{\partial x^q}{\partial \overline{x}^j}\right),$$

where, (A^p_q) denotes the matrix of A^p_q and other symbols have similar meanings. Taking determinants of both sides of the above equality, we get

$$\left|\overline{A}^i_j\right| = \left|\frac{\partial \overline{x}^i}{\partial x^p}\right|\,\left|A^p_q\right|\,\left|\frac{\partial x^q}{\partial \overline{x}^j}\right|;\ \text{as } |AB| = |A||B|$$

or

$$\left|\overline{A}^i_j\right| = \left|\frac{\partial \overline{x}^i}{\partial x^p}\right|\left|\frac{\partial x^q}{\partial \overline{x}^j}\right|\,\left|A^p_q\right|$$

or

$$\left|\overline{A}^i_j\right| = JJ'\,\left|A^p_q\right| = \left|A^p_q\right|;\ \text{where } J = \left|\frac{\partial \overline{x}^i}{\partial x^p}\right| \text{ and } J' = \left|\frac{\partial x^q}{\partial \overline{x}^j}\right|.$$

From this it follows that $\left|A^i_j\right|$ is an invariant.

EXAMPLE 1.8.3 *If* u^i *is an arbitrary contravariant vector and* $a_{ij}u^iu^j$ *is an invariant, show that* $a_{ij} + a_{ji}$ *is a covariant tensor of second order.*

Solution: Let u^i be an arbitrary contravariant vector. Since, $a_{ij}u^iu^j$ is an invariant, we have, $\overline{a}_{ij}\,\overline{u}^i\,\overline{u}^j = a_{ij}u^iu^j$. Applying tensor law of transformation, we get

$$\begin{aligned}\overline{a}_{ij}\,\overline{u}^i\,\overline{u}^j &= a_{ij}\,\frac{\partial x^i}{\partial \overline{x}^p}\,\overline{u}^p\,\frac{\partial x^j}{\partial \overline{x}^q}\,\overline{u}^q\\ &= a_{ij}\,\frac{\partial x^i}{\partial \overline{x}^p}\,\frac{\partial x^j}{\partial \overline{x}^q}\,\overline{u}^p\,\overline{u}^q.\end{aligned}$$

Replacing the dummy indices $i, j; p, q$ by $p, q; i, j$ respectively, we get

$$\overline{a}_{ij}\,\overline{u}^i\,\overline{u}^j = a_{pq}\,\frac{\partial x^p}{\partial \overline{x}^i}\,\frac{\partial x^q}{\partial \overline{x}^j}\,\overline{u}^i\,\overline{u}^j.$$

Interchanging the suffixes i and j, we get

$$\overline{a}_{ji}\overline{u}^j\overline{u}^i = a_{pq}\frac{\partial x^p}{\partial \overline{x}^j}\frac{\partial x^q}{\partial \overline{x}^i}\overline{u}^j\overline{u}^i.$$

On interchanging the dummy indices p, q, we get

$$\overline{a}_{ji}\overline{u}^j\overline{u}^i = a_{qp}\frac{\partial x^q}{\partial \overline{x}^j}\frac{\partial x^p}{\partial \overline{x}^i}\overline{u}^j\overline{u}^i$$

or

$$\overline{a}_{ji}\overline{u}^i\overline{u}^j = a_{qp}\frac{\partial x^p}{\partial \overline{x}^i}\frac{\partial x^q}{\partial \overline{x}^j}\overline{u}^i\overline{u}^j;\ \text{as}\ \overline{u}^j\overline{u}^i = \overline{u}^i\overline{u}^j.$$

Adding properly, we get

$$(\overline{a}_{ij} + \overline{a}_{ji})\,\overline{u}^i\overline{u}^j = \left[(a_{pq} + a_{qp})\,\frac{\partial x^p}{\partial \overline{x}^i}\frac{\partial x^q}{\partial \overline{x}^j}\right]\overline{u}^i\overline{u}^j$$

or

$$\left[(\overline{a}_{ij} + \overline{a}_{ji}) - (a_{pq} + a_{qp})\,\frac{\partial x^p}{\partial \overline{x}^i}\frac{\partial x^q}{\partial \overline{x}^j}\right]\overline{u}^i\overline{u}^j = 0.$$

Since $\overline{u}^i$ and $\overline{u}^j$ are arbitrary, we have

$$\overline{a}_{ij} + \overline{a}_{ji} - a_{pq} + a_{qp}\,\frac{\partial x^p}{\partial \overline{x}^i}\,\frac{\partial x^q}{\partial \overline{x}^j} = 0$$

or

$$\overline{a}_{ij} + \overline{a}_{ji} = (a_{pq} + a_{qp})\,\frac{\partial x^p}{\partial \overline{x}^i}\,\frac{\partial x^q}{\partial \overline{x}^j}.$$

This confirms the covariant second rank tensor law of transformation. Therefore, $a_{ij} + a_{ji}$ is a covariant tensor of order two.

EXAMPLE 1.8.4 *If a_{ij} is a covariant tensor such that $|a_{ij}| \neq 0$, determine whether the determinant $|a_{ij}|$ is an invariant.*

Solution: Since a_{ij} is a covariant tensor of second order, we have

$$\overline{a}_{ij} = \frac{\partial x^p}{\partial \overline{x}^i} \frac{\partial x^q}{\partial \overline{x}^j} a_{pq}$$

or

$$|\overline{a}_{ij}| = \left|\frac{\partial x^p}{\partial \overline{x}^i}\right| \left|\frac{\partial x^q}{\partial \overline{x}^j}\right| |a_{pq}|$$

or

$$|\overline{a}_{ij}| = \left|\frac{\partial x^p}{\partial \overline{x}^i}\right|^2 |a_{pq}|\,; \text{ as } \left|\frac{\partial x^p}{\partial \overline{x}^i}\right| = \left|\frac{\partial x^q}{\partial \overline{x}^j}\right|$$

or

$$|\overline{a}_{ij}| = J^2 |a_{pq}|\,; \text{ where, } J = \left|\frac{\partial x^p}{\partial \overline{x}^i}\right|,$$

where J is the Jacobian of transformation $x^i = x^i\left(\overline{x}^1, \overline{x}^2, \ldots, \overline{x}^N\right); i = 1, 2, \ldots, N$. From this result, it follows that $|a_{ij}|$ is not, in general, an invariant due to the presence of the term J^2 in the right-hand side. In this case, $|a_{ij}|$ is said to be a *relative invariant* of weight 2.

EXAMPLE 1.8.5 *If a_{ij} are components of a covariant tensor of second order and λ^i, μ^j are components of two contravariant vector, show that $a_{ij}\lambda^i\mu^j$ is an invariant.*

Solution: Since a_{ij} are components of a covariant tensor of type $(0, 2)$, we have, $\overline{a}_{ij} = \frac{\partial x^p}{\partial \overline{x}^i} \frac{\partial x^q}{\partial \overline{x}^j} a_{pq}$. We have to show that $a_{ij}\lambda^i\mu^j$ is an invariant. Now,

$$\begin{aligned}
\overline{a}_{ij}\overline{\lambda}^i\overline{\mu}^j &= \frac{\partial x^p}{\partial \overline{x}^i} \frac{\partial x^q}{\partial \overline{x}^j} a_{pq} \frac{\partial \overline{x}^i}{\partial x^r} \lambda^r \frac{\partial \overline{x}^j}{\partial x^s} \mu^s \\
&= \frac{\partial x^p}{\partial \overline{x}^i} \frac{\partial x^q}{\partial \overline{x}^j} \frac{\partial \overline{x}^i}{\partial x^r} \frac{\partial \overline{x}^j}{\partial x^s} a_{pq}\lambda^r\mu^s \\
&= \frac{\partial x^p}{\partial \overline{x}^i} \frac{\partial \overline{x}^i}{\partial x^r} \frac{\partial x^q}{\partial \overline{x}^j} \frac{\partial \overline{x}^j}{\partial x^s} a_{pq}\lambda^r\mu^s \\
&= \delta^p_r \delta^q_s a_{pq}\lambda^r\mu^s = \delta^p_r a_{pq}\delta^q_s \mu^s \lambda^r \\
&= a_{rq}\mu^q\lambda^r = a_{rp}\lambda^r\mu^q.
\end{aligned}$$

Replacing the dummy indices r and p by i and j, we get

$$\overline{a}_{ij}\overline{\lambda}^i\overline{\mu}^j = a_{ij}\lambda^i\mu^j,$$

from which it follows that $a_{ij}\lambda^i\mu^j$ is an invariant.

EXAMPLE 1.8.6 *If $a^{ij}_k\lambda_i\mu_j\nu^k$ is a scalar invariant, λ_i, μ_j and ν^k are vectors, show that a^{ij}_k is a mixed tensor of type $(2, 1)$.*

Solution: Since $a_k^{ij}\lambda_i\mu_j\nu^k$ is a scalar invariant, so,

$$\overline{a}_k^{ij}\overline{\lambda}_i\overline{\mu}_j\overline{\nu}^k = a_k^{ij}\lambda_i\mu_j\nu^k = a_p^{\alpha\beta}\lambda_\alpha\mu_\beta\nu^p$$

or

$$\overline{a}_k^{ij}\frac{\partial x^\alpha}{\partial \overline{x}^i}\lambda_\alpha\frac{\partial x^\beta}{\partial \overline{x}^j}\mu_\beta\frac{\partial \overline{x}^k}{\partial x^p}\nu^p = a_p^{\alpha\beta}\lambda_\alpha\mu_\beta\nu^p$$

or

$$\left[\overline{a}_k^{ij}\frac{\partial x^\alpha}{\partial \overline{x}^i}\frac{\partial x^\beta}{\partial \overline{x}^j}\frac{\partial \overline{x}^k}{\partial x^p} - a_p^{\alpha\beta}\right]\lambda_\alpha\mu_\beta\nu^p = 0$$

or

$$\overline{a}_k^{ij}\frac{\partial x^\alpha}{\partial \overline{x}^i}\frac{\partial x^\beta}{\partial \overline{x}^j}\frac{\partial \overline{x}^k}{\partial x^p} = a_p^{\alpha\beta}.$$

From this relation it follows that, a_k^{ij} is a mixed tensor of type $(2, 1)$.

EXAMPLE 1.8.7 *If f is an invariant, determine whether $\frac{\partial^2 f}{\partial x^p \partial x^q}$ is a tensor.*

Solution: Since f is an invariant, so $f = \overline{f}$. Let f be a scalar function of co-ordinates x^i. Consider a co-ordinate transformation $x^i \to \overline{x}^i$, i.e. $\overline{x}^i = \overline{x}^i(x^k)$. Evidently,

$$\frac{\partial f}{\partial x^i} = \frac{\partial \overline{f}}{\partial \overline{x}^p}\frac{\partial \overline{x}^p}{\partial x^i};\ \text{as } f = \overline{f}.$$

This is a covariant law of transformation. Hence, the gradient of a scalar function f, i.e. $\frac{\partial f}{\partial x^i}$ is a covariant vector. Now,

$$\begin{aligned}\frac{\partial^2 f}{\partial x^i \partial x^j} &= \frac{\partial \overline{f}}{\partial \overline{x}^p}\frac{\partial^2 \overline{x}^p}{\partial x^j \partial x^i} + \frac{\partial}{\partial x^j}\left(\frac{\partial \overline{f}}{\partial \overline{x}^p}\right)\frac{\partial \overline{x}^p}{\partial x^i}\\ &= \frac{\partial \overline{f}}{\partial \overline{x}^p}\frac{\partial^2 \overline{x}^p}{\partial x^j \partial x^i} + \frac{\partial^2 \overline{f}}{\partial \overline{x}^q \partial \overline{x}^p}\frac{\partial \overline{x}^q}{\partial x^j}\frac{\partial \overline{x}^p}{\partial x^i}.\end{aligned}$$

Therefore, if f is an invariant, $\frac{\partial^2 f}{\partial x^p \partial x^q}$ is not a tensor due to the presence of the first term in the right-hand side.

1.9 Addition and Subtraction of Tensors

Two tensors can be added or subtracted provided they are of the same rank and similar character. Note that, these two binary operations relate to tensors at same point.

Theorem 1.9.1 *If A_k^{ij} and B_k^{ij} be tensors, then their sum and difference are tensors.*

Proof: Let A^{ij}_k and B^{ij}_k be tensors so that they satisfy the tensor law of transformations namely,

$$\overline{A}^{ij}_k = \frac{\partial \overline{x}^i}{\partial x^p} \frac{\partial \overline{x}^j}{\partial x^q} \frac{\partial x^r}{\partial \overline{x}^k} A^{pq}_r$$

and

$$\overline{B}^{ij}_k = \frac{\partial \overline{x}^i}{\partial x^p} \frac{\partial \overline{x}^j}{\partial x^q} \frac{\partial x^r}{\partial \overline{x}^k} B^{pq}_r.$$

(i) Let the sum of the two tensors A^{ij}_k and B^{ij}_k be defined as

$$A^{ij}_k + B^{ij}_k = C^{ij}_k$$

and the algebraic operation by which the sum is obtained is called *addition of tensors.* We have to show that C^{ij}_k is a tensor. Now,

$$\overline{A}^{ij}_k + \overline{B}^{ij}_k = \frac{\partial \overline{x}^i}{\partial x^p} \frac{\partial \overline{x}^j}{\partial x^q} \frac{\partial x^r}{\partial \overline{x}^k} \left(A^{pq}_r + B^{pq}_r\right)$$

or

$$\overline{C}^{ij}_k = \frac{\partial \overline{x}^i}{\partial x^p} \frac{\partial \overline{x}^j}{\partial x^q} \frac{\partial x^r}{\partial \overline{x}^k} C^{pq}_r.$$

This shows that C^{ij}_k satisfies the tensor law of transformation and hence C^{ij}_k is a tensor. In general, if $A^{i_1 i_2 \cdots i_p}_{j_1 j_2 \cdots j_q}$ and $B^{i_1 i_2 \ldots i_p}_{j_1 j_2 \cdots j_q}$ are components of two tensors of type (p, q), then the sum

$$A^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q} + B^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q}$$

are the components of another tensor of type (p, q).

(ii) Let the difference of the two tensors A^{ij}_k and B^{ij}_k be defined as

$$A^{ij}_k - B^{ij}_k = D^{ij}_k$$

and the algebraic operation by which the sum is obtained, is called *subtraction of tensors.* Now,

$$\overline{A}^{ij}_k - \overline{B}^{ij}_k = \frac{\partial \overline{x}^i}{\partial x^p} \frac{\partial \overline{x}^j}{\partial x^q} \frac{\partial x^r}{\partial \overline{x}^k} \left(A^{pq}_r - B^{pq}_r\right)$$

or

$$\overline{D}^{ij}_k = \frac{\partial \overline{x}^i}{\partial x^p} \frac{\partial \overline{x}^j}{\partial x^q} \frac{\partial x^r}{\partial \overline{x}^k} D^{pq}_r.$$

This shows that D^{ij}_k satisfies the tensor law of transformation and hence D^{ij}_k is a tensor. In general, if $A^{i_1 i_2 \cdots i_p}_{j_1 j_2 \cdots j_q}$ and $B^{i_1 i_2 \cdots i_p}_{j_1 j_2 \cdots j_q}$ are components of two tensors of type (p, q), then the difference

$$A^{i_1 i_2 \cdots i_p}_{j_1 j_2 \cdots j_q} - B^{i_1 i_2 \cdots i_p}_{j_1 j_2 \cdots j_q}$$

are the components of another tensor of type (p, q).

1.10 Multiplication by a Scalar

Let A_k^{ij} be the components of the tensor in a co-ordinate system (x^i) and $\overline{A}_k^{ij}$ be its components in another system $(\overline{x}^i)$. Let the scalar be denoted by ϕ and $\overline{\phi}$ in the co-ordinate systems (x^i) and $(\overline{x}^i)$, respectively. Then,

$$\begin{aligned}\overline{\phi}\,\overline{A}_k^{ij} &= \overline{\phi}\frac{\partial \overline{x}^i}{\partial x^p}\frac{\partial \overline{x}^j}{\partial x^q}\frac{\partial x^r}{\partial \overline{x}^k}A_r^{pq}\\ &= \phi\frac{\partial \overline{x}^i}{\partial x^p}\frac{\partial \overline{x}^j}{\partial x^q}\frac{\partial x^r}{\partial \overline{x}^k}A_r^{pq}; \quad \text{as} \quad \overline{\phi} = \phi\\ &= \frac{\partial \overline{x}^i}{\partial x^p}\frac{\partial \overline{x}^j}{\partial x^q}\frac{\partial x^r}{\partial \overline{x}^k}\phi A_r^{pq}.\end{aligned}$$

From this it follows that ϕA_k^{ij} are the components of a tensor of type $(2, 1)$. This tensor is called the *product of the tensor* and the scalar under consideration and the algebraic operation by which it is obtained, is called the *multiplication of the tensor by the scalar.*

In general, if $A_{j_1 \cdots j_p}^{i_1 \cdots i_p}$ are components of the tensor and ϕ be a scalar, then $\phi A_{j_1 \cdots j_p}^{i_1 \cdots i_p}$ are the components of a tensor of the type (p, q), called the *multiplication of the tensor by a scalar.* Note that, this operation relates to a tensor and a scalar at the same point.

Result 1.10.1 (Zero tensor): The components of a tensor may be all zero in a co-ordinate system. Let the components $A_{j_1 \cdots j_p}^{i_1 \cdots i_p}$ of a tensor of the type (p, q) be all zero in a co-ordinate system (x^i). Denote its components in another system $(\overline{x}^i)$ by $\overline{A}_{j_1 \cdots j_p}^{i_1 \cdots i_p}$. Then by the tensor law of transformation, we have

$$\begin{aligned}\overline{A}_{j_1 \cdots j_p}^{i_1 \cdots i_p} &= \frac{\partial \overline{x}^{i_1}}{\partial x^{t_1}} \cdots \frac{\partial \overline{x}^{i_p}}{\partial x^{t_p}}\frac{\partial x^{r_1}}{\partial \overline{x}^{j_1}} \cdots \frac{\partial x^{r_q}}{\partial \overline{x}^{j_q}}A_{r_1 \cdots r_q}^{t_1 \cdots t_p}\\ &= 0; \quad \text{as by condition } A_{r_1 \cdots r_q}^{t_1 \cdots t_p} = 0.\end{aligned}$$

Thus, if the components of a tensor are all zero in one co-ordinate system, then they are also zero in every other co-ordinate system. A tensor whose components are all zero in every co-ordinate system is called a *zero tensor.*

Result 1.10.2 (Equality of two tensors): Two tensors of the same type are said to be equal in the same co-ordinate system if they have the same contravariant rank and the same covariant rank and every component of one is equal to the corresponding component of the other.

Thus, if $A_{j_1 j_2 \cdots j_q}^{i_1 i_2 \cdots i_p}$ and $B_{j_1 j_2 \cdots j_q}^{i_1 i_2 \cdots i_p}$ are components of two equal tensors of type (p, q) in the same co-ordinate system, then

$$A_{j_1 j_2 \cdots j_q}^{i_1 i_2 \cdots i_p} = B_{j_1 j_2 \cdots j_q}^{i_1 i_2 \cdots i_p}.$$

Let $A^{i_1 i_2 \cdots i_p}_{j_1 j_2 \cdots j_q}$ and $B^{i_1 i_2 \cdots i_p}_{j_1 j_2 \cdots j_q}$ are components of two equal tensors in a co-ordinate system $\left(x^i\right)$. Hence, the difference

$$A^{i_1 i_2 \cdots i_p}_{j_1 j_2 \cdots j_q} - B^{i_1 i_2 \cdots i_p}_{j_1 j_2 \cdots j_q}$$

are the components of a zero tensor in the co-ordinate system (x^i). Therefore, the difference of the two tensors under consideration must be a zero tensor in every other co-ordinate system. This means that the tensors are equal in every other co-ordinate system. Thus two tensors of the same type are said to be equal if their components are equal to each other in every co-ordinate system.

Result 1.10.3 Any linear combination of tensors of the same type and rank is again a tensor of the same type and rank.

1.11 Outer Multiplication

Let A^i_{jk} and B^m_n be tensors of type $(1,2)$ and $(1,1)$, respectively, then

$$\overline{A}^i_{jk} = \frac{\partial \overline{x}^i}{\partial x^r} \frac{\partial x^s}{\partial \overline{x}^j} \frac{\partial x^t}{\partial \overline{x}^k} A^r_{st}$$

and

$$\overline{B}^m_n = \frac{\partial \overline{x}^m}{\partial x^u} \frac{\partial x^v}{\partial \overline{x}^n} B^u_v.$$

Let the product of two tensors A^i_{jk} and B^m_n be defined as

$$A^i_{jk} B^m_n = C^{im}_{jkn}, \quad \text{say.} \tag{1.53}$$

Since A^i_{jk} has N^3 components, B^m_n has N^2 components, so, C^{im}_{jkn} has N^S components. Now,

$$\begin{aligned} \overline{A}^i_{jk} \overline{B}^m_n &= \frac{\partial \overline{x}^i}{\partial x^r} \frac{\partial x^s}{\partial \overline{x}^j} \frac{\partial x^t}{\partial \overline{x}^k} A^r_{st} \frac{\partial \overline{x}^m}{\partial x^u} \frac{\partial x^v}{\partial \overline{x}^n} B^u_v \\ &= \frac{\partial \overline{x}^i}{\partial x^r} \frac{\partial x^s}{\partial \overline{x}^j} \frac{\partial x^t}{\partial \overline{x}^k} \frac{\partial \overline{x}^m}{\partial x^u} \frac{\partial x^v}{\partial \overline{x}^n} A^r_{st} B^u_v \end{aligned}$$

or

$$\overline{C}^{im}_{jkn} = \frac{\partial \overline{x}^i}{\partial x^r} \frac{\partial x^s}{\partial \overline{x}^j} \frac{\partial x^t}{\partial \overline{x}^k} \frac{\partial \overline{x}^m}{\partial x^u} \frac{\partial x^v}{\partial \overline{x}^n} C^{ru}_{stv}.$$

From this relation it follows that $C^{im}_{jkn} = A^i_{jk} B^m_n$ are the components of a tensor of type $(2,3)$. The tensor $A^i_{jk} B^m_n$ is called the *open or outer product* or *Kronecker product* of the tensors A^i_{jk} and B^m_n and its rank is higher than that of each of the tensors from which it is obtained and the operation by which this tensor is obtained, is called *outer multiplication.* In general, if we multiply a tensor $A^{i_1 i_2 \cdots i_l}_{j_1 j_2 \cdots j_m}$, of the type (l, m) by a tensor $B^{p_1 p_2 \cdots p_r}_{q_1 q_2 \cdots q_s}$, of the type (r, s), where r and s not being both zero, then the product obtained is

$$A^{i_1 i_2 \cdots i_l}_{j_1 j_2 \cdots j_m} B^{p_1 p_2 \cdots p_r}_{q_1 q_2 \cdots q_s}, \tag{1.54}$$

of rank $(l + r, m + s)$, this product is called the *open or outer product* of two tensors.

Result 1.11.1 In taxing the out product of any number of tensors, care should be taken to use distinct indices. For example, it should be wrong to write the outer product of A^i_j, B_k and C^{lm}_{kp} as $A^i_j B_k C^{lm}_{kp}$, because the covariant index k is repeated.

Result 1.11.2 The operation of outer product/multiplication relates to tensors of any two types [the type $(0, 0)$ being excluded] at the same point.

Result 1.11.3 The outer product of two tensors is a tensor whose order is sum of the orders of the two tensors. This provides us with an easy method of forming tensors of higher rank and of any variance (co or contra).

Result 1.11.4 Let C^i_j be the open product of two vectors A^i and B_j, then $C^i_j = A^i B_j$ is a mixed tensor of order two. But every mixed tensor of order two is not necessarily the tensor product of contravariant vector and a covariant vector. Note that, every tensor can not be written as a product of two tensors of lower rank. For this reason division of tensors is not always possible.

1.12 Contraction

If we set in a mixed tensor one covariant and one contravariant suffixes equal, the process is called *contraction.* Let A^{ij}_{klm} be a mixed tensor of type $(2, 3)$, then by tensor law of transformation,

$$\overline{A}^{ij}_{klm} = \frac{\partial \overline{x}^i}{\partial x^p} \frac{\partial \overline{x}^j}{\partial x^q} \frac{\partial x^r}{\partial \overline{x}^k} \frac{\partial x^s}{\partial \overline{x}^l} \frac{\partial x^t}{\partial \overline{x}^m} A^{pq}_{rst}.$$

Replacing the lower index l by the upper index i and taking summation over i, we get

$$\begin{aligned}
\overline{A}^{ij}_{kim} &= \frac{\partial \overline{x}^i}{\partial x^p} \frac{\partial \overline{x}^j}{\partial x^q} \frac{\partial x^r}{\partial \overline{x}^k} \frac{\partial x^s}{\partial \overline{x}^i} \frac{\partial x^t}{\partial \overline{x}^m} A^{pq}_{rst} \\
&= \frac{\partial \overline{x}^i}{\partial x^p} \frac{\partial x^s}{\partial \overline{x}^i} \frac{\partial \overline{x}^j}{\partial x^q} \frac{\partial x^r}{\partial \overline{x}^k} \frac{\partial x^t}{\partial \overline{x}^m} A^{pq}_{rst} \\
&= \delta^s_p \frac{\partial \overline{x}^j}{\partial x^q} \frac{\partial x^r}{\partial \overline{x}^k} \frac{\partial x^t}{\partial \overline{x}^m} A^{pq}_{rst} = \frac{\partial \overline{x}^j}{\partial x^q} \frac{\partial x^r}{\partial \overline{x}^k} \frac{\partial x^t}{\partial \overline{x}^m} A^{pq}_{rpt}.
\end{aligned}$$

If we denote A^{pq}_{rpt} by B^q_{rt} and $\overline{A}^{ij}_{kim}$ by $\overline{B}^j_{km}$, then the above relation can be written as

$$\overline{B}^j_{km} = \frac{\partial \overline{x}^j}{\partial x^q}\frac{\partial x^r}{\partial \overline{x}^k}\frac{\partial x^t}{\partial \overline{x}^m} B^q_{rt},$$

which shows that B^j_{km}, i.e. A^{ij}_{kim} is a tensor of type $(1, 2)$. The tensor A^{ij}_{kim} is called a *contracted tensor* of the given tensor and the operation by which it is obtained, is called *contraction.* Therefore, contraction reduces rank of tensor by two. In general, if $A^{i_1 i_2 \cdots i_l}_{j_1 j_2 \cdots j_m}$, of the type $(l, m); l \neq 0, m \neq 0$, then the quantities obtained by replacing any one upper index i_p and one lower index j_q by the same index i_p and performing summation over i_p, are the components of a tensor of type $(l - 1, m - 1)$.

(i) A tensor can be repeatedly contracted. Thus the tensor A^{ijk}_{lm} of total rank 5, on contraction, gives the tensor A^{ijk}_{im} of total rank 3, which can be further contracted go give the tensor A^{ijk}_{ij} or A^{ijk}_{ik} of contravariant rank 1.

(ii) It should be evident that the inner product of tensors can be thought of as their outer product followed by contraction. Thus, the inner product $A^{ij}_k B^k_q = C^{ij}_k$ can be obtained by first taking the outer product $A^{ij}_k B^p_q = D^{ijp}_{kq}$, then contracting this tensor by equating the indices p and k, and finally identifying C^{ij}_q with D^{ijp}_{pq}.

(iii) If two similar indices of a tensor are equated, the resulting entity is not a tensor. Thus, if D^{ijp}_{kq} is a tensor, D^{ijp}_{kq} and D^{ijp}_{kk} are not tensors.

Note 1.12.1 Contraction is to be operated with respect to an upper index and a lower index and not with respect to two indices of the same kind.

Note 1.12.2 Contraction of m pairs of indices of a tensor of type (p, q) yields a tensor of type $(p - m, q - m)$, whose rank is less than that of original tensor by $2m$. Thus contraction can lower the rank of a tensor by an even number only. Let us consider a mixed tensor A^i_j of the type $(1, 1)$, then by tensor law of transformation, we get

$$\overline{A}^i_j = \frac{\partial \overline{x}^i}{\partial x^p}\frac{\partial x^q}{\partial \overline{x}^j} A^p_q.$$

Contracting with respect to i and j, we get,

$$\overline{A}^i_i = \frac{\partial \overline{x}^i}{\partial x^p}\frac{\partial x^q}{\partial \overline{x}^i} A^p_q = \delta^q_p A^p_q = A^p_p = A^i_i,$$

which is an invariant. Thus, contraction of a pair of indices of a tensor of type $(1, 1)$ yields a tensor of type $(0, 0)$, i.e. an invariant.

1.13 Inner Multiplication

Inner multiplication is a combination of outer multiplication and contraction. If an outer product of two tensors be contracted with respect to an upper index of one factor and lower index of other, then a tensor is obtained which is called an *inner product* of two tensors.

Let A_r^p and B_t^{qs} be the components of mixed tensors of the type $(1,1)$ and (2, 1), respectively. Then the outer product of these two tensors is $A_r^p B_t^{qs}$. The inner product of these tensors are given by $A_r^p B_t^{rs}$. By tensor law of transformation, we get

$$\begin{aligned}\overline{A}_r^p\ \overline{B}_t^{rs} &= \frac{\partial \overline{x}^p}{\partial x^i}\frac{\partial x^j}{\partial \overline{x}^r}A_j^i\frac{\partial \overline{x}^r}{\partial x^k}\frac{\partial \overline{x}^s}{\partial x^l}\frac{\partial x^m}{\partial \overline{x}^t}B_m^{kl}\\ &= \frac{\partial \overline{x}^p}{\partial x^i}\frac{\partial x^j}{\partial \overline{x}^r}\frac{\partial \overline{x}^r}{\partial x^k}\frac{\partial \overline{x}^s}{\partial x^l}\frac{\partial x^m}{\partial \overline{x}^t}A_j^iB_m^{kl}\\ &= \frac{\partial \overline{x}^p}{\partial x^i}\delta_k^j\frac{\partial \overline{x}^s}{\partial x^l}\frac{\partial x^m}{\partial \overline{x}^t}A_j^iB_m^{kl}\\ &= \frac{\partial \overline{x}^p}{\partial x^i}\frac{\partial \overline{x}^s}{\partial x^l}\frac{\partial x^m}{\partial \overline{x}^t}A_k^iB_m^{kl}.\end{aligned}$$

But this is the law of transformation of the mixed tensor of rank 3. Thus, the inner product $A_r^p B_t^{rs}$ is a mixed tensor of rank 3. Consider the following two particular cases:

(i) The outer product A_iB^{jk} of the tensors A_i and B^{jk}, when contracted for indices i and j, produces the tensor A_iB^{ik}, which is of the type $(1,0)$. This tensor is an inner product of the tensors A_i and B^{jk}. Another inner product A_iB^{ji} can be obtained by contracting for the indices i and k in the outer product A_iB^{jk}.

(ii) The outer product A_iB^j of the vectors A_i and B^j, when contracted for the indices i and j, produces the tensor A_iB^i of type $(0,0)$, namely a scalar. This inner product is called the scalar product of vectors A_i and B^j, because it is a scalar. This scalar product is an invariant, i.e. it has the same value in any set of co-ordinates.

If we set in a product of two tensors one contravariant and one covariant suffixes equal, the process is called inner multiplication and the resulting tensor is called the inner product of two tensors. For example,

$$A_k^{ij}B_{pqr}^k;\ A_k^{ij}B_{ipr}^s;\ A_k^{ij}B_{pjr}^s$$

all the inner products of the tensors A_k^{ij} and B_{pqr}^s. No index should occur more than twice. For example, it should be wrong to write the inner product as $A_k^{ij}B_q^i$ because two contravariant indices has been equated, or as $A_k^{ik}B_k^k$, because k is repeated four times.

Note: Each of the above mentioned algebraic operations on tensor or tensors produces again a tensor at the same point. These operations constitute what is called the tensor algebra of V_N.

1.14 Quotient Law of Tensors

The name quotient law is in a certain sense appropriate because the application of this law produces a tensor from two tensors just as the operation of division of two numbers produces a number, namely, their quotient. If the result of taking an inner product of a given set of functions with a particular type of tensor of arbitrary components is known to be a tensor, then the given functions will form the components of a tensor.

Let a quantity $A(p,q,r)$ be such that in the co-ordinate system $\left(x^i\right)$,

$$A(p,q,r)B_r^{qs} = C_p^s,$$

where B_r^{qs} is an arbitrary tensor and C_p^s is a tensor. In the $\left(\overline{x}^i\right)$ co-ordinate system, this is transformed to

$$\overline{A}(i,j,k)\overline{B}_k^{jl} = \overline{C}_i^l.$$

Applying tensor law of transformation, we get

$$\overline{A}(i,j,k)B_r^{qt}\frac{\partial\overline{x}^j}{\partial x^q}\frac{\partial\overline{x}^l}{\partial x^t}\frac{\partial x^r}{\partial\overline{x}^k} = C_p^s\frac{\partial\overline{x}^l}{\partial x^s}\frac{\partial x^p}{\partial\overline{x}^i}$$

or

$$\overline{A}(i,j,k)B_r^{qt}\frac{\partial\overline{x}^j}{\partial x^q}\frac{\partial\overline{x}^l}{\partial x^t}\frac{\partial x^r}{\partial\overline{x}^k}\frac{\partial x^s}{\partial\overline{x}^l}\frac{\partial\overline{x}^i}{\partial x^p} = C_p^s$$

or

$$\overline{A}(i,j,k)B_r^{qt}\frac{\partial\overline{x}^j}{\partial x^q}\frac{\partial\overline{x}^l}{\partial x^t}\frac{\partial x^s}{\partial\overline{x}^l}\frac{\partial x^r}{\partial\overline{x}^k}\frac{\partial\overline{x}^i}{\partial x^p} = C_p^s$$

or

$$\overline{A}(i,j,k)B_r^{qt}\frac{\partial\overline{x}^j}{\partial x^q}\delta_t^s\frac{\partial x^r}{\partial\overline{x}^k}\frac{\partial\overline{x}^i}{\partial x^p} = C_p^s$$

or

$$\overline{A}(i,j,k)B_r^{qs}\frac{\partial\overline{x}^j}{\partial x^q}\frac{\partial x^r}{\partial\overline{x}^k}\frac{\partial\overline{x}^i}{\partial x^p} = A(p,q,r)B_r^{qs}$$

or

$$\left[\overline{A}(i,j,k)\frac{\partial\overline{x}^i}{\partial x^p}\frac{\partial\overline{x}^j}{\partial x^q}\frac{\partial x^r}{\partial\overline{x}^k} - A(p,q,r)\right]B_r^{qs} = 0.$$

Since B_r^{qs} is arbitrary so the expression within the third bracket is zero and consequently,

$$A(p,q,r) = \overline{A}(i,j,k)\frac{\partial\overline{x}^i}{\partial x^p}\frac{\partial\overline{x}^j}{\partial x^q}\frac{\partial x^r}{\partial\overline{x}^k},$$

which confirms the tensor law of transformation. Thus $A(p, q, r)$ is a tensor of the type A^r_{pq}. In general, let $A^{i_1\cdots i_p}_{j_1\cdots j_q k}$ be a set of quantities where k, the i's and j's take the values from 1 to N. Let u^k be an arbitrary vector. If the inner product $B^{i_1\cdots i_p}_{j_1\cdots j_q}$ given by

$$B^{i_1\cdots i_p}_{j_1\cdots j_q} = A^{i_1\cdots i_p}_{j_1\cdots j_q k} u^k$$

is a tensor, then $A^{i_1\cdots i_p}_{j_1\cdots j_q k}$ is a tensor. It is important in the use of the quotient law that the tensor with which inner product is taken should be an arbitrary tensor. The following statements are useful criteria or 'tests' for tensor character; they may all be derived as special cases of the quotient law:

(i) If $A_i B^i \equiv E$ is invariant for all contravariant vectors B^i, then A_i is a covariant vector.

(ii) If $A_{ij} B^i \equiv C_j$ are components of a covariant vector for all contravariant vectors B^i, then A_{ij} is a covariant tensor of order two.

(iii) If $A_{ij} B^i C^j \equiv E$ is invariant for all contravariant vectors B^i and C^i, then A_{ij} is a covariant tensor of order two.

(iv) If A_{ij} is symmetric and $A_{ij} B^i B^j \equiv E$ is invariant for all contravariant vectors B^i, then A_{ij} is a covariant tensor of order two.

EXAMPLE 1.14.1 *Using quotient law of tensor, prove that Kronecker delta is a mixed tensor of rank 2.*

Solution: Let A^k be an arbitrary vector. Using the definition of Kronecker delta, we have, $\delta^i_j A^j = A^i$.

Thus, we see that the inner product of δ^i_j with an arbitrary vector A^k is a contravariant vector of rank 1. Hence, by quotient law δ^i_j is also a tensor. We see that, δ^i_j has one subscript and one superscript, so that δ^i_j is a mixed tensor of rank 2.

Deduction 1.14.1 Tensor equations: Much of the importance of tensors in mathematical physics and engineering resides in the fact that if a tensor equation or identity is true in one co-ordinate system, then it is true in all co-ordinate systems.

There are some simple rules for checking the correctness of indices in a tensor equation.

(i) A free index should match in all terms throughout the equation.

(ii) A dummy index should match in each term of the equation separately.

(iii) No index should occur more than twice in any term.

(iv) When a co-ordinate differential such as ∂x^i occurs, is a term i is to be regarded as a contravariant index if ∂x^i occurs in the numerator and as a covariant index if it occurs in the denominator. Thus, in an expression such as $\frac{\partial x^i}{\partial \overline{x}^j}$, i is a contravariant index, while j is a covariant index.

1.15 Symmetric Tensor

If in a co-ordinate system two contravariant or covariant indices of a tensor can be interchanged without altering the tensor, then it is said to be symmetric with respect to these indices in the co-ordinate system. So, a tensor A_{ij} is said to be symmetric if

$$A_{ij} = A_{ji}.$$

For a general tensor of arbitrary rank, symmetry can be defined for a pair of similar indices. For example, a tensor A_{ijk} is symmetric in the suffixes j and k if

$$A^{lmp}_{ijk} = A^{lmp}_{ikj}.$$

It is important to specify the positions of the indices rather than the indices themselves.

Property 1.15.1 Symmetric property remains unchanged by tensor law of transformation, i.e. if a tensor is symmetric with respect to two contravariant or covariant indices in any co-ordinate system, then it remains so with respect to these two indices in any other co-ordinate system.

Proof: Let a tensor A_{ij} be symmetric in one co-ordinate system $\left(x^i\right)$, i.e. $A_{ij} = A_{ji}$ and $\overline{A}_{ij}$ in another co-ordinate system $\left(\overline{x}^i\right)$. Now,

$$\begin{aligned}\overline{A}_{ij} &= \frac{\partial x^p}{\partial \overline{x}^i}\frac{\partial x^q}{\partial \overline{x}^j}A_{pq} = \frac{\partial x^p}{\partial \overline{x}^i}\frac{\partial x^q}{\partial \overline{x}^j}A_{qp}\\ &= \frac{\partial x^q}{\partial \overline{x}^j}\frac{\partial x^p}{\partial \overline{x}^i}A_{qp} = \overline{A}_{ji}.\end{aligned}$$

This shows that symmetry with respect to i and j also holds in the system $\left(\overline{x}^i\right)$. Similar result may be obtained by tracing the case of a covariant tensor or a mixed tensor. Thus the property of symmetry is an intrinsic property of a tensor and is independent of the choice of the co-ordinate system.

Note 1.15.1 It is to be noted that symmetry cannot, in general, be defined for a tensor with respect to two indices of which one is contravariant and the other is covariant, except the tensor δ^i_j, which has the interesting property that it is symmetric in i and j and this symmetry is preserved under co-ordinate transformation. Thus,

$$\delta^i_j = \delta^j_i \text{ and } \overline{\delta}^i_j = \overline{\delta}^j_i; \text{ as } \delta^i_j = \overline{\delta}^i_j.$$

Property 1.15.2 In an N dimensional space, a symmetric covariant tensor of second order has atmost $\frac{N(N+1)}{2}$ different components.

Proof: Let A_{ij} be a symmetric covariant tensor of second order, then it has N^2 components in V_N. These components are

$$\begin{array}{cccc} A_{11} & A_{12} & \cdots & A_{1N} \\ A_{21} & A_{22} & \cdots & A_{2N} \\ \vdots & \vdots & \cdots & \vdots \\ A_{N1} & A_{N2} & \cdots & A_{NN} \end{array}.$$

These components are of two types:

(i) Those in which the indices i and j are the same, i.e. the components along the diagonal. The maximum number of distinct components of this type is N.

(ii) Those in which the indices i and j are different, i.e. the components along the nondiagonal. Hence, the maximum number of components of this type is

$$= N^2 - N = N(N-1).$$

But due to symmetry of A_{ij}, (the components above and below the diagonal) the maximum number of distinct components of this type is $\frac{N(N-1)}{2}$.

Therefore, the maximum number of independent components is given by

$$N + \frac{N(N-1)}{2} = \frac{N}{2}(N+1).$$

EXAMPLE 1.15.1 *Assume $\phi = a_{jk}A^jA^k$, show that $\phi = b_{jk}A^jA^k$, where b_{jk} is symmetric.*

Solution: In the given relation $\phi = a_{jk}A^jA^k$, interchanging the indices k and j, we get, $\phi = a_{kj}A^kA^j$. Therefore,

$$2\phi = (a_{jk} + a_{kj})\, A^jA^k$$

or

$$\phi = \frac{1}{2}(a_{jk} + a_{kj})\, A^jA^k = b_{jk}A^jA^k, \text{ say}$$

where, $b_{jk} = \frac{1}{2}(a_{jk} + a_{kj})$. Also,

$$b_{jk} = \frac{1}{2}(a_{jk} + a_{kj}) = \frac{1}{2}(a_{kj} + a_{jk}) = b_{kj}.$$

Therefore, b_{jk} is symmetric.

EXAMPLE 1.15.2 *If the tensors a_{ij} and b_{ij} are symmetric, and u^i, v^i are components of contravariant vectors satisfying the equations*

$$\left.\begin{aligned}(a_{ij} - kb_{ij})\, u^i = 0 \\ (a_{ij} - k'b_{ij})\, v^i = 0\end{aligned}\right\};\ i, j = 1, 2, \ldots, n \text{ and } k \neq k',$$

show that $b_{ij}u^iv^j = 0$ and $a_{ij}u^iv^j = 0$.

Solution: Since a_{ij} and b_{ij} are symmetric, so $a_{ij} = a_{ji}, b_{ij} = b_{ji}$. Multiplying the first equation by v^j and the second by u^j, respectively, and subtracting, we get

$$a_{ij}u^iv^j - a_{ij}v^iu^j - kb_{ij}u^iv^j + k'b_{ij}v^iu^j = 0$$

or

$$a_{ij}u^iv^j - a_{ji}v^iu^j - kb_{ij}u^iv^j + k'b_{ji}v^iu^j = 0$$

or

$$a_{ij}u^iv^j - a_{ij}u^iv^j - kb_{ij}u^iv^j + k'b_{ij}u^iv^j = 0$$

(interchanging dummy indices i, j)

or

$$-(k - k')b_{ij}u^iv^j = 0 \Rightarrow b_{ij}u^iv^j = 0 \text{ as } k \neq k'.$$

Multiplying the first equation by v^j, we get

$$a_{ij}u^iv^j - kb_{ij}u^iv^j = 0.$$

or

$$a_{ij}u^iv^j = 0;\ \text{as } b_{ij}u^iv^j = 0.$$

EXAMPLE 1.15.3 *If a_{ij} is symmetric tensor and b_i is a vector and*

$$a_{ij}b_k + a_{jk}b_i + a_{ki}b_j = 0,$$

then prove that $a_{ij} = 0$ or $b_k = 0$.

Solution: The equation is

$$a_{ij}b_k + a_{jk}b_i + a_{ki}b_j = 0; \overline{a}_{ij}\overline{b}_k + \overline{a}_{jk}\overline{b}_i + \overline{a}_{ki}\overline{b}_j = 0.$$

Using tensor law of transformation, we get

$$a_{pq}\frac{\partial x^p}{\partial \overline{x}^i}\frac{\partial x^q}{\partial \overline{x}^j}b_r\frac{\partial x^r}{\partial \overline{x}^k} + a_{pq}\frac{\partial x^p}{\partial \overline{x}^j}\frac{\partial x^q}{\partial \overline{x}^k}b_r\frac{\partial x^r}{\partial \overline{x}^i} + a_{pq}\frac{\partial x^p}{\partial \overline{x}^k}\frac{\partial x^q}{\partial \overline{x}^i}b_r\frac{\partial x^r}{\partial \overline{x}^j} = 0$$

or

$$a_{pq}b_r\left[\frac{\partial x^p}{\partial \overline{x}^i}\frac{\partial x^q}{\partial \overline{x}^j}\frac{\partial x^r}{\partial \overline{x}^k} + \frac{\partial x^p}{\partial \overline{x}^j}\frac{\partial x^q}{\partial \overline{x}^k}\frac{\partial x^r}{\partial \overline{x}^i} + \frac{\partial x^p}{\partial \overline{x}^k}\frac{\partial x^q}{\partial \overline{x}^i}\frac{\partial x^r}{\partial \overline{x}^j}\right] = 0$$

or

$$a_{pq}b_r = 0 \Rightarrow a_{pq} = 0 \quad \text{or} \quad b_r = 0$$

or

$$a_{ij} = 0 \quad \text{or} \quad b_k = 0.$$

1.16 Skew-symmetric Tensor

If by interchanging every pair of contravariant or covariant indices of a tensor each of its components is altered in sign, but not in magnitude, then the tensor is said to be skew-symmetric or anti-symmetric with respect to these indices in the co-ordinate system. Therefore, a tensor A_{ij} is said to be anti-symmetric if

$$A_{ij} = -A_{ji}.$$

Similarly, a tensor A_{ijk} is anti-symmetric in the suffixes j and k if

$$A_{ijk} = -A_{ikj}.$$

Antisymmetry of an arbitrary tensor can be defined for any pair of similar indices.

Property 1.16.1 If a tensor is skew-symmetric with respect to a pair of contravariant or covariant indices in any co-ordinate system, then it remains so with respect to these two indices in any other co-ordinate system.

Proof: Let a tensor A_{ij} be skew-symmetric in one co-ordinate system $\left(x^i\right)$, i.e. $A_{ij} = -A_{ji}$ and $\overline{A}_{ij}$ in another co-ordinate system $\left(\overline{x}^i\right)$. Now,

$$\begin{aligned} \overline{A}_{ij} &= \frac{\partial x^p}{\partial \overline{x}^i}\frac{\partial x^q}{\partial \overline{x}^j}A_{pq} = -\frac{\partial x^p}{\partial \overline{x}^i}\frac{\partial x^q}{\partial \overline{x}^j}A_{qp} \\ &= -\frac{\partial x^q}{\partial \overline{x}^j}\frac{\partial x^p}{\partial \overline{x}^i}A_{qp} = -\overline{A}_{ji}. \end{aligned}$$

This shows that anti-symmetry with respect to i and j also holds in the system $\left(\overline{x}^i\right)$.

Property 1.16.2 In an N dimensional space, a skew-symmetric covariant tensor of second order has atmost $\frac{N(N-1)}{2}$ different components.

Proof: Let A_{ij} be a skew-symmetric covariant tensor of second order, then it has N^2 components in V_N. These components are

$$\begin{matrix} 0 & A_{12} & \cdots & A_{1N} \\ A_{21} & 0 & \cdots & A_{2N} \\ \vdots & \vdots & \cdots & \vdots \\ A_{N1} & A_{N2} & \cdots & 0 \end{matrix}.$$

These components are of two types:

(i) Those in which the indices i and j are the same. In this case

$$A_{ii} = -A_{ii} \Rightarrow A_{ii} = 0; i = 1, 2, \ldots, N.$$

(ii) Those in which the indices i and j are different, i.e. the components along the non-diagonal. Hence, the maximum number of components of this type is

$$= N^2 - N = N(N-1).$$

But due to anti-symmetry of A_{ij} (the components above and below the diagonal) the maximum number of distinct components of this type is $\frac{N(N-1)}{2}$.

Therefore, the maximum number of independent components is given by

$$0 + \frac{N(N-1)}{2} = \frac{N}{2}(N-1).$$

Property 1.16.3 If A^{ij} and B^{pq} are skew-symmetric tensors, then outer product is symmetric tensor.

Proof: Since A^{ij} and B^{pq} are skew-symmetric, so by definition,

$$A^{ji} = -A^{ij} \text{ and } B^{qp} = -B^{pq}.$$

If the outer product of A^{ij} and B^{pq} be C^{ijpq}, then,

$$C^{ijpq} = A^{ij}B^{pq}.$$

Thus

$$C^{ijpq} = \left(-A^{ji}\right)\left(-B^{qp}\right) = A^{ji}B^{qp} = C^{jiqp},$$

shows that $C^{ijpq} =$ the outer product of the skew-symmetric tensors A^{ij} and B^{pq} is symmetric tensor.

EXAMPLE 1.16.1 *If A_{ijk} is completely skew-symmetric and the indices run from 1 to N, show that the number of distinct non-vanishing components of A_{ijk} is $\frac{N(N-1)(N-2)}{6}$.*

Solution: A tensor A_{ijk} is anti-symmetric in the suffixes i and j if $A_{ijk} = -A_{jik}$. This tensor has

$$= \frac{N}{2}(N-1) \cdot N = \frac{N^2}{2}(N-1)$$

independent components. A tensor A_{ijk} is anti-symmetric in the suffixes i, j and k if

$$A_{ijk} = -A_{jik}, A_{ijk} = -A_{kji} \text{ and } A_{ijk} = -A_{ikj}.$$

Taking $i = j = k$, then the tensor A_{ijk} is of the type A_{iii} and in this case

$$A_{iii} = -A_{iii} \Rightarrow A_{iii} = 0; \text{ for all } i.$$

In this case, the number of independent components of A_{ijk} is zero. When A_{ijk} is of the type A_{iik}, then

$$A_{iik} = -A_{iik} \Rightarrow A_{iik} = 0.$$

In the case, the number of independent components of A_{ijk} is zero. This means that the number of independent components of A_{ijk} is

$${}^{N}C_3 = \frac{N(N-1)(N-2)}{3!}.$$

Thus, the non-vanishing components of A_{ijk} is $\frac{N(N-1)(N-2)}{6}$. For the symmetric tensor the maximum number of non-vanishing components are

$${}^{N}C_1 + {}^{N}P_2 + {}^{N}C_3 = \frac{N(N+1)(N+2)}{6}.$$

EXAMPLE 1.16.2 *If A_i be the component of a covariant vector, show that $\frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i}$ are components of a skew-symmetric covariant tensor of rank 2.*

Solution: Since A_i be the component of a covariant vector, by law of transformation $\overline{A}_i = \frac{\partial x^k}{\partial \overline{x}^i} A_k$. Differentiating it with respect to $\overline{x}^j$ partially,

$$\frac{\partial \overline{A}_i}{\partial \overline{x}^j} = \frac{\partial}{\partial \overline{x}^j}\left(\frac{\partial x^k}{\partial \overline{x}^i} A_k\right) = \frac{\partial^2 x^k}{\partial \overline{x}^j \partial \overline{x}^i} A_k + \frac{\partial x^k}{\partial \overline{x}^i} \frac{\partial A_k}{\partial \overline{x}^j}$$

$$= \frac{\partial^2 x^k}{\partial \overline{x}^j \partial \overline{x}^i} A_k + \frac{\partial x^k}{\partial \overline{x}^i} \frac{\partial x^l}{\partial \overline{x}^j} \frac{\partial A_k}{\partial x^l}. \qquad \text{(i)}$$

Similarly,

$$\frac{\partial \overline{A}_j}{\partial \overline{x}^i} = \frac{\partial^2 x^k}{\partial \overline{x}^i \partial \overline{x}^j} A_k + \frac{\partial x^k}{\partial \overline{x}^j} \frac{\partial x^l}{\partial \overline{x}^i} \frac{\partial A_k}{\partial x^l}.$$

Interchanging the dummy indices k and l, we get

$$\frac{\partial \overline{A}_j}{\partial \overline{x}^i} = \frac{\partial^2 x^k}{\partial \overline{x}^i \partial \overline{x}^j} A_k + \frac{\partial x^k}{\partial \overline{x}^i} \frac{\partial x^l}{\partial \overline{x}^j} \frac{\partial A_l}{\partial x^k}. \qquad \text{(ii)}$$

Subtracting, (i) and (ii) we get

$$\frac{\partial \overline{A}_i}{\partial \overline{x}^j} - \frac{\partial \overline{A}_j}{\partial \overline{x}^i} = \frac{\partial x^k}{\partial \overline{x}^i} \frac{\partial x^l}{\partial \overline{x}^j} \left(\frac{\partial A_k}{\partial x^l} - \frac{\partial A_l}{\partial x^k}\right).$$

This is the law of transformation of covariant tensor of rank 2. Therefore, $\frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i}$ are components of a covariant tensor of rank 2. Now,

$$P_{ij} = \frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i} = -\left(\frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j}\right) = -P_{ji}.$$

Thus, $\frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i}$ are components of a skew-symmetric covariant tensor of rank 2.

EXAMPLE 1.16.3 *The components of a tensor of type* $(0,2)$ *can be expressed uniquely as a sum of a symmetric tensor and a skew-symmetric tensor of the same type.*

Solution: Let a_{ij} be the components of a tensor of type $(0,2)$. Now, a_{ij} can be written in the form

$$\begin{aligned} a_{ij} &= \frac{1}{2}(a_{ij}+a_{ji}) + \frac{1}{2}(a_{ij}-a_{ji}) \\ &= A_{ij} + B_{ij}, \text{ say,} \end{aligned}$$

where

$$A_{ij} = \frac{1}{2}(a_{ij}+a_{ji}) \text{ and } B_{ij} = \frac{1}{2}(a_{ij}-a_{ji}).$$

Since a_{ij} is a tensor of type $(0,2)$, a_{ji} is also a tensor of type $(0,2)$. Since addition, subtraction of two tensors of the same rank and scalar multiplication with a tensor is a tensor, similar character, both A_{ij} and B_{ij} are tensors of type $(0,2)$. Now,

$$\begin{aligned} A_{ji} &= \frac{1}{2}(a_{ji}+a_{ij}) = \frac{1}{2}(a_{ij}+a_{ji}) = A_{ij} \\ B_{ji} &= \frac{1}{2}(a_{ji}-a_{ij}) = -\frac{1}{2}(a_{ij}-a_{ji}) = -B_{ij}. \end{aligned}$$

Thus, A_{ij} is symmetric and B_{ij} is skew-symmetric. Therefore, the components of a tensor of type $(0,2)$ can be expressed as a sum of a symmetric tensor and a skew-symmetric tensor of the same type.

Uniqueness: Now, we have to show that the representation is unique. For this, let $a_{ij} = C_{ij} + D_{ij}$, where C_{ij} is symmetric and D_{ij} is skew-symmetric. Now,

$$\begin{aligned} & a_{ij} = C_{ij} + D_{ij} \Rightarrow a_{ji} = C_{ij} - D_{ij} \\ \Rightarrow\ & C_{ij} = \frac{1}{2}(a_{ij}+a_{ji}) = A_{ij}; \quad D_{ij} = \frac{1}{2}(a_{ji}-a_{ij}) = B_{ij}. \end{aligned}$$

Thus, the representation is unique. Therefore, every tensor of type $(0,2)$ can be expressed uniquely as a sum of a symmetric tensor and a skew-symmetric tensor of the same type.

EXAMPLE 1.16.4 *If* $a_{ij}(\neq 0)$ *are the components of a covariant tensor of order two such that* $ba_{ij} + ca_{ji} = 0$, *where* b *and* c *are non-zero scalars, show that either* $b = c$ *and* a_{ij} *is skew-symmetric or* $b = -c$ *and* a_{ij} *is symmetric.*

Solution: The given relation $ba_{ij} + ca_{ji} = 0$ can be written as $ba_{ij} = -ca_{ji}$. Multiplying both sides by b, we get

$$b^2 a_{ij} = -bca_{ij} = -c\,(ba_{ji}) = -c\,(ca_{ij}) = c^2 a_{ij}$$

or

$$(b^2 - c^2)a_{ij} = 0 \Rightarrow b^2 - c^2 = 0; \text{ as } a_{ij} \neq 0$$

or

$$b = \pm c, \text{ as } a_{ij} \neq 0.$$

When $b = c$, then from the given relation we get

$$ca_{ij} = -ca_{ji} \Rightarrow a_{ij} = -a_{ji},$$

which shows that a_{ij} is skew-symmetric. Again, when $b = -c$, it follows from the given relation that:

$$-ca_{ij} = -ca_{ji} \Rightarrow a_{ij} = a_{ji},$$

which shows that a_{ij} is symmetric.

EXAMPLE 1.16.5 *If a_{ij} is a skew-symmetric tensor, prove that*

$$\left(\delta^i_j \delta^k_l + \delta^i_l \delta^k_j\right) a_{ik} = 0.$$

Solution: Since a_{ij} is a skew-symmetric tensor, so, $a_{ij} = -a_{ji}$. Now,

$$\begin{aligned} \text{LHS } &= \left(\delta^i_j \delta^k_l + \delta^i_l \delta^k_j\right) a_{ik} = \delta^i_j \delta^k_l a_{ik} + \delta^i_l \delta^k_j a_{ik} \\ &= \delta^i_j a_{il} + \delta^i_l a_{ij} = a_{jl} + a_{lj} = a_{jl} + (-a_{jl}) = 0. \end{aligned}$$

EXAMPLE 1.16.6 *If a tensor a_{ijk} is symmetric in the first two indices from the left and skew-symmetric in the second and third indices from the left, show that $a_{ijk} = 0$.*

Solution: Using the definition of a_{ijk}, we have

$$\begin{aligned} a_{ijk} &= a_{jik}; \text{ symmetric with respect to } i, j \\ &= -a_{jki}; \text{ skew-symmetric with respect to } i, k \\ &= -a_{kji}; \text{ symmetric with respect to } j, k \\ &= a_{kij}; \text{ skew-symmetric with respect to } j, i \\ &= a_{ikj}; \text{ symmetric with respect to } k, i \\ &= -a_{ijk}; \text{ skew-symmetric with respect to } k, j \end{aligned}$$

or

$$2a_{ijk} = 0; \text{ i.e. } a_{ijk} = 0.$$

1.17 Pseudo-Tensors

Pseudo tensors have been studied and it is used somewhere in mechanics.

(i) In V_N, consider N vectors $a_i, b_i, c_i, \ldots$. The outer product

$$a_i b_j c_k, \ldots$$

has N^N components, but only $N!$ distinct components.

(ii) In the Euclidean E^3, consider two vectors a_i, b_j and form

$$c_{ij} = a_i b_j - a_j b_i.$$

Then,

$$(c_{ij}) = \begin{pmatrix} 0 & c_{12} & c_{13} \\ c_{21} & 0 & c_{23} \\ c_{31} & c_{32} & 0 \end{pmatrix} = \begin{pmatrix} 0 & c_{12} & -c_{31} \\ -c_{12} & 0 & c_{23} \\ c_{31} & -c_{23} & 0 \end{pmatrix}.$$

This has only three independent components $c_{23}, -c_{31}, c_{12}$. They are the components of the cross-product of the theory of vectors. They form an axial vector.

(iii) In a V_4, consider two vectors

$$\begin{cases} a & b & c & d \\ u & v & w & t \end{cases}.$$

From these, we construct six determinants such as

$$\begin{vmatrix} a & b \\ u & v \end{vmatrix} = av - bu.$$

More generally, in a V_N consider two vectors a_i and b_j. We then form

$$c_{ij} = a_i b_j - a_j b_i.$$

Then the number of independent components is $\frac{1}{2}N(N-1) > N$, if $N > 3$. A set of two vectors will be called a bi-vector, to which we associate c_{ij}.

(iv) In a V_N, consider N elementary displacements

$$d_1 x^i, d_2 x^i, \ldots, d_N x^i.$$

The indices which affect the letter d have no tensorial significance, they are simply labels which distinguish the vectors. Consider the determinant

$$\Delta = \begin{vmatrix} d_1 x^1 & d_1 x^2 & \ldots & d_1 x^N \\ d_2 x^1 & d_2 x^2 & \ldots & d_2 x^N \\ \vdots & \vdots & \ldots & \vdots \\ d_N x^1 & d_N x^2 & \ldots & d_N x^N \end{vmatrix}.$$

If we exchange the vectors, in any manner, Δ will preserve its value, but only change its sign. There is only one independent Δ.

We have seen that in V_4 there are exactly six independent components of the tensor c_{ij}. If we pass to any other system of co-ordinates, we may consider the law of transformation of this set of six components to the corresponding set of components in the new system of co-ordinates. We say that these six components form a pseudo-tensor.

1.18 Reciprocal Tensor of a Tensor

Let a_{ik} be a symmetric tensor of type $(0, 2)$ satisfying the condition $|a_{ik}| \neq 0$. Let b^{ij} be the cofactor of a_{ij} in $|a_{ij}|$ divided by $|a_{ij}|$, i.e.

$$b^{ij} = \frac{\text{cofactor of } a_{ij} \text{ in } |a_{ij}|}{|a_{ij}|}. \tag{1.55}$$

From the theory of determinants, we get,

$$a_{ij} b^{ik} = \begin{cases} 1; & \text{when } k = j \\ 0; & \text{when } k \neq j \end{cases} \tag{1.56}$$

or

$$a_{ij} b^{ik} = \delta^k_j.$$

Let ξ^i be an arbitrary contravariant vector and let $B_i = a_{ij}\xi^j$, then according to the definition of inner product B_i is an arbitrary vector, as ξ^i is so. Now,

$$B_i b^{ik} = a_{ij}\xi^j b^{ik} = \xi^j a_{ij} b^{ik} = \xi^j \delta^k_j = \xi^k.$$

Applying quotient law to the equation $B_i b^{ik} = \xi^k$, we conclude that b^{ik} is a contravariant tensor of type $(2, 0)$. The tensor b^{ik} is symmetric because a_{ik} is so. Thus, from the symmetric tensor a_{ij} of type $(0, 2)$, we get a symmetric tensor b^{ij} of type $(2, 0)$. This tensor b^{ij} is called the *reciprocal or conjugate tensor* of the tensor a_{ij}.

Result 1.18.1 If b^{ij} is the *reciprocal tensor* of the tensor a_{ij}, then a_{ij} is the *reciprocal tensor* of the tensor b^{ij}.

Proof: Since $a_{ij} b^{ik} = \delta^k_j$, it follows that:

$$|a_{ij}| \left| b^{ik} \right| = \left| \delta^k_j \right| = 1 \Rightarrow \left| b^{ik} \right| \neq 0.$$

Let us define another tensor c_{ij} as

$$c_{ij} = \frac{\text{cofactor of } b^{ij} \text{ in } |b^{ij}|}{|b^{ij}|}.$$

From the theory of determinants, we get, $c_{ij}b^{ik} = \delta^k_j$. Multiplying both sides by a_{lk}, we get

$$c_{ij}b^{ik}a_{lk} = \delta^k_j a_{lk}$$

or

$$c_{ij}\delta^i_l = a_{lj} \Rightarrow c_{lj} = a_{lj}. \tag{1.57}$$

Since c_{ij} is the reciprocal tensor of b^{ij}, it follows from (1.57) that a_{ij} is the reciprocal tensor of b^{ij}. Thus, if b^{ij} is the *reciprocal tensor* of the tensor a_{ij}, then a_{ij} is the *reciprocal tensor* of the tensor b^{ij}. Hence, if the relation $a_{ij}b^{ik} = \delta^k_j$ is satisfied, then we say that a_{ij} and b^{ij} are *mutually reciprocal tensors.*

EXAMPLE 1.18.1 *If a_{ij}, b_{kl} are components of two symmetric tensors in an N dimensional space, such that $|b_{kl}| \neq 0$ and*

$$a_{ij}b_{kl} - a_{il}b_{jk} + a_{jk}b_{il} - a_{kl}b_{ij} = 0,$$

prove that $a_{ij} = \lambda b_{ij}$, where λ is some scalar.

Solution: Since b_{kl} are components of two symmetric tensors and $|b_{kl}| \neq 0$, we can get the reciprocal tensor c^{ij}, the cofactor of b_{ij} in $|b_{ij}|$, such that $b_{ij}c^{ik} = \delta^k_j$. Now, multiplying both sides of the given relation by c^{kl} we get

$$a_{ij}c^{kl}b_{kl} - a_{il}c^{kl}b_{jk} + a_{jk}c^{kl}b_{il} - a_{kl}c^{kl}b_{ij} = 0$$

or

$$a_{ij}N - a_{il}\delta^l_j + a_{jk}\delta^k_i - \rho b_{ij} = 0; \rho = a_{kl}c^{kl}$$

or

$$Na_{ij} - a_{ij} + a_{ji} - \rho b_{ij} = 0$$

or

$$Na_{ij} - \rho b_{ij} = 0; \text{ as } a_{ij} = a_{ji}$$

or

$$a_{ij} = \frac{\rho}{N}b_{ij} = \lambda b_{ij},$$

where $\lambda = \frac{\rho}{N}$ is a scalar because ρ is so.

1.19 Relative Tensor

A system of order $p+q$ whose components $a^{i_1\cdots i_p}_{j_1\cdots j_p}$ in a co-ordinate system $\left(x^i\right)$ transform according to the following formula, when referred to another co-ordinate system $\left(\overline{x}^i\right)$:

$$\overline{a}^{i_1 i_2\cdots i_p}_{j_1 j_2\cdots j_p} = J^\omega \frac{\partial \overline{x}^{i_1}}{\partial x^{t_1}} \frac{\partial \overline{x}^{i_2}}{\partial x^{t_2}} \cdots \frac{\partial \overline{x}^{i_p}}{\partial x^{t_p}} \frac{\partial x^{r_1}}{\partial \overline{x}^{j_1}} \cdots \frac{\partial x^{r_q}}{\partial \overline{x}^{j_q}} a^{t_1 t_2\cdots t_p}_{r_1 r_2\cdots r_q}, \tag{1.58}$$

where J is the Jacobian of transformation (1.2) is called a relative tensor of weight w.

The sets of quantities $a^{t_1 t_2 \ldots t_p}_{r_1 r_2 \ldots r_q}$ obeying this law of transformation (1.58) are called the components of a relative tensor of weight w. In addition, form the linear and homogeneous character of this transformation (1.58) it follows that if all components of a relative tensor vanish in one co-ordinate system, they vanish in every co-ordinate system.

An immediate corollary of this is that a tensor equation involving relative tensors when true is one co-ordinate system is valid in all co-ordinate systems. In this case the relative tensors on two sides of equations must be same weight.

(i) Relative tensors of the same type and weight may be added, and the sum is relative tensor of the same type and weight.
(ii) Relative tensors may be multiplied, the weight of the product being the sum of the weights of tensors entering in the product.
(iii) The operation of contraction on a relative tensor yields a relative tensor of the same weight as the original tensor.

To distinguish mixed tensors, considered in the preceding sections, from relative tensors, the term absolute tensor is frequently used to designate the former. We shall encounter several relative tensors in applications of tensor theory.

A function $f(x^1, x^2, \ldots, x^N)$, represents a scalar in the X-reference frame whenever in the Y-reference frame determined by the transformation

$$x^i = x^i(y^1, y^2, \ldots, y^N),$$

the scalar is given by the formula

$$g(y^1, y^2, \ldots, y^N) = f[x^1(y), x^2(y), \ldots, x^N(y)].$$

We will encounter functions $f(x)$ which transform in accordance with the more general law, namely

$$g(y^1, y^2, \ldots, y^N) = f[x^1(y), x^2(y), \ldots, x^N(y)] \left|\frac{\partial x^i}{\partial y^j}\right|^w, \tag{1.59}$$

where $\left|\frac{\partial x^i}{\partial y^i}\right|$ denotes the Jacobian of the transformation and w is a constant. The formula (1.59) determines a class of invariant functions known as relative scalars of weight w.

A relative scalar of weight zero is the scalar. Sometimes a scalar of weight zero is called an absolute scalar. If the weight of a relative tensor is zero, then the relative tensor is called an absolute tensor.

A relative scalar of weight 1 is called scalar density. A relative tensor of weight 1 is called a tensor density. The reason for this terminology may be seen from the expression for the total mass of a distribution of matter of density $p(x_1, x_2, x_3)$, the co-ordinates x^i being rectangular Cartesian.

EXAMPLE 1.19.1 *If a_{ij} is a covariant symmetric tensor of order two and $|a_{ij}| = a$, show that $\sqrt{a}$ is a tensor density.*

Solution: Since a_{ij} is a covariant tensor of order two, we have

$$\overline{a}_{ij} = \frac{\partial x^p}{\partial \overline{x}^i}\frac{\partial x^q}{\partial \overline{x}^j} a_{pq}$$

or

$$|\overline{a}_{ij}| = \left|\frac{\partial x^p}{\partial \overline{x}^i}\right| \left|\frac{\partial x^q}{\partial \overline{x}^j}\right| |a_{pq}| = \left|\frac{\partial x^p}{\partial \overline{x}^i}\right|^2 a$$

or

$$\overline{a} = \left|\frac{\partial x^p}{\partial \overline{x}^i}\right|^2 a; \text{ where } |\overline{a}_{ij}| = \overline{a}$$

or

$$\sqrt{\overline{a}} = \left|\frac{\partial x^p}{\partial \overline{x}^i}\right| \sqrt{a} = J^1\sqrt{a}; J = \left|\frac{\partial x^p}{\partial \overline{x}^i}\right|.$$

From this relation it follows that $\sqrt{a}$ is a relative tensor of weight 1. In other words $\sqrt{a}$ is a tensor density.

EXAMPLE 1.19.2 *Show that the equations of transformation of a relative tensor possess the group property.*

Solution: Let A_{ij} be a relative tensor of weight ω. Consider the co-ordinate transformations

$$x^i \to \overline{x}^i \to \overline{\overline{x}}^i$$
$$A_{ij} \to \overline{A}_{ij} \to \overline{\overline{A}}_{ij}$$

In case of transformation $x^i \to \overline{x}^i$, we have

$$\overline{A}_{\alpha\beta} = \frac{\partial x^p}{\partial \overline{x}^\alpha}\frac{\partial x^q}{\partial \overline{x}^\beta}\left|\frac{\partial x}{\partial \overline{x}}\right|^\omega A_{pq}.$$

In case of transformation $\overline{x}^i \to \overline{\overline{x}}^i$, we have

$$\begin{aligned}\overline{\overline{A}}_{ij} &= \frac{\partial \overline{x}^\alpha}{\partial \overline{\overline{x}}^i} \frac{\partial \overline{x}^\beta}{\partial \overline{\overline{x}}^j} \left| \frac{\partial \overline{x}^i}{\partial \overline{\overline{x}}^j} \right|^\omega \overline{A}_{\alpha\beta} \\ &= A_{pq} \frac{\partial x^p}{\partial \overline{x}^\alpha} \frac{\partial x^q}{\partial \overline{x}^\beta} \frac{\partial \overline{x}^\alpha}{\partial \overline{\overline{x}}^i} \frac{\partial \overline{x}^\beta}{\partial \overline{\overline{x}}^j} \left| \frac{\partial \overline{x}^i}{\partial \overline{\overline{x}}^j} \right|^\omega \left| \frac{\partial x}{\partial \overline{x}} \right|^\omega \\ &= A_{pq} \frac{\partial x^p}{\partial \overline{\overline{x}}^i} \frac{\partial x^q}{\partial \overline{\overline{x}}^j} \left| \frac{\partial x^i}{\partial \overline{x}^k} \frac{\partial \overline{x}^k}{\partial \overline{\overline{x}}^j} \right|^\omega = A_{pq} \frac{\partial x^p}{\partial \overline{\overline{x}}^i} \frac{\partial x^q}{\partial \overline{\overline{x}}^j} \left| \frac{\partial x^i}{\partial \overline{\overline{x}}^j} \right|^\omega\end{aligned}$$

or

$$\overline{\overline{A}}_{ij} = A_{pq} \frac{\partial x^p}{\partial \overline{\overline{x}}^i} \frac{\partial x^q}{\partial \overline{\overline{x}}^j} \left| \frac{\partial x}{\partial \overline{\overline{x}}} \right|^\omega.$$

This proves that, if we make the direct transformation from $x^i \to \overline{\overline{x}}^i$, we get the same law of transformation. Therefore, relative law of transformations possess the group property.

EXAMPLE 1.19.3 *Prove that the scalar product of a relative covariant vector of weight ω_1 and a contravariant vector of weight ω_2 is a relative scalar of weight $\omega_1 + \omega_2$.*

Solution: Let A^i be the components of relative contravariant vector of weight ω_1, then

$$\overline{A}^i = A^p \frac{\partial \overline{x}^i}{\partial x^p} \left| \frac{\partial x}{\partial \overline{x}} \right|^{\omega_1} = A^p \frac{\partial \overline{x}^i}{\partial x^p} J^{\omega_1}$$

and B_i be the components of relative covariant vector of weight ω_2, then

$$\overline{B}_i = B_q \frac{\partial x^q}{\partial \overline{x}^i} \left| \frac{\partial x}{\partial \overline{x}} \right|^{\omega_2} = B_q \frac{\partial x^q}{\partial \overline{x}^i} J^{\omega_2}.$$

We have to show that the scalar product $A^i B_i$ is a relative scalar of weight $\omega_1 + \omega_2$. For this, we have

$$\begin{aligned}\overline{A}^i \overline{B}_i &= A^p B^q \frac{\partial \overline{x}^i}{\partial x^p} \frac{\partial x^q}{\partial \overline{x}^i} J^{\omega_1 + \omega_2} \\ &= A^p B^q \frac{\partial x^q}{\partial x^p} J^{\omega_1 + \omega_2} \\ &= A^p B^q \delta_p^q J^{\omega_1 + \omega_2} = A^p B^p J^{\omega_1 + \omega_2}.\end{aligned}$$

From this, it follows that $A^i B_i$ is a relative scalar of weight $\omega_1 + \omega_2$.

EXAMPLE 1.19.4 *If A^{ij} and A_{ij} are components of symmetric relative tensors of weight ω, show that*

$$\left|\overline{A}^{ij}\right| = \left|A^{ij}\right| \left|\frac{\partial x}{\partial \overline{x}}\right|^{w-2} \quad \textit{and} \quad \left|\overline{A}_{ij}\right| = \left|A_{ij}\right| \left|\frac{\partial x}{\partial \overline{x}}\right|^{w+2}.$$

Solution: Since A^{ij} are components of symmetric relative tensors of weight ω, by definition,

$$\overline{A}^{ij} = A^{\alpha\beta} \frac{\partial \overline{x}^i}{\partial x^\alpha} \frac{\partial \overline{x}^j}{\partial x^\beta} \left|\frac{\partial x}{\partial \overline{x}}\right|^w.$$

Taking modulus of both sides and noting that $\left|\frac{\partial \overline{x}^i}{\partial x^\alpha}\right| = \left|\frac{\partial \overline{x}^j}{\partial x^\beta}\right| = \left|\frac{\partial \overline{x}}{\partial x}\right|$, we get

$$\begin{aligned} \left|\overline{A}^{ij}\right| &= \left|A^{\alpha\beta}\right| \left|\frac{\partial \overline{x}}{\partial x}\right| \left|\frac{\partial \overline{x}}{\partial x}\right| \left|\frac{\partial x}{\partial \overline{x}}\right|^w \\ &= \left|A^{\alpha\beta}\right| \left|\frac{\partial x}{\partial \overline{x}}\right|^{-2} \left|\frac{\partial x}{\partial \overline{x}}\right|^w; \text{ as } \left|\frac{\partial \overline{x}}{\partial x}\right| = \left|\frac{\partial x}{\partial \overline{x}}\right|^{-1} \\ &= \left|A^{ij}\right| \left|\frac{\partial x}{\partial \overline{x}}\right|^{w-2}; \quad \text{as } \left|A^{\alpha\beta}\right| = \left|A^{ij}\right|. \end{aligned}$$

Since A_{ij} are components of symmetric relative tensors of weight w. By definition,

$$\overline{A}_{ij} = A_{\alpha\beta} \frac{\partial x^\alpha}{\partial \overline{x}^i} \frac{\partial x^\beta}{\partial \overline{x}^j} \left|\frac{\partial x}{\partial \overline{x}}\right|^w$$

or

$$\left|\overline{A}_{ij}\right| = \left|A_{\alpha\beta}\right| \left|\frac{\partial x}{\partial \overline{x}}\right| \left|\frac{\partial x}{\partial \overline{x}}\right| \left|\frac{\partial x}{\partial \overline{x}}\right|^w = \left|A_{ij}\right| \left|\frac{\partial x}{\partial \overline{x}}\right|^{w+2}.$$

1.20 Cartesian Tensors

A tensor of Euclidean space E^n, obtained by orthogonal transformation of coordinate axes, is called a Cartesian tensor. Thus a Cartesian tensor of rank r in a three-dimensional Euclidean space is a set of 3^r components which transform according to the rule

$$\overline{A}^{i_1 i_2 \cdots i_r}_{j_1 j_2 \cdots j_r} = \frac{\partial \overline{x}^{i_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial \overline{x}^{i_r}}{\partial x^{\alpha_r}} \frac{\partial x^{\beta_1}}{\partial \overline{x}^{j_1}} \cdots \frac{\partial x^{\beta_r}}{\partial \overline{x}^{j_r}} A^{\alpha_1 \alpha_2 \cdots \alpha_r}_{\beta_1 \beta_2 \cdots \beta_r} \tag{1.60}$$

only under orthogonal co-ordinate transformations

$$T : \overline{x}^i = a^i_j x^j; \; (a^i_j) \text{ is orthogonal} \tag{1.61}$$

so that $|a^i_j| \neq 0$. This is a weaker condition than that imposed on general tensor. Since a general tensor satisfies Eq. (1.60), for all co-ordinate transformation, it is clear that a general tensor is also a Cartesian tensor but a Cartesian tensor is not necessarily a general tensor.

1.20.1 Affine Tensor

A transformation of the form

$$T : \overline{x}^i = a^i_j x^j; |a^i_j| \neq 0 \tag{1.62}$$

takes a rectangular co-ordinate system (x^i) into a system $(\overline{x}^i)$ having oblique axes. Tensors corresponding to admissible co-ordinate changes, Eq. (1.62), are called *affine tensors.* Thus, affine tensors are defined on the class of all such oblique co-ordinate systems. Since the Jacobian matrices of T and T^{-1} are

$$J = \left[\frac{\partial \overline{x}^i}{\partial x^j}\right]_{rr} = \left[a^i_j\right]_{rr} \text{ and } J^{-1} = \left[\frac{\partial x^i}{\partial \overline{x}^j}\right]_{rr} = \left[b^i_j\right]_{rr}$$

the laws for affine tensors are,

$$\begin{aligned} \text{Contravariant } &: \overline{A}^i = a^i_p A^p; \overline{A}^{ij} = a^i_p a^j_q A^{pq}, \ldots \\ \text{Covariant } &: \overline{A}_i = b^p_i A_p; \overline{A}_{ij} = b^p_i b^q_j A_{pq}, \ldots \\ \text{Mixed } &: \overline{A}^i_j = a^i_p b^q_j A^p_q; \overline{A}^i_{jk} = a^i_l b^m_j b^n_k A^l_{mn}, \ldots \end{aligned} \tag{1.63}$$

Under the less stringent condition Eq. (1.63), more objects can qualify as tensors than before. From Eq. (1.62), we have

$$\frac{\partial \overline{x}^i}{\partial x^j} = \frac{\partial x^j}{\partial \overline{x}^i} = a^i_j$$

indicates that the distinction between covariance and contravariance vanishes. Thus, we can use all indices as subscripts, so long as we are confining ourselves to orthogonal co-ordinate transformations of the type of Eq. (1.61). The transformation law for a Cartesian tensor thus reduces to

$$\overline{A}^{i_1 i_2 \cdots i_r} = a^{i_1}_{j_1} a^{i_2}_{j_2} \cdots a^{i_r}_{j_r} A^{j_1 j_2 \cdots j_r}. \tag{1.64}$$

Equation (1.61) suggests that x^i is also a Cartesian vector, though it is not a general vector. Since the co-ordinate differentials dx^i constitute a general vector, hence also a Cartesian vector.

The Kronecker delta δ^i_j and the fully anti-symmetric tensor ε_{ijk} are also Cartesian tensors in which all indices can be written as covariant indices. This follows from the fact that:

$$\overline{\delta}^p_q = a^p_i a^j_q \delta^i_j = a^p_i a^i_q = \delta^p_q. \tag{1.65}$$

Thus δ^i_j is a general tensor. Note that, the quotient law is valid for Cartesian tensors also.

Thus, we conclude that an ordinary position vector (x^i) becomes an affine tensor and the partial derivatives of a tensor define an affine tensor.

1.20.2 Isotropic Tensor

A Cartesian tensor whose components remain unchanged under a rotation of axes is called *isotropic tensor.* We say that an isotropic tensor transforms into itself under orthogonal transformations.

Since the only values the Kronecker delta symbol takes are 1 and 0, it is seen that, it is an isotropic tensor, that is, has the same components in any co-ordinate system.

A scalar is an isotropic tensor of rank zero, as it remains the same value in all co-ordinate systems. Let $\mathbf{u} = (u_1, u_2, u_3)$ be a vector and $A = [a_{ij}]$ an arbitrary orthogonal transformation. Let, $\overline{\mathbf{u}} = (\overline{u}_1, \overline{u}_2, \overline{u}_3)$ be the transformed vector, then,

$$\overline{\mathbf{u}} = A\mathbf{u}; A = \text{ transformed matrix.} \tag{1.66}$$

But if $\mathbf{u}$ is an isotropic vector, we must have

$$\overline{\mathbf{u}} = \mathbf{u};\ \text{i.e. } \overline{u}_i = u_i \quad \text{for} \quad i = 1, 2, 3. \tag{1.67}$$

Thus, Eq. (1.66) reduces to

$$A\mathbf{u} = \mathbf{u} \Rightarrow (A - I)\mathbf{u} = \underline{0}. \tag{1.68}$$

where $\underline{0}$ is the null matrix. If this is to be true for every orthogonal matrix A, it is clear that the only solution of Eq. (1.68) is $\mathbf{u} = \underline{0}$. Thus, there is no isotropic tensor of rank 1 except the null vector.

1.21 Exercises

1. (a) If $x^i = a^i_p y^p$ and $y^i = b^i_q z^q$, show that $x^i = a^i_q b^q_p z^p$.

   (b) Show that the expression $b^{ij} y_i y_j$ becomes in terms of x variable as $c_{ij} x_i x_j$ if $y_i = c_{ij} x_j$ and $b^{ij} c_{ik} = \delta^j_k$.

2. (a) If $A = a_p x^p$ for all values of independent variables $x^1, x^2, \ldots, x^N$ and a_ps are constants, show that

$$\frac{\partial}{\partial x^j}(a_p x^p) = a_j.$$

(b) Calculate

(i) $\frac{\partial}{\partial x^k}\left(a_{ij}x^j\right)$, (ii) $\frac{\partial}{\partial x^k}[a_{ij}x^i(x^j)^2]; a_{ij} = a_{ji}$, (iii) $\frac{\partial}{\partial x^l}\left(a_{ijk}x^i x^j x^k\right)$;

where a_{ijk} are constants.

(c) Find the following partial derivative if a_{ij} are constants:

$$\frac{\partial}{\partial x^k}\left(a_{11}x^1 + a_{12}x^2 + a_{13}x^3\right);\ k = 1, 2, 3.$$

(d) Using the relation $\frac{\partial x^p}{\partial x^q} = \delta_{pq}$, show that

$$\frac{\partial}{\partial x^k}\left(a_{ij}x^i x^j\right) = (a_{ik} + a_{ki})x^i.$$

3. Write all the terms in each of the following sums expressed in summation convention:

(a) $a_{ijk}u^k;\ k = 1, 2, \ldots, N.$
(b) $\delta_{ij}u^i u^j;\ i, j = 1, 2, \ldots, N.$
(c) $a_{ijk}u^i u^j u^k;\ i, j, k = 1, 2, \ldots, N.$

4. Evaluate each of the following (range of indices 1 to N):

(a) $\delta^i_j A^j$ and $\delta^i_j A^{jk}$.
(b) $\delta^p_q A^{st}_p$ and $\delta^i_j \delta^k_l A^{jl}$.
(c) $a^j a_i \delta^i_j$ and $\delta^i_k \delta^k_l \delta^l_i$.
(d) $\delta^i_j \delta^j_l \delta^l_k \delta^k_i$ and $\delta_{ij}\delta^{ij}$.

5. Show that the expression $b^{ij}y_i y_j$ becomes in terms of x variables as $c_{ij}x_i x_j$, if $y_i = c_{ij}x_j$ and $b^{ij}c_{ik} = \delta^j_k$.

6. Explain with examples why Kronecker deltas are called substitution operator in tensor analysis.

7. Suppose that the following transformation connects the (x^i) and $(\overline{x}^i)$ co-ordinate system $\overline{x}^1 = e^{x^1+x^2}$; $\overline{x}^2 = e^{x^1-x^2}$. Calculate the Jacobian matrix $J, |J|$ and J^{-1}. Calculate also $\overline{J}$.

8. Show that for independent functions $\overline{x}^i = \overline{x}^i\left(x^1, x^2, \ldots, x^N\right)$,

$$\frac{\partial \overline{x}^i}{\partial x^r}\frac{\partial x^r}{\partial \overline{x}^j} = \delta^i_j.$$

Take the partial derivative with respect to x^k, to establish the formula,

$$\frac{\partial^2 \overline{x}^i}{\partial x^k \partial x^r}\frac{\partial x^r}{\partial \overline{x}^j} = -\frac{\partial^2 \overline{x}^r}{\partial \overline{x}^s \partial \overline{x}^j}\frac{\partial \overline{x}^i}{\partial x^r}\frac{\partial \overline{x}^s}{\partial x^k}.$$

9. Discuss the transformations in which the co-ordinates $\overline{x}^i$ are rectangular Cartesian is E^3:

$$\overline{x}^1 = x^1\cos x^2;\ \overline{x}^2 = x^1\sin x^2;\ \overline{x}^3 = x^3,$$

where x^1, x^2, x^3 are in cylindrical co-ordinate system.

10. (a) Discuss the transformations in which the co-ordinates $\overline{x}^i$ are rectangular Cartesian is E^3:

$$\overline{x}^1 = \frac{1}{\sqrt{6}}x^1 + \frac{2}{\sqrt{6}}x^2 + \frac{1}{\sqrt{6}}x^3;\ \overline{x}^2 = \frac{1}{\sqrt{2}}x^1 - \frac{1}{\sqrt{3}}x^2 + \frac{1}{\sqrt{3}}x^3;$$
$$\overline{x}^3 = \frac{1}{\sqrt{2}}x^1 - \frac{1}{\sqrt{2}}x^3.$$

Write this system of equations in tensor form.

(b) If $f(x^1, x^2, \ldots, x^N)$ is a homogeneous function of degree m, prove that

$$\frac{\partial f}{\partial x^i}x^i = mf.$$

11. Show that the cylindrical co-ordinates of the points whose Cartesian co-ordinates (i) $(4, 8, 3)$ (ii) $(0, 1, 1)$ (iii) $(0, -3, -3)$ (iv) $(-2, 3, 2)$
are given by
(i) $\left(4\sqrt{5}, \tan^{-1}2, 3\right)$ (ii) $\left(1, \frac{\pi}{2}, 1\right)$ (iii) $\left(3, \frac{\pi}{2}, -3\right)$ (iv) $\left(\sqrt{13}, \tan^{-1}\left(-\frac{3}{2}\right), 2\right)$.
Also find the spherical polar co-ordinates in each case.

12. Show that the Cartesian co-ordinates of the points whose cylindrical co-ordinates (i) $\left(6, \frac{\pi}{3}, 2\right)$ (ii) $\left(2\sqrt{3}, -\frac{\pi}{4}, 3\right)$ (iii) $\left(8, \frac{2\pi}{3}, -4\right)$ (iv) $\left(4, \frac{\pi}{6}, 1\right)$
are given by
(i) $\left(3, 3\sqrt{2}, 2\right)$ (ii) $(2, -2, 3)$ (iii) $\left(-4, 4\sqrt{3}, -4\right)$ (iv) $\left(2\sqrt{3}, 2, 1\right)$.
Hence, find the spherical polar co-ordinates in each case.

13. Show that the cylindrical co-ordinates of the points whose spherical co-ordinates
(i) $\left(4, \frac{\pi}{4}, \frac{\pi}{6}\right)$ (ii) $\left(6, \frac{\pi}{3}, \frac{2\pi}{3}\right)$ (iii) $\left(8, \frac{2\pi}{3}, \frac{\pi}{3}\right)$ (iv) $\left(2, \frac{2\pi}{3}, \frac{5\pi}{6}\right)$
are given by
(i) $\left(2\sqrt{2}, \frac{\pi}{6}, 2\sqrt{2}\right)$ (ii) $\left(3\sqrt{3}, -\frac{\pi}{3}, 3\right)$ (iii) $\left(4\sqrt{3}, \frac{\pi}{3}, -4\right)$ (iv) $\left(\sqrt{3}, -\frac{\pi}{6}, -1\right)$.

14. Show that the spherical co-ordinates of the points whose cylindrical co-ordinates
(i) $\left(4, \frac{\pi}{2}, 3\right)$ (ii) $\left(1, \frac{5\pi}{6}, -2\right)$ (iii) $\left(7, \frac{2\pi}{3}, -4\right)$ (iv) $\left(3, -\frac{\pi}{4}, 2\right)$.
are given by
(i) $\left(5, \cos^{-1}\frac{1}{5}, \frac{\pi}{2}\right)$ (ii) $\left(\sqrt{5}, \cos^{-1}\frac{-2}{\sqrt{5}}, \frac{-\pi}{6}\right)$ (iii) $\left(\sqrt{65}, \cos^{-1}\frac{-4}{\sqrt{65}}, \frac{-\pi}{3}\right)$
(iv) $\left(\sqrt{13}, \cos^{-1}\frac{2}{\sqrt{13}}, \frac{-\pi}{4}\right)$.
15. Show that $e_{ijk}e^{ijk} = 6$, where e_{ijk} and e^{ijk} are e-systems of third order, if $i, j, k = 1, 2, 3$.
16. Expand for $N = 2$
(i) $e^{ij}a_i^1a_j^2$ (ii) $e^{ij}a_i^2a_j^1$ (iii) $e^{\alpha\beta}a_\alpha^i a_\beta^j = e^{ij}|a|$.
17. Verify that
(i) $\delta_{\alpha\beta}^{ij}a^{\alpha\beta} = a^{ij} - a^{ji}$.
(ii) $\delta_{\alpha\beta\gamma}^{ijk}a^{\alpha\beta\gamma} = a^{ijk} - a^{ikj} + a^{jki} - a^{jik} + a^{kij} - a^{kji}$.
18. Prove that

$$e_{k\alpha\beta}e^{kij} = \delta_{\alpha\beta}^{ij} = \begin{vmatrix} \delta_\alpha^i & \delta_\beta^i \\ \delta_\alpha^j & \delta_\beta^j \end{vmatrix} \text{ and } \delta_{\alpha\beta\gamma}^{ijk} = \begin{vmatrix} \delta_\alpha^i & \delta_\beta^i & \delta_\gamma^i \\ \delta_\alpha^j & \delta_\beta^j & \delta_\gamma^j \\ \delta_\alpha^k & \delta_\beta^k & \delta_\gamma^k \end{vmatrix}.$$

19. Suppose that two sets of functions u^i and $\overline{u}^i$ $(i = 1, 2, \ldots, N)$ are connected by the relations

$$\overline{u}^i = \frac{\partial \overline{x}^i}{\partial x^j}u^j;\ i = 1, 2, \ldots, N.$$

Prove that $u^k = \frac{\partial x^k}{\partial \overline{x}^j}\overline{u}^j$.
20. If the components of a contravariant vector in $\left(x^i\right)$ co-ordinate system are $(8, 4)$, show that its components in $\left(\overline{x}^i\right)$ co-ordinate system are $(24, 52)$, where $\overline{x}^1 = 3x^1$ and $\overline{x}^2 = 5x^1 + 3x^2$.
21. If the components of a contravariant vector in $\left(x^i\right)$ co-ordinate system are $(2, 1, 1)$, show that its components in $\left(\overline{x}^i\right)$ co-ordinate system are $(5, 5, 5)$, where,

$$\overline{x}^1 = 3x^1 - 3x^2 + 2x^3,\ \overline{x}^2 = 2x^2 + 3x^3 \text{ and } \overline{x}^3 = x^1 + x^2 + 2x^3.$$

22. Show that if λ and μ are invariants and A^i and B^i are components of a contravariant vectors, the vector defined in all co-ordinate systems by $(\lambda A^i + \mu B^i)$ is a contravariant vector. Using it verify that $2A^i + 3B^i$ is also a contravariant vector.
23. The components of a contravariant vector in the (x) co-ordinate system are 2 and 3. Find its components in the $(\overline{x})$ co-ordinate system if
$$\overline{x}^1 = 3(x^1)^2 \text{ and } \overline{x}^2 = 5(x^1)^2 + 3(x^2)^2.$$
24. If the components of a contravariant tensor of type $(2, 0)$ in
$$V_2 : \left\{ \left(x^1, x^2\right) : x^1, x^2 \in \Re \right\}$$
are $A^{11} = 1, A^{12} = 0 = A^{21}, A^{22} = 1$, find $\overline{A}^{ij}$ in
$$\overline{V}_2 : \left\{ \left(\overline{x}^1, \overline{x}^2\right) : \overline{x}^1, \overline{x}^2 \in \Re \right\}$$
where functional relation between co-ordinate systems are $\overline{x}^1 = (x^1)^2, \overline{x}^2 = (x^2)^2$.
25. (a) Show that the component of tangent vector of a smooth curve in N dimensional space are components of a contravariant vector.
    (b) Prove that the gradient of an arbitrary differentiable function is a covariant vector.
    (c) If a vector has components $\dfrac{d^2x}{dt^2}, \dfrac{d^2y}{dt^2}$ is rectangular Cartesian co-ordinates, show that they are
$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2; \frac{d^2\theta}{dt^2} + \frac{2}{r}\frac{dr}{dt}\frac{d\theta}{dt}$$
    is polar co-ordinates.
    (d) Obtain the components of the gradient of a scalar field in terms of polar co-ordinates in a two dimensional space.
26. (a) Show that if the transformation $T : y^i = a^i_j x^j$ is orthogonal, then the distinction between the covariant and contravariant laws disappears.
    (b) Prove that there is no distinction between contravariant and covariant vectors when we restrict ourselves to rectangular Cartesian transformation of co-ordinates.
27. Prove that the transformation of covariant vectors form a group.
28. Prove that the transformation of the tensors of the type $(1, 1)$ form a group.

29. If a covariant vector has components $\left(\frac{x^2}{x^1}, \frac{x^1}{x^2}\right)$ in rectangular Cartesian co-ordinates x^1, x^2; prove that its components in polar co-ordinates $(\overline{x}^1, \overline{x}^2)$ are

$$\sin \overline{x}^2 + \cos \overline{x}^2; \quad -\overline{x}^1 \frac{\sin^2 \overline{x}^2}{\cos \overline{x}^2} + \overline{x}^1 \frac{\cos^2 \overline{x}^2}{\sin \overline{x}^2}.$$

30. If a covariant vector has components $\left(x^1\right)^2, \frac{\left(x^1\right)^3}{x^2}$ in rectangular Cartesian co-ordinates x^1, x^2; prove that its components in polar co-ordinates $(\overline{x}^1, \overline{x}^2)$ are

$$2\left(\overline{x}^1\right)^2 \cos^2 \overline{x}^2; \quad -\left(\overline{x}^1\right)^3 \sin \overline{x}^2 \cos^2 \overline{x}^2 + \left(\overline{x}^1\right)^3 \frac{\cos^4 \overline{x}^2}{\sin \overline{x}^2}.$$

31. If X, Y, Z are the components of a covariant vector in rectangular Cartesian co-ordinates x, y, z in E^3 show that the components of the vector in spherical polar co-ordinates are

$$\begin{aligned}
&X \cos \overline{x}^2 \sin \overline{x}^3 + Y \sin \overline{x}^2 \sin \overline{x}^3 + Z \cos \overline{x}^3;\\
&Y \overline{x}^1 \sin \overline{x}^2 \cos \overline{x}^3 - Z \overline{x}^1 \sin \overline{x}^3;\\
&-X \overline{x}^1 \sin \overline{x}^2 \sin \overline{x}^3 + Y \overline{x}^1 \cos \overline{x}^2 \sin \overline{x}^3.
\end{aligned}$$

32. IF X, Y, Z are the components of a contravariant vector in rectangular Cartesian co-ordinates x, y, z in E^3 show that the components of the vector in spherical polar co-ordinates are

$$\begin{aligned}
&X \sin \overline{x}^2 \cos \overline{x}^3 + Y \sin \overline{x}^2 \sin \overline{x}^3 + Z \cos \overline{x}^3;\\
&-X \overline{x}^1 \cos \overline{x}^2 \cos \overline{x}^3 - Y \overline{x}^1 \sin \overline{x}^2 \sin \overline{x}^3 - \frac{1}{\left(\overline{x}^1\right)^2} Z;\\
&-\frac{\sin \overline{x}^2}{\overline{x}^1 \sin \overline{x}^3} X + \frac{\cos \overline{x}^2}{\overline{x}^1 \sin \overline{x}^3} Y.
\end{aligned}$$

33. Prove that if A^i_{jkl} is a tensor such that in the (x^i) co-ordinate system, $A^i_{jkl} = 3A^i_{ljk}$, then $\overline{A}^i_{jkl} = 3TA^i_{ljk}$ in all co-ordinate systems.

34. (a) A covariant tensor has components $xy, 2y - z^2, xz$ in rectangular co-ordinates. Determine its components in spherical polar co-ordinates.
    (b) In orthogonal Cartesian co-ordinate system a contravariant vector is given by $(1, 1, 1)$. Find its components in cylindrical co-ordinate system.

35. (a) Prove that ε_{ijk} and ε^{ijk} are covariant and contravariant tensors of order three, respectively.

(b) Prove that δ_{ij} and δ^{ij} have different components in different co-ordinate systems.

36. (a) Show that an inner product of tensors A^i_{jk} and B^{lm}_n is a tensor of type (2, 2). How many inner products are possible in this case?
    (b) If C^i_{jk} is an arbitrary mixed tensor and $B(i,j,k)C^i_{jk}$ an invariant, prove that $B(i,j,k)$ is a tensor of type B^{jk}_i.
37. Prove that $\dfrac{\partial A_i}{\partial x^j}$ is not a tensor though A_i is a covariant vector. Hence show that $\dfrac{\partial A_i}{\partial x^j} - \dfrac{\partial A_j}{\partial x^j}$ is a covariant tensor of type (0, 2).
38. If the relation $b^i_j v_i = 0$ holds for any arbitrary covariant vector v_i, show that $b^i_j = 0$.
39. If the relation $a^{ij} v_i v_j = 0$ holds for any arbitrary covariant vector v_i, show that $a^{ij} + a^{ji} = 0$.
40. If the relation $a_{ij} v^i v^j = b_{ij} v^i v^j$ holds for any arbitrary values of v^i, show that $a_{ij} + a_{ji} = b_{ij} + b_{ji}$. If a_{ij} and b_{ij} are symmetric tensors, then further show that $a_{ij} = b_{ij}$.
    Hints: Take $c_{ij} = a_{ij} - b_{ij}$, then the given relation becomes $c_{ij} v^i v^j = 0$.
41. If the equality $a^i_j v_i = \beta v_j$ holds for every covariant vector v_i, where β is a scalar, show that $a^i_j = \beta \delta^i_j$.
    Hints: Take $b^i_j = a^i_j - \beta \delta^i_j$, then the relation $a^i_j v_i = \beta \delta^i_j v_i$ becomes $b^i_j v_i = 0$.
42. If the equality $a^i_j v^j = \beta v^i$ holds for every contravariant vector v^i, where β is a scalar, show that $a^i_j = \beta \delta^i_j$.
43. (a) If A^i_{jk} is a mixed tensor, then prove that $C_j = A^i_{ji}$ is a covariant vector.
    (b) Show that if $A(i,j,k)B^i C^j D_k$ is a scalar for arbitrary vectors B^i, C^j, D_k, then $A(i,j,k)$ is a tensor.
    (c) Assume that $X(i,j)B^j = C_i$, where B^j is an arbitrary contravariant vector and C_i, is a covariant vector. Show that $X(i,j)$ is a tensor. What is its type?
44. If the relation $a_{hijk} \lambda^h \mu^i \lambda^j \mu^k = 0$, where λ^i and μ^j are components of two arbitrary contravariant vectors, then

$$a_{hijk} + a_{hkji} + a_{jihk} + a_{jkhi} = 0.$$

45. If the relation $a_{ijk} \lambda^i \lambda^j \lambda^k = 0$ holds for any arbitrary contravariant vector λ^i, show that

$$a_{ijk} + a_{jki} + a_{kij} + a_{jik} + a_{kji} + a_{ikj} = 0.$$

46. If A_{ij} is a symmetric tensor and $B_{ij} = A_{ji}$, show that B_{ij} is a symmetric tensor.

47. If $a^{ij}u_iu_j$ is an invariant for an arbitrary covariant vectors u_i, show that $a^{ij}+a^{ji}$ is a contravariant tensor of second order.
48. If a_{ij} are components of a covariant tensor of second order and λ^i, μ^j are components of two contravariant vectors, show that $a_{ij}\lambda^i\mu^j$ is an invariant.
49. If $a_{ij}u^iu^j$ is an invariant, where u^i is an arbitrary contravariant vector. If a_{ij} is a symmetric tensor and $u^i = \lambda^i + \mu^i$, show that $a_{ij}\lambda^i\mu^j$ is an invariant.
50. (a) If f is an invariant scalar function, determine whether $\frac{\partial f}{\partial x^i}$ are components of a covariant vector.
    (b) Show that the second derivatives of a scalar field f, i.e. $\dfrac{\partial^2 f}{\partial x^i \partial x^j}$, are not the components of a second order tensor.
51. (a) Prove that contraction of a mixed tensor A^i_j is a scalar invariant.
    (b) Prove that any contraction of a tensor A^i_{jk} results in a covariant vector.
    (c) Show that if A^{ij}_{kl} are tensor components, A^{ij}_{ij} is an invariant.
    (d) Prove that the contraction of a tensor of order (2, 3) is a tensor of order (1, 2).
52. (a) Verify that the outer product of a contravariant vector and a covariant vector is a mixed tensor of order two.
    (b) If A^i and B^j are two contravariant vectors, then the N^2 quantities A^iB^j are the components of a contravariant tensor of order two.
    (c) If A_{ij} is a covariant tensor and B^i is a contravariant vector, prove that $A_{ij}B^i$ is a covariant vector.
53. Prove that the inner product of covariant and contravariant vectors is a scalar invariant.
54. If v^i is an arbitrary contravariant vector and $a_{ij}(i,j = 1, 2, \ldots, N)$ are N^2 functions such that $a_{ij}v^j$ are components of a covariant vector, what can be said about a_{ij}, justify your answer.
55. If (a_{ij}) be a matrix defined in a given co-ordinate system along with corresponding matrices in other co-ordinates such that $B_j = a_{ij}\xi^i$ is a covariant vector for any arbitrary contravariant vector ξ^i, show that a_{ij} is a covariant tensor.
56. If a_{ij} is symmetric tensor and b_i is a vector and
$$a_{ij}b_k + a_{jk}b_i + a_{ki}b_j = 0,$$
then prove that $a_{ij} = 0$ or $b_k = 0$.
57. If a_{ij} is a component of a covariant symmetric tensor and b_i is a non-zero covariant vector such that
$$a_{ij}b_k + a_{ik}b_j + a_{ki}b_j = 0,$$
then prove that $a_{ij} = 0$.

58. Suppose that x components A^i_{jk} of a mixed tensor of order three are symmetric in the subscripts j and k. Show that $\overline{x}$ components $\overline{A}^i_{jk}$ of the same mixed tensor of order three are also symmetric in the subscripts j and k.
59. If the relation $a_{ijk}\lambda^i\lambda^j\lambda^k = 0$ holds for any arbitrary contravariant vector λ^i, where a_{ijk} is a symmetric tensor in i and j, show that

$$a_{ijk} + a_{jki} + a_{kij} = 0.$$

60. (a) If a_{ij} is a component of a symmetric covariant tensor and u^i, v^i are two contravariant vectors, show that $a_{ij}u^iv^j$ is an invariant.
    (b) If $A(i,j)dx^idx^j$ is an invariant for an arbitrary vector dx^i and $A(i,j)$ is symmetric, show that $A(i,j)$ is a tensor A_{ij}.
    (c) It is given that $A(i,j,k)B^{jk} = \xi^i$, where B^{jk} is an arbitrary symmetric tensor and ξ^i is an arbitrary contravariant vector. Show that $A(i,j,k) + A(i,k,j)$ is a tensor. Hence deduce that, if $A(i,j,k)$ is symmetric in j and k, then $A(i,j,k)$ is a tensor.
61. If a_{ij} is a skew-symmetric covariant tensor of rank 2 and A^j is an arbitrary contravariant vector, prove that $a_{ij}A^iA^j = 0$.
62. If A_{ijk} is completely symmetric and the indices run from 1 to N, show that the number of distinct components of A_{ijk} is $\frac{N(N+1)(N+2)}{6}$.
63. If a_{ij} and b_{ij} are components of two covariant tensors in an N dimensional space, where $b_{ij} = b_{ji}$ and $|b_{ij}| \neq 0$, satisfying

$$a_{ij}b_{kj} - a_{ij}b_{jk} + a_{kj}b_{ij} - a_{kj}b_{ij} = 0,$$

prove that $a_{ij} = a_{ji}$ for all i and j.
64. The square of the element of arc ds appears in the form: $ds^2 = g_{ij}dx^idx^j$. Let T be the admissible transformation of co-ordinates $x^i = x^i\left(y^1, y^2, \ldots, y^N\right)$, then $ds^2 = h_{ij}dy^idy^j$. Prove that $|g_{ij}|$ is a relative scalar of weight 2.
65. If a^{ij} are the components of a contravariant tensor and b_{ij} are the components of a symmetric tensor such that $b = |b_{ij}| \neq 0$, show that $\sqrt{b}a^{ij}$ are the components of a tensor density.
66. If a^{ij} is a contravariant tensor such that $\left|a^{ij}\right| \neq 0$, show that $\left|a^{ij}\right|$ is a relative invariant of weight -2.
67. If a_{ij} is a symmetric tensor such that $\left|a^{ij}\right| \neq 0$ and b^{ij} is the cofactor of a_{ij} in $\left|a^{ij}\right|$, prove that b^{ij} is a relative tensor of weight 2.
68. (a) Verify the following formulas for the permutation symbols e_{ij} and e_{ijk} (for distinct values of the indices only):

$$e_{ij} = \frac{j-i}{|j-i|};\ e_{ijk} = \frac{(j-i)(k-i)(k-j)}{|j-i||k-i||k-j|}.$$

(b) Prove the general formula,

$$e_{i_1 i_2 \cdots i_n} = \frac{(i_2 - i_1)(i_3 - i_1)\cdots(i_n - i_1)(i_3 - i_2)\cdots(i_n - i_2)\cdots(i_n - i_{n-1})}{|i_2 - i_1||i_3 - i_1|\cdots|i_n - i_1||i_3 - i_2|\cdots|i_n - i_2|\cdots|i_n - i_{n-1}|}.$$

69. Prove that

(a) $\varepsilon_{iks}\varepsilon_{mps} = \delta_{im}\delta_{kp} - \delta_{ip}\delta_{km}$.
(b) $\varepsilon_{iks}\varepsilon_{mks} = 2\delta_{im}$.
(c) Apply contraction on δ^i_j and find its value.

70. Prove that ε_{ijk} and ε^{ijk} are relative tensors of weight –1 and 1, respectively. Also show that they are associated.
71. If a_i is any vector, show that $\varepsilon_{ijk}a_j a_k = 0$.
72. Show that the determinant of a square matrix $A = [a_{ij}]_{N\times N}$ can be expressed using the fully antisymmetric tensor of rank N as

$$\det A = \varepsilon_{ijk\ldots p}\; a_{1i}a_{2j}a_{3k}\ldots a_{Np};$$

where $\{i, j, k, \ldots, p\}$ is set of N indices. Further show that

$$\varepsilon_{ijk\ldots p}\; a_{ri}a_{sj}a_{tk}\ldots a_{zp} = (\det A)\varepsilon_{rst,\ldots,z},$$

where $\{r, s, t, \ldots, z\}$ is another set of N indices.
73. Show that

$$\varepsilon_{ijk}\varepsilon^{ist} = \begin{vmatrix} \delta^s_j & \delta^t_j \\ \delta^s_k & \delta^t_k \end{vmatrix}.$$

Hence deduce that

(i) $\varepsilon_{ijk}\varepsilon^{ijt} = 2\delta^t_k$, (ii) $\varepsilon_{ijk}\varepsilon^{ijk} = 3!$.

Chapter 2

Riemannian Metric

In Chapter 1 we have considered some algebraic operations on tensors in V_N which constitute the so-called *tensor algebra* in V_N. Each of these operations on a tensor or tensors produces again a tensor.

The notion of distance (or metric) is fundamental in applied mathematics. Frequently, the distance concept most useful in a particular application is non-Euclidean (under which the Pythagorean relation for geodesic right triangles is not valid). Tensor calculus provides a natural tool for the investigation of general formulas of distance; it studies not only non-Euclidean metrices but also the forms assumed by the Euclidean metric in particular co-ordinate system.

A space which admits an object called an *affine transformation* possesses sufficient structure to permit the operation of tensor calculus within it. It is known that a Riemannian space is necessarily endowed with an affine connection. Therefore, for the development of tensor calculus we can either consider a V_N endowed with an affine connection or can consider a Riemannian space. In this chapter we consider the alternative for the development of tensor calculus. This calculus has an important application in physics, specially in the theory of relativity.

Calculus texts often contain derivations of arc-length formulas for polar co-ordinates that apparently apply only to that one co-ordinate system. Here we develop a coincise method for obtaining the arc length formula for any admissible co-ordinate system.

2.1 The Metric Tensor

Let us consider a space of N dimensions. Let us consider a displacement vector dx^i; $i = 1, 2, 3, \ldots, N$ determined by a pair of neighbouring points x^i and $x^i + dx^i$. The distance ds between the two adjacent points whose co-ordinates in any system are x^i $(i = 1, 2, 3, \ldots, N)$ and $x^i + dx^i$ is given by the quadratic formula

$$ds^2 = g_{ij}dx^i dx^j;\ i, j = 1, 2, 3, \ldots, N \tag{2.1}$$

where the coefficients g_{ij} are arbitrary functions of co-ordinates x^i such that $g = |g_{ij}| \neq 0$. This quadratic differential form $g_{ij}dx^i dx^j$, we expresses the distance between two neighbouring points, is called a *metric* or a *Riemannian metric* or *line element*. The coefficient g_{ij} in Eq. (2.1) is called *metric tensor* or *fundamental tensor of the Riemannian metric.*

(i) The quadratic form $g_{ij}dx^i dx^j$ is positive definite, if it is positive for all the values of the differentials dx^i, not all equal to zero.

(ii) Since the distance between the two continuous points is independent of the coordinate system, the line element ds is an invariant. ds is called the element of arc in V_N.

(iii) The tensor is called the metric tensor, because, all essential metric properties of Euclidean space are completely determined by this tensor. We have introduced a metric since we are now able to define the measure ds of an elementary displacement.

(iv) If ds is real, it is called an elementary distance. In relativity, ds may be imaginary and is called elementary interval.

(v) The signature of ds^2 is the difference between the number of positive squares and the number of negative squares. Signature is invariant in a transformation of variables (follows from "the theorem of inertia" of vector space).

(vi) Every three-dimensional Euclidean space E^3 referred to an orthogonal Cartesian system can be written as

$$ds^2 = \delta_{ij} dx^i dx^j;\ i, j = 1, 2, 3 \text{ and } \delta_{ij} = g_{ij},$$

where δ^i_j is Kronecker delta defined in Eq. (1.1). Somtimes, $ds^2 = e g_{ij} dx^i dx^j$ where the numerical factor e, called the indicator, equal to $+1$ or -1 so that ds^2 is always non-negative. Thus if all the coefficients g_{ij} are independent of x^i, the space becomes Euclidean space.

An N dimensional space characterised by this metric is called *Riemannian space* of N dimensions and is denoted by V_N. Geometry based on this metric is called *Riemannian geometry* of N dimensions. Now, we are going to establish the nature of g_{ij}.

Theorem 2.1.1 *In a Riemannian space, the fundamental tensor g_{ij} is a covariant symmetrical tensor of order two.*

Proof: The metric is given by Eq. (2.1). Let us consider a covariant transformation from x^i to $\overline{x}^i (i = 1, 2, 3, \ldots, N)$ given by

$$\overline{x}^i = \overline{x}^i \left(x^1, x^2, \ldots, x^N\right) \tag{2.2}$$

so that the metric $ds^2 = g_{ij} dx^i dx^j$ transforms to $ds^2 = \overline{g}_{ij} d\overline{x}^i d\overline{x}^j$.

Step 1: Here, we have to show that dx^i is a contravariant vector. Now,

$$\overline{x}^i = \overline{x}^i \left(x^1, x^2, \ldots, x^N\right) \text{ or } d\overline{x}^i = \frac{\partial \overline{x}^i}{\partial x^p} dx^p.$$

It is the law of transformation of contravariant vector. So, dx^i is a contravariant vector.

Step 2: To show that g_{ij} is a covariant tensor of rank 2. Since dx^i and dx^j are contravariant vectors, we get, by tensor law of transformation

$$d\overline{x}^i = \frac{\partial \overline{x}^i}{\partial x^p} dx^p \text{ and } d\overline{x}^j = \frac{\partial \overline{x}^j}{\partial x^q} dx^q. \tag{2.3}$$

Since ds^2 is invariant under co-ordinate transformation, we have

$$g_{ij} dx^i dx^j = \overline{g}_{ij} d\overline{x}^i d\overline{x}^j$$

or

$$g_{ij} dx^i dx^j = \overline{g}_{ij} \frac{\partial \overline{x}^i}{\partial x^p} \frac{\partial \overline{x}^j}{\partial x^q} dx^p dx^q; \qquad \text{by Eq. (2.3)}$$

or

$$g_{pq} dx^p dx^q = \overline{g}_{ij} \frac{\partial \overline{x}^i}{\partial x^p} \frac{\partial \overline{x}^j}{\partial x^q} dx^p dx^q$$

(Changing the dummy indices i, j by p, q)
or

$$\left[g_{pq} - \overline{g}_{ij} \frac{\partial \overline{x}^i}{\partial x^p} \frac{\partial \overline{x}^j}{\partial x^q} \right] dx^p dx^q = 0.$$

Since dx^p and dx^q are arbitrary vectors, we get

$$g_{pq} - \overline{g}_{ij} \frac{\partial \overline{x}^i}{\partial x^p} \frac{\partial \overline{x}^j}{\partial x^q} = 0 \Rightarrow g_{pq} = \overline{g}_{ij} \frac{\partial \overline{x}^i}{\partial x^p} \frac{\partial \overline{x}^j}{\partial x^q}. \tag{2.4}$$

This is a second rank covariant tensor law of transformation. Therefore, g_{ij} is covariant tensor of rank 2.

Step 3: Finally, we have to show that g_{ij} is symmetric. Now, g_{ij} can be written as

$$g_{ij} = \frac{1}{2} (g_{ij} + g_{ji}) + \frac{1}{2} (g_{ij} - g_{ji}) = A_{ij} + B_{ij}; \quad \text{say,} \tag{2.5}$$

where

$$A_{ij} = \frac{1}{2} (g_{ij} + g_{ji}) \text{ and } B_{ij} = \frac{1}{2} (g_{ij} - g_{ji}).$$

Since linear combination of two tensors is also a tensor of same rank, so A_{ij} is a symmetric covariant tensor and B_{ij} is anti-symmetric covariant tensor of rank 2. Therefore,

$$g_{ij} dx^i dx^j = A_{ij} dx^i dx^j + B_{ij} dx^i dx^j$$

or

$$(g_{ij} - A_{ij}) dx^i dx^j = B_{ij} dx^i dx^j. \tag{2.6}$$

Interchanging the dummy indices i and j in $B_{ij}dx^i dx^j$, we get

$$\begin{aligned}B_{ij}dx^i dx^j &= B_{ji}dx^j dx^i \\ &= -B_{ij}dx^j dx^i;\ \text{Since } B_{ij} \text{ is anti-symmetric} \\ &= -B_{ij}dx^i dx^j\end{aligned}$$

or

$$2B_{ij}dx^i dx^j = 0 \qquad \text{i.e.} \qquad B_{ij}dx^i dx^j = 0.$$

Therefore, from Eq. (2.6) we get

$$(g_{ij} - A_{ij})\, dx^i dx^j = 0.$$

Since dx^i and dx^j are arbitrary, we conclude that

$$\begin{aligned}g_{ij} - A_{ij} = 0 &\Rightarrow g_{ij} = A_{ij} \\ &\Rightarrow g_{ij} \text{ is symmetric as } A_{ij} \text{ is so.}\end{aligned}$$

Therefore, the coefficient g_{ij} of the Riemannian metric form a symmetric tensor of type $(0, 2)$. The tensor g_{ij} is called fundamental covariant tensor of V_N. Since g_{ij} are symmetric, the number of independent components of the metric tensor g_{ij} cannot exceed $\frac{1}{2}N(N + 1)$.

In an N-dimensional space, a co-ordinate system in terms of which $g_{ij} = 0$ for $i \neq j$ is called an orthogonal co-ordinate system. Further, a system in which $g_{ii} = 1$ for $1 \leq i \leq N$ (no summation over i) and $g_{ij} = 0$ for $i \neq j$ is called a Cartesian co-ordinate system.

Theorem 2.1.2 *The line element $g_{ij}dx^i dx^j$ is an invariant.*

Proof: Let us consider a co-ordinate transformation from x^i to $\overline{x}^i$ given by

$$x^i = x^i\left(\overline{x}^1, \overline{x}^2, \ldots, \overline{x}^N\right);\ i = 1, 2, \ldots, N.$$

Since g_{ij} is a covariant tensor of rank 2, we have

$$\overline{g}_{ij} = g_{pq}\frac{\partial x^p}{\partial \overline{x}^i}\frac{\partial x^q}{\partial \overline{x}^j} \Rightarrow \overline{g}_{pq} = g_{ij}\frac{\partial x^i}{\partial \overline{x}^p}\frac{\partial x^j}{\partial \overline{x}^q}.$$

Interchanging the dummy indices i, j by p, q

$$\left[\overline{g}_{pq} - g_{ij}\frac{\partial x^i}{\partial \overline{x}^p}\frac{\partial x^j}{\partial \overline{x}^q}\right] = 0$$

or

$$\left[\overline{g}_{pq} - g_{ij}\frac{\partial x^i}{\partial \overline{x}^p}\frac{\partial x^j}{\partial \overline{x}^q}\right] d\overline{x}^p d\overline{x}^q = 0$$

or

$$\overline{g}_{pq} d\overline{x}^p d\overline{x}^q = g_{ij}\frac{\partial x^i}{\partial \overline{x}^p}\frac{\partial x^j}{\partial \overline{x}^q} d\overline{x}^p d\overline{x}^q$$

or

$$\overline{g}_{pq} d\overline{x}^p d\overline{x}^q = g_{ij}\frac{\partial x^i}{\partial \overline{x}^p} d\overline{x}^p \frac{\partial x^j}{\partial \overline{x}^q} d\overline{x}^q \Rightarrow \overline{g}_{pq} d\overline{x}^p d\overline{x}^q = g_{ij} dx^i dx^j.$$

From this, we conclude that, $g_{ij}dx^i dx^j$ is invariant.

EXAMPLE 2.1.1 *Prove that invariance of the volume element dV where*

$$V = \iint \cdots \int \sqrt{g}\, dx^1 dx^2 \cdots dx^n$$

of a finite region R of V_N bounded by a closed V_{N-1}.

Solution: Let us consider a co-ordinate transformation from x^i to $\overline{x}^i (i = 1, 2, \ldots, n)$. Since g_{ij} is a covariant tensor of rank 2, we have

$$\overline{g}_{pq} = \frac{\partial x^i}{\partial \overline{x}^p}\frac{\partial x^j}{\partial \overline{x}^q} g_{ij}.$$

Taking determinant of both sides,

$$\left|\overline{g}_{pq}\right| = \left|\frac{\partial x^i}{\partial \overline{x}^p}\right| \left|\frac{\partial x^j}{\partial \overline{x}^q}\right| |g_{ij}|$$

or

$$\overline{g} = gJ^2 \Rightarrow \sqrt{\frac{\overline{g}}{g}} = J; \text{ where } J = \left|\frac{\partial x}{\partial \overline{x}}\right|.$$

Since the transformation from x^i to $\overline{x}^i$ exists, we have

$$dx^1 dx^2 \cdots dx^n = \left|\frac{\partial x}{\partial \overline{x}}\right| d\overline{x}^1 d\overline{x}^2 \cdots d\overline{x}^n.$$

Hence

$$\sqrt{\frac{\overline{g}}{g}} = \frac{dx^1 dx^2 \cdots dx^n}{d\overline{x}^1 d\overline{x}^2 \cdots d\overline{x}^n}$$

or

$$\sqrt{\overline{g}}\, d\overline{x}^1 d\overline{x}^2 \cdots d\overline{x}^n = \sqrt{g}\, dx^1 dx^2 \cdots dx^n.$$

This shows that the volume element $dV = \sqrt{g}\, dx^1 dx^2 \cdots dx^n$ is an invariant.

2.1.1 Fundamental Contravariant Tensor

Let g^{ij} be the components of the reciprocal or the conjugate tensor of g_{ij}, given by

$$g^{ij} = \frac{\text{cofactor of } g_{ij} \text{ in } g}{g}; \text{ where } g = |g_{ij}| \neq 0, \tag{2.7}$$

then the tensor g^{ij} is called the *contravariant fundamental tensor* of V_N. g_{ij} is called first fundamental tensor and g^{ij} is the second fundamental tensor.

Property 2.1.1 The properties of reciprocal tensor g^{ij} are (i) $g_{ij}g^{kj} = \delta_i^k$, (ii) $g^{ij}g_{ij} = N$ and (iii) g^{ij} is also a symmetric contravariant tensor of rank 2.

Proof: (i) Let the cofactor of g_{ij} in g be denoted by $\xi(i,j)$. From properties of determinants, we have

$$g_{ij}\xi(i,j) = g; \qquad g = |g_{ij}|$$

or

$$g_{ij}\frac{\xi(i,j)}{g} = 1 \Rightarrow g_{ij}g^{ij} = 1; \text{ using Eq. (2.7)},$$

where the summation is taken over i and j. Now,

$$g_{ij}\xi(k,j) = 0 \Rightarrow g_{ij}\frac{\xi(k,j)}{g} = 0; \text{ as } g \neq 0$$

or

$$g_{ij}g^{kj} = 0 \text{ if } k \neq i, \text{ using Eq. (2.7)}.$$

Therefore, we conclude that

$$\begin{aligned} g_{ij}g^{kj} &= 1; \qquad \text{if } i = k \\ &= 0; \qquad \text{if } i \neq k \end{aligned}$$

and hence $g_{ij}g^{kj} = \delta_i^k$.
(ii) Using the above property, we get

$$\begin{aligned} g_{ij}g^{ij} &= \delta_j^i = \delta_1^1 + \delta_2^1 + \cdots + \delta_2^2 + \cdots + \delta_N^N \\ &= \delta_1^1 + \delta_2^2 + \cdots + \delta_N^N \\ &= N \text{ as } \delta_j^i = 1, \text{ if } i = j \text{ and } 0, \text{ if } i \neq j. \end{aligned}$$

(iii) Using property (i), we have

$$\begin{aligned} g_{ij}g^{kj} &= \delta_i^k \Rightarrow |g_{ij}| \left|g^{kj}\right| = \left|\delta_i^k\right| = 1 \\ &\Rightarrow \left|g^{kj}\right| \neq 0. \end{aligned}$$

Let us define a tensor r_{ij} as

$$r_{ij} = \frac{\text{cofactor of } g^{ij} \text{ in } \left|g^{ij}\right|}{\left|g^{ij}\right|} \tag{2.8}$$

then

$$r_{ij}g^{ik} = \delta_j^k.$$

Multiplying both sides by g_{lk}, we get

$$\begin{aligned} r_{ij}g^{ik}g_{lk} &= \delta_j^k g_{lk} = g_{ij} \\ &\Rightarrow r_{ij}\delta_l^i = g_{lj} \Rightarrow r_{lj} = g_{lj} \\ &\Rightarrow r_{ij} = g_{ij}. \end{aligned}$$

Since r_{ij} is a reciprocal tensor of g_{ij}, it follows that g_{ij} is the reciprocal tensor of g^{ij}. Since g_{ij} is symmetric tensor of rank 2, so, g^{ij} is also a symmetric tensor of rank 2. This tensor g^{ij} is called the conjugate metric tensor or fundamental contravariant tensor of the type $(2,0)$.

EXAMPLE 2.1.2 *Find the expression of metric, the matrix and component of first and second fundamental tensors in spherical co-ordinates.*

Solution: Let E^3 be covered by orthogonal Cartesian co-ordinates x^i and consider a transformation

$$x^1 = y^1 \sin y^2 \cos y^3,\ x^2 = y^1 \sin y^2 \sin y^3,\ x^3 = y^1 \cos y^2$$

where the y^i are spherical polar co-ordinates. The metric in Euclidean space E^3, referred to Cartesian co-ordinates is given by

$$ds^2 = \left(dx^1\right)^2 + \left(dx^2\right)^2 + \left(dx^3\right)^2.$$

Comparing this, with Eq. (2.1), we see that

$$g_{11} = g_{22} = g_{33} = 1 \text{ and } g_{ij} = 0;\ \text{for } i \neq j.$$

The fundamental symmetric tensor $\overline{g}_{ij}$ in spherical co-ordinates are given by

$$\overline{g}_{11} = \frac{\partial x^i}{\partial y^1}\frac{\partial x^j}{\partial y^1}g_{ij} = \left(\frac{\partial x^1}{\partial y^1}\right)^2 g_{11} + \left(\frac{\partial x^2}{\partial y^1}\right)^2 g_{22} + \left(\frac{\partial x^3}{\partial y^1}\right)^2 g_{33}$$

$$= \left(\sin y^2 \cos y^3\right)^2 \times 1 + \left(\sin y^2 \sin y^3\right)^2 \times 1 + \left(\cos y^2\right)^2 \times 1$$
$$= \sin^2 y^2 \left(\cos^2 y^3 + \sin^2 y^3\right) + \cos^2 y^2 = 1.$$

$$\overline{g}_{22} = \frac{\partial x^i}{\partial y^2}\frac{\partial x^j}{\partial y^2} g_{ij} = \left(\frac{\partial x^1}{\partial y^2}\right)^2 g_{11} + \left(\frac{\partial x^2}{\partial y^2}\right)^2 g_{22} + \left(\frac{\partial x^3}{\partial y^2}\right)^2 g_{33}$$
$$= \left(y^1 \cos y^2 \cos y^3\right)^2 \times 1 + \left(y^1 \cos y^2 \sin y^3\right)^2 \times 1 + \left(-y^1 \sin y^2\right)^2 \times 1$$
$$= \left(y^1\right)^2 \cos^2 y^2 \left(\cos^2 y^3 + \sin^2 y^3\right) + \left(y^1\right)^2 \sin^2 y^2$$
$$= \left(y^1\right)^2 \left[\cos^2 y^2 + \sin^2 y^2\right] = \left(y^1\right)^2.$$

$$\overline{g}_{33} = \frac{\partial x^i}{\partial y^3}\frac{\partial i^j}{\partial y^3} g_{ij} = \left(\frac{\partial x^1}{\partial y^3}\right)^2 g_{11} + \left(\frac{\partial x^2}{\partial y^3}\right)^2 g_{22} + \left(\frac{\partial x^3}{\partial y^3}\right)^2 g_{33}$$
$$= \left(-y^1 \sin y^2 \sin y^3\right)^2 \times 1 + \left(y^1 \sin y^2 \cos y^3\right)^2 \times 1 + 0 \times 1$$
$$= \left(y^1\right)^2 \sin^2 y^2 \left[\sin^2 y^3 + \cos^2 y^2\right] = \left(y^1\right)^2 \sin^2 y^2.$$

$$\overline{g}_{12} = \frac{\partial x^i}{\partial y^1}\frac{\partial x^j}{\partial y^2} g_{ij} = \frac{\partial x^1}{\partial y^1}\frac{\partial x^1}{\partial y^2} g_{11} + \frac{\partial x^2}{\partial y^1}\frac{\partial x^2}{\partial y^2} g_{22} + \frac{\partial x^3}{\partial y^1}\frac{\partial x^3}{\partial y^2} g_{33}$$
$$= \left(\sin y^2 \cos y^3\right)\left(y^1 \cos y^2 \cos y^3\right) + \left(\sin y^2 \sin y^3\right)\left(y^1 \cos y^2 \sin y^3\right)$$
$$+ \left(\cos y^2\right)\left(-y^1 \sin y^2\right)$$
$$= y^1 \sin y^2 \cos y^2 \left(\cos^2 y^3 + \sin^2 y^3\right) - y^1 \sin y^2 \cos y^2$$
$$= y^1 \sin y^2 \cos y^2 - y^1 \sin y^2 \cos y^2 = 0.$$

$$\overline{g}_{13} = \frac{\partial x^i}{\partial y^1}\frac{\partial x^j}{\partial y^3} g_{ij} = \frac{\partial x^1}{\partial y^1}\frac{\partial x^1}{\partial y^3} g_{11} + \frac{\partial x^2}{\partial y^1}\frac{\partial x^2}{\partial y^3} g_{22} + \frac{\partial x^3}{\partial y^1}\frac{\partial x^3}{\partial y^3} g_{33}$$
$$= \left(\sin y^2 \cos y^3\right)\left(-y^1 \sin y^2 \sin y^3\right) + \left(\sin y^2 \sin y^3\right)\left(y^1 \sin y^2 \cos y^3\right)$$
$$+ \left(\cos y^2\right) \cdot 0 = -y^1 \sin^2 y^2 \sin y^3 \cos y^3 + y^1 \sin^2 y^2 \sin y^3 \cos y^3 = 0 = \overline{g}_{31}.$$

$$\overline{g}_{23} = \frac{\partial x^i}{\partial y^2}\frac{\partial x^j}{\partial y^3} g_{ij} = \frac{\partial x^1}{\partial y^2}\frac{\partial x^1}{\partial y^3} g_{11} + \frac{\partial x^2}{\partial y^2}\frac{\partial x^2}{\partial y^3} g_{22} + \frac{\partial x^3}{\partial y^2}\frac{\partial x^3}{\partial y^3} g_{33}$$
$$= \left(y^1 \cos y^2 \cos y^3\right)\left(-y^1 \sin y^2 \sin y^3\right) + \left(y^1 \cos y^2 \sin y^3\right)\left(y^1 \sin y^2 \cos y^3\right)$$
$$+ \left(-y^1 \sin y^2\right) \cdot 0$$
$$= -\left(y^1\right)^2 \sin y^2 \cos y^2 \sin y^3 \cos y^3 + \left(y^1\right)^2 \sin y^2 \cos y^2 \sin y^3 \cos y^3 = 0 = \overline{g}_{32}.$$

Thus, the symmetric metric tensor $\overline{g}_{ij}$ in spherical polar co-ordinates can be written in matrix form as

$$[\overline{g}_{ij}] = \begin{bmatrix} \overline{g}_{11} & \overline{g}_{12} & \overline{g}_{13} \\ \overline{g}_{21} & \overline{g}_{22} & \overline{g}_{23} \\ \overline{g}_{31} & \overline{g}_{32} & \overline{g}_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & (y^1)^2 & 0 \\ 0 & 0 & (y^1)^2 \sin^2 y^2 \end{bmatrix}.$$

The expression for the metric in spherical polar co-ordinates is given by

$$\begin{aligned} ds^2 &= \overline{g}_{ij} dy^i dy^j; \;\; i, j = 1, 2, 3 \\ &= \overline{g}_{11} (dy^1)^2 + \overline{g}_{22} (dy^2)^2 + \overline{g}_{33} (dy^3)^2 \\ &= (dy^1)^2 + (y^1)^2 (dy^2)^2 + (y^1)^2 (\sin y^2)^2 (dy^3)^2 . \end{aligned}$$

Let $\overline{g} = |\overline{g}_{ij}|$, then it is given by

$$\overline{g} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & (y^1)^2 & 0 \\ 0 & 0 & (y^1)^2 \sin^2 y^2 \end{vmatrix} = (y^1)^4 \sin^2 y^2 \neq 0.$$

Therefore, the conjugate or reciprocal symmetric tensor $\overline{g}^{ij}$ are given by

$$\overline{g}^{11} = \frac{\text{cofactor of } \overline{g}_{11} \text{ in } \overline{g}}{\overline{g}} = \frac{1}{(y^1)^4 \sin^2 y^2} \begin{vmatrix} (y^1)^2 & 0 \\ 0 & (y^1)^2 \sin^2 y^2 \end{vmatrix} = 1.$$

$$\overline{g}^{22} = \frac{\text{cofactor of } \overline{g}_{22} \text{ in } \overline{g}}{\overline{g}} = \frac{1}{(y^1)^4 \sin^2 y^2} \begin{vmatrix} 1 & 0 \\ 0 & (y^1)^2 \sin^2 y^2 \end{vmatrix} = \frac{1}{(y^1)^2}.$$

$$\overline{g}^{33} = \frac{\text{cofactor of } \overline{g}_{33} \text{ in } \overline{g}}{\overline{g}} = \frac{1}{(y^1)^4 \sin^2 y^2} \begin{vmatrix} 1 & 0 \\ 0 & (y^1)^2 \end{vmatrix} = \frac{1}{(y^1)^2 \sin^2 y^2}.$$

Similarly, $\overline{g}^{ij} = 0$, for $i \neq j$. Hence, the reciprocal tensor $\overline{g}^{ij}$ can be represented in matrix form as

$$[\overline{g}^{ij}] = \begin{bmatrix} \overline{g}^{11} & \overline{g}^{12} & \overline{g}^{13} \\ \overline{g}^{21} & \overline{g}^{22} & \overline{g}^{23} \\ \overline{g}^{31} & \overline{g}^{32} & \overline{g}^{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \dfrac{1}{(y^1)^2} & 0 \\ 0 & 0 & \dfrac{1}{(y^1)^2 \sin^2 y^2} \end{bmatrix}.$$

EXAMPLE 2.1.3 *Find the expression of metric and component of first and second fundamental tensor in cylindrical co-ordinates.*

Solution: Let x^i be the orthogonal Cartesian co-ordinates and y^i be the cylindrical co-ordinates, then the transformation formula is

$$x^1 = y^1 \cos y^2,\ x^2 = y^1 \sin y^2,\ x^3 = y^3.$$

The metric in Cartesian co-ordinates is given by

$$ds^2 = \left(dx^1\right)^2 + \left(dx^2\right)^2 + \left(dx^3\right)^2.$$

Comparing this, with Eq. (2.1), we see that

$$g_{11} = g_{22} = g_{33} = 1 \text{ and } g_{ij} = 0;\ \text{for } i \neq j.$$

Let g_{ij} and $\overline{g}_{ij}$ be the symmetric metric tensors in Cartesian co-ordinates and cylindrical co-ordinates, respectively. On transformation

$$\overline{g}_{pq} = g_{ij}\frac{\partial x^i}{\partial y^p}\frac{\partial x^j}{\partial y^q};\ i,j = 1,2,3,$$

where g_{ij} is covariant tensor of rank 2. Thus,

$$\begin{aligned}\overline{g}_{11} &= \frac{\partial x^i}{\partial y^1}\frac{\partial x^j}{\partial y^1}g_{ij} = \left(\frac{\partial x^1}{\partial y^1}\right)^2 g_{11} + \left(\frac{\partial x^2}{\partial y^1}\right)^2 g_{22} + \left(\frac{\partial x^3}{\partial y^1}\right)^2 g_{33} \\ &= \left(\cos y^2\right)^2 \times 1 + \left(\sin y^2\right)^2 \times 1 + 0 \times 1 = 1.\end{aligned}$$

$$\begin{aligned}\overline{g}_{22} &= \frac{\partial x^i}{\partial y^2}\frac{\partial x^j}{\partial y^2}g_{ij} = \left(\frac{\partial x^1}{\partial y^2}\right)^2 g_{11} + \left(\frac{\partial x^2}{\partial y^2}\right)^2 g_{22} + \left(\frac{\partial x^3}{\partial y^2}\right)^2 g_{33} \\ &= \left(-y^1 \sin y^2\right)^2 \times 1 + \left(y^1 \cos y^2\right)^2 \times 1 + 0 \times 1 = \left(y^1\right)^2. \\ \overline{g}_{33} &= \frac{\partial x^i}{\partial y^3}\frac{\partial x^j}{\partial y^3}g_{ij} = \left(\frac{\partial x^1}{\partial y^3}\right)^2 g_{11} + \left(\frac{\partial x^2}{\partial y^3}\right)^2 g_{22} + \left(\frac{\partial x^3}{\partial y^3}\right)^2 g_{33} \\ &= 0 \times 1 + 0 \times 1 + 1 \times 1 = 1. \\ \overline{g}_{12} &= \frac{\partial x^i}{\partial y^1}\frac{\partial x^j}{\partial y^2}g_{ij} = \frac{\partial x^1}{\partial y^1}\frac{\partial x^1}{\partial y^2}g_{11} + \frac{\partial x^2}{\partial y^1}\frac{\partial x^2}{\partial y^2}g_{22} + \frac{\partial x^3}{\partial y^1}\frac{\partial x^3}{\partial y^2}g_{33} \\ &= \cos y^2 \cdot \left(-y^1 \sin y^2\right)^2 \cdot 1 + \sin y^2 \cdot \left(y^1 \cos y^2\right)^2 \cdot 1 + 0 \cdot 0 \cdot 1 = 0 = \overline{g}_{21}. \\ \overline{g}_{13} &= \frac{\partial x^i}{\partial y^1}\frac{\partial x^j}{\partial y^3}g_{ij} = \frac{\partial x^1}{\partial y^1}\frac{\partial x^1}{\partial y^3}g_{11} + \frac{\partial x^2}{\partial y^1}\frac{\partial x^2}{\partial y^3}g_{22} + \frac{\partial x^3}{\partial y^1}\frac{\partial x^3}{\partial y^3}g_{33} \\ &= \cos y^2 \times 0 \times 1 + \sin y^2 \times 0 \times 1 + 0 \times 1 \times 1 = 0 = \overline{g}_{31}. \\ \overline{g}_{23} &= \frac{\partial x^i}{\partial y^2}\frac{\partial x^j}{\partial y^3}g_{ij} = \frac{\partial x^1}{\partial y^2}\frac{\partial x^1}{\partial y^3}g_{11} + \frac{\partial x^2}{\partial y^2}\frac{\partial x^2}{\partial y^3}g_{22} + \frac{\partial x^3}{\partial y^2}\frac{\partial x^3}{\partial y^3}g_{33} = 0 = \overline{g}_{32}.\end{aligned}$$

Thus, the metric tensors in cylindrical co-ordinates can be written in matrix form as

$$\left[\overline{g}_{ij}\right] = \begin{bmatrix} \overline{g}_{11} & \overline{g}_{12} & \overline{g}_{13} \\ \overline{g}_{21} & \overline{g}_{22} & \overline{g}_{23} \\ \overline{g}_{31} & \overline{g}_{32} & \overline{g}_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \left(y^1\right)^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The expression for the metric in cylindrical co-ordinates is given by

$$\begin{aligned} ds^2 = \overline{g}_{ij} dy^i dy^j &= \overline{g}_{11}\left(dy^1\right)^2 + \overline{g}_{22}\left(dy^2\right)^2 + \overline{g}_{33}\left(dy^3\right)^2 \\ &= \left(dy^1\right)^2 + \left(y^1\right)^2\left(dy^2\right)^2 + \left(dy^3\right)^2. \end{aligned}$$

Let $\overline{g} = \left|\overline{g}_{ij}\right|$, then it is given by

$$\overline{g} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \left(y^1\right)^2 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \left(y^1\right)^2 \neq 0.$$

Therefore, the conjugate or reciprocal symmetric tensor $\overline{g}^{ij}$ are given by

$$\begin{aligned} \overline{g}^{11} &= \frac{\text{cofactor of } \overline{g}_{11} \text{ in } \overline{g}}{\overline{g}} = \frac{1}{\left(y^1\right)^2}\begin{vmatrix} \left(y^1\right)^2 & 0 \\ 0 & 1 \end{vmatrix} = 1 \\ \overline{g}^{22} &= \frac{\text{cofactor of } \overline{g}_{22} \text{ in } \overline{g}}{\overline{g}} = \frac{1}{\left(y^1\right)^2}\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = \frac{1}{\left(y^1\right)^2} \\ \overline{g}^{33} &= \frac{\text{cofactor of } \overline{g}_{33} \text{ in } \overline{g}}{\overline{g}} = \frac{1}{\left(y^1\right)^2}\begin{vmatrix} 1 & 0 \\ 0 & \left(y^1\right)^2 \end{vmatrix} = 1. \end{aligned}$$

Similarly, $\overline{g}^{ij} = 0$, for $i \neq j$. Hence, the reciprocal tensor $\overline{g}^{ij}$ can be represented in matrix form as

$$\left[\overline{g}^{ij}\right] = \begin{bmatrix} \overline{g}^{11} & \overline{g}^{12} & \overline{g}^{13} \\ \overline{g}^{21} & \overline{g}^{22} & \overline{g}^{23} \\ \overline{g}^{31} & \overline{g}^{32} & \overline{g}^{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \dfrac{1}{\left(y^1\right)^2} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

EXAMPLE 2.1.4 *If the metric is given by*

$$ds^2 = 5\left(dx^1\right)^2 + 3\left(dx^2\right)^2 + 4\left(dx^3\right)^2 - 6dx^1dx^2 + 4dx^2dx^3$$

evaluate g and g^{ij}.

Solution: Comparing the given metric with (2.1), we have

$$g_{11} = 5,\ g_{22} = 3,\ g_{33} = 4,\ g_{12} = g_{21} = -3,\ g_{23} = g_{32} = 2 \quad \text{and} \quad g_{13} = g_{31} = 0.$$

If $g = |g_{ij}|$, then it is given by

$$g = \begin{vmatrix} 5 & -3 & 0 \\ -3 & 3 & 2 \\ 0 & 2 & 4 \end{vmatrix} = 4 \neq 0.$$

Now, the conjugate or reciprocal tensor g^{ij} are given by

$$g^{11} = \frac{\text{cofactor of } g_{11} \text{ in } g}{g} = \frac{1}{4}\begin{vmatrix} 3 & 2 \\ 2 & 4 \end{vmatrix} = 2$$

$$g^{22} = \frac{\text{cofactor of } g_{22} \text{ in } g}{g} = \frac{1}{4}\begin{vmatrix} 5 & 0 \\ 0 & 4 \end{vmatrix} = 5$$

$$g^{33} = \frac{\text{cofactor of } g_{33} \text{ in } g}{g} = \frac{1}{4}\begin{vmatrix} 5 & -3 \\ -3 & 3 \end{vmatrix} = \frac{3}{2}$$

$$g^{12} = \frac{\text{cofactor of } g_{12} \text{ in } g}{g} = \frac{-1}{4}\begin{vmatrix} -3 & 2 \\ 0 & 4 \end{vmatrix} = 3 = g^{21}$$

$$g^{13} = \frac{\text{cofactor of } g_{13} \text{ in } g}{g} = \frac{1}{4}\begin{vmatrix} -3 & 3 \\ 0 & 2 \end{vmatrix} = -\frac{3}{2} = g^{31}$$

$$g^{23} = \frac{\text{cofactor of } g_{23} \text{ in } g}{g} = \frac{-1}{4}\begin{vmatrix} 5 & 3 \\ 0 & 2 \end{vmatrix} = -\frac{5}{2} = g^{32}.$$

Therefore, the reciprocal tensor g^{ij} can be represented in the matrix form as

$$\left[g^{ij}\right] = \begin{bmatrix} g^{11} & g^{12} & g^{13} \\ g^{21} & g^{22} & g^{23} \\ g^{31} & g^{32} & g^{33} \end{bmatrix} = \begin{bmatrix} 2 & 3 & -\frac{3}{2} \\ 3 & 5 & -\frac{5}{2} \\ -\frac{3}{2} & -\frac{5}{2} & \frac{3}{2} \end{bmatrix}.$$

EXAMPLE 2.1.5 *Let g_{mn} and g^{mn} be the fundamental metric tensors and the reciprocal tensors respectively. Show that*

$$g^{mn}\frac{\partial}{\partial x^s}g_{mn} + g_{mn}\frac{\partial}{\partial x^s}g^{mn} = 0$$

and

$$\frac{\partial \log g}{\partial x^s} = g^{mn}\frac{\partial}{\partial x^s}g_{mn} = -g_{mn}\frac{\partial}{\partial x^s}g^{mn};\ \text{where } g = |g_{mn}|.$$

Solution: Let g^{ij} be the components of the reciprocal or the conjugate tensor of g_{ij}, given by

$$g^{ij} = \frac{\text{cofactor of } g_{ij} \text{ in } g}{g}; \text{ where, } g = |g_{ij}| \neq 0,$$

then g^{ij} are the *contravariant fundamental tensor* of V_N. From the property of reciprocal tensor g^{ij}, as $g_{mn}g^{mn} = N$ we get, after differentiation with respect to x^s,

$$g^{mn}\frac{\partial}{\partial x^s}g_{mn} + g_{mn}\frac{\partial}{\partial x^s}g^{mn} = 0.$$

Let the cofactor of g_{mn} in g be denoted by $\xi(m,n)$. From properties of determinants, we have

$$g_{mn}\xi(m,n) = g \Rightarrow \frac{\partial g}{\partial g_{mn}} = \xi(m,n); g = |g_{mn}|.$$

Also, using the relation $g_{mn}\xi(m,n) = g$, we get

$$g^{ms}g_{mn}\xi(m,n) = g^{ms}g \Rightarrow \xi(m,s) = gg^{ms}.$$

Differentiating with respect to x^s we get

$$\frac{\partial g}{\partial x^s} = \frac{\partial g}{\partial g_{mn}}\frac{\partial g_{mn}}{\partial x^s} = \xi(m,n)\frac{\partial g_{mn}}{\partial x^s}$$

or

$$\frac{\partial g}{\partial x^s} = gg^{mn}\frac{\partial g_{mn}}{\partial x^s}; \text{ as } \xi(m,s) = gg^{mn}$$

or

$$g^{mn}\frac{\partial g_{mn}}{\partial x^s} = \frac{1}{g}\frac{\partial g}{\partial x^s} = \frac{\partial \log g}{\partial x^s}$$

or

$$\frac{\partial \log g}{\partial x^s} = -g_{mn}\frac{\partial}{\partial x^s}g^{mn}; \text{ as } g^{mn}\frac{\partial}{\partial x^s}g_{mn} + g_{mn}\frac{\partial}{\partial x^s}g^{mn} = 0.$$

Thus, we get

$$\frac{\partial \log g}{\partial x^s} = g^{mn}\frac{\partial}{\partial x^s}g_{mn} = -g_{mn}\frac{\partial}{\partial x^s}g^{mn}; \text{ where } g = |g_{mn}|.$$

EXAMPLE 2.1.6 *Prove that in a V_N,*

$$(g_{hj}g_{ik} - g_{hk}g_{ij})\,g^{hj} = (N-1)g_{ik},$$

where g_{ij} and g^{ij} have their usual meanings.

Solution: Here, g_{ij} is the fundamental metric tensor and g^{ij} is the reciprocal tensor of g_{ij}. Using the properties we get

$$\begin{aligned}(g_{hj}g_{ik} - g_{hk}g_{ij})\, g^{hj} &= g_{hj}g^{hj}g_{ik} - g_{hk}g^{hj}g_{ij} \\ &= Ng_{ik} - \delta^j_k g_{ij} = Ng_{ik} - g_{ik} = (N-1)g_{ik}.\end{aligned}$$

Hence, the result follows.

EXAMPLE 2.1.7 *For a V_2 in which $g_{11} = E, g_{12} = F, g_{22} = G$, prove that*

$$g = EG - F^2,\ g^{11} = \frac{G}{g},\ g^{12} = -\frac{F}{g},\ g^{22} = \frac{E}{g}.$$

Solution: Since g_{ij} is a symmetric covariant tensor of rank 2, so, $g_{12} = g_{21} = F$. Now, in V_2,

$$g = |g_{ij}| = \begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix} = \begin{vmatrix} E & F \\ F & G \end{vmatrix} = EG - F^2 \neq 0.$$

The reciprocal tensor g^{ij} are given by

$$\begin{aligned} g^{11} &= \frac{\text{cofactor of } g_{11} \text{ in } g}{g} = \frac{G}{g} = \frac{G}{EG - F^2} \\ g^{22} &= \frac{\text{cofactor of } g_{22} \text{ in } g}{g} = \frac{E}{g} = \frac{E}{EG - F^2} \\ g^{12} &= \frac{\text{cofactor of } g_{12} \text{ in } g}{g} = -\frac{F}{g} = -\frac{F}{EG - F^2} = g^{21}. \end{aligned}$$

Result 2.1.1 Assume that a matrix field $\mathbf{g} = (g_{ij})$ exists satisfying in all (admissible) co-ordinate systems (x^i) and in some (open) region of space.

(i) All second order partial derivatives of the g_{ij} exist and are continuous.

(ii) g_{ij} is symmetric, i.e. $g_{ij} = g_{ji}$.

(iii) $\mathbf{g} = (g_{ij})$ is nonsingular, i.e. $|g_{ij}| \neq 0$.

(iv) The differential form (2.1) and hence, the distance concept generated by g_{ij} is invariant with respect to change of co-ordinates.

Sometimes, particularly in geometric applications of tensors, a property stronger *(iii)* above is assumed: $\mathbf{g} = (g_{ij})$ is positive definite.

Under this property, $|g_{ij}|$ and $g_{11}, g_{22}, \ldots, g_{NN}$ are all positive. Furthermore, the inverse matrix field $\mathbf{g}^{-1}$ is also positive definite.

2.1.2 Length of a Curve

Consider a continuous curve in a Riemannian V_N. Curve is continuous implies that the co-ordinates of any current point on it are expressible as functions of some parameter t (say). Let s denote arc length of the curve measured from a fixed point P_0 on the curve. The length ds of the arc between the points, whose co-ordinates are x^i and dx^i, given by Eq. (2.1).

Let L denote arc length of the curve between the points P_1 and P_2 on the curve which corresponds to the two values t_1 and t_2 of the parameter t. Then

$$L = \int_{P_1}^{P_2} ds = \int_{t_1}^{t_2} \left(g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} \right)^{1/2} dt. \tag{2.9}$$

If $g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} = 0$ along the curve, then the points P_1 and P_2 are zero distance, despite of the fact that they are not coincident. Such a curve is called *minimal* or *null curve.* If ds^2 is positive definite, null curves will not exist.

A curve is null if it or any of its subarcs has zero length. Here, a subarc is understood to be nontrivial; i.e. it consists of more than one point and corresponds to an interval $c \le t \le d$, where $c < d$. A curve is *null at a point* if for some value of the parameter t the tangent vector is a null vector; i.e. $g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} = 0$. The set of t values at which the curve is null is known as the *null set of the curve.* In the space-time continuum of relativity certain lines of length zero are identified as the world-lines of light.

Theorem 2.1.3 *Formula (2.9) for arc length does not depend on the particular parameterisation of the curve.*

Proof: Given a curve $\mathcal{C} : x^i = x^i(t);\ a \le t \le b$, suppose that $\mathcal{C} : x^i = x^i(\overline{t});\ \overline{a} \le \overline{t} \le \overline{b}$ is a different parameterisation, where $\overline{t} = \phi(t)$, with $\phi'(t) > 0$ and $\overline{a} = \phi(a), \overline{b} = \phi(b)$. Then by the chain rule and substitution rule for integrals,

$$\begin{aligned} L &= \int_a^b \left(g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} \right)^{1/2} dt = \int_a^b \left(g_{ij} \frac{dx^i}{d\overline{t}} \frac{dx^j}{d\overline{t}} \right)^{1/2} \phi'(t) dt \\ &= \int_{\overline{a}}^{\overline{b}} \left(g_{ij} \frac{dx^i}{d\overline{t}} \frac{dx^j}{d\overline{t}} \right)^{1/2} d\overline{t} = \overline{L}. \end{aligned}$$

This shows that, formula (2.9) for arc length does not depend on the particular parameterisation of the curve.

EXAMPLE 2.1.8 *A curve in spherical co-ordinates x^i is given by*

$$x^1 = t, x^2 = \sin^{-1}\left(\frac{1}{t}\right),\ x^3 = 2\sqrt{t^2 - 1}.$$

Find the length of arc for $1 \le t \le 2$.

Solution: In the spherical co-ordinate, the metric is given by

$$\begin{aligned} ds^2 &= (dx^1)^2 + (x^1)^2(dx^2)^2 + (x^1 \sin x^2)^2(dx^3)^2 \\ &= (dt)^2 + t^2\left(-\frac{dt}{t\sqrt{t^2-1}}\right)^2 + \left[t\cdot\frac{1}{t}\right]^2\left(\frac{2t}{t\sqrt{t^2-1}}dt\right)^2 \\ &= \frac{5t^2}{t^2-1}(dt)^2 \Rightarrow ds = \sqrt{5}\frac{t}{\sqrt{t^2-1}}dt. \end{aligned}$$

Thus, the required length of the arc, $1 \le t \le 2$ is given by

$$\int_{t_1}^{t_2} ds = \sqrt{5}\int_1^2 \frac{t}{\sqrt{t^2-1}}dt = \sqrt{15} \text{ units.}$$

EXAMPLE 2.1.9 *Find the length of arc for $1 \le t \le 2$ for the curve $x^1 = 1, x^2 = t$, if the metric is that of the hyperbolic plane ($x^2 > 0$): $g_{11} = g_{22} = \frac{1}{(x^2)^2}$; $g_{12} = g_{21} = 0$.*

Solution: For the given hyperbolic plane the metric is given by

$$ds^2 = \frac{1}{(x^2)^2}\left[(dx^1)^2 + (dx^2)^2\right] = \frac{1}{t^2}(dt)^2.$$

Thus, the required length of the arc, $1 \le t \le 2$ is given by

$$L = \int_1^2 \frac{1}{t}dt = \log 2 \text{ units.}$$

EXAMPLE 2.1.10 *Under the metric $ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 - (dx^4)^2$, consider a curve given by*

$$x^1 = 3\cos t,\ x^2 = 3\sin t,\ x^3 = 4t \text{ and } x^4 = 5t.$$

Find the length of arc for $0 \le t \le 1$.

Solution: For the given metric, we have

$$\begin{aligned} ds^2 &= (dx^1)^2 + (dx^2)^2 + (dx^3)^2 - (dx^4)^2 \\ &= \left[9\sin^2 t + 9\cos^2 t + 16 - 25\right](dt)^2 = 0. \end{aligned}$$

Thus, the required length of the arc, $0 \le t \le 1$ is given by

$$L = \int_0^1 0 ds = 0.$$

Thus, according to the definition the given curve is a null curve.

2.2 Associated Tensors

One of the fundamental concepts of tensor calculus resides in the 'raising' or 'lowering' of indices in tensors.

In E^N the covariance or contravariance of a tensor was a quality which it was impossible to change. With the introduction of a metric in E^N this barrier falls in V_N. We have at our disposal the fundamental tensors g_{ij} and g^{ij} which allow us a number of combinations.

Let A^i and B_i be a contravariant and covariant vector, respectively, in (x^i) system. Define two vectors A_i and B^i as follows:

$$A_i = g_{ij}A^j \text{ and } B^i = g^{ij}B_j. \tag{2.10}$$

Then the associate to a contravariant vector A^j is formed by lowering its index by the fundamental metric tensor g_{ij} and the associate to a given covariant vector B_j is formed by raising its index by the conjugate metric tensor. Now,

$$g^{ij}A_j = g^{ij}g_{jk}A^k = \delta^i_k A^k = A^i. \tag{2.11}$$

The procedure of raising and lowering indices is clearly reversible. From Eq. (2.11) it follows that the associate to A_i is A^i. Consequently, if A_i is the associate to A^i, then A^i is the associate to A_i. Thus, A_i and A^i are mutually associated and so they are *associate vectors.*

Next, we consider tensors of order greater than one. Any index of such a tensor can be lowered or raised by the fundamental tensors as in the case of vectors. Consider a tensor A_{ij} and form the following inner products:

$$A^i_{\bullet j} = g^{ik}A_{kj};\ A^j_{i\bullet} = g^{jk}A_{ik} \quad \text{and} \quad A^{ij} = g^{ik}g^{jl}A_{kl}. \tag{2.12}$$

The tensor $A^i_{\bullet j}$, $A^j_{i\bullet}$ and A^{ij} are called *associates* to the tensor A_{ij}. It is to be noted that any two of the four tensors $A_{ij}, A^i_{\bullet j}, A^j_{i\bullet}, A^{ij}$ may be formed from each other by lowering and raising indices. For example,

$$A^i_{\bullet j} = g^{ik}A_{kj} = g^{ik}g_{mj}A^m_{k\bullet};\ \text{as } g_{il}A^j_{i\bullet} = A_{il}$$

and so on. In general, $g^{ik}A_{jk} = A^i_{j\bullet}$ and $g^{ik}A_{kj} = A^i_{\bullet j}$ are different. But they are identical, whenever $A_{ij} = A_{ji}$ and it is denoted by A^i_j.

Similarly, consider a tensor A_{ijk} and form the following inner products

$$g^{mi}A_{ijk} = A^m_{\bullet jk},\ g^{mj}A_{ijk} = A^m_{i\bullet k} \quad \text{and} \quad g^{mk}A_{ijk} = A^m_{ij\bullet} \tag{2.13}$$

All these tensors are associated with the tensor A_{ijk}. Operating on these tensors with g^{ij} again, we can form another associated tensor.

Let $A^{i_1\cdots i_p}_{j_1\cdots j_q}$ be a given tensor of order (p,q), i.e. of rank $(p+q)$. A tensor obtained by the process of inner multiplication of $A^{i_1\cdots i_p}_{j_1\cdots j_q}$ with either of the fundamental metric tensor g_{ij} or its reciprocal g^{ij} are called *associated tensors* of the given tensor.

For a particular case, from the fundamental tensor g_{ij}, we get

$$g^{i}_{\bullet j} = g^{ik} g_{kj} = \delta^i_j;\ g^{i}_{j\bullet} = g^{ik} g_{jk} = \delta^i_j$$

and

$$g^{ij} = g^{ik} g^{jl} g_{kl}.$$

Since $g^{i}_{\bullet j}$ and $g^{j}_{i\bullet}$ are equal, it is not necessary to distinguish between them, so that we can write g^i_j. Since the metric tensor g_{ij} in an E^n with rectangular Cartesian co-ordinates is δ_{ij}, the Eq. (2.10) becomes

$$A_i = \delta_{ij} A^j = A^i$$

in these co-ordinates. This circumstance shows that in V_N covariant and contravariant vectors A_i and A^i which are connected by the equality (2.10) may not be considered as two distinct objects existing independent of co-ordinate system. The reason is that if we assume the contrary, then we come to the contradiction that in rectangular Cartesian system we will not find them as distinct. These circumstances show the propriety of an agreement to regard pairs of tensors like A_i, A^i, g_{ij}, g^{ij}, A_{ij}, A^{ij} as different types of components of the same tensor of corresponding order.

EXAMPLE 2.2.1 *If A_i and B_j are two covariant vectors, show that*

$$g^{ij}\left(A_i B_j - A_j B_i\right) = 0,$$

where g^{ij} is the contravariant fundamental tensor.

Solution: Using the definition of associated vectors, we have

$$\begin{aligned} g^{ij}\left(A_i B_j - A_j B_i\right) &= g^{ij} A_i B_j - g^{ij} A_j B_i \\ &= A^j B_j - A^i B_i = A^i B_i - A^i B_i = 0, \end{aligned}$$

where, we have changed the dummy index j by i. Hence the result follows.

EXAMPLE 2.2.2 *If $g_{pq} A^q = B_p$, then show that $A^p = g^{pq} B_q$.*

Solution: Since in $g_{pq} A^q$ the q is dummy, therefore, there is summation over q. Now, giving the values $1, 2, \ldots, N$ we have following N linear equations:

$$\begin{aligned} g_{11} A^1 + g_{12} A^2 + \cdots + g_{1N} A^N &= B_1 \\ g_{21} A^1 + g_{22} A^2 + \cdots + g_{2N} A^N &= B_2 \\ \vdots \qquad \vdots \qquad\qquad \vdots \qquad & \quad \vdots \\ g_{N1} A^1 + g_{N2} A^2 + \cdots + g_{NN} A^N &= B_N. \end{aligned}$$

Solving these equations by the method of determinants, we obtain

$$\begin{aligned}
A^1 &= g^{11}B_1 + g^{12}B_2 + \cdots + g^{1N}B_N \\
A^2 &= g^{21}B_1 + g^{22}B_2 + \cdots + g^{2N}B_N \\
\vdots &\quad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
A^N &= g^{N1}B_1 + g^{N2}B_2 + \cdots + g^{NN}B_N.
\end{aligned}$$

These equations may be written as $A^p = g^{pq}B_q$. Hence, the result follows.

EXAMPLE 2.2.3 *Prove the following relations:*

$$(i)\ A_{ij}B^{ij} = A^{ij}B_{ij} \quad and \quad (ii)\ A_{kl}B^{li} = A^i_k B^l_i.$$

Solution: (i) We have, by definition,

$$A_{kl} = g_{ki}g_{lj}A^{ij} \ \text{ and } \ B^{kl} = g^{ki}g^{lj}B_{ij}.$$

Using this result we get

$$\begin{aligned}
A_{kl}B^{kl} &= g_{ki}g_{lj}A^{ij}g^{ki}g^{lj}B_{ij} = g^{ki}g_{ki}g^{lj}g_{lj}A^{ij}B_{ij} \\
&= A^{ij}B_{ij}; \ \text{ as } \ g^{ki}g_{ki} = 1,
\end{aligned}$$

or

$$A_{ij}B^{ij} = A^{ij}B_{ij}.$$

(ii) According to the definition,

$$g^{ki}B^l_i = B^{kl} \Rightarrow g^k_i g^{ki}B^l_i = B^{li} \text{ and } g^l_i A^i_k = A_{kl} \Rightarrow g^l_i g_{li}A^i_k = A_{ki},$$

where, $g^i_k = \delta^i_k$. Therefore,

$$A_{ki}B^{li} = g^i_l g_{il}A^i_k g^i_k g^{ki}B^l_i = g_{ii}A^i_k g^{ii}B^l_i = A^i_k B^l_i.$$

EXAMPLE 2.2.4 *Express the relationship between following pairs of associated tensors:*

$$(i) B^{jkl} \text{ and } B_{pqr}, \quad (ii) B^{\bullet k}_{j\bullet l} \text{ and } B^{qkr}, \quad (iii) B^{p\bullet rs\bullet}_{\bullet q\bullet\bullet t} \text{ and } B^{\bullet\bullet\bullet st}_{jqk\bullet\bullet}$$

Solution: (i) According to the definition,

$$B^{jkl} = g^{jp}g^{kq}g^{lr}B_{pqr} \text{ and } B_{pqr} = g_{jp}g_{kq}g_{lr}B^{jkl}.$$

(ii) Using the definition,

$$B^{\bullet k}_{j\bullet l} = g_{jq}g_{lr}B^{qkr} \Rightarrow B^{qkr} = g^{jq}g^{lr}B^{\bullet k}_{j\bullet l}.$$

(iii) Using the definition we get

$$B^{p\bullet rs\bullet}_{\bullet q\bullet\bullet t} = g^{pj}g^{rk}g_{tl}B^{\bullet\bullet\bullet sl}_{jqk}$$

or

$$B^{\bullet\bullet\bullet sl}_{jqk} = g_{pj}g_{rk}g^{tl}B^{p\bullet rs\bullet}_{\bullet q\bullet\bullet t}.$$

EXAMPLE 2.2.5 *Let the vectors u^i, v^i be defined by $u^i = g^{ij}u_j$ and $v^i = g^{ij}v_j$. Show that*

$$u_i = g_{ij}u^j, u^i v_i = u_i v^i \text{ and } u^i g_{ij} u^j = u_i g^{ij} u_j.$$

Solution: Given that $u^i = g^{ij}u_j$ and $v^i = g^{ij}v_j$. Therefore,

$$g_{ik}u^i = g_{ik}g^{ij}u_j = \delta^j_k u_j = u_k$$

or

$$u_k = g_{ik}u^i = g_{ik}u^j$$

or

$$u_i = g_{ji}u^j = g_{ij}u^j; \text{ as } g_{ij} = g_{ji}.$$

Now, we have to show that $u^i v_i = u_i v^i$. For this

$$u^i v_i = \left(g^{ij}u_j\right)v_i = \left(g^{ij}v_i\right)u_j = v^j u_j = v^i u_i = u_i v^i.$$

Lastly, we have to deduce that $u^i g_{ij} u^j = u_i g^{ij} u_j$. For this

$$u^i g_{ij} u^j = \left(u^i g_{ij}\right)u^j = u_j u^j = \left(u_i g^{ij}\right)u_j = u_i g^{ij} u_j.$$

Thus, the results are proved.

2.2.1 Magnitude of a Vector

The *magnitude or length* A of a contravariant vector A^i in a curvilinear co-ordinate system E^3 is defined as

$$A^2 = g_{ij}A^i A^j; \text{ i.e. } A = \sqrt{g_{ij}A^i A^j}. \tag{2.14}$$

Equation (2.14) can be written in the form,

$$A^2 = A^i g_{ij} A^j = A^i A_i = A_i g^{ij} A_j = g^{ij} A_i A_j.$$

Similarly, the *magnitude* or *length* B of a covariant vector B_i is defined as

$$B^2 = g^{ij} B_i B_j. \tag{2.15}$$

Equation (2.15) can be written in the form,

$$B^2 = g^{ij}B_iB_j = B_ig^{ij}B_j = B_iB^i = g_{ij}B^iB^j.$$

Thus, it follows that the magnitude of the two associated vectors A^i, A_i and B_i, B^i are same. A vector with unity as magnitude is called a *unit vector.* In this case,

$$g_{ij}A^iA^j = 1;\ g^{ij}B_iB_j = 1. \tag{2.16}$$

A vector whose magnitude is zero is called a *null vector.* In that case,

$$g_{ij}A^iA^j = 0 = g^{ij}B_iB_j. \tag{2.17}$$

A null vector should be distinguished from a zero vector each of whose component is zero. So it is different from a zero vector.

EXAMPLE 2.2.6 *Show that the magnitude of two associated vectors is same.*

Solution: Let A and B be magnitudes of associate vectors A^i and A_i respectively, then by definition of magnitude,

$$A^2 = g_{ij}A^iA^j \ \text{ and } \ B^2 = g^{ij}A_iA_j.$$

We have to show that $A = B$. Using the definition of associate vectors, we have

$$A^2 = \left(g_{ij}A^i\right)A^j = A_jA^j$$
$$B^2 = \left(g^{ij}A_i\right)A_j = A^jA_j$$
$$\Rightarrow A^2 = B^2;\ \text{i.e. } A = B.$$

Thus the magnitude of associate vectors A^i and A_i are equal. Therefore, A^i and A_i are referred to as contravariant and covariant components, respectively, of the same vector $\overrightarrow{A}$. Also, it is clear that

$$A^2 = g_{ij}A^iA^j = g^{ij}A_iA_j = A^iA_i.$$

This result is of vital importance.

EXAMPLE 2.2.7 *Show that* $\frac{dx^i}{ds}$ *is a unit contravariant vector.*

Solution: From the definition of Riemannian metric, we have

$$ds^2 = g_{ij}dx^idx^j \Rightarrow 1 = g_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds},$$

which according to the definition of unit vector, $\frac{dx^i}{ds}$ is a contravariant vector of magnitude with the unit vector, $\frac{dx^i}{ds}$ is defined as unit tangent vector to a some curve $\mathcal{C}$ in Riemannian V_N.

EXAMPLE 2.2.8 *Show that in the* V_4 *with line element*

$$ds^2 = -\left(dx^1\right)^2 - \left(dx^2\right)^2 - \left(dx^3\right)^2 + c^2\left(dx^4\right)^2$$

the vector $\left(-1, 0, 0, \frac{1}{c}\right)$ *is a null vector.*

Solution: Let $\left(-1, 0, 0, \frac{1}{c}\right)$ be the components of a contravariant vector A^i in V_4, then $A^1 = -1, A^2 = 0, A^3 = 0$ and $A^4 = \frac{1}{c}$. Now, comparing the given metric with Eq. (2.1), we get, $g_{11} = -1 = g_{22} = g_{33}, g_{44} = c^2$ and $g_{ij} = 0$ for $i \neq j$. Therefore,

$$\begin{aligned} g_{ij}A^iA^j &= g_{11}A^1A^1 + g_{22}A^2A^2 + g_{33}A^3A^3 + g_{44}a^4A^4 \\ &= (-1)\cdot(-1)\cdot(-1) + (-1)\cdot 0\cdot 0 + (-1)\cdot 0\cdot 0 + \frac{1}{c^2}\cdot c\cdot c \\ &= -1 + 0 + 0 + 1 = 0. \end{aligned}$$

Hence, A^i is a null vector in V_4. Note that, its components are not all zero, so it is different from a zero vector.

EXAMPLE 2.2.9 *Show that in the* V_4 *with line element*

$$ds^2 = -\left(dx^1\right)^2 - \left(dx^2\right)^2 - \left(dx^3\right)^2 + c^2\left(dx^4\right)^2$$

the vector $\left(1, 0, 0, \frac{\sqrt{2}}{c}\right)$ *is a unit vector.*

Solution: Let $\left(1, 0, 0, \frac{\sqrt{2}}{c}\right)$ be the components of a contravariant vector A^i in V_4, then $A^1 = 1, A^2 = 0, A^3 = 0$ and $A^4 = \frac{\sqrt{2}}{c}$. Now, comparing the given metric with Eq. (2.1), we get $g_{11} = -1 = g_{22} = g_{33}, g_{44} = c^2$ and $g_{ij} = 0$ for $i \neq j$. Therefore,

$$\begin{aligned} g_{ij}A^iA^j &= g_{11}A^1A^1 + g_{22}A^2A^2 + g_{33}A^3A^3 + g_{44}A^4A^4 \\ &= (-1)\cdot 1\cdot 1 + (-1)\cdot 0\cdot 0 + (-1)\cdot 0\cdot 0 + c^2\cdot\frac{\sqrt{2}}{c}\cdot\frac{\sqrt{2}}{c} \\ &= -1 + 0 + 0 + 2 = 1. \end{aligned}$$

Hence, A^i is a unit vector in V_4.

EXAMPLE 2.2.10 *Prove that the length of a vector is invariant.*

Solution: Using the transformation formula for contravariant vector and covariant tensor of type $(0, 2)$, we get from Eq. (2.14),

$$\begin{aligned} A^2 = g_{ij}A^iA^j &= \overline{g}_{mn}\frac{\partial\overline{x}^m}{\partial x^i}\frac{\partial\overline{x}^n}{\partial x^j}\overline{A}^s\frac{\partial x^i}{\partial\overline{x}^s}A^l\frac{\partial x^j}{\partial\overline{x}^l} \\ &= \overline{g}_{mn}\overline{\delta}_s^m\overline{\delta}_l^n\overline{A}^s\overline{A}^l = \overline{g}_{mn}\overline{A}^m\overline{A}^l. \end{aligned}$$

Similarly, from Eq. (2.15), we get

$$B^2 = g^{ij} B_i B_j = \overline{g}^{mn} \overline{B}_m \overline{B}_n.$$

Thus, the length of a vector is an invariant.

EXAMPLE 2.2.11 *If* $A^i = \frac{1}{\sqrt{g_{pq} B^p B^q}} B^i$, *where* B^i *is a contravariant vector and* g_{ij} *is the fundamental tensor, show that* A^i *is a unit vector.*

Solution: According to the definition,

$$g_{ij} A^i A^j = g_{ij} \frac{1}{\sqrt{g_{pq} B^p B^q}} B^i \frac{1}{\sqrt{g_{pq} B^p B^q}} B^j = \frac{g_{ij} B^i B^j}{g_{pq} B^p B^q} = 1.$$

Therefore, A^i is a unit vector.

2.2.2 Angle Between Two Vectors

Let A^i and B^i be any two non-null contravariant vectors, then the angle θ between them is defined by the formula,

$$\cos\theta = \frac{g_{ij} A^i B^j}{\sqrt{g_{ij} A^i A^j} \sqrt{g_{ij} B^i B^j}}; \; 0 \le \theta \le \pi. \tag{2.18}$$

The angle θ between two non-null covariant vectors A_i and B_i is given by

$$\cos\theta = \frac{g_{ij} A_i B_j}{\sqrt{g_{ij} A_i A_j} \sqrt{g_{ij} B_i B_j}}; \; 0 \le \theta \le \pi. \tag{2.19}$$

If any one the vectors considered is the null vector, the angle is not defined. If A and B are two non-null vectors with $A^i, A_i; B^i, B_i$ as respective contravariant and covariant components, then the angle θ between A and B is given by

$$\cos\theta = \frac{A^i B_i}{\sqrt{A^j A_j} \sqrt{B^k B_k}}. \tag{2.20}$$

Note that, if two vectors are such that one of them is a null vector or both of them are so, then the angle between them is not defined.

EXAMPLE 2.2.12 *Show that the angle between the vectors* $(1, 0, 0, 0)$ *and* $\left(\sqrt{2}, 0, 0, \frac{\sqrt{3}}{c}\right)$, *c being constant, in a space with line element given by*

$$ds^2 = -\left(dx^1\right)^2 - \left(dx^2\right)^2 - \left(dx^3\right)^2 + c^2 \left(dx^4\right)^2$$

is not real.

Solution: The quantities $g_{ij}A^iA^j$, $g_{ij}B^iB^j$ and $g_{ij}A^iB^j$ are given by

$$\begin{aligned} g_{ij}A^iA^j &= g_{11}A^1A^1 + g_{22}A^2A^2 + g_{33}A^3A^3 + g_{44}A^4A^4 \\ &= -1 \cdot 1 \cdot 1 + (-1) \cdot 0 \cdot 0 + (-1) \cdot 0 \cdot 0 + c^2 \cdot 0 \cdot 0 = -1. \\ g_{ij}B^iB^j &= g_{11}B^1B^1 + g_{22}B^2B^2 + g_{33}B^3B^3 + g_{44}B^4B^4 \\ &= -1 \cdot \sqrt{2} \cdot \sqrt{2} + (-1) \cdot 0 \cdot 0 + (-1) \cdot 0 \cdot 0 + c^2 \cdot \frac{\sqrt{3}}{c} \cdot \frac{\sqrt{3}}{c} = 1. \end{aligned}$$

and

$$\begin{aligned} g_{ij}A^iB^j &= g_{11}A^1B^1 + g_{22}A^2B^2 + g_{33}A^3B^3 + g_{44}A^4B^4 \\ &= -1 \cdot 1 \cdot \sqrt{2} + (-1) \cdot 0 \cdot 0 + (-1) \cdot 0 \cdot 0 + c^2 \cdot 0 \cdot \frac{\sqrt{3}}{c} = -\sqrt{2}. \end{aligned}$$

If θ be the angle between A^i and B^i, then from Eq. (2.18) is given by

$$\cos\theta = \frac{g_{ij}A^iB^j}{\sqrt{g_{ij}A^iA^j}\sqrt{g_{ij}B^iB^j}} = \frac{-\sqrt{2}}{\sqrt{-1}\sqrt{1}}.$$

Thus, the angle between the two vectors is not real.

2.2.3 Orthogonality of Two Vectors

If $\theta = \frac{\pi}{2}$, the vectors are called orthogonal. Therefore, two non-null vectors A^i and B^i are said to be orthogonal, if

$$g_{ij}A^iB^j = 0. \tag{2.21}$$

Similarly, two vectors A_i and B_i are said to be orthogonal if

$$g^{ij}A_iB_j = 0. \tag{2.22}$$

It follows from Eqs. (2.21) and (2.22) that the angle between two non-null orthogonal vectors A^i, B^i is $\pi/2$ and that between two non-null orthogonal vectors A_i, B_i is also $\pi/2$. According to the definition of orthogonality given by Eqs. (2.21) and (2.22) it follows that a null vector A^i or A_i is self-orthogonal.

EXAMPLE 2.2.13 *If u^i and v^i are orthogonal unit vectors, show that*

$$(g_{hj}g_{ki} - g_{hk}g_{ji})\, u^hv^iu^jv^k = 1.$$

Solution: Since u^i and v^i are orthogonal unit vectors, so by definition (2.19),

$$g_{ij}u^iu^j = 1, g_{ij}v^iv^j = 1 \quad \text{and} \quad g_{ij}u^iv^j = 0.$$

Using these results, we get

$$\begin{aligned} \text{LHS} &= (g_{hj}g_{ki} - g_{hk}g_{ji})\, u^hv^iu^jv^k \\ &= g_{hj}u^hu^jg_{ki}v^kv^i - g_{hk}u^hv^kg_{ji}u^jv^i = 1\cdot 1 - 0\cdot 0 = 1. \end{aligned}$$

EXAMPLE 2.2.14 *If θ be the angle between two non-null vectors A^i and B^i at a point, prove that*

$$\sin^2\theta = \frac{(g_{ij}g_{pq} - g_{ip}g_{jq})\, A^iB^pA^jB^q}{(g_{ij}A^iA^j)\,(g_{pq}B^pB^q)}.$$

Solution: Let θ be the angle between two non-null vectors A^i and B^i at a point, then by definition (2.19),

$$\cos\theta = \frac{g_{ij}A^iB^j}{\sqrt{g_{ij}A^iA^j}\sqrt{g_{pq}B^pB^q}}.$$

Using the result $\cos^2\theta = 1 - \sin^2\theta$, we get

$$\begin{aligned} \sin^2\theta = 1 - \cos^2\theta &= 1 - \frac{g_{ij}A^iB^jg_{pq}A^pB^q}{(g_{ij}A^iA^j)\,(g_{pq}B^pB^q)} \\ &= \frac{g_{ij}g_{pq}A^iA^jB^pB^q - g_{ij}g_{pq}A^iA^pB^jB^q}{(g_{ij}A^iA^j)\,(g_{pq}B^pB^q)} \\ &= \frac{g_{ij}g_{pq}A^iB^pA^jB^q - g_{ip}g_{jq}A^iB^pA^jB^q}{(g_{ij}A^iA^j)\,(g_{pq}B^pB^q)} \\ &\quad \text{(Replacing the dummy indices } j \text{ and } p \text{ by } p \text{ and } j) \\ &= \frac{(g_{ij}g_{pq} - g_{ip}g_{jq})\, A^iB^pA^jB^q}{(g_{ij}A^iA^j)\,(g_{pq}B^pB^q)}. \end{aligned}$$

EXAMPLE 2.2.15 *Show that under the metric for polar co-ordinates, the vectors $A^i = \left(\frac{3}{5}, \frac{4}{5x^1}\right)$ and $B^i = \left(-\frac{4}{5}, \frac{3}{5x^1}\right)$ are orthogonal.*

Solution: Using matrices, we have

$$g_{ij}A^iA^j = \begin{pmatrix} \dfrac{3}{5} & \dfrac{4}{5x^1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & (x^1)^2 \end{pmatrix} \begin{pmatrix} \dfrac{3}{5} \\ \dfrac{4}{5x^1} \end{pmatrix} = \frac{9}{25} + \frac{16x^1}{25x^1} = 1.$$

Likewise, $g_{ij}B^iB^j = 1$. Now,

$$g_{ij}A^iB^j = \begin{pmatrix} \frac{3}{5} & \frac{4}{5x^1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & (x^1)^2 \end{pmatrix} \begin{pmatrix} -\frac{4}{5} \\ \frac{3}{5x^1} \end{pmatrix} = -\frac{12}{25} + \frac{12x^1}{25x^1} = 0.$$

Thus, under the metric for polar co-ordinates in V_2, the vectors $A^i = \left(\frac{3}{5}, \frac{4}{5x^1}\right)$ and $B^i = \left(-\frac{4}{5}, \frac{3}{5x^1}\right)$ are orthogonal.

EXAMPLE 2.2.16 *If A^i and B^i are two non-null vectors such that*

$$g_{ij}U^iU^j = g_{ij}V^iV^j;\ U^i = A^i + B^i \text{and } V^i = A^i - B^i,$$

show that A^i and B^i are orthogonal.

Solution: Since, $U^i = A^i + B^i$ and $V^i = A^i - B^i$, we have

$$g_{ij}U^iU^j = g_{ij}V^iV^j$$

or

$$g_{ij}(A^i + B^i)(A^j + B^j) = g_{ij}(A^i - B^i)(A^j - B^j)$$

or

$$g_{ij}A^iA^j + g_{ij}B^iB^j + 2g_{ij}A^iB^j = g_{ij}A^iA^j + g_{ij}B^iB^j - 2g_{ij}A^iB^j$$

or

$$4g_{ij}A^iB^j = 0 \Rightarrow g_{ij}A^iB^j = 0.$$

Hence, A^i and B^j are orthogonal.

EXAMPLE 2.2.17 *If a_{ij} is a symmetric tensor of type $(0, 2)$ and A^i, B^i are unit vectors orthogonal to a vector C^i satisfying the conditions*

$$(a_{ij} - \kappa_1 g_{ij})A^i + \lambda_1 g_{ij}C^i = 0$$

and

$$(a_{ij} - \kappa_2 g_{ij})B^i + \lambda_2 g_{ij}C^i = 0,$$

where, $\kappa_1 \neq \kappa_2$, show that A^i and B^i are orthogonal and $a_{ij}A^iB^j = 0$.

Solution: Since A^i, B^i are unit vectors orthogonal to a vector C^i, so,

$$g_{ij}A^iA^j = 1;\ g_{ij}B^iB^j = 1;\ g_{ij}A^iC^j = 0;\ g_{ij}B^iC^j = 0.$$

Multiplying the given first relation by B^j we get

$$(a_{ij} - \kappa_1 g_{ij})A^iB^j + \lambda_1 g_{ij}C^iB^j = 0$$

or

$$a_{ij}A^iB^j - \kappa_1 g_{ij}A^iB^j = 0. \tag{i}$$

Similarly, multiplying the given second relation by A^j we get

$$(a_{ij} - \kappa_2 g_{ij})B^iA^j + \lambda_2 g_{ij}C^iA^j = 0$$

or

$$a_{ij}B^iA^j - \kappa_2 g_{ij}B^iA^j = 0$$

or

$$a_{ji}A^jB^i - \kappa_2 g_{ji}A^jB^i = 0;\ \text{as } a_{ij}, g_{ij} \text{ are symmetric.}$$

Replacing the dummy indices i and j by j and i, respectively, we get

$$a_{ij}A^iB^j - \kappa_2 g_{ij}A^iB^j = 0. \tag{ii}$$

Subtracting (ii) from (i), we get

$$(\kappa_2 - \kappa_1)g_{ij}A^iB^j = 0 \Rightarrow g_{ij}A^iB^j = 0;\ \text{as } \kappa_2 \neq \kappa_1.$$

Hence, A^i and B^j are orthogonal. In virtue of the last result $g_{ij}A^iB^j = 0$, it follows from (i) that $a_{ij}A^iB^j = 0$.

EXAMPLE 2.2.18 *Show that under the metric for cylindrical co-ordinates, the contravariant vectors* $\mathbf{A} = \left[0, 1, 2bx^2\right]$ *and* $\mathbf{B} = \left[0, -2bx^2, (x^1)^2\right]$ *are orthogonal. Interpret geometrically along* $x^1 = a, x^2 = t, x^3 = bt^2$.

Solution: Using matrices for cylindrical co-ordinates, we have

$$g_{ij}A^iB^j = \begin{pmatrix} 0 & 1 & 2bx^2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & (x^1)^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -2bx^2 \\ (x^1)^2 \end{pmatrix} = 0.$$

The geometric interpretation is that $x^1 = a, x^2 = t, x^3 = bt^2$, for real t, represents a sort of variable-pitch helix on the right cylinder $r = a$, having tangent field $\mathbf{A}$. Therefore, any solution of

$$\frac{du^1}{dx} = B^1 = 0;\ \frac{dx^2}{du} = B^2 = -2bx^2;\ \frac{dx^3}{du} = B^3 = a^2$$

will represent a curve on that cylinder that is orthogonal to this pseudo-helix.

2.3 Some Loci

Let us suppose that the metric is positive definite.

2.3.1 Co-ordinate Curve

A curve along which only one co-ordinate varies is called a *co-ordinate curve.* If only the particular co-ordinate x^i (i being a particular integer) varies, the curve is called the x^i curve. Along this curve $dx^j = 0$, when $j \neq i$.

Angle between two co-ordinate curves: The angle between two co-ordinate curves is defined as the angle between their tangents. The x^j co-ordinate curve is defined by

$$x^i = c^i;\ \text{for every } i,$$

except $i = j$. Differentiating,

$$dx^i = 0;\ \text{for every } i,$$

except $i = j$ and $dx^j \neq 0$. Hence,

$$dx^i = \left(0, 0, \ldots, dx^j, \ldots, 0, 0\right).$$

Let A^i and B^i be the tangent vectors to the x^p co-ordinate curve and x^q co-ordinate curve respectively. Therefore,

$$A^i = dx^i = (0, 0, \ldots, A^p, \ldots, 0, 0)$$
$$B^i = dx^i = (0, 0, \ldots, B^q, \ldots, 0, 0).$$

Let θ be the angle between the two co-ordinate curves, then,

$$\cos\theta = \frac{g_{ij}A^iA^j}{\sqrt{g_{ij}A^iA^j}\sqrt{g_{ij}B^iB^j}} \tag{2.23}$$
$$= \frac{g_{pq}A^pB^q}{\sqrt{g_{pp}A^pA^p}\sqrt{g_{qq}B^qB^q}};\ \text{no summation on } p, q$$
$$= \frac{g_{pq}A^pB^q}{\sqrt{g_{pp}g_{qq}}A^pB^q} = \frac{g_{pq}}{\sqrt{g_{pp}g_{qq}}}.$$

The tangent field to a family of smooth curves is a contravariant vector, so that Eq. (2.23) yields the geometrical. The angle θ_{ij} between x^i co-ordinate curve and x^j co-ordinate curve is given by

$$\cos\theta_{ij} = \frac{g_{ij}}{\sqrt{g_{ii}g_{jj}}}.$$

If the co-ordinate curves of parameter x^i and x^j are orthogonal, then,

$$\theta_{ij} = \frac{\pi}{2} \Rightarrow \cos\theta_{ij} = \cos\frac{\pi}{2} = 0$$
$$\Rightarrow \frac{g_{ij}}{\sqrt{g_{ii}g_{jj}}} = 0,\ \text{i.e. } g_{ij} = 0.$$

Therefore, the x^i co-ordinate curve and x^j co-ordinate curve are orthogonal if $g_{ij} = 0$. In a general co-ordinate syst em, if A^i and B^i are the tangent vectors to two families of curves, then the families are mutually orthogonal if and only if $g_{ij}A^iB^j = 0$.

Result 2.3.1 The physical components of a vector A^p or A_p denoted by A_u, A_v and A_w are the projections of the vector on the tangents to the co-ordinates curves, and are given in the case of orthogonal co-ordinates by

$$A_u = \sqrt{g_{11}}A^1 = \frac{1}{\sqrt{g_{11}}}A^1; A_v = \sqrt{g_{22}}A^2 = \frac{1}{\sqrt{g_{22}}}A^2;$$
$$A_w = \sqrt{g_{33}}A^3 = \frac{1}{\sqrt{g_{33}}}A^3.$$

Result 2.3.2 The angle between the two parametric lines through a surface point is

$$\cos\theta = g_{12}/\sqrt{g_{11}g_{22}}.$$

The two families of parametric lines from an orthogonal net if and only if $g_{12} = 0$ at every point of $\mathcal{S}$.

EXAMPLE 2.3.1 *Prove that for the surface*

$$x^1 = a\sin u^1 \cos u^2;\ \ x^2 = a\sin u^1 \sin u^2;\ \ x^3 = a\cos u^1,$$

the co-ordinates curves are orthogonal where (x^i) *are orthogonal Cartesian co-ordinates and* a *is a constant.*

Solution: Let E^3 be covered by orthogonal Cartesian co-ordinates x^i, and consider a transformation $x^1 = a\sin u^1 \cos u^2$; $x^2 = a\sin u^1 \sin u^2$; $x^3 = a\cos u^1$ $a =$ constant. The metric in Euclidean space E^3, referred to Cartesian co-ordinates is given by

$$ds^2 = \left(dx^1\right)^2 + \left(dx^2\right)^2 + \left(dx^3\right)^2.$$

Comparing this, with Eq. (2.1), we see that $g_{11} = g_{22} = g_{33} = 1$ and $g_{ij} = 0$; for $i \neq j$. The fundamental symmetric tensor $\overline{g}_{ij}; i \neq j$ is given by

$$\begin{aligned}\overline{g}_{12} = \frac{\partial x^i}{\partial u^1}\frac{\partial x^j}{\partial u^2}g_{ij} &= \frac{\partial x^1}{\partial u^1}\frac{\partial x^1}{\partial u^2}g_{11} + \frac{\partial x^2}{\partial u^1}\frac{\partial u^2}{\partial u^2}g_{22} + \frac{\partial x^3}{\partial u^1}\frac{\partial x^3}{\partial u^2}g_{33} \\ &= a\cos u^1\cos u^2\left(-a\sin u^1\sin u^2\right) + a\cos u^1\sin u^2\cdot a\sin u^1\cos u^2 + \left(-a\sin u^1\right)\cdot 0 \\ &= 0.\end{aligned}$$

Similarly, it can be shown that $\overline{g}_{13} = 0$ and $\overline{g}_{23} = 0$. Therefore, the co-ordinate curves are orthogonal.

EXAMPLE 2.3.2 *Show that each member of the family of curves given in polar co-ordinates by* $e^{1/r} = a(\sec\theta + \tan\theta)$; $a \geq 0$ *is orthogonal to each of the curves (limacons of Pascal)* $r = \sin\theta + c; c \geq 0$.

Solution: In polar co-ordinates $x^1 = r, x^2 = \theta$, and with curve parameter t, the first curve becomes

$$\frac{1}{x^1} = \log a + \log|\sec t + \tan t|;\ x^2 = t.$$

With curve parameter u, the second curve becomes $x^1 = \sin u + c$, $x^2 = u$. Therefore,

$$A^i = \left(\frac{dx^1}{dt}, \frac{dx^2}{dt}\right) = \left[-(x^1)^2 \sec x^2, 1\right]$$

$$B^i = \left(\frac{dx^1}{du}, \frac{dx^2}{du}\right) = \left(\cos x^2, 1\right).$$

Now the Euclidean metric tensor in polar co-ordinates is given by

$$\begin{aligned} g_{ij}A^iB^j &= g_{11}A^1B^1 + g_{22}A^2B^2 + 0 \cdot (A^1B^2 + A^2B^1) \\ &= 1[-(x^1)^2 \sec x^2] \cos x^2 + (x^1)^2 \cdot 1 \cdot 1 = -(x^1)^2 + (x^1)^2 = 0. \end{aligned}$$

2.3.2 Hypersurface

Let $t^1, t^2, \ldots, t^M$ be M parameters. The N equations

$$x^i = x^i\left(t^1, t^2, \ldots, t^M\right);\ i = 1, 2, \ldots, N; M < N \tag{2.24}$$

defines M dimensional subspace V_M of V_N. If we eliminate the M parameters $t^1, t^2, \ldots, t^M$, from these N equations we shall get $(N - M)$ equations in x^is which represent M dimensional curve in V_N. In particular if $M = N - 1$, we get only one equation as

$$N - M = N - (N - 1) = 1,$$

in x^is which represent $N - 1$ dimensional curve in V_N. This particular curve is called *hypersurface* in V_N. Thus, a family of hypersurfaces of V_N is determined by

$$\phi(x^i) = \phi\left(x^1, x^2, \ldots, x^N\right) = \text{ constant}, \tag{2.25}$$

where $\phi(x^i)$ is a scalar function of co-ordinates x^i. Thus a hypersurface is obtained if x^i are functions of $(N - 1)$ independent parameters.

A parametric hypersurface is a hypersurface on which one particular co-ordinate x^i (say) is constant, while the others vary. Let us call it the x^i-hypersurface, with equation $x^i = c =$ constant.

Angle between two hypersurfaces: The angle between two curves is defined as the angle between their normals at their point of intersections. Let

$$\phi(x^i) = \phi\left(x^1, x^2, \ldots, x^N\right) = \text{ constant} \tag{2.26}$$

$$\psi(x^i) = \psi\left(x^1, x^2, \ldots, x^N\right) = \text{ constant} \tag{2.27}$$

represent two families of hypersurfaces. Now differentiating (2.26) we get

$$\frac{\partial \phi}{\partial x^i} dx^i = \phi_i dx^i = 0; \text{ where } \phi_i = \frac{\partial \phi}{\partial x^i}. \tag{2.28}$$

These partial derivatives are, by definition, components of a covariant vector. The relation (2.28) shows that, at any point, the vector ϕ_i is orthogonal to all displacements dx^i at P, on the surface and hence dx^i is in the tangential to hypersurface (2.26). Thus, gradient vector ϕ_i at any point of the hypersurface is normal to the hypersurface at the point.

Similarly, $\psi_i = \frac{\partial \psi}{\partial x^i}$ is the normal to the hypersurface (2.27). Let ω be the angle between the hypersurfaces (2.26) and (2.27), then by definition ω is the angle between their respective normals. Hence, the required angle ω is given by

$$\cos\omega = \frac{g^{ij}\phi_i\psi_j}{\sqrt{g^{ij}\phi_i\phi_j}\sqrt{g^{ij}\psi_i\psi_j}} = \frac{g^{ij}\dfrac{\partial \phi}{\partial x^i}\dfrac{\partial \psi}{\partial x^j}}{\sqrt{g^{ij}\dfrac{\partial \phi}{\partial x^i}\dfrac{\partial \phi}{\partial x^j}}\sqrt{g^{ij}\dfrac{\partial \psi}{\partial x^i}\dfrac{\partial \psi}{\partial x^j}}}. \tag{2.29}$$

In particular, let the hypersurfaces (2.26) and (2.27) be taken as co-ordinate hypersurfaces of parameters say x^p and x^q, respectively, then we have

$$\phi = x^p = \text{ constant} \tag{2.30}$$

$$\psi = x^q = \text{ constant.} \tag{2.31}$$

Then the angle ω between (2.30) and (2.31) is given by

$$\cos\omega = \frac{g^{ij}\dfrac{\partial x^p}{\partial x^i}\dfrac{\partial x^q}{\partial x^j}}{\sqrt{g^{ij}\dfrac{\partial x^p}{\partial x^i}\dfrac{\partial x^p}{\partial x^j}}\sqrt{g^{ij}\dfrac{\partial x^q}{\partial x^i}\dfrac{\partial x^q}{\partial x^j}}} = \frac{g^{ij}\delta_i^p\delta_j^q}{\sqrt{g^{ij}\delta_i^p\delta_j^p}\sqrt{g^{ij}\delta_j^q\delta_j^q}} = \frac{g^{pq}}{\sqrt{g^{pp}g^{qq}}}. \tag{2.32}$$

The angle ω_{ij} between the co-ordinate hypersurfaces $x^i =$ constant and $x^j =$ constant is given by

$$\cos\omega_{ij} = \frac{g^{ij}}{\sqrt{g^{ii}g^{jj}}}. \tag{2.33}$$

If $g_{ij} = 0$ when $i \neq j$, the parametric curves are orthogonal to each other and so also are the parametric hypersurfaces. Therefore, if the co-ordinate hypersurfaces x^i and x^j are orthogonal, then

$$\omega_{ij} = \frac{\pi}{2} \Rightarrow \cos \omega_{ij} = 0 \Rightarrow g^{ij} = 0. \tag{2.34}$$

Hence, the co-ordinate hypersurfaces of parameters x^i and x^j are orthogonal if $g^{ij} = 0$.

N **ply orthogonal system of hypersurfaces:** If in a V_N, there are N families of hypersurfaces such that, at every point, each hypersurface is orthogonal to the $N-1$ hypersurfaces of the other families which pass through that point, they are said to form as *N ply orthogonal system of hypersurfaces.*

Theorem 2.3.1 *The necessary and sufficient condition for the existence of an N ply orthogonal system of co-ordinate hypersurfaces is that the fundamental form must be of the form*

$$ds^2 = g_{11}\left(dx^1\right)^2 + g_{22}\left(dx^2\right)^2 + \cdots + g_{NN}\left(dx^N\right)^2 = g_{ii}\left(dx^i\right)^2. \tag{2.35}$$

Proof: **Condition necessary:** Let us suppose that co-ordinate hypersurfaces form an N ply orthogonal system of hypersurfaces. Since the co-ordinate hypersurfaces form an N ply orthogonal system of hypersurfaces, we have

$$g^{ij} = 0; \text{ for every } i, j = 1, 2, \ldots, N \text{ and } i \neq j. \tag{2.36}$$

If $\triangle$ denotes the determinant of $|g^{ij}|$, then,

$$\triangle = \frac{1}{g} = \begin{vmatrix} g^{11} & 0 & \cdots & 0 \\ 0 & g^{22} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & g^{NN} \end{vmatrix} \Rightarrow \triangle = \frac{1}{g} = g^{11}g^{22}\ldots g^{NN} \neq 0.$$

Since $\triangle \neq 0$, so none of the quantities $g^{11}, g^{22}, \ldots, g^{NN}$ are zero. Now,

$$g_{ij} = \frac{\text{cofactor of } g^{ij} \text{ in } \triangle}{\triangle}$$

or

$$\triangle g_{ij} = \text{ cofactor of } g^{ij} \text{ in } \triangle$$

or

$$\frac{g_{ij}}{g} = \text{ cofactor of } g^{ij} \text{ in } \triangle$$

or

$$\frac{g_{ij}}{g} = 0;\ \text{for } i \neq j \Rightarrow g_{ij} = 0;\ \text{for } i \neq j.$$

Thus, the fundamental form is of the form

$$ds^2 = g_{ij}dx^i dx^j = g_{11}\left(dx^1\right)^2 + g_{22}\left(dx^2\right)^2 + \cdots + g_{NN}\left(dx^N\right)^2 = g_{ii}\left(dx^i\right)^2.$$

Condition sufficient: Suppose the fundamental form is given by

$$ds^2 = g_{11}\left(dx^1\right)^2 + g_{22}\left(dx^2\right)^2 + \cdots + g_{NN}\left(dx^N\right)^2 = g_{ii}\left(dx^i\right)^2,$$

then we have to show that co-ordinate hypersurfaces form an N ply orthogonal system of co-ordinate hypersurfaces. Comparing the given fundamental form Eq. (2.35) with Eq. (2.1) we see that

$$g_{ij} = 0;\ \text{for every } i, j = 1, 2, \ldots, N \text{ and } i \neq j \text{ and } g_{ij} \neq 0;\ \text{for } i = j.$$

Also, the reciprocal tensor g^{ij} is given by

$$g^{ij} = \frac{\text{cofactor of } g_{ij} \text{ in } g}{g};\ g = |g_{ij}| \Rightarrow g^{ij} = \frac{0}{g} = 0;\ \text{for } i \neq j,$$

which is necessary and sufficient condition for orthogonality. This proves the sufficient condition.

Deduction 2.3.1 Here, we to show that an arbitrary V_N does not admit an N ply orthogonal system of hypersurfaces. Suppose that an arbitrary V_N admits N ply orthogonal system of hypersurfaces. The fundamental form in this case is given by

$$ds^2 = g_{11}\left(dx^1\right)^2 + g_{22}\left(dx^2\right)^2 + \cdots + g_{NN}\left(dx^N\right)^2 = g_{ii}\left(dx^i\right)^2.$$

Let an N ply orthogonal system of hypersurfaces is given by the hypersurfaces

$$\phi_i = c_i;\ i = 1, 2, \ldots, N$$

where c_i are constants. Now, the condition that the hypersurfaces determine an N ply orthogonal system of hypersurfaces is

$$g_{ij}\frac{\partial \phi_p}{\partial x^i}\frac{\partial \phi_q}{\partial x^j} = 0;\ \text{for } p \neq q; i, j = 1, 2, \ldots, N.$$

These equations admit $N(N-1)/2$ simultaneous partial differential equations to find N unknowns. In other equations admit N solutions, we have

$$\frac{N(N-1)}{2} = N.$$

These equations are consistent if

$$\frac{N(N-1)}{2} \not> N \Rightarrow N \leq 3.$$

This is in contradiction to our supposition that diagonalisation is possible for all values of N. Hence, an arbitrary V_N does not admit an N ply orthogonal system of hypersurfaces.

Definition 2.3.1 (Congruence of curves): A congruence of curves in a V_N is a family of curves, one of which passes through each point of V_N.

2.3.3 Orthogonal Ennuple

An orthogonal ennuple in a Riemannian V_N consists of N mutually orthogonal congruence of curves. Consider N unit tangents $\lambda^i_{h|}; h = 1, 2, \ldots, N$ to congruence $\lambda_{h|}; h = 1, 2, \ldots, N$ of an orthogonal ennuple in a Riemannian V_N. The subscript h followed by an upright bar simply distinguishes one congruence from other. It does not denote tensor suffix. The quantities $\lambda_{h|i}; i = 1, 2, \ldots, N$ and $\lambda^i_{h|}$ denote the covariant and contravariant components of $\lambda_{h|}$, respectively.

Suppose any two congruences of orthogonal ennuple are $\lambda_{h|}$ and $\lambda_{k|}$ so that

$$\lambda_{h|}\lambda_{k|} = \delta^h_k, \text{ and } g_{ij}\lambda^i_{h|}\lambda^j_{k|} = \delta^h_k$$

$$\Rightarrow g_{ij}\lambda^i_{h|}\lambda^j_{h|} = 1 \text{ and } g_{ij}\lambda^i_{h|}\lambda^j_{k|} = 0.$$

Let us define,

$$\lambda^i_{h|} = \frac{\text{cofactor of } \lambda_{h|i} \text{ in the determinant } |\lambda_{h|i}|}{|\lambda_{h|i}|}.$$

Also, from the determinant property, we have

$$\sum_{h=1}^{N} \lambda^i_{h|}\lambda_{h|i} = \delta^i_j. \tag{2.37}$$

Multiplying Eq. (2.37) by g^{jk}, we have,

$$\sum_{h=1}^{N} \lambda^i_{h|}\lambda_{h|i}g^{jk} = \delta^i_j g^{jk}$$

or

$$g^{ik} = \sum_{h=1}^{N} \lambda^i_{h|}\lambda^k_{h|} \Rightarrow g^{ij} = \sum_{h=1}^{N} \lambda^i_{h|}\lambda^j_{h|}. \tag{2.38}$$

Multiplying Eq. (2.37) by g_{ik}, we have

$$\sum_{h=1}^{N} \lambda^i_{h|} \lambda_{h|i} g_{ik} = \delta^i_j g_{ik}$$

or

$$g_{jk} = \sum_{h=1}^{N} \lambda_{h|k} \lambda_{h|j} \Rightarrow g_{ij} = \sum_{h=1}^{N} \lambda_{k|i} \lambda_{h|j}. \tag{2.39}$$

Equations (2.38) and (2.39) give the values of the fundamental tensors g_{ij} and g^{ij} in terms of the components of the unit tangent $\lambda_{h|}; h = 1, 2, \ldots, N$ to an orthogonal ennuple.

Deduction 2.3.2 Let us consider a vector **u** which can be written in the form

$$u^i = \sum_{h=1}^{N} C_h \lambda^i_{h|}, \tag{2.40}$$

where C_h are constants to be determined. Multipling Eq. (2.40) by $\lambda_{k|i}$, we get

$$u^i \lambda_{k|i} = \sum_{h=1}^{N} C_h e^i_{h|} \lambda_{k|i} = \sum_{h=1}^{N} C_h \delta^h_k = C_k$$

$$\Rightarrow C_k = \text{ Projection of } u^i \text{ on } \lambda_{k|i} = u^i \lambda_{k|i}.$$

Therefore,

$$u^i = \sum_{h=1}^{N} u^j \lambda_{h|j} \lambda^i_{k|}, \text{ from (2.40)}.$$

If u denotes the magnitude of the vector **u**, then,

$$\begin{aligned}
\mathbf{u}^2 = u^2 = u^i u_i &= \left(\sum_{h=1}^{N} C_h \lambda^i_{h|} \right) \left(\sum_{k=1}^{N} C_k \lambda^i_{k|} \right) \\
&= \sum_{h=1}^{N} \sum_{k=1}^{N} C_h C_k \lambda^i_{h|} \lambda^i_{k|} = \sum_{h=1}^{N} \sum_{k=1}^{N} C_h C_k \delta^h_k \\
&= \sum_{h=1}^{N} C_h C_h = \sum_{h=1}^{N} C^2_h.
\end{aligned} \tag{2.41}$$

Equation (2.41) shows that

$$u = 0 \Longleftrightarrow u^2 = 0 \Longleftrightarrow C_h = 0.$$

Therefore, if the magnitude of a vector u is zero iff all the projections of the vector **u** on N mutually orthogonal directions $\lambda^i_{h|}$ are zero.

2.4 Affine Co-ordinates

Let us consider the transformation from a given co-ordinate system (x^i) to a rectangular system $(\bar{x}^i)$. The Jacobian matrix for such transformation is $J = \left(\frac{\partial \bar{x}^i}{\partial x^j}\right)$. Then the matrix $G = (g_{ij})$ of the Euclidean metric tensor in the (x^i) system is $G = J^T J$.

Our task, to find $G = (g_{ij})$ for the three dimensional affine co-ordinates (x^i). From section 1.20.1, we see that position vectors are contravariant affine vectors—in particular, the unit vectors

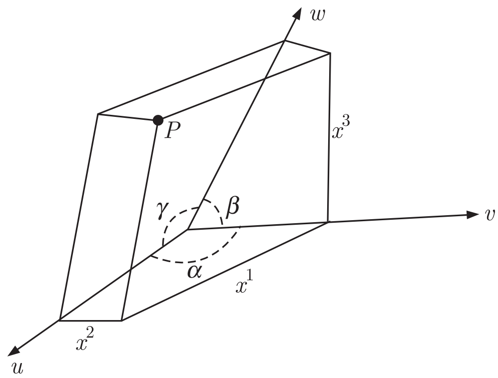

Figure 2.1: Oblique axes.

$$u = (\delta^i_1), \quad v = (\delta^i_2) \quad w = (\delta^i_3)$$

along the oblique axes (Figure 2.1). Now, using Eq. (2.18), we obtain

$$\cos\alpha = \frac{g_{ij}\,\delta^i_1\,\delta^i_2}{\sqrt{g_{pq}\,\delta^p_1\,\delta^q_1}\sqrt{g_{rs}\,\delta^r_2\,\delta^s_2}} = \frac{g_{12}}{\sqrt{g_{11}}\sqrt{g_{22}}} = g_{12}$$

as obviously, $g_{11} = g_{22} = g_{33} = 1$. Likewise,

$$\cos\beta = g_{13}, \quad \text{and} \quad \cos\gamma = g_{23}$$

The complete symmetric matrix is

$$G = (g_{ij}) = \begin{bmatrix} 1 & \cos\alpha & \cos\beta \\ \cos\alpha & 1 & \cos\gamma \\ \cos\beta & \cos\gamma & 1 \end{bmatrix}$$

The corresponding metric is given by

$$ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 + 2\cos\alpha dx^1 dx^2 + 2\cos\beta dx^1 dx^3 + 2\cos\gamma dx^2 dx^3.$$

Note that the matrix G defining the Enclidean metric is non diagonal in affine co-ordinates.

2.5 Curvilinear Co-ordinates

In Chapter 1, we have discussed cylindrical and spherical polar co-ordinates in the three-dimensional Euclidean space E^3. In this section, we shall study a type of co-ordinates in E^3 of which above two polar co-ordinates will be a particular one. Such co-ordinates are called *curvilinear co-ordinates* (Figure 2.2). Consider a general functional transformation T defined by

$$T : x^i = x^i\left(y^1, y^2, y^3\right); \; i = 1, 2, 3 \tag{2.42}$$

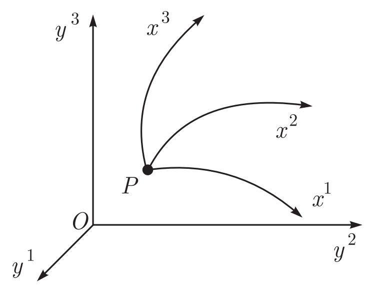

Figure 2.2: Curvilinear co-ordinates.

such that each single valued x^i is a continuously differentiable function of $\left(y^1, y^2, y^3\right)$ in some region R of E^3. Since x^1, x^2, x^3 must be independent, the Jacobian

$$J = \frac{\partial\left(x^1, x^2, x^3\right)}{\partial\left(y^1, y^2, y^3\right)} = \begin{vmatrix} \frac{\partial x^1}{\partial y^1} & \frac{\partial x^1}{\partial y^2} & \frac{\partial x^1}{\partial y^3} \\ \frac{\partial x^2}{\partial y^1} & \frac{\partial x^2}{\partial y^2} & \frac{\partial x^2}{\partial y^3} \\ \frac{\partial x^3}{\partial y^1} & \frac{\partial x^3}{\partial y^2} & \frac{\partial x^3}{\partial y^3} \end{vmatrix} \neq 0. \tag{2.43}$$

Under these conditions y^1, y^2, y^3 can be obtained as single valued functions of x^1, x^2, x^3 with continuous partial derivatives of the first order. Thus the inverse transformation

$$T^{-1}: y^i = y^i\left(x^1, x^2, x^3\right);\ i = 1, 2, 3 \tag{2.44}$$

will then be single-valued and the transformations T and T^{-1} establish one-to-one correspondence between the sets of the values $\left(x^1, x^2, x^3\right)$ and $\left(y^1, y^2, y^3\right)$. The co-ordinates $\left(y^1, y^2, y^3\right)$ are called the *curvilinear co-ordinate* system in R of E^3. The curvilinear co-ordinates y^1, y^2, y^3 in E^3 are related to the rectangular Cartesian co-ordinates x^1, x^2, x^3 by the formula (2.42).

2.5.1 Co-ordinate Surfaces

Let one of y^1, y^2, y^3 be kept fixed, say $y^1 = c^1 =$ constant in T of Eq. (2.42), where c^1 is a constant and let y^2, y^3 be allowed to vary. Then, $P\left(x^1, x^2, x^3\right)$ will satisfy the relations

$$x^i = x^i\left(c^1, y^2, y^3\right);\ i = 1, 2, 3,$$

which defines a surface and the point $P\left(x^1, x^2, x^3\right)$ will lie on the surface which will be denoted by $y^1 = c^1$. If the constant is now allowed to assume different values, we get a one-parameter family of surfaces. Similarly,

$$x^i = x^i\left(y^1, c^2, y^3\right) \text{ and } x^i = x^i\left(y^1, y^2, c^3\right);\ i = 1, 2, 3$$

define two families of surfaces $y^2 = c^2$ and $y^3 = c^3$. Each of the surfaces $y^1 = c^1$, $y^2 = c^2$ and $y^3 = c^3$ is called a *co-ordinate surface of the curvilinear co-ordinate system* and their intersections pair-by-pair are the *co-ordinate lines* (Figure 2.3). There will be

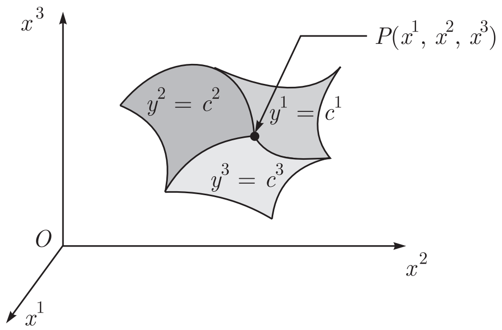

Figure 2.3: Co-ordinate surfaces.

three families of such surfaces corresponding to different values of c^1, c^2, c^3. Through a given point $P\left(x^1, x^2, x^3\right)$, there pass three co-ordinate surfaces corresponding to fixed values of c^1, c^2, c^3. The condition that the Jacobian $J \neq 0$ in the region under consideration expresses the fact that the surfaces

$$y^1 = c^1,\ y^2 = c^2,\ y^3 = c^3 \tag{2.45}$$

intersect in one and only one point.

2.5.2 Co-ordinate Curves

Let two of the co-ordinates y^1, y^2, y^3 be kept fixed in T, say $y^2 = c^2$, $y^3 = c^3$, where c^2 and c^3 are constants, and y^1 be allowed to vary. Then, $P\left(x^1, x^2, x^3\right)$ will satisfy the relations

$$x^i = x^i\left(y^1, c^2, c^3\right);\ i = 1, 2, 3.$$

Since x^1, x^2, x^3 are functions of only one variable, it follows that $P\left(x^1, x^2, x^3\right)$ will lie on a curve, called a *co-ordinate curve* (Figure 2.4). This co-ordinate curve

$$y^2 = c^2 \text{ and } y^3 = c^3$$

is called the y^1 *curve.*

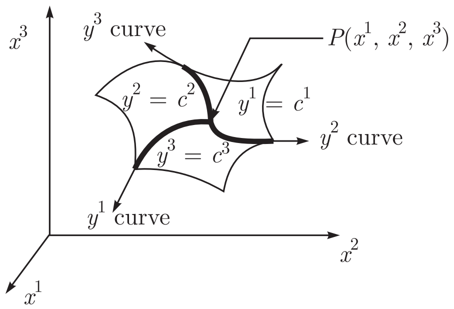

Figure 2.4: Co-ordinate surfaces.

Thus, the line of intersection of $y^2 = c^2$, $y^3 = c^3$ is the y^1 co-ordinate line because along this line the variable y^1 is the only one that is changing. Therefore, the y^1 curve lies on both the surfaces $y^2 = c^2$ and $y^3 = c^3$. Similarly, we can define y^2 and y^3 curve. It is to be noted that through a given point $P\left(x^1, x^2, x^3\right)$, there pass three co-ordinate curves corresponding to fixed values c^1, c^2, c^3.

EXAMPLE 2.5.1 *Show that* $\left(x^1, x^2, x^3\right)$ *defined by the transformations*

$$T : x^1 = u^1 \sin u^2 \cos u^3;\ x^2 = u^1 \sin u^2 \sin u^3;\ x^3 = u^1 \cos u^2,$$

are curvilinear co-ordinates. Also find the co-ordinate surfaces and curves.

Solution: Consider a co-ordinate system defined by the given transformation, the Jacobian of transformation is given by

$$J = \begin{vmatrix} \sin u^2 \cos u^3 & u^1 \cos u^2 \cos u^3 & -u^1 \sin u^2 \sin u^3 \\ \sin u^2 \sin u^3 & u^1 \cos u^2 \sin u^3 & u^1 \sin u^2 \cos u^3 \\ \cos u^2 & -u^1 \sin u^2 & 0 \end{vmatrix} = (u^1)^2 \sin u^2 \neq 0.$$

Hence, the inverse transformation exists and T^{-1} is given by

$$u^1 = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2};\ u^2 = \tan^{-1} \frac{\sqrt{(x^1)^2 + (x^2)^2}}{x^3};\ u^3 = \tan^{-1}\left(\frac{x^2}{x^1}\right)$$

if, $u^1 > 0$, $0 < u^2 < \pi$, $0 \leq u^3 < 2\pi$. Here x^1, x^2, x^3 are the rectangular Cartesian co-ordinates of a point P and u^1, u^2, u^3 be its spherical co-ordinates.

Therefore, the given co-ordinate system are curvilinear co-ordinate system. This is the familiar spherical co-ordinate system Figure 2.5. The co-ordinate surfaces are given by $u^1 = \text{constant} = \sqrt{c_1}$, say, where c_1 is constant and u^2, u^3 are allowed to vary then

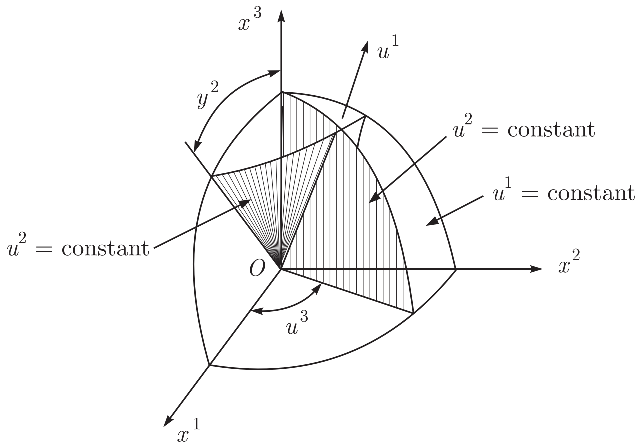

Figure 2.5: Spherical co-ordinates.

$$(x^1)^2 + (x^2)^2 + (x^3)^2 = c_1,$$

which represents the equation of the sphere with centre at origin and radius is $\sqrt{c_1}$. Thus $u^1 = c_1$ represents a surface. Next, let u^2 be kept fixed, say $u^2 = c_2$, where c_2 is a constant then

$$\tan^{-1} \frac{\sqrt{(x^1)^2 + (x^2)^2}}{x^3} = c_2 \text{ say}$$

or

$$(x^1)^2 + (x^2)^2 = (x^3)^2 \tan^2 c_2,$$

which are circular cones whose vertex is the origin and axis is the x^3 axis (Figure 2.5). Thus $u^2 = c_2$ represents a surface. Lastly, we keep u^3 fixed, say $u^3 = c_3 =$ constant, then

$$\tan^{-1}\left(\frac{x^2}{x^1}\right) = c_3 \Rightarrow x^2 = x^1 \tan c_3,$$

which are planes containing the x^3 axis. Thus $u^3 = c^3$ represents a surface. Each of the Surfaces $u^1 = \sqrt{c_1}, u_2 = c_2, u_3 = c_3$, is called a co-ordinate surface. There will be three families of these surfaces corresponding to different values of c_1, c_2, c_3. Through a given point $P(u^1, u^2, u^3)$ there pass three co-ordinate surfaces corresponding to fixed values of c_1, c_2, c_3. It is to be noted that of the three surfaces through a point, $u^1 = \sqrt{c_1}$ is a sphere, $u^2 = c_2$ is a cone and $u^3 = c_3$ is a plane through the x^3-axis.

Let two of the co-ordinates x^1, x^3 be kept fixed, say $x^1 = c_1$ and $x^3 = c_3$, where c_1 and c_2 are constants and x^2 be allowed to vary. Then

$$x^1 = c_1 \sin c_2 \cos u^3, x^2 = c_1 \sin c_2 \sin u^3, x^3 = c_1 \cos c_2.$$

Hence

$$\begin{aligned}(x^1)^2 + (x^2)^2 &= c_1^2 \sin^2 c_2(\cos^2 u^3 + \sin^2 u^3)\\ &= c_1^2 \sin^2 c_2 = \lambda^2; \lambda = c_1 \sin c_2 = \text{constant}.\end{aligned}$$

Thus

$$(x^1)^2 + (x^2)^2 = \lambda^2, x^3 = \mu \tag{i}$$

from which it follows that the point $P(u^1, u^2, u^3)$ is the intersection of a cylinder having x^3-axis as its axis and a plane parallel to the $x^1 - x^2$ plane. Thus (i) represents a circle in the plane $x^3 = \mu$ (Figure 2.5).

Thus (i) is a curve which is called a co-ordinate curve. This curve is called the u^3-curve. Next, let x^1 and x^3 be kept fixed, say $x^1 = c_1$ and $x^3 = c_3$ where c_1 and c_3 are constants and let x^2 be allowed to vary. Then

$$(x^1)^2 + (x^2)^2 + (x^3)^2 = c_1^2 \text{ and } x^2 = x^1 \tan c_3. \tag{ii}$$

From (ii) it follows that the point $P(u^1, u^2, u^3)$ lies on the intersection of a sphere and a plane through the x^3-axis. This intersection is a great circle.

Thus $x^1 = c_1, x^3 = c_3$ is a curve. This curve is called another co-ordinate curve. It is called the u^2-curve. Lastly, let x^2 and x^3 be kept fixed, say $x^2 = c_2$ and $x^3 = c_3$ and let x^1 be allowed to vary. Then

$$\cos c_2 = \frac{x^3}{x^1} = \frac{x^3}{\sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}}; \frac{x^2}{x^1} = \tan c_3$$

or

$$(x^1)^2 + (x^2)^2 = (\lambda^2 - 1)(x^3)^2; \quad x^2 = x^1 \tan c_3 \tag{iii}$$

where $\lambda = \sec c_2$ = a constant. From (iii) it follows that the point $P(x^1, x^2, x^3)$ lies on the intersection of a cone with the origin as its vertex and a plane through the u^3-axis. This intersection is therefore a straight line passing through the origin.

Thus $x^2 = c_2, x^3 = c_3$ is a curve. This curve is called a third co-ordinate curve. It is called the u^1-curve.

Through a given point $P(u^1, u^2, u^3)$ there pass three co-ordinate curves corresponding to fixed values of c_1, c_2, c_3. It is to be noted that of these three curves, two are circles and the remaining one is a straight line through the origin.

EXAMPLE 2.5.2 *Find the co-ordinate surface, defined by the transformation*

$$T : x^1 = u^1 \cos u^2, x^2 = u^1 \sin u^2, x^3 = u^3.$$

Also find the co-ordinate lines.

Solution: The Jacobian of transformation is given by

$$J = \begin{vmatrix} \cos u^2 & -u^1 \sin u^2 & 0 \\ \sin u^2 & u^1 \cos u^2 & 0 \\ 0 & 0 & 1 \end{vmatrix} = u^1 \neq 0.$$

Hence the inverse transformation exists and is given by

$$T^{-1} : u^1 = \sqrt{(x^1)^2 + (x^2)^2}, \ u^2 = \tan^{-1} \frac{x^2}{x^1}, \ u^3 = x^3$$

if $u^1 \geq 0, 0 \leq u^2 < 2\pi, -\infty < u^3 < \infty$. This co-ordinate system defines a cylindrical co-ordinate system (Figure 2.6). The co-ordinate surface are given by

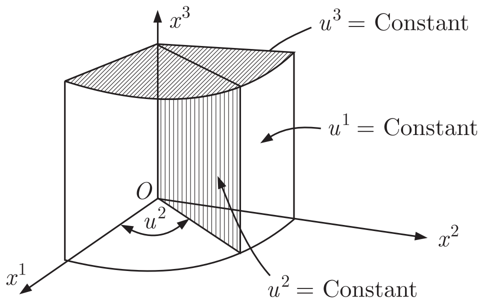

Figure 2.6: Cylindrical co-ordinates.

$$u^1 = \text{constant} = \sqrt{c_1}, \text{ say}$$

i.e.

$$(x^1)^2 + (x^2)^2 = c_1$$

which are circles. Now, $u^2 = \text{constant}$, i.e.

$$\tan^{-1}\frac{x^2}{x^1} = c_2, \text{ say } \Rightarrow x^2 = x_1 \tan c_2$$

which is a straight line. Also $u^3 = \text{constant}$ gives $x^3 = \text{constant} = c_3$ (say), which is a plane parallel to the x^1x^2-plane.

2.5.3 Line Element

Here, we have to obtain the line element of E^3 in curvilinear co-ordinates. Let $P\left(y^1, y^2, y^3\right)$ and $Q\left(y^1 + dy^1, y^2 + dy^2, y^3 + dy^3\right)$ be two neighbouring points in a region R of E^3 in which a curvilinear co-ordinate system

$$x^i = x^i\left(y^1, y^2, y^3\right);\ i = 1, 2, 3$$

is defined. The Euclidean distance between a pair of such points is determined by the quadratic form

$$\begin{aligned} ds^2 &= \left[\left(y^1 + dy^1\right) - y^1\right]^2 + \left[\left(y^2 + dy^2\right) - y^2\right]^2 + \left[\left(y^3 + dy^3\right) - y^3\right]^2 \\ &= \left(dy^1\right)^2 + \left(dy^2\right)^2 + \left(dy^3\right)^2 = \sum_i \left(dy^i\right)^2 \\ &= \left[\frac{\partial y^h}{\partial x^i}\frac{\partial y^h}{\partial x^j}\right] dx^i dx^j;\ \text{as } dy^i = \frac{\partial y^i}{\partial x^j} dx^j \\ &= g_{ij} dx^i dx^j;\ \text{where } g_{ij} = \frac{\partial y^h}{\partial x^i}\frac{\partial y^h}{\partial x^j}; i, j = 1, 2, 3. \end{aligned} \tag{2.46}$$

This is the elementary arc length in curvilinear co-ordinate system. Obviously, g_{ij} is symmetric. Moreover, by quotient law of tensor, since ds^2 is an invariant and the vector dx^i is arbitrary, we call g_{ij} the *fundamental metric tensor.* Denoted by g, the determinant $g = |g_{ij}|$; is positive in R since $g_{ij}dx^i dx^j$ is a positive definite form. Hence, we can introduce the conjugate symmetric tensor g^{ij}, defined Eq. (2.7) as

$$g^{ij} = \frac{\text{cofactor of } g_{ij} \text{ in } g}{g};\ \text{where } g = |g_{ij}| \neq 0. \tag{2.47}$$

Hence, g^{ij} is a symmetric $(2, 0)$ tensor conjugate to g_{ij}. The tensor g_{ij} plays an important role in deriving metric properties of the space E^3.

2.5.4 Length of a Vector

Consider a contravariant vector A^i in a curvilinear co-ordinate system. Now, we form the invariant

$$A = \left[g_{ij} A^i A^j\right]^{1/2}. \tag{2.48}$$

In orthogonal Cartesian co-ordinates $g_{ij} = \delta_{ij}$, and we get $A = \sqrt{A^i A^i}$. Therefore, in the orthogonal Cartesian frame the invariant Eq. (2.48) assumes the form

$$A = \left[(A^1)^2 + (A^2)^2 + (A^3)^2\right]^{1/2}.$$

We see that A represents the length of the vector A^i. Similarly, the length of the covariant vector A_i is defined by the formula

$$A = \left[g^{ij} A_i A_j\right]^{1/2}. \tag{2.49}$$

A vector whose length is 1 is called a unit vector. From Eq. (2.46), we see that

$$1 = g_{ij} \frac{dx^i}{ds} \frac{dx^j}{ds}. \tag{2.50}$$

It follows from Eq. (2.50) that $\frac{dx^i}{ds}$ is a contravariant vector, g_{ij} is a tensor of type $(0, 2)$ and 1 is an invariant. Hence, if we write $\lambda^i = \frac{dx^i}{ds}$, Eq. (2.50) can be written as

$$g_{ij} \lambda^i \lambda^j = 1.$$

Therefore, the vector with components λ^i is a unit vector. If $x^i = y^i$, i.e. the co-ordinate system is Cartesian, then

$$\frac{dx^1}{ds} = \lambda^1, \frac{dx^2}{ds} = \lambda^2, \frac{dx^3}{ds} = \lambda^3$$

are precisely the direction cosines of the displacement vector $\left(dx^1, dx^2, dx^3\right)$. Accordingly, we take the vector λ^i to define the direction in space relative to a curvilinear co-ordinate system X (Figure 2.7). Let $ds_{(1)}$ denote the element of arc along x^1 curve

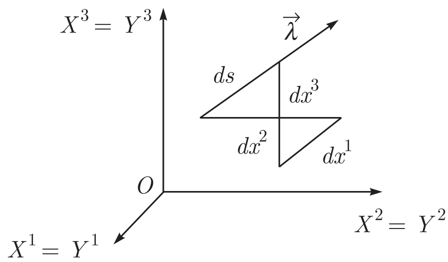

Figure 2.7: Length of a vector.

at $P\left(x^1, x^2, x^3\right)$. Along the x^1 curve, $x^2 =$ constant and $x^3 =$ constant, so that $dx^2 = 0$ and $dx^3 = 0$. Therefore, the length of the elementary arc measured along the co-ordinate curves of the curvilinear co-ordinate system is given by

$$ds_{(1)}^2 = g_{11} dx^1 dx^1 = g_{11} (dx^1)^2 \Rightarrow ds_{(1)} = \sqrt{g_{11}} dx^1,$$

where $s_{(1)}$ denotes the arc length along x^1 curve (Figure 2.8). Similarly,

$$ds_{(2)} = \sqrt{g_{22}}dx^2 \quad \text{and} \quad ds_{(3)} = \sqrt{g_{33}}dx^3.$$

From developments of this section we see that the metric properties of E^3 referred to a curvilinear co-ordinate system x, are completely determined by the tensor g_{ij}. Accordingly, this tensor is called the metric tensor and the quadratic form $ds^2 = g_{ij}dx^i dx^j$ is termed as fundamental quadratic form.

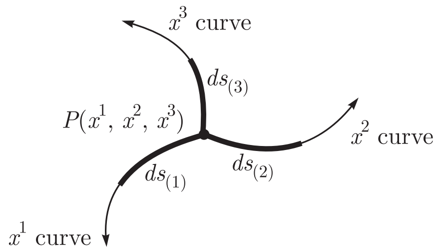

Figure 2.8: Curvilinear co-ordinate system.

2.5.5 Angle between Two Vectors

Let A^i and B^i be any two non-null contravariant vectors. Then from the definition of the length of a vector, the angle θ between them is defined by the formula,

$$\cos\theta = \frac{g_{ij}A^iB^j}{\sqrt{g_{ij}A^iA^j}\sqrt{g_{ij}B^iB^j}}; \quad 0 \le \theta \le \pi.$$

$$\Rightarrow AB\cos\theta = g_{ij}A^iB^j,$$

where A and B represent the lengths of the vectors A^i and B^i respectively. Let A_i and B_i be any two non-null contravariant vectors. Then from the definition of the length of a vector, the angle θ between them is defined by the formula,

$$\cos\theta = \frac{g^{ij}A_iB_j}{\sqrt{g^{ij}A_iA_j}\sqrt{g^{ij}B_iB_j}}; \quad 0 \le \theta \le \pi.$$

If x^1, x^2, x^3 are curvilinear co-ordinates of a point P and $\lambda^i = \frac{dx^i}{ds}; i = 1, 2, 3$, then the unit vector λ^i is defined to be a *direction* at a point P. Thus, if λ^i and λ^j are two directions at a given point, then the angle θ between them is given by

$$\cos\theta = g_{ij}\lambda^i\lambda^j.$$

EXAMPLE 2.5.3 *Prove that the angles θ_{12}, θ_{23} and θ_{31} between the co-ordinate curves in a three dimensional co-ordinate system are given by*

$$\cos\theta_{12} = \frac{g_{12}}{\sqrt{g_{11}g_{22}}}; \cos\theta_{23} = \frac{g_{23}}{\sqrt{g_{22}g_{33}}} \textit{ and } \cos\theta_{31} = \frac{g_{31}}{\sqrt{g_{33}g_{11}}}.$$

Solution: The square of the elementary arc length ds between two neighbouring points $P(x^1, x^2, x^3)$ and $Q(x^1 + dx^1, x^2 + dx^2, x^3 + dx^3)$ is given by

$$ds^2 = g_{ij} dx^i dx^j; \;\; i, j = 1, 2, 3.$$

Along the x^1 curve, $x^2 =$ constant and $x^3 =$ constant, so the length of the elementary arc measured along the co-ordinate curves of the curvilinear co-ordinate system is given by

$$ds_{(1)}^2 = g_{11} dx^1 dx^1 = g_{11}(dx^1)^2 \Rightarrow ds_{(1)} = \sqrt{g_{11}} dx^1,$$

where $s_{(1)}$ denotes the arc length along x^1 curve (Figure 2.9). Similarly,

$$ds_{(2)} = \sqrt{g_{22}} dx^2 \text{ and } ds_{(3)} = \sqrt{g_{33}} dx^3.$$

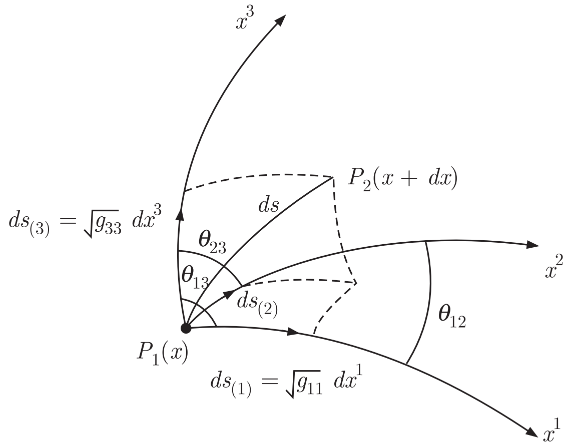

Figure 2.9: Angle between co-ordinate curves.

Now, the displacement vectors along x^1 curve, x^2 curve, x^3 curve are, respectively, given by $(dx^1, 0, 0)$, $(0, dx^2, 0)$, and $(0, 0, dx^3)$ and the length of the displacement vectors are $\sqrt{g_{11}} dx^1$, $\sqrt{g_{22}} dx^2$ and $\sqrt{g_{33}} dx^3$ respectively. Let θ_{ij} be the angle between the x^i and x^j co-ordinate curves; $i, j = 1, 2, 3$ and $i \neq j$, then

$$\cos\theta_{12} = \frac{g_{12} dx^1 dx^2}{\sqrt{g_{11} dx^1 dx^1}\sqrt{g_{22} dx^2 dx^2}} = \frac{g_{12}}{\sqrt{g_{11} g_{22}}}.$$

Similarly, we obtain

$$\cos\theta_{13} = \frac{g_{13}}{\sqrt{g_{11} g_{33}}} \text{ and } \cos\theta_{23} = \frac{g_{13}}{\sqrt{g_{22} g_{33}}}.$$

We see that, for an orthogonal system, $\theta_{12} = \theta_{23} = \theta_{31} = 90°$. Using the fact that $g_{ij} = g_{ji}$, it follows that, $g_{12} = g_{23} = g_{31} = 0$. Thus, the necessary and sufficient condition that the curvilinear co-ordinate system to be orthogonal is that $g_{ij} = 0$ for $i \neq j; i, j = 1, 2, 3$ and $g_{ii} \neq 0$ for all $i = 1, 2, 3$ at every point of the region R of E^3.

Result 2.5.1 Let us consider the co-ordinate curves of the curvilinear co-ordinate system. Along the x^1 curve, x^2 = constant and x^3 = constant, so the length of the elementary arc measured along the co-ordinate curves of the curvilinear co-ordinate system is given by

$$ds_{(1)}^2 = g_{11}dx^1dx^1 = g_{11}(dx^1)^2 \Rightarrow \frac{dx^1}{ds_{(1)}} = \frac{1}{\sqrt{g_{11}}},$$

where $s_{(1)}$ denotes the arc length along x^1 curve. Let $\xi^i_{(1)}, \xi^i_{(2)}, \xi^i_{(3)}$ be the unit vectors along the directions of the tangents to the co-ordinate curves at P. Then,

$$\xi^i_{(1)} = \frac{1}{\sqrt{g_{11}}}\delta^i_{(1)};\ \xi^i_{(2)} = \frac{1}{\sqrt{g_{22}}}\delta^i_{(2)};\ \xi^i_{(3)} = \frac{1}{\sqrt{g_{33}}}\delta^i_{(3)}.$$

It is to be noted that the angle between two co-ordinate curves is defined as the angle between their tangents.

2.5.6 Reciprocal Base System

For the desired interpretation of some results of tensor analysis in E^3 referred to a curvilinear co-ordinate system in the language and notation of ordinary vector analysis referred to curvilinear co-ordinate system, it is necessary to introduce the notation of reciprocal base system.

Now, we shall find the nature of the vector in E^3 referred to curvilinear co-ordinate system. Let a Cartesian system of axes be determined by a set of orthogonal base vectors $\mathbf{b_1}, \mathbf{b_2}, \mathbf{b_3}$ (Figure 2.10). Then the position vector $\mathbf{r}$ of any point $P\left(y^1, y^2, y^3\right)$ can be represented in the form

$$\mathbf{r} = y^1\mathbf{b_1} + y^2\mathbf{b_2} + y^3\mathbf{b_3} = y^i\mathbf{b_i};\ i = 1, 2, 3. \tag{2.51}$$

Since the base vectors $\mathbf{b_i}$ are independent of the position of the point $P\left(y^1, y^2, y^3\right)$, we deduce from Eq. (2.51) that

$$d\mathbf{r} = dy^1\mathbf{b_1} + dy^2\mathbf{b_2} + dy^3\mathbf{b_3} = dy^i\mathbf{b_i};\ i = 1, 2, 3. \tag{2.52}$$

By definition the square of the elementary arc length ds between two neighbouring points $P(y^1, y^2, y^3)$ and $Q(y^1 + dy^1, y^2 + dy^2, y^3 + dy^3)$ is $ds^2 = d\mathbf{r} \cdot d\mathbf{r}$. Using this result, Eq. (2.52) becomes

$$ds^2 = \mathbf{b_i} \cdot \mathbf{b_j} dy^i dy^j = \delta_{ij} dy^i dy^j = dy^i dy^i,$$

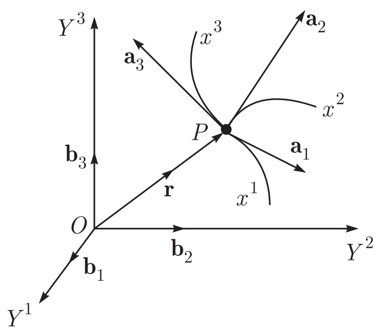

Figure 2.10: Reciprocal base system.

which is the expression for the square of the element of arc in orthogonal Cartesian co-ordinates. Let a set of equations of transformation

$$x^i = x^i(y^1, y^2, y^3); \;\; i = 1, 2, 3,$$

define a curvilinear co-ordinate system X. The position vector $\mathbf{r}$ can now be regarded as a function of co-ordinates x^i, and we write

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial x^i} dx^i,$$

and

$$\begin{aligned} ds^2 = d\mathbf{r} \cdot d\mathbf{r} &= \frac{\partial \mathbf{r}}{\partial x^i} \cdot \frac{\partial \mathbf{r}}{\partial x^j} dx^i dx^j \\ &= g_{ij} dx^i dx^j; \;\; g_{ij} = \frac{\partial \mathbf{r}}{\partial x^i} \cdot \frac{\partial \mathbf{r}}{\partial x^j} = \mathbf{a}_i \cdot \mathbf{a}_j \end{aligned}$$

where, $\mathbf{a}_i = \frac{\partial \mathbf{r}}{\partial x^i}$. Now, $\frac{\partial \mathbf{r}}{\partial x^1}$, $\frac{\partial \mathbf{r}}{\partial x^2}$ and $\frac{\partial \mathbf{r}}{\partial x^3}$ represent geometrically the respective tangent vectors to the x^1 curve, x^2 curve and x^3 curve at a point P. We observe that the base vectors $\mathbf{a}_i$ are no longer independent of the co-ordinates $\left(x^1, x^2, x^3\right)$. The base vectors cannot be, in general, taken to be unit vector or orthogonal as $\mathbf{a}_i \cdot \mathbf{a}_j = g_{ij} \neq 1$ and $\mathbf{a}_i \cdot \mathbf{a}_j = g_{ij} = 0$ is not given. Now, any vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ with initial point P are called *base vectors in curvilinear co-ordinate system.* So, we can write

$$d\mathbf{r} = \mathbf{a}_i dx^i \text{ and } g_{ij} = \mathbf{a}_i \cdot \mathbf{a}_j.$$

Now, any vector with initial point at P can be uniquely expressed as linear combination of the base vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$. The use of covariant notation for the base vectors $\mathbf{a}_i$ and $\mathbf{b}_i$ can be justified by observing from Eq. (2.52) that

$$\mathbf{a}_j dx^j = \mathbf{b}_i dy^i = \mathbf{b}_i \frac{\partial y^i}{\partial x^j} dx^j \Rightarrow \mathbf{a}_j = \frac{\partial y^i}{\partial x^j} \mathbf{b}_i$$

as dx^j is arbitrary. This is the law of transformation of the components of covariant vectors. Now, any vector $\mathbf{A}$ can be written in the form $\mathbf{A} = kd\mathbf{r}$, where k is a suitable scalar. Therefore,

$$\begin{aligned}\mathbf{A} &= \frac{\partial \mathbf{r}}{\partial x^i}\left(kdx^i\right); \text{ as } d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial x^i}dx^i \\ &= \mathbf{a}_i A^i; \text{ where } A^i = kdx^i.\end{aligned}$$

The numbers A^i are the contravariant components of the vector $\mathbf{A}$, and the vectors $A^1\mathbf{a}_1, A^2\mathbf{a}_2, A^3\mathbf{a}_3$ form the edges of the parallelepiped whose diagonal is $\mathbf{A}$. Since the $\mathbf{a}_i$ are not unit vectors in general, we see that the lengths of the edges of the parallelepiped, or the physical components of $\mathbf{A}$, are determined by the formulas

$$A^1\sqrt{g_{11}},\ A^2\sqrt{g_{22}},\ A^3\sqrt{g_{33}}$$

since $g_{11} = \mathbf{a}_1 \cdot \mathbf{a}_1$, $g_{22} = \mathbf{a}_2 \cdot \mathbf{a}_2$ and $g_{33} = \mathbf{a}_3 \cdot \mathbf{a}_3$. Let $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ be base vectors at a point P in the region R of E^3 and let $\mathbf{a}^1, \mathbf{a}^2, \mathbf{a}^3$ be three independent vectors at P such that

$$\mathbf{a}^i \cdot \mathbf{a}_j = \delta^i_j; \text{ where } \delta^i_j = \text{ Kronecker delta.}$$

Then the three vectors $\mathbf{a}^1, \mathbf{a}^2, \mathbf{a}^3$ are called the *reciprocal base vectors* of the base vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ at each point P of R in E^3. Let us define the non-coplanar vectors

$$\mathbf{a}^1 = \frac{\mathbf{a}_2 \times \mathbf{a}_3}{[\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3]},\ \mathbf{a}^2 = \frac{\mathbf{a}_3 \times \mathbf{a}_1}{[\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3]} \text{ and } \mathbf{a}^3 = \frac{\mathbf{a}_1 \times \mathbf{a}_2}{[\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3]}, \tag{2.53}$$

where $\mathbf{a}_2 \times \mathbf{a}_3$, etc., denote the vector product of $\mathbf{a}_2$ and $\mathbf{a}_3$, and $[\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3]$ is the triple scalar product $\mathbf{a}_1 \cdot \mathbf{a}_2 \times \mathbf{a}_3$. It is obvious from definition Eq. (2.53) that $\mathbf{a}_i \cdot \mathbf{a}_j = \delta^i_j$, and

$$[\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3] = \varepsilon_{ijk}\frac{\partial y^i}{\partial x^1}\frac{\partial y^j}{\partial x^2}\frac{\partial y^k}{\partial x^3} = \begin{vmatrix} \frac{\partial y^1}{\partial x^1} & \frac{\partial y^2}{\partial x^1} & \frac{\partial y^3}{\partial x^1} \\ \frac{\partial y^1}{\partial x^2} & \frac{\partial y^2}{\partial x^2} & \frac{\partial y^3}{\partial x^2} \\ \frac{\partial y^1}{\partial x^3} & \frac{\partial y^2}{\partial x^3} & \frac{\partial y^3}{\partial x^3} \end{vmatrix}$$

or

$$[\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3]^2 = \begin{vmatrix} \frac{\partial y^1}{\partial x^1} & \frac{\partial y^2}{\partial x^1} & \frac{\partial y^3}{\partial x^1} \\ \frac{\partial y^1}{\partial x^2} & \frac{\partial y^2}{\partial x^2} & \frac{\partial y^3}{\partial x^2} \\ \frac{\partial y^1}{\partial x^3} & \frac{\partial y^2}{\partial x^3} & \frac{\partial y^3}{\partial x^3} \end{vmatrix} \begin{vmatrix} \frac{\partial y^1}{\partial x^1} & \frac{\partial y^2}{\partial x^1} & \frac{\partial y^3}{\partial x^1} \\ \frac{\partial y^1}{\partial x^2} & \frac{\partial y^2}{\partial x^2} & \frac{\partial y^3}{\partial x^2} \\ \frac{\partial y^1}{\partial x^3} & \frac{\partial y^2}{\partial x^3} & \frac{\partial y^3}{\partial x^3} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial y^p}{\partial x^1}\frac{\partial y^p}{\partial x^1} & \frac{\partial y^p}{\partial x^1}\frac{\partial y^p}{\partial x^2} & \frac{\partial y^p}{\partial x^1}\frac{\partial y^p}{\partial x^3} \\ \frac{\partial y^p}{\partial x^2}\frac{\partial y^p}{\partial x^1} & \frac{\partial y^p}{\partial x^2}\frac{\partial y^p}{\partial x^2} & \frac{\partial y^p}{\partial x^2}\frac{\partial y^p}{\partial x^3} \\ \frac{\partial y^p}{\partial x^3}\frac{\partial y^p}{\partial x^1} & \frac{\partial y^p}{\partial x^3}\frac{\partial y^p}{\partial x^2} & \frac{\partial y^p}{\partial x^3}\frac{\partial y^p}{\partial x^3} \end{vmatrix} = \begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix} = g$$

or

$$[\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3] = \sqrt{g};\ \ g = |g_{ij}|, \tag{2.54}$$

that the triple scalar products $\left[\mathbf{a}^1\mathbf{a}^2\mathbf{a}^3\right]$ and $[\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3]$ are reciprocally related, so that $\left[\mathbf{a}^1\mathbf{a}^2\mathbf{a}^3\right] = \frac{1}{\sqrt{g}}$. Therefore, the vectors $\mathbf{a}^1, \mathbf{a}^2, \mathbf{a}^3$ are non-coplanar. Moreover,

$$\mathbf{a}_1 = \frac{\mathbf{a}^2 \times \mathbf{a}^3}{[\mathbf{a}^1\mathbf{a}^2\mathbf{a}^3]},\ \mathbf{a}_2 = \frac{\mathbf{a}^3 \times \mathbf{a}^1}{[\mathbf{a}^1\mathbf{a}^2\mathbf{a}^3]} \text{ and } \mathbf{a}_3 = \frac{\mathbf{a}^1 \times \mathbf{a}^2}{[\mathbf{a}^1\mathbf{a}^2\mathbf{a}^3]} \tag{2.55}$$

with the aid of Eq. (2.53). In view of this it is natural to call the system of vectors $\mathbf{a}^1, \mathbf{a}^2, \mathbf{a}^3$ the *reciprocal base system.* Using the reciprocal base system, we have $d\mathbf{r} = \mathbf{a}^i dx_i$, where the dx_i are the appropriate components of $d\mathbf{r}$. Therefore,

$$\begin{aligned} ds^2 &= d\mathbf{r} \cdot d\mathbf{r} = \left(\mathbf{a}^i dx_i\right) \cdot \left(\mathbf{a}^j dx_j\right) \\ &= \mathbf{a}^i \cdot \mathbf{a}^j dx_i dx_j = g^{ij} dx_i dx_j,\ \text{ where } g^{ij} = \mathbf{a}^i \cdot \mathbf{a}^j = g^{ji}. \end{aligned} \tag{2.56}$$

Using the system of base vectors determined by Eq. (2.53), an arbitrary vector $\mathbf{A}$ with initial point P can be expressed as

$$\mathbf{A} = \mathbf{a}^i A_i = \mathbf{a}_i A^i,$$

where A_1, A_2, A_3 are suitable scalars (Figure 2.12). Thus the covariant components of a vector $\mathbf{A}$ with base vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ has contravariant components A^1, A^2, A^3 and the corresponding covariant components will be determined by reciprocal base vectors $\mathbf{a}^1, \mathbf{a}^2, \mathbf{a}^3$ and will be given by A_1, A_2, A_3.

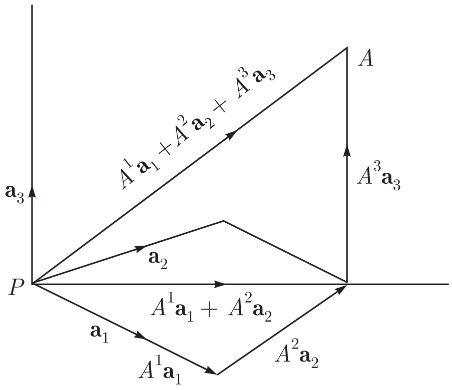

Figure 2.11: Representation of A.

The scalar product of the vector $A_i a^i$ with the base vector a_j, and note that the later is directed along the x^i co-ordinate line, we get

$$A_i a^i \cdot a_j = A_i \delta^i_j = A_j.$$

Thus $\dfrac{A_j}{\sqrt{g_{jj}}}$ (no summation on j) is the length of the orthogonal projection of the vector A on the tangent to the x^j co-ordinate curve at the point P (Figure 2.11), whereas $\dfrac{A_j}{\sqrt{g_{jj}}}$ is the length of the edge of the parallelepiped whose diagonal is the vector A. Since

$$A = a_i A^i = a^i A_i,$$

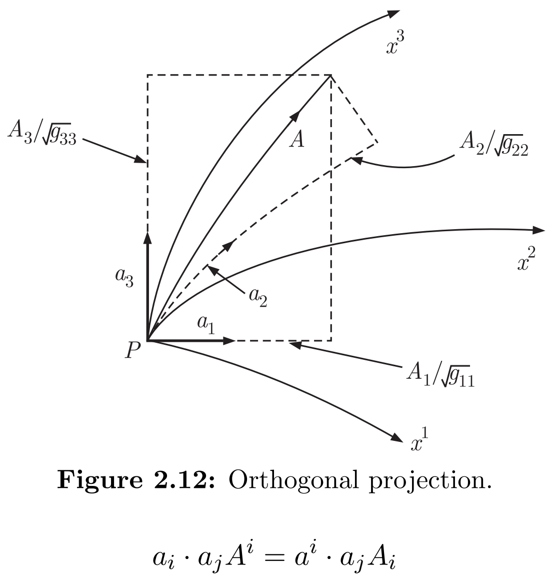

Figure 2.12: Orthogonal projection.

We have,

$$a_i \cdot a_j A^i = a^i \cdot a_j A_i$$

or

$$g_{ij}A^i = d_i^j A_i = A_j.$$

We see that the vector obtained by lowering the index in A^i is precisely the co-variant ventor A_i. The two sets of quantities A^i and A_i are thus seen to represent the same vector A referred to two different base systems. Thus, the distinction between the covariant and contravariant components of A disappears whenever the base vectors are orthogonal.

2.5.7 Partial Derivative

Let $\mathbf{A}$ be a vector localised at some point $P\left(y^1, y^2, y^3\right)$ of E^3 referred to an orthogonal Cartesian frame Y. If at each point of some region R about P we have a uniquely defined vector $\mathbf{A}$, we refer to the totality of vectors $\mathbf{A}$ in R as a *vector field.* We suppose that the components of $\mathbf{A}$ are continuously differentiable functions of u^i in R, and, if we introduce a curvilinear system of co-ordinates X by means of transformation

$$T : x^i = x^i\left(y^1, y^2, y^3\right),$$

the corresponding components $A^i(x)$ will be continuously differentiable functions of the point $P\left(x^1, x^2, x^3\right)$ determined by the position vector $\mathbf{r}\left(x^1, x^2, x^3\right)$. We will be concerned with the calculation of the vector change $\triangle\mathbf{A}$ in $\mathbf{A}$ as the point $P\left(x^1, x^2, x^3\right)$ assumes the different position

$$P'\left(x^1 + \triangle x^1, x^2 + \triangle x^2, x^3 + \triangle x^3\right).$$

Using the system of base vectors determined by (2.53), an arbitrary vector $\mathbf{A}$ with initial point P can be expressed as

$$\mathbf{A} = \mathbf{a}_i A^i;\ \mathbf{a}_i = \frac{\partial \mathbf{r}}{\partial x^i}, \tag{2.57}$$

where A^1, A^2, A^3 are suitable scalars. Therefore,

$$\begin{aligned}\mathbf{A} &= \left(A^i + \triangle A^i\right)(\mathbf{a}_i + \triangle\mathbf{a}_i) - A^i\mathbf{a}_i \\ &= \triangle A^i\mathbf{a}_i + A^i\triangle\mathbf{a}_i + \triangle A^i\triangle\mathbf{a}_i.\end{aligned}$$

As in ordinary calculus we denote the principal part of the change by $d\mathbf{A}$ and write

$$d\mathbf{A} = \mathbf{a}_i dA^i + A^i d\mathbf{a}_i. \tag{2.58}$$

This formula states that the differential change in $\mathbf{A}$ arises from two sources

(i) Change in the components A^i as the values $\left(x^1, x^2, x^3\right)$ are changed.

(ii) Change in the base vectors $\mathbf{a}_i$ as the position of the point $\left(x^1, x^2, x^3\right)$ is altered.

The partial derivative of $\mathbf{A}$ with respect to x^j is defined as the limit of the quotient,

$$\lim_{\triangle x^j \to 0} \frac{\triangle \mathbf{A}}{\triangle x^j} = \frac{\partial \mathbf{A}}{\partial x^j},$$

and it follows from the expression for the increment $\triangle \mathbf{A}$ that:

$$\frac{\partial \mathbf{A}}{\partial x^j} = \frac{\partial A^i}{\partial x^j}\mathbf{a}_i + \frac{\partial \mathbf{a}_i}{\partial x^j}A^i. \tag{2.59}$$

Now, we find the components of the vector $\frac{\partial \mathbf{A}}{\partial x^j}$ referred to a basis $\mathbf{a}_i$. Since $g_{ij} = \mathbf{a}_i \cdot \mathbf{a}_j$, hence,

$$\frac{\partial g_{ij}}{\partial x^k} = \frac{\partial \mathbf{a}_i}{\partial x^k} \cdot \mathbf{a}_j + \frac{\partial \mathbf{a}_j}{\partial x^k} \cdot \mathbf{a}_i.$$

Permuting the indices in this formula, we get

$$\frac{\partial g_{ik}}{\partial x^j} = \frac{\partial \mathbf{a}_i}{\partial x^j} \cdot \mathbf{a}_k + \frac{\partial \mathbf{a}_k}{\partial x^j} \cdot \mathbf{a}_i; \text{ and } \frac{\partial g_{jk}}{\partial x^i} = \frac{\partial \mathbf{a}_j}{\partial x^i} \cdot \mathbf{a}_k + \frac{\partial \mathbf{a}_k}{\partial x^i} \cdot \mathbf{a}_j.$$

If we assume that T is of class C^2, then,

$$\frac{\partial \mathbf{a}_i}{\partial x^j} = \frac{\partial \mathbf{a}_j}{\partial x^i}; \text{ as } \mathbf{a}_i = \frac{\partial \mathbf{r}}{\partial x^i}$$

and

$$\frac{\partial \mathbf{a}_i}{\partial x^j} = \frac{\partial}{\partial x^j}\left(\frac{\partial \mathbf{r}}{\partial x^i}\right) = \frac{\partial}{\partial x^i}\left(\frac{\partial \mathbf{r}}{\partial x^j}\right) = \frac{\partial \mathbf{a}_j}{\partial x^i}.$$

Using this relation, we get

$$\frac{\partial \mathbf{a}_i}{\partial x^j} \cdot \mathbf{a}_k = \frac{1}{2}\left\{\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right\} = [ij, k]$$

or

$$\frac{\partial \mathbf{a}_i}{\partial x^j} = [ij, k]\mathbf{a}^k$$

or

$$\frac{\partial \mathbf{a}_i}{\partial x^j} \cdot \mathbf{a}^\alpha = [ij, k]\mathbf{a}^k \cdot \mathbf{a}^\alpha = [ij, k]g^{k\alpha} = \left\{ \begin{matrix} \alpha \\ i\ \mathrm{J} \end{matrix} \right\}$$

or

$$\frac{\partial \mathbf{a}_i}{\partial x^j} = \left\{ \begin{matrix} \alpha \\ i\ \ j \end{matrix} \right\} \cdot \mathbf{a}_\alpha.$$

Substituting this result in Eq. (2.59), we get

$$\frac{\partial \mathbf{A}}{\partial x^j} = \frac{\partial A^i}{\partial x^j}\mathbf{a}_i + \left\{ \begin{matrix} \alpha \\ i \ \ j \end{matrix} \right\} \cdot \mathbf{a}_\alpha A^i = \left[\frac{\partial A^\alpha}{\partial x^j} + \left\{ \begin{matrix} \alpha \\ i \ \ j \end{matrix} \right\} A^i \right] \mathbf{a}_\alpha = A^\alpha_{,j}\mathbf{a}_\alpha, \qquad (2.60)$$

where $A^\alpha_{,j}$ means the covariant derivative of A^α with respect to the metric tensor of E^3. Therefore, Eq. (2.60) can be written as

$$\frac{\partial \mathbf{A}}{\partial x^j} = A^\alpha_{,j}\mathbf{a}_\alpha. \qquad (2.61)$$

In virtue of Eq. (2.61) we can state the following interpretation of the covariant derivative of a vector A^i.

Interpretation: If a vector $\mathbf{A}$ of E^3 has contravariant components A^i referred to a basis $\mathbf{a}_i$, then the covariant derivative of the vector $\mathbf{A}^i$ with respect to the metric tensor of E^3 is a vector whose components are those of the vector $\frac{\partial \mathbf{A}}{\partial x^j}$ referred to the basis $\mathbf{a}_i$.

EXAMPLE 2.5.4 *If g_{ij} is the metric tensor of Euclidean space E^3 in curvilinear co-ordinates x^i, and y^i are rectangular Cartesian co-ordinates, show that*

$$\frac{\partial y^i}{\partial x^p} = g_{pq}\frac{\partial x^q}{\partial y^i}.$$

Solution: We know, $g_{pq} = \frac{\partial y^j}{\partial x^p}\frac{\partial y^j}{\partial x^q}$. Multiplying both sides by $\frac{\partial x^q}{\partial y^i}$, we get

$$g_{pq}\frac{\partial x^q}{\partial y^i} = \frac{\partial y^j}{\partial x^p}\frac{\partial y^j}{\partial x^q}\frac{\partial x^q}{\partial y^i} = \frac{\partial y^j}{\partial x^p}\delta^j_i = \frac{\partial y^i}{\partial x^p}.$$

Therefore, $\frac{\partial y^i}{\partial x^p} = g_{pq}\frac{\partial x^q}{\partial y^i}$.

EXAMPLE 2.5.5 *If g^{ij} is the conjugate metric tensor of Euclidean space E^3 in curvilinear co-ordinates x^i, and y^i are rectangular Cartesian co-ordinates, show that*

$$\frac{\partial x^p}{\partial y^i} = g^{pq}\frac{\partial y^i}{\partial x^q}.$$

Solution: Since g^{ij} is the conjugate metric tensor of Euclidean space E^3 in curvilinear co-ordinates x^i, and y^i are rectangular Cartesian co-ordinates, we find according to the law of transformation of a $(2,0)$ tensor we have, $g^{pq} = \delta^{ij}\frac{\partial x^p}{\partial y^i}\frac{\partial x^q}{\partial y^j}$. Multiplying both sides by $\frac{\partial y^i}{\partial x^q}$, we get

$$\begin{aligned} g^{pq}\frac{\partial y^i}{\partial x^q} &= \delta^{ij}\frac{\partial x^p}{\partial y^i}\frac{\partial x^q}{\partial y^j}\frac{\partial y^i}{\partial x^q} \\ &= \delta^{ij}\frac{\partial x^p}{\partial y^i}\frac{\partial y^i}{\partial x^q}\frac{\partial x^q}{\partial y^j} = \delta^{ij}\delta^p_q\frac{\partial y^i}{\partial x^q} = \frac{\partial x^p}{\partial y^i}. \end{aligned}$$

EXAMPLE 2.5.6 *If y^i and x^i are rectangular Cartesian and curvilinear co-ordinates, respectively, show that in E^3,*

$$[ij,k] = \frac{\partial^2 y^p}{\partial x^i \partial x^j}\frac{\partial y^p}{\partial x^k} \text{ and } \left\{ \begin{matrix} i \\ j \quad k \end{matrix} \right\} = \frac{\partial^2 y^p}{\partial x^j \partial x^k}\frac{\partial x^i}{\partial y^p}.$$

Solution: Here, y^i and x^i are rectangular cartesian and curvilinear co-ordinates respectively. Using the relation $g_{ik} = \frac{\partial y^p}{\partial x^i}\frac{\partial y^p}{\partial x^k}$, we get

$$\frac{\partial g_{ik}}{\partial x^j} = \frac{\partial^2 y^p}{\partial x^j \partial x^i}\frac{\partial y^p}{\partial x^k} + \frac{\partial y^p}{\partial x^i}\frac{\partial^2 y^p}{\partial x^j \partial x^k}.$$

Similarly

$$\frac{\partial g_{jk}}{\partial x^i} = \frac{\partial^2 y^p}{\partial x^i \partial x^j}\frac{\partial y^p}{\partial x^k} + \frac{\partial y^p}{\partial x^j}\frac{\partial^2 y^p}{\partial x^i \partial x^k}$$

and

$$\frac{\partial g_{ij}}{\partial x^k} = \frac{\partial^2 y^p}{\partial x^k \partial x^i}\frac{\partial y^p}{\partial x^j} + \frac{\partial y^p}{\partial x^i}\frac{\partial^2 y^p}{\partial x^k \partial x^j}.$$

Therefore, the Christoffel symbol of first kind is given by

$$[ij,k] = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right) = \frac{\partial^2 y^p}{\partial x^i \partial x^j}\frac{\partial y^p}{\partial x^k}.$$

Now, the Christoffel symbol of second kind is given by

$$\left\{ \begin{matrix} k \\ i \quad j \end{matrix} \right\} = g^{km}[ij,m] = g^{km}\frac{1}{2}\left(\frac{\partial g_{im}}{\partial x^j} + \frac{\partial g_{jm}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^m}\right)$$
$$= g^{km}\frac{\partial^2 y^p}{\partial x^i \partial x^j}\frac{\partial y^p}{\partial x^m} = \frac{\partial^2 y^p}{\partial x^i \partial x^j} g^{km}\frac{\partial y^p}{\partial x^m} = \frac{\partial^2 y^p}{\partial x^j \partial x^k}\frac{\partial x^i}{\partial y^p}.$$

EXAMPLE 2.5.7 *Show that the area of the parallelogram constructed on the base vectors $\mathbf{a}_2$ and $\mathbf{a}_3$ is $\sqrt{gg^{11}}$, where g_{ij} and g^{ij} are the metric and conjugate metric tensors in curvilinear co-ordinate system and $g = |g_{ij}|$.*

Solution: Using the relation $g^{ij} = \mathbf{a}^i \cdot \mathbf{a}^j$, we get

$$g^{11} = \mathbf{a}^1 \cdot \mathbf{a}^1 = \left|\mathbf{a}^1\right|^2 \Rightarrow \left|\mathbf{a}^1\right| = \sqrt{g^{11}}.$$

Also, from Eqs. (2.53) and (2.54) we get the relations

$$\mathbf{a}^1 = \frac{\mathbf{a}_2 \times \mathbf{a}_3}{[\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3]} \text{ and } [\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3] = \sqrt{g}$$

$$\Rightarrow \sqrt{g}\mathbf{a}^1 = \mathbf{a}_2 \times \mathbf{a}_3$$

$$\Rightarrow |\mathbf{a}_2 \times \mathbf{a}_3| = \left|\sqrt{g}\mathbf{a}^1\right| = \sqrt{g}\left|\mathbf{a}^1\right| = \sqrt{g}\sqrt{g^{11}}.$$

Area of the parallelogram constructed on the base vectors $\mathbf{a}_2$ and $\mathbf{a}_3$ is given by

$$|\mathbf{a}_2 \times \mathbf{a}_3| = \sqrt{g}\sqrt{g^{11}} = \sqrt{gg^{11}}.$$

Similarly, the area of the parallelogram constructed on the base vectors $\mathbf{a}_3$ and $\mathbf{a}_1$ is $\sqrt{gg^{22}}$ and on the base vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ is $\sqrt{gg^{33}}$. If V be the volume of the parallelopiped constructed on the base vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$, then

$$V = \mathbf{a}_1 \cdot (\mathbf{a}_2 \times \mathbf{a}_3) = [\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3] = \sqrt{g}.$$

Let x^i denote the curvilinear co-ordinates. The volume dV of the element of the parallelopiped constructed on the vectors $\mathbf{a}_1 dx^1, \mathbf{a}_2 dx^2, \mathbf{a}_3 dx^3$ is given by

$$dV = \left(\mathbf{a}_1 dx^1\right) \cdot \left(\mathbf{a}_2 dx^2 \times \mathbf{a}_3 dx^3\right) = \sqrt{g} dx^1 dx^2 dx^3.$$

EXAMPLE 2.5.8 *Find the physical components of the vector with components A^i in spherical polar co-ordinates.*

Solution: The expression for the metric in spherical polar co-ordinates is given by

$$ds^2 = \left(dx^1\right)^2 + \left(x^1\right)^2\left(dx^2\right)^2 + \left(x^1\right)^2\left(\sin x^2\right)^2\left(dx^3\right)^2.$$

Therefore, in spherical polar co-ordinates (x^i), $g_{11} = 1, g_{22} = (x^1)^2$ and $g_{33} = (x^1 \sin x^2)^2$, $g_{ij} = 0$, for $i \neq j$. If $\overline{A}_i$ denote the physical components, then using the relation

$$\overline{A}_i = \frac{g_{ij}A^j}{\sqrt{g_{ii}}} = \frac{g_{i1}A^1 + g_{i2}A^2 + g_{i3}A^3}{\sqrt{g_{ii}}},$$

we get

$$\overline{A}_1 = \frac{g_{11}A^1}{\sqrt{g_{11}}} = A^1; \overline{A}_2 = \frac{g_{22}A^2}{\sqrt{g_{22}}} = \sqrt{g_{22}}A^2 = x^1A^2$$

and

$$\overline{A}_3 = \frac{g_{33}A^3}{\sqrt{g_{33}}} = \sqrt{g_{33}}A^3 = (x^1 \sin x^2)A^3.$$

Hence, the required physical components of the vector with components A^i are A^1, x^1A^2, $(x^1 \sin x^2)A^3$ in spherical polar co-ordinates.

2.6 Exercises

1. If g_{ij} is the metric tensor in a Riemannian space and g^{ij} its reciprocal, show that,
   (a) g^{ij} is a symmetrical contravariant tensor
   (b) and $g^{i\alpha}g_{\alpha j} = \delta^i_j$.

2. In V_2, find the quantities g^{ij} where $g_{ij} = i + j$.
3. Prove that the maximum number of independent components of the metric g_{ij} in V_N is $\frac{1}{2}N(N+1)$.
4. Find g and the reciprocal tensors, if the metric is given by
   (i) $ds^2 = 5(dx^1)^2 + 4(dx^2)^2 - 3(dx^3)^2 + 4dx^1dx^2 - 6dx^2dx^3$.
   (ii) $ds^2 = a(du)^2 + b(dv)^2 + c(dw)^2 + 2fdvdw + 2gdwdu + 2hdudv$.
   (iii) $ds^2 = -a(dx^1)^2 - b(dx^2)^2 - c(dx^3)^2 + d(dx^4)^2$.
   (iv) $ds^2 = 3(dx^1)^2 + 2(dx^2)^2 + 2(dx^3)^2 - 4dx^1dx^2$.
   (v) $ds^2 = (dx^1)^2 - 2(dx^2)^2 + 3(dx^3)^2 - 8dx^2dx^3$.
   (vi) $ds^2 = (dx^1)^2 + 2\cos\alpha dx^1dx^2 + (dx^2)^2$.
5. (a) Find the metric for the surface of sphere of constant radius a interms of spherical polar co-ordinates.
   (b) Show that in a case of an orthogonal co-ordinate system, a covariant component of a tensor is related only to the corresponding contravariant component of a tensor.
   (c) Show that with respect to a cartesian co-ordinate system, the distinction among the contravariant, the covariant and the mixed components of a tensor vanishes.
6. Show that
   (a) if $y^1 = a\cos u, y^2 = a\sin u, y^3 = v$, $ds^2 = a^2(du)^2 + (dv)^2$.
   (b) if $y^1 = u\cos v, y^2 = u\sin v, y^3 = av$, $ds^2 = (du)^2 + [(u)^2 + a^2](dv)^2$.
   (c) if $y^1 = u, y^2 = v, y^3 = \psi(u, v)$,
   $ds^2 = (1 + \psi_1^2)(du)^2 + 2\psi_1\psi_2 dudv + (1 + \psi_2)(dv)^2$.
7. Show that, if the relation between the Cartesian co-ordinates (x^1, x^2, x^3) and
   (a) the parabolic cylindrical co-ordinates $(\overline{x}^1, \overline{x}^2, \overline{x}^3)$ be

$$x^1 = \frac{1}{2}\left[(\overline{x}^1)^2 - (\overline{x}^2)^2\right];\; x^2 = \overline{x}^1\overline{x}^2;\; x^3 = \overline{x}^3,$$

   the metric is given by

$$ds^2 = \left[(\overline{x}^1)^2 + (\overline{x}^2)^2\right]\left\{(d\overline{x}^1)^2 + (d\overline{x}^2)^2\right\} + (d\overline{x}^3)^2.$$

   (b) the paraboloidal co-ordinates (y^1, y^2, y^3) be

$$x^1 = y^1y^2\cos y^3, x^2 = y^1y^2\sin y^3, x^3 = \frac{1}{2}[(y^1)^2 - (y^2)^2],$$

   then the metric is given by

$$ds^2 = \left[(y^1)^2 + (y^2)^2\right]\left\{(dy^1)^2 + (dy^2)^2\right\} + (y^1y^2)^2(dy^3)^2.$$

(c) the elliptic cylindrical co-ordinates (y^1, y^2, y^3) be

$$x^1 = a\cosh y^1 \cos y^2, x^2 = a\sinh y^1 \sin y^2, x^3 = y^3,$$

then the metric is given by

$$ds^2 = a^2\left[(\sinh y^1)^2 + (\sin y^2)^2\right]\{(dy^1)^2 + (dy^2)^2\} + (dy^3)^2.$$

(d) the oblate spherical co-ordinates (y^1, y^2, y^3) be

$$x^1 = a\cosh y^1 \cos y^2 \cos y^3, x^2 = a\cosh y^1 \cos y^2 \sin y^3, x^3 = a\sinh y^1 \sin y^2,$$

then the metric is given by

$$ds^2 = a^2\left[(\sinh y^1)^2 + (\sin y^2)^2\right]\{(dy^1)^2 + (dy^2)^2\} + a^2\cosh^2 y^1 \cos^2 y^2 (dy^3)^2.$$

8. (a) Prove that, in V_n,
   (i) $(g_{hj}g_{ik} - g_{hi}g_{jk})\, g^{hj} = (N-1)g_{ik}$
   (ii) $\frac{\partial \phi}{\partial x^j}(g_{hk}g_{il} - g_{hl}g_{ik})\, g^{hj} = \frac{\partial \phi}{\partial x^k} g_{il} - \frac{\partial \phi}{\partial x^i} g_{ik}$, if ϕ is invariant

   where g_{ij} and g^{ij} have their usual meanings.

   (b) If the Jacobian matrix of the transformation from a given co-ordinate system (x^i) to a rectangular system $(\bar{x}_i)$ is $J = \left(\frac{\partial \bar{x}^i}{\partial x^j}\right)$, then prove that the matrix $G = (g_{ij})$ of the Euclidean metric tensor in (x^i) system in $G = J^T J$.

9. (a) Show that the length of the arc of the curve $x^1 = 3t$; $x^2 = e^t$; $0 \le t \le 2$ is 10 for the metric components $g_{11} = 2; g_{12} = g_{21} = (x^2)^{-1}; g_{22} = (x^2)^{-2}$.
   (b) A curve in cylindrical co-ordinates x^i is given by $x^1 = a\cos t$; $x^2 = a\sin t$; $x^3 = bt$, where a and b are positive constants. Show that the length of arc for $0 \le t < c$ is $c\sqrt{a^2 + b^2}$.
   (c) A curve in spherical co-ordinates x^i is given by $x^1 = t$; $x^2 = a\sin^{-1}\frac{1}{t}$; $x^3 = \sqrt{t^2 - 1}$. Show that the length of arc for $1 \le t \le 2$ is $\sqrt{6}$.
   (d) Using the Euclidean metric for polar co-ordinates, compute the length of arc for the curve $x^1 = 2a\cos t; x^2 = t$; $0 \le t \le \frac{\pi}{2}$, and interpret geometrically.
   (e) Calculate the length of the curve $x^1 = 3 - t, x^2 = 6t + 3, x^3 = \log t$ for $1 \le t \le e$, for the metric $\begin{pmatrix} 12 & 4 & 0 \\ 4 & 1 & 1 \\ 0 & 1 & (x^1)^2 \end{pmatrix}$.

10. Show that the tensors g_{pq}, g^{pq} and δ^p_q are associated tensors.

11. Prove that the associated tensors of A^{ij} are A^j_m and A_{mn}.

12. Prove that the relationship between the pairs of
    (a) the associated tensors A^{pq} and $A_j^{\bullet q}$ is
$$A^{pq} = g^{pj} A_j^{\bullet q}.$$
    (b) the associated tensors $A_{\bullet q}^{p \bullet r}$ and A_{jql} is
$$A_{\bullet q}^{p \bullet r} = g^{pj} g^{rl} A_{jql}.$$
    (c) the associated tensors $A_{pq}^{\bullet\bullet r}$ and $A_{\bullet\bullet l}^{jk}$ is
$$A_{pq}^{\bullet\bullet r} = g_{pj} g_{qk} g^{rl} A_{\bullet\bullet l}^{jk}.$$
    (d) associated tensors of B^{jkl} and B_{pqr} is
$$B^{jkl} = g^{jp} g^{kq} g^{lr} B_{pqr}.$$
13. Show that for an orthogonal co-ordinate system
$$g_{11} = \frac{1}{g_{11}}, \; g_{22} = \frac{1}{g_{22}} \qquad \text{and} \qquad g_{33} = \frac{1}{g_{33}}$$
    Considering an N-dimensional Euclidean space E_N with rectangular Cartesian co-ordinates as a particular case of V_N, show that in E_N there is no distinction between covariant and contravariant vectors.
14. Illustrate the concept of an associate vector in a Riemannian space. If in a two-dimensional Riemannian space, the components of a metric tensor are $g_{11} = 1$, $g_{12} = 0$, $g_{22} = r^2$, find the components of the associated tensor.
15. Let E^3 be covered by orthogonal Cartesian co-ordinates x^i and let $x^i = a_j^i y^j$ where $|a_j^i| \neq 0 (i, j = 1, 2, 3)$ represent a linear transformation of co-ordinates. Determine the metric coefficients $g_{ij}(y)$. Discuss the case when the transformation is orthogonal.
16. Let g_{ij} and g^{ij} be reciprocal symmetric tensor of the second order and u_i, v_i be component of covariant vectors. If u^i and v^i are defined by
$$u^i = g^{ij} u_j, \; v^i = g^{ij} v_j; \; i, j = 1, 2, \ldots, N.$$
    Show that $u_i = g_{ij} u^j, u^i v_i = u_i v^i$ and $u^i g_{ij} u^j = u_i g^{ij} u_j$.
17. (a) Prove that the necessary and sufficient condition that the curvilinear co-ordinate system to be orthogonal is that $g_{ij} = 0$ for $i \neq j;\; i, j = 1, 2, 3$ and $g_{ii} \neq 0$ for all $i = 1, 2, 3$ at every point of the region R of E^3.
    (b) Under the Euclidean metric for spherical co-ordinates, determine a particular family of curves that intersect $x^1 = a, x^2 = bt, x^3 = t$ orthogonally.

18. Show that in the V_4 with line element

$$ds^2 = -\left(dx^1\right)^2 - \left(dx^2\right)^2 - \left(dx^3\right)^2 + c^2\left(dx^4\right)^2$$

each of the following vectors is unit vector (i) $\left(1, 1, 0, \frac{\sqrt{3}}{c}\right)$ and (ii) $\left(\sqrt{2}, 0, 0, \frac{\sqrt{3}}{c}\right)$.

19. If A^p is a vector field, then show that the corresponding unit vector is

$$A^p/\sqrt{A^p A_q} \quad \text{or} \quad A^p/\sqrt{g_{pq} A^p A_q}.$$

20. If $A_i = \frac{1}{\sqrt{g^{pq} B_p B_q}}$, where B_i is a covariant vector, show A_i is unit vector.
21. If A and B are orthogonal vectors of length l, prove that

$$\left(g_{hj} g_{ik} - g_{hk} g_{ij}\right) A^h A^j B^i B^k = -l^4.$$

22. Prove that the magnitude of two associated vectors is equal. Prove also that the relation of a vector and its associated vector is reciprocal.
23. (a) Prove that the angle between the two non-null vectors is invariant.
    (b) Show that the angle between two contravariant vectors is real when the Riemannian metric is positive definite.
    (c) Show that is a Cartesian co-ordinate system, the contravariant and the covariant components of a vector are identical.
24. Show that the cosines of the angles which the three-dimensional unit vector U^i make with the co-ordinate curves are given by

$$U_1/\sqrt{g_{11}}, \; U_2/\sqrt{g_{22}}, \; U_3/\sqrt{g_{33}}.$$

25. Show that $\left(x^1, x^2, x^3\right)$ defined by the transformations

$$T : x^1 = u^1 \cos u^2; \; x^2 = u^1 \sin u^2; \; x^3 = u^3,$$

where x^1, x^2, x^3 are rectangular Cartesian co-ordinates are curvilinear co-ordinates and also find the co-ordinate curves and surfaces.

26. If λ^i is a unit vector, show that the cosines of the angles which its direction makes with the co-ordinates curves are

$$\frac{\lambda^1}{\sqrt{g_{11}}}; \; \frac{\lambda^2}{\sqrt{g_{22}}}; \; \frac{\lambda^3}{\sqrt{g_{33}}}.$$

27. Find a necessary and sufficient condition for two contravariant vectors u^i and v^i defined in a Riemannian space, to be orthogonal.

28. If θ be the angle between two non-null vectors A^i and B^i at a point, prove that

$$\sin^2\theta = \frac{(g_{ij}g_{pq} - g_{ip}g_{jq})\,A^iB^pA^jB^q}{(g_{ij}A^iA^j)\,(g_{pq}B^pB^q)}$$

Hence, show that, if A^i and B^i are orthogonal unit vectors then

$$(g_{ij}g_{pq} - g_{ip}g_{jp})\,A^iB^pA^jB^q = 1.$$

29. If a_{ij} are components of a symmetric covariant tensor and u^i, v^i are unit vectors orthogonal to w and

$$(a_{ij} - \alpha g_{ij})u^i + \lambda w_j = 0$$
$$(a_{ij} - \beta g_{ij})v^i + P\lambda w_j = 0;\ \text{where } \alpha \neq \beta$$

then prove that u^i and v^i are orthogonal and $a_{ij}u^iv^j = 0$.

30. If A^i and B^i are two unit vectors. Prove that they are inclined at a constant angle iff $A^i_{,k}B_i + B^i_{,k}A_i = 0$.
31. Define the angle between two vectors at a point in a Riemannian space. Show that it is an invariant under a co-ordinate transformation.
32. Define the magnitude of any covariant vector in a Riemannian space. Prove that the square of the magnitude of a covariant vector is the scalar product of the vector and its associative contravariant vector.
33. Find the form of the line element ds^2 of V_N when its co-ordinates hypersurfaces form an N ply orthogonal system.
34. (a) Show that an arbitrary V_N does not admit an N ply orthogonal system of hypersurfaces.
    (b) Prove that the magnitude of any vector u is zero if the projections of u on $\lambda_{h|}$ are all zero.
35. (a) If g^{ij} is the conjugate metric tensor of Euclidean space E^3 in curvilinear co-ordinates x^i, and y^i are rectangular Cartesian co-ordinates, show that $\frac{\partial x^p}{\partial y^i} = g^{pq}\frac{\partial y^i}{\partial x^q}$.
    (b) If g^{ij} is the conjugate metric tensor of Euclidean space E^3 in curvilinear co-ordinates x^i, and y^i are rectangular Cartesian co-ordinates, show that $g^{ij} = \frac{\partial x^i}{\partial y^r}\frac{\partial x^j}{\partial y^r}$.
36. If $A = A_i\mathbf{a}^i$ show that $\frac{\partial A}{\partial x^j} = A_{i,j}\mathbf{a}^i$.
37. Prove that

$$\left(\mathbf{a}^i \times \mathbf{a}_j\right)\cdot \mathbf{a}_k = g^{ip}e_{pjk},$$

where e_{ijk} has its usual meaning.

38. If $\mathbf{a}_i$ and $\mathbf{a}^i$ are base and reciprocal base vectors of a curvilinear co-ordinate system, show that
$$\mathbf{a}_1 = \frac{\mathbf{a}^2 \times \mathbf{a}^3}{[\mathbf{a}^1\mathbf{a}^2\mathbf{a}^3]}, \mathbf{a}_2 = \frac{\mathbf{a}^3 \times \mathbf{a}^1}{[\mathbf{a}^1\mathbf{a}^2\mathbf{a}^3]} \text{ and } \mathbf{a}_3 = \frac{\mathbf{a}^1 \times \mathbf{a}^2}{[\mathbf{a}^1\mathbf{a}^2\mathbf{a}^3]}.$$

39. If y^i are rectangular Cartesian co-ordinates, show that in E_3,
$$[\alpha\beta, \gamma] = \frac{\partial^2 y^i}{\partial x^\alpha \partial x^\beta}\frac{\partial y^i}{\partial x^\gamma} \text{ and } \left\{ \begin{matrix} \gamma \\ \alpha \;\; \beta \end{matrix} \right\} = \frac{\partial^2 y^i}{\partial x^\alpha \partial x^\beta}\frac{\partial x^\gamma}{\partial y^i}.$$

40. Prove that the area of the parallelogram constructed on the base vectors $\mathbf{a}_3$ and $\mathbf{a}_1$ is $\sqrt{gg^{22}}$ and on the base vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ is $\sqrt{gg^{33}}$.
41. Find the physical components of the vector with components A_i in (i) spherical polar co-ordinates (ii) in cylindrical polar co-ordinates.

Chapter 3

Christoffel's Symbols and Covariant Differentiation

We now consider two expressions due to Elwin Bruno Christoffel involving the derivatives of the components of the fundamental tensors g_{ij} and g^{ij}. In fact, the operation of partial differentiation on a tensor does not always produce a tensor. A new operation of differentiation may be introduced with the help of two functions formed in terms of the partial derivatives of the components of the fundamental tensor.

3.1 Christoffel Symbols

Here, we consider two expressions due to Christoffel involving of the components of g_{ij}, which will prove useful in the development of the calculus of tensors. The Christoffel symbol of first kind, (N^3 functions) denoted by $[ij, k]$, is defined as

$$[ij,k] = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right); \; i,j,k = 1,2,\ldots,N \tag{3.1}$$

which is also called the *Christoffel 3 index symbols of the first kind.* The *Christoffel 3 index symbols of the second kind*, denoted by $\left\{ \begin{matrix} k \\ i \;\; j \end{matrix} \right\}$ or Γ^k_{ij} is defined as

$$\left\{ \begin{matrix} k \\ i \;\; j \end{matrix} \right\} = g^{km}[ij,m] = \frac{1}{2}g^{km}\left(\frac{\partial g_{im}}{\partial x^j} + \frac{\partial g_{jm}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^m}\right), \tag{3.2}$$

where g^{ij} is the reciprocal tensor for the fundamental metric tensor g_{ij}. Either kind of these symbols in V_N is a set of functions of co-ordinates x^i in a given co-ordinate system (x^i). Note that, in the Christoffel symbol, the contraction always takes place at the third index. For example,

$$g^{mj}[ij,k] \neq \left\{ \begin{matrix} m \\ i \;\; k \end{matrix} \right\}.$$

In a V_N, it is always possible to choose a co-ordinate system such that all the Christoffel symbols vanish at a particular point P_0. Such a co-ordinate system is called a geodesic co-ordinate system with the point P_0 as pole.

There are N distinct Christoffel symbols of each kind for each independent g_{ij}. Since g_{ij} is symmetric tensor of rank 2 and has $\frac{1}{2}N(N+1)$ independent components, so, the number of independent components of Christoffel's symbols are

$$N \cdot \frac{1}{2}N(N+1) = \frac{1}{2}N^2(N+1).$$

EXAMPLE 3.1.1 *Find the Christoffel symbols of the second kind for the V_2 with line element*

$$ds^2 = a^2\left(dx^1\right)^2 + a^2 \sin^2 x^1 \left(dx^2\right)^2,$$

where a is a constant.

Solution: Comparing the given metric with Eq. (2.1), we get

$$g_{11} = a^2, g_{22} = a^2 \sin^2 x^1, g_{12} = 0 = g_{21}.$$

$$\Rightarrow g = \begin{vmatrix} a^2 & 0 \\ 0 & a^2 \sin^2 x^1 \end{vmatrix} = a^4 \sin^2 x^1.$$

The reciprocal tensors g^{ij} for the tensor g_{ij} are given by

$$g^{11} = \frac{\text{cofactor of } g_{11} \text{ in } g}{g} = \frac{a^2 \sin^2 x^1}{a^4 \sin^2 x^1} = \frac{1}{a^2}$$

$$g^{22} = \frac{\text{cofactor of } g_{22} \text{ in } g}{g} = \frac{a^2}{a^4 \sin^2 x^1} = \frac{1}{a^2 \sin^2 x^1}$$

$$g^{12} = \frac{\text{cofactor of } g_{12} \text{ in } g}{g} = 0 = g^{21}.$$

Now, the Christoffel symbols of first kind are

$$[11,1] = \frac{1}{2}\left(\frac{\partial g_{11}}{\partial x^1} + \frac{\partial g_{11}}{\partial x^1} - \frac{\partial g_{11}}{\partial x^1}\right) = \frac{1}{2}\frac{\partial g_{11}}{\partial x^1} = 0$$

$$[11,2] = \frac{1}{2}\left(\frac{\partial g_{12}}{\partial x^1} + \frac{\partial g_{12}}{\partial x^1} - \frac{\partial g_{11}}{\partial x^2}\right) = -\frac{1}{2}\frac{\partial g_{11}}{\partial x^2} = 0$$

$$[12,1] = \frac{1}{2}\left(\frac{\partial g_{21}}{\partial x^1} + \frac{\partial g_{11}}{\partial x^2} - \frac{\partial g_{21}}{\partial x^1}\right) = \frac{1}{2}\frac{\partial g_{11}}{\partial x^2} = 0 = [21,1]$$

$$[12,2] = \frac{1}{2}\left(\frac{\partial g_{12}}{\partial x^2} + \frac{\partial g_{22}}{\partial x^1} - \frac{\partial g_{12}}{\partial x^2}\right) = \frac{1}{2}\frac{\partial g_{22}}{\partial x^1} = a^2 \sin x^1 \cos x^1$$

$$[22,1] = \frac{1}{2}\left(\frac{\partial g_{21}}{\partial x^2} + \frac{\partial g_{21}}{\partial x^2} - \frac{\partial g_{22}}{\partial x^1}\right) = -\frac{1}{2}\frac{\partial g_{22}}{\partial x^1} = -a^2 \sin x^1 \cos x^1$$

$$[22,2] = \frac{1}{2}\left(\frac{\partial g_{22}}{\partial x^2} + \frac{\partial g_{22}}{\partial x^2} - \frac{\partial g_{22}}{\partial x^2}\right) = \frac{1}{2}\frac{\partial g_{22}}{\partial x^2} = 0.$$

The Christoffel symbols of second kind are

$$\left\{ {1 \atop 1\ 1} \right\} = g^{1k}[11,k] = g^{11}[11,1] + g^{12}[11,2] = \frac{1}{a^2} \times 0 + 0 \times 0 = 0.$$

$$\left\{ {1 \atop 1\ 2} \right\} = g^{1k}[12,k] = g^{11}[12,1] + g^{12}[12,2]$$

$$= \frac{1}{a^2} \times 0 + 0 \times a^2 \sin x^1 \cos x^1 = 0 = \left\{ {1 \atop 2\ 1} \right\}.$$

$$\left\{ {1 \atop 2\ 2} \right\} = g^{1k}[22,k] = g^{11}[22,1] + g^{12}[22,2]$$

$$= \frac{1}{a^2} \times \left(-a^2 \sin x^1 \cos^1\right) + 0 \times 0 = -\sin x^1 \cos^1.$$

$$\left\{ {2 \atop 1\ 1} \right\} = g^{2k}[11,k] = g^{21}[11,1] + g^{22}[11,2]$$

$$= 0 \times 0 + \frac{1}{a^2 \sin^2 x^1} \times 0 = 0.$$

$$\left\{ {2 \atop 1\ 2} \right\} = g^{2k}[12,k] = g^{21}[12,1] + g^{22}[12,2]$$

$$= 0 \times 0 + \frac{1}{a^2 \sin^2 x^1} \times a^2 \sin x^1 \cos x^1 = \cot x^1 = \left\{ {2 \atop 2\ 1} \right\}.$$

$$\left\{ {2 \atop 2\ 2} \right\} = g^{2k}[22,k] = g^{21}[22,1] + g^{22}[22,2]$$

$$= 0 \times \left(-a^2 \sin x^1 \cos^1\right) + \frac{1}{a^2 \sin^2 x^1} \times 0 = 0.$$

EXAMPLE 3.1.2 *Calculate the Christoffel symbols* $\left\{ {k \atop i\ j} \right\}$ *corresponding to the metric* $ds^2 = du^2 + f^2 dv^2$, *where* f *is a function of* u *and* v.

Solution: Comparing the given metric with Eq. (2.1), we get,

$$g_{11} = 1, g_{22} = f^2, g_{12} = 0 = g_{21} \Rightarrow g = \begin{vmatrix} 1 & 0 \\ 0 & f^2 \end{vmatrix} = f^2.$$

The reciprocal tensors g^{ij} for the tensor g_{ij} are given by

$$g^{11} = 1;\ g^{22} = \frac{1}{f^2};\ g^{12} = 0 = g^{21}.$$

Now, the Christoffel symbols of first kind are

$$[11,1] = \frac{1}{2}\frac{\partial g_{11}}{\partial u} = 0;\ [11,2] = -\frac{1}{2}\frac{\partial g_{11}}{\partial v} = 0$$
$$[12,1] = \frac{1}{2}\frac{\partial g_{11}}{\partial v} = 0 = [21,1];\ [12,2] = \frac{1}{2}\frac{\partial g_{22}}{\partial u} = f\frac{\partial f}{\partial u} = [21,2]$$
$$[22,1] = -\frac{1}{2}\frac{\partial g_{22}}{\partial u} = f\frac{\partial f}{\partial u};\ [22,2] = \frac{1}{2}\frac{\partial g_{22}}{\partial v} = f\frac{\partial f}{\partial v}.$$

The Christoffel symbols of second kind are

$$\begin{Bmatrix} 1 \\ 1\ 1 \end{Bmatrix} = g^{1k}[11,k] = g^{11}[11,1] + g^{12}[11,2] = 0.$$
$$\begin{Bmatrix} 1 \\ 1\ 2 \end{Bmatrix} = g^{1k}[12,k] = g^{11}[12,1] + g^{12}[12,2] = 0 = \begin{Bmatrix} 1 \\ 2\ 1 \end{Bmatrix}.$$
$$\begin{Bmatrix} 1 \\ 2\ 2 \end{Bmatrix} = g^{1k}[22,k] = g^{11}[22,1] + g^{12}[22,2] = -f\frac{\partial f}{\partial u}.$$
$$\begin{Bmatrix} 2 \\ 1\ 1 \end{Bmatrix} = g^{2k}[11,k] = g^{21}[11,1] + g^{22}[11,2] = 0.$$
$$\begin{Bmatrix} 2 \\ 1\ 2 \end{Bmatrix} = g^{2k}[12,k] = g^{21}[12,1] + g^{22}[12,2] = \frac{1}{f}\frac{\partial f}{\partial u} = \begin{Bmatrix} 2 \\ 2\ 1 \end{Bmatrix}.$$
$$\begin{Bmatrix} 2 \\ 2\ 2 \end{Bmatrix} = g^{2k}[22,k] = g^{21}[22,1] + g^{22}[22,2] = \frac{1}{f}\frac{\partial f}{\partial u}.$$

3.1.1 Properties of the Christoffel Symbols

In this section we proceed to deduce several properties and identities involving Christoffel symbols, which will prove useful to us in the sequel.

Property 3.1.1 The Christoffel symbols of First and Second kind defined in Eqs. (3.1) and (3.2) are symmetric with respect to the indices i and j.

Proof: In the definition of Christoffel symbol (3.1) of first kind, interchanging of i and j, provides

$$\begin{aligned}[][ji,k] &= \frac{1}{2}\left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ji}}{\partial x^k}\right) \\ &= \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right); \text{ as } g_{ij} \text{ is symmetric} \\ &= [ij,k].\end{aligned}$$

Thus, $[ij,k] = [ji,k]$, shows that Christoffel symbol of first kind defined by Eq. (3.1) is symmetric with respect to the indices i and j. In the definition of Christoffel symbol of second kind (3.2), interchanging of i and j, we get.

$$\begin{Bmatrix} k \\ j \;\; i \end{Bmatrix} = g^{km}[ji,m] = g^{km}[ij,m] = \begin{Bmatrix} k \\ i \;\; j \end{Bmatrix}.$$

Therefore, the 3 index Christoffel symbol of second kind defined in Eq. (3.2) is symmetric with respect to i and j.

Property 3.1.2 The necessary and sufficient condition that all the Christoffel symbols vanish at a point is that g_{ij} are constants.

Proof: Let g_{ij} be constants, at a point $P\left(x^i\right)$, then

$$\frac{\partial g_{ij}}{\partial x^k} = 0, \frac{\partial g_{ik}}{\partial x^j} = 0 \quad \text{and} \quad \frac{\partial g_{jk}}{\partial x^i} = 0.$$

Using definition (3.1) of Christoffel symbol of first kind, we get

$$[ij,k] = \frac{1}{2}\left[\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right] = 0,$$

it follows that they will be all zero at the point P. From the definition (3.2) of Christoffel symbol of second kind, we get

$$\begin{Bmatrix} k \\ i \;\; j \end{Bmatrix} = g^{km}[ij,m] = 0; \text{ as } [ij,m] = 0,$$

at P. Therefore, the condition is necessary. Conversely, if the Christoffel symbols vanish at a point, we have

$$\frac{1}{2}\left[\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right] = 0.$$

Since the co-ordinates x^i are independent, the above relation holds, if

$$\frac{\partial g_{ij}}{\partial x^k} = 0, \frac{\partial g_{ik}}{\partial x^j} = 0 \quad \text{and} \quad \frac{\partial g_{jk}}{\partial x^i} = 0.$$

This means that, g_{pq} is independent of x^i for all i, i.e. g_{ij} are constants at the point $\left(x^i\right)$. Therefore, the condition is sufficient. Thus, in any particular co-ordinate system, the Christoffel symbols uniformly vanish if and only if the metric tensor has constant components in that system.

EXAMPLE 3.1.3 *Is it true that if all $[ij,k]$ vanish in any co-ordinate system, then the metric tensor has constant components in every co-ordinate system?*

Solution: By Property 3.1.2, the conclusion would be valid if the $[ij,k]$ vanished in every co-ordinate system. But $[ij,k]$ is not a tensor, and the conclusion is false. For instance, all $[ij,k] = 0$ for the Euclidean metric in rectangular co-ordinates, but $g_{22} = (x^1)^2$ in spherical co-ordinates.

Property 3.1.3 To establish $[ij,m] = g_{km}\begin{Bmatrix} k \\ i\ \ j \end{Bmatrix}$.

Proof: We see from the defining formula (3.2) that we can pass from the symbol of the first kind $[ij,m]$ to the symbol $\begin{Bmatrix} k \\ i\ \ j \end{Bmatrix}$ by forming the inner product $g^{km}[ij,m]$. Therefore, the inner multiplication of $\begin{Bmatrix} k \\ i\ \ j \end{Bmatrix}$ with g_{km} gives

$$\begin{aligned} g_{km}\begin{Bmatrix} k \\ i\ \ j \end{Bmatrix} &= g_{km}g^{kp}[ij,p]; \text{ as } \begin{Bmatrix} k \\ i\ \ j \end{Bmatrix} = g^{kp}[ij,p] \\ &= \delta^p_m[ij,p]; \text{ as } g_{km}g^{kp} = \delta^p_m \\ &= [ij,m]. \end{aligned}$$

This is the relation between the two symbols. The formulas

$$\begin{Bmatrix} k \\ i\ \ j \end{Bmatrix} = g^{km}[ij,m] \text{ and } [ij,m] = g_{km}\begin{Bmatrix} k \\ i\ \ j \end{Bmatrix}$$

are easy to remember if it is noted that the operation of inner multiplication of $[ij,m]$ with g^{km} raises the index and replaces the square brackets by the braces. The multiplication of $\begin{Bmatrix} k \\ i\ \ j \end{Bmatrix}$ by g_{km}, on the other hand, lowers the index and replaces the braces by the square brackets.

Property 3.1.4 To establish $[ij,k] + [jk,i] = \frac{\partial g_{ik}}{\partial x^j}$.

Proof: Using definition (3.1) of Christoffel symbol of first kind, we get

$$[ij,k]+[jk,i]=\frac{1}{2}\left(\frac{\partial g_{jk}}{\partial x^i}+\frac{\partial g_{ik}}{\partial x^j}-\frac{\partial g_{ij}}{\partial x^k}\right)+\frac{1}{2}\left(\frac{\partial g_{ki}}{\partial x^j}+\frac{\partial g_{ji}}{\partial x^k}-\frac{\partial g_{jk}}{\partial x^i}\right)$$

$$=\frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^j}+\frac{\partial g_{ki}}{\partial x^j}\right)=\frac{\partial g_{ki}}{\partial x^j}=\frac{\partial g_{ik}}{\partial x^j}=\partial_j g_{ik}.$$

This is an expression for the partial derivative of the fundamental tensor g_{ij} in terms of the symbols of first kind. In the similar manner, we get

$$[jk,i]-[ij,k]=\frac{\partial g_{ij}}{\partial x^k}-\frac{\partial g_{jk}}{\partial x^i}.$$

Property 3.1.5 To establish $\partial_j g_{ik}=\frac{\partial g^{ik}}{\partial x^j}=-g^{hk}\left\{\begin{matrix} i \\ hj\end{matrix}\right\}-g^{hi}\left\{\begin{matrix} k \\ hj\end{matrix}\right\}$.

Proof: The formula for the partial derivatives of the contravariant tensor g^{ik} can be obtained by differentiating the identity $g_{pm}g^{mi}=\delta^i_p$, where δ^i_p is the Kronecker delta, with respect to x^j, we get

$$\frac{\partial}{\partial x^j}\left(g_{pm}g^{mi}\right)=0\Rightarrow\frac{\partial\left(g_{pm}\right)}{\partial x^j}g^{mi}+g_{pm}\frac{\partial\left(g^{mi}\right)}{\partial x^j}=0$$

or

$$g^{mi}\frac{\partial g_{pm}}{\partial x^j}=-g_{pm}\frac{\partial g^{mi}}{\partial x^j}.$$

To solve this system of equations for $\frac{\partial g^{mi}}{\partial x^j}$, we multiply both sides by g^{pk}, we get

$$g^{mi}g^{pk}\frac{\partial g_{pm}}{\partial x^j}=-g^{pk}g_{pm}\frac{\partial g^{mi}}{\partial x^j}$$

or

$$g^{mi}g^{pk}[pj,m]+g^{mi}g^{pk}[jm,p]=-g^{pk}g_{pm}\frac{\partial g^{mi}}{\partial x^j}$$

$$\left(\text{since } \frac{\partial g_{pm}}{\partial x^j}=[pj,m]+[jm,p]\right)$$

or

$$g^{pk}\left\{\begin{matrix} i \\ p\ \ j\end{matrix}\right\}+g^{mi}\left\{\begin{matrix} k \\ m\ \ j\end{matrix}\right\}=-\delta^k_m\frac{\partial g^{mi}}{\partial x^j};\ \text{as } g^{pk}g_{pm}=\delta^k_m$$

or

$$g^{pk}\left\{\begin{matrix} i \\ p\ \ j\end{matrix}\right\}+g^{mi}\left\{\begin{matrix} k \\ m\ \ j\end{matrix}\right\}=-\frac{\partial g^{ik}}{\partial x^j}$$

or

$$\frac{\partial g^{ik}}{\partial x^j}=-g^{hk}\left\{\begin{matrix} i \\ h\ \ j\end{matrix}\right\}-g^{hi}\left\{\begin{matrix} k \\ h\ \ j\end{matrix}\right\},$$

where we have to replace dummy indices p and m by h.

Property 3.1.6 To establish $\left\{ {i \atop i\ j} \right\} = \left\{ {i \atop j\ i} \right\} = \frac{\partial}{\partial x^j}\left(\log \sqrt{g}\right)$, where $g = |g_{ij}| \neq 0$.

Proof: According to the definition of reciprocal tensor, we have

$$g^{ij} = \frac{\text{cofactor of } g_{ij} \text{ in } |g_{ij}|}{|g_{ij}|} = \frac{G^{ij}}{g},$$

where G^{ij} denotes the cofactor of g_{ij} in $g = |g_{ij}|$. Since, $g = |g_{ij}|$, we get

$$g = \begin{vmatrix} g_{11} & g_{12} & \cdots & g_{1N} \\ g_{21} & g_{22} & \cdots & g_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ g_{N1} & g_{N2} & \cdots & g_{NN} \end{vmatrix}.$$

The derivative of a determinant is obtained by differentiating each row of it separately and keeping the other rows the same. Therefore, differentiating $g = |g_{ij}|$ with respect to x^j, we have

$$\frac{\partial g}{\partial x^j} = \begin{vmatrix} \frac{\partial g_{11}}{\partial x^j} & \frac{\partial g_{12}}{\partial x^j} & \cdots & \frac{\partial g_{1N}}{\partial x^j} \\ g_{21} & g_{22} & \cdots & g_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ g_{N1} & g_{N2} & \cdots & g_{NN} \end{vmatrix} + \cdots + \begin{vmatrix} g_{11} & g_{12} & \cdots & g_{1N} \\ g_{21} & g_{22} & \cdots & g_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial g_{N1}}{\partial x^j} & \frac{\partial g_{N2}}{\partial x^j} & \cdots & \frac{\partial g_{NN}}{\partial x^j} \end{vmatrix}$$

Clearly, cofactor of $\frac{\partial g_{11}}{\partial x^j}$ = cofactor of g_{11} in g, so summing the resulting determinants obtained, we get

$$\begin{aligned} \frac{\partial g}{\partial x^j} &= G^{ik}\frac{\partial g_{ik}}{\partial x^j} = gg^{ik}\frac{\partial g_{ik}}{\partial x^j} = gg^{ik}\left([ij,k] + [kj,i]\right) \\ &= gg^{ik}[ij,k] + gg^{ik}[kj,i] = g\left\{ {i \atop i\ j} \right\} + g\left\{ {k \atop k\ j} \right\}. \end{aligned}$$

Replacing the dummy index k by i, we get

$$\frac{\partial g}{\partial x^j} = g\left\{ {i \atop i\ j} \right\} + g\left\{ {i \atop i\ j} \right\} = 2g\left\{ {i \atop i\ j} \right\}$$

or

$$\left\{ {i \atop i\ j} \right\} = \frac{1}{2g}\frac{\partial g}{\partial x^j} = \frac{\partial}{\partial x^j}\left(\log \sqrt{g}\right) = \frac{1}{2}\frac{\partial}{\partial x^j}\log g = \frac{1}{2}\partial_j \log g.$$

The quantities $\left\{ {i \atop i\ j} \right\}$ are sometimes called *contracted Christoffel symbols.* The derived formula for the derivative of the logarithm of the determinant $g = |g_{ij}|$ plays an important role in tensor calculus.

EXAMPLE 3.1.4 *If* $|g_{ij}| \neq 0$, *show that*

$$g_{\alpha\beta}\frac{\partial}{\partial x^j}\left\{\begin{matrix}\beta \\ i\ \ k\end{matrix}\right\} = \frac{\partial}{\partial x^j}[ik,\alpha] - \left\{\begin{matrix}\beta \\ i\ \ k\end{matrix}\right\}([\beta j,\alpha] + [\alpha j,\beta]).$$

Solution: By definition of the Christoffel symbol of second kind, we get $\left\{\begin{matrix}\beta \\ i\ \ k\end{matrix}\right\} = g^{\beta\alpha}[ik,\alpha]$. Multiplying innerly by $g_{\alpha\beta}$, we get

$$g_{\alpha\beta}\left\{\begin{matrix}\beta \\ i\ \ k\end{matrix}\right\} = g_{\alpha\beta}g^{\beta\alpha}[ik,\alpha]$$

or

$$g_{\alpha\beta}\left\{\begin{matrix}\beta \\ i\ \ k\end{matrix}\right\} = [ik,\alpha];\ \text{as}\ g_{\alpha\beta}g^{\beta\alpha} = 1.$$

Differentiating this relation with respect to x^j partially, we get,

$$\frac{\partial}{\partial x^j}\left(g_{\alpha\beta}\left\{\begin{matrix}\beta \\ i\ \ k\end{matrix}\right\}\right) = \frac{\partial}{\partial x^j}[ik,\alpha]$$

or

$$g_{\alpha\beta}\frac{\partial}{\partial x^j}\left\{\begin{matrix}\beta \\ i\ \ k\end{matrix}\right\} + \left\{\begin{matrix}\beta \\ i\ \ k\end{matrix}\right\}\frac{\partial g_{\alpha\beta}}{\partial x^j} = \frac{\partial}{\partial x^j}[ik,\alpha]$$

or

$$g_{\alpha\beta}\frac{\partial}{\partial x^j}\left\{\begin{matrix}\beta \\ i\ \ k\end{matrix}\right\} + \left\{\begin{matrix}\beta \\ i\ \ k\end{matrix}\right\}([\beta j,\alpha] + [\alpha j,\beta]) = \frac{\partial}{\partial x^j}[ik,\alpha],$$

as

$$\frac{\partial g_{\alpha\beta}}{\partial x^j} = [\beta j,\alpha] + [\alpha j,\beta]$$

or

$$g_{\alpha\beta}\frac{\partial}{\partial x^j}\left\{\begin{matrix}\beta \\ i\ \ k\end{matrix}\right\} = \frac{\partial}{\partial x^j}[ik,\alpha] - \left\{\begin{matrix}\beta \\ i\ \ k\end{matrix}\right\}([\beta j,\alpha] + [\alpha j,\beta]).$$

EXAMPLE 3.1.5 *Evaluate the Christoffel symbols of both kinds for spaces, where,*

$$g_{ij} = 0,\ if\ i \neq j.$$

Solution: The definition of Christoffel symbols of first kind is given by Eq. (3.1). We consider the following four cases:

Case 1: Let $i = j = k$, then Eq. (3.1) becomes

$$[ij,k] = [ii,i] = \frac{1}{2}\left(\frac{\partial g_{ii}}{\partial x^i} + \frac{\partial g_{ii}}{\partial x^i} - \frac{\partial g_{ii}}{\partial x^i}\right) = \frac{1}{2}\frac{\partial g_{ii}}{\partial x^i}.$$

Case 2: Let $i = k \neq j$, then, we get from above definition

$$[ij, k] = [ij, i] = \frac{1}{2}\left(\frac{\partial g_{ji}}{\partial x^i} + \frac{\partial g_{ii}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^i}\right) = \frac{1}{2}\frac{\partial g_{ii}}{\partial x^j}.$$

Case 3: Let $i = j \neq k$, then, we get from above definition

$$[ij, k] = [ii, k] = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^i} - \frac{\partial g_{ii}}{\partial x^k}\right) = -\frac{1}{2}\frac{\partial g_{ii}}{\partial x^k};\ \text{as } g_{ik} = 0 \text{ for } i \neq k.$$

Case 4: If $i \neq j \neq k$ and noting $g_{ij} = 0$ if $i \neq j$, we get $[ij, k] = 0$.
In the above such four results, no summation is applied. Now, we know, if co-ordinate system is orthogonal, then $g^{ij} = \frac{1}{g_{ij}}$, where no summation is used. The definition of Christoffel symbol of second kind is given by Eq. (3.2). Clearly if $k \neq h$,

$$\begin{Bmatrix} k \\ i\ \ j \end{Bmatrix} = 0;\ \text{as } g^{kh} = 0,$$

and if $k = h$,

$$\begin{Bmatrix} k \\ i\ \ j \end{Bmatrix} = g^{kk}[ij, k] = \frac{1}{g_{kk}}[ij, k].$$

Using the above cases, we get the following four subcases for the Christoffel symbol of second kind:

Case 1: If $i = j = k$, then we have

$$\begin{Bmatrix} i \\ i\ \ i \end{Bmatrix} = \frac{1}{g_{ii}}[ii, i] = \frac{1}{2g_{ii}}\frac{\partial g_{ii}}{\partial x^i} = \frac{\partial}{\partial x^i}\left(\log\sqrt{g_{ii}}\right).$$

Case 2: If $i = k \neq j$, then we have

$$\begin{Bmatrix} i \\ i\ \ j \end{Bmatrix} = \frac{1}{g_{ii}}[ij, i] = \frac{1}{2g_{ii}}\frac{\partial g_{ii}}{\partial x^j} = \frac{\partial}{\partial x^j}\left(\log\sqrt{g_{ii}}\right).$$

Case 3: If $i = j \neq k$, then we have

$$\begin{Bmatrix} k \\ i\ \ i \end{Bmatrix} = \frac{1}{g_{kk}}[ii, k] = -\frac{1}{2g_{kk}}\frac{\partial g_{ii}}{\partial x^k}.$$

Case 4: If $i \neq j \neq k$, then we have $\begin{Bmatrix} k \\ i\ \ j \end{Bmatrix} = 0.$

EXAMPLE 3.1.6 *Calculate the non-vanishing Christoffel symbols corresponding to the metric*

$$ds^2 = -(dx^1)^2 - (dx^2)^2 - (dx^3)^2 + f(x^1, x^2, x^3)(dx^4)^2.$$

Solution: Comparing the given metric with Eq. (2.1), we get

$$g_{11} = -1 = g_{22} = g_{33}; g_{44} = f(x^1, x^2, x^3);\; g_{ij} = 0;\; \text{for } i \neq j.$$

$$\Rightarrow g = \begin{vmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & f \end{vmatrix} = -f.$$

For orthogonal co-ordinates $g^{ij} = \frac{1}{g^{ij}}$ (no summation). So, the reciprocal tensors g^{ij} for g_{ij} are given by

$$g^{11} = -1 = g^{22} = g^{33}; g^{44} = \frac{1}{f}; g^{ij} = 0;\; \text{for } i \neq j.$$

Since g^{11}, g^{22}, g^{33} are constants, from Example 3.1.5, the non-vanishing Christoffel symbols of the two kinds are

$$[14, 4] = [41, 4] = \frac{1}{2}\frac{\partial g_{44}}{\partial x^1} = \frac{1}{2}\frac{\partial f}{\partial x^1}; \left\{ \begin{matrix} 4 \\ 1\;\; 4 \end{matrix} \right\} = \left\{ \begin{matrix} 4 \\ 4\;\; 1 \end{matrix} \right\} = \frac{\partial}{\partial x^1} \log \sqrt{f}.$$

$$[24, 4] = [42, 4] = \frac{1}{2}\frac{\partial g_{44}}{\partial x^2} = \frac{1}{2}\frac{\partial f}{\partial x^2}; \left\{ \begin{matrix} 4 \\ 2\;\; 4 \end{matrix} \right\} = \left\{ \begin{matrix} 4 \\ 4\;\; 2 \end{matrix} \right\} = \frac{\partial}{\partial x^2} \log \sqrt{f}.$$

$$[34, 4] = [43, 4] = \frac{1}{2}\frac{\partial g_{44}}{\partial x^3} = \frac{1}{2}\frac{\partial f}{\partial x^1}; \left\{ \begin{matrix} 4 \\ 3\;\; 4 \end{matrix} \right\} = \left\{ \begin{matrix} 4 \\ 4\;\; 3 \end{matrix} \right\} = \frac{\partial}{\partial x^3} \log \sqrt{f}.$$

$$[44, 1] = -\frac{1}{2}\frac{\partial g_{44}}{\partial x^1} = -\frac{1}{2}\frac{\partial f}{\partial x^1}; \left\{ \begin{matrix} 1 \\ 4\;\; 4 \end{matrix} \right\} = -\frac{1}{2g_{11}}\frac{\partial g_{44}}{\partial x^1} = \frac{1}{2}\frac{\partial f}{\partial x^1}.$$

$$[44, 2] = -\frac{1}{2}\frac{\partial g_{44}}{\partial x^2} = -\frac{1}{2}\frac{\partial f}{\partial x^1}; \left\{ \begin{matrix} 2 \\ 4\;\; 4 \end{matrix} \right\} = -\frac{1}{2g_{11}}\frac{\partial g_{44}}{\partial x^2} = \frac{1}{2}\frac{\partial f}{\partial x^2}.$$

$$[44, 3] = -\frac{1}{2}\frac{\partial g_{44}}{\partial x^3} = -\frac{1}{2}\frac{\partial f}{\partial x^3}; \left\{ \begin{matrix} 3 \\ 4\;\; 4 \end{matrix} \right\} = -\frac{1}{2g_{11}}\frac{\partial g_{44}}{\partial x^3} = \frac{1}{2}\frac{\partial f}{\partial x^3}.$$

EXAMPLE 3.1.7 *Calculate the non-vanishing Christoffel symbols of first and second kind (in cylindrical co-ordinates) for the V_3 corresponding to the line element*

$$ds^2 = \left(dx^1\right)^2 + \left(x^1\right)^2 \left(dx^2\right)^2 + \left(dx^3\right)^2.$$

Solution: Comparing the given metric with Eq. (2.1), we get

$$g_{11} = 1,\; g_{22} = \left(x^1\right)^2,\; g_{33} = 1,\; g_{ij} = 0 \text{ for } i \neq j.$$

$$\Rightarrow g = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \left(x^1\right)^2 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \left(x^1\right)^2.$$

The reciprocal tensors g^{ij} for the metric tensor g_{ij} are given by

$$g^{11} = \frac{\text{cofactor of } g_{11} \text{ in } g}{g} = \frac{\left(x^1\right)^2}{\left(x^1\right)^2} = 1;\; g^{22} = \frac{\text{cofactor of } g_{22} \text{ in } g}{g} = \frac{1}{\left(x^1\right)^2};$$

$$g^{33} = \frac{\text{cofactor of } g_{33} \text{ in } g}{g} = \frac{\left(x^1\right)^2}{\left(x^1\right)^2} = 1;\; g^{12} = \frac{\text{cofactor of } g_{12} \text{ in } g}{g} = 0 = g^{21};$$

and

$$g^{13} = 0 = g^{31};\; g^{23} = 0 = g^{32}.$$

Since $g_{ij} = 0$ for $i \neq j$, according to Theorem 3.1.5, we get

$$[ij, k] = 0 \text{ and } \left\{ \begin{matrix} i \\ j \;\; k \end{matrix} \right\} = 0;\; \text{for } i \neq j \neq k$$

$$\Rightarrow [12, 3] = [21, 3] = 0;\; [23, 1] = [32, 1] = 0;\; [13, 2] = [31, 2] = 0;$$

$$\text{and } \left\{ \begin{matrix} 1 \\ 2 \;\; 3 \end{matrix} \right\} = 0; \left\{ \begin{matrix} 2 \\ 1 \;\; 3 \end{matrix} \right\} = 0; \left\{ \begin{matrix} 3 \\ 1 \;\; 2 \end{matrix} \right\} = 0.$$

For $i = j = k$, the Christoffel symbols $[ii, i]$ of first kind are given by

$$[11, 1] = \frac{1}{2}\left(\frac{\partial g_{11}}{\partial x^1} + \frac{\partial g_{11}}{\partial x^1} - \frac{\partial g_{11}}{\partial x^1}\right) = \frac{1}{2}\frac{\partial g_{11}}{\partial x^1} = 0.$$

$$[22, 2] = \frac{1}{2}\left(\frac{\partial g_{22}}{\partial x^2} + \frac{\partial g_{22}}{\partial x^2} - \frac{\partial g_{22}}{\partial x^2}\right) = \frac{1}{2}\frac{\partial g_{22}}{\partial x^2} = 0.$$

$$[33, 3] = \frac{1}{2}\left(\frac{\partial g_{33}}{\partial x^3} + \frac{\partial g_{33}}{\partial x^3} - \frac{\partial g_{33}}{\partial x^3}\right) = \frac{1}{2}\frac{\partial g_{33}}{\partial x^3} = 0.$$

For $i = k \neq j$, the Christoffel symbols $[ij, i]$ of first kind are given by

$$[12, 1] = \frac{1}{2}\left(\frac{\partial g_{21}}{\partial x^1} + \frac{\partial g_{11}}{\partial x^2} - \frac{\partial g_{21}}{\partial x^1}\right) = \frac{1}{2}\frac{\partial g_{11}}{\partial x^2} = 0 = [21, 1].$$

$$[13, 1] = \frac{1}{2}\left(\frac{\partial g_{11}}{\partial x^3} + \frac{\partial g_{31}}{\partial x^1} - \frac{\partial g_{13}}{\partial x^1}\right) = \frac{1}{2}\frac{\partial g_{11}}{\partial x^3} = 0 = [31, 1].$$

$$[12, 2] = \frac{1}{2}\left(\frac{\partial g_{12}}{\partial x^2} + \frac{\partial g_{22}}{\partial x^1} - \frac{\partial g_{12}}{\partial x^2}\right) = \frac{1}{2}\frac{\partial g_{22}}{\partial x^1} = 2x^1 = [21, 2].$$

$$[23, 2] = \frac{1}{2}\left(\frac{\partial g_{22}}{\partial x^3} + \frac{\partial g_{32}}{\partial x^2} - \frac{\partial g_{23}}{\partial x^2}\right) = \frac{1}{2}\frac{\partial g_{22}}{\partial x^3} = 0 = [32, 2].$$

$$[13, 3] = \frac{1}{2}\left(\frac{\partial g_{13}}{\partial x^3} + \frac{\partial g_{33}}{\partial x^1} - \frac{\partial g_{13}}{\partial x^3}\right) = \frac{1}{2}\frac{\partial g_{33}}{\partial x^1} = 0 = [31, 3].$$

$$[23, 3] = \frac{1}{2}\left(\frac{\partial g_{23}}{\partial x^3} + \frac{\partial g_{33}}{\partial x^2} - \frac{\partial g_{23}}{\partial x^3}\right) = \frac{1}{2}\frac{\partial g_{33}}{\partial x^2} = 0 = [32, 3].$$

For $i = j \neq k$, the Christoffel symbols $[ii, k]$ of first kind are given by

$$[11,2] = \frac{1}{2}\left(\frac{\partial g_{12}}{\partial x^1} + \frac{\partial g_{12}}{\partial x^1} - \frac{\partial g_{11}}{\partial x^2}\right) = -\frac{1}{2}\frac{\partial g_{11}}{\partial x^2} = 0.$$

$$[11,3] = \frac{1}{2}\left(\frac{\partial g_{13}}{\partial x^1} + \frac{\partial g_{13}}{\partial x^1} - \frac{\partial g_{11}}{\partial x^3}\right) = -\frac{1}{2}\frac{\partial g_{11}}{\partial x^3} = 0.$$

$$[22,1] = \frac{1}{2}\left(\frac{\partial g_{21}}{\partial x^2} + \frac{\partial g_{21}}{\partial x^2} - \frac{\partial g_{22}}{\partial x^1}\right) = -\frac{1}{2}\frac{\partial g_{22}}{\partial x^1} = -x^1.$$

$$[22,3] = \frac{1}{2}\left(\frac{\partial g_{23}}{\partial x^2} + \frac{\partial g_{23}}{\partial x^2} - \frac{\partial g_{22}}{\partial x^3}\right) = -\frac{1}{2}\frac{\partial g_{22}}{\partial x^3} = 0.$$

$$[33,1] = \frac{1}{2}\left(\frac{\partial g_{31}}{\partial x^3} + \frac{\partial g_{31}}{\partial x^3} - \frac{\partial g_{33}}{\partial x^1}\right) = -\frac{1}{2}\frac{\partial g_{33}}{\partial x^1} = 0.$$

$$[33,2] = \frac{1}{2}\left(\frac{\partial g_{32}}{\partial x^3} + \frac{\partial g_{32}}{\partial x^3} - \frac{\partial g_{33}}{\partial x^2}\right) = -\frac{1}{2}\frac{\partial g_{33}}{\partial x^2} = 0.$$

For $i = j = k$, the Christoffel symbols of second kind are given by

$$\begin{aligned}
\left\{ {1 \atop 1\ \ 1} \right\} &= g^{1k}[11,k] = g^{11}[11,1] + g^{12}[11,2] + g^{13}[11,3] \\
&= 1 \times 0 + 0 \times 0 + 0 \times 0 = 0. \\
\left\{ {2 \atop 2\ \ 2} \right\} &= g^{2k}[22,k] = g^{21}[22,1] + g^{22}[22,2] + g^{23}[22,3] \\
&= 0 \times \left(-x^1\right) + \frac{1}{\left(x^1\right)^2} \times 0 + 0 \times 0 = 0. \\
\left\{ {3 \atop 3\ \ 3} \right\} &= g^{3k}[33,k] = g^{31}[33,1] + g^{32}[33,2] + g^{33}[33,3] \\
&= 0 \cdot \left(-x^1 \sin x^2\right) + 0 \cdot \left[-\left(x^1\right)^2 \sin x^2 \cos x^2\right] + \frac{1}{\left(x^1\right)^2 \sin^2 x^2} \cdot 0 = 0.
\end{aligned}$$

For $i = k \neq j$, the Christoffel symbols of second kind are given by

$$\begin{aligned}
\left\{ {1 \atop 1\ \ 2} \right\} &= g^{1k}[12,k] = g^{11}[12,1] + g^{12}[12,2] + g^{13}[12,3] \\
&= 1 \times 0 + 0 \times x^1 + 0 \times 0 = 0 = \left\{ {1 \atop 2\ \ 1} \right\}.
\end{aligned}$$

$$\left\{ {1 \atop 1\ 3} \right\} = g^{1k}[13,k] = g^{11}[13,1] + g^{12}[13,2] + g^{13}[13,3]$$

$$= 1 \times 0 + 0 \times 0 + 0 \times x^1 \sin^2 x^2 = 0 = \left\{ {1 \atop 3\ 1} \right\}.$$

$$\left\{ {2 \atop 1\ 2} \right\} = g^{2k}[12,k] = g^{21}[12,1] + g^{22}[12,2] + g^{23}[12,3]$$

$$= 0 \times 0 + \frac{1}{(x^1)^2} \times x^1 + 0 \times 0 = \frac{1}{(x^1)^2} = \left\{ {2 \atop 2\ 1} \right\}.$$

$$\left\{ {3 \atop 1\ 3} \right\} = g^{3k}[13,k] = g^{31}[13,1] + g^{32}[13,2] + g^{33}[13,3]$$

$$= 0 \times 0 + 0 \times 0 + \frac{1}{(x^1)^2 \sin^2 x^2} \times x^1 \sin^2 x^2 = \frac{1}{x^1} = \left\{ {3 \atop 3\ 1} \right\}.$$

$$\left\{ {3 \atop 2\ 3} \right\} = g^{3k}[23,k] = g^{31}[23,1] + g^{32}[23,2] + g^{33}[23,3]$$

$$= 0 \times 0 + 0 \times 0 + \frac{(x^1)^2 \sin x^2 \cos x^2}{(x^1)^2 \sin^2 x^2} = \cot x^2 = \left\{ {3 \atop 3\ 2} \right\}.$$

$$\left\{ {1 \atop 2\ 2} \right\} = g^{1k}[22,k] = g^{11}[22,1] + g^{12}[22,2] + g^{13}[22,3]$$

$$= 1 \times \left(-x^1\right) + 0 \times 0 + 0 \times 0 = -x^1.$$

$$\left\{ {1 \atop 2\ 3} \right\} = g^{1k}[23,k] = g^{11}[23,1] + g^{12}[23,2] + g^{13}[23,3]$$

$$= 1 \times 0 + 0 \times 0 + 0 \times \left(x^1\right)^2 \sin x^2 = 0 = \left\{ {1 \atop 3\ 2} \right\}.$$

$$\left\{ {1 \atop 3\ 3} \right\} = g^{1k}[33,k] = g^{11}[33,1] + g^{12}[33,2] + g^{13}[33,3]$$

$$= 1 \times \left(-x^1 \sin^2 x^2\right)^2 + 0 \times 0 + 0 \times 0 = -x^1 \sin^2 x^2.$$

$$\left\{ {2 \atop 1\ 1} \right\} = g^{2k}[11,k] = g^{21}[11,1] + g^{22}[11,2] + g^{23}[11,3].$$

$$= 0 \times 0 + \frac{1}{(x^1)^2} \times 0 + 0 \times 0.$$

$$\left\{ {2 \atop 2\ 3} \right\} = g^{2k}[23,k] = g^{21}[23,1] + g^{22}[23,2] + g^{23}[23,3]$$

$$= 0 \times 0 + \frac{1}{(x^1)^2} \times 0 + 0 \times \left(-x^1\right)^2 \sin x^2 = 0 = \left\{ {2 \atop 3\ 2} \right\}.$$

$$\left\{ \begin{matrix} 2 \\ 3\ \ 3 \end{matrix} \right\} = g^{2k}[33,k] = g^{21}[33,1] + g^{22}[33,2] + g^{23}[33,3]$$

$$= 0 \times \left(-x^1 \sin^2 x^2\right) + \frac{-\left(x^1\right)^2 \sin x^2 \cos x^2}{\left(x^1\right)^2} + 0.0 = -\sin x^2 \cos x^2.$$

$$\left\{ \begin{matrix} 3 \\ 1\ \ 1 \end{matrix} \right\} = g^{3k}[11,k] = g^{31}[11,1] + g^{32}[11,2] + g^{33}[11,3]$$

$$= 1 \times 0 + 0 \times 0 + \frac{1}{\left(x^1\right)^2 \sin^2 x^2} \times 0 = 0.$$

$$\left\{ \begin{matrix} 3 \\ 1\ \ 2 \end{matrix} \right\} = g^{3k}[12,k] = g^{31}[12,1] + g^{32}[12,2] + g^{33}[12,3]$$

$$= 0 \times 0 + 0 \times x^1 + \frac{1}{\left(x^1\right)^2 \sin^2 x^2} \times 0 = 0 = \left\{ \begin{matrix} 3 \\ 2\ \ 1 \end{matrix} \right\}.$$

EXAMPLE 3.1.8 *Calculate the non-vanishing Christoffel symbols of first and second kind (in spherical co-ordinates) corresponding to the metric*

$$ds^2 = \left(dx^1\right)^2 + \left(x^1\right)^2 \left(dx^2\right)^2 + \left(x^1\right)^2 \sin^2 x^2 \left(dx^3\right)^2.$$

Solution: Comparing the given metric, with Eq. (2.1), we get $g_{11} = 1,\ g_{22} = \left(x^1\right)^2$, $g_{33} = \left(x^1\right)^2 \sin^2 x^2$ and $g_{ij} = 0$ for $i \neq j$. For orthogonal co-ordinates $g^{ij} = \frac{1}{g_{ij}}$(no summation). Therefore, $g = (x^1)^4 \sin^2 x^2$ and

$$g^{11} = \frac{1}{g_{11}} = 1,\ g^{22} = \frac{1}{g_{22}} = \frac{1}{\left(x^1\right)^2};\ g^{33} = \frac{1}{g_{33}} = \frac{1}{\left(x^1\right)^2 \sin^2 x^2},$$

and $g^{ij} = 0$ for $i \neq j$. The Christoffel symbols of first kind are given by

$$[11,1] = \frac{1}{2}\left(\frac{\partial g_{11}}{\partial x^1} + \frac{\partial g_{11}}{\partial x^1} - \frac{\partial g_{11}}{\partial x^1}\right) = \frac{1}{2}\frac{\partial g_{11}}{\partial x^1} = 0.$$

$$[11,2] = \frac{1}{2}\left(\frac{\partial g_{12}}{\partial x^1} + \frac{\partial g_{12}}{\partial x^1} - \frac{\partial g_{11}}{\partial x^2}\right) = \frac{\partial g_{12}}{\partial x^1} - \frac{1}{2}\frac{\partial g_{11}}{\partial x^2} = 0.$$

$$[12,1] = \frac{1}{2}\left(\frac{\partial g_{11}}{\partial x^2} + \frac{\partial g_{21}}{\partial x^1} - \frac{\partial g_{12}}{\partial x^1}\right) = \frac{1}{2}\frac{\partial g_{11}}{\partial x^2} = 0 = [21,1].$$

$$[12,2] = \frac{1}{2}\left(\frac{\partial g_{12}}{\partial x^2} + \frac{\partial g_{22}}{\partial x^1} - \frac{\partial g_{12}}{\partial x^2}\right) = \frac{1}{2}\frac{\partial g_{22}}{\partial x^1} = x^1 = [21,2].$$

$$[12,3] = \frac{1}{2}\left(\frac{\partial g_{13}}{\partial x^2} + \frac{\partial g_{23}}{\partial x^1} - \frac{\partial g_{12}}{\partial x^3}\right) = 0 = [21,3].$$

$$[11,3] = \frac{1}{2}\left(\frac{\partial g_{13}}{\partial x^1} + \frac{\partial g_{13}}{\partial x^1} - \frac{\partial g_{11}}{\partial x^3}\right) = \frac{\partial g_{13}}{\partial x^1} - \frac{1}{2}\frac{\partial g_{11}}{\partial x^3} = 0.$$

$$[13,1] = \frac{1}{2}\left(\frac{\partial g_{11}}{\partial x^3} + \frac{\partial g_{31}}{\partial x^1} - \frac{\partial g_{13}}{\partial x^1}\right) = \frac{1}{2}\frac{\partial g_{11}}{\partial x^3} = 0 = [31,1].$$

$$[13,3] = \frac{1}{2}\left(\frac{\partial g_{13}}{\partial x^3} + \frac{\partial g_{33}}{\partial x^1} - \frac{\partial g_{13}}{\partial x^3}\right) = \frac{1}{2}\frac{\partial g_{33}}{\partial x^1} = x^1 \sin^2 x^2 = [31,3].$$

$$[22,2] = \frac{1}{2}\left(\frac{\partial g_{22}}{\partial x^2} + \frac{\partial g_{22}}{\partial x^2} - \frac{\partial g_{22}}{\partial x^2}\right) = \frac{1}{2}\frac{\partial g_{22}}{\partial x^2} = 0.$$

$$[22,3] = \frac{1}{2}\left(\frac{\partial g_{23}}{\partial x^2} + \frac{\partial g_{23}}{\partial x^2} - \frac{\partial g_{22}}{\partial x^3}\right) = \frac{1}{2}\frac{\partial g_{22}}{\partial x^3} = 0.$$

$$[22,1] = \frac{1}{2}\left(\frac{\partial g_{21}}{\partial x^2} + \frac{\partial g_{21}}{\partial x^2} - \frac{\partial g_{22}}{\partial x^1}\right) = -\frac{1}{2}\frac{\partial g_{22}}{\partial x^1} = -x^1.$$

$$[23,1] = \frac{1}{2}\left(\frac{\partial g_{21}}{\partial x^3} + \frac{\partial g_{31}}{\partial x^2} - \frac{\partial g_{23}}{\partial x^1}\right) = 0 = [32,1].$$

$$[23,3] = \frac{1}{2}\left(\frac{\partial g_{23}}{\partial x^3} + \frac{\partial g_{33}}{\partial x^2} - \frac{\partial g_{23}}{\partial x^3}\right) = \frac{1}{2}\frac{\partial g_{33}}{\partial x^2} = \left(x^1\right)^2 \sin x^2 \cos x^2 = [32,3].$$

$$[33,3] = \frac{1}{2}\left(\frac{\partial g_{33}}{\partial x^3} + \frac{\partial g_{33}}{\partial x^3} - \frac{\partial g_{33}}{\partial x^3}\right) = \frac{1}{2}\frac{\partial g_{33}}{\partial x^3} = 0.$$

$$[33,1] = \frac{1}{2}\left(\frac{\partial g_{31}}{\partial x^3} + \frac{\partial g_{31}}{\partial x^3} - \frac{\partial g_{33}}{\partial x^1}\right) = -\frac{1}{2}\frac{\partial g_{33}}{\partial x^1} = -x^1 \sin^2 x^2.$$

$$[33,2] = \frac{1}{2}\left(\frac{\partial g_{32}}{\partial x^3} + \frac{\partial g_{32}}{\partial x^3} - \frac{\partial g_{33}}{\partial x^2}\right) = \frac{1}{2}\frac{\partial g_{33}}{\partial x^2} = -\left(x^1\right)^2 \sin x^2 \cos x^2.$$

$$[32,2] = \frac{1}{2}\left(\frac{\partial g_{32}}{\partial x^2} + \frac{\partial g_{22}}{\partial x^3} - \frac{\partial g_{32}}{\partial x^2}\right) = \frac{1}{2}\frac{\partial g_{22}}{\partial x^3} = 0 = [23,2].$$

$$[31,2] = \frac{1}{2}\left(\frac{\partial g_{32}}{\partial x^1} + \frac{\partial g_{12}}{\partial x^3} - \frac{\partial g_{31}}{\partial x^2}\right) = 0 = [13,2].$$

The Christoffel symbols of second kind are given by

$$\begin{Bmatrix} 1 \\ 1\ \ 1 \end{Bmatrix} = g^{1k}[11,k] = g^{11}[11,1] + g^{12}[11,2] + g^{13}[11,3]$$
$$= 1 \times 0 + 0 \times 0 + 0 \times 0 = 0.$$

$$\left\{ {1 \atop 1\ 2} \right\} = g^{1k}[12,k] = g^{11}[12,1] + g^{12}[12,2] + g^{13}[12,3]$$

$$= 1 \times 0 + 0 \times x^1 + 0 \times 0 = 0 = \left\{ {1 \atop 2\ 1} \right\}.$$

$$\left\{ {1 \atop 2\ 2} \right\} = g^{1k}[22,k] = g^{11}[22,1] + g^{12}[22,2] + g^{13}[22,3]$$

$$= 1 \times \left(-x^1\right) + 0 \times 0 + 0 \times 0 = -x^1.$$

$$\left\{ {1 \atop 1\ 3} \right\} = g^{1k}[13,k] = g^{11}[13,1] + g^{12}[13,2] + g^{13}[13,3]$$

$$= 1 \times 0 + 0 \times 0 + 0 \times x^1 \sin^2 x^2 = 0 = \left\{ {1 \atop 3\ 1} \right\}.$$

$$\left\{ {1 \atop 2\ 3} \right\} = g^{1k}[23,k] = g^{11}[23,1] + g^{12}[23,2] + g^{13}[23,3]$$

$$= 1 \times 0 + 0 \times 0 + 0 \times \left(x^1\right)^2 \sin x^2 = 0 = \left\{ {1 \atop 3\ 2} \right\}.$$

$$\left\{ {1 \atop 3\ 3} \right\} = g^{1k}[33,k] = g^{11}[33,1] + g^{12}[33,2] + g^{13}[33,3]$$

$$= 1 \times \left(-x^1 \sin^2 x^2\right) + 0 \times \left[-\left(x^1\right)^2 \sin x^2 \cos x^2\right] + 0 \times 0 = -x^1 \sin^2 x^2.$$

$$\left\{ {2 \atop 1\ 1} \right\} = g^{2k}[11,k] = g^{21}[11,1] + g^{22}[11,2] + g^{23}[11,3]$$

$$= 0 \times 0 + \frac{1}{(x^1)^2} \times 0 + 0 \times 0 = 0.$$

$$\left\{ {2 \atop 1\ 2} \right\} = g^{2k}[12,k] = g^{21}[12,1] + g^{22}[12,2] + g^{23}[12,3].$$

$$= 0 \times 0 + \frac{1}{(x^1)^2} \times x^1 + 0 \times 0 = \frac{1}{x^1} = \left\{ {2 \atop 2\ 1} \right\}.$$

$$\left\{ {2 \atop 1\ 3} \right\} = g^{2k}[13,k] = g^{21}[13,1] + g^{22}[13,2] + g^{23}[13,3]$$

$$= a \times 0 + \frac{1}{(x^1)^2} \times 0 + 0 \times x^1 \sin^2 x^2 = 0 = \left\{ {2 \atop 3\ 1} \right\}.$$

$$\left\{ \begin{matrix} 2 \\ 2 \ \ 2 \end{matrix} \right\} = g^{2k}[22, k] = g^{21}[22, 1] + g^{22}[22, 2] + g^{23}[22, 3]$$

$$= 0 \times \left(-x^1\right) + \frac{1}{(x^1)^2} \times 0 + 0 \times 0 = 0.$$

$$\left\{ \begin{matrix} 2 \\ 2 \ \ 3 \end{matrix} \right\} = g^{2k}[23, k] = g^{21}[23, 1] + g^{22}[23, 2] + g^{23}[23, 3]$$

$$= 0 \times 0 + \frac{1}{(x^1)^2} \times 0 + 0 \times \left(-x^1\right)^2 \sin x^2 \cos x^2 = 0 = \left\{ \begin{matrix} 2 \\ 3 \ \ 2 \end{matrix} \right\}.$$

$$\left\{ \begin{matrix} 2 \\ 3 \ \ 3 \end{matrix} \right\} = g^{2k}[33, k] = g^{21}[33, 1] + g^{22}[33, 2] + g^{23}[33, 3]$$

$$= 0 \times \left(-x^1 \sin^2 x^2\right) + \frac{-\left(x^1\right)^2 \sin x^2 \cos x^2}{(x^1)^2} + 0 \times 0 = -\sin x^2 \cos x^2.$$

$$\left\{ \begin{matrix} 3 \\ 1 \ \ 1 \end{matrix} \right\} = g^{3k}[11, k] = g^{31}[11, 1] + g^{32}[11, 2] + g^{33}[11, 3]$$

$$= 1 \times 0 + 0 \times 0 + \frac{1}{(x^1)^2 \sin^2 x^2} \times 0 = 0.$$

$$\left\{ \begin{matrix} 3 \\ 1 \ \ 2 \end{matrix} \right\} = g^{3k}[12, k] = g^{31}[12, 1] + g^{32}[12, 2] + g^{33}[12, 3]$$

$$= 0 \times 0 + 0 \times x^1 + \frac{1}{(x^1)^2 \sin^2 x^2} \times 0 = 0 = \left\{ \begin{matrix} 3 \\ 2 \ \ 1 \end{matrix} \right\}.$$

$$\left\{ \begin{matrix} 3 \\ 1 \ \ 3 \end{matrix} \right\} = g^{3k}[13, k] = g^{31}[13, 1] + g^{32}[13, 2] + g^{33}[13, 3]$$

$$= 0 \times 0 + 0 \times 0 + \frac{1}{(x^1)^2 \sin^2 x^2} \times x^1 \sin^2 x^2 = \frac{1}{x^1} = \left\{ \begin{matrix} 3 \\ 1 \ \ 3 \end{matrix} \right\}.$$

$$\left\{ \begin{matrix} 3 \\ 2 \ \ 3 \end{matrix} \right\} = g^{3k}[23, k] = g^{31}[23, 1] + g^{32}[23, 2] + g^{33}[23, 3]$$

$$= 0 \times 0 + 0 \times 0 + \frac{\left(x^1\right)^2 \sin x^2 \cos x^2}{(x^1)^2 \sin^2 x^2} = \cot x^2 = \left\{ \begin{matrix} 3 \\ 3 \ \ 2 \end{matrix} \right\}.$$

$$\left\{ \begin{matrix} 3 \\ 3 \ \ 3 \end{matrix} \right\} = g^{3k}[33, k] = g^{31}[33, 1] + g^{32}[33, 2] + g^{33}[33, 3]$$

$$= 0 \times \left(-x^1 \sin x^2\right) + 0 \cdot \left[-\left(x^1\right)^2 \sin x^2 \cos x^2\right] + \frac{1}{(x^1)^2 \sin^2 x^2} \cdot 0 = 0.$$

We can verify Theorem 3.1.5. When $i \neq j \neq k$, then both the first kind $[ij,k] = 0$ and second kind $\left\{ \begin{matrix} k \\ i \;\; j \end{matrix} \right\} = 0$. Therefore, the non-vanishing Christoffel symbols of first kind are

$$[12,2] = x^1 = [21,2];\ [13,3] = x^1 \sin^2 x^2 = [31,3].$$

$$[22,1] = -x^1;\ [23,3] = \left(x^1\right)^2 \sin x^2 \cos x^2.$$

$$[33,1] = -x^1 \sin^2 x^2;\ [33,2] = -\left(x^1\right)^2 \sin x^2 \cos x^2.$$

The non-vanishing Christoffel symbols of second kind are

$$\left\{ \begin{matrix} 1 \\ 2 \;\; 2 \end{matrix} \right\} = -x^1; \left\{ \begin{matrix} 1 \\ 3 \;\; 3 \end{matrix} \right\} = -x^1 \sin^2 x^2; \left\{ \begin{matrix} 2 \\ 1 \;\; 2 \end{matrix} \right\} = \left\{ \begin{matrix} 2 \\ 2 \;\; 1 \end{matrix} \right\} = \frac{1}{x^1}$$

$$\left\{ \begin{matrix} 2 \\ 3 \;\; 3 \end{matrix} \right\} = -\sin x^2 \cos x^2; \left\{ \begin{matrix} 3 \\ 1 \;\; 3 \end{matrix} \right\} = \left\{ \begin{matrix} 3 \\ 3 \;\; 1 \end{matrix} \right\} = \frac{1}{x^1}; \left\{ \begin{matrix} 3 \\ 2 \;\; 3 \end{matrix} \right\} = \left\{ \begin{matrix} 3 \\ 3 \;\; 2 \end{matrix} \right\} = \cot x^2.$$

EXAMPLE 3.1.9 *If a^{ij} are components of a symmetric tensor, show that*

$$a^{jk}[ij,k] = \frac{1}{2} a^{jk} \frac{\partial g_{jk}}{\partial x^i},$$

where g_{jk} have their usual meaning.

Solution: Here, we use the relation

$$[ji,k] + [ki,j] = \frac{\partial g_{jk}}{\partial x^i},$$

for the Christoffel symbol of first kind. Multiplying both sides by a^{jk}, we get

$$a^{jk}[ji,k] + a^{jk}[ki,j] = a^{jk} \frac{\partial g_{jk}}{\partial x^i}$$

or

$$a^{jk}[ji,k] + a^{kj}[ki,j] = a^{jk} \frac{\partial g_{jk}}{\partial x^i};\ \text{as } a^{jk} = a^{kj}$$

or

$$a^{jk}[ji,k] + a^{jk}[ki,j] = a^{jk} \frac{\partial g_{jk}}{\partial x^i}.$$

Replacing the dummy indices k and j in the second term by j and k, respectively, we get

$$2a^{jk}[ji,k] = a^{jk} \frac{\partial g_{jk}}{\partial x^i}$$

or

$$2a^{jk}[ij,k] = a^{jk} \frac{\partial g_{jk}}{\partial x^i};\ \text{as } [ji,k] = [ij,k]$$

Therefore

$$a^{jk}[ij,k] = \frac{1}{2} a^{jk} \frac{\partial g_{jk}}{\partial x^i}.$$

EXAMPLE 3.1.10 *Show that the maximum number of independent components of the Christoffel symbols in a* V_N,*a* N *dimensional Riemannian space is* $\frac{1}{2}N^2(N+1)$.

Solution: Since the dimension of the Riemannian space V_N is N, the i and j of the fundamental tensor g_{ij} varies from 1 to N. Thus, g_{ij} has N^2 components.

As g_{ij} is symmetric, the maximum number of distinct components, where $i \neq j$ are $\frac{1}{2}(N^2 - N)$. Therefore, total number of distinct components of g_{ij} has atmost

$$N + \frac{1}{2}(N^2 - N) = \frac{1}{2}N(N+1)$$

components. Since k varies from 1 to N in $\frac{\partial g_{ij}}{\partial x^k}$, the number of independent components will be

$$N\frac{1}{2}N(N+1) = \frac{N^2}{2}(N+1).$$

Since the Christoffel symbols $[ij, k]$ and $\left\{ {l \atop i\ j} \right\}$ of first and second kinds, respectively, are the linear combination of the terms $\frac{\partial g_{ij}}{\partial x^k}$, so the maximum number of independent components is $\frac{N^2}{2}(N+1)$.

EXAMPLE 3.1.11 *Prove that the following*

$$\frac{\partial}{\partial x^j}\left(\sqrt{g}g^{ij}\right) + \sqrt{g}\left\{ {i \atop j\ k} \right\} g^{jk} = 0.$$

Solution: Here we use the properties of Christoffel symbols. We have

$$\begin{aligned}
\frac{\partial}{\partial x^j}\left(\sqrt{g}g^{ij}\right) &= \frac{\partial\sqrt{g}}{\partial x^j}g^{ij} + \sqrt{g}\frac{\partial g^{ij}}{\partial x^j} \\
&= \frac{\partial\sqrt{g}}{\partial x^j}g^{ij} + \sqrt{g}\left[-g^{jp}\left\{ {i \atop p\ j} \right\} - g^{it}\left\{ {j \atop t\ j} \right\}\right] \\
&= \frac{\partial\sqrt{g}}{\partial x^j}g^{ij} - \sqrt{g}\left\{ {i \atop p\ j} \right\} g^{jp} - \sqrt{g}g^{it}\left\{ {j \atop t\ j} \right\} \\
&= \frac{\partial\sqrt{g}}{\partial x^j}g^{ij} - \sqrt{g}\left\{ {i \atop p\ j} \right\} g^{pj} - \sqrt{g}g^{it}\frac{1}{2g}\frac{\partial g}{\partial x^t} \\
&= \frac{1}{2\sqrt{g}}\frac{\partial g}{\partial x^i}g^{ij} - \frac{1}{2\sqrt{g}}\frac{\partial g}{\partial x^t}g^{it} - \sqrt{g}\left\{ {i \atop p\ j} \right\} g^{pj} \\
&= -\sqrt{g}\left\{ {i \atop p\ j} \right\} g^{pj}.
\end{aligned}$$

Therefore,

$$\frac{\partial}{\partial x^j}\left(\sqrt{g}g^{ij}\right) + \sqrt{g}\begin{Bmatrix} i \\ j \quad k \end{Bmatrix} g^{jk} = 0.$$

EXAMPLE 3.1.12 *If A^{ij} is a skew-symmetric tensor, show that* $A^{ik}\begin{Bmatrix} i \\ j \quad k \end{Bmatrix} = 0.$

Solution: Since A^{ij} is a skew-symmetric, so, by definition $A^{ki} = -A^{ik}$. Therefore,

$$A^{ik}\begin{Bmatrix} i \\ j \quad k \end{Bmatrix} = A^{ki}\begin{Bmatrix} i \\ k \quad j \end{Bmatrix},$$

where, we have to replace the dummy indices k by j and j by k. Therefore,

$$\begin{aligned} A^{ik}\begin{Bmatrix} i \\ j \quad k \end{Bmatrix} &= -A^{ik}\begin{Bmatrix} i \\ k \quad j \end{Bmatrix}; \text{ as } A^{ki} = -A^{ik}. \\ &= -A^{ik}\begin{Bmatrix} i \\ j \quad k \end{Bmatrix}; \quad \text{as} \quad \begin{Bmatrix} i \\ k \quad j \end{Bmatrix} = \begin{Bmatrix} i \\ j \quad k \end{Bmatrix} \end{aligned}$$

or

$$2A^{ik}\begin{Bmatrix} i \\ j \quad k \end{Bmatrix} = 0, \quad \text{i.e.} \quad A^{ik}\begin{Bmatrix} i \\ j \quad k \end{Bmatrix} = 0.$$

3.1.2 Transformation of Christoffel Symbols

Here, we have to calculate the transformation formula of Christoffel symbols. Let the fundamental tensors g_{ij}, g^{ij} and the symbols $[ij,k], \begin{Bmatrix} k \\ i \quad j \end{Bmatrix}$ are defined in x^i co-ordinate systems. Let these quantities be $\overline{g}_{ij}, \overline{g}^{ij}$ $\overline{[ij,k]}, \overline{\begin{Bmatrix} k \\ i \quad j \end{Bmatrix}}$ in $\overline{x}^i$ co-ordinate systems.

Law of transformation of Christoffel symbols of the first kind: From the tensor law of transformation of covariant tensor g_{ij} of type $(0,2)$, we see that

$$\overline{g}_{lm} = \frac{\partial x^i}{\partial \overline{x}^l}\frac{\partial x^j}{\partial \overline{x}^m} g_{ij}.$$

The transformation law for the $[ij,k]$ can be inferred from that for the g_{ij}. Differentiating partially with respect to $\overline{x}^n$, we get

$$\frac{\partial \overline{g}_{lm}}{\partial \overline{x}^n} = \frac{\partial x^i}{\partial \overline{x}^l}\frac{\partial x^j}{\partial \overline{x}^m}\frac{\partial g_{ij}}{\partial x^k}\frac{\partial x^k}{\partial \overline{x}^n} + \frac{\partial^2 x^i}{\partial \overline{x}^l \partial \overline{x}^n}\frac{\partial x^j}{\partial \overline{x}^m} g_{ij} + \frac{\partial x^i}{\partial \overline{x}^l}\frac{\partial^2 x^j}{\partial \overline{x}^m \partial \overline{x}^n} g_{ij}. \tag{3.3}$$

Similarly differentiating the transformation law with respect to $\overline{x}^l$, we get

$$\begin{aligned}\frac{\partial \overline{g}_{mn}}{\partial \overline{x}^l} &= \frac{\partial x^i}{\partial \overline{x}^m}\frac{\partial x^j}{\partial \overline{x}^n}\frac{\partial g_{ij}}{\partial x^k}\frac{\partial x^k}{\partial \overline{x}^l} + \frac{\partial^2 x^i}{\partial \overline{x}^m \partial \overline{x}^l}\frac{\partial x^j}{\partial \overline{x}^n} g_{ij} + \frac{\partial x^i}{\partial \overline{x}^m}\frac{\partial^2 x^j}{\partial \overline{x}^n \partial \overline{x}^l} g_{ij} \\ &= \frac{\partial x^i}{\partial \overline{x}^l}\frac{\partial x^j}{\partial \overline{x}^m}\frac{\partial x^k}{\partial \overline{x}^n}\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial^2 x^j}{\partial \overline{x}^m \partial \overline{x}^l}\frac{\partial x^k}{\partial \overline{x}^n} g_{jk} + \frac{\partial^2 x^k}{\partial \overline{x}^n \partial \overline{x}^i}\frac{\partial x^j}{\partial \overline{x}^m} g_{jk},\end{aligned} \tag{3.4}$$

where, we have to replace the dummy indices i, j, k by j, k, i, respectively. Further,

$$\begin{aligned}\frac{\partial \overline{g}_{ln}}{\partial \overline{x}^m} &= \frac{\partial x^i}{\partial \overline{x}^l}\frac{\partial x^j}{\partial \overline{x}^n}\frac{\partial g_{ij}}{\partial x^k}\frac{\partial x^k}{\partial \overline{x}^m} + \frac{\partial^2 x^i}{\partial \overline{x}^l \partial \overline{x}^m}\frac{\partial x^j}{\partial \overline{x}^n} g_{ij} + \frac{\partial^2 x^j}{\partial \overline{x}^n \partial \overline{x}^m}\frac{\partial x^i}{\partial \overline{x}^l} g_{ij} \\ &= \frac{\partial x^i}{\partial \overline{x}^l}\frac{\partial x^j}{\partial \overline{x}^m}\frac{\partial x^k}{\partial \overline{x}^n}\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial^2 x^i}{\partial \overline{x}^l \partial \overline{x}^m}\frac{\partial x^k}{\partial \overline{x}^n} g_{jk} + \frac{\partial^2 x^k}{\partial \overline{x}^m \partial \overline{x}^n}\frac{\partial x^i}{\partial \overline{x}^l} g_{ik},\end{aligned} \tag{3.5}$$

where we have to replace the dummy indices j and k by k and j, respectively. Adding Eqs. (3.4) and (3.5) and subtracting Eq. (3.3) from the result thus obtained, we get

$$\begin{aligned}&\frac{\partial \overline{g}_{mn}}{\partial \overline{x}^l} + \frac{\partial \overline{g}_{nl}}{\partial \overline{x}^m} - \frac{\partial \overline{g}_{lm}}{\partial \overline{x}^n} \\ &= \left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right)\frac{\partial x^i}{\partial \overline{x}^l}\frac{\partial x^j}{\partial \overline{x}^m}\frac{\partial x^k}{\partial \overline{x}^n} + \left(\frac{\partial x^k}{\partial \overline{x}^n}\frac{\partial^2 x^j}{\partial \overline{x}^m \partial \overline{x}^l} + \frac{\partial x^j}{\partial \overline{x}^m}\frac{\partial^2 x^k}{\partial \overline{x}^n \partial \overline{x}^l}\right) g_{jk} \\ &\quad + \left(\frac{\partial x^k}{\partial \overline{x}^n}\frac{\partial^2 x^i}{\partial \overline{x}^l \partial \overline{x}^m} + \frac{\partial x^i}{\partial \overline{x}^l}\frac{\partial^2 x^k}{\partial \overline{x}^n \partial \overline{x}^m}\right) g_{ik} - \left(\frac{\partial x^j}{\partial \overline{x}^m}\frac{\partial^2 x^i}{\partial \overline{x}^l \partial \overline{x}^n} + \frac{\partial x^i}{\partial \overline{x}^l}\frac{\partial^2 x^j}{\partial \overline{x}^m \partial \overline{x}^n}\right) g_{ij} \\ &= \left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right)\frac{\partial x^i}{\partial \overline{x}^l}\frac{\partial x^j}{\partial \overline{x}^m}\frac{\partial x^k}{\partial \overline{x}^n} + \left(\frac{\partial x^i}{\partial \overline{x}^n}\frac{\partial^2 x^j}{\partial \overline{x}^m \partial \overline{x}^l} + \frac{\partial x^j}{\partial \overline{x}^m}\frac{\partial^2 x^i}{\partial \overline{x}^n \partial \overline{x}^l}\right) g_{ji} \\ &\quad + \left(\frac{\partial x^j}{\partial \overline{x}^n}\frac{\partial^2 x^i}{\partial \overline{x}^l \partial \overline{x}^m} + \frac{\partial x^i}{\partial \overline{x}^l}\frac{\partial^2 x^j}{\partial \overline{x}^n \partial \overline{x}^m}\right) g_{ij} - \left(\frac{\partial x^j}{\partial \overline{x}^m}\frac{\partial^2 x^i}{\partial \overline{x}^l \partial \overline{x}^n} + \frac{\partial x^i}{\partial \overline{x}^l}\frac{\partial^2 x^j}{\partial \overline{x}^m \partial \overline{x}^n}\right) g_{ij}\end{aligned} \tag{3.6}$$

(replacing k by i and k by j in the second and third term)

$$\begin{aligned}&= \left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right)\frac{\partial x^i}{\partial \overline{x}^l}\frac{\partial x^j}{\partial \overline{x}^m}\frac{\partial x^k}{\partial \overline{x}^n} + \frac{\partial x^i}{\partial \overline{x}^n}\frac{\partial^2 x^j}{\partial \overline{x}^m \partial \overline{x}^l} g_{ij} + \frac{\partial x^j}{\partial \overline{x}^n}\frac{\partial^2 x^i}{\partial \overline{x}^l \partial \overline{x}^m} g_{ji} \\ &= \left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right)\frac{\partial x^i}{\partial \overline{x}^l}\frac{\partial x^j}{\partial \overline{x}^m}\frac{\partial x^k}{\partial \overline{x}^n} + 2\frac{\partial x^i}{\partial \overline{x}^n}\frac{\partial^2 x^j}{\partial \overline{x}^l \partial \overline{x}^m} g_{ij}.\end{aligned}$$

Therefore,

$$\overline{[lm, n]} = [ij, k]\frac{\partial x^i}{\partial \overline{x}^l}\frac{\partial x^j}{\partial \overline{x}^m}\frac{\partial x^k}{\partial \overline{x}^n} + \frac{\partial x^i}{\partial \overline{x}^n}\frac{\partial^2 x^j}{\partial \overline{x}^l \partial \overline{x}^m} g_{ij}. \tag{3.7}$$

In the second and third relations of Eq. (3.7), we have to replace the dummy indices k by i and k by j, respectively. Equation (3.7) gives the law of transformation of the components, $[ij, k]$ of the Christoffel symbols of the first kind from one co-ordinate system to another. From this law it follows that the components $[ij,k]$ do not transform like a tensor due to the presence of the second term in the right-hand side of Eq. (3.7). This provides that Christoffel symbol of the first kind (or, connection coefficient) does not follow the tensor law of transformation and hence it is not a tensor.

Law of transformation of Christoffel symbols of the second kind: Since g^{ij} is a tensor of type $(2,0)$, we have the transformation law

$$\overline{g}^{np} = g^{rs}\frac{\partial \overline{x}^n}{\partial x^r}\frac{\partial \overline{x}^p}{\partial x^s}. \tag{3.8}$$

Inner multiplication of both sides of Eq. (3.7) by the corresponding side of Eq. (3.8) gives

$$\overline{g}^{np}\overline{[lm,n]} = g^{rs}\frac{\partial \overline{x}^n}{\partial x^r}\frac{\partial \overline{x}^p}{\partial x^s}[ij,k]\frac{\partial x^i}{\partial \overline{x}^l}\frac{\partial x^j}{\partial \overline{x}^m}\frac{\partial x^k}{\partial \overline{x}^n} + g_{ij}\frac{\partial x^i}{\partial \overline{x}^n}\frac{\partial^2 x^j}{\partial \overline{x}^l \partial \overline{x}^m}g^{rs}\frac{\partial \overline{x}^n}{\partial x^r}\frac{\partial \overline{x}^p}{\partial x^s}$$

or

$$\begin{aligned}
\overline{\left\{ \begin{matrix} p \\ l\ \ m \end{matrix} \right\}} &= g^{rs}\frac{\partial x^k}{\partial x^r}[ij,k]\frac{\partial \overline{x}^p}{\partial x^s}\frac{\partial x^i}{\partial \overline{x}^l}\frac{\partial x^j}{\partial \overline{x}^m} + g_{ij}\frac{\partial x^i}{\partial x^r}\frac{\partial \overline{x}^p}{\partial x^s}g^{rs}\frac{\partial^2 x^j}{\partial \overline{x}^l \partial \overline{x}^m} \\
&= g^{ks}[ij,k]\frac{\partial \overline{x}^p}{\partial x^s}\frac{\partial x^i}{\partial \overline{x}^l}\frac{\partial x^j}{\partial \overline{x}^m} + g_{rj}g^{rs}\frac{\partial \overline{x}^p}{\partial x^s}\frac{\partial^2 x^j}{\partial \overline{x}^l \partial \overline{x}^m}; \quad \text{as} \quad \frac{\partial x^k}{\partial x^r} = \delta^k_r \\
&= \left\{ \begin{matrix} s \\ i\ \ j \end{matrix} \right\}\frac{\partial \overline{x}^p}{\partial x^s}\frac{\partial x^i}{\partial \overline{x}^l}\frac{\partial x^j}{\partial \overline{x}^m} + \delta^s_j\frac{\partial \overline{x}^p}{\partial x^s}\frac{\partial^2 x^j}{\partial \overline{x}^l \partial \overline{x}^m} \\
&= \left\{ \begin{matrix} s \\ i\ \ j \end{matrix} \right\}\frac{\partial \overline{x}^p}{\partial x^s}\frac{\partial x^i}{\partial \overline{x}^l}\frac{\partial x^j}{\partial \overline{x}^m} + \frac{\partial \overline{x}^p}{\partial x^j}\frac{\partial^2 x^j}{\partial \overline{x}^l \partial \overline{x}^m}.
\end{aligned} \tag{3.9}$$

Equation (3.9) gives the law of transformation of the components $\left\{ \begin{matrix} l \\ i\ \ j \end{matrix} \right\}$ of the Christoffel symbols of the second kind. From this law it follows that the components do not transform like a tensor due to presence of the second term in the right-hand side of Eq. (3.9). Therefore, from the form of Eq. (3.9), it is clear that the set of Christoffel symbols is a third order covariant affine tensor but is not a general tensor. Sometimes, it is called Christoffel connection.

Result 3.1.1 Here we have deduced a result relative to the Christoffel connection which will play a vital role in the matter of introducing a new kind of differentiation in V_N. Inner multiplication of Eq. (3.9) by $\frac{\partial x^r}{\partial \overline{x}^p}$ gives,

$$\begin{aligned}\frac{\partial x^r}{\partial \overline{x}^p}\overline{\begin{Bmatrix} p \\ l\ \ m \end{Bmatrix}} &= \begin{Bmatrix} s \\ i\ \ j \end{Bmatrix}\frac{\partial \overline{x}^p}{\partial x^s}\frac{\partial x^i}{\partial \overline{x}^l}\frac{\partial x^j}{\partial \overline{x}^m}\frac{\partial x^r}{\partial \overline{x}^p} + \frac{\partial x^r}{\partial \overline{x}^p}\frac{\partial \overline{x}^p}{\partial x^j}\frac{\partial^2 x^j}{\partial \overline{x}^l \partial \overline{x}^m} \\ &= \begin{Bmatrix} s \\ i\ \ j \end{Bmatrix}\delta^r_s \frac{\partial x^i}{\partial \overline{x}^l}\frac{\partial x^j}{\partial \overline{x}^m} + \delta^r_j \frac{\partial^2 x^j}{\partial \overline{x}^l \partial \overline{x}^m} \\ &= \begin{Bmatrix} r \\ i\ \ j \end{Bmatrix}\frac{\partial x^i}{\partial \overline{x}^l}\frac{\partial x^j}{\partial \overline{x}^m} + \frac{\partial^2 x^r}{\partial \overline{x}^l \partial \overline{x}^m}\end{aligned}$$

or

$$\frac{\partial^2 x^r}{\partial \overline{x}^l \partial \overline{x}^m} = \frac{\partial x^r}{\partial \overline{x}^p}\overline{\begin{Bmatrix} p \\ l\ \ m \end{Bmatrix}} - \frac{\partial x^i}{\partial \overline{x}^l}\frac{\partial x^j}{\partial \overline{x}^m}\begin{Bmatrix} r \\ i\ \ j \end{Bmatrix}, \tag{3.10}$$

which is the second derivative of x's with respect to $\overline{x}$'s in terms of symbols of second kind and first derivative. This important formula (3.10) were first deduced in as entirely different way by Christoffel in a memoir concerned with the study of equivalence of quadratic differential forms. We will make use formula (3.10) to define the operations of tensorial differentiation in V_N. Needless to say, Eq. (3.10) holds barred and unbarred co-ordinates are interchanged.

EXAMPLE 3.1.13 *Find the most general three-dimensional transformation $x^i = x^i(\overline{x}^1, \overline{x}^2, \overline{x}^3)$ of co-ordinates such that (x^i) is rectangular and $(\overline{x}^i)$ is any other co-ordinate system for which the Christoffel symbols are*

$$\overline{\begin{Bmatrix} 1 \\ 1\ \ 1 \end{Bmatrix}} = 1; \quad \overline{\begin{Bmatrix} 2 \\ 2\ \ 2 \end{Bmatrix}} = 2; \quad \overline{\begin{Bmatrix} 3 \\ 3\ \ 3 \end{Bmatrix}} = 3,$$

and all other components are zero.

Solution: Since $\begin{Bmatrix} r \\ i\ \ j \end{Bmatrix} = 0$, relation (3.10) reduces to the system of linear partial differential equation with constant coefficients as

$$\frac{\partial^2 x^r}{\partial \overline{x}^l \partial \overline{x}^m} = \frac{\partial x^r}{\partial \overline{x}^p}\overline{\begin{Bmatrix} p \\ l\ \ m \end{Bmatrix}}. \tag{i}$$

It is simplest first to solve the intermediate, first order system

$$\frac{\partial \overline{u}^r_m}{\partial \overline{x}^l} = \overline{u}^r_p \overline{\begin{Bmatrix} p \\ l\ \ m \end{Bmatrix}}; \quad \text{where } \overline{u}^r_p = \frac{\partial x^r}{\partial \overline{x}^p} \tag{ii}$$

or

$$\frac{\partial \overline{u}_m}{\partial \overline{x}^l} = \overline{u}_p \overline{\begin{Bmatrix} p \\ l \ \ m \end{Bmatrix}}, \tag{iii}$$

where temporarily replace $\overline{u}_p^r$ by $\overline{u}_p$ and x^r by x as the first systems (ii) for $r = 1, 2, 3$ are the same. For $m = 1$, (iii) becomes,

$$\frac{\partial \overline{u}_1}{\partial \overline{x}^1} = \overline{\begin{Bmatrix} 1 \\ 1 \ \ 1 \end{Bmatrix}}\overline{u}_1 + \overline{\begin{Bmatrix} 2 \\ 1 \ \ 1 \end{Bmatrix}}\overline{u}_2 + \overline{\begin{Bmatrix} 3 \\ 1 \ \ 1 \end{Bmatrix}}\overline{u}_3 = \overline{u}_1$$

$$\frac{\partial \overline{u}_1}{\partial \overline{x}^2} = \overline{\begin{Bmatrix} 1 \\ 2 \ \ 1 \end{Bmatrix}}\overline{u}_1 + \overline{\begin{Bmatrix} 2 \\ 2 \ \ 1 \end{Bmatrix}}\overline{u}_2 + \overline{\begin{Bmatrix} 3 \\ 2 \ \ 1 \end{Bmatrix}}\overline{u}_3 = 0$$

$$\frac{\partial \overline{u}_1}{\partial \overline{x}^3} = \overline{\begin{Bmatrix} 1 \\ 3 \ \ 1 \end{Bmatrix}}\overline{u}_1 + \overline{\begin{Bmatrix} 2 \\ 3 \ \ 1 \end{Bmatrix}}\overline{u}_2 + \overline{\begin{Bmatrix} 3 \\ 3 \ \ 1 \end{Bmatrix}}\overline{u}_3 = 0.$$

Thus, $\overline{u}_1$ is a function of $\overline{x}^1$ alone, and the first differential equation integrates to give,

$$\overline{u}_1 = b_1 e^{\overline{x}^1}; \text{ where } b_1 = \text{ constant.}$$

In the same way, we find for $j = 2$ and $j = 3$,

$$\overline{u}_2 = b_2 e^{2\overline{x}^2}; \ \overline{u}_3 = b_3 e^{3\overline{x}^3}; \text{ where } b_2, b_3 = \text{ constants.}$$

Since $\frac{\partial x}{\partial \overline{x}^i} = \overline{u}_i$, with the above solutions, we get,

$$\frac{\partial x}{\partial \overline{x}^1} = b_1 e^{\overline{x}^1}; \ \frac{\partial x}{\partial \overline{x}^2} = b_2 e^{2\overline{x}^2}; \ \frac{\partial x}{\partial \overline{x}^3} = b_3 e^{3\overline{x}^1}. \tag{iv}$$

Solution of the first equation of (iv) gives,

$$x = b_1 e^{\overline{x}^1} + \psi(\overline{x}^1, \overline{x}^1)$$

and the second and third equations of (iv) give,

$$\frac{\partial \psi}{\partial \overline{x}^2} = b_2 e^{2\overline{x}^2} \Rightarrow \psi = a_2 e^{2\overline{x}^2} + \varphi(\overline{x}^3)$$

$$\frac{\partial \varphi}{\partial \overline{x}^3} = b_3 e^{3\overline{x}^3} \Rightarrow \varphi = a_3 e^{3\overline{x}^3} + a_4.$$

This means that, with $a_1 = b_1$,

$$x = a_1 e^{\overline{x}^1} + a_2 e^{2\overline{x}^2} + a_3 e^{3\overline{x}^3} + a_4,$$

so that the general solution of (i) becomes,

$$x^k = a_1^k e^{\overline{x}^1} + a_2^k e^{2\overline{x}^2} + a_3^k e^{3\overline{x}^3} + a_4^k,$$

where $k = 1, 2, 3$.

Result 3.1.2 The Christoffel symbols are objects different from tensors, because their components do not transform according to the law corresponding to that for a tensor of type $(0,3)$ or $(1,2)$.

Result 3.1.3 We have provided that Christoffel symbols are not tensor quantities. But in some very special cases of linear transformation of co-ordinates were $\frac{\partial^2 x^j}{\partial \overline{x}^l \partial \overline{x}^m} = 0$, Eqs. (3.7) and (3.9) reduce to give tensor law of transformations and symbols behave like tensors. Consider a linear transformation

$$x^j = a_l^j \overline{x}^l + b^j,$$

where a_l^j and b^j are constants. Thus, relations (3.7) and (3.9) becomes

$$\overline{[lm,n]} = [ij,k] \frac{\partial x^i}{\partial \overline{x}^l} \frac{\partial x^j}{\partial \overline{x}^m} \frac{\partial x^k}{\partial \overline{x}^n}$$

and

$$\overline{\begin{Bmatrix} p \\ l \;\; m \end{Bmatrix}} = \begin{Bmatrix} s \\ i \;\; j \end{Bmatrix} \frac{\partial \overline{x}^p}{\partial x^s} \frac{\partial x^i}{\partial \overline{x}^l} \frac{\partial x^j}{\partial \overline{x}^m}.$$

Thus, the second term of Eqs. (3.7) and (3.9) will vanish identically. If the co-ordinate transformation is of affine i.e. $x^j = a_l^j \overline{x}^l + b^j$, where a_l^j and b^j are constants, then the Christoffel symbols are tensors. For this Christoffel symbols are sometimes called affine tensors of rank 3.

Theorem 3.1.1 *The transformation of Christoffel's symbols from a group, i.e. possess the transitive property.*

Proof: Let the co-ordinates x^i be transformed to the co-ordinate system $\overline{x}^i$ and $\overline{x}^i$ be transformed to $\overline{\overline{x}}^i$. When the co-ordinates x^i be transformed to $\overline{x}^i$, the law of transformation of Christoffel's symbols of second kind (3.9) is

$$\overline{\begin{Bmatrix} k \\ i \;\; j \end{Bmatrix}} = \begin{Bmatrix} s \\ p \;\; q \end{Bmatrix} \frac{\partial \overline{x}^k}{\partial x^s} \frac{\partial x^p}{\partial \overline{x}^i} \frac{\partial x^q}{\partial \overline{x}^j} + \frac{\partial \overline{x}^k}{\partial x^s} \frac{\partial^2 x^s}{\partial \overline{x}^i \partial \overline{x}^j}. \tag{3.11}$$

When co-ordinate $\overline{x}^i$ be transformed to $\overline{\overline{x}}^i$, then

$$\begin{aligned}
\overline{\overline{\begin{Bmatrix} r \\ u \;\; v \end{Bmatrix}}} &= \overline{\begin{Bmatrix} k \\ i \;\; j \end{Bmatrix}} \frac{\partial \overline{x}^i}{\partial \overline{\overline{x}}^u} \frac{\partial \overline{x}^j}{\partial \overline{\overline{x}}^v} \frac{\partial \overline{\overline{x}}^r}{\partial \overline{x}^k} + \frac{\partial \overline{\overline{x}}^r}{\partial \overline{x}^k} \frac{\partial^2 \overline{x}^k}{\partial \overline{\overline{x}}^u \partial \overline{\overline{x}}^v} \\
&= \begin{Bmatrix} s \\ p \;\; q \end{Bmatrix} \frac{\partial \overline{x}^k}{\partial x^s} \frac{\partial x^p}{\partial \overline{x}^i} \frac{\partial x^q}{\partial \overline{x}^j} \frac{\partial \overline{x}^i}{\partial \overline{\overline{x}}^u} \frac{\partial \overline{x}^j}{\partial \overline{\overline{x}}^v} \frac{\partial \overline{\overline{x}}^r}{\partial \overline{x}^k} \\
&\quad + \frac{\partial^2 x^s}{\partial \overline{x}^i \partial \overline{x}^j} \frac{\partial \overline{x}^k}{\partial x^s} \frac{\partial \overline{x}^i}{\partial \overline{\overline{x}}^u} \frac{\partial \overline{x}^j}{\partial \overline{\overline{x}}^v} \frac{\partial \overline{\overline{x}}^r}{\partial \overline{x}^k} + \frac{\partial \overline{\overline{x}}^r}{\partial \overline{x}^k} \frac{\partial^2 \overline{x}^k}{\partial \overline{\overline{x}}^u \partial \overline{\overline{x}}^v} \\
&= \begin{Bmatrix} s \\ p \;\; q \end{Bmatrix} \frac{\partial x^p}{\partial \overline{\overline{x}}^u} \frac{\partial x^q}{\partial \overline{\overline{x}}^v} \frac{\partial \overline{\overline{x}}^r}{\partial x^s} + \frac{\partial^2 x^s}{\partial \overline{x}^i \partial \overline{x}^j} \frac{\partial \overline{\overline{x}}^r}{\partial x^s} \frac{\partial \overline{x}^i}{\partial \overline{\overline{x}}^u} \frac{\partial \overline{x}^j}{\partial \overline{\overline{x}}^v} + \frac{\partial \overline{\overline{x}}^r}{\partial \overline{x}^k} \frac{\partial^2 \overline{x}^k}{\partial \overline{\overline{x}}^u \partial \overline{\overline{x}}^v}. \quad (3.12)
\end{aligned}$$

Differentiating with respect to $\overline{\overline{x}}^v$, of the relation $\frac{\partial x^s}{\partial \overline{x}^i}\frac{\partial \overline{x}^i}{\partial \overline{\overline{x}}^u} = \frac{\partial x^s}{\partial \overline{\overline{x}}^u}$, we get

$$\frac{\partial}{\partial \overline{\overline{x}}^v}\left(\frac{\partial x^s}{\partial \overline{x}^i}\right)\frac{\partial \overline{x}^i}{\partial \overline{\overline{x}}^u} + \frac{\partial x^s}{\partial \overline{x}^i}\frac{\partial}{\partial \overline{\overline{x}}^v}\left(\frac{\partial \overline{x}^i}{\partial \overline{\overline{x}}^u}\right) = \frac{\partial^2 x^s}{\partial \overline{\overline{x}}^u \partial \overline{\overline{x}}^v}$$

or

$$\frac{\partial^2 x^s}{\partial \overline{x}^i \partial \overline{x}^j}\frac{\partial \overline{x}^j}{\partial \overline{\overline{x}}^v}\frac{\partial \overline{x}^i}{\partial \overline{\overline{x}}^u} + \frac{\partial x^s}{\partial \overline{x}^i}\frac{\partial^2 \overline{x}^i}{\partial \overline{\overline{x}}^u \partial \overline{\overline{x}}^v} = \frac{\partial^2 x^s}{\partial \overline{\overline{x}}^u \partial \overline{\overline{x}}^v}$$

or

$$\frac{\partial^2 x^s}{\partial \overline{x}^i \partial \overline{x}^j}\frac{\partial \overline{x}^j}{\partial \overline{\overline{x}}^v}\frac{\partial \overline{x}^i}{\partial \overline{\overline{x}}^u}\frac{\partial \overline{\overline{x}}^r}{\partial x^s} + \frac{\partial x^s}{\partial \overline{x}^i}\frac{\partial^2 \overline{x}^i}{\partial \overline{\overline{x}}^u \partial \overline{\overline{x}}^v}\frac{\partial \overline{\overline{x}}^r}{\partial x^s} = \frac{\partial^2 x^s}{\partial \overline{\overline{x}}^u \partial \overline{\overline{x}}^v}\frac{\partial \overline{\overline{x}}^r}{\partial x^s}$$

or

$$\frac{\partial^2 x^s}{\partial \overline{x}^i \partial \overline{x}^j}\frac{\partial \overline{x}^j}{\partial \overline{\overline{x}}^v}\frac{\partial \overline{x}^i}{\partial \overline{\overline{x}}^u}\frac{\partial \overline{\overline{x}}^r}{\partial x^s} + \frac{\partial^2 \overline{x}^k}{\partial \overline{\overline{x}}^u \partial \overline{\overline{x}}^v}\frac{\partial \overline{\overline{x}}^r}{\partial \overline{x}^k} = \frac{\partial^2 x^s}{\partial \overline{\overline{x}}^u \partial \overline{\overline{x}}^v}\frac{\partial \overline{\overline{x}}^r}{\partial x^s}, \tag{3.13}$$

where we have to replace the dummy index i by k in the second term on the left hand side. Using this relation (3.13), from (3.12) we get

$$\overline{\overline{\left\{\begin{matrix} r \\ u \quad v \end{matrix}\right\}}} = \left\{\begin{matrix} s \\ p \quad q \end{matrix}\right\}\frac{\partial \overline{x}^p}{\partial \overline{\overline{x}}^u}\frac{\partial \overline{x}^q}{\partial \overline{\overline{x}}^v}\frac{\partial \overline{\overline{x}}^r}{\partial \overline{x}^s} + \frac{\partial^2 x^s}{\partial \overline{\overline{x}}^u \partial \overline{\overline{x}}^v}\frac{\partial \overline{\overline{x}}^r}{\partial x^s}.$$

This shows that if we make direct transformation from x^i to $\overline{\overline{x}}^i$, we get the same law of transformation. This property is called transformation of Christoffel's symbols form a group.

EXAMPLE 3.1.14 *Let the Christoffel symbols formed from the symmetric tensors* $a_{ij}(x)$ *and* $b_{ij}(x)$ *be* ${}_a\left\{\begin{matrix} i \\ j \quad k \end{matrix}\right\}$ *and* ${}_b\left\{\begin{matrix} i \\ j \quad k \end{matrix}\right\}$, *prove that the quantities*

$${}_a\left\{\begin{matrix} i \\ j \quad k \end{matrix}\right\} - {}_b\left\{\begin{matrix} i \\ j \quad k \end{matrix}\right\}$$

are components of a mixed tensor of rank 3.

Solution: From the Eq. (3.9), we get

$$\overline{\left\{\begin{matrix} p \\ l \quad m \end{matrix}\right\}} = \left[\left\{\begin{matrix} r \\ i \quad k \end{matrix}\right\}\frac{\partial x^i}{\partial \overline{x}^l}\frac{\partial x^k}{\partial \overline{x}^m} + \frac{\partial^2 x^r}{\partial \overline{x}^l \partial \overline{x}^m}\right]\frac{\partial \overline{x}^p}{\partial x^q}.$$

Hence, we may write the two results as follows, on using the above:

$$\overline{\left\{\begin{matrix} p \\ l \quad m \end{matrix}\right\}}_a = \left[\left\{\begin{matrix} r \\ i \quad k \end{matrix}\right\}_a\frac{\partial x^i}{\partial \overline{x}^l}\frac{\partial x^k}{\partial \overline{x}^m} + \frac{\partial^2 x^r}{\partial \overline{x}^l \partial \overline{x}^m}\right]\frac{\partial \overline{x}^p}{\partial x^q}$$

and

$$\overline{\begin{Bmatrix} p \\ l \;\; m \end{Bmatrix}_b} = \left[\begin{Bmatrix} r \\ i \;\; k \end{Bmatrix}_b \frac{\partial x^i}{\partial \overline{x}^l}\frac{\partial x^k}{\partial \overline{x}^m} + \frac{\partial^2 x^r}{\partial \overline{x}^l \partial \overline{x}^m}\right]\frac{\partial \overline{x}^p}{\partial x^q}.$$

Subtracting, we obtain

$$\left[\overline{\begin{Bmatrix} p \\ l \;\; m \end{Bmatrix}_a} - \overline{\begin{Bmatrix} p \\ l \;\; m \end{Bmatrix}_b}\right] = \left[\begin{Bmatrix} r \\ i \;\; k \end{Bmatrix}_a - \begin{Bmatrix} r \\ i \;\; k \end{Bmatrix}_b\right]\frac{\partial x^i}{\partial \overline{x}^l}\frac{\partial x^k}{\partial \overline{x}^m}\frac{\partial \overline{x}^p}{\partial x^r},$$

which is of the form

$$\overline{A}^p_{lm} = \frac{\partial x^i}{\partial \overline{x}^l}\frac{\partial x^k}{\partial \overline{x}^m}\frac{\partial \overline{x}^p}{\partial x^r}A^r_{ik}.$$

This equation shows that

$$\begin{Bmatrix} i \\ j \;\; k \end{Bmatrix}_a - \begin{Bmatrix} i \\ j \;\; k \end{Bmatrix}_b \equiv A^i_{jk},$$

say, represents the components of a mixed tensor of type $(1, 2)$.

EXAMPLE 3.1.15 *If B_i are the components of a covariant vector, determine whether $\Gamma^i_{jk} = \begin{Bmatrix} i \\ j \;\; k \end{Bmatrix} + 2\delta^i_j B_k$ are components of a tensor.*

Solution: The law of transformation of Christoffel symbols of the second kind is

$$\overline{\begin{Bmatrix} i \\ j \;\; k \end{Bmatrix}} = \begin{Bmatrix} r \\ t \;\; s \end{Bmatrix}\frac{\partial \overline{x}^i}{\partial x^r}\frac{\partial x^t}{\partial \overline{x}^j}\frac{\partial x^s}{\partial \overline{x}^k} + \frac{\partial \overline{x}^i}{\partial x^m}\frac{\partial^2 x^m}{\partial \overline{x}^j \partial \overline{x}^k}.$$

From the given relation for Γ^i_{jk}, we get

$$\begin{aligned}
\overline{\Gamma^i_{jk}} &= \overline{\begin{Bmatrix} i \\ j \;\; k \end{Bmatrix}} + 2\overline{\delta^i_j}\,\overline{B}_k = \begin{Bmatrix} r \\ t \;\; s \end{Bmatrix}\frac{\partial \overline{x}^i}{\partial x^r}\frac{\partial x^t}{\partial \overline{x}^j}\frac{\partial x^s}{\partial \overline{x}^k} + \frac{\partial \overline{x}^i}{\partial x^m}\frac{\partial^2 x^m}{\partial \overline{x}^j \partial \overline{x}^k} + 2\delta^i_j\frac{\partial x^p}{\partial \overline{x}^k}B_p \\
&= \left[\begin{Bmatrix} r \\ t \;\; s \end{Bmatrix} + 2\delta^r_t B_s\right]\frac{\partial \overline{x}^i}{\partial x^r}\frac{\partial x^t}{\partial \overline{x}^j}\frac{\partial x^s}{\partial \overline{x}^k} + \frac{\partial \overline{x}^i}{\partial x^m}\frac{\partial^2 x^m}{\partial \overline{x}^j \partial \overline{x}^k} \\
&= \Gamma^r_{ts}\frac{\partial \overline{x}^i}{\partial x^r}\frac{\partial x^t}{\partial \overline{x}^j}\frac{\partial x^s}{\partial \overline{x}^k} + \frac{\partial \overline{x}^i}{\partial x^m}\frac{\partial^2 x^m}{\partial \overline{x}^j \partial \overline{x}^k}.
\end{aligned}$$

From this relation it follows that Γ^i_{jk} are not the components of a tensor due to the presence of the second term in its right-hand side.

3.2 Covariant Differentiation

In Chapter 1, we have already studied the algebraic operations of tensors, consisting the so-called *tensor algebra* of V_N. These algebras are such that when applied to tensors, they produce again tensors. But regarding differentiation the matter is somewhat different, because although the partial differentiation of an invariant produces a tensor of type $(0, 1)$; i.e. a covariant vector, partial differentiation of a tensor of rank ≥ 1 does not, in general, produce a tensor. The necessity therefore, arises to introduce a new kind of differentiation which when applied to a tensor will produce a tensor. Such a differentiation, called *covariant differentiation* will be considered in this section.

It is to be noted that the word 'covariant' was also used to mean, independent of the choice of co-ordinates, as in the principle of covariance of the general theory of relativity which asserts that the laws of Physics must be independent of the space-time co-ordinates. It seems more plausible that the name covariant differentiation is just due to this property.

Covariant differentiation of covariant vectors: Partial differentiation of the transformation law (1.43) of a covariant vector A_i with respect to $\overline{x}^j$, we get

$$\begin{aligned}\frac{\partial \overline{A}_i}{\partial \overline{x}^j} &= \frac{\partial x^k}{\partial \overline{x}^i}\frac{\partial A_k}{\partial x^l}\frac{\partial x^l}{\partial \overline{x}^j} + \frac{\partial^2 x^k}{\partial \overline{x}^i \partial \overline{x}^j}A_k; \text{ using chain rule,}\\ &= \frac{\partial x^k}{\partial \overline{x}^i}\frac{\partial x^l}{\partial \overline{x}^j}\frac{\partial A_k}{\partial x^l} + \frac{\partial x^k}{\partial \overline{x}^p}\overline{\begin{Bmatrix} p \\ i\ \ j\end{Bmatrix}}A_k - \frac{\partial x^m}{\partial \overline{x}^i}\frac{\partial x^n}{\partial \overline{x}^j}\begin{Bmatrix} k \\ m\ \ n\end{Bmatrix}A_k; \text{ from (3.10)}\end{aligned}$$

or

$$\frac{\partial \overline{A}_i}{\partial \overline{x}^j} - \overline{\begin{Bmatrix} p \\ i\ \ j\end{Bmatrix}}\frac{\partial x^k}{\partial \overline{x}^p}A_k = \frac{\partial A_k}{\partial x^l}\frac{\partial x^k}{\partial \overline{x}^i}\frac{\partial x^l}{\partial \overline{x}^j} - \begin{Bmatrix} k \\ m\ \ n\end{Bmatrix}\frac{\partial x^m}{\partial \overline{x}^i}\frac{\partial x^n}{\partial \overline{x}^j}A_k.$$

Replacing the dummy indices k and l in the first term of the right-hand side by t and s, respectively, and the dummy indices k, m, n in the second term of the right-hand side by r, t and s, respectively, we get

$$\begin{aligned}\frac{\partial \overline{A}_i}{\partial \overline{x}^j} - \overline{\begin{Bmatrix} p \\ i\ \ j\end{Bmatrix}}\frac{\partial x^k}{\partial \overline{x}^p}A_k &= \frac{\partial A_t}{\partial x^s}\frac{\partial x^t}{\partial \overline{x}^i}\frac{\partial x^s}{\partial \overline{x}^j} - \begin{Bmatrix} r \\ t\ \ s\end{Bmatrix}\frac{\partial x^t}{\partial \overline{x}^i}\frac{\partial x^s}{\partial \overline{x}^j}A_r\\ &= \left[\frac{\partial A_t}{\partial x^s} - \begin{Bmatrix} r \\ t\ \ s\end{Bmatrix}A_r\right]\frac{\partial x^t}{\partial \overline{x}^i}\frac{\partial x^s}{\partial \overline{x}^j}. \qquad (3.14)\end{aligned}$$

From Eq. (3.14) it follows that the N^2 function $\frac{\partial A_t}{\partial x^s} - \begin{Bmatrix} r \\ t\ \ s\end{Bmatrix}A_r$ containing the partial derivatives of a covariant vector and the Christoffel symbols of second kind are the components of a covariant tensor of type $(0, 2)$, known as the covariant derivative, denoted by $A_{t,s}$. This tensor is defined to be the covariant derivative of a covariant vector with components A_i of type $(0,1)$. Thus,

$$A_{t,s} = \frac{\partial A_t}{\partial x^s} - \left\{ \begin{matrix} r \\ t \;\; s \end{matrix} \right\} A_r. \tag{3.15}$$

In other words, when the components of $\frac{\partial A_t}{\partial x^s}$ are corrected by subtracting certain linear combinations of the combinations A_t, the result is a tensor (and not just an affine tensor).

(i) The two covariant indices are noted t and, s to emphasise that the second index arose from an operation with respect to the sth co-ordinate.

(ii) The covariant derivative and the partial derivative coincide when the g_{ij} are constants (as in a rectangular co-ordinate system).

(iii) The function $A_{i,j}$ is said to be the jth covariant derivative of the vector A_i.

EXAMPLE 3.2.1 *Show that if the covariant derivative of a covariant vector is symmetric, then the vector is gradient.*

Solution: If the covariant derivative of a covariant vector is symmetric, then $A_{i,j} = A_{j,i}$. Hence, from Eq. (3.15), we get

$$\frac{\partial A_i}{\partial x^j} - \left\{ \begin{matrix} m \\ i \;\; j \end{matrix} \right\} A_m = \frac{\partial A_j}{\partial x^i} - \left\{ \begin{matrix} m \\ j \;\; i \end{matrix} \right\} A_m.$$

Therefore

$$\frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i} = 0 \quad \Rightarrow \quad \frac{\partial A_i}{\partial x^j} dx^j = \frac{\partial A_j}{\partial x^i} dx^j$$

or

$$dA_i = \frac{\partial}{\partial x^i}\left(A_j dx^j\right)$$

or

$$A_i = \int \frac{\partial}{\partial x^i}\left(A_j dx^j\right) = \frac{\partial}{\partial x^i} \int A_j dx^j.$$

But $\int A_j dx^j$ is a scalar quantity, let it be ϕ. Hence, we get from above

$$A_i = \frac{\partial \phi}{\partial x^i} = \phi_{,i} = \text{ grad}\phi.$$

Covariant differentiation of covariant tensors: Now, let A_{ij} be the components of a tensor of the type $(0,2)$, then by, tensor law of transformation,

$$\overline{A}_{pq} = \frac{\partial x^i}{\partial \overline{x}^p} \frac{\partial x^j}{\partial \overline{x}^q} A_{ij}.$$

Differentiating this relation with respect to $\overline{x}^r$, we get

$$\frac{\partial \overline{A}_{pq}}{\partial \overline{x}^r} = \frac{\partial A_{ij}}{\partial x^h}\frac{\partial x^h}{\partial \overline{x}^r}\frac{\partial x^i}{\partial \overline{x}^p}\frac{\partial x^j}{\partial \overline{x}^q} + A_{ij}\frac{\partial^2 x^i}{\partial \overline{x}^r \partial \overline{x}^p}\frac{\partial x^j}{\partial \overline{x}^q} + A_{ij}\frac{\partial x^i}{\partial \overline{x}^p}\frac{\partial^2 x^j}{\partial \overline{x}^r \partial \overline{x}^q}$$

$$= \frac{\partial A_{ij}}{\partial x^h}\frac{\partial x^h}{\partial \overline{x}^r}\frac{\partial x^i}{\partial \overline{x}^p}\frac{\partial x^j}{\partial \overline{x}^q} + A_{ij}\frac{\partial x^j}{\partial \overline{x}^q}\left[\frac{\partial x^i}{\partial \overline{x}^s}\overline{\left\{\begin{matrix} s \\ r\ \ p\end{matrix}\right\}} - \frac{\partial x^m}{\partial \overline{x}^r}\frac{\partial x^n}{\partial \overline{x}^p}\left\{\begin{matrix} i \\ m\ \ n\end{matrix}\right\}\right]$$

$$+A_{ij}\frac{\partial x^i}{\partial \overline{x}^p}\left[\frac{\partial x^j}{\partial \overline{x}^t}\overline{\left\{\begin{matrix} t \\ r\ \ q\end{matrix}\right\}} - \frac{\partial x^u}{\partial \overline{x}^r}\frac{\partial x^v}{\partial \overline{x}^q}\left\{\begin{matrix} j \\ u\ \ v\end{matrix}\right\}\right]; \text{using Eq. (3.10)}$$

$$= \overline{\left\{\begin{matrix} s \\ r\ \ p\end{matrix}\right\}}\frac{\partial x^j}{\partial \overline{x}^q}\frac{\partial x^i}{\partial \overline{x}^s}A_{ij} - \overline{\left\{\begin{matrix} t \\ r\ \ q\end{matrix}\right\}}\frac{\partial x^i}{\partial \overline{x}^p}\frac{\partial x^j}{\partial \overline{x}^t}A_{ij} + \frac{\partial A_{ij}}{\partial x^h}\frac{\partial x^h}{\partial \overline{x}^r}\frac{\partial x^i}{\partial \overline{x}^p}\frac{\partial x^j}{\partial \overline{x}^q}$$

$$-\left\{\begin{matrix} i \\ m\ \ n\end{matrix}\right\}\frac{\partial x^m}{\partial \overline{x}^r}\frac{\partial x^n}{\partial \overline{x}^p}\frac{\partial x^j}{\partial \overline{x}^q}A_{ij} - \left\{\begin{matrix} j \\ u\ \ v\end{matrix}\right\}\frac{\partial x^i}{\partial \overline{x}^p}\frac{\partial x^v}{\partial \overline{x}^q}\frac{\partial x^u}{\partial \overline{x}^r}A_{ij}.$$

Replacing the dummy indices i, m, n in the second term of the right-hand side by t, h, i respectively and the dummy indices j, u, v in the third term of this side by t, h and j respectively, we get

$$\frac{\partial \overline{A}_{pq}}{\partial \overline{x}^r} = \overline{\left\{\begin{matrix} s \\ r\ \ p\end{matrix}\right\}}\frac{\partial x^j}{\partial \overline{x}^q}\frac{\partial x^i}{\partial \overline{x}^s}A_{ij} - \overline{\left\{\begin{matrix} t \\ r\ \ q\end{matrix}\right\}}\frac{\partial x^i}{\partial \overline{x}^p}\frac{\partial x^j}{\partial \overline{x}^t}A_{ij} + \frac{\partial A_{ij}}{\partial x^h}\frac{\partial x^h}{\partial \overline{x}^r}\frac{\partial x^i}{\partial \overline{x}^p}\frac{\partial x^j}{\partial \overline{x}^q}$$

$$-A_{tj}\left\{\begin{matrix} t \\ h\ \ i\end{matrix}\right\}\frac{\partial x^i}{\partial \overline{x}^p}\frac{\partial x^j}{\partial \overline{x}^q}\frac{\partial x^h}{\partial \overline{x}^r} - A_{it}\left\{\begin{matrix} t \\ h\ \ j\end{matrix}\right\}\frac{\partial x^i}{\partial \overline{x}^p}\frac{\partial x^j}{\partial \overline{x}^q}\frac{\partial x^h}{\partial \overline{x}^r}$$

or

$$\frac{\partial \overline{A}_{pq}}{\partial \overline{x}^r} - \overline{\left\{\begin{matrix} s \\ r\ \ p\end{matrix}\right\}}\overline{A}_{sq} - \overline{\left\{\begin{matrix} t \\ r\ \ q\end{matrix}\right\}}\overline{A}_{pt}$$

$$= \left[\frac{\partial A_{ij}}{\partial x^h} - A_{tj}\left\{\begin{matrix} t \\ i\ \ h\end{matrix}\right\} - A_{it}\left\{\begin{matrix} t \\ j\ \ h\end{matrix}\right\}\right]\frac{\partial x^i}{\partial \overline{x}^p}\frac{\partial x^j}{\partial \overline{x}^q}\frac{\partial x^h}{\partial \overline{x}^r}. \qquad (3.16)$$

From Eq. (3.16) it follows that $\frac{\partial A_{ij}}{\partial x^h} - A_{tj}\left\{\begin{matrix} t \\ i\ \ h\end{matrix}\right\} - A_{it}\left\{\begin{matrix} t \\ j\ \ h\end{matrix}\right\}$ are the N^2 components of a tensor of the type $(0, 3)$. This tensor is defined to be the covariant derivative of the covariant tensor with components A_{ij} and is denoted by $A_{ij,h}$. Thus,

$$A_{ij,h} = \frac{\partial A_{ij}}{\partial x^h} - A_{tj}\left\{\begin{matrix} t \\ i\ \ h\end{matrix}\right\} - A_{it}\left\{\begin{matrix} t \\ j\ \ h\end{matrix}\right\}. \qquad (3.17)$$

Thus, the covariant derivative of a tensor of the type $(0, 2)$ is a tensor of type $(0, 3)$.

Covariant differentiation of contravariant vectors: Let A^i be the component of contravariant vector, then by the law of transformation Eq. (1.40), we get, $A^k = \frac{\partial x^k}{\partial \overline{x}^i}\overline{A}^i$. Differentiating both sides with respect to x^j we get

$$\begin{aligned}\frac{\partial A^k}{\partial x^j} &= \frac{\partial \overline{A}^i}{\partial \overline{x}^p}\frac{\partial \overline{x}^p}{\partial x^j}\frac{\partial x^k}{\partial \overline{x}^i} + \overline{A}^i \frac{\partial^2 x^k}{\partial \overline{x}^i \partial \overline{x}^p}\frac{\partial \overline{x}^p}{\partial x^j} \\ &= \frac{\partial \overline{A}^i}{\partial \overline{x}^p}\frac{\partial \overline{x}^p}{\partial x^j}\frac{\partial x^k}{\partial \overline{x}^i} + \overline{A}^i \frac{\partial \overline{x}^p}{\partial x^j}\left[\frac{\partial x^k}{\partial \overline{x}^m}\overline{\begin{Bmatrix} m \\ i \;\; p \end{Bmatrix}} - \frac{\partial x^t}{\partial \overline{x}^i}\frac{\partial x^s}{\partial \overline{x}^p}\begin{Bmatrix} k \\ t \;\; s \end{Bmatrix}\right]; \text{using Eq. (3.10)} \\ &= \frac{\partial \overline{A}^i}{\partial \overline{x}^p}\frac{\partial \overline{x}^p}{\partial x^j}\frac{\partial x^k}{\partial \overline{x}^i} + \overline{A}^i \frac{\partial \overline{x}^p}{\partial x^j}\frac{\partial x^k}{\partial \overline{x}^m}\overline{\begin{Bmatrix} m \\ i \;\; p \end{Bmatrix}} - A^t \begin{Bmatrix} k \\ t \;\; j \end{Bmatrix}\end{aligned}$$

or

$$\frac{\partial A^k}{\partial x^j} + A^t \begin{Bmatrix} k \\ t \;\; j \end{Bmatrix} = \left[\frac{\partial \overline{A}^i}{\partial \overline{x}^p} + \overline{A}^m \overline{\begin{Bmatrix} i \\ m \;\; p \end{Bmatrix}}\right]\frac{\partial \overline{x}^p}{\partial x^j}\frac{\partial x^k}{\partial \overline{x}^i},$$

where we have to replace the dummy index i by m and m by i. Multiplying both sides by $\frac{\partial x^i}{\partial \overline{x}^p}\frac{\partial \overline{x}^j}{\partial x^k}$ we get

$$\begin{aligned}\frac{\partial x^i}{\partial \overline{x}^p}\frac{\partial \overline{x}^j}{\partial x^k}\left[\frac{\partial A^k}{\partial x^j} + A^t \begin{Bmatrix} k \\ t \;\; j \end{Bmatrix}\right] &= \frac{\partial x^i}{\partial \overline{x}^p}\frac{\partial \overline{x}^j}{\partial x^k}\frac{\partial \overline{x}^p}{\partial x^j}\frac{\partial x^k}{\partial \overline{x}^i}\left[\frac{\partial \overline{A}^i}{\partial \overline{x}^p} + \overline{A}^m \overline{\begin{Bmatrix} i \\ m \;\; p \end{Bmatrix}}\right] \\ &= \frac{\partial \overline{A}^i}{\partial \overline{x}^p} + \overline{A}^m \overline{\begin{Bmatrix} i \\ m \;\; p \end{Bmatrix}}; \quad \text{as} \quad \frac{\partial x^i}{\partial \overline{x}^p}\frac{\partial \overline{x}^j}{\partial x^k}\frac{\partial \overline{x}^p}{\partial x^j}\frac{\partial x^k}{\partial \overline{x}^i} = 1\end{aligned}$$

or

$$\frac{\partial \overline{A}^i}{\partial \overline{x}^p} + \overline{A}^m \overline{\begin{Bmatrix} i \\ m \;\; p \end{Bmatrix}} = \frac{\partial x^i}{\partial \overline{x}^p}\frac{\partial \overline{x}^j}{\partial x^k}\left[\frac{\partial A^k}{\partial x^j} + A^t \begin{Bmatrix} k \\ t \;\; j \end{Bmatrix}\right]. \tag{3.18}$$

From Eq. (3.18) it follows that $\frac{\partial A^k}{\partial x^j} + A^t \begin{Bmatrix} k \\ t \;\; j \end{Bmatrix}$ are the N^2 components of a mixed tensor of type $(1,1)$ and is denoted by $A^k_{,j}$. This tensor is defined to be the covariant derivative of the contravariant vector with components A^i. Therefore,

$$A^k_{,j} = \frac{\partial A^k}{\partial x^j} + A^t \begin{Bmatrix} k \\ t \;\; j \end{Bmatrix}. \tag{3.19}$$

Similarly, the covariant derivative of a contravariant tensor of order two is given by the formula

$$A^{ik}_{,j} = \frac{\partial A^{ik}}{\partial x^j} + \begin{Bmatrix} i \\ j \;\; l \end{Bmatrix} A^{lk} + \begin{Bmatrix} k \\ j \;\; l \end{Bmatrix} A^{il}. \tag{3.20}$$

From Eq. (3.20) it follows that, the covariant derivative of a tensor of type $(2,0)$ is a tensor of type $(2,1)$.

EXAMPLE 3.2.2 *A fluid in motion in a plane has the velocity vector field given by $v^i = (x^2, y^2)$ in Cartesian co-ordinates. Find the covariant derivative of the vector field in polar co-ordinates.*

Solution: Let us choose the usual polar co-ordinates and let u^i denote the velocity vector in polar co-ordinates. A simple transformation from the Cartesian components v^i to the polar components u^i according to Eq. (1.40) gives,

$$u^1 = r^2(\cos^3\theta + \sin^3\theta);\ u^2 = r\sin\theta\cos\theta(\sin\theta - \cos\theta).$$

Let us take $x^1 = r$ and $x^2 = \theta$, then the non-vanishing Christoffel symbols of the second kind are

$$\left\{\begin{matrix} 1 \\ 2\ \ 2 \end{matrix}\right\} = -r;\quad \left\{\begin{matrix} 2 \\ 1\ \ 2 \end{matrix}\right\} = \left\{\begin{matrix} 2 \\ 2\ \ 1 \end{matrix}\right\} = \frac{1}{r}.$$

The covariant derivative of u^i is given by (3.19) as

$$u^i_{,j} = \frac{\partial u^i}{\partial x^j} + u^k \left\{\begin{matrix} i \\ j\ \ k \end{matrix}\right\}.$$

Thus, the components of the covariant derivatives in polar co-ordinates are found to be

$$\begin{aligned}
u^1_{,1} &= \frac{\partial u^1}{\partial x^1} + u^k \left\{\begin{matrix} 1 \\ 1\ \ k \end{matrix}\right\} = \frac{\partial u^1}{\partial x^1} + u^1 \left\{\begin{matrix} 1 \\ 1\ \ 1 \end{matrix}\right\} + u^2 \left\{\begin{matrix} 1 \\ 1\ \ 2 \end{matrix}\right\} \\
&= \frac{\partial u^1}{\partial x^1} = 2r(\cos^3\theta + \sin^3\theta). \\
u^1_{,2} &= \frac{\partial u^1}{\partial x^2} + u^k \left\{\begin{matrix} 1 \\ 2\ \ k \end{matrix}\right\} = \frac{\partial u^1}{\partial x^2} + u^1 \left\{\begin{matrix} 1 \\ 1\ \ 1 \end{matrix}\right\} + u^2 \left\{\begin{matrix} 1 \\ 2\ \ 2 \end{matrix}\right\} \\
&= 2r^2\sin\theta\cos\theta(\sin\theta - \cos\theta). \\
u^2_{,1} &= \frac{\partial u^2}{\partial x^1} + u^k \left\{\begin{matrix} 2 \\ 1\ \ k \end{matrix}\right\} = 2\sin\theta\cos\theta(\sin\theta - \cos\theta). \\
u^2_{,2} &= \frac{\partial u^2}{\partial x^2} + u^k \left\{\begin{matrix} 2 \\ 2\ \ k \end{matrix}\right\} = 2r\sin\theta\cos\theta(\sin\theta + \cos\theta).
\end{aligned}$$

Covariant differentiation of tensor of type (1,1): Let A^i_j be a mixed tensor of rank 2, so by tensor law of transformation,

$$\overline{A}^i_j = A^\alpha_\beta \frac{\partial \overline{x}^i}{\partial x^\alpha}\frac{\partial x^\beta}{\partial \overline{x}^j} \Rightarrow \overline{A}^i_j \frac{\partial x^\alpha}{\partial \overline{x}^i} = A^\alpha_\beta \frac{\partial x^\beta}{\partial \overline{x}^j}.$$

Differentiating both sides partially with respect to $\overline{x}^k$, we get

$$\frac{\partial \overline{A}^i_j}{\partial \overline{x}^k}\frac{\partial x^\alpha}{\partial \overline{x}^i} + \overline{A}^i_j \frac{\partial^2 x^\alpha}{\partial \overline{x}^i \partial \overline{x}^k} = \frac{\partial A^\alpha_\beta}{\partial x^\gamma}\frac{\partial x^\beta}{\partial \overline{x}^j}\frac{\partial x^\gamma}{\partial \overline{x}^k} + A^\alpha_\beta \frac{\partial^2 x^\beta}{\partial \overline{x}^j \partial \overline{x}^k}$$

or

$$\frac{\partial \overline{A}^i_j}{\partial \overline{x}^k}\frac{\partial x^\alpha}{\partial \overline{x}^i} + \overline{A}^i_j \left[\overline{\begin{Bmatrix} h \\ i \;\; k \end{Bmatrix}} \frac{\partial x^\alpha}{\partial \overline{x}^h} - \begin{Bmatrix} \alpha \\ m \;\; \gamma \end{Bmatrix} \frac{\partial x^m}{\partial \overline{x}^i}\frac{\partial x^\gamma}{\partial \overline{x}^k} \right] = \frac{\partial A^\alpha_\beta}{\partial x^\gamma}\frac{\partial x^\beta}{\partial \overline{x}^j}\frac{\partial x^\gamma}{\partial \overline{x}^k}$$
$$+ A^\alpha_\beta \left[\overline{\begin{Bmatrix} h \\ j \;\; k \end{Bmatrix}} \frac{\partial x^\beta}{\partial \overline{x}^h} - \begin{Bmatrix} \beta \\ n \;\; \gamma \end{Bmatrix} \frac{\partial x^\beta}{\partial \overline{x}^j}\frac{\partial x^\gamma}{\partial \overline{x}^k} \right]; \text{ from Eq. (3.10)}$$

or

$$\frac{\partial \overline{A}^i_j}{\partial \overline{x}^k}\frac{\partial x^\alpha}{\partial \overline{x}^i} + \overline{A}^i_j \overline{\begin{Bmatrix} h \\ i \;\; k \end{Bmatrix}} \frac{\partial x^\alpha}{\partial \overline{x}^h} - A^\alpha_\beta \overline{\begin{Bmatrix} h \\ j \;\; k \end{Bmatrix}} \frac{\partial x^\beta}{\partial \overline{x}^h}$$
$$= \frac{\partial A^\alpha_\beta}{\partial x^\gamma}\frac{\partial x^\beta}{\partial \overline{x}^j}\frac{\partial x^\gamma}{\partial \overline{x}^k} + \overline{A}^i_j \frac{\partial x^\alpha}{\partial \overline{x}^i}\frac{\partial x^\gamma}{\partial \overline{x}^k} \begin{Bmatrix} \alpha \\ m \;\; \gamma \end{Bmatrix} - A^\alpha_\beta \frac{\partial x^\beta}{\partial \overline{x}^j}\frac{\partial x^\gamma}{\partial \overline{x}^k} \begin{Bmatrix} \beta \\ n \;\; \gamma \end{Bmatrix}$$

or

$$\frac{\partial \overline{A}^i_j}{\partial \overline{x}^k}\frac{\partial x^\alpha}{\partial \overline{x}^i} + \overline{A}^i_j \overline{\begin{Bmatrix} h \\ i \;\; k \end{Bmatrix}} \frac{\partial x^\alpha}{\partial \overline{x}^h} - A^i_j \overline{\begin{Bmatrix} h \\ j \;\; k \end{Bmatrix}} \frac{\partial x^\alpha}{\partial \overline{x}^i}$$
$$= \frac{\partial A^\alpha_\beta}{\partial x^\beta}\frac{\partial x^\beta}{\partial \overline{x}^j}\frac{\partial x^\gamma}{\partial \overline{x}^k} + \overline{A}^m_n \frac{\partial x^m}{\partial \overline{x}^j}\frac{\partial x^\gamma}{\partial \overline{x}^k} \begin{Bmatrix} \alpha \\ m \;\; \gamma \end{Bmatrix} - A^\alpha_\beta \frac{\partial x^m}{\partial \overline{x}^j}\frac{\partial x^\gamma}{\partial \overline{x}^k} \begin{Bmatrix} \beta \\ n \;\; \gamma \end{Bmatrix}$$

or

$$\left[\frac{\partial \overline{A}^i_j}{\partial \overline{x}^k} + \overline{A}^h_j \overline{\begin{Bmatrix} i \\ h \;\; k \end{Bmatrix}} \frac{\partial x^\alpha}{\partial \overline{x}^h} - A^i_h \overline{\begin{Bmatrix} h \\ j \;\; k \end{Bmatrix}} \right] \frac{\partial x^\alpha}{\partial \overline{x}^i}$$
$$= \left[\frac{\partial A^m_\beta}{\partial x^\gamma} + A^\alpha_\beta \begin{Bmatrix} \alpha \\ m \;\; \gamma \end{Bmatrix} - A^\alpha_m \begin{Bmatrix} m \\ \beta \;\; \gamma \end{Bmatrix} \right] \frac{\partial x^\beta}{\partial \overline{x}^j}\frac{\partial x^\gamma}{\partial \overline{x}^k}. \tag{3.21}$$

If we write,

$$A^\alpha_{\beta,\gamma} = \frac{\partial A^\alpha_\beta}{\partial x^\gamma} + A^m_\beta \begin{Bmatrix} \alpha \\ m \;\; \gamma \end{Bmatrix} - A^\alpha_m \begin{Bmatrix} m \\ \beta \;\; \gamma \end{Bmatrix},$$

then Eq. (3.21) becomes

$$\overline{A}^i_{j,k} \frac{\partial x^\alpha}{\partial \overline{x}^i} = A^\alpha_{\beta,\gamma} \frac{\partial x^\beta}{\partial \overline{x}^j}\frac{\partial x^\gamma}{\partial \overline{x}^k} \Rightarrow \overline{A}^i_{j,k} = \frac{\partial x^\beta}{\partial \overline{x}^j}\frac{\partial x^\gamma}{\partial \overline{x}^k}\frac{\partial \overline{x}^i}{\partial x^\alpha} A^\alpha_{\beta,\gamma} \tag{3.22}$$

which confirms the tensor law of transformation. Equation (3.22) declares that $A^{\alpha}_{\beta,\gamma}$ is a mixed tensor of rank 3. In general, the covariant derivative of a tensor of type (p, q) is given by

$$A^{i_1\cdots i_p}_{j_1\cdots j_q,k} = \frac{\partial A^{i_1\cdots i_p}_{j_1\cdots j_q}}{\partial x^k} + A^{\alpha i_2\cdots i_p}_{j_1\cdots j_q}\left\{\begin{matrix} i_1 \\ \alpha \;\; k\end{matrix}\right\} + \cdots + A^{\alpha i_2\cdots i_{p-1}\alpha}_{j_1\cdots j_q}\left\{\begin{matrix} i_p \\ \alpha \;\; k\end{matrix}\right\}$$
$$- A^{\alpha i_1\cdots i_p}_{\beta j_2\cdots j_q}\left\{\begin{matrix} \beta \\ j_1 \;\; k\end{matrix}\right\} - \cdots - A^{\alpha i_1\cdots i_p}_{j_1\cdots j_{q-1}\beta}\left\{\begin{matrix} \beta \\ j_q \;\; k\end{matrix}\right\}. \tag{3.23}$$

In fact, the covariant differentiation of sums and products of tensors obey the same rule as that of ordinary differentiation.

Covariant differentiation of an invariant: The covariant derivative of an invariant (tensor of rank zero) or scalar ϕ is defined to be the partial derivative of ϕ and is denoted by $\phi_{,j}$. Thus,

$$\phi_{,j} = \frac{\partial \phi}{\partial x^j}. \tag{3.24}$$

Since $\frac{\partial \phi}{\partial x^j}$ is a covariant vector, the covariant derivative of an invariant is a tensor. Thus, the covariant derivative of a tensor of type $(0,0)$ is a tensor of type $(0,1)$. Equation (3.24) shows that the covariant derivative of a tensor of rank zero is indentical with its ordinary derivative.

Result: Ordinary differentiation of the product of two functions satisfies the distributive law. For example, $(uv)' = u'v + uv'$, where u and v are functions of some variable and the primes denote differentiation with respect to the variable. It can be shown that covariant differentiation of the product of two or more tensors also satisfies the same distributive law.

Theorem 3.2.1 (Ricci Theorem): *The fundamental tensors and the Kronecker delta behave in covariant differentiation as through they were constants.*

Proof: We have to show that $g_{ij,k} = 0, g^{ik}_{,j} = 0$ and $\delta^i_{k,j} = 0$.

(i)
$$\begin{aligned} g_{ij,k} &= \frac{\partial g_{ij}}{\partial x^k} - \left\{\begin{matrix} \alpha \\ i \;\; k\end{matrix}\right\} g_{\alpha j} - \left\{\begin{matrix} \alpha \\ j \;\; k\end{matrix}\right\} g_{i\alpha} \\ &= \frac{\partial g_{ij}}{\partial x^k} - [ik, j] - [jk, i] \\ &= \frac{\partial g_{ij}}{\partial x^k} - ([ik, j] + [jk, i]) = \frac{\partial g_{ij}}{\partial x^k} - \frac{\partial g_{ij}}{\partial x^k} = 0. \end{aligned}$$

The statement that the covariant derivative of the fundamental tensor g_{ij} is zero, is known as Ricci's lemma.

(ii) The covariant differentiation of Kronecker delta δ^i_j with respect to x^k is given by

$$\begin{aligned}\delta^i_{j,k} &= \frac{\partial \delta^i_j}{\partial x^m} - \begin{Bmatrix} \alpha \\ j \;\; k \end{Bmatrix} \delta^i_\alpha + \begin{Bmatrix} i \\ \alpha \;\; k \end{Bmatrix} \delta^\alpha_j \\ &= \frac{\partial \delta^i_j}{\partial x^m} - \begin{Bmatrix} i \\ j \;\; k \end{Bmatrix} + \begin{Bmatrix} i \\ j \;\; k \end{Bmatrix} \\ &= \frac{\partial \delta^i_j}{\partial x^m} = 0; \text{ for } \delta^i_j = 1 \text{ or } 0 = \text{ constant.}\end{aligned}$$

(iii) From the property of reciprocal tensor, we have $g_{ij}g^{jk} = \delta^k_i$. Differentiating partially with respect to x^m, we obtain

$$\frac{\partial}{\partial x^m}\left(g_{ij}g^{jk}\right) = 0; \text{ i.e. } \frac{\partial g_{ij}}{\partial x^m} g^{jk} + g_{ij}\frac{\partial g^{jk}}{\partial x^m} = 0$$

or

$$g_{ij}\frac{\partial g^{jk}}{\partial x^m} + g^{jk}\left([im, j] + [jm, i]\right) = 0$$

or

$$g_{ij}g^{li}\frac{\partial g^{jk}}{\partial x^m} + g^{li}g^{jk}[im, j] + g^{li}g^{jk}[jn, i] = 0$$

or

$$\delta^l_j \frac{\partial g^{jk}}{\partial x^m} + g^{li}\begin{Bmatrix} k \\ i \;\; m \end{Bmatrix} + g^{jk}\begin{Bmatrix} l \\ j \;\; m \end{Bmatrix} = 0$$

or

$$\frac{\partial}{\partial x^m}\left(\delta^l_j g^{jk}\right) + g^{li}\begin{Bmatrix} k \\ i \;\; m \end{Bmatrix} + g^{jk}\begin{Bmatrix} l \\ j \;\; m \end{Bmatrix} = 0; \text{ as } \frac{\partial \delta^l_j}{\partial x^m} = 0$$

or

$$\frac{\partial g^{lk}}{\partial x^m} + g^{li}\begin{Bmatrix} k \\ i \;\; m \end{Bmatrix} + g^{jk}\begin{Bmatrix} l \\ j \;\; m \end{Bmatrix} = 0$$

or

$$g^{lk}_{,m} = 0; \text{ i.e. } g^{ij}_{,k} = 0.$$

Thus, we conclude that the metric tensors are covariant constants with respect to Christoffel symbols. From Ricci's theorem we conclude that the fundamental tensors may be taken outside the sign of covariant differentiation. Because of property of the metric tensor and its inverse, the operation of covariant differentiation commutes with those of raising and lowering indices. For example,

$$A^i_{j,k} = \left(g^{ip}A_{pj}\right)_{,k} = g^{ip}A_{pj,k}.$$

EXAMPLE 3.2.3 *Write down the expression for $A^i_{,j}$, show that $A^i_{,i}$ is an invariant.*

Solution: The covariant derivative of the contravariant vector A^i with respect to x^j is denoted by $A^i_{,j}$ whose transformation law is

$$A^i_{,j} = \frac{\partial \overline{x}^m}{\partial x^j}\frac{\partial x^i}{\partial \overline{x}^k}\overline{A}^k_{,m}.$$

By contraction, we have

$$A^i_{,i} = \frac{\partial \overline{x}^m}{\partial x^i}\frac{\partial x^i}{\partial \overline{x}^k}\overline{A}^k_{,m} = \frac{\partial \overline{x}^m}{\partial \overline{x}^k}\overline{A}^k_{,m} = \delta^m_k \overline{A}^k_{,m} = \overline{A}^k_{,k};$$

which shows that $A^i_{,i}$ is an invariant.

EXAMPLE 3.2.4 *If at a specified point, the derivatives of the g_{ij} with respect to x^p are all zero, prove that the components of covariant derivative at that point are same as ordinary derivatives.*

Solution: We are given that at a point P, $\frac{\partial g_{ij}}{\partial x^p} = 0$ for all values of i, j, p. Consider a contravariant tensor A^{ij}. Then,

$$A^{ij}_{,p} = \frac{\partial A^{ij}}{\partial x^p} + A^{ik}\begin{Bmatrix} i \\ k \quad p \end{Bmatrix} + \begin{Bmatrix} j \\ k \quad p \end{Bmatrix} A^{ij} = \frac{\partial A^{ij}}{\partial x^p} \text{ at } P;$$

as the Christoffel symbols of second kind $\begin{Bmatrix} j \\ k \quad p \end{Bmatrix}$ contains term of the type $\frac{\partial g_{ij}}{\partial x^p}$. This shows that, if at a specified point, the derivatives of the g_{ij} with respect to x^p are all zero, then the components of covariant derivative at that point are same as ordinary derivatives.

EXAMPLE 3.2.5 *If A^i is a contravariant vector, show that $A^i_{,i} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(\sqrt{g}A^i\right)$; where, $g = |g_{ij}| > 0$.*

Solution: The covariant derivative of the contravariant vector A^i with respect to x^j is given by

$$A^i_{,j} = \frac{\partial A^i}{\partial x^j} + \begin{Bmatrix} i \\ r \quad j \end{Bmatrix} A^r.$$

By contraction, we have

$$\begin{aligned} A^i_{,i} &= \frac{\partial A^i}{\partial x^i} + \begin{Bmatrix} i \\ r \quad i \end{Bmatrix} A^r = \frac{\partial A^i}{\partial x^i} + A^r \frac{\partial}{\partial x^r}\left(\log \sqrt{g}\right) \\ &= \frac{\partial A^i}{\partial x^i} + \frac{A^i}{\sqrt{g}}\frac{\partial}{\partial x^i}\sqrt{g}; \text{ replacing the dummy index } r \text{ by } i \\ &= \frac{1}{\sqrt{g}}\left[\sqrt{g}\frac{\partial A^i}{\partial x^i} + A^i \frac{\partial}{\partial x^i}\sqrt{g}\right] = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(\sqrt{g}A^i\right). \end{aligned}$$

EXAMPLE 3.2.6 *Show that* $A^{ij}_{,i} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(\sqrt{g}A^{ij}\right)$, *where* A^{ij} *is a skew-symmetric tensor.*

Solution: The covariant derivative of the contravariant tensor A^{ij} with respect to x^k as in Eq. (3.20), is given by

$$A^{ij}_{,k} = \frac{\partial A^{ij}}{\partial x^k} + \begin{Bmatrix} i \\ k \;\; \alpha \end{Bmatrix} A^{\alpha j} + \begin{Bmatrix} j \\ k \;\; \alpha \end{Bmatrix} A^{i\alpha}.$$

Given that A^{ij} is a skew-symmetric tensor. Therefore,

$$A^{jk}\begin{Bmatrix} i \\ j \;\; k \end{Bmatrix} = A^{kj}\begin{Bmatrix} i \\ k \;\; j \end{Bmatrix} = A^{kj}\begin{Bmatrix} i \\ j \;\; k \end{Bmatrix} = -A^{jk}\begin{Bmatrix} i \\ j \;\; k \end{Bmatrix}$$

$$\Rightarrow A^{jk}\begin{Bmatrix} i \\ j \;\; k \end{Bmatrix} = 0.$$

By contraction, we have

$$\begin{aligned}
A^{ij}_{,i} &= \frac{\partial A^{ij}}{\partial x^i} + \begin{Bmatrix} i \\ i \;\; \alpha \end{Bmatrix} A^{\alpha j} + \begin{Bmatrix} j \\ i \;\; \alpha \end{Bmatrix} A^{i\alpha} \\
&= \frac{\partial A^{ij}}{\partial x^i} + \begin{Bmatrix} i \\ i \;\; \alpha \end{Bmatrix} A^{\alpha j};\ \text{as } A^{ij} \text{ is skew-symmetric} \\
&= \frac{\partial A^{ij}}{\partial x^i} + A^{\alpha j}\frac{\partial}{\partial x^\alpha}\left(\log\sqrt{g}\right) = \frac{\partial A^{ij}}{\partial x^i} + \frac{A^{\alpha j}}{\sqrt{g}}\frac{\partial}{\partial x^\alpha}\left(\sqrt{g}\right) \\
&= \frac{1}{\sqrt{g}}\left[\sqrt{g}\frac{\partial A^{ij}}{\partial x^i} + A^{\alpha j}\frac{\partial}{\partial x^\alpha}\left(\sqrt{g}\right)\right] = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^\alpha}\left(\sqrt{g}A^{ij}\right) = \frac{\partial}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(\sqrt{g}A^{ij}\right),
\end{aligned}$$

where, we have to replace the dummy index α by i.

EXAMPLE 3.2.7 *Find the covariant derivative of* $A^j_k B^{lm}_n$ *with respect to* x^q.

Solution: According to the definition of covariant derivative, we get

$$\begin{aligned}
\left(A^j_k B^{lm}_n\right)_q &= \frac{\partial\left(A^j_k B^{lm}_n\right)}{\partial x^q} - \begin{Bmatrix} s \\ k \;\; q \end{Bmatrix} A^j_s B^{lm}_n - \begin{Bmatrix} s \\ n \;\; q \end{Bmatrix} A^j_k B^{lm}_s \\
&\quad + \begin{Bmatrix} j \\ q \;\; s \end{Bmatrix} A^s_k B^{lm}_n + \begin{Bmatrix} l \\ q \;\; s \end{Bmatrix} A^j_k B^{sm}_n + \begin{Bmatrix} m \\ q \;\; s \end{Bmatrix} A^j_k B^{ls}_n \\
&= \left(\frac{\partial A^j_k}{\partial x^q} - \begin{Bmatrix} s \\ k \;\; q \end{Bmatrix} A^j_s + \begin{Bmatrix} j \\ q \;\; s \end{Bmatrix} A^s_k\right) B^{lm}_n \\
&\quad + A^k_j\left(\frac{\partial B^{lm}_n}{\partial x^q} - \begin{Bmatrix} s \\ n \;\; q \end{Bmatrix} B^{lm}_s + \begin{Bmatrix} l \\ q \;\; s \end{Bmatrix} B^{sm}_n + \begin{Bmatrix} m \\ q \;\; s \end{Bmatrix} B^{ls}_n\right) \\
&= A^j_{k,q} B^{lm}_n + A^j_k B^{lm}_{n,q}.
\end{aligned}$$

This example shows that the covariant derivatives of a product of tensors obey rules like those of ordinary derivatives of products in elementary calculus.

EXAMPLE 3.2.8 *If A^{ijk} is a skew-symmetric tensor, show that $\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^k}\left(\sqrt{g}A^{ijk}\right)$ is a tensor.*

Solution: According to the covariant derivative formula and contraction, we have

$$\begin{aligned}
A^{ijl}_{,i} &= \frac{\partial A^{ijl}}{\partial x^l} + \left\{ \begin{matrix} l \\ \alpha \ \ l \end{matrix} \right\} A^{\alpha jl} + \left\{ \begin{matrix} j \\ \alpha \ \ l \end{matrix} \right\} A^{i\alpha l} + \left\{ \begin{matrix} l \\ \alpha \ \ l \end{matrix} \right\} A^{ij\alpha} \\
&= \frac{\partial A^{ijl}}{\partial x^l} + \left\{ \begin{matrix} l \\ \alpha \ \ l \end{matrix} \right\} A^{ij\alpha}; \quad \text{as} \quad \left\{ \begin{matrix} j \\ \alpha \ \ l \end{matrix} \right\} A^{i\alpha l} = 0 \\
&= \frac{\partial A^{ijl}}{\partial x^l} + A^{ij\alpha}\frac{\partial}{\partial x^\alpha}(\log\sqrt{g}) = \frac{\partial A^{ijl}}{\partial x^l} + \frac{A^{ij\alpha}}{\sqrt{g}}\frac{\partial}{\partial x^\alpha}(\sqrt{g}) \\
&= \frac{1}{\sqrt{g}}\left[\sqrt{g}\frac{\partial A^{ijl}}{\partial x^l} + A^{ij\alpha}\frac{\partial}{\partial x^\alpha}(\sqrt{g})\right] \\
&= \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^\alpha}\left(\sqrt{g}A^{ij\alpha}\right) = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^k}\left(\sqrt{g}A^{ijk}\right)
\end{aligned}$$

Since the left hand side is a tensor, the right-hand side is also.

EXAMPLE 3.2.9 *If A_{ij} is a symmetric tensor, show that $A_{ij,k}$ is symmetric with respect to indices i and j.*

Solution: The covariant derivative of a contravariant vector A_{ij} with respect to x^k is given by

$$A_{ij,k} = \frac{\partial A_{ij}}{\partial x^k} - \left\{ \begin{matrix} \alpha \\ i \ \ k \end{matrix} \right\} A_{\alpha j} - \left\{ \begin{matrix} \alpha \\ j \ \ k \end{matrix} \right\} A_{i\alpha}.$$

Since A_{ij} is a tensor of the type $(0,2)$, A_{ji} is so. Hence,

$$\begin{aligned}
A_{ji,k} &= \frac{\partial A_{ji}}{\partial x^k} - \left\{ \begin{matrix} \alpha \\ j \ \ k \end{matrix} \right\} A_{\alpha i} - \left\{ \begin{matrix} \alpha \\ i \ \ k \end{matrix} \right\} A_{j\alpha} \\
&= \frac{\partial A_{ij}}{\partial x^k} - \left\{ \begin{matrix} \alpha \\ i \ \ k \end{matrix} \right\} A_{\alpha j} - \left\{ \begin{matrix} \alpha \\ j \ \ k \end{matrix} \right\} A_{i\alpha}
\end{aligned}$$

as A_{ij} is skew-symmetric and $\left\{ \begin{matrix} i \\ j \ \ k \end{matrix} \right\}$ is symmetric with respect to the indices j and k. From these two results, we get

$$A_{ij,k} = A_{ji,k} \Rightarrow A_{ij,k} \text{ is symmetric in } i \text{ and } j.$$

EXAMPLE 3.2.10 *Let A^{ij} be a symmetric tensor. Prove that the covariant derivative of A_i^j with respect to x^k is given by*

$$A_{i,k}^j = \frac{\partial A_i^j}{\partial x^k} - \left\{ \begin{matrix} \alpha \\ i \;\; k \end{matrix} \right\} A_\alpha^j + \left\{ \begin{matrix} j \\ \alpha \;\; k \end{matrix} \right\} A_i^\alpha .$$

Hence calculate $A_{i,j}^j$.

Solution: Since A^{ij} be a symmetric tensor, so $A^{ij} = A^{ji}$. Putting $j = k$, we get

$$\begin{aligned} A_{i,j}^j &= \frac{\partial A_i^j}{\partial x^j} + \left\{ \begin{matrix} j \\ \alpha \;\; j \end{matrix} \right\} A_i^\alpha - \left\{ \begin{matrix} \alpha \\ i \;\; j \end{matrix} \right\} A_\alpha^j \\ &= \frac{\partial A_i^j}{\partial x^j} + \frac{\partial \left(\log \sqrt{g}\right)}{\partial x^\alpha} A_i^\alpha - g^{l\alpha}[ij, l] A_\alpha^j \\ &= \frac{\partial A_i^j}{\partial x^j} + \frac{A_i^j}{\sqrt{g}} \frac{\partial}{\partial x^j} (\sqrt{g}) - A^{jl}[ij, l]; \text{ as } A^{ij} \text{ is symmetric} \\ &= \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^j} \left(A_i^j \sqrt{g} \right) - A^{jk}[ij, k]. \end{aligned}$$

This expression for $A_{i,j}^j$ does not require any calculation of Christoffel symbols. But, we know

$$\begin{aligned} A^{jk}[ij, k] &= \frac{1}{2} A^{jk} \left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k} \right) \\ &= \frac{1}{2} \left(A^{jk} \frac{\partial g_{jk}}{\partial x^i} + A^{kj} \frac{\partial g_{ji}}{\partial x^k} - A^{jk} \frac{\partial g_{ij}}{\partial x^k} \right), \end{aligned}$$

where we have to interchange the dummy suffixes j and k for the second term. Therefore,

$$A^{jk}[ij, k] = \frac{1}{2} \left(A^{jk} \frac{\partial g_{jk}}{\partial x^i} + A^{jk} \frac{\partial g_{ij}}{\partial x^k} - A^{jk} \frac{\partial g_{ij}}{\partial x^k} \right) = \frac{1}{2} A^{jk} \frac{\partial g_{jk}}{\partial x^i}$$

(since $A^{jk} = A^{kj}$ and $g_{ij} = g_{ji}$).

Using the result, we get

$$A_{i,j}^j = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^j} \left(A_i^j \sqrt{g} \right) - \frac{1}{2} A^{jk} \frac{\partial g_{jk}}{\partial x^i}.$$

This the expression for $A_{i,j}^j$ in terms of the metric tensors g_{ij}.

3.3 Gradient, Divergence and Curl

The most frequently occurring equations in theoretical physics are expressed with the help of a small number of operators called *gradient, divergence, Curl and Laplacian.* In this section, we are introducing these notations.

3.3.1 Divergence

Divergence of a contravariant vector: Let A^i be a contravariant vector. Then its covariant derivatives $A^i_{,j}$, given by

$$A^i_{,j} = \frac{\partial A^i}{\partial x^j} + \left\{ \begin{matrix} i \\ \alpha \ \ j \end{matrix} \right\} A^\alpha, \tag{3.25}$$

is a tensor of type $(1,1)$. If we now contract the indices i and j, we get the tensor $A^i_{,i}$, which is a tensor of the type $(0,0)$, i.e. an invariant. This invariant is called *the divergence* of the contravariant vector A^i and is denoted by divA^i, i.e.

$$\text{div}A^i = \nabla_i A^i = A^i_{,i}. \tag{3.26}$$

Note that, the divergence of a vector is an invariant. Now, we derive an expression for the divergence of a contravariant vector A^i. Now

$$\begin{aligned} \text{div}A^i = A^i_{,i} &= \frac{\partial A^i}{\partial x^i} + \left\{ \begin{matrix} i \\ \alpha \ \ i \end{matrix} \right\} A^\alpha \\ &= \frac{\partial A^i}{\partial x^i} + \frac{\partial}{\partial x^\alpha} (\log \sqrt{g})\, A^\alpha \\ &= \frac{\partial A^i}{\partial x^i} + \frac{\partial}{\partial x^i} (\log \sqrt{g})\, A^i = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left(\sqrt{g} A^i\right), \end{aligned} \tag{3.27}$$

where $g = |g_{ij}|$. Taking $\sqrt{g}A^i = B^i$, we get

$$\text{div}A^i = \frac{1}{\sqrt{g}} \frac{\partial B^i}{\partial x^i} = \frac{1}{\sqrt{g}} \left[\frac{\partial B^1}{\partial x^1} + \frac{\partial B^2}{\partial x^2} + \cdots + \frac{\partial B^N}{\partial x^N} \right].$$

The advantage of this formula is that it does not require the calculation of the Christoffel symbols.

Result 3.3.1 Let $\mathbf{A}$ and $\mathbf{B}$ be two arbitrary vectors, then, $\text{div}(\mathbf{A}+\mathbf{B}) = \text{div}\mathbf{A} + \text{div}\mathbf{B}$.

Proof: We know from Eq. (3.27) that

$$\text{div}A^i = A^i_{,i} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left(\sqrt{g} A^i\right),$$

where $g = |g_{ij}|$. Putting $A^i + B^i$ in this relation, we get

$$\begin{aligned} \text{div}\left(A^i + B^i\right) &= \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left[\sqrt{g}\left(A^i + B^i\right)\right] \\ &= \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(\sqrt{g}A^i\right) + \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(\sqrt{g}B^i\right) \\ &= \text{div}\mathbf{A} + \text{div}\mathbf{B}. \end{aligned}$$

Divergence of a covariant vector: Let A_i is a covariant vector, then $g^{jk}A_{j,k}$ is an invariant. This invariant is defined to be the divergence of the vector A_i and is denoted by divA_i. Thus

$$\text{div}A_i = g^{jk}A_{j,k}. \tag{3.28}$$

According to definition Eq. (3.28), we get

$$\begin{aligned} \text{div}A_i = g^{jk}A_{j,k} &= \left(g^{jk}A_j\right)_{,k}; \quad \text{as } g^{jk}_{,k} = 0 \\ &= A^k_{,k} = \text{div}A^k = \text{div}A^i. \end{aligned}$$

Thus, if A_i and A^i are, respectively, the covariant and contravariant components of the same vector $\overrightarrow{A}$, then div $A^i = \text{div}A_i$.

EXAMPLE 3.3.1 *Prove that*

$$divA^{ij} = \nabla_j A^{ij} = A^{ij}_{,j} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^j}\left(A^{ij}\sqrt{g}\right) + \left\{\begin{matrix} j \\ \alpha \;\; k \end{matrix}\right\} A^{i\alpha}.$$

Find the expression for div A^{ij}, when A^{ij} is skew-symmetric.

Solution: The covariant derivative of A^{ij} with respect to x^k is given by

$$A^{ij}_{,k} = \frac{\partial A^{ij}}{\partial x^k} + \left\{\begin{matrix} i \\ \alpha \;\; k \end{matrix}\right\} A^{\alpha j} + \left\{\begin{matrix} j \\ \alpha \;\; k \end{matrix}\right\} A^{i\alpha}.$$

Putting $j = k$, we get

$$\begin{aligned}
A^{ij}_{,j} &= \frac{\partial A^{ij}}{\partial x^j} + \left\{ \begin{matrix} i \\ \alpha \;\; j \end{matrix} \right\} A^{\alpha j} + \left\{ \begin{matrix} j \\ \alpha \;\; i \end{matrix} \right\} A^{i\alpha} \\
&= \frac{\partial A^{ij}}{\partial x^j} + \left\{ \begin{matrix} i \\ j \;\; k \end{matrix} \right\} A^{jk} + \frac{\partial}{\partial x^\alpha} (\log \sqrt{g})\, A^{i\alpha} \\
&= \frac{\partial A^{ij}}{\partial x^j} + \frac{A^{i\alpha}}{\sqrt{g}} \frac{\partial}{\partial x^\alpha} (\sqrt{g}) + \left\{ \begin{matrix} i \\ j \;\; k \end{matrix} \right\} A^{jk} \\
&= \frac{\partial A^{ij}}{\partial x^j} + \frac{A^{ij}}{\sqrt{g}} \frac{\partial}{\partial x^j} (\sqrt{g}) + \left\{ \begin{matrix} i \\ j \;\; k \end{matrix} \right\} A^{jk} \\
&= \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^j} \left(A^{ij} \sqrt{g} \right) + \left\{ \begin{matrix} i \\ j \;\; k \end{matrix} \right\} A^{jk},
\end{aligned}$$

which gives the first result. If A^{ij} is skew-symmetric, $A^{jk} = -A^{kj}$. On interchanging the dummy indices j and k we get

$$\begin{aligned}
A^{jk} \left\{ \begin{matrix} i \\ j \;\; k \end{matrix} \right\} &= A^{kj} \left\{ \begin{matrix} i \\ k \;\; j \end{matrix} \right\} = -A^{jk} \left\{ \begin{matrix} i \\ k \;\; j \end{matrix} \right\} \\
&= -A^{jk} \left\{ \begin{matrix} i \\ j \;\; k \end{matrix} \right\}; \text{ as } \left\{ \begin{matrix} i \\ j \;\; k \end{matrix} \right\} = \left\{ \begin{matrix} i \\ k \;\; j \end{matrix} \right\}
\end{aligned}$$

or

$$A^{jk} \left\{ \begin{matrix} i \\ j \;\; k \end{matrix} \right\} + A^{jk} \left\{ \begin{matrix} i \\ j \;\; k \end{matrix} \right\} = 0 \quad \Rightarrow \quad A^{jk} \left\{ \begin{matrix} i \\ j \;\; k \end{matrix} \right\} = 0.$$

Thus, when A^{ij} is skew-symmetric, the expression for divergence becomes

$$\text{div} A^{ij} = \nabla_j A^{ij} = A^{ij}_{,j} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^j} \left(A^{ij} \sqrt{g} \right).$$

EXAMPLE 3.3.2 *Express the divergence of a vector A^i in terms of its physical components for (i) cylindrical co-ordinates and (ii) spherical co-ordinates.*

Solution: (i) In the cylindrical co-ordinate, the metric is given by

$$ds^2 = (dx^1)^2 + (x^1)^2 (dx^2)^2 + (dx^3)^2,$$

where $g_{11} = 1, g_{22} = (x^1)^2, g_{33} = 1$ and $g_{ij} = 0$ for $i \neq j$. Thus,

$$g = |g_{ij}| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & (x^1)^2 & 0 \\ 0 & 0 & 1 \end{vmatrix} = (x^1)^2.$$

Let the physical components be denoted by A_1, A_2 and A_3, then they are given by

$$A_1 = \sqrt{g_{11}}A^1 = A^1;\ A_2 = \sqrt{g_{22}}A^2 = x^1 A^2 \quad \text{and} \quad A_3 = \sqrt{g_{33}}A^3 = A^3.$$

Therefore, the divergence of A^i is given by

$$\begin{aligned}\text{div}A^i &= \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^k}\left(\sqrt{g}A^k\right) = \frac{1}{x^1}\left[\frac{\partial}{\partial x^1}\left(x^1A^1\right) + \frac{\partial}{\partial x^2}\left(x^1A^2\right) + \frac{\partial}{\partial x^3}\left(x^1A^3\right)\right]\\ &= \frac{1}{x^1}\left[\frac{\partial}{\partial x^1}\left(x^1A_1\right) + \frac{\partial A_2}{\partial x^2} + \frac{\partial}{\partial x^3}\left(x^1A_3\right)\right].\end{aligned}$$

(ii) In the spherical co-ordinates, the metric is given by

$$ds^2 = (dx^1)^2 + (x^1)^2(dx^2)^2 + (x^1\sin x^2)^2(dx^3)^2,$$

where $g_{11} = 1, g_{22} = (x^1)^2, g_{33} = (x^1\sin x^2)^2$ and $g_{ij} = 0$ for $i \neq j$. Thus,

$$g = |g_{ij}| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & (x^1)^2 & 0 \\ 0 & 0 & (x^1\sin x^2)^2 \end{vmatrix} = (x^1)^4(\sin x^2)^2.$$

Let the physical components be denoted by A_1, A_2 and A_3, then they are given by

$$A_1 = \sqrt{g_{11}}A^1 = A^1;\ A_2 = \sqrt{g_{22}}A^2 = x^1A^2$$

and

$$A_3 = \sqrt{g_{33}}A^3 = x^1\sin x^2 A^3.$$

Therefore, the divergence of A^i is given by,

$$\begin{aligned}\text{div}A^i &= \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^k}\left(\sqrt{g}A^k\right) = \frac{1}{\left(x^1\right)^2\sin x^2}\left[\frac{\partial}{\partial x^1}\left\{\left(x^1\right)^2\sin x^2A^1\right\}\right.\\ &\quad\left. + \frac{\partial}{\partial x^2}\left\{\left(x^1\right)^2\sin x^2A^2\right\} + \frac{\partial}{\partial x^3}\left\{\left(x^1\right)^2\sin x^2A^3\right\}\right]\\ &= \frac{1}{(x^1)^2}\frac{\partial}{\partial x^1}\left\{(x^1)^2A_1\right\} + \frac{1}{x^1\sin x^2}\frac{\partial}{\partial x^2}\left(\sin x^2A_2\right) + \frac{1}{x^1\sin x^2}\frac{\partial A_3}{\partial x^3}.\end{aligned}$$

EXAMPLE 3.3.3 *Prove that in a* V_2 *with line element* $ds^2 = (dx^1)^2 + (x^1)^2(dx^2)^2$, *the divergence of the covariant vector with components* $x^1\cos 2x^2, -(x^1)^2\sin 2x^2$ *is zero.*

Solution: Comparing the given metric with Eq. (2.1), we get $g_{11} = 1, g_{22} = (x^1)^2$ and $g_{ij} = 0$ for $i \neq j$. Thus,

$$g = |g_{ij}| = \begin{vmatrix} 1 & 0 \\ 0 & (x^1)^2 \end{vmatrix} = (x^1)^2.$$

The reciprocal tensors are given by $g^{11} = 1$; $g^{22} = \frac{1}{(x^1)^2}$ and $g^{12} = 0 = g^{21}$. The physical components be denoted by $A_1 = x^1 \cos 2x^2$ and $A_2 = -(x^1)^2 \sin 2x^2$, and so,

$$A_1 = \sqrt{g_{11}}A^1 = A^1; \; A_2 = \sqrt{g_{22}}A^2 = x^1 A^2.$$

Therefore, the divergence of A^i is given by

$$\begin{aligned}
\text{div}A^i &= \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^k}\left(\sqrt{g}A^k\right) = \frac{1}{\sqrt{g}}\left[\frac{\partial}{\partial x^1}\left(\sqrt{g}A^1\right) + \frac{\partial}{\partial x^2}\left(\sqrt{g}A^2\right)\right] \\
&= \frac{1}{x^1}\left[\frac{\partial}{\partial x^1}\left(x^1 A_1\right) + \frac{\partial}{\partial x^2}\left(\frac{x^1}{x^2}A^2\right)\right] \\
&= \frac{1}{x^1}\left[\frac{\partial}{\partial x^1}\left\{(x^1)^2 \cos 2x^2\right\} + \frac{\partial}{\partial x^2}\left\{-(x^1)^2 \sin 2x^2\right\}\right] \\
&= \frac{1}{x^1}\left[2x^1 \cos 2x^2 - 2x^1 \cos 2x^2\right] = 0.
\end{aligned}$$

EXAMPLE 3.3.4 *Express the divergence theorem in tensor form.*

Solution: If V is the volume bounded by a closed surface $\mathcal{S}$ and $\overrightarrow{A}$ is a vector function of position with continuous derivatives, then

$$\int\int\int_V \overrightarrow{\nabla} \cdot \overrightarrow{V} dV = \int\int_{\mathcal{S}} \overrightarrow{A} \cdot \hat{n} ds,$$

where $\hat{n}$ is the positive normal to $\mathcal{S}$. Let A^k define a tensor field of rank 1 and let ν_k denote the outward drawn unit normal to any point of a closed surface $\mathcal{S}$ bounding a volume V. Then the divergence theorem states that

$$\int\int\int_V A^k_{,k} dV = \int\int_{\mathcal{S}} A^k \nu_k ds.$$

For N dimensional space the triple integral is replaced by an N-tuple integral, and the double integral by an $N-1$ tuple integral. The invariant $A^k_{,k}$ is the divergence of A^k. The invariant $A^k \nu_k$ is the scalar product of A^k and ν_k, analogous to $\overrightarrow{A} \cdot \hat{n}$ in the vector notation.

EXAMPLE 3.3.5 *If A is the magnitude of A^i, prove that $A_{,j} = A_{i,j}\frac{A^i}{A}$.*

Solution: If A be the magnitude of A^i, then by definition, $A^2 = g_{ik}A^iA^k$, therefore,

$$\begin{aligned}2AA_{,j} &= (g_{ik})_{,j}A^iA^k + g_{ik}\left(A^iA^k\right)_{,j}\\ &= g_{ik}\left[A^i_{,j}A^k + A^iA^k_{,j}\right];\ \text{as } g_{ik,j} = 0\\ &= g_{ik}A^i_{,j}A^k + g_{ki}A^k_{,j}A^i = 2g_{ki}A^k_{,j} = 2\left(g_{ki}A^k\right)_{,j}A^i = 2A_{i,j}A^i\end{aligned}$$

or

$$AA_{,j} = A_{i,j}A^i \Rightarrow A_{,j} = A_{i,j}\frac{A^i}{A}.$$

3.3.2 Gradient of an Invariant

The partial derivative of an invariant ϕ is a covariant vector, which is called the gradient of ϕ and is denoted by $\text{grad}\,\phi$ or $\nabla\phi$. Thus,

$$\text{grad}\,\phi = \nabla\phi = \phi_{,j}, \tag{3.29}$$

which is a covariant tensor of rank 1.

Result 3.3.2 Let $\mathbf{A}$ be an arbitrary vector and ϕ be a scalar function, then,

$$\text{div}(\phi\mathbf{A}) = \text{grad }(\phi)\,\mathbf{A} + \phi\text{div}\,\mathbf{A}.$$

Proof: We know from Eq. (3.27) that, $\text{div}\,A^i = A^i_{,i} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(\sqrt{g}A^i\right)$, where $g = |g_{ij}|$. Therefore,

$$\begin{aligned}\text{div }\left(\phi A^i\right) = \left(\phi A^i\right)_{,i} &= \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(\sqrt{g}\phi A^i\right)\\ &= \frac{1}{\sqrt{g}}\left[\frac{\partial}{\partial x^i}(\sqrt{g}A^i)\phi + \sqrt{g}A^i\frac{\partial\phi}{\partial x^i}\right]\\ &= \frac{\partial\phi}{\partial x^i}A^i + \phi\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(\sqrt{g}A^i\right) = \text{grad}\,(\phi)\,\mathbf{A} + \phi\,\text{div}\,\mathbf{A}.\end{aligned}$$

EXAMPLE 3.3.6 *If A^i is a contravariant vector such that $A^i_{,k} = a_kA^i$, where a_k is a covariant vector, show that a_k is a gradient.*

Solution: By using the given condition, we have

$$g_{ij}A^iA^ja_k = g_{ij}\left(a_kA^i\right)A^j = g_{ij}A^i_{,k}A^j.$$

Again,

$$\begin{aligned}\left(g_{ij}A^iA^j\right)_{,k} &= g_{ij,k}A^iA^j + g_{ij}\left(A^i_{,k}A^j + A^iA^j_{,k}\right)\\ &= g_{ij}\left(A^i_{,k}A^j + A^iA^j_{,k}\right);\ \text{as } g_{ij,k}=0.\end{aligned}$$

Now, replacing the dummy indices j and i in the second term of the right-hand side by i and j, respectively, we get

$$\begin{aligned}\left(g_{ij}A^iA^j\right)_{,k} &= g_{ij}A^i_{,k}A^j + g_{ji}A^jA^i_{,k}\\ &= g_{ij}A^i_{,k}A^j + g_{ij}A^jA^i_{,k} = 2g_{ij}A^i_{,k}A^j;\ \text{as } g_{ij}=g_{ji}\\ \Rightarrow g_{ij}A^i_{,k}A^j &= \frac{1}{2}\left(g_{ij}A^iA^j\right)_{,k} = g_{ij}A^iA^ja_k\\ \Rightarrow \phi a_k &= \frac{1}{2}\phi_{,k};\ \text{where } \phi = g_{ij}A^iA^j = \text{ an invariant}\end{aligned}$$

or

$$a_k = \frac{1}{2\phi}\frac{\partial\phi}{\partial x^k} = \frac{\partial}{\partial x^k}\log\phi^{1/2}.$$

So, a_k is the gradient of the invariant $\log\phi^{1/2}$, i.e. a_k is a gradient.

3.3.3 Laplacian Operator

Let ϕ be the scalar function of co-ordinates x^i, then divergence of grad ϕ is defined to be *Laplacian of an invariant* ϕ and it is denoted by $\nabla^2\phi$. Thus,

$$\text{div grad } \phi = \text{div}\,\nabla\phi = \nabla^2\phi. \tag{3.30}$$

Now, we derive an expression for the Laplacian in terms of co-ordinates x^i.

(i) If ϕ is a scalar function of co-ordinates x^i, then we know

$$\text{grad } \phi = \nabla\phi = \phi_{,j} = \frac{\partial\phi}{\partial x^j}.$$

Also, if A_i is a covariant vector, then the divergence is given by

$$\begin{aligned}\text{div } A_i = A^i_{,i} &= \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(A^i\sqrt{g}\right)\\ &= \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(g^{ij}A_j\sqrt{g}\right) = \text{div } A^i.\end{aligned}$$

Now, set $A_i = \phi_{,i}$, so that A_i is a covariant vector, and thus $g^{ij}A_j = A^i$ is a contravariant vector. Therefore

$$\text{div } \phi_{,i} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(g^{ij}\phi_{,j}\sqrt{g}\right)$$

or

$$\text{div grad}\,\phi = \text{div }(g^{ij}\phi, j) = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(\sqrt{g}g^{ij}\phi_{,j}\right)$$

or

$$\nabla^2\phi = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^k}\left(\sqrt{g}g^{kj}\frac{\partial\phi}{\partial x^j}\right).$$

This expression for $\nabla^2\phi$ does not include any Christoffel symbol.

(ii) By definition of divergence, the contraction of covariant derivative of $g^{ij}\phi_{,i}$ is called the div $\phi_{,i}$, i.e. div grad ϕ. Thus,

$$\begin{aligned}\nabla^2\phi &= \text{div grad}\,\phi = g^{ij}\phi_{,ij} = g^{ij}\phi_{i,j} = \frac{\delta\phi_{,i}}{\delta x^j}\\ &= g^{ij}\left[\frac{\partial\phi_i}{\partial x^j} - \begin{Bmatrix} k \\ i\;\; j\end{Bmatrix}\phi_{,k}\right] = g^{ij}\left[\frac{\partial}{\partial x^j}\left(\frac{\partial\phi}{\partial x^i}\right) - \begin{Bmatrix} k \\ i\;\; j\end{Bmatrix}\frac{\partial\phi}{\partial x^k}\right]\\ &= g^{ij}\left[\frac{\partial^2\phi}{\partial x^i\partial x^j} - \begin{Bmatrix} k \\ i\;\; j\end{Bmatrix}\frac{\partial\phi}{\partial x^k}\right],\end{aligned}$$

where ϕ is a scalar function of co-ordinates x^i. Interchanging i and j, we get

$$\begin{aligned}\phi_{,ji} &= \frac{\partial^2\phi}{\partial x^j\partial x^i} - \begin{Bmatrix} k \\ j\;\; i\end{Bmatrix}\frac{\partial\phi}{\partial x^k} = \frac{\partial^2\phi}{\partial x^j\partial x^i} - \begin{Bmatrix} k \\ i\;\; j\end{Bmatrix}\frac{\partial\phi}{\partial x^k};\ \text{as}\ \begin{Bmatrix} k \\ j\;\; i\end{Bmatrix} = \begin{Bmatrix} k \\ i\;\; j\end{Bmatrix}\\ &= \frac{\partial^2\phi}{\partial x^j\partial x^i} - \begin{Bmatrix} k \\ i\;\; j\end{Bmatrix}\frac{\partial\phi}{\partial x^k};\ \text{as}\ \frac{\partial^2\phi}{\partial x^i\partial x^j} = \frac{\partial^2\phi}{\partial x^i\partial x^j}\\ &= \phi_{,ij}.\end{aligned}$$

This shows that $\phi_{,ij}$ is symmetric. This expression requires the calculations of Christoffel symbols.

EXAMPLE 3.3.7 *Express $\nabla^2\phi$ in (i) cylindrical co-ordinates and (ii) spherical co-ordinates.*

Solution:

(i) In the cylindrical co-ordinate, the metric is given by,

$$ds^2 = (dx^1)^2 + (x^1)^2(dx^2)^2 + (dx^3)^2,$$

where $g_{11} = 1, g_{22} = (x^1)^2, g_{33} = 1$ and $g_{ij} = 0$ for $i \neq j$. Thus, $g = |g_{ij}| = (x^1)^2$. The reciprocal tensors are $g^{11} = 1; g^{22} = \frac{1}{x^1}; g^{33} = 1$. Therefore, the expression for $\nabla^2\phi$ in cylindrical co-ordinates is given by

$$\begin{aligned}\nabla^2\phi &= \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^k}\left(\sqrt{g}g^{kj}\frac{\partial\phi}{\partial x^j}\right)\\ &= \frac{1}{x^1}\left[\frac{\partial}{\partial x^1}\left(x^1\cdot 1\frac{\partial\phi}{\partial x^1}\right) + \frac{\partial}{\partial x^2}\left(x^1\frac{1}{(x^1)^2}\frac{\partial\phi}{\partial x^2}\right) + \frac{\partial}{\partial x^3}\left(x^1\cdot 1\frac{\partial\phi}{\partial x^3}\right)\right]\\ &= \frac{1}{x^1}\frac{\partial}{\partial x^1}\left(x^1\frac{\partial\phi}{\partial x^1}\right) + \frac{1}{(x^1)^2}\frac{\partial^2\phi}{(\partial x^2)^2} + \frac{\partial^2\phi}{(\partial x^3)^2}.\end{aligned}$$

(ii) In the spherical co-ordinates, the metric is given by,

$$ds^2 = (dx^1)^2 + (x^1)^2(dx^2)^2 + (x^1\sin x^2)^2(dx^3)^2,$$

where $g_{11} = 1, g_{22} = (x^1)^2, g_{33} = (x^1\sin x^2)^2$ and $g_{ij} = 0$ for $i \neq j$. Thus, $g = |g_{ij}| = (x^1)^4(\sin x^2)^2$. The reciprocal tensors are given by $g^{11} = 1; g^{22} = \frac{1}{(x^1)^2}; g^{33} = \frac{1}{(x^1\sin x^2)^2}$. Therefore, the expression for $\nabla^2\phi$ in spherical co-ordinates is given by

$$\begin{aligned}\nabla^2\phi &= \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^k}\left(\sqrt{g}g^{kj}\frac{\partial\phi}{\partial x^j}\right) = \frac{1}{(x^1)^2\sin^2 x^2}\left[\frac{\partial}{\partial x^1}\left\{\left(x^1\right)^2\sin x^2\cdot 1\cdot\frac{\partial\phi}{\partial x^1}\right\}\right.\\ &\left.+\frac{\partial}{\partial x^2}\left\{\frac{\left(x^1\right)^2\sin x^2}{(x^1)^2}\frac{\partial\phi}{\partial x^2}\right\} + \frac{\partial}{\partial x^3}\left\{\frac{\left(x^1\right)^2\sin x^2}{\left(x^1\right)^2\sin x^2}\frac{\partial\phi}{\partial x^3}\right\}\right]\\ &= \frac{1}{(x^1)^2}\frac{\partial}{\partial x^1}\left((x^1)^2\frac{\partial\phi}{\partial x^1}\right) + \frac{1}{(x^1)^2\sin x^2}\frac{\partial}{\partial x^2}\left(\sin x^2\frac{\partial\phi}{\partial x^2}\right) + \frac{1}{(x^1\sin x^2)^2}\frac{\partial^2\phi}{(\partial x^3)^2}.\end{aligned}$$

3.3.4 Curl

Curl of a covariant vector: A skew-symmetric tensor of type $(0, 2)$ formed from a covariant vector assumes particular importance in E_3, where it can be identified with a vector. We now introduce such a skew-symmetric tensor in V_N which is called the *curl of a covariant vector.*

Let us consider a covariant vector A_i. Then $A_{i,j}$ is a tensor of type $(0, 2)$. Hence, $A_{j,i}$ is also a tensor of type $(0, 2)$. Consequently, $A_{i,j} - A_{j,i}$ is a tensor of type $(0, 2)$, which evidently skew-symmetric. This skew-symmetric tensor is called *the curl* or *the rotation* or *the rotator* of the vector A_i and is denoted by curl A_i or $RotA_i$. Thus,

$$\text{curl } A_i = A_{i,j} - A_{j,i} = \frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i}. \tag{3.31}$$

Now, we derive an expression for the curl of a covariant vector A_i. We know,

$$A_{i,j} = \frac{\partial A_i}{\partial x^j} - \left\{ \begin{matrix} k \\ i \;\; j \end{matrix} \right\} A_k$$

or

$$A_{j,i} = \frac{\partial A_j}{\partial x^i} - \left\{ \begin{matrix} k \\ j \;\; i \end{matrix} \right\} A_k.$$

Therefore, the tensor $A_{i,j} - A_{j,i}$ of rank 2 is given by,

$$\begin{aligned} \text{curl } A_i = A_{i,j} - A_{j,i} &= \frac{\partial A_i}{\partial x^j} - \left\{ \begin{matrix} k \\ i \;\; j \end{matrix} \right\} A_k - \frac{\partial A_j}{\partial x^i} + \left\{ \begin{matrix} k \\ j \;\; i \end{matrix} \right\} A_k \\ &= \frac{\partial A_i}{\partial x^j} - \left\{ \begin{matrix} k \\ i \;\; j \end{matrix} \right\} A_k, - \frac{\partial A_j}{\partial x^i} + \left\{ \begin{matrix} k \\ i \;\; j \end{matrix} \right\} A_k \\ &= \frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i}. \end{aligned}$$

Hence, is calculating the curl, we may replace covariant derivative by ordinary derivative.

Theorem 3.3.1 *A necessary and sufficient condition that the curl of a vector field vanishes is that the vector field be gradient.*

Proof: Let A_i be a covariant vector. Let the curl of the vector A_i vanish, so that

$$\text{curl } A_i = A_{i,j} - A_{j,i} = \frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i} = 0.$$

We have to show that $A_i = \nabla\phi$, where ϕ is a scalar. Now,

$$\begin{aligned} & A_{i,j} - A_{j,i} = \frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i} = 0 \\ & \Rightarrow \frac{\partial A_i}{\partial x^j} dx^j = \frac{\partial A_j}{\partial x^i} dx^j \\ & \quad \Rightarrow dA_i = \frac{\partial}{\partial x^i}\left(A_j dx^j\right);\ \text{as } dA_i = \frac{\partial A_j}{\partial x^i} dx^j \\ & \quad \Rightarrow A_i = \int \frac{\partial}{\partial x^i}\left(A_j dx^j\right) = \frac{\partial}{\partial x^i}\int A_j dx^j;\ \text{integrating.} \end{aligned}$$

But $\int A_j dx^j$ is a scalar quantity, let $\int A_j dx^j = \phi$, a scalar, then,

$$A_i = \frac{\partial \phi}{\partial x^i} = \nabla\phi.$$

Hence, A_i is the gradient of ϕ, i.e. A_i is a gradient.

Conversely, suppose that a vector A_i is such that $A_i = \nabla\phi$, where ϕ is a scalar. We have shown that curl grad $\phi = 0$. Now

$$A_i = \nabla\phi \Rightarrow A_i = \frac{\partial\phi}{\partial x^i}$$

or

$$\frac{\partial A_i}{\partial x^j} = \frac{\partial^2\phi}{\partial x^j \partial x^i} = \frac{\partial^2\phi}{\partial x^i \partial x^j}$$

or

$$\frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i} = \frac{\partial^2\phi}{\partial x^i \partial x^j} - \frac{\partial^2\phi}{\partial x^j \partial x^i} = 0$$

or

$$\text{curl } A_i = A_{i,j} - A_{j,i} = \frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i} = 0.$$

From this theorem, we conclude that, a necessary and sufficient condition that the covariant derivative of a covariant vector be symmetric is that the vector field be gradient.

EXAMPLE 3.3.8 *If A_{ij} is the curl of a covariant vector, prove that*

$$A_{ij,k} + A_{jk,i} + A_{ki,j} = 0.$$

Solution: Let A_{ij} be the components of the curl of a covariant vector B_i, so that

$$A_{ij} = \text{curl } B_i = B_{i,j} - B_{j,i}.$$

Interchanging i and j, we get

$$A_{ji} = B_{j,i} - B_{i,j} = -(B_{i,j} - B_{j,i}) = -A_{ij}$$

or

$$A_{ij} + A_{ji} = 0.$$

This proves that A_{ij} is skew-symmetric. Therefore,

$$\begin{aligned} A_{ij} &= \text{curl } B_i = B_{i,j} - B_{j,i} \\ &= \frac{\partial B_i}{\partial x^j} - \left\{ \begin{matrix} \alpha \\ i \ \ j \end{matrix} \right\} B_\alpha - \left[\frac{\partial B_j}{\partial x^i} - \left\{ \begin{matrix} \alpha \\ j \ \ i \end{matrix} \right\} B_\alpha \right] \\ &= \frac{\partial B_i}{\partial x^j} - \frac{\partial B_j}{\partial x^i}; \text{ as } \left\{ \begin{matrix} \alpha \\ i \ \ j \end{matrix} \right\} = \left\{ \begin{matrix} \alpha \\ j \ \ i \end{matrix} \right\}. \end{aligned}$$

or

$$\frac{\partial A_{ij}}{\partial x^k} = \frac{\partial^2 B_i}{\partial x^k \partial x^j} - \frac{\partial^2 B_j}{\partial x^k \partial x^i}.$$

Using the expressions for $A_{ij,k}$, we get

$$\begin{aligned} A_{ij,k} &= \frac{\partial A_{ij}}{\partial x^k} - \left\{ \begin{matrix} \alpha \\ i \;\; k \end{matrix} \right\} A_{\alpha j} - \left\{ \begin{matrix} \alpha \\ j \;\; k \end{matrix} \right\} A_{ik} \\ &= \frac{\partial^2 B_i}{\partial x^k \partial x^j} - \frac{\partial^2 B_j}{\partial x^k \partial x^i} - \left\{ \begin{matrix} \alpha \\ i \;\; k \end{matrix} \right\} A_{\alpha j} - \left\{ \begin{matrix} \alpha \\ j \;\; k \end{matrix} \right\} A_{ik}. \end{aligned}$$

Taking the sum of this and similar equations obtained by cyclic permutation of i, j, k, we obtain,

$$\begin{aligned} & A_{ij,k} + A_{jk,i} + A_{ki,j} \\ &\quad = \frac{\partial^2 B_i}{\partial x^k \partial x^j} - \frac{\partial^2 B_j}{\partial x^k \partial x^i} - \left\{ \begin{matrix} \alpha \\ i \;\; k \end{matrix} \right\} A_{\alpha j} - \left\{ \begin{matrix} \alpha \\ j \;\; k \end{matrix} \right\} A_{ik} \\ &\qquad + \frac{\partial^2 B_j}{\partial x^i \partial x^k} - \frac{\partial^2 B_k}{\partial x^i \partial x^j} - \left\{ \begin{matrix} \alpha \\ j \;\; i \end{matrix} \right\} A_{\alpha k} - \left\{ \begin{matrix} \alpha \\ k \;\; i \end{matrix} \right\} A_{j\alpha} \\ &\qquad + \frac{\partial^2 B_j}{\partial x^j \partial x^i} - \frac{\partial^2 B_i}{\partial x^j \partial x^k} - \left\{ \begin{matrix} \alpha \\ k \;\; j \end{matrix} \right\} A_{\alpha i} - \left\{ \begin{matrix} \alpha \\ i \;\; j \end{matrix} \right\} A_{k\alpha} \\ &\quad = 0; \text{ as } A_{ij} \text{ is skew-symmetric and } \frac{\partial^2 B_i}{\partial x^j \partial x^k} = \frac{\partial^2 B_i}{\partial x^k \partial x^j}. \end{aligned}$$

This result equivalently can be written as

$$\frac{\partial A_{ij}}{\partial x^k} + \frac{\partial A_{jk}}{\partial x^i} + \frac{\partial A_{ki}}{\partial x^j} = 0$$

or

$$\nabla_k A_{ij} + \nabla_i A_{jk} + \nabla_j A_{ki} = 0.$$

Curl of a covariant vector in V_3: Let A_i be a covariant vector, then $A_{i,k}$ is a tensor of type $(0, 2)$. Then the inner product

$$\varepsilon^{ikl} A_{l,k} = B^i, \text{ (say); } \varepsilon^{ijk} = \text{ permutation tensor}$$

a contravariant vector of type $(1, 0)$, is called *the curl of the vector* A_i. Thus, in V_3,

$$\begin{aligned} & \text{curl } A_i = B^i = \varepsilon^{ikl} A_{l,k} \\ & \quad \Rightarrow B^i = \frac{1}{\sqrt{g}} e^{ikl} A_{l,k}; \text{ as } \varepsilon^{ijk} = \frac{1}{\sqrt{g}} e^{ikl}. \end{aligned} \tag{3.32}$$

This Eq. (3.32) can be written in the component form as

$$B^1 = \frac{1}{\sqrt{g}} e^{1kl} A_{l,k} = \frac{1}{\sqrt{g}} e^{123} A_{3,2} + \frac{1}{\sqrt{g}} e^{132} A_{2,3} = \frac{1}{\sqrt{g}} (A_{3,2} - A_{2,3});$$

$$B^2 = \frac{1}{\sqrt{g}} e^{2kl} A_{l,k} = \frac{1}{\sqrt{g}} e^{231} A_{1,3} + \frac{1}{\sqrt{g}} e^{213} A_{3,1} = \frac{1}{\sqrt{g}} (A_{1,3} - A_{3,1});$$

$$B^3 = \frac{1}{\sqrt{g}} e^{3kl} A_{l,k} = \frac{1}{\sqrt{g}} e^{312} A_{2,1} + \frac{1}{\sqrt{g}} e^{321} A_{1,2} = \frac{1}{\sqrt{g}} (A_{2,1} - A_{1,2}).$$

Using relation (3.31), the components of curl A_i can be written in the form

$$\frac{1}{\sqrt{g}} (A_{3,2} - A_{2,3}), \frac{1}{\sqrt{g}} (A_{1,3} - A_{3,1}), \frac{1}{\sqrt{g}} (A_{2,1} - A_{1,2})$$

or

$$\frac{1}{\sqrt{g}} \left(\frac{\partial A_3}{\partial x^2} - \frac{\partial A_2}{\partial x^3} \right), \frac{1}{\sqrt{g}} \left(\frac{\partial A_1}{\partial x^3} - \frac{\partial A_3}{\partial x^1} \right), \frac{1}{\sqrt{g}} \left(\frac{\partial A_2}{\partial x^1} - \frac{\partial A_1}{\partial x^2} \right).$$

Deduction 3.3.1 Vector product of two covariant vectors in V_3: Let A_i and B_i be two covariant vectors of a V_3. Then the product

$$\varepsilon^{ijk} A_j B_k = C^i \text{ (say)},$$

a tensor of type $(1, 0)$, i.e. a covariant vector, is called the *vector product or cross product* of the vectors A_i and B_i. Now,

$$C^i = \varepsilon^{ijk} A_j B_k = \frac{1}{\sqrt{g}} e^{ijk} A_j B_k. \tag{3.33}$$

This Eq. (3.33) can be written in the component form as

$$C^1 = \frac{1}{\sqrt{g}} e^{123} A_2 B_3 + \frac{1}{\sqrt{g}} e^{132} A_3 B_2 = \frac{1}{\sqrt{g}} (A_2 B_3 - A_3 B_2);$$

$$C^2 = \frac{1}{\sqrt{g}} e^{231} A_3 B_1 + \frac{1}{\sqrt{g}} e^{213} A_1 B_3 = \frac{1}{\sqrt{g}} (A_3 B_1 - A_1 B_3);$$

$$C^3 = \frac{1}{\sqrt{g}} e^{312} A_1 B_2 + \frac{1}{\sqrt{g}} e^{321} A_2 B_1 = \frac{1}{\sqrt{g}} (A_1 B_2 - A_2 B_1).$$

Thus, the components of the vector product of two covariant vectors A_i and B_j can be written in the form

$$\frac{1}{\sqrt{g}} (A_2 B_3 - A_3 B_2); \ \frac{1}{\sqrt{g}} (A_3 B_1 - A_1 B_3); \ \frac{1}{\sqrt{g}} (A_1 B_2 - A_2 B_1)$$

If A_i and B_i be the covariant components of the vectors $\overrightarrow{A}$ and $\overrightarrow{B}$, then their vector products, denoted by $\overrightarrow{A} \times \overrightarrow{B}$, is given by $\overrightarrow{A} \times \overrightarrow{B} = \varepsilon^{ijk} A_j B_k$. Hence, in a V_3

$$\overrightarrow{A} \times \overrightarrow{B} = \left[\frac{1}{\sqrt{g}}(A_2B_3 - A_3B_2);\ \frac{1}{\sqrt{g}}(A_3B_1 - A_1B_3);\ \frac{1}{\sqrt{g}}(A_1B_2 - A_2B_1)\right]. \quad (3.34)$$

EXAMPLE 3.3.9 *Express Stokes' theorem in tensor form.*

Solution: Let $\mathcal{S}$ be an open, two-sided surface bounded by a closed, non-intersecting simple closed curve $\mathcal{C}$. If $\overrightarrow{A}$ has continuous derivatives,

$$\oint_{\mathcal{C}} \overrightarrow{A} \cdot d\overrightarrow{r} = \int\int_{\mathcal{S}} \left(\nabla \times \overrightarrow{A}\right) \cdot \hat{n} ds = \int\int_{\mathcal{S}} \left(\nabla \times \overrightarrow{A}\right) \cdot d\overrightarrow{s} \cdot$$

We consider the covariant derivative $A_{i,j}$ of the vector A_i and from the contravariant vector $B^i = -\varepsilon^{ijk} A_{j,k}$. We shall define the vector $\overrightarrow{B}$ to the curl of $\overrightarrow{A}$. Also,

$$\left(\nabla \times \overrightarrow{A}\right) \cdot \hat{n} = B^i \nu_i = -\varepsilon^{ijk} A_{j,k} \nu_i.$$

Let $\frac{dx^p}{ds}$ be the unit tangent vector to the closed curve $\mathcal{C}$ and ν^p is the positive unit normal to the surface $\mathcal{S}$, which has $\mathcal{C}$ as a boundary. Then Stokes' theorem can be written in the form

$$\oint_{\mathcal{C}} A_p \frac{dx^p}{ds} ds = -\int\int_{\mathcal{S}} \varepsilon^{pqr} A_{q,r} \nu_p ds.$$

The integral $\oint_{\mathcal{C}} A_p dx^p$ is called the *circulation* of $\overrightarrow{A}$ along the contour $\mathcal{C}$.

3.4 Intrinsic Derivative

In Section 3.2, we have introduced covariant differentiation in a Riemannian space. Such a concept is regarded as a generalisation of partial differentiation in Euclidean space with orthogonal Cartesian co-ordinates. In this section, we introduce another kind of differentiation of tensors, called intrinsic differentiation or absolute differentiation, which is the generalisation of ordinary differentiation. Following McConnell we will make free use of intrinsic differentiation in the treatment of geometry of curves and surfaces.

Let $\mathcal{C}$ be a certain space curve described by the parametric equations in V_N, a Riemannian space as,

$$\mathcal{C}: x^i = x^i(t);\ i = 1, 2, \ldots, N, \quad (3.35)$$

where $x^i(t)$ are N functions of a single parameter t, which obey certain continuity condition. In general, it will be sufficient that derivatives exist up to any order required.

Let $A^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q}$ be a tensor field of type (p, q) defined on the points of $\mathcal{C}$, i.e. whose components are continuous functions of the parameter t along $\mathcal{C}$. Then the covariant derivative of this tensor is given by

$$A^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q, k} = \frac{\partial}{\partial x^k} A^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q} + A^{\alpha i_2 \ldots i_p}_{j_1 j_2 \ldots j_q} \begin{Bmatrix} i_1 \\ \alpha \;\; k \end{Bmatrix} + \cdots + A^{i_1 i_2 \ldots \alpha}_{j_1 j_2 \ldots j_q} \begin{Bmatrix} i_p \\ \alpha \;\; k \end{Bmatrix}$$
$$- A^{i_1 i_2 \ldots i_p}_{\beta j_2 \ldots j_q} \begin{Bmatrix} \beta \\ j_1 \;\; k \end{Bmatrix} - \cdots - A^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_{q-1} \beta} \begin{Bmatrix} \beta \\ j_p \;\; k \end{Bmatrix}. \quad (3.36)$$

This is the generalisation of the directional derivative of the classical theory of vectors. For an Euclidean space with orthogonal Cartesian co-ordinates $\begin{Bmatrix} i \\ j \;\; k \end{Bmatrix} = 0$, so Eq. (3.36) becomes

$$A^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q, k} = \frac{\partial}{\partial x^k} A^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q}. \quad (3.37)$$

Let $\frac{dx^k}{dt}$ be the tangent vector of the curve $\mathcal{C}$, then the *absolute* or *intrinsic derivative* of such a tensor with respect to t, along the curve $\mathcal{C}$, denoted by $\frac{\delta}{\delta t} A^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q}$ is defined by

$$\frac{\delta}{\delta t} A^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q} = A^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q, k} \frac{dx^k}{dt}, \quad (3.38)$$

where $(,)$ denotes the covariant differentiation. Since $A^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q, k}$ is a tensor and dx^k is a tensor (and hence $\frac{dx^k}{dt}$ is a tensor), it follows from the quotient law that the intrinsic derivative defined by Eq. (3.38) is a tensor. Accordingly, the intrinsic derivative of a tensor is a tensor of the same order and type as the original tensor.

Using Eqs. (3.38) and (3.19), the *absolute* or *intrinsic derivative* of a tensor of type $(1, 0)$ with components A^i along $\mathcal{C}$ is given by

$$\frac{\delta A^i}{\delta t} = A^i_{,k} \frac{dx^k}{dt} = \left[\frac{\partial A^i}{\partial x^k} + A^m \begin{Bmatrix} i \\ m \;\; k \end{Bmatrix} \right] \frac{dx^k}{dt}$$
$$= \frac{dA^i}{dt} + A^m \begin{Bmatrix} i \\ m \;\; k \end{Bmatrix} \frac{dx^k}{dt}. \quad (3.39)$$

Similarly, the *absolute* or *intrinsic derivative* of a tensor of type $(0, 1)$ with components A_i along $\mathcal{C}$ is given by

$$\frac{\delta A_i}{\delta t} = A_{i,k} \frac{dx^k}{dt} = \frac{dA_i}{dt} - A_m \begin{Bmatrix} m \\ i \;\; k \end{Bmatrix} \frac{dx^k}{dt}. \quad (3.40)$$

Equations (3.39) and (3.40) are also sometimes written in the differential form

$$\delta A^i = dA^i + \left\{ {i \atop k\ j} \right\} A^j dx^k; \ \delta A_i = dA_i - \left\{ {j \atop k\ i} \right\} A_j dx^k. \tag{3.41}$$

From the definition of intrinsic derivatives, it follows that the familiar rules for differentiation of sums, products, etc., remain valid for the process of intrinsic differentiation. Intrinsic derivatives to tensors of rank greater than one is defined as

$$\frac{\delta A^i_{jk}}{\delta t} \equiv \frac{dA^i_{jk}}{dt} + \left\{ {i \atop \alpha\ \beta} \right\} A^\alpha_{jk} \frac{dx^\beta}{dt} - \left\{ {\alpha \atop j\ \beta} \right\} A^i_{\alpha k} \frac{dx^\beta}{dt} - \left\{ {\alpha \atop k\ \beta} \right\} A^i_{j\alpha} \frac{dx^\beta}{dt}. \tag{3.42}$$

Intrinsic derivatives of higher order are easily defined. For example,

$$\frac{\delta^2 A^i_j}{\delta t^2} = \frac{\delta}{\delta t}\left(\frac{\delta A^i_j}{\delta t}\right) = \left(A^i_{j,k}\frac{dx^k}{dt}\right)_{,l}\frac{dx^l}{dt}. \tag{3.43}$$

In general, intrinsic differentiation is not commutative. The intrinsic differentiation is useful in the study of differential geometry of curves and surfaces due to the fact that this differentiation of a tensor produces again a tensor.

Result 3.4.1 Let $I(x^i)$ is a scalar field, so that $I = I(t)$ along the curve $\mathcal{C}$. So I is an invariant. The intrinsic derivative of I along $\mathcal{C}$ is given by

$$\frac{\delta I}{\delta t} = I_{,k}\frac{dx^k}{dt} = \frac{\partial I}{\partial x^k}\frac{dx^k}{dt} = \frac{dI}{dt},$$

which is invariant. That is the intrinsic derivative of scalar field or invariant coincides with the ordinary derivative along the curve $\mathcal{C}$.

Result 3.4.2 Using Eq. (3.38), the intrinsic derivative along the curve $\mathcal{C}$ of the fundamental tensors g_{ij} is given by

$$\frac{\delta}{\delta t}g^{ij} = g^{ij}_{,k}\frac{dx^k}{dt} = 0; \ \text{as } g^{ij}_{,k} = 0$$

and

$$\frac{\delta}{\delta t}g_{ij} = g_{ij,k}\frac{dx^k}{dt} = 0; \ \text{as } g_{ij,k} = 0.$$

The intrinsic derivative of the Kronecker delta is given by

$$\frac{\delta}{\delta t}\delta^i_j = \delta^i_{j,k}\frac{dx^k}{dt} = 0.$$

Thus, the intrinsic derivatives along any curves of the fundamental tensors and the Kronecker delta are zero. Since $\frac{\delta g_{ij}}{\delta t} = 0$, the fundamental tensors g_{ij} and g^{ij} can be taken outside the sign of intrinsic differentiation.

Result 3.4.3 Uniqueness of the intrinsic derivative: The tensor derivative from a given tensor A^i that coincides with the ordinary derivative $\frac{dA^i}{dt}$ along some curve $\mathcal{C}$ in a rectangular co-ordinate system is the absolute derivative of A^i along that curve $\mathcal{C}$.

EXAMPLE 3.4.1 *If A^i is a vector field defined along a curve such that $\frac{\delta A^i}{\delta t} = 0$, show that $\frac{\delta A_i}{\delta t} = 0$, where A_i is the associate to A^i.*

Solution: If A_i is the associate to the contravariant vector A^i, then by definition, $A_j = g_{ij}A^i$. Now, the intrinsic derivative of A_i is given by

$$\frac{\delta A_j}{\delta t} = \frac{\delta}{\delta t}\left(g_{ij}A^i\right) = \frac{\delta g_{ij}}{\delta t}A^i + g_{ij}\frac{\delta A^i}{\delta t}$$

$$= 0A^i + g_{ij}0 = 0; \text{ as } \frac{\delta g_{ij}}{\delta t} = 0.$$

EXAMPLE 3.4.2 *If the intrinsic derivative of a non-null vector A^i along a curve Γ vanishes at all points of Γ, show that the magnitude of A^i is constant along Γ.*

Solution: By the given condition, we have $\frac{\delta A^i}{\delta t} = 0$. Now,

$$\frac{\delta}{\delta t}\left(g_{ij}A^iA^j\right) = \left(\frac{\delta}{\delta t}g_{ij}\right)A^iA_j + g_{ij}\left(\frac{\delta}{\delta t}A^iA_j\right)$$

$$= 0 + g_{ij}\left(\frac{\delta A^i}{\delta t}A_j + A^i\frac{\delta A_j}{\delta t}\right) = 0; \text{ as } \frac{\delta g_{ij}}{\delta t} = 0; \frac{\delta A^i}{\delta t} = 0.$$

From the relation it follows that $g_{ij}A^iA^j$, the square of the magnitude of the vector A^i, is constant along the curve. In other words, the magnitude of the vector A^i is constant along the curve.

EXAMPLE 3.4.3 *Show that*

$$\frac{d}{dt}\left(g_{ij}A^iA^j\right) = 2g_{ij}A^i\frac{\delta A^j}{\delta t}.$$

Solution: Since $g_{ij}A^iA^j$ is an invariant, we have

$$\frac{d}{dt}\left(g_{ij}A^iA^j\right) = \frac{\delta}{\delta t}\left(g_{ij}A^iA^j\right) = \frac{\delta}{\delta t}g_{ij}\left(A^iA^j\right) + g_{ij}\frac{\delta}{\delta t}\left(A^iA^j\right)$$

$$= 0\left(A^iA^j\right) + g_{ij}\left(\frac{\delta A^i}{\delta t}A^j + A^i\frac{\delta A^j}{\delta t}\right); \text{ as } \frac{\delta g_{ij}}{\delta t} = 0$$

$$= g_{ij}\frac{\delta A^i}{\delta t}A^j + g_{ji}A^j\frac{\delta A^i}{\delta t}$$

$$= g_{ij}\frac{\delta A^i}{\delta t}A^j + g_{ij}A^j\frac{\delta A^i}{\delta t} = 2g_{ij}A^i\frac{\delta A^j}{\delta t}.$$

EXAMPLE 3.4.4 *Show that intrinsic differentiation of the product of two or more tensors satisfies the distributive law, i.e.*

$$\frac{d}{dt}\left(g_{ij}A^iB^j\right) = g_{ij}\frac{\delta A^i}{\delta t}B^j + g_{ij}A^i\frac{\delta B^j}{\delta t}.$$

Hence show that

$$\frac{d}{dt}\left(A^iB_i\right) = \frac{\delta A^i}{\delta t}B_i + A^i\frac{\delta B_i}{\delta t}.$$

Solution: Since $g_{ij}A^iB^j$ is a scalar, its ordinary derivative with respect to t is its intrinsic derivative. Thus,

$$\begin{aligned}\frac{d}{dt}\left(g_{ij}A^iB^j\right) &= \frac{\delta}{\delta t}\left(g_{ij}A^iB^j\right) = \frac{\delta}{\delta t}g_{ij}\left(A^iB^j\right) + g_{ij}\frac{\delta}{\delta t}\left(A^iB^j\right)\\ &= 0\left(A^iB^j\right) + g_{ij}\left(\frac{\delta A^i}{\delta t}B^j + A^i\frac{\delta B^j}{\delta t}\right); \text{ as } \frac{\delta g_{ij}}{\delta t} = 0\\ &= g_{ij}\frac{\delta A^i}{\delta t}B^j + g_{ji}A^i\frac{\delta B^j}{\delta t} = g_{ij}\frac{\delta A^i}{\delta t}B^j + g_{ij}A^i\frac{\delta B^j}{\delta t}\\ &= g_{ij}\frac{\delta A^i}{\delta t}B^j + g_{ij}A^i\frac{\delta B^j}{\delta t}.\end{aligned}$$

According to the definition of associate vector, we have, $g_{ij}B^j = B_i$. Thus, the expression becomes

$$\frac{d}{dt}\left(A^iB_i\right) = \frac{\delta A^i}{\delta t}B_i + A^i\frac{\delta B_i}{\delta t}$$

as required. This is the distributive law of intrinsic derivative.

EXAMPLE 3.4.5 *Show that*

$$\frac{\delta}{\delta t}\left(\frac{dx^i}{dt}\right) = \frac{d^2x^i}{dt^2} + \left\{\begin{matrix} i \\ j\ \ k\end{matrix}\right\}\frac{dx^j}{dt}\frac{dx^k}{dt}.$$

Solution: We know $\frac{dx^i}{dt}$ are the components of a contravariant vector. According to definition (3.38), we have

$$\begin{aligned}\frac{\delta}{\delta t}\left(\frac{dx^i}{dt}\right) &= \left(\frac{dx^i}{dt}\right)_{,k}\left(\frac{dx^k}{dt}\right) = \left[\frac{\partial}{\partial x^k}\left(\frac{dx^i}{dt}\right) + \left\{\begin{matrix} i \\ j\ \ k\end{matrix}\right\}\frac{dx^j}{dt}\right]\frac{dx^k}{dt}\\ &= \frac{\partial}{\partial x^k}\left(\frac{dx^i}{dt}\right)\frac{dx^k}{dt} + \left\{\begin{matrix} i \\ j\ \ k\end{matrix}\right\}\frac{dx^j}{dt}\frac{dx^k}{dt}\\ &= \frac{d^2x^i}{dt^2} + \left\{\begin{matrix} i \\ j\ \ k\end{matrix}\right\}\frac{dx^j}{dt}\frac{dx^k}{dt}.\end{aligned}$$

EXAMPLE 3.4.6 *Find the expression of acceleration for the particle.*

Solution: Let a particle moves along a curve $x^k = x^k(t)$, where t is the parameter time t. In the co-ordinate system x^k, the velocity at any point has components $\frac{dx^k}{dt}$. In rectangular co-ordinates, the acceleration vector of a particle is the time derivative of its velocity vector, or the second time derivative of its position $x^k(t)$ as

$$a^k = \frac{dv^k}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x^k}{dt^2}.$$

The length of this vector at time t is the instantaneous acceleration of the particle

$$a = \sqrt{g_{ij}a^i a^j}.$$

It is not in general a tensor and so cannot represent the physical quantity in all co-ordinate systems. Since $v^k = \frac{dx^k}{dt}$ is a covariant tensor, we define the acceleration a^k as the intrinsic derivative of the velocity. Since derivatives are taken along the particle trajectory, the natural generalisation of a^k is

$$\begin{aligned} a^k = \frac{\delta v^k}{\delta t} &= \frac{dv^k}{dt} + \left\{ \begin{matrix} k \\ q \ \ p \end{matrix} \right\} v^p \frac{dx^q}{dt} \\ &= \frac{d^2x^k}{dt^2} + \left\{ \begin{matrix} k \\ p \ \ q \end{matrix} \right\} \frac{dx^p}{dt}\frac{dx^q}{dt}. \end{aligned}$$

Thus, $a^k = \frac{\delta v^k}{\delta t}$ is a covariant tensor of rank 1 and $a = \sqrt{|\delta_{ij}a^i a^j|}$.

EXAMPLE 3.4.7 *A particle is in motion along the circular is given perimetrically in spherical co-ordinates by* $x^1 = b, x^2 = \frac{\pi}{4}, x^3 = \omega t$, $t =$ *time. Find the acceleration.*

Solution: The metric components along the circle are $g_{11} = 1; g_{22} = (x^1)^2 = b^2; g_{33} = (x^1)^2 \sin^2 x^2 = \frac{b^2}{2}$. For the given problem, the non-vanishing Christoffel symbols are,

$$\left\{ \begin{matrix} 1 \\ 2 \ \ 2 \end{matrix} \right\} = -x^1 = b; \left\{ \begin{matrix} 1 \\ 3 \ \ 3 \end{matrix} \right\} = -x^1 \sin^2 x^2 = -\frac{b}{2}$$

$$\left\{ \begin{matrix} 2 \\ 1 \ \ 2 \end{matrix} \right\} = \frac{1}{x^1} = \frac{1}{b}; \left\{ \begin{matrix} 2 \\ 3 \ \ 3 \end{matrix} \right\} = -\sin x^2 \cos x^2 = -\frac{1}{2}$$

$$\left\{ \begin{matrix} 3 \\ 1 \ \ 3 \end{matrix} \right\} = \frac{1}{x^1} = \frac{1}{b}; \left\{ \begin{matrix} 3 \\ 2 \ \ 3 \end{matrix} \right\} = \cot x^2 = 1.$$

Therefore, the components of acceleration are

$$a^1 = \frac{d^2x^1}{dt^2} + \left\{ \begin{matrix} 1 \\ p \;\; q \end{matrix} \right\} \frac{dx^p}{dt}\frac{dx^q}{dt}$$

$$= 0 + \left\{ \begin{matrix} 1 \\ 2 \;\; 2 \end{matrix} \right\} \left(\frac{dx^2}{dt}\right)^2 + \left\{ \begin{matrix} 1 \\ 3 \;\; 3 \end{matrix} \right\} \left(\frac{dx^3}{dt}\right)^2 = -\frac{b\omega^2}{2}.$$

$$a^2 = \frac{d^2x^2}{dt^2} + \left\{ \begin{matrix} 2 \\ p \;\; q \end{matrix} \right\} \frac{dx^p}{dt}\frac{dx^q}{dt}$$

$$= 0 + 2\left\{ \begin{matrix} 2 \\ 1 \;\; 2 \end{matrix} \right\} \frac{dx^1}{dt}\frac{dx^2}{dt} + \left\{ \begin{matrix} 2 \\ 3 \;\; 3 \end{matrix} \right\} \left(\frac{dx^3}{dt}\right)^2 = -\frac{\omega^2}{2}.$$

$$a^3 = \frac{d^2x^2}{dt^2} + \left\{ \begin{matrix} 3 \\ p \;\; q \end{matrix} \right\} \frac{dx^p}{dt}\frac{dx^q}{dt}$$

$$= 0 + 2\left\{ \begin{matrix} 3 \\ 1 \;\; 3 \end{matrix} \right\} \frac{dx^1}{dt}\frac{dx^3}{dt} + 2\left\{ \begin{matrix} 3 \\ 2 \;\; 3 \end{matrix} \right\} \frac{dx^2}{dt}\frac{dx^3}{dt} = 0.$$

The length of this vector at time t is the instantaneous acceleration of the particle

$$a = \sqrt{g_{ij}a^i a^j} = \frac{b\omega^2}{\sqrt{2}}.$$

3.5 Exercises

1. Show that, if $g_{ij} = 0$ for $i \neq j$, then $\left\{ \begin{matrix} k \\ i \;\; j \end{matrix} \right\} = 0$ whenever i, j and k are distinct.
2. Show that, if $g_{ij} = 0$ for $i \neq j$, then

$$\left\{ \begin{matrix} i \\ i \;\; i \end{matrix} \right\} = \frac{1}{2}\frac{\partial}{\partial x^i}\log g_{ii}; \quad \left\{ \begin{matrix} i \\ i \;\; j \end{matrix} \right\} = \frac{1}{2}\frac{\partial}{\partial x^j}\log g_{ij}; \quad \left\{ \begin{matrix} i \\ j \;\; j \end{matrix} \right\} = -\frac{1}{2g_{ii}}\frac{\partial g_{jj}}{\partial x^i},$$

   where we suspend the summation convention and suppose that $i \neq j$.
3. Calculate the Christoffel symbols in rectangular co-ordinates.
4. Surface of a sphere in two-dimensional Riemannian space is given by

$$ds^2 = (x^1)^2(dx^2)^2 + (x^1)^2 \sin^2 x^2 (dx^3)^2.$$

   Compute the Christoffel symbols of first and second kinds.
5. Calculate the non-zero Christoffel symbols corresponding to the metric
   (i) $ds^2 = (dx^1)^2 + \left[(x^2)^2 - (x^1)^2\right](dx^2)^2$.
   (ii) $ds^2 = (dx^1)^2 + G(x^1, x^2)(dx^2)^2$.

(iii) $ds^2 = (dx^1)^2 + f(x^1, x^2)(dx^2)^2 + (dx^3)^2$.
(iv) $ds^2 = -a(dx^1)^2 - b(dx^2)^2 - c(dx^3)^2 + d(dx^4)^2$.
(v) $ds^2 = \left[(x^1)^2 + (x^2)^2\right]\{(dx^1)^2 + (dx^2)^2\} + (dx^3)^2$.
(vi) $ds^2 = \left[(y^1)^2 + (y^2)^2\right]\{(dy^1)^2 + (dy^2)^2\} + (y^1y^2)^2(dy^3)^2$.
(vii) $ds^2 = a^2\left[(\sinh y^1)^2 + (\sin y^2)^2\right]\{(dy^1)^2 + (dy^2)^2\} + (dy^3)^2$.
(viii) $ds^2 = f(u, v)(du)^2 + h(u, v)(dv)^2$.
(ix) $ds^2 = -e^{-\lambda}dr^2 - r^2d\theta^2 - r^2sin^2\theta d\phi^2 + e^{\mu}dt^2$.

6. If A^{pq} is a symmetric tensor, show that, $A^{qm}[iq, m] = \frac{1}{2}A^{qm}\frac{\partial g_{qm}}{\partial x^i}$.
7. If $g_{ij} \neq 0$, show that

$$g_{\alpha\beta}\frac{\partial}{\partial x^j}\begin{Bmatrix} \beta \\ i \;\; k \end{Bmatrix} = \frac{\partial}{\partial x^j}[ik, \alpha] - \begin{Bmatrix} \beta \\ i \;\; k \end{Bmatrix}([\beta j, \alpha] + [\alpha j, \beta]).$$

8. If $y^i = a^i_j x^j$ is a transformation from a set of orthogonal Cartesian variables y^i to a set of oblique Cartesian co-ordinates x^i covering E_3, what are the metric coefficients g_{ij} in $ds^2 = g_{ij}dx^idx^j$?
9. If y^i are rectangular Cartesian co-ordinates and x^i are curvilinear co-ordinates, prove that the Christoffel symbols of the second kind $\begin{Bmatrix} i \\ j \;\; k \end{Bmatrix}$ are given by

$$\begin{Bmatrix} i \\ j \;\; k \end{Bmatrix} = \frac{\partial^2 y^p}{\partial x^j \partial x^k}\frac{\partial x^i}{\partial y^p}.$$

10. (a) Show that for affine transformation of co-ordinates $x^i = a^i_p\overline{x}^p + b^i; i, p = 1, 2, \ldots, N$ the Christoffel symbols possess tensor character.
    (b) How do the Christoffel symbols of the first and the second kinds transform under co-ordinate transformations?
11. If A^{pqr} is a skew-symmetric tensor, then show that

$$A^{pqr}\begin{Bmatrix} l \\ p \;\; q \end{Bmatrix} = A^{pqr}\begin{Bmatrix} l \\ q \;\; r \end{Bmatrix} = A^{pqr}\begin{Bmatrix} l \\ p \;\; r \end{Bmatrix} = 0.$$

12. If $\Gamma^m_{np} = \begin{Bmatrix} m \\ n \;\; p \end{Bmatrix} + 2\delta^m_n A_k$; where A_k is a covariant vector, then show that

$$g_{tn}\Gamma^t_{mp} + g_{tm}\Gamma^t_{np} - 4g_{mn}A_k = \frac{\partial g_{mn}}{\partial x^k}.$$

13. If A_i are the components of a covariant vector, then show that $\begin{Bmatrix} i \\ j \;\; k \end{Bmatrix} + 2\delta^i_j A_k$ are not components of a tensor.
14. Show that

$$\frac{\partial^2 \overline{x}^r}{\partial x^k \partial x^l} = \begin{Bmatrix} i \\ k \;\; l \end{Bmatrix}\frac{\partial \overline{x}^r}{\partial x^i} - \overline{\begin{Bmatrix} r \\ s \;\; t \end{Bmatrix}}\frac{\partial \overline{x}^s}{\partial x^k}\frac{\partial \overline{x}^t}{\partial x^l}.$$

15. Prove that the following expressions are tensors
    (i) $A^{ij}_{,k} = \frac{\partial A^{ij}}{\partial x^k} + \left\{ \begin{matrix} i \\ \alpha \; k \end{matrix} \right\} A^{\alpha j} + \left\{ \begin{matrix} j \\ \alpha \; k \end{matrix} \right\} A^{i\alpha}.$
    (ii) $A^i_{j,k} = \frac{\partial A^i_j}{\partial x^k} - \left\{ \begin{matrix} \alpha \\ j \; k \end{matrix} \right\} A^i_\alpha + \left\{ \begin{matrix} i \\ \alpha \; k \end{matrix} \right\} A^\alpha_j.$
    (iii) $A_{ij,k} = \frac{\partial A_{ij}}{\partial x^k} - \left\{ \begin{matrix} \alpha \\ i \; k \end{matrix} \right\} A_{\alpha j} + \left\{ \begin{matrix} \alpha \\ j \; k \end{matrix} \right\} A_{i\alpha}.$
    (iv) $A^r_{ijk,l} = \frac{\partial A^r_{ijk}}{\partial x^l} - \left\{ \begin{matrix} \alpha \\ i \; l \end{matrix} \right\} A^r_{\alpha jk} + \left\{ \begin{matrix} \alpha \\ j \; l \end{matrix} \right\} A^r_{i\alpha k} - \left\{ \begin{matrix} \alpha \\ k \; l \end{matrix} \right\} A^r_{ij\alpha} + \left\{ \begin{matrix} r \\ \alpha \; l \end{matrix} \right\} A^\alpha_{ijk}.$
16. Prove that the covariant derivative with respect to x^q of
    (a) the tensor A^{jk}_l is

$$A^{jk}_{l,q} = \frac{\partial A^{jk}_l}{\partial x^q} - \left\{ \begin{matrix} \alpha \\ l \; q \end{matrix} \right\} A^{jk}_\alpha + \left\{ \begin{matrix} j \\ q \; \alpha \end{matrix} \right\} A^{\alpha k}_l + \left\{ \begin{matrix} k \\ q \; \alpha \end{matrix} \right\} A^{j\alpha}_l.$$

    (b) the tensor A^{jk}_{lm} is

$$A^{jk}_{lm,q} = \frac{\partial A^{jk}_{lm}}{\partial x^q} - \left\{ \begin{matrix} \alpha \\ l \; q \end{matrix} \right\} A^{jk}_{\alpha m} - \left\{ \begin{matrix} \alpha \\ m \; q \end{matrix} \right\} A^{jk}_{l\alpha} + \left\{ \begin{matrix} j \\ q \; \alpha \end{matrix} \right\} A^{\alpha k}_{lm} + \left\{ \begin{matrix} k \\ q \; \alpha \end{matrix} \right\} A^{j\alpha}_{lm}.$$

    (c) the tensor A^{jkl}_m is

$$A^{jkl}_{m,q} = \frac{\partial A^{jkl}_m}{\partial x^q} - \left\{ \begin{matrix} \alpha \\ m \; q \end{matrix} \right\} A^{jkl}_\alpha + \left\{ \begin{matrix} j \\ q \; \alpha \end{matrix} \right\} A^{\alpha kl}_m + \left\{ \begin{matrix} k \\ q \; \alpha \end{matrix} \right\} A^{j\alpha l}_m + \left\{ \begin{matrix} l \\ q \; \alpha \end{matrix} \right\} A^{jk\alpha}_m.$$

    (d) the tensor A^{jk}_{lmn} is

$$A^{jk}_{lmn,q} = \frac{\partial A^{jk}_{lmn}}{\partial x^q} - \left\{ \begin{matrix} \alpha \\ l \; q \end{matrix} \right\} A^{jk}_{\alpha mn} - \left\{ \begin{matrix} \alpha \\ m \; q \end{matrix} \right\} A^{jk}_{l\alpha n} - \left\{ \begin{matrix} \alpha \\ n \; q \end{matrix} \right\} A^{jk}_{lm\alpha}$$
$$+ \left\{ \begin{matrix} j \\ q \; \alpha \end{matrix} \right\} A^{\alpha k}_{lmn} + \left\{ \begin{matrix} k \\ q \; \alpha \end{matrix} \right\} A^{j\alpha}_{lmn}.$$

17. If A is the magnitude of A^i, show that $A_{,j} = \frac{1}{A}\left(A_{i,j}A^i\right)$.
18. Find covariant derivative of (a) $g_{jk}A^k$ (b) A^jB_k and (c) $\delta^j_kA_j$ with respect to x^q.
19. Prove that the covariant derivative of an arbitrary tensor is a tensor of which the covariant order exceeds that of the original tensor by exactly one.
20. (a) Prove that ${}_a\left\{ \begin{matrix} k \\ i \; j \end{matrix} \right\} - {}_b\left\{ \begin{matrix} k \\ i \; j \end{matrix} \right\}$ are components of a tensor of rank 3, where the two Christoffel symbols formed from the symmetric tensors $a_{ij}(x)$ and $b_{ij}(x)$.

(b) Prove that in the covariant derivative of a contravariant vector, the order is increased by one covariantly.

21. If a_{ij} is a skew-symmetric tensor, show that $a_{ij,k}$ is skew-symmetric in the indices i and j.
22. If A_{ij} is a skew-symmetric tensor such that $A_{ij,k} = A_{ik,j}$, prove that $A_{ij,k} = 0$.
23. Prove that,

$$\frac{\partial}{\partial x^k}\left(\sqrt{g}g^{jk}\right) + \sqrt{g}\left\{ {i \atop k\ \ m} \right\} g^{km} = 0.$$

24. Prove that

$$A^{ij}_{,j} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^j}\left(A^{ij}\sqrt{g}\right) + A^{jk}\left\{ {i \atop j\ \ k} \right\},$$

where A^{ij} is a tensor of type $(2,0)$. What will happen, if A^{ij} is skew-symmetric?

25. If A^{ij} is a symmetric tensor and $A^j_i = A^{jk}g_{ik}$, prove that

$$A^j_{i,j} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^j}\left(A^j_i\sqrt{g}\right) - \frac{1}{2}A^{jk}\frac{\partial}{\partial x^i}g_{jk}.$$

26. Show that the operation of raising or lowering of indices can be performed either before or after covariant differentiation.
27. If A_{ij} is a symmetric tensor such that $A_{ij,k} = A_{ik,j}$, show that $A_{ij,k}$ is a symmetric tensor.
28. Prove that $A^j_{i,j} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^j}\left(A^j_i\sqrt{g}\right) + A^j_k\left\{ {k \atop i\ \ j} \right\}$.
29. If curl of a covariant vector A_i vanishes identically, then A_i is gradient.
30. If A_{ij} is the curl of a covariant vector, show that $A_{ij,k} + A_{jk,i} + A_{ki,j} = 0$. Show further that this expression is equivalent to

$$\frac{\partial A_{ij}}{\partial x^k} + \frac{\partial A_{jk}}{\partial x^i} + \frac{\partial A_{ki}}{\partial x^j} = 0.$$

31. (a) If $A_{ij} = B_{i,j} - B_{j,i}$, prove that $A_{ij,k} + A_{jk,i} + A_{ki,j} = 0$.

    (b) If A_{ij} is an antisymmetric tensor, show that $A_{ij,k} + A_{jk,i} + A_{ki,j}$ is a tensor which is antisymmetric is any pair of indices.
32. If A^{pq} is a skew-symmetric tensor, show that $A^{pq}_{,p} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^p}\left(\sqrt{g}A^{pq}\right)$.
33. If A^{ijk} is a skew-symmetric tensor, show that $\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^k}\left(\sqrt{g}A^{ijk}\right)$ is a tensor.
34. Let **A** and **B** be two arbitrary vectors and ϕ, ψ are scalar functions. Then show that

(i) curl $(\mathbf{A}+\mathbf{B}) = curl\mathbf{A} + curl\mathbf{B}$.
(ii) grad $(\phi\psi) = \phi \operatorname{grad} \psi + \psi \operatorname{grad} \phi$
(iii) $\nabla^2(\phi\psi) = \phi\nabla^2\psi + \psi\nabla^2\phi + 2\nabla\phi\nabla\psi$.
(iv) $\operatorname{curl}(\phi\mathbf{A}) = \phi$ curl $\mathbf{A} + \mathbf{A} \times \nabla\phi$.

35. If A^i is a vector field along a curve such that $\frac{\delta A^i}{\delta t} = 0$, show that $\frac{\delta A_i}{\delta t}$ is also zero.
36. Prove that in co-ordinate systems in which the g_{ij} are constants, absolute differentiation reduces to ordinary differentiation.
37. If $A_i = g_{ij}A^j$, show that $A_{j,k} = g_{i\alpha}A^{\alpha}_{,k}$.
38. Show that

$$\frac{\partial}{\partial x^k}\left(g_{ij}A^iB^j\right) = A_{i,k}B^i + A_iB_{i,k}.$$

39. Prove that if A is the magnitude of A^i, then $A_{,j} = A_{i,j}A^i/A$.
40. (a) The velocity vector field of a fluid in motion in a plane is $v^i = (x, 2y)$ is Cartesian co-ordinates. Find its covariant derivative in polar co-ordinates.
    (b) Find the intrinsic derivative of this vector field along the spiral $r = a\theta + b; a > 0, b > 0$.
41. Prove in general that intrinsic differentiation of the product of two or more tensors satisfies the distributive law.
42. Find the Christoffel symbols of:
    (i) A surface of revolution represented in the form $x(u^1, u^2) = (u^2\cos u^1, u^2\sin u^1, h(u^2))$.
    (ii) A surface represented in the form $x_3 = F(x_1, x_2)$.

Chapter 4

Riemannian Geometry

The covariant derivative of a tensor, in general, is a tensor. If the resulting tensor is again subjected to covariant differentiation, then we again get a tensor. This tensor obtained after two operations is called the *second covariant derivative* of the original tensor. The tensor obtained by covariant differentiation of the second covariant derivative is called the *third covariant derivative* of the original tensor and so on.

In case of an invariant, the operation of covariant differentiation is commutative. But for a tensor of order greater than or equal to one the operation is not, however, commutative. This is due to the peculiarity of the environment in which the operation is undertaken, namely Riemannian space.

The characteristic peculiarity of such a space consists in a certain tensor is called the *curvature tensor* whose components can be expressed with the help of the components of the fundamental tensors. The following discussion will show why the operation of covariant differentiation is not, in general, commutative in a Riemannian space and will reveal the role played by the curvature tensor in this matter.

4.1 Riemann–Christoffel Tensor

Here, we will investigate the commutative problem with respect to covariant differentiation. Let B_i be an arbitrary covariant vector, then its covariant derivative with respect to x^j is given by

$$B_{i,j} = \frac{\partial B_i}{\partial x^j} - \begin{Bmatrix} \alpha \\ i \;\; j \end{Bmatrix} B_\alpha, \tag{4.1}$$

which is a covariant tensor of rank 2. Further differentiating covariantly $B_{i,j}$ with respect to x^k, we get

$$\left(B_{i,j}\right)_{,k} = B_{i,jk} = \frac{\partial B_{i,j}}{\partial x^k} - \begin{Bmatrix} \alpha \\ i \;\; k \end{Bmatrix} B_{\alpha,j} - \begin{Bmatrix} \alpha \\ j \;\; k \end{Bmatrix} B_{i,\alpha} \tag{4.2}$$

and is known as the second covariant derivative of the given covariant vector. Using Eq. (4.1), we get

$$\begin{aligned}
B_{i,jk} &= \frac{\partial}{\partial x^k}\left[\frac{\partial B_i}{\partial x^j} - \left\{\begin{matrix}\alpha \\ i\ \ j\end{matrix}\right\} B_\alpha\right] \\
&\quad - \left[\frac{\partial B_\alpha}{\partial x^j} - \left\{\begin{matrix}\beta \\ \alpha\ \ j\end{matrix}\right\} B_\beta\right]\left\{\begin{matrix}\alpha \\ i\ \ k\end{matrix}\right\} - \left[\frac{\partial B_i}{\partial x^\alpha} - \left\{\begin{matrix}\beta \\ i\ \ \alpha\end{matrix}\right\} B_\beta\right]\left\{\begin{matrix}\alpha \\ j\ \ k\end{matrix}\right\} \\
&= \frac{\partial^2 B_i}{\partial x^j \partial x^k} - \left\{\begin{matrix}\alpha \\ i\ \ j\end{matrix}\right\}\frac{\partial B_\alpha}{\partial x^k} - \left\{\begin{matrix}\alpha \\ i\ \ k\end{matrix}\right\}\frac{\partial B_\alpha}{\partial x^j} - \left\{\begin{matrix}\alpha \\ j\ \ k\end{matrix}\right\}\frac{\partial B_i}{\partial x^\alpha} \\
&\quad - B_\alpha \frac{\partial}{\partial x^k}\left\{\begin{matrix}\alpha \\ i\ \ j\end{matrix}\right\} + B_\beta \left\{\begin{matrix}\beta \\ \alpha\ \ j\end{matrix}\right\}\left\{\begin{matrix}\alpha \\ i\ \ k\end{matrix}\right\} + B_\beta \left\{\begin{matrix}\beta \\ i\ \ \alpha\end{matrix}\right\}\left\{\begin{matrix}\alpha \\ j\ \ k\end{matrix}\right\}.
\end{aligned}$$

Interchanging the dummy indices α and β in last two terms, we get

$$\begin{aligned}
&= \frac{\partial^2 B_i}{\partial x^j \partial x^k} - \left\{\begin{matrix}\alpha \\ i\ \ j\end{matrix}\right\}\frac{\partial B_\alpha}{\partial x^k} - \left\{\begin{matrix}\alpha \\ i\ \ k\end{matrix}\right\}\frac{\partial B_\alpha}{\partial x^j} - \left\{\begin{matrix}\alpha \\ j\ \ k\end{matrix}\right\}\frac{\partial B_i}{\partial x^\alpha} \\
&\quad - B_\alpha \frac{\partial}{\partial x^k}\left\{\begin{matrix}\alpha \\ i\ \ j\end{matrix}\right\} + B_\alpha \left\{\begin{matrix}\alpha \\ \beta\ \ j\end{matrix}\right\}\left\{\begin{matrix}\beta \\ i\ \ k\end{matrix}\right\} + B_\alpha \left\{\begin{matrix}\alpha \\ i\ \ \beta\end{matrix}\right\}\left\{\begin{matrix}\beta \\ j\ \ k\end{matrix}\right\}.
\end{aligned} \tag{4.3}$$

Interchanging j and k in Eq. (4.3), we have

$$\begin{aligned}
B_{i,kj} &= \frac{\partial^2 B_i}{\partial x^k \partial x^j} - \left\{\begin{matrix}\alpha \\ i\ \ k\end{matrix}\right\}\frac{\partial B_\alpha}{\partial x^j} - \left\{\begin{matrix}\alpha \\ i\ \ j\end{matrix}\right\}\frac{\partial B_\alpha}{\partial x^k} - \left\{\begin{matrix}\alpha \\ k\ \ j\end{matrix}\right\}\frac{\partial B_i}{\partial x^\alpha} \\
&\quad - B_\alpha \frac{\partial}{\partial x^j}\left\{\begin{matrix}\alpha \\ i\ \ k\end{matrix}\right\} + B_\alpha \left\{\begin{matrix}\alpha \\ \beta\ \ k\end{matrix}\right\}\left\{\begin{matrix}\beta \\ i\ \ j\end{matrix}\right\} + B_\alpha \left\{\begin{matrix}\alpha \\ i\ \ \beta\end{matrix}\right\}\left\{\begin{matrix}\beta \\ k\ \ j\end{matrix}\right\}.
\end{aligned} \tag{4.4}$$

Subtracting Eq. (4.4) from Eq. (4.3), we get

$$\begin{aligned}
B_{i,jk} - B_{i,kj} &= \left[-\frac{\partial}{\partial x^k}\left\{\begin{matrix}\alpha \\ i\ \ j\end{matrix}\right\} + \frac{\partial}{\partial x^j}\left\{\begin{matrix}\alpha \\ i\ \ k\end{matrix}\right\}\right. \\
&\quad \left. + \left\{\begin{matrix}\alpha \\ \beta\ \ j\end{matrix}\right\}\left\{\begin{matrix}\beta \\ i\ \ k\end{matrix}\right\} - \left\{\begin{matrix}\alpha \\ \beta\ \ k\end{matrix}\right\}\left\{\begin{matrix}\beta \\ i\ \ j\end{matrix}\right\}\right] B_\alpha.
\end{aligned} \tag{4.5}$$

This shows that the covariant differentiation is not commutative in a non-Euclidean space. Since B_i is an arbitrary covariant vector, it follows from the quotient law that the expression in the square brackets of Eq. (4.5) is a mixed tensor of the fourth order, contravariant order one and covariant order three. If we write, N^4 components R^{α}_{ijk} as

$$R^{\alpha}_{ijk} = -\frac{\partial}{\partial x^k}\left\{\begin{matrix}\alpha\\ i\ \ j\end{matrix}\right\} + \frac{\partial}{\partial x^j}\left\{\begin{matrix}\alpha\\ i\ \ k\end{matrix}\right\} + \left\{\begin{matrix}\alpha\\ \beta\ \ j\end{matrix}\right\}\left\{\begin{matrix}\beta\\ i\ \ k\end{matrix}\right\} - \left\{\begin{matrix}\alpha\\ \beta\ \ k\end{matrix}\right\}\left\{\begin{matrix}\beta\\ i\ \ j\end{matrix}\right\} \tag{4.6}$$

$$= \begin{vmatrix} \dfrac{\partial}{\partial x^j} & \dfrac{\partial}{\partial x^k} \\ \left\{\begin{matrix}\alpha\\ i\ \ j\end{matrix}\right\} & \left\{\begin{matrix}\alpha\\ i\ \ k\end{matrix}\right\} \end{vmatrix} + \begin{vmatrix} \left\{\begin{matrix}\alpha\\ \beta\ \ j\end{matrix}\right\} & \left\{\begin{matrix}\alpha\\ \beta\ \ k\end{matrix}\right\} \\ \left\{\begin{matrix}\beta\\ i\ \ j\end{matrix}\right\} & \left\{\begin{matrix}\beta\\ i\ \ k\end{matrix}\right\} \end{vmatrix}, \tag{4.7}$$

then Eq. (4.5) can be written in the form

$$B_{i,jk} - B_{i,kj} = B_\alpha R^{\alpha}_{ijk}. \tag{4.8}$$

Since $B_{i,jk}$ is a tensor of third order, so the left-hand side of Eq. (4.8) is a covariant tensor of rank 3. But B_α is an arbitrary vector and hence, it follows from quotient law that R^{α}_{ijk}, of the type $(1, 3)$ is a mixed tensor of order four.

The tensor R^{α}_{ijk} is called *Riemann–Christoffel tensor* of the second kind or *the curvature tensor* of the Riemannian space. The symbols R^{α}_{ijk} are referred to as *Riemann's symbol of the second kind.* It is formed exclusively from the fundamental tensor g_{ij} and its derivatives up to and including the second order. This tensor does not depend on the choice of the vector B_i.

Property 4.1.1 The necessary and sufficient condition that the covariant differentiation of all vectors be commutative is that the Riemann tensor R^{α}_{ijk} vanishes identically.

Proof: If the left-hand side of Eq. (4.8) is to vanish, i.e. the order of covariant differentiation is to be immaterial, then $R^{\alpha}_{ijk} = 0$. Since B_i is arbitrary, in general $R^{\alpha}_{ijk} \neq 0$, so that the order of covariant differentiation is no immaterial. Thus, it is clear from Eq. (4.8) that the necessary and sufficient condition for the validity of inversion of the order of covariant differentiation is that the tensor R^{α}_{ijk} vanishes identically.

Property 4.1.2 The Riemann–Christoffel curvature tensor R^{α}_{ijk} is skew-symmetric with respect to the indices j and k.

Proof: The Riemann–Christoffel curvature tensor R^{α}_{ijk} is given by Eq. (4.6). Interchanging j and k, in the expression for R^{α}_{ijk}, we get

$$R^{\alpha}_{ikj} = -\frac{\partial}{\partial x^j}\left\{\begin{matrix}\alpha\\ i\ \ k\end{matrix}\right\} + \frac{\partial}{\partial x^k}\left\{\begin{matrix}\alpha\\ i\ \ j\end{matrix}\right\} + \left\{\begin{matrix}\alpha\\ \beta\ \ k\end{matrix}\right\}\left\{\begin{matrix}\beta\\ i\ \ j\end{matrix}\right\} - \left\{\begin{matrix}\alpha\\ \beta\ \ j\end{matrix}\right\}\left\{\begin{matrix}\beta\\ i\ \ k\end{matrix}\right\}.$$

$$= -\left[\frac{\partial}{\partial x^j}\left\{\begin{matrix}\alpha\\ i\ \ k\end{matrix}\right\} - \frac{\partial}{\partial x^k}\left\{\begin{matrix}\alpha\\ i\ \ j\end{matrix}\right\} + \left\{\begin{matrix}\alpha\\ \beta\ \ j\end{matrix}\right\}\left\{\begin{matrix}\beta\\ i\ \ k\end{matrix}\right\} - \left\{\begin{matrix}\alpha\\ \beta\ \ k\end{matrix}\right\}\left\{\begin{matrix}\beta\\ i\ \ j\end{matrix}\right\}\right] = -R^{\alpha}_{ijk}.$$

This shows that the Riemann–Christoffel curvature tensor R^{α}_{ijk} is skew-symmetric with respect to the indices j and k.

Property 4.1.3 The curvature tensor R^{α}_{ijk} satisfies *cyclic property,* i.e.

$$R^{\alpha}_{ijk} + R^{\alpha}_{jki} + R^{\alpha}_{kij} = 0; \text{ i.e. } R^{\alpha}_{[ijk]} = 0.$$

Proof: The Riemannian curvature tensor R^{α}_{ijk} is given by Eq. (4.6). Taking the sum of this and two similar equations obtained by cyclic permutation of i, j, k we obtain

$$\begin{aligned}
R^{\alpha}_{[ijk]} &= R^{\alpha}_{ijk} + R^{\alpha}_{jki} + R^{\alpha}_{kij} \\
&= -\frac{\partial}{\partial x^k}\left\{ \begin{matrix} \alpha \\ i \;\; j \end{matrix} \right\} + \frac{\partial}{\partial x^j}\left\{ \begin{matrix} \alpha \\ i \;\; k \end{matrix} \right\} + \left\{ \begin{matrix} \alpha \\ \beta \;\; j \end{matrix} \right\}\left\{ \begin{matrix} \beta \\ i \;\; k \end{matrix} \right\} - \left\{ \begin{matrix} \alpha \\ \beta \;\; k \end{matrix} \right\}\left\{ \begin{matrix} \beta \\ i \;\; j \end{matrix} \right\} \\
&\quad -\frac{\partial}{\partial x^i}\left\{ \begin{matrix} \alpha \\ j \;\; k \end{matrix} \right\} + \frac{\partial}{\partial x^k}\left\{ \begin{matrix} \alpha \\ j \;\; i \end{matrix} \right\} + \left\{ \begin{matrix} \alpha \\ \beta \;\; k \end{matrix} \right\}\left\{ \begin{matrix} \beta \\ j \;\; i \end{matrix} \right\} - \left\{ \begin{matrix} \alpha \\ \beta \;\; i \end{matrix} \right\}\left\{ \begin{matrix} \beta \\ j \;\; k \end{matrix} \right\} \\
&\quad -\frac{\partial}{\partial x^j}\left\{ \begin{matrix} \alpha \\ k \;\; i \end{matrix} \right\} + \frac{\partial}{\partial x^i}\left\{ \begin{matrix} \alpha \\ k \;\; j \end{matrix} \right\} + \left\{ \begin{matrix} \alpha \\ \beta \;\; i \end{matrix} \right\}\left\{ \begin{matrix} \beta \\ k \;\; j \end{matrix} \right\} - \left\{ \begin{matrix} \alpha \\ \beta \;\; j \end{matrix} \right\}\left\{ \begin{matrix} \beta \\ k \;\; i \end{matrix} \right\} \\
&= 0; \quad \text{as } \left\{ \begin{matrix} \alpha \\ i \;\; j \end{matrix} \right\} = \left\{ \begin{matrix} \alpha \\ j \;\; i \end{matrix} \right\}, \text{ etc.}
\end{aligned}$$

Hence, the cyclic property is established. Thus, if contravariant index α of the tensor R^{α}_{ijk} is held fixed while the remaining three indices are cyclically permuted and the components added, the result is zero.

Property 4.1.4 The Riemann–Christoffel curvature tensor of the second kind can be contracted in two ways: one of these leads to a zero tensor and the other to a symmetric tensor.

Proof: Starting from the curvature tensor R^{α}_{ijk}, we get the three different contracted tensors $R^{\alpha}_{\alpha jk}$, $R^{\alpha}_{i\alpha k}$ and $R^{\alpha}_{ij\alpha}$ by contracting two indices. From Eq. (4.6), we have

$$\begin{aligned}
R^{\alpha}_{\alpha jk} &= -\frac{\partial}{\partial x^k}\left\{ \begin{matrix} \alpha \\ \alpha \;\; j \end{matrix} \right\} + \frac{\partial}{\partial x^j}\left\{ \begin{matrix} \alpha \\ \alpha \;\; k \end{matrix} \right\} + \left\{ \begin{matrix} \alpha \\ \beta \;\; j \end{matrix} \right\}\left\{ \begin{matrix} \beta \\ \alpha \;\; k \end{matrix} \right\} - \left\{ \begin{matrix} \alpha \\ \beta \;\; k \end{matrix} \right\}\left\{ \begin{matrix} \beta \\ \alpha \;\; j \end{matrix} \right\} \\
&= -\frac{\partial}{\partial x^k}\left[\frac{\partial}{\partial x^j}(\log\sqrt{g})\right] + \frac{\partial}{\partial x^j}\left[\frac{\partial}{\partial x^k}(\log\sqrt{g})\right] \\
&\quad + \left\{ \begin{matrix} \alpha \\ \beta \;\; k \end{matrix} \right\}\left\{ \begin{matrix} \beta \\ \alpha \;\; j \end{matrix} \right\} - \left\{ \begin{matrix} \alpha \\ \beta \;\; k \end{matrix} \right\}\left\{ \begin{matrix} \beta \\ \alpha \;\; j \end{matrix} \right\} \\
&= -\frac{\partial^2}{\partial x^k \partial x^j}(\log\sqrt{g}) + \frac{\partial^2}{\partial x^j \partial x^k}(\log\sqrt{g}) = 0.
\end{aligned}$$

Thus, contraction of R^{α}_{ijk} with respect to the suffixes α and i leads to a zero tensor. Using the skew-symmetric property, we have $R^{\alpha}_{i\alpha k} = -R^{\alpha}_{ik\alpha}$. Lastly,

$$\begin{aligned} R^{\alpha}_{ij\alpha} &= \frac{\partial}{\partial x^j}\begin{Bmatrix} \alpha \\ i \;\; \alpha \end{Bmatrix} - \frac{\partial}{\partial x^\alpha}\begin{Bmatrix} \alpha \\ i \;\; j \end{Bmatrix} + \begin{Bmatrix} \alpha \\ \beta \;\; j \end{Bmatrix}\begin{Bmatrix} \beta \\ i \;\; \alpha \end{Bmatrix} - \begin{Bmatrix} \alpha \\ \beta \;\; \alpha \end{Bmatrix}\begin{Bmatrix} \beta \\ i \;\; j \end{Bmatrix} \\ &= \frac{\partial}{\partial x^j}\left[\frac{\partial}{\partial x^i}(\log\sqrt{g})\right] \\ &\quad -\frac{\partial}{\partial x^\alpha}\begin{Bmatrix} \alpha \\ i \;\; j \end{Bmatrix} + \begin{Bmatrix} \alpha \\ \beta \;\; j \end{Bmatrix}\begin{Bmatrix} \beta \\ \alpha \;\; i \end{Bmatrix} - \begin{Bmatrix} \alpha \\ \beta \;\; \alpha \end{Bmatrix}\begin{Bmatrix} \beta \\ j \;\; i \end{Bmatrix}, \end{aligned} \tag{4.9}$$

which is not identically zero. Thus, of the three contracted tensors the last one only needs consideration, because the second one is the negative of the last and the first one is identically zero. The contracted tensor $R^{\alpha}_{ij\alpha}$ is not identically zero.

4.1.1 Ricci Tensor

The Riemann–Christoffel tensor R^{α}_{ijk} can be contracted in three ways with respect to α and any one of its lower indices, i.e. $R^{\alpha}_{\alpha jk}$, $R^{\alpha}_{i\alpha k}$ and $R^{\alpha}_{ij\alpha}$. The contracted tensor $R^{\alpha}_{ij\alpha}$, which is not identically zero is called the *Ricci tensor of the first kind* and its components are denoted by R_{ij}. Now, using Eq. (4.9) and write $R_{ij} = R^{\alpha}_{ij\alpha}$, the *Ricci tensor of the first kind* is defined by

$$R_{ij} = \frac{\partial^2}{\partial x^j \partial x^i}(\log\sqrt{g}) - \frac{\partial}{\partial x^\alpha}\begin{Bmatrix} \alpha \\ i \;\; j \end{Bmatrix} + \begin{Bmatrix} \alpha \\ \beta \;\; j \end{Bmatrix}\begin{Bmatrix} \beta \\ \alpha \;\; i \end{Bmatrix} - \begin{Bmatrix} \alpha \\ \beta \;\; \alpha \end{Bmatrix}\begin{Bmatrix} \beta \\ j \;\; i \end{Bmatrix}, \tag{4.10}$$

which is a tensor of the type $(0, 2)$, plays an important role in Einstein's theory of gravitation. From Eq. (4.10) it follows that:

$$\begin{aligned} R_{ji} = R^{\alpha}_{ji\alpha} &= \frac{\partial^2}{\partial x^i \partial x^j}(\log\sqrt{g}) - \frac{\partial}{\partial x^\alpha}\begin{Bmatrix} \alpha \\ j \;\; i \end{Bmatrix} + \begin{Bmatrix} \alpha \\ \beta \;\; i \end{Bmatrix}\begin{Bmatrix} \beta \\ \alpha \;\; j \end{Bmatrix} - \begin{Bmatrix} \alpha \\ \beta \;\; \alpha \end{Bmatrix}\begin{Bmatrix} \beta \\ i \;\; j \end{Bmatrix} \\ &= \frac{\partial^2}{\partial x^j \partial x^i}(\log\sqrt{g}) - \frac{\partial}{\partial x^\alpha}\begin{Bmatrix} \alpha \\ i \;\; j \end{Bmatrix} + \begin{Bmatrix} \alpha \\ \beta \;\; j \end{Bmatrix}\begin{Bmatrix} \beta \\ \alpha \;\; i \end{Bmatrix} - \begin{Bmatrix} \alpha \\ \beta \;\; \alpha \end{Bmatrix}\begin{Bmatrix} \beta \\ j \;\; i \end{Bmatrix} = R_{ij}, \end{aligned}$$

where we will replace the dummy indices β and α in the third term of the right-hand side by α and β, respectively, and $\begin{Bmatrix} \alpha \\ i \;\; j \end{Bmatrix} = \begin{Bmatrix} \alpha \\ j \;\; i \end{Bmatrix}$, etc. Thus,

$$R_{ji} = R^{\alpha}_{ji\alpha} = R^{\alpha}_{ij\alpha} = R_{ij}.$$

This shows that the Ricci tensor R_{ij} is symmetric. Also,

$$R^{\alpha}_{i\alpha k} = -R^{\alpha}_{ik\alpha} - R_{ik} \text{ and } R^{\alpha}_{\alpha jk} = 0.$$

The *Ricci tensor of the second kind* R^i_j is given by

$$R^i_j = g^{i\alpha} R_{\alpha j}. \tag{4.11}$$

The *scalar curvature* or curvature invariant R is defined by

$$R = R^i_i = g^{i\alpha} R_{\alpha i} \tag{4.12}$$

EXAMPLE 4.1.1 *Find the scalar curvature of a sphere in E^3 with constant radius a.*

Solution: In E^3 the line element of a sphere of constant radius a can be taken in spherical polar co-ordinates as

$$ds^2 = a^2(dx^1)^2 + a^2 \sin^2 x^1 (dx^2)^2.$$

For the given metric, we have $g_{11} = a^2$; $g_{12} = 0 = g_{21}$; $g_{22} = a^2 \sin^2 x^1$. Since $g = a^4 \sin^2 x^1$, so the reciprocal tensors g^{ij} are given by

$$g^{11} = \frac{1}{a^2}; \quad g^{12} = 0 = g^{21}; \quad g^{22} = \frac{1}{a^2 \sin^2 x^1}.$$

The non-vanishing Christoffel symbols of second kind are

$$\left\{ {1 \atop 2\ 2} \right\} = -\sin x^1 \cos x^1; \quad \left\{ {2 \atop 1\ 2} \right\} = \left\{ {2 \atop 2\ 1} \right\} = \cot x^1.$$

Using formula Eq. (4.10), the Ricci tensors of first kind are given by

$$R_{11} = \frac{\partial^2}{(\partial x^1)^2}(\log \sqrt{g}) - \frac{\partial}{\partial x^\alpha}\left\{ {\alpha \atop 1\ 1} \right\} + \left\{ {\alpha \atop \beta\ 1} \right\}\left\{ {\beta \atop \alpha\ 1} \right\} - \left\{ {\alpha \atop \beta\ \alpha} \right\}\left\{ {\beta \atop 1\ 1} \right\}$$

$$= -\text{cosec}^2 x^1 + \left\{ {2 \atop 2\ 1} \right\}\left\{ {2 \atop 2\ 1} \right\} = -\text{cosec}^2 x^1 + \cot^2 x^1 = -1.$$

$$R_{22} = \frac{\partial^2}{(\partial x^2)^2}(\log \sqrt{g}) - \frac{\partial}{\partial x^\alpha}\left\{ {\alpha \atop 2\ 2} \right\} + \left\{ {\alpha \atop \beta\ 2} \right\}\left\{ {\beta \atop \alpha\ 2} \right\} - \left\{ {\alpha \atop \beta\ \alpha} \right\}\left\{ {\beta \atop 2\ 2} \right\}$$

$$= \cos 2x^1 + \left\{ {1 \atop 2\ 2} \right\}\left\{ {1 \atop 1\ 2} \right\} + \left\{ {1 \atop 2\ 2} \right\}\left\{ {2 \atop 1\ 2} \right\} + \left\{ {2 \atop 1\ 2} \right\}\left\{ {1 \atop 2\ 2} \right\}$$

$$- \left\{ {2 \atop 1\ 2} \right\}\left\{ {1 \atop 2\ 2} \right\} = -\sin^2 x^1.$$

Therefore, the scalar curvature R is given by Eq. (4.12) as

$$R = g^{11} R_{11} + g^{22} R_{22} = \frac{1}{a^2} \cdot (-1) + \frac{1}{a^2 \sin^2 x^1} \cdot (-\sin^2 x^1) = -\frac{2}{a^2}.$$

Hence, the required scalar curvature $= -\frac{2}{a^2}$. Using Eq. (4.6), the Riemannian curvature tensors R^{α}_{ijk} can be written as

$$R^1_{212} = -\frac{\partial}{\partial x^2}\left\{\begin{matrix} 1 \\ 2\ \ 1 \end{matrix}\right\} + \frac{\partial}{\partial x^1}\left\{\begin{matrix} 1 \\ 2\ \ 2 \end{matrix}\right\} + \left\{\begin{matrix} 1 \\ \beta\ \ 1 \end{matrix}\right\}\left\{\begin{matrix} \beta \\ 2\ \ 2 \end{matrix}\right\} - \left\{\begin{matrix} 1 \\ \beta\ \ 2 \end{matrix}\right\}\left\{\begin{matrix} \beta \\ 2\ \ 1 \end{matrix}\right\}.$$

$$= \frac{\partial}{\partial x^1}\left\{\begin{matrix} 1 \\ 2\ \ 2 \end{matrix}\right\} - \left\{\begin{matrix} 1 \\ 2\ \ 2 \end{matrix}\right\}\left\{\begin{matrix} 2 \\ 2\ \ 1 \end{matrix}\right\} = \sin^2 x^1.$$

$$R^2_{212} = -\frac{\partial}{\partial x^2}\left\{\begin{matrix} 2 \\ 2\ \ 1 \end{matrix}\right\} + \frac{\partial}{\partial x^1}\left\{\begin{matrix} 2 \\ 2\ \ 2 \end{matrix}\right\} + \left\{\begin{matrix} 2 \\ \beta\ \ 1 \end{matrix}\right\}\left\{\begin{matrix} \beta \\ 2\ \ 2 \end{matrix}\right\} - \left\{\begin{matrix} 2 \\ \beta\ \ 2 \end{matrix}\right\}\left\{\begin{matrix} \beta \\ 2\ \ 1 \end{matrix}\right\} = 0.$$

The only non-vanishing covariant curvature tensor is given by

$$R_{1212} = g_{1\alpha}R^{\alpha}_{212} = g_{11}R^1_{212} + g_{12}R^2_{212} = g_{11}R^1_{212} = a^2\sin^2 x^1.$$

Thus, the Riemannian curvature κ given by

$$\kappa = \frac{R_{1212}}{g} = \frac{R_{1212}}{g_{11}g_{22}} = \frac{a^2\sin^2 x^1}{a^4\sin^2 x^1} = \frac{1}{a^2}.$$

Thus, if κ and R are, respectively, the curvature and scalar curvature of a sphere of constant radius a, then $2\kappa + R = 0$.

4.1.2 Covariant Curvature Tensor

The completely *covariant curvature tensor* R_{hijk} of rank 4 is defined as

$$R_{hijk} = g_{h\alpha}R^{\alpha}_{ijk}. \tag{4.13}$$

The associated tensor R_{hijk} is referred to as the *covariant Riemann–Christoffel tensor* or the *Riemann–Christoffel tensor of the first kind.* Now, we have

$$g_{h\alpha}\frac{\partial}{\partial x^k}\left\{\begin{matrix} \alpha \\ i\ \ j \end{matrix}\right\} = \frac{\partial}{\partial x^k}\left[g_{h\alpha}\frac{\partial}{\partial x^k}\left\{\begin{matrix} \alpha \\ i\ \ j \end{matrix}\right\}\right] - \left\{\begin{matrix} \alpha \\ i\ \ j \end{matrix}\right\}\frac{\partial g_{h\alpha}}{\partial x^k}$$

$$= \frac{\partial}{\partial x^k}[ij, h] - \left\{\begin{matrix} \alpha \\ i\ \ j \end{matrix}\right\}\frac{\partial g_{h\alpha}}{\partial x^k}$$

and

$$g_{h\alpha}\frac{\partial}{\partial x^j}\left\{\begin{matrix} \alpha \\ i\ \ k \end{matrix}\right\} = \frac{\partial}{\partial x^j}[ik, h] - \left\{\begin{matrix} \alpha \\ i\ \ k \end{matrix}\right\}\frac{\partial g_{h\alpha}}{\partial x^j}. \tag{4.14}$$

Therefore, using Eqs. (4.6) and (4.14), the expression for R_{hijk} as Eq. (4.13) becomes

$$R_{hijk} = g_{h\alpha}\left[-\frac{\partial}{\partial x^k}\left\{\begin{matrix}\alpha\\ i\ \ j\end{matrix}\right\}+\frac{\partial}{\partial x^j}\left\{\begin{matrix}\alpha\\ i\ \ k\end{matrix}\right\}+\left\{\begin{matrix}\alpha\\ \beta\ \ j\end{matrix}\right\}\left\{\begin{matrix}\beta\\ i\ \ k\end{matrix}\right\}-\left\{\begin{matrix}\alpha\\ \beta\ \ k\end{matrix}\right\}\left\{\begin{matrix}\beta\\ i\ \ j\end{matrix}\right\}\right]$$

$$= -\frac{\partial}{\partial x^k}[ij,h]+\left\{\begin{matrix}\alpha\\ i\ \ j\end{matrix}\right\}\frac{\partial g_{h\alpha}}{\partial x^k}+\frac{\partial}{\partial x^j}[ik,h]-\left\{\begin{matrix}\alpha\\ i\ \ k\end{matrix}\right\}\frac{\partial g_{h\alpha}}{\partial x^j}$$

$$+g_{h\alpha}\left\{\begin{matrix}\alpha\\ \beta\ \ j\end{matrix}\right\}\left\{\begin{matrix}\beta\\ i\ \ k\end{matrix}\right\}-g_{h\alpha}\left\{\begin{matrix}\alpha\\ \beta\ \ k\end{matrix}\right\}\left\{\begin{matrix}\beta\\ i\ \ j\end{matrix}\right\}$$

$$= -\frac{1}{2}\frac{\partial}{\partial x^k}\left(\frac{\partial g_{jh}}{\partial x^i}+\frac{\partial g_{ih}}{\partial x^j}-\frac{\partial g_{ij}}{\partial x^h}\right)+\left\{\begin{matrix}\alpha\\ i\ \ j\end{matrix}\right\}([\alpha k,h]+[hk,\alpha])$$

$$+\frac{1}{2}\frac{\partial}{\partial x^j}\left(\frac{\partial g_{ih}}{\partial x^k}+\frac{\partial g_{hk}}{\partial x^j}-\frac{\partial g_{ik}}{\partial x^h}\right)-\left\{\begin{matrix}\alpha\\ i\ \ k\end{matrix}\right\}([\alpha j,h]+[hj,\alpha])$$

$$+\left\{\begin{matrix}\beta\\ i\ \ k\end{matrix}\right\}[\beta j,h]-\left\{\begin{matrix}\beta\\ i\ \ j\end{matrix}\right\}[\beta k,h].$$

$$= \frac{1}{2}\left(\frac{\partial^2 g_{ij}}{\partial x^h\partial x^k}+\frac{\partial^2 g_{hk}}{\partial x^i\partial x^j}-\frac{\partial^2 g_{hj}}{\partial x^i\partial x^k}-\frac{\partial^2 g_{ik}}{\partial x^h\partial x^j}\right)$$

$$+\left\{\begin{matrix}\alpha\\ i\ \ k\end{matrix}\right\}[\alpha j,h]-\left\{\begin{matrix}\alpha\\ i\ \ j\end{matrix}\right\}[\alpha k,h].$$

$$= \frac{1}{2}\left(\frac{\partial^2 g_{ij}}{\partial x^h\partial x^k}+\frac{\partial^2 g_{hk}}{\partial x^i\partial x^j}-\frac{\partial^2 g_{hj}}{\partial x^i\partial x^k}-\frac{\partial^2 g_{ik}}{\partial x^h\partial x^j}\right)$$

$$+\left\{\begin{matrix}\alpha\\ i\ \ j\end{matrix}\right\}[hk,\alpha]-\left\{\begin{matrix}\alpha\\ i\ \ k\end{matrix}\right\}[hj,\alpha] \tag{4.15}$$

$$= \frac{1}{2}\left(\frac{\partial^2 g_{ij}}{\partial x^h\partial x^k}+\frac{\partial^2 g_{hk}}{\partial x^i\partial x^j}-\frac{\partial^2 g_{hj}}{\partial x^i\partial x^k}-\frac{\partial^2 g_{ik}}{\partial x^h\partial x^j}\right)$$

$$+\left\{\begin{matrix}\alpha\\ i\ \ j\end{matrix}\right\}\left\{\begin{matrix}\beta\\ h\ \ k\end{matrix}\right\}g_{\alpha\beta}-\left\{\begin{matrix}\beta\\ h\ \ j\end{matrix}\right\}\left\{\begin{matrix}\alpha\\ i\ \ k\end{matrix}\right\}g_{\alpha\beta}. \tag{4.16}$$

Therefore, R_{hijk} can also be written in the form

$$R_{hijk} = -\frac{\partial}{\partial x^k}[ij,h]+\frac{\partial}{\partial x^j}[ik,h]+\left\{\begin{matrix}\alpha\\ i\ \ j\end{matrix}\right\}[hk,\alpha]-[hj,\alpha]\left\{\begin{matrix}\alpha\\ i\ \ k\end{matrix}\right\}$$

$$= \begin{vmatrix}\dfrac{\partial}{\partial x^j} & \dfrac{\partial}{\partial x^k}\\ [ij,h] & [ik,h]\end{vmatrix}+\begin{vmatrix}\left\{\begin{matrix}\alpha\\ i\ \ j\end{matrix}\right\} & \left\{\begin{matrix}\alpha\\ i\ \ k\end{matrix}\right\}\\ [hj,\alpha] & [hk,\alpha]\end{vmatrix}. \tag{4.17}$$

Using the property $g_{ij}g^{kj} = \delta_i^k$ and Eq. (4.13), the Ricci tensor of first kind can be written as

$$R_{ij} = R^{\alpha}_{ij\alpha} = g^{kh}R_{ihkj}. \tag{4.18}$$

Property 4.1.5 The covariant curvature tensor R_{hijk} is skew-symmetric with respect to h, i and j, k; i.e.

$$R_{hijk} = -R_{ihjk} \text{ and } R_{hijk} = -R_{hikj}$$

and symmetric in two pairs of indices $R_{hijk} = R_{jkhi}$. We say that two components of a curvature tensor are not distinct if they are equal or differ only by their signs.

Proof: We shall prove these properties one by one. Here, we use expression (4.15) for the covariant curvature tensor R_{hijk}. Interchanging i and h, we get

$$\begin{aligned} R_{ihjk} &= \frac{1}{2}\left(\frac{\partial^2 g_{hj}}{\partial x^i \partial x^k} + \frac{\partial^2 g_{ik}}{\partial x^h \partial x^j} - \frac{\partial^2 g_{hk}}{\partial x^i \partial x^j} - \frac{\partial^2 g_{ij}}{\partial x^h \partial x^k}\right) \\ &\quad + \left\{ {\alpha \atop h \ j} \right\}[ik, \alpha] - \left\{ {\alpha \atop h \ k} \right\}[ij, \alpha] \\ &= -\frac{1}{2}\left(\frac{\partial^2 g_{ij}}{\partial x^h \partial x^k} + \frac{\partial^2 g_{hk}}{\partial x^i \partial x^j} - \frac{\partial^2 g_{hj}}{\partial x^i \partial x^k} - \frac{\partial^2 g_{ik}}{\partial x^h \partial x^j}\right) \\ &\quad + \left\{ {\beta \atop h \ j} \right\}\left\{ {\alpha \atop i \ k} \right\} g_{\alpha\beta} - \left\{ {\alpha \atop i \ j} \right\}\left\{ {\beta \atop h \ k} \right\} g_{\alpha\beta}, \end{aligned}$$

where we have interchanged the dummy indices α and β. Therefore, by using Eq. (4.15), we have $R_{ihjk} = -R_{hijk}$. Similarly, interchange j and k in Eq. (4.15) and proceed as above, we get, $R_{hijk} = -R_{hikj}$. Interchanging h and j in Eq. (4.15), we get

$$\begin{aligned} R_{jihk} &= \frac{1}{2}\left(\frac{\partial^2 g_{ih}}{\partial x^j \partial x^k} + \frac{\partial^2 g_{jk}}{\partial x^i \partial x^h} - \frac{\partial^2 g_{jh}}{\partial x^i \partial x^k} - \frac{\partial^2 g_{ik}}{\partial x^j \partial x^h}\right) \\ &\quad + \left\{ {\alpha \atop i \ h} \right\}\left\{ {\beta \atop j \ k} \right\} g_{\alpha\beta} - \left\{ {\beta \atop j \ h} \right\}\left\{ {\alpha \atop i \ k} \right\} g_{\alpha\beta}. \end{aligned}$$

Now, interchanging k and i, we get

$$\begin{aligned} R_{jkhi} &= \frac{1}{2}\left(\frac{\partial^2 g_{kh}}{\partial x^j \partial x^k} + \frac{\partial^2 g_{jk}}{\partial x^i \partial x^h} - \frac{\partial^2 g_{jh}}{\partial x^k \partial x^i} - \frac{\partial^2 g_{ki}}{\partial x^j \partial x^h}\right) \\ &\quad + \left\{ {\alpha \atop k \ h} \right\}\left\{ {\beta \atop j \ i} \right\} g_{\alpha\beta} - \left\{ {\beta \atop j \ h} \right\}\left\{ {\alpha \atop k \ i} \right\} g_{\alpha\beta}. \end{aligned}$$

Interchanging the dummy indices α and β in the third term, we get

$$R_{jkhi} = \frac{1}{2}\left(\frac{\partial^2 g_{kh}}{\partial x^j \partial x^k} + \frac{\partial^2 g_{jk}}{\partial x^i \partial x^h} - \frac{\partial^2 g_{jh}}{\partial x^k \partial x^i} - \frac{\partial^2 g_{ki}}{\partial x^j \partial x^h}\right)$$

$$+ \left\{ \begin{matrix} \alpha \\ i \ \ j \end{matrix} \right\} \left\{ \begin{matrix} \beta \\ h \ \ k \end{matrix} \right\} g_{\alpha\beta} - \left\{ \begin{matrix} \beta \\ h \ \ j \end{matrix} \right\} \left\{ \begin{matrix} \alpha \\ i \ \ k \end{matrix} \right\} g_{\alpha\beta} = R_{hijk}.$$

This shows that, the curvature tensor R_{hijk} is symmetric in two pairs of indices.

Property 4.1.6 The covariant curvature tensor R_{hijk} satisfies *cyclic property,* i.e.

$$R_{hijk} + R_{hjki} + R_{hkji} = 0.$$

Proof: The covariant curvature tensor R_{hijk} is given by Eq. (4.16,) from which we get,

$$R_{hjki} = \frac{1}{2}\left(\frac{\partial^2 g_{jk}}{\partial x^h \partial x^i} + \frac{\partial^2 g_{hi}}{\partial x^j \partial x^k} - \frac{\partial^2 g_{ji}}{\partial x^h \partial x^k} - \frac{\partial^2 g_{hk}}{\partial x^i \partial x^j}\right)$$

$$+ \left\{ \begin{matrix} \alpha \\ j \ \ k \end{matrix} \right\} \left\{ \begin{matrix} \beta \\ h \ \ i \end{matrix} \right\} g_{\alpha\beta} - \left\{ \begin{matrix} \alpha \\ j \ \ i \end{matrix} \right\} \left\{ \begin{matrix} \beta \\ h \ \ k \end{matrix} \right\} g_{\alpha\beta}.$$

$$R_{hkji} = \frac{1}{2}\left(\frac{\partial^2 g_{ki}}{\partial x^j \partial x^h} + \frac{\partial^2 g_{jh}}{\partial x^k \partial x^i} - \frac{\partial^2 g_{kj}}{\partial x^h \partial x^i} - \frac{\partial^2 g_{hi}}{\partial x^k \partial x^j}\right)$$

$$+ \left\{ \begin{matrix} \alpha \\ k \ \ i \end{matrix} \right\} \left\{ \begin{matrix} \beta \\ h \ \ j \end{matrix} \right\} g_{\alpha\beta} - \left\{ \begin{matrix} \alpha \\ k \ \ j \end{matrix} \right\} \left\{ \begin{matrix} \beta \\ h \ \ i \end{matrix} \right\} g_{\alpha\beta}.$$

Taking the sum of the equations, we obtain $R_{hijk} + R_{hjki} + R_{hkji} = 0$. This property of R_{hijk} is called *cyclic property.*

Property 4.1.7 The curvature tensor R_{hijk} satisfies the *differential property*

$$R^{\alpha}_{ijk,m} + R^{\alpha}_{ikm,j} + R^{\alpha}_{imj,k} = 0;\ \text{i.e.}\ R_{hijk,l} + R_{hikl,j} + R_{hilj,k} = 0.$$

Proof: Let us choose a system of geodesic co-ordinates with the pole at P_0, then at P_0 Christoffel symbols vanish and first covariant derivative reduce to corresponding ordinary partial derivatives, i.e. at P_0,

$$[ki, j] = 0;\ \ \left\{ \begin{matrix} j \\ i \ \ k \end{matrix} \right\} = 0.$$

Differentiating Eq. (4.6) for the expression of Riemann–Christoffel symbol R^{α}_{ijk}, covariantly with respect to x^m and then imposing condition of geodesic co-ordinates with the pole at P_0, we get

$$R^{\alpha}_{ijk,m} = \frac{\partial}{\partial x^m}\left(R^{\alpha}_{ijk}\right) = -\frac{\partial^2}{\partial x^m \partial x^k}\left\{ \begin{matrix} \alpha \\ i \ \ j \end{matrix} \right\} + \frac{\partial^2}{\partial x^m \partial x^j}\left\{ \begin{matrix} \alpha \\ i \ \ k \end{matrix} \right\}.$$

Cyclic interchange of j, k and m gives us two relations

$$R^{\alpha}_{ikm,j} = -\frac{\partial^2}{\partial x^j \partial x^m} \left\{ \begin{matrix} \alpha \\ i \ \ k \end{matrix} \right\} + \frac{\partial^2}{\partial x^j \partial x^k} \left\{ \begin{matrix} \alpha \\ i \ \ m \end{matrix} \right\}$$

and

$$R^{\alpha}_{imj,k} = -\frac{\partial^2}{\partial x^k \partial x^j} \left\{ \begin{matrix} \alpha \\ i \ \ m \end{matrix} \right\} + \frac{\partial^2}{\partial x^k \partial x^m} \left\{ \begin{matrix} \alpha \\ i \ \ j \end{matrix} \right\}.$$

Taking the sum of the equations, we obtain

$$R^{\alpha}_{ijk,m} + R^{\alpha}_{ikm,j} + R^{\alpha}_{imj,k} = 0. \tag{4.19}$$

Multiplying innerly Eq. (4.19) by $g_{h\alpha}$ and summing over α, we get

$$R_{hijk,l} + R_{hikl,j} + R_{hilj,k} = 0. \tag{4.20}$$

Now, each term of Eqs. (4.19) and (4.20) is a tensor. Therefore, Eqs. (4.19) and (4.20) are tensorial equations, true at the pole P_0 of the geodesic system of co-ordinates. Thus, it holds for every co-ordinate system at pole. Further any point can be chosen as a pole in the geodesic co-ordinate system. This means that these equations hold in every co-ordinate system and at every points of space (since P_0 is an arbitrary point of V_N). They are, therefore, called *Bianchi identities.* Equations (4.19) and (4.20) are, respectively, called *Bianchi first identity* and *Bianchi second identity.*

Property 4.1.8 The number of distinct non-vanishing components of the covariant curvature tensor does not exceed $\frac{1}{12}N^2(N^2-1)$ and that is independent from the rest.

Proof: Here, we have to show that the covariant curvature tensor R_{hijk} has $\frac{1}{12}N^2$ (N^2-1) distinct non-vanishing components. Now, the covariant curvature tensor R_{hijk} satisfies four properties namely

(i) First skew-symmetry: $R_{hijk} = -R_{ihjk}$,
(ii) Second skew-symmetry: $R_{hijk} = -R_{hikj}$,
(iii) Block symmetry: $R_{hijk} = R_{jkih}$,
(iv) Cyclic property: $R_{hijk} + R_{hjki} + R_{hkij} = 0$.

Due to these properties all the N^4 components of the tensor R_{hijk} of fourth order are not independent. The following four cases may arise:

Case 1: *When R_{hijk} is of the form R_{hhhh}.* Here all the four indices are same. Now, by skew-symmetric property, we observe

$$R_{hhhh} = -R_{hhhh} \Rightarrow 2R_{hhhh} = 0 \Rightarrow R_{hhhh} = 0.$$

Hence R_{hhhh} has no component.

Case 2: *When R_{hijk} is of the form R_{hjhj}.* There are two distinct suffixes h and j, where $h, j = 1, 2, \ldots, N$. Now, the index j can be chosen in N different ways. Then giving to j particular value, the index h can be given remaining $(N-1)$ values. Hence, the number of ways in which h and j can be chosen are $N(N-1)$. But by skew-symmetric property we have

$$R_{hjhj} = -Rjhhj \quad \text{or} \quad R_{hjhj} = R_{jhjh},$$

i.e. by interchanging indices h and j, we have the same components. Thus, due to this property, the number $N(N-1)$, is reduced to $\frac{1}{2}N(N-1)$. Clearly by symmetric property, there is no reduction. Also, by cyclic property there is no reduction since

$$R_{hjhj} + R_{hhjj} + R_{hjjh} = 0$$
$$\Rightarrow -R_{hjjh} + 0 + R_{hjjh} = 0 \Rightarrow 0 = 0,$$

i.e. cyclic property is itself satisfied. Hence, in this case total number of distinct non-vanishing components of R_{hijk} are $\frac{1}{2}N(N-1)$.

Case 3: *When R_{hijk} is of the form R_{hihk}.* As in case 2, the indices h, i, k can be chosen in $N(N-1)(N-2)$ ways. Clearly due to skew-symmetric property there is no reduction in number of components. Now, due to the symmetric property $R_{hihk} = R_{hkhi}$, the number $N(N-1)(N-2)$ reduced to $\frac{N}{2}(N-1)(N-2)$. Lastly by cyclic property, we have

$$R_{hihk} + R_{hhki} + R_{hkih} = 0$$
$$\Rightarrow R_{hkhi} + 0 + R_{hkih} = 0$$
$$\Rightarrow -R_{hkih} + R_{hkih} = 0 \Rightarrow 0 = 0,$$

i.e. cyclic property is itself satisfied. Hence, there is no reduction due to cyclic property also. Therefore, in this case, the total number of distinct non-vanishing components of R_{hijk} are $\frac{1}{2}N(N-1)(N-2)$.

Case 4: *When R_{hijk} has all the four distinct suffixes.* The indices h, i, j, k can be chosen in $N(N-1)(N-2)(N-3)$ ways. By skew-symmetric property, we have

$$R_{hijk} = -R_{ihjk} = R_{ihkj} \text{ and } R_{hijk} = -R_{hikj} = R_{ihkj}.$$

Therefore, the number $N(N-1)(N-2)(N-3)$ is reduced to $\frac{1}{2^2}N(N-1)(N-2)(N-3)$. Due to the symmetric property $R_{hijk} = R_{jkhi}$, the number $\frac{1}{2^2}N(N-1)(N-2)(N-3)$ is reduced to $\frac{1}{2^3}N(N-1)(N-2)(N-3)$. Lastly, by cyclic property, we have

$$R_{hijk} + R_{hjki} + R_{hkij} = 0$$
$$\Rightarrow R_{hjki} = -(R_{hijk} + R_{hkij}).$$

Thus, there is a relation between three components and hence, two components are independent out of three components. Hence, above number is reduced to

$$\frac{2}{3}\cdot\frac{1}{2^3}N(N-1)(N-2)(N-3)=\frac{1}{12}N(N-1)(N-2)(N-3),$$

which are the total number of distinct non-vanishing components of R_{hijk}.

Combining the above four cases, the total number of distinct non-vanishing components of R_{hijk} are

$$\begin{aligned}&=0+\frac{1}{2}N(N-1)+\frac{1}{12}N(N-1)(N-2)+\frac{1}{12}N(N-1)(N-2)(N-3)\\&=\frac{1}{12}N(N-1)\left[6+(N-2)(N+3)\right]=\frac{1}{12}N^2(N^2-1).\end{aligned}$$

EXAMPLE 4.1.2 *Calculate the components R_{hijk} of the Riemannian tensor for the metric*

$$ds^2=(dx^1)^2-(x^2)^{-2}(dx^2)^2.$$

Solution: For the given metric, we have $g_{11}=1$; $g_{12}=0=g_{21}$; $g_{22}=-(x^2)^{-2}$. Since $g=-(x^2)^{-2}$, so the reciprocal tensors g^{ij} are given by

$$g^{11}=1;\;\; g^{12}=0=g^{21};\;\; g^{22}=(x^2)^2.$$

The only non-vanishing Christoffel symbols of second kind is $\left\{\begin{matrix}2\\2\;2\end{matrix}\right\}=-(x^2)^{-1}$. Using Eq. (4.6), the Riemannian curvature tensors R^{α}_{ijk} can be written as

$$R^1_{212}=-\frac{\partial}{\partial x^2}\left\{\begin{matrix}1\\2\;1\end{matrix}\right\}+\frac{\partial}{\partial x^1}\left\{\begin{matrix}1\\2\;2\end{matrix}\right\}+\left\{\begin{matrix}1\\\beta\;1\end{matrix}\right\}\left\{\begin{matrix}\beta\\2\;2\end{matrix}\right\}-\left\{\begin{matrix}1\\\beta\;2\end{matrix}\right\}\left\{\begin{matrix}\beta\\2\;1\end{matrix}\right\}=0.$$

$$R^2_{212}=-\frac{\partial}{\partial x^2}\left\{\begin{matrix}2\\2\;1\end{matrix}\right\}+\frac{\partial}{\partial x^1}\left\{\begin{matrix}2\\2\;2\end{matrix}\right\}+\left\{\begin{matrix}2\\\beta\;1\end{matrix}\right\}\left\{\begin{matrix}\beta\\2\;2\end{matrix}\right\}-\left\{\begin{matrix}2\\\beta\;2\end{matrix}\right\}\left\{\begin{matrix}\beta\\2\;1\end{matrix}\right\}=0.$$

Using Eq. (4.13), the only non-vanishing covariant curvature tensor is given by

$$R_{1212}=g_{1\alpha}R^{\alpha}_{212}=g_{11}R^1_{212}+g_{12}R^2_{212}=0.$$

EXAMPLE 4.1.3 *For the metric of spherical co-ordinates*

$$ds^2=\left(dx^1\right)^2+\left(x^1\right)^2\left(dx^2\right)^2+\left(x^1\right)^2\sin^2 x^2\left(dx^3\right)^2.$$

Calculate the non-zero components of R_{hijk}, if any.

Solution: For $N=3$, there are six potentially non-zero components

(i) $R_{1212}, \quad R_{1313}, \quad R_{2323}$
(ii) $R_{1213}, \quad R_{1232}(= R_{2123}), \quad R_{1323}(= R_{3132})$.
The non-vanishing Christoffel symbols of second kind are

$$\left\{ \begin{matrix} 1 \\ 2\ 2 \end{matrix} \right\} = -x^1; \quad \left\{ \begin{matrix} 1 \\ 3\ 3 \end{matrix} \right\} = -x^1 \sin^2 x^2; \quad \left\{ \begin{matrix} 2 \\ 1\ 2 \end{matrix} \right\} = \left\{ \begin{matrix} 2 \\ 2\ 1 \end{matrix} \right\} = \frac{1}{x^1}$$

$$\left\{ \begin{matrix} 2 \\ 3\ 3 \end{matrix} \right\} = -\sin x^2 \cos x^2; \quad \left\{ \begin{matrix} 3 \\ 1\ 3 \end{matrix} \right\} = \left\{ \begin{matrix} 3 \\ 3\ 1 \end{matrix} \right\} = \frac{1}{x^1}; \left\{ \begin{matrix} 3 \\ 2\ 3 \end{matrix} \right\} = \left\{ \begin{matrix} 3 \\ 3\ 2 \end{matrix} \right\} = \cot x^2.$$

Using Eq. (4.6), the Riemannian curvature tensors R^{α}_{ijk} can be written as

$$R^1_{212} = -\frac{\partial}{\partial x^2} \left\{ \begin{matrix} 1 \\ 2\ 1 \end{matrix} \right\} + \frac{\partial}{\partial x^1} \left\{ \begin{matrix} 1 \\ 2\ 2 \end{matrix} \right\} + \left\{ \begin{matrix} 1 \\ \beta\ 1 \end{matrix} \right\} \left\{ \begin{matrix} \beta \\ 2\ 2 \end{matrix} \right\} - \left\{ \begin{matrix} 1 \\ \beta\ 2 \end{matrix} \right\} \left\{ \begin{matrix} \beta \\ 2\ 1 \end{matrix} \right\}$$

$$= \frac{\partial}{\partial x^1} \left\{ \begin{matrix} 1 \\ 2\ 2 \end{matrix} \right\} - \left\{ \begin{matrix} 2 \\ 2\ 1 \end{matrix} \right\} \left\{ \begin{matrix} 1 \\ 2\ 2 \end{matrix} \right\} = -1 - (-x^1)\frac{1}{x^1} = 0.$$

$$R^1_{313} = -\frac{\partial}{\partial x^1} \left\{ \begin{matrix} 1 \\ 3\ 1 \end{matrix} \right\} + \frac{\partial}{\partial x^1} \left\{ \begin{matrix} 1 \\ 3\ 3 \end{matrix} \right\} + \left\{ \begin{matrix} 1 \\ \beta\ 1 \end{matrix} \right\} \left\{ \begin{matrix} \beta \\ 3\ 3 \end{matrix} \right\} - \left\{ \begin{matrix} 1 \\ \beta\ 3 \end{matrix} \right\} \left\{ \begin{matrix} \beta \\ 3\ 1 \end{matrix} \right\}$$

$$= \frac{\partial}{\partial x^1} \left\{ \begin{matrix} 1 \\ 3\ 3 \end{matrix} \right\} - \left\{ \begin{matrix} 3 \\ 3\ 1 \end{matrix} \right\} \left\{ \begin{matrix} 1 \\ 3\ 3 \end{matrix} \right\} = -\sin^2 x^2 - \frac{1}{x^1}(-x^1 \sin^2 x^2) = 0.$$

$$R^2_{323} = -\frac{\partial}{\partial x^3} \left\{ \begin{matrix} 2 \\ 3\ 2 \end{matrix} \right\} + \frac{\partial}{\partial x^2} \left\{ \begin{matrix} 2 \\ 3\ 3 \end{matrix} \right\} + \left\{ \begin{matrix} 2 \\ \beta\ 2 \end{matrix} \right\} \left\{ \begin{matrix} \beta \\ 3\ 3 \end{matrix} \right\} - \left\{ \begin{matrix} 2 \\ \beta\ 3 \end{matrix} \right\} \left\{ \begin{matrix} \beta \\ 3\ 2 \end{matrix} \right\}$$

$$= \frac{\partial}{\partial x^2} \left\{ \begin{matrix} 2 \\ 3\ 3 \end{matrix} \right\} + \left\{ \begin{matrix} 1 \\ 3\ 3 \end{matrix} \right\} \left\{ \begin{matrix} 2 \\ 1\ 2 \end{matrix} \right\} - \left\{ \begin{matrix} 3 \\ 3\ 2 \end{matrix} \right\} \left\{ \begin{matrix} 2 \\ 3\ 3 \end{matrix} \right\}$$

$$= -\cos 2x^2 - \sin^2 x^2 + \cos^2 x^2 = 0.$$

$$R^1_{323} = -\frac{\partial}{\partial x^3} \left\{ \begin{matrix} 1 \\ 3\ 2 \end{matrix} \right\} + \frac{\partial}{\partial x^2} \left\{ \begin{matrix} 1 \\ 3\ 3 \end{matrix} \right\} + \left\{ \begin{matrix} 1 \\ \beta\ 2 \end{matrix} \right\} \left\{ \begin{matrix} \beta \\ 3\ 3 \end{matrix} \right\} - \left\{ \begin{matrix} 1 \\ \beta\ 3 \end{matrix} \right\} \left\{ \begin{matrix} \beta \\ 2\ 2 \end{matrix} \right\}$$

$$= \frac{\partial}{\partial x^2} \left\{ \begin{matrix} 1 \\ 3\ 3 \end{matrix} \right\} + \left\{ \begin{matrix} 2 \\ 3\ 3 \end{matrix} \right\} \left\{ \begin{matrix} 1 \\ 2\ 2 \end{matrix} \right\} - \left\{ \begin{matrix} 3 \\ 3\ 2 \end{matrix} \right\} \left\{ \begin{matrix} 1 \\ 3\ 3 \end{matrix} \right\}$$

$$= -2x^1 \sin x^2 \cos x^2 + x^1 \sin x^2 \cos x^2 + \cot x^2 (x^1 \sin^2 x^2) = 0.$$

$$R^1_{213} = -\frac{\partial}{\partial x^3} \left\{ \begin{matrix} 1 \\ 2\ 1 \end{matrix} \right\} + \frac{\partial}{\partial x^1} \left\{ \begin{matrix} 1 \\ 2\ 3 \end{matrix} \right\} + \left\{ \begin{matrix} 1 \\ \beta\ 1 \end{matrix} \right\} \left\{ \begin{matrix} \beta \\ 2\ 3 \end{matrix} \right\} - \left\{ \begin{matrix} 1 \\ \beta\ 3 \end{matrix} \right\} \left\{ \begin{matrix} \beta \\ 2\ 1 \end{matrix} \right\} = 0.$$

$$R^1_{232} = -\frac{\partial}{\partial x^2} \left\{ \begin{matrix} 1 \\ 2\ 3 \end{matrix} \right\} + \frac{\partial}{\partial x^3} \left\{ \begin{matrix} 1 \\ 2\ 2 \end{matrix} \right\} + \left\{ \begin{matrix} 1 \\ \beta\ 3 \end{matrix} \right\} \left\{ \begin{matrix} \beta \\ 2\ 2 \end{matrix} \right\} - \left\{ \begin{matrix} 1 \\ \beta\ 2 \end{matrix} \right\} \left\{ \begin{matrix} \beta \\ 2\ 3 \end{matrix} \right\} = 0.$$

Since none of R^{α}_{ijk} is non-zero, therefore, $R_{hijk} = 0$, for all i, j, k, l.

EXAMPLE 4.1.4 *Prove that in a Riemannian space div* $\left(R^{\alpha}_{ijk}\right) = R_{ij,k} - R_{ik,j}$.

Solution: Here, we will use the Bianchi identity Eq. (4.19), $R^{\alpha}_{ijk,m} + R^{\alpha}_{ikm,j} + R^{\alpha}_{imj,k} = 0$. Contracting α and m in the above relation, we get

$$R^{\alpha}_{ijk,\alpha} + R^{\alpha}_{ik\alpha,j} + R^{\alpha}_{i\alpha j,k} = 0$$

or

$$\begin{aligned} R^{\alpha}_{ijk,\alpha} &= -R^{\alpha}_{i\alpha j,k} - R^{\alpha}_{ik\alpha,j} = R^{\alpha}_{ij\alpha,k} - R_{ik,j}; \quad \text{as } R^{\alpha}_{ik\alpha} = R_{ik} \\ &= R_{ij,k} - R_{ik,j}. \end{aligned}$$

In other words, div $\left(R^{\alpha}_{ijk}\right) = R_{ij,k} - R_{ik,j}$.

EXAMPLE 4.1.5 *If* $R_{ij,k} = 2B_kR_{ij} + B_iR_{kj} + B_jR_{ik}$, *prove that* $B_k = \frac{\partial}{\partial x^k}\left(\log\sqrt{R}\right)$, *for any covariant vector* B_i.

Solution: The given relation is $R_{ij,k} = 2B_kR_{ij} + B_iR_{kj} + B_jR_{ik}$. Using the definition of associated tensor $B^j = g^{ij}B_i$ and the scalar curvature $R = g^{ij}R_{ij}$, we get

$$\begin{aligned} g^{ij}R_{ij,k} &= 2B_kg^{ij}R_{ij} + g^{ij}B_iR_{kj} + g^{ij}B_jR_{ik} \\ &= 2B_kR + B^jR_{kj} + B^iR_{ik} \end{aligned}$$

or

$$R_{,k} = 2B_kR + B^iR_{ki} + B^iR_{ik} = 2B_kR + 2B^iR_{ik}; \quad R_{ij} = R_{ji}. \qquad \text{(i)}$$

Also, from the given condition, we have

$$R_{ij,k} - R_{ik,j} = R_{ij} - B_jR_{ik}$$

or

$$g^{ij}R_{ij,k} - g^{ij}R_{ik,j} = B_kg^{ij}R_{ij} - B_jg^{ij}R_{ik}$$

or

$$R_{,k} - \frac{1}{2}R^j_{k,j} = B_kR - B^iR_{ik}$$

or

$$R_{,k} - \frac{1}{2}R_{,k} = B_kR - B^iR_{ik} \Rightarrow R_{,k} = 2B_kR - 2B^iR_{ik}. \qquad \text{(ii)}$$

Therefore, from Eqs. (i) and (ii) it follows that:

$$B^iR_{ik} = 0 \Rightarrow R_{,k} = 2B_kR \quad \text{from (i)}$$

or

$$B_k = \frac{1}{2R}R_{,k} = \frac{1}{2R}\;\frac{\partial R}{\partial x^k} = \frac{\partial}{\partial x^k}\left(\log\sqrt{R}\right).$$

4.2 Riemannian Curvature

Expression for Riemannian curvature: We are given two unit vectors p^i and q^i defined at a point P_0 of V_N. Consider the vector $t^i = \alpha p^i + \beta q^i$, where α and β are parameters. This vector determines pencils of direction of p^i. Similarly, one and only one geodesic will pass through in the direction of q^i. These two geodesics through P_0 determine a two-dimensional geodesic surface through P_0 determined by the orientation of the unit vectors p^i and q^i. Call this surface by the name $\mathcal{S}$.

The Gaussian curvature for the surface $\mathcal{S}$ at P_0 is called *Riemannian curvature* of V_N at P_0 determined by the orientation of p^i and q^i. Introduce Riemannian co-ordinates y^i with the origin at P_0. The equation of the surface $\mathcal{S}$ interms of y^i is

$$y^i = \left(\alpha p^i + \beta q^i\right) s, \tag{4.21}$$

where s denotes the arc length measured from P_0 to any point P along the geodesic Eq. (4.21) through P_0 in the direction of t^i. By taking $\alpha s = u^1$ and $\beta s = u^2$, three parameters namely, α, β and s can be reduced to two parameters u^1 and u^2. Here, u^1 and u^2 are taken as co-ordinates of any current point on the surface $\mathcal{S}$ defined by

$$y^i = u^1 p^i + u^2 q^i, \tag{4.22}$$

where p^i and q^i being fixed. Let $ds^2 = a_{ij} du^i du^j$, be the metric for the surface $\mathcal{S}$. Let $\left\{ \begin{matrix} k \\ i \;\; j \end{matrix} \right\}_g$ and $\left\{ \begin{matrix} \gamma \\ \alpha \;\; \beta \end{matrix} \right\}_a$ be the Christoffel symbols of the second kind corresponding to the co-ordinates y^i and u^a, respectively, then

$$[\alpha\beta, \gamma]_a = g_{km} \frac{\partial y^i}{\partial u^\alpha} \frac{\partial y^j}{\partial u^\beta} \frac{\partial y^k}{\partial u^\gamma} \left\{ \begin{matrix} m \\ i \;\; j \end{matrix} \right\}_g. \tag{4.23}$$

Let $R_{\alpha\beta\gamma\delta}$ and R_{hijk} be the curvature tensors corresponding to the metrices $a_{\alpha\beta} du^\alpha du^\beta$ and $g_{ij} dy^i dy^j$, where $\alpha, \beta, \gamma, \delta$ take values 1 and 2; i and j take values from 1 to N. The number of independent components of $\overline{R}_{\alpha\beta\gamma\delta}$ is $\frac{2^2(2^2-1)}{12} = 1$. We have,

$$\begin{aligned}
\overline{R}'_{\alpha\beta\gamma\delta} &= \frac{\partial u^\alpha}{\partial \overline{u}^1} \frac{\partial u^\beta}{\partial \overline{u}^2} \frac{\partial u^\gamma}{\partial \overline{u}^1} \frac{\partial u^\delta}{\partial \overline{u}^2} \overline{R}_{\alpha\beta\gamma\delta} = \frac{\partial u^1}{\partial \overline{u}^1} \frac{\partial u^\beta}{\partial \overline{u}^2} \frac{\partial u^\gamma}{\partial \overline{u}^1} \frac{\partial u^\delta}{\partial \overline{u}^2} \overline{R}_{1\beta\gamma\delta} + \frac{\partial u^2}{\partial \overline{u}^1} \frac{\partial u^\beta}{\partial \overline{u}^2} \frac{\partial u^\gamma}{\partial \overline{u}^1} \frac{\partial u^\delta}{\partial \overline{u}^2} \overline{R}_{2\beta\gamma\delta} \\
&= \frac{\partial u^1}{\partial \overline{u}^1} \frac{\partial u^2}{\partial \overline{u}^2} \frac{\partial u^\gamma}{\partial \overline{u}^1} \frac{\partial u^\delta}{\partial \overline{u}^2} \overline{R}_{12\gamma\delta} + \frac{\partial u^2}{\partial \overline{u}^1} \frac{\partial u^1}{\partial \overline{u}^2} \frac{\partial u^\gamma}{\partial \overline{u}^1} \frac{\partial u^\delta}{\partial \overline{u}^2} \overline{R}_{21\gamma\delta} \\
&= \frac{\partial u^1}{\partial \overline{u}^1} \frac{\partial u^2}{\partial \overline{u}^2} \frac{\partial u^1}{\partial \overline{u}^1} \frac{\partial u^2}{\partial \overline{u}^2} \overline{R}_{1212} + \frac{\partial u^1}{\partial \overline{u}^1} \frac{\partial u^2}{\partial \overline{u}^2} \frac{\partial u^2}{\partial \overline{u}^1} \frac{\partial u^1}{\partial \overline{u}^2} \overline{R}_{1221} \\
&\quad + \frac{\partial u^2}{\partial \overline{u}^1} \frac{\partial u^1}{\partial \overline{u}^2} \frac{\partial u^1}{\partial \overline{u}^1} \frac{\partial u^2}{\partial \overline{u}^2} \overline{R}_{2121} + \frac{\partial u^2}{\partial \overline{u}^1} \frac{\partial u^1}{\partial \overline{u}^2} \frac{\partial u^2}{\partial \overline{u}^1} \frac{\partial u^1}{\partial \overline{u}^2} \overline{R}_{2121}
\end{aligned}$$

$$= \left[\left(\frac{\partial u^1}{\partial \overline{u}^1}\frac{\partial u^2}{\partial \overline{u}^2}\right)^2 - 2\frac{\partial u^1}{\partial \overline{u}^1}\frac{\partial u^2}{\partial \overline{u}^2}\frac{\partial u^2}{\partial \overline{u}^1}\frac{\partial u^1}{\partial \overline{u}^2} + \left(\frac{\partial u^2}{\partial \overline{u}^1}\frac{\partial u^1}{\partial \overline{u}^2}\right)^2\right]\overline{R}_{1212}$$

$$= J^2\overline{R}_{2121};\ \text{where } J = \begin{vmatrix} \frac{\partial u^1}{\partial \overline{u}^1} & \frac{\partial u^1}{\partial \overline{u}^2} \\ \frac{\partial u^2}{\partial \overline{u}^1} & \frac{\partial u^2}{\partial \overline{u}^2} \end{vmatrix} = \left|\frac{\partial u}{\partial \overline{u}}\right|. \tag{4.24}$$

Since a_{ij} is a covariant tensor, by Eq. (1.45), we get

$$\overline{a}_{\alpha\beta} = \frac{\partial u^\gamma}{\partial \overline{u}^\alpha}\frac{\partial u^\delta}{\partial \overline{u}^\beta}a_{\gamma\delta} \Rightarrow |\overline{a}_{\alpha\beta}| = \left|\frac{\partial u^\gamma}{\partial \overline{u}^\alpha}\right|\left|\frac{\partial u^\delta}{\partial \overline{u}^\beta}\right|a_{\gamma\delta} \Rightarrow \overline{a} = J^2 a. \tag{4.25}$$

Dividing Eq. (4.24) by Eq. (4.25), we get

$$\frac{R'_{1212}}{\overline{a}} = \frac{R_{1212}}{\overline{a}} = \kappa(\text{say}). \tag{4.26}$$

This κ, which is an invariant for transformation of co-ordinates, is defined as *Gaussian curvature* of the surface $\mathcal{S}$ at P_0. Hence, κ is the Riemannian curvature of $\mathcal{S}$ at P_0. Since Riemannian co-ordinate y^i with the origin at P_0 behave as geodesic co-ordinates with the pole at P_0. Thus, at P_0, $\left\{\begin{matrix} k \\ i\ \ j \end{matrix}\right\}_g$ and $\left\{\begin{matrix} \gamma \\ \alpha\ \ \beta \end{matrix}\right\}_a = 0$. Therefore, we have

$$\overline{R}_{\alpha\beta\gamma\delta} = \left(-\frac{\partial}{\partial u^\delta}[\beta\gamma,\alpha]_a + \frac{\partial}{\partial u^\gamma}[\beta\delta,\alpha]_a\right)\ \text{at } P_0$$

or

$$\overline{R}_{1212} = -\frac{\partial}{\partial u^2}[21,1]_a + \frac{\partial}{\partial u^1}[22,1]_a \text{ at } P_0. \tag{4.27}$$

or

$$\kappa = \frac{1}{a}\left(-\frac{\partial}{\partial u^2}[21,1]_a + \frac{\partial}{\partial u^1}[22,1]_a\right),\ \text{by Eq. (4.26)}. \tag{4.28}$$

This is an expression for Riemannian curvature at P_0.

Formula for Riemannian curvature: Here, we have to derive a formula for Riemannian curvature interms of covariant curvature tensor of V_N. We know that

$$[\alpha\beta,\gamma]_a = [ij,k]_g\frac{\partial y^i}{\partial u^\alpha}\frac{\partial y^j}{\partial u^\beta}\frac{\partial y^k}{\partial u^\gamma} + g_{ij}\frac{\partial^2 y^i}{\partial u^\alpha u^\beta}\frac{\partial y^j}{\partial u^\gamma}$$

$$= [ij,k]_g\frac{\partial y^i}{\partial u^\alpha}\frac{\partial y^j}{\partial u^\beta}\frac{\partial y^k}{\partial u^\gamma};\ \text{by Eq. (4.22)}. \tag{4.29}$$

In particular, we have,

$$[21,1]_a = [ij,k]_g \frac{\partial y^i}{\partial u^2} \frac{\partial y^j}{\partial u^1} \frac{\partial y^k}{\partial u^1} = [ij,k]_g q^i p^j p^k$$

or

$$\begin{aligned} \frac{\partial}{\partial u^2}[21,1]_a &= \frac{\partial}{\partial u^2}[ij,k]_g q^i p^j p^k = q^i p^j p^k q^h \frac{\partial}{\partial y^h}[ij,k]_g \\ &= q^i p^j p^h q^k \frac{\partial}{\partial y^k}[ij,h]_g; \text{ interchanging } h \text{ and } k. \end{aligned} \tag{4.30}$$

By virtue of Eq. (4.29), we get

$$[22,1]_a = [ij,k]_g \frac{\partial y^i}{\partial u^2} \frac{\partial y^j}{\partial u^2} \frac{\partial y^k}{\partial u^1} = [ij,k]_g q^i q^j q^k$$

or

$$\begin{aligned} \frac{\partial}{\partial u^2}[22,1]_a &= q^i q^j p^k p^h \frac{\partial}{\partial y^h}[ij,k]_g \\ &= q^i q^k p^j p^h \frac{\partial}{\partial y^h}[ik,j]_g; \text{ interchanging } j \text{ and } k \\ &= q^i q^k p^h p^j \frac{\partial}{\partial y^j}[ik,h]_g; \text{ interchanging } h \text{ and } j. \end{aligned} \tag{4.31}$$

Thus using Eqs. (4.30) and (4.31), from Eq. (4.27), the expression for $\overline{R}_{1212}$ at P_0 becomes

$$\overline{R}_{1212} = p^h q^i p^j q^k \left(-\frac{\partial}{\partial y^k}[ij,h]_g + \frac{\partial}{\partial y^j}[ik,h]_g \right) = p^h q^i p^j q^k R_{hijk}, \text{ at } P_0. \tag{4.32}$$

Using the relation $a_{\alpha\beta} = g_{ij} \frac{\partial y^i}{\partial u^\alpha} \frac{\partial y^j}{\partial u^\beta}$, we get

$$\begin{aligned} a_{11} &= g_{ij} \frac{\partial y^i}{\partial u^1} \frac{\partial y^j}{\partial u^1} = g_{ij} p^i p^j = g_{hj} p^h p^j. \\ a_{22} &= g_{ij} \frac{\partial y^i}{\partial u^2} \frac{\partial y^j}{\partial u^2} = g_{ij} q^i q^j = g_{hk} q^i q^k. \\ a_{12} &= g_{ij} \frac{\partial y^i}{\partial u^1} \frac{\partial y^j}{\partial u^2} = g_{ij} p^i p^j g_{hk} p^h q^k = a_{21}. \\ a &= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = p^h q^i p^j q^k \left[g_{hj} g_{ik} - g_{ij} g_{kh} \right]. \end{aligned} \tag{4.33}$$

Dividing Eq. (4.32) by Eq. (4.33), we get

$$\kappa = \frac{\overline{R}_{1212}}{a} = \frac{p^h q^i p^j q^k R_{hijk}}{p^h q^i p^j q^k \left[g_{hj} g_{ik} - g_{ij} g_{kh}\right]} = \frac{H_{hijk} R_{hijk}}{G_{hijk} H_{hijk}}, \tag{4.34}$$

where the N^4 functions H_{hijk} and G_{hijk} are given by

$$H_{hijk} = \begin{vmatrix} p^h & p^i \\ q^h & q^i \end{vmatrix} \ \begin{vmatrix} p^j & p^k \\ q^j & q^k \end{vmatrix} \tag{4.35}$$

and

$$G_{hijk} = g_{hj}\ g_{ik} - g_{ij}\ g_{kh}. \tag{4.36}$$

Equation (4.34) is the required expression for Riemannian curvature κ of V_N at P_0 determined by the orientation of the unit vectors p^i and q^j at P_0. If the Riemannian curvature at P_0 does not change with the orientation of a 2-flat through P_0, then the co-ordinates of P_0 is called *isotropic.* Thus, if $N = 2$, Eq. (4.34) reduces to

$$\kappa = \frac{R_{1212}}{g_{11} g_{22} - g_{12}^2} = \frac{R_{1212}}{g}$$

shows that all points of a two-dimensional Riemannian space are isotropic. Thus, at a given point in Riemannian 2-space, the curvature is determined by the g_{ij} and their derivatives, and is independent of the directions of P and Q.

Theorem 4.2.1 *If at each point, the Riemannian curvature of a space is independent of the orientation chosen, then it is constant throughout the space.*

Proof: Let κ be the Riemannian curvature of V_N at P_0 determined by the orientation of the unit vectors p^i and q^j, then it is given by

$$\kappa = \frac{\overline{R}_{1212}}{a} = \frac{p^h q^i p^j q^k R_{hijk}}{p^h q^i p^j q^k \left[g_{hj} g_{ik} - g_{ij} g_{hk}\right]}.$$

If κ is independent of orientation determined by p^i and q^i, then

$$\kappa = \frac{R_{hijk}}{(g_{hj} g_{ik} - g_{ij} g_{hk})} \Rightarrow R_{hijk} = (-g_{ij} g_{hk} + g_{hj} g_{ik})\kappa. \tag{4.37}$$

Here κ is constant throughout the space V_N. If $N = 2$, then at any point of V_2 there is only one orientation, which is same at every point and the theorem is obvious.

So, we consider the case of V_N, where $N \geq 3$. Since g_{ij} are constants with respect to the covariant differentiation, so covariant differentiation of Eq. (4.37) yields

$$R_{hijk,l} = (-g_{ij} g_{hk} + g_{hj} g_{ik})\kappa_{,l}. \tag{4.38}$$

where $\kappa_{,l}$ denotes the partial derivative of κ with respect to xs (as κ is scalar). Taking the sum of Eq. (4.38) and two similar equations obtained by cyclic permutation of the suffixes j, k and l

$$(-g_{ij}g_{hk} + g_{hj}g_{ik})\kappa_{,l} + (-g_{ik}g_{hl} + g_{il}g_{hk})\kappa_{,j} + (-g_{il}g_{hj} + g_{ij}g_{hl})\kappa_{,k}$$
$$= R_{hijk,l} + R_{hikl,j} + R_{hilj,k} = 0; \text{ by Eq. (4.20).} \tag{4.39}$$

Since $N \geq 3$ and therefore we can give three distinct values to the suffixes j, k and l. Multiplying Eq. (4.39) by g^{hj} and using the fact that $g^{hj}g_{hl} = \delta^j_l$, we get

$$(-\delta^h_i g_{hk} + Ng_{ik})\kappa_{,l} + (-g_{ik}\delta^j_l + g_{il}\delta^j_k)\kappa_{,j} + (-Ng_{il} + \delta^h_i g_{hl})\kappa_{,k} = 0$$

or

$$(N-1)g_{ik}\kappa_{,l} + (1-N)g_{il}\kappa_{,k} = 0; \text{ as } \delta^i_j = 0 \text{ for } i \neq j$$

or

$$g_{ik}\kappa_{,l} - g_{il}\kappa_{,k} = 0; \text{ as } N \neq 1$$

or

$$g^{ik}g_{ik}\kappa_{,l} - g^{ik}g_{il}\kappa_{,k} = 0; \text{ multiplying } g^{ik}$$

or

$$N\kappa_{,l} - \kappa_{,k} = 0; \text{ as } g^{ik}g_{ik} = N \text{ and } g^{ik}g_{il} = \delta^k_j$$

or

$$(N-1)\kappa_{,l} = 0 \Rightarrow \kappa_{,l} = \frac{\partial \kappa}{\partial x^l} = 0; \text{ as } N \neq 1.$$

This implies that the partial derivatives of κ with respect to xs are all zero. Hence, κ is a constant at P_0. But P_0 be an arbitrary point of V_N. Hence, the Riemannian curvature κ is constant throughout the space V_N. This is known as *Schurt's theorem.*

EXAMPLE 4.2.1 *Calculate κ for the Riemannian metric*

$$ds^2 = (x^1)^{-2}(dx^1)^2 - (x^1)^{-2}(dx^2)^2.$$

Solution: For the given metric, we have $g_{11} = (x^1)^{-2}$; $g_{12} = 0 = g_{21}$; $g_{22} = -(x^1)^{-2}$. Since $g = -(x^1)^{-4}$, so the reciprocal tensors g^{ij} are given by

$$g^{11} = (x^1)^2; \quad g^{12} = 0 = g^{21}; \quad g^{22} = -(x^1)^2.$$

The non-vanishing Christoffel symbols of second kind are

$$\left\{ \begin{matrix} 1 \\ 1 \ \ 1 \end{matrix} \right\} = -\frac{1}{x^1} = \left\{ \begin{matrix} 1 \\ 2 \ \ 2 \end{matrix} \right\}; \quad \left\{ \begin{matrix} 2 \\ 1 \ \ 2 \end{matrix} \right\} = \left\{ \begin{matrix} 2 \\ 2 \ \ 1 \end{matrix} \right\} = -\frac{1}{x^1}.$$

Using Eq. (4.6), the Riemannian curvature tensors R^{α}_{ijk} can be written as

$$R^1_{212} = -\frac{\partial}{\partial x^2}\left\{\begin{matrix}1\\2\ 1\end{matrix}\right\} + \frac{\partial}{\partial x^1}\left\{\begin{matrix}1\\2\ 2\end{matrix}\right\} + \left\{\begin{matrix}1\\\beta\ 1\end{matrix}\right\}\left\{\begin{matrix}\beta\\2\ 2\end{matrix}\right\} - \left\{\begin{matrix}1\\\beta\ 2\end{matrix}\right\}\left\{\begin{matrix}\beta\\2\ 1\end{matrix}\right\}.$$

$$= \frac{\partial}{\partial x^1}\left\{\begin{matrix}1\\2\ 2\end{matrix}\right\} + \left\{\begin{matrix}1\\1\ 1\end{matrix}\right\}\left\{\begin{matrix}1\\2\ 2\end{matrix}\right\} - \left\{\begin{matrix}1\\2\ 2\end{matrix}\right\}\left\{\begin{matrix}2\\2\ 1\end{matrix}\right\}.$$

$$= \frac{1}{(x^1)^2} - \frac{1}{x^1}\cdot\left(-\frac{1}{x^1}\right) - \left(-\frac{1}{x^1}\right)\cdot\left(-\frac{1}{x^1}\right) = \frac{1}{(x^1)^2}.$$

$$R^2_{212} = -\frac{\partial}{\partial x^2}\left\{\begin{matrix}2\\2\ 1\end{matrix}\right\} + \frac{\partial}{\partial x^1}\left\{\begin{matrix}2\\2\ 2\end{matrix}\right\} + \left\{\begin{matrix}2\\\beta\ 1\end{matrix}\right\}\left\{\begin{matrix}\beta\\2\ 2\end{matrix}\right\} - \left\{\begin{matrix}2\\\beta\ 2\end{matrix}\right\}\left\{\begin{matrix}\beta\\2\ 1\end{matrix}\right\} = 0.$$

Using Eq. (4.13), the only non-vanishing covariant curvature tensor is given by

$$R_{1212} = g_{1\alpha}R^{\alpha}_{212} = g_{11}R^1_{212} + g_{12}R^2_{212} = g_{11}R^1_{212}.$$

Thus, the Riemannian curvature κ given by Eq. (4.34) as

$$\kappa = \frac{R_{1212}}{g} = \frac{g_{11}R^1_{212}}{g_{11}g_{22}} = \frac{R^1_{212}}{g_{22}} = -1.$$

EXAMPLE 4.2.2 *Evaluate the Riemannian curvature at any point* (x^i) *of Riemannian 3 space in the directions* $\mathbf{P} = (1,0,0)$ *and* $\mathbf{Q} = (0,1,1)$ *if the metric is given by*

$$g_{11} = 1;\quad g_{22} = 2x^1;\quad g_{33} = 2x^2;\quad g_{ij} = 0,\ \textit{for } i \neq j.$$

Hence, calculate R_{ij}, R^i_j *and* R.

Solution: Since $g = 4x^1x^2$, so the reciprocal tensors g^{ij} are given by

$$g^{11} = 1;\quad g^{22} = \frac{1}{2x^1};\quad g^{33} = \frac{1}{2x^2};\quad g^{ij} = 0 = g^{ji}, i \neq j.$$

The non-vanishing Christoffel symbols of second kind are

$$\left\{\begin{matrix}1\\2\ 2\end{matrix}\right\} = -1;\quad \left\{\begin{matrix}2\\1\ 2\end{matrix}\right\} = \left\{\begin{matrix}2\\2\ 1\end{matrix}\right\} = \frac{1}{2x^1},$$

$$\left\{\begin{matrix}2\\3\ 3\end{matrix}\right\} = -\frac{1}{2x^1};\quad \left\{\begin{matrix}3\\2\ 3\end{matrix}\right\} = \left\{\begin{matrix}3\\3\ 2\end{matrix}\right\} = \frac{1}{2x^2}.$$

Since $N = 3$, only six components of the Riemannian tensor need to be considered: R_{1212}, R_{1313}, R_{2323}, R_{1213}, R_{1232}, R_{1323}. Using Eq. (4.6), the Riemannian curvature tensors R^{α}_{ijk} can be written as

$$\begin{aligned} R^1_{212} &= -\frac{\partial}{\partial x^2}\begin{Bmatrix} 1 \\ 2\ \ 1 \end{Bmatrix} + \frac{\partial}{\partial x^1}\begin{Bmatrix} 1 \\ 2\ \ 2 \end{Bmatrix} + \begin{Bmatrix} 1 \\ \beta\ \ 1 \end{Bmatrix}\begin{Bmatrix} \beta \\ 2\ \ 2 \end{Bmatrix} - \begin{Bmatrix} 1 \\ \beta\ \ 2 \end{Bmatrix}\begin{Bmatrix} \beta \\ 2\ \ 1 \end{Bmatrix} \\ &= \frac{\partial}{\partial x^1}\begin{Bmatrix} 1 \\ 2\ \ 2 \end{Bmatrix} - \begin{Bmatrix} 1 \\ 2\ \ 2 \end{Bmatrix}\begin{Bmatrix} 2 \\ 2\ \ 1 \end{Bmatrix} = \frac{1}{2x^1} \end{aligned}$$

and similarly, we get

$$R^1_{313} = 0, \quad R^2_{323} = \frac{1}{4x^1x^2}, \quad R^1_{213} = 0, \quad R^2_{123} = 0, \quad R^3_{132} = \frac{1}{4x^1x^2}.$$

Using Eq. (4.13), three non-zero terms of covariant curvature tensor R_{ijkl} of rank 4 are given by

$$R_{1212} = g_{11}R^1_{212} = \frac{1}{2x^1}; \quad R_{2323} = g_{22}R^2_{323} = \frac{1}{2x^2}$$

and

$$R_{3132} = g_{33}R^3_{132} = \frac{1}{2x^1}.$$

Since $\mathbf{g} = (g_{ij})$ is diagonal, all the non-zero G_{ijkl} will be derivable from the relation,

$$G_{ijij} = g_{ii}g_{jj}; \quad i < j \text{ and no summation.}$$

Thus, the non-vanishing G_{ijkl} are given by

$$G_{1212} = g_{11}g_{22} = 2x^1; \quad G_{1313} = g_{11}g_{33} = 2x^2; \quad G_{2323} = g_{22}g_{33} = 4x^1x^2.$$

For $\begin{bmatrix} \mathbf{P} \\ \mathbf{Q} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$, the coefficients H_{hijk} are given by Eq. (4.35). Therefore,

$$H_{1212} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}^2 = 1; \qquad H_{2323} = \begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix}^2 = 0;$$

$$H_{3132} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}\begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix} = 0; \quad H_{1313} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}^2 = 1.$$

Thus, the Riemannian curvature κ given by Eq. (4.34) as

$$\begin{aligned} \kappa &= \frac{R_{1212}H_{1212} + R_{2323}H_{2323} + R_{3132}H_{3132} + R_{1332}H_{1332}}{G_{1212}H_{1212} + G_{2323}H_{2323} + G_{3132}H_{3132} + G_{1332}H_{1332}} \\ &= \frac{\left(1/2x^1\right)\cdot 1 + \left(1/2x^2\right)\cdot 0 + 2\left(1/2x^1\right)\cdot 0}{(2x^1)\cdot 1 + (2x^2)\cdot 0 + 2\,(4x^1x^2)\cdot 0} = \frac{1}{4x^1(x^1 + x^2)}. \end{aligned}$$

Using formula (4.10), the Ricci tensors of first kind are given by

$$R_{ij} = R^{\alpha}_{ij\alpha} = R^1_{ij1} + R^2_{ij2} + R^3_{ij3}$$
$$= g^{11}R_{1ij1} + g^{22}R_{2ij2} + g^{33}R_{3ij3}.$$

Therefore, the non-zero Ricci tensors are given by

$$R_{11} = g^{22}R_{2112} = -\frac{1}{4(x^1)^2}; \quad R_{22} = g^{11}R_{1221} + g^{33}R_{3223} = -\frac{1}{2x^1} - \frac{1}{4(x^1)^2}$$
$$R_{33} = g^{22}R_{2332} = -\frac{1}{4x^1x^2}; \quad R_{12} = g^{33}R_{3123} = -\frac{1}{4x^1x^2} = g^{33}R_{3213} = R_{21}.$$

Using formula (4.11), the Ricci tensors of first kind are given by

$$R^i_j = g^{i\alpha}R_{\alpha j} = g^{i1}R_{1j} + g^{i2}R_{2j} + g^{i3}R_{3j}.$$

So

$$R^1_1 = g^{11}R_{11} + g^{12}R_{21} + g^{13}R_{31} = -\frac{1}{4(x^1)^2}.$$
$$R^2_2 = g^{12}R_{12} + g^{22}R_{22} + g^{23}R_{32} = \frac{1}{2x^1}\left[-\frac{1}{2x^1} - \frac{1}{4(x^1)^2}\right].$$
$$R^3_3 = g^{31}R_{13} + g^{32}R_{23} + g^{33}R_{33} = \frac{1}{2x^2}\left[-\frac{1}{4x^1x^2}\right].$$

Therefore, the total curvature R is given by Eq. (4.12) as

$$R = R^i_j = R^1_1 + R^2_2 + R^3_3$$
$$= -\frac{1}{4(x^1)^2} - \frac{1}{4(x^1)^2} - \frac{1}{8x^1(x^2)^2} - \frac{1}{8x^1(x^2)^2} = -\frac{x^1 + 2(x^2)^2}{4(x^1x^2)^2}.$$

Deduction 4.2.1 Isotropic points: If the Riemannian curvature at x does not change with the orientation of a 2-flat through x, then x is called *isotropic.* All points of a two-dimensional Riemannian space are isotropic.

EXAMPLE 4.2.3 *Find the isotropic points in the Riemannian space R^3 with metric*

$$g_{11} = 1,\ g_{22} = (x^1)^2 + 1 = g_{33},\ g_{ij} = 0;\ \textit{for } i \neq j$$

and calculate the Riemannian curvature κ at that points.

Solution: Since $g = [(x^1)^2 + 1]^2$, so the reciprocal tensors g^{ij} are given by

$$g^{11} = 1; \quad g^{ij} = 0 = g^{ji}; \quad g^{22} = \frac{1}{(x^1)^2 + 1} = g^{33}.$$

The non-vanishing Christoffel symbols of second kind are,

$$\left\{ \begin{matrix} 1 \\ 2\ 2 \end{matrix} \right\} = -x^1 = \left\{ \begin{matrix} 1 \\ 3\ 3 \end{matrix} \right\}; \ \left\{ \begin{matrix} 2 \\ 2\ 1 \end{matrix} \right\} = \frac{x^1}{(x^1)^2+1} = \left\{ \begin{matrix} 3 \\ 3\ 1 \end{matrix} \right\}.$$

Using Eq. (4.6), the Riemannian curvature tensors R^{α}_{ijk} can be written as

$$\begin{aligned}
R^1_{212} &= -\frac{\partial}{\partial x^2}\left\{ \begin{matrix} 1 \\ 2\ 1 \end{matrix} \right\} + \frac{\partial}{\partial x^1}\left\{ \begin{matrix} 1 \\ 2\ 2 \end{matrix} \right\} + \left\{ \begin{matrix} 1 \\ \beta\ 1 \end{matrix} \right\}\left\{ \begin{matrix} \beta \\ 2\ 2 \end{matrix} \right\} - \left\{ \begin{matrix} 1 \\ \beta\ 2 \end{matrix} \right\}\left\{ \begin{matrix} \beta \\ 2\ 1 \end{matrix} \right\} \\
&= \frac{\partial}{\partial x^1}\left\{ \begin{matrix} 1 \\ 2\ 2 \end{matrix} \right\} - \left\{ \begin{matrix} 1 \\ 2\ 2 \end{matrix} \right\}\left\{ \begin{matrix} 2 \\ 2\ 1 \end{matrix} \right\} = -1 - \frac{x^1}{(x^1)^2+1}(-x^1) = -\frac{1}{(x^1)^2+1}. \\
R^1_{313} &= -\frac{\partial}{\partial x^3}\left\{ \begin{matrix} 1 \\ 3\ 1 \end{matrix} \right\} + \frac{\partial}{\partial x^1}\left\{ \begin{matrix} 1 \\ 3\ 3 \end{matrix} \right\} + \left\{ \begin{matrix} 1 \\ \beta\ 1 \end{matrix} \right\}\left\{ \begin{matrix} \beta \\ 3\ 3 \end{matrix} \right\} - \left\{ \begin{matrix} 1 \\ \beta\ 3 \end{matrix} \right\}\left\{ \begin{matrix} \beta \\ 3\ 1 \end{matrix} \right\} \\
&= \frac{\partial}{\partial x^1}\left\{ \begin{matrix} 1 \\ 3\ 3 \end{matrix} \right\} - \left\{ \begin{matrix} 1 \\ 3\ 3 \end{matrix} \right\}\left\{ \begin{matrix} 3 \\ 3\ 1 \end{matrix} \right\} = -1 - \frac{x^1}{(x^1)^2+1}(-x^1) = -\frac{1}{(x^1)^2+1}. \\
R^2_{323} &= \left\{ \begin{matrix} 1 \\ 3\ 3 \end{matrix} \right\}\left\{ \begin{matrix} 2 \\ 1\ 2 \end{matrix} \right\} = -\frac{(x^1)^2}{(x^1)^2+1}. \\
R^1_{213} &= R^2_{123} = R^3_{132} = 0.
\end{aligned}$$

Using Eq. (4.13), the non-vanishing covariant curvature tensor R_{hijk} are given by

$$R_{1212} = g_{11}R^1_{212} = -\left[(x^1)^2+1\right]^{-1} = R_{1313} = g_{11}R^1_{313}; R_{2323} = g_{22}R^2_{323} = -(x^1)^2.$$

Using Eq. (4.36), the non-vanishing tensors G_{hijk} are given by

$$G_{1212} = g_{11}g_{22} = (x^1)^2+1 = G_{1313} = g_{11}g_{33}; G_{2323} = g_{22}g_{33} = \left[(x^1)^2+1\right]^2.$$

If the curvature κ is to be independent of H_{hijk}, defined in Eq. (4.35) (which vary with the direction of the 2 flat), then $(x^1)^2 = 1$, i.e. $x^1 = \pm 1$. Therefore, the isotropic points compose two surfaces, on which the curvature has the value given by

$$\begin{aligned}
\kappa &= \frac{-\left[(x^1)^2+1\right]^{-1}H_{1212} - \left[(x^1)^2+1\right]^{-1}H_{1313} - (x^1)^2 H_{2323}}{\left[(x^1)^2+1\right]H_{1212} + \left[(x^1)^2+1\right]H_{1313} + \left[(x^1)^2+1\right]^2 H_{2323}} \\
&= -\left[(x^1)^2+1\right]^{-2}\frac{H_{1212}+H_{1313}+(x^1)^2\left[(x^1)^2+1\right]H_{2323}}{H_{1212}+H_{1313}+\left[(x^1)^2+1\right]H_{2323}} \\
&= -\left[1+1\right]^{-2}\cdot 1 = -\frac{1}{4}.
\end{aligned}$$

EXAMPLE 4.2.4 *Show that every point of R^3 is isotropic for the metric*

$$ds^2 = (x^1)^{-2}(dx^1)^2 + (x^1)^{-2}(dx^2)^2 + (x^1)^{-2}(dx^3)^2.$$

Solution: For the given metric, we have $g_{11} = (x^1)^{-2} = g_{22} = g_{33}$; $g_{ij} = 0 = g_{ji}$; for $i \neq j$. Since $g = (x^1)^{-6}$, so the reciprocal tensors g^{ij} are given by

$$g^{11} = (x^1)^2 = g^{22} = g^{33}; \quad g^{ij} = 0 = g^{ji}; \quad \text{for } i \neq j.$$

The non-vanishing Christoffel symbols of second kind are

$$\left\{ {1 \atop 1\ 1} \right\} = -\frac{1}{x^1} = \left\{ {2 \atop 1\ 2} \right\} = \left\{ {3 \atop 1\ 3} \right\}; \quad \left\{ {1 \atop 2\ 2} \right\} = \frac{1}{x^1} = \left\{ {1 \atop 3\ 3} \right\}.$$

Using Eq. (4.6), the Riemannian curvature tensors R^{α}_{ijk} can be written as

$$\begin{aligned}
R^1_{212} &= -\frac{\partial}{\partial x^2}\left\{ {1 \atop 2\ 1} \right\} + \frac{\partial}{\partial x^1}\left\{ {1 \atop 2\ 2} \right\} + \left\{ {1 \atop \beta\ 1} \right\}\left\{ {\beta \atop 2\ 2} \right\} - \left\{ {1 \atop \beta\ 2} \right\}\left\{ {\beta \atop 2\ 1} \right\}. \\
&= \frac{\partial}{\partial x^1}\left\{ {1 \atop 2\ 2} \right\} - \left\{ {1 \atop 2\ 2} \right\}\left\{ {1 \atop 1\ 1} \right\} - \left\{ {2 \atop 2\ 1} \right\}\left\{ {1 \atop 2\ 2} \right\} = -\frac{1}{(x^1)^2}. \\
R^1_{313} &= -\frac{\partial}{\partial x^3}\left\{ {1 \atop 3\ 1} \right\} + \frac{\partial}{\partial x^1}\left\{ {1 \atop 3\ 3} \right\} + \left\{ {1 \atop \beta\ 1} \right\}\left\{ {\beta \atop 3\ 3} \right\} - \left\{ {1 \atop \beta\ 3} \right\}\left\{ {\beta \atop 3\ 1} \right\}. \\
&= \frac{\partial}{\partial x^1}\left\{ {1 \atop 3\ 3} \right\} + \left\{ {1 \atop 3\ 3} \right\}\left\{ {1 \atop 1\ 1} \right\} - \left\{ {3 \atop 3\ 1} \right\}\left\{ {1 \atop 3\ 3} \right\} = -\frac{1}{(x^1)^2}. \\
R^2_{323} &= \left\{ {1 \atop 3\ 3} \right\}\left\{ {2 \atop 1\ 2} \right\} = -\frac{1}{(x^1)^2}; \quad R^1_{213} = R^2_{123} = R^3_{132} = 0.
\end{aligned}$$

Using Eq. (4.13), the non-vanishing covariant curvature tensors R_{hijk} are given by

$$R_{1212} = R_{1313} = R_{2323} = -(x^1)^{-4}; \; G_{1212} = G_{1313} = G_{2323} = (x^1)^{-4}.$$

The Riemannian curvature κ is given by

$$\begin{aligned}
\kappa &= \frac{R_{1212}H_{1212} + R_{1313}H_{1313} + R_{2323}H_{2323}}{G_{1212}H_{1212} + G_{1313}H_{1313} + G_{2323}H_{2323}} \\
&= \frac{\left[-(x^1)^{-4}\right]}{[(x^1)^{-4}]}\frac{H_{1212} + H_{1313} + H_{2323}}{H_{1212} + H_{1313} + H_{2323}} = -1.
\end{aligned}$$

Thus, we see that this Riemannian space is more than just isotropic; it is a space of constant curvature.

4.2.1 Space of Constant Curvature

If the Riemannian curvature κ is constant in a space V_N, then that space V_N is said to be of *constant curvature.*

Let us consider an Euclidean space S_N of N dimensions. We have to show that $R_{hijk} = 0$. In the case of S_N, the metric tensor g_{ij} is either 1 or 0. Hence,

$$[ij, k] = 0 = \left\{ \begin{matrix} k \\ i \;\; j \end{matrix} \right\}. \tag{4.40}$$

Using Eq. (4.17), the covariant curvature tensors R_{hijk} are given by

$$R_{hijk} = \begin{vmatrix} \dfrac{\partial}{\partial x^j} & \dfrac{\partial}{\partial x^k} \\ [ij, h] & [ik, h] \end{vmatrix} + \begin{vmatrix} \left\{ \begin{matrix} \alpha \\ i \;\; j \end{matrix} \right\} & \left\{ \begin{matrix} \alpha \\ i \;\; k \end{matrix} \right\} \\ [hj, \alpha] & [hk, \alpha] \end{vmatrix} = 0; \quad \text{by Eq. (4.40)}.$$

Conversely, let $R_{hijk} = 0$ in a space V_N, we have to show that the space V_N is an Euclidean space S_N. The Riemannian curvature κ at any point P of V_N is given by

$$\kappa = \frac{p^h q^i p^j q^k R_{hijk}}{p^h q^i p^j q^k \left[g_{hj} g_{ik} - g_{ij} g_{hk} \right]} = 0; \text{ as } R_{hijk} = 0.$$

Consequently, geodesics through P are straight lines. But P is an arbitrary point of V_N. Hence, geodesics through every point of V_N are straight lines. This will happen only if V_N is S_N. Therefore, a necessary and sufficient condition that a space V_N be an Euclidean space of N dimensions is that the curvature tensor vanishes.

4.2.2 Zero Curvature

A fundamental question has run unanswered through preceding sections: How can one tell whether a given metrisation of R^N is an Euclidean or not?

Let us consider a specified co-ordinate system (x^i) in which a Riemannian metric $\mathbf{g} = (g_{ij})$. Now, consider a co-ordinate system in which $\overline{g}_{ij} = \delta_{ij}$, which is a rectangular system. Suppose that a rectangular system $(\overline{x}^i)$ does exist. Then $\widetilde{\kappa} = 0$, since all Christoffel symbols vanish in $(\overline{x}^i)$. But the Riemannian curvature is an invariant, so that $\kappa = 0$ in the original co-ordinate (x^i) as well. Moreover,

$$g_{ij} u^i u^j = \overline{u}^i \overline{u}^j \geq 0.$$

Thus, we see that a Riemannian metric (g_{ij}) is the Euclidean metric if the Riemannian curvature κ is zero at all points and the metric is positive definite. To prove the converse portion, we set up a system of first order partial differential equations for N

rectangular co-ordinates $\overline{x}^i$ as functions of the given co-ordinates $x^j (j = 1, 2, \ldots, N)$. The system that immediately comes to mind is $G = J^T J$, or,

$$\frac{\partial \overline{x}^k}{\partial x^i} \frac{\partial \overline{x}^k}{\partial x^j} = g_{ij} \left(x^1, x^2, \ldots, x^N \right). \tag{4.41}$$

But the partial differential equation Eq. (4.41) is generally intractable because of its nonlinearity. Instead, we select the linear system that results when barred and unbarred co-ordinates are interchanged in Eq. (3.10) and then the $\overline{\left\{ \begin{matrix} i \\ j \ \ k \end{matrix} \right\}}$ are

$$\frac{\partial^2 \overline{x}^k}{\partial x^i \partial x^j} = \left\{ \begin{matrix} r \\ i \ \ j \end{matrix} \right\} \frac{\partial \overline{x}^k}{\partial x^r}. \tag{4.42}$$

Setting $\omega \equiv \overline{x}^k$ and $u_i \equiv \frac{\partial \overline{x}^k}{\partial x^i}$ yields the desired first order system

$$\frac{\partial \omega}{\partial x^i} = u_i \text{ and } \frac{\partial u_i}{\partial x^j} = \left\{ \begin{matrix} r \\ i \ \ j \end{matrix} \right\} u_r. \tag{4.43}$$

A Riemannian metric $\mathbf{g} = (g_{ij})$, specified in a co-ordinate system (x^i), is the Euclidean metric, if, under some permissible co-ordinate transformation Eq. (1.2) $\overline{\mathbf{g}} = (\delta_{ij})$.

EXAMPLE 4.2.5 *Show that R^3 under the following metric is Euclidean.*

$$ds^2 = [(x^1)^2 + (x^2)^2](dx^1)^2 + [(x^1)^2 + (x^2)^2](dx^2)^2 + (dx^3)^2.$$

Solution: For the given metric, we have $g_{11} = (x^1)^2 + (x^2)^2 = g_{22}$ $g_{33} = 1$; $g_{ij} = 0 = g_{ji}; j \neq i$. Since $g_{33} =$ constant, and g_{11} and g_{22} independent of x^3, so $\left\{ \begin{matrix} i \\ j \ \ k \end{matrix} \right\} = 0$, whenever i, j or k equals to 3. Consequently, of the six independent components of the Riemannian tensor, R_{1212} is possibly non-zero. Since $g = [(x^1)^2 + (x^2)^2]^2$, so the reciprocal tensors g^{ij} are given by

$$g^{11} = \frac{1}{(x^1)^2 + (x^2)^2}; \quad g^{22} = \frac{1}{(x^1)^2 + (x^2)^2}; \quad g^{33} = 1.$$

The non-vanishing Christoffel symbols of second kind are

$$\left\{ \begin{matrix} 1 \\ 1 \ \ 1 \end{matrix} \right\} = \frac{x^1}{(x^1)^2 + (x^2)^2} = \left\{ \begin{matrix} 2 \\ 1 \ \ 2 \end{matrix} \right\}; \quad \left\{ \begin{matrix} 1 \\ 2 \ \ 2 \end{matrix} \right\} = -\frac{x^1}{(x^1)^2 + (x^2)^2}$$

$$\left\{ \begin{matrix} 1 \\ 1 \ \ 2 \end{matrix} \right\} = \frac{x^2}{(x^1)^2 + (x^2)^2} = \left\{ \begin{matrix} 2 \\ 2 \ \ 2 \end{matrix} \right\}; \quad \left\{ \begin{matrix} 2 \\ 1 \ \ 1 \end{matrix} \right\} = -\frac{x^2}{(x^1)^2 + (x^2)^2}.$$

Using Eq. (4.6), the Riemannian curvature tensors R^{α}_{ijk} can be written as

$$\begin{aligned} R^1_{212} &= -\frac{\partial}{\partial x^2}\left\{\begin{matrix} 1 \\ 2\ \ 1 \end{matrix}\right\} + \frac{\partial}{\partial x^1}\left\{\begin{matrix} 1 \\ 2\ \ 2 \end{matrix}\right\} + \left\{\begin{matrix} 1 \\ \beta\ \ 1 \end{matrix}\right\}\left\{\begin{matrix} \beta \\ 2\ \ 2 \end{matrix}\right\} - \left\{\begin{matrix} 1 \\ \beta\ \ 2 \end{matrix}\right\}\left\{\begin{matrix} \beta \\ 2\ \ 1 \end{matrix}\right\}. \\ &= \frac{-\sqrt{g} + x^1(2x^1)}{g} - \frac{\sqrt{g} - x^2(2x^2)}{g} + \frac{-x^1}{\sqrt{g}} \cdot \frac{x^1}{\sqrt{g}} + \frac{x^2}{\sqrt{g}}\frac{x^2}{\sqrt{g}} - \frac{x^2}{\sqrt{g}}\frac{x^2}{\sqrt{g}} - \frac{x^1}{\sqrt{g}}\frac{-x^1}{\sqrt{g}} \\ &= 0. \end{aligned}$$

Consequently, $R_{1212} = 0 = \kappa$. As the metric is clearly positive definite, so the space is Euclidean.

EXAMPLE 4.2.6 *Consider the two-dimensional metric* $g_{11} = 0$; $g_{12} = 0 = g_{21}$; $g_{22} = (x^2)^2$. *Find the general solution in* $(\overline{x}^i)$ *co-ordinate system.*

Solution: Since $g = (x^2)^2$, so the reciprocal tensors g^{ij} are given by

$$g^{11} = 1;\ \ g^{12} = 0 = g^{21};\ \ g^{22} = (x^2)^2.$$

The only non-vanishing Christoffel symbol of second kind is $\left\{\begin{matrix} 2 \\ 2\ \ 2 \end{matrix}\right\} = \frac{1}{x^2}$, so the metric is obviously positive definite. Using Eq. (4.6), the Riemannian curvature tensors R^{α}_{ijk} can be written as

$$R^1_{212} = -\frac{\partial}{\partial x^2}\left\{\begin{matrix} 1 \\ 2\ \ 1 \end{matrix}\right\} + \frac{\partial}{\partial x^1}\left\{\begin{matrix} 1 \\ 2\ \ 2 \end{matrix}\right\} + \left\{\begin{matrix} 1 \\ \beta\ \ 1 \end{matrix}\right\}\left\{\begin{matrix} \beta \\ 2\ \ 2 \end{matrix}\right\} - \left\{\begin{matrix} 1 \\ \beta\ \ 2 \end{matrix}\right\}\left\{\begin{matrix} \beta \\ 2\ \ 1 \end{matrix}\right\} = 0.$$

$$R^2_{212} = -\frac{\partial}{\partial x^2}\left\{\begin{matrix} 2 \\ 2\ \ 1 \end{matrix}\right\} + \frac{\partial}{\partial x^1}\left\{\begin{matrix} 2 \\ 2\ \ 2 \end{matrix}\right\} + \left\{\begin{matrix} 2 \\ \beta\ \ 1 \end{matrix}\right\}\left\{\begin{matrix} \beta \\ 2\ \ 2 \end{matrix}\right\} - \left\{\begin{matrix} 2 \\ \beta\ \ 2 \end{matrix}\right\}\left\{\begin{matrix} \beta \\ 2\ \ 1 \end{matrix}\right\} = 0.$$

Using Eq. (4.13), the covariant curvature tensor is given by

$$R_{1212} = g_{1\alpha}R^{\alpha}_{212} = g_{11}R^1_{212} + g_{12}R^2_{212} = 0.$$

Thus, the Riemannian curvature κ is $\kappa = \frac{R_{1212}}{g} = 0$. Let us introduce,

$$f_1 = \frac{\partial \overline{x}^1}{\partial x^1};\ f_2 = \frac{\partial \overline{x}^1}{\partial x^2};\ f_3 = \frac{\partial \overline{x}^2}{\partial x^1};\ f_4 = \frac{\partial \overline{x}^2}{\partial x^2},$$

then the system of Eq. (4.41) becomes,

$$\begin{aligned} &f_1^2 + f_3^2 = 1;\ \ f_1 f_2 + f_3 f_4 = 0;\ \ f_2^2 + f_4^2 = (x^2)^2 \\ \Rightarrow\ &f_1 = f_1;\ \ f_2 = x^2\sqrt{1 - f_1^2};\ \ f_3 = -\sqrt{1 - f_1^2};\ \ f_4 = x^2 f_1. \end{aligned}$$

Therefore, the system of equations becomes two simple first order systems in $\overline{x}^1$ alone and $\overline{x}^2$ alone

$$\mathrm{I}: \frac{\partial \overline{x}^1}{\partial x^1} = f_1; \qquad \frac{\partial \overline{x}^1}{\partial x^2} = x^2\sqrt{1 - f_1^2}$$

$$\mathrm{II}: \frac{\partial \overline{x}^2}{\partial x^1} = -\sqrt{1 - f_1^2}; \qquad \frac{\partial \overline{x}^2}{\partial x^2} = x^2 f_1.$$

The unknown function f_1 is determined by the requirements that the two conditions I and II be compatible

$$\frac{\partial f_1}{\partial x^2} = \frac{\partial}{\partial x^1}\left(x^2\sqrt{1 - f_1^2}\right) \text{ and } \frac{\partial}{\partial x^2}\left(-\sqrt{1 - f_1^2}\right) = \frac{\partial}{\partial x^1}\left(x^2 f_1\right).$$

The only function satisfying these two compatibility condition is

$$f_1 = \text{ constant } = \cos\alpha$$

and I and II immediately integrate to give

$$\overline{x}^1 = x^1\cos\alpha + \frac{1}{2}(x^2)^2\sin\alpha + c$$

$$\overline{x}^2 = -x^1\sin\alpha + \frac{1}{2}(x^2)^2\cos\alpha + d,$$

where c and d are constants and we use, of course, free to set $\alpha = c = d = 0$. From Eq. (4.43) we have to solve,

$$\frac{\partial \omega}{\partial x^1} = u_1; \quad \frac{\partial \omega}{\partial x^2} = u_2$$

$$\frac{\partial u_1}{\partial x^1} = 0, \frac{\partial u_1}{\partial x^2} = 0; \quad \frac{\partial u_2}{\partial x^1} = 0, \frac{\partial u_2}{\partial x^2} = u_2\left\{\begin{matrix} 2 \\ 2\ \ 2 \end{matrix}\right\} = \frac{u_2}{x^2}.$$

$$\Rightarrow \omega = a_1x^1 + a_2(x^2)^2 + a_3; \quad a_1, a_2, a_3 = \text{ constant}$$

$$\Rightarrow \overline{x}^k = a_1^k x^1 + a_2^k(x^2)^2 + a_3^k; \quad a_1^k = \text{ constant},$$

which is similar to the solution set $\overline{x}^1$ and $\overline{x}^2$ as above.

Deduction 4.2.2 The quasilinear first-order system,

$$\frac{\partial u_\lambda}{\partial x^j} = F_{\lambda j}(u_0, u_1, \ldots, u_m;\ x^1, x^2, \ldots, x^n); \ \lambda = 0, 1, \ldots, m; j = 1, 2, \ldots, n$$

where the functions $F_{\lambda j}$ are of the differentiability class C^1, has a nontrivial solution for the u_λ, bounded over some region of R^n, if and only if

$$\frac{\partial F_{\lambda j}}{\partial u_\nu}F_{\nu k} + \frac{\partial F_{\lambda j}}{\partial x^k} = \frac{\partial F_{\lambda k}}{\partial u_\nu}F_{\nu j} + \frac{\partial F_{\lambda k}}{\partial x^j}; \ \ \lambda = 0, 1, \ldots, m; \ \ 1 \le j < k \le n,$$

where ν summations run from 0 to m. The conditions $R_{hijk} = 0$ are the sufficient condition for *compatibility.*

4.2.3 Einstein Tensor

The covariant tensor

$$\Gamma_{ij} = R_{ij} - \frac{1}{2}Rg_{ij} \tag{4.44}$$

is called *Einstein tensor.* The mixed tensor

$$\Gamma^i_j = R^i_j - \frac{1}{2}\delta^i_j R; \quad g^{\alpha i}R_{\alpha j} = R^i_j \text{ and } g^{ij}R_{ij} = R \tag{4.45}$$

is called the *Einstein tensor.* This tensor is widely used in the general theory of relativity.

EXAMPLE 4.2.7 *Prove that for any Riemannian metric, the divergence of the Einstein tensor is zero at all points.*

Solution: The *Einstein tensor* Γ^i_j is given by

$$\Gamma^i_j = R^i_j - \frac{1}{2}\ \delta^i_j\ \ R,$$

where $g^{\alpha i}R_{\alpha j} = R^i_j$ and $g^{ij}R_{ij} = R$. Covariant differentiation of the *Einstein tensor* Γ^i_j with respect to x^k is

$$\begin{aligned}\Gamma^i_{j,k} &= R^i_{j,k} - \frac{1}{2}\delta^i_{j,k}R - \frac{1}{2}\delta^i_j R_{,k}\\ &= R^i_{j,k} - \frac{1}{2}\delta^i_j R_{,k}; \text{ as } \delta^i_{j,k} = 0.\end{aligned}$$

Hence, the divergence of the *Einstein tensor* Γ^i_j is given by

$$\begin{aligned}\Gamma^i_{j,i} &= R^i_{j,i} - \frac{1}{2}\delta^i_j R_{,i}\\ &= R^i_{j,i} - \frac{1}{2}R_{,j} = \frac{1}{2}R_{,j} - \frac{1}{2}R_{,j} = 0.\end{aligned}$$

Thus, Einstein tensor is divergence free. This equation plays an important role in the theory of relativity.

4.2.4 Flat Riemannian Spaces

We know, covariant differentiation of all vectors is commutative if and only if the Riemann–Christoffel curvature tensor R^α_{ijk} is identically zero. The covariant curvature tensors of fourth order R_{hijk} are zero, then all the components of the tensor R_{hijk} are zero and vice versa. A Riemannian space whose curvature tensor is identically zero is called a *flat space.*

If in a neighbourhood N of a point O of V_N, there exists a co-ordinate system with respect to which the metric tensor has constant values, then V_N is said to be *locally flat* at O. The Riemannian space V_N is said to be *locally flat,* if the metric tensor g_{ij} has constant values throughout V_N.

A space is Euclidean (or flat) if it is possible to find a Cartesian co-ordinate system everywhere in it; if this is not possible, the space is non-Euclidean (or curved).

Theorem 4.2.2 *The necessary and sufficient condition that a space V_N be locally flat (or flat) in the neighbourhood of O is that the curvature tensor be identically zero.*

Proof: First let V_N be locally flat in the neighbourhood of O, so that g_{ij} = constant for all i and j. Thus,

$$\frac{\partial g_{ij}}{\partial x^k} = 0 \Rightarrow \begin{Bmatrix} k \\ i \;\; j \end{Bmatrix} = 0,$$

in the neighbourhood of O. The Riemann–Christoffel symbol R^{α}_{ijk} is given by

$$\begin{aligned} R^{\alpha}_{ijk} &= -\frac{\partial}{\partial x^k}\begin{Bmatrix} \alpha \\ i \;\; j \end{Bmatrix} + \frac{\partial}{\partial x^j}\begin{Bmatrix} \alpha \\ i \;\; k \end{Bmatrix} - \begin{Bmatrix} \beta \\ i \;\; j \end{Bmatrix}\begin{Bmatrix} \alpha \\ \beta \;\; k \end{Bmatrix} + \begin{Bmatrix} \beta \\ i \;\; k \end{Bmatrix}\begin{Bmatrix} \alpha \\ \beta \;\; j \end{Bmatrix} \\ &= 0; \quad \text{in the neighbourhood of } O. \end{aligned}$$

Conversely, let $R^{\alpha}_{ijk} = 0$ in the neighbourhood of O. To prove that V_N is locally flat, it is enough to prove that g_{ij} = constant in the neighbourhood of O. Given co-ordinate system x^i, let us choose co-ordinate system $\overline{x}^i$ such that

$$\begin{Bmatrix} \alpha \\ \beta \;\; \gamma \end{Bmatrix} = \frac{\partial^2 \overline{x}^l}{\partial x^\alpha \partial x^\beta}\frac{\partial x^\gamma}{\partial \overline{x}^l}. \tag{4.46}$$

The tensor law of transformation for the Christoffel symbol is

$$\overline{\begin{Bmatrix} k \\ i \;\; j \end{Bmatrix}} = \begin{Bmatrix} \gamma \\ \alpha \;\; \beta \end{Bmatrix}\frac{\partial x^\alpha}{\partial \overline{x}^i}\frac{\partial x^\beta}{\partial \overline{x}^j}\frac{\partial x^k}{\partial \overline{x}^\gamma} + \frac{\partial^2 x^l}{\partial \overline{x}^i \partial \overline{x}^j}\frac{\partial \overline{x}^k}{\partial x^l}$$

or

$$\begin{Bmatrix} k \\ i \;\; j \end{Bmatrix} = \overline{\begin{Bmatrix} \gamma \\ \alpha \;\; \beta \end{Bmatrix}}\frac{\partial \overline{x}^\alpha}{\partial x^i}\frac{\partial \overline{x}^\beta}{\partial x^j}\frac{\partial x^k}{\partial \overline{x}^\gamma} + \frac{\partial^2 \overline{x}^l}{\partial x^i \partial x^j}\frac{\partial x^k}{\partial \overline{x}^l}$$

or

$$\frac{\partial^2 \overline{x}^l}{\partial x^i \partial x^j}\frac{\partial x^k}{\partial \overline{x}^l} = \begin{Bmatrix} k \\ i \;\; j \end{Bmatrix} - \overline{\begin{Bmatrix} \gamma \\ \alpha \;\; \beta \end{Bmatrix}}\frac{\partial \overline{x}^\alpha}{\partial x^i}\frac{\partial \overline{x}^\beta}{\partial x^j}\frac{\partial x^k}{\partial \overline{x}^\gamma}$$

or

$$\overline{\begin{Bmatrix} \gamma \\ \alpha \ \ \beta \end{Bmatrix}} \frac{\partial \overline{x}^\alpha}{\partial x^i} \frac{\partial \overline{x}^\beta}{\partial x^j} \frac{\partial x^k}{\partial \overline{x}^\gamma} = 0$$

or

$$\overline{\begin{Bmatrix} \gamma \\ \alpha \ \ \beta \end{Bmatrix}} \delta_l^\alpha \delta_m^\beta \delta_\gamma^k = 0; \text{ multiplying } \frac{\partial x^i}{\partial \overline{x}^l} \frac{\partial x^j}{\partial \overline{x}^m} \frac{\partial \overline{x}^p}{\partial x^k}$$

or

$$\overline{\begin{Bmatrix} \beta \\ l \ \ m \end{Bmatrix}} = 0; \quad \delta_l^\alpha = \begin{cases} 1; & \text{for } \alpha = l \\ 0; & \text{for } \alpha \neq l \end{cases}, \text{ etc.}$$

or

$$\overline{R}_{ijk}^h = 0; \quad \text{i.e.} \quad R_{ijk}^h = 0.$$

Since if a tensor vanishes in one co-ordinate system, then it vanishes in all systems. It follows that Eq. (4.46) is a solution of $R_{ijk}^h = 0$, which is given. Hence, co-ordinate system $\overline{x}^i$ exists, and

$$\frac{\partial \overline{g}_{ij}}{\partial \overline{x}^k} = 0 \Rightarrow \overline{g}_{ij} = \text{ constant.}$$

Consequently, a co-ordinate system $\overline{x}^i$ exists relative to which $\overline{g}_{ij}$ is constant. Hence, V_N is locally flat.

EXAMPLE 4.2.8 *Determine whether the following metric is flat and/or Euclidean:*

$$ds^2 = (dx^1)^2 - (x^2)^2(dx^2)^2; \quad N = 2.$$

Solution: For the given metric, we have $g_{11} = 1$; $g_{12} = 0 = g_{21}$; $g_{22} = -(x^2)^{-2}$. Since $g = -(x^2)^2$, so the reciprocal tensors g^{ij} are given by

$$g^{11} = 1; \quad g^{12} = 0 = g^{21}; \quad g^{22} = -\frac{1}{(x^2)^2}.$$

Since the metric is not positive definite, it cannot be Euclidean. The only non-vanishing Christoffel symbol of second kind is $\begin{Bmatrix} 2 \\ 2 \ \ 2 \end{Bmatrix} = (x^2)^3$. Using Eq. (4.6), the Riemannian curvature tensors R_{ijk}^α can be written as

$$R_{212}^1 = -\frac{\partial}{\partial x^2}\begin{Bmatrix} 1 \\ 2 \ \ 1 \end{Bmatrix} + \frac{\partial}{\partial x^1}\begin{Bmatrix} 1 \\ 2 \ \ 2 \end{Bmatrix} + \begin{Bmatrix} 1 \\ \beta \ \ 1 \end{Bmatrix}\begin{Bmatrix} \beta \\ 2 \ \ 2 \end{Bmatrix} - \begin{Bmatrix} 1 \\ \beta \ \ 2 \end{Bmatrix}\begin{Bmatrix} \beta \\ 2 \ \ 1 \end{Bmatrix} = 0.$$

$$R_{212}^2 = -\frac{\partial}{\partial x^2}\begin{Bmatrix} 2 \\ 2 \ \ 1 \end{Bmatrix} + \frac{\partial}{\partial x^1}\begin{Bmatrix} 2 \\ 2 \ \ 2 \end{Bmatrix} + \begin{Bmatrix} 2 \\ \beta \ \ 1 \end{Bmatrix}\begin{Bmatrix} \beta \\ 2 \ \ 2 \end{Bmatrix} - \begin{Bmatrix} 2 \\ \beta \ \ 2 \end{Bmatrix}\begin{Bmatrix} \beta \\ 2 \ \ 1 \end{Bmatrix} = 0.$$

Using Eq. (4.13), the covariant curvature tensor is given by

$$R_{1212} = g_{1\alpha} R^{\alpha}_{212} = g_{11} R^{1}_{212} + g_{12} R^{2}_{212} = 0.$$

Hence, the given space is flat.

EXAMPLE 4.2.9 *Consider the Riemannian metric*

$$ds^2 = (dx^1)^2 + 4(x^2)^2(dx^2)^2 + 4(x^3)^2(dx^3)^2 - 4(dx^4)^2(dx^4)^2.$$

(a) Calculate the Riemannian curvature. (b) Find the solution $(\overline{x}^i)$*, by considering that the space is flat.*

Solution: (a) For the given metric, we have $g_{11} = 1$; $g_{22} = 4(x^2)^2; g_{33} = 4(x^3)^2$ and $g_{44} = -4(x^4)^2$. Since $g = -64(x^2x^3x^4)^2$, so the reciprocal tensors g^{ij} are given by

$$g^{11} = 1;\;\; g^{22} = \frac{1}{4(x^2)^2};\;\; g^{33} = \frac{1}{4(x^3)^2};\;\; g^{44} = -\frac{1}{4(x^4)^2}.$$

The non-vanishing Christoffel symbols of second kind are

$$\begin{Bmatrix} 2 \\ 2\;\;2 \end{Bmatrix} = \frac{1}{x^2};\;\; \begin{Bmatrix} 3 \\ 3\;\;3 \end{Bmatrix} = \frac{1}{x^3} \text{ and } \begin{Bmatrix} 4 \\ 4\;\;4 \end{Bmatrix} = \frac{1}{x^4}.$$

Because $\begin{Bmatrix} i \\ j\;\;k \end{Bmatrix} = 0$ unless $i = j = k$, the partial derivative terms drop out of Eq. (4.6), leaving

$$\begin{aligned} R^{\alpha}_{ijk} &= \begin{Bmatrix} r \\ i\;\;k \end{Bmatrix}\begin{Bmatrix} \alpha \\ r\;\;j \end{Bmatrix} - \begin{Bmatrix} r \\ i\;\;j \end{Bmatrix}\begin{Bmatrix} \alpha \\ r\;\;k \end{Bmatrix} \\ &= \begin{Bmatrix} \alpha \\ \alpha\;\;\alpha \end{Bmatrix}\begin{Bmatrix} \alpha \\ \alpha\;\;\alpha \end{Bmatrix} - \begin{Bmatrix} \alpha \\ \alpha\;\;\alpha \end{Bmatrix}\begin{Bmatrix} \alpha \\ \alpha\;\;\alpha \end{Bmatrix} = 0; \text{ (not summed)}, \end{aligned}$$

which in turn implies that $R_{hijk} = 0$ and $\kappa = 0$.
(b) For the above calculated Christoffel symbols, system (4.43) becomes,

$$\frac{\partial u_1}{\partial x^1} = 0,\;\; \frac{\partial u_2}{\partial x^2} = \frac{u_2}{x^2},\;\; \frac{\partial u_3}{\partial x^3} = \frac{u_3}{x^3},\;\; \frac{\partial u_4}{\partial x^4} = \frac{u_4}{x^4},$$

with $\frac{\partial u_i}{\partial x^j} = 0$ for $i \neq j$. Integrating,

$$\begin{aligned} &u_1 = f_1\left(x^2, x^3, x^4\right),\;\; u_2 = x^2 f_2\left(x^1, x^3, x^4\right), \\ &u_3 = x^3 f_3\left(x^1, x^2, x^4\right),\;\; u_4 = x^4 f_4\left(x^1, x^2, x^3\right) \end{aligned}$$

for arbitrary functions f_i. But the remaining part of Eq. (4.43), $\frac{\partial \omega}{\partial x^i} = u_i$, gives rise to the compatibility relations $\frac{\partial u_i}{\partial x^j} = \frac{\partial u_j}{\partial x^i}$, which is satisfied only if $f_i = c_i =$ constant. Therefore,

$$\omega = a_1 x^1 + a_2(x^2)^2 + a_3(x^3)^2 + a_4(x^4)^2 + a_5$$

and the transformation must be of the general form,

$$\overline{x}^k = a_1^k x^1 + a_2^k (x^2)^2 + a_3^k (x^3)^2 + a_4^k (x^4)^2 + a_5^k; \quad a_i^k \text{ constants.}$$

EXAMPLE 4.2.10 *For the Euclidean space under the following metric:*

$$ds^2 = [(x^1)^2 + (x^2)^2](dx^1)^2 + [(x^1)^2 + (x^2)^2](dx^2)^2 + (dx^3)^2$$

exhibit a transformation from the co-ordinate system (x^i) to a rectangular system $(\overline{x}^i)$.

Solution: For the given metric, we have $g_{11} = (x^1)^2 + (x^2)^2 = g_{22}\, g_{33} = 1$; $g_{ij} = 0 = g_{ji}; j \neq i$. Since $g = [(x^1)^2 + (x^2)^2]^2$, so the reciprocal tensors g^{ij} are given by

$$g^{11} = \frac{1}{(x^1)^2 + (x^2)^2}; \quad g^{22} = \frac{1}{(x^1)^2 + (x^2)^2}; \quad g^{33} = 1.$$

The non-vanishing Christoffel symbols of second kind are

$$\begin{Bmatrix} 1 \\ 1\ 1 \end{Bmatrix} = \frac{x^1}{(x^1)^2 + (x^2)^2} = \begin{Bmatrix} 2 \\ 1\ 2 \end{Bmatrix}; \quad \begin{Bmatrix} 1 \\ 2\ 2 \end{Bmatrix} = -\frac{x^1}{(x^1)^2 + (x^2)^2}$$
$$\begin{Bmatrix} 1 \\ 1\ 2 \end{Bmatrix} = \frac{x^2}{(x^1)^2 + (x^2)^2} = \begin{Bmatrix} 2 \\ 2\ 2 \end{Bmatrix}; \quad \begin{Bmatrix} 2 \\ 1\ 1 \end{Bmatrix} = -\frac{x^2}{(x^1)^2 + (x^2)^2}.$$

For the above calculated Christoffel symbols, system Eq. (4.43) becomes,

$$\frac{\partial u_1}{\partial x^1} = \frac{x^1 u_1 - x^2 u_2}{\sqrt{g}}, \quad \frac{\partial u_1}{\partial x^2} = \frac{x^2 u_1 + x^1 u_2}{\sqrt{g}}, \quad \frac{\partial u_1}{\partial x^3} = 0, \tag{i}$$

$$\frac{\partial u_2}{\partial x^1} = \frac{x^2 u_1 + x^1 u_2}{\sqrt{g}}, \quad \frac{\partial u_2}{\partial x^2} = \frac{-x^1 u_1 + x^2 u_2}{\sqrt{g}}, \quad \frac{\partial u_2}{\partial x^3} = 0, \tag{ii}$$

$$\frac{\partial u_3}{\partial x^1} = 0, \quad \frac{\partial u_3}{\partial x^2} = 0, \quad \frac{\partial u_3}{\partial x^3} = 0. \tag{iii}$$

Thus, u_1 and u_2 are functions of x^1, x^2 alone, and $u_3 =$ constant. Since the g_{ij} are all polynomials of degree 2 in x^1, x^2, use the method of undetermined coefficients, assuming polynomial forms

$$u_i = a_i(x^1)^2 + b_i x^1 x^2 + c_i(x^2)^2 + d_i x^1 + e_i x^2 + f_i; \quad i = 1, 2.$$

The compatibility relations $\frac{\partial u_1}{\partial x^2} = \frac{\partial u_2}{\partial x^1}$ implied by the second part of Eq. (i) and the first part of Eq. (ii) require,

$$b_1 = 2a_2; \quad 2c_1 = b_2; \quad e_1 = d_2.$$

Similarly, the compatibility relations $\frac{\partial u_1}{\partial x^1} = -\frac{\partial u_2}{\partial x^2}$ imply

$$2a_1 = -b_2; \quad b_1 = -2c_2; \quad d_1 = -e_2.$$

From the first part of Eq. (i), we get

$$a_1 = 0 = a_2; \;\; c_1 = b_2, b_1 = -c_2, d_1 = -e_2, f_1 = 0 = -f_2.$$
$$\Rightarrow b_1 = b_2 = c_1 = c_2 = 0.$$

Thus, the solution can be written in the form

$$u_1 = ax^1 + bx^2; \quad u_2 = bx^1 - ax^2; \quad u_3 = c,$$

where we are redenoting the constants d_1 and e_1. The first part of Eq. (4.43) gives

$$\frac{\partial \omega}{\partial x^1} = ax^1 + bx^2; \quad \frac{\partial \omega}{\partial x^2} = bx^1 - ax^2; \quad \frac{\partial \omega}{\partial x^3} = c,$$

or

$$\omega = \frac{a}{2}(x^1)^2 + bx^1x^2 - \frac{a}{2}(x^2)^2 + cx^3 + d$$

or

$$\overline{x}^k = \frac{a^k}{2}(x^1)^2 + b^k x^1 x^2 - \frac{a^k}{2}(x^2)^2 + c^k x^3 \text{ with } d = 0.$$

It is clear that we take $c^1 = c^2 = 0 = a^3 = b^3$ and $c^3 = 1$;

$$\overline{x}^1 = \frac{1}{2}a^1(x^1)^2 + b^1x^1x^2 - \frac{1}{2}a^1(x^2)^2$$
$$\overline{x}^2 = \frac{1}{2}a^2(x^1)^2 + b^2x^1x^2 - \frac{1}{2}a^2(x^2)^2; \;\; \overline{x}^3 = x^3.$$

Now, the Jacobian matrix J satisfies $J^TJ = G$. Here

$$J = \begin{bmatrix} a^1(x^1)^2 + b^1x^1x^2 & b^1x^1x^2 - \frac{1}{2}a^1(x^2)^2 & 0 \\ a^2(x^1)^2 + b^2x^1x^2 & b^2x^1x^2 - \frac{1}{2}a^2(x^2)^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow (a^1)^2 + (a^2)^2 = 1, a^1b^1 + a^2b^2 = 0, (b^1)^2 + (b^2)^2 = 1; \quad \text{as } J^TJ = G.$$

So, take $a^1 = 0, a^2 = 1, b^2 = 0, b^1 = 1$. Finally, the transformation is given by

$$\overline{x}^1 = x^1x^2; \;\; \overline{x}^2 = \frac{1}{2}[(x^1)^2 - (x^2)^2], \;\; \overline{x}^3 = x^3.$$

Deduction 4.2.3 Let the metric tensors g_{ij} are constants, then all the partial derivatives of g_{ij} are zero. Consequently, all Christoffel symbols will vanish and all $R_{ijkl} = 0$, i.e. $\kappa = 0$. Thus, by Theorem 4.2.2, the space is flat. If $\overline{x} = Ax$, then $J = A$ and

$$G = J^T \overline{G} J = A^T \overline{G} A.$$

However, since G is real and symmetric, its eigenvectors form an orthogonal matrix which we now choose as A, with

$$AGA^{-1} = AGA^T = D; \quad D = \text{ diagonal matrix}$$

of eigenvalues of G. Hence, $\overline{G} = AGA^T = D$. Hence, if the metric tensor is constant, the space is flat and the transformation $\overline{x} = Ax$, where A is a rank N matrix of eigenvalues $G = (g_{ij})$, diagonalises the metric, i.e. $\overline{g}_{ij} = 0$ for $i \neq j$.

EXAMPLE 4.2.11 *Find the signature of the flat metric*

$$ds^2 = 4(dx^1)^2 + 5(dx^2)^2 - 2(dx^4)^2 - 4dx^2dx^3 - 4dx^2dx^4 - 10dx^3dx^4.$$

Solution: For the given metric, we have $g_{11} = 4$; $g_{22} = 5, g_{33} = -2, g_{44} = 2, g_{23} = -2$, $g_{24} = -2$ and $g_{34} = -5$. Now, we find the eigenvalues of $G = (g_{ij})$. The characteristic equation is

$$|G - \lambda I| = \begin{vmatrix} 4-\lambda & 0 & 0 & 0 \\ 0 & 5-\lambda & -2 & -2 \\ 0 & -2 & -2-\lambda & -5 \\ 0 & -2 & -5 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow -(4-\lambda)(5-\lambda)(37-\lambda^2) = 0$$

$$\Rightarrow \lambda = 4, 5, \pm\sqrt{37}.$$

This means that there is a transformation which changes the metric into the form

$$\begin{aligned} ds^2 &= 4(dx^1)^2 + 5(dx^2)^2 + \sqrt{37}\left[(dx^3)^2 - (dx^4)^2\right] \\ &= (d\overline{x}^1)^2 + (d\overline{x}^2)^2 + (d\overline{x}^3)^2 - (d\overline{x}^4)^2, \end{aligned}$$

with the obvious change of co-ordinates. Hence, the signature is $(+ + + -)$, or some permutation thereof.

4.2.5 Projective Curvature Tensor

The *projective curvature tensor* or *Weyl tensor,* denoted by W_{hijk}, is defined by

$$W_{hijk} = R_{hijk} + \frac{1}{1-N}(g_{kl}R_{hj} - g_{kh}R_{ij}), \tag{4.47}$$

where R_{hijk} is the *Riemannian curvature tensor* and R_{ij} is the *Ricci tensor.*

Theorem 4.2.3 *A necessary and sufficient condition that a Riemannian space $V_N (N > 3)$ to be of constant Riemannian curvature is that the Weyl tensor vanishes identically throughout V_N.*

Proof: Let κ be the constant Riemannian curvature of V_N given by

$$\kappa = \frac{p^h q^i p^j q^k R_{hijk}}{p^h q^i p^j q^k \left[g_{hj} g_{ik} - g_{ij} g_{hk}\right]} = \text{ constant.}$$

Since κ is constant, it is independent of the orientation determined by p^i and q^j and hence

$$\kappa = \frac{R_{hijk}}{g_{hj} g_{ik} - g_{ij} g_{hk}} \Rightarrow R_{hijk} = (-g_{ij} g_{hk} + g_{hj} g_{ik})\kappa$$

or

$$g^{hk} R_{hijk} = g^{hk}(-g_{ij} g_{hk} + g_{hj} g_{ik})\kappa$$

or

$$R_{ij} = g^{hk}(-N g_{ij} + \delta^k_j g_{ik})\kappa = (1 - N) g_{ij} \kappa$$

or

$$g^{ij} R_{ij} = (1 - N) g^{ij} g_{ij} \kappa$$

or

$$R = (1 - N) N \kappa \Rightarrow R_{ij} = \frac{R}{N} g_{ij}. \tag{4.48}$$

Since κ is constant, Eq. (4.48) shows that R is constant. Now, the *Weyl tensor* W_{hijk} is given by

$$\begin{aligned} W_{hijk} &= R_{hijk} + \frac{1}{1-N}(g_{ki} R_{hj} - g_{kh} R_{ij}) \\ &= R_{hijk} + \frac{1}{1-N}\left[g_{ki} \frac{1}{N} R g_{hj} - g_{kh} \frac{1}{N} R g_{ij}\right] \\ &= R_{hijk} + \frac{R}{N(1-N)}\left[g_{ki} g_{hj} - g_{hk} g_{ij}\right] \\ &= R_{hijk} + \frac{R}{N(1-N)} \frac{1}{\kappa} R_{hijk} \\ &= R_{hijk} + \kappa \frac{1}{\kappa} R_{hijk} = 2 R_{hijk} = 0; \text{ as } \kappa \text{ is constant.} \end{aligned}$$

Thus, the condition is necessary. Conversely, let the *Weyl tensor* $W_{hijk} = 0$, we have to prove that $\kappa =$ constant. Since $W_{hijk} = 0$, we have

$$R_{hijk} + \frac{1}{1-N}\left[g_{ik} R_{hj} - g_{hk} R_{ij}\right] = 0$$

or

$$g^{hk}R_{hijk} + \frac{1}{1-N}\left[g^{hk}g_{ik}R_{hj} - g^{hk}g_{hk}R_{ij}\right] = 0$$

or

$$R_{ij} + \frac{1}{1-N}\left[\delta_i^h R_{hj} - NR_{ij}\right] = 0$$

or

$$R_{ij} + \frac{1}{1-N}\left[R_{ij} - NR_{ij}\right] = 0$$

or

$$R_{ij} + \frac{1-N}{1-N}R_{ij} = 0 \Rightarrow R_{ij} = 0$$

or

$$R_{ij} = g^{hk}R_{hijk} = 0 \Rightarrow \text{ either } g^{hk} = 0 \text{ or } R_{hijk} = 0.$$

If $R_{hijk} = 0$, then the Riemann curvature $\kappa = 0$. If $g^{hk} = 0$, then $g_{hk} = 0$ and hence,

$$\kappa = \frac{p^h q^i p^j q^k R_{hijk}}{p^h q^i p^j q^k \left[g_{hj}g_{ik} - 0g_{ij}\right]} = \frac{p^h q^i p^j q^k R_{hijk}}{p^h q^i p^j q^k g_{hj}g_{ik}}$$

$$= \frac{p^h q^i p^j q^k R_{hijk}}{\left(p^h p^j g_{hj}\right)\left(q^i q^k g_{ik}\right)}$$

$$= \frac{p^h q^i p^j q^k R_{hijk}}{p^2 q^2}; \; p^2 = 1 = g_{hj}p^h p^j; \;\; q^2 = 1 = g_{ik}q^i q^k$$

or

$$\kappa = p^h q^i p^j q^k R_{hijk} = \text{constant as } R_{ij} = 0.$$

Since 0 is constant, in either case κ is a constant. Thus, the condition is necessary.

4.2.6 Uniform Vector Field

The construction of a field of parallel vectors is possible only when the Riemann–Christoffel curvature tensor vanishes. Here, we have to prove that, when Riemann–Christoffel tensor vanishes the differential equations

$$A_{\mu,\nu} = \frac{\partial A_\mu}{\partial x^\nu} - \left\{ \begin{matrix} a \\ \mu \;\; \nu \end{matrix} \right\} A_a = 0$$

are integrable. Let A_μ be a covariant vector, then by definition

$$\frac{\partial A_\mu}{\partial x^\nu} - A_\alpha \;\; \left\{ \begin{matrix} \alpha \\ \mu \;\; \nu \end{matrix} \right\} = A_{\mu,\nu} = 0, \tag{4.49}$$

where we suppose that, $A_{\mu,\nu} = 0$. We want to show that Eq. (4.49) can be integrated only when curvature tensor $R^a_{ijk} = 0$. From Eq. (4.49) we get,

$$\frac{\partial A_\mu}{\partial x^\nu}\ dx^\nu = A_\alpha \begin{Bmatrix} \alpha \\ \mu\ \ \nu \end{Bmatrix} dx^\nu$$

or

$$\int \frac{\partial A_\mu}{\partial x^\nu}\ dx^\nu = \int A_\alpha \begin{Bmatrix} \alpha \\ \mu\ \ \nu \end{Bmatrix} dx^\nu$$

or

$$\int dA_\mu = A_\mu = \int A_\alpha \begin{Bmatrix} \alpha \\ \mu\ \ \nu \end{Bmatrix} dx^\nu. \tag{4.50}$$

This means that Eq. (4.49) is integrable only when the right hand side of Eq. (4.50) is integrable, for which the condition is $A_\alpha \begin{Bmatrix} \alpha \\ \mu\ \ \nu \end{Bmatrix} dx^\nu$ must be perfect differential. So, we can write,

$$A_\alpha \begin{Bmatrix} \alpha \\ \mu\ \ \nu \end{Bmatrix} dx^\nu = dB_\mu = \frac{\partial B_\mu}{\partial x^\nu} dx^\nu$$

or

$$\left[A_\alpha \begin{Bmatrix} \alpha \\ \mu\ \ \nu \end{Bmatrix} - \frac{\partial B_\mu}{\partial x^\nu}\right] dx^\nu = 0$$

or

$$A_\alpha \begin{Bmatrix} \alpha \\ \mu\ \ \nu \end{Bmatrix} = \frac{\partial B_\mu}{\partial x^\nu}; \quad \text{as } dx^\nu \text{ is arbitrary.} \tag{4.51}$$

Differentiating Eq. (4.51) with respect to x^σ, we get

$$\frac{\partial A_\alpha}{\partial x^\sigma} \begin{Bmatrix} \alpha \\ \mu\ \ \nu \end{Bmatrix} + A_\alpha \frac{\partial}{\partial x^\sigma} \begin{Bmatrix} \alpha \\ \mu\ \ \nu \end{Bmatrix} = \frac{\partial^2 B_\mu}{\partial x^\sigma \partial x^\nu}. \tag{4.52}$$

Interchanging ν and σ we get

$$\frac{\partial A_\alpha}{\partial x^\nu} \begin{Bmatrix} \alpha \\ \mu\ \ \sigma \end{Bmatrix} + A_\alpha \frac{\partial}{\partial x^\nu} \begin{Bmatrix} \alpha \\ \mu\ \ \sigma \end{Bmatrix} = \frac{\partial^2 B_\mu}{\partial x^\nu \partial x^\sigma}. \tag{4.53}$$

Subtracting Eq. (4.53) from Eq. (4.52), we get

$$\frac{\partial A_\alpha}{\partial x^\sigma} \begin{Bmatrix} \alpha \\ \mu\ \ \nu \end{Bmatrix} + A_\alpha\ \frac{\partial}{\partial x^\sigma} \begin{Bmatrix} \alpha \\ \mu\ \ \nu \end{Bmatrix} - \frac{\partial A_\alpha}{\partial x^\nu}\ \begin{Bmatrix} \alpha \\ \mu\ \ \sigma \end{Bmatrix} - A_\alpha \frac{\partial}{\partial x^\nu} \begin{Bmatrix} \alpha \\ \mu\ \ \sigma \end{Bmatrix} = 0$$

or

$$\left\{\begin{matrix} \alpha \\ \mu \ \ \nu \end{matrix}\right\} A_c \left\{\begin{matrix} c \\ \alpha \ \ \sigma \end{matrix}\right\} + A_\alpha \frac{\partial}{\partial x^\sigma} \left\{\begin{matrix} \alpha \\ \mu \ \ \nu \end{matrix}\right\} - \left\{\begin{matrix} \alpha \\ \mu \ \ \sigma \end{matrix}\right\} A_c \left\{\begin{matrix} c \\ \alpha \ \ \nu \end{matrix}\right\}$$

$$-A_c \frac{\partial}{\partial x^\nu} \left\{\begin{matrix} c \\ \mu \ \ \sigma \end{matrix}\right\} = 0 \text{ by Eq. (4.49)}$$

or

$$\left\{\begin{matrix} \alpha \\ \mu \ \ \nu \end{matrix}\right\} A_c \left\{\begin{matrix} c \\ \alpha \ \ \sigma \end{matrix}\right\} + A_c \frac{\partial}{\partial x^\sigma} \left\{\begin{matrix} c \\ \mu \ \ \nu \end{matrix}\right\} - \left\{\begin{matrix} \alpha \\ \mu \ \ \sigma \end{matrix}\right\} \left\{\begin{matrix} c \\ \alpha \ \ \nu \end{matrix}\right\} A_c - A_c \frac{\partial}{\partial x^\nu} \left\{\begin{matrix} c \\ \mu \ \ \sigma \end{matrix}\right\} = 0$$

or

$$\left[\left\{\begin{matrix} \alpha \\ \mu \ \ \nu \end{matrix}\right\} \left\{\begin{matrix} c \\ \alpha \ \ \sigma \end{matrix}\right\} + \frac{\partial}{\partial x^\sigma} \left\{\begin{matrix} c \\ \mu \ \ \nu \end{matrix}\right\} - \left\{\begin{matrix} \alpha \\ \mu \ \ \sigma \end{matrix}\right\} \left\{\begin{matrix} c \\ \alpha \ \ \nu \end{matrix}\right\} - \frac{\partial}{\partial x^\nu} \left\{\begin{matrix} c \\ \mu \ \ \sigma \end{matrix}\right\}\right] A_c = 0$$

or

$$\left[-\frac{\partial}{\partial x^\nu} \left\{\begin{matrix} c \\ \mu \ \ \sigma \end{matrix}\right\} + \frac{\partial}{\partial x^\sigma} \left\{\begin{matrix} c \\ \mu \ \ \nu \end{matrix}\right\} - \left\{\begin{matrix} \alpha \\ \mu \ \ \sigma \end{matrix}\right\} \left\{\begin{matrix} c \\ \alpha \ \ \nu \end{matrix}\right\} + \left\{\begin{matrix} \alpha \\ \mu \ \ \nu \end{matrix}\right\} \left\{\begin{matrix} c \\ \alpha \ \ \sigma \end{matrix}\right\}\right] A_c = 0$$

or

$$R^c_{\mu\sigma\nu} A_c = 0, \quad \text{i.e. } R^c_{\mu\sigma\nu} = 0; \text{ as } A_c \text{ is arbitrary.}$$

Thus, we see that the RHS of Eq. (4.50) is integrable only when the curvature tensor vanishes. Consequently, Eq. (4.49) is integrable only when $R^c_{\mu\sigma\nu} = 0$.

Now, we can carry the vector A_μ to any point by parallel displacement. This gives a unique result independent of the path of transfer. If a vector is displaced in this way we obtain a uniform vector field. Thus, the construction of uniform vector field is only possible when the curvature tensor vanishes.

4.3 Einstein Space

Einstein space is defined as a space which is homogeneous with regard to the Ricci tensor R_{ij}. Thus, a space for which

$$R_{ij} = \lambda g_{ij}; \quad \lambda = \text{invariant at all points} \tag{4.54}$$

at every point of the space, then that space is called *Einstein space.* Inner multiplication of g^{ij} shows that

$$g^{ij} R_{ij} = \lambda g^{ij} g_{ij} \Rightarrow R = \lambda N. \tag{4.55}$$

Thus for an Einstein space,

$$R_{ij} = \lambda g_{ij} = \frac{1}{N} R g_{ij}. \tag{4.56}$$

Hence, for an Einstein space Eq. (4.56) holds at every point of the space.

Theorem 4.3.1 *A space of constant curvature is an Einstein space.*

Proof: Let the Riemannian curvature κ at P of V_N for the orientation determined by p^i and q^i, is given in Eq. (4.34) as

$$\kappa = \frac{p^h q^i p^j q^k R_{hijk}}{p^h q^i p^j q^k \left[g_{hj} g_{ik} - g_{ij} g_{kh}\right]}.$$

Since κ is constant and independent of the orientation so,

$$\kappa = \frac{R_{hijk}}{g_{hj} g_{ik} - g_{ij} g_{kh}}$$

or

$$R_{hijk} = \kappa \left[g_{hj} g_{ik} - g_{ij} g_{kh}\right].$$

Multiplying by g^{hk}, we get

$$\kappa g^{hk} \left[g_{hj} g_{ik} - g_{ij} g_{kh}\right] = g^{hk} R_{hijk}$$

or

$$\kappa \left(\delta_i^h g_{hj} - N g_{ij}\right) = R_{ij}; \quad \text{since, } g^{hk} R_{hijk} = R_{ij}$$

or

$$\kappa \left(g_{ij} - N g_{ij}\right) = R_{ij}$$

or

$$\kappa \left(1 - N\right) g_{ij} g^{ij} = R_{ij} g^{ij}; \text{ multiplying by } g^{ij}$$

or

$$\kappa N(1 - N) = R; \quad \text{as } R_{ij} g^{ij} = R.$$

Therefore,

$$R_{ij} = (1 - N) g_{ij} \frac{R}{N(1 - N)} = \frac{R}{N} g_{ij}.$$

This is the necessary and sufficient condition for the space V_N to be Einstein space.

EXAMPLE 4.3.1 *If* $g^{ik} R_{kj} = R_j^i$ *and* $g^{ij} R_{ij} = R$, *show that* $R^i_{j,i} = \frac{1}{2} \frac{\partial R}{\partial x^j}$. *Hence, show that for an Einstein space, the scalar curvature R is constant.*

Solution: Here, we use the Bianchi identity, Eq. (4.20), $R_{pkjt,i} + R_{pkti,j} + R_{pkij,t} = 0$. The covariant differentiation of the relation $g^{ik} R_{kj} = R_j^i$ with respect to x^l is given by

$$R^i_{j,l} = g^{ik}_{,l} R_{kj} + g^{ik} R_{kj,l} = g^{ik} R_{kj,l}; \quad \text{as } g^{ik}_{,l} = 0.$$

Thus, the divergence of R^i_j is given by

$$\begin{aligned}
R^i_{j,i} &= g^{ik}R_{kj,i} = g^{ik}R^t_{kjt,i};\ \text{ as } R^t_{kit} = R_{kj}\\
&= g^{ik}\left(g^{pt}R_{pkjt}\right)_{,i};\quad \text{as } g^{pt}R_{pkjt} = R^t_{kjt}\\
&= g^{ik}g^{pt}R_{pkjt,i};\quad \text{as } g^{pt}_{,i} = 0\\
&= -g^{ik}g^{pt}\left[R_{pkti,j} + R_{pkij,t}\right] = -g^{pt}\left[g^{ik}R_{pkti,j} + g^{ik}R_{pkij,t}\right]\\
&= -g^{pt}\left[g^{ki}R_{kpti,j} + g^{ik}R_{kpji,t}\right]\\
&\quad\ \text{as } R_{kpti} = -R_{pkti} \text{ and } R_{kpji} = R_{pkij}\\
&= -g^{pt}\left[-R^i_{pti,j} + R^i_{pji,t}\right] = -g^{pt}\left[-R_{pt,j} + R_{pj,t}\right]\\
&= \left(g^{pt}R_{pt}\right)_{,j} - g^{pt}R_{pj,t} = R_{,j} - R^t_{j,t} = R_{,j} - R^i_{j,i}\\
&\quad\ \text{replacing the dummy index } t \text{ by } i
\end{aligned}$$

or

$$2R^i_{j,i} = R_{,j} = \frac{\partial R}{\partial x^j} \Rightarrow R^i_{j,i} = \frac{1}{2}\frac{\partial R}{\partial x^j}.$$

For an Einstein space, $R_{ij} = \frac{R}{N}g_{ij}$, therefore,

$$g^{ki}R_{ij} = \frac{R}{N}g_{ij}g^{ki} \Rightarrow R^k_j = \frac{R}{N}\delta^k_j$$

or

$$R^k_{j,k} = \frac{1}{N}R_{,k}\delta^k_j = \frac{1}{N}R_{,j}$$

or

$$\frac{1}{2}R_{,j} = \frac{1}{N}R_{,j} \Rightarrow R_{,j} = 0;\quad \text{for } N > 2.$$

Therefore, for an Einstein space, the scalar curvature R is constant.

EXAMPLE 4.3.2 *For a V_2 referred to an orthogonal system of parametric curves ($g_{12} = 0$), show that*

$$\begin{aligned}
R_{12} = 0,&\quad R_{11}g_{22} = R_{22}g_{11} = R_{1221}\\
R = g^{ij}R_{ij} = \frac{2R_{1221}}{g_{11}g_{22}},&\quad \textit{consequently, } R_{ij} = \frac{R}{2}g_{ij}.
\end{aligned}$$

Hence, show that V_2 is an Einstein space.

Solution: Given that, in a V_2, $g_{12} = 0$ so that $g^{12} = 0$. Also,

$$g^{ij} = \frac{1}{g_{ij}} \Rightarrow g^{11} = \frac{1}{g_{11}}, \quad g^{22} = \frac{1}{g_{22}}.$$

The metric of V_2 is given by

$$\begin{aligned} ds^2 &= g_{ij}dx^i dx^j; \quad i, j = 1, 2 \\ &= g_{11}(dx^1)^2 + g_{22}(dx^2)^2; \quad \text{as } g_{12} = 0. \end{aligned}$$

Also, we know that $g^{hk}R_{hijk} = R_{ij}$ and

$$g = \begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix} = \begin{vmatrix} g_{11} & 0 \\ 0 & g_{22} \end{vmatrix} = g_{11}g_{22}.$$

(a) Here, we have to show that $R_{12} = 0$. For this,

$$\begin{aligned} R_{12} &= g^{hk}R_{h12k} = g^{h1}R_{h121} + g^{h2}R_{h122} = g^{h1}R_{h121}; \quad \text{as } R_{h122} = 0 \\ &= g^{11}R_{1121} + g^{21}R_{2121} = 0; \quad \text{as } g^{12} = 0 \text{ and } R_{1121} = 0. \end{aligned}$$

(b) Using the relation $g^{hk}R_{hijk} = R_{ij}$ we get

$$R_{11} = g^{hk}R_{h11k} = g^{2k}R_{211k} = g^{22}R_{2112} = \frac{R_{2112}}{g_{22}}.$$

$$R_{22} = g^{hk}R_{h22k} = g^{1k}R_{122k} = g^{11}R_{1221} = \frac{R_{1221}}{g_{11}}.$$

Since $R_{2112} = R_{1221}$, it follows that, $R_{11}g_{22} = R_{1221} = R_{22}g_{11}$. Therefore, in a two-dimensional Riemannian space the components of a Ricci tensor are proportional to the components of a metric tensor.

(c) From the definition of R, we get

$$\begin{aligned} R &= g^{ij}R_{ij} = g^{i1}R_{i1} + g^{i2}R_{i2} \\ &= g^{11}R_{11} + g^{22}R_{22}; \quad \text{as } g^{12} = 0 \\ &= \frac{R_{11}}{g_{11}} + \frac{R_{22}}{g_{22}} = \frac{R_{1221}}{g_{22}g_{11}} + \frac{R_{1221}}{g_{11}g_{22}} \\ &= \frac{2R_{1221}}{g_{11}g_{22}}; \quad \text{as } R_{2112} = R_{1221}. \end{aligned}$$

(d) From (c), we see that

$$R = \frac{2R_{1221}}{g_{11}g_{22}} = \frac{2R_{1221}}{g}; \quad \text{as } g = g_{11}g_{22}$$
$$\Rightarrow R_{1221} = \frac{1}{2}Rg.$$

Therefore, the relation in (b) can be written in the form

$$R_{11}g_{22} = \frac{1}{2}Rg = R_{22}g_{11}$$

from which it follows that

$$R_{11} = \frac{Rg}{2g_{22}} = \frac{Rg_{11}g_{22}}{2g_{22}} = \frac{1}{2}Rg_{11}$$
$$R_{22} = \frac{Rg}{2g_{11}} = \frac{Rg_{11}g_{22}}{2g_{11}} = \frac{1}{2}Rg_{22}$$
$$R_{12} = \frac{1}{2}Rg_{12} = 0; \quad \text{as } g_{12} = 0.$$

Consequently, we get $R_{ij} = \frac{R}{2}g_{ij}$. This shows that V_2 is an Einstein space.

4.4 Mean Curvature

Let $\lambda^i_{h|}$ be the components of unit vector in a direction at a point P of a V_N. Let $\lambda^i_{k|}$ be the components of unit vector forming an orthogonal ennuple. One vector out of N vectors $\lambda^i_{k|}$ is $\lambda^i_{h|}$ and hence, we can say that $\lambda^i_{k|}$ are components of $N-1$ vectors where $h \neq k$ and $k = 1, 2, \ldots, N$.

Let the Riemannian curvature at P of V_N for the orientation determined by $\lambda^i_{h|}$ and $\lambda^i_{k|}$ $(k \neq h)$ be denoted by K_{hk} and is given by

$$K_{hk} = \frac{\lambda^p_{h|}\lambda^q_{k|}\lambda^r_{h|}\lambda^s_{k|}R_{pqrs}}{\lambda^p_{h|}\lambda^q_{k|}\lambda^r_{h|}\lambda^s_{k|}\left(g_{pr}g_{qs} - g_{ps}g_{qr}\right)}$$
$$= \frac{\lambda^p_{h|}\lambda^q_{k|}\lambda^r_{h|}\lambda^s_{k|}R_{pqrs}}{\left(\lambda^p_{h|}\lambda^r_{h|}g_{pr}\right)\left(\lambda^q_{k|}\lambda^s_{k|}g_{qs}\right) - \left(\lambda^p_{h|}\lambda^s_{k|}g_{ps}\right)\left(\lambda^q_{k|}\lambda^r_{h|}g_{qr}\right)}. \tag{4.57}$$

Since unit vectors $\lambda^i_{h|}$ and $\lambda^i_{k|}$ are orthogonal, so,

$$\lambda^p_{h|}\lambda^r_{h|}g_{pr} = 1 \text{ and } \lambda^p_{h|}\lambda^s_{k|}g_{ps} = 0.$$

Therefore, from Eq. (4.57), we get

$$K_{hk} = \frac{\lambda^p_{h|}\lambda^q_{k|}\lambda^r_{h|}\lambda^s_{k|}R_{pqrs}}{1.1 - 0.0} = \lambda^p_{h|}\lambda^q_{k|}\lambda^r_{k|}\lambda^s_{k|}R_{pqrs}. \tag{4.58}$$

Since the right-hand side of Eq. (4.58) vanishes for $h = k$, and so we say K_{hh} is equal to zero. Let $M_h = \sum_{k=1}^{N} K_{hk}$. Hence from Eq. (4.58) we have,

$$\begin{aligned}
\sum_{k=1}^{N} K_{hk} = M_h &= \sum_{k=1}^{N} \lambda^p_{h|}\lambda^q_{k|}\lambda^r_{h|}\lambda^s_{k|}R_{pqrs} \\
&= \lambda^p_{h|}\lambda^q_{h|}\sum_{k=1}^{N} \lambda^q_{k|}\lambda^s_{k|}R_{pqrs} \\
&= \lambda^p_{h|}\lambda^r_{h|}g^{qs}R_{pqrs};\ \text{as}\ \sum_{k=1}^{N}\lambda^q_{k|}\lambda^s_{k|} = g^{qs} \\
&= -\lambda^p_{h|}\lambda^r_{h|}g^{qs}R_{qprs} = -\lambda^p_{h|}\lambda^r_{h|}R^s_{pqs} = -\lambda^p_{h|}\lambda^r_{h|}R_{pr}.
\end{aligned} \tag{4.59}$$

This shows that M_h is independent of $(N-1)$ orthogonal direction choosen to complete an orthogonal ennuple. Here, M_h is defined as mean curvature or Riccian curvature of V_N for the direction $\lambda^p_{h|}$. Summing Eq. (4.58) over the N mutually orthogonal directions, we get

$$\sum_{h=1}^{N} M_h = -\sum_{h=1}^{N} \lambda^p_{h|}\lambda^r_{h|}R_{pr} = -g^{pr}R_{pr} = -R,$$

where R is the scalar curvature. This proves that the sum of mean curvatures for N mutual orthogonal directions is independent of the directions chosen to complete an orthogonal ennuple and has the value $-R$.

4.4.1 Ricci's Principal Directions

Let us suppose that $\lambda^i_{h|}$ is not a unit vector and therefore, the mean curvature M_h in this case is given by

$$M_h = -\frac{R_{ij}\lambda^i_{h|}\lambda^j_{h|}}{g_{ij}\lambda^i_{h|}\lambda^j_{h|}} \Rightarrow (R_{ij} + M_h g_{ij})\,\lambda^i_{h|}\lambda^j_{h|} = 0.$$

Differentiating this equation with respect to $\lambda^i_{h|}$, we get

$$\frac{\partial M_h}{\partial \lambda^i_{h|}} g_{jk}\lambda^j_{h|}\lambda^k_{h|} + 2\,(R_{ij} + M_h g_{ij})\,\lambda^j_{h|} = 0.$$

For extreme value of M_h, we must have $\frac{\partial M_h}{\partial \lambda^i_{h|}} = 0$, and so

$$(R_{ij} + M_h g_{ij})\, \lambda^j_{h|} = 0,$$

which are called *Ricci's principal direction* of the space and they are principal directions of Ricci tensor R_{ij}.

EXAMPLE 4.4.1 *For a two-dimensional manifold, prove that* $K_{pq} = -\frac{R}{2}$.

Solution: In V_2, we know R_{1212} is the only non-vanishing Riemann–Christoffel tensor given by

$$R_{1212} = \frac{1}{2}Rg; R = \text{ total curvature.}$$

Let the Riemannian curvature at P of V_N for the orientation determined by $\lambda^i_{p|}$ and $\lambda^i_{q|}$ $(q \neq p)$ be denoted by K_{pq} and from Eq. (4.57) we get,

$$\begin{aligned} K_{pq} &= \frac{\lambda^h_{p|}\lambda^i_{q|}\lambda^j_{p|}\lambda^k_{q|} R_{hijk}}{\lambda^h_{p|}\lambda^i_{q|}\lambda^j_{p|}\lambda^k_{q|}\,(g_{hj}g_{ik} - g_{hk}g_{ij})} \\ &= \frac{\lambda^1_{p|}\lambda^2_{q|}\lambda^1_{p|}\lambda^2_{q|} R_{1212}}{\lambda^1_{p|}\lambda^2_{q|}\lambda^1_{p|}\lambda^2_{q|}\,(g_{11}g_{22} - g_{12}g_{21})} = \frac{R_{1212}}{g_{12}^2 - g_{11}g_{22}} = \frac{R_{1212}}{-g} = -\frac{1}{2}R. \end{aligned}$$

4.5 Geodesics in a V_N

Consider a curve $\Gamma : x^i = x^i(t)$ in a V_N, an N-dimensional Riemannian space, on it and let $x^i(t)$ be the co-ordinates of a general point P on it. Let the points A and B of the path correspond, respectively, to the value t_1 and t_2 of the parameter.

Consider two points on the path close to each other with co-ordinates x^i and $x^i + dx^i$. Let the parameter take values t and $t + dt$ corresponding to these two points, respectively. The interval ds between the two points is given by

$$ds^2 = g_{ij}dx^i dx^j = g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}dt^2. \tag{4.60}$$

The total interval between A and B along the path therefore,

$$\begin{aligned} S &= \int_{t_1}^{t_2} \sqrt{g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}}dt \\ &= \int_A^B F(x^i, \dot{x}^i)dt, \text{ say; where } F(x^i, \dot{x}^i) = \sqrt{g_{ij}\dot{x}^i\dot{x}^j}. \end{aligned} \tag{4.61}$$

In order Γ is a geodesic, integral Eq. (4.61) should be stationary. In order that integral Eq. (4.61) is stationary, we have the Euler-Lagrange condition,

$$\frac{\partial F}{\partial \dot{x}^l} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}^l}\right) = 0$$

or

$$\frac{1}{2\dot{s}}\frac{\partial g_{ij}}{\partial x^l}\dot{x}^i\dot{x}^j - \left[-\frac{\ddot{s}}{\dot{s}}g_{li}\dot{x}^i + \frac{1}{\dot{s}}\frac{\partial g_{li}}{\partial \dot{x}^j}\dot{x}^j\dot{x}^i + \frac{1}{\dot{s}}g_{li}\ddot{x}^i\right] = 0$$

or

$$g_{li}\ddot{x}^i - \frac{\ddot{s}}{\dot{s}}g_{li}\dot{x}^i + [li, j]\dot{x}^i\dot{x}^j = 0. \tag{4.62}$$

Multiplying this relation by g^{lm} and perform summation over l, we get

$$\ddot{x}^m - \frac{\ddot{s}}{\dot{s}}\dot{x}^m + \left\{ \begin{matrix} m \\ i \ \ j \end{matrix} \right\}\dot{x}^i\dot{x}^j = 0. \tag{4.63}$$

Equation (4.63) of geodesic can be further simplified by taking $s = t$, so that $\dot{s} = 1$, $\ddot{s} = 0$, and therefore, Eq. (4.63) becomes

$$\frac{d^2x^i}{ds^2} + \left\{ \begin{matrix} i \\ j \ \ k \end{matrix} \right\}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0, \tag{4.64}$$

where s itself is chosen as the parameter. This, of course, represents a set of N coupled differential equations. Equation (4.64) is known as geodesic equation in V_N.

EXAMPLE 4.5.1 *Show that on the surface of a sphere all great circles are geodesics while no other circle is a geodesic.*

Solution: The components of the metric tensor on the surface of a sphere of radius a are $g_{11} = a^2, g_{22} = a^2\sin^2\theta, g_{12} = g_{21} = 0$ and $g = a^4\sin^2\theta$ so that the metric is $ds^2 = a^2(d\theta)^2 + a^2\sin^2\theta(d\phi)^2$. The non-vanishing Christoffel symbols are

$$\left\{ \begin{matrix} 2 \\ 1 \ \ 2 \end{matrix} \right\} = \cot\theta = \left\{ \begin{matrix} 2 \\ 2 \ \ 1 \end{matrix} \right\}; \ \left\{ \begin{matrix} 1 \\ 2 \ \ 2 \end{matrix} \right\} = -\sin\theta\cos\theta.$$

Thus, the geodesic equation reduces to (with $\theta = x^1, \phi = x^2$)

$$\frac{d^2\theta}{ds^2} - \sin\theta\cos\theta\left(\frac{d\phi}{ds}\right)^2 = 0 \tag{i}$$

$$\frac{d^2\phi}{ds^2} + 2\cot\theta\frac{d\theta}{ds}\frac{d\phi}{ds} = 0. \tag{ii}$$

(a) Consider a great circle on the surface of the sphere and choose the z axis ($\theta = 0$) to be normal to the plane of the circle, so that the great circle is the equator. The parametric equation of the great circle is

$$\theta = \frac{\pi}{2}, \quad \phi = cs + d; \quad c \neq 0, \tag{iii}$$

where c and d are independent of s, θ and ϕ. It is evident that Eq. (iii) satisfy Eqs. (i) and (ii), so that the great circle is a geodesic. Since the choice of the polar axis $\theta = 0$ is arbitrary, it follows that any great circle is a geodesic.

(b) Consider a circle on the sphere such that the plane of the circle does not pass through the centre of the sphere. Again, choose the direction $\theta = 0$ to be normal to the plane of the circle so that the parametric equation of the circle is

$$\theta = \theta_0, \ \phi = ps + q; \ \theta_0 \in (0, \pi) - \left\{\frac{\pi}{2}\right\}; \ p \neq 0, \tag{iv}$$

where p and q are independent of s, θ and ϕ. On substitution, it is seen that Eq. (ii) is satisfied but Eq. (i) is reduces to

$$p^2 \sin\theta_0 \cos\theta_0 = 0.$$

The conditions on θ_0 and p of Eq. (iv) show that this equation cannot be satisfied, proving that the circle is not a geodesic.

Now, we find the lines of shortest distance of the sphere. From Eq. (4.61), we have,

$$S = a\int_A^B F(x^i, \dot{x}^i)dt,$$

where

$$F(x^i, \dot{x}^i) = \sqrt{g_{ij}\dot{x}^i\dot{x}^j} = \sqrt{1 + \dot{\phi}^2 \sin^2\theta}.$$

In order Γ is a geodesic, then Euler–Lagrange equation reduces to,

$$F\dot{\phi} = \text{constant} \ \Rightarrow \frac{\dot{\phi}\sin^2\theta}{\sqrt{1 + \dot{\phi}^2\sin^2\theta}} = c$$

$$\Rightarrow \dot{\phi}^2 = \frac{c^2}{\sin^4\theta - c^2\sin^2\theta} = \frac{c^2 \text{cosec}^4\theta}{(1 - c^2) - c^2\cot^2\theta}$$

$$\Rightarrow \dot{\phi} = \frac{k \ \text{cosec}^2\theta}{\sqrt{1 - k^2\cot^2\theta}}; k^2 = \frac{c^2}{1 - c^2}$$

$$\Rightarrow \phi = \alpha - \sin^{-1}(k\cot\theta); \quad \text{i.e.}\ \sin(\alpha - \phi) = k\cot\theta.$$

Let us verify these curves are great circles. Multiplied last equation by $\alpha \sin\theta$, we get

$$(\alpha \sin\theta \cos\phi) \sin\alpha - (\alpha \sin\theta \sin\phi) \cos\alpha = k\alpha \cos\theta$$

or

$$x \sin\alpha - y \cos\alpha = kz.$$

This represents a plane through the centre of the sphere, and the section of the sphere by this plane is a great circle.

4.5.1 Parallel Vector Fields

In an Euclidean space, a vector field $A_i(x^j)$ is said to be a field of parallel vectors if the components A_i are constants, i.e. $A_{i,j} = 0$, i.e. or, suppose that a vector A_i defined at a point x^j undergoes a displacement dx^j to the point $x^j + dx^j$, where, the vector becomes $A_i + \partial A_i$. We shall say that A_i undergoes a parallel displacement if $\frac{\partial A_i}{\partial x^j} = 0$. For example, a uniform electric field is a field of parallel vectors; the gravitational field of the earth in a sufficiently small region of space is a field of parallel vectors.

Here, our object is to generalise the concept of parallel displacement of surface vectors and tensors in Riemannian space, due to Levi Civita. Let us consider a space curve Γ, total of points, whose co-ordinates satisfy equation of the form

$$\Gamma: \; x^i = x^i(t), \tag{4.65}$$

where x^i (belongs to the class C^1) are functions of a single parameter t. Let a vector A be localised at some point P of Γ whose components A^i are functions of t. If we construct at every point of Γ, a vector equal to A in magnitude and parallel to it in direction, we obtain what is known as a parallel field of vectors along the curve Γ. We have shown in Chapter 3 that if A_i is a vector, its co-ordinate derivative $\frac{\partial A_i}{\partial x^j}$ is not a tensor, but the intrinsic derivative $\frac{\delta A_i}{\delta t}$ along the curve Eq. (4.65) is a tensor. To retain the tensor character, therefore, we must define the parallel displacement of a tensor in a Riemannian space differently. If A^i is a solution of the system of differential equation

$$\frac{\delta A^i}{\delta t} = \frac{dA^i}{dt} + \left\{ \begin{matrix} i \\ j \;\; k \end{matrix} \right\} \; A^j \frac{dx^k}{ds} \tag{4.66}$$

then the vectors A^i are said to form a parallel vector field along the curve Γ given by Eq. (4.65). We say that if the vector A is a parallel field along Γ, then the vectors A do not change along the curve. Thus, the vector A suffers a parallel displacement along Γ if the intrinsic derivative of the tensor along the curve vanishes, (Figure 4.1), i.e.

$$A^i_{,j} \frac{dx^i}{dt} = \frac{\delta A^i}{\delta t} = 0. \tag{4.67}$$

Conversely, we can show that every solution of system Eq. (4.67) yields a parallel vector field along Γ. Indeed, from the theory of differential equations it is known that

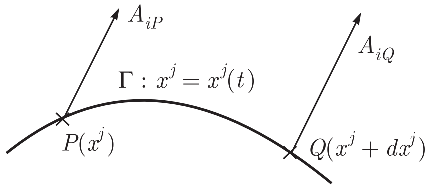

Figure 4.1: Parallel vectors A_{iP} and A_{iQ}.

this system of first order differential equations has a unique solution when the values of the components A^i are specified at a point of Γ. But the vector field formed by constructing a family of vectors of fixed lengths, parallel to a given vector, satisfies the system. Hence, every solution of Eq. (4.67) satisfies the initial conditions must form a parallel field along Γ.

In other words, we say that A^i is parallely displaced with respect to Riemannian space along the curve Γ, if Eq. (4.67) holds at all points of Γ. Thus, we have a field of parallel vectors, if,

$$A^i_{,j}\frac{dx^i}{dt} = 0 \Rightarrow g_{ik}\left(A^i_{,j}\frac{dx^j}{dt}\right) = 0 \Rightarrow \left(g_{ik}A^i\right)_{,j}\frac{dx^j}{dt} = 0$$

$$\Rightarrow A_{k,j}\frac{dx^j}{dt} = 0; \text{i.e.} \quad A_{i,j}\frac{dx^j}{dt} = 0$$

or

$$\left[\frac{dA^i}{dx^j} + \left\{ \begin{matrix} i \\ \alpha \ \ j \end{matrix} \right\} A^\alpha\right]\frac{dx^j}{dt} = 0$$

or

$$\frac{dA^i}{dt} + \left\{ \begin{matrix} i \\ \alpha \ \ j \end{matrix} \right\} A^\alpha \frac{dx^j}{dt} = 0$$

or

$$dA^i = -\left\{ \begin{matrix} i \\ j \ \ k \end{matrix} \right\} A^j dx^k \tag{4.68}$$

along the curve. The converse follows, as previously, for the existence and uniqueness of solution of such systems of differential equations. Similarly, the condition for a parallel displacement of a covariant vector A_i is

$$dA_i = \left\{ \begin{matrix} j \\ i \ \ k \end{matrix} \right\} A_j dx^k. \tag{4.69}$$

Also, it follows from the observation that $A_{i,k} = g_{i\alpha}A^{\alpha}_{,k}$ whenever $A_i = g_{ij}A^j$. Thus, the increments in A^i and A_i due to a displacement dx^j along the curve are given by Eqs. (4.68) and (4.69), respectively.

Result 4.5.1 Now, consider a point $P(x^i)$ and two displacements dx^i and δx^i from P giving rise to the points $Q(x^i + dx^i)$ and $R(x^i + \delta x^i)$, respectively, as seen in Figure 4.2.

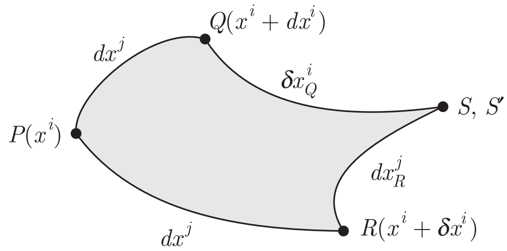

Figure 4.2: Generalised infinitesimal parallelogram.

Imagine the segment $PR = \delta x^i$ being parallely displaced such that as the end point P goes to Q, the other end R goes to S. The displacement QS will be

$$QS \equiv \delta x^i_Q = \delta x^i + d(\delta x^i), \tag{4.70}$$

which makes the co-ordinates of the points S,

$$S \equiv \left[x^i + dx^i + \delta x^i_Q\right] = \left[x^i + dx^i + \delta x^i + d(\delta x^i)\right], \tag{4.71}$$

where by Eq. (4.68), we have

$$d(\delta x^i) = -\left\{ \begin{matrix} i \\ j \quad k \end{matrix} \right\}_P \delta x_j dx^k. \tag{4.72}$$

The subscript P on the Christoffel symbol means is to be evaluated at P. Similarly, imagine the segment $PQ = dx^i$ being parallely displaced such that as the point P goes to R, the other end Q goes to S'. The displacement RS' will be given by

$$RS = dx^i_R = \delta x^i + d(\delta x^i), \tag{4.73}$$

which makes the co-ordinates of the point S'

$$S' \equiv \left[x^i + dx^i + \delta x^i_R\right] = \left[x^i + \delta x^i + dx^i + \delta(dx^i)\right], \tag{4.74}$$

$$\text{where,} \quad \delta(dx^i) = -\left\{ \begin{matrix} i \\ j \quad k \end{matrix} \right\}_P dx_j \delta x^k = -\left\{ \begin{matrix} i \\ j \quad k \end{matrix} \right\}_P dx_k \delta x^j, \tag{4.75}$$

which follows by interchanging j and k and using the symmetric property of the Christoffel symbol. Equations (4.72) and (4.75) show that $d(\delta x^i) = \delta(dx^i)$ indicating that the point S' coincides with S. The closed figure $PQRS$ can be regarded as the generalised infinitesimal parallelogram with opposite sides parallel to each other in the Riemannian space.

Result 4.5.2 (**Curvature of Riemannian space**). Here we have introduced the curvature tensor by considering the parallel displacement of a vector along two different paths. Consider an infinitesimal parallelogram PQRS (Figure 4.3) as described with adjacent sides $PQ = dx^i$ and $PR = \delta x^i$. Imagine a constravariant vector A_p^i defined at the point P being parally displaced is the following two ways:

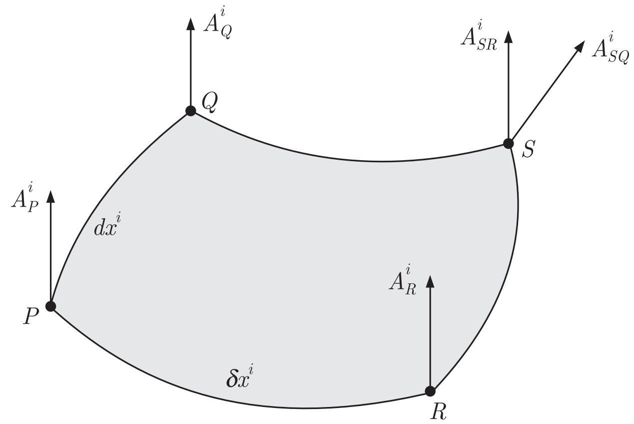

Figure 4.3: Curvature of Riemannian space.

(i) Displace A_p^i parallely from P to Q resulting A_Q^i and then displace A_Q^i parallely from Q to S giving the vector A_{SQ}^i.

(ii) Displace A_p^i parallely from P to R giving A_R^i parallely from R to S resulting the vector A_{SR}^i.

The vector A_Q^i obtained by a parallel displacement of A_P^i is given by

$$A_Q^i = A_P^i + dA_P^i; \quad \text{where } dA_P^i = -\begin{Bmatrix} i \\ j \;\; k \end{Bmatrix}_P A_P^i dx^k. \tag{i}$$

This vector is parallely displaced from Q to S resulting the vector A_{SQ}^i which will be given by

$$A_{SQ}^i \;=\; A_Q^i \;+\; \delta A_Q^i \;=\; A_Q^i \;-\; \begin{Bmatrix} i \\ l \;\; h \end{Bmatrix}_Q A_Q^l \,\delta x^h. \tag{ii}$$

The Christoffel symbol depends on the metric tensor which in turn is a function of co-ordinates. For a small displacement, we can therefore write

$$\begin{Bmatrix} i \\ l \quad h \end{Bmatrix}_Q = \begin{Bmatrix} i \\ l \quad h \end{Bmatrix}_P + \begin{Bmatrix} i \\ l \quad h \end{Bmatrix}_{P,m} dx^m \tag{iii}$$

where $\left\{ \begin{matrix} i \\ l \quad h \end{matrix} \right\}_{P,m} = \left[\frac{\partial}{\partial x^m} \left\{ \begin{matrix} i \\ l \quad h \end{matrix} \right\} \right]_{at\ P}$. Substituting Eqs. (iii) and (i) in Eq. (ii), we have

$$A^i_{SQ} = A^i_P - \left\{ \begin{matrix} i \\ j \quad k \end{matrix} \right\}_P A^j_P\, dx^k - \left[\left\{ \begin{matrix} i \\ l \quad h \end{matrix} \right\}_P + \left\{ \begin{matrix} i \\ l \quad h \end{matrix} \right\}_{P,m} dx^m \right]$$

$$\left[A^l_P - \left\{ \begin{matrix} l \\ j \quad k \end{matrix} \right\}_P A^j_P\, dx^k \right] \delta x^h$$

$$= A^i - \left\{ \begin{matrix} i \\ j \quad k \end{matrix} \right\} A^j\, dx^k - \left\{ \begin{matrix} i \\ l \quad h \end{matrix} \right\} A^l \delta x^h - \left\{ \begin{matrix} i \\ l \quad h \end{matrix} \right\}_{,m} A^l\, dx^m\, \delta x^h$$

$$+ \left\{ \begin{matrix} i \\ l \quad h \end{matrix} \right\} \left\{ \begin{matrix} l \\ j \quad k \end{matrix} \right\} A^j\, dx^k\, dx^h + O\left[(dx^k)^3 \right], \qquad (iv)$$

where we have dropped the subscript P from the right hand side of (iv) because all the quantities are to be evaluated at P. The vector A^i_{SR} obtained by a parallel displacement along the path PRS will be simply given by an interchange of dx^i and dx^i in the expression for A^i_{SQ}:

$$A^i_{SR} = A^i - \left\{ \begin{matrix} i \\ j \quad k \end{matrix} \right\} A^j\, \delta x^k - \left\{ \begin{matrix} i \\ l \quad h \end{matrix} \right\} A^l dx^h - \left\{ \begin{matrix} i \\ l \quad h \end{matrix} \right\}_{,m} A^l\, \delta x^m\, dx^h$$

$$+ \left\{ \begin{matrix} l \\ l \quad h \end{matrix} \right\} \left\{ \begin{matrix} l \\ j \quad k \end{matrix} \right\} A^j \delta x^k\, dx^h + O\left[(dx^k)^3 \right]. \qquad (v)$$

Subtracting (iv) from (v) we get

$$A^i_{SR} - A^i_{SR} = \left\{ \begin{matrix} i \\ l \quad h \end{matrix} \right\}_{,m} A^l\, dx^m\, \delta x^h - \left\{ \begin{matrix} i \\ l \quad h \end{matrix} \right\}_{,m} A^l\, \delta x^m\, dx^h$$

$$+ \left\{ \begin{matrix} i \\ l \quad h \end{matrix} \right\} \left\{ \begin{matrix} l \\ j \quad k \end{matrix} \right\} A^j\, \delta x^k\, dx^h - \left\{ \begin{matrix} i \\ l \quad h \end{matrix} \right\} \left\{ \begin{matrix} l \\ j \quad k \end{matrix} \right\} A^j\, dx^k\, \delta x^h. \qquad (vi)$$

Replacing l by j and m by k in the first two terms on the right hand side of (vi) and interchanging k and h in the second and third terms, the Eq. (vi) can be written as

$$A^i_{SR} - A^i_{SQ} = R^i_{jhk}\, A^j\, dx^k\, \delta x^h, \qquad \text{(vii)}$$

where

$$R^i_{jkh} = \left\{ \begin{matrix} i \\ j \quad h \end{matrix} \right\}_{,k} - \left\{ \begin{matrix} i \\ j \quad k \end{matrix} \right\}_{,h} + \left\{ \begin{matrix} i \\ l \quad k \end{matrix} \right\} \left\{ \begin{matrix} i \\ j \quad h \end{matrix} \right\} - \left\{ \begin{matrix} i \\ l \quad h \end{matrix} \right\} \left\{ \begin{matrix} l \\ j \quad k \end{matrix} \right\} \qquad (viii)$$

Since A^j, dx^k and δx^h are arbitrary vectors, if follows from quotient law that R^i_{jkh} is a tensor of rank four, known as the Riemann-Christoffel curvature tensor. It is independent of the vector A^i and depends only on the metric tensor and its first and second derivatives.

It follows that if a tensor is parallely displaced along a closed curve until we come back to the starting point, the resulting vector will not necessarily be the same as the original vector in a non-Euclidean space.

EXAMPLE 4.5.2 *If A^i and B^i are two vectors of constant magnitudes and undergo parallel displacements along a curve then show that they are inclined at a constant angle.*

Solution: Let the two vectors A^i and B^i be of constant magnitudes, so that their lengths do not change as we move along the curve Γ. Let these vectors suffer parallel displacement along a curve Γ, so by hypothesis

$$\frac{\delta A^i}{\delta t} = 0 = \frac{\delta B^i}{\delta t},$$

at each point of Γ. The angle θ between the vectors A^i and B^i, is given by

$$\cos\theta = g_{ij}A^iB^j.$$

The angle θ between the vectors A^i and B^i remains fixed as the parameter t is allowed to change. Here, $g_{ij}A^iB^j$ is an invariant. Differentiating intrinsically with respect to t, we get

$$\begin{aligned}\frac{\delta}{\delta t}(\cos\theta) &= \frac{\delta}{\delta t}\left(g_{ij}A^iB^j\right) = \frac{d}{dt}\left(g_{ij}A^iB^j\right)\\ &= g_{ij}\frac{\delta A^i}{\delta t}B^j + g_{ij}A^i\frac{\delta B^j}{\delta t} = g_{ij}\cdot 0\cdot B^j + g_{ij}A^i\cdot 0 = 0\end{aligned}$$

or

$$\frac{d}{dt}(\cos\theta) = 0 \Rightarrow \cos\theta = \text{constant, i.e. } \theta = \text{constant.}$$

Thus, we conclude that $g_{ij}A^iB^j$ is constant along Γ. It follows directly from the result that, if $A^i = B^i$, then $g_{ij}A^iA^j = A^2$ is constant along Γ and this implies $\theta =$ constant.

EXAMPLE 4.5.3 *Prove that geodesics are autoparallel curves.*

Solution: To prove the result, it is enough to prove that the unit tangent vector, say λ^i, suffers a parallel displacement along a geodesic curve Γ. Let $\Gamma : x^i = x^i(t)$ be a geodesic in a Riemannian space. Then the co-ordinates of the points on the geodesic satisfy the geodesic equation,

$$\frac{d^2x^i}{dt^2} + \left\{ \begin{matrix} i \\ j \;\; k \end{matrix} \right\} \frac{dx^j}{dt}\frac{dx^k}{dt} = 0$$

or

$$\frac{d\lambda^i}{dt} + \left\{ \begin{matrix} i \\ j \;\; k \end{matrix} \right\} \lambda^j \frac{dx^k}{dt} = 0$$

or

$$\left[\frac{\partial \lambda^i}{\partial x^k} + \left\{ \begin{matrix} i \\ j \;\; k \end{matrix} \right\} \lambda^j \right] \frac{dx^k}{dt} = 0$$

or

$$\lambda^i_{,k}\frac{dx^k}{dt} = 0; \text{ i.e. } \lambda^i_{,j}\frac{dx^j}{dt} = 0,$$

at each point on Γ. This result shows that the unit vectors form a system of parallel vectors along a geodesic. Hence, it follows from the definition of parallel displacement that the field of tangent vectors to a geodesic is a field of parallel vectors.

EXAMPLE 4.5.4 *Any vector of constant magnitude which undergoes a parallel displacement along a geodesic is inclined at a constant angle to the curve.*

Solution: Let a vector A^i of constant magnitude undergo a parallel displacement along a geodesic Γ, so that,

$$\frac{\delta A^i}{\delta t} = A^i_{,j}\frac{dx^j}{dt} = 0$$

at each point of Γ. Let λ^i be a unit tangent vector to the curve Γ, so that,

$$\frac{\delta \lambda^i}{\delta t} = \lambda^i_{,j}\frac{dx^j}{dt} = 0$$

at each point of Γ, since geodesics are autoparallel curves. Let θ be the angle between the vector λ^i and a curve Γ at any point of Γ, i.e. θ is the angle between A^i and λ^i, then,

$$A^i\lambda_i = A \cdot 1 \cdot \cos\theta$$

or

$$\frac{d}{dt}(A\cos\theta) = \frac{d}{dt}(A^i\lambda_i) = (A^i\lambda_i)_{,j}\frac{dx^i}{dt}$$

or

$$-A\sin\theta\frac{d\theta}{dt} = A^i_{,j}\frac{dx^j}{dt}\lambda_i + A^i\lambda_{i,j}\frac{dx^j}{dt}$$
$$= 0t_1 + A^i 0 = 0.$$

or

$$\sin\theta \frac{d\theta}{dt} = 0; \quad \text{as } A \neq 0,$$

or

$$\sin\theta = 0 \quad \text{or } \frac{d\theta}{dt} = 0 \Rightarrow \theta = 0 \quad \text{or } \theta = \text{ constant.}$$

EXAMPLE 4.5.5 *If the intrinsic derivative of a vector vanishes, then show that the magnitude of the vector is constant.*

Solution: Let the intrinsic derivative of a vector A^i be zero so that

$$\frac{\delta A^i}{\delta t} = A^i_{,j}\frac{dx^j}{dt} = 0.$$

We have to show that $A^2 = A^i A_i$ is constant. Differentiating both sides with respect to a parameter t and noting that the intrinsic derivative of a scalar is ordinary derivative, we have

$$\begin{aligned}\frac{dA^2}{dt} &= \frac{d}{dt}\left(A^i A_i\right) = \left(A^i A_i\right)_{,j}\frac{dx^j}{ds}\\ &= A^i_{,j}\frac{dx^j}{dt}A_i + A^i A_{i,j}\frac{dx^j}{ds} = 0A_i + A^i 0 = 0\end{aligned}$$

or

$$A^2 = \text{constant } \Rightarrow A = \text{constant.}$$

Thus, if the derived vector of a given vector A in the direction of a given curve Γ vanishes at all points of a curve, then the magnitude of the vector is constant. If A^i forms a field of parallel vectors along the curve $x^i = x(t)$, then we have

$$\frac{\delta A^i}{\delta t} = A^i_{,j}\frac{dx^j}{dt} = 0.$$

Then similarly, as above we can show that $A =$ constant. Thus, the magnitude of a vector remain invariant under parallel displacement.

EXAMPLE 4.5.6 *Show that the parallel displacement of a vector taken all around a circle on the surface of a sphere does not lead back to the same vector except when the circle is a great circle, that is, a geodesic.*

Solution: Choose spherical polar co-ordinates $x^1 = \theta$ and $x^2 = \phi$ on the surface of the sphere. The non-vanishing Christoffel symbols are,

$$\left\{\begin{matrix} 2 \\ 1\ \ 2 \end{matrix}\right\} = \cot\theta = \left\{\begin{matrix} 2 \\ 2\ \ 1 \end{matrix}\right\}; \ \left\{\begin{matrix} 1 \\ 2\ \ 2 \end{matrix}\right\} = -\sin\theta\cos\theta.$$

Choose a small circle normal to the polar axis so that its parametric equation is

$$\theta = \alpha, \phi = ps + q; \quad p \neq 0 \text{ with } \frac{d\theta}{dt} = 0.$$

Consider a vector $A^i = (a^1, A^2)$ at a point on the circle which may be taken to be $\phi = 0$. When the vector is displaced parallely along the circle, the change in its components is given, from Eq. (4.69), by

$$\frac{dA^1}{dt} = -\left\{ \begin{matrix} 1 \\ j \quad k \end{matrix} \right\} A^j \frac{dx^k}{dt} = A^2 \sin\alpha \cos\alpha \frac{d\phi}{dt} \tag{i}$$

$$\frac{dA^2}{dt} = -\left\{ \begin{matrix} 2 \\ j \quad k \end{matrix} \right\} A^j \frac{dx^k}{dt} = -A \cot\alpha \frac{d\phi}{dt}. \tag{ii}$$

Now, for any function f of θ and ϕ through t, we have

$$\frac{df}{dt} = \frac{\partial f}{\partial \theta}\frac{d\theta}{dt} + \frac{\partial f}{\partial \phi}\frac{d\phi}{dt} = \frac{\partial f}{\partial \phi}\frac{d\phi}{dt},$$

which reduces Eqs. (i) and (ii) to

$$\frac{\partial A^1}{\partial \phi} = A^2 \sin\alpha\cos\alpha; \quad \frac{\partial A^2}{\partial \phi} = -A^1 \cot\alpha. \tag{iii}$$

Differentiating once again with respect to ϕ and substituting one in the other, Eq. (iii) give rise to the decoupled equations

$$\frac{\partial^2 A^1}{\partial \phi^2} = -A^1 \cos^2\alpha; \quad \frac{\partial^2 A^2}{\partial \phi^2} = -A^2 \cos^2\alpha, \tag{iv}$$

whose solution, consistent with Eq. (iii), is

$$A^1 = \sin\alpha\,[c\cos(\phi\cos\alpha) + d\sin(\phi\cos\alpha)]$$
$$A^2 = d\cos(\phi\cos\alpha) - c\sin(\phi\cos\alpha).$$

At $\phi = 0$, the components of the vector are given by

$$A^1(\phi = 0) = c\sin\alpha, \quad A^2(\phi = 0) = d, \tag{v}$$

whereas, after parallel displacement once around the circle, the components become,

$$A^1(\phi = 2\pi) = \sin\alpha\,[c\cos(2\pi\cos\alpha) + d\sin(2\pi\cos\alpha)]$$
$$A^2(\phi = 2\pi) = d\cos(2\pi\cos\alpha) - c\sin(2\pi\cos\alpha),$$

which are not the same as those given by Eq. (v) except when $\alpha = \frac{\pi}{2}$, i.e. when the displacement is along the geodesic.

4.6 Exercises

1. Obtain the geodesic equations in cylindrical co-ordinates and spherical polar co-ordinates for the three-dimensional Euclidean space.
2. Consider a great circle on the surface of a sphere and choose the polar axis to lie in the plane of the great circle is a meridian. With this choice of co-ordinates, show that any meridian on a sphere is a geodesic.
3. Show that parallel displacement of a vector along any closed curve in an Euclidean space leads back to the original vector.
4. Prove that cyclic property of the covariant curvature tensor by setting a geodesic co-ordinate system.
5. Verify that, $R_{ijkl,m} + R_{ijlm,k} + R_{ijmk,l} = 0$.
6. In a two-dimensional Riemannian space, the components of the Ricci tensor are proportional to the components of the metric tensor.
7. Given the relation
$$A^i_j = R^i_j + \delta^i_j(\lambda R + \mu)$$
where λ, μ are constants. Show that for the value of $\lambda = -\frac{1}{2}$, $A^i_{j,i} = 0$.
8. Let in a V_2 space the line element is $ds^2 = 2f(x^1, x^2)dx^1dx^2$. Find R_{1212} for the space. Prove that the space is flat, if
$$f\frac{\partial^2 f}{\partial x^1 \partial x^2} = \frac{\partial f}{\partial x^1}\frac{\partial f}{\partial x^2}.$$
Show also if f is independent of at least one of the two co-ordinates x^1 and x^2, then the space is flat.
9. Prove that a necessary and sufficient condition for a Riemannian $V_N (N > 2)$, to be of constant Riemannian curvature is that the Weyl tensor vanishes identically throughout V_N.
10. Prove that the mean curvature in the direction λ_i at a point of a V_N is the sum of $N-1$ Riemmanian curvatures along the direction pairs consisting of the direction and $N-1$ other directions forming with this directions an orthogonal frame.
11. Prove that the sum of mean curvatures of a V_N for a mutually orthogonal directions at a point, is independent of the ennuple chosen. Obtain the value of this sum.
12. Prove that a Riemannian metric (g_{ij}) is the Euclidean metric if and only if the Riemannian curvature κ is zero at all points and the metric is positive definite.
13. (a) Calculate the Riemannian curvature for the metric
$$ds^2 = (dx^1)^2 - 2x^1(dx^2)^2.$$

(b) Show that the components R_{1212} of the curvature tensor for a two-dimensional space with metric $ds^2 = dx^2 + f(x,y)dy^2$ equals

$$-\frac{1}{2}\frac{\partial^2 f}{\partial x^2} + \frac{1}{4f}\left(\frac{\partial f}{\partial x}\right)^2.$$

(c) A two-dimensional space have the metric

$$ds^2 = f(u,v)du^2 + h(u,v)dv^2.$$

Find the components R_{1212} of the curvature tensor for the space.

14. Show that the Riemannian curvature at any point (x^i) of Riemannian 3 space in the direction $P = (0,1,0)$ and $Q = (1,1,0)$ is $\frac{1}{4}(x^1)^{-2}$, if the metric is given by

$$g_{11} = 1;\ g_{22} = 2x^1;\ g_{33} = 2x^2;\ g_{ij} = 0,\ \text{for } i \neq j.$$

15. Prove that, in a V_2,

$$R_{ijkl} = -\frac{R}{2}\left[g_{ik}g_{jl} - g_{il}g_{jk}\right].$$

16. Consider the metric $ds^2 = (dx^1)^2 - (x^1)^2[(dx^2)^2 + (dx^3)^2]$. Show that $R_{1212} = 2$ and that, therefore, the space is not flat.
17. Prove that a Riemannian space is an Einstein space if

$$R^{\alpha}_{ijk} = \mu\left(\delta^{\alpha}_{j} g_{jk} - \delta^{\alpha}_{k} g_{ij}\right),$$

where μ is a constant. Hence, show that for an Einstein space, the scalar curvature R is constant.
18. Prove Schur's theorem. If all points in some neighbourhood in a Riemannian R^N are isotropic and $N \geq 3$, then κ is a constant throughout that neighbourhood.
19. Verify that $ds^2 = (dx^1)^2 + (x^1)^2(dx^2)^2$ represents the Euclidean metric in polar co-ordinates.
20. Determine whether the following metric is flat and/or Euclidean:
    (a) $ds^2 = (dx^1)^2 - (x^1)^2(dx^2)^2; N = 2.$
    (b) $ds^2 = (dx^1)^2 + (x^3)^2(dx^2)^2 + (dx^3)^2.$
21. Find the isotropic points for the Riemannian metric

$$ds^2 = (\log x^2)\left[(dx^1)^2 + (dx^2)^2 + (dx^3)^2\right];\quad x^2 > 1,$$

and find the curvature κ at those points.

22. Show that R^3 under the metric

$$g_{11} = e^{x^2}; \quad g_{22} = 1; \quad g_{33} = e^{x^2}; \quad g_{ij} = 0; \text{ for } i \neq j$$

has constant Riemannian curvature with all points isotropic, find that curvature.

23. If Γ_i^j is the Einstein tensor, prove that
    (a) $\Gamma^j_{i,j} = \left(R_i^j - \frac{1}{2}\delta_i^j R\right)_{,j} = 0.$
    (b) The Einstein invariant $\Gamma = \Gamma_i^i$, vanishes if the space is flat.

24. For a two-dimensional space (V_2), show that

$$gR_{ij} = -g_{ij}R_{1212} \quad \text{and} \quad gR = -2R_{1212}.$$

Hence, deduce that every V_2 is an Einstein space.

25. The metric of the V_2 formed by the surface of a sphere of radius r is

$$ds^2 = r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

in spherical polar co-ordinates. Show that the surface of a sphere is a surface of constant curvature.

26. In an Euclidean V_4, prove that the hypersurface, $x^1 = c\sin u^1 \sin u^2 \sin u^3$, $x^2 = c\sin u^1 \sin u^2 \cos u^3$, $x^3 = c\sin u^1 \cos u^2$, $x^4 = c\cos u^1$ has a constant curvature c^{-2}.
27. Show that an Einstein space V_3 has constant curvature.
28. In a space in which the relation

$$R_{ij}g_{kl} - R_{il}g_{jk} + R_{jk}g_{il} - R_{kl}g_{ij} = 0$$

holds, show that the space is an Einstein space.

29. Find the lines of shortest distance of a sphere of radius a.
30. Show that the magnitude of a vector and angle between two vectors remain invariant under parallel displacement.
31. If two unit vectors are such that at all points of a given curve Γ, their intrinsic derivatives along Γ are zero, show that they are inclined at a constant angle.
32. Show that the field of tangent vectors to a geodesic is a field of parallel vectors.

Chapter 5

Geometry of Space Curve

For the study of a curve by the method of calculus, its parametric representation is covariant. In this chapter, three vectors (tangent, normal, binormal) of fundamental importance to curve theory are discussed. A curve is determined uniquely except for position by measures of two notions associated with it, called its *curvature* and *torsion*. For a given curve there is a set of important formulas dealing with its curvature and torsion. These formulas are called *Serret–Frenet formulae,* which will be established in the next theory.

5.1 Curve Theory

A curve $\mathcal{C}$ in E^3 is the image of a class C^3 mapping, r, from an interval of real numbers into E^3. The moving frame of the curve

$$\mathcal{C}\colon \mathbf{r} = \mathbf{r}(s) = \big(x^1(s), x^2(s), x^3(s)\big) \tag{5.1}$$

with the arbitrary parameter s, has been defined as an image of the real number for the arc length parameterization. Let s be the arc length. Then,

$$\begin{aligned}
\mathbf{r}' = \mathbf{T} = \frac{d\mathbf{r}}{ds} &= \Big(x^{1'}(s), x^{2'}(s), x^{3'}(s)\Big) \\
\mathbf{r}'' = \frac{d^2\mathbf{r}}{ds^2} &= \Big(x^{1''}(s), x^{2''}(s), x^{3''}(s)\Big) = \frac{d\mathbf{T}}{ds} = \kappa\mathbf{N} \\
\mathbf{r}''' = \frac{d^3\mathbf{r}}{ds^3} &= \Big(x^{1'''}(s), x^{2'''}(s), x^{3'''}(s)\Big) \\
&= \frac{d}{ds}(\kappa\mathbf{N}) = \kappa'\mathbf{N} + \kappa\frac{d\mathbf{N}}{ds} = \kappa'\mathbf{N} + \kappa\,(-\kappa\mathbf{T} + \tau\mathbf{B}).
\end{aligned}$$

We have at once for the unit tangent vector,

$$\hat{\mathbf{T}} = \hat{\mathbf{r}}' = \frac{1}{|\mathbf{r}'|}\mathbf{r}' = \frac{1}{|\mathbf{r}'|}\Big(x^{1'}(s), x^{2'}(s), x^{3'}(s)\Big).$$

Since $\mathbf{r}'' = \frac{d\mathbf{T}}{ds} = \kappa\hat{\mathbf{N}}$, where $\mathbf{N}$ is the normal vector, we have $|\mathbf{r}''| = \kappa$, and so, the unit normal vector is given by

$$\hat{\mathbf{N}} = \frac{1}{|\mathbf{r}''|}\mathbf{r}'' = \frac{1}{|\mathbf{r}''|}\left(x^{1''}(s), x^{2''}(s), x^{3''}(s)\right).$$

κ is called the curvature. The unit vector $\hat{N}$ is perpendicular to $\hat{T}$ and in the plane of the tangents at P and a consecutive point. This plane, containing two consecutive tangents and three consecutive points at P, is called the plane of curvature or the osculating plane at P. If $\mathbf{R}$ is any point in this plane, then the equation of the osculating plane

$$[\mathbf{R} - \mathbf{r}, \hat{T}, \hat{N}] = 0, i.e.\ [\mathbf{R} - \mathbf{r}, \mathbf{r}', \mathbf{r}''] = 0.$$

The unit vectors $\hat{T}$ and $\hat{N}$ are perpendicular to each other, and their plane is the plane of curvature. The straight line through P parallel to $\hat{N}$ is called the principal normal at P (Figure 5.1). If $\mathbf{R}$ be the current point on the line, its equation is

$$\mathbf{R} = \mathbf{r} + u\hat{N}.$$

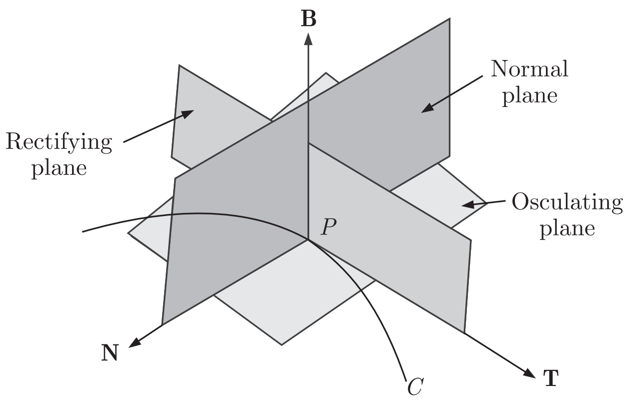

Figure 5.1: Planes.

The circle of curvature at P is the circle passing through three points on the curve ultimately coincident at P. Its radius $P\left(=\frac{1}{\kappa}\right)$ is called the radius of curvature, and its centre C the centre of curvature. This circle is clearly lies on the osculating plane at P, and its curvature is the same as that of the curve at P, for it has two consecutive tangents in common with the centre. The centre of curvature C lies on the principal normal, and

$$\overrightarrow{PC} = \rho\hat{\mathbf{N}} = \frac{1}{\kappa}\hat{\mathbf{N}}$$

The direction cosines of $\hat{N}$ can be written as

$$\hat{\mathbf{N}} = \rho\frac{d\hat{\mathbf{T}}}{ds} = \rho\, \mathbf{r}''.$$

Finally, for the binormal vector, we have

$$\mathbf{B} = \mathbf{T} \times \mathbf{N} = \mathbf{r}' \times \frac{1}{|\mathbf{r}''|}\mathbf{r}'' = \frac{1}{|\mathbf{r}''|}\left(\mathbf{r}' \times \mathbf{r}''\right).$$

The positive direction along the binormal is taken as that of $\mathbf{B}$, just as the positive direction along the principal normal is that of $\hat{\mathbf{N}}$. The equation of the binormal is

$$\mathbf{R} = \mathbf{r} + u\mathbf{B}.$$

Since $\mathbf{B}$ is a vector of constant length, it follows that $\frac{d\mathbf{B}}{ds}$ is perpendicular to $\mathbf{B}$. Moreover differentiating the relation $\mathbf{T.B} = 0$ with respect to s we find

$$\mathbf{kN} \cdot \mathbf{B} \; + \; \mathbf{T} \cdot \frac{\mathrm{d}\mathbf{B}}{\mathrm{ds}} \; = \; 0$$

$\Rightarrow \;\; \mathbf{T} \cdot \frac{\mathrm{d}\mathbf{B}}{\mathrm{ds}} \; = \; 0,$ as $\mathbf{N}$ is perpendicular to $\mathbf{B}$. This equation shows that $\frac{d\mathbf{B}}{ds}$ is perpendicular to $\mathbf{T}$, which is perpendicular to $\mathbf{B}$, and must therefore parallel to $\mathbf{N}$. We may then write

$$\frac{\mathrm{d}\mathbf{B}}{\mathrm{ds}} \; = \; -\,\tau\mathbf{N}.$$

In this equation $\frac{d\mathbf{T}}{ds} \; = \; \kappa\mathbf{N}$, the scalar κ measures the arc-rate of turning of the unit vector $\mathbf{T}$, so here τ measures the arc-rate of turning of the unit vector $\mathbf{B}$. This rate of turning of binormal is called the torsion of the curve at the point P. It is of course the rate of rotation of the osculating plane. The negative sign indicates that the torsion is regarded as positive when the rotation of the binormal as s increase is in the same sense as that of a right handed screw travelling in the direction of $\mathbf{T}$. Now

$$\begin{aligned}\frac{d\mathbf{N}}{ds} \; &= \; \frac{d}{ds}(\mathbf{B} \; \times \; \mathbf{T}) \; = \; -\tau\mathbf{N} \; \times \; \mathbf{T} \; + \; \mathbf{B} \; \times \; (\kappa\mathbf{N}) \\ &= \; \tau\mathbf{B} \; - \; \kappa\mathbf{T}.\end{aligned}$$

The formulas for $\frac{d\mathbf{T}}{ds}, \frac{d\mathbf{N}}{ds}$ and $\frac{d\mathbf{B}}{ds}$ gather together gives

$$\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N}; \frac{d\mathbf{N}}{ds} = \tau\mathbf{B} - \kappa\mathbf{T}; \frac{d\mathbf{B}}{ds} = -\tau\mathbf{N}.$$

These equations are the vector equivalents of the Serret–Frenet formulae for the derivatives of the direction cosines of the tangent, the principal normal and the binormal. Therefore

$$\frac{\delta}{\delta s}\begin{pmatrix}\mathbf{T}\\ \mathbf{N}\\ \mathbf{B}\end{pmatrix}\begin{pmatrix}0 & \kappa & 0\\ -\kappa & 0 & \tau\\ 0 & -\tau & 0\end{pmatrix}\begin{pmatrix}\mathbf{T}\\ \mathbf{N}\\ \mathbf{B}\end{pmatrix}$$

which is the Serret–Frenet formulae in the matrix form. The coefficient matrix is Skew-symmetric.

Now, we have to calculate $[\mathbf{r}', \mathbf{r}'', \mathbf{r}''']$ as

$$\begin{aligned}\left[\mathbf{r}', \mathbf{r}'', \mathbf{r}'''\right] &= \left[\mathbf{T}, \kappa\mathbf{N}, \kappa'\mathbf{N} + \kappa\left(-\kappa\mathbf{T} + \tau\mathbf{B}\right)\right] \\ &= [\mathbf{T}, \kappa\mathbf{N}, \kappa\tau\mathbf{B}] = \kappa^2\tau\,[\mathbf{T}, \mathbf{N}, \mathbf{B}] = \kappa^2\tau \\ \Rightarrow\ \tau = \frac{1}{\kappa^2}&\left[\mathbf{r}', \mathbf{r}'', \mathbf{r}'''\right] = \frac{1}{|\mathbf{r}''|^2}\left[\mathbf{r}', \mathbf{r}'', \mathbf{r}'''\right].\end{aligned}$$

An alternative formula giving the square of the torsion, may be deduced from the expression for r''' found above. On squaring this and diving throughout by κ^2 we obtain the result

$$\tau^2 \;=\; \frac{1}{\kappa^2}\,r'''^2 \;-\; \kappa^2 \;-\; \left(\frac{\kappa'}{\kappa}\right)^2.$$

By analogy with the relation that the radius of curvature is equal to the reciprocal of the curvature, it is customary to speak of the reciprocal of the torsion, and to denote it by σ. So $\sigma = \frac{1}{\tau}$. But there is no circle of torsion or centre of torsion associated with the curve in the same way as the circle and centre of curvature.

Let the equation of the curve be

$$\mathcal{C} : \mathbf{r} = \mathbf{r}(t) = \left(x^1(t), x^2(t), x^3(t)\right), \tag{5.2}$$

where t is a parameter. Therefore,

$$\begin{aligned}\dot{\mathbf{r}} &= \frac{d\mathbf{r}}{dt} = \left(\dot{x}^1(t), \dot{x}^2(t), \dot{x}^3(t)\right) = \frac{d\mathbf{r}}{ds}\frac{ds}{dt} = \mathbf{r}'\dot{s} = \dot{s}\mathbf{T} \\ \ddot{\mathbf{r}} &= \frac{d^2\mathbf{r}}{dt^2} = \left(\ddot{x}^1(t), \ddot{x}^2(t), \ddot{x}^3(t)\right) = \frac{d}{dt}\left(\dot{s}\mathbf{T}\right) = \ddot{s}\mathbf{T} + \kappa\dot{s}^2\mathbf{N} \\ \dddot{\mathbf{r}} &= \frac{d^3\mathbf{r}}{dt^3} = \left(\dddot{x}^1(t), \dddot{x}^2(t), \dddot{x}^3(t)\right) = \frac{d}{dt}\left(\ddot{s}\mathbf{T} + \kappa\dot{s}^2\mathbf{N}\right) \\ &= \frac{d\mathbf{T}}{ds}\dot{s}\ddot{s} + \mathbf{T}\dddot{s} + \dot{\kappa}\dot{s}^2\mathbf{N} + \kappa\left(-\kappa\mathbf{T} + \tau\mathbf{B}\right)\dot{s}^3 + 2\kappa\dot{s}\ddot{s}\mathbf{N}.\end{aligned}$$

Since $\mathbf{T}\cdot\ddot{\mathbf{r}} = \ddot{s}$, so, $\ddot{s} = \frac{\dot{\mathbf{r}}\cdot\ddot{\mathbf{r}}}{|\dot{\mathbf{r}}|}$. Therefore,

$$\ddot{\mathbf{r}} = \ddot{s}\mathbf{T} + \kappa\dot{s}^2\mathbf{N} = \frac{\dot{\mathbf{r}}\cdot\ddot{\mathbf{r}}}{|\dot{\mathbf{r}}|}\frac{\dot{\mathbf{r}}}{|\dot{\mathbf{r}}|} + \kappa\dot{s}^2\mathbf{N}$$

or

$$\kappa\mathbf{N} = \frac{1}{|\dot{\mathbf{r}}|^2}\left[\ddot{\mathbf{r}} - \frac{(\mathbf{r}\cdot\ddot{\mathbf{r}})}{|\dot{\mathbf{r}}|^2}\dot{\mathbf{r}}\right]$$

or

$$\kappa = \frac{1}{|\dot{\mathbf{r}}|^2}\left[|\ddot{\mathbf{r}}| - \frac{(\mathbf{r}\cdot\ddot{\mathbf{r}})}{|\dot{\mathbf{r}}|}\right] = \frac{|\dot{\mathbf{r}}\times\ddot{\mathbf{r}}|}{|\dot{\mathbf{r}}|^3}.$$

Therefore, the normal vector is given by

$$\mathbf{N} = \frac{1}{\kappa |\dot{\mathbf{r}}|^2}\left[|\ddot{\mathbf{r}}| - \frac{(\mathbf{r}\cdot\ddot{\mathbf{r}})}{|\dot{\mathbf{r}}|}\dot{\mathbf{r}}\right].$$

Now, we have to calculate $\left[\dot{\mathbf{r}}, \ddot{\mathbf{r}}, \dddot{\mathbf{r}}\right]$ as

$$\left[\dot{\mathbf{r}}, \ddot{\mathbf{r}}, \dddot{\mathbf{r}}\right] = \left[\dot{s}\mathbf{T}, \kappa\dot{s}^2\mathbf{N}, \kappa\tau\dot{s}^3\mathbf{B}\right] = \kappa^2\tau\dot{s}^6\left[\mathbf{T}, \mathbf{N}, \mathbf{B}\right] = \kappa^2\tau\dot{s}^6.$$

or

$$\tau = \frac{\left[\dot{\mathbf{r}}, \ddot{\mathbf{r}}, \dddot{\mathbf{r}}\right]}{\kappa^2\dot{s}^6} = \frac{\left[\dot{\mathbf{r}}, \ddot{\mathbf{r}}, \dddot{\mathbf{r}}\right]}{|\dot{\mathbf{r}}\times\ddot{\mathbf{r}}|^2}\frac{|\dot{\mathbf{r}}|^6}{|\dot{\mathbf{r}}|^6} = \frac{\left[\dot{\mathbf{r}}, \ddot{\mathbf{r}}, \dddot{\mathbf{r}}\right]}{|\dot{\mathbf{r}}\times\ddot{\mathbf{r}}|^2},$$

which is the expression for torsion.

5.2 Plane Curve

When the curve $x^i = f^i(s)$ lies on the plane, then it is defined as a *plane curve.* Every planner curve has a principal normal vector. A necessary and sufficient condition for a curve $x^i = f^i(s)$ lies on a plane is

$$\phi = \begin{vmatrix} f^{1'} & f^{2'} & f^{3'} \\ f^{1''} & f^{2''} & f^{3''} \\ f^{1'''} & f^{2'''} & f^{3'''} \end{vmatrix} = 0.$$

Let the curve $x^i = f^i(s)$ lie on the plane

$$a_1x^1 + a_2x^2 + a_3x^3 + a_0 = 0$$

or

$$a_1f^1(s) + a_2f^2(s) + a_3f^3(s) + a_0 = 0.$$

Differentiate with respect to s thrice, we have

$$a_1f^{1'}(s) + a_2f^{2'}(s) + a_3f^{3'}(s) = 0$$

$$a_1f^{1''}(s) + a_2f^{2''}(s) + a_3f^{3''}(s) = 0$$

$$a_1f^{1'''}(s) + a_2f^{2'''}(s) + a_3f^{3'''}(s) = 0.$$

Eliminating a_1, a_2, a_3 from the above three equations, we get

$$\phi = \begin{vmatrix} f^{1'} & f^{2'} & f^{3'} \\ f^{1''} & f^{2''} & f^{3''} \\ f^{1'''} & f^{2'''} & f^{3'''} \end{vmatrix} = 0.$$

Therefore, the condition is necessary. To prove the sufficient condition, we have $\phi = 0$. So the rank of ϕ may be $0, 1, 2$.

Case 1: Let rank $\phi = 0$, i.e. all the elements of ϕ are zero, i.e.

$$f^{1'}(s) = f^{2'}(s) = f^{3'}(s) = 0$$

or

$$f^1(s) = f^2(s) = f^3(s) = \text{ constant}$$

or

$$x^1(s) = k_1; x^2(s) = k_2; x^3(s) = k_3.$$

Thus, the curve $x^i = f^i(s)$ reduces to a point. Therefore, the given curve is a plane curve. (Trivial case).

Case 2: Let the rank of ϕ be 1, then any row will be proportional to any other row of ϕ, i.e.

$$\frac{f^{1''}}{f^{1'}} = \frac{f^{2''}}{f^{2'}} = \frac{f^{3''}}{f^{3'}}$$

or

$$\log f^{1'} = \psi(s) + c_1 \Rightarrow f^{1'} = e^{\psi(s)+c_1} = a_1 e^{\psi(s)}$$

or

$$x^i = f^i(s) = a_i \int e^{\psi(s)} ds + b_i = a_i \xi(s) + b_i$$

or

$$\frac{x^1 - b_1}{a_1} = \frac{x^2 - b_2}{a_2} = \frac{x^3 - b_3}{a_3} = \xi(s),$$

which is an equation of a line passing through b_i. Hence, the curves is a plane curve.

Case 3: Let the rank of ϕ be 2, then at least one of the minors of order two does not vanish. We assume that

$$b_1 = \begin{vmatrix} f^{2'} & f^{3'} \\ f^{2''} & f^{3''} \end{vmatrix}, b_2 = \begin{vmatrix} f^{3'} & f^{1'} \\ f^{3''} & f^{1''} \end{vmatrix}, b_3 = \begin{vmatrix} f^{1'} & f^{2'} \\ f^{1''} & f^{2''} \end{vmatrix}.$$

So, there exists three numbers b_1, b_2, b_3 such that

$$b_1 f^{1'}(s) + b_2 f^{2'}(s) + b_3 f^{3'}(s) = 0 \qquad \text{(i)}$$

$$b_1 f^{1''}(s) + b_2 f^{2''}(s) + b_3 f^{3''}(s) = 0 \qquad \text{(ii)}$$

$$b_1 f^{1'''}(s) + b_2 f^{2'''}(s) + b_3 f^{3'''}(s) = 0. \qquad \text{(iii)}$$

Differentiating with respect to s, we get from Eq. (i),

$$b_i f^{i''}(s) + b_i' f^{i'}(s) = 0 \Rightarrow b_i' f^{i'}(s) = 0; \text{ using Eq. (ii).}$$

Similarly, differentiate Eq. (ii) with respect to s and using Eq. (iii), we get, $b_i' f^{i''}(s) = 0$. So, there exists a unique solution to the linear equations

$$b_1' f^{1'}(s) + b_2' f^{2'}(s) + b_3' f^{3'}(s) = 0$$
$$b_1' f^{1''}(s) + b_2' f^{2''}(s) + b_3' f^{3''}(s) = 0.$$

Solving, we get

$$\frac{b_1'}{\begin{vmatrix} f^{2'} & f^{3'} \\ f^{2''} & f^{3''} \end{vmatrix}} = \frac{b_2'}{\begin{vmatrix} f^{3'} & f^{1'} \\ f^{3''} & f^{1''} \end{vmatrix}} = \frac{b_3'}{\begin{vmatrix} f^{1'} & f^{2'} \\ f^{1''} & f^{2''} \end{vmatrix}}$$

$$\Rightarrow \frac{b_1'}{b_1} = \frac{b_2'}{b_2} = \frac{b_3'}{b_3} = \psi'(s); \text{ say,}$$

or

$$\log b_i = \psi(s) + c_i \Rightarrow b_i = e^{\psi(s)+c_i} = a_i e^{\psi(s)}.$$

But $b_i f^{i'} = 0$, from Eq. (i) replacing b_i, we get

$$a_i e^{\psi(s)} f^{i'} = 0 \Rightarrow a_i f^{i'} = 0; \text{ as } e^{\psi(s)} \neq 0$$

or

$$a_1 f^{1'} + a_2 f^{2'} + a_3 f^{3'} = 0$$

or

$$a_1 f^1 + a_2 f^2 + a_3 f^3 + a_0 = 0.$$

Hence, the curve is a plane curve. When the curve is not plane, then it is called skew curve.

EXAMPLE 5.2.1 *Determine $\zeta(t)$ such that the curve $x^1 = a\cos t, x^2 = a\sin t$, $x^3 = \zeta(t)$ is a plane curve and what is the nature of the conic?*

Solution: Since the given curve is a plane curve, we have

$$\begin{vmatrix} x^{1'} & x^{2'} & x^{3'} \\ x^{1''} & x^{2''} & x^{3''} \\ x^{1'''} & x^{2'''} & x^{3'''} \end{vmatrix} = \begin{vmatrix} -a\sin t & a\cos t & \zeta' \\ -a\cos t & -a\sin t & \zeta'' \\ a\sin t & -a\cos t & \zeta''' \end{vmatrix} = 0$$

or

$$\zeta''' + \zeta' = 0 \Rightarrow \zeta(s) = a^1 + b\cos t + c\sin t.$$

The nature of the curve will be intersected between the cylinder $x^2 + y^2 = a^2$ and the plane $lx + my + nz = k$.

5.3 Twisted Curve

A curve in E^3 which does not lie on a plane is called *twisted curve.* Thus, the twisted curve is not a plane curve.

EXAMPLE 5.3.1 *Find the curvature and the torsion of the circular helix*

$$\mathbf{r} = \left(a\cos\frac{s}{c}, a\sin\frac{s}{c}, \frac{bs}{c}\right);\; c = \sqrt{a^2+b^2},$$

where s is arc length.

Solution: By differentiation with respect to the arc length, the tangent vector is given by

$$\mathbf{T} = \mathbf{r}' = \left(-\frac{a}{c}\sin\frac{s}{c}, \frac{a}{c}\cos\frac{s}{c}, \frac{s}{c}\right).$$

The normal vector $\mathbf{N}$ is given by

$$\mathbf{N} = \mathbf{r}'' = \left(-\cos\frac{s}{c}, -\sin\frac{s}{c}, 0\right) \text{ as } c = \sqrt{a^2+b^2}.$$

Finally, the binormal vector is given by

$$\mathbf{B} = \mathbf{T}\times\mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\frac{a}{c}\sin\frac{s}{c} & \frac{a}{c}\cos\frac{s}{c} & \frac{s}{c} \\ -\cos\frac{s}{c} & -\sin\frac{s}{c} & 0 \end{vmatrix} = \left(\frac{b}{c}\sin\frac{s}{c}, -\frac{b}{c}\cos\frac{s}{c}, \frac{a}{c}\right).$$

Therefore, the curvature and the torsion of the circular helix are given by,

$$\kappa = \frac{a}{c^2}\cos^2\frac{s}{c} + \frac{a}{c^2}\sin^2\frac{s}{c} + 0 = \frac{a}{c^2}$$

$$\tau = \frac{b}{c^2}\cos^2\frac{s}{c} + \frac{b}{c^2}\sin^2\frac{s}{c} + 0 = \frac{b}{c^2}.$$

Thus the curvature and the torsion are both constant, and therefore their ratio is constant. The principal normal intersects the axis of the cylinder orthogonally and the tangent and binormal are inclined at constant angles to the fixed direction of the generators.

5.4 Space Curve Theory

In this section we describe the geometry of the space curve. Formulae analogous to those of Frenet also occur in the theory of surfaces. This section contains a set of three remarkable formulas, generally known as *Serret–Frenet formulae,* which characterise, in the small, all essential geometric properties of space curves.

5.4.1 Serret–Frenet Formulae

In this section we will investigate the theory of twisted curves. Let the curve Γ be given by the equation

$$\Gamma : \; x^i = x^i(s), \tag{5.3}$$

where the parameter s measures the arc distance along the curve. As arc length s along a curve remains invariant, we consider s as the parameter to study the geometry of the space curves. The square of the length of the elements of arc of Γ is given by

$$ds^2 = g_{ij} dx^i dx^j$$

or

$$1 = g_{ij} \frac{dx^i}{ds} \frac{dx^j}{ds}, \tag{5.4}$$

showing that $\frac{dx^i}{ds}$ is a unit vector. Let P be a given point with co-ordinates (x^i) and Q be a neighbouring point with co-ordinates $(x^i + dx^i)$ on Γ, corresponding to an increment ds in the arc (Figure 5.2). Then the vector $\lim_{Q \to P} \frac{\overrightarrow{PQ}}{ds}$ is called the *tangent vector* and we shall denote it by λ^i. Thus, Eq. (5.4) can be written as

$$1 = g_{ij} \lambda^i \lambda^j; \quad \text{where } \lambda^i = \frac{dx^i}{ds}. \tag{5.5}$$

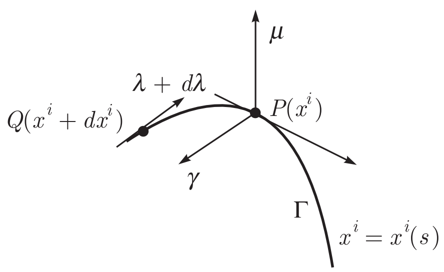

Figure 5.2: Tangent, normal, binormal.

These are precisely the direction cosines of the tangent vector to the curve Γ. We shall assume that curve Γ is of class C^2, so that it has continuously turning tangent at all points of Γ. Taking the intrinsic derivative of Eq. (5.5) with respect to s, we have

$$g_{ij} \frac{\delta \lambda^i}{\delta s} \lambda^j + g_{ij} \lambda^i \frac{\delta \lambda^j}{\delta s} = 0$$

or

$$g_{ij}\lambda^i \frac{\delta \lambda^j}{\delta s} = 0.$$

Thus, either $\frac{\delta \lambda^j}{\delta s}$ vanishes or is orthogonal to the tangent vector at some point of a curve, is said to be a *normal vector* of that curve at that point. Hence, the condition that a vector μ^i be normal to Γ at P is

$$g_{ij}\lambda^i \mu^j = 0. \tag{5.6}$$

Thus, if $\frac{\delta \lambda^i}{\delta s}$ does not vanish, we denote the unit vector codirectional with $\frac{\delta \lambda^j}{\delta s}$ by μ^j and write it as

$$\mu^j = \frac{1}{\kappa} \frac{\delta \lambda^j}{\delta s}, \tag{5.7}$$

where $\kappa > 0$ is so chosen to make μ^j a unit vector. The vector μ^i, determined by Eq. (5.7) is called the *principal normal vector* to the curve Γ at P and κ is called the *curvature* of the curve Γ at that point P. The plane determined by the tangent λ and the normal μ is called the *osculating plane* to the curve Γ at P.

Since μ is a unit vector, so the quadratic relation

$$g_{ij}\mu^i \mu^j = 1 \tag{5.8}$$

is satisfied. Differentiating Eq. (5.8) intrinsically, we get

$$g_{ij}\frac{\delta \mu^i}{\delta s}\mu^j + g_{ij}\mu^i \frac{\delta \mu^j}{\delta s} = 0 \quad \Rightarrow \quad g_{ij}\mu^i \frac{\delta \mu^j}{\delta s} = 0. \tag{5.9}$$

Taking the intrinsic derivative of the orthogonal relation Eq. (5.5) we get

$$g_{ij}\frac{\delta \lambda^i}{\delta s}\mu^j + g_{ij}\lambda^i \frac{\delta \mu^j}{\delta s} = 0$$

or

$$\begin{aligned} g_{ij}\lambda^i \frac{\delta \mu^j}{\delta s} &= -g_{ij}\frac{\delta \lambda^i}{\delta s}\mu^j = -g_{ij}\kappa \mu^i \mu^j; \quad \text{from Eq. (5.7)} \\ &= -\kappa; \quad \text{from Eq. (5.8)} \\ &= -\kappa g_{ij}\lambda^i \lambda^j; \quad \text{from Eq. (5.5)} \end{aligned}$$

or

$$g_{ij}\lambda^i \left(\frac{\delta \mu^j}{\delta s} + \kappa \lambda^j \right) = 0. \tag{5.10}$$

Also, from Eq. (5.6) we can write $g_{ij}\mu^i\lambda^j = 0$, as $g_{ij} = g_{ji}$, i.e.

$$\kappa g_{ij}\mu^i\lambda^j = 0$$

or

$$g_{ij}\mu^i\frac{\delta\mu^j}{\delta s} + \kappa g_{ij}\mu^i\lambda^j = 0$$

or

$$g_{ij}\mu^i\left(\frac{\delta\mu^j}{\delta s} + \kappa\lambda^j\right) = 0. \tag{5.11}$$

From Eqs. (5.10) and (5.11) we see that, the vector $\left(\frac{\delta\mu^j}{\delta s} + \kappa\lambda^j\right)$ is orthogonal to both the tangent vector λ^i and the principal normal vector μ^i. Hence, we choose a vector γ^i such that

$$\gamma^i = \frac{1}{\tau}\left(\frac{\delta\mu^i}{\delta s} + \kappa\lambda^i\right) \tag{5.12}$$

and τ is chosen to make γ, a unit vector. The constant τ thus introduced is called the *torsion* of the curve. Note that, it may be positive or negative in contrast to the curvature which is essentially positive, being the magnitude of a vector.

The sign of τ is not always positive but we agree to choose the sign of τ in such a way that (λ, μ, γ) form a right-hand system, i.e.

$$\varepsilon_{ijk}\lambda^i\mu^j\gamma^k = +1, \tag{5.13}$$

where

$$\varepsilon_{ijk} = \sqrt{g}e_{ijk}; \quad \text{and } g = \left|\frac{\partial y^i}{\partial x^j}\right|^2, \tag{5.14}$$

so that the left-hand member of Eq. (5.13) is an invariant. The vector γ is called the *binormal* of Γ at P. Also, using Eqs. (5.13) and (5.14), we get

$$\sqrt{g}e_{ijk}\lambda^i\mu^j\gamma^k = 1$$

or

$$e^{pqr}\sqrt{g}e_{ijk}\lambda^i\mu^j\gamma^k = e^{pqr}$$

or

$$\delta_i^p\delta_j^q\delta_k^r\sqrt{g}\lambda^i\mu^j\gamma^k = e^{pqr}$$

or

$$\lambda^p\mu^q\gamma^r = \frac{1}{\sqrt{g}}e^{pqr} = \frac{1}{\sqrt{g}}e^{rpq}$$

or

$$g_{ip}\lambda^i g_{jq}\mu^j \lambda^p \mu^q \gamma^r = g_{ip}\lambda^i g_{jq}\mu^j \varepsilon^{pqr}; \quad \text{by Eq. (5.14)}$$

or

$$\gamma^r = \varepsilon^{pqr}\lambda_p\mu_q \text{ i.e. } \gamma^i = \varepsilon^{ijk}\lambda_j\mu_k. \tag{5.15}$$

Differentiating Eq. (5.15) intrinsically, we get

$$\begin{aligned}\frac{\delta\gamma^i}{\delta s} &= \varepsilon^{ijk}\frac{\delta\lambda_j}{\delta s}\mu_k + \varepsilon^{ijk}\lambda_j\frac{\delta\mu^k}{\delta s}\\ &= \varepsilon^{ijk}\kappa\mu_j\mu_k + \varepsilon^{ijk}\lambda_i\left[\tau\gamma_k - \kappa\lambda_k\right],\end{aligned}$$

which reduces on account of skew symmetry of ε^{ijk} to

$$\frac{\delta\gamma^i}{\delta s} = \tau\varepsilon^{ijk}\lambda_j\gamma_k.$$

But λ_i, μ_i and γ_i form a right-handed system of unit vectors, so $\varepsilon^{ijk}\lambda_j\gamma_k = -\mu^i$. Therefore,

$$\frac{\delta\gamma_i}{\delta s} = -\tau\mu^i. \tag{5.16}$$

Equations (5.7), (5.12) and (5.16) are called the *Frenet formulae* for space curve. On account of their importance in the theory of curves, we group these formulae together for convenient reference in the form

$$\left.\begin{aligned}\frac{\delta\lambda^i}{\delta s} &= \kappa\mu^i\\ \frac{\delta\mu^i}{\delta s} &= -\kappa\lambda^i + \tau\gamma^i\\ \frac{\delta\gamma^i}{\delta s} &= -\tau\mu^i\end{aligned}\right\}. \tag{5.17}$$

Equation (5.17) can be written in the matrix form as

$$\frac{\delta}{\delta s}\begin{pmatrix}\lambda^i\\ \mu^i\\ \gamma^i\end{pmatrix} = \begin{pmatrix}0 & \kappa & 0\\ -\kappa & 0 & \tau\\ 0 & -\tau & 0\end{pmatrix}\begin{pmatrix}\lambda^i\\ \mu^i\\ \gamma^i\end{pmatrix}.$$

The coefficient matrix is skew symmetric. Formulae analogous to those of Frenet also occur in the theory of surfaces.

Note 5.4.1 If the curve lies on a plane, then it is called a *plane curve,* otherwise it is called a *skew curve.*

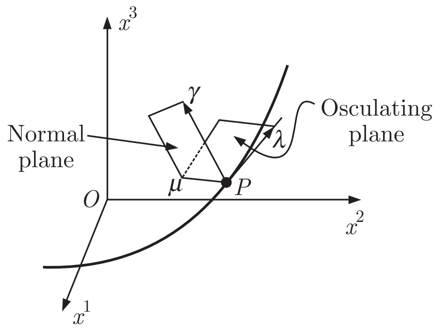

Figure 5.3: Osculating, normal rectifying plane.

Note 5.4.2 The plane determined by the tangent λ and principal normal μ is called the *osculating plane* at a point $P(x_0^i)$ of a space curve (Figure 5.3). It's equation is

$$g_{ij}\left(x^i - x_0^i\right)\gamma^j = 0,$$

where x^i is any point on the plane. The plane determined by principal normal μ and binormal γ is called the *normal plane* at $P(x_0^i)$ and its equation is given by

$$g_{ij}\left(x^i - x_0^i\right)\lambda^j = 0.$$

The plane determined by binormal ν and tangent λ is called the *rectifying plane* at $P(x_0^i)$ and its equation is given by (Figure 5.3)

$$g_{ij}\left(x^i - x_0^i\right)\mu^j = 0.$$

EXAMPLE 5.4.1 *Show that*

(i) $\kappa = \left(g_{mn}\dfrac{\delta\lambda^m}{\delta s}\dfrac{\delta\lambda^n}{\delta s}\right)^{1/2}$

(ii) $\tau = \varepsilon_{rst}\lambda^r\mu^s\dfrac{\delta\mu^t}{\delta s}$

(iii) $\gamma_i = \frac{1}{\tau}\left(\dfrac{\delta\mu_i}{\delta s} + \kappa\lambda_i\right)$

(iv) $\kappa\gamma_i = \varepsilon_{ijk}\lambda^i\dfrac{\delta\lambda^k}{\delta s}$

(v) $\mu_i = \frac{1}{\kappa}\dfrac{\delta\lambda_i}{\delta s}$,

where the symbols have their usual meanings.

Solution:

(i) We know that

$$\mu^m = \frac{1}{\kappa}\frac{\delta\lambda^m}{\delta s}; \quad \frac{\delta\lambda^m}{\delta s} = \kappa\mu^m,$$

where $\kappa > 0$ is so chosen to make μ^m a unit vector. Thus,

$$g_{mn}\frac{\delta\lambda^m}{\delta s}\mu^n = \kappa g_{mn}\mu^m\mu^n$$

or

$$g_{mn}\frac{\delta\lambda^m}{\delta s}\mu^n = \kappa \text{ as } g_{mn}\mu^m\mu^n = 1$$

or

$$\kappa = g_{mn}\frac{\delta\lambda^m}{\delta s}\frac{1}{\kappa}\frac{\delta\lambda^n}{\delta s}$$

or

$$\kappa^2 = g_{mn}\frac{\delta\lambda^m}{\delta s}\frac{\delta\lambda^n}{\delta s}$$

or

$$\kappa = \left(g_{mn}\frac{\delta\lambda^m}{\delta s}\frac{\delta\lambda^n}{\delta s}\right)^{1/2}; \quad \text{as } \kappa > 0.$$

(ii) From the second Frenet formula we have

$$\frac{\delta\mu^t}{\delta s} = -\kappa\lambda^t + \tau\gamma^t$$

or

$$\begin{aligned}
\varepsilon_{rst}\lambda^r\mu^s\frac{\delta\mu^t}{\delta s} &= -\varepsilon_{rst}\lambda^r\mu^s\left(\kappa\lambda^t + \tau\gamma^t\right)\\
&= -\kappa\varepsilon_{rst}\lambda^r\mu^s\lambda^t + \tau\varepsilon_{rst}\lambda^r\mu^s\gamma^t\\
&= 0 + \tau\gamma^t\gamma_t; \text{ as } \varepsilon_{rst}\lambda^r\mu^t = 0\\
&= \tau; \text{ as } \gamma^t\gamma_t = 1,
\end{aligned}$$

which is the required result.

(iii) Follows from Eq. (5.12).

(iv) Using relation Eq. (5.17), we get $\frac{\delta\lambda^k}{\delta s} = \kappa\mu^k$, i.e.

$$\varepsilon_{ijk}\lambda^j\frac{\delta\lambda^k}{\delta s} = \varepsilon_{ijk}\lambda^j\kappa\mu^k = \kappa\varepsilon_{ijk}\lambda^j\mu^k = \kappa\gamma^i, \text{ from Eq. (5.15).}$$

(v) We see that

$$\frac{1}{\kappa}\frac{\delta\lambda_i}{\delta s} = \frac{1}{\kappa}\frac{\delta}{\delta s}\left(g_{ij}\lambda^j\right) = \frac{1}{\kappa}g_{ij}\frac{\delta\lambda^j}{\delta s} = \frac{1}{\kappa}g_{ij}\kappa\mu^j = \mu_i.$$

EXAMPLE 5.4.2 *Show that*

$$g_{ij}\frac{\delta\mu^i}{\delta s}\frac{\delta\mu^j}{\delta s} = \kappa^2 + \tau^2.$$

Solution: Using Eq. (5.17), we get

$$\begin{aligned} g_{ij}\frac{\delta\mu^i}{\delta s}\frac{\delta\mu^j}{\delta s} &= g_{ij}\left(-\kappa\lambda^i + \tau\gamma^i\right)\left(-\kappa\lambda^j + \tau\gamma^j\right) \\ &= g_{ij}\left(\kappa^2\lambda^i\lambda^j - \kappa\tau\lambda^i\gamma^j - \kappa\tau\lambda^j\gamma^i + \tau^2\gamma^i\gamma^j\right) \\ &= \kappa^2 g_{ij}\lambda^i\lambda^j - \kappa\tau g_{ij}\lambda^i\gamma^j - \kappa\tau g_{ij}\lambda^j\gamma^i + \tau^2 g_{ij}\gamma^i\gamma^j \\ &= \kappa^2\cdot 1 - \kappa\tau\cdot 0 - \kappa\tau\cdot 0 + \tau^2\cdot 1 = \kappa^2 + \tau^2. \end{aligned}$$

EXAMPLE 5.4.3 *Show that*

(i) $\dfrac{\delta^2\lambda^i}{\delta s^2} = \dfrac{d\kappa}{ds}\mu^i + \kappa\left(\tau\gamma^i - \kappa\lambda^i\right)$

(ii) $\dfrac{\delta^2\mu^i}{\delta s^2} = \dfrac{d\tau}{ds}\gamma^i - (\kappa^2 + \tau^2)\mu^i - \dfrac{d\kappa}{ds}\lambda^i$

(iii) $\dfrac{\delta^2\gamma^r}{\delta s^2} = \tau\left(\kappa\lambda^r - \tau\gamma^r\right) - \dfrac{d\tau}{ds}\mu^r,$

where the symbols have their usual meanings.

Solution:

(i) Using Eq. (5.7), we get, $\frac{\delta\lambda^i}{\delta s} = \kappa\mu^i$; where $\kappa > 0$. Differentiating intrinsically with respect to the parameter s we get

$$\begin{aligned} \frac{\delta^2\lambda^i}{\delta s^2} &= \frac{\delta}{\delta s}\left(\kappa\mu^i\right) = \frac{d}{ds}\left(\kappa\mu^i\right) + \kappa\mu^j\begin{Bmatrix} i \\ j\ \ k\end{Bmatrix}\frac{dx^k}{ds} \\ &= \frac{d\kappa}{ds}\mu^i + \kappa\frac{d\mu^i}{ds} + \kappa\mu^j\begin{Bmatrix} i \\ j\ \ k\end{Bmatrix}\frac{dx^k}{ds} \\ &= \frac{d\kappa}{ds}\mu^i + \kappa\left[\frac{d\mu^i}{ds} + \mu^j\begin{Bmatrix} i \\ j\ \ k\end{Bmatrix}\frac{dx^k}{ds}\right] \\ &= \frac{d\kappa}{ds}\mu^i + \kappa\frac{\delta\mu^i}{\delta s} = \frac{d\kappa}{ds}\mu^i + \kappa\left[-\kappa\lambda^i + \tau\gamma^i\right];\ \text{from Eq. (5.17)} \\ \frac{\delta^2\lambda^i}{\delta s^2} &= \frac{d\kappa}{ds}\mu^i + \kappa\left(\tau\gamma^i - \kappa\lambda^i\right), \end{aligned}$$

which is the result as required.

(ii) Using Eq. (5.17), we get $\frac{\delta\mu^i}{\delta s} = \tau\gamma^i - \kappa\lambda^i$.

$$\begin{aligned}\frac{\delta^2\mu^i}{\delta s^2} &= \frac{\delta}{\delta s}\left(\tau\gamma^i\right) - \frac{\delta}{\delta s}\left(\kappa\lambda^i\right)\\ &= \frac{d}{ds}\left(\tau\gamma^i\right) + \tau\gamma^j\left\{\begin{matrix} i \\ j\ \ k\end{matrix}\right\}\frac{dx^k}{ds} - \left[\frac{d}{ds}\left(\kappa\lambda^i\right) + \kappa\lambda^j\left\{\begin{matrix} i \\ j\ \ k\end{matrix}\right\}\frac{dx^k}{ds}\right]\\ &= \frac{d\tau}{ds}\gamma^i + \tau\left[\frac{d\gamma^i}{ds} + \gamma^j\left\{\begin{matrix} i \\ j\ \ k\end{matrix}\right\}\frac{dx^k}{ds}\right] - \frac{d\kappa}{ds}\lambda^i - \kappa\left[\frac{d\lambda^i}{ds} + \lambda^j\left\{\begin{matrix} i \\ j\ \ k\end{matrix}\right\}\frac{dx^k}{ds}\right]\\ &= \frac{d\tau}{ds}\gamma^i + \tau\frac{\delta\gamma^i}{\delta s} - \frac{d\kappa}{ds}\lambda^i - \kappa\frac{\delta\lambda^i}{\delta s}\\ &= \frac{d\tau}{ds}\gamma^i + \tau(-\tau\mu^i) - \frac{d\kappa}{ds}\lambda^i - \kappa\kappa\mu^i = \frac{d\tau}{ds}\gamma^i - (\kappa^2+\tau^2)\mu^i - \frac{d\kappa}{ds}\lambda^i.\end{aligned}$$

(iii) Using Eq. (5.17), we get $\frac{\delta\gamma^i}{\delta s} = -\tau\mu^i$. Differentiating intrinsically with respect to the parameter s we get

$$\begin{aligned}\frac{\delta^2\gamma^i}{\delta s^2} &= -\frac{\delta}{\delta s}\left(\tau\mu^i\right) = -\frac{d}{ds}\left(\tau\mu^i\right) - \tau\mu^j\left\{\begin{matrix} i \\ j\ \ k\end{matrix}\right\}\frac{dx^k}{ds}\\ &= -\frac{d\tau}{ds}\mu^i - \tau\left[\frac{d\mu^i}{ds} + \mu^j\left\{\begin{matrix} i \\ j\ \ k\end{matrix}\right\}\frac{dx^k}{ds}\right]\\ &= -\frac{d\tau}{ds}\mu^i - \tau\frac{\delta\mu^i}{\delta s}\\ &= -\frac{d\tau}{ds}\mu^i - \tau\left[-\kappa\lambda^i + \tau\gamma^i\right];\ \text{from Eq. (5.17)}\\ &= \tau\left(\kappa\lambda^i - \tau\gamma^i\right) - \frac{d\tau}{ds}\mu^i.\end{aligned}$$

EXAMPLE 5.4.4 *Show that*

$$\tau = \frac{1}{\kappa^2}\varepsilon_{ijk}\lambda^i\frac{\delta\lambda^i}{\delta s}\frac{\delta^2\lambda^k}{\delta s^2},$$

where the notations have their usual meaning.

Solution: Using Eq. (5.7), we get $\frac{\delta\lambda^i}{\delta s} = \kappa\mu^i$; where, $\kappa > 0$. Differentiating intrinsically with respect to the parameter s we get,

$$\frac{\delta^2\lambda^i}{\delta s^2} = \frac{\delta\kappa}{\delta s}\mu^i + \kappa\frac{\delta\mu^i}{\delta s} = \frac{\delta\kappa}{\delta s}\mu^i + \kappa\left(\tau\gamma^i - \kappa\lambda^i\right).$$

Using the Frenet formula [Eq. (5.17)], we get,

$$\begin{aligned}
&\frac{1}{\kappa^2}\varepsilon_{ijk}\lambda^i\frac{\delta\lambda^i}{\delta s}\frac{\delta^2\lambda^k}{\delta s^2}\\
&=\frac{1}{\kappa^2}\varepsilon_{ijk}\lambda^i\kappa\mu^j\left[\frac{\delta\kappa}{\delta s}\mu^k+\kappa\left(\tau\gamma^k-\kappa\lambda^k\right)\right]\\
&=\frac{1}{\kappa}\left[\frac{\delta\kappa}{\delta s}\varepsilon_{ijk}\lambda^i\mu^j\mu^k+\kappa\tau\varepsilon_{ijk}\lambda^i\mu^j\gamma^k-\kappa^2\varepsilon_{ijk}\lambda^i\mu^j\lambda^k\right]\\
&=\frac{1}{\kappa}(0+\kappa\tau-0)=\tau.
\end{aligned}$$

EXAMPLE 5.4.5 *Find the curvature and torsion at any point of a given curve (circle)* Γ, *given by*

$$\Gamma\colon x^1=a, x^2=t, x^3=0,$$

where $ds^2=\left(dx^1\right)^2+(x^1)^2\left(dx^2\right)^2+\left(dx^3\right)^2$.

Solution: The square of the element of arc in cylindrical co-ordinates as

$$ds^2=\left(dx^1\right)^2+(x^1)^2\left(dx^2\right)^2+\left(dx^3\right)^2.$$

For the given metric, the non-vanishing Christoffel symbols are

$$\left\{\begin{matrix}1\\2\;\;2\end{matrix}\right\}=-x^1,\left\{\begin{matrix}2\\1\;\;2\end{matrix}\right\}=\left\{\begin{matrix}2\\2\;\;1\end{matrix}\right\}=\frac{1}{x^1};$$

and $g_{11}=1=g_{33}$ and $g_{22}=(x^1)^2=a^2$. The tangent vector λ^i is given by

$$\lambda^i=\frac{dx^i}{ds}=\left(\frac{dx^1}{ds},\frac{dx^2}{ds},\frac{dx^3}{ds}\right)=\left(0,\frac{dt}{ds},0\right).$$

As λ is a unit vector, so $g_{ij}\lambda^i\lambda^j=1$, i.e.

$$g_{11}\lambda^1\lambda^1+g_{22}\lambda^2\lambda^2+g_{33}\lambda^3\lambda^3=1$$

or

$$a^2\left(\frac{dt}{ds}\right)^2=1\Rightarrow\frac{dt}{ds}=\frac{1}{a}.$$

Therefore

$$\lambda^i=\left(0,\frac{dt}{ds},0\right)=\left(0,\frac{1}{a},0\right).$$

Now, from the Serret–Frenet formula [Eq. (5.7)], i.e. $\kappa\mu^i = \frac{\delta\lambda^i}{\delta s}$, we get,

$$\begin{aligned}
\kappa\mu^1 &= \frac{\delta\lambda^1}{\delta s} = \frac{d\lambda^1}{ds} + \left\{ {1 \atop j\ \ k} \right\} \lambda^j \frac{dx^k}{ds} \\
&= \frac{d\lambda^1}{ds} + \left\{ {1 \atop 2\ \ 2} \right\} \lambda^2 \frac{dx^2}{ds} = 0 + (-x^1)\frac{1}{a}\frac{dt}{ds} = -\frac{1}{a}. \\
\kappa\mu^2 &= \frac{\delta\lambda^2}{\delta s} = \frac{d\lambda^2}{ds} + \left\{ {2 \atop j\ \ k} \right\} \lambda^j \frac{dx^k}{ds} \\
&= 0 + \left\{ {2 \atop 1\ \ 2} \right\} \lambda^1 \frac{dx^2}{ds} + \left\{ {2 \atop 2\ \ 1} \right\} \lambda^2 \frac{dx^1}{ds} \\
&= \frac{1}{x^1} 0 \lambda^2 + \frac{1}{x^1}\frac{1}{a}\lambda^1 = 0.
\end{aligned}$$

Similarly, $\kappa\mu^3 = 0$. Since μ is a unit vector, so,

$$g_{ij}\mu^i\mu^j = 1; \quad \text{i.e.} \quad g_{11}\mu^1\mu^1 + g_{22}\mu^2\mu^2 + g_{33}\mu^3\mu^3 = 1$$

or

$$1 \cdot \frac{1}{\kappa a} \cdot \frac{1}{\kappa a} + 0 + 0 \cdot 0 \cdot 0 = 1 \Rightarrow \kappa = \frac{1}{a}; \text{ as } \kappa > 0.$$

Consequently,

$$\kappa\mu^1 = -\frac{1}{a} \Rightarrow \mu^1 = -1.$$

So, $\mu^i = (-1, 0, 0)$. From Eq. (5.17), we get $\frac{\delta\mu^i}{\delta s} = -\kappa\lambda^i + \tau\gamma^i; i = 1, 2, 3$, thus,

$$\begin{aligned}
\tau\gamma^1 - \kappa\lambda^1 = \frac{\delta\mu^1}{\delta s} &= \frac{d\mu^1}{ds} + \left\{ {1 \atop j\ \ k} \right\} \mu^j \frac{dx^k}{ds} \\
&= 0 + \left\{ {1 \atop 2\ \ 2} \right\} \mu^2\lambda^2 = 0.
\end{aligned}$$

Therefore, $\tau\gamma^1 = \kappa\lambda^1 = 0$. Now

$$\begin{aligned}
\tau\gamma^2 - \kappa\lambda^2 &= \frac{\delta\mu^2}{\delta s} = \frac{d\mu^2}{ds} + \left\{ {2 \atop j\ \ k} \right\} \mu^j \frac{dx^k}{ds} \\
&= 0 + \left\{ {2 \atop 1\ \ 2} \right\} \mu^1\lambda^2 + \left\{ {2 \atop 2\ \ 1} \right\} \mu^2\lambda^1 \\
&= \frac{1}{a}(-1)\frac{1}{a} + 0 = -\frac{1}{a^2}
\end{aligned}$$

or

$$\tau\gamma^2 = \kappa\lambda^2 - \frac{1}{a^2} = \frac{1}{a}\frac{1}{a} - \frac{1}{a^2} = 0.$$

and

$$\tau\gamma^3 - \kappa\lambda^3 = \frac{\delta\mu^3}{\delta s} = \frac{d\mu^3}{ds} + \left\{ \begin{matrix} 3 \\ j \quad k \end{matrix} \right\} \mu^j \frac{dx^k}{ds} = 0.$$

Therefore $\tau\gamma^3 = \kappa\lambda^3$. Since γ is a unit binormal vector, so

$$\tau\gamma^1 = \tau\gamma^2 = \tau\gamma^3 = 0 \Rightarrow \tau = 0.$$

To determine the component of γ, using $\gamma^i = \varepsilon^{ijk}\lambda_j\mu_k$ we get

$$\begin{aligned}
\gamma^1 &= \varepsilon^{1jk}\lambda_j\mu_k = \varepsilon^{123}\lambda_2\mu_3 + \varepsilon^{132}\lambda_3\mu_2 = 0 \\
\gamma^2 &= \varepsilon^{2jk}\lambda_j\mu_k = \varepsilon^{231}\lambda_3\mu_1 + \varepsilon^{213}\lambda_1\mu_3 = 0 \\
\gamma^3 &= \varepsilon^{3jk}\lambda_j\mu_k = \varepsilon^{312}\lambda_1\mu_2 + \varepsilon^{321}\lambda_2\mu_1 \\
&= \varepsilon^{321}\lambda_2\mu_1 = \frac{1}{g}e^{321}\lambda_2\mu_1; g = a^2.
\end{aligned}$$

Now, using the relation $\lambda_2 = g_{2m}\lambda^m$ and $\mu_1 = g_{1m}\mu^m$, we get

$$\begin{aligned}
\lambda_2 &= g_{2m}\lambda^m = g_{22}\lambda^2 = a^2 \cdot \frac{1}{a} = a \\
\mu_1 &= g_{1m}\mu^m = g_{11}\mu^1 = -1.
\end{aligned}$$

Thus

$$\gamma^3 = \frac{1}{a}(-1)a(-1) = 1.$$

Hence, the component of the binormal $\gamma = (0, 0, 1)$.

Deduction 5.4.1 Let x^i are curvilinear co-ordinates and $\left\{ \begin{matrix} i \\ j \quad k \end{matrix} \right\}$ are Christoffel symbols corresponding to the metric tensor of E^3. According to definition (3.38), we have

$$\frac{\delta\lambda^i}{\delta s} = \lambda^i_{,k}\frac{dx^k}{ds},$$

where the comma (,) notation denotes covariant differentiation with respect to the metric tensor. Hence,

$$\begin{aligned}
\frac{\delta\lambda^i}{\delta s} &= \left[\frac{\delta\lambda^i}{\delta x^k} + \lambda^p \left\{ \begin{matrix} i \\ p \quad k \end{matrix} \right\}\right] \frac{dx^k}{ds} \\
&= \frac{d\lambda^i}{ds} + \lambda^p \left\{ \begin{matrix} i \\ p \quad k \end{matrix} \right\} \frac{dx^k}{ds} = \frac{d^2x^i}{ds^2} + \left\{ \begin{matrix} i \\ p \quad k \end{matrix} \right\} \frac{dx^k}{ds}\frac{dx^p}{ds}.
\end{aligned}$$

Thus, the formula $\frac{\delta\lambda^i}{\delta s} = \kappa\mu^i$ can be written as

$$\frac{d\lambda^i}{ds} + \lambda^p \left\{ \begin{matrix} i \\ j \quad k \end{matrix} \right\} \frac{dx^j}{ds} = \kappa\mu^i$$

or

$$\frac{d^2x^i}{ds^2} + \left\{ \begin{matrix} i \\ j \quad k \end{matrix} \right\} \frac{dx^j}{ds}\frac{dx^k}{ds} = \kappa\mu^i.$$

Similarly, we have,

$$\frac{\delta\mu^i}{\delta s} = \frac{d\mu^i}{ds} + \left\{ \begin{matrix} i \\ j \quad k \end{matrix} \right\} \mu^j \frac{dx^k}{ds}$$

and

$$\frac{\delta\nu^i}{\delta s} = \frac{d\nu^i}{ds} + \left\{ \begin{matrix} i \\ j \quad k \end{matrix} \right\} \nu^j \frac{dx^k}{ds}. \tag{5.18}$$

These are the Serret–Frenet formulas in curvilinear co-ordinates E^3.

EXAMPLE 5.4.6 *Find the equation of the straight line in curvilinear co-ordinate system E^3.*

Solution: Let Γ be a straight line in E^3 and (x^i) be the curvilinear co-ordinates of any point on it. Then the direction of the tangent vector of Γ is fixed. Hence,

$$\frac{\delta\lambda^i}{\delta s} = 0; \text{ i.e. } \kappa\mu^i = 0.$$

Since μ^i is not zero, it follows that $\kappa = 0$. Hence, from Eq. (5.18) we get

$$\frac{d^2x^i}{ds^2} + \left\{ \begin{matrix} i \\ j \quad k \end{matrix} \right\} \frac{dx^j}{ds}\frac{dx^k}{ds} = 0.$$

Since this result holds for any point $P(x^i)$ on Γ, it is the equation of Γ in curvilinear co-ordinates E^3.

EXAMPLE 5.4.7 *Show that a space curve is a straight line if and only if its curvature is zero at all points of it.*

Solution: First, suppose that the curvature $\kappa = 0$ at all points of a space curve Γ. Then by the Frenet formula,

$$\frac{\delta\lambda^i}{\delta s} = \kappa\mu^i \Rightarrow \frac{\delta\lambda^i}{\delta s} = 0$$

for all s. This means that λ^i is a fixed direction. Hence Γ is a straight line.

Conversely, let Γ be a straight line, then the direction of the tangent vector of Γ is fixed. Hence,

$$\frac{\delta\lambda^i}{\delta s} = 0 \Rightarrow \kappa\mu^i = 0.$$

Since μ^i is not zero, it follows that the $\kappa = 0$. From this example, we can say that the curvature of a curve measures how much the curve differ from being a straight line.

EXAMPLE 5.4.8 *Prove that a space curve is a plane curve if and only if its torsion is zero at all points of it.*

Solution: First, suppose that a space curve Γ is a plane curve. Then the osculating plane at every point Γ is the plane of Γ. Hence, the binormal γ^i has a fixed direction, so $\frac{\delta\gamma^i}{\delta s} = 0$. Using Frenet formula we have

$$\frac{\delta\gamma^i}{\delta s} = -\tau\mu^i \Rightarrow \tau = 0 \text{ as } \mu^i \neq 0.$$

Next, we suppose that for a space curve Γ, $\tau = 0$ at every point of it. Then $\frac{\delta\gamma^i}{\delta s} = 0$, which means that the binormal vector γ^i has a fixed direction. Hence, the curve Γ is a plane curve. From this example, we can say that the torsion of a curve measures how far the curve departs from lying in a plane.

EXAMPLE 5.4.9 *If* $(a\lambda^r + b\mu^r + c\gamma^r)$ *forms a parallel vector field along* Γ*, prove that*

$$\frac{da}{ds} - \kappa b = 0; \frac{db}{ds} + \kappa a - \tau c = 0; \frac{dc}{ds} + \tau b = 0,$$

where the notations have their usual meanings.

Solution: Let $A^r = a\lambda^r + b\mu^r + c\gamma^r$. This vector A^r forms a parallel vector field along Γ, if,

$$\frac{\delta A^i}{\delta s} = A^i_{,j}\frac{dx^j}{ds} = 0$$

or

$$\frac{\delta}{\delta s}\left(a\lambda^i + b\mu^i + c\gamma^i\right) = 0$$

or

$$\frac{da}{ds}\lambda^i + \frac{db}{ds}\mu^i + \frac{dc}{ds}\gamma^i + a\frac{\delta\lambda^i}{\delta s} + b\frac{\delta\mu^i}{\delta s} + c\frac{\delta\gamma^i}{\delta s} = 0$$

or

$$\frac{da}{ds}\lambda^i + \frac{db}{ds}\mu^i + \frac{dc}{ds}\gamma^i + a\kappa\mu^i + b\left(\tau\gamma^i - \kappa\lambda^i\right) + c(-\tau\mu^i) = 0$$

or

$$\left(\frac{da}{ds} - \kappa b\right)\lambda^i + \left(\frac{db}{ds} + \kappa a - \tau c\right)\mu^i + \left(\frac{dc}{ds} + \tau b\right)\gamma^i = 0.$$

Since λ^i, μ^i and γ^i are arbitrary, we get

$$\frac{da}{ds} - \kappa b = 0; \frac{db}{ds} + \kappa a - \tau c = 0; \frac{dc}{ds} + \tau b = 0.$$

Thus the curvature and the torsion are both constant, and therefore their ratio is constant. The principal normal intersects the axis of the cylinder orthogonally an the tangent and binormal are inclined at constant argles to the fixed direction of the generators.

5.5 Bertrand Curves

Saint–Venant proposed and Bertrand solved the problem of finding the curves whose principal normals are also the principal normals of another curve. If the two curves Γ and $\overline{\Gamma}$ given by (Figure 5.4)

$$\Gamma: \ x^i = x^i(s) \text{ and } \overline{\Gamma}: \ \overline{x}^i = \overline{x}^i(s), \tag{5.19}$$

have a common principal normal, at any of their points, the one is called the *Bertrand of another* or one is called the *Bertrand associate* of another.

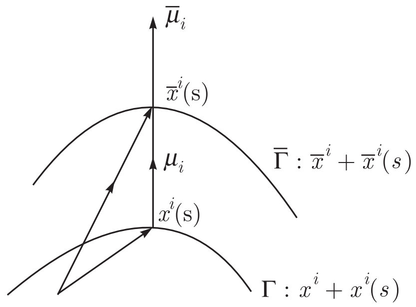

Figure 5.4: Bertrand curve.

EXAMPLE 5.5.1 *If Γ is a plane curve, find a curve $\bar{\Gamma}$ such that Γ and $\bar{\Gamma}$ are Bertrand curves.*

Solution: Let E be a plane evolute of Γ, then all involutes $\bar{\Gamma}$ of E have the same principal normal as the given curve $\bar{\Gamma}$, because by definition of the involute, these curves intersect orthogonally any tangent to E (Figure 5.5). Γ and any of the curve $\bar{\Gamma}$ are therefore Bertrand curves. Moreover, the distance measured along the common principal normal, between corresponding points of any two of these involutes is constant. Two plane curves are therefore called plane parallel curves if they have common principal normals.

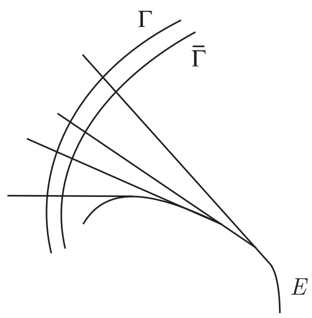

Figure 5.5: Plane curve.

EXAMPLE 5.5.2 *Find the necessary and sufficient condition for two curves* Γ *and* $\bar{\Gamma}$ *are Betrand.*

Solution: Let: $\Gamma : x^i = x^i(s)$ and

$$\bar{\Gamma} = \bar{x}^i = \bar{x}^i(s) = x^i(s) + a(s)\mu^i \tag{5.20}$$

Differentiating with respect to s intrinsically, we get

$$\begin{aligned}\frac{\delta\overline{x}^i}{\delta s} &= \frac{\delta x^i}{\delta s} + a^1(s)\mu^i + a(s)\frac{\delta\mu^i}{\delta s}\\ &= \lambda^i + a^1(s)\mu^i + a(s)\left[\tau\gamma_1^i - \kappa\lambda^i\right]\end{aligned}$$

or

$$\overline{\lambda}^i\frac{d\overline{s}}{ds} = \lambda^i + a^1(s)\mu^i + a(s)\left[\tau\gamma_1^i - \kappa\lambda^i\right]. \tag{5.21}$$

Since the two curves are Bertrand, so $\overline{\mu}^i$ is parallel to μ^i. Taking the inner product of Eq. (5.21) with μ^i we get

$$\overline{\mu}^i\overline{\lambda}^i\frac{d\overline{s}}{ds} = \lambda^i\mu^i + a^1(s)\mu^i\mu^i + a(s)\left[\tau\gamma_1^i\mu^i - \kappa\lambda^i\mu^i\right]$$

or

$$0 = 0 + a^1(s)1 + a(s)[\tau\cdot 0 - \kappa\cdot 0] \Rightarrow a(s) = \text{ constant},$$

which is the necessary condition that the two curves are Bertrand curves. Also, from Eq. (5.21), we get

$$\overline{\lambda}^i\frac{d\overline{s}}{ds} = \lambda^i + a(s)\left[\tau\gamma_1^i - \kappa\lambda^i\right] = \lambda^i[1 - a\kappa] + a\tau\gamma^i.$$

Further, refer to the curve $\overline{\Gamma}$, we have

$$\frac{d}{ds}(\lambda^i\cdot\overline{\lambda}^i) = \kappa\mu^i\overline{\lambda}^i + \lambda^i(\overline{\kappa}\mu^i)\frac{d\overline{s}}{ds} = 0 \Rightarrow \lambda^i\cdot\overline{\lambda}^i = \text{ constant}.$$

Thus, the tangents to the two curves are inclined at a constant angle. But the principal normals coincide, and therefore, the binormals of the two curves are inclined at the same constant angle. Let θ be the inclination of $\overline{\gamma}^i$ to γ^i, then θ is constant. If the curve Γ is not plane, i.e. $\tau \neq 0$, then from Eq. (5.21), we get

$$\overline{\lambda}^i = \lambda^i[1 - a\kappa]\frac{ds}{d\overline{s}} + a\tau\gamma^i\frac{ds}{d\overline{s}}. \tag{5.22}$$

λ^i lies in the plane $\lambda^i\gamma^i$, i.e. on the rectifying plane. Moreover, it is obvious from the Figure (5.6) that

$$\overline{\lambda}^i = \cos\theta\lambda^i + \sin\theta\gamma^i. \tag{5.23}$$

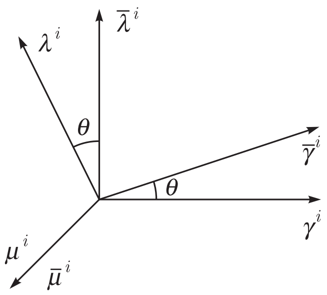

Figure 5.6: The triad.

Comparing Eqs. (5.22) and (5.23), we get

$$\frac{1 - a\kappa}{\cos\theta} = \frac{a\tau}{\sin\theta} = \frac{d\overline{s}}{ds} \Rightarrow 1 - a\kappa = a\tau\cot\theta. \tag{5.24}$$

Differentiating Eq. (5.23) with respect to s, we get

$$\begin{aligned}\overline{\kappa\mu}\frac{d\overline{s}}{ds} &= -\sin\theta\lambda^i\frac{d\theta}{ds} + \cos\theta\lambda\mu^i + \cos\theta\gamma^i\frac{d\theta}{ds} - \tau\mu^i\sin\theta \\ &= \mu^i(\kappa\cos\theta - \tau\sin\theta) + \left(\gamma^i\cos\theta - \lambda^i\sin\theta\right)\frac{d\theta}{ds}.\end{aligned}$$

Since the curves are Bertrand curve, so $\overline{\mu}^i$ and μ^i are parallel and equal. Therefore,

$$\begin{aligned}\left(\gamma^i\cos\theta - \lambda^i\sin\theta\right)\frac{d\theta}{ds} &= 0 \\ \Rightarrow \frac{d\theta}{ds} = 0 \Rightarrow \theta &= \text{ constant} \\ \Rightarrow 1 - a\kappa = a\tau\cot\theta &= b\tau(\text{say}),\end{aligned}$$

where $b = a\cot\theta =$ constant. Thus, we get a linear equation in κ and τ with constant coefficients as $a\kappa + b\tau = 1$. This is the necessary condition that the two curves are Bertrand curves when the curves is not plane.

We shall now show that if X and Y satisfying $a\kappa + b\tau = 1$, where a, b are constants, then Γ is Bertrand of $\overline{\Gamma}$. Differentiating Eq. (5.20) intrinsically with respect to s, we get

$$\begin{aligned}\overline{\lambda}^i \frac{d\overline{s}}{ds} &= \lambda^i[1 - a\kappa] + a\tau\gamma^i = \lambda^i b\tau + a\tau\gamma^i \\ &= \tau\left(b\lambda^i + a\gamma^i\right);\ \text{as } a\kappa + b\tau = 1\end{aligned}$$

or

$$\overline{\lambda}^i = \tau\left(b\lambda^i + a\gamma^i\right)\frac{ds}{d\overline{s}} = d\left(b\lambda^i + a\gamma^i\right).$$

Since $\overline{\lambda}^i$ is unit vector, so

$$1 = a^2 d^2 + b^2 d^2 \Rightarrow d = \frac{1}{\sqrt{a^2 + b^2}} = \text{ constant.}$$

Therefore

$$\overline{\lambda}^i = p\lambda^i + q\gamma^i;$$

where p and q are constants. Differentiating with respect to s, we get

$$\begin{aligned}\overline{\kappa\mu}\frac{d\overline{s}}{ds} &= p\kappa\mu^i - q\tau\mu^i = \mu^i(p\kappa - q\tau) \\ \Rightarrow \overline{\mu}^i &\text{ is parallel to } \mu^i \Rightarrow \overline{\mu}^i = \mu^i.\end{aligned}$$

Hence, the curves are Bertrand curves. Thus, the condition is sufficient.

Deduction 5.5.1 If the curve is a plane curve, then $\tau = 0$ and so

$$\overline{\lambda}^i \frac{d\overline{s}}{ds} = \lambda^i[1 - a\kappa], \tag{5.25}$$

i.e. $\overline{\lambda}^i$ is parallel to λ^i. Since $\frac{d\overline{s}}{ds}$ and $(1 - a\kappa)$ are constants, $\overline{\lambda}^i$ and λ^i are both constant vectors and $\overline{\lambda}^i = \lambda^i$. Differentiating intrinsically, with respect to s, we get,

$$\frac{\delta\overline{x}^i}{\delta\overline{s}}\frac{d\overline{s}}{ds} = \frac{\delta\lambda^i}{\delta s} \Rightarrow \overline{\kappa\mu}_i \frac{d\overline{s}}{ds} = \kappa\mu^i,$$

which shows that $\overline{\mu}^i$ is parallel to μ^i. Thus $a =$ constant is the necessary and sufficient condition that the two curves Γ and $\overline{\Gamma}$ are Bertrand curves, if Γ is a plane curve.

Deduction 5.5.2 There is a linear relation with constant coefficients between the curvature and torsion of Γ. On comparing, we get

$$\cos\theta = (1 - a\tau)\frac{ds}{d\overline{s}};\ \sin\theta = a\tau\frac{ds}{d\overline{s}}.$$

Now, the relation between the curves Γ and $\overline{\Gamma}$ is clearly a reciprocal one. Hence,

$$\cos\theta = (1 + a\tau)\frac{d\overline{s}}{ds};\ \sin\theta = a\tau\frac{d\overline{s}}{ds}.$$

On multiplying together of the above equations we obtain the relation

$$\tau\overline{\tau} = \frac{1}{a^2}\sin^2\theta.$$

This shows that the torsions of the two curves have the same sign, and their product is constant. This theorem is due to Schell. Also,

$$\cos^2\theta = (1 - a\tau)(1 + a\tau)$$

This shows that if P, P_1 are corresponding points on the two conjugate Bertrand curves, and O, O_1 their centres of curvature, the cross ratio of the range (POP_1O_1) is constant and equal to $\sec^2\theta$. This theorem is due to Mannheim.

5.6 Helix

A space curve traced on the surface of a cylinder, and cutting the generators at a constant angle, is called a *helix*. Therefore, the tangent vector at every point to a helix makes a constant angle with the fixed direction. Let

$$\Gamma:\ x^i = x^i(s) \tag{5.26}$$

be an equation of a helix. Let the tangent vector λ^i to the helix Γ makes a constant angle α with a fixed direction a^i (parallel to the generators of the cylinder), where a^i is a unit vector (Figure 5.7). Then,

$$\cos\alpha = g_{ij}\lambda^i a^j; 0 < |\alpha| \le \frac{1}{2}\pi. \tag{5.27}$$

Differentiating Eq. (5.27) intrinsically with respect to s, we get

$$g_{ij}\frac{\delta\lambda^i}{\delta s}a^j = 0$$

or

$$g_{ij}\kappa\mu^i a^j = 0;\ \text{as}\ \frac{\delta\lambda^i}{\delta s} = \kappa\mu^i$$

or

$$g_{ij}\mu^i a^j = 0;\ \text{as}\ \kappa > 0.$$

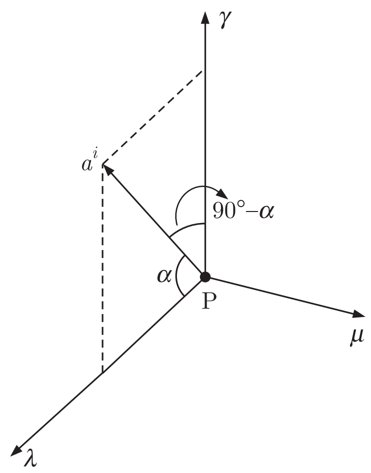

Figure 5.7: Helix.

So, a^i is orthogonal to μ^i. Thus, since the curvature of the helix does not vanish, the principal normal is everywhere perpendicular to the generators. Hence, the fixed direction of the generators is parallel to the plane of μ^i and γ^i; and since it makes a constant angle with λ^i, it makes a constant angle $90° - \alpha$ with γ^i.

An important property of all helices is that *the curvature and torsion are in a constant ratio.* Since the angle between a^i and γ^i is $90° - \alpha$, so,

$$g_{ij}\gamma^i a^j = \cos(90° - \alpha) = \sin\alpha. \tag{5.28}$$

Differentiate the relation $g_{ij}\mu^i a^j = 0$, we get

$$g_{ij}\left(\tau\gamma^i - \kappa\lambda^i\right)a^i = 0.$$

Thus, λ^i is perpendicular to $\tau\gamma^i - \kappa\lambda^i$. Since a^i must lie in the plane of μ^i and γ^i, we can find scalars a and b such that

$$a^i = a\lambda^i + b\gamma^i \tag{5.29}$$

or

$$g_{ij}a^i\lambda^j = ag_{ij}\lambda^i\lambda^j + bg_{ij}\gamma^i\lambda^j$$

or

$$\cos\alpha = a; \text{as } g_{ij}\lambda^i\lambda^j = 1 \text{ and } g_{ij}\gamma^i\lambda^j = 0.$$

Similarly, from Eq. (5.29) we get,

$$g_{ij}a^i\gamma^j = ag_{ij}\lambda^i\gamma^j + bg_{ij}\gamma^i\gamma^j$$

or

$$\sin\alpha = b; \text{ as } g_{ij}\lambda^i\gamma^j = 0 \text{ and } g_{ij}\gamma^i\gamma^j = 1.$$

Thus, Eq. (5.29) can be written as

$$a^i = \lambda^i \cos\alpha + \gamma^i \sin\alpha. \tag{5.30}$$

Differentiating Eq. (5.30) intrinsically with respect to s we get

$$\frac{\delta a^i}{\delta s} = \frac{\lambda a^i}{\delta s}\cos\alpha + \frac{\gamma a^i}{\delta s}\sin\alpha;\ \text{as } \alpha = \text{ constant}$$

or

$$0 = \kappa\mu^i \cos\alpha - \tau\mu^i \sin\alpha$$

or

$$\kappa\cos\alpha - \tau\sin\alpha = 0;\ \text{as } \mu^i \neq 0.$$

or

$$\frac{\kappa}{\tau} = \tan\alpha = \text{ constant}.$$

But a^i is parallel to the plane of λ^i and γ^i, and must therefore be parallel to the vector $\tau\lambda^i + \kappa\gamma^i$, which is inclined to λ^i at an angle $\tan^{-1}\frac{\kappa}{\tau}$. But this angle is constant. Therefore, the curvature and torsion are in a constant ratio.

Conversely, we may prove that *a curve whose curvature and torsion are in a constant ratio is a helix.* We suppose that for a space curve different from a straight line $\frac{\kappa}{\tau}$ is a constant, i.e. $\tau = c\kappa; c =$ constant. Then using Frenet formula we get

$$\frac{\delta\gamma^i}{\delta s} = -\kappa c\mu^i = -c\kappa\mu^i = -c\frac{\delta\lambda^i}{\delta s}$$

or

$$\frac{\delta}{\delta s}\left(\gamma^i + c\lambda^i\right) = 0.$$

Therefore, $\gamma^i + c\lambda^i$ is a constant vector, say a^i. Now,

$$\begin{aligned} g_{ij}a^i\lambda^j &= g_{ij}\left(\gamma^i + c\lambda^i\right)\lambda^j \\ &= g_{ij}\gamma^i\lambda^j + cg_{ij}\lambda^i\lambda^j = 0 + c = c. \end{aligned}$$

This shows that the tangent vector λ^i makes a constant angle with a fixed direction a^i, and the curve is therefore a general helix.

5.6.1 Cylindrical Helix

The *cylindrical helix* is a curve upon a cylinder which cuts the generator of the cylinder at a constant angle α with a fixed direction. Let us take z axis (x^3) as the axis of the cylindrical. Then the equation of the cylinder

$$x^i = \rho^i(t);\ i = 1, 2.$$

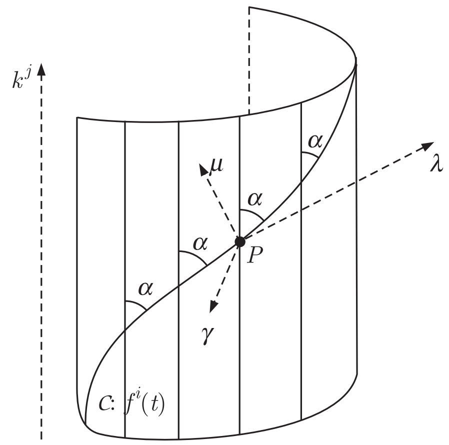

Figure 5.8: Cylindrical helix.

Any curve on the cylinder can be defined by $\mathcal{C}$: $x^i = f^i(t)$; $i = 1, 2, 3$ (Figure 5.8). Let α be the angle at any point P on the curve made by the tangent at P and the generator of this cylinder passing through P. Let the unit vector along the generator is

$$\overrightarrow{k} = (0, 0, 1) = (k^1, k^2, k^3)$$

and the unit tangent to the curve at P is

$$\begin{aligned}\lambda^i &= \frac{\delta x^i}{\delta s} = \frac{\delta x^i/\delta t}{\delta s/\delta t} = \frac{\delta x^i/\delta t}{|\delta x^i/\delta f|} \\ &= \frac{\left[f^1(t), f^2(t), f^3(t)\right]}{[\{f^1(t)\}^2 + \{f^2(t)\}^2 + \{f^3(t)\}^2]^{1/2}}.\end{aligned}$$

Hence, by given condition of the helix,

$$\cos\alpha = g_{ij}\lambda^i k^j = \frac{\dot{f}^3(t)}{\left[\{\dot{f}^1(t)\}^2 + \{\dot{f}^2(t)\}^2 + \{\dot{f}^3(t)\}^2\right]^{1/2}}$$

or

$$\{\dot{f}^3(t)\}^2 = \cos^2\alpha\left[\{\dot{f}^1(t)\}^2 + \{\dot{f}^2(t)\}^2 + \{\dot{f}^3(t)\}^2\right]$$

or

$$\{\dot{f}^3(t)\}^2\sin^2\alpha = \cos^2\alpha\left[\{\dot{f}^1(t)\}^2 + \{\dot{f}^2(t)\}^2\right]$$

or

$$\dot{f}^3(t) = \cot\alpha\left[\{\dot{f}^1(t)\}^2 + \{\dot{f}^2(t)\}^2\right]^{1/2}$$

or

$$f^3(t) = \cot\alpha \int \left[\{\dot{f}^1(t)\}^2 + \{\dot{f}^2(t)\}^2\right]^{1/2} + \text{ constant}$$

or

$$f^3(t) = x^3(t) = \cot\alpha \int \left[\{\dot{x}^1\}^2 + \{\dot{x}^2\}^2\right]^{1/2} + \text{ constant.}$$

Therefore, the equation of the cylindrical helix can be written as

$$\left.\begin{array}{l} x^1 = f^1(t),\ x^2 = f^2(t) \\ x^3 = f^3(t) = \cot\alpha \int \left[\{\dot{x}^1\}^2 + \{\dot{x}^2\}^2\right]^{1/2} + \text{ constant.} \end{array}\right\} \tag{5.31}$$

5.6.2 Circular Helix

If a space curve Γ lies on a circular cylinder and the tangent at each point of it makes a constant angle with the axis of the cylinder, then the curve is a helix. Such a helix is called a circular helix. Thus, a particular case of the cylindrical helix is circular helix, which can be defined by $x^1 = a\cos t$ and $x^2 = a\sin t$. Hence, by Eq. (5.33), we get

$$\begin{aligned} x^3(\alpha) &= \cot\alpha \int_0^t \sqrt{a^2(\cos^2 t + \sin^2 t)}d\alpha \\ &= a\cot\alpha t = bt, \text{ where } b = a\cot\alpha = \text{ constant.} \end{aligned}$$

Thus, the parametric equation of the circular helix can be written as

$$C : x^1 = a\cos t,\quad x^2 = a\sin t,\quad x^3 = bt;\quad b \neq 0. \tag{5.32}$$

This curve [Eq. (5.32)] can be defined by giving a constant rotation about the axis of the cylinder and a constant translation about the same axis (Figure 5.9).

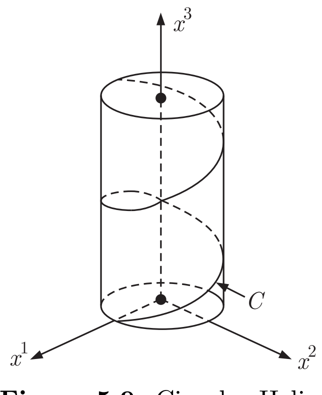

Figure 5.9: Circular Helix.

The circular helix is a twisted curve. The orthogonal projection of the helix into the x^1x^2-plane is the circle

$$(x^1)^2 + (x^2)^2 - a^2 = 0, x^3 = 0,$$

which is the intersection of this plane with the cylinder of revolution on which the helix lies. Projecting the helix orthogonally into the x^1x^3-plane, we obtain the sine curve

$$x^2 - a\sin\frac{x^3}{b} = 0, x^1 = 0.$$

A cosine curve is obtained by projecting the helix orthogonally into the x^1x^3-plane. Now

$$\frac{d\vec{r}}{dt} = (-a\sin t, a\cos t, b).$$

Therefore the arc $s(t)$ is given by

$$s(t) = \int_o^t \left[\frac{d\vec{r}}{dt}\cdot\frac{d\vec{r}}{dt}\right]^{1/2} dt = \int_o^t (a^2+b^2)^{1/2} dt$$

$$= \sqrt{a^2+b^2}\ \ t = wt; \quad \text{where } w = \sqrt{a^2+b^2}.$$

Thus the parametric representation of the circular helix with s as parameter is given by

$$\vec{r} = \left(a\cos\left(\frac{s}{w}\right),\ a\sin\left(\frac{s}{w}\right), \frac{b}{w}\right); \quad w = \sqrt{a^2+b^2}.$$

The unit tangent vector to this curve is of the form

$$\vec{\lambda} = \frac{d\vec{r}}{ds} = \left(-\frac{a}{w}\sin t, \frac{a}{w}\cos t, \frac{b}{w}\right); \quad w = \sqrt{a^2+b^2}.$$

The osculating plane of the circular helix can be represented in the form

$$\begin{vmatrix} z_1 - a\cos t & -a\sin t & -a\cos t \\ z_2 - a\sin t & a\cos t & -a\sin t \\ z_3 - bt & b & 0 \end{vmatrix} = 0$$

which is equivalent to

$$z_1 b\sin t - z_2 b\cos t + (z_3 - bt)a = 0$$

The osculating plane passes through the tangent to the helix and passes through the straight line $z^*(e) = (c\cos t, c\sin t, bt)$ where t is fixed and c is a real parameter. This line is parallel to x^1x^2-plane. It passes through the axis of the cylinder on which the helix lies as well as through a point P of the helix. Moreover, it is orthogonal to the tangent to the helix at P. Now the curvature and torsion of the circular helix are

$$\kappa = \left|\frac{d\vec{r}}{ds}\right| = \sqrt{\frac{d^2\vec{r}}{ds^2}\cdot\frac{d^2\vec{r}}{ds^2}} = \frac{a}{a^2+b^2}$$

$$\tau = \frac{\left|\left[\dfrac{d\vec{r}}{ds},\dfrac{d^2\vec{r}}{ds^2},\dfrac{d^3\vec{r}}{ds^3}\right]\right|}{\left(\dfrac{d\vec{r}}{ds}\cdot\dfrac{d\vec{r}}{ds}\right)\left(\dfrac{d^2\vec{r}}{ds^2}\cdot\dfrac{d^2\vec{r}}{ds^2}\right)-\left(\dfrac{d\vec{r}}{ds}\cdot\dfrac{d^2\vec{r}}{ds^2}\right)^2} = \frac{b}{a^2+b^2}.$$

The circular helix $\mathbf{r} = (a\cos t, a\sin t, bt)$ is said to be right-handed (Figure 5.10) if $b > 0$, where the curvature $k = \frac{a}{a^2+b^2}$. As the torsion $\tau = \frac{b}{a^2+b^2}$ so the curve is right-handed when $\tau > 0$.

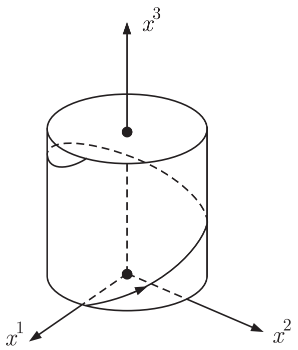

Figure 5.10: Circular Helix $\tau > 0$.

Similarly, the circular helix $\mathbf{r} = (a\cos t, a\sin t, bt)$ is said to be left-handed if $b < 0$, that is $\tau < 0$ (Figure 5.11).

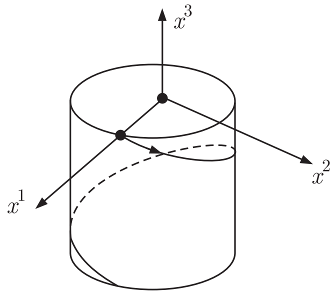

Figure 5.11: Circular Helix $\tau < 0$.

If b tends to 0, the radius of curvature $\rho = \frac{1}{\kappa}$ tends to the radius a of the circle of intersection between x^1x^2-plane and the cylinder of revolution on which the helix lies.

EXAMPLE 5.6.1 *Prove that*

$$\varepsilon_{ijk}\frac{\delta\lambda^i}{\delta s}\frac{\delta^2\lambda^j}{\delta s^2}\frac{\delta^3\lambda^k}{\delta s^3} = \kappa^5\frac{d}{ds}\left(\frac{\tau}{\kappa}\right).$$

Hence, a curve is a helix if and only if

$$\varepsilon_{ijk}\frac{\delta\lambda^i}{\delta s}\frac{\delta^2\lambda^j}{\delta s^2}\frac{\delta^3\lambda^k}{\delta s^3} = 0.$$

Solution: Using Eq. (5.7), we get

$$\frac{\delta\lambda^i}{\delta s} = \kappa\mu^i; \text{ where } \kappa > 0.$$

Differentiating intrinsically with respect to the parameter s we get

$$\frac{\delta^2\lambda^i}{\delta s^2} = \frac{\delta\kappa}{\delta s}\mu^i + \kappa\frac{\delta\mu^i}{\delta s} = \frac{\delta\kappa}{\delta s}\mu^i + \kappa\left(\tau\gamma^i - \kappa\lambda^i\right).$$

$$\begin{aligned}
\frac{\delta^3\lambda^i}{\delta s^3} &= \frac{\delta^2\kappa}{\delta s^2}\mu^i + \frac{\delta\kappa}{\delta s}\frac{\delta\mu^i}{\delta s} + \frac{\delta\kappa}{\delta s}\tau\nu^i \\
&\quad + \kappa\frac{\delta\tau}{\delta s}\gamma^i + \kappa\tau\frac{\delta\gamma^i}{\delta s} - 2\frac{\delta\kappa}{\delta s}\kappa\lambda^i - \kappa^2\frac{\delta\lambda^i}{\delta s} \\
&= \frac{\delta^2\kappa}{\delta s^2}\mu^i + \frac{\delta\kappa}{\delta s}\left(\tau\gamma^i - \kappa\lambda^i\right) + \left(\frac{\delta\kappa}{\delta s}\tau + \kappa\frac{\delta\tau}{\delta s}\right)\nu^i \\
&\quad + \kappa\tau\left(-\tau\mu^i\right) - 2\frac{\delta\kappa}{\delta s}\kappa\lambda^i - \kappa^2\left(\kappa\mu^i\right) \\
&= \left(\frac{\delta^2\kappa}{\delta s^2} - \kappa\tau^2 - \kappa^3\right)\mu^i + \left(2\frac{\delta\kappa}{\delta s}\tau + \kappa\frac{\delta\tau}{\delta s}\right)\gamma^i + \left(-3\kappa\frac{\delta\kappa}{\delta s}\right)\lambda^i.
\end{aligned}$$

Therefore the given expression becomes,

$$\begin{aligned}
&\varepsilon_{ijk}\frac{\delta\lambda^i}{\delta s}\frac{\delta^2\lambda^j}{\delta s^2}\frac{\delta^3\lambda^k}{\delta s^3} \\
&= \varepsilon_{ijk}\kappa\mu^i\left[\frac{\delta\kappa}{\delta s}\mu^j + \kappa\left(\tau\gamma^j - \kappa\lambda^j\right)\right]\left[\left(\frac{\delta^2\kappa}{\delta s^2} - \kappa\tau^2 - \kappa^3\right)\mu^k\right. \\
&\quad \left. + \left(2\frac{\delta\kappa}{\delta s}\tau + \kappa\frac{\delta\tau}{\delta s}\right)\gamma^k + \left(-3\kappa\frac{\delta\kappa}{\delta s}\right)\lambda^k\right] \\
&= \left(\kappa\frac{\delta\kappa}{\delta s}\varepsilon_{ijk}\mu^i\mu^j + \kappa^2\tau\varepsilon_{ijk}\mu^i\gamma^j - \kappa^3\varepsilon_{ijk}\mu^i\lambda^j\right) \\
&\quad \times\left[\left(\frac{\delta^2\kappa}{\delta s^2} - \kappa\tau^2 - \kappa^3\right)\mu^k + \left(2\frac{\delta\kappa}{\delta s}\tau + \kappa\frac{\delta\tau}{\delta s}\right)\gamma^k + \left(-3\kappa\frac{\delta\kappa}{\delta s}\right)\lambda^k\right] \\
&= \left(0 + \kappa^2\tau\lambda^k + \kappa^3\gamma^k\right)\left[\left(\frac{\delta^2\kappa}{\delta s^2} - \kappa\tau^2 - \kappa^3\right)\mu^k + \left(2\frac{\delta\kappa}{\delta s}\tau + \kappa\frac{\delta\tau}{\delta s}\right)\gamma^k\right. \\
&\quad \left. + \left(-3\kappa\frac{\delta\kappa}{\delta s}\right)\lambda^k\right]
\end{aligned}$$

$$= \left(\frac{\delta^2\kappa}{\delta s^2} - \kappa\tau^2 - \kappa^3\right)\left(\kappa^2\tau\lambda^k\mu^k + \kappa^3\mu^k\gamma^k\right) + \left(2\frac{\delta\kappa}{\delta s}\tau + \kappa\frac{\delta\tau}{\delta s}\right)$$

$$\times\left(\kappa^2\tau\lambda^k\gamma^k + \kappa^3\gamma^k\gamma^k\right) - 3\kappa\frac{\delta\kappa}{\delta s}\left(\kappa^2\tau\lambda^k\lambda^k + \kappa^3\gamma^k\lambda^k\right)$$

$$= \kappa^3\left(2\frac{\delta\kappa}{\delta s}\tau + \kappa\frac{\delta\tau}{\delta s}\right) - 3\kappa\frac{\delta\kappa}{\delta s}(\kappa^2\tau)$$

$$= \kappa^3\left(\kappa\frac{\delta\tau}{\delta s} - \tau 3\kappa\frac{\delta\kappa}{\delta s}\right) = \kappa^5\frac{1}{\kappa^2}\left(\kappa\frac{\delta\tau}{\delta s} - \tau 3\kappa\frac{\delta\kappa}{\delta s}\right) = \kappa^5\frac{d}{ds}\left(\frac{\tau}{\kappa}\right).$$

Thus the result is established. Now,

$$\varepsilon_{ijk}\frac{\delta\lambda^i}{\delta s}\frac{\delta^2\lambda^j}{\delta s^2}\frac{\delta^3\lambda^k}{\delta s^3} = 0$$

or

$$\left(\tau\lambda^i + \kappa\gamma^i\right)\mu^i(\kappa^2 + \tau^2) = 0$$

or

$$\tau\lambda^i + \kappa\gamma^i = 0;\ \text{as } \kappa^2 + \tau^2 \neq 0 \text{ and } \mu^i \text{ is arbitrary}$$

or

$$g_{ij}\left(\tau\lambda^i + \kappa\gamma^i\right) = 0.$$

If a^i is the fixed direction, then by hypothesis $g_{ij}a^i\lambda^j = \cos\alpha$ and $g_{ij}a^i\gamma^j = \sin\alpha$. Therefore,

$$g_{ij}a^i\left(\tau\lambda^i + \kappa\gamma^i\right) = 0$$

or

$$\kappa\cos\alpha + \tau\sin\alpha = 0$$

or

$$\frac{\kappa}{\tau} = -\tan\alpha = \text{ constant.}$$

Hence, the curve is a helix.

EXAMPLE 5.6.2 *Prove that a circular helix is a Bertrand curve.*

Solution: We know the circular helix is a twisted curve. The principal normals to a circular helix Γ intersect the axis of rotation of the cylinder Z on which Γ lies at right angles. The points of intersection of these normals with any cylinder Z^*, coaxial with Z, determine a circular helix $\overline{\Gamma}$ (Figure 5.12). Obviously Γ and $\overline{\Gamma}$ are Bertrand curves, and there exist infinitely many helices $\overline{\Gamma}$ of this type which, together with Γ, are Bertrand curves.

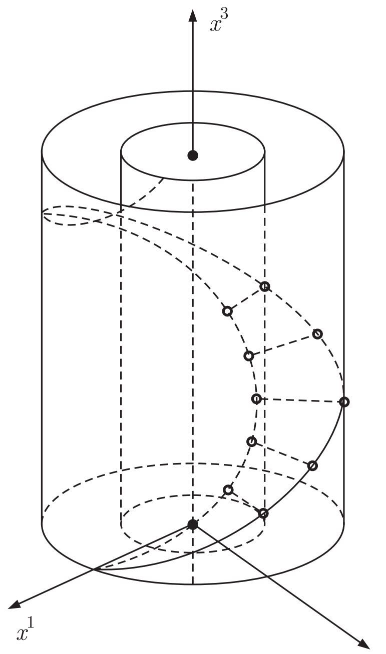

Figure 5.12: Coaxial circular helices $\tau < 0$.

If two curves $\Gamma : x^i(s), k > 0$ and $\overline{\Gamma} : \overline{x}^i(s)$ are Bertrand curves then, $\overline{\Gamma}$ can be represented in the form

$$\bar{x}^i(s) = x^i(s) + a(s)\mu^i(s)$$

where $\mu^i(s)$ is the unit principal normal vector to Γ and the scalar $a(s)$ is the distance of a point P^* of $\overline{\Gamma}$ from the corresponding point P of Γ. Now $a(s)$ is constant, that is, is dependent of S. Hence the circular helix is a Bertrand curve.

EXAMPLE 5.6.3 *If the tangent and binormal to a space curve make angles θ and ϕ with a fixed direction, show that*

$$\frac{\kappa}{\tau} = -\frac{\sin\theta}{\sin\phi}\frac{d\theta}{d\phi}.$$

Solution: Let the tangent vector with components λ^i to a space curve $\mathcal{C}$ make an angle θ with a fixed direction c^i, where the vector with components c^i is a unit vector and the binormal μ^i makes an angle ϕ with c^i. Then,

$$g_{ij}\lambda^i c^j = \cos\theta \ \text{ and } g_{ij}\mu^i c^j = \cos\phi.$$

Since θ and ϕ are invariants,

$$\frac{\delta\theta}{\delta s} = \frac{d\theta}{ds} \ \text{ and } \frac{\delta\phi}{\delta s} = \frac{d\phi}{ds}.$$

Taking intrinsic derivative of both the sides of the equation $g_{ij}\lambda^i c^j = \cos\theta$, we get

$$g_{ij}\left(\kappa\lambda^i\right)c^j = \frac{\delta}{\delta s}\cos\theta = \frac{d}{ds}(\cos\theta)$$

or

$$\kappa g_{ij}\lambda^i c^j = -\sin\theta\frac{d\theta}{ds}.$$

Similarly, from the relation $g_{ij}\mu^i c^j = \cos\phi$, we get

$$-\tau g_{ij}\lambda^i c^j = -\sin\phi\frac{d\phi}{ds}.$$

Combining these two results, we get

$$\frac{\kappa}{\tau} = -\frac{\sin\theta}{\sin\phi}\frac{d\theta}{d\phi}.$$

EXAMPLE 5.6.4 *A curve* Γ *is defined in cylindrical co-ordinates* x^i *as follows:*

$$x^1 = a,\ x^2 = t,\ x^3 = bt; b \neq 0$$

where a *and* b *are constants of which* a *is positive and* t *is a function of the natural parameter* s*. Find the curvature and torsion of* Γ*.*

Solution: The components of the tangent vector λ^i are given by

$$\lambda^1 = \frac{dx^1}{ds} = 0; \lambda^2 = \frac{dx^2}{ds} = \frac{dt}{ds}; \lambda^3 = \frac{dx^3}{ds} = b\frac{dt}{ds}.$$

Since $g_{ij}\lambda^i\lambda^j = 1$, we get

$$g_{11}\lambda^1\lambda^1 + g_{22}\lambda^2\lambda^2 + g_{33}\lambda^3\lambda^3 = 1$$

or

$$0 + \left(x^1\right)^2\left(\frac{dt}{ds}\right)^2 + b^2\left(\frac{dt}{ds}\right)^2 = 1$$

or

$$(a^2 + b^2)\left(\frac{dt}{ds}\right)^2 = 1 \Rightarrow \left(\frac{dt}{ds}\right)^2 = \frac{1}{a^2 + b^2}.$$

From the relation

$$\kappa\mu^i = \frac{d\lambda^i}{ds} + \left\{\begin{matrix} i \\ j\ \ k \end{matrix}\right\}\lambda^j\frac{dx^k}{ds},$$

we get

$$\kappa\mu^1 = \frac{d\lambda^1}{ds} + \left\{ \begin{matrix} 1 \\ 2 \;\; 2 \end{matrix} \right\} \lambda^2 \frac{dx^2}{ds} = -x^1 \left(\frac{dt}{ds}\right)^2 = -\frac{a}{a^2+b^2}$$

$$\kappa\mu^2 = \frac{d\lambda^2}{ds} + \left\{ \begin{matrix} 2 \\ 2 \;\; 1 \end{matrix} \right\} \lambda^2 \frac{dx^1}{ds} + \left\{ \begin{matrix} 2 \\ 1 \;\; 2 \end{matrix} \right\} \lambda^1 \frac{dx^2}{ds} = 0$$

$$\kappa\mu^3 = 0.$$

Using $g_{ij}\mu^i\mu^j = 1$, we get

$$g_{11}\mu^1\mu^1 + g_{22}\mu^2\mu^2 + g_{33}\mu^3\mu^3 = 1$$

or

$$-\frac{a}{\kappa(a^2+b^2)} . -\frac{a}{\kappa(a^2+b^2)} = 1 \Rightarrow \kappa = \frac{a}{a^2+b^2}.$$

So $\mu^1 = -1,\ \mu^2 = 0,$ and $\mu^3 = 0$. From the relation

$$\frac{d\mu^i}{ds} + \left\{ \begin{matrix} i \\ j \;\; k \end{matrix} \right\} \mu^j \frac{dx^k}{ds} = -\kappa\lambda^i + \tau\gamma^i,$$

we get

$$\frac{d\mu^1}{ds} + \left\{ \begin{matrix} 1 \\ 2 \;\; 2 \end{matrix} \right\} \mu^2 \frac{dx^2}{ds} = -\kappa\lambda^1 + \tau\gamma^2$$

or

$$0 + 0 = \tau\gamma^1 \Rightarrow \tau\gamma^1 = 0.$$

Similarly, $\tau\gamma^2 = \dfrac{-b^2}{a(a^2+b^2)^{3/2}}$ and $\tau\gamma^3 = \dfrac{-ab}{(a^2+b^2)^{3/2}}$. Using $g_{ij}\gamma^i\gamma^j = 1$, we get

$$g_{11}\gamma^1\gamma^1 + g_{22}\gamma^2\gamma^2 + g_{33}\gamma^3\gamma^3 = 1$$

or

$$0 + a^2 \frac{-b^2}{\tau a(a^2+b^2)^{3/2}} \cdot \frac{-b^2}{\tau a(a^2+b^2)^{3/2}} + \frac{-ab}{\tau(a^2+b^2)^{3/2}} \cdot \frac{-ab}{\tau(a^2+b^2)^{3/2}} = 1$$

or

$$\tau^2 = \frac{b^2(a^2+b^2)}{(a^2+b^2)^3} = \frac{b^2}{(a^2+b^2)^2} \Rightarrow \tau = \frac{b}{a^2+b^2}.$$

Thus, for the given helix, both the curvature κ and the torsion τ are constants. Next, we shall show that κ and τ are constants, thus, there exists a unique circular helix. Let $\kappa = l$ and $\tau = m$ be constants. Choose

$$l = \frac{a}{a^2+b^2} \quad \text{and} \quad m = \frac{b}{a^2+b^2},$$

where a and b are constants. Therefore,

$$\frac{l}{a} = \frac{m}{b} = \frac{1}{a^2 + b^2} \Rightarrow a = l(a^2 + b^2),\ b = m(a^2 + b^2).$$

Also

$$l^2 + m^2 = \frac{a^2 + b^2}{(a^2 + b^2)^2} = \frac{1}{a^2 + b^2}$$

So

$$\frac{a}{l} = \frac{b}{m} = a^2 + b^2 = \frac{1}{l^2 + m^2}.$$

Let us form a curve

$$\Gamma\colon x^i(t) = \left\{ \frac{l}{l^2 + m^2} \cos t, \frac{m}{l^2 + m^2} \sin t, \frac{m}{l^2 + m^2} t \right\},$$

which will satisfy all the necessary conditions for a circular helix.

5.7 Spherical Indicatrix

The locus of a point, whose position vector is equal to unit tangent λ^i of a given curve, is called the *spherical indicatrix of the tangent* to the curve. Such a locus lies on the surface of a unit sphere; hence the name. Let the two curves Γ and $\overline{\Gamma}$ be given by

$$\Gamma\colon\ x^i = x^i(s) \text{ and } \overline{\Gamma}\colon\ \overline{x}^i = \overline{x}^i(s).$$

of class 2 with non-vanishing curvature. We assume that the tangent principal normal and the binormal vectors undergo a parallel displacement (Figure 5.13) and become bound at the origion O is space. Then the terminal points of these vectors $\lambda(s), \mu(s)$ are $\gamma(s)$ lie on a unit sphere S with centre O and generate, is general, three curves on S which are called tangent indicatrix, the principal normal indicatrix and the binormal indicatrix, respectively, of the curve Γ.

Thus, according to the definition of tangent indicatrix, $\overline{x}^i(s) = \lambda^i$, therefore,

$$\overline{\lambda}^i = \frac{d\overline{x}^i}{d\overline{s}} = \frac{d\lambda^i}{ds}\frac{ds}{d\overline{s}} = \kappa\mu^i \frac{ds}{d\overline{s}}$$

showing that the tangent to the spherical indicatrix is parallel to the principal normal at the corresponding point of the given curve.

The linear element $d\overline{s}_1, d\overline{s}_N$ and $d\overline{s}_B$ of these indicatrices or spherical images can be obtained by means of Eq. (5.17) for the Serret-Frenet formula. Since $\lambda(s), \mu(s)$ and

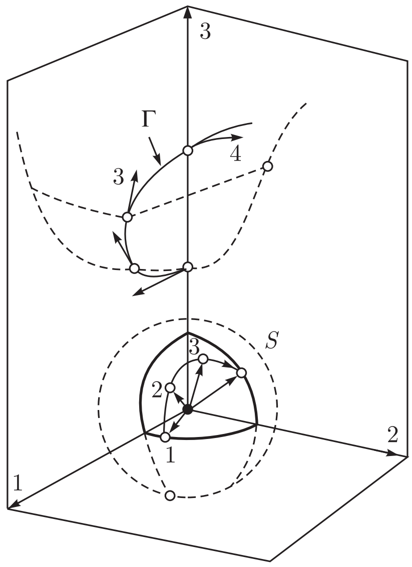

Figure 5.13: Curve Γ and corresponding tangent indicatrix.

$\gamma(s)$ are the vector functions representing these curves we find

$$d\bar{s}_T^2 = \frac{d\lambda^i}{ds}\frac{d\lambda^j}{ds}ds^2 = k^2\mu^i\mu^j ds^2 = k^2 ds^2$$

$$d\bar{s}_N^2 = \frac{d\mu^i}{ds}\frac{d\mu^j}{ds}ds^2 = (-k\lambda^i + \tau\gamma^i)(-k\lambda^i + \tau\gamma^i)ds^2 = (k^2 + \tau^2)ds^2$$

$$d\bar{s}_B^2 = \frac{d\gamma^i}{ds}\frac{d\gamma^i}{ds}ds^2 = (\tau\mu^i)(\tau\mu^j)ds^2 = \tau^2 ds^2.$$

Moreover we obtain the Equation of Lancret $ds_N^2 = ds_T^2 + ds_B^2$. The expression $\sqrt{ds_T^2 + ds_B^2}$ is sometimes called the third and total curvature of a curve.

We may measure $\bar{s}$ so that $\overline{\lambda}^i = \mu^i$ and therefore $\frac{d\bar{s}}{ds} = \kappa$. For the curvature $\overline{\kappa}$ of the indicatrix, on differentiating the relation $\overline{\lambda}^i = \mu^i$, we find the formula

$$\overline{\kappa\mu}^i = \frac{d\mu^i}{ds}\frac{ds}{d\bar{s}} = \frac{1}{\kappa}\left(\tau\gamma^i - \kappa\lambda^i\right).$$

Squaring both sides we obtain the result

$$\overline{\kappa}^2 = \frac{1}{\kappa^2}(\kappa^2 + \tau^2),$$

so that the curvature of the indicatrix is the ratio of the screw curvature to the circular curvature of the curve. The unit binormal of the indicatrix is

$$\overline{\gamma}^i = \varepsilon^{ijk}\lambda_j\mu_k = \frac{1}{\kappa\overline{\kappa}}(\tau\lambda^i + \kappa\gamma^i).$$

Using the result

$$\varepsilon_{ijk}\frac{\delta\lambda^i}{\delta s}\frac{\delta^2\lambda^j}{\delta s^2}\frac{\delta^3\lambda^k}{\delta s^3} = \kappa^3\left(\kappa\frac{\delta\tau}{\delta s} - \tau\frac{\delta\kappa}{\delta s}\right)$$

the torsion can be obtained as

$$\overline{\kappa}^2\overline{\tau}\left(\frac{d\overline{s}}{ds}\right)^6 = \varepsilon_{ijk}\frac{\delta\lambda^i}{\delta s}\frac{\delta^2\lambda^j}{\delta s^2}\frac{\delta^3\lambda^k}{\delta s^3} = \kappa^3\left(\kappa\frac{\delta\tau}{\delta s} - \tau\frac{\delta\kappa}{\delta s}\right)$$
$$\Rightarrow \overline{\tau} = \frac{1}{\kappa(\kappa^2+\tau^2)}\left(\kappa\frac{\delta\tau}{\delta s} - \tau\frac{\delta\kappa}{\delta s}\right).$$

Similarly, the spherical indicatrix of the binormal of the given curve is the locus of a point whose position vector is $\overline{x}^i = \gamma^i$. Therefore,

$$\overline{\lambda}^i = \frac{d\gamma^i}{ds}\frac{ds}{d\overline{s}} = -\tau\mu^i\frac{ds}{d\overline{s}}.$$

We may measure $\overline{s}$ so that $\overline{\lambda}^i = -\mu^i$, and therefore $\frac{d\overline{s}}{ds} = \tau$. To find the curvature differentiate the equation $\overline{\lambda}^i = -\mu^i$, we get

$$\overline{\kappa\mu}^i = \frac{d}{ds}(-\mu^i)\frac{ds}{d\overline{s}} = \frac{1}{\tau}(\kappa\lambda^i - \tau\gamma^i),$$

giving the direction of the principal normal. On squaring this result we get

$$\overline{\kappa}^2 = \frac{1}{\tau^2}(\kappa^2+\tau^2).$$

Thus, the curvature of the indicatrix is the ratio of the screw curvature to the torsion of the given curve. The unit binormal is

$$\overline{\gamma}^i = \frac{1}{\tau\overline{\kappa}}(\tau\lambda^i + \kappa\gamma^i)$$

and the torsion, found as in the previous case, is equal to

$$\overline{\tau} = \frac{1}{\tau(\kappa^2+\tau^2)}\left(-\kappa\frac{\delta\tau}{\delta s} + \tau\frac{\delta\kappa}{\delta s}\right).$$

Different curves may have the same spherical images. Simple examples illustrating this fact are circles in the same plane, with arbitrary radius and centre, and also circular helices on coaxial cylinders (Figure 5.12) for which the ratio $a:b$ is the same.

5.8 Exercises

1. If the principal normal of a space curve vanishes identically, show that its equation in curvilinear co-ordinates x^i is given by

$$\frac{d^2x^i}{ds^2} + \left\{ \begin{matrix} i \\ j \quad k \end{matrix} \right\} \frac{dx^j}{ds}\frac{dx^k}{ds} = 0.$$

2. If ε^{ijk} is the covariant permutation tensor, show that
   (i) $\varepsilon^{ijk}\mu_i\gamma_j = \lambda^k = -\varepsilon^{ijk}\gamma_i\mu_j$
   (ii) $\varepsilon^{ijk}\gamma_i\lambda_j = \mu^k = -\varepsilon^{ijk}\lambda_i\gamma_j$
   (iii) $\varepsilon^{ijk}\lambda_i\mu_j = \gamma^k = -\varepsilon^{ijk}\mu_i\lambda_j.$

3. Show that
   (i) $g_{ij}\dfrac{\delta\lambda^i}{\delta s}\dfrac{\delta\lambda^j}{\delta s} = -\kappa\tau$
   (ii) $g_{ij}\dfrac{\delta\mu^i}{\delta s}\dfrac{\delta\mu^j}{\delta s} = \kappa^2 + \tau^2$
   (iii) $g_{ij}\dfrac{\delta\gamma^i}{\delta s}\dfrac{\delta\gamma^j}{\delta s} = \tau^2;$
   (iv) $g_{ij}\dfrac{\delta\lambda^i}{\delta s}\dfrac{\delta\mu^j}{\delta s} = 0.$

4. Prove that,

$$\varepsilon_{ijk}\frac{\delta\gamma^i}{\delta s}\frac{\delta^2\gamma^j}{\delta s^2}\frac{\delta^3\gamma^k}{\delta s^3} = \tau^5\frac{d}{ds}\left(\frac{\kappa}{\tau}\right).$$

5. Prove that
   (i) $-\kappa = g_{ij}\dfrac{\delta\mu^i}{\delta s}\lambda^j = -g_{ij}\mu^i\dfrac{\delta\lambda^j}{\delta s}$
   (ii) $\tau = g_{ij}\dfrac{\delta\mu^i}{\delta s}\gamma^j = -g_{ij}\mu^i\dfrac{\delta\gamma^j}{\delta s}$
   (iii) $\dfrac{\delta\mu^i}{\delta s} = \left(g_{ij}\dfrac{\delta\mu^i}{\delta s}\lambda^j\right)\lambda^i + \left(g_{ij}\dfrac{\delta\mu^i}{\delta s}\mu^j\right)\mu^i + \left(g_{ij}\dfrac{\delta\mu^i}{\delta s}\gamma^j\right)\gamma^i$
   (iv) $g_{ij}\dfrac{\delta\gamma^i}{\delta s}\lambda^j = 0;\ (v) g_{ij}\gamma^i\dfrac{\delta\lambda^j}{\delta s} = 0;\ (vi) g_{ij}\dfrac{\delta\mu^i}{\delta s}\mu^j = 0.$

6. If $a^i = \tau\lambda^i + \kappa\gamma^j$, prove that

$$\frac{\delta\lambda^k}{\delta s} = \varepsilon^{ijk}a_i\lambda_j;\ \frac{\delta\mu^k}{\delta s} = \varepsilon^{ijk}a_i\mu_j;\ \frac{\delta\gamma^k}{\delta s} = \varepsilon^{ijk}a_i\gamma_j.$$

7. Prove that a space curve is helix if and only if
$$\varepsilon_{ijk}\frac{\delta\lambda^i}{\delta s}\frac{\delta^2\lambda^j}{\delta s^2}\frac{\delta^3\lambda^k}{\delta s^3}=0.$$
8. Find the curvature and torsion at any point of a given curve Γ, given by
$$\Gamma\colon x^1=a, x^2=t, x^3=ct,$$
where $ds^2=\left(dx^1\right)^2+(x^1)^2\left(dx^2\right)^2+\left(dx^3\right)^2$, a, c are non-zero constants. Also, show that the ratio of the curvature and torsion is a constant.
9. Show that the ratio of the curvature to the torsion of a space curve different from a straight line is a non-zero constant if and only if the curve is a helix.
10. If the curvature and torsion of a space curve are both constants, prove that the curve is a circular helix.
11. If κ and τ are the curvature and torsion of a circular helix given in cylindrical co-ordinates x^i by
$$x^1=a,\ x^2=t,\ x^3=bt; b\neq 0,$$
where $a(>0)$ and b are non-zero constants, show that
(a) $a\kappa+b\tau=1$.
(b) $a=\dfrac{\kappa}{\kappa^2+\tau^2}, b=\dfrac{\tau}{\kappa^2+\tau^2}$.
12. Show that the curvature and torsion of a helix are in a constant ratio to the curvature of the plane section of the cylinder perpendicular to the generators.
13. Defining a Bertrand curve as a space curve for which $a\kappa+b\tau=1$, where a and b are non-zero constants with $a>0$, prove that a circular helix is a Bertrand curve.
14. Prove that the principal normal of a cylindrical helix is everywhere perpendicular to the generators of the cylinder.
15. Prove that a circular helix C is the only twisted curve for which more than one curve C^* exists such that C and C^* are Bertrand curves.
16. Prove that the product of the torsions of Bertrand curves is constant.
17. Find the curves of which
(a) the tangent indicatrix
(b) the binormal indicatrix
degenerates a point. What does it mean when a spherical image of a closed curve?
18. Investigate the spherical images of a circular helix.

Chapter 6

Intrinsic Geometry of Surface

We will investigate the differential geometry of surfaces by means of tensor calculus. A great advantage of this method lies in the fact that it can immediately be generalized to Riemannian spaces of higher dimension which have assumed increasing importance during the last few decades. In addition, many aspects of the theory of surfaces are simplified when treated with the aid of tensor calculus which thus leads to a better and deeper insight into several problems of differential geometry.

In this chapter we will study the geometric properties of surfaces imbedded in a three-dimensional Euclidean space by means of differential geometry. The reasoning which will lead us to a definition of a surface is similar to that which led us to the concept of a curve. It will be shown that certain of these properties can be phrased independently of the space in which the surface is immersed and that they are concerned solely with the structure of the differential quadratic form for the element of an arc of a curve drawn on a surface. All such properties of surfaces are termed as the *intrinsic properties* and the geometry based on the study of this differential quadratic form is called the *intrinsic geometry of the surface.*

6.1 Curvilinear Co-ordinates on a Surface

A surface often arises as the locus of a point P which satisfies some restrictions. We find it convenient to refer the space in which surface is imbedded to a set of orthogonal Cartesian axes Y and regarded the locus of points satisfying the equation

$$F\left(y^1, y^2, y^3\right) = 0 \tag{6.1}$$

as an analytical implicit or constraint definition of a surface $\mathcal{S}$. We suppose that only two of the variables y^i in Eq. (6.1) are independent and that the specification of, say y^1 and y^2 in some region of the Y^1Y^2-plane determines uniquely a real number y^3 such that the $F\left(y^1, y^2, y^3\right)$ reduces to 0. If we suppose that $F\left(y^1, y^2, y^3\right)$, regarded as a function of three independent variables, is of class C^1 in some region R about the point $P_0\left(y_0^1, y_0^2, y_0^3\right)$ with $\left(\frac{\partial F}{\partial y^3}\right)_{P_0} \neq 0$ and $F\left(y_0^1, y_0^2, y_0^3\right) = 0$, then the fundamental theorem on implicit function guarantees the existence of a unique solution $y^3 = f\left(y^1, y^2\right)$, such that $y_0^3 = f\left(y_0^1, y_0^2\right)$.

We are now going to investigate the properties of a surface in its relation to the surrounding space. Consequently, we are dealing with two distinct system of co-ordinates namely, the three curvilinear co-ordinates for the surrounding space which we denote by $y^i;\ i = 1, 2, 3$ and the two curvilinear co-ordinates of the surface which we denote by $u^\alpha;\ \alpha = 1, 2$. A surface $\mathcal{S}$ is defined, in general, due to Gauss, as the set of points whose co-ordinates are functions of two independent parameters. Let us denote the equation of a surface $\mathcal{S}$ embedded in E^3 by

$$\mathcal{S}\colon y^1 = y^1(u^1, u^2);\ y^2 = y^2(u^1, u^2);\ y^3 = y^3(u^1, u^2)$$

or

$$\mathcal{S}\colon y^i = y^i\left(u^1, u^2\right); i = 1, 2, 3 \tag{6.2}$$

where $u_1^1 \le u^1 \le u_2^1$ and $u_1^2 \le u^2 \le u_2^2$ are parameters and the y^i are real functions of class C^1 in the region of definition of the independent parameters u^1, u^2. The function $y^i(u^1, u^2)$ are single-valued and continuous, and are here assumed to possess continuous partial derivatives of the rth order. In this case the surface is said to be of class r. In order to reconcile these two different definitions we shall require that the functions $y^i\left(u^1, u^2\right)$ be such that the Jacobian matrix

$$J = \begin{bmatrix} \dfrac{\partial y^1}{\partial u^1} & \dfrac{\partial y^2}{\partial u^1} & \dfrac{\partial y^3}{\partial u^1} \\ \dfrac{\partial y^1}{\partial u^2} & \dfrac{\partial y^2}{\partial u^2} & \dfrac{\partial y^3}{\partial u^2} \end{bmatrix} \tag{6.3}$$

be of rank 2, so that not all the determinants of the second order selected from this matrix vanish identically in the region of definition of parameters u^i. The three second order matrices

$$J_1 = \begin{bmatrix} \dfrac{\partial y^2}{\partial u^1} & \dfrac{\partial y^2}{\partial u^2} \\ \dfrac{\partial y^3}{\partial u^1} & \dfrac{\partial y^3}{\partial u^2} \end{bmatrix};\ J_2 = \begin{bmatrix} \dfrac{\partial y^3}{\partial u^1} & \dfrac{\partial y^3}{\partial u^2} \\ \dfrac{\partial y^1}{\partial u^1} & \dfrac{\partial y^1}{\partial u^2} \end{bmatrix};\ J_3 = \begin{bmatrix} \dfrac{\partial y^1}{\partial u^1} & \dfrac{\partial y^1}{\partial u^2} \\ \dfrac{\partial y^2}{\partial u^1} & \dfrac{\partial y^2}{\partial u^2} \end{bmatrix}$$

are called *permuted submatrices* of J. This requirement ensures that it is possible to solve two equations in Eq. (6.2) for u^1 and u^2 in terms of some pair of variables y^i, and the substitution of these solutions in the remaining equation leads to an equation of the form $y^3 = y^3\left(y^1, y^2\right)$. A representation of the form [Eq. (6.2)] is said to be an *allowable representation,* and parameters u^1, u^2 are frequently called curvilinear co-ordinates.

Note that, if any two determinants formed from the matrix [Eq. (6.3)] vanish identically, then the third one also vanishes, provided that the surface $\mathcal{S}$ is not a plane parallel to one of the co-ordinate planes.

6.1.1 Nature of Surface Co-ordinates

Since u^1 and u^2 are independent variables, the locus defined by Eq. (6.2) in two-dimensional, and these equations give the co-ordinates y^i of a point on the surface when u^1 and u^2 are assigned particular values. This point of view leads one to consider the surface as a two-dimensional manifold $\mathcal{S}$ imbedded in a three-dimensional enveloping space E_3. We can also show surface without reference to the surrounding space, and consider parameters u^1 and u^2 as co-ordinate points in the surface. A familiar example of this is the use of the latitude and longitude as co-ordinates of points on the surface of the earth.

The intersection of a pair of co-ordinate curves obtained by setting $u^1 = u^1_0; u^2 = u^2_0$ determines a point P_0. The variables u^1, u^2 determining the point of $\mathcal{S}$ are called the *curvilinear* or *Gaussian co-ordinates* of the surface (Figure 6.1).

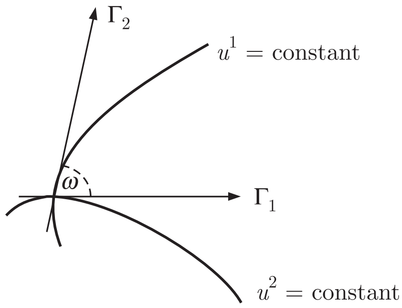

Figure 6.1: Curvilinear co-ordinates.

Obviously, the parametric representation of a surface is not unique and there are infinitely many curvilinear co-ordinate systems which can be used to locate points on a given surface $\mathcal{S}$. For example,

$$x^1 = u^1 + u^2;\ x^2 = u^1 - u^2;\ x^3 = 4u^1u^2$$

and

$$x^1 = v^1 \cosh v^2;\ x^2 = v^1 \sinh v^2;\ x^3 = (v^1)^2$$

represent the same surface

$$(x^1)^2 - (x^2)^2 = x^3;\ \ \text{hyperbolic paraboloid.}$$

The two representation may be related by the parametric transformation

$$v^1 = 2\sqrt{u^1u^2};\ \ v^2 = \frac{1}{2}\log\left(\frac{u^1}{u^2}\right).$$

Let us examine the geometric significance of these co-ordinates (u^1, u^2). If in Eq. (6.2) of the surface $\mathcal{S}$ we put, $u^1 = c$, a constant and let u^2 only vary, we get a locus given by a single variable parameter (one-dimensional manifold). Such a curve

$$\mathcal{S}: y^i = y^i\left(c, u^2\right); i = 1, 2, 3 \tag{6.4}$$

is called a u^2 curve, characterised by the equation

$$u^2 \text{ curve: } u^1 = \text{ constant.}$$

For different values of u^1, we shall get a family of u^1 curves covering the whole surface. Similarly, for constant value of u^2 we get a family of curves, each called an u^1 curve. In this way we get a family of u^1 curves and a family of u^1 curves, each family covering the whole surface. Thus, u^1 curve is characterised by the equation

$$u^1 \text{ curve: } u^2 = \text{ constant.}$$

Both the families are called *family of co-ordinate curves.* Together they are called the *co-ordinate net.* For example, on the surface of revolution $x = u\cos\phi$, $y = u\sin\phi, z = f(u)$ the parametric curves are

$$u \text{ curve: } x = u\cos c_1, y = u\sin c_1, z = f(u);\ c_1 = \text{ constant.}$$
$$\phi \text{ curve: } x = c_2\cos\phi, y = c_2\sin\phi, z = f(c_2);\ c_2 = \text{ constant.}$$

EXAMPLE 6.1.1 *A surface $\mathcal{S}$ is generated by a curve rotating about a fixed straight line. Find a parametric representation of $\mathcal{S}$.*

Solution: A surface $\mathcal{S}$ generated by a curve rotating about a fixed straight line A is called a *surface of revolution.* A is called the axis of the surface $\mathcal{S}$. A surface of revolution $\mathcal{S}$, not a cylinder, can be represented in the form (Figure 6.2).

$$\begin{aligned}\mathbf{r} &= \left(x^1, x^2, x^3\right) = \left[x^1\left(u^1, u^2\right), x^2\left(u^1, u^2\right), x^3\left(u^1, u^2\right)\right]\\ &= \left(u^1\cos u^2, u^1\sin u^2, f(u^1)\right).\end{aligned}$$

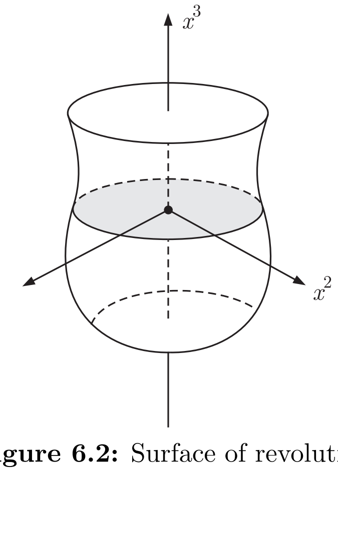

Figure 6.2: Surface of revolution.

This representation is allowable when $u^1 > 0$ and $\frac{df}{du^1} < \infty$, i.e. at any point of the rotating curve $\mathcal{C}$ which does not lie on the axis A of $\mathcal{S}$ or at which the tangent to $\mathcal{C}$ is not parallel to A. The curves $u^2 =$ constant are called the *meridians*, which are the curves of intersection of $\mathcal{S}$ and the planes passing through the x^3 axis and the curves $u^1 =$ constant are called *parallels*, which are the circles parallel to the x^1x^2 plane.

EXAMPLE 6.1.2 *Find a parametric representation of a right conoid.*

Solution: A surface $\mathcal{S}$ is called a *right conoid* if it can be generated by a moving straight line G intersecting a fixed straight line G_0 so that G and G_0 are always orthogonal. Let us choose the co-ordinates in space so that the straight line G_0 coincides with x^3 axis. Then G is parallel to the x^1x^2 plane, and a right conoid $\mathcal{S}$ generated by G can be represented in the form (Figure 6.3)

$$\begin{aligned}\mathbf{r} &= \left(x^1, x^2, x^3\right) = \left[x^1\left(u^1, u^2\right), x^2\left(u^1, u^2\right), x^3\left(u^1, u^2\right)\right] \\ &= \left(u^1 \cos u^2, u^1 \sin u^2, f(u^2)\right).\end{aligned}$$

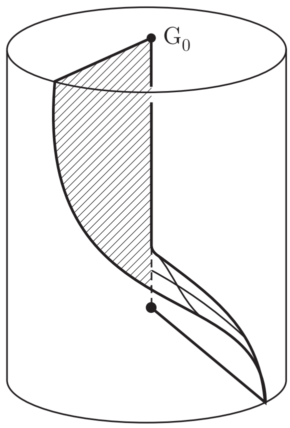

Figure 6.3: Right conoid.

This representation is allowable at any point at which $\dfrac{df}{du^2} < \infty$ and $(u^1)^2 + \left(\dfrac{df}{du^2}\right)^2 > 0$. The co-ordinate curves $u^2 =$ constant are called straight lines; each value of u^2 corresponds to a certain position of G in space. Any co-ordinate line $u^1 =$ constant is the locus of all points of $\mathcal{S}$ which are at the same distance from G_0.

If in particular f is a linear function of u^2, say $f = cu^2 + d$, the corresponding right conoid is called a *helicoid.* In this case, the curves $u^1 =$ constant are *circular helices.*

EXAMPLE 6.1.3 *Find a parametric representation of a general helicoid.*

Solution: A helicoid is a surface generated by the screw motion of a curve about a fixed line, the axis. The various positions of the generating curve thus obtained by first translating it through a distance a parallel to the axis and then rotating it through an angle u^2 about the axis, where $\frac{a}{u^2}$ has a constant value. The constant $\frac{2\pi a}{u^2}$ is the pitch of the helicoids, being the distance translated in one complete revolution. If a twisted curve $\mathcal{C}$ rotates about a fixed axis A and, at the same time, is displaced parallel to A so that the velocity of displacement is always proportional to the angular velocity of rotation, then $\mathcal{C}$ generates a surface $\mathcal{S}$ which is called *general helicoid.* The intersection M of a helicoid $\mathcal{S}$ and a plane passing through the axis A of $\mathcal{S}$ is called a *meridian* of $\mathcal{S}$. The meridians are congruent curves. Consequently, any general helicoid can be generated by a plane curve. We choose as axis A the x^3 axis of the co-ordinate space. Then M can be represented in the form $x^3 = f(u^1)$, where u^1 is the distance of the points of M from A. We assume that at the start of the motion M lies in the x^1x^2 plane. Let u^2 denote the angle of rotation. The displacement of M has the direction of the x^3 axis and is proportional to u^2. We may thus represent $\mathcal{S}$ in the form

$$\begin{aligned}\mathbf{r} &= \left(x^1, x^2, x^3\right) = \left[x^1\left(u^1, u^2\right), x^2\left(u^1, u^2\right), x^3\left(u^1, u^2\right)\right] \\ &= \left(u^1 \cos u^2, u^1 \sin u^2, cu^2 + f(u^1)\right).\end{aligned}$$

where C is a constant, and $2\pi c$ is called the pitch of the helicoidal motion. If $c \neq 0$ this local representation is allowable at any point at which the tangent to M is not parallel to A. The curves u^1 = constant are circular helices; the curves u^2 = constant are the meridians of the helicoid.

EXAMPLE 6.1.4 *Find the parametric representation of the sphere.*

Solution: Here we are dealing with a surface of a sphere of radius r, and that three mutually perpendicular diameters are chosen as co-ordinate axes. The sphere of radius r with centre at (0, 0, 0) can be represented in the form Eq. (6.1) by $\sum_{i=1}^{n}(x^i)^2 - r^2 = 0$. The latitude u^2 of a point P on the surface may be defined as the inclination of the radius through P to the x^1x^2 plane, and the longitude u^1 as the inclination of the plane containing P and the x^3-axis to the x^3x^1-plane. From this we can obtain a representation of the form (6.2)

$$x^3 = \pm\sqrt{r^2 - (x^1)^2 - (x^2)^2},$$

depending on the choice of the sign this is a representation of one of the two hemispheres $x^3 \geq 0$ and $x^3 \leq 0$. The parametric representation of the sphere under consideration is (Figure 6.4)

$$\mathbf{r} = (x^1, x^2, x^3) = (r \cos u^2 cos u^1, r \cos u^2 \sin u^1, r \sin u^2)$$

or, at length

$$x^1 = r\cos u^2 cosu^1, x^2 = r\cos u^2 \sin u^1, x^3 = r\sin u^2$$

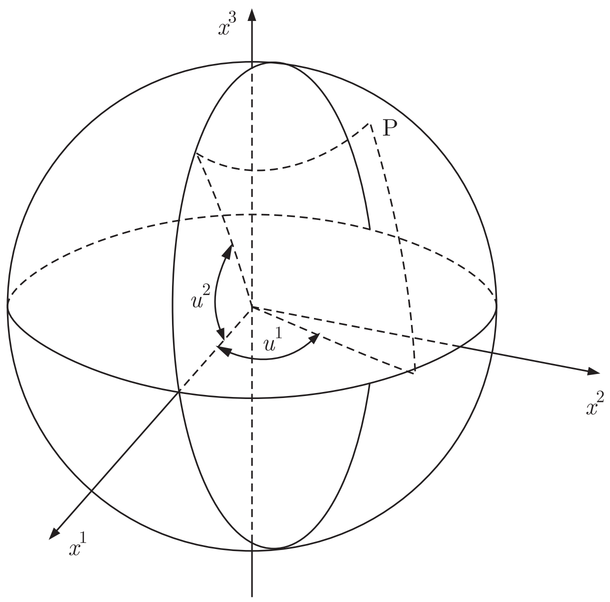

Figure 6.4: Parametric representation of sphere.

where $0 \leq u^1 < 2\pi, -\frac{1}{2}\pi \leq u^2 \leq \frac{1}{2}\pi$ are polar angles. This co-ordinate system is used especially in geography for determining the latitude and longitude of points on the globe. Indeed, the co-ordinate curves u^1 = constant and u^2 = constant are the great circles 'meridians' and 'parallels', respectively. The 'equator' is given by $u^2 = 0$ and the 'poles' by $u^2 = \pm\frac{1}{2}\pi$. At the poles the corresponding matrix

$$J = \begin{pmatrix} -r\cos u^2 \sin u^1 & r\cos u^2 \cos u^1 & 0 \\ -r\sin u^2 \cos u^1 & -r\sin u^2 \sin u^1 & r\cos u^2 \end{pmatrix}$$

is of rank 1, i.e. these points are singular with respect to the representation. Every co-ordinate curve u^1 = const. Passes through these points, are the curves $u^2 = \pm\frac{1}{2}\pi$ degenerate into points. If these two systems of curves cut each other at right angles, we say the parametric curves are orthogonal.

EXAMPLE 6.1.5 *Find the parametric representation of a cone of revolution.*

Solution: A cone of revolution with apex at (0, 0, 0) and with x^3 axis as axis of revolution can be represented in the form

$$a^2\left[(x^1)^2 + (x^2)^2\right] - (x^3)^2 = 0$$

The resulting representation of the form

$$x^3 = \pm a\sqrt{(x^1)^2 + (x^2)^2}$$

represents one of the two portions $x^3 \geq 0$ and $x^3 \leq 0$ of this Cone, depending on the choice of the sign of the square root. Now,

$$r = (x^1, x^2, x^3) = (u^1 cosu^2, u^1 sinu^2, au^1)$$

is the parametric representation (Figure 6.5) of the cone.

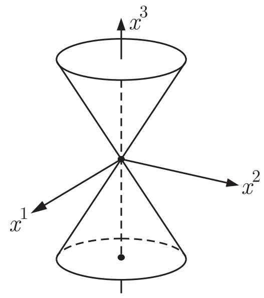

Figure 6.5: Cone of revolution.

The curve u^1 = constant are circles parallel to x_1x_2 plane while the curves u^2 = constant are the generating straight lines of the cone. The corresponding matrix

$$J = \begin{pmatrix} \cos u^2 & \sin u^2 & a \\ -u^1 \sin u^2 & u^1 \cos u^2 & 0 \end{pmatrix}$$

is of rank 1 at $u^1 = 0$; the apex is a singular point of the cone.

EXAMPLE 6.1.6 *Find the parametric representation of an elliptical helix.*

Solution: An elliptical helix (Figure 6.6) is a helix lying on an elliptical cylinder $\dfrac{(x^1)^2}{a^2} + \dfrac{(x^2)^2}{b^2} = 1$ in the $x^1x^2x^3$ space. The parametric representation of the elliptical helix under consideration is given by

$$\mathbf{r} = (x^1, x^2, x^3) = (a\cos t, b\sin t, ct)$$

where c is defined as pitch. At length

$$x^1 = a\cos t, x^2 b\sin t, x^3 = ct.$$

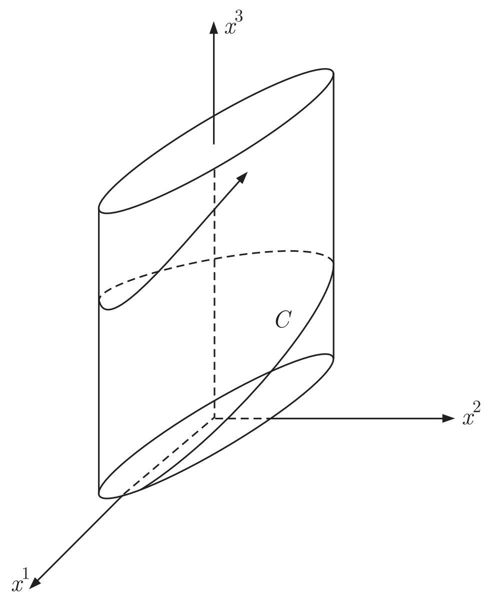

Figure 6.6: Elliptical helix.

6.1.2 Regular Surfaces

Surfaces are generally encountered in the calculus in the form $z = F(x, y)$; i.e. as graphs of two variable functions in three-dimensional space. A surface $\mathcal{S}$ in E^2 is the image of a C^3 vector function,

$$\mathbf{r}(x^1, x^2) = [f(x^1, x^2), g(x^1, x^2), h(x^1, x^2)],$$

which maps some region V of E^2 into E^3. The co-ordinate breakdown of the mapping $\mathbf{r}$,

$$x = f(x^1, x^2) \quad y = g(x^1, x^2) \quad z = h(x^1, x^2) \tag{6.5}$$

is called the Gaussian form or representation of $\mathcal{S}$. Point P is a *regular point* of $\mathcal{S}$ if $\frac{\partial \mathbf{r}}{\partial x^1} \times \frac{\partial \mathbf{r}}{\partial x^2} \neq \mathbf{0}$ at P'; otherwise, P is a *singular point.* If every point of $\mathcal{S}$ is a regular point, then $\mathcal{S}$ is a *regular surface.*

EXAMPLE 6.1.7 *Show that $x^1 = u^1 \cos u^2$, $x^2 = u^1 \sin u^2$, $x^3 = u^1 u^2$ is a regular surface.*

Solution: The condition which is necessary for

$$\mathbf{r} = \left(x^1, x^2, x^3\right) = \left[x^1\left(u^1, u^2\right), x^2\left(u^1, u^2\right), x^3\left(u^1, u^2\right)\right]$$

to be a surface is that $\mathbf{r}_{\mathbf{u}^1} \times \mathbf{r}_{\mathbf{u}^2} \neq \mathbf{0}$. Here,

$$\mathbf{r} = \left(x^1, x^2, x^3\right) = \left(u^1 \cos u^2, u^1 \sin u^2, u^1 u^2\right).$$

Hence,

$$\mathbf{r}_{\mathbf{u}^1} = \left(\frac{\partial x^1}{\partial u^1}, \frac{\partial x^2}{\partial u^1}, \frac{\partial x^3}{\partial u^1}\right) = \left(\cos u^2, \sin u^2, u^2\right)$$

$$\mathbf{r}_{\mathbf{u}^2} = \left(\frac{\partial x^1}{\partial u^2}, \frac{\partial x^2}{\partial u^2}, \frac{\partial x^3}{\partial u^2}\right) = \left(-u^1 \sin u^2, u^1 \cos u^2, u^1\right)$$

$$\Rightarrow \mathbf{r}_{\mathbf{u}^1} \times \mathbf{r}_{\mathbf{u}^2} = \left(u^1 \sin u^2 - u^1 u^2 \cos u^2, u^1 u^2 \sin u^2 + u^1 \cos u^2, u^1\right) \neq 0.$$

Therefore, $x^1 = u^1 \cos u^2$, $x^2 = u^1 \sin u^2$, $x^3 = u^1 u^2$ is a regular surface.

6.2 Intrinsic Geometry

In Section 6.1 we see that the properties of surfaces that can be described without reference to the space in which the surface is imbedded are termed as *intrinsic properties.* A study of intrinsic properties is made to depend on a certain quadratic differential form describing the metric character of the surface. A transformation T of space co-ordinate from one system $\left(x^i\right)$ to another system $\left(\overline{x}^i\right)$ will be written as

$$T\colon x^i = x^i\left(\overline{x}^1, \overline{x}^2, \overline{x}^3\right) \tag{6.6}$$

a transformation of Gaussian surface co-ordinates will be

$$u^\alpha = u^\alpha\left(\overline{u}^1, \overline{u}^2\right). \tag{6.7}$$

We will now determine the element of arc of such a curve. Suppose that all functions appearing here are of class C^2 in the region of their definition. Let P and Q be two neighbouring points on the surface $\mathcal{S}$ with co-ordinates u^α and $u^\alpha + du^\alpha$, respectively, then from Eq. (6.2), we get

$$y^i = y^i\left(u^1, u^2\right) \text{ so that } dy^i = \frac{\partial y^i}{\partial u^\alpha} du^\alpha \tag{6.8}$$

where the y^i are the orthogonal Cartesian co-ordinates covering the space E^3 in which the surface $\mathcal{S}$ is imbedded, and a curve $\mathcal{C}$ on $\mathcal{S}$ defined by

$$\mathcal{C}\colon u^\alpha = u^\alpha(t); t_1 \leq t \leq t_2; \text{ so that, } du^\alpha = \frac{du^\alpha}{dt} dt, \tag{6.9}$$

where the u^αs are the Gaussian co-ordinates covering $\mathcal{S}$. Viewed from the surrounding space, the curve defined by Eq. (6.9) is a curve in a three-dimensional Euclidean space and its element of arc is given by the formula

$$ds^2 = dy^i dy^i = \frac{\partial y^i}{\partial u^\alpha}\frac{\partial y^i}{\partial u^\beta} du^\alpha du^\beta$$
$$= a_{\alpha\beta} du^\alpha du^\beta \text{(say); where } a_{\alpha\beta} = \frac{\partial y^i}{\partial u^\alpha}\frac{\partial y^i}{\partial u^\beta}. \tag{6.10}$$

The expression for ds^2, namely Eq. (6.10), is the square of the linear element of $\mathcal{C}$ lying on the surface $\mathcal{S}$, and the right-hand member of Eq. (6.10), i.e. $a_{\alpha\beta}du^\alpha du^\beta$ is called the *metric form* or *first fundamental quadratic form* of the surface. $a_{\alpha\beta}$s are functions of u's and is called the coefficients of the first fundamental form. The first fundamental form of a regular surface is positive definite. This form (6.10) enable us to measure arc lengths, angles, and areas on a surface; so it defines a metric on the surface.

The length of arc of the curve defined by Eq. (6.7) is given by

$$S = \int_{t_1}^{t_2} \sqrt{a_{\alpha\beta}\dot{u}^\alpha \dot{u}^\beta} dt, \text{ where } \dot{u}^\alpha = \frac{du^\alpha}{dt}. \tag{6.11}$$

Since in a non-trivial case $ds^2 > 0$, it follows at once from Eq. (6.10) upon setting. $u^2 =$ constant and $u^1 =$ constant in turn, that

$$ds^2_{(1)} = a_{11}\left(du^1\right)^2 \text{ and } ds^2_{(2)} = a_{22}\left(du^2\right)^2.$$

Thus, a_{11} and a_{22} are positive functions of u^1 and u^2.

Theorem 6.2.1 *Show that $a_{\alpha\beta}$ is a symmetric covariant tensor of order two.*

Proof: Consider a transformation of surface co-ordinates

$$u^\alpha = u^\alpha\left(\overline{u}^1, \overline{u}^2\right) \text{ so that } du^\alpha = \frac{\partial u^\alpha}{\partial \overline{u}^\gamma} d\overline{u}^\gamma \tag{6.12}$$

with the Jacobian $J = \left|\frac{\partial u^\alpha}{\partial \overline{u}^\beta}\right| \neq 0$. The square of the linear element, i.e. ds^2 of $\mathcal{C}$ lying on the surface $\mathcal{S}$, is given by the expression

$$ds^2 = a_{\alpha\beta} du^\alpha du^\beta = \frac{\partial y^i}{\partial u^\alpha}\frac{\partial y^i}{\partial u^\beta} du^\alpha du^\beta.$$

Also

$$ds^2 = \overline{a}_{\gamma\delta} d\overline{u}^\gamma d\overline{u}^\delta = \frac{\partial y^i}{\partial \overline{u}^\gamma}\frac{\partial y^i}{\partial \overline{u}^\delta} d\overline{u}^\gamma d\overline{u}^\delta.$$

Due to invariant property, we get

$$\frac{\partial y^i}{\partial u^\alpha}\frac{\partial y^i}{\partial u^\beta}du^\alpha du^\beta = \frac{\partial y^i}{\partial \overline{u}^\gamma}\frac{\partial y^i}{\partial \overline{u}^\delta}d\overline{u}^\gamma d\overline{u}^\delta$$

or

$$\frac{\partial y^i}{\partial u^\alpha}\frac{\partial y^i}{\partial u^\beta}du^\alpha du^\beta = \frac{\partial y^i}{\partial \overline{u}^\gamma}\frac{\partial y^i}{\partial \overline{u}^\delta}\frac{\partial \overline{u}^\gamma}{\partial u^\alpha}\frac{\partial \overline{u}^\delta}{\partial u^\beta}du^\alpha du^\beta$$

or

$$\left[\frac{\partial y^i}{\partial u^\alpha}\frac{\partial y^i}{\partial u^\beta} - \frac{\partial y^i}{\partial \overline{u}^\gamma}\frac{\partial y^i}{\partial \overline{u}^\delta}\frac{\partial \overline{u}^\gamma}{\partial u^\alpha}\frac{\partial \overline{u}^\delta}{\partial u^\beta}\right] du^\alpha du^\beta = 0$$

or

$$\left[a_{\alpha\beta} - \overline{a}_{\gamma\delta}\frac{\partial \overline{u}^\gamma}{\partial u^\alpha}\frac{\partial \overline{u}^\delta}{\partial u^\beta}\right] du^\alpha du^\beta = 0$$

Since du^α, du^β are arbitrary tensors, we have

$$a_{\alpha\beta} - \overline{a}_{\gamma\delta}\frac{\partial \overline{u}^\gamma}{\partial u^\alpha}\frac{\partial \overline{u}^\delta}{\partial u^\beta} = 0 \Rightarrow \overline{a}_{\gamma\delta} = \frac{\partial u^\alpha}{\partial \overline{u}^\gamma}\frac{\partial u^\beta}{\partial \overline{u}^\delta}a_{\alpha\beta},$$

which shows that $a_{\alpha\beta}$ is a tensor of order two. Now, we have to show that $a_{\alpha\beta}$ is symmetric tensor. Let

$$\begin{aligned} a_{\alpha\beta} &= \frac{1}{2}\left(a_{\alpha\beta} + a_{\beta\alpha}\right) + \frac{1}{2}\left(a_{\alpha\beta} - a_{\beta\alpha}\right) \\ &= A_{\alpha\beta} + B_{\alpha\beta};\ \text{say} \end{aligned}$$

where

$$A_{\alpha\beta} = \frac{1}{2}\left(a_{\alpha\beta} - a_{\beta\alpha}\right)\ \text{and}\ B_{\alpha\beta} = \frac{1}{2}\left(a_{\alpha\beta} + a_{\beta\alpha}\right).$$

It is clear that $A_{\alpha\beta}$ is symmetric and $B_{\alpha\beta}$ is skew symmetric. Now,

$$a_{\alpha\beta}du^\alpha du^\beta = A_{\alpha\beta}du^\alpha du^\beta + B_{\alpha\beta}du^\alpha du^\beta$$

or

$$\left(a_{\alpha\beta} - A_{\alpha\beta}\right) du^\alpha du^\beta = B_{\alpha\beta}du^\alpha du^\beta \tag{6.13}$$

or

$$\left(a_{\alpha\beta} - A_{\alpha\beta}\right) du^\alpha du^\beta = -B_{\beta\alpha}du^\alpha du^\beta; B_{\alpha\beta}\ \text{is antisymmetric}$$

or

$$\left(a_{\alpha\beta} - A_{\alpha\beta}\right) du^\alpha du^\beta = -B_{\alpha\beta}du^\beta du^\alpha;$$
interchanging the dummy indices α, β

or

$$\left(a_{\alpha\beta} - A_{\alpha\beta}\right) du^\alpha du^\beta = -B_{\alpha\beta}du^\alpha du^\beta. \tag{6.14}$$

From Eqs. (6.13) and (6.14) we get

$$2B_{\alpha\beta}du^\alpha du^\beta = 0 \Rightarrow B_{\alpha\beta} = 0$$

as du^α and du^β are arbitrary tensors. Therefore, $a_{\alpha\beta} = A_{\alpha\beta}$, where $a_{\alpha\beta}$ is a symmetric tensor. Thus, we see that the set of quantities $a_{\alpha\beta}$ represents a symmetric covariant tensor of rank 2, with respect to the admissible transformation Eq. (6.12) of surface co-ordinates. The tensor $a_{\alpha\beta}$ is called the *covariant metric tensor of the surface* and are also called *fundamental magnitudes of the first order.*

Definition 6.2.1 Since the form Eq. (6.10) is positive definite, the determinant

$$a = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0$$

and we can define the reciprocal tensor $a^{\alpha\beta}$ by

$$a^{\alpha\beta} = \frac{\text{cofactor of } a_{\alpha\beta} \text{ in } a}{a}$$

with the property $a^{\alpha\beta}a_{\beta\gamma} = \delta^\alpha_\gamma$. Also, $a^{\alpha\beta}$ is a contravariant tensor of order two. The contravariant tensor $a^{\alpha\beta}$ is called the *contravariant metric tensor.* a is called the discriminant of the form.

Since our space admits of arbitrary contravariant vectors, let λ^α be any arbitrary contravariant vector defined at any general point on the surface. Then the equation

$$\lambda_\alpha = a_{\alpha\beta}\lambda^\beta \tag{6.15}$$

is uniquely solvable if $|a_{\alpha\beta}| \neq 0$. This shows that the correspondence is one to one. Since λ_α here is a covariant vector, we find that our space admits of arbitrary covariant vector. Taking inner product with $a^{\alpha\gamma}$ of Eq. (6.15) we get

$$a^{\alpha\gamma}\lambda_\alpha = a^{\alpha\gamma}a_{\alpha\beta}\lambda^\beta = \delta^\gamma_\beta\lambda^\beta = \lambda^\gamma$$

or

$$a^{\alpha\gamma}\lambda_\alpha\lambda_\gamma = \lambda^\gamma\lambda_\gamma = |\lambda|^2. \tag{6.16}$$

This relation shows that $a^{\alpha\gamma}$ is that contravariant tensor of second order. Interchanging the dummy indices α and γ in Eq. (6.16) we get

$$a^{\gamma\alpha}\lambda_\gamma\lambda_\alpha = |\lambda|^2.$$

Therefore, $a^{\alpha\gamma}$ is a symmetric tensor.

EXAMPLE 6.2.1 *Find the first fundamental form of a plane with respect to polar co-ordinates.*

Solution: Let $x^1 = u^1 \cos u^2, x^2 = u^1 \sin u^2$, where x^1, x^2 are Cartesian co-ordinates and u^1, u^2 are polar co-ordinates. Thus, the parametric representation is given by

$$x^1 = u^1 \cos u^2, x^2 = u^1 \sin u^2, x^3 = 0$$

i.e.

$$\mathbf{r} = (x^1, x^2, x^3) = \left(u^1 \cos u^2, u^1 \sin u^2, 0\right).$$

Thus, the coefficients $a_{\alpha\beta}$ of the first fundamental form are given by

$$a_{11} = \left(\frac{\partial x^1}{\partial u^1}\right)^2 + \left(\frac{\partial x^2}{\partial u^1}\right)^2 + \left(\frac{\partial x^3}{\partial u^1}\right)^2 = 1.$$

$$a_{22} = \left(\frac{\partial x^1}{\partial u^2}\right)^2 + \left(\frac{\partial x^2}{\partial u^2}\right)^2 + \left(\frac{\partial x^3}{\partial u^2}\right)^2 = (u^1)^2.$$

$$\begin{aligned} a_{12} &= \frac{\partial x^i}{\partial u^1}\frac{\partial x^i}{\partial u^2} = \frac{\partial x^1}{\partial u^1}\frac{\partial x^1}{\partial u^2} + \frac{\partial x^2}{\partial u^1}\frac{\partial x^2}{\partial u^2} + \frac{\partial x^3}{\partial u^1}\frac{\partial x^3}{\partial u^2} \\ &= \cos u^2 \cdot (-u^1 \sin u^2) + \sin u^2 (u^1 \cos u^2) + 0 = 0 = a_{21}. \end{aligned}$$

Therefore, the first fundamental form for the surface is given by

$$\begin{aligned} ds^2 &= a_{ij} du^i du^j = a_{11}(du^1)^2 + 2a_{12} du^1 du^2 + a_{22}(du^2)^2 \\ &= (du^1)^2 + (u^1)^2 (du^2)^2. \end{aligned}$$

EXAMPLE 6.2.2 *Find the first fundamental form of the right helicoid.*

Solution: The parametric representation of the right helicoid is given by

$$x^1 = u^1 \cos u^2, x^2 = u^1 \sin u^2, x^3 = cu^2$$

i.e.

$$\mathbf{r} = (x^1, x^2, x^3) = \left(u^1 \cos u^2, u^1 \sin u^2, cu^2\right),$$

where c is a constant. Now,

$$\begin{aligned} &\frac{\partial x^1}{\partial u^1} = -\cos u^2; \ \frac{\partial x^2}{\partial u^1} = \sin u^2; \ \frac{\partial x^3}{\partial u^1} = 0 \\ &\frac{\partial x^1}{\partial u^2} = -u^1 \sin u^2; \ \frac{\partial x^2}{\partial u^2} = u^1 \cos u^2; \ \frac{\partial x^3}{\partial u^2} = c. \end{aligned}$$

Thus, the coefficients $a_{\alpha\beta}$ of the first fundamental form are given by

$$a_{11} = \left(\frac{\partial x^1}{\partial u^1}\right)^2 + \left(\frac{\partial x^2}{\partial u^1}\right)^2 + \left(\frac{\partial x^3}{\partial u^1}\right)^2 = 1.$$

$$a_{22} = \left(\frac{\partial x^1}{\partial u^2}\right)^2 + \left(\frac{\partial x^2}{\partial u^2}\right)^2 + \left(\frac{\partial x^3}{\partial u^2}\right)^2 = (u^1)^2 + c^2.$$

$$\begin{aligned} a_{12} &= \frac{\partial x^i}{\partial u^1}\frac{\partial x^i}{\partial u^2} = \frac{\partial x^1}{\partial u^1}\frac{\partial x^1}{\partial u^2} + \frac{\partial x^2}{\partial u^1}\frac{\partial x^2}{\partial u^2} + \frac{\partial x^3}{\partial u^1}\frac{\partial x^3}{\partial u^2} \\ &= \cos u^2(-u^1 \sin u^2) + \sin u^2(u^1 \cos u^2) + 0 \cdot c = 0 = a_{21}. \end{aligned}$$

Therefore, the first fundamental form for the surface is given by

$$\begin{aligned} ds^2 &= a_{ij}du^i du^j = a_{11}(du^1)^2 + 2a_{12}du^1du^2 + a_{22}(du^2)^2 \\ &= (du^1)^2 + [(u^1)^2 + c^2](du^2)^2. \end{aligned}$$

The metric for R^2 corresponding to the right helicoid is non-Euclidean. Now, the parameters u^1, u^2 which are actual polar co-ordinates in the x^1x^2 plane of E^3, (Figure 6.7) formerly keep that significance when the plane is considered abstractly as parameter space. This is an instance of the formal use of familiar co-ordinate system in non-Euclidean space.

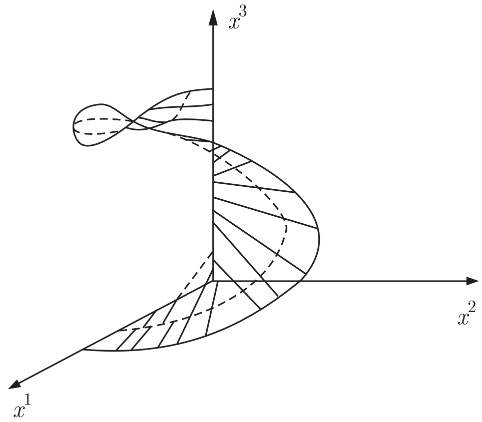

Figure 6.7: Right helicoid.

EXAMPLE 6.2.3 *Find the first fundamental form for any surface of revolution, and specialise to a right circular cone (Figure 6.8).*

Solution: The Gaussian form of a surface of revolution about z axis is

$$\mathbf{r}(x^1, x^2) = \left(f(x^1)\cos x^2, f(x^1)\sin x^2, g(x^1)\right); \ f(x^1) > 0.$$

From this parametric representation, we get

$$\frac{\partial \mathbf{r}}{\partial x^1} = \left(f'(x^1)\cos x^2, f'(x^1)\sin x^2, g'(x^1)\right); \ \frac{\partial \mathbf{r}}{\partial x^2} = \left(-f(x^1)\sin x^2, f(x^1)\cos x^2, 0\right).$$

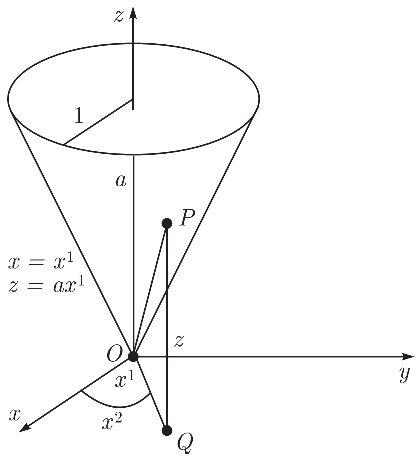

Figure 6.8: Right circular cone.

Thus, the coefficients $a_{\alpha\beta}$ of the first fundamental form are given by

$$a_{11} = \frac{\partial \mathbf{r}}{\partial x^1}.\frac{\partial \mathbf{r}}{\partial x^1} = (f')^2 + (g')^2; \ a_{22} = \frac{\partial \mathbf{r}}{\partial x^2}.\frac{\partial \mathbf{r}}{\partial x^2} = f^2;$$
$$a_{12} = \frac{\partial \mathbf{r}}{\partial x^1}.\frac{\partial \mathbf{r}}{\partial x^2} = 0 = a_{21}.$$

Therefore, the first fundamental form for the surface is given by

$$\begin{aligned} ds^2 &= a_{11}(dx^1)^2 + 2a_{12}dx^1dx^2 + a_{22}(dx^2)^2 \\ &= \left((f')^2 + (g')^2\right)(dx^1)^1 + f^2(dx^2)^2. \end{aligned}$$

For a right circular cone, $f(x^1) = x^1$ and $g(x^1) = ax^1$; hence,

$$ds^2 = \left(1 + a^2\right)(dx^1)^1 + (x^1)^2(dx^2)^2.$$

EXAMPLE 6.2.4 *Find the first fundamental form for the catenoid $x = a\cosh x^1$, $z = ax^1$ and compute the length of the curve $x^1 = t, x^2 = t$; $0 \le t \le \log(1+\sqrt{2})$.*

Solution: For that case, $f(x^1) = a\cosh x^1$, $g(x^1) = ax^1$. Therefore, the first fundamental form for the surface is given by

$$ds^2 = a^2\cosh^2 x^1 \left[(dx^1)^2 + (dx^2)^2\right].$$

Thus, the required length of the arc, $0 \leq t \leq \log(1+\sqrt{2})$ is given by

$$L = a\sqrt{2}\int_0^{\log(1+\sqrt{2})} \cosh t dt = a\sqrt{2}\sinh[\log(1+\sqrt{2})] = a\sqrt{2} \text{ units.}$$

EXAMPLE 6.2.5 *Find the first fundamental form of the surface is given in Monge's form $x^3 = f(u^1, u^2)$, where the co-ordinates u^1, u^2 may be taken as parameters.*
Solution: The parametric representation of the right helicoid is given by

$$x^1 = u^1, \;\; x^2 = u^2, \;\; x^3 = f(u^1, u^2)$$

i.e.

$$\mathbf{r} = (x^1, x^2, x^3) = \left(u^1, u^2, f(u^1, u^2)\right),$$

From this parametric representation, we get

$$\frac{\partial x^1}{\partial u^1} = 1; \; \frac{\partial x^2}{\partial u^1} = 0; \; \frac{\partial x^3}{\partial u^1} = \frac{\partial f}{\partial u^1} = f_1$$
$$\frac{\partial x^1}{\partial u^2} = 0; \; \frac{\partial x^2}{\partial u^2} = 1; \; \frac{\partial x^3}{\partial u^2} = \frac{\partial f}{\partial u^2} = f_2,$$

where $\frac{\partial f}{\partial u^i} = f_i$. Thus, the coefficients $a_{\alpha\beta}$ of the first fundamental form are given by

$$a_{11} = \frac{\partial x^i}{\partial u^1}\frac{\partial x^i}{\partial u^1} = \left(\frac{\partial x^1}{\partial u^1}\right)^2 + \left(\frac{\partial x^2}{\partial u^1}\right)^2 + \left(\frac{\partial x^3}{\partial u^1}\right)^2 = 1 + f_1^2.$$
$$a_{22} = \frac{\partial x^i}{\partial u^2}\frac{\partial x^i}{\partial u^2} = \left(\frac{\partial x^1}{\partial u^2}\right)^2 + \left(\frac{\partial x^2}{\partial u^2}\right)^2 + \left(\frac{\partial x^3}{\partial u^2}\right)^2 = 1 + f_2^2.$$
$$a_{12} = \frac{\partial x^i}{\partial u^1}\frac{\partial x^i}{\partial u^2} = \frac{\partial x^1}{\partial u^1}\frac{\partial x^1}{\partial u^2} + \frac{\partial x^2}{\partial u^1}\frac{\partial x^2}{\partial u^2} + \frac{\partial x^3}{\partial u^1}\frac{\partial x^3}{\partial u^2} = f_1 f_2 = a_{21}.$$

Therefore, the first fundamental form for the surface is given by

$$ds^2 = a_{ij}du^i du^j = a_{11}(du^1)^2 + 2a_{12}du^1du^2 + a_{22}(du^2)^2$$
$$= (1+f_1^2)(du^1)^1 + 2f_1f_2du^1du^2 + [1+f_2^2](du^2)^2.$$

Note 6.2.1 Along with the first fundamental form, tensor calculus enters the picture. For the intrinsic properties of a particular surface $\mathcal{S}$ in E^3 (the properties defined by measurements of distance on the surface) are all implicit in Eq. (6.10), which can be interpreted as a particular Riemannian metrication of the parameter plane. Thus, the study of intrinsic properties of surfaces becomes the tensor analysis of Riemannian matrices in R^2 and this may be conducted without any reference of E^3 whatever. Observe that the metrices under consideration will all be positive definite but not necessarily Euclidean. Accordingly, we shall drop the designation E^2 for the parameter plane, which shall henceforth be referred to general co-ordinates.

6.2.1 Angle Between Two Intersecting Curves

Here, we have to find the angle between two intersecting curves on a surface. The equation of a curve $\mathcal{C}$ drawn on the surface $\mathcal{S}$ can be written in the form

$$\mathcal{C}\colon u^\alpha = u^\alpha(t); t_1 \le t \le t_2. \tag{6.17}$$

Since the $u^\alpha(t)$ is assumed to be of class C^2, the curve $\mathcal{C}$ has a continuously turning tangent. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two such curves intersecting at the point P of $\mathcal{S}$. We take the equations of $\mathcal{S}$, referred to orthogonal Cartesian axes y^i, in the form

$$y^i = y^i\left(u_1, u_2\right);\ i = 1, 2, 3 \tag{6.18}$$

and denote the direction cosines of the tangent lies to $\mathcal{C}_1$ and $\mathcal{C}_2$ at P by ξ^i and η^i, respectively, then

$$\xi^i = \frac{\partial y^i}{\partial u^\alpha}\frac{du^\alpha}{dS_{(1)}} = \frac{dy^i}{dS_{(1)}} \text{ and } \eta^i = \frac{\partial y^i}{\partial u^\beta}\frac{du^\beta}{dS_{(2)}} = \frac{dy^i}{dS_{(2)}},$$

where the subscripts 1 and 2 refer to the elements of arc of $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively. We can write the unit vectors in the directions of the tangents $\mathcal{C}_1$ and $\mathcal{C}_2$ as

$$\lambda^\alpha = \frac{du^\alpha}{dS_{(1)}};\ \ \mu^\alpha = \frac{du^\alpha}{dS_{(2)}} \text{ and } \xi^i = \frac{\partial y^i}{\partial u^\alpha}\lambda^\alpha;\ \ \eta^i = \frac{\partial y^i}{\partial u^\beta}\mu^\beta. \tag{6.19}$$

The cosine of the angle θ between $\mathcal{C}_1$ and $\mathcal{C}_2$, calculated by a geometer in the enveloping space E^3, (Figure 6.9) is

$$\begin{aligned}\cos\theta = \xi^i\eta^i &= \frac{\partial y^i}{\partial u^\alpha}\frac{\partial y^i}{\partial u^\beta}\lambda^\alpha\mu^\beta \\ &= a_{\alpha\beta}\lambda^\alpha\mu^\beta = a_{\alpha\beta}\frac{du^\alpha}{dS_{(1)}}\frac{du^\beta}{dS_{(2)}}.\end{aligned} \tag{6.20}$$

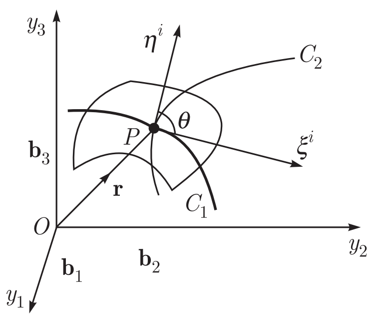

Figure 6.9: Angle between two intersecting curves.

In two-dimensional manifold, the skew-symmetric e-systems can be defined by formula (1.5) as $e_{11} = e_{22} = e^{11} = e^{22} = 0; e^{12} = -e^{21} = e_{12} = -e_{21} = 1$ and since these systems are relative tensors, the expressions

$$\varepsilon_{\alpha\beta} = \sqrt{a} e_{\alpha\beta} \text{ and } \varepsilon^{\alpha\beta} = \frac{1}{\sqrt{a}} e^{\alpha\beta} \tag{6.21}$$

are absolute tensors. Let θ be the angle between two unit vectors $\lambda^\alpha, \mu^\alpha$, then,

$$\begin{aligned}
\sin^2\theta &= 1 - \cos^2\theta = \left(\xi^i\xi^i\right)\left(\eta^i\eta^i\right) - \left(\xi^i\eta^i\right)^2 \\
&= \begin{vmatrix} \xi^1 & \eta^1 \\ \xi^2 & \eta^2 \end{vmatrix}^2 + \begin{vmatrix} \xi^2 & \eta^2 \\ \xi^3 & \eta^3 \end{vmatrix}^2 + \begin{vmatrix} \xi^3 & \eta^3 \\ \xi^1 & \eta^1 \end{vmatrix}^2; \text{ by Lagrange identity} \\
&= \left[\begin{vmatrix} \frac{\partial y^1}{\partial u^1} & \frac{\partial y^1}{\partial u^2} \\ \frac{\partial y^2}{\partial u^1} & \frac{\partial y^2}{\partial u^2} \end{vmatrix}^2 + \begin{vmatrix} \frac{\partial y^2}{\partial u^1} & \frac{\partial y^2}{\partial u^2} \\ \frac{\partial y^3}{\partial u^1} & \frac{\partial y^3}{\partial u^2} \end{vmatrix}^2 + \begin{vmatrix} \frac{\partial y^3}{\partial u^1} & \frac{\partial y^3}{\partial u^2} \\ \frac{\partial y^1}{\partial u^1} & \frac{\partial y^1}{\partial u^2} \end{vmatrix}^2 \right] \begin{vmatrix} \lambda^1 & \mu^1 \\ \lambda^2 & \mu^2 \end{vmatrix}^2 \\
&= \left(J_1^2 + J_2^2 + J_3^2\right)\left(\lambda^1\mu^2 - \lambda^2\mu^1\right)^2 = a\left(\lambda^1\mu^2 - \lambda^2\mu^1\right)^2
\end{aligned}$$

or

$$\sin\theta = \sqrt{a}\left(\lambda^1\mu^2 - \lambda^2\mu^1\right) = \sqrt{a}\; e_{\alpha\beta}\lambda^\alpha\mu^\beta = \varepsilon_{\alpha\beta}\lambda^\alpha\mu^\beta. \tag{6.22}$$

Equation (6.22) $\varepsilon_{\alpha\beta}\lambda^\alpha\mu^\beta = \sin\theta$, is numerically equal to the area of the parallelogram constructed on the unit vectors λ^α and μ^α. It follows from this result that a necessary and sufficient condition for the orthonormality of two surface unit vectors λ^α and μ^α is $\left|\varepsilon_{\alpha\beta}\lambda^\alpha\mu^\beta\right| = 1$.

If the curves $\mathcal{C}_1$ and $\mathcal{C}_2$ are orthogonal,

$$a_{\alpha\beta}\lambda^\alpha\mu^\beta = 0 \tag{6.23}$$

Also, if the surface vectors λ^α and μ^β are taken along the co-ordinate curves

$$\lambda^1 = \frac{1}{\sqrt{a_{11}}}, \lambda^2 = 0, \quad \mu^1 = 0, \text{ and } \mu^2 = \frac{1}{\sqrt{a_{22}}},$$

then from Eq. (6.23) the co-ordinate curves will form an orthogonal net if and only if $a_{12} = 0$ at every point of the surface. If in particular, the vectors $\lambda^\alpha, \mu^\beta$ are taken along the parametric curves, then, for the u'-curve, $du^2 = 0$. Consequently, Eq. (6.10) reduces to $dS_{(1)}^2 = a_{11}\left(du^1\right)^2$ and hence, the unit vector $\lambda_{(1)}^\alpha$ along u^1 curve is

$$\lambda_{(1)}^\alpha = \left(\frac{du^1}{dS_{(1)}}, \frac{du^2}{dS_{(1)}}\right) = \left(\frac{1}{\sqrt{a_{11}}}, 0\right) = \frac{1}{\sqrt{a_{11}}}\delta_{(1)}^\alpha.$$

Similarly, the unit vector $\mu_{(2)}^\alpha$ along u^2 curve is

$$\mu_{(2)}^\alpha = \left(\frac{du^1}{dS_{(2)}}, \frac{du^2}{dS_{(2)}}\right) = \left(\frac{1}{\sqrt{a_{22}}}, 0\right) = \frac{1}{\sqrt{a_{22}}}\delta_{(2)}^\alpha.$$

EXAMPLE 6.2.6 *Show that the arc length lying on the plane $z = 0$ is the same in both the systems.*

Solution: Let P and Q be two points on the plane $z = 0$, whose co-ordinates are say $P\left(x^i\right) = P\left(x^1, x^2, x^3\right)$ and $Q\left(x^i + dx^i\right) = Q\left(x^1 + dx^1, x^2 + dx^2, x^3 + dx^3\right)$. Now, the symmetric tensors a_{ij} are given by

$$a_{11} = \frac{\partial x^i}{\partial u^1}\frac{\partial x^i}{\partial u^1} = \left(\frac{\partial x^1}{\partial u^1}\right)^2 + \left(\frac{\partial x^2}{\partial u^1}\right)^2 + \left(\frac{\partial x^3}{\partial u^1}\right)^2 = 1.$$

$$a_{12} = \frac{\partial x^i}{\partial u^1}\frac{\partial x^i}{\partial u^2} = \frac{\partial x^1}{\partial u^1}\frac{\partial x^1}{\partial u^2} + \frac{\partial x^2}{\partial u^1}\frac{\partial x^2}{\partial u^2} + \frac{\partial x^3}{\partial u^1}\frac{\partial x^3}{\partial u^2} = 0 = a_{21}.$$

$$a_{22} = \frac{\partial x^i}{\partial u^2}\frac{\partial x^i}{\partial u^2} = \left(\frac{\partial x^1}{\partial u^2}\right)^2 + \left(\frac{\partial x^2}{\partial u^2}\right)^2 + \left(\frac{\partial x^3}{\partial u^2}\right)^2 = 1.$$

Now, the square of the linear element is given by

$$\begin{aligned} ds^2 &= a_{\alpha\beta}du^\alpha du^\beta = a_{11}\left(du^1\right)^2 + 2a_{12}du^1du^2 + a_{22}\left(du^2\right)^2 \\ &= a_{11}\left(dx^1\right)^2 + 2a_{12}dx^1dx^2 + a_{22}\left(dx^2\right)^2; \\ &\qquad\qquad \text{as } u^1 = x^1, u^2 = x^2, u^3 = 0 \\ &= 1\left(dx^1\right)^2 + 1\left(dx^2\right)^2 = \left(dx^1\right)^2 + \left(dx^2\right)^2 = dx^idx^i. \end{aligned}$$

Therefore, the arc length lying on the plane $z = 0$ is the same in both the systems.

EXAMPLE 6.2.7 *If θ be the angle between the parametric curves lying on a surface, immersed in E_3, show that*

$$\cos\theta = \frac{a_{12}}{\sqrt{a_{11}}\sqrt{a_{22}}} \text{ and } \sin\theta = \frac{a}{\sqrt{a_{11}}\sqrt{a_{22}}}$$

and hence, show that the parametric curves on a surface are orthogonal iff $a_{12} = 0$.

Solution: Since θ be the angle between the parametric curves, so,

$$\begin{aligned} \cos\theta &= a_{\alpha\beta}\lambda^\alpha_{(1)}\lambda^\beta_{(2)} = a_{11}\lambda^1_{(1)}\lambda^1_{(2)} + a_{22}\lambda^2_{(1)}\lambda^2_{(2)} + a_{12}\lambda^1_{(1)}\lambda^2_{(2)} + a_{21}\lambda^2_{(1)}\lambda^1_{(2)} \\ &= a_{11}\frac{1}{\sqrt{a_{11}}} \times 0 + a_{22} \times 0 \times \frac{1}{\sqrt{a_{22}}} + a_{12} \times \frac{1}{\sqrt{a_{11}}} \times \frac{1}{\sqrt{a_{22}}} + a_{21} \times 0 \times 0 \\ &= \frac{a_{12}}{\sqrt{a_{11}}\sqrt{a_{22}}}. \end{aligned}$$

Using the identity $\cos^2\theta = 1 - \sin^2\theta$ we get

$$\sin^2\theta = 1 - \cos^2\theta = 1 - \frac{a_{12}^2}{a_{11}a_{22}} = \frac{a_{11}a_{22} - a_{12}^2}{a_{11}a_{22}} = \frac{a}{a_{11}a_{22}}$$

$$\Rightarrow \sin\theta = \frac{\sqrt{a}}{\sqrt{a_{11}}\sqrt{a_{22}}}.$$

If the parametric curves on the surface S are orthogonal, i.e. if $\theta = \frac{\pi}{2}$, then,

$$\frac{\sqrt{a}}{\sqrt{a_{11}}\sqrt{a_{22}}} = \sin\frac{\pi}{2} = 1$$

so that a_{11} and $a_{22} \neq 0$ and so $a > 0$ and so,

$$\cos\frac{\pi}{2} = 0 = \frac{a_{12}}{\sqrt{a_{11}}\sqrt{a_{22}}} \Rightarrow a_{12} = 0.$$

EXAMPLE 6.2.8 *Prove that the parametric curves on a surface given by* $x^1 = a\sin u\cos v, x^2 = a\sin u\sin v, x^3 = a\cos u$ *form an orthogonal system.*

Solution: For the given surface, the symmetric covariant tensor $a_{\alpha\beta}$ of order two are given by

$$\begin{aligned}
a_{11} &= \left(\frac{\partial x^1}{\partial u}\right)^2 + \left(\frac{\partial x^2}{\partial u}\right)^2 + \left(\frac{\partial x^3}{\partial u}\right)^2 \\
&= (a\cos u\cos v)^2 + (a\cos u\sin v)^2 + (-a\sin u)^2 = a^2. \\
a_{22} &= \left(\frac{\partial x^1}{\partial v}\right)^2 + \left(\frac{\partial x^2}{\partial v}\right)^2 + \left(\frac{\partial x^3}{\partial v}\right)^2 \\
&= (-a\sin u\sin v)^2 + (a\sin u\cos v)^2 + 0^2 = a^2\sin^2 u \\
a_{12} &= \frac{\partial x^1}{\partial u}\frac{\partial x^1}{\partial v} + \frac{\partial x^2}{\partial u}\frac{\partial x^2}{\partial v} + \frac{\partial x^3}{\partial u}\frac{\partial x^3}{\partial v} \\
&= a\cos u\cos v \times (-a\sin u\sin v) + a\cos u\sin v \\
&\quad \times (a\sin u\cos v) + (-a\sin u) \times 0 = 0 = a_{21}.
\end{aligned}$$

Since $a_{12} = 0$, so the co-ordinate curves are orthogonal. Let λ^α and λ^β be taken along the parametric curves. For the u-curve, $dv = 0$ and for the v-curve, $du = 0$. Thus, the unit vector $\lambda^\alpha_{(1)}$ along the u-curve is

$$\lambda^\alpha_{(1)} = \left(\frac{du}{ds_{(1)}}, \frac{dv}{ds_{(1)}}\right) = \left(\frac{1}{\sqrt{a_{11}}}, 0\right) = \left(\frac{1}{a}, 0\right)$$

and the unit vector $\lambda^\beta_{(2)}$ along the v-curve is

$$\lambda^\beta_{(2)} = \left(\frac{du}{ds_{(2)}}, \frac{dv}{ds_{(2)}}\right) = \left(0, \frac{1}{\sqrt{a_{22}}}\right) = \left(0, \frac{1}{a\sin u}\right).$$

If θ be the angle between the parametric curves, then

$$\begin{aligned}\cos\theta &= a_{\alpha\beta}\lambda^\alpha_{(1)}\lambda^\beta_{(2)} = a_{11}\lambda^1_{(1)}\lambda^1_{(2)} + a_{22}\lambda^2_{(1)}1\lambda^2_{(2)} + a_{12}\lambda^1_{(1)}\lambda^2_{(2)}\\ &= a^2 \times \frac{1}{a} \times 0 + a^2\sin^2 u \times 0 \times \frac{1}{a\sin u} + 0 \times \frac{1}{a} \times \frac{1}{a\sin u} + 0 \times 0 \times 0\\ &= 0 \Rightarrow \theta = \frac{\pi}{2}.\end{aligned}$$

Thus, the parametric curves on the surface given by $x^1 = a\sin u\cos v, x^2 = a\sin u\sin v,$ $x^3 = a\cos u$ form an orthogonal system.

6.2.2 Element of Surface Area

Let $\mathbf{r}$ denotes the position vector of any point P on the surface $\mathcal{S}$, and the $\mathbf{b_i}$ are the unit vectors directed along the orthogonal co-ordinate axes Y, then from Eq. (6.18) of the surface $\mathcal{S}$ can be written in the vector form as

$$\mathbf{r}\left(u_1, u_2\right) = \mathbf{b_i} y^i\left(u_1, u_2\right).$$

According to the representation of the surface $\mathcal{S}$, it follows that

$$\begin{aligned}ds^2 = d\mathbf{r}\cdot d\mathbf{r} &= \frac{\partial \mathbf{r}}{\partial u^\alpha}\cdot\frac{\partial \mathbf{r}}{\partial u^\beta} du^\alpha du^\beta\\ &= a_{\alpha\beta} du^\alpha du^\beta; \quad \text{where, } a_{\alpha\beta} = \frac{\partial \mathbf{r}}{\partial u^\alpha}\cdot\frac{\partial \mathbf{r}}{\partial u^\beta}.\end{aligned}$$

Setting $\frac{\partial \mathbf{r}}{\partial u^\alpha} = \mathbf{a}_\alpha$, where $\mathbf{a_1}$ and $\mathbf{a_2}$ are obviously tangent vectors to the co-ordinate curves, we see that

$$a_{11} = \mathbf{a_1}\cdot\mathbf{a_1};\ a_{12} = \mathbf{a_1}\cdot\mathbf{a_2};\ a_{22} = \mathbf{a_2}\cdot\mathbf{a_2}.$$

In the notation of Eq. (6.19) the space components of $\mathbf{a_1}$ and $\mathbf{a_2}$ are ξ^i and η^i, respectively. Let us define an element of area $d\sigma$ of the surface $\mathcal{S}$ by the formula, (Figure 6.10)

$$\begin{aligned}d\sigma &= \left|\mathbf{a_1}\times\mathbf{a_2}\right| du^1 du^2\\ &= \sqrt{a_{11}a_{22} - a_{12}^2}\, du^1 du^2 = \sqrt{a}\ du^1 du^2.\end{aligned}$$

This formula has widely used to calculate the volume element.

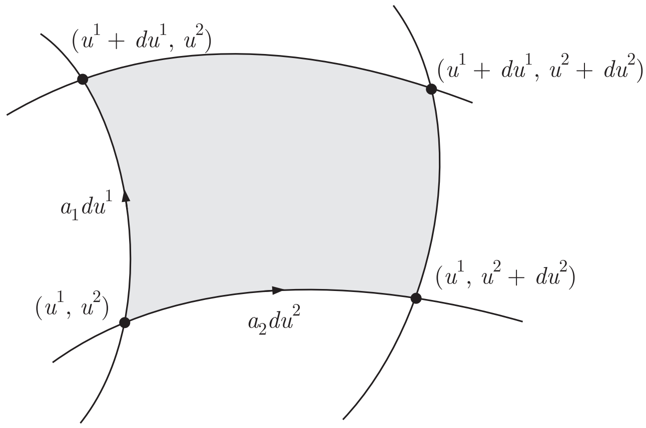

Figure 6.10: Element of area.

EXAMPLE 6.2.9 *Find the element of area of the surface of radius* r*, if the equations of the surface are given in the form*

$$y^1 = r \sin u^1 \cos u^2, y^2 = r \sin u^1 \sin u^2, y^3 = r \cos u^1,$$

where the y^i *are orthogonal Cartesian co-ordinates.*

Solution: The parametric representation of the surface of radius r is given by

$$y^1 = r \sin u^1 \cos u^2, y^2 = r \sin u^1 \sin u^2, y^3 = r \cos u^1$$

i.e.

$$\mathbf{r} = (y^1, y^2, y^3) = \left(r \sin u^1 \cos u^2, r \sin u^1 \sin u^2, r \cos u^1\right).$$

From this parametric representation, we get

$$\frac{\partial y^1}{\partial u^1} = r \cos u^1 \cos u^2; \ \frac{\partial y^2}{\partial u^1} = r \cos u^1 \sin u^2; \ \frac{\partial y^3}{\partial u^1} = -r \sin u^1$$

$$\frac{\partial y^1}{\partial u^2} = -r \sin u^1 \sin u^2; \ \frac{\partial y^2}{\partial u^2} = r \sin u^1 \cos u^2; \ \frac{\partial y^3}{\partial u^2} = 0.$$

Thus, the coefficients $a_{\alpha\beta}$ of the first fundamental form are given by

$$\begin{aligned} a_{11} &= \left(\frac{\partial y^1}{\partial u^1}\right)^2 + \left(\frac{\partial y^2}{\partial u^1}\right)^2 + \left(\frac{\partial y^3}{\partial u^1}\right)^2 \\ &= \left(r \cos u^1 \cos u^2\right)^2 + \left(r \cos u^1 \sin u^2\right)^2 + \left(-r \sin u^1\right)^2 = r^2. \\ a_{22} &= \left(\frac{\partial y^1}{\partial u^2}\right)^2 + \left(\frac{\partial y^2}{\partial u^2}\right)^2 + \left(\frac{\partial y^3}{\partial u^2}\right)^2 = r^2 \sin^2 u^1. \end{aligned}$$

$$
\begin{aligned}
a_{12} &= \frac{\partial y^1}{\partial u^1}\frac{\partial y^1}{\partial u^2} + \frac{\partial y^2}{\partial u^1}\frac{\partial y^2}{\partial u^2} + \frac{\partial y^3}{\partial u^1}\frac{\partial y^3}{\partial u^2} \\
&= \left(r\cos u^1 \cos u^2\right)\cdot\left(-r\sin u^1 \sin u^2\right) + \left(r\cos u^1 \sin u^2\right)\cdot\left(r\sin u^1\cos u^2\right) \\
&\quad + \left(-r\sin u^1\right)\cdot 0 = 0 = a_{21}.
\end{aligned}
$$

Therefore,

$$
a = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} r^2 & 0 \\ 0 & r^2\sin^2 u^1 \end{vmatrix} = r^4 \sin^2 u^1.
$$

The element of area $d\sigma$ of the surface $\mathcal{S}$ of radius r by the formula,

$$
d\sigma = \sqrt{a}\;\; du^1\;\; du^2 = r^2 \sin u^1 du^1 du^2.
$$

6.3 Geodesic on a Surface

Here, we will discuss the problem of finding curves of minimum length or shortest arc joining a pair of given points on the surface. We will carry out our calculation for the case of the n-dimensional Riemannian manifolds, since our results will be of interest not only in connection with the geometry of surfaces but also in the study of dynamical trajectories. Obviously this is a problem of calculus of variation, first considered by J. Bernoulli.

Let metric properties of the n-dimensional manifold R^n be determined by

$$
ds^2 = g_{ij} du^i du^j;\;\; ;\;\; i, j = 1, 2, \ldots, n, \tag{6.24}
$$

where $g_{ij} = g_{ji}$ are specified functions on the variables u^i. Let us suppose that the form Eq. (6.24) is positive definite and the functions g_{ij} are of class C^2. Let $\mathcal{C}$ be a curve given by (Figure 6.11)

$$
\mathcal{C}: u^i = u^i(t);\quad t_1 \le t \le t_2
$$

and the length of the curve between two points P and Q on it be given by

$$
S = \int_{t_1}^{t_2} \sqrt{g_{\alpha\beta}\dot{u}^\alpha \dot{u}^\beta} dt;\;\; \alpha, \beta = 1, 2, \cdots, n. \tag{6.25}
$$

The extremals of the functional Eq. (6.25) will be termed as *geodesics* in R^n. Let $\overline{\mathcal{C}}$ be any curve in the neighbourhood of $\mathcal{C}$ joining P and Q and let it be given by

$$
\overline{\mathcal{C}} : \overline{u}^i = u^i(t) + \varepsilon w^i(t),
$$

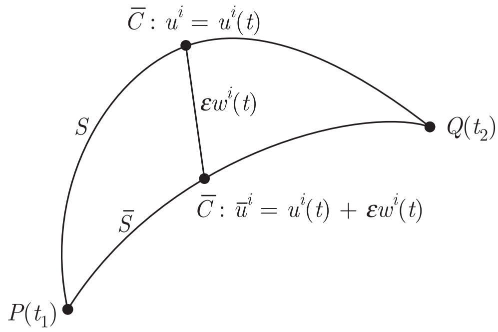

Figure 6.11: Geodestic on a surface.

where w^i is a function of t such that $w^i(t) = 0$ at $P(t_1)$ and $Q(t_2)$ and ε is a number of infinitesimal order. The arc length between P and Q along $\overline{\mathcal{C}}$ is given by

$$\overline{S} = \int_{t_1}^{t_2} \sqrt{g_{\alpha\beta}\dot{\overline{u}}^{\alpha}\ \dot{\overline{u}}^{\beta}}\,dt.$$

Let us consider the integral,

$$I = \int_{t_1}^{t_2} \phi(u^\alpha, u^\alpha) dt,$$

where ϕ is a function and

$$\begin{aligned}\overline{I} &= \int_{t_1}^{t_2} \phi\left(u^\alpha + \varepsilon w^\alpha, u^\alpha + \varepsilon w^\alpha\right)\ dt\\ &= \int_{t_1}^{t_2} \phi(u^\alpha, u^\alpha)\ dt + \varepsilon \int_{t_1}^{t_2} \left[w^\alpha \frac{\partial \phi}{\partial u^\alpha} + \dot{w}^\alpha \frac{\partial \phi}{\partial \dot{u}^\alpha}\right] dt,\end{aligned}$$

by Taylor's theorem, on neglecting the other terms. Therefore, the increment $\overline{I} - I$ is given by

$$\begin{aligned}\overline{I} - I &= \varepsilon \int_{t_1}^{t_2} \left[w^\alpha \frac{\partial \phi}{\partial u^\alpha} + \dot{w}^\alpha \frac{\partial \phi}{\partial \dot{u}^\alpha}\right] dt;\ \alpha = 1, 2\\ &= \varepsilon \int_{t_1}^{t_2} w^\alpha \frac{\partial \phi}{\partial u^\alpha} dt + \varepsilon \int_{t_1}^{t_2} \frac{\partial \phi}{\partial \dot{u}^\alpha} \dot{w}^\alpha dt,\\ &= \varepsilon \int_{t_1}^{t_2} w^\alpha \frac{\partial \phi}{\partial u^\alpha}\ dt + \varepsilon \left[\frac{\partial \phi}{\partial \dot{u}^\alpha} w^\alpha\right]_{t_1}^{t_2} - \varepsilon \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial \phi}{\partial \dot{u}^\alpha}\right) w^\alpha dt,\\ &= \varepsilon \int_{t_1}^{t_2} \left[\frac{\partial \phi}{\partial u^\alpha} - \frac{d}{dt}\left(\frac{\partial \phi}{\partial \dot{u}^\alpha}\right)\right] w^\alpha dt.\end{aligned} \tag{6.26}$$

If $\mathcal{C}$ is a geodesic, then the increment $\overline{I} - I$ must be zero for all neighbouring curves through P and Q, i.e. Eq. (6.26) must vanish for all arbitrary values of the vector w^α along $\mathcal{C}$. Thus,

$$\frac{d}{dt}\left(\frac{\partial\phi}{\partial\dot{u}^\alpha}\right) - \frac{\partial\phi}{\partial u^\alpha} = 0, \tag{6.27}$$

which is called *Euler's* or *Lagrange's* equations for ϕ. In our case,

$$\phi(u^\alpha, \dot{u}^\alpha) = \sqrt{g_{\alpha\beta}\dot{u}^\alpha\dot{u}^\beta}$$

$$\Rightarrow \frac{\partial\phi}{\partial u^\gamma} = \frac{1}{2}\left(g_{\alpha\beta}\dot{u}^\alpha\dot{u}^\beta\right)^{-1/2}\frac{\partial g_{\alpha\beta}}{\partial u^\gamma}\dot{u}^\alpha\dot{u}^\beta$$

and

$$\frac{\partial\phi}{\partial\dot{u}^\gamma} = \frac{1}{2}\left(g_{\alpha\beta}\dot{u}^\alpha\dot{u}^\beta\right)^{-1/2} g_{\alpha\gamma}\dot{u}^\alpha.$$

Hence, the Euler's equation [Eq. (6.27)] gives,

$$\frac{d}{dt}\left[\frac{g_{\alpha\gamma}\dot{u}^\alpha}{\sqrt{g_{\alpha\beta}\dot{u}^\alpha\dot{u}^\beta}}\right] - \frac{1}{2\sqrt{g_{\alpha\beta}\dot{u}^\alpha\dot{u}^\beta}}\frac{\partial g_{\alpha\beta}}{\partial u^\gamma}\dot{u}^\alpha\dot{u}^\beta = 0$$

or

$$\frac{d}{dt}\left[\frac{g_{\alpha\gamma}\dot{u}^\alpha}{ds/dt}\right] - \frac{1}{2ds/dt}\frac{\partial g_{\alpha\beta}}{\partial u^\gamma}\dot{u}^\alpha\dot{u}^\beta = 0$$

or

$$g_{\alpha\gamma}\ddot{u}^\alpha + \frac{\partial g_{\alpha\gamma}}{\partial u^\beta}\dot{u}^\alpha\dot{u}^\beta - \frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial u^\gamma}\dot{u}^\alpha\dot{u}^\beta = \frac{1}{ds/dt}g_{\alpha\gamma}\dot{u}^\alpha\frac{d^2s}{dt^2}$$

or

$$g_{\alpha\gamma}\ddot{u}^\alpha + \frac{1}{2}\left(\frac{\partial g_{\alpha\gamma}}{\partial u^\beta} + \frac{\partial g_{\beta\gamma}}{\partial u^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial u^\gamma}\right)\dot{u}^\alpha\dot{u}^\beta = \frac{1}{ds/dt}g_{\alpha\gamma}\dot{u}^\alpha\frac{d^2s}{dt^2}$$

or

$$g_{\alpha\gamma}\ddot{u}^\alpha + [\alpha\beta,\gamma]\dot{u}^\alpha\dot{u}^\beta = g_{\alpha\gamma}\dot{u}^\alpha\frac{d^2s/dt^2}{ds/dt}. \tag{6.28}$$

These are the desired equations of geodesics on an arbitrary surface the Eq. (6.28) of geodesics for the curves of minimal length will play a role similar to that of straight lines in a plane. Also, this can be written as

$$g^{\gamma\delta}g_{\alpha\gamma}\ddot{u}^\alpha + g^{\gamma\delta}[\alpha\beta,\gamma]\dot{u}^\alpha\dot{u}^\beta = g^{\gamma\delta}g_{\alpha\gamma}\ \ \dot{u}^\alpha\frac{d^2s/dt^2}{ds/dt}$$

or

$$\ddot{u}^\delta + \left\{\begin{matrix}\delta\\ \alpha\ \ \beta\end{matrix}\right\}\dot{u}^\alpha\dot{u}^\beta = \frac{d^2s/dt^2}{ds/dt}\dot{u}^\delta. \tag{6.29}$$

If we choose the parameter t to be the arc length s of the curve, i.e. if we get $s = t$, then,

$$\frac{ds}{dt} = \phi = \sqrt{g_{\alpha\beta}\dot{u}^\alpha \dot{u}^\beta} = 1 \text{ and } \frac{d^2 s}{dt^2} = \frac{d\phi}{dt} = 0$$

then Eq. (6.28) reduces to

$$g_{\alpha\gamma}\ddot{u}^\alpha + [\alpha\beta, \gamma]\dot{u}^\alpha \dot{u}^\beta = 0, \tag{6.30}$$

where '.' denotes the differentiation with respect to the arc parameter s. Multiplying Eq. (6.30) by $g^{\delta\gamma}$ and sum, we obtain a simple form of the equations of geodesics in R^n as

$$g^{\delta\gamma} g_{\alpha\gamma}\ddot{u}^\alpha + g^{\delta\gamma}[\alpha\beta, \gamma]\dot{u}^\alpha \dot{u}^\beta = 0,$$

or

$$\ddot{u}^\delta + \left\{ \begin{matrix} \delta \\ \alpha \ \ \beta \end{matrix} \right\} \dot{u}^\alpha \dot{u}^\beta = 0$$

or

$$\frac{d^2 u^\delta}{ds^2} + \left\{ \begin{matrix} \delta \\ \alpha \ \ \beta \end{matrix} \right\} \frac{du^\alpha}{ds}\frac{du^\beta}{ds} = 0; \ \delta, \alpha, \beta = 1, 2, \cdots, n, \tag{6.31}$$

which is the differential equation of a geodesic in R^n. Since Eq. (6.31) is an ordinary second order differential equation it possesses a unique solution when the values $u^i(s)$ and the first derivatives $\frac{du^i}{ds}$ are prescribed arbitrarily at a given point $u^i(s_0)$. If we regard a given surface $\mathcal{S}$ as a Riemannian two-dimensional manifold R^2, covered by Gaussian co-ordinates u^α, then Eq. (6.31) takes the form

$$\frac{d^2 u^\delta}{ds^2} + \left\{ \begin{matrix} \delta \\ \alpha \ \ \beta \end{matrix} \right\} \frac{du^\alpha}{ds}\frac{du^\beta}{ds} = 0; \ \delta, \alpha, \beta = 1, 2. \tag{6.32}$$

Hence, at each point of $\mathcal{S}$ there exists a unique geodesic with an arbitrarily prescribed direction $\lambda^\alpha = \frac{du^\alpha}{ds}$. Thus, if there exists a unique solution $u^\alpha(s)$, passing through two given points on $\mathcal{S}$, then the curve $u^\alpha(s)$ is the curve of shortest length joining these points. The solution of the system (6.32) of second order differential equations will define the geodesics $u^i = u^i(s)$.

Deduction 6.3.1 (Necessary and sufficient condition for a geodesic): Let us introduce

$$T(u^\alpha, \dot{u}^\alpha) = \frac{1}{2} g_{\alpha\beta}\dot{u}^\alpha \dot{u}^\beta, U^\alpha \equiv \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}^\alpha}\right) - \frac{\partial T}{\partial u^\alpha} = \frac{1}{2T}\frac{dT}{dt}\frac{\partial T}{\partial \dot{u}^\alpha}, \tag{6.33}$$

where $\alpha, \beta = 1, 2$. The expression U^α so defined are important in relation to any curve, whether it is a geodesic or not. Now,

$$\dot{u}^\alpha U^\alpha = \frac{d}{dt}\left(\dot{u}^\alpha \frac{\partial T}{\partial \dot{u}^\alpha}\right) - \ddot{u}^\alpha \frac{\partial T}{\partial \dot{u}^\alpha} - \dot{u}^\alpha \frac{\partial T}{\partial u^\alpha} = \frac{d}{dt}(2T) - \frac{dT}{dt} = \frac{dT}{dt}, \tag{6.34}$$

where T is a function of $u^\alpha \dot{u}^\alpha$ homogeneous of degree 2 in $\dot{u}^\alpha$. Since also the expression for U^α on the right how side of Eq. (6.33) satisfy the same identity, i.e.

$$\dot{u}^\alpha \left(\frac{1}{2T} \frac{dT}{dt} \frac{\partial T}{\partial \dot{u}^\alpha} \right) = \frac{1}{2T} \frac{dT}{dt} \left(\dot{u}^\alpha \frac{dT}{d\dot{u}^\alpha} \right) = \frac{dT}{dt},$$

it follows that the equations in Eq. (6.33) are not independent, they are therefore equivalent to only one equation for the unknowns $u^\alpha(t)$. Eliminating $\frac{dT}{dt}$ in Eq. (6.33) we get

$$u^\alpha \frac{\partial T}{\partial \dot{u}^\beta} - u^\beta \frac{dT}{\partial \dot{u}^\alpha} = 0 : \alpha, \beta = 1, 2. \tag{6.35}$$

This, then, is necessary for a geodesic. To prove that it is also sufficient, suppose that it is satisfied by functions $u^\alpha(t)$, whose first derivatives do not vanish simultaneously at any point. The $\frac{\partial T}{\partial \dot{u}^\alpha}$ cannot vanish together. Hence

$$U^\alpha = \mu \frac{\partial T}{\partial \dot{u}^\alpha}; \quad \text{for some } \mu,$$

and then from Eq. 6.34 we get

$$\frac{dT}{dt} = \mu \left(\dot{u}^\alpha \frac{\partial T}{\partial \dot{u}^\alpha} \right) = 2T\mu \Rightarrow \mu = \frac{1}{2T} \frac{dT}{dt}.$$

The function $u^\alpha(t)$ therefore satisfy Eq. (6.33).

Deduction 6.3.2 Intrinsic curvature: The intrinsic curvature of a curve $\mathcal{C}$ in the surface $\mathcal{S}$ is the function

$$\widetilde{\kappa} = \sqrt{g_{ij} b^i b^j}; \; b^i = \frac{\delta}{\delta s} \left(\frac{dx^i}{ds} \right),$$

where the intrinsic curvature vector b^i is given by

$$b^i = \frac{\delta}{\delta s} \left(\frac{dx^i}{ds} \right) = \frac{d^2 x^i}{ds^2} + \left\{ \begin{matrix} i \\ p \;\; q \end{matrix} \right\} \frac{dx^p}{ds} \; \frac{dx^q}{ds}.$$

Intrinsic curvature can be shown to be the instantaneous rate of change of the angle between the tangent vector of $\mathcal{C}$ and another vector in the tangent space that is transported parallely along the curve. A curve on a surface is a geodesic if and only if its intrinsic curvature $\widetilde{\kappa}$ is identically zero.

EXAMPLE 6.3.1 *Prove that the curves of the family* $\frac{v^3}{u^2} = constant$ *are geodesics on the surface with metric* $v^2 du^2 - 2uvdudv + 2u^2 dv^2; u > 0, v > 0$.

Solution: Let $\dfrac{v^3}{u^2} = c(> 0)$ so that the parametric form of the equation of the curve is $u = ct^3, v = ct^2$. Construct.

$$T = \frac{1}{2}\left(v^2\dot{u}^2 - 2uv\dot{u}\dot{v} + 2u^2\dot{v}^2\right)$$

Therefore,

$$\begin{aligned}
&\frac{\partial T}{\partial u} = -v\dot{u}\dot{v} + 2u\dot{v}^2 = 2c^3t^5,\ \ \frac{\partial T}{\partial v} = v\dot{u}^2 - u\dot{u}\dot{v} = 3c^3t^6\\
&\frac{\partial T}{\partial \dot{u}} = v^2\dot{u} - uv\dot{v} = c^3t^6,\ \ \frac{\partial T}{\partial \dot{v}} = -uv\dot{u} - 2u^2\dot{v} = c^3t^7\\
&\dot{U}^1 = \frac{d}{dt}(c^3t^6) - 2c^3t^5 = 4c^3t^5,\ \ V^2 = \frac{d}{dt}(c^3t^7) - 3c^3t^6 = 4c^3t^6
\end{aligned}$$

Since $V\dfrac{\partial T}{\partial \dot{u}} - U\dfrac{\partial T}{\partial \dot{v}} = 0$, i.e. the curve is a geodesic for every value of c.

EXAMPLE 6.3.2 *Show that the geodesics are straight lines when the co-ordinates are Cartesian.*

Solution: Consider the Euclidean space E^3 of 3 dimensions. In this case, the metric is $ds^2 = g_{ij}dx^\alpha dx^\beta$, where the tensor g_{ij} is denoted by a_{ij} and $g_{ij} = a_{ij} = \delta^i_j$. Thus, the components of the Christofell symbols are given by

$$g_{ij} = \delta^i_j \Rightarrow \left\{\begin{matrix} k \\ i\ \ j \end{matrix}\right\} = 0 = [ij, k],$$

relative to E^n. The differential equation of a geodesic in E^n, with Euclidean co-ordinates x^i is

$$\frac{d^2x^i}{ds^2} + \left\{\begin{matrix} i \\ j\ \ k \end{matrix}\right\}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0$$

or

$$\frac{d^2x^i}{ds^2} = 0 \Rightarrow x^i = a^is + b^i,\quad s = \text{ arc length},$$

where a^i and b^i are constants with $g_{ij}a^ia^j = 1$. Thus, from each point $\mathbf{x} = \mathbf{b}$ of space there emanates a geodesic ray in every direction (unit vector) $\mathbf{a}$. This equation can be written in the form

$$\frac{x^1 - b^1}{a^1} = \frac{x^1 - b^1}{a^1} = \frac{x^1 - b^1}{a^1}(= s).$$

The equation $x^i = a^is + b^i$ represents the equation of straight lines in E^3. Thus in affine co-ordinates, where all g_{ij} are constants and all Christoffel symbols vanish, the geodesics are straight lines.

EXAMPLE 6.3.3 *Find the differential equations for the geodesic in spherical co-ordinates.*

Solution: The relation between Cartesian and spherical co-ordinates is

$$x^1 = u^1 \sin u^2 \cos u^3, x^2 = u^1 \sin u^2 \sin u^3, x^3 = u^1 \cos u^2.$$

The expression for the metric in spherical co-ordinates is given by

$$ds^2 = \left(du^1\right)^2 + \left(u^1\right)^2 \left(du^2\right)^2 + \left(u^1\right)^2 \sin^2 u^2 \left(du^3\right)^2.$$

Comparing the given metric, with $ds^2 = a_{ij} du^i du^j$, we get

$$a_{11} = 1; a_{22} = \left(u^1\right)^2; a_{33} = \left(u^1\right)^2 \sin^2 u^2 \text{ and } a_{ij} = 0 \text{ for } i \neq j.$$

$$\Rightarrow a = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = (u^1)^4 \sin^2 u^2.$$

Therefore, the reciprocal tensors are given by

$$a^{11} = \frac{\text{cofactor of } a_{11} \text{ in } a}{a} = 1;\ a^{22} = \frac{\text{cofactor of } a_{22} \text{ in } a}{a} = \frac{1}{(u^1)^2}.$$

$$a^{33} = \frac{\text{cofactor of } a_{33} \text{ in } a}{a} = \frac{1}{\left(u^1\right)^2 \sin^2 u^2}.$$

$$a^{12} = 0 = a^{21};\ a^{13} = 0 = a^{31};\ a^{23} = 0 = a^{32}.$$

The non-vanishing Christofell symbols of the second kind are

$$\left\{ \begin{matrix} 1 \\ 2\ 2 \end{matrix} \right\} = -u^1;\ \left\{ \begin{matrix} 2 \\ 1\ 2 \end{matrix} \right\} = \frac{1}{(u^1)^2} = \left\{ \begin{matrix} 2 \\ 2\ 1 \end{matrix} \right\};\ \left\{ \begin{matrix} 2 \\ 3\ 3 \end{matrix} \right\} = -\sin u^2 \cos u^2;$$

$$\left\{ \begin{matrix} 3 \\ 1\ 3 \end{matrix} \right\} = \frac{1}{u^1} = \left\{ \begin{matrix} 3 \\ 3\ 1 \end{matrix} \right\};\ \left\{ \begin{matrix} 1 \\ 3\ 3 \end{matrix} \right\} = -u^1 \sin^2 u^2;\ \left\{ \begin{matrix} 3 \\ 2\ 3 \end{matrix} \right\} = \cot u^2 = \left\{ \begin{matrix} 3 \\ 3\ 2 \end{matrix} \right\}.$$

The differential equation of a geodesic for the surface is given by

$$\frac{d^2u^i}{ds^2} + \left\{ \begin{matrix} i \\ j\ k \end{matrix} \right\} \frac{du^j}{ds} \frac{du^k}{ds} = 0;\ i, j, k = 1, 2, 3. \tag{6.36}$$

For $i = 1$, the differential equation becomes

$$\frac{d^2u^1}{ds^2} + \left\{ \begin{matrix} 1 \\ j\ k \end{matrix} \right\} \frac{du^j}{ds} \frac{du^k}{ds} = 0$$

or

$$\frac{d^2u^1}{ds^2} + \left\{ \begin{matrix} 1 \\ 1 \;\; k \end{matrix} \right\} \frac{du^1}{ds}\frac{du^k}{ds} + \left\{ \begin{matrix} 1 \\ 2 \;\; k \end{matrix} \right\} \frac{du^2}{ds}\frac{du^k}{ds} + \left\{ \begin{matrix} 1 \\ 3 \;\; k \end{matrix} \right\} \frac{du^3}{ds}\frac{du^k}{ds} = 0$$

or

$$\frac{d^2u^1}{ds^2} + \left\{ \begin{matrix} 1 \\ 1 \;\; 1 \end{matrix} \right\} \left(\frac{du^1}{ds}\right)^2 + 2\left\{ \begin{matrix} 1 \\ 1 \;\; 2 \end{matrix} \right\} \frac{du^1}{ds}\frac{du^2}{ds} + 2\left\{ \begin{matrix} 1 \\ 1 \;\; 3 \end{matrix} \right\} \frac{du^1}{ds}\frac{du^3}{ds}$$
$$+ \left\{ \begin{matrix} 1 \\ 2 \;\; 2 \end{matrix} \right\} \left(\frac{du^2}{ds}\right)^2 + 2\left\{ \begin{matrix} 1 \\ 2 \;\; 3 \end{matrix} \right\} \frac{du^2}{ds}\frac{du^3}{ds} + \left\{ \begin{matrix} 1 \\ 3 \;\; 3 \end{matrix} \right\} \left(\frac{du^3}{ds}\right)^2 = 0$$

or

$$\frac{d^2u^1}{ds^2} - u^1 \left(\frac{du^2}{ds}\right)^2 - u^1 \sin^2 u^2 \left(\frac{du^3}{ds}\right)^2 = 0$$

For $i = 2$, the differential equation becomes

$$\frac{d^2u^2}{ds^2} + \left\{ \begin{matrix} 2 \\ j \;\; k \end{matrix} \right\} \frac{du^j}{ds}\frac{du^k}{ds} = 0$$

or

$$\frac{d^2u^2}{ds^2} + \left\{ \begin{matrix} 2 \\ 1 \;\; k \end{matrix} \right\} \frac{du^1}{ds}\frac{du^k}{ds} + \left\{ \begin{matrix} 2 \\ 2 \;\; k \end{matrix} \right\} \frac{du^2}{ds}\frac{du^k}{ds} + \left\{ \begin{matrix} 2 \\ 3 \;\; k \end{matrix} \right\} \frac{du^3}{ds}\frac{du^k}{ds} = 0$$

or

$$\frac{d^2u^2}{ds^2} + \left\{ \begin{matrix} 2 \\ 1 \;\; 1 \end{matrix} \right\} \left(\frac{du^1}{ds}\right)^2 + 2\left\{ \begin{matrix} 2 \\ 1 \;\; 2 \end{matrix} \right\} \frac{du^1}{ds}\frac{du^2}{ds} + 2\left\{ \begin{matrix} 2 \\ 1 \;\; 3 \end{matrix} \right\} \frac{du^1}{ds}\frac{du^3}{ds}$$
$$+ \left\{ \begin{matrix} 2 \\ 2 \;\; 2 \end{matrix} \right\} \left(\frac{du^2}{ds}\right)^2 + 2\left\{ \begin{matrix} 2 \\ 2 \;\; 3 \end{matrix} \right\} \frac{du^2}{ds}\frac{du^3}{ds} + \left\{ \begin{matrix} 2 \\ 3 \;\; 3 \end{matrix} \right\} \left(\frac{du^3}{ds}\right)^2 = 0$$

or

$$\frac{d^2u^2}{ds^2} + \frac{2}{(u^1)^2}\frac{du^1}{ds}\frac{du^2}{ds} - \sin u^2 \cos u^2 \left(\frac{du^3}{ds}\right)^2 = 0.$$

For $i = 3$, the differential equation becomes

$$\frac{d^2u^3}{ds^2} + \left\{ \begin{matrix} 3 \\ j \;\; k \end{matrix} \right\} \frac{du^j}{ds}\frac{du^k}{ds} = 0$$

or

$$\frac{d^2u^3}{ds^2} + \left\{ \begin{matrix} 3 \\ 1 \;\; k \end{matrix} \right\} \frac{du^1}{ds}\frac{du^k}{ds} + \left\{ \begin{matrix} 3 \\ 2 \;\; k \end{matrix} \right\} \frac{du^2}{ds}\frac{du^k}{ds} + \left\{ \begin{matrix} 3 \\ 3 \;\; k \end{matrix} \right\} \frac{du^3}{ds}\frac{du^k}{ds} = 0$$

or

$$\frac{d^2u^3}{ds^2} + \left\{\begin{matrix} 3 \\ 1\ 1 \end{matrix}\right\}\left(\frac{du^1}{ds}\right)^2 + 2\left\{\begin{matrix} 3 \\ 1\ 2 \end{matrix}\right\}\frac{du^1}{ds}\frac{du^2}{ds} + 2\left\{\begin{matrix} 3 \\ 1\ 3 \end{matrix}\right\}\frac{du^1}{ds}\frac{du^3}{ds}$$

$$+\left\{\begin{matrix} 3 \\ 2\ 2 \end{matrix}\right\}\left(\frac{du^2}{ds}\right)^2 + 2\left\{\begin{matrix} 3 \\ 2\ 3 \end{matrix}\right\}\frac{du^2}{ds}\frac{du^3}{ds} + \left\{\begin{matrix} 3 \\ 3\ 3 \end{matrix}\right\}\left(\frac{du^3}{ds}\right)^2 = 0$$

or

$$\frac{d^2u^3}{ds^2} + \frac{2}{u^1}\frac{du^1}{ds}\frac{du^3}{ds} + \cot u^2 \frac{du^2}{ds}\frac{du^3}{ds} = 0.$$

EXAMPLE 6.3.4 *Find the differential equations for the geodesic in cylindrical co-ordinates.*

Solution: The relation between Cartesian and cylindrical co-ordinates is

$$x^1 = u^1 \cos u^2, x^2 = u^1 \sin u^2, x^3 = u^3.$$

The expression for the metric in cylindrical co-ordinates is given by

$$ds^2 = \left(du^1\right)^2 + \left(u^1\right)^2\left(du^2\right)^2 + \left(du^3\right)^2,$$

from which, the coefficients $a_{\alpha\beta}$ of the first fundamental form are given by

$$a_{11} = 1;\ a_{22} = (u^1)^2;\ a_{33} = 1 \text{ and } a_{ij} = 0;\ \text{for } i \neq j,$$

so that $a = (u^1)^2$. The reciprocal tensors $a^{\alpha\beta}$ are given by

$$a^{11} = 1;\ a^{22} = \frac{1}{(u^1)^2};\ a^{33} = 1$$

$$a^{12} = 0 = a^{21};\ a^{13} = 0 = a^{31};\ a^{23} = 0 = a^{32}.$$

The non-vanishing Christofell symbols of the second kind are

$$\left\{\begin{matrix} 1 \\ 2\ 2 \end{matrix}\right\} = -u^1;\ \left\{\begin{matrix} 2 \\ 1\ 2 \end{matrix}\right\} = \frac{1}{u^1} = \left\{\begin{matrix} 2 \\ 2\ 1 \end{matrix}\right\}.$$

The differential equation of a geodesic for the surface is given by Eq. (6.36). For $i = 1$, the differential equation becomes

$$\frac{d^2u^1}{ds^2} + \left\{\begin{matrix} 1 \\ j\ k \end{matrix}\right\}\frac{du^j}{ds}\frac{du^k}{ds} = 0$$

or

$$\frac{d^2u^1}{ds^2} + \left\{\begin{matrix} 1 \\ 1\ k \end{matrix}\right\}\frac{du^1}{ds}\frac{du^k}{ds} + \left\{\begin{matrix} 1 \\ 2\ k \end{matrix}\right\}\frac{du^2}{ds}\frac{du^k}{ds} + \left\{\begin{matrix} 1 \\ 3\ k \end{matrix}\right\}\frac{du^3}{ds}\frac{du^k}{ds} = 0$$

or

$$\frac{d^2u^1}{ds^2} + \left\{ \begin{matrix} 1 \\ 1\ \ 1 \end{matrix} \right\} \left(\frac{du^1}{ds} \right)^2 + 2 \left\{ \begin{matrix} 1 \\ 1\ \ 2 \end{matrix} \right\} \frac{du^1}{ds}\frac{du^2}{ds} + 2 \left\{ \begin{matrix} 1 \\ 1\ \ 3 \end{matrix} \right\} \frac{du^1}{ds}\frac{du^3}{ds}$$

$$+ \left\{ \begin{matrix} 1 \\ 2\ \ 2 \end{matrix} \right\} \left(\frac{du^2}{ds} \right)^2 + 2 \left\{ \begin{matrix} 1 \\ 2\ \ 3 \end{matrix} \right\} \frac{du^2}{ds}\frac{du^3}{ds} + \left\{ \begin{matrix} 1 \\ 3\ \ 3 \end{matrix} \right\} \left(\frac{du^3}{ds} \right)^2 = 0$$

or

$$\frac{d^2u^1}{ds^2} - u^1 \left(\frac{du^2}{ds} \right)^2 = 0$$

For $i = 2$, the differential equation becomes

$$\frac{d^2u^2}{ds^2} + \left\{ \begin{matrix} 2 \\ j\ \ k \end{matrix} \right\} \frac{du^j}{ds}\frac{du^k}{ds} = 0$$

or

$$\frac{d^2u^2}{ds^2} + \left\{ \begin{matrix} 2 \\ 1\ \ k \end{matrix} \right\} \frac{du^1}{ds}\frac{du^k}{ds} + \left\{ \begin{matrix} 2 \\ 2\ \ k \end{matrix} \right\} \frac{du^2}{ds}\frac{du^k}{ds} + \left\{ \begin{matrix} 2 \\ 3\ \ k \end{matrix} \right\} \frac{du^3}{ds}\frac{du^k}{ds} = 0$$

or

$$\frac{d^2u^2}{ds^2} + \left\{ \begin{matrix} 2 \\ 1\ \ 1 \end{matrix} \right\} \left(\frac{du^1}{ds} \right)^2 + 2 \left\{ \begin{matrix} 2 \\ 1\ \ 2 \end{matrix} \right\} \frac{du^1}{ds}\frac{du^2}{ds} + 2 \left\{ \begin{matrix} 2 \\ 1\ \ 3 \end{matrix} \right\} \frac{du^1}{ds}\frac{du^3}{ds}$$

$$+ \left\{ \begin{matrix} 2 \\ 2\ \ 2 \end{matrix} \right\} \left(\frac{du^2}{ds} \right)^2 + 2 \left\{ \begin{matrix} 2 \\ 2\ \ 3 \end{matrix} \right\} \frac{du^2}{ds}\frac{du^3}{ds} + \left\{ \begin{matrix} 2 \\ 3\ \ 3 \end{matrix} \right\} \left(\frac{du^3}{ds} \right)^2 = 0$$

or

$$\frac{d^2u^2}{ds^2} + \frac{2}{u^1}\frac{du^1}{ds}\frac{du^2}{ds} = 0.$$

EXAMPLE 6.3.5 *Find the differential equations of the geodesic for the metric*

$$ds^2 = (dx^1)^2 + \{(x^2)^2 - (x^1)^2\}(dx^2)^2.$$

Solution: The square of the elementary arc length is given by

$$ds^2 = (dx^1)^2 + \{(x^2)^2 - (x^1)^2\}(dx^2)^2.$$

The coefficients $a_{\alpha\beta}$ of the first fundamental form are given by

$$a_{11} = 1;\ a_{22} = (x^2)^2 - (x^1)^2 \text{ and } a_{12} = 0 = a_{21}.$$

so that $a = (x^2)^2 - (x^1)^2$. The reciprocal tensors $a^{\alpha\beta}$ are given by

$$a^{11} = 1;\ a^{22} = \frac{1}{(x^2)^2 - (x^1)^2};\ a^{12} = 0 = a^{21}.$$

The non-vanishing Christofell symbols of the second kind are

$$\left\{ \begin{matrix} 1 \\ 2\ \ 2 \end{matrix} \right\} = x^1;\ \left\{ \begin{matrix} 2 \\ 1\ \ 2 \end{matrix} \right\} = -\frac{x^1}{(x^2)^2 - (x^1)^2};\ \left\{ \begin{matrix} 2 \\ 2\ \ 2 \end{matrix} \right\} = \frac{x^2}{(x^2)^2 - (x^1)^2}.$$

The differential equation of a geodesic for the surface is given by

$$\frac{d^2x^i}{ds^2} + \left\{ \begin{matrix} i \\ j\ \ k \end{matrix} \right\} \frac{dx^j}{ds}\frac{dx^k}{ds} = 0;\ i, j, k = 1, 2. \tag{6.37}$$

For $i = 1$, the differential equation becomes

$$\frac{d^2x^1}{ds^2} + \left\{ \begin{matrix} 1 \\ j\ \ k \end{matrix} \right\} \frac{dx^j}{ds}\frac{dx^k}{ds} = 0$$

or

$$\frac{d^2x^1}{ds^2} + \left\{ \begin{matrix} 1 \\ 1\ \ k \end{matrix} \right\} \frac{dx^1}{ds}\frac{dx^k}{ds} + \left\{ \begin{matrix} 1 \\ 2\ \ k \end{matrix} \right\} \frac{dx^2}{ds}\frac{dx^k}{ds} = 0$$

or

$$\frac{d^2x^1}{ds^2} + \left\{ \begin{matrix} 1 \\ 1\ \ 1 \end{matrix} \right\} \left(\frac{dx^1}{ds}\right)^2 + 2\left\{ \begin{matrix} 1 \\ 1\ \ 2 \end{matrix} \right\} \frac{dx^1}{ds}\frac{dx^2}{ds} + \left\{ \begin{matrix} 1 \\ 2\ \ 2 \end{matrix} \right\} \left(\frac{dx^2}{ds}\right)^2 = 0$$

or

$$\frac{d^2x^1}{ds^2} + x^1 \left(\frac{dx^2}{ds}\right)^2 = 0.$$

For $i = 2$, the differential equation becomes

$$\frac{d^2x^2}{ds^2} + \left\{ \begin{matrix} 2 \\ j\ \ k \end{matrix} \right\} \frac{dx^j}{ds}\frac{dx^k}{ds} = 0$$

or

$$\frac{d^2x^2}{ds^2} + \left\{ \begin{matrix} 2 \\ 1\ \ k \end{matrix} \right\} \frac{dx^1}{ds}\frac{dx^k}{ds} + \left\{ \begin{matrix} 2 \\ 2\ \ k \end{matrix} \right\} \frac{dx^2}{ds}\frac{dx^k}{ds} = 0$$

or

$$\frac{d^2x^2}{ds^2} + \left\{ \begin{matrix} 2 \\ 1\ \ 1 \end{matrix} \right\} \left(\frac{dx^1}{ds}\right)^2 + 2\left\{ \begin{matrix} 2 \\ 1\ \ 2 \end{matrix} \right\} \frac{dx^1}{ds}\frac{dx^2}{ds} + \left\{ \begin{matrix} 2 \\ 2\ \ 2 \end{matrix} \right\} \left(\frac{dx^2}{ds}\right)^2 = 0$$

or

$$\frac{d^2x^2}{ds^2} - \frac{2x^1}{(x^2)^2 - (x^1)^2}\frac{dx^1}{ds}\frac{dx^2}{ds} + \frac{x^2}{(x^2)^2 - (x^1)^2}\left(\frac{dx^2}{ds}\right)^2 = 0,$$

which are the required equations of the geodesic.

EXAMPLE 6.3.6 *Find the geodesics on the surface*

$$x^1 = u^1 \cos u^2, x^2 = u^1 \sin u^2, x^3 = 0,$$

embedded in E^3*, where* x^i *are orthogonal Cartesian co-ordinates.*

Solution: The square of the elementary arc length is given by

$$\begin{aligned}(ds)^2 &= (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \\ &= \left[du^1 \cos u^2 + u^1(-\sin u^2)du^2\right]^2 + \left[du^1 \sin u^2 + u^1 \cos u^2 du^2\right]^2 + 0 \\ &= (du^1)^2 + (u^1)^2(du^2)^2.\end{aligned}$$

The coefficients $a_{\alpha\beta}$ of the first fundamental form are given by

$$a_{11} = 1;\ a_{22} = (u^1)^2 \text{ and } a_{12} = 0 = a_{21},$$

so that $a = (u^1)^2$. The reciprocal tensors $a^{\alpha\beta}$ are given by

$$a^{11} = 1;\ a^{22} = \frac{1}{(u^1)^2};\ a^{12} = 0 = a^{21}.$$

The non-vanishing Christofell symbols of the second kind are

$$\left\{ \begin{matrix} 1 \\ 2\ 2 \end{matrix} \right\} = -u^1;\ \left\{ \begin{matrix} 2 \\ 1\ 2 \end{matrix} \right\} = \frac{1}{u^1} = \left\{ \begin{matrix} 2 \\ 2\ 1 \end{matrix} \right\}.$$

The differential equation of a geodesic for the surface is given by Eq. (6.37). For $i = 1$, the differential equation becomes

$$\frac{d^2u^1}{ds^2} + \left\{ \begin{matrix} 1 \\ j\ k \end{matrix} \right\} \frac{du^j}{ds}\frac{du^k}{ds} = 0$$

or

$$\frac{d^2u^1}{ds^2} + \left\{ \begin{matrix} 1 \\ 1\ k \end{matrix} \right\} \frac{du^1}{ds}\frac{du^k}{ds} + \left\{ \begin{matrix} 1 \\ 2\ k \end{matrix} \right\} \frac{du^2}{ds}\frac{du^k}{ds} = 0$$

or

$$\frac{d^2u^1}{ds^2} + \left\{ \begin{matrix} 1 \\ 1\ 1 \end{matrix} \right\} \left(\frac{du^1}{ds}\right)^2 + 2\left\{ \begin{matrix} 1 \\ 1\ 2 \end{matrix} \right\} \frac{du^1}{ds}\frac{du^2}{ds} + \left\{ \begin{matrix} 1 \\ 2\ 2 \end{matrix} \right\} \left(\frac{du^2}{ds}\right)^2 = 0$$

or

$$\frac{d^2u^1}{ds^2} - u^1 \left(\frac{du^2}{ds}\right)^2 = 0. \qquad \text{(i)}$$

For $i = 2$, the differential equation becomes

$$\frac{d^2u^2}{ds^2} + \left\{ \begin{matrix} 2 \\ j\ k \end{matrix} \right\} \frac{du^j}{ds}\frac{du^k}{ds} = 0$$

or

$$\frac{d^2u^2}{ds^2} + \left\{ \begin{matrix} 2 \\ 1 \quad k \end{matrix} \right\} \frac{du^1}{ds}\frac{du^k}{ds} + \left\{ \begin{matrix} 2 \\ 2 \quad k \end{matrix} \right\} \frac{du^2}{ds}\frac{du^k}{ds} = 0$$

or

$$\frac{d^2u^2}{ds^2} + \left\{ \begin{matrix} 2 \\ 1 \quad 1 \end{matrix} \right\} \left(\frac{du^1}{ds}\right)^2 + 2\left\{ \begin{matrix} 2 \\ 1 \quad 2 \end{matrix} \right\} \frac{du^1}{ds}\frac{du^2}{ds} + \left\{ \begin{matrix} 2 \\ 2 \quad 2 \end{matrix} \right\} \left(\frac{du^2}{ds}\right)^2 = 0$$

or

$$\frac{d^2u^2}{ds^2} + \frac{2}{u^1}\frac{du^1}{ds}\frac{du^2}{ds} = 0; \text{ i.e. } \frac{d^2u^2}{ds^2}\Big/\frac{du^2}{ds} + \frac{2}{u^1}\frac{du^1}{ds} = 0$$

or

$$\log\left(\frac{du^2}{ds}\right) + 2\log u^1 = \log c; \; c = \text{ constant}$$

or

$$\log\left[(u^1)^2\frac{du^2}{ds}\right] = \log c \Rightarrow \frac{du^2}{ds} = \frac{c}{(u^1)^2}. \tag{ii}$$

Therefore, from Eq. (i) we get

$$\frac{d^2u^1}{ds^2} - u^1\frac{c^2}{(u^1)^4} = 0; \text{ i.e. } 2\frac{du^1}{ds}\frac{d^2u^1}{ds^2} = 2\frac{c^2}{(u^1)^3}\frac{du^1}{ds}$$

or

$$\left(\frac{du^1}{ds}\right)^2 = -\frac{c^2}{(u^1)^2} + c_1^2; c_1 = \text{ constant}$$

$$= c_1^2\left[1 - \frac{c^2}{c_1^2(u^1)^2}\right] = c_1^2\frac{(u^1)^2 - d^2}{(u^1)^2}; d = \frac{c}{c_1}$$

or

$$\frac{u^1 \; du^1}{(u^1)^2 - d^2} = c_1 ds \Rightarrow (u^1)^2 = (c_1 s + e)^2 + d^2,$$

where e is an integration constant. Therefore, from Eq. (ii) we get

$$\frac{du^2}{ds} = \frac{c}{(u^1)^2} = \frac{c}{d^2}\frac{1}{1 + \left(\frac{c_1 s + e}{d}\right)^2}$$

or

$$u^2 = \frac{c}{d^2}\frac{1}{c_1}\tan^{-1}\left(\frac{c_1 s + e}{d}\right)$$

or

$$u^2 = \frac{1}{p}\tan^{-1}\left(\frac{c_1 s + e}{d}\right); \; p = \frac{d^2 c_1}{c}$$

or

$$d\tan(pu^2) = c_1 s + e,$$

which is the required equation of the geodesic.

EXAMPLE 6.3.7 *On a right helicoid of pitch $2\pi a$, a geodesic makes as angle α with a generator at a point distant c from the axis $(0 < \alpha < \frac{\pi}{2}, c > 0)$. Find the condition that the geodesics meets the axis.*

Solution: The first fundamental form of the right helicoid is given by

$$ds^2 = du^2 + (u^2 + a^2)dv^2.$$

The non-vanishing Christofell symbols of the second kind are

$$\left\{ \begin{matrix} 1 \\ 2 \;\; 2 \end{matrix} \right\} = -u, \left\{ \begin{matrix} 2 \\ 2 \;\; 1 \end{matrix} \right\} = \frac{u}{u^2 + a^2} = \left\{ \begin{matrix} 2 \\ 1 \;\; 2 \end{matrix} \right\}.$$

The differential equation of geodesics are

$$\frac{d^2u}{ds^2} - u\left(\frac{dv}{ds}\right)^2 = 0 \quad \text{and} \quad \frac{d^2v}{ds^2} + \frac{du}{u^2 + a^2}\frac{du}{ds}\frac{dv}{ds} = 0.$$

Thus the first integral of the geodesic equations is

$$\frac{dv}{du} = \pm \frac{k}{\{(u^2 + a^2)(u^2 + a^2 - k^2)\}^{1/2}},$$

where k is an arbitrary positive constant. Further integration in general requires elliptic functions.

The given point is $(c, 0)$ for a suitable choice of axes, and α is the angle between the direction $(1, 0)$ and $\left(\frac{du}{ds}, \frac{dv}{ds}\right)$ at this point, then

$$\tan\alpha = (a_{11}a_{22} - a_{12}^2)^{1/2}\frac{dv/ds}{du/ds} = k(c^2 + a^2 - k^2)^{-1/2}$$
$$\Rightarrow \quad k = (c^2 + a^2)^{1/2}\sin\alpha.$$

There are two geodesics satisfying the given initial conditions, but it will be sufficient to consider the one for which $\frac{dv}{du} < 0$ initially. Now we consider the following three cases:

(i) $k^2 > a^2$, i.e. $c\tan\alpha > a$. Since which $\frac{dv}{du} < 0$ initially, u deereases as v increases until

$$u = (k^2 - \alpha^2)^{1/2} = (c^2\sin^2\alpha - a^2\cos^2\alpha)^{1/2}.$$

As v continues to increase, the sign of which $\frac{dv}{du}$ changes and u increases indefinitely. The least distance from the axis is therefore $(c^2 sin^2\alpha - a^2\cos^2\alpha)^{1/2}$.

(ii) $k^2 < a^2$, i.e. $c\tan\alpha < a$. In this case which $\dfrac{dv}{du} < 0$ for all v, and u decreases indefinitely as v increases. There is a point on the curve at which $u = 0$, i.e. the curve meets the axis.

(iii) $k^2 = a^2$, i.e. $c\tan\alpha = a$. In this special case

$$\frac{dv}{du} = \frac{-a}{u\sqrt{u^2 + a^2}} \Rightarrow v = -\beta + \sin h^1\left(\frac{a}{u}\right)$$

where $\beta = \sin h^{-1}(a/c)$, as $v = 0$, when $u = c$. The geodesic is therefore given by

$$u \sin h(v + \beta) = a, \beta = \sin h^{-1}(a/c).$$

As v increases, the curve approaches the axis without reaching it. In the opposite sense, $u \to \infty$ as $v \to -\beta$, showing that the generator $v = -\beta$ is an asymptote.

6.4 Geodesic Co-ordinates

If a Riemannian space is Euclidean, a co-ordinate system exists in which the components of the metric tensors g_{ij} are constant throughout the space and hence, $\frac{\partial g_{ij}}{\partial x^k} = 0$; for all i, j, k. Consequently, the vanishing of the Christoffel symbols, as Eq. (3.1). This is not true for any Riemannian space. But if in a Riemannian space, there exists a co-ordinate system, in fact infinitely many, with respect to which the Christoffel symbols vanish at a given point, then, that system is called a *geodesic co-ordinate system* and the point is called the *pole of the given system.*

Let us consider a surface $\mathcal{S}$ whose curvilinear co-ordinates are u^1, u^2 and also consider a point $P(u_0^1, u_0^2)$ on $\mathcal{S}$. If v^α; $\alpha = 1, 2$ are the co-ordinates of some net on $\mathcal{S}$, then we consider a transformation

$$u^\alpha = u^\alpha\left(v^1, v^2\right);\ \alpha = 1, 2. \tag{6.38}$$

The second derivative formula yields the relation,

$$\frac{\partial^2 u^\alpha}{\partial v^\lambda \partial v^\mu} + \left\{ \begin{matrix} \alpha \\ \beta\ \ \gamma \end{matrix} \right\}_u \frac{\partial u^\beta}{\partial v^\lambda}\frac{\partial u^\gamma}{\partial v^\mu} = \left\{ \begin{matrix} \gamma \\ \lambda\ \ \mu \end{matrix} \right\}_v \frac{\partial u^\alpha}{\partial v^\gamma}, \tag{6.39}$$

where $\left\{ \begin{matrix} \alpha \\ \beta\ \ \gamma \end{matrix} \right\}_u$ and $\left\{ \begin{matrix} \gamma \\ \lambda\ \ \mu \end{matrix} \right\}_v$ are the Christoffel symbols in u co-ordinate system and v co-ordinate system, respectively. However, if there exists a transformation of co-ordinates Eq. (6.38) for which $\left\{ \begin{matrix} \gamma \\ \lambda\ \ \mu \end{matrix} \right\}_v$ vanish at P, then for that particular point

$$\frac{\partial^2 u^\alpha}{\partial v^\lambda \partial v^\mu} + \left\{ \begin{matrix} \alpha \\ \beta\ \ \gamma \end{matrix} \right\}_u \frac{\partial u^\beta}{\partial v^\lambda}\frac{\partial u^\gamma}{\partial v^\mu} = 0. \tag{6.40}$$

Next, we exhibit next a solution of the Eq. (6.40) yielding transformation Eq. (6.38) to a co-ordinate system v^α in which the Christoffel symbols vanish at $P(u_0^1, u_0^2)$ on $\mathcal{S}$. Let us take a second degree polynomial

$$u^\alpha = u_P^\alpha + v^\alpha - \left\{ \begin{matrix} \alpha \\ \lambda \;\; \mu \end{matrix} \right\}_P v^\lambda v^\mu, \tag{6.41}$$

where u_P^α is the value of u^α at P. On differentiation, from Eq. (6.41), we get

$$\frac{\partial u^\alpha}{\partial v^\mu} = \delta_\mu^\alpha - \left\{ \begin{matrix} \alpha \\ \lambda \;\; \mu \end{matrix} \right\}_P v^\lambda \text{ and } \frac{\partial^2 u^\alpha}{\partial v^\lambda \partial v^\mu} = -\left\{ \begin{matrix} \alpha \\ \lambda \;\; \mu \end{matrix} \right\}_P$$

$$\left(\frac{\partial u^\alpha}{\partial v^\mu}\right)_P = \delta_\mu^\alpha \text{ and } \left(\frac{\partial^2 u^\alpha}{\partial v^\lambda \partial v^\mu}\right)_P = -\left\{ \begin{matrix} \alpha \\ \lambda \;\; \mu \end{matrix} \right\}_P. \tag{6.42}$$

Since the Jacobian determinant $\left|\left(\frac{\partial u^\alpha}{\partial v^l}\right)_P\right| = |\delta_\mu^\alpha| = 1 \neq 0$, so the given transformation is permissible in the neighbourhood of $P(u_0^1, u_0^2)$ on $\mathcal{S}$. Using the formula,

$$u^\alpha_{,\lambda\mu} = \frac{\partial^2 u^\alpha}{\partial v^\lambda \partial v^\mu} - \left\{ \begin{matrix} k \\ \lambda \;\; \mu \end{matrix} \right\} \frac{\partial u^\alpha}{\partial v^k},$$

we see that at $P(u_0^1, u_0^2)$ on $\mathcal{S}$,

$$\begin{aligned}(u^\alpha_{,\lambda\mu})_P &= \left(\frac{\partial^2 u^\alpha}{\partial v^\lambda \partial v^\mu}\right)_P - \left\{ \begin{matrix} k \\ \lambda \;\; \mu \end{matrix} \right\}_P \left(\frac{\partial u^\alpha}{\partial v^k}\right)_P \\ &= \left\{ \begin{matrix} \alpha \\ \lambda \;\; \mu \end{matrix} \right\}_P - \left\{ \begin{matrix} k \\ \lambda \;\; \mu \end{matrix} \right\}_P \delta_k^\alpha = 0. \end{aligned} \tag{6.43}$$

Therefore, the values in Eq. (6.42) satisfies Eq. (6.40) at P. From Eq. (6.41), we see that at P, the new co-ordinates, given by $v^\alpha = 0$ are the geodesic co-ordinates.

EXAMPLE 6.4.1 *Prove that the co-ordinate system u^α defined by*

$$u^\alpha = a_m^j (v^m - u_0^m) + \frac{1}{2} a_h^j \left\{ \begin{matrix} h \\ l \;\; m \end{matrix} \right\}_P (v^l - u_0^l)(v^m - u_0^m)$$

is a geodesic co-ordinate, where the coefficients a_m^j being constants and are such that their determinant do not vanish.

Solution: The necessary and sufficient condition that a system of co-ordinates be geodesic with pole at P are that their second covariant derivatives with respect to the metric of the space all vanish at P. From the given relation,

$$u^\alpha = a_m^j (v^m - u_0^m) + \frac{1}{2} a_h^j \left\{ \begin{matrix} h \\ l \;\; m \end{matrix} \right\}_P (v^l - u_0^l)(v^m - u_0^m),$$

we get after differentiation,

$$\begin{aligned}\frac{\partial u^\alpha}{\partial v^l} &= a^j_m \delta^m_l + \frac{1}{2} a^j_h \left\{ \begin{matrix} h \\ l \ \ m \end{matrix} \right\}_P (v^m - u^m_0) + \frac{1}{2} a^j_h \left\{ \begin{matrix} h \\ l \ \ m \end{matrix} \right\}_P (v^l - u^l_0)\delta^m_l \\ &= a^j_l + \frac{1}{2} a^j_h \left\{ \begin{matrix} h \\ l \ \ m \end{matrix} \right\}_P (v^m - u^m_0). \\ \Rightarrow \left(\frac{\partial u^\alpha}{\partial v^l}\right)_P &= a^j_l.\end{aligned}$$

Since the Jacobian determinant $\left|\left(\frac{\partial u^\alpha}{\partial v^l}\right)_P\right| = |a^j_l| \neq 0$, so the given transformation is permissible in the neighbourhood of $P(u^1_0, u^2_0)$ on $\mathcal{S}$. Again differentiating with respect to v^m, we get

$$\left(\frac{\partial^2 u^\alpha}{\partial v^m \partial v^l}\right)_P = a^j_h \left\{ \begin{matrix} h \\ l \ \ m \end{matrix} \right\}_P.$$

Using the formula,

$$u^\alpha_{,lm} = \frac{\partial^2 u^\alpha}{\partial v^l \partial v^m} - \left\{ \begin{matrix} h \\ l \ \ m \end{matrix} \right\} \frac{\partial u^\alpha}{\partial v^h},$$

we see that at $P(u^1_0, u^2_0)$ on $\mathcal{S}$,

$$\begin{aligned}(u^\alpha_{,lm})_P &= \left(\frac{\partial^2 u^\alpha}{\partial v^l \partial v^m}\right)_P - \left\{ \begin{matrix} h \\ l \ \ m \end{matrix} \right\}_P \left(\frac{\partial u^\alpha}{\partial v^h}\right)_P \\ &= a^j_h \left\{ \begin{matrix} h \\ l \ \ m \end{matrix} \right\}_P - a^j_h \left\{ \begin{matrix} h \\ l \ \ m \end{matrix} \right\}_P = 0. \qquad (6.44)\end{aligned}$$

This shows that the new co-ordinate system u^α defined in the given relation is a geodesic co-ordinate system with the pole at P.

6.5 Parallel Vector Fields on a Surface

The concept of parallel vector fields along a curve $\mathcal{C}$ imbedded in E^3 was generalised by Levi-Civita to curves imbedded in an n-dimensional Riemannian manifolds. As an illustration of the usefulness of the concept, consider a surface $\mathcal{S}$ immersed in E^3 and a curve $\mathcal{C}$ on $\mathcal{S}$, whose equations are taken in the form

$$\mathcal{C} : u^\alpha = u^\alpha(t);\ t_1 \le t \le t_2;\ \alpha = 1, 2 \qquad (6.45)$$

where u^1, u^2 are the curvilinear co-ordinates covered by the surface $\mathcal{S}$. Let the metric properties of the surface $\mathcal{S}$ are governed by the co-ordinates of the first fundamental form $a_{\alpha\beta}$. Let A^α be a surface vector field defined along $\mathcal{C}$, then the surface intrinsic derivative of A^α along $\mathcal{C}$ is

$$\frac{\delta A^\alpha}{\delta t} = \frac{dA^\alpha}{dt} + \left\{ \begin{matrix} \alpha \\ \beta \quad \gamma \end{matrix} \right\} A^\beta \frac{du^\gamma}{dt}, \tag{6.46}$$

which is identical with the form of Eq. (3.39) defining the parallel vector field along a space curve. Thus, if $\frac{\delta A^\alpha}{\delta t} = 0$, the differential Eq. (6.46) becomes

$$\frac{dA^\alpha}{dt} + \left\{ \begin{matrix} \alpha \\ \beta \quad \gamma \end{matrix} \right\} A^\beta \frac{du^\gamma}{dt} = 0, \tag{6.47}$$

which determines a unique vector field when the components of the vector are specified at an arbitrary point of $\mathcal{C}$ as the definition of the parallel vector field along a curve $\mathcal{C}$ on the surface $\mathcal{S}$. If the parameter t is chosen as the arc length s and if A^α is taken as the unit tangent vector to $\mathcal{C}$, i.e. if we take

$$A^\alpha = \frac{du^\alpha}{ds} = \lambda^\alpha; \text{ with } a_{\alpha\beta}\lambda^\alpha\lambda^\beta = 1,$$

then Eq. (6.47) reduces to

$$\frac{d}{dt}\left(\frac{du^\alpha}{ds}\right) + \left\{ \begin{matrix} \alpha \\ \beta \quad \gamma \end{matrix} \right\} \frac{du^\beta}{ds}\frac{du^\gamma}{ds} = 0$$

or

$$\frac{d^2u^\alpha}{ds^2} + \left\{ \begin{matrix} \alpha \\ \beta \quad \gamma \end{matrix} \right\} \frac{du^\beta}{ds}\frac{du^\gamma}{ds} = 0, \tag{6.48}$$

which is the equation of a geodesic on the surface $\mathcal{S}$. From uniqueness of the solution of Eq. (6.48), it follows that the property of tangency of a parallel vector field to a surface curve is both a necessary and sufficient condition for a geodesic. Also, Eq. (6.47) can be written in the form

$$\frac{d}{dt}\left(\frac{du^\alpha}{ds}\right) + \left\{ \begin{matrix} \alpha \\ \beta \quad \gamma \end{matrix} \right\} \frac{du^\beta}{ds}\frac{du^\gamma}{ds} = 0$$

or

$$\frac{\partial}{\partial u^\beta}\left(\frac{du^\alpha}{ds}\right)\frac{du^\beta}{ds} + \left\{ \begin{matrix} \alpha \\ \beta \quad \gamma \end{matrix} \right\} \frac{du^\beta}{ds}\frac{du^\gamma}{ds} = 0$$

or

$$\left[\frac{\partial}{\partial u^\beta}\left(\frac{du^\alpha}{ds}\right) + \left\{ \begin{matrix} \alpha \\ \beta \quad \gamma \end{matrix} \right\} \frac{du^\gamma}{ds}\right]\frac{du^\beta}{ds} = 0$$

or

$$\left(\frac{du^\beta}{ds}\right)_{,\beta}\frac{du^\beta}{ds} = 0.$$

Therefore, the vector obtained by the parallel propagation of the tangent vector to a geodesic always remains tangent to the geodesic.

6.5.1 Geodesic Parallels

The 'field of geodesics' was introduced by Weierstrass. A one-parameter family of geodesics on a surface S is said to be a field of geodesics in a portion S' of S if through every point of S' there passes (just once) exactly one of those geodesics. For example, a family of parallel straight lines is a field of geodesics in a plane. The generating straight lines of a cylinder constitute a field of geodesics.

Suppose a family of geodesics is given, and that a parameter system is chosen so that the geodesics of the family are the curves v = constant and their orthogonal trajectories are the curves u = constant. Since the co-ordinates are orthogonal, we have $a_{12} = 0$. Since the curves v = constant are geodesics we have

$$(\chi_g)_{v=\text{constant}} = 0 \;\;\Rightarrow\;\; -\frac{1}{\sqrt{a_{22}}}\frac{\partial}{\partial v}(\log\sqrt{a_{11}}) = 0 \;\;\Rightarrow\;\; \frac{\partial a_{11}}{\partial v} = 0,$$

that is a_{11} depends on u only. The metric is of the form

$$ds^2 = a_{11}(u)\,du^2 + a_{22}(u,v)\,dv^2.$$

Consider the distance between any two orthogonal trajectories, say $u = a$ and $u = b$, measured only the geodesic $v = c$. Along $v = c$, $dv = 0$, so

$$ds = \sqrt{a_{11}(u)}\,du.$$

Therefore, the length of a geodesic intercepted between the trajectories $u = a$ and $u = b$ is

$$\int_a^b \sqrt{a_{11}(u)}\,du = c \text{ dependent}.$$

The distance is thus the same along whichever geodesic v = constant it is measured and is called geodesic distance between two curves. Because of this property, the orthogonal trajectories u = constant are called geodesic parallels.

Hence, if we take $\int\sqrt{a_{11}}\,du$ as a new parameter u, the first fundamental form is given by the expression

$$ds^2 = (du)^2 + a_{22}(u,v)(dv)^2 = (du)^2 + a(dv)^2,$$

which is characteristic for geodesic co-ordinates. In this metric the parameter u can be specialized by taking it to be the distance from some fixed parallel to the parallel determined by u, the distance being meased along any geodesic $v = c$, i.e. $a_{11} = 1$. Since a_{11} is now equal to unity, the length of an element of arc of a geodesic is du.

Consider the two points P, Q in which a geodesic is cut by the parallels $u = a, u = b$. The length of the arc of the geodesic joining the two points is $(b - a)$. For any other curve joining them the length of arc is

$$\int_P^Q ds = \int_P^Q \sqrt{(du)^2 + a(dv)^2} > \int_a^b du,$$

since a is positive. Thus the distance is least in the case of geodesic. Hence, for any given family of geodesics, a parameter system can be chosen so that the metric takes the form $(du)^2 + a(dv)^2$. The given geodesics are the parametric curves $v =$ constant and their orthogonal trajectories are $u =$ constant, u being the measured along a geodesic from some fixed parallel.

EXAMPLE 6.5.1 *Prove that the geodesic is an auto parallel curve.*

Solution: An autoparallel curve is a curve whose tangent vector field constituted by the tangents at each point of the curve is a parallel vector field. The differential equation of the geodesic $\mathcal{C}$ is given by

$$\frac{d^2x^\alpha}{ds^2} + \frac{dx^\gamma}{ds}\frac{dx^\beta}{ds}\begin{Bmatrix} \alpha \\ \beta \ \gamma \end{Bmatrix} = 0$$

or

$$\frac{\partial \lambda^\alpha}{\partial s} + \lambda^\gamma \frac{dx^\beta}{ds}\begin{Bmatrix} \alpha \\ \gamma \ \beta \end{Bmatrix} = 0$$

or

$$\left[\frac{\partial \lambda^\alpha}{\partial x^\beta} + \lambda^\gamma \begin{Bmatrix} \alpha \\ \gamma \ \beta \end{Bmatrix}\right]\frac{dx^\beta}{ds} = 0 \Rightarrow \lambda^\alpha_{,\beta}\lambda^\beta = 0,$$

which shows that the curve $\mathcal{C}$ possesses the property that the tangents at all its points are parallel. Also, $\lambda^\alpha_{,\beta}\lambda^\beta = 0$ shows that the unit tangent $\lambda^\alpha = \frac{dx^\alpha}{ds}$ suffers a parallel displacement along a geodesic, i.e. a geodesic is an autoparallel curve.

6.6 Gaussian Curvature

The meaning of the curvature tensors will be explained here in connexion with parallel displacement and absolute differentiation.

Let us consider a surface $\mathcal{S}$, embedded in E^3 whose fundamental tensors are $a_{\alpha\beta}$. On the surface $\mathcal{S}$, where the metric is Eq. (6.10), with u^1, u^2 as the Gaussian coordinates for the surface $\mathcal{S}$, the Riemann-Christoffel curvature tensor or simply the curvature tensor is given by

$$R^{\alpha}_{\beta\gamma\delta} = \frac{\partial}{\partial u^{\gamma}}\begin{Bmatrix} \alpha \\ \beta \ \ \delta \end{Bmatrix} - \frac{\partial}{\partial u^{\delta}}\begin{Bmatrix} \alpha \\ \beta \ \ \gamma \end{Bmatrix} + \begin{Bmatrix} \sigma \\ \beta \ \ \delta \end{Bmatrix}\begin{Bmatrix} \alpha \\ \sigma \ \ \gamma \end{Bmatrix} - \begin{Bmatrix} \sigma \\ \beta \ \ \gamma \end{Bmatrix}\begin{Bmatrix} \alpha \\ \sigma \ \ \delta \end{Bmatrix}$$

$$= \begin{vmatrix} \frac{\partial}{\partial u^{\gamma}} & \frac{\partial}{\partial u^{\delta}} \\ \begin{Bmatrix} \alpha \\ \beta \ \ \gamma \end{Bmatrix} & \begin{Bmatrix} \alpha \\ \beta \ \ \delta \end{Bmatrix} \end{vmatrix} + \begin{vmatrix} \begin{Bmatrix} \sigma \\ \beta \ \ \delta \end{Bmatrix} & \begin{Bmatrix} \sigma \\ \beta \ \ \gamma \end{Bmatrix} \\ \begin{Bmatrix} \alpha \\ \sigma \ \ \delta \end{Bmatrix} & \begin{Bmatrix} \alpha \\ \sigma \ \ \gamma \end{Bmatrix} \end{vmatrix} \tag{6.49}$$

and the associated tensor is given by

$$R_{\sigma\beta\gamma\delta} = a_{\sigma\alpha} \ \ R^{\alpha}_{\beta\gamma\delta}, \tag{6.50}$$

where $R_{\sigma\beta\gamma\delta}$ are the components of the Riemann curvature tensor for the surface $\mathcal{S}$, given by

$$R_{\sigma\beta\gamma\delta} = \frac{\partial}{\partial u^{\gamma}}[\beta\delta, \sigma] - \frac{\partial}{\partial u^{\delta}}[\beta\gamma, \sigma] + \begin{Bmatrix} \lambda \\ \beta \ \ \gamma \end{Bmatrix}[\sigma\delta, \lambda] - \begin{Bmatrix} \lambda \\ \beta \ \ \delta \end{Bmatrix}[\sigma\gamma, \lambda]. \tag{6.51}$$

Note that, this associated tensor $R_{\sigma\beta\gamma\delta}$ defined in Eq. (6.50) in skew symmetric in the first two and last two indices. Therefore,

$$R_{\sigma\sigma\gamma\delta} = R_{\sigma\beta\gamma\gamma} = 0, \text{ i.e. } R_{1212} = -R_{2112} = -R_{1221} = R_{2121}.$$

Hence, every non-vanishing components of the Riemann curvature tensor for a surface is R_{1212} or to its negative. Let a surface $\mathcal{S}$ be immersed in E_3 with metric tensor $a_{\alpha\beta}$. Define a quantity by

$$\kappa = \frac{R_{1212}}{a} = \frac{1}{a}a_{1\alpha}R^{\alpha}_{212}; \quad \text{where } a = |a_{\alpha\beta}| \neq 0 \tag{6.52}$$
$$= \frac{1}{a}a_{1\alpha}\left[\frac{\partial}{\partial u^1}\begin{Bmatrix} \alpha \\ 2 \ \ 2 \end{Bmatrix} - \frac{\partial}{\partial u^2}\begin{Bmatrix} \alpha \\ 2 \ \ 1 \end{Bmatrix} + \begin{Bmatrix} \sigma \\ 2 \ \ 2 \end{Bmatrix}\begin{Bmatrix} \alpha \\ \sigma \ \ 1 \end{Bmatrix} - \begin{Bmatrix} \sigma \\ 2 \ \ 1 \end{Bmatrix}\begin{Bmatrix} \alpha \\ \sigma \ \ 2 \end{Bmatrix}\right].$$

Such a quantity κ is called the *total curvature* or the *Gaussian curvature* of the surface $\mathcal{S}$. The Eq. (6.49) is a representation of the Gaussian curvature k in terms of the Christoffel symbols. Since only the metric tensor $a_{\alpha\beta}$ and its derivatives are involved in expression (6.52) for κ, the properties described by κ are intrinsic properties of the surface $\mathcal{S}$. Let us introduce an e-system $e_{\alpha\beta}, e^{\alpha\beta}$ defined by Eq. (1.5), and the permutation tensor by Eq. (6.21). With the help of ε tensor, Eq. (6.52) can also be written in the form

$$R_{1212} = a\kappa = \sqrt{a} \ \ e_{12}\sqrt{a} \ \ e_{12}\kappa = \varepsilon_{12}\varepsilon_{12}\kappa.$$

In general

$$R_{\alpha\beta\gamma\delta} = \kappa\varepsilon_{\alpha\beta} \;\; \varepsilon_{\gamma\delta}. \tag{6.53}$$

Also, we have

$$\begin{aligned}
\varepsilon^{\alpha\beta}\varepsilon_{\alpha\beta} &= \varepsilon^{1\beta}\varepsilon_{1\beta} + \varepsilon^{2\beta}\varepsilon_{2\beta} = \varepsilon^{11}\varepsilon_{11} + \varepsilon^{12}\varepsilon_{12} + \varepsilon^{21}\varepsilon_{21} + \varepsilon^{22}\varepsilon_{22} \\
&= \sqrt{a}e_{11}\frac{e^{11}}{\sqrt{a}} + \sqrt{a}e_{12}\frac{e^{12}}{\sqrt{a}} + \sqrt{a}e_{21}\frac{e^{21}}{\sqrt{a}} + \sqrt{a}e_{22}\frac{e^{22}}{\sqrt{a}} \\
&= 0 + 1 + (-1)\cdot(-1) + 0 = 2.
\end{aligned}$$

Therefore, from Eq. (6.53) we obtain

$$\varepsilon^{\alpha\beta}\varepsilon^{\gamma\delta}R_{\alpha\beta\gamma\delta} = \varepsilon^{\alpha\beta}\varepsilon^{\gamma\delta} \;\; \kappa \;\; \varepsilon_{\alpha\beta} \;\; \varepsilon_{\gamma\delta} = 4\kappa$$

or

$$\kappa = \frac{1}{4}R_{\alpha\beta\gamma\delta}\varepsilon^{\alpha\beta}\varepsilon^{\gamma\delta} \tag{6.54}$$

from which it is evident that the Gaussian curvature κ is an invariant. Again, from (6.53), we get

$$\begin{aligned}
a^{\alpha\delta}a^{\beta\gamma}R_{\alpha\beta\gamma\delta} &= a^{11}a^{\beta\gamma}R_{\beta 11\gamma} + a^{22}a^{\beta\gamma}R_{\beta 22\gamma} + a^{12}a^{\beta\gamma}R_{\beta 12\gamma} + a^{21}a^{\beta\gamma}R_{\beta 21\gamma} \\
&= a^{11}a^{22}R_{2112} + a^{12}a^{21}R_{2121} + a^{21}a^{12}R_{1212} + a^{22}a^{11}R_{1221} \\
&= -a^{11}a^{22}R_{1212} + a^{12}a^{21}R_{1212} + a^{21}a^{12}R_{1212} - a^{22}a^{11}R_{1221} \\
&= 2R_{1212}\left(a^{12}a^{21} - a^{11}a^{22}\right) \\
&= -2R_{1212}\left[\frac{a_{22}}{a}\frac{a_{11}}{a} - \left(-\frac{a_{12}}{a}\right)\left(-\frac{a_{12}}{a}\right)\right] \\
&= -2R_{1212}\left(\frac{a}{a^2}\right) = -2\kappa;\; a = |a_{\alpha\beta}| = a_{11}a_{22} - a_{12}^2
\end{aligned}$$

or

$$\kappa = -\frac{1}{2}\;\; R;\quad \text{where } R = a^{\alpha\delta}a^{\beta\gamma}R_{\alpha\beta\gamma\delta} = a^{\alpha\beta}R_{\alpha\beta}. \tag{6.55}$$

$R_{\alpha\beta} = R^{\gamma}_{\alpha\beta\gamma} = a^{\lambda\gamma}R_{\lambda\alpha\beta\gamma}$ is the Ricci tensor. The invariant R in Eq. (6.55) is sometimes defined as the *scalar curvature* or *Einstein curvature* of the surface $\mathcal{S}$. A surface on which holds

$$R_{\alpha\beta\gamma\delta} = \rho(a_{\alpha\delta}a_{\beta\gamma} - a_{\alpha\gamma}a_{\beta\delta}) \tag{6.56}$$

where ρ is a scalar, is called a *surface of constant curvature.* In particular, if $\rho = 0$, the surface is called a *flat surface.*

Geometrical properties which are expressible in terms of the first fundamental form, may be called *intrinsic Properties.* Since only metric coefficients $a_{\alpha\beta}$ are involved in the definition of κ in Eq. (6.55), the properties of κ are intrinsic properties of the surface.

EXAMPLE 6.6.1 *If the co-ordinate system is orthogonal, show that*

$$\kappa = -\frac{1}{2\sqrt{a}}\left[\frac{\partial}{\partial u^1}\left(\frac{1}{\sqrt{a}}\frac{\partial a_{22}}{\partial u^1}\right) + \frac{\partial}{\partial u^2}\left(\frac{1}{\sqrt{a}}\frac{\partial a_{11}}{\partial u^2}\right)\right].$$

Solution: If the system of co-ordinates is orthogonal, then $a_{12} = 0 = a_{21}$. Therefore,

$$a = \begin{vmatrix} a_{11} & 0 \\ 0 & a_{22} \end{vmatrix};\ a^{11} = \frac{1}{a_{11}};\ a^{22} = \frac{1}{a_{22}}.$$

The non-vanishing Christoffel symbols of the first kind are

$$[11,1] = \frac{1}{2}\frac{\partial a_{11}}{\partial u^1};\ [12,1] = \frac{1}{2}\frac{\partial a_{11}}{\partial u^2} = -[11,2]$$

$$[22,1] = -\frac{1}{2}\frac{\partial a_{22}}{\partial u^1} = -[12,2];\ [22,2] = \frac{1}{2}\frac{\partial a_{22}}{\partial u^2}.$$

The non-vanishing Christofell symbols of the second kind are

$$\left\{ {1 \atop 1\ 1} \right\} = \frac{1}{2a_{11}}\frac{\partial a_{11}}{\partial u^1};\ \left\{ {1 \atop 1\ 2} \right\} = \frac{1}{2a_{11}}\frac{\partial a_{11}}{\partial u^2};\ \left\{ {1 \atop 2\ 2} \right\} = -\frac{1}{2a_{11}}\frac{\partial a_{22}}{\partial u^1};$$

$$\left\{ {2 \atop 1\ 1} \right\} = -\frac{1}{2a_{22}}\frac{\partial a_{11}}{\partial u^2};\ \left\{ {2 \atop 1\ 2} \right\} = \frac{1}{2a_{22}}\frac{\partial a_{22}}{\partial u^1};\ \left\{ {2 \atop 2\ 2} \right\} = \frac{1}{2a_{22}}\frac{\partial a_{22}}{\partial u^2}.$$

Using formula (6.51), the Riemann tensor R_{1212} is given by

$$\begin{aligned} R_{1212} &= \frac{\partial}{\partial u^1}[22,1] - \frac{\partial}{\partial u^2}[21,1] + \left\{ {\lambda \atop 2\ 1} \right\}[12,\lambda] - \left\{ {\lambda \atop 2\ 2} \right\}[11,\lambda] \\ &= \frac{\partial}{\partial u^1}[22,1] - \frac{\partial}{\partial u^2}[21,1] + \left\{ {1 \atop 2\ 1} \right\}[12,1] + \left\{ {2 \atop 2\ 1} \right\}[12,2] \\ &\quad - \left\{ {1 \atop 2\ 2} \right\}[11,1] - \left\{ {2 \atop 2\ 2} \right\}[11,2] \end{aligned}$$

$$= -\frac{1}{2}\frac{\partial}{\partial u^1}\left(\frac{\partial a_{22}}{\partial u^1}\right) - \frac{1}{2}\frac{\partial}{\partial u^2}\left(\frac{\partial a_{22}}{\partial u^2}\right) + \frac{1}{2a_{11}}\frac{\partial a_{11}}{\partial u^2}.\frac{1}{2}\frac{\partial a_{11}}{\partial u^2}$$

$$+\frac{1}{2a_{22}}\frac{\partial a_{22}}{\partial u^1}\cdot\frac{1}{2}\frac{\partial a_{22}}{\partial u^1} + \frac{1}{2a_{11}}\frac{\partial a_{22}}{\partial u^1}.\frac{1}{2}\frac{\partial a_{11}}{\partial u^1} + \frac{1}{2a_{22}}\frac{\partial a_{22}}{\partial u^2}\cdot\frac{1}{2}\frac{\partial a_{11}}{\partial u^2}$$

$$= -\frac{1}{2}\left[\frac{\partial^2 a_{11}}{(\partial u^2)^2} + \frac{\partial^2 a_{22}}{(\partial u^1)^2}\right] + \frac{1}{4a_{11}}\left[\left(\frac{\partial a_{11}}{\partial u^2}\right)^2 + \frac{\partial a_{11}}{\partial u^1}\frac{\partial a_{22}}{\partial u^1}\right]$$

$$+\frac{1}{4a_{22}}\left[\left(\frac{\partial a_{22}}{\partial u^1}\right)^2 + \frac{\partial a_{11}}{\partial u^2}\frac{\partial a_{22}}{\partial u^2}\right]$$

$$= -\frac{1}{2\sqrt{a_{11}a_{22}}}\left[\frac{\partial}{\partial u^1}\left(\frac{1}{\sqrt{a_{11}a_{22}}}\frac{\partial a_{22}}{\partial u^1}\right) + \frac{\partial}{\partial u^2}\left(\frac{1}{\sqrt{a_{11}a_{22}}}\frac{\partial a_{11}}{\partial u^2}\right)\right]$$

$$= -\frac{1}{2\sqrt{a}}\left[\frac{\partial}{\partial u^1}\left(\frac{1}{\sqrt{a}}\frac{\partial a_{22}}{\partial u^1}\right) + \frac{\partial}{\partial u^2}\left(\frac{1}{\sqrt{a}}\frac{\partial a_{11}}{\partial u^2}\right)\right].$$

This is a representation of the Gaussian curvature κ in terms of the coefficients of the first fundamental form.

EXAMPLE 6.6.2 *Calculate the Gaussian curvature for the surface with metric*

$$ds^2 = a^2\sin^2 u^1(du^2)^2 + a^2(du^1)^2.$$

Solution: Comparing the given metric with Eq. (6.10), we get $a_{11} = a^2$, $a_{22} = a^2\sin^2 u^1$ and $a_{12} = 0 = a_{21}$, so that

$$\triangle = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a^4\sin^2 u^1.$$

The non-vanishing Christoffel symbols of first kind are

$$[22, 1] = -\frac{a^2}{2}\sin 2u^1;\ \ [12, 2] = \frac{a^2}{2}\sin 2u^1 = [21, 2].$$

The reciprocal or conjugate tensors are

$$a^{11} = \frac{a^2\sin^2 u^1}{a^4\sin^2 u^1} = \frac{1}{a^2};\ a^{22} = \frac{a^2}{a^4\sin^2 u^1} = \frac{1}{a^2\sin^2 u^1}$$

and $a^{12} = 0 = a^{21}$. The non-vanishing Christoffel symbols of second kind are

$$\left\{\begin{matrix} 1 \\ 2\ \ 2 \end{matrix}\right\} = a^{1k}[22, k] = a^{11}[22, 1] + a^{12}[22, 2] = -\frac{1}{2}\sin 2u^1$$

$$\left\{\begin{matrix} 1 \\ 2\ \ 1 \end{matrix}\right\} = a^{2k}[21, k] = a^{21}[21, 1] + a^{22}[21, 2] = \cot u^1 = \left\{\begin{matrix} 1 \\ 1\ \ 2 \end{matrix}\right\}.$$

The Riemann tensor R_{1212} is given by

$$\begin{aligned}
R_{1212} &= a_{1\alpha}R^{\alpha}_{212} = a_{11}R^{1}_{212} + a_{12}R^{2}_{212} = a_{11}R^{1}_{212}\\
&= a_{11}\left[\frac{\partial}{\partial u^1}\left\{\begin{matrix}1\\2\ 2\end{matrix}\right\} - \frac{\partial}{\partial u^2}\left\{\begin{matrix}1\\2\ 1\end{matrix}\right\} + \left\{\begin{matrix}\sigma\\2\ 2\end{matrix}\right\}\left\{\begin{matrix}1\\\sigma\ 1\end{matrix}\right\} - \left\{\begin{matrix}\sigma\\2\ 1\end{matrix}\right\}\left\{\begin{matrix}1\\\sigma\ 2\end{matrix}\right\}\right]\\
&= a^2\left[\frac{\partial}{\partial u^1}\left(-\frac{1}{2}\sin 2u^1\right) - \frac{\partial}{\partial u^2}\left(\cot u^1\right) + \left\{\begin{matrix}1\\2\ 2\end{matrix}\right\}\left\{\begin{matrix}1\\1\ 1\end{matrix}\right\} + \left\{\begin{matrix}2\\2\ 2\end{matrix}\right\}\left\{\begin{matrix}1\\2\ 1\end{matrix}\right\}\right.\\
&\qquad \left. - \left\{\begin{matrix}1\\2\ 1\end{matrix}\right\}\left\{\begin{matrix}1\\2\ 1\end{matrix}\right\} - \left\{\begin{matrix}2\\2\ 1\end{matrix}\right\}\left\{\begin{matrix}1\\2\ 2\end{matrix}\right\}\right]\\
&= a^2\left[-\cos 2u^1 - \cot u^1 \times \left(-\frac{1}{2}\sin 2u^1\right)\right] = a^2\sin^2 u^1.
\end{aligned}$$

The *total curvature* or the *Gaussian curvature* of the surface $\mathcal{S}$ is given by

$$\kappa = \frac{R_{1212}}{\triangle} = \frac{a^2\sin^2 u^1}{a^4\sin^2 u^1} = \frac{1}{a^2}.$$

EXAMPLE 6.6.3 *For a surface of revolution defined by* $x^1 = u^1\cos u^2,\ x^2 = u^1\sin u^2,\ x^3 = f(u^1)$, *where* f *is of class* C^2, *find the Gaussian curvature* κ.

Solution: The parametric representation of the right helicoid is given by

$$x^1 = u^1\cos u^2,\quad x^2 = u^1\sin u^2,\quad x^3 = f(u^1)$$

i.e.

$$\mathbf{r} = (x^1, x^2, x^3) = \left(u^1\cos u^2, u^1\sin u^2, f(u^1)\right).$$

From this parametric representation, we get

$$\begin{aligned}
&\frac{\partial x^1}{\partial u^1} = \cos u^2;\ \frac{\partial x^2}{\partial u^1} = \sin u^2;\ \frac{\partial x^3}{\partial u^1} = \frac{\partial f}{\partial u^1} = f_1\\
&\frac{\partial x^1}{\partial u^2} = -u^1\sin u^2;\ \frac{\partial x^2}{\partial u^2} = u^1\cos u^2;\ \frac{\partial x^3}{\partial u^2} = \frac{\partial f}{\partial u^2} = 0,
\end{aligned}$$

where $\frac{\partial f}{\partial u^1} = f_1$. Thus, the coefficients $a_{\alpha\beta}$ of the first fundamental form are given by

$$\begin{aligned}
a_{11} &= \left(\frac{\partial x^1}{\partial u^1}\right)^2 + \left(\frac{\partial x^2}{\partial u^1}\right)^2 + \left(\frac{\partial x^3}{\partial u^1}\right)^2 = 1 + f_1^2.\\
a_{22} &= \left(\frac{\partial x^1}{\partial u^2}\right)^2 + \left(\frac{\partial x^2}{\partial u^2}\right)^2 + \left(\frac{\partial x^3}{\partial u^2}\right)^2 = (u^1)^2.\\
a_{12} &= \frac{\partial x^1}{\partial u^1}\frac{\partial x^1}{\partial u^2} + \frac{\partial x^2}{\partial u^1}\frac{\partial x^2}{\partial u^2} + \frac{\partial x^3}{\partial u^1}\frac{\partial x^3}{\partial u^2}\\
&= \cos u^2(-u^1\sin u^2) + (u^1\cos u^2)\sin u^2 + f_1\cdot 0 = 0 = a_{21}.
\end{aligned}$$

Therefore, $a = (1 + f_1^2)(u^1)^2$. The reciprocal tensors $a^{\alpha\beta}$ are given by

$$a^{11} = \frac{1}{1 + f_1^2};\ a^{22} = \frac{1}{(u^1)^2};\ a^{12} = 0 = a^{21}.$$

The non-vanishing Christoffel symbols of the first kind are

$$[11,1] = \frac{1}{2}\frac{\partial a_{11}}{\partial u^1} = f_1 f_2;\ [12,2] = \frac{1}{2}\frac{\partial a_{22}}{\partial u^1} = u^1 = [21,2]$$
$$[22,1] = -\frac{1}{2}\frac{\partial a_{22}}{\partial u^1} = -u^1,$$

where we use the notations $f_1 = \frac{df}{du^1}$ and $f_2 = \frac{d^2 f}{(du^1)^2}$. The non-vanishing Christoffel symbols of the second kind are

$$\left\{ \begin{matrix} 1 \\ 1\ 1 \end{matrix} \right\} = a^{11}[11,1] = \frac{f_1 f_2}{1 + f_1^2};\ \left\{ \begin{matrix} 1 \\ 2\ 2 \end{matrix} \right\} = a^{11}[22,1] = -\frac{u^1}{1 + f_1^2}$$
$$\left\{ \begin{matrix} 2 \\ 2\ 1 \end{matrix} \right\} = a^{22}[21,2] = \frac{1}{u^1} = \left\{ \begin{matrix} 2 \\ 1\ 2 \end{matrix} \right\}.$$

The Riemann tensor R_{1212} is given by

$$\begin{aligned} R_{1212} &= a_{1\alpha} R^{\alpha}_{212} = a_{11} R^1_{212} + a_{12} R^2_{212} = a_{11} R^1_{212} \\ &= a_{11}\left[\frac{\partial}{\partial u^1}\left\{ \begin{matrix} 1 \\ 2\ 2 \end{matrix} \right\} - \frac{\partial}{\partial u^2}\left\{ \begin{matrix} 1 \\ 2\ 1 \end{matrix} \right\} + \left\{ \begin{matrix} \sigma \\ 2\ 2 \end{matrix} \right\}\left\{ \begin{matrix} 1 \\ \sigma\ 1 \end{matrix} \right\} - \left\{ \begin{matrix} \sigma \\ 2\ 1 \end{matrix} \right\}\left\{ \begin{matrix} 1 \\ \sigma\ 2 \end{matrix} \right\}\right] \\ &= a_{11}\left[\frac{\partial}{\partial u^1}\left(\frac{-u^1}{1 + f_1^2}\right) + \left\{ \begin{matrix} 1 \\ 2\ 2 \end{matrix} \right\}\left\{ \begin{matrix} 1 \\ 1\ 1 \end{matrix} \right\} + \left\{ \begin{matrix} 2 \\ 2\ 2 \end{matrix} \right\}\left\{ \begin{matrix} 1 \\ 2\ 1 \end{matrix} \right\}\right. \\ &\quad \left. - \left\{ \begin{matrix} 1 \\ 2\ 1 \end{matrix} \right\}\left\{ \begin{matrix} 1 \\ 2\ 1 \end{matrix} \right\} - \left\{ \begin{matrix} 2 \\ 2\ 1 \end{matrix} \right\}\left\{ \begin{matrix} 1 \\ 2\ 2 \end{matrix} \right\}\right] \\ &= (1 + f_1^2)\left[-\frac{1}{1 + f_1^2} + \frac{2u^1 f_1 f_2}{(1 + f_1^2)^2} + \left(\frac{-u^1}{1 + f_1^2}\right)\left(\frac{f_1 f_2}{1 + f_1^2}\right) - \frac{1}{u^1}\left(\frac{-u^1}{1 + f_1^2}\right)\right] \\ &= (1 + f_1^2)\frac{u^1 f_1 f_2}{(1 + f_1^2)^2} = \frac{u^1 f_1 f_2}{1 + f_1^2}. \end{aligned}$$

The *total curvature* or the *Gaussian curvature* of the surface $\mathcal{S}$ is given by

$$\kappa = \frac{R_{1212}}{\triangle} = \frac{u^1 f_1 f_2}{1 + f_1^2} \times \frac{1}{(1 + f_1^2)(u^1)^2} = \frac{f_1 f_2}{u^1[1 + (f_1)^2]^2}.$$

6.7 Isometry

The properties of surfaces (i.e. the lengths of curves, angle between intersecting curves), which have been already stated are expressed completely by means of first fundamental quadratic form ds^2 as given in Eq. (6.10). These properties constitute a body of what is known as the *intrinsic geometry of surfaces.* We have seen that intrinsic property of a surface depend on the metric tensor of the surface and its derivatives. But the metric of the surface is a local property and it may happen that two surfaces have the same metric.

Two surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$ be such that there exists a co-ordinate system with respect to which the linear element on $\mathcal{S}_1$ and $\mathcal{S}_2$ are characterised by the same metric tensors $a_{\alpha\beta}$, then, they are said to be isometric, and the transformation of parameters is called an *isometry.* For example, the surfaces of the cylinder and cone are isometric with the Euclidean plane, since these surfaces can be rolled out, or developed, on the plane without changing the lengths of arc elements, and hence without altering the measurements of angles and areas.

The problem of finding surfaces isometric to a plane is of special importance. For this we now introduce the concept of a ruled surface and the developable surface. A surface is called a ruled surface (Figure 6.12) if it contains (at least) one one-parameter

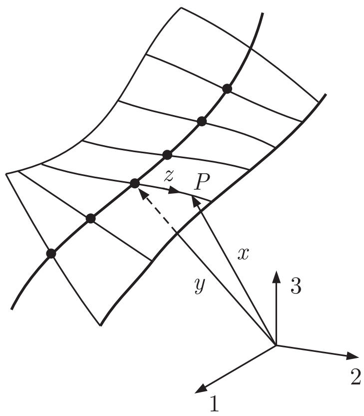

Figure 6.12: Portion of a ruled surface.

family of straight lines which can be chosen as co-ordinate curves on the surface. The straight lines are called generators of a ruled surface. One of the examples of a ruled surface is the hyperbolic paraboloid $\dfrac{(x^1)^2}{a^2} - \dfrac{(x^2)^2}{b^2} - x^3 = 0$ (Figure 6.13).

Consequently a ruled surface may be generated by a continuous motion of a straight line in space. Such a motion is completely determined if the path $y(s)$ (with arc length s) of a point of the moving line (Figure 6.12) is given and also the direction of the line for every values of s, for example, by a unit vector $z(s)$. A ruled surface may therefore

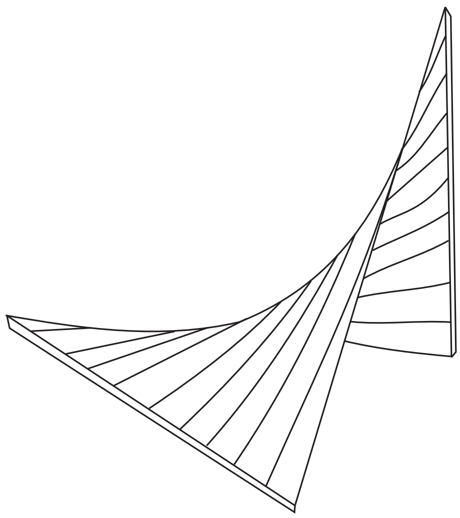

Figure 6.13: Hyperbolic paraboloid.

be represented in the form

$$x(s,t) = y(s) + tz(s). \tag{6.57}$$

Obviously the co-ordinate t is the (directed) distance of the points of this surface Eq. (6.57) from the centre $y(s)$, measured along the corresponding generator.

A surface which is isometric to a plane is called a *developable surface* or simply *developable.* A developable surface is a special ruled surface with the property that it has the same targent plane at all points on one and same generator. The tangent surface is always a developable surface. The principal normal surface and the binormal surface are developable surfaces if and only if the corresponding curve is plane. Developable surfaces are of particular importance because they are the only surfaces which can be mapped isometrically into a plane. The Gaussian curvature enables us to determine the circumstances under which a given surface is developable or not. The following theorem shows that, a developable surface is a surface which can be developed in a plane:

Theorem 6.7.1 *Prove the $\kappa = 0$ is the necessary and sufficient condition for a surface to be a developable.*

Proof: The Gaussian curvature of the total curvature κ of the surface $\mathcal{S}$ is given by

$$\kappa = \frac{R_{\alpha\beta\gamma\delta}}{a}; \quad a = |a_{\alpha\beta}|,$$

where the tensor $R_{\alpha\beta\gamma\delta}$ is skew symmetric in the first two and last two indices. Now, when a surface $\mathcal{S}$ is isometric with the Euclidean plane, there exists on $\mathcal{S}$ a co-ordinate system with respect to which

$$a_{11} = a_{22} = 1;\ a_{12} = 0 = a_{21} \Rightarrow a = 1 \neq 0.$$

In this case, the Riemann curvature tensor $R_{\alpha\beta\gamma\delta} = 0$ for the surface $\mathcal{S}$, in this particular co-ordinate system, and since $R_{\alpha\beta\gamma\delta}$ is a tensor, it must vanish in every co-ordinate system. Thus $R_{\alpha\beta\gamma\delta} = 0$, and hence the Gaussian curvature $\kappa = 0$ for the surface $\mathcal{S}$.

Conversely, let $\kappa = 0$, i.e. the Riemann tensors $R_{\alpha\beta\gamma\delta}$ vanishes at all points of the surface. A necessary and sufficient condition that a symmetric tensor $a_{\alpha\beta}$ with $|a_{\alpha\beta}| \neq 0$ reduces under a suitable transformation of co-ordinates to a tensor $h_{\alpha\beta}$, where $h_{\alpha\beta}$ are constants, is that the Riemann curvature tensor formed from the $a_{\alpha\beta}$s be a zero tensor.

Hence, it guarantees that there exists co-ordinate systems on the surface such that $a_{11} = a_{22} = 1, a_{12} = 0$. Hence, the surface is isometric with the Euclidean plane.

EXAMPLE 6.7.1 *Prove that the following surfaces:*

$$\mathcal{S}_1 : y^1 = v^1 \cos v^2, y^2 = v^1 \sin v^2, y^3 = a \cosh^{-1} \frac{v^1}{a}$$
$$\mathcal{S}_2 : y^1 = u^1 \cos u^2, y^2 = u^1 \sin u^2, y^3 = au^2$$

are isomorphic but non developable.

Solution: The first surface $\mathcal{S}_1$ is the catenoid obtained by revolving the catenary $y^2 = \cosh\left(\frac{y^3}{a}\right)$ about the y^3 axis. Now,

$$\begin{aligned}
a_{11} &= \left(\frac{\partial y^i}{\partial v^1}\right)^2 = \left(\frac{\partial y^1}{\partial v^1}\right)^2 + \left(\frac{\partial y^2}{\partial v^1}\right)^2 + \left(\frac{\partial y^3}{\partial v^1}\right)^2 \\
&= (\cos v^2)^2 + (\sin v^2)^2 + \left(\frac{a}{\sqrt{(v^1/a)^2 - 1}}\frac{1}{a}\right)^2 = \frac{(v^1)^2}{(v^1)^2 - a^2}. \\
a_{22} &= \left(\frac{\partial y^i}{\partial v^2}\right)^2 = \left(\frac{\partial y^1}{\partial v^2}\right)^2 + \left(\frac{\partial y^2}{\partial v^2}\right)^2 + \left(\frac{\partial y^3}{\partial v^2}\right)^2 \\
&= (-v^1 \sin v^2)^2 + (v^1 \cos v^2)^2 + 0 = (v^1)^2. \\
a_{12} &= \frac{\partial y^1}{\partial v^1}\frac{\partial y^1}{\partial v^2} + \frac{\partial y^2}{\partial v^1}\frac{\partial y^2}{\partial v^2} + \frac{\partial y^3}{\partial v^1}\frac{\partial y^3}{\partial v^2} \\
&= \cos v^2(-v^1 \sin v^2) + \sin v^2(v^1 \cos v^2) + \frac{a}{\sqrt{(v^1)^2 - a^2}}.0 = 0 = a_{21}.
\end{aligned}$$

Thus, the first fundamental quadratic form for the surface $\mathcal{S}_1$ becomes,

$$ds^2 = a_{\alpha\beta}dv^\alpha dv^\beta = a_{11}(dv^1)^2 + a_{22}(dv^2)^2 + 2a_{12}dv^1dv^2$$
$$= \frac{(v^1)^2}{(v^1)^2 - a^2}(dv^1)^2 + (v^1)^2(dv^2)^2$$
$$= \frac{(v^1)^2}{(v^1)^2 - a^2}(dv^1)^2 + [a^2 + \{(v^1)^2 - a^2\}](dv^2)^2.$$

The second surface $\mathcal{S}_2$ is a right helicoid. Now,

$$a_{11} = \left(\frac{\partial y^i}{\partial u^1}\right)^2 = \left(\frac{\partial y^1}{\partial u^1}\right)^2 + \left(\frac{\partial y^2}{\partial u^1}\right)^2 + \left(\frac{\partial y^3}{\partial u^1}\right)^2$$
$$= (\cos u^2)^2 + (\sin u^2)^2 + 0 = 1.$$
$$a_{22} = \left(\frac{\partial y^i}{\partial u^2}\right)^2 = \left(\frac{\partial y^1}{\partial u^2}\right)^2 + \left(\frac{\partial y^2}{\partial u^2}\right)^2 + \left(\frac{\partial y^3}{\partial u^2}\right)^2$$
$$= (-u^1 \sin u^2)^2 + (u^1 \cos u^2)^2 + a^2 = (u^1)^2 + a^2.$$
$$a_{12} = \frac{\partial y^1}{\partial u^1}\frac{\partial y^1}{\partial u^2} + \frac{\partial y^2}{\partial u^1}\frac{\partial y^2}{\partial u^2} + \frac{\partial y^3}{\partial u^1}\frac{\partial y^3}{\partial u^2}$$
$$= \cos u^2(-u^1 \sin u^2) + \sin u^2(u^1 \cos u^2) + 0 \cdot a = 0.$$

Thus the first fundamental quadratic form for the surface $\mathcal{S}_2$ is given by

$$ds^2 = a_{\alpha\beta}du^\alpha du^\beta = a_{11}(du^1)^2 + a_{22}(du^2)^2 + 2a_{12}du^1du^2$$
$$= (du^1)^2 + [a^2 + (u^1)^2](du^2)^2.$$

Now, if we set $(v^1)^2 - a^2 = (u^1)^2$ and $v^2 = u^2$, then the two surfaces have the same metric. Thus, the surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$ are isometric.

Now, we calculate the Gaussian curvature for the first surface $\mathcal{S}_1$. The reciprocal tensors for the first fundamental quadratic form are

$$a^{11} = \frac{(v^1)^2 - a^2}{(v^1)^2};\ a^{22} = \frac{1}{(v^1)^2};\ a^{12} = 0 = a^{21}.$$

The non-vanishing Christoffel symbols of first kind are

$$[11, 1] = \frac{1}{2}\left[\frac{\partial a_{11}}{\partial v^1} + \frac{\partial a_{11}}{\partial v^1} - \frac{\partial a_{11}}{\partial v^1}\right] = \frac{1}{2}\frac{\partial a_{11}}{\partial v^1} = -\frac{a^2v^1}{[(v^1)^2 - a^2]^2}$$

$$[22, 1] = \frac{1}{2}\left[\frac{\partial a_{21}}{\partial v^2} + \frac{\partial a_{21}}{\partial v^2} - \frac{\partial a_{22}}{\partial v^1}\right] = -\frac{1}{2}\frac{\partial a_{22}}{\partial v^1} = -v^1$$

$$[12, 2] = \frac{1}{2}\left[\frac{\partial a_{12}}{\partial v^2} + \frac{\partial a_{22}}{\partial v^1} - \frac{\partial a_{12}}{\partial v^2}\right] = \frac{1}{2}\frac{\partial a_{22}}{\partial v^1} = v^1 = [21, 2].$$

The non-vanishing Christoffel symbols of second kind are

$$\left\{ {1 \atop 1\ 1} \right\} = a^{1k}[11,k] = a^{11}[11,1] + a^{12}[11,2]$$

$$= -\frac{(v^1)^2 - a^2}{(v^1)^2} \times \frac{a^2 v^1}{[(v^1)^2 - a^2]^2} = -\frac{a^2}{v^1[(v^1)^2 - a^2]}$$

$$\left\{ {1 \atop 2\ 2} \right\} = a^{1k}[22,k] = a^{11}[22,1] + a^{12}[22,2]$$

$$= \frac{(v^1)^2 - a^2}{(v^1)^2} \times (-v^1) = -\frac{(v^1)^2 - a^2}{v^1}$$

$$\left\{ {2 \atop 2\ 1} \right\} = a^{2k}[21,k] = a^{21}[21,1] + a^{22}[21,2] = \frac{v^1}{(v^1)^2} = \frac{1}{v^1} = \left\{ {2 \atop 1\ 2} \right\}.$$

The Riemann tensor R_{1212} is given by

$$R_{1212} = a_{1\alpha} R^{\alpha}_{212} = a_{11} R^1_{212} + a_{12} R^2_{212} = a_{11} R^1_{212}$$

$$= a_{11} \left[\frac{\partial}{\partial v^1} \left\{ {1 \atop 2\ 2} \right\} - \frac{\partial}{\partial v^2} \left\{ {1 \atop 2\ 1} \right\} + \left\{ {\sigma \atop 2\ 2} \right\} \left\{ {1 \atop \sigma\ 1} \right\} - \left\{ {\sigma \atop 2\ 1} \right\} \left\{ {1 \atop \sigma\ 2} \right\} \right]$$

$$= a_{11} \left[\frac{\partial}{\partial v^1} \left(-\frac{(v^1)^2 - a^2}{(v^1)^2} \right) - 0 + \left\{ {1 \atop 2\ 2} \right\} \left\{ {1 \atop 1\ 1} \right\} + \left\{ {2 \atop 2\ 2} \right\} \left\{ {1 \atop 2\ 1} \right\} \right.$$

$$\left. - \left\{ {1 \atop 2\ 1} \right\} \left\{ {1 \atop 2\ 1} \right\} - \left\{ {2 \atop 2\ 1} \right\} \left\{ {1 \atop 2\ 2} \right\} \right]$$

$$= \frac{(v^1)^2}{(v^1)^2 - a^2} \left[-\frac{2a^2}{(v^1)^3} + \frac{a^2}{(v^1)^3} + \frac{(v^1)^2 - a^2}{(v^1)^3} \right] = \frac{(v^1)^2 - 2a^2}{v^1[(v^1)^2 - a^2]}.$$

The *Gaussian curvature* of the surface $\mathcal{S}_1$ is given by

$$\kappa = \frac{R_{1212}}{a} = \frac{(v^1)^2 - 2a^2}{v^1[(v^1)^2 - a^2]} \times \frac{(v^1)^2 - a^2}{(v^1)^4} = \frac{(v^1)^2 - 2a^2}{(v^1)^5} \neq 0.$$

Thus, the surface $\mathcal{S}_1$ is non-developable. Now, the reciprocal tensors for the first fundamental form to $\mathcal{S}_2$ are

$$a^{11} = 1;\ a^{22} = \frac{1}{a^2 + (u^1)^2},\quad a^{12} = 0 = a^{21}.$$

The non-vanishing Christoffel symbols of first and second kinds are

$$[22,1] = \frac{u^1}{[a^2 + (u^1)^2]^2},\quad [12,2] = -\frac{u^1}{[a^2 + (u^1)^2]^2} = [21,2].$$

$$\left\{ {1 \atop 2\ 2} \right\} = \frac{u^1}{[a^2 + (u^1)^2]^2};\ \left\{ {2 \atop 2\ 1} \right\} = -\frac{u^1}{[a^2 + (u^1)^2]^3} = \left\{ {2 \atop 1\ 2} \right\}.$$

The Riemann tensor R_{1212} is given by

$$\begin{aligned}
R_{1212} &= a_{1\alpha}R^{\alpha}_{212} = a_{11}R^1_{212} + a_{12}R^2_{212} = a_{11}R^1_{212} \\
&= a_{11}\left[\frac{\partial}{\partial u^1}\begin{Bmatrix}1\\2\ 2\end{Bmatrix} - \frac{\partial}{\partial u^2}\begin{Bmatrix}1\\2\ 1\end{Bmatrix} + \begin{Bmatrix}\sigma\\2\ 2\end{Bmatrix}\begin{Bmatrix}1\\\sigma\ 1\end{Bmatrix} - \begin{Bmatrix}\sigma\\2\ 1\end{Bmatrix}\begin{Bmatrix}1\\\sigma\ 2\end{Bmatrix}\right] \\
&= \frac{\partial}{\partial u^1}\left(\frac{u^1}{[a^2+(u^1)^2]^2}\right) - 0 + \begin{Bmatrix}1\\2\ 2\end{Bmatrix}\begin{Bmatrix}1\\1\ 1\end{Bmatrix} + \begin{Bmatrix}2\\2\ 2\end{Bmatrix}\begin{Bmatrix}1\\2\ 1\end{Bmatrix} \\
&\quad - \begin{Bmatrix}1\\2\ 1\end{Bmatrix}\begin{Bmatrix}1\\2\ 1\end{Bmatrix} - \begin{Bmatrix}2\\2\ 1\end{Bmatrix}\begin{Bmatrix}1\\2\ 2\end{Bmatrix} \\
&= \frac{a^2-3(u^1)^2}{[a^2+(u^1)^2]^3} + \frac{(u^1)^2}{[a^2+(u^1)^2]^5}.
\end{aligned}$$

The *Gaussian curvature* of the surface $\mathcal{S}_2$ is given by

$$\kappa = \frac{R_{1212}}{a} = \frac{a^2-3(u^1)^2}{[a^2+(u^1)^2]^2} + \frac{(u^1)^2}{[a^2+(u^1)^2]^4} \neq 0.$$

Thus, the surface $\mathcal{S}_2$ is non-developable.

EXAMPLE 6.7.2 *Determine whether the surface with the metric* $ds^2 = (u^2)^2(du^1)^2 + (u^1)^2(du^2)^2$ *is a developable or not?*

Solution: From the given metric, we see that $a_{11} = (u^2)^2, a_{22} = (u^1)^2, a_{12} = 0 = a_{21}$, so that $a = (u^1u^2)^2$. Now, the reciprocal tensors $a^{\alpha\beta}$ are given by

$$a^{11} = \frac{1}{(u^2)^2};\ a^{22} = \frac{1}{(u^1)^2},\quad a^{12} = 0 = a^{21}.$$

The non-vanishing Christoffel symbols of first and second kinds are

$$[11,2] = -u^2, [12,1] = u^2 = [21,1], [12,2] = u^1 = [21,2], [22,1] = -u^1.$$

$$\begin{Bmatrix}1\\2\ 1\end{Bmatrix} = \frac{1}{u^2} = \begin{Bmatrix}1\\1\ 2\end{Bmatrix};\quad \begin{Bmatrix}2\\2\ 1\end{Bmatrix} = \frac{1}{u^1} = \begin{Bmatrix}2\\1\ 2\end{Bmatrix}; \begin{Bmatrix}1\\2\ 2\end{Bmatrix} = -\frac{u^1}{(u^2)^2}.$$

The Riemann tensor R_{1212} is given by

$$\begin{aligned}
R_{1212} &= a_{1\alpha}R^{\alpha}_{212} = a_{11}R^1_{212} + a_{12}R^2_{212} = a_{11}R^1_{212} \\
&= a_{11}\left[\frac{\partial}{\partial u^1}\begin{Bmatrix}1\\2\ 2\end{Bmatrix} - \frac{\partial}{\partial u^2}\begin{Bmatrix}1\\2\ 1\end{Bmatrix} + \begin{Bmatrix}\sigma\\2\ 2\end{Bmatrix}\begin{Bmatrix}1\\\sigma\ 1\end{Bmatrix} - \begin{Bmatrix}\sigma\\2\ 1\end{Bmatrix}\begin{Bmatrix}1\\\sigma\ 2\end{Bmatrix}\right] \\
&= (u^2)^2\left[\frac{\partial}{\partial u^1}\left(-\frac{u^1}{(u^2)^2}\right) - \frac{\partial}{\partial u^2}\left(\frac{1}{u^2}\right) + \begin{Bmatrix}1\\2\ 2\end{Bmatrix}\begin{Bmatrix}1\\1\ 1\end{Bmatrix} + \begin{Bmatrix}2\\2\ 2\end{Bmatrix}\begin{Bmatrix}1\\2\ 1\end{Bmatrix}\right. \\
&\quad \left. - \begin{Bmatrix}1\\2\ 1\end{Bmatrix}\begin{Bmatrix}1\\2\ 1\end{Bmatrix} - \begin{Bmatrix}2\\2\ 1\end{Bmatrix}\begin{Bmatrix}1\\2\ 2\end{Bmatrix}\right] = (u^2)^2 \times 0 = 0.
\end{aligned}$$

The *Gaussian curvature* for the given surface is given by

$$\kappa = \frac{R_{1212}}{a} = \frac{0}{(u^1u^2)^2} = 0.$$

Thus, the surface is developable.

EXAMPLE 6.7.3 *Show that the surface given by $x^1 = f_1(u^1)$, $x^2 = f_2(u^1)$, $x^3 = u^2$ is a developable, where f_1, f_2 are differentiable functions.*

Solution: The first fundamental forms for the given surfaces are given by

$$a_{11} = \left(\frac{\partial x^i}{\partial u^1}\right)^2 = \left(\frac{\partial x^1}{\partial u^1}\right)^2 + \left(\frac{\partial x^2}{\partial u^1}\right)^2 + \left(\frac{\partial x^3}{\partial u^1}\right)^2 = (f_1')^2 + (f_2')^2.$$

$$a_{22} = \left(\frac{\partial x^i}{\partial u^2}\right)^2 = \left(\frac{\partial x^1}{\partial u^2}\right)^2 + \left(\frac{\partial x^2}{\partial u^2}\right)^2 + \left(\frac{\partial x^3}{\partial u^2}\right)^2 = 1.$$

$$a_{12} = \frac{\partial x^1}{\partial u^1}\frac{\partial x^1}{\partial u^2} + \frac{\partial x^2}{\partial u^1}\frac{\partial x^2}{\partial u^2} + \frac{\partial x^3}{\partial u^1}\frac{\partial x^3}{\partial u^2} = 0 = a_{21}.$$

Therefore, $a = (f_1')^2 + (f_2')^2$. The reciprocal tensors are given by

$$a^{11} = \frac{1}{(f_1')^2 + (f_2')^2};\ a^{22} = 1;\ a^{12} = 0 = a^{21}.$$

The only non-vanishing Christoffel symbols of first kind and second kinds are

$$[11, 1] = f_1'f_1'' + f_2'f_2'';\quad \left\{ {1 \atop 1\ 1} \right\} = \frac{f_1'f_1'' + f_2'f_2''}{(f_1')^2 + (f_2')^2}.$$

The Riemann tensor R_{1212} is given by

$$\begin{aligned} R_{1212} &= a_{1\alpha}R^{\alpha}_{212} = a_{11}R^1_{212} + a_{12}R^2_{212} = a_{11}R^1_{212} \\ &= a_{11}\left[\frac{\partial}{\partial u^1}\left\{ {1 \atop 2\ 2} \right\} - \frac{\partial}{\partial u^2}\left\{ {1 \atop 2\ 1} \right\} + \left\{ {\sigma \atop 2\ 2} \right\}\left\{ {1 \atop \sigma\ 1} \right\} - \left\{ {\sigma \atop 2\ 1} \right\}\left\{ {1 \atop \sigma\ 2} \right\}\right] \\ &= a_{11}\left[\frac{\partial}{\partial u^1}0 - \frac{\partial}{\partial u^2}0 + \left\{ {1 \atop 2\ 2} \right\}\left\{ {1 \atop 1\ 1} \right\} + \left\{ {2 \atop 2\ 2} \right\}\left\{ {1 \atop 2\ 1} \right\}\right. \\ &\quad \left. - \left\{ {1 \atop 2\ 1} \right\}\left\{ {1 \atop 1\ 2} \right\} - \left\{ {2 \atop 2\ 1} \right\}\left\{ {1 \atop 2\ 2} \right\}\right] = 0. \end{aligned}$$

The *Gaussian curvature* for the given surface is given by

$$\kappa = \frac{R_{1212}}{a} = \frac{0}{(f_1')^2 + (f_2')^2} = 0.$$

Thus, the surface is developable.

6.8 Geodesic Curvature

In this section we will describe the geodesic curvature of surface curves on a surface $\mathcal{S}$ of class ≥ 2. Here, using the intrinsic geometry of surfaces we will derive a formula describing the behaviour of the tangent vector to a surface curve. Let $\mathcal{C}$ be a surface curve given by a parametric representation

$$\mathcal{C}\colon u^\alpha = u^\alpha(s), \tag{6.58}$$

where s measures arc distance along the curve $\mathcal{C}$. Then, at an arbitrary point P of the curve, we have the condition

$$a_{\alpha\beta}\frac{du^\alpha}{ds}\frac{du^\beta}{ds} = 1. \tag{6.59}$$

The quantities $\frac{du^1}{ds}$; $\frac{du^2}{ds}$ obviously determine a tangent vector λ^α to $\mathcal{C}$, and so writing the unit values λ^α for $\frac{du^\alpha}{ds}$ we have from Eq. (6.59) that

$$a_{\alpha\beta}\frac{du^\alpha}{ds}\frac{du^\beta}{ds} = a_{\alpha\beta}\lambda^\alpha\lambda^\beta = 1. \tag{6.60}$$

Differentiate the quadratic relation (6.60) intrinsically with respect to s, we get

$$a_{\alpha\beta}\lambda^\alpha\frac{\delta\lambda^\beta}{\delta s} = 0$$

from which it follows that either the surface vector $\frac{\delta\lambda^\beta}{\delta s}$ is orthogonal to λ^α or $\frac{\delta\lambda^\beta}{\delta s} = 0$. If $\frac{\delta\lambda^\beta}{\delta s} \neq 0$, we introduce a unit surface vector η^β normal to λ^β (codirectional with $\frac{\delta\lambda^\beta}{\delta s}$), (Figure 6.14), so that

$$\frac{\delta\lambda^\beta}{\delta s} = \chi_g\eta^\beta, \tag{6.61}$$

where χ_g is a suitable scalar, positive or negative. We choose the sense of η such that the rotation $(\lambda^\alpha, \eta^\alpha)$ is positive, i.e. $\xi_{\alpha\beta}\lambda^\alpha\eta^\beta = 1$. The scalar χ_g is called *the geodesic curvature* of the curve $\mathcal{C}$ on the surface. The geodesic curvature was introduced by Lionville.

(i) Now we give the geometrical significance of the geodesic curvature χ_g. Let $\mathcal{C}'$ be the orthogonal projection of $\mathcal{C}$ on the tangent plane to $\mathcal{S}$ at P (Figure 6.14). The geodesic curvature χ_g of $\mathcal{C}$ at P is defined as the curvature of the projected curve $\dot{C}$ at P, taken with a suitable sign.

(ii) The sign of χ_g is defined as follows: Suppose that the curvature of $\mathcal{C}'$ at P is not zero. Then χ_g is positive sign if the centre of curvature $\mathcal{C}'$ at P lies in the direction of the unit vector $\nu = \eta \times \lambda$, where η is the unit normal vector to $\mathcal{S}$ at P and λ is the tangent vector to $\mathcal{C}$ at P. χ_g is negative sign if that centre of curvature lies in the direction opposite to the vector ν. The sign of χ_g thus depends on the orientations of $\mathcal{S}$ and $\mathcal{C}$. (In Figure 6.14 , χ_g is negative).

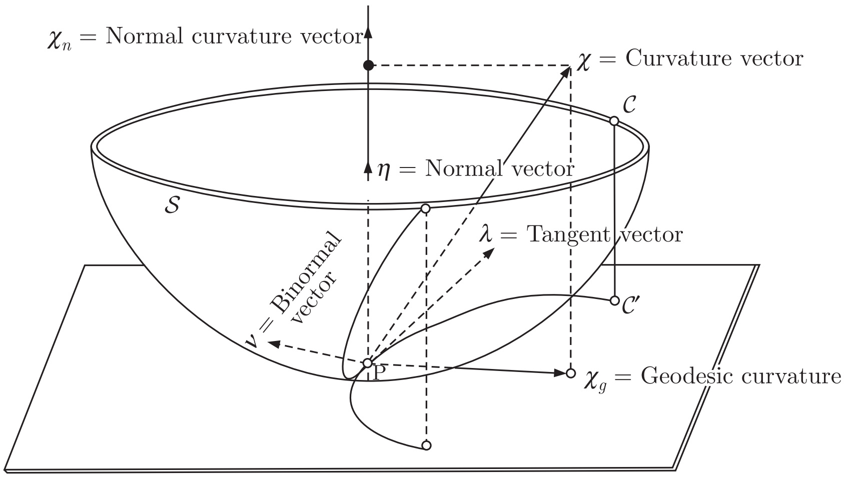

Figure 6.14: Geodesic curvature.

Since $\lambda^\alpha = \varepsilon^{\alpha\beta}\eta_\beta$ and $\eta^\beta = \varepsilon^{\alpha\beta}\lambda_\alpha$, taking the intrinsic derivative, we get

$$\begin{aligned}\frac{\delta\eta^\beta}{\delta s} &= \varepsilon^{\alpha\beta}\frac{\delta\lambda_\alpha}{\delta s} = \varepsilon^{\alpha\beta}\frac{\delta}{\delta s}\left(a_{\alpha\beta}\lambda^\beta\right)\\ &= \varepsilon^{\alpha\beta}a_{\alpha\beta}\frac{\delta\lambda^\beta}{\delta s} = \varepsilon^{\alpha\beta}a_{\alpha\beta}\chi_g\eta^\beta\end{aligned}$$

or

$$\frac{\delta\eta^\beta}{\delta s} = \chi_g\varepsilon^{\alpha\beta}\eta_\alpha = -\chi_g\varepsilon^{\beta\alpha}\eta_\alpha = -\chi_g\lambda^\beta$$

or

$$\frac{\delta\eta^\alpha}{\delta s} = -\chi_g\lambda^\alpha. \tag{6.62}$$

We may refer this pair of Eqs. (6.61) and (6.62) as the Serret–Frenet formulae for the curve $\mathcal{C}$ relative to the surface. The transversion of $\frac{\delta\lambda_\alpha}{\delta s} = \chi_g\eta_\alpha$ by η_α, we get

$$\begin{aligned}\chi_g &= \frac{\delta\lambda_\alpha}{\delta s}\lambda^\alpha = \frac{\delta\lambda_\alpha}{\delta s}\left(\varepsilon^{\alpha\beta}\lambda_\beta\right) = \varepsilon^{\alpha\beta}\lambda_\beta\frac{\delta\lambda_\alpha}{\delta s}\\ &= \frac{1}{\sqrt{a}}e^{\alpha\beta}\lambda_\alpha\frac{\delta\lambda_\alpha}{\delta s} = \frac{1}{\sqrt{a}}\begin{vmatrix}\lambda_1 & \lambda_2\\ \dfrac{\delta\lambda_1}{\delta s} & \dfrac{\delta\lambda_2}{\delta s}\end{vmatrix}.\end{aligned}$$

Alternatively, using $\dfrac{\delta\lambda^\alpha}{\delta s} = \chi_g \eta^\alpha$, we get

$$\chi_g = \varepsilon_{\alpha\beta}\lambda^\alpha \frac{\delta\lambda^\beta}{\delta s} = \frac{1}{\sqrt{a}}\begin{vmatrix} \lambda^1 & \lambda^2 \\ \dfrac{\delta\lambda^1}{\delta s} & \dfrac{\delta\lambda^2}{\delta s} \end{vmatrix}$$

$$= \sqrt{a}\begin{vmatrix} \dfrac{du^1}{ds} & \dfrac{du^2}{ds} \\ \dfrac{d^2u^1}{ds^2} + \begin{Bmatrix} 1 \\ \beta\ \ \gamma \end{Bmatrix}\dfrac{du^\beta}{ds}\dfrac{du^\gamma}{ds} & \dfrac{d^2u^2}{ds^2} + \begin{Bmatrix} 2 \\ \beta\ \ \gamma \end{Bmatrix}\dfrac{du^\beta}{ds}\dfrac{du^\gamma}{ds} \end{vmatrix}. \tag{6.63}$$

These expressions are known as *Beltrami's formula* for geodesics curvature.

Theorem 6.8.1 *The necessary and sufficient condition for a curve on a surface to be a geodesic is that its geodesic curvature is zero.*

Proof: The differential equation of the geodesic is

$$\frac{d^2u^\alpha}{ds^2} + \begin{Bmatrix} \alpha \\ \beta\ \ \gamma \end{Bmatrix}\frac{du^\beta}{ds}\frac{du^\gamma}{ds} = 0$$

or

$$\frac{d}{ds}\left(\frac{du^\alpha}{ds}\right) + \begin{Bmatrix} \alpha \\ \beta\ \ \gamma \end{Bmatrix}\frac{du^\beta}{ds}\frac{du^\gamma}{ds} = 0$$

or

$$\frac{\partial}{\partial u^\beta}\left(\frac{du^\alpha}{ds}\right)\frac{du^\beta}{ds} + \begin{Bmatrix} \alpha \\ \beta\ \ \gamma \end{Bmatrix}\frac{du^\beta}{ds}\frac{du^\gamma}{ds} = 0$$

or

$$\left[\frac{\partial}{\partial u^\beta}\left(\frac{du^\alpha}{ds}\right) + \begin{Bmatrix} \alpha \\ \beta\ \ \gamma \end{Bmatrix}\frac{du^\gamma}{ds}\right]\frac{du^\beta}{ds} = 0$$

or

$$\left(\frac{du^\alpha}{ds}\right)_{,\beta}\frac{du^\beta}{ds} = 0.$$

According to the intrinsic derivative of a tensor of type $(1, 0)$ with component u^α, the above relation becomes

$$\frac{\delta}{\delta s}\left(\frac{du^\alpha}{ds}\right) = 0 \Rightarrow \frac{\delta}{\delta s}\lambda^\alpha = 0$$

$$\Rightarrow \chi_g \eta^\alpha = 0;\ \text{as}\ \frac{\delta\lambda^\alpha}{\delta s} = \chi_g\eta^\alpha$$

$$\Rightarrow \chi_g = 0 \text{ as } \eta^\alpha \text{ is a unit vector.}$$

Hence, the curve on a surface is to be a geodesic if the geodesic curvature is zero.

Conversely, let $\chi_g = 0$, identically, then we have

$$\frac{\delta}{\delta s}\left(\frac{du^\alpha}{ds}\right) = \frac{\delta}{\delta s}\lambda^\alpha = \chi_g \eta^\alpha = 0$$
$$\Rightarrow \frac{du^\alpha}{ds} = A \Rightarrow u^\alpha = As + B.$$

This shows that, a geodesic of a surface is, therefore, analogous to a straight line in a plane. Therefore, a curve $\mathcal{C}$ on a surface $\mathcal{S}$ is a geodesic curve or geodesic if its geodesic curvature χ_g vanishes identically. Straight lines on any surface are geodesics.

EXAMPLE 6.8.1 *Find the geodesic curvature of the small circle*

$$\mathcal{C}: u^1 = constant = u_0^1 \neq 0, u^2 = u^2$$

on the surface of the sphere $\mathcal{S}$.

Solution: On the surface of sphere

$$\mathcal{S}: y^1 = a\cos u^1 \cos u^2, y^2 = a\cos u^1 \sin u^2, y^3 = a \sin u^1.$$

If the arc length s of $\mathcal{C}$ is measured from the plane $u_2 = 0$, we have $u^2 = \dfrac{s}{a\cos u_0^1}$, and the equation of $\mathcal{C}$ can be written in the form

$$u^1 = u_0^1, \quad u^2 = \frac{s}{a\cos u_0^1},$$

whose metric is $ds^2 = a^2\left(du^1\right)^2 + a^2\left(\cos u^1\right)^2\left(du^2\right)^2$, where the metric coefficients of s are $a_{11} = a^2, a_{12} = 0$ and $a_{22} = a^2\cos^2 u^1$. The components of the unit tangent vector $\lambda^\alpha = \frac{du^\alpha}{ds}$ along $\mathcal{C}$ are given by

$$\lambda^\alpha = \left(\lambda^1, \lambda^2\right) = \left(\frac{du^1}{ds}, \frac{du^2}{ds}\right) = \left(0, \frac{1}{a\cos u_0^1}\right).$$

The only non-vanishing Christoffel symbols are

$$\left\{ \begin{matrix} 1 \\ 2\ \ 2 \end{matrix} \right\} = a^{11}[22,1] = \frac{1}{a^{11}}\frac{\partial a_{22}}{\partial u^1} = \cos u_0^1 \sin u_0^1$$
$$\left\{ \begin{matrix} 2 \\ 1\ \ 2 \end{matrix} \right\} = a^{22}[12,2] = \frac{1}{a^{22}}\frac{\partial a_{22}}{\partial u^1} = -\tan u_0^1 = \left\{ \begin{matrix} 2 \\ 2\ \ 1 \end{matrix} \right\}.$$

Using the definition of intrinsic derivative, we get

$$\frac{\delta\lambda^1}{\delta s} = \frac{d\lambda^1}{ds} + \begin{Bmatrix} 1 \\ \alpha\ \ \beta \end{Bmatrix} \lambda^\alpha \frac{du^\beta}{ds} = \frac{d\lambda^1}{ds} + \begin{Bmatrix} 1 \\ 1\ \ 1 \end{Bmatrix} \lambda^1 \frac{du^1}{ds} + \begin{Bmatrix} 1 \\ 1\ \ 2 \end{Bmatrix} \lambda^1 \frac{du^2}{ds}$$

$$+ \begin{Bmatrix} 1 \\ 2\ \ 1 \end{Bmatrix} \lambda^2 \frac{du^1}{ds} + \begin{Bmatrix} 1 \\ 2\ \ 2 \end{Bmatrix} \lambda^2 \frac{du^2}{ds} = \begin{Bmatrix} 1 \\ 2\ \ 2 \end{Bmatrix} \lambda^2\lambda^2$$

$$\frac{\delta\lambda^2}{\delta s} = \frac{d\lambda^2}{ds} + \begin{Bmatrix} 2 \\ \alpha\ \ \beta \end{Bmatrix} \lambda^\alpha \frac{du^\beta}{ds} = 0 + \begin{Bmatrix} 2 \\ 2\ \ 2 \end{Bmatrix} \lambda^2\lambda^2$$

Using the formula (6.62) we get

$$\chi_g \eta^1 = \frac{\delta\lambda^1}{\delta s} = \begin{Bmatrix} 1 \\ 2\ \ 2 \end{Bmatrix} \lambda^2\lambda^2 = \cos u_0^1 \sin u_0^1 \left(\frac{1}{a\cos u_0^1}\right)^2 = \frac{1}{a^2}\tan u_0^1$$

$$\chi_g \eta^2 = \begin{Bmatrix} 2 \\ 2\ \ 2 \end{Bmatrix} \lambda^2\lambda^2 = 0.$$

Since $\mathcal{C}$ is not a geodesic, $\chi_g \neq 0$ and we conclude that $\eta^2 = 0$. But η^2 is a unit vector so that $a_{\alpha\beta}\eta^\alpha\eta^\beta = 1$, i.e.

$$a_{11}\eta^1\eta^1 + 2a_{12}\eta^1\eta^2 + a_{22}\eta^2\eta^2 = 1$$

or

$$a^2\eta^1\eta^1 = 1 \Rightarrow \eta^1 = \frac{1}{a}.$$

Here χ_g is given by

$$\chi_g = \frac{1}{\eta^1}\frac{1}{a^2}\tan u_0^1 = \frac{1}{a}\tan u_0^1.$$

Hence, the geodesic curvature is $\frac{1}{a}\tan u_0^1$.

EXAMPLE 6.8.2 *Consider the surface of the right circular cone*

$$\mathcal{C}\colon x^1 = u^1\cos u^2, x^2 = u^1\sin u^2, x^3 = u^1,$$

where u^1, u^2 *are the curvilinear co-ordinates on* $\mathcal{S}$ *and the curve* $\mathcal{C}$ *whose equation taken in the form* $C\colon u^1 = a, u^2 = \frac{s}{a}$*, where* s *is the arc parameter. Show that* $\chi_g = \frac{\sqrt{2}}{2\sqrt{a}}$.

Solution: Here, the metric is given by

$$ds^2 = 2\left(du^1\right)^2 + \left(u^1\right)^2\left(du^2\right)^2,$$

where the metric coefficients of $\mathcal{S}$ are $a_{11} = 2, a_{22} = \left(u^1\right)^2, a_{33} = 0$ and $a_{12} = 0$ so that $a = 2\left(u^1\right)^2$. The conjugate tensors are given by

$$a^{11} = \frac{1}{a_{11}} = 2, a^{22} = \frac{1}{\left(u^1\right)^2} \text{ and } a^{33} = 0; a^{12} = 0 = a^{21}.$$

The only non-vanishing Christoffel symbols of second kind are

$$\left\{ {1 \atop 2\ \ 2} \right\} = -\frac{1}{2a_{11}} \frac{\partial a_{22}}{\partial u^1} = -u^1 = -a.$$

Now, the components of the unit tangent vector $\lambda^\alpha = \frac{du^\alpha}{ds}$ along $\mathcal{C}$ are given by

$$\lambda^\alpha = \left(\lambda^1, \ \ \lambda^2\right) = \left(\frac{du^1}{ds}, \frac{du^2}{ds}\right) = \left(0, \frac{1}{a}\right).$$

Using the definition of intrinsic derivative, we get

$$\begin{aligned}
\frac{\delta\lambda^1}{\delta s} &= \frac{d\lambda^1}{ds} + \left\{ {1 \atop \alpha\ \ \beta} \right\} \lambda^\alpha \frac{du^\beta}{ds} \\
&= \frac{d\lambda^1}{ds} + \left\{ {1 \atop 1\ \ 1} \right\} \lambda^1 \frac{du^1}{ds} + \left\{ {1 \atop 2\ \ 2} \right\} \lambda^2 \frac{du^2}{ds} + \left\{ {1 \atop 2\ \ 1} \right\} \lambda^2 \frac{du^1}{ds} + \left\{ {1 \atop 1\ \ 2} \right\} \lambda^1 \frac{du^2}{ds} \\
&= \left\{ {1 \atop 2\ \ 2} \right\} \lambda^2 \frac{du^2}{ds} = -\frac{1}{2a} \\
\frac{\delta\lambda^2}{\delta s} &= \frac{d\lambda^2}{ds} + \left\{ {2 \atop \alpha\ \ \beta} \right\} \lambda^\alpha \frac{du^\beta}{ds} \\
&= \frac{d\lambda^2}{ds} + \left\{ {2 \atop 1\ \ 1} \right\} \lambda^1 \frac{du^1}{ds} + \left\{ {2 \atop 2\ \ 2} \right\} \lambda^2 \frac{du^2}{ds} + \left\{ {2 \atop 1\ \ 2} \right\} \lambda^2 \frac{du^1}{ds} + \left\{ {2 \atop 2\ \ 1} \right\} \lambda^1 \frac{du^2}{ds} = 0.
\end{aligned}$$

Using formula (6.62), we get

$$\chi_g \eta^1 = \frac{\delta\lambda^1}{\delta s} = -\frac{1}{2a} \text{ and } \chi_g \eta^2 = \frac{\delta\lambda^2}{\delta s} = 0.$$

Since C is not a geodesic $\chi_g \neq 0$ and we conclude that $\eta^2 = 0$. But η^2 is a unit vector so that $a_{\alpha\beta}\eta^\alpha\eta^\beta = 1$, i.e.

$$a_{11}\eta^1\eta^1 + a_{22}\eta^2\eta^2 + 2a_{12}\eta^1\eta^2 = 1$$

or

$$\begin{aligned}
2\left(\eta^1\right)^2 + \left(u^1\right)^2\left(\eta^2\right)^2 = 1 &\Rightarrow \left(\eta^1\right)^2 = \frac{1}{2} \\
\Rightarrow \chi_g = \frac{1}{\eta_1} \times \left(-\frac{1}{2a}\right) &= \frac{1}{\sqrt{2}} \frac{1}{2a} = \frac{1}{a2\sqrt{2}}.
\end{aligned}$$

EXAMPLE 6.8.3 *Show that the geodesic curvature of the curve $u = c$ on a surface with metric $\phi^2(du)^2 + \mu^2(dv)^2$ is $\frac{1}{\phi\mu}\frac{\partial\mu}{\partial u}$.*

Solution: For the given metric $ds^2 = \phi^2(du)^2 + \mu^2(dv)^2$, the metric coefficients are $a_{11} = \phi^2, a_{22} = \mu^2$ and $a_{12} = 0$ and so, $a = \phi^2\mu^2$. Thus, the conjugate tensors are given by

$$a^{11} = \frac{a_{22}}{a} = \frac{1}{\phi^2};\ a^{22} = \frac{a_{11}}{a} = \frac{1}{\mu^2};\ a^{12} = 0 = a^{21}.$$

The Christoffel symbols of first kind are

$$[11, 1] = \frac{1}{2}\frac{\partial a_{11}}{\partial u^1} = \phi\frac{d\phi}{du};\ [22, 2] = \frac{1}{2}\frac{\partial a_{22}}{\partial u^2} = \mu\frac{d\mu}{dv}$$

$$[11, 2] = \frac{1}{2}\left[\frac{\partial a_{12}}{\partial u^1} + \frac{\partial a_{12}}{\partial u^1} - \frac{\partial a_{11}}{\partial u^2}\right] = -\phi\frac{d\phi}{dv}$$

$$[21, 1] = \frac{1}{2}\left[\frac{\partial a_{21}}{\partial u^1} + \frac{\partial a_{11}}{\partial u^2} - \frac{\partial a_{21}}{\partial u^1}\right] = \frac{1}{2}\frac{\partial a_{11}}{\partial u^2} = \phi\frac{d\phi}{dv} = [12, 1]$$

$$[22, 1] = \frac{1}{2}\left[\frac{\partial a_{21}}{\partial u^2} + \frac{\partial a_{21}}{\partial u^2} - \frac{\partial a_{22}}{\partial u^1}\right] = -\frac{1}{2}\frac{\partial a_{22}}{\partial u^1} = -\mu\frac{d\mu}{du}$$

$$[12, 2] = \frac{1}{2}\left[\frac{\partial a_{12}}{\partial u^2} + \frac{\partial a_{22}}{\partial u^1} - \frac{\partial a_{12}}{\partial u^2}\right] = \frac{1}{2}\frac{\partial a_{22}}{\partial u^1} = \mu\frac{d\mu}{du} = [21, 2].$$

The Christoffel symbols of second kind are

$$\begin{Bmatrix} 1 \\ 1\ \ 1 \end{Bmatrix} = a^{1k}[11, k] = a^{11}[11, 1] = \frac{1}{\phi}\frac{d\phi}{du}$$

$$\begin{Bmatrix} 1 \\ 2\ \ 1 \end{Bmatrix} = a^{1k}[21, k] = a^{11}[21, 1] + a^{12}[21, 2] = \frac{1}{\phi}\frac{d\phi}{dv} = \begin{Bmatrix} 1 \\ 1\ \ 2 \end{Bmatrix}$$

$$\begin{Bmatrix} 1 \\ 2\ \ 2 \end{Bmatrix} = a^{1k}[22, k] = a^{11}[22, 1] + a^{12}[22, 2] = -\frac{\mu}{\phi^2}\frac{d\mu}{du}$$

$$\begin{Bmatrix} 2 \\ 2\ \ 1 \end{Bmatrix} = a^{2k}[21, k] = a^{21}[21, 1] + a^{22}[21, 2] = \frac{1}{\mu}\frac{d\mu}{du} = \begin{Bmatrix} 2 \\ 1\ \ 2 \end{Bmatrix}$$

$$\begin{Bmatrix} 2 \\ 2\ \ 2 \end{Bmatrix} = a^{2k}[22, k] = a^{21}[22, 1] + a^{22}[22, 2] = \frac{1}{\mu}\frac{d\mu}{dv}$$

$$\begin{Bmatrix} 2 \\ 1\ \ 1 \end{Bmatrix} = a^{2k}[11, k] = a^{21}[11, 1] + a^{22}[11, 2] = -\frac{\phi}{\mu^2}\frac{d\phi}{dv}$$

Let χ_g be the geodesic curvature, then from Beltrami's formula (6.63), we have

$$\chi_g = \sqrt{a}\left[\frac{du^1}{ds}\left(\frac{d^2u^\alpha}{ds^2} + \begin{Bmatrix} 2 \\ \beta\ \gamma \end{Bmatrix}\frac{du^\beta}{ds}\frac{du^\gamma}{ds}\right) - \frac{du^2}{ds}\left(\frac{d^2u^\alpha}{ds^2} + \begin{Bmatrix} 1 \\ \beta\ \gamma \end{Bmatrix}\frac{du^\beta}{ds}\frac{du^\gamma}{ds}\right)\right]$$

$$= \sqrt{a}\left[\frac{du}{ds}\left(\frac{d^2v}{ds^2} + \begin{Bmatrix} 2 \\ 1\ 1 \end{Bmatrix}\left(\frac{du}{ds}\right)^2 + \begin{Bmatrix} 2 \\ 2\ 2 \end{Bmatrix}\left(\frac{dv}{ds}\right)^2 + 2\begin{Bmatrix} 2 \\ 1\ 2 \end{Bmatrix}\frac{du}{ds}\frac{dv}{ds}\right)\right.$$

$$\left. -\frac{dv}{ds}\left(\frac{d^2u}{ds^2} + \begin{Bmatrix} 1 \\ 1\ 1 \end{Bmatrix}\left(\frac{du}{ds}\right)^2 + \begin{Bmatrix} 1 \\ 2\ 2 \end{Bmatrix}\left(\frac{dv}{ds}\right)^2 + 2\begin{Bmatrix} 1 \\ 1\ 2 \end{Bmatrix}\frac{du}{ds}\frac{dv}{ds}\right)\right]$$

Since for the v parametric curve, $u = c =$ constant, so,

$$\chi_g = -\sqrt{a}\ \frac{dv}{ds}\begin{Bmatrix} 1 \\ 2\ 2 \end{Bmatrix}\left(\frac{dv}{ds}\right)^2 = -\sqrt{a}\begin{Bmatrix} 1 \\ 2\ 2 \end{Bmatrix}\left(\frac{dv}{ds}\right)^3$$

$$= -\sqrt{a}\begin{Bmatrix} 1 \\ 2\ 2 \end{Bmatrix}\frac{1}{a_{22}\sqrt{a_{22}}};\ \text{as}\ \frac{dvdv}{ds^2} = \frac{dvdv}{a_{22}dvdv} = \frac{1}{a_{22}}$$

$$= -\phi\mu \times \left[-\frac{1}{2}\frac{\partial}{\partial u}(\mu^2)\right] \times \frac{1}{\mu^3} = \frac{1}{\phi\mu}\frac{\partial \mu}{\partial u}.$$

EXAMPLE 6.8.4 *Show that the condition that the parametric curves u^1 and u^2 on a portion of a surface be geodesic are respectively*

$$\begin{Bmatrix} 2 \\ 1\ 1 \end{Bmatrix} = 0\ \textit{and}\ \begin{Bmatrix} 1 \\ 2\ 2 \end{Bmatrix} = 0.$$

Solution: Let $\chi_g^{(1)}$ and $\chi_g^{(2)}$ be the geodesic curvatures of the u^1 and u^2 parametric curves, respectively. Then from Beltrami's formula (6.63), we get

$$\chi_g^{(1)} = \sqrt{a}\left[\frac{du^1}{ds}\left(\frac{d^2u^2}{ds^2} + \begin{Bmatrix} 2 \\ 1\ 1 \end{Bmatrix}\left(\frac{du^1}{ds}\right)^2 + \begin{Bmatrix} 2 \\ 2\ 2 \end{Bmatrix}\left(\frac{du^2}{ds}\right)^2 + 2\begin{Bmatrix} 2 \\ 1\ 2 \end{Bmatrix}\frac{du^1}{ds}\frac{du^2}{ds}\right)\right.$$

$$\left. -\frac{du^2}{ds}\left(\frac{d^2u^1}{ds^2} + \begin{Bmatrix} 1 \\ 1\ 1 \end{Bmatrix}\left(\frac{du^1}{ds}\right)^2 + \begin{Bmatrix} 1 \\ 2\ 2 \end{Bmatrix}\left(\frac{du^2}{ds}\right)^2 + 2\begin{Bmatrix} 1 \\ 1\ 2 \end{Bmatrix}\frac{du^1}{ds}\frac{du^2}{ds}\right)\right].$$

Since for the u^1 parametric curve, $u^2 = c_1 =$ constant, so,

$$\chi_g^{(1)} = \sqrt{a}\frac{du^1}{ds}\begin{Bmatrix} 2 \\ 1\ 1 \end{Bmatrix}\left(\frac{du^1}{ds}\right)^2 = \sqrt{a}\begin{Bmatrix} 2 \\ 1\ 1 \end{Bmatrix}\left(\frac{du^1}{ds}\right)^3$$

$$= \sqrt{a}\begin{Bmatrix} 2 \\ 1\ 1 \end{Bmatrix}\frac{1}{a_{11}\sqrt{a_{11}}};\ \text{as}\ \frac{du^1du^1}{ds^2} = \frac{du^1du^1}{a_{11}du^1du^1} = \frac{1}{a_{11}}.$$

Similarly, for the u^2 parametric curve, $u^1 = c_2 =$ constant, so,

$$\chi_g^{(2)} = -\sqrt{a}\left\{\begin{matrix} 1 \\ 2\ \ 2 \end{matrix}\right\}\frac{1}{a_{22}\sqrt{a_{22}}}.$$

We can derive these results by using the formula

$$\chi_g = \varepsilon_{\alpha\beta}\lambda^{\alpha}\frac{\delta\lambda^{\beta}}{\delta s};\ \text{ where } \lambda_{(1)}^{\alpha} = \left(\frac{1}{\sqrt{a_{11}}}, 0\right) \text{ and } \lambda_{(2)}^{\alpha} = \left(0, \frac{1}{\sqrt{a_{22}}}\right).$$

If the geodesic curvature vanishes identically, then the curve is a geodesic. Thus, from the relations

$$\chi_g^{(1)} = \sqrt{a}\left\{\begin{matrix} 2 \\ 1\ \ 1 \end{matrix}\right\}\frac{1}{a_{11}\sqrt{a_{11}}} \text{ and } \chi_g^{(2)} = -\sqrt{a}\left\{\begin{matrix} 1 \\ 2\ \ 2 \end{matrix}\right\}\frac{1}{a_{22}\sqrt{a_{22}}}$$

we get that parametric curves are geodesic if and only if

$$\left\{\begin{matrix} 2 \\ 1\ \ 1 \end{matrix}\right\} = 0 \text{ and } \left\{\begin{matrix} 1 \\ 2\ \ 2 \end{matrix}\right\} = 0;\ \text{ as } a = |a_{\alpha\beta}| \neq 0.$$

EXAMPLE 6.8.5 *Show that the condition that the parametric curves on a portion of a surface*

$$r = (c\cos u^2 \cos u^1, c\cos u^2 \sin u^1, c\sin u^2)$$

be geodesic.

Solution: In the representation of the sphere $r(u^1, u^2)$, the co-ordinates u^1, u^2 are orthogonal. We have

$$a_{11} = c^2\cos^2 u^2,\ a_{22} = c^2,\ \frac{\partial a_{22}}{\partial u^1} = 0, \text{ and } a_{12} = 0 = a_{21}.$$

Therefore, the Christoffel symbol $\left\{\begin{matrix} 1 \\ 2\ \ 2 \end{matrix}\right\}$ is given by

$$\left\{\begin{matrix} 1 \\ 2\ \ 2 \end{matrix}\right\} = \frac{1}{2a}\left[a_{22}\left(2\frac{\partial a_{12}}{\partial u^2} - \frac{\partial a_{22}}{\partial u^1}\right) - a_{12}\frac{\partial a_{22}}{\partial u^2}\right]$$

$$= \frac{1}{2c^4\cos^2 u^2}\left[c^2(2\cdot 0 - 0) - 0.0\right] = 0$$

that is, the curves $u^1 =$ constant are geodesics. Now,

$$\left\{\begin{matrix} 2 \\ 1\ \ 1 \end{matrix}\right\} = \frac{1}{2a}\left[a_{11}\left(2\frac{\partial a_{12}}{\partial u^1} - \frac{\partial a_{11}}{\partial u^2}\right) - a_{12}\frac{\partial a_{11}}{\partial u^1}\right]$$

$$= \frac{1}{2c^4\cos^2 u^2}\left[c^2\cos^2 u^2(2\cdot 0 + 2c^2\cos u^2\sin u^2) - 0\cdot 0\right]$$

$$= \cos u^2 \sin u^2 = 0;\ \text{ if } u^2 = 0$$

that is, along the equator which is the only geodesic of the curves $u^2 =$ constant. At the poles $u^2 = \dfrac{\pi}{2}$ and also $\left\{ \begin{matrix} 2 \\ 1 \quad 1 \end{matrix} \right\} = 0$ but

$$x^1 = c\cos u^2 \cos u^1, x^2 = c\cos u^2 \sin u^1, x^3 = c\sin u^2.$$

is not valid at these points. Furthermore we see from

$$(\chi_g)_{u^1=\text{constant}} = \frac{1}{\sqrt{a_{11}}}\frac{\partial}{\partial u^1}(\log\sqrt{a_{22}})$$
$$(\chi_g)_{u^2=\text{constant}} = -\frac{1}{\sqrt{a_{22}}}\frac{\partial}{\partial u^2}(\log\sqrt{a_{11}})$$

that the co-ordinate curves of orthogonal co-ordinates are geodesics if and only if a_{11} does not depend on u^2 and a_{22} does not depend on u^1.

6.9 Exercises

1. On the surface of revolution $\mathbf{r} = (u\cos\phi, u\sin\phi, f(u))$ what are the parametric curves for $u =$ constant and what are the curves $\phi =$ constant.

2. On the right helicoid given by $\mathbf{r} = (u\cos\phi, u\sin\phi, c\phi)$, show that the parametric curves are circular helices and straight lines.

3. On the hyperboloid of one sheet

$$\frac{x}{a} = \frac{\lambda+\mu}{1+\lambda\mu};\ \frac{y}{b} = \frac{1-\lambda\mu}{1+\lambda\mu};\ \frac{z}{c} = \frac{\lambda-\mu}{1+\lambda\mu},$$

   the parametric curves are the generators. What curves are represented by $\lambda = \mu$ and by $\lambda\mu =$ constant?

4. What types of surface are determined by the following representations:
   (a) $r = (x^1, x^2, x^3) = (0, u^1, u^2)$,
   (b) $r = (x^1, x^2, x^3) = (u^1 + u^2, u^1 + u^2, u^1)$,
   (c) $r = (x^1, x^2, x^3) = (a\cos u^1, a\sin u^1, u^2)$.
   Investigate the behaviour of the corresponding matrices and find representation of the form (6.1).

5. Find a parametric representation of the cylinder generated by a straight line G which moves along a curve.

$$\mathcal{C} : x(s) = (h_1(s), h_2(s), h_3(s))$$

   and is parallel to the x^3 axis; consider the corresponding matrix.

6. Find a representation of the form (6.1) of the following surfaces

   (a) Ellipsoid: $\mathbf{r} = (a\cos u^2 cos u^1, bcos u^2 \sin u^1, c\sin u^2)$.
   (b) Elliptic paraboloid: $\mathbf{r} = (au^1 \cos u^2, bu^1 \sin u^2, (u^1)^2))$.
   (c) Hyperbolic paraboloid : $\mathbf{r} = (au^1 \cosh u^2, bu^1 \sinh u^2, (u^1)^2)$.
   (d) Hyperboloid of two sheets : $\mathbf{r} = (a\sinh u^1 \cos u^2, b\sinh u^1 \sin u^2, c\cosh u^1)$.

   What kind of co-ordinate curves do you have in each case?

7. Show that the first fundamental form for

   (a) the surface $x^1 = a\cos u^1, x^2 = a\sin u^1, x^3 = u^2$ is given by
   $$ds^2 = a^2(du^1)^2 + (du^2)^2.$$
   (b) the surface $x^1 = u^1 \cos u^2,\ x^2 = u^1 \sin u^2,\ x^3 = 0$ is given by
   $$ds^2 = (du^1)^2 + (u^1)^2(du^2)^2.$$
   (c) the surface $x^1 = c\cos u^1 \sin u^2,\ x^2 = c\sin u^1 \sin u^2,\ x^3 = c\cos u^2$ is given by
   $$ds^2 = c^2 \sin^2 u^2 (du^1)^2 + c^2 \cos^2 u^2 (du^2)^2.$$
   (d) the surface of revolution $x^1 = u^1 \cos u^2,\ x^2 = u^1 \sin u^2,\ x^3 = f(u^1)$ is given by
   $$ds^2 = (1 + f_1^2)(du^1)^2 + (u^1)^2(du^2)^2.$$
   (e) the paraboloid $x^1 = u^1, x^2 = u^2,\ x^3 = (u^1)^2 - (u^2)^2$ is given by
   $$ds^2 = [1 + 4(u^1)^2](du^1)^2 - 4u^1u^2du^1du^2 + [1 + 4(u^2)^2](du^2)^2.$$
   (f) the sphere $x^1 = c\cos u^1 \cos u^2, x^2 = c\cos u^1 \sin u^2, x^3 = c\sin u^1$ is given by
   $$ds^2 = c^2(du^1)^2 + c^2 \cos^2 u^1 (du^2)^2.$$
   (g) a plane with respect to polar co-ordinates.

8. Prove that on the surface given by
   $$x^1 = a(u^1 + u^2);\ x^2 = b(u^1 - u^2);\ x^3 = u^1u^2$$
   the parametric curves are straight lines.

9. Determine the first fundamental form of the cylinder

$$\mathbf{r} = (x^1, x^2, x^3) = \big(h_1(u^1), h_2(u^1), u^2\big)$$

and of a cylinder of revolution.

10. Prove that at regular points of a surface the first fundamental form is positive definite.

11. If θ be the angle between the parametric curves lying on a surface, immersed in E_3, show that

$$\sin\theta = \frac{\sqrt{a}}{\sqrt{a_{11}}\sqrt{a_{22}}} \text{ and } \tan\theta = \frac{\sqrt{a}}{a_{12}}$$

and hence, show that the parametric curves on a surface are orthogonal if and only if $a_{12} = 0$.

12. Show that the area of the anchor ring

$$\mathbf{r} = ((b + a\cos u)\cos v, (b + a\cos u)\sin v, a\sin v)$$

where $0 < u < 2\pi, 0 < v < 2\pi$ is $4\pi^2 ab$.

13. Prove that a necessary and sufficient condition that a surface $\mathcal{S}$ be isometric with the Euclidean plane is that the Riemann tensor (or the Gaussian curvature of $\mathcal{S}$) be identically zero.

14. (a) Prove that $\varepsilon_{\alpha\beta}$ and $\varepsilon^{\alpha\beta}$ can be obtained from the other by raising or lowering of indices with the help of the metric tensors.

    (b) If $\lambda^\alpha, \mu^\alpha$ are two unit vectors the rotation $\lambda^\alpha, \mu^\alpha$ is positive, show that

$$\sin\theta = \varepsilon_{\alpha\beta}\lambda^\alpha\mu^\alpha.$$

    (c) If $\lambda^\alpha, \mu^\alpha$ are two parallel vector fields, show that $\varepsilon_{\alpha\beta}\lambda^\alpha\mu^\alpha$ remains constant.

15. Prove that the vector obtained by parallel propagation of the tangent vector to a geodesic always remains tangent to the geodesic.

16. Find the differential equations of the geodesic in (i) Rectangular co-ordinates; (ii) Spherical co-ordinates; (iii) Cylindrical co-ordinates.

17. Show that an elliptic helix is not in general a geodesic on an elliptic cylinder.

18. Show that the differential equations of the geodesic for

(a) the metric $ds^2 = (dx^1)^2 + [(x^1)^2 + c^2](dx^2)^2$ are

$$\frac{d^2x^1}{ds^2} - x^1\left(\frac{dx^2}{ds}\right)^2 = 0; \frac{d^2x^2}{ds^2} + \frac{2x^1}{(x^1)^2 + c^2}\frac{dx^1}{ds}\frac{dx^2}{ds} = 0.$$

(b) the metric $ds^2 = (du)^2 + (\sin u)^2(dv)^2$ are

$$\frac{d^2u}{ds^2} - \sin u \cos u\left(\frac{dv}{ds}\right)^2 = 0; \frac{d^2v}{ds^2} + 2\cot u\frac{du}{ds}\frac{dv}{ds} = 0.$$

19. Obtain the differential equations of geodesics for metric

(i) $ds^2 = f(x)dx^2 + dy^2 + dz^2 + \dfrac{1}{f(x)}dt^2.$

(ii) $ds^2 = e^{-2kt}\left[dx^2 + dy^2 + dz^2\right] + dt^2.$

20. Show that the geodesics of a sphere of radius a determined by the equation

$$y^1 = a\cos u^1 \cos u^2;\ y^2 = a\cos u^1 \sin u^2;\ y^3 = a\sin u^1,$$

are great circles, i.e. the great circles on sphere are geodesics.

21. Determine the components of curvature tensors

(i) of a surface of revolution $r = (u^2\cos u^1, u^2\sin u^1, f(u^2))$,

(ii) of a right conoid $r = (u^2\cos u^1, u^2\sin u^1, f(u^1))$,

(iii) of a surface represented in the form $x_3 = f(x_1, x_2)$.

22. On the surface generated by the tangents to a twisted curve, find the differential equation of the curves which cut the generators at a constant angle α.

23. If v^α are geodesic co-ordinates in the neighbourhood of a point if they are subjected to the transformation

$$u^\alpha = v^\alpha + \frac{1}{6}C^\alpha_{\beta\gamma\lambda}\ v^\beta v^\gamma v^\lambda,$$

where C's are constants, then show that u^α are geodesic co-ordinates in the neighbourhood of the origin O.

24. (a) If X^α is a vector field satisfying $\frac{\delta X^\alpha}{\delta t} = 0$, show that $a_{\alpha\beta}X^\alpha X^\beta$ remains constant along the curve. Deduce that the magnitudes of the parallel vector fields are equal.

(b) Prove that the unit tangent vector $\lambda^\alpha = \frac{du^\alpha}{ds}$ of a geodesic is a parallel vector field along the geodesic.

(c) Prove that the necessary and sufficient condition that a system of co-ordinates be geodesic with pole at P are that their second covariant derivatives with respect to the metric of the space all vanish at P.

(d) Prove that the geodesic curvature of a curve C on a surface S depends on the first fundamental form of S only.

25. Prove that the co-ordinate system u^α defined by

$$u^\alpha = v^\alpha - u_0^\alpha + \frac{1}{2}\left\{ \begin{matrix} \alpha \\ \beta \quad \gamma \end{matrix} \right\}_P (v^\beta - u_0^\beta)(v^\gamma - u_0^\gamma),$$

is a geodesic co-ordinate system with the pole at $P(u_0^1, u_0^2)$.

26. Prove that the catenoid,

$$ds^2 = a^2 \cosh^2 x^1 (dx^1)^2 + a^2 \cosh^2 x^2 (dx^2)^2$$

and the right helicoid

$$ds^2 = (dy^1)^2 + [(y^1)^2 + a^2](dy^2)^2$$

are locally isometric.

27. Prove that, if the system of co-ordinates is orthogonal, the non-vanishing Christofell symbols of the second kind are

$$\left\{ \begin{matrix} 1 \\ 1 \quad 1 \end{matrix} \right\} = \frac{1}{2a_{11}} \frac{\partial a_{11}}{\partial u^1}; \quad \left\{ \begin{matrix} 1 \\ 1 \quad 2 \end{matrix} \right\} = \frac{1}{2a_{11}} \frac{\partial a_{11}}{\partial u^2}; \quad \left\{ \begin{matrix} 1 \\ 2 \quad 2 \end{matrix} \right\} = -\frac{1}{2a_{11}} \frac{\partial a_{22}}{\partial u^1}$$

$$\left\{ \begin{matrix} 2 \\ 1 \quad 1 \end{matrix} \right\} = -\frac{1}{2a_{22}} \frac{\partial a_{11}}{\partial u^2}; \quad \left\{ \begin{matrix} 2 \\ 1 \quad 2 \end{matrix} \right\} = \frac{1}{2a_{22}} \frac{\partial a_{22}}{\partial u^1}; \quad \left\{ \begin{matrix} 2 \\ 2 \quad 2 \end{matrix} \right\} = \frac{1}{2a_{22}} \frac{\partial a_{22}}{\partial u^2}.$$

28. Prove that for a helicoids of non-zero pitch the sections by planes containing the axis are geodesics if and only if these sections are straight lines.

29. Prove that, on a general surface, a necessary and sufficient condition that the curve $v = c$ be a geodesic is $EE_2 + FE_1 - 2EF_1 = 0$ for all values of u, where $E = r_1^2, F = r_1.r_2, G = r_2^2$.

30. Prove that if a surface admits two orthogonal families of geodesics, it is isometric with the plane.

31. Show that, if the line element is of the form $ds^2 = (du^1)^2 + a_{22}(du^2)^2$, the Gaussian curvature is

$$\kappa = -\frac{1}{\sqrt{a_{22}}} \frac{\partial^2}{(\partial u^1)^2} \sqrt{a_{22}}.$$

32. Prove that the Gaussian curvature is an invariant.

33. Find the Gaussian curvature of the surface given (in Monge form) by $z = f(x, y)$, where x and y are taken as parameters.

34. Show that the Gaussian curvature is $-\frac{1}{\lambda}\frac{\partial^2\lambda}{\partial u^2}$ for a surface with the metric

$$ds^2 = (du)^2 + \lambda^2(dv)^2$$

35. Show that for the sphere of radius c, with the equation of the form

$$x^1 = c\sin u^1 \cos u^2; x^2 = c\sin u^1 \sin u^2;\ x^3 = c\cos u^1,$$

where c is a constant, the total curvature is $\kappa = \frac{1}{c^2}$.

36. Show that the surface given by

$$x^1 = f_1(u^1),\ x^2 = f_2(u^1),\ x^3 = u^2$$

is a developable, where f_1, f_2 are differentiable functions.

37. Determine whether the surface of a helicoid given by

$$x^1 = u^1 \cos u^2;\ x^2 = u^1 \sin u^2; x^3 = cu^1$$

is developable, where c is a constant and u^1, u^2 are curvilinear co-ordinates of the surface.

38. Show that the surface with the metric
    (a) $ds^2 = (du^1)^2 + [(du^1)^2 + c^2](du^2)^2$ is not developable.
    (b) $ds^2 = (u^2)^2(du^1)^2 + (u^1)^2(du^2)$ is developable.

39. Find the equations of the surface of revolution for which $ds^2 = du^2 + (a^2 - u^2)dv^2$.

40. Prove that a ruled surface $x(s,t) = y(s) + tz(s)$ is a developable surface if and only if $|[\dot{y}, z, \dot{z}]| = 0$, where a dot denotes the derivatives with respect to the arc length s of the curve $y(s)$ and [.,.,.] denotes the scalar triple product.

41. Find an analytic expression of the Gaussian curvature of a ruled surface and prove that a ruled surface is developable surface if and only if its Gaussian curvature vanishes.

42. Show that
    (i) $\chi_g = -\varepsilon_{\alpha\beta}\eta^\beta \frac{\delta u^\alpha}{ds} = \varepsilon_{\alpha\beta}\lambda^\alpha \frac{\delta u^\beta}{ds}$.

(ii) $\chi_g^2 = a_{\alpha\beta}\dfrac{\delta\lambda^\alpha}{\delta s}\dfrac{\delta\lambda^\beta}{\delta s}$; and $\chi_g = \varepsilon_{\alpha\beta}\lambda^\alpha\dfrac{\delta\lambda^\beta}{\delta s}$

(iii) $\dfrac{d^2u^\alpha}{\delta s^2} + \left\{\begin{matrix} \alpha \\ \beta\ \ \gamma \end{matrix}\right\}\dfrac{du^\beta}{\delta s}\dfrac{du^\gamma}{\delta s} = -\chi_g\varepsilon^{\alpha\beta}a_{\beta\gamma}\dfrac{du^\gamma}{\delta s}$.

43. Show that the geodesic curvature of the curve $u = c$ with the metric
$$ds^2 = \lambda^2(du)^2 + \mu^2(dv)^2 \text{ is } \frac{1}{\lambda\mu}\frac{\partial\mu}{\partial u}.$$

44. When the parametric curves are orthogonal, prove that
$$\chi_g^{(1)} = -\frac{1}{2\sqrt{a_{22}}}\frac{\partial \log a_{11}}{\partial u^2} \text{ and } \chi_g^{(2)} = \frac{1}{2\sqrt{a_{11}}}\frac{\partial \log a_{22}}{\partial u^1}.$$

45. Deduce that
$$\chi_g^{(1)} = \frac{1}{\sqrt{a}}\left[\frac{\partial}{\partial u^1}\left(\frac{a_{12}}{\sqrt{a_{11}}}\right) - \frac{\partial\sqrt{a_{11}}}{\partial u^2}\right];\ \chi_g^{(2)} = \frac{1}{\sqrt{a}}\left[-\frac{\partial}{\partial u^2}\left(\frac{a_{12}}{\sqrt{a_{22}}}\right) + \frac{\partial\sqrt{a_{22}}}{\partial u^1}\right].$$

46. Prove that the geodesic curvatures for the co-ordinate curves are
$$\chi_g^{(1)} = \frac{\sqrt{a}}{(a_{11})^{3/2}}\left\{\begin{matrix} 2 \\ 1\ \ 1 \end{matrix}\right\} \text{ and } \chi_g^{(2)} = -\frac{\sqrt{a}}{(a_{22})^{3/2}}\left\{\begin{matrix} 1 \\ 2\ \ 2 \end{matrix}\right\}.$$

47. Prove that the necessary and sufficient condition for a curve on a surface to be a geodesic is that its geodesic curvature be zero.

48. Show that the formula for the Gaussian curvature κ can be written in the form
$$\kappa = \frac{1}{2\sqrt{a}}\left[\frac{\partial}{\partial u^1}\left\{\frac{a_{12}}{a_{11}\sqrt{a}}\frac{\partial a_{11}}{\partial u^2} - \frac{1}{\sqrt{a}}\frac{\partial a_{22}}{\partial u^1}\right\}\right.$$
$$\left. + \frac{\partial}{\partial u^2}\left\{\frac{2}{\sqrt{a}}\frac{\partial a_{12}}{\partial u^1} - \frac{1}{\sqrt{a}}\frac{\partial a_{11}}{\partial u^2} - \frac{a_{12}}{a_{11}\sqrt{a}}\frac{\partial a_{11}}{\partial u^1}\right\}\right].$$

49. Prove that any circular helix C on a cylinder of revolution S is a geodesic on C.

50. Prove that if a geodesic on a surface of revolution cuts the meridians at a constant angle, the surface is a right cylinder.

51. Prove that the meridians of a ruled helicoid are geodesics.

52. Prove that the orthogonal trajectories of the helices on a helicoid are geodesics.

53. Show that if a geodesic on a surface of revolution cuts the medians at a constant angle, the surface is a right cylinder.

Chapter 7

Surfaces in Space

With the exception of occasional references to the surrounding space, our study of geometry of surfaces was carried out from the point of view of a two-dimensional being whose universe is determined by the surface parameters u^1 and u^2. The treatment of surfaces in space presented in this chapter was based entirely on the study of the first quadratic differential form. In this chapter, the geometric shape and properties of a surface have discussed in the neighbourhood of any of its points. Of curse, this problem is of fundamental importance.

7.1 The Tangent Vector

We are now going to investigate the properties of a surface in its relation to the surrounding space. Consequently, we are dealing with two distinct system of co-ordinates namely, the three orthogonal Cartesian co-ordinates for the surrounding space which we denote by $y^i; i = 1, 2, 3$ and the two curvilinear co-ordinates of the surface which we denote by $u^\alpha; \alpha = 1, 2$. Let us denote the equation of a surface $\mathcal{S}$ embedded in E^3 by

$$\begin{aligned} &y^1 = y^1(u^1, u^2);\ y^2 = y^2(u^1, u^2);\ y^3 = y^3(u^1, u^2) \\ &y^i = y^i(u^1, u^2);\ i = 1, 2, 3. \end{aligned} \tag{7.1}$$

The line element of arc ds of a curve lying on $\mathcal{S}$ is determined by the formula,

$$ds^2 = a_{\alpha\beta} du^\alpha du^\beta;\ \text{where } a_{\alpha\beta} = \frac{\partial y^i}{\partial u^\alpha}\frac{\partial y^i}{\partial u^\beta}. \tag{7.2}$$

The choice of Cartesian variables y^i in the space enveloping the surface is clearly not essential, and we could have equally well referred the points of E^3 to a curvilinear co-ordinate system X related to Y by the transformation

$$x^i = x^i(y^1, y^2, y^3).$$

Now, relative to the frame X, the line element in E^3 is given by

$$ds^2 = g_{mn}dx^m dx^n; \quad \text{where, } g_{mn} = \frac{\partial y^k}{\partial x^m}\frac{\partial y^k}{\partial x^n} \tag{7.3}$$

and the set of Eq. (7.1) for the surface $\mathcal{S}$ can be written as

$$\mathcal{S} : x^i = x^i(u^1, u^2); \; i = 1, 2, 3. \tag{7.4}$$

If we take a small displacement du^α on the surface, the corresponding component of the displacement in space are given by

$$dx^i = \frac{\partial x^i}{\partial u^\alpha} du^\alpha \tag{7.5}$$

and hence expression (7.3) for the surface element of arc assumes the form

$$ds^2 = g_{ij}dx^i dx^j = g_{ij}\frac{\partial x^i}{\partial u^\alpha}\frac{\partial x^j}{\partial u^\beta} \; du^\alpha du^\beta. \tag{7.6}$$

A comparison of Eq. (7.6) with Eq. (7.2) leads to the conclusion that

$$a_{\alpha\beta} = g_{ij}\frac{\partial x^i}{\partial u^\alpha}\frac{\partial x^j}{\partial u^\beta}; \; i, j = 1, 2, 3 \text{ and } \alpha, \beta = 1, 2. \tag{7.7}$$

Now, dx^i is a space vector and is surface invariant, i.e. its components are unaltered Gaussian co-ordinates alone one transformed. Similarly, du^α is a surface vector and is also a space invariant. Hence, if we regard [Eq. (7.5)] first from the point of view of a transformation of space co-ordinates and then from the point of view of a transformation of surface co-ordinates and then from the point of view of a transformation of surface co-ordinates, we see that $\frac{\partial x^i}{\partial u^\alpha}$ is a contravariant space vector and also a covariant surface vector, so that, it may be represented by

$$dx^i = \frac{\partial x^i}{\partial u^\alpha} du^\alpha = x^i_\alpha du^\alpha; \quad \text{where, } x^i_\alpha = \frac{\partial x^i}{\partial u^\alpha}, \tag{7.8}$$

where the indices properly describe the tensor character of this set of quantities.

Let $\mathbf{r}$ be the position vector of an arbitrary point P on $\mathcal{S}$. The point P is determined by a pair of Gaussian co-ordinates (u^1, u^2), or by a triplet of space co-ordinates (x^1, x^2, x^3). Accordingly, the vector $\mathbf{r}$ can be viewed as a function of the space variables x^i satisfying Eq. (7.4). Thus,

$$\frac{\partial \mathbf{r}}{\partial u^\alpha} = \frac{\partial \mathbf{r}}{\partial x^i}\frac{\partial x^i}{\partial u^\alpha}. \tag{7.9}$$

But $\frac{\partial \mathbf{r}}{\partial x^i}$ are the base vectors $\mathbf{b}_i$ at P, associated with the curvilinear co-ordinate system X, whereas $\frac{\partial \mathbf{r}}{\partial u^\alpha}$ are the base vectors $\mathbf{a}_\alpha$ at P relative to the Gaussian co-ordinate system U. Hence, Eq. (7.9) yields

$$\mathbf{a}_\alpha = \mathbf{b}_i \frac{\partial x^i}{\partial u^\alpha}. \tag{7.10}$$

It is clear from this representation that

$$\mathbf{a}_1 = \mathbf{b}_i \frac{\partial x^i}{\partial u^1} \text{ and } \mathbf{a}_2 = \mathbf{b}_i \frac{\partial x^i}{\partial u^2},$$

so that $\frac{\partial x^i}{\partial u^\alpha} \equiv x^i_\alpha$, $(\alpha = 1, 2)$, are the contravariant components of the surface base vectors $\mathbf{a}_\alpha$ referred to the base systems $\mathbf{b}_i$. Thus, the set of quantities

$$x^i_1 : \left(\frac{\partial x^1}{\partial u^1}, \frac{\partial x^2}{\partial u^1}, \frac{\partial x^3}{\partial u^1} \right) \text{ and } x^i_2 : \left(\frac{\partial x^1}{\partial u^2}, \frac{\partial x^2}{\partial u^2}, \frac{\partial x^3}{\partial u^2} \right)$$

transform in a contravariant manner relative to the transformation of space co-ordinates x^i. Now the three surface vectors

$$x^1_\alpha : \left(\frac{\partial x^1}{\partial u^1}, \frac{\partial x^1}{\partial u^2} \right), x^2_\alpha : \left(\frac{\partial x^2}{\partial u^1}, \frac{\partial x^2}{\partial u^2} \right) \text{ and } x^3_\alpha : \left(\frac{\partial x^3}{\partial u^1}, \frac{\partial x^3}{\partial u^2} \right)$$

transform according to the covariant law with respect to the transformation of Gaussian surface co-ordinates u^α. Indeed, consider a transformation $u^\alpha = u^\alpha \left(\overline{u}^1, \overline{u}^2 \right)$; then Eq. (7.4) of $\mathcal{S}$ go over into $x^i = x^i \left(\overline{u}^1, \overline{u}^2 \right)$ and

$$\frac{\partial x^i}{\partial u^\alpha} = \frac{\partial x^i}{\partial \overline{u}^\beta} \frac{\partial \overline{u}^\beta}{\partial u^\alpha}. \tag{7.11}$$

But $\frac{\partial x^i}{\partial \overline{u}^\beta} = \overline{x}^i_\beta$, and Eq. (7.11) yields the covariant law

$$x^i_\alpha = \frac{\partial \overline{u}^\beta}{\partial u^\alpha} \bar{x}^i_\beta; \;\; i = 1, 2, 3.$$

Let ds be an element of arc joining a pair of points $P(u^1, u^2)$ and $Q(u^1 + du^1, u^2 + du^2)$ on $\mathcal{S}$. The direction of the line element ds is given by the direction parameters $\frac{du^\alpha}{ds} = \lambda^\alpha$. The same direction can be specified by an observer in the enveloping space by means of three parameters $\frac{dx^i}{ds} = \lambda^i$, and from the Eq. (7.5), we find that

$$\lambda^i = \frac{dx^i}{ds} = \frac{\partial x^i}{\partial u^\alpha} \; \frac{du^\alpha}{ds}; \text{ i.e. } \lambda^i = x^i_\alpha \lambda^\alpha. \tag{7.12}$$

This formula tells us that any surface vector A^α can be viewed as a space vector with components A^i determined by

$$A^i = x^i_\alpha A^\alpha. \tag{7.13}$$

We shall refer to a vector A^i determined by Eq. (7.13) as a *tangent vector to the surface* $\mathcal{S}$.

7.2 The Normal Line

An entity that provides a characterisation of the shape of the surface as it appears from the enveloping surface is the normal line to the surface.
Let **A** and **B** be a pair of surface vectors drawn at some point P on the surface $\mathcal{S}$, such that the rotation $\mathbf{A}, \mathbf{B}$ is positive. The unit normal vector $\boldsymbol{\xi}$ to the surface $\mathcal{S}$ is orthogonal to both **A** and **B** and is so oriented that (**A**,**B**,ξ) from a right-handed system (Figure 7.1), i.e.

$$\varepsilon_{ijk} A^i B^j \xi^k = 1 \tag{7.14}$$

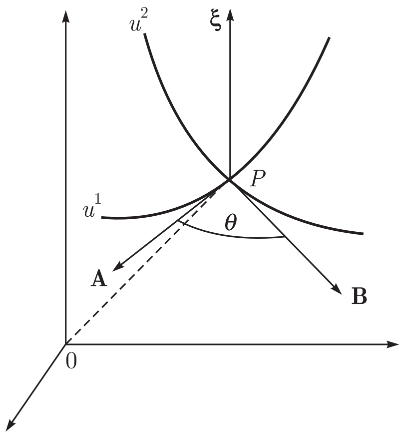

Figure 7.1: The normal lines to the surface.

where

$$\boldsymbol{\xi} = \frac{1}{|\mathbf{A} \times \mathbf{B}|} (\mathbf{A} \times \mathbf{B}) = \frac{1}{AB \sin\theta} (\mathbf{A} \times \mathbf{B})$$

where θ is the angle between **A** and **B**. We call the vector $\boldsymbol{\xi}$ the unit normal vector to the surface $\mathcal{S}$ at P. Clearly, $\boldsymbol{\xi}$ is a function of co-ordinates $\left(u^1, u^2\right)$, and as the point $P\left(u^1, u^2\right)$ is displaced to a new position $Q\left(u^1 + du^1, u^2 + du^2\right)$, the vector $\boldsymbol{\xi}$ undergoes a change

$$d\boldsymbol{\xi} = \frac{\partial \boldsymbol{\xi}}{\partial u^\alpha} du^\alpha = \boldsymbol{\xi}_\alpha \;\; du^\alpha, \text{ say,} \tag{7.15}$$

whereas the position vector **r** is changed to the amount

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u^\beta} du^\beta = \mathbf{r}_\beta du^\beta, \text{ say.} \tag{7.16}$$

The behaviour of the normal line as its foot displaced along the surface depends on the shape of the surface, and it occurred to Gauss to describe certain properties of surfaces with the aid of a quadratic form that depends in a fundamental way on the behaviour of the normal line. Let us form the scalar product

$$\begin{aligned} d\boldsymbol{\xi} \cdot d\mathbf{r} &= \frac{\partial \boldsymbol{\xi}}{\partial u^\alpha} \cdot \frac{\partial \mathbf{r}}{\partial u^\beta} du^\alpha du^\beta = \boldsymbol{\xi_\alpha} \cdot \boldsymbol{r_\beta} du^\alpha du^\beta \\ &= -b_{\alpha\beta} du^\alpha du^\beta, \text{ say }, \end{aligned} \tag{7.17}$$

if we define $b_{\alpha\beta}$ as

$$\begin{aligned} b_{\alpha\beta} &= -\frac{1}{2}\left(\frac{\partial \boldsymbol{\xi}}{\partial u^\alpha} \cdot \frac{\partial \mathbf{r}}{\partial u^\beta} + \frac{\partial \boldsymbol{\xi}}{\partial u^\beta} \cdot \frac{\partial \mathbf{r}}{\partial u^\alpha}\right) \\ &= -\frac{1}{2}\left(\boldsymbol{\xi_\alpha} \cdot \boldsymbol{r_\beta} + \boldsymbol{\xi_\beta} \cdot \boldsymbol{r_\alpha}\right) \end{aligned} \tag{7.18}$$

The left-hand side of Eq. (7.17), being the scalar product of two vectors, is an invariant. Thus, the second fundamental form is invariant with respect to any allowable co-ordinate transformation which preserves the sense of $\boldsymbol{\xi}$. Since the co-ordinate differentials du^α have a contravariant transformation behaviour and the second fundamental form is invariant, the coefficients $b_{\alpha\beta}$ of this form are the components of a covariant tensor of second order with respect to co-ordinate transformations. From the quotient law, $b_{\alpha\beta}$ is a covariant tensor of order $(0, 2)$ and from Eq. (7.18) it is symmetric with respect to α and β. The quadratic form

$$B \equiv b_{\alpha\beta} du^\alpha du^\beta \tag{7.19}$$

introduced by Gauss, is called the *second fundamental quadratic form of the surface.* This differential form Eq. (7.19) plays an important part in the study of the geometry of the surface when they are viewed from the surrounding space.

Now, we shall write Eq. (7.14) in terms of the components x^i_α of the base vectors. The Eq. (7.13) tells us

$$A^i = x^i_\alpha A^\alpha; \ B^j = x^j_\beta B^\beta.$$

If θ be the angle between A^i and B^j, we can write

$$AB \sin\theta = \varepsilon_{\alpha\beta} A^\alpha B^\beta.$$

Let the contravariant components of the unit surface normal $\boldsymbol{\xi}$ by ξ^i and the covariant components ξ_i is, then

$$\xi_i = \frac{\varepsilon_{ijk} x^j_\alpha A^\alpha x^k_\beta B^\beta}{\varepsilon_{\alpha\beta} A^\alpha B^\beta}$$

or

$$\left(\xi_i \varepsilon_{\alpha\beta} - \varepsilon_{ijk} x^j_\alpha x^k_\beta\right) A^\alpha B^\beta = 0.$$

This relation is valid for all surface vectors. Since, A^α and B^β are arbitrary, we have

$$\xi_i \varepsilon_{\alpha\beta} = \varepsilon_{ijk} x^j_\alpha x^k_\beta.$$

Multiplying this relation through by $\varepsilon^{\alpha\beta}$, and noting $\varepsilon^{\alpha\beta}\varepsilon_{\alpha\beta} = 2$, we get

$$\xi_i \varepsilon_{\alpha\beta}\varepsilon^{\alpha\beta} = \varepsilon^{\alpha\beta}\varepsilon_{ijk} x^j_\alpha x^k_\beta$$

or

$$\xi_i \cdot 2 = \varepsilon^{\gamma\delta}\varepsilon_{ijk} x^j_\gamma x^k_\delta$$

or

$$\xi_i = \frac{1}{2}\varepsilon^{\gamma\delta}\varepsilon_{ijk} x^j_\gamma x^k_\delta.$$

It is clear from the structure of this formula that ξ_i is a space vector which does not depend on the choice of surface co-ordinates. This fact is also obvious from purely geometrical considerations.

EXAMPLE 7.2.1 *Calculate the second fundamental form for the right helicoid given by* $\mathbf{r} = (u\cos v, u\sin v, cv)$.

Solution: Since $\mathbf{r} = (u\cos v, u\sin v, cv)$, so

$$\mathbf{A} = \frac{d\mathbf{r}}{du} = (\cos v, \sin v, 0) \text{ and } \mathbf{B} = \frac{d\mathbf{r}}{dv} = (-u\sin v, u\cos v, c).$$

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & c \end{vmatrix} = c\sin v\hat{i} - c\cos v\hat{j} + u\hat{k}.$$

Thus, the normal vector is given by

$$\boldsymbol{\xi} = \frac{1}{|\mathbf{A}\times\mathbf{B}|}(\mathbf{A}\times\mathbf{B}) = \frac{1}{\sqrt{c^2+u^2}}(c\sin v, -c\cos v, u).$$

The symmetric covariant tensor $b_{\alpha\beta}$ are given by

$$b_{11} = -\frac{1}{2}\left[\frac{\partial\boldsymbol{\xi}}{\partial u^1}\cdot\frac{\partial\mathbf{r}}{\partial u^1} + \frac{\partial\boldsymbol{\xi}}{\partial u^1}\cdot\frac{\partial\mathbf{r}}{\partial u^1}\right] = -\frac{\partial\boldsymbol{\xi}}{\partial u^1}\cdot\frac{\partial\mathbf{r}}{\partial u^1} = -\frac{\partial\boldsymbol{\xi}}{\partial u}\cdot\frac{\partial\mathbf{r}}{\partial u}$$

$$= -\left(\frac{cu\sin v}{(c^2+u^2)^{\frac{3}{2}}}, \frac{cu\cos v}{(c^2+u^2)^{\frac{3}{2}}}, \frac{c^2}{(c^2+u^2)^{\frac{3}{2}}}\right)\cdot(\cos v, \sin v, 0) = 0$$

$$b_{22} = -\frac{\partial \boldsymbol{\xi}}{\partial v} \cdot \frac{\partial \mathbf{r}}{\partial v} = \frac{1}{\sqrt{c^2+u^2}} (c\cos v, c\sin v, 0) \cdot (-u\sin v, u\cos u, c)$$

$$= -\frac{1}{\sqrt{c^2+u^2}} (-uc\sin v\cos v + uc\sin v\cos v + 0) = 0$$

$$b_{12} = -\frac{1}{2}\left[\frac{\partial \boldsymbol{\xi}}{\partial u^1} \cdot \frac{\partial \mathbf{r}}{\partial u^2} + \frac{\partial \boldsymbol{\xi}}{\partial u^1} \cdot \frac{\partial \mathbf{r}}{\partial u^2}\right] = -\frac{1}{2}\left[\frac{\partial \boldsymbol{\xi}}{\partial u} \cdot \frac{\partial \mathbf{r}}{\partial v} + \frac{\partial \boldsymbol{\xi}}{\partial v} \cdot \frac{\partial \mathbf{r}}{\partial u}\right]$$

$$= -\frac{1}{2}\left[\left(\frac{-cu\sin v}{(u^2+c^2)^{\frac{3}{2}}}, \frac{cu\cos v}{(u^2+c^2)^{\frac{3}{2}}}, \frac{c^2}{(u^2+c^2)^{\frac{3}{2}}}\right) \cdot (-u\sin v, u\cos v, c)\right.$$

$$\left. + \frac{1}{\sqrt{u^2+c^2}} (c\cos v, c\sin v, 0) \cdot (\cos v, \sin v, 0)\right]$$

$$= -\frac{1}{2}\left[\frac{cu^2\sin^2 v + cu^2\cos^2 v + c^3}{(u^2+c^2)^{\frac{3}{2}}} + \frac{c\cos^2 v + c\sin^2 v}{\sqrt{u^2+c^2}}\right]$$

$$= -\frac{1}{2}\left[\frac{c}{\sqrt{u^2+c^2}} + \frac{c}{\sqrt{u^2+c^2}}\right] = -\frac{c}{(u^2+c^2)^{\frac{1}{2}}} = b_{21}.$$

Thus, the second fundamental quadratic form becomes

$$B \equiv b_{11}(du)^2 + 2b_{12}dudv + b_{22}(dv)^2 = 2b_{12}dudv \equiv -\frac{2c}{(u^2+c^2)^{\frac{1}{2}}}dudv.$$

EXAMPLE 7.2.2 *Calculate the second fundamental form for the paraboloid given by* $\mathbf{r} = \left(u, v, u^2 - v^2\right)$.

Solution: Since $\mathbf{r} = \left(u, v, u^2 - v^2\right)$, so $\mathbf{A} = (1, 0, 2u)$, $\mathbf{B} = (0, 1, -2v)$ and

$$\mathbf{A} \times \mathbf{B} = -2u\hat{i} - 2v\hat{j} + \hat{k}.$$

Thus, the unit normal vector is given by

$$\boldsymbol{\xi} = \frac{1}{|\mathbf{A} \times \mathbf{B}|} (\mathbf{A} \times \mathbf{B}) = \frac{1}{\sqrt{4u^2+4v^2+1}}(-2u, -2v, 1).$$

The symmetric covariant tensors $b_{\alpha\beta}$ are given by

$$b_{11} = -\frac{\partial \boldsymbol{\xi}}{\partial u} \cdot \frac{\partial \mathbf{r}}{\partial u} = \frac{1}{(4u^2+4v^2+1)^{\frac{3}{2}}} \left(2\left\{4v^2+1\right\}, -8uv, 4u\right) \cdot (1, 0, 2u)$$

$$= \frac{2\left(4u^2+1\right) + 8u^2}{(4u^2+4v^2+1)^{\frac{3}{2}}} = \frac{2}{\sqrt{4u^2+4v^2+1}}$$

$$b_{22} = -\frac{\partial \boldsymbol{\xi}}{\partial v} \cdot \frac{\partial \mathbf{r}}{\partial v} = \frac{1}{(4u^2+4v^2+1)^{\frac{3}{2}}} \left(-8uv, -2(4u^2+1), 4v\right) \cdot (0, 1, -2v)$$

$$= \frac{-2\left(4u^2+1\right) - 8v^2}{(4u^2+4v^2+1)^{\frac{3}{2}}} = -\frac{2}{\sqrt{4u^2+4v^2+1}}$$

$$b_{12} = -\frac{1}{2}\left[\frac{\partial \boldsymbol{\xi}}{\partial u} \cdot \frac{\partial \mathbf{r}}{\partial v} + \frac{\partial \boldsymbol{\xi}}{\partial v} \cdot \frac{\partial \mathbf{r}}{\partial u}\right] = 0 = b_{21}.$$

Thus the second fundamental quadratic form becomes

$$B \equiv b_{11}(du)^2 + b_{22}(dv)^2 + 2b_{12}dudv \equiv \frac{2}{\sqrt{4u^2+4v^2+1}}\left[(du)^2 - (dv)^2\right].$$

EXAMPLE 7.2.3 *Show a surface of revolution is regular and exhibit the unit surface normal.*

Solution: The Gaussian form of a surface of revolution about z axis is

$$\mathbf{r}(x^1, x^2) = \left(f(x^1)\cos x^2, f(x^1)\sin x^2, g(x^1)\right); \; f(x^1) > 0.$$

From this parametric representation, we get

$$\mathbf{A} = \frac{\partial \mathbf{r}}{\partial x^1} = \left(f'(x^1)\cos x^2, f'(x^1)\sin x^2, g'(x^1)\right);$$

$$\mathbf{B} = \frac{\partial \mathbf{r}}{\partial x^2} = \left(-f(x^1)\sin x^2, f(x^1)\cos x^2, 0\right).$$

so

$$\mathbf{A} \times \mathbf{B} = \left(-fg'\cos x^2, -fg'\sin x^2, ff'\right).$$

Therefore, the norm is $|\mathbf{A} \times \mathbf{B}| = f\sqrt{f'^2 + g'^2}$. Now $f = f(x^1) \neq 0$; further, the generating curve is regular, which means that, with $t = x^1$, the tangent vector of the curve,

$$\left(\frac{dx}{dt}, 0, \frac{dz}{dt}\right) = (f', 0, g')$$

is non-null and $f'^2 + g'^2 \neq 0$. Therefore, $f\sqrt{f'^2 + g'^2} \neq 0$ and the surface is regular. Thus, the unit surface normal vector is given by

$$\boldsymbol{\xi} = \frac{1}{|\mathbf{A} \times \mathbf{B}|}(\mathbf{A} \times \mathbf{B}) = \frac{1}{f\sqrt{f'^2+g'^2}}\left(-fg'\cos x^2, -fg'\sin x^2, ff'\right)$$

$$= \left(-\frac{g'}{\sqrt{f'^2+g'^2}}\cos x^2, -\frac{g'}{\sqrt{f'^2+g'^2}}\sin x^2, \frac{f'}{\sqrt{f'^2+g'^2}}\right).$$

EXAMPLE 7.2.4 *Prove that the tangent planes to a cone of revolution at the points of its generating straight lines coincide.*

Solution: The parametric representation to a cone of revolution is

$$r = (x^1, x^2, x^3) = (u^1 \cos u^2, u^1 \sin u^2, au^1).$$

The first fundamental form of the surface, $a_{\alpha\beta}$ are given by

$$a_{11} = \left(\frac{\partial x^1}{\partial u^1}\right)^2 + \left(\frac{\partial x^2}{\partial u^1}\right)^2 + \left(\frac{\partial x^3}{\partial u^1}\right)^2 = 1 + a^2$$
$$a_{22} = \left(\frac{\partial x^1}{\partial u^2}\right)^2 + \left(\frac{\partial x^2}{\partial u^2}\right)^2 + \left(\frac{\partial x^3}{\partial u^2}\right)^2 = (u^1)^2$$
$$a_{12} = \frac{\partial x^1}{\partial u^1}\frac{\partial x^1}{\partial u^2} + \frac{\partial x^2}{\partial u^1}\frac{\partial x^2}{\partial u^2} + \frac{\partial x^3}{\partial u^1}\frac{\partial x^3}{\partial u^2} = 0.$$

Now **A** and **B** vectors are given by

$$\mathbf{A} = \frac{\partial \mathbf{r}}{\partial u^1} = (\cos u^2, \sin u^2, a)$$
$$\mathbf{B} = \frac{\partial \mathbf{r}}{\partial u^2} = (-u^1 \sin u^2, u^1 \cos u^2, 0).$$

Thus, the unit surface normal is given by

$$\boldsymbol{\xi} = \frac{1}{|\mathbf{A} \times \mathbf{B}|}(\mathbf{A} \times \mathbf{B}) = \left(-\frac{a \cos u^2}{\sqrt{1+a^2}}, -\frac{a \sin u^2}{\sqrt{1+a^2}}, \frac{1}{\sqrt{1+a^2}}\right),$$

that is, the normal direction to the cone is independent of u^1. Since the curves u^2 = constant are the generating straight lines of the cone, $\boldsymbol{\xi}$ is constant along any of those generators; hence the tangent planes to the cone at the points of any generating straight line coincide.

7.3 Tensor Derivative

In this section, we are introducing new tensors by differentiation of given vector fields. Consider a curve $\mathcal{C}$ lying on a given surface $\mathcal{S}$. If A^i is a component of a space vector, defined along the curve $\mathcal{C}$. If t is a parameter along $\mathcal{C}$, the intrinsic derivative $\frac{\delta A^i}{\delta t}$ of A^i is given by

$$\frac{\delta A^i}{\delta t} = \frac{dA^i}{dt} + {}_g\left\{ {i \atop j \ \ k} \right\} A^j \frac{dx^k}{dt}. \tag{7.20}$$

The Christoffel symbols of second kind in Eq. (7.20) refer to the space co-ordinates x^i and are formed from the metric coefficients g_{ij}. This is indicated by the prefix g on the symbol. Again, if we consider a surface vector A^α, defined along the same curve $\mathcal{C}$, we can form the intrinsic derivative with respect to the surface variables, namely,

$$\frac{\delta A^\alpha}{\delta t} = \frac{dA^\alpha}{dt} + {}_a\begin{Bmatrix} \alpha \\ \beta \;\; \gamma \end{Bmatrix} A^\beta \frac{du^\gamma}{dt}, \tag{7.21}$$

where the Christoffel symbols of second kind in Eq. (7.21) are formed from the metric coefficients $a_{\alpha\beta}$ associated with the Gaussian surface co-ordinates u^α. The corresponding formulas for the intrinsic derivatives of the covariant vectors with components A_i and A_α are

$$\frac{\delta A_i}{\delta t} = \frac{dA_i}{dt} - {}_g\begin{Bmatrix} k \\ i \;\; j \end{Bmatrix} A_k \frac{dx^i}{dt}$$

and

$$\frac{\delta A_\alpha}{\delta t} = \frac{dA_\alpha}{dt} - {}_a\begin{Bmatrix} \gamma \\ \alpha \;\; \beta \end{Bmatrix} A_\gamma \frac{du^\beta}{dt} \tag{7.22}$$

A geometric interpretation of these formulas is at hand where A^i and A^α are such that

$$\frac{\delta A^i}{\delta t} = 0 \;\text{ and }\; \frac{\delta A^\alpha}{\delta t} = 0. \tag{7.23}$$

In the first case of Eq. (7.23) the vectors A^i form a parallel field with respect to $\mathcal{C}$, considered as a space curve, where as in the later case of Eq. (7.23), the vectors A^α form a parallel field with respect to $\mathcal{C}$ regarded as a surface curve.

Consider a tensor field T^i_α, which is a contravariant vector with respect to a transformation of space co-ordinates x^i and a covariant vector relative to a transformation of the surface co-ordinates u^α. For example, a field of this type is a tensor x^i_α introduced in Eq. (7.8). If T^i_α is defined along a surface curve $\mathcal{C}$, with t as parameter, then T^i_α is a function of t. We form an invariant

$$\phi(t) = T^i_\alpha A_i B^\alpha,$$

where a parallel vector field A_i along $\mathcal{C}$, regarded as a space curve, and a parallel vector field B^α along $\mathcal{C}$, viewed as a surface curve. The derivative of $\phi(t)$ with respect to the parameter t is given by

$$\frac{d\phi(t)}{dt} = \frac{dT^i_\alpha}{dt} A_i B^\alpha + T^i_\alpha \frac{dA_i}{dt} B^\alpha + T^i_\alpha A_i \frac{dB^\alpha}{dt}, \tag{7.24}$$

which is obviously an invariant relative to both the space and surface co-ordinates. But since the fields $A_i(t)$ and $B^\alpha(t)$ are parallel,

$$\frac{dA_i}{dt} = {}_g\begin{Bmatrix} k \\ i \;\; j \end{Bmatrix} A_k \frac{dx^i}{dt} \;\text{ and }\; \frac{dB^\alpha}{dt} = -{}_a\begin{Bmatrix} \alpha \\ \beta \;\; \gamma \end{Bmatrix} B^\beta \frac{du^\gamma}{dt}$$

therefore, Eq. (7.24) becomes

$$\frac{d\phi}{dt} = \left[\frac{dT^i_\alpha}{dt} + {}_g\begin{Bmatrix} i \\ j \ \ k \end{Bmatrix} T^j_\alpha \frac{dx^k}{dt} - {}_a\begin{Bmatrix} \mu \\ \alpha \ \ \gamma \end{Bmatrix} T^i_\mu \frac{du^\gamma}{dt}\right] A_i B^\alpha. \tag{7.25}$$

Since this is invariant for an arbitrary choice of parallel fields A_i and B^α, by quotient law, the expression in the bracket of Eq. (7.25) is a tensor of the same character as T^i_α with respect to the parameter t, and we write

$$\frac{\delta T^i_\alpha}{\delta t} = \frac{dT^i_\alpha}{dt} + {}_g\begin{Bmatrix} i \\ j \ \ k \end{Bmatrix} T^j_\alpha \frac{dx^k}{dt} - {}_a\begin{Bmatrix} \mu \\ \alpha \ \ \gamma \end{Bmatrix} T^i_\mu \frac{du^\gamma}{dt}. \tag{7.26}$$

If the field T^i_α is defined over the entire surface $\mathcal{S}$, its components are functions of us and we have a tensor field over the surface. Also, if $\mathcal{C}$ is any curve on the surface, we can argue that

$$\begin{aligned}\frac{\delta T^i_\alpha}{\delta t} &= \left[\frac{\partial T^i_\alpha}{\partial u^\gamma} + {}_g\begin{Bmatrix} i \\ j \ \ k \end{Bmatrix} T^j_\alpha \frac{\partial x^k}{\partial u^\gamma} - {}_a\begin{Bmatrix} \mu \\ \gamma \ \ \alpha \end{Bmatrix} T^i_\mu\right] \frac{du^\gamma}{dt} \\ &= \left[\frac{\partial T^i_\alpha}{\partial u^\gamma} + {}_g\begin{Bmatrix} i \\ j \ \ k \end{Bmatrix} T^j_\alpha x^k_\gamma - {}_a\begin{Bmatrix} \mu \\ \gamma \ \ \alpha \end{Bmatrix} T^i_\mu\right] \frac{du^\gamma}{dt}\end{aligned}$$

is a tensor field. Since $\frac{du^\gamma}{dt}$ is an arbitrary surface vector (for $\mathcal{C}$ is arbitrary) and since the right-hand side of this last equation is a tensor, we conclude that

$$T^i_{\alpha,\gamma} = \frac{\partial T^i_\alpha}{\partial u^\gamma} + {}_g\begin{Bmatrix} i \\ j \ \ k \end{Bmatrix} T^j_\alpha x^k_\gamma - {}_a\begin{Bmatrix} \mu \\ \alpha \ \ \gamma \end{Bmatrix} T^i_\mu \tag{7.27}$$

is a tensor which is singly contravariant in the space co-ordinates and doubly covariant in the surface co-ordinates. We shall call $T^i_{\alpha,\gamma}$, the tensor derivative of T^i_α with respect to u^γ. Similarly, the tensor derivative of $T^i_{\alpha\beta}$ with respect to u^γ is given by

$$T^i_{\alpha\beta,\gamma} = \frac{\partial T^i_\alpha}{\partial u^\gamma} + {}_g\begin{Bmatrix} i \\ j \ \ k \end{Bmatrix} T^j_{\alpha\beta} x^k_\gamma - {}_a\begin{Bmatrix} \mu \\ \beta \ \ \gamma \end{Bmatrix} T^i_{\alpha\mu} - {}_a\begin{Bmatrix} \mu \\ \alpha \ \ \gamma \end{Bmatrix} T^i_{\mu\beta}. \tag{7.28}$$

If the surface co-ordinates at any point of the surface are geodesics and the space co-ordinates are orthogonal Cartesian, we see that at that point the tensor derivatives reduce to ordinary derivatives. This leads us to conclude that the operations of tensor differentiation of products and sums follow the usual rules and that the tensor derivatives of $g_{ij}, a_{\alpha\beta}, \varepsilon_{\alpha\beta}$ and their associated tensors vanish. Accordingly, they behave as constants in the tensor differentiation.

7.3.1 The Second Fundamental Form of a Surface

Before we introduce the new quadratic form, we must point out the relations, descriptions in the study of surfaces. Let us take the tensor derivative of the tensor x^i_α, representing the components of the surface base vectors $\mathbf{a}_\alpha$, we find from Eq. (7.27) that

$$x^i_{\alpha,\beta} = \frac{\partial^2 x^i}{\partial u^\alpha \ \partial u^\beta} + {}_g\left\{ \begin{matrix} i \\ j \quad k \end{matrix} \right\} x^j_\alpha x^k_\beta - {}_a\left\{ \begin{matrix} \mu \\ \alpha \quad \beta \end{matrix} \right\} x^i_\mu \tag{7.29}$$

from which it follows that $x^i_{\alpha,\beta} = x^i_{\beta,\alpha}$. Tensor differentiation of the relation $a_{\alpha\beta}$ $= g_{mn}x^m_\alpha x^n_\beta$ gives

$$g_{mn}x^m_{\alpha,\gamma}x^n_\beta + g_{mn}x^m_\alpha x^n_{\beta,\gamma} = 0. \tag{7.30}$$

Interchanging α, β, γ cyclically, we get

$$g_{mn}x^m_{\beta,\alpha}x^n_\gamma + g_{mn}x^m_\beta x^n_{\gamma,\alpha} = 0 \tag{7.31}$$

and

$$g_{mn}x^m_{\gamma,\beta} \ \ x^n_\alpha + g_{mn}x^m_\gamma x^n_{\alpha,\beta} = 0. \tag{7.32}$$

Adding, Eqs. (7.31) and (7.32) and subtracting (7.30), we get

$$g_{mn}x^m_{\alpha,\beta}x^n_\gamma = 0; \quad \text{as } x^m_{\alpha,\beta} = x^m_{\beta,\alpha}. \tag{7.33}$$

This orthogonality relation interpreted geometrically, states that $x^i_{\alpha,\beta}$, from the point of view of the space co-ordinates is a space vector normal to the surface, and hence, it is codirectional with the normal vector ξ^i. Consequently, there must exist a set of functions $\eta_{\alpha\beta}$ such that

$$x^i_{\alpha,\beta} = \eta_{\alpha\beta}\xi^i, \tag{7.34}$$

where $\eta_{\alpha\beta} = b_{\alpha\beta}$. The quantities $b_{\alpha\beta}$ are the components of a symmetric surface tensor, and the differential quadratic form

$$B \equiv b_{\alpha\beta}du^\alpha du^\beta \tag{7.35}$$

is the desired second fundamental form. Equation (7.34) is known as *Gauss's formula.*

Deduction 7.3.1 Equivalence of the two definitions of the second fundamental quadratic form: Let us consider a surface $\mathcal{S}$ embedded in E^3 with the first fundamental form given by

$$ds^2 = a_{\alpha\beta}du^\alpha du^\beta$$

where (u^1, u^2) are the Gaussian co-ordinates for the surface $\mathcal{S}$ defined by

$$x^i = x^i(u^1, u^2);\ i = 1, 2, 3;$$

x^i being the Cartesian co-ordinates of E^3. Let $\mathbf{r}$ be the position vector of any point P whose surface co-ordinates are (u^1, u^2) and space co-ordinates are (x^1, x^2, x^3). Now,

$$\mathbf{a}_\alpha = \frac{\partial \mathbf{r}}{\partial u^\alpha};\quad \alpha = 1, 2;$$

are the surface base vectors and

$$\mathbf{b}_\alpha = \frac{\partial \mathbf{r}}{\partial x^\alpha};\quad \alpha = 1, 2, 3;$$

are the space base vectors and they are usually related by

$$\mathbf{a}_\alpha = \frac{\partial \mathbf{r}}{\partial u^\alpha} = \frac{\partial \mathbf{r}}{\partial x^i}\frac{\partial x^i}{\partial u^\alpha} = x^i_\alpha \mathbf{b}_i. \tag{7.36}$$

Let $\boldsymbol{\xi}$ be the unit vector at P normal to the tangent plane, so, we get

$$\boldsymbol{\xi} = \xi^i b_i \text{ and } \boldsymbol{\xi} \cdot \boldsymbol{\xi} = 1;\ \boldsymbol{\xi} \cdot \mathbf{a}_\alpha = 0. \tag{7.37}$$

Since the vectors $\boldsymbol{\xi}$ and $\mathbf{a}_\alpha$ are orthogonal, so

$$\boldsymbol{\xi} \cdot \frac{\partial \mathbf{r}}{\partial u^\alpha} = 0 \quad \text{and} \quad \boldsymbol{\xi} \cdot \frac{\partial \mathbf{r}}{\partial u^\beta} = 0.$$

Differentiating these two scalar products with respect to u^α and u^β, respectively, and then adding, we get

$$\frac{\partial \boldsymbol{\xi}}{\partial u^\beta} \cdot \frac{\partial \mathbf{r}}{\partial u^\alpha} + \boldsymbol{\xi} \cdot \frac{\partial^2 \mathbf{r}}{\partial u^\beta \partial u^\alpha} + \frac{\partial \boldsymbol{\xi}}{\partial u^\alpha} \cdot \frac{\partial \mathbf{r}}{\partial u^\beta} + \boldsymbol{\xi} \cdot \frac{\partial^2 \mathbf{r}}{\partial u^\alpha u^\beta} = 0$$

or

$$\frac{1}{2}\left[\frac{\partial \boldsymbol{\xi}}{\partial u^\beta} \cdot \frac{\partial \mathbf{r}}{\partial u^\alpha} + 2\boldsymbol{\xi} \cdot \frac{\partial^2 \mathbf{r}}{\partial u^\alpha \partial u^\beta} + \frac{\partial \boldsymbol{\xi}}{\partial u^\alpha} \cdot \frac{\partial \mathbf{r}}{\partial u^\beta}\right] = 0$$

or

$$-\frac{1}{2}\left[\frac{\partial \boldsymbol{\xi}}{\partial u^\beta} \cdot \frac{\partial \mathbf{r}}{\partial u^\alpha} + \frac{\partial \boldsymbol{\xi}}{\partial u^\alpha} \cdot \frac{\partial \mathbf{r}}{\partial u^\beta}\right] = \boldsymbol{\xi} \cdot \frac{\partial^2 \mathbf{r}}{\partial u^\alpha \partial u^\beta}$$

or

$$b_{\alpha\beta} = \boldsymbol{\xi} \cdot \frac{\partial^2 \mathbf{r}}{\partial u^\alpha \partial u^\beta} \quad \text{using Eq. (7.18).} \tag{7.38}$$

Now, differentiating the relation $a_\alpha = x^i_\alpha b_i$ with respect to u^β, we get

$$\frac{\partial^2 \mathbf{r}}{\partial u^\alpha \partial u^\beta} = b_i \frac{\partial x^i_\alpha}{\partial u^\beta} + \frac{\partial b_i}{\partial u^\beta} x^i_\alpha = b_i \frac{\partial^2 x^i_\alpha}{\partial u^\alpha \partial u^\beta} + \frac{\partial b_i}{\partial x^j} x^i_\alpha x^j_\beta$$

$$= b_h \left[\frac{\partial^2 x^h}{\partial u^\alpha \partial u^\beta} + {}_g\left\{ \begin{matrix} h \\ i \quad j \end{matrix} \right\} x^i_\alpha x^j_\beta \right], \tag{7.39}$$

where in the last step, we made use of formula

$$\frac{\partial b_i}{\partial x^j} = {}_g\left\{ \begin{matrix} h \\ i \quad j \end{matrix} \right\} b_h$$

for the derivatives of the base vector b_i. Using Eq. (7.29),

$$x^i_{\alpha,\beta} = \frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta} + {}_g\left\{ \begin{matrix} i \\ j \quad k \end{matrix} \right\} x^j_\alpha x^k_\beta - {}_a\left\{ \begin{matrix} \mu \\ \alpha \quad \beta \end{matrix} \right\} x^i_\mu$$

expression (7.39) reduces to

$$\frac{\partial^2 \mathbf{r}}{\partial u^\alpha \partial u^\beta} = \left[x^h_{\alpha,\beta} + {}_a\left\{ \begin{matrix} \gamma \\ \alpha \quad \beta \end{matrix} \right\} x^h_\gamma \right] b_h$$

$$= b_{\alpha\beta} \xi^h b_h + {}_a\left\{ \begin{matrix} \gamma \\ \alpha \quad \beta \end{matrix} \right\} x^h_\gamma b_h; \quad \text{as } x^i_{\alpha,\beta} = b_{\alpha\beta} \xi^i$$

$$= b_{\alpha\beta} \boldsymbol{\xi} + {}_a\left\{ \begin{matrix} \gamma \\ \alpha \quad \beta \end{matrix} \right\} x^h_\gamma \mathbf{a}_\gamma; \quad \text{using Eq. (7.37).} \tag{7.40}$$

Multiplying Eq. (7.39) scalarly by $\boldsymbol{\xi}$, we get

$$\boldsymbol{\xi} \cdot \frac{\partial^2 \mathbf{r}}{\partial u^\alpha \partial u^\beta} = b_{\alpha\beta} \boldsymbol{\xi} \cdot \boldsymbol{\xi} + {}_a\left\{ \begin{matrix} \gamma \\ \alpha \quad \beta \end{matrix} \right\} x^h_\gamma \mathbf{a}_\gamma \cdot \boldsymbol{\xi}$$

$$= b_{\alpha\beta} + 0 = b_{\alpha\beta}; \text{ as } \mathbf{a}_\gamma \cdot \boldsymbol{\xi} = 0 \tag{7.41}$$

This establishes the equivalence of the two definitions Eqs. [(7.38) and (7.41)] of the second fundamental quadratic form.

Deduction 7.3.2 The integrability conditions: In order to get insight into the significance of the tensor $b_{\alpha\beta}$, let us examine the Gauss's formula Eq. (7.34), as

$$x^i_{\alpha,\beta} = b_{\alpha\beta} \;\; \xi^i,$$

where

$$x^i_{\alpha,\beta} = \frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta} + {}_g\left\{ \begin{matrix} i \\ j \quad k \end{matrix} \right\} x^j_\alpha x^k_\beta - {}_a\left\{ \begin{matrix} \delta \\ \alpha \quad \beta \end{matrix} \right\} x^i_\delta$$

$$\eta_i = \frac{1}{2} \varepsilon^{\alpha\beta} \varepsilon_{ijk} x^j_\alpha x^k_\beta \text{ and } x^i_\alpha = \frac{\partial x^i}{\partial u^\alpha}.$$

If we insert these expressions in Eq. (7.34), we obtain a set of second order partial differential equations, in which the dependent variables x^i are the functions of the surface co-ordinates u^α. The coefficients in the differential equations are functions of the metric coefficients g_{ij} of the manifold in which the surface $\mathcal{S}$, defined by

$$x^i = x^i(u^1, u^2);\; i = 1, 2 \tag{7.42}$$

are immersed; they are also functions of

$$a_{\alpha\beta} = g_{ij} \frac{\partial x^i}{\partial u^\alpha} \frac{\partial x^j}{\partial u^\beta}$$

and $b_{\alpha\beta}$. If Eq. (7.42) is given, we can compute $a_{\alpha\beta}$ and $b_{\alpha\beta}$, insert the approximate expressions in Eq. (7.34) and, of course, Eq. (7.34) will be satisfied identically.

Conversely, if the functions $a_{\alpha\beta}$ and $b_{\alpha\beta}$ are given, Eq. (7.34) will become the equation of conditions and in general they will have no solutions yielding Eq. (7.42) of the surface $\mathcal{S}$. In order that the terms $a_{\alpha\beta}$ and $b_{\alpha\beta}$ are related to some surface, it is necessary that x^i satisfy the conditions of integrability

$$\frac{\partial^2 x^i_\alpha}{\partial u^\gamma \partial u^\beta} = \frac{\partial^2 x^i_\alpha}{\partial u^\beta \;\partial u^\gamma}, \tag{7.43}$$

whenever the function x^α_i are continuously differentiable of degree 2. Equation (7.43) is equivalent to

$$x^i_{\alpha,\beta\gamma} - x^i_{\alpha,\gamma\beta} = R^\mu_{\alpha\beta\gamma} x^i_\mu, \tag{7.44}$$

where $R^\mu_{\alpha\beta\gamma}$ is the Riemann second kind curvature tensor, of type $(1, 3)$, formed with the coefficients $a_{\alpha\beta}$ of the first fundamental quadratic form. Equation (7.44) involve third order partial derivatives of the co-ordinates x^i and we shall assume from now on that the functions entering in Eq. (7.42) are continuously thrice differentiable.

We shall see that the conditions of integrability Eq. (7.44) impose certain restrictions on the possible choice of functions $b_{\alpha\beta}$ and $a_{\alpha\beta}$. These restrictive conditions are known as the equations of *Gauss* and *Codazzi.*

7.4 Structure Formulas for Surfaces

Two fundamental sets of relationships involve the parts of the moving triad $(\mathbf{A}, \mathbf{B}, \boldsymbol{\xi})$ of a surface at any point P. Any vector bound at P can be represented as a linear combinations of those vectors. If the derive of those vectors exist, then the corresponding combinations are called the formulae of Frenet.

7.4.1 Equations of Weingarten

Weingarten's formula is an expression for the derivatives of the unit normal vector ξ^i to the surface $\mathcal{S}$. Since ξ^i, the contravariant components of the normal vector $\boldsymbol{\xi}$ is a unit vector, so,

$$g_{ij}\xi^i\xi^j = 1. \tag{7.45}$$

Here we have to show that the partial derivatives of this vector with respect to the co-ordinates u^1 and u^2 can be represented as a linear combination of vectors. Using the tensor derivative formula

$$\xi^i_{,\alpha} = \frac{\partial \xi^i}{\partial u^\alpha} + {}_g\left\{ \begin{matrix} i \\ j \quad k \end{matrix} \right\} \xi^i x^k_\alpha$$

we have from Eq. (7.45)

$$g_{ij}\xi^i_{,\alpha}\xi^j + g_{ij}\xi^i\xi^j_{,\alpha} = 0 \quad \Rightarrow \quad g_{ij}\xi^i\xi^j_{,\alpha} = 0. \tag{7.46}$$

Geometrically, Eq. (7.46) shows that $\xi^j_{,\alpha}$, from the point of view of space co-ordinates, considered as a space vector, is orthogonal to the unit normal ξ^i, and hence, it lies in the tangent plane to the surface. Accordingly, it can be expressed as a linear form in the base vectors x^i_α, as

$$\xi^i_{,\alpha} = c^\beta_\alpha x^i_\beta, \tag{7.47}$$

where c^β_αs are to be determined. Since ξ^i is normal to the surface, the tensor derivative of the orthogonality relation $g_{ij}x^i_\alpha\xi^j = 0$ gives

$$g_{ij}x^i_{\alpha,\beta}\xi^j + g_{ij}x^i_\alpha\xi^j_{,\beta} = 0$$

or

$$g_{ij}b_{\alpha\beta}\xi^i\xi^j + g_{ij}x^i_\alpha\xi^j_{,\beta} = 0; \text{ as } x^i_{\alpha,\beta} = b_{\alpha\beta}\xi^i$$

or

$$b_{\alpha\beta} + g_{ij}x^i_\alpha c^\gamma_\beta x^j_\gamma = 0$$

or

$$b_{\alpha\beta} + a_{\alpha\gamma}c^\gamma_\beta = 0; \quad \text{as } a_{\alpha\beta} = g_{ij}x^i_\alpha x^j_\beta$$

or

$$a^{\alpha\gamma}b_{\alpha\beta} + a^{\alpha\gamma}\, a_{\alpha\gamma}c^\gamma_\beta = 0$$

or

$$c^\gamma_\beta = -a^{\alpha\gamma}\, b_{\alpha\beta}, \text{ as } a^{\alpha\beta}a_{\beta\gamma} = \delta^\alpha_\gamma. \tag{7.48}$$

Consequently, Eq. (7.47) reduces to

$$\xi^i_{,\alpha} = -a^{\gamma\beta} b_{\gamma\alpha} x^i_{\beta}. \tag{7.49}$$

These equations are known as *Weingarten's formula,* which can be used to derive the equations of Gauss and Codazzi. We have noticed that the formulae of Weingarten are the analogue of the formulae of Frenet. If we write

$$C_{\alpha\beta} = g_{ij}\ \xi^i_{,\alpha}\ \xi^j_{,\beta}, \tag{7.50}$$

we see that $C_{\alpha\beta}$ is symmetric tensor of order $(0, 2)$ and we call the quadratic form

$$C \equiv C_{\alpha\beta} du^\alpha du^\beta \tag{7.51}$$

the third fundamental form of the surface. Using Weingarten formula Eq. (7.49), we get

$$\begin{aligned} C_{\alpha\beta} = g_{ij}\ \xi^i_{,\alpha}\xi^j_{,\beta} &= g_{ij}\left(a^{\eta\gamma} b_{\eta\alpha} x^i_{\gamma}\right)\left(a^{\mu\nu} b_{\mu\beta} x^j_{\nu}\right) \\ &= a_{\gamma\nu} a^{\eta\gamma} b_{\eta\alpha} a^{\mu\nu} b_{\mu\beta} = a^{\eta\mu} b_{\eta\alpha} b_{\mu\beta}. \end{aligned}$$

This is a relation between three fundamental forms on a surface. This relation also shows that the third fundamental form is not actually a fundamental form, because this can be obtained from first and second fundamental forms.

Deduction 7.4.1 From the Weingarten formula Eq. (7.49), we get

$$\xi^i_{,\alpha} = -a^{\beta\gamma} b_{\beta\alpha} x^i_{\gamma} = 0; \quad \text{if } b_{\alpha\beta} = 0.$$

Hence, ξ^i is a constant vector. Hence, the surface is a plane. Therefore, if $b_{\alpha\beta} = 0$, then the normal vector to the surface is constant which means that the surface is a plane.

7.4.2 Equations of Gauss and Codazzi

The question arises whether, if functions $a_{\alpha\beta}(u^1, u^2)$ and $b_{\alpha\beta}(u^1, u^2)$ are given, there always exists a surface such that the given functions are the coefficients of the corresponding fundamental forms. In the case of functions of class ≥ 3 the answer to this problem is negative unless certain integrability conditions are satisfied which we will now derive. Let us consider the tensor derivative of Eq. (7.34), we get

$$\begin{aligned} x^i_{\alpha,\beta\gamma} &= b_{\alpha\beta,\gamma}\ \ \xi^i + b_{\alpha\beta}\xi^i_{,\gamma} \\ &= b_{\alpha\beta,\gamma}\ \ \xi^i - b_{\alpha\beta} a^{\mu\sigma} b_{\mu\gamma} x^i_{\sigma};\ \text{using Eq. (7.49)} \end{aligned}$$

or

$$
\begin{aligned}
x^i_{\alpha,\beta\gamma} - x^i_{\alpha,\gamma\beta} &= (b_{\alpha\beta,\gamma} - b_{\alpha\gamma,\beta})\xi^i + b_{\alpha\gamma}a^{\mu\sigma}b_{\mu\beta}x^i_\sigma - b_{\alpha\beta}a^{\mu\sigma}b_{\mu\gamma}x^i_\alpha \\
&= (b_{\alpha\beta,\gamma} - b_{\alpha\gamma,\beta})\xi^i + a^{\mu\sigma}(b_{\alpha\gamma}b_{\mu\beta} - b_{\alpha\beta}b_{\mu\gamma})x^i_\sigma
\end{aligned}
$$

or

$$(b_{\alpha\beta,\gamma} - b_{\alpha\gamma,\beta})\xi^i - a^{\mu\sigma}(b_{\alpha\beta}b_{\mu\gamma} - b_{\alpha\gamma}b_{\mu\beta})x^i_\sigma \equiv R^\sigma_{\alpha\beta\gamma}x^i_\sigma, \tag{7.52}$$

if we write

$$x^i_{\alpha,\beta\gamma} - x^i_{\alpha,\gamma\beta} = R^\sigma_{\alpha\beta\gamma}x^i_\sigma,$$

where $R^\sigma_{\alpha\beta\gamma}$ is the *Riemann curvature of the surface.* Multiplying Eq. (7.52) by ξ_i and using the fact that $\xi^i\xi_i = 1$ and $x^i_\sigma\xi_i = 0$, we get

$$b_{\alpha\beta,\gamma} - b_{\alpha\gamma,\beta} = 0. \tag{7.53}$$

These equations are called the *Codazzi equations* of the surface, which will constitute all integrability conditions of the formulae of Weingarten. Again, multiplying Eq. (7.52) by $g_{ij}x^j_\rho$, we get

$$a_{\rho\sigma}R^\sigma_{\alpha\beta\gamma} = a_{\rho\sigma}b_{\alpha\gamma}a^{\mu\sigma}b_{\mu\beta} - a_{\rho\sigma}b_{\alpha\beta}a^{\mu\sigma}b_{\mu\gamma}$$

or

$$R_{\rho\alpha\beta\gamma} = b_{\rho\beta}b_{\alpha\gamma} - b_{\rho\gamma}b_{\alpha\beta}. \tag{7.54}$$

These equations are called the *Gauss equations of surface.* Since α, β assume values $1, 2$ and $b_{\alpha\beta} = b_{\beta\alpha}$, we see that there are two independent equations of Codazzi given by

$$b_{\alpha\alpha,\beta} - b_{\alpha\beta,\alpha} = 0;\ \alpha \neq \beta \tag{7.55}$$

or

$$\frac{\partial b_{\alpha\alpha}}{\partial u^\beta} - \frac{\partial b_{\alpha\beta}}{\partial u^\alpha} - b_{\alpha\mu}\left\{ \begin{matrix} \mu \\ \alpha\ \ \beta \end{matrix} \right\} + b_{\mu\beta}\left\{ \begin{matrix} \mu \\ \alpha\ \ \alpha \end{matrix} \right\} = 0;\ \alpha \neq \beta$$

and only one independent equation of Gauss is

$$b = b_{11}b_{22} - b_{12}^2 = R_{1212};\quad \text{where, } b = |b_{ij}|. \tag{7.56}$$

This equation relates the coefficients $b_{\alpha\beta}$ and $a_{\alpha\beta}$ in the two fundamental quadratic forms. Thus, it follows from the theory of partial differential equation that if the tensors $a_{\alpha\beta}$ and $b_{\alpha\beta}$ are the fundamental tensors of the surface $\mathcal{S} : x^i = x^i\left(u^1, u^2\right)$, Eqs. (7.53) and (7.56) are satisfied. Conversely, if the two sets of functions $a_{\alpha\beta}$ and $b_{\alpha\beta}$ satisfying Eqs. (7.53) and (7.56) are prescribed, and if $a_{\alpha\beta}du^\alpha du^\beta$ is a positive definite form, then the surface $\mathcal{S}$ is determined (locally) to within a rigid body space.

Deduction 7.4.2 It is clear that, when $\beta = \gamma$, we get identities from $b_{\alpha\beta,\gamma} - b_{\alpha\gamma,\beta} = 0$ of the form $0 = 0$. Again the equation with $\beta = p$ and $\gamma = q$ is equivalent to the equation $\beta = q$ and $\gamma = p$. Hence, there are only two independent equations, as

$$b_{11,2} - b_{12,1} = 0 \text{ and } b_{21,2} - b_{22,1} = 0.$$

These two equations are equivalent to

$$b_{\alpha\alpha,\beta} - b_{\alpha\beta,\alpha} = 0;\ \alpha \neq \beta.$$

This shows that, $b_{\alpha\beta,\gamma} - b_{\alpha\gamma,\beta} = 0$ is equivalent to $b_{\alpha\alpha,\beta} - b_{\alpha\beta,\alpha} = 0;\ \alpha \neq \beta$.

Deduction 7.4.3 From Codazzi equations, we get $b_{\alpha\beta,\gamma} - b_{\alpha\gamma,\beta} = 0;\ \alpha \neq \beta$. Let $b_{\alpha\beta} = \lambda a_{\alpha\beta}$, then

$$b_{\alpha\beta,\gamma} = \lambda_{,\gamma} a_{\alpha\beta} \text{ and } b_{\alpha\gamma,\beta} = \lambda_{,\beta} a_{\alpha\gamma}$$

or

$$\lambda_{,\gamma} a_{\alpha\beta} - \lambda_{,\beta} a_{\alpha\gamma} = 0$$

or

$$\lambda_{,\gamma} a_{\alpha\beta} a^{\alpha\beta} - \lambda_{,\beta} a_{\alpha\gamma} a^{\alpha\beta} = 0$$

or

$$2\lambda_{,\gamma} - \lambda_{,\beta}\delta^{\beta}_{\gamma} = 0 \Rightarrow \lambda_{,\gamma} = 0,$$

which implies that λ is a constant. Thus, if $b_{\alpha\beta} = \lambda a_{\alpha\beta}$, where λ is a scalar, then λ is a constant.

7.4.3 Curvatures of a Surface

From Eq. (6.52), the total curvature κ of a surface is given by

$$\kappa = \frac{R_{1212}}{a};\ a = a_{11}a_{22} - a_{12}^2 = |a_{ij}|. \tag{7.57}$$

Using the Gauss equation [Eq. (7.56)], the total curvature κ is given by

$$\kappa = \frac{b_{11}b_{22} - b_{12}^2}{a_{11}a_{22} - a_{12}^2} = \frac{b}{a}. \tag{7.58}$$

Thus, the Gaussian curvature is equal to the quotient of the determinants of the second and first fundamental quadratic forms. A surface $\mathcal{S}$ is said to be *developable* if $\kappa = 0$ identically at each point of the surface. A developable surface is a surface which can be developed into a plane. We now define another important invariant H, given by the formula

$$H \equiv \frac{1}{2} a^{\alpha\beta} b_{\alpha\beta}, \tag{7.59}$$

which is called the *mean curvature of the surface.* The invariants κ, H are connected with the ordinary curvatures of certain curves formed by taking normal sections of the surface (Section 8.2).

7.4.4 Minimal Surfaces

If at every point of a surface of class ≥ 2, the total curvature

$$H \equiv \frac{1}{2} a^{\alpha\beta} b_{\alpha\beta} = 0, \tag{7.60}$$

then the surface $\mathcal{S}$ is called a minimal surface. Minimal surfaces derive their name from the fact that they are surfaces of minimum area satisfying given boundary conditions. They are illustrated by the shapes of thin soap films in equilibrium, with the air pressure the same on both sides. If this property of least area be taken as defining minimal surfaces, the use of the calculus of variations leads to the vanishing of the first curvature as an equivalent property. Later, we have to prove the following very important properties:

(i) Minimal surfaces cannot have elliptical points.
(ii) The lines of curvature on a minimal surfaces form an isometric system.
(iii) The asymptotic lines on a minimal surface form an orthogonal system.
(iv) Asymptotic lines on a minimal surface form on orthogonal system.

Theorem 7.4.1 (Theorema Egregium of Gauss)*: The Gaussian curvature κ is an intrinsic property of a surface, depending only on the first fundamental form and their derivatives but is independent of the second fundamental form.*

Proof: The total curvature κ is given by

$$\kappa = \frac{b}{a} = \frac{b_{11}b_{22} - b_{12}^2}{a_{11}a_{22} - a_{12}^2}.$$

Thus, the total curvature, as obtained, depends upon the first fundamental form as well as the second fundamental form. But we have the equations of Gauss,

$$R_{\alpha\beta\gamma\delta} = b_{\alpha\gamma}b_{\beta\delta} - b_{\alpha\delta}b_{\beta\gamma};$$

from which it follows that $R_{1212} = b_{11}b_{22} - b_{12}^2$. Therefore, $\kappa = \frac{R_{1212}}{a}$. But the Riemann curvature tensor depends only upon the first fundamental form $a_{\alpha\beta}$ together with their derivatives. Hence, the theorem.

The practical question of whether inhabitants of a fog-enshrouded planet could, solely by measuring distances on the surface of the planet, determine its curvature, is answered in the affirmative by this theorem. Now,

(i) If two surfaces are locally isometric, then this theorem shows that, the Gaussian curvatures are identical.

(ii) Minding's theorem states that, if two surfaces are the same constant Gaussian curvature, then they are locally isometric.

(iii) In the case of constant Gaussian curvature, Beltrami's theorem tells us that there is a parameterisation for $\mathcal{S}$ for which the first fundamental form takes the form

Metric	Gaussian curvature
$ds^2 = a^2(dx^1)^2 + a^2 \sinh^2 x^1 (dx^2)^2$	$\kappa = -\frac{1}{a^2} < 0$
$ds^2 = (dx^1)^2 + (dx^2)^2$	$\kappa = 0$
$ds^2 = a^2(dx^1)^2 + a^2 \sin^2 x^1 (dx^2)^2$	$\kappa = \frac{1}{a^2} > 0$

(iv) From Eq. (7.54) we see that in case of surfaces of vanishing curvature (for example planes, cones, cylinders) all components of the curvature tensor are zero.

EXAMPLE 7.4.1 *Show that*

$$b_{\alpha\beta} = g_{ij} x^i_{\alpha,\beta} \xi^j = \frac{1}{2} \varepsilon^{\gamma\delta} \varepsilon_{ijk} x^i_{\alpha,\beta} x^j_\gamma x^k_\delta.$$

Solution: Equation (7.13) tells us $A^i = x^i_\alpha A^\alpha$; $B^j = x^j_\beta B^\alpha$. If θ be the angle between the vectors A^i and B^j, we can write

$$AB \sin\theta = \varepsilon_{\alpha\beta} A^\alpha B^\beta.$$

If ξ is the unit surface normal, then

$$\xi_i = \frac{\varepsilon_{ijk} x^j_\alpha A^\alpha x^k_\beta B^\beta}{\varepsilon_{\alpha\beta} A^\alpha B^\beta} \quad \Rightarrow \quad \left(\xi_i \varepsilon_{\alpha\beta} - \varepsilon_{ijk} x^j_\alpha x^k_\beta\right) A^\alpha B^\beta = 0.$$

Since A^α and B^β are arbitrary, we have, $\xi_i \varepsilon_{\alpha\beta} = \varepsilon_{ijk} x^j_\alpha x^k_\beta$ and therefore

$$\xi_i \varepsilon_{\alpha\beta} \varepsilon^{\alpha\beta} = \varepsilon^{\alpha\beta} \varepsilon_{ijk} x^j_\alpha x^k_\beta$$

or

$$\xi_i \cdot 2 = \varepsilon^{\gamma\delta} \varepsilon_{ijk} x^j_\gamma x^k_\delta \quad \Rightarrow \quad \xi_i = \frac{1}{2} \varepsilon^{\gamma\delta} \varepsilon_{ijk} x^j_\gamma x^k_\delta.$$

Using Eq. (7.34) as $x^i_{\alpha,\beta} = b_{\alpha\beta} \xi^i$, the *Gauss's formula*, we get

$$b_{\alpha\beta} = x^i_{\alpha,\beta} \xi_i = x^i_{\alpha,\beta} g_{ij} \xi^j = g_{ij} x^i_{\alpha,\beta} \xi^j.$$

Also, using the Eq. (7.34), we get

$$b_{\alpha\beta} = g_{ij} x^i_{\alpha,\beta} \xi_i = \frac{1}{2} \varepsilon_{ijk} x^j_\gamma x^k_\delta x^i_{\alpha,\beta}.$$

Combining these results, we get

$$b_{\alpha\beta} = g_{ij}x^i_{\alpha,\beta}\xi^j = \frac{1}{2}\varepsilon^{\gamma\delta}\varepsilon_{ijk}x^i_{\alpha,\beta}x^j_\gamma x^k_\delta.$$

EXAMPLE 7.4.2 *Prove that* $C - 2HB + \kappa A = 0$, *where the notations have their usual meaning.*

Solution: The notations have the following expressions:

C: The third fundamental form of the surface $\equiv c_{\alpha\beta}du^\alpha du^\beta$.
B: The second fundamental form of the surface $\equiv b_{\alpha\beta}du^\alpha du^\beta$.
A: The first fundamental form of the surface $\equiv a_{\alpha\beta}du^\alpha du^\beta$.
κ: The Gaussian curvature $= \frac{R_{1212}}{a}$.
H: The mean curvature of the surface $= \frac{1}{2}a^{\alpha\beta}b_{\alpha\beta}$.
Let us take the Gauss's formula,

$$R_{\rho\alpha\beta\gamma} = b_{\rho\beta}b_{\alpha\gamma} - b_{\rho\gamma}b_{\alpha\beta} \text{ and } R_{\rho\alpha\beta\gamma} = \kappa\varepsilon_{\rho\alpha}\varepsilon_{\beta\gamma},$$

we get

$$\kappa\varepsilon_{\rho\alpha}\varepsilon_{\beta\gamma} = b_{\rho\beta}b_{\alpha\gamma} - b_{\rho\gamma}b_{\alpha\beta}.$$

Multiplying both sides by $a^{\rho\gamma}$, we get

$$\kappa\varepsilon_{\rho\alpha}\varepsilon_{\beta\gamma}a^{\rho\gamma} = a^{\rho\gamma}b_{\rho\beta}b_{\alpha\gamma} - a^{\rho\gamma}b_{\rho\gamma}b_{\alpha\beta}$$

or

$$-\kappa a_{\alpha\beta} = a^{\rho\gamma}b_{\rho\beta}b_{\gamma\alpha} - a^{\rho\gamma}b_{\rho\gamma}b_{\alpha\beta};\ \text{as } a_{\alpha\beta} = -a^{\rho\gamma}\varepsilon_{\rho\alpha}\varepsilon_{\beta\gamma}$$

or

$$\kappa a_{\alpha\beta} = 2Hb_{\alpha\beta} - c_{\alpha\beta};\ \text{as } c_{\alpha\beta} = a^{\rho\gamma}b_{\rho\gamma}b_{\alpha\beta}$$

or

$$c_{\alpha\beta} - 2Hb_{\alpha\beta} + \kappa a_{\alpha\beta} = 0$$

or

$$c_{\alpha\beta}du^\alpha du^\beta - 2Hb_{\alpha\beta}du^\alpha du^\beta + \kappa a_{\alpha\beta}du^\alpha du^\beta = 0$$

or

$$C - 2HB + \kappa A = 0.$$

EXAMPLE 7.4.3 *Show that*
(i) $a^{\alpha\beta}c_{\alpha\beta} = 4H^2 - 2\kappa$. *(ii)* $g_{ij}\xi^i\xi^j_{,\alpha\beta} = -c_{\alpha\beta}$.

Solution: (i) Using the relation $\kappa a_{\alpha\beta} = 2Hb_{\alpha\beta} - c_{\alpha\beta}$, we get

$$\kappa a_{\alpha\beta}\lambda^\alpha\lambda^\beta = 2Hb_{\alpha\beta}\lambda^\alpha\lambda^\beta - c_{\alpha\beta}\lambda^\alpha\lambda^\beta$$

or

$$\kappa = 2Hb_{\alpha\beta}\lambda^\alpha\lambda^\beta - c_{\alpha\beta}\lambda^\alpha\lambda^\beta.$$

Also, using the relation $\kappa a_{\alpha\beta} = 2Hb_{\alpha\beta} - c_{\alpha\beta}$, we get

$$c_{\alpha\beta}a^{\alpha\beta} - 2Hb_{\alpha\beta}a^{\alpha\beta} + \kappa a_{\alpha\beta}a^{\alpha\beta} = 0$$

or

$$a^{\alpha\beta}c_{\alpha\beta} - 2H \cdot 2H + \kappa \cdot 2 = 0; \text{ as } a_{\alpha\beta}a^{\alpha\beta} = 2$$

or

$$c_{\alpha\beta}a^{\alpha\beta} = 4H^2 - 2\kappa.$$

(ii) We know that, $g_{ij}\xi^i\xi^j_{,\alpha} = 0$. Taking the tensor derivative of both sides, we get

$$g_{ij}\xi^i_{,\beta}\xi^j_{,\alpha} + g_{ij}\xi^i\xi^j_{,\alpha\beta} = 0$$

or

$$g_{ij}\xi^i\xi^j_{,\alpha\beta} = -g_{ij}\xi^i_{,\beta}\xi^j_{,\alpha} = -c_{\alpha\beta}.$$

EXAMPLE 7.4.4 *If $b^{\gamma\delta}$ is the cofactor of $b_{\gamma\delta}$ in $|b_{\gamma\delta}|$, divided by $|b_{\gamma\delta}|$, show that*

$$x^i_\alpha = -a_{\alpha\delta}b^{\delta\gamma}\xi^i_{,\gamma}.$$

Solution: From the definition,

$$b^{\gamma\delta} = \frac{\text{cofactor of } b_{\gamma\delta} \text{ in } |b_{\gamma\delta}|}{|b_{\gamma\delta}|}$$

with the property $b^{\gamma\delta}b_{\alpha\delta} = \delta^\gamma_\alpha$. Hence,

$$-a_{\alpha\delta}b^{\delta\gamma}x^i_{,\gamma} = -a_{\alpha\delta}b^{\delta\gamma}\left(-a^{\beta\sigma}b_{\beta\gamma}x^i_\sigma\right) = a_{\alpha\delta}a^{\beta\sigma}b^{\sigma\gamma}b_{\beta\gamma}x^i_\sigma$$
$$= a_{\alpha\mu}a^{\beta\sigma}\delta^\mu_\beta x^i_\sigma = a_{\alpha\mu}a^{\mu\sigma}x^i_\sigma = \delta^\sigma_\alpha x^i_\sigma = x^i_\alpha.$$

EXAMPLE 7.4.5 *Find the Gaussian curvature at the point $(2, 0, 1)$ of the surface $(x^1)^2 - (x^2)^2 = 4x^3$, where x^1, x^2, x^3 are rectangular Cartesian co-ordinates.*

Solution: Let the parametric representation of the surface is given by

$$x^1 = u^1 + u^2, x^2 = u^1 - u^2, x^3 = u^1u^2$$

i.e.

$$\mathbf{r} = (x^1, x^2, x^3) = (u^1 + u^2, u^1 - u^2, u^1u^2).$$

Therefore, at $(2, 0, 1)$, we have

$$u^1 = \frac{1}{2}(x^1 + x^2) = 1 \text{ and } u^2 = \frac{1}{2}(x^1 - x^2) = 1.$$

For the first fundamental form of the surface [Eq. (7.2)] the tensors $a_{\alpha\beta}$ are given by

$$\begin{aligned}
a_{11} &= \frac{\partial x^i}{\partial u^1}\frac{\partial x^i}{\partial u^1} = \left(\frac{\partial x^1}{\partial u^1}\right)^2 + \left(\frac{\partial x^2}{\partial u^1}\right)^2 + \left(\frac{\partial x^3}{\partial u^1}\right)^2 \\
&= 1^2 + 1^2 + (u^2)^2 = 2 + (u^2)^2. \\
a_{22} &= \frac{\partial x^i}{\partial u^2}\frac{\partial x^i}{\partial u^2} = \left(\frac{\partial x^1}{\partial u^2}\right)^2 + \left(\frac{\partial x^2}{\partial u^2}\right)^2 + \left(\frac{\partial x^3}{\partial u^2}\right)^2 \\
&= 1^2 + (-1)^2 + (u^1)^2 = 2 + (u^1)^2. \\
a_{12} &= \frac{\partial x^i}{\partial u^1}\frac{\partial x^i}{\partial u^2} = \frac{\partial x^1}{\partial u^1}\frac{\partial x^1}{\partial u^2} + \frac{\partial x^2}{\partial u^1}\frac{\partial x^2}{\partial u^2} + \frac{\partial x^3}{\partial u^1}\frac{\partial x^3}{\partial u^2} \\
&= 1 \cdot 1 + 1 \cdot (-1) + u^2 \cdot u^1 = u^1u^2 = a_{21}.
\end{aligned}$$

Therefore,

$$a = \begin{vmatrix} 2 + (u^2)^2 & 0 \\ 0 & 2 + (u^2)^2 \end{vmatrix} = 2\{(u^1)^2 + (u^2)^2 + 2\}.$$

For the given parametric surface, $\mathbf{A} = (1, 1, u^2)$, $\mathbf{B} = (1, -1, u^1)$ and so

$$\mathbf{A} \times \mathbf{B} = (u^1 + u^2)\hat{i} + (u^2 - u^1)\hat{j} - 2\hat{k}.$$

Therefore, the normal vector $\boldsymbol{\xi}$ is given by

$$\boldsymbol{\xi} = \frac{1}{|\mathbf{A} \times \mathbf{B}|}(\mathbf{A} \times \mathbf{B}) = \frac{1}{\sqrt{2\{(u^1)^2 + (u^2)^2 + 2\}}}(u^1 + u^2, u^2 - u^1, -2).$$

Thus the covariant tensors $b_{\alpha\beta}$ for the second fundamental form are given by

$$\begin{aligned}
b_{11} = \frac{-1}{\sqrt{2}K^{\frac{3}{2}}}\left((u^2)^2 + 2 - u^1u^2, -\{(u^2)^2 + 2 + u^1u^2\}, 2u^1\right) \cdot (1, 1, u^2), \\
[\text{where } K = (u^1)^2 + (u^2)^2 + 1];
\end{aligned}$$

$$= \frac{-1}{\sqrt{2}K^{\frac{3}{2}}}\left[(u^2)^2 + 2 - u^1u^2 - (u^2)^2 - 2 - u^1u^2 + 2u^1u^2\right] = 0.$$

$$b_{22} = \frac{-1}{\sqrt{2}K^{\frac{3}{2}}}\left((u^1)^2 + 2 - u^1u^2, \{(u^1)^2 + 2 + u^1u^2\}, 2u^2\right)\cdot(1, -1, u^1)$$

$$= \frac{-1}{\sqrt{2}K^{\frac{3}{2}}}\left[(u^1)^2 + 2 - u^1u^2 - (u^1)^2 - 2 - u^1u^2 + 2u^1u^2\right] = 0.$$

$$b_{12} = -\frac{1}{2}\left[\frac{\partial \boldsymbol{\xi}}{\partial u^1}\cdot\frac{\partial \mathbf{r}}{\partial u^2} + \frac{\partial \boldsymbol{\xi}}{\partial u^2}\cdot\frac{\partial \mathbf{r}}{\partial u^1}\right]$$

$$= -\frac{2}{2\sqrt{2}}\left[\frac{(u^2)^2 + 1 + (u^1)^2}{\{(u^1)^2 + (u^2)^2 + 1\}^{\frac{3}{2}}} + \frac{(u^1)^2 + 1 + (u^2)^2}{\{(u^1)^2 + (u^2)^2 + 1\}^{\frac{3}{2}}}\right]$$

$$= -\frac{2}{\sqrt{2\{(u^1)^2 + (u^2)^2 + 2\}}}.$$

Therefore,

$$b = \begin{vmatrix} 0 & \frac{-2}{\sqrt{2\{(u^1)^2+(u^2)^2+2\}}} \\ \frac{-2}{\sqrt{2\{(u^1)^2+(u^2)^2+2\}}} & 0 \end{vmatrix} = -\frac{2}{(u^1)^2 + (u^2)^2 + 2}.$$

Thus, the Gaussian curvature κ is given by

$$\kappa = \frac{b}{a} = -\frac{1}{[(u^1)^2 + (u^2)^2 + 2]^2} = -\frac{1}{16}; \text{ at } (u^1, u^2) = (1, 1).$$

EXAMPLE 7.4.6 *Find the Gaussian and mean curvature of the surface $x^1x^2 = x^3$.*

Solution: Let the parametric representation of the surface (hyperbolic paraboloid) is given by, $x^1 = u^1, x^2 = u^2$ then, $x^3 = u^1u^2$. i.e.

$$\mathbf{r} = (x^1, x^2, x^3) = (u^1, u^2, u^1u^2).$$

The symmetric covariant tensors $a_{\alpha\beta}$ of the first fundamental form are given by

$$a_{11} = \left(\frac{\partial x^1}{\partial u^1}\right)^2 + \left(\frac{\partial x^2}{\partial u^1}\right)^2 + \left(\frac{\partial x^3}{\partial u^1}\right)^2 = 1 + (u^2)^2.$$

$$a_{22} = \left(\frac{\partial x^1}{\partial u^2}\right)^2 + \left(\frac{\partial x^2}{\partial u^2}\right)^2 + \left(\frac{\partial x^3}{\partial u^2}\right)^2 = 1 + (u^1)^2.$$

$$a_{12} = \frac{\partial x^1}{\partial u^1}\frac{\partial x^1}{\partial u^2} + \frac{\partial x^2}{\partial u^1}\frac{\partial x^2}{\partial u^2} + \frac{\partial x^3}{\partial u^1}\frac{\partial x^3}{\partial u^2} = u^1u^2 = a_{21}.$$

Therefore,

$$a = \begin{vmatrix} 1+(u^2)^2 & u^1u^2 \\ u^1u^2 & 1+(u^1)^2 \end{vmatrix} = 1+(u^1)^2+(u^2)^2.$$

The reciprocal tensors are given by

$$a^{11} = \frac{1+(u^1)^2}{1+(u^1)^2+(u^2)^2};\ a^{22} = \frac{1+(u^2)^2}{1+(u^1)^2+(u^2)^2};\ a^{12} = \frac{-u^1u^2}{1+(u^1)^2+(u^2)^2} = a^{21}.$$

Since $\mathbf{r} = (x^1, x^2, x^3) = (u^1, u^2, u^1u^2)$, we have, $\mathbf{A} = (1, 0, u^2)$, $\mathbf{B} = (0, 1, u^1)$ and so

$$\mathbf{A} \times \mathbf{B} = -u^2\hat{\imath} - u^1\hat{\jmath} + \hat{k}.$$

Therefore, the normal vector $\boldsymbol{\xi}$ is given by

$$\boldsymbol{\xi} = \frac{1}{|\mathbf{A}\times\mathbf{B}|}(\mathbf{A}\times\mathbf{B}) = \frac{1}{\sqrt{(u^1)^2+(u^2)^2+1}}(-u^2, -u^1, 1).$$

The symmetric covariant tensors $b_{\alpha\beta}$ are given by

$$\begin{aligned} b_{11} &= -\frac{\partial\boldsymbol{\xi}}{\partial u^1}\cdot\frac{\partial\mathbf{r}}{\partial u^1} = \frac{-1}{\{(u^1)^2+(u^2)^2+1\}^{\frac{3}{2}}}\left(u^1u^2, -\{(u^2)^2+1\}, -u^1\right)\cdot(1, 0, u^2) \\ &= \frac{-1}{\{(u^1)^2+(u^2)^2+1\}^{\frac{3}{2}}}\left[u^1u^2 - u^1u^2\right] = 0. \\ b_{22} &= -\frac{\partial\boldsymbol{\xi}}{\partial u^2}\cdot\frac{\partial\mathbf{r}}{\partial u^2} = \frac{-1}{\{(u^1)^2+(u^2)^2+1\}^{\frac{3}{2}}}\left[-\{(u^1)^2+1\}, -u^1u^2, u^2\right]\cdot(0, 1, u^1) \\ &= \frac{1}{\{(u^1)^2+(u^2)^2+1\}^{\frac{3}{2}}}\left[u^1u^2 - u^1u^2\right] = 0. \\ b_{12} &= -\frac{1}{2}\left[\frac{\partial\boldsymbol{\xi}}{\partial u^1}\cdot\frac{\partial\mathbf{r}}{\partial u^2} + \frac{\partial\boldsymbol{\xi}}{\partial u^2}\cdot\frac{\partial\mathbf{r}}{\partial u^1}\right] \\ &= -\frac{1}{2}\left[-\frac{(u^2)^2+1+(u^1)^2}{\{(u^1)^2+(u^2)^2+1\}^{\frac{3}{2}}} - \frac{(u^1)^2+1+(u^2)^2}{\{(u^1)^2+(u^2)^2+1\}^{\frac{3}{2}}}\right] \\ &= \frac{1}{\sqrt{(u^1)^2+(u^2)^2+1}} = b_{21}. \end{aligned}$$

Therefore,

$$b = \begin{vmatrix} 0 & \dfrac{1}{\sqrt{(u^1)^2+(u^2)^2+1}} \\ \dfrac{1}{\sqrt{(u^1)^2+(u^2)^2+1}} & 0 \end{vmatrix} = -\frac{1}{(u^1)^2+(u^2)^2+1}.$$

The Gaussian curvature κ is given by the formula,

$$\kappa = \frac{b}{a} = -\frac{1}{\left[(u^1)^2 + (u^2)^2 + 1\right]^2} < 0.$$

Therefore, the mean curvature H is given by

$$\begin{aligned} H &= \frac{1}{2}a^{\alpha\beta}b_{\alpha\beta} = \frac{1}{2}\left[a^{11}b_{11} + a^{22}b_{22} + 2a^{12}b_{12}\right] \\ &= \frac{1}{2}\left[0 + 0 + \frac{-u^1u^2}{1 + (u^1)^2 + (u^2)^2} \times \frac{1}{\sqrt{(u^1)^2 + (u^2)^2 + 1}}\right] \\ &= \frac{-u^1u^2}{\{1 + (u^1)^2 + (u^2)^2\}^{3/2}}. \end{aligned}$$

EXAMPLE 7.4.7 *Find the mean curvature of the surface* $\mathbf{r} = (u, v, u^2 - v^2)$.

Solution: For the first fundamental form of the surface [Eq. (7.2)], $a_{\alpha\beta}$ are given by

$$a_{11} = \left(\frac{\partial x^1}{\partial u}\right)^2 + \left(\frac{\partial x^2}{\partial u}\right)^2 + \left(\frac{\partial x^3}{\partial u}\right)^2 = 1 + 4u^2.$$

$$a_{22} = \left(\frac{\partial x^1}{\partial v}\right)^2 + \left(\frac{\partial x^2}{\partial v}\right)^2 + \left(\frac{\partial x^3}{\partial v}\right)^2 = 1 + 4v^2.$$

$$a_{12} = \frac{\partial x^1}{\partial u}\frac{\partial x^1}{\partial v} + \frac{\partial x^2}{\partial u}\frac{\partial x^2}{\partial v} + \frac{\partial x^3}{\partial u}\frac{\partial x^3}{\partial v} = -4uv = a_{21}.$$

Therefore,

$$a = (1 + 4u^2)(1 + 4v^2) - 16u^2v^2 = 1 + 4u^2 + 4v^2.$$

The reciprocal tensors are given by

$$a^{11} = \frac{1 + 4v^2}{1 + 4u^2 + 4v^2};\ a^{22} = \frac{1 + 4u^2}{1 + 4u^2 + 4v^2};\ a^{12} = \frac{4uv}{1 + 4u^2 + 4v^2} = a^{21}.$$

For the second fundamental form, the symmetric covariant tensors are given by

$$b_{11} = \frac{2}{\sqrt{1 + 4u^2 + 4v^2}};\ b_{22} = \frac{-2}{\sqrt{1 + 4u^2 + 4v^2}} \text{ and } b_{12} = 0 = b_{21}.$$

Therefore, the mean curvature H is given by

$$\begin{aligned} H &= \frac{1}{2}a^{\alpha\beta}b_{\alpha\beta} = \frac{1}{2}\left[a^{11}b_{11} + a^{22}b_{22} + 2a^{12}b_{12}\right] \\ &= \frac{1}{2}\left[\frac{2(1 + 4v^2)}{(1 + 4u^2 + 4v^2)^{3/2}} - \frac{2(1 + 4u^2)}{(1 + 4u^2 + 4v^2)^{3/2}} + 2 \cdot \frac{4uv}{1 + 4u^2 + 4v^2} \cdot 0\right] \\ &= \frac{1 + 4v^2 - 1 - 4u^2}{(1 + 4u^2 + 4v^2)^{3/2}} = \frac{4(v^2 - u^2)}{(1 + 4u^2 + 4v^2)^{3/2}}. \end{aligned}$$

EXAMPLE 7.4.8 *Show that the right helicoid given by*

$$x^1 = u^1 \cos u^2;\ x^2 = u^1 \sin u^2;\ x^3 = cu^2$$

is a minimal surface.

Solution: For the first fundamental form of the surface [Eq. (7.2)], $a_{\alpha\beta}$ are given by

$$a_{11} = (\cos u^2)^2 + (\sin u^2)^2 + 0 = 1.$$

$$a_{22} = (-u^1 \sin u^2)^2 + (u^1 \cos u^2)^2 + (c)^2 = (u^1)^2 + c^2.$$

$$\begin{aligned} a_{12} &= \frac{\partial x^1}{\partial u^1}\frac{\partial x^1}{\partial u^2} + \frac{\partial x^2}{\partial u^1}\frac{\partial x^2}{\partial u^2} + \frac{\partial x^3}{\partial u^1}\frac{\partial x^3}{\partial u^2} \\ &= \cos u^2(-u^1 \sin u^2) + \sin u^2(u^1 \cos u^2) + 0 = 0 = a_{21}. \end{aligned}$$

Therefore, $a = c^2 + (u^1)^2$. The reciprocal tensors are given by

$$a^{11} = \frac{c^2 + (u^1)^2}{c^2 + (u^1)^2} = 1;\ a^{22} = \frac{1}{c^2 + (u^1)^2};\ a^{12} = 0 = a^{21}.$$

Now, we calculate the tensors of second fundamental form. Since $\mathbf{r} = (x^1, x^2, x^3) = (u^1 \cos u^2, u^1 \sin u^2, cu^2)$, so,

$$\mathbf{A} = (\cos u^2, \sin u^2, 0) \text{ and } \mathbf{B} = (-u^1 \sin u^2, u^1 \cos u^2, c)$$

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos u^2 & \sin u^2 & 0 \\ -u^1 \sin u^2 & u^1 \cos u^2 & c \end{vmatrix} = c \sin u^2 \hat{i} - c \cos u^2 \hat{j} + u^1 \hat{k}$$

$$\boldsymbol{\xi} = \frac{1}{|\mathbf{A} \times \mathbf{B}|}(\mathbf{A} \times \mathbf{B}) = \frac{1}{\sqrt{c^2 + (u^1)^2}}(c \sin u^2, -c \cos u^2, u^1).$$

Thus, the covariant tensors $b_{\alpha\beta}$ for the second fundamental form are given by

$$b_{11} = 0,\ b_{22} = 0 \text{ and } b_{12} = -\frac{c}{\sqrt{c^2 + (u^1)^2}} = b_{21}.$$

Therefore, the mean curvature H is given by

$$\begin{aligned} H &= \frac{1}{2}a^{\alpha\beta}b_{\alpha\beta} = \frac{1}{2}\left[a^{11}b_{11} + a^{22}b_{22} + 2a^{12}b_{12}\right] \\ &= \frac{1}{2}\left[1 \times 0 + \frac{1}{c^2 + (u^1)^2} \times 0 + 2 \times 0 \times \left(\frac{-c}{\sqrt{c^2 + (u^1)^2}}\right)\right] = 0. \end{aligned}$$

Since the mean curvature $H = 0$, at every point of the given surface, so, it is a minimal surface.

EXAMPLE 7.4.9 *Calculate the fundamental magnitudes for the conicoid*

$$x^1 = u^1 \cos u^2;\ x^2 = u^1 \sin u^2;\ x^3 = f(u^2),$$

where u^1, u^2 *are parameters. Find also Gaussian and mean curvature.*

Solution: The parametric representation of the surface is given by

$$\mathbf{r} = (x^1, x^2, x^3) = \left(u^1 \cos u^2;\ u^1 \sin u^2;\ f(u^2)\right),$$

The first fundamental magnitudes $a_{\alpha\beta}$ are given by

$$\begin{aligned}
a_{11} &= \left(\frac{\partial x^1}{\partial u^1}\right)^2 + \left(\frac{\partial x^2}{\partial u^1}\right)^2 + \left(\frac{\partial x^3}{\partial u^1}\right)^2 = (\cos u^2)^2 + (\sin u^2)^2 + 0^2 = 1,\\
a_{22} &= \left(\frac{\partial x^1}{\partial u^2}\right)^2 + \left(\frac{\partial x^2}{\partial u^2}\right)^2 + \left(\frac{\partial x^3}{\partial u^2}\right)^2\\
&= (-u^1 \sin u^2)^2 + (u^1 \cos u^2)^2 + f_1^2 = (u^1)^2 + f_1^2,\\
a_{12} &= \frac{\partial x^1}{\partial u^1}\frac{\partial x^1}{\partial u^2} + \frac{\partial x^2}{\partial u^1}\frac{\partial x^2}{\partial u^2} + \frac{\partial x^3}{\partial u^1}\frac{\partial x^3}{\partial u^2} = 0 = a_{21},
\end{aligned}$$

where $f_1 = \frac{\partial f}{\partial u^2}$. Therefore,

$$a = \begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & (u^1)^2 + f_1^2 \end{vmatrix} = (u^1)^2 + f_1^2.$$

The reciprocal tensors are given by

$$a^{11} = \frac{(u^1)^2 + f_1^2}{(u^1)^2 + f_1^2} = 1;\ a^{22} = \frac{1}{(u^1)^2 + f_1^2};\ a^{12} = \frac{0}{(u^1)^2 + f_1^2} = 0 = a^{21}.$$

Since $\mathbf{r} = (x^1, x^2, x^3) = \left(u^1 \cos u^2;\ u^1 \sin u^2;\ f(u^2)\right)$, we have

$$\mathbf{A} = (\cos u^2, \sin u^2, 0) \text{ and } \mathbf{B} = (-u^1 \sin u^2, u^1 \cos u^2, f_1)$$

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos u^2 & \sin u^2 & 0 \\ -u^1 \sin u^2 & u^1 \cos u^2 & f_1 \end{vmatrix} = f_1 \sin u^2 \hat{i} + f_1 \cos u^2 \hat{j} + u^1 \hat{k}$$

Therefore, the normal vector $\boldsymbol{\xi}$ is given by

$$\boldsymbol{\xi} = \frac{1}{|\mathbf{A} \times \mathbf{B}|}(\mathbf{A} \times \mathbf{B}) = \frac{1}{\sqrt{(u^1)^2 + f_1^2}}\left(f_1 \sin u^2, -f_1 \cos u^2, u^1\right).$$

The symmetric covariant tensors $b_{\alpha\beta}$ are given by

$$b_{11} = \frac{\partial \boldsymbol{\xi}}{\partial u^1} \cdot \frac{\partial \mathbf{r}}{\partial u^1} = (0,0,0).\frac{1}{\sqrt{(u^1)^2 + f_1^2}}\left(f_1 \sin u^2, -f_1 \cos u^2, u^1\right) = 0$$

$$\begin{aligned} b_{22} &= \frac{\partial \boldsymbol{\xi}}{\partial u^2} \cdot \frac{\partial \mathbf{r}}{\partial u^2} = (-\sin u^2, \cos u^2, 0).\frac{1}{\sqrt{(u^1)^2 + f_1^2}}\left(f_1 \sin u^2, -f_1 \cos u^2, u^1\right) \\ &= -\frac{f_1}{\sqrt{(u^1)^2 + f_1^2}} \end{aligned}$$

$$\begin{aligned} b_{12} &= \left(-u^1 \cos u^2, -u^1 \sin u^2, f_2\right).\frac{1}{\sqrt{(u^1)^2 + f_1^2}}\left(f_1 \sin u^2, -f_1 \cos u^2, u^1\right) \\ &= \frac{u^1 f_2}{\sqrt{(u^1)^2 + f_1^2}} = b_{21}, \end{aligned}$$

where $f_2 = \frac{\partial^2 f}{(\partial u^2)^2}$. Therefore,

$$b = \begin{vmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{vmatrix} = \begin{vmatrix} 0 & \dfrac{u^1 f_2}{\sqrt{(u^1)^2 + f_1^2}} \\ \dfrac{u^1 f_2}{\sqrt{(u^1)^2 + f_1^2}} & -\frac{f_1}{\sqrt{(u^1)^2+f_1^2}} \end{vmatrix} = -\frac{\left(u^1 f_2\right)^2}{(u^1)^2 + f_1^2}.$$

The Gaussian curvature κ is given by the formula

$$\kappa = \frac{b}{a} = -\frac{\left(u^1 f_2\right)^2}{\left[(u^1)^2 + f_1^2\right]^2} < 0.$$

Therefore, the mean curvature H is given by

$$\begin{aligned} H &= \frac{1}{2} a^{\alpha\beta} b_{\alpha\beta} = \frac{1}{2}\left[a^{11} b_{11} + a^{22} b_{22} + 2a^{12} b_{12}\right] \\ &= \frac{1}{2}\left[0 + \frac{1}{(u^1)^2 + f_1^2} \times \left(\frac{-f_1}{\sqrt{(u^1)^2 + f_1^2}}\right) + 0\right] = \frac{-f_1}{2\{(u^1)^2 + f_1^2\}^{3/2}}. \end{aligned}$$

7.5 Exercises

1. Prove that the space vector perpendicular to λ^i and tangent to the surface is

$$\mu^i = x^i_\alpha \mu^\alpha,$$

   where μ^i is the unit normal vector in the surface.

2. (i) Prove that for any curve on a surface $\kappa^2 = \chi_g^2 + \kappa_{(n)}^2$.
(ii) Prove that the covariant derivative $b_{\alpha\beta,\gamma}$ can be written in the form

$$b_{\alpha\beta,\gamma} = \frac{\partial b_{\alpha\alpha}}{\partial u^\gamma} - {}_a\left\{ \begin{matrix} \mu \\ \alpha \;\; \gamma \end{matrix} \right\} b_{\mu\beta} - {}_a\left\{ \begin{matrix} \mu \\ \beta \;\; \gamma \end{matrix} \right\} b_{\alpha\mu}.$$

3. Show that (i) $b_{\alpha\beta} = -g_{ij}\xi^i_{,\alpha} x^j_\beta$; (ii) $a^{\alpha\beta} x^r_{\alpha,\beta} = 2H\xi^r$.
4. If the space co-ordinates are rectangular Cartesian, show that

$$b_{\alpha\beta} = \frac{1}{\sqrt{a}} \varepsilon_{ijk} \frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta} x_1^j x_2^k.$$

5. Show that a surface is a sphere if and only if its second fundamental form is a non-zero constant multiple of the first fundamental form.
6. Show that the second fundamental form for the

(i) paraboloid $\mathbf{r} = (u, v, u^2 - v^2)$ is given by

$$B \equiv \frac{2}{\sqrt{4u^2 + 4v^2 + 1}} [(du)^2 - (dv)^2].$$

(ii) helicoid $\mathbf{r} = (u \cos v, u \sin v, f(u) + cv)$ is given by

$$B \equiv \frac{1}{\sqrt{u^2 + f_1^2 u^2 + c^2}} [u f_{11} (du)^2 - 2c du dv + f_1 u^2 (dv)^2].$$

(iii) the surface of revolution $\mathbf{r} = (u \cos \phi, u \sin \phi, f(u))$ is given by

$$B \equiv \frac{1}{\sqrt{1 + f_1^2}} [f_{11} (du)^2 + u f_1 (d\phi)^2].$$

7. Determine the second fundamental form of a surface represented in the form $x_3 = f(x_1, x_2)$.
8. Prove by direct calculation that the coefficients of the second fundamental form are the components of a covariant tensor of second order with respect to co-ordinate transformations which preserve the sense of the unit normal vector ζ.
9. Taking x, y as parameters, calculate the fundamental magnitudes to the surface

$$2z = ax^2 + 2hxy + by^2$$

and show that the normal to the surface is given by

$$\boldsymbol{\xi} = \frac{1}{\sqrt{1 + (ax + hy)^2 + (hx + by)^2}} (-ax - hy, -hx - by, 1).$$

10. Show that the mean curvature for the

 (i) paraboloid $\mathbf{r} = (u, v, u^2 - v^2)$ is given by

$$H = \frac{4(v^2 - u^2)}{[4u^2 + 4v^2 + 1]^{3/2}}.$$

 (ii) hyperbolic paraboloid $x^1 x^2 = x^3$ is given by

$$H = -\frac{x^3}{[(x^1)^2 + (x^2)^2 + 1]^{3/2}}.$$

 (iii) helicoid $\mathbf{r} = (u \cos v, u \sin v, f(u) + cv)$ is given by

$$H = \frac{1}{2} \frac{(1 + f_1^2)u^2 f_1 + 2c^2 f_{11} + u f_{11}(u^2 + c^2)}{[u^2 + f_1^2 u^2 + c^2]^{3/2}}.$$

 (iv) right helicoid $\mathbf{r} = (x^1, x^2, x^3) = \left(u^1 \cos u^2; u^1 \sin u^2; cu^2\right)$ is zero.

11. Show that the Gaussian curvature for the

 (i) paraboloid $\mathbf{r} = (u, v, u^2 - v^2)$ is given by

$$\kappa = -\frac{4}{[4u^2 + 4v^2 + 1]^2}.$$

 (ii) the surface of revolution $\mathbf{r} = (u \cos \phi, u \sin \phi, f(u))$ is given by

$$\kappa = \frac{f_1 f_{11}}{u(1 + f_1^2)^2}.$$

 (iii) helicoid $\mathbf{r} = (u \cos v, u \sin v, f(u) + cv)$ is given by

$$\kappa = \frac{u^3 f_1 f_{22} - c^2}{[u^2 + f_1^2 u^2 + c^2]^2}.$$

 (iv) right helicoid $\mathbf{r} = (x^1, x^2, x^3) = \left(u^1 \cos u^2; u^1 \sin u^2; cu^2\right)$ is given by

$$\kappa = -\frac{c^2}{(u^2 + c^2)^2}.$$

 (v) the anchor ring

$$\mathbf{r} = (x^1, x^2, x^3) = \left((b + a \cos u^1) \cos u^2; (b + a \cos u^1) \sin u^2; a \sin u^1\right)$$

 is given by $\kappa = \dfrac{\cos u^1}{a(b + a \cos u^1)}$.

12. Calculate the fundamental magnitudes and unit normals for the

    (i) $x^1 = u^1 \cos u^2;\ x^2 = u^1 \sin u^2;\ x^3 = cu^2,$
    (ii) $x^1 = u^1 \cos u^2;\ x^2 = u^1 \sin u^2;\ x^3 = f(u^1),$
    (iii) $x^1 = u^1 \cos u^2;\ x^2 = u^1 \sin u^2;\ x^3 = f(u^1) + cu^2,$

    where u^1, u^2 are parameters. Find also Gaussian and mean curvature.
13. Determine the discriminant b of the second fundamental form of a sphere.
14. Prove that a surface for which at every point all coefficients of the second fundamental form vanish is a plane.
15. If two curves on a surface cut at right angles, show that the sum of their geodesic torsion is zero.
16. Prove that a surface, all of whose points are umbilics, is a sphere or a plane.
17. Prove that at each non-umbilical point of a surface, there exist two mutually orthogonal directions for which the normal curvature attains its extreme values.
18. Find the equation for the principal curvatures of the surface
    $$x^1 = u^1,\ x^2 = u^2,\ x^3 = f(u^1, u^2),$$
    the x co-ordinates being rectangular Cartesian.
19. Show that the equation of the principal curvatures for

    (i) surface of revolution are $\kappa_{(1)} = \frac{f_{11}}{(1+f_1^2)^{3/2}}$ and $\kappa_{(2)} = \frac{f_1}{u(1+f_1^2)^{3/2}}$.
    (ii) surface $(x^1)^2 + (x^2)^2 + (x^3)^2 = 1$ are -1 and 1.

20. If two conjugate directions make angles θ_1 and θ_2 with a principal direction, show that
    $$\tan\theta_1 \tan\theta_2 = -\frac{\kappa_1}{\kappa_2}.$$
21. If a geodesic on a surface is a plane curve, show that it is also a line of curvature and conversely.
22. Prove that the lines of curvature on a surface are given by
    $$\varepsilon^{\gamma\mu} a_{\alpha\gamma} a_{\beta\mu} du^\alpha du^\beta = 0.$$
23. Show that the lines of curvature of the helicoid
    $$x^1 = u^1 \cos u^2, x^2 = u^1 \sin u^2, x^3 = f(u^1) + cu^2$$
    are given by
    $$c\left[1 + f_1^2 + u^1 f_1 f_{11}\right](du^1)^2 - c\left[(u^1)^2 + c^2 + (u^1)^2 f_1^2\right](du^2)^2 + \left[\{(u^1)^2 + c^2\}u^1 f_{11} - (1 + f_1^2)(u^1)^2 f_1\right] du^1 du^2 = 0.$$

24. Show that the lines of curvature for the surface catenoid, given by

$$x^1 = u^1 \cos u^2,\ x^2 = u^1 \sin u^2,\ x^3 = c \cosh^{-1} \frac{u^1}{c},$$

where c is a constant, are the parametric curves.

25. If the co-ordinate curves are lines of curvature, prove that $a_{12} = 0 = b_{12}$ and conversely.

26. Calculate the Christoffel symbols of the second kind for the right helicoid, show that circular helices on the surface are geodesics.

27. Calculate κ, H for two different parameterisation of the paraboloid $z = a(x^2 + y^2)$:

    (i) as the surface of revolution for which $f = x^1, g = a(x^1)^2$;
    (ii) as the surface $\mathbf{r} = (y^1, y^2, a(y^1)^2 + a(y^2)^2)$.

    Interpret the results.

28. Find the values of the mean curvature H and Gaussian curvature κ, when parametric curves are asymptotic lines.

29. If θ be the angle between the asymptotic directions, find the value of $\tan \frac{\theta}{2}$.

30. Prove that on a surface is given by

$$x^1 = a(u^1 + u^2), x^2 = b(u^1 - u^2),\ x^3 = u^1 u^2,$$

where a and b are constants, the parametric curves are asymptotic lines.

31. Show that the parametric curves are asymptotic lines of the surface

$$x^1 = u^1 \cos u^2, x^2 = u^1 \sin u^2, x^3 = au^2.$$

Hints: For the surface we have to show that

$$b_{11} = 0, b_{22} = 0 \text{ and } b_{12} = -\frac{a}{\sqrt{a^2 + (u^1)^2}}.$$

32. Prove that for an asymptotic direction the normal curvature is zero.

33. Show that if every point of a surface S is parabolic, then S is developable.

Chapter 8

Curves on a Surface

In Chapter 7, we assumed the surface co-ordinates to be u^α and the space co-ordinates x^i, then the equation of a surface be given by

$$\mathcal{S} : x^i = x^i(u^1, u^2). \tag{8.1}$$

If the co-ordinates u^α are given as functions of a parameter then the point represented by these co-ordinates describes a curve on the surface as the parameter varies. We shall take the arc length s of the curve as the arc parameter and consequently, the equation of a smooth curve $\mathcal{C}$ lying on the surface $\mathcal{S}$ is given by

$$\mathcal{C} : \quad u^\alpha = u^\alpha(s). \tag{8.2}$$

We see from Eq. (8.1) that the space co-ordinates of any point of the curve are also functions of the arc parameter s and we obtain the space co-ordinates x^i in the form

$$\mathcal{S} : x^i = x^i(s) = x^i\left(u^1(s), u^2(s)\right), \tag{8.3}$$

which is the equation of $\mathcal{C}$ regarded as a space curve. The properties of $\mathcal{C}$ can then be studied with the aid of the Frenet–Serret formula, by analyzing the rates of change of the unit tangent vector $\boldsymbol{\lambda}$, the unit principal normal $\boldsymbol{\mu}$ and the unit binormal $\boldsymbol{\nu}$. Then its curvature κ and its torsion τ are connected with these vectors by the Frenet formulae [Eq. (5.17)] as

$$\left.\begin{aligned} \frac{\delta\lambda^i}{\delta s} &= \kappa\mu^i \\ \frac{\delta\mu^i}{\delta s} &= -\kappa\lambda^i + \tau\gamma^i \\ \frac{\delta\gamma^i}{\delta s} &= -\tau\mu^i \end{aligned}\right\} \tag{8.4}$$

Considering the curve as a curve on the surface we shall denote its unit tangent vector by λ^α and its unit normal vector in the surface by ζ^α.

8.1 Curves Theory on a Surface

If we regard $\mathcal{C}$ as a surface curve, defined by Eq. (8.2), the components λ^α of the unit tangent vector $\boldsymbol{\lambda}$ are related to the space components λ^i of the same vector by the formulas

$$\lambda^i = \frac{dx^i}{ds} = \frac{\partial x^i}{\partial u^\alpha}\frac{du^\alpha}{ds} \equiv x^i_\alpha \lambda^\alpha; \text{ where } \lambda^\alpha = \frac{du^\alpha}{ds}. \tag{8.5}$$

If ζ^α is the unit normal to $\mathcal{C}$ in the tangent plane to the surface (Figure 8.1) and χ_g is the geodesic curvature of $\mathcal{C}$, then from Eq. (6.61) we get

$$\frac{\delta\lambda^\alpha}{\delta s} = \chi_g \zeta^\alpha.$$

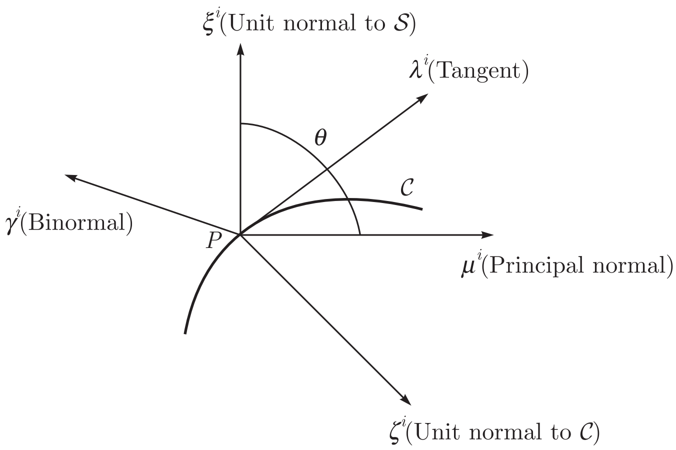

Figure 8.1: Tangent and normals on the surface.

If we differentiate Eq. (8.5) intrinsically with respect to s, we get

$$\frac{\delta\lambda^i}{\delta s} = x^i_{\alpha,\beta}\lambda^\alpha \frac{du^\beta}{ds} + x^i_\alpha \frac{\delta\lambda^\alpha}{\delta s}$$

or

$$\chi\mu^i = x^i_{\alpha,\beta}\lambda^\alpha\lambda^\beta + \chi_g x^i_\alpha \zeta^\alpha,$$

where we use the Frenet formula $\frac{\delta\lambda^i}{\delta s} = \chi\mu^i$. Using the Gauss's formula $x^i_{\alpha,\beta} = b_{\alpha\beta}\xi^i$ and the space components ζ^i of $\boldsymbol{\zeta}$ as $\zeta^i = x^i_\alpha \zeta^\alpha$, the Eq. (8.5) becomes

$$\chi\mu^i = b_{\alpha\beta}\lambda^\alpha\lambda^\beta\xi^i + \chi_g\zeta^i, \tag{8.6}$$

where ξ^i is the unit normal to the surface $\mathcal{S}$. Formula (8.6) states that the principal normal $\boldsymbol{\mu}$ to the curve $\mathcal{C}$ lies on the plane of the vectors $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$. Since $\boldsymbol{\xi}$, $\boldsymbol{\zeta}$ and $\boldsymbol{\lambda}$ are orthogonal and $\boldsymbol{\xi} \times \boldsymbol{\zeta} = \boldsymbol{\lambda}$, we have,

$$\varepsilon_{ijk}\xi^j\zeta^k = \lambda_i. \tag{8.7}$$

Let θ be the angle between $\boldsymbol{\mu}$ and $\boldsymbol{\xi}$. Since the tangent vector $\boldsymbol{\lambda}$ is orthogonal to the plane of $\boldsymbol{\xi}$ and $\boldsymbol{\mu}$, we have

$$\boldsymbol{\mu} \times \boldsymbol{\xi} = -\sin\theta\boldsymbol{\lambda}, \text{ i.e. } \varepsilon_{ijk}\mu^j\xi^k = -\sin\theta\lambda_i. \tag{8.8}$$

On multiplying by χ, the foregoing Eq. (8.8) becomes

$$\varepsilon_{ijk}\chi\mu^j\xi^k = -\chi\sin\theta\lambda_i$$

or

$$\varepsilon_{ijk}\left(b_{\alpha\beta}\lambda^\alpha\lambda^\beta\xi^j + \chi_g\zeta^j\right)\xi^k = -\chi\sin\theta\lambda_i; \text{ using Eq. (8.6)}$$

or

$$\chi_g = \chi\sin\theta; \text{ as } \varepsilon_{ijk}\xi^j\xi^k = 0 \text{ and } \varepsilon_{ijk}\zeta^j\xi^k = -\lambda_i. \tag{8.9}$$

Also, as θ be the angle between the principal normal μ^i and the surface normal ξ^i, then the angle between μ^i and ζ^i is $\frac{\pi}{2} - \theta$, therefore,

$$\cos\theta = \mu^i\xi^j \text{ and } \cos\left(\frac{\pi}{2} - \theta\right) = \mu^i\zeta^j.$$

Therefore, using Eq. (8.6), we get

$$\chi\mu^i\xi^j = b_{\alpha\beta}\lambda^\alpha\lambda^\beta\xi^i\xi^j + \chi_g\zeta^i\xi^j$$

or

$$\chi\cos\theta = b_{\alpha\beta}\lambda^\alpha\lambda^\beta\xi^i\xi^j + \chi_g\zeta^i\xi^j = b_{\alpha\beta}\lambda^\alpha\lambda^\beta. \tag{8.10}$$

From Eq. (8.10) we see that the quantity $b_{\alpha\beta}\lambda^\alpha\lambda^\beta$ is the same for all curves on the surface $\mathcal{S}$ which have the same tangent vector λ^α at P. In particular, it has the same value for the curve formed by the intersection of the normal plane containing $\boldsymbol{\xi}$ and $\boldsymbol{\lambda}$. But for every normal plane section the angle θ is either 0 or π radians, so that for the normal plane section

$$\chi\cos\theta = \chi \text{ or } -\chi$$

and consequently, $\chi\cos\theta$ is also the same for all such curves. Since $b_{\alpha\beta}\lambda^\alpha\lambda^\beta$ is an invariant, the value of $\chi\cos\theta$ for every curve $\mathcal{C}$ tangent to $\boldsymbol{\lambda}$ is equal to the curvature

$\chi_{(n)}$ of the normal section in the direction $\boldsymbol{\lambda}$. The curvature $\chi_{(n)}$ is called the *normal curvature* of the surface $\mathcal{S}$ in the direction $\boldsymbol{\lambda}$. Thus Eq. (8.10) can be written as

$$\chi_{(n)} = \chi \cos\theta = b_{\alpha\beta}\lambda^\alpha\lambda^\beta. \tag{8.11}$$

Therefore, from Eq. (8.11), we state that, for all curves on a surface, which have the same tangent vector, the quantity $\chi\cos\theta$ has the same value, where θ is the angle between the surface normal and the principal normal, χ being the curvature of the curve. Accordingly, Eq. (8.6) can be written in the form

$$\chi\mu^i = \chi_{(n)}\xi^i + \chi_g\zeta^i. \tag{8.12}$$

Equation (8.12) states that $\chi_{(n)}$ and χ_g are the components of the curvature $\chi\mu^i$ in the direction of the vectors ξ^i and ζ^i.

Result 8.1.1 The radius of curvature $\rho = \frac{1}{\chi}$ of any curve at a given point on the surface is equal to the product of the radius of curvature $\rho_{(n)} = \frac{1}{\chi_{(n)}}$ of the corresponding normal section at that point by cosine of the angle between the normal to the surface and the principal normal to the curve. This is known as *Meusnier's theorem.* Mathematically, we have

$$\rho = \pm\rho_{(n)}\cos\theta. \tag{8.13}$$

If $\mathcal{S}$ is a sphere, every normal section is a great circle of the sphere, and if $\mathcal{C}$ is any circle drawn on the sphere, then the result [Eq. (8.13)] becomes obvious from elementary geometric considerations.

If $ds^2 = a_{\alpha\beta}du^\alpha du^\beta$ and $\lambda^\alpha = \frac{du^\alpha}{ds}$, Eq. (8.11) can be written in the form

$$\begin{aligned}\chi_{(n)} &= b_{\alpha\beta}\lambda^\alpha\lambda^\beta = b_{\alpha\beta}\frac{du^\alpha}{ds}\frac{du^\beta}{ds}\\ &= \frac{b_{\alpha\beta}du^\alpha du^\beta}{ds^2} = \frac{b_{\alpha\beta}du^\alpha du^\beta}{a_{\alpha\beta}du^\alpha du^\beta} = \frac{\mathcal{B}}{\mathcal{A}}.\end{aligned} \tag{8.14}$$

Result 8.1.2 If the surface is a plane, the normal curvature $\chi_{(n)} = 0$ at all points of the plane, and if it is a sphere $\chi_{(n)} = \frac{1}{R}$, where R is the radius of the sphere. Accordingly, we conclude from Eq. (8.14) that for the plane $b_{\alpha\beta} = 0$. Also, for the sphere

$$b_{\alpha\beta}du^\alpha du^\beta = \frac{1}{R}a_{\alpha\beta}du^\alpha du^\beta \tag{8.15}$$

so that $a_{\alpha\beta} = Rb_{\alpha\beta}$ at all points of the sphere and for all directions du^α. Therefore, for a sphere $b_{\alpha\beta} = \frac{1}{R}a_{\alpha\beta}$, i.e. $b_{\alpha\beta}$ and $a_{\alpha\beta}$ are proportional.

Deduction 8.1.1 Since ζ^i is perpendicular to λ^i, it lies in the plane containing ξ^i and μ^i. Also, it is tangent to the surface and hence the angle between μ^i and ζ^i is $\frac{\pi}{2} - \theta$. If we multiply Eq. (8.6) by ζ^i, we get

$$\chi \cos\left(\frac{\pi}{2} - \theta\right) = \chi_g \Rightarrow \chi_g = \chi \sin\theta, \tag{8.16}$$

which is the formula connecting the geodesic curvature of a surface curve with its curvature.

Deduction 8.1.2 From Eq. (8.14), we have

$$\chi_{(n)} = \frac{b_{\alpha\beta} du^\alpha \ du^\beta}{a_{\alpha\beta} du^\alpha \ du^\beta} = \frac{\mathcal{B}}{\mathcal{A}}.$$

Squaring and adding Eqs. (8.11) and (8.16), we obtain

$$\chi_{(n)}^2 + \chi_g^2 = (\chi\cos\theta)^2 + (\chi\sin\theta)^2 = \chi^2.$$

Also, from the Eq. (8.12), it follows that:

$$g_{ij}\chi\mu^i\mu^j = g_{ij} b_{\alpha\beta}\lambda^\alpha\lambda^\beta\xi^i\mu^j + \chi_g g_{ij}\zeta^i\mu^j$$

or

$$\begin{aligned}\chi &= b_{\alpha\beta}\lambda^\alpha\lambda^\beta\cos\theta + \chi_g\cos\left(\frac{\pi}{2} - \theta\right)\\ &= \chi_{(n)}\cos\theta + \chi_g\sin\theta.\end{aligned}$$

Deduction 8.1.3 On using relations (8.9) and (8.10), we get from Eq. (8.6)

$$\chi\mu^i = b_{\alpha\beta}\lambda^\alpha\lambda^\beta\xi^i + \chi_g\zeta^i$$

or

$$\chi\mu^i = \chi\cos\theta\xi^i + \chi\sin\theta\zeta^i$$

or

$$\mu^i = \cos\theta\xi^i + \sin\theta\zeta^i.$$

Deduction 8.1.4 If the surface is a plane, then any normal section at any point of the plane is a straight line and, therefore, its curvature is zero and hence $\chi_{(n)} = 0$. Therefore,

$$b_{\alpha\beta} du^\alpha du^\beta = 0,$$

for all directions du^α, i.e. $b_{\alpha\beta} = 0$. Thus, for a plane $b_{\alpha\beta} = 0$.

Deduction 8.1.5 The normal sections of a sphere are great circles, any other plane section is a small circle. At any point the circles of curvature are identical with the normal sections and therefore lie on the sphere.

EXAMPLE 8.1.1 *Find the normal curvature of the right helicoid.*

Solution: The parametric representation of the right helicoid is given by

$$x^1 = u^1 \cos u^2; \; x^2 = u^1 \sin u^2; \;\; x^3 = cu^2$$

i.e.

$$\mathbf{r} = (x^1, x^2, x^3) = \left(u^1 \cos u^2; \; u^1 \sin u^2; cu^2\right).$$

The symmetric covariant second order tensors $a_{\alpha\beta}$ for the first fundamental form of the surface are given by

$$a_{11} = 1; \;\; a_{22} = (u^1)^2 + c^2 \text{ and } a_{12} = 0 = a_{21}.$$

$$\Rightarrow |a_{\alpha\beta}| = \begin{vmatrix} 1 & 0 \\ 0 & c^2 + (u^1)^2 \end{vmatrix} = c^2 + (u^1)^2 > 0.$$

The symmetric covariant tensors $b_{\alpha\beta}$ for the second fundamental form are given by

$$b_{11} = 0, b_{22} = 0 \text{ and } b_{12} = -\frac{c}{\sqrt{c^2 + (u^1)^2}}.$$

$$\Rightarrow |b_{\alpha\beta}| = \begin{vmatrix} 0 & -\dfrac{c}{\sqrt{c^2 + (u^1)^2}} \\ -\dfrac{c}{\sqrt{c^2 + (u^1)^2}} & 0 \end{vmatrix} = -\frac{c^2}{c^2 + (u^1)^2}.$$

Therefore, the Gaussian curvature is given by

$$\kappa = \frac{|b_{\alpha\beta}|}{|a_{\alpha\beta}|} = -\frac{c^2}{[c^2 + (u^1)^2]^2} < 0.$$

Thus, the given surface is a surface of negative curvature. Also, it is a minimal surface (as $H = 0$). Thus, the principal curvatures $\chi_{(1)}$ and $\chi_{(2)}$ are related as

$$\chi_{(1)} + \chi_{(2)} = 2H = 0 \text{ and } \;\; \chi_{(1)} \cdot \chi_{(2)} = \kappa = -\frac{c^2}{[c^2 + (u^1)^2]^2}.$$

Now, the normal curvature $\chi_{(n)}$ of the given right helicoid is given by

$$\chi_{(n)} = \frac{b_{\alpha\beta}du^\alpha du^\beta}{a_{\alpha\beta}du^\alpha du^\beta} = \frac{b_{11}(du^1)^2 + b_{22}(du^2)^2 + 2b_{12}du^1du^2}{a_{11}(du^1)^2 + a_{22}(du^2)^2 + 2a_{12}du^1du^2}$$

$$= \frac{\left[-2c/\sqrt{c^2+(u^1)^2}\right]du^1du^2}{(du^1)^2 + \{c^2+(u^1)^2\}(du^2)^2}.$$

$$= -\frac{1}{\sqrt{c^2+(u^1)^2}}\frac{2cdu^1du^2}{[(du^1)^2 + \{c^2+(u^1)^2\}(du^2)^2]}.$$

EXAMPLE 8.1.2 *Prove that the normal curvatures in the directions of the co-ordinate curves are* $\frac{b_{11}}{a_{11}}$ *and* $\frac{b_{22}}{a_{22}}$*, respectively.*

Solution: Let the co-ordinate curve or parametric curve be taken, then for u^1 curve; i.e. $du^2 = 0$, the normal curvature $\chi_{(n1)}$ is given by

$$\chi_{(n1)} = \frac{b_{\alpha\beta}du^\alpha du^\beta}{a_{\alpha\beta}du^\alpha du^\beta} = \frac{b_{11}(du^1)^2 + b_{22}(du^2)^2 + 2b_{12}du^1du^2}{a_{11}(du^1)^2 + a_{22}(du^2)^2 + 2a_{12}du^1du^2}$$

$$= \frac{b_{11}(du^1)^2}{a_{11}(du^1)^2} = \frac{b_{11}}{a_{11}};\ \text{as } du^1 \neq 0.$$

The normal curvature $\chi_{(n2)}$ along the u^2 curve is given by

$$\chi_{(n2)} = \frac{b_{\alpha\beta}du^\alpha du^\beta}{a_{\alpha\beta}du^\alpha du^\beta} = \frac{b_{11}(du^1)^2 + b_{22}(du^2)^2 + 2b_{12}du^1du^2}{a_{11}(du^1)^2 + a_{22}(du^2)^2 + 2a_{12}du^1du^2}$$

$$= \frac{b_{22}(du^2)^2}{a_{22}(du^2)^2} = \frac{b_{22}}{a_{22}};\ \text{as } du^2 \neq 0.$$

EXAMPLE 8.1.3 *If a curve is a geodesic on the surface, prove that it is either a straight line or its principal normal is orthogonal to the surface at every point and conversely.*

Solution: From the formula, $\chi_g = \chi\sin\theta$, we get for a geodesic

$$\chi_g = \chi\sin\theta = 0 \Rightarrow \text{ either } \chi = 0 \text{ or } \sin\theta = 0.$$

Therefore, when $\chi = 0$, the curve is a straight line. If $\sin\theta = 0$, then $\theta = 0$ or π. If $\theta = 0$ or π, then the principal normal and the surface normal are colinear. Consequently, the principal normal is orthogonal to the surface at every point.

EXAMPLE 8.1.4 *Given the paraboloid of revolution* $S; x^3 = (x^1)^2 + (x^2)^2$*. Determine the radius and centre of the circle of curvature of the normal section of* S *at a point* $P : x^3 = x_0^3$ *whose tangent at* P *is parallel to the* x^1x^2*-plane.*

Solution: The intersection of the paraboloid S and the plane $E : x^3 = x_0^3$ is a circle (Figure 8.2) of radius $\rho = \sqrt{x_0^3} = r_0$. This is not a normal section of S. Let θ be the angle between E and the normal to S, this can be obtained

$$\tan\left[\frac{1}{2}\pi - \theta\right] = \tan\alpha = 2r_0$$

or

$$\cos\theta = \sin\alpha = \frac{\tan\alpha}{\sqrt{1+\tan^2\alpha}} = \frac{2r_0}{\sqrt{1+4r_0^2}}$$

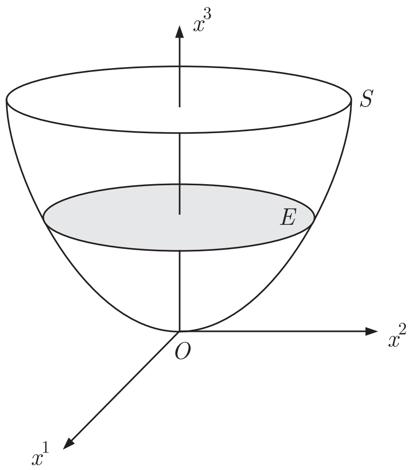

Figure 8.2: Paraboloid of revolution.

According to Meusnier's theorem we thus obtain

$$\rho_{(n)} = \frac{\rho}{\cos\theta} = \frac{1}{2}\sqrt{1+4r_0^2}$$

The centre of curvature of the normal section under consideration lies on the axis of revolution.

8.2 Principal Curvatures

The point of intersection of consecutive normals along a line of curvature at P is called a *centre of curvature* of the surface; and its distance from P, measured in the direction of the unit normal, is called a *(principal) radius of curvature* of the surface. The reciprocal of the principal radius of curvature is called a *principal curvature.*

Here, we will concern with the determination of the principal curvatures of a surface. In Section 8.1 we considered the normal curvature $\chi_{(n)}$ at an arbitrary point

P of a surface as a function of the direction of the tangent to the normal sections at P as Eq. (8.14)

$$\chi_{(n)} = \frac{b_{\alpha\beta} du^\alpha du^\beta}{ds^2} = \frac{b_{\alpha\beta} du^\alpha du^\beta}{a_{\alpha\beta} du^\alpha du^\beta} = \frac{\mathcal{B}}{\mathcal{A}}.$$

The normal curvature $\chi_{(n)}$ at a point depends on the directions of $\lambda^\alpha = \frac{du^\alpha}{ds}$. We shall now find out those directions $\lambda^\alpha = \frac{du^\alpha}{ds}$ (i.e. those values of $du^2 : du^1$) on the surface for which the normal curvature $\chi_{(n)}$ has the extreme values. These directions are called the *principal directions* at the given point and the corresponding values for $\chi_{(n)}$ are called *principal normal curvatures* of the surface at P. Let us denote $\chi_{(n)}$ by $\chi_{(p)}$ for the principal curvatures.

Since the vector λ^α is a unit vector, $\chi_{(n)}$ has to be maximised subject to the constraint $a_{\alpha\beta}\lambda^\alpha\lambda^\beta = 1$. The necessary condition for the extremum is

$$b_{\alpha\beta}\lambda^\beta + \vartheta a_{\alpha\beta}\lambda^\beta = 0; \vartheta = \text{ Lagrangian parameter}$$

or

$$b_{\alpha\beta}\lambda^\alpha\lambda^\beta + \vartheta a_{\alpha\beta}\lambda^\alpha\lambda^\beta = 0$$

or

$$\chi_{(p)} + \vartheta 1 = 0 \Rightarrow \vartheta = -\chi_{(p)}.$$

Thus, the equation, for the determination of directions yielding extreme values of $\chi_{(n)}$ as $\chi_{(p)}$, can be written as

$$\left(b_{\alpha\beta} - \chi_{(p)} a_{\alpha\beta}\right)\lambda^\beta = 0; \quad \alpha = 1, 2. \tag{8.17}$$

The above set of homogeneous equations will possess non-trivial solutions for λ^β if and only if, the values of $\chi_{(p)}$ are the roots of the determinant equation

$$\left|b_{\alpha\beta} - \chi_{(p)} a_{\alpha\beta}\right| = 0$$

or

$$\begin{vmatrix} b_{11} - \chi_{(p)} a_{11} & b_{12} - \chi_{(p)} a_{12} \\ b_{21} - \chi_{(p)} a_{21} & b_{22} - \chi_{(p)} a_{22} \end{vmatrix} = 0$$

or

$$\begin{aligned} & b_{11}b_{22} - (a_{11}b_{22} + a_{22}b_{11})\chi_{(p)} + \chi_{(p)}^2 a_{11}a_{22} \\ & \qquad -b_{12}b_{21} + (a_{12}b_{21} + a_{21}b_{12})\chi_{(p)} - \chi_{(p)}^2 a_{12}a_{21} = 0 \end{aligned}$$

or

$$(b_{11}b_{22} - b_{12}b_{21}) + \chi^2_{(p)}(a_{11}a_{22} - a_{12}a_{21})$$
$$+\chi_{(p)}(a_{12}b_{21} + a_{21}b_{12} - a_{11}b_{22} - a_{22}b_{11}) = 0.$$

or

$$\chi^2_{(p)} - \frac{a_{11}b_{22} + b_{11}a_{22} - 2a_{12}b_{12}}{a} + \frac{b}{a} = 0; \quad b = |b_{\alpha\beta}|, a = |a_{\alpha\beta}|. \tag{8.18}$$

The roots of the quadratic Eq. (8.18) determine those directions for which the normal curvature $\chi_{(n)}$ becomes extreme. Those directions are called the principal directions of normal curvature (or curvature directions) at the point P under consideration. The centres of curvature of the corresponding normal sections are called the centres of principal curvature of the surface $\mathcal{S}$ at P.

Also, since $a_{11} = aa^{22}, a_{12} = -aa^{21}$, $a_{21} = -aa^{12}$ and $a_{22} = aa^{11}$, so the above equation can be written in the form

$$a\chi^2_{(p)} - (aa^{22}b_{22} + aa^{11}b_{11} + aa^{21}b_{21} + aa^{12}b_{12})\chi_{(p)} + b = 0$$

or

$$a\chi^2_{(p)} - aa^{\alpha\beta}b_{\alpha\beta}\chi_{(p)} + b = 0$$

or

$$\chi^2_{(p)} - a^{\alpha\beta}b_{\alpha\beta}\chi_{(p)} + \frac{b}{a} = 0$$

or

$$\chi^2_{(p)} - 2H\chi_{(p)} + \kappa = 0, \tag{8.19}$$

where $H = \frac{1}{2}a^{\alpha\beta}b_{\alpha\beta}$ is the mean curvature and $\kappa = \frac{b}{a}$ is the Gaussian curvature. We shall denote the two roots of $\chi_{(p)}$ by $\chi_{(1)}$ and $\chi_{(2)}$ and call them the *principal curvature* of the surface and the directions corresponding to $\chi_{(1)}$ and $\chi_{(2)}$ are called *principal directions* of the surface. Note that, the principal directions on the surfaces are real.

Definition 8.2.1 Those portions of the surface on which the two principal curvatures $\chi_{(1)}$ and $\chi_{(2)}$ have the same sign are said to be *synclasic.* The surface of a sphere or of an ellipsoid as synclasic at all points. On the other hand if the principal curvatures have opposite signs on any part of the surface, this part is said to be *anticlastic.* The surface of a hyperbolic paraboloid is anticlastic at all points.

Result 8.2.1 The principal directions are real and orthogonal.

Proof: The reality of the principal directions is obvious geometrically. From Eq. (8.19) it is clear that, the principal curvatures $\chi_{(1)}$ and $\chi_{(2)}$ are related to the mean and Gaussian curvatures by the formulas

$$\chi_{(1)} + \chi_{(2)} = 2H \text{ and } \chi_{(1)}\chi_{(2)} = \kappa. \tag{8.20}$$

From Eq. (8.17), it follows that, the principal directions $\lambda^{\beta}_{(1)}$, $\lambda^{\beta}_{(2)}$, say, corresponding to $\chi_{(1)}$ and $\chi_{(2)}$, respectively, are determined by

$$\left.\begin{aligned}\left(b_{\alpha\beta} - \chi_{(1)}a_{\alpha\beta}\right)\lambda^{\beta}_{(1)} = 0\\ \left(b_{\alpha\beta} - \chi_{(2)}a_{\alpha\beta}\right)\lambda^{\beta}_{(2)} = 0\end{aligned}\right\} \tag{8.21}$$

Multiplying the first part of Eq. (8.21) by $\lambda^{\alpha}_{(2)}$ and second part of Eq. (8.21) by $\lambda^{\alpha}_{(2)}$ and subtracting, we get

$$\left(\chi_{(2)} - \chi_{(1)}\right) a_{\alpha\beta} \;\; \lambda^{\alpha}_{(1)}\lambda^{\beta}_{(2)} = 0. \tag{8.22}$$

A point at which $\chi_{(1)} = \chi_{(2)}$ is called an *umbilic point* or naval point. At an umbilic point, Eq. (8.19) has coincident roots, i.e. $H^2 = \kappa$, so that, it can be written in the form

$$\left[a^{\alpha\beta}b_{\alpha\beta}\right]^2 = 4\frac{b}{a}$$

or

$$\left[a^{11}b_{11} + a^{22}b_{22} + a^{21}b_{21} + a^{12}b_{12}\right]^2 = 4\frac{b}{a}$$

or

$$4a(a_{11}b_{12} - a_{12}b_{11})^2 + [a_{11}(a_{11}b_{22} - a_{22}b_{11}) - 2a_{12}(a_{11}b_{12} - a_{12}b_{11})]^2 = 0.$$

Since $a_{ij}dx^i dx^j$ is positive definite, a is positive, and thus, we have

$$\begin{aligned} a_{11}b_{12} - a_{12}b_{11} = a_{11}b_{22} - a_{22}b_{11} = 0\\ \Rightarrow \frac{b_{11}}{a_{11}} = \frac{b_{12}}{a_{12}} = \frac{b_{22}}{a_{22}}.\end{aligned} \tag{8.23}$$

Thus, at an umbilic we have the above condition Eq. (8.23). From Eq. (8.14), since

$$\chi_{(n)} = \frac{b_{\alpha\beta}du^{\alpha}du^{\beta}}{a_{\alpha\beta}du^{\alpha}du^{\beta}},$$

so, $\chi_{(n)}$ is independent of the direction $\frac{du^{\alpha}}{ds}$, i.e. at an umbilic, the normal curvature is the same in every directions. At all other points where $\chi_{(1)} \neq \chi_{(2)}$, we have

$$a_{\alpha\beta}\lambda^{\alpha}_{(1)}\lambda^{\beta}_{(2)} = 0. \tag{8.24}$$

Therefore, at each non-umbilical point of a surface there exist two mutually orthogonal directions for which the normal curvature attains its extreme values.

If the principal curvatures at any point of a surface are equal in magnitude and opposite in sign and the indication is a rectangular hyperbola then it is a minimal surface.

EXAMPLE 8.2.1 *Find the principal curvature of the surface defined by*

$$x^1 = u^1; x^2 = u^2; \quad x^3 = f(u^1, u^2).$$

Solution: The parametric representation of the given curve is given by

$$x^1 = u^1;\ x^2 = u^2;\ x^3 = f(u^1, u^2)$$

i.e.

$$\mathbf{r} = (x^1, x^2, x^3) = \left(u^1, u^2, f(u^1, u^2)\right).$$

The symmetric covariant second order tensors $a_{\alpha\beta}$ for the first fundamental form of the surface are given by

$$a_{11} = \left(\frac{\partial x^i}{\partial u^1}\right)^2 = 1^2 + 0^2 + \left(\frac{\partial f}{\partial u^1}\right)^2 = 1 + f_1^2.$$

$$a_{22} = \left(\frac{\partial x^i}{\partial u^2}\right)^2 = 0^2 + 1^2 + \left(\frac{\partial f}{\partial u^2}\right)^2 = 1 + f_2^2.$$

$$a_{12} = \frac{\partial x^i}{\partial u^1}\frac{\partial x^i}{\partial u^2} = 1.0 + 0.1 + \frac{\partial f}{\partial u^1}\frac{\partial f}{\partial u^2} = f_1 f_2 = a_{21}$$

where $f_1 = \frac{\partial f}{\partial u^1}$ and $f_2 = \frac{\partial f}{\partial u^2}$. Therefore,

$$a = |a_{\alpha\beta}| = \begin{vmatrix} 1 + f_1^2 & f_1 f_2 \\ f_1 f_2 & 1 + f_2^2 \end{vmatrix} = 1 + f_1^2 + f_2^2.$$

The reciprocal tensors are given by

$$a^{11} = \frac{1 + f_2^2}{1 + f_1^2 + f_2^2}; \quad a^{22} = \frac{1 + f_1^2}{1 + f_1^2 + f_2^2}; \quad a^{12} = -\frac{f_1 f_2}{1 + f_1^2 + f_2^2} = a^{21}.$$

Now, we calculate the tensors of second fundamental form. Since

$$\mathbf{r} = (x^1, x^2, x^3) = \left(u^1, u^2, f(u^1, u^2)\right),$$

so

$$\mathbf{A} = (1, 0, f_1) \text{ and } \mathbf{B} = (0, 1, f_2); \quad \mathbf{A} \times \mathbf{B} = (-f_1, -f_2, 1)$$

$$\boldsymbol{\xi} = \frac{1}{|\mathbf{A} \times \mathbf{B}|} (\mathbf{A} \times \mathbf{B}) = \frac{1}{\sqrt{f_1^2 + f_2^2 + 1}} (-f_1, -f_2, 1).$$

Thus, the covariant tensors $b_{\alpha\beta}$ for the second fundamental form are given by

$$\begin{aligned} b_{11} = -\frac{\partial \boldsymbol{\xi}}{\partial u^1} \cdot \frac{\partial \mathbf{r}}{\partial u^1} &= -(f_1^2 + f_2^2 + 1)^{-1/2}[(-f_{11}, -f_{12}, 0) \cdot (1, 0, f_1) \\ &\quad -(f_1^2 + f_2^2 + 1)(f_1 f_{11} + f_2 f_{12})(-f_1, -f_2, 1) \cdot (1, 0, f_1)] \\ &= \frac{f_{11}}{\sqrt{f_1^2 + f_2^2 + 1}}. \\ b_{22} = -\frac{\partial \boldsymbol{\xi}}{\partial u^2} \cdot \frac{\partial \mathbf{r}}{\partial u^2} &= -(f_1^2 + f_2^2 + 1)^{-1/2}[(-f_{21}, -f_{22}, 0) \cdot (0, 1, f_2) \\ &\quad -(f_1^2 + f_2^2 + 1)(f_1 f_{21} + f_2 f_{22})(f_1, f_2, -1) \cdot (0, 1, f_2)] \\ &= \frac{f_{22}}{\sqrt{f_1^2 + f_2^2 + 1}}. \end{aligned}$$

$$\begin{aligned} b_{12} &= -\frac{1}{2}\left[\frac{\partial \boldsymbol{\xi}}{\partial u^1} \cdot \frac{\partial \mathbf{r}}{\partial u^2} + \frac{\partial \boldsymbol{\xi}}{\partial u^2} \cdot \frac{\partial \mathbf{r}}{\partial u^1}\right] = -\frac{1}{2}(f_1^2 + f_2^2 + 1)^{-1/2}[\{(-f_{11}, -f_{12}, 0) \cdot (0, 1, f_2) \\ &\quad +(f_1^2 + f_2^2 + 1)(f_1 f_{11} + f_2 f_{12})(-f_1, -f_2, 1) \cdot (0, 1, f_2)\} \\ &\quad +\{(-f_{21}, -f_{22}, 0) \cdot (1, 0, f_1) + (f_1^2 + f_2^2 + 1)(f_1 f_{21} + f_2 f_{22})(f_1, f_2, -1) \cdot (1, 0, f_1)\}] \\ &= \frac{f_{12}}{\sqrt{f_1^2 + f_2^2 + 1}} = b_{21}. \end{aligned}$$

The symmetric covariant tensors $b_{\alpha\beta}$ for the second fundamental form are given by

$$b_{11} = \frac{f_{11}}{\sqrt{f_1^2 + f_2^2 + 1}}, b_{22} = \frac{f_{22}}{\sqrt{f_1^2 + f_2^2 + 1}} \text{ and } b_{12} = \frac{f_{12}}{\sqrt{f_1^2 + f_2^2 + 1}} = b_{21}.$$

$$\Rightarrow b = |b_{\alpha\beta}| = \begin{vmatrix} \frac{f_{11}}{P} & \frac{f_{12}}{P} \\ \frac{f_{12}}{P} & \frac{f_{22}}{P} \end{vmatrix} = \frac{f_{11} f_{22} - f_{12}^2}{1 + f_1^2 + f_2^2}; \quad P = \sqrt{f_1^2 + f_2^2 + 1},$$

where $f_i = \frac{\partial f}{\partial u^i}$ and $f_{ij} = \frac{\partial^2 f}{\partial u^i \partial u^j}$. The equation giving the principal curvatures is

$$\chi_\rho^2 - a^{\alpha\beta} b_{\alpha\beta} \chi_\rho + \frac{b}{a} = 0$$

or

$$\chi_\rho^2 - \left[(f_{11}+f_{22}) - \frac{f_1^2 f_{11} + f_2^2 f_{22} + 2f_1 f_2 f_{12}}{1+f_1^2+f_2^2}\right]\chi_\rho + f_{11}f_{22} - f_{12}^2 = 0.$$

This is a quadratic equation in χ_ρ which gives the two values of χ_ρ. At an umbilic, we have

$$\frac{b_{11}}{a_{11}} = \frac{b_{12}}{a_{12}} = \frac{b_{22}}{a_{22}} \Rightarrow \frac{f_{11}}{1+f_1^2} = \frac{f_{12}}{f_1 f_2} = \frac{f_{22}}{1+f_2^2}.$$

EXAMPLE 8.2.2 *Find the principal curvature of the surface defined by*

$$x^1 = u^1 \cos u^2;\; x^2 = u^1 \sin u^2;\; x^3 = f(u^1).$$

Find the condition that it is a minimal surface.

Solution: The parametric representation of the surface of revolution is given by

$$x^1 = u^1 \cos u^2; x^2 = u^1 \sin u^2;\; x^3 = f(u^1)$$

i.e.

$$\mathbf{r} = (x^1, x^2, x^3) = \left(u^1 \cos u^2; u^1 \sin u^2; f(u^1)\right).$$

The first fundamental magnitudes $a_{\alpha\beta}$ of the surface are given by

$$a_{11} = 1 + f_1^2;\;\; a_{22} = (u^1)^2 \text{ and } a_{12} = 0 = a_{21}.$$

$$\Rightarrow a = |a_{\alpha\beta}| = \begin{vmatrix} 1+f_1^2 & 0 \\ 0 & (u^1)^2 \end{vmatrix} = (u^1)^2(1+f_1^2),$$

where $f_1 = \frac{\partial f}{\partial u^1}$. The reciprocal tensors are given by

$$a^{11} = \frac{1}{1+f_1^2};\; a^{22} = \frac{1}{(u^1)^2};\; a^{12} = 0 = a^{21}.$$

The symmetric covariant tensors $b_{\alpha\beta}$ for the second fundamental form are given by

$$b_{11} = \frac{f_{11}}{\sqrt{1+f_1^2}}, b_{22} = \frac{u^1 f_1}{\sqrt{1+f_1^2}} \text{ and } b_{12} = 0 = b_{21}.$$

$$\Rightarrow |b_{\alpha\beta}| = \begin{vmatrix} \dfrac{f_{11}}{\sqrt{1+f_1^2}} & 0 \\ 0 & \dfrac{u^1 f_1}{\sqrt{1+f_1^2}} \end{vmatrix} = \frac{u^1 f_1 f_{11}}{1+f_1^2},$$

where $f_1 = \frac{\partial f}{\partial u^1}$ and $f_{11} = \frac{\partial^2 f}{\partial u^1 \partial u^1}$. The equation giving the principal curvatures is

$$\chi_\rho^2 - a^{\alpha\beta} b_{\alpha\beta} \chi_\rho + \frac{b}{a} = 0$$

or

$$\chi_\rho^2 - \left[\frac{f_{11}}{(1+f_1^2)^{3/2}} + \frac{f_1}{u^1\sqrt{1+f_1^2}}\right]\chi_\rho + \frac{f_1 f_{11}}{u^1(1+f_1^2)^2} = 0.$$

This is a quadratic equation in χ_ρ which gives the following two values of χ_ρ:

$$\chi_{(1)} = \frac{f_{11}}{(1+f_1^2)^{3/2}}; \quad \chi_{(2)} = \frac{f_1}{u^1(1+f_1^2)^{1/2}}.$$

If ρ_1 and ρ_2 be the corresponding radii of curvatures then

$$\rho_1 = \frac{1}{\chi_{(1)}} = \frac{(1+f_1^2)^{3/2}}{f_{11}}; \quad \rho_2 = \frac{1}{\chi_{(2)}} = \frac{u^1(1+f_1^2)^{1/2}}{f_1}.$$

The condition for the surface to be minimal is that

$$2H = \chi_{(1)} + \chi_{(2)} = 0$$

or

$$\frac{f_{11}}{(1+f_1^2)^{3/2}} + \frac{f_1}{u^1(1+f_1^2)^{1/2}} = 0$$

or

$$f_1(1+f_1^2) + u^1 f_{11} = 0.$$

EXAMPLE 8.2.3 *Find the principal curvature of the surface defined by*

$$x^1 = u^1\cos u^2; x^2 = u^1\sin u^2; \quad x^3 = a\log\left(u^1 + \sqrt{(u^1)^2 - a^2}\right).$$

Prove that it is a minimal surface and is the only minimal surface.

Solution: From example 8.2.2 we see that, the parametric representation of the surface of revolution is given by

$$x^1 = u^1\cos u^2; \; x^2 = u^1\sin u^2; \; x^3 = f(u^1),$$

where $f(u^1) = a\log\left(u^1 + \sqrt{(u^1)^2 - a^2}\right)$. Therefore,

$$f_1 = \frac{\partial f}{\partial u^1} = \frac{a}{\sqrt{(u^1)^2 - a^2}}; \quad f_{11} = \frac{\partial^2 f}{(\partial u^1)^2} = -\frac{a u^1}{[(u^1)^2 - a^2]^{3/2}}.$$

Therefore, $\chi_{(1)}$ and $\chi_{(2)}$ are given by

$$\chi_{(1)} = \frac{f_{11}}{(1+f_1^2)^{3/2}} = \frac{-a}{(u^1)^2}; \; \chi_{(2)} = \frac{f_1}{u^1(1+f_1^2)^{1/2}} = \frac{a}{(u^1)^2}.$$

Since $2H = \chi_{(1)} + \chi_{(2)} = 0$, the surface is a minimal surface. Now, we will prove that the given surface of revolution is the minimal surface. Let the equation of any surface of revolution be

$$x^1 = u^1 \cos u^2; x^2 = u^1 \sin u^2; \quad x^3 = f(u^1).$$

As in example 8.2.2, the condition for the minimal surface is

$$f_1(1 + f_1^2) + u^1 f_{11} = 0$$

or

$$u^1 \frac{d^2 f}{d(u^1)^2} + \frac{df}{du^1}\left[1 + \left(\frac{df}{du^1}\right)^2\right] = 0$$

or

$$\frac{dp}{p(1+p^2)} + \frac{du}{u} = 0; \quad \text{where, } p = \frac{df}{du^1}$$

or

$$\log p - \frac{1}{2}\log(1+p^2) + \log u^1 = \text{ constant}$$

or

$$\frac{p^2(u^1)^2}{1+p^2} = a^2 \text{ (say)} \Rightarrow p = \frac{df}{du^1} = \frac{a}{\sqrt{(u^1)^2 - a^2}}$$

or

$$x^3 = f(u^1) = a\cosh^{-1}\frac{u^1}{a} + c_1 = a\log\left(u^1 + \sqrt{(u^1)^2 - a^2}\right) + c_1.$$

Choosing $x^3 = a \log a$ when $u^1 = a$ so that $c_1 = 0$, so,

$$\begin{aligned} x^3 = f(u^1) &= a\log\left(u^1 + \sqrt{(u^1)^2 - a^2}\right) \\ &= a\cosh^{-1}\frac{\sqrt{(x^1)^2 + (x^2)^2}}{a} \end{aligned}$$

or

$$\sqrt{(x^1)^2 + (x^2)^2} = a\cosh\frac{x^3}{a}.$$

Thus, except for position in space, the only surface of revolution is

$$x^1 = u^1 \cos u^2; x^2 = u^1 \sin u^2; \quad x^3 = a\log\left(u^1 + \sqrt{(u^1)^2 - a^2}\right)$$

or $\sqrt{(x^1)^2 + (x^2)^2} = a\cosh\frac{x^3}{a}$ is the only minimal surface formed by revolution of catenary $x^1 = a\cosh\frac{x^3}{a}$ about x^1 axis, i.e. directrix.

EXAMPLE 8.2.4 *Find the equation for the principal curvatures of the surface for the right helicoid.*

Solution: The parametric representation of the right helicoid is given by

$$x^1 = u^1 \cos u^2;\ x^2 = u^1 \sin u^2;\ x^3 = cu^2$$

i.e.

$$\mathbf{r} = (x^1, x^2, x^3) = \left(u^1 \cos u^2;\ u^1 \sin u^2; cu^2\right).$$

The symmetric covariant second order tensors $a_{\alpha\beta}$ for the first fundamental form of the surface are given by

$$a_{11} = 1;\ a_{22} = (u^1)^2 + c^2 \text{ and } a_{12} = 0 = a_{21}.$$

$$\Rightarrow |a_{\alpha\beta}| = \begin{vmatrix} 1 & 0 \\ 0 & c^2 + (u^1)^2 \end{vmatrix} = c^2 + (u^1)^2.$$

The reciprocal tensors are given by

$$a^{11} = \frac{c^2 + (u^1)^2}{c^2 + (u^1)^2} = 1; a^{22} = \frac{1}{c^2 + (u^1)^2}; a^{12} = 0 = a^{21}.$$

The symmetric covariant tensors $b_{\alpha\beta}$ for the second fundamental form are given by

$$b_{11} = 0, b_{22} = 0 \text{ and } b_{12} = -\frac{c}{\sqrt{c^2 + (u^1)^2}}.$$

$$\Rightarrow |b_{\alpha\beta}| = \begin{vmatrix} 0 & -\frac{c}{\sqrt{c^2+(u^1)^2}} \\ -\frac{c}{\sqrt{c^2+(u^1)^2}} & 0 \end{vmatrix} = -\frac{c^2}{c^2 + (u^1)^2}.$$

The equation giving the principal curvatures is

$$\chi_\rho^2 - a^{\alpha\beta} b_{\alpha\beta} \chi_\rho + \frac{b}{a} = 0$$

or

$$\chi_\rho^2 - \left[a^{11}b_{11} + a^{22}b_{22} + 2a^{12}b_{12}\right] \chi_\rho + \frac{b}{a} = 0$$

or

$$\chi_\rho^2 - \left[1 \cdot 0 + \frac{1}{c^2 + (u^1)^2} \cdot 0 + 0 \cdot \left(-\frac{c^2}{c^2 + (u^1)^2}\right)\right] \chi_\rho + \frac{b}{a} = 0$$

or

$$\chi_\rho^2 - \frac{c^2}{[c^2 + (u^1)^2]^2} = 0 \Rightarrow \chi_\rho = \pm \frac{c}{c^2 + (u^1)^2}.$$

This is a quadratic equation in χ_ρ which gives the following two values of χ_ρ:

$$\chi_{(1)} = \frac{c}{c^2 + (u^1)^2};\ \chi_{(2)} = -\frac{c}{c^2 + (u^1)^2}.$$

Since $2H = \chi_{(1)} + \chi_{(2)} = 0$, the surface is a minimal surface.

EXAMPLE 8.2.5 *Calculate the principal curvatures for the conicoid*

$$x^1 = u^1 \cos u^2;\ x^2 = u^1 \sin u^2;\ x^3 = f(u^2),$$

where u^1, u^2 are parameters. Hence, find the Gaussian and the mean curvature.

Solution: The parametric representation of the surface is given by

$$\mathbf{r} = (x^1, x^2, x^3) = \left(u^1 \cos u^2;\ \ u^1 \sin u^2;\ \ f(u^2)\right).$$

The first fundamental magnitudes $a_{\alpha\beta}$ are given by

$$a_{11} = 1;\ a_{22} = (u^1)^2 + f_1^2;\ a_{12} = 0 = a_{21},$$

where $f_1 = \frac{\partial f}{\partial u^2}$. Therefore, $a = (u^1)^2 + f_1^2$. The reciprocal tensors are given by

$$a^{11} = \frac{(u^1)^2 + f_1^2}{(u^1)^2 + f_1^2} = 1;\ \ a^{22} = \frac{1}{(u^1)^2 + f_1^2};\ \ a^{12} = \frac{0}{(u^1)^2 + f_1^2} = 0 = a^{21}.$$

The symmetric covariant tensors $b_{\alpha\beta}$ are given by

$$b_{11} = 0;\ b_{22} = -\frac{f_1}{\sqrt{(u^1)^2 + f_1^2}};\ b_{12} = \frac{u^1 f_2}{\sqrt{(u^1)^2 + f_1^2}} = b_{21},$$

where $f_2 = \frac{\partial^2 f}{(\partial u^2)^2}$ so, $b = \frac{-\left(u^1 f_2\right)^2}{(u^1)^2 + f_1^2}$. The equation giving the principal curvatures is

$$\chi_\rho^2 - a^{\alpha\beta} b_{\alpha\beta} \chi_\rho + \frac{b}{a} = 0$$

or

$$\chi_\rho^2 + \left[\frac{f_1}{\{(u^1)^2 + f_1^2\}^{3/2}}\right] \chi_\rho - \frac{\left(u^1 f_2\right)^2}{\left[(u^1)^2 + f_1^2\right]^2} = 0.$$

This is a quadratic equation in χ_ρ which gives the two values $\chi_{(1)}$ and $\chi_{(2)}$. The mean curvature H is given by

$$H = \frac{1}{2}[\chi_{(1)} + \chi_{(2)}] = \frac{1}{2} \frac{-f_1}{\{(u^1)^2 + f_1^2\}^{3/2}}$$

and the Gaussian curvature κ is given by the formula,

$$\kappa = \chi_{(1)} \chi_{(2)} = -\frac{\left(u^1 f_2\right)^2}{\left[(u^1)^2 + f_1^2\right]^2}.$$

EXAMPLE 8.2.6 *Show that the surface* $4c^2(x^3)^2 = ((x^1)^2 - 2c^2)((x^2)^2 - 2c^2)$ *has a line of umbilic lying on the surface* $(x^1)^2 + (x^2)^2 + (x^3)^2 = 4c^2$.

Solution: Let $(x^1)^2 - 2c^2 = (u^1)^2$ and $(x^2)^2 - 2c^2 = (u^2)^2$, then the parametric representation of the surface is given by

$$\mathbf{r} = (x^1, x^2, x^3) = \left(\sqrt{2c^2 + (u^1)^2};\ \sqrt{2c^2 + (u^2)^2};\ \frac{u^1 u^2}{2c}\right).$$

The first fundamental magnitudes $a_{\alpha\beta}$ are given by

$$a_{11} = \frac{(u^1)^2}{4c^2 + (u^1)^2} + \frac{(u^2)^2}{4c^2};\ a_{22} = \frac{(u^2)^2}{4c^2 + (u^2)^2} + \frac{(u^1)^2}{4c^2};\ a_{12} = \frac{u^1 u^2}{4c^2} = a_{21},$$

where $f_1 = \frac{\partial f}{\partial u^2}$. The symmetric covariant tensors $b_{\alpha\beta}$ are given by

$$b_{11} = \frac{-c(u^2)^2}{a(2c^2 + (u^1)^2)^{3/2}(2a^2 + (u^2)^2)^{1/2}};$$

$$b_{22} = \frac{-c(u^1)^2}{a(2c^2 + (u^1)^2)^{1/2}(2a^2 + (u^2)^2)^{3/2}};$$

$$b_{12} = \frac{-cu^2}{2ca\sqrt{(2c^2 + (u^1)^2)(2a^2 + (u^2)^2)}},$$

where $a = |a_{\alpha\beta}|$. At an umbilic, we have $\frac{b_{11}}{a_{11}} = \frac{b_{12}}{a_{12}} = \frac{b_{22}}{a_{22}}$, i.e. $b_{11}a_{22} = b_{22}a_{11}$, and so,

$$\left[\frac{(u^1)^2}{2c^2 + (u^2)^2} + \frac{(u^2)^2}{4c^2}\right]\frac{(u^1)^2}{2c^2 + (u^2)^2} = \left[\frac{(u^2)^2}{2c^2 + (u^2)^2} + \frac{(u^1)^2}{4c^2}\right]\frac{(u^2)^2}{2c^2 + (u^1)^2}$$

or

$$\frac{(u^1)^4 - (u^2)^4}{(2c^2 + (u^1)^2)(2c^2 + (u^2)^2)} + \frac{(u^1)^2(u^2)^2}{4c^2}\left[\frac{1}{2c^2 + (u^2)^2} - \frac{1}{2c^2 + (u^1)^2}\right] = 0$$

or

$$(u^1)^4 - (u^2)^4 + \frac{(u^1)^2(u^2)^2}{4c^2}\left[(u^1)^2 - (u^2)^2\right] = 0$$

or

$$(u^1)^2 + (u^2)^2 + \frac{(u^1 u^2)^2}{4c^2} = 0 \Rightarrow (x^1)^2 + (x^2)^2 + (x^3)^2 = 4c^2.$$

Thus, the umbilic lie on the surface $(x^1)^2 + (x^2)^2 + (x^3)^2 = 4c^2$.

8.2.1 Surfaces of Positive and Negative Curvatures

In order to determine the shape of a surface $\mathcal{S}$ in a neighbourhood of any of its points we first considered arbitrary curves on $\mathcal{S}$. We enabled us to restrict of $\mathcal{S}$ and planes which are orthogonal to the tangent plane to $\mathcal{S}$ at a point P under consideration. That investigation led us to the introduction of the normal curvature [Eq. (8.14)]

$$\chi_{(n)} = \frac{b_{\alpha\beta} du^\alpha du^\beta}{a_{\alpha\beta} du^\alpha du^\beta} = \frac{\mathcal{B}}{\mathcal{A}}.$$

$|\chi_{(n)}|$ is the curvature of the normal section of $\mathcal{S}$ at P whose tangent direction is $du^2 : du^1$. We will now see that there are three different possible forms of $\mathcal{S}$ in a neighbourhood of a point at which the second fundamental form does not vanish identically. For this purpose we consider the behaviour of $\chi_{(n)}$ as a function of the direction $du^2 : du^1$ of the tangent to the normal sections at P.

We know, the first fundamental form [Eq. (7.2)] is positive definite, hence the sign of $\chi_{(n)}$ depends on the second fundamental form only.

(i) A surface is called a *surface of positive curvature,* if at all points, the Gaussian curvature $\kappa > 0$, i.e.

$$b = b_{11}b_{22} - b_{12}^2 > 0; \quad \text{as } a > 0 \text{ on the surface.} \tag{8.25}$$

That is the second fundamental form is positive definite. In this case, $\chi_{(n)}$ has the same sign for all possible directions of the normal sections at P, i.e. the centres of curvature of all normal sections lie on the same side of the surface $\mathcal{S}$. The point P on the surface $\mathcal{S}$ is called *elliptic.* Figure 8.3 shows the shape of a surface S in the neighbourhood of an elliptic point.

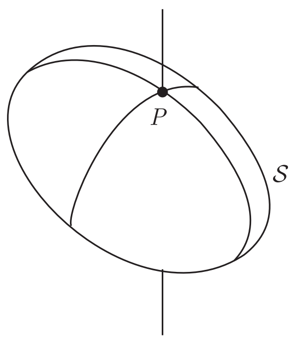

Figure 8.3: Elliptic point.

(ii) A surface is called a *surface of negative curvature*, if at all points, the Gaussian curvature $\kappa < 0$, i.e.

$$b = b_{11}b_{22} - b_{12}^2 < 0; \quad \text{as } a > 0 \text{ on the surface.} \tag{8.26}$$

If $b < 0$ at a point P of a surface $\chi_{(n)}$ does not maintain the same sign for all directions $du^2 : du^1$. More precisely, there exists two real asymptotic directions for which $\chi_{(n)} = 0$. These directions separate the directions for which $\chi_{(n)}$ is positive from which $\chi_{(n)}$ is negative, P is then called a *hyperbolic or saddle point* of the surface $\mathcal{S}$. Figure 8.4 shows the shape of a surface in the neighbourhood of a hyperbolic point.

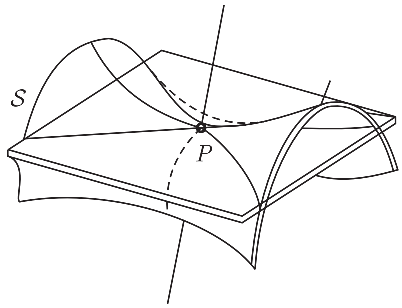

Figure 8.4: Hyperbolic point.

This figure shows also the tangent plane at that point.

(iii) If $b = 0$ at a point P of a surface $\mathcal{S}$, $\chi_{(n)}$ does not change sign, but there is exactly one direction where $\chi_{(n)} = 0$, i.e. exactly one real asymptotic direction. P is then called a *parabolic point* or flat point or planner point of the surface $\mathcal{S}$. Figure 8.5 shows the shape of a surface S in the neighbourhood of a parabolic point.

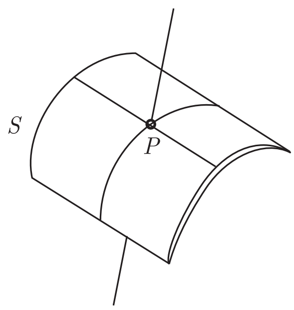

Figure 8.5: Parabolic point.

b of second fundamental form	Name	Number of real asymptotic directions	Figures
$b > 0$	Elliptic point	0	8.3
$b = 0$	Parabolic point	1	8.5
$b < 0$	Hyperbolic point	2	8.4

Since the conditions $b >= 0$ corresponds to a geometric property of the surface, they must be invariant with respect to any allowable co-ordinate transformation.

Deduction 8.2.1 Consider the section of the surface by a plane parallel and infinitely close to the tangent plane at the point P (Figure 8.6). Suppose the surface is synclastic in the neighbourhood of P. Then near P it lies entirely on one side of the tangent plane. Let the plane be taken on this (concave) side of the surface, parallel to the tangent plane at P, and at an infinitesimal distance from it, whose measure is h in the direction of the unit normal $\boldsymbol{\mu}$. Let the principal radii of curvature be α, β, then

$$\alpha\chi_{(1)} = 1 \text{ and } \beta\chi_{(2)} = 1.$$

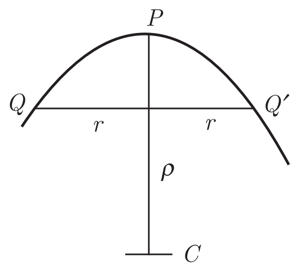

Figure 8.6: Dupin's indicatrix.

Thus, h has the same sign as α and β. Consider also any normal plane QPQ' through P, cutting the former plane in QQ'. Then if ρ is the radius of curvature of the normal section, and $2r$ the length of QQ', we have $r^2 = 2h\rho$ to the first order. If θ is the inclination of the normal section to the principal direction, Euler's formula gives

$$\frac{1}{\alpha}\cos^2\theta + \frac{1}{\beta}\sin^2\theta = \frac{1}{\rho} = \frac{2h}{r^2},$$

If then we write $\varsigma = r\cos\theta$ and $\upsilon = r\sin\theta$, we have

$$\frac{\varsigma^2}{a} + \frac{\upsilon^2}{b} = 2h.$$

Thus, the section of the surface by the plane parallel to the tangent plane at P, and indefinitely close to it, is similar and similarly situated to the ellipse

$$\frac{\varsigma^2}{|\alpha|} + \frac{\upsilon^2}{|\beta|} = 1, \tag{8.27}$$

whose axes are tangents to the lines of curvature at P. This ellipse is called the *Dupin's indicatrix* at the point P, and P is said to be an *elliptic point.* It is sometimes described as a point of positive curvature, because the second curvature κ is positive.

Let the principal radii, α and β, have opposite signs, and the surface lies partly on the other side of the tangent plane at P. In this case the Gaussian curvature κ

is negative at P, so that the surface is anticlastic in the neighbourhood. Two planes parallel to this tangent plane, one on either side, and equidistant from it, cut the surface in the conjugate hyperbola

$$\frac{\varsigma^2}{a} + \frac{\upsilon^2}{b} = \pm 2h.$$

These are similar and similarly situated to the conjugate hyperbolas

$$\frac{\varsigma^2}{\alpha} + \frac{\upsilon^2}{\beta} = \pm 1, \tag{8.28}$$

which constitute the indicatrix at P. The point P is then called a *hyperbolic point*, or a point of negative curvature. The normal curvature is zero in the directions of the asymptotes. When the Gaussian curvature κ is zero at the point P it is called a *parabolic point*. One of the principal curvatures is zero, and the indicatrix is a pair of parallel straight lines. Consequently, the principal directions bisect the angles between directions corresponding to the same normal curvature, in particular the angles between the asymptotic direction.

EXAMPLE 8.2.7 *Show that the right helicoid is a surface of negative curvature.*

Solution: The parametric representation of the right helicoid is given by

$$x^1 = u^1 \cos u^2;\ x^2 = u^1 \sin u^2;\ x^3 = cu^2$$

i.e.

$$\mathbf{r} = (x^1, x^2, x^3) = \left(u^1 \cos u^2;\ u^1 \sin u^2;\ cu^2\right).$$

The symmetric covariant second order tensors $a_{\alpha\beta}$ for the first fundamental form of the surface are given by

$$a_{11} = 1;\ a_{22} = (u^1)^2 + c^2 \text{ and } a_{12} = 0 = a_{21}.$$

$$\Rightarrow |a_{\alpha\beta}| = \begin{vmatrix} 1 & 0 \\ 0 & c^2 + (u^1)^2 \end{vmatrix} = c^2 + (u^1)^2 > 0.$$

The symmetric covariant tensors $b_{\alpha\beta}$ for the second fundamental form are given by

$$b_{11} = 0, b_{22} = 0 \text{ and } b_{12} = -\frac{c}{\sqrt{c^2 + (u^1)^2}}.$$

$$\Rightarrow |b_{\alpha\beta}| = \begin{vmatrix} 0 & -\frac{c}{\sqrt{c^2+(u^1)^2}} \\ -\frac{c}{\sqrt{c^2+(u^1)^2}} & 0 \end{vmatrix} = -\frac{c^2}{c^2 + (u^1)^2}.$$

Therefore, the Gaussian curvature is given by

$$\kappa = \frac{|b_{\alpha\beta}|}{|a_{\alpha\beta}|} = -\frac{c^2}{[c^2 + (u^1)^2]^2} < 0.$$

Thus, the given surface is a surface of negative curvature.

EXAMPLE 8.2.8 *Find the nature of a point on a unit sphere*

$$(x^1)^2 + (x^2)^2 + (x^3)^2 = 1.$$

Solution: The parametric representation of a unit sphere is given by

$$x^1 = \cos u^1 \cos u^2;\ x^2 = \sin u^1 \cos u^2;\ x^3 = \sin u^2$$

i.e.

$$\mathbf{r} = (x^1, x^2, x^3) = \left(\cos u^1 \cos u^2; \sin u^1 \cos u^2; \sin u^2\right).$$

The first fundamental magnitudes $a_{\alpha\beta}$ of the surface are given by

$$a_{11} = \cos^2 u^2;\ a_{22} = 1 \text{ and } a_{12} = 0 = a_{21}.$$

$$\Rightarrow a = |a_{\alpha\beta}| = \begin{vmatrix} \cos^2 u^2 & 0 \\ 0 & 1 \end{vmatrix} = \cos^2 u^2 > 0.$$

The symmetric covariant tensors $b_{\alpha\beta}$ for the second fundamental form are given by

$$b_{11} = -\cos^2 u^2, b_{12} = 0 = b_{21} \text{ and } b_{22} = -1.$$

$$\Rightarrow b = |b_{\alpha\beta}| = \begin{vmatrix} -\cos^2 u^2 & 0 \\ 0 & -1 \end{vmatrix} = \cos^2 u^2.$$

Therefore, the Gaussian curvature is given by

$$\kappa = \frac{b}{a} = \frac{\cos^2 u^2}{\cos^2 u^2} = 1 > 0.$$

Since $\kappa > 0$, it follows that the nature of the point on the sphere is elliptic.

8.2.2 Isometric Lines

Suppose that, in terms of the parameters u, v the square of the linear element of the surface has the form

$$ds^2 = \lambda(du^2 + dv^2), \tag{8.29}$$

where λ is a function of u, v or a constant. Then the parameters curves are orthogonal because $a_{12} = 0$. Further, the lengths of elements of the parametric curves are $\sqrt{\lambda}du$

and $\sqrt{\lambda}dv$, and these equal if $du = dv$. Thus, the parametric curves corresponding to the values $u, u + du, v, v + dv$ bound a small square provided $du = dv$. In this way the surface may be mapped out into small squares by means of the parametric curves, the sides of any one square corresponding to equal increments in u and v.

More generally, if the square of the linear element has the form

$$ds^2 = \lambda(U du^2 + V dv^2), \tag{8.30}$$

where U is a function of u only and V is a function of v only, we may change the parameters to ϕ, φ by the transformation

$$d\phi = \sqrt{U} du; \quad d\varphi = \sqrt{V} dv.$$

This does not alter the parametric curves; for the curves $u =$ constant are identical with the curves $\phi =$ constant; and similarly the curves $v =$ constant are identical with the curves $\varphi =$ constant. Equation (8.30) then becomes

$$ds^2 = \lambda(d\phi^2 + d\varphi^2), \tag{8.31}$$

which is of the same form as Eq. (8.29). Whenever the square of the linear element has the form Eq. (8.30), so that, without alteration of the parametric curves, it may be reduced to the form Eq. (8.29), the parametric curves are called *isometric lines*, and the parameters *isometric parameters.*

In the form Eq. (8.29) the fundamental magnitudes a_{11} and a_{22} are equal; but in the more general form Eq. (8.30) they are such that

$$\frac{a_{11}}{a_{22}} = \frac{U}{V} \text{ and, therefore, } \frac{\partial^2}{\partial u \partial v} \log \frac{a_{11}}{a_{22}} = 0. \tag{8.32}$$

Either of these equations, in conjunction with $a_{12} = 0$, expresses the condition that the parametric variables may be isometric. If it is satisfied, ds^2 has the form Eq. (8.30) and may therefore be reduced to the form Eq. (8.29).

EXAMPLE 8.2.9 *Prove that isometric curves is afforded by the meridians and parallels on a surface of revolution, given by*

Solution: With the usual notation, let $\mathcal{S}$ is a surface of revolution, given by

$$x^1 = u \cos v; \; x^2 = u \sin v; \; x^3 = f(u); \text{ where } f(u) \in C^2.$$

The first order magnitudes are given by

$$a_{11} = 1 + f_1^2; \; a_{22} = u^2 \text{ and } a_{12} = 0 = a_{21},$$

where $f_1 = \frac{\partial f}{\partial u}$. In terms of the parameters u, v the square of the linear element of the surface has the form

$$ds^2 = (1 + f_1^2)du^2 + u^2 dv^2 = u^2 \left[\frac{1 + f_1^2}{u^2} du^2 + dv^2 \right],$$

which is of the form Eq. (8.30). The parametric curves are the meridians $v =$ constant and the parallels $u =$ constant. If we make the transformation

$$d\omega = \frac{1}{u}\sqrt{1 + f_1^2}du,$$

the curves $\omega =$ constant are the same as the parallels, and the square of the linear element becomes

$$ds^2 = u^2(d\omega^2 + dv^2),$$

which is of the form Eq. (8.29). Thus, the meridians and parallels of a surface of revolution are isometric lines.

8.3 Lines of Curvature

A curve drawn on a surface, and possessing the property that the normals to the surface at consecutive points intersect, is called a *line of curvature*. Thus, for a *line of curvature*, a curve on a surface such that the tangent line to it at every point is directed along a principal direction. The differential equation for which the lines of curvature on $\mathcal{S}$ are the integral curves from directly from Eq. (8.17), as

$$b_{\alpha\beta}\lambda^\beta = \chi_{(p)} a_{\alpha\beta}\lambda^\beta.$$

Therefore,

$$b_{1\beta}\lambda^\beta = \chi_{(p)} a_{1\beta}\lambda^\beta; \text{ when } \alpha = 1$$

$$b_{2\beta}\lambda^\beta = \chi_{(p)} a_{2\beta}\lambda^\beta; \text{ when } \alpha = 2.$$

If we eliminate $\chi_{(p)}$ from these equations and set $\lambda^\beta = \frac{du^\beta}{ds}$, we get

$$\frac{b_{1\beta}}{a_{1\beta}} \frac{du^\beta}{du^\beta} = \frac{b_{2\beta}}{a_{2\beta}} \frac{du^\beta}{du^\beta}$$

or

$$(b_{11}a_{12} - b_{12}a_{11})(du^1)^2 + (b_{11}a_{22} - b_{22}a_{11})du^1 du^2$$
$$+(b_{12}a_{22} - b_{22}a_{12})(du^2)^2 = 0. \tag{8.33}$$

This quadratic Eq. (8.33) represents the *lines of curvature* of the surface, showing that at every point of the surface there are two principal directions. Thus, at each point of a surface there exist two mutually orthogonal directions for which the normal curvature attains its extreme values.

It follows from the above that the direction of a line of curvature at any point is the principal direction at that point. Through each point on the surface pass two lines of curvature cutting each other at right angles; and on the surface there are two system of lines of curvature whose differential equation is Eq. (8.33).

Deduction 8.3.1 The existence of (real) asymptotic curves on a surface $\mathcal{S}$ depends on the geometric shape of $\mathcal{S}$ the lines of curvature are always real. At each point of $\mathcal{S}$ where either $b_{\alpha\beta}du^\alpha du^\beta \neq 0$ or is not proportional to $a_{\alpha\beta}du^\alpha du^\beta$, the Eq. (8.33) specifies two orthogonal directions

$$\frac{du^2}{du^1} = \phi_\alpha\left(u^1, u^2\right);\ \alpha = 1, 2, \tag{8.34}$$

which coincide with directions of the principal curvatures. Each equation in [Eq. (8.34)] determines a family of curves on $\mathcal{S}$ covering the surface without gaps. These two families of curves are orthogonal, and, if they are taken as a parametric net on $\mathcal{S}$, the first fundamental form has the form

$$(ds)^2 = \overline{a}_{11}(d\overline{u}^1)^2 + \overline{a}_{22}(d\overline{u}^2)^2.$$

Accordingly, Eq. (8.33) in the co-ordinate system $\overline{u}^\alpha$ takes the form

$$-\overline{b}_{12}\overline{a}_{11}(d\overline{u}^1)^2 + (\overline{b}_{11}\overline{a}_{22} - \overline{b}_{22}\overline{a}_{11})d\overline{u}^1 d\overline{u}^2 + \overline{b}_{12}\overline{a}_{22}(d\overline{u}^2)^2 = 0$$

and its solutions are

$$\overline{u}^1 = \text{ constant}, \overline{u}^2 = \text{ constant}.$$

If we take $d\overline{u}^1 \neq 0$ and $d\overline{u}^2 = 0$, we see that $\overline{b}_{12} = 0$, since $\overline{a}_{11} = 0$. Thus, a necessary condition for the net of lines of curvature to be orthogonal is that $a_{12} = 0 = b_{12}$. Consequently we may always choose co-ordinates u^1, u^2 on S so that the lines of curvature are the co-ordinate curves of this system which is allowable at any point of S which is not umbilic. Conversely, if $a_{12} = 0 = b_{12}$, then Eq. (8.33) has the solutions $u^1 =$ constant and $u^2 =$ constant, so that the co-ordinate lines are the lines of curvature.

Note that, for every orthogonal net on a plane or a sphere $a_{12} = 0 = b_{12}$. Formula (8.14), for the normal curvature, when the co-ordinate system is taken to be the net of lines of curvature becomes

$$\chi_{(n)} = \frac{b_{11}(du^1)^2 + b_{22}(du^2)^2}{a_{11}(du^1)^2 + a_{22}(du^2)^2}.$$

If we set $du^1 = 0, du^2 \neq 0$, and $du^2 = 0, du^1 \neq 0$, we get

$$\chi_{(1)} = \frac{b_{11}}{a_{11}};\ \chi_{(2)} = \frac{b_{22}}{a_{22}}$$

for the curvatures of the co-ordinate lines $u^1 =$ constant and $u^2 =$ constant. The lines of curvature on $\mathcal{S}$ should not be confused with the normal sections of $\mathcal{S}$. The normal sections are necessarily plane curves, whereas the lines of curvature ordinarily are not plane curves. Thus the lines of curvature of any (real) surface S of class ≥ 3 are real curves.

Deduction 8.3.2 A point on a surface is called *umbilic* if at that point $b_{\alpha\beta} = \lambda a_{\alpha\beta}$, where λ is a scalar. In particular, if $\lambda = 0$, the point is called a *planer point.* Note that, at an umbilic point the normal curvature $\kappa_{(n)}$ is the same for all directions. If $\mathcal{S}$ has no umbilics the lines of curvature form an orthogonal net everywhere on $\mathcal{S}$.

Theorem 8.3.1 (Rodrigue's formula): *A line of curvature is characterised by*

$$\frac{\partial \xi^i}{\partial s} + \chi_{(n)}\frac{dx^i}{ds} = 0,$$

where $\chi_{(n)}$ is the principal curvature of the surface.

Proof: From Weingarten formula Eq. (7.49), we have

$$\xi^i_{,\alpha} = -a^{\beta\gamma}\ b_{\beta\gamma}x^i_\gamma$$

or

$$\xi^i_{,\alpha}\frac{du^\alpha}{ds} = -a^{\beta\gamma}b_{\beta\gamma}x^i_\gamma\frac{du^\alpha}{ds}$$

or

$$\frac{\partial \xi^i}{\partial s} = -a^{\beta\gamma}b_{\beta\gamma}x^i_\gamma\lambda^\alpha. \tag{8.35}$$

Again, we know that for a line of curvature,

$$b_{\alpha\beta}\lambda^\beta = \chi_{(n)}a_{\alpha\beta}\lambda^\beta;\text{i.e. } b_{\alpha\beta}\lambda^\alpha = \chi_{(n)}a_{\alpha\beta}\lambda^\alpha,$$

as $a_{\alpha\beta}$ and $b_{\alpha\beta}$ are symmetric. Using result (8.35), we get

$$\begin{aligned}\frac{\partial \xi^i}{\partial s} &= -a^{\beta\gamma}\chi_{(n)}a_{\alpha\beta}\lambda^\alpha x^i_\gamma\\ &= -\delta^\gamma_\alpha\chi_{(n)}\lambda^\alpha x^i_\gamma = -\chi_{(n)}\lambda^\alpha x^i_\alpha\\ &= -\chi_{(n)}\frac{\partial u^\alpha}{\partial s}\ \frac{\partial x^i}{\partial u^\alpha} = -\chi_{(n)}\frac{dx^i}{ds}\end{aligned}$$

or

$$\frac{\partial \xi^i}{\partial s} + \chi_{(n)} \ \frac{dx^i}{ds} = 0.$$

Hence, the theorem is proved. Conversely, let the relation

$$\frac{\partial \xi^i}{\partial s} + \chi_{(n)} \frac{dx^i}{ds} = 0$$

holds. Using Weingarten formula Eq. (7.49), we have

$$\frac{\partial \xi^i}{\partial s} = -a^{\beta\gamma} b_{\beta\gamma} x^i_\gamma \lambda^\alpha$$

or

$$-a^{\beta\gamma} b_{\beta\gamma} x^i_\gamma \lambda^\alpha + \chi_{(n)} \ \frac{dx^i}{ds} = 0.$$

Taking the inner product with $g_{ik} x^k_p$, we get

$$-a^{\beta\gamma} b_{\beta\gamma} g_{ik} x^k_p x^i_\gamma \lambda^\alpha + \chi_{(n)} g_{ik} x^k_p \frac{dx^i}{ds} = 0$$

or

$$-a^{\beta\gamma} b_{\beta\alpha} a_{p\gamma} \lambda^\alpha + \chi_{(n)} g_{ki} x^i_p \frac{dx^k}{ds} = 0.$$

Interchanging the dummy indices i and k in the second term, we get

$$-b_{p\alpha} \lambda^\alpha + \chi_{(n)} g_{ik} x^i_p \frac{\partial x^k}{\partial u^\alpha} \frac{du^\alpha}{ds} = 0$$

or

$$-b_{p\alpha} \lambda^\alpha + \chi_{(n)} g_{ik} x^i_p x^k_\alpha \lambda^\alpha = 0$$

or

$$-b_{p\alpha} \lambda^\alpha + \chi_{(n)} a_{p\alpha} \lambda^\alpha = 0$$

or

$$\left(\chi_{(n)} a_{p\alpha} - b_{p\alpha}\right) \lambda^\alpha = 0,$$

which is the equation of the line of curvature. The Rodrigues formulae are characteristic for the lines of curvature.

EXAMPLE 8.3.1 *Show that the parametric curves are the lines of curvature if and only if* $a_{12} = 0 = b_{12}$.

Solution: Let us assume that the parametric curves be the lines of curvature. Then they form an orthogonal net and will satisfy Eq. (8.33). Now, for u^1 curve, $du^2 = 0$ and hence from Eq. (8.33), we get

$$b_{11}a_{12} - a_{11}b_{12} = 0; \quad \text{as } du^1 \neq 0$$

and for u^2 curve, $du^1 = 0$ and hence from Eq. (8.33), we get

$$b_{12}a_{22} - a_{12}b_{22} = 0; \quad \text{as } du^2 \neq 0.$$

Multiplying first equation by a_{22} and second by a_{11} and adding, we get

$$a_{12}\left(a_{22}b_{11} - a_{11}b_{22}\right) = 0$$
$$\Rightarrow \quad \text{either } a_{12} = 0 \text{ or } \frac{a_{11}}{b_{11}} = \frac{a_{22}}{b_{22}}.$$

For the parametric curve, $a_{22}b_{11} \neq a_{11}b_{22}$ and so $a_{12} = 0$. The condition $a_{12} = 0$ is that of orthogonality satisfied by all lines of curvature. Similarly, multiplying first equation by b_{22} and second by b_{11} and adding, we get

$$b_{12}\left(a_{22}b_{11} - a_{11}b_{22}\right) = 0 \Rightarrow b_{12} = 0.$$

Thus $b_{12} = 0$, is the necessary and sufficient condition that the parametric curves form a conjugate system. Conversely, let $a_{12} = 0 = b_{12}$, then Eq. (8.33) reduces to

$$\left(a_{22}b_{11} - a_{11}b_{22}\right) du^1 du^2 = 0.$$

If the lines of curvature exist, then $a_{22}b_{11} - a_{11}b_{22} \neq 0$ and the curves are the solution of the equation $du^1du^2 = 0$. Hence,

$$\text{when} \quad du^1 \neq 0; \ du^2 = 0 \Rightarrow u^2 = \text{ constant.}$$
$$\text{when} \quad du^2 \neq 0; \ du^1 = 0 \Rightarrow u^1 = \text{ constant.}$$

Hence, the parametric curves are the lines of curvature. Thus, if in particular the co-ordinates are chosen so that the co-ordinate curves are lines of curvature on the surfaces S then $a_{12} = b_{12} = 0$ and therefore the Rodrigues formulae becomes

$$\frac{\partial \xi^i}{\partial S} = -\frac{b_{ii}}{a_{ii}}\frac{dx^i}{ds}(\text{no summation}); \ i = 1, 2.$$

EXAMPLE 8.3.2 *Show that the lines of curvature on a minimal surface form an isometric systems.*

Solution: If the parametric curves are the lines of curvature, then $a_{12} = 0 = b_{12}$. Now, for the minimal surface,

$$\begin{aligned} H &= \frac{1}{2} a^{\alpha\beta} b_{\alpha\beta} = 0 \\ \Rightarrow H &= \frac{1}{2a} \left[a_{22} b_{11} + a_{11} b_{22} \right] = 0; \quad \text{as } a_{12} = 0 = b_{12} \\ \Rightarrow b_{11} &= b_{22} = 0; \quad \text{as } a \neq 0 \\ \Rightarrow b_{\alpha\beta} &= 0. \end{aligned}$$

Therefore, the surface is a plane, i.e. the surface is isomorphic with Euclidean plane.

Theorem 8.3.2 (Euler's theorem on normal curvature): *If the lines of curvature are not indeterminant at a given point P on the surface and if θ is the angle between a given direction and a principal direction at P, then the normal curvature is given by the formula*

$$\chi_{(n)} = \chi_{(1)} \cos^2\theta + \chi_{(2)} \sin^2\theta.$$

Proof: We assume that P is not an umbilic. If the parametric curves are taken as the lines of curvature, then the principal curvatures are given by

$$\chi_{(1)} = \frac{b_{11}}{a_{11}} \text{ and } \chi_{(2)} = \frac{b_{22}}{a_{22}}.$$

Let θ be the angle between a given direction $(\delta u^1, \delta u^2)$ and a principal direction at a given point P. Since the co-ordinate curves are orthogonal, we have

$$\cos\theta = \sqrt{a_{11}} \frac{du^1}{ds} \quad \text{and} \quad \sin\theta = \sqrt{a_{22}} \frac{du^2}{ds}.$$

Also, let $\chi_{(1)}, \chi_{(2)}$ be the principal curvatures at P. If the parametric curves are taken as lines of curvature, then the normal curvature at P is given by

$$\begin{aligned} \chi_{(n)} &= \frac{b_{11}(du^1)^2 + b_{22}(du^2)^2}{ds^2} = b_{11} \left(\frac{du^1}{ds} \right)^2 + b_{22} \left(\frac{du^2}{ds} \right)^2 \\ &= \frac{b_{11}}{a_{11}} \cos^2\theta + \frac{b_{22}}{a_{22}} \sin^2\theta = \chi_{(1)} \cos^2\theta + \chi_{(2)} \sin^2\theta. \end{aligned}$$

This is *Euler's theorem on normal curvature.* This theorem tells us that, the normal curvature corresponding to any direction can be simply represented in terms of the principal curvatures. The Gaussian and total curvature in this case is given by

$$\kappa = \frac{b_{11}}{a_{11}} \frac{b_{22}}{a_{22}} \quad \text{and} \quad H = \frac{1}{2} \left(\frac{b_{11}}{a_{11}} + \frac{b_{22}}{a_{22}} \right).$$

From this we conclude the lines of curvature on a minimal surface form an isometric system.

Deduction 8.3.3 Let $\chi_{(n_1)}$ and $\chi_{(n_2)}$ denote normal curvatures in two orthogonal directions on the surface and θ be the angle between the first direction $du^2 = 0$; thus, the angle between the second direction and the principal direction $du^1 = 0$ will be $\frac{\pi}{2} + \theta$. Thus, from Euler's theorem, we have

$$\begin{aligned}\chi_{(n_1)} &= \chi_{(1)} \cos^2 \theta + \chi_{(2)} \sin^2 \theta, \\ \chi_{(n_2)} &= \chi_{(1)} \cos^2 \left(\frac{\pi}{2} + \theta\right) + \chi_{(2)} \sin^2 \left(\frac{\pi}{2} + \theta\right) \\ &= \chi_{(1)} \sin^2 \theta + \chi_{(2)} \cos^2 \theta.\end{aligned}$$

Now, adding, we get

$$\chi_{(n_1)} + \chi_{(n_2)} = \chi_{(1)} + \chi_{(2)}.$$

Thus, the sum of the normal curvatures in two orthogonal directions is equal to the sum of the principal curvatures at that point. This is known as *Dupin's theorem.*

Result: The theorem of enter and the theorem of Meusnier give complete information on the curvature of any curve on a surface.

EXAMPLE 8.3.3 *Find the equations for the lines of curvature for the surface given by* $\mathbf{r} = (u \cos v, u \sin v, cv)$.

Solution: For the given surface, the symmetric covariant second order tensors $a_{\alpha\beta}$ for the first fundamental form are given by

$$a_{11} = 1;\ a_{22} = u^2 + c^2;\ a_{12} = 0 = a_{21};$$

and the second fundamental magnitudes $b_{\alpha\beta}$ are given by

$$b_{11} = 0;\quad b_{22} = 0;\ b_{12} = -\frac{c}{\sqrt{u^2 + c^2}} = b_{21}.$$

Since $a_{12} = 0 = b_{12}$, this given parametric curve has the line of curvature. The lines of curvature are given by

$$\begin{aligned}&(b_{11}a_{12} - b_{12}a_{11})(du)^2 + (b_{11}a_{22} - b_{22}a_{11})du\ \ dv \\ &\quad +(b_{12}a_{22} - b_{22}a_{12})(dv)^2 = 0\end{aligned}$$

or

$$-b_{12}a_{11}(du)^2 + b_{12}a_{22})(dv)^2 = 0$$

or

$$\frac{c}{\sqrt{u^2+c^2}}(du)^2 - \frac{c}{\sqrt{u^2+c^2}}(u^2+c^2)(dv)^2 = 0$$

or

$$\frac{du}{\sqrt{u^2+c^2}} = \pm dv \Rightarrow v = \pm \sinh^{-1}\frac{u}{c} + k,$$

where k is a constant.

EXAMPLE 8.3.4 *Given an ellipsoid of revolution, whose surface is determined by*

$$x^1 = a\cos u^1 \sin u^2, \quad x^2 = a\sin u^1 \sin u^2, x^3 = c\cos u^2$$

where $a^2 > c^2$. *Find* $a_{\alpha\beta}$, $b_{\alpha\beta}$ *and* $\kappa = \chi_{(1)}\chi_{(2)}$. *Discuss the lines of curvature.*

Solution: The parametric representation of the ellipsoid of revolution is given by

$$x^1 = a\cos u^1 \sin u^2, x^2 = a\sin u^1 \sin u^2, x^3 = c\cos u^2$$

i.e.

$$\mathbf{r} = (x^1, x^2, x^3) = \left(a\cos u^1 \sin u^2, a\sin u^1 \sin u^2, c\cos u^2\right).$$

The symmetric covariant second order tensors $a_{\alpha\beta}$ for the first fundamental form of the surface of the ellipsoid of revolution are given by

$$\begin{aligned}
a_{11} &= \left(\frac{\partial x^1}{\partial u^1}\right)^2 + \left(\frac{\partial x^2}{\partial u^1}\right)^2 + \left(\frac{\partial x^3}{\partial u^1}\right)^2 \\
&= (-a\sin u^1 \sin u^2)^2 + (a\cos u^1 \sin u^2)^2 + 0^2 = a^2\sin^2 u^2. \\
a_{22} &= \left(\frac{\partial x^1}{\partial u^2}\right)^2 + \left(\frac{\partial x^2}{\partial u^2}\right)^2 + \left(\frac{\partial x^3}{\partial u^2}\right)^2 \\
&= (a\cos u^1 \cos u^2)^2 + (a\sin u^1 \cos u^2)^2 + (-c\sin u^2)^2 \\
&= a^2\cos^2 u^2 + c^2\sin^2 u^2. \\
a_{12} &= \frac{\partial x^1}{\partial u^1}\frac{\partial x^1}{\partial u^2} + \frac{\partial x^2}{\partial u^1}\frac{\partial x^2}{\partial u^2} + \frac{\partial x^3}{\partial u^1}\frac{\partial x^3}{\partial u^2} = 0 = a_{21}.
\end{aligned}$$

Therefore,

$$a = \begin{vmatrix} a^2\sin^2 u^2 & 0 \\ 0 & a^2\cos^2 u^2 + c^2\sin^2 u^2 \end{vmatrix} = a^2\sin^2 u^2(a^2\cos^2 u^2 + c^2\sin^2 u^2).$$

To calculate the tensors of second fundamental form, we have

$$\begin{aligned}
\mathbf{A} &= (-a\sin u^1 \sin u^2, a\cos u^1 \sin u^2, 0) \text{ and} \\
\mathbf{B} &= (a\cos u^1 \cos u^2, a\sin u^1 \cos u^2, -c\sin u^2) \\
\mathbf{A}\times\mathbf{B} &= \left(-ac\cos u^1 \sin u^2, ac\sin u^1 \sin u^2, -a^2\sin u^2 \cos u^2\right).
\end{aligned}$$

The normal vector is given by

$$\boldsymbol{\xi} = \frac{1}{\sqrt{a^2\cos^2 u^2 + c^2\sin^2 u^2}}\left(-ac\cos u^1\sin u^2, ac\sin u^1\sin u^2, -a^2\sin u^2\cos u^2\right).$$

Thus, the covariant tensors $b_{\alpha\beta}$ for the second fundamental form are given by

$$b_{11} = \frac{ac\sin^2 u^2}{\sqrt{a^2\cos^2 u^2 + c^2\sin^2 u^2}}; b_{22} = \frac{ac}{\sqrt{a^2\cos^2 u^2 + c^2\sin^2 u^2}};\ b_{12} = 0.$$

$$\Rightarrow b = b_{11}b_{22} - b_{12}^2 = \frac{a^2c^2\sin^2 u^2}{a^2\cos^2 u^2 + c^2\sin^2 u^2}.$$

Therefore, the Gaussian curvature κ is given by

$$\kappa = \frac{b}{a} = \frac{a^2c^2\sin^2 u^2}{a^2\cos^2 u^2 + c^2\sin^2 u^2} \times \frac{1}{a^2\sin^2 u^2(a^2\cos^2 u^2 + c^2\sin^2 u^2)}$$

$$= \frac{c^2}{\left(a^2\cos^2 u^2 + c^2\sin^2 u^2\right)^2}.$$

Since $a_{12} = 0, b_{12} = 0$, so the equation of the lines of curvature is given by

$$(b_{11}a_{22} - b_{22}a_{11})\, du^1du^2 = 0,$$
$$(a^2 - c^2)\sin^4 u^2 du^1du^2 = 0.$$

EXAMPLE 8.3.5 *Show that any curve on a sphere is a line of curvature.*

Solution: The parametric representation of the sphere of radius a is given by

$$x^1 = a\sin u^1\cos u^2,\ \ x^2 = a\sin u^1\sin u^2, x^3 = a\cos u^2$$

i.e.

$$\mathbf{r} = (x^1, x^2, x^3) = \left(a\sin u^1\cos u^2, a\sin u^1\sin u^2, a\cos u^2\right).$$

For the given surface, the symmetric covariant second order tensors $a_{\alpha\beta}$ for the first fundamental form are given by

$$a_{11} = a^2;\ a_{22} = a^2\sin^2 u^1;\ a_{12} = 0 = a_{21}$$

and the symmetric covariant second order tensors $b_{\alpha\beta}$ for the second fundamental form are given by

$$b_{11} = -a;\ b_{22} = -a\sin^2 u^1;\ b_{12} = 0 = b_{21}.$$

Since $a_{12} = 0 = b_{12}$, this given parametric curve has the line of curvature. In fact,

$$\frac{b_{11}}{a_{11}} = \frac{b_{12}}{a_{12}} = \frac{b_{22}}{a_{22}} = -\frac{1}{a},$$

i.e. $b_{\alpha\beta}$ are proportional to $a_{\alpha\beta}$. Now,

$$\begin{aligned}&(b_{11}a_{12} - b_{12}a_{11})(du^1)^2 + (b_{11}a_{22} - b_{22}a_{11})du^1du^2\\ &\quad +(b_{12}a_{22} - b_{22}a_{12})(du^2)^2 = \left(-a^3\sin^2 u^1 + a^3\sin^2 u^1\right) du^1du^2 = 0.\end{aligned}$$

This shows that the equation of the lines of curvature reduces to an identity. Hence, any curve on a sphere is a line of curvature.

EXAMPLE 8.3.6 *Given a surface of revolution $\mathcal{S}$*

$$x^1 = u^1\cos u^2; x^2 = u^1\sin u^2; \quad x^3 = f(u^1)$$

with $f(u^1)$ of class C^2. Prove that the lines of curvature on $\mathcal{S}$ are the meridians $u^2 =$ constant and the parallels $u^1 =$ constant.

Solution: The parametric representation of the right helicoid is given by

$$x^1 = u^1\cos u^2;\ x^2 = u^1\sin u^2;\ x^3 = f(u^1)$$

i.e.

$$\mathbf{r} = (x^1, x^2, x^3) = \left(u^1\cos u^2;\ u^1\sin u^2;\ f(u^1)\right).$$

The symmetric covariant second order tensors $a_{\alpha\beta}$ for the first fundamental form of the surface are given by

$$a_{11} = 1 + f_1^2;\ \ a_{22} = (u^1)^2 \text{ and } a_{12} = 0 = a_{21}$$

$$\Rightarrow a = \begin{vmatrix} 1+f_1^2 & 0 \\ 0 & (u^1)^2 \end{vmatrix} = (u^1)^2(1+f_1^2),$$

where $f_1 = \frac{\partial f}{\partial u^1}$. The symmetric covariant tensors $b_{\alpha\beta}$ for the second fundamental form are given by

$$b_{11} = \frac{f_{11}}{\sqrt{1+f_1^2}}, b_{22} = \frac{u^1 f_1}{\sqrt{1+f_1^2}} \text{ and } b_{12} = 0 = b_{21}.$$

$$\Rightarrow |b_{\alpha\beta}| = \begin{vmatrix} \dfrac{f_{11}}{\sqrt{1+f_1^2}} & 0 \\ 0 & \dfrac{u^1 f_1}{\sqrt{1+f_1^2}} \end{vmatrix} = \frac{u^1 f_1 f_{11}}{1+f_1^2},$$

where $f_1 = \frac{\partial f}{\partial u^1}$ and $f_{11} = \frac{\partial^2 f}{\partial u^1 \partial u^1}$. Equation (8.33) giving the principal curvatures is

$$(b_{11}a_{12} - b_{12}a_{11})(du^1)^2 + (b_{11}a_{22} - b_{22}a_{11})du^1 du^2$$
$$+(b_{12}a_{22} - b_{22}a_{12})(du^2)^2 = 0$$

or

$$\left[(1+f_1^2)\,\frac{u^1 f_1}{\sqrt{1+f_1^2}} + \left(u^1\right)^2 \frac{f_{11}}{\sqrt{1+f_1^2}}\right] du^1 du^2 = 0$$

or

$$du^1 du^2 = 0 \Rightarrow u^2 = \text{constant, or, } u^1 = \text{ constant.}$$

EXAMPLE 8.3.7 *Find the differential equation of lines of curvature of the helicoid*

$$x^1 = u^1 \cos u^2;\ x^2 = u^1 \sin u^2;\ x^3 = f(u^1) + cu^2,$$

where u^1, u^2 are parameters.

Solution: The parametric representation of the surface is given by

$$\mathbf{r} = (x^1, x^2, x^3) = \left(u^1 \cos u^2;\quad u^1 \sin u^2;\quad f(u^1) + cu^2\right).$$

The first fundamental magnitudes $a_{\alpha\beta}$ are given by

$$a_{11} = \left(\frac{\partial x^1}{\partial u^1}\right)^2 + \left(\frac{\partial x^2}{\partial u^1}\right)^2 + \left(\frac{\partial x^3}{\partial u^1}\right)^2 = 1 + f_1^2,$$

$$a_{22} = \left(\frac{\partial x^1}{\partial u^2}\right)^2 + \left(\frac{\partial x^2}{\partial u^2}\right)^2 + \left(\frac{\partial x^3}{\partial u^2}\right)^2 = (u^1)^2 + c^2,$$

$$a_{12} = \frac{\partial x^1}{\partial u^1}\frac{\partial x^1}{\partial u^2} + \frac{\partial x^2}{\partial u^1}\frac{\partial x^2}{\partial u^2} + \frac{\partial x^3}{\partial u^1}\frac{\partial x^3}{\partial u^2} = cf_1 = a_{21},$$

where $f_1 = \frac{\partial f}{\partial u^1}$. Therefore,

$$a = \begin{vmatrix} 1 + f_1^2 & cf_1 \\ cf_1 & (u^1)^2 + c^2 \end{vmatrix} = (u^1)^2(1 + f_1^2) + c^2.$$

The reciprocal tensors are given by

$$a^{11} = \frac{(u^1)^2 + c^2}{(u^1)^2(1 + f_1^2) + c^2};\ a^{22} = \frac{1 + f_1^2}{(u^1)^2(1 + f_1^2) + c^2};$$

$$a^{12} = \frac{-cf_1}{(u^1)^2(1 + f_1^2) + c^2} = a^{21}.$$

Since $\mathbf{r} = (x^1, x^2, x^3) = \left(u^1 \cos u^2;\ \ u^1 \sin u^2;\ \ f(u^1) + cu^2\right)$, we have

$$\mathbf{A} = (\cos u^2, \sin u^2, f_1) \text{ and } \mathbf{B} = (-u^1 \sin u^2, u^1 \cos u^2, c)$$
$$\mathbf{A} \times \mathbf{B} = (c \sin u^2 - u^1 f_1 \cos u^2)\hat{\imath} + (-c \cos u^2 - u^1 f_1 \sin u^2)\hat{\jmath} + u^1 \hat{k}.$$

Therefore, the unit normal vector $\boldsymbol{\xi}$ is given by

$$\boldsymbol{\xi} = \frac{1}{\sqrt{(u^1)^2(1+f_1^2)+c^2}} \left(c \sin u^2 - u^1 f_1 \cos u^2, -c \cos u^2 - u^1 f_1 \sin u^2, u^1\right).$$

The symmetric covariant tensors $b_{\alpha\beta}$ are given by

$$b_{11} = \frac{u^1 f_2}{\sqrt{(u^1)^2(1+f_1^2)+c^2}}; b_{22} = \frac{(u^1)^2 f_1}{\sqrt{(u^1)^2(1+f_1^2)+c^2}}; b_{12} = \frac{-c}{\sqrt{(u^1)^2(1+f_1^2)+c^2}},$$

where $f_2 = \frac{\partial^2 f}{(\partial u^2)^2}$. Therefore,

$$b = \begin{vmatrix} \dfrac{u^1 f_2}{\sqrt{(u^1)^2(1+f_1^2)+c^2}} & \dfrac{-c}{\sqrt{(u^1)^2(1+f_1^2)+c^2}} \\ \dfrac{-c}{\sqrt{(u^1)^2(1+f_1^2)+c^2}} & \dfrac{(u^1)^2 f_1}{\sqrt{(u^1)^2(1+f_1^2)+c^2}} \end{vmatrix} = \frac{(u^1)^3 f_1 f_2 - c^2}{(u^1)^2(1+f_1^2)+c^2}.$$

The equation giving the principal curvatures is

$$\chi_\rho^2 - a^{\alpha\beta} b_{\alpha\beta} \chi_\rho + \frac{b}{a} = 0$$

or

$$\chi_\rho^2 + \left[\frac{(1+f_1^2)(u^1)^2 f_1 + 2c^2 f_1 + ((u^1)^2 + c^2)u^1 f_2}{\left[(u^1)^2(1+f_1^2)+c^2\right]^{3/2}}\right] \chi_\rho + \frac{(u^1)^3 f_1 f_2 - c^2}{\left[(u^1)^2(1+f_1^2)+c^2\right]^2} = 0.$$

This is a quadratic equation in χ_ρ giving the two values $\chi_{(1)}$ and $\chi_{(2)}$. The mean curvature H is given by

$$H = \frac{1}{2}[\chi_{(1)} + \chi_{(2)}] = \frac{1}{2}\left[\frac{(1+f_1^2)(u^1)^2 f_1 + 2c^2 f_1 + ((u^1)^2 + c^2)u^1 f_2}{\left[(u^1)^2(1+f_1^2)+c^2\right]^{3/2}}\right]$$

and the Gaussian curvature κ is given by the formula,

$$\kappa = \chi_{(1)}\chi_{(2)} = \frac{(u^1)^3 f_1 f_2 - c^2}{\left[(u^1)^2(1+f_1^2)+c^2\right]^2}.$$

From this relation, we see that, along the helix $u^1 =$ constant, the value of κ is constant and is independent of parameter u^2. The differential equation giving the directions of the lines of curvatures is

$$(b_{11}a_{12} - b_{12}a_{11})(du^1)^2 + (b_{11}a_{22} - b_{22}a_{11})du^1du^2 \\ +(b_{12}a_{22} - b_{22}a_{12})(du^2)^2 = 0$$

or

$$c[u^1f_1f_2 + 1 + f_1^2](du^1)^2 + [u^1f_2\{(u^1)^2 + c^2\} - (u^1)^2f_1(1 + f_1^2)]du^1du^2 \\ -c[(u^1)^2(1 + f_1^2) + c^2](du^2)^2 = 0.$$

If the meridians are lines of curvature, i.e. curves $u^2 =$ constant are lines of curvature then $du^2 = 0$ and so, we get

$$c[u^1f_1f_2 + 1 + f_1^2](du^1)^2 = 0$$

or

$$u^1f_1f_2 + 1 + f_1^2 = 0 \text{ as } c \neq 0; du^1 \neq 0,$$

which is the required condition for the meridians to be lines of curvatures.

EXAMPLE 8.3.8 *Prove that the line of curvature of the paraboloid $x^1x^2 = ax^3$ lie on the surface*

$$\sinh^{-1}\frac{x^1}{a} \pm \sinh^{-1}\frac{x^2}{a} = c,$$

where c is an constant. Hence find the equation of the cone passes through a line of curvature of the paraboloid $x^1x^2 = ax^3$.

Solution: If the parameter x^3 can be written in the form

$$x^3 = f(x^1, x^2) = \frac{x^1x^2}{a}$$

then the parametric equation of the paraboloid $x^1x^2 = ax^3$ can be written in the form

$$\mathbf{r} = (x^1,\ x^2,\ x^3) = \left(x^1;\ x^2;\ f(x^1, x^2)\right).$$

The first fundamental magnitudes $a_{\alpha\beta}$ are given by

$$a_{11} = \left(\frac{\partial x^1}{\partial u^1}\right)^2 + \left(\frac{\partial x^2}{\partial u^1}\right)^2 + \left(\frac{\partial x^3}{\partial u^1}\right)^2 = 1 + f_1^2,$$

$$a_{22} = \left(\frac{\partial x^1}{\partial u^2}\right)^2 + \left(\frac{\partial x^2}{\partial u^2}\right)^2 + \left(\frac{\partial x^3}{\partial u^2}\right)^2 = 1 + f_2^2,$$

$$a_{12} = \frac{\partial x^1}{\partial u^1}\frac{\partial x^1}{\partial u^2} + \frac{\partial x^2}{\partial u^1}\frac{\partial x^2}{\partial u^2} + \frac{\partial x^3}{\partial u^1}\frac{\partial x^3}{\partial u^2} = f_1f_2 = a_{21},$$

where $f_1 = \frac{\partial f}{\partial u^1}$ and $f_2 = \frac{\partial f}{\partial u^2}$. Therefore,

$$a = \begin{vmatrix} 1+f_1^2 & f_1 f_2 \\ f_1 f_2 & 1+f_2^2 \end{vmatrix} = 1 + f_1^2 + f_2^2.$$

Since $\mathbf{r} = \left(x^1;\ \ x^2;\ \ f(x^1, x^2)\right)$, we have $\mathbf{A} = (1, 0, f_1)$ and $\mathbf{B} = (0, 1, f_2)$

$$\mathbf{A} \times \mathbf{B} = -f_1 \hat{\imath} - f_2 \hat{\jmath} + \hat{k}.$$

Therefore, the unit normal vector $\boldsymbol{\xi}$ is given by

$$\boldsymbol{\xi} = \frac{1}{\sqrt{1+f_1^2+f_2^2}} \left(-f_1, -f_2, 1\right).$$

The symmetric covariant tensors $b_{\alpha\beta}$ are given by

$$b_{11} = \frac{f_{11}}{\sqrt{1+f_1^2+f_2^2}};\ b_{22} = \frac{f_{22}}{\sqrt{1+f_1^2+f_2^2}};\ b_{12} = \frac{f_{12}}{\sqrt{1+f_1^2+f_2^2}},$$

where $f_{ij} = \frac{\partial^2 f}{\partial u^i \partial u^j}$. Therefore,

$$b = \begin{vmatrix} \dfrac{f_{11}}{\sqrt{1+f_1^2+f_2^2}} & \dfrac{f_{12}}{\sqrt{1+f_1^2+f_2^2}} \\ \dfrac{f_{12}}{\sqrt{1+f_1^2+f_2^2}} & \dfrac{f_{22}}{\sqrt{1+f_1^2+f_2^2}} \end{vmatrix} = \frac{f_{11} f_{22} - f_{12}^2}{1+f_1^2+f_2^2}.$$

The differential equation giving the directions of the lines of curvatures is

$$(b_{11}a_{12} - b_{12}a_{11})(dx^1)^2 + (b_{11}a_{22} - b_{22}a_{11})dx^1 dx^2$$
$$+(b_{12}a_{22} - b_{22}a_{12})(dx^2)^2 = 0$$

or

$$\frac{1}{a}\left(1 + \frac{(x^2)^2}{a^2}\right)(dx^1)^2 - \frac{1}{a}\left(1 + \frac{(x^1)^2}{a^2}\right)(dx^2)^2 = 0$$

or

$$\frac{dx^1}{\sqrt{a^2 + (x^1)^2}} \pm \frac{dx^2}{\sqrt{a^2 + (x^2)^2}} = 0$$

or

$$\sinh^{-1}\frac{x^1}{a} \pm \sinh^{-1}\frac{x^2}{a} = c,$$

where c is a constant. This equation can be written as

$$\sinh^{-1}\left[\frac{x^1}{a}\sqrt{1 + \frac{(x^2)^2}{a^2}} \pm \frac{x^2}{a}\sqrt{1 + \frac{(x^1)^2}{a^2}}\right] = c$$

or

$$\frac{x^1}{a}\sqrt{1+\frac{(x^2)^2}{a^2}} \pm \frac{x^2}{a}\sqrt{1+\frac{(x^1)^2}{a^2}} = \sinh c = p;\ \text{(say)}$$

or

$$\frac{x^3}{x^2}\sqrt{1+\frac{(x^3)^2}{(x^1)^2}} \pm \frac{x^3}{x^1}\sqrt{1+\frac{(x^3)^2}{(x^2)^2}} = p$$

or

$$x^3\left[\sqrt{(x^1)^2+(x^3)^2} \pm \sqrt{(x^2)^2+(x^3)^2}\right] = px^1x^2.$$

This is a homogeneous second degree equation and hence represents a cone passes through the paraboloid $x^1x^2 = ax^3$.

EXAMPLE 8.3.9 *Find the principal directions and the principal curvatures on the surface*

$$x^1 = a(u^1+u^2),\ x^2 = b(u^1-u^2),\ x^3 = u^1u^2.$$

Solution: The parametric representation of the surface is given by

$$\mathbf{r} = (x^1, x^2, x^3) = \left(a(u^1+u^2),\ \ b(u^1-u^2), u^1u^2\right).$$

The first fundamental magnitudes $a_{\alpha\beta}$ are given by

$$a_{11} = \left(\frac{\partial x^1}{\partial u^1}\right)^2 + \left(\frac{\partial x^2}{\partial u^1}\right)^2 + \left(\frac{\partial x^3}{\partial u^1}\right)^2 = a^2+b^2+\left(u^2\right)^2.$$

$$a_{22} = \left(\frac{\partial x^1}{\partial u^2}\right)^2 + \left(\frac{\partial x^2}{\partial u^2}\right)^2 + \left(\frac{\partial x^3}{\partial u^2}\right)^2 = a^2+b^2+\left(u^1\right)^2.$$

$$a_{12} = \frac{\partial x^1}{\partial u^1}\frac{\partial x^1}{\partial u^2} + \frac{\partial x^2}{\partial u^1}\frac{\partial x^2}{\partial u^2} + \frac{\partial x^3}{\partial u^1}\frac{\partial x^3}{\partial u^2} = a^2-b^2+u^1u^2 = a_{21}.$$

Therefore,

$$\triangle = 4a^2b^2 + a^2\left(u^1-u^2\right)^2 + b^2(u^1+u^2)^2.$$

The reciprocal tensors are given by

$$a^{11} = \frac{a^2+b^2+\left(u^1\right)^2}{\triangle};\ \ a^{22} = \frac{a^2+b^2+\left(u^2\right)^2}{\triangle};\ \ a^{12} = \frac{a^2-b^2+u^1u^2}{\triangle} = a^{21}.$$

Since $\mathbf{r} = (x^1, x^2, x^3) = \left(a(u^1+u^2),\ b(u^1-u^2), u^1u^2\right)$, we have

$$\mathbf{A} = (a, b, u^2) \text{ and } \mathbf{B} = (a, -b, u^1)$$

$$\mathbf{A}\times\mathbf{B} = \left(b(u^1+u^2), a(u^2-u^1), -2ab\right).$$

Therefore, the unit normal vector $\boldsymbol{\xi}$ is given by

$$\boldsymbol{\xi} = \frac{1}{\sqrt{\triangle}}\left(b(u^1+u^2),\ a(u^2-u^1),\ \ -2ab\right).$$

The symmetric covariant tensors $b_{\alpha\beta}$ are given by

$$b_{11} = \frac{\partial \boldsymbol{\xi}}{\partial u^1}\cdot\frac{\partial \mathbf{r}}{\partial u^1} = 0;\ \ b_{22} = \frac{\partial \boldsymbol{\xi}}{\partial u^2}\cdot\frac{\partial \mathbf{r}}{\partial u^2} = 0;\ \ b_{12} = \frac{-2ab}{\sqrt{\triangle}}.$$

$$\text{Therefore, } b^* = -\frac{4a^2b^2}{\triangle}.$$

The equation giving the principal curvatures is

$$\chi_\rho^2 - a^{\alpha\beta}b_{\alpha\beta}\chi_\rho + \frac{b^*}{\triangle} = 0$$

or

$$\chi_\rho^2 - \frac{4ab}{\triangle^{3/2}}\left[a^2-b^2+u^1u^2\right]\chi_\rho - \frac{4a^2b^2}{\triangle^2} = 0.$$

This is a quadratic equation in χ_ρ giving the two values $\chi_{(1)}$ and $\chi_{(2)}$. The first curvature H is given by,

$$H = \frac{1}{2}[\chi_{(1)}+\chi_{(2)}] = \frac{2ab}{\triangle^{3/2}}\left[a^2-b^2+u^1u^2\right]$$

and the specific curvature κ is given by the formula,

$$\kappa = \chi_{(1)}\chi_{(2)} = -\frac{4a^2b^2}{\triangle^2}.$$

The differential equation giving the directions of the lines of curvatures is

$$(b_{11}a_{12}-b_{12}a_{11})(du^1)^2 + (b_{11}a_{22}-b_{22}a_{11})du^1du^2 + (b_{12}a_{22}-b_{22}a_{12})(du^2)^2 = 0$$

or

$$\left[a^2+b^2+\left(u^2\right)^2\right](du^1)^2 - \left[a^2+b^2+\left(u^1\right)^2\right](du^2)^2 = 0$$

or

$$\frac{du^1}{\sqrt{a^2+b^2+(u^1)^2}} = \pm\frac{du^2}{\sqrt{a^2+b^2+(u^2)^2}}$$

or

$$\log\left(u^1+\sqrt{a^2+b^2+(u^1)^2}\right) = \pm\log\left(u^2+\sqrt{a^2+b^2+(u^2)^2}\right) + C.$$

8.3.1 Null Lines

The *null lines* (or minimal curves) on a surface are defined as the curves of zero length. Therefore, they are imaginary on a real surface, and their importance is chiefly analytic. The differential equation of the null lines is obtained by equating to zero the square of the line element. It is therefore,

$$a_{11}du^2 + 2a_{12}dudv + a_{22}dv^2 = 0. \tag{8.36}$$

If the parametric curves are null lines, this equation must be equivalent to $dudv = 0$. Hence $a_{11} = 0, a_{12} = 0$ and $a_{12} \neq 0$. These are the necessary and sufficient conditions that the parametric curves be null lines. In this case the square of the line element has the form $ds^2 = \lambda dudv$, where λ is a function of u, v or a constant; and the parameters u, v are then said to be *symmetric.*

When the parametric curves are null lines, so that $a_{11} = 0, a_{22} = 0$, $a = -a_{12}^2$, the differential equation of the lines of curvature is

$$b_{11}du^2 - b_{22}dv^2 = 0.$$

Therefore, the Gauss curvature κ and the mean curvature H are, respectively, given by

$$\kappa = \frac{b_{11}b_{22} - b_{12}^2}{-a_{12}^2};\ H = \frac{2b_{12}}{a_{12}}.$$

8.3.2 Conjugate Directions

Let Q be a point on the surface adjacent to P, and let PR be the line of intersection of the tangent plane at P and Q. Then, as Q tends to coincidence with P, the limiting directions of PQ and PR are said to be *conjugate directions* at P. Thus, the characteristic of the tangent plane, as the point of contact moves along a given curve, is the tangent line in the direction conjugate to that of the curve at a point of contact.

At a point $P \in \mathcal{S}$, let two directions (du^1, du^2) and $(\delta u^1, \delta u^2)$ satisfy the relation

$$b_{11}du^1\delta u^1 + b_{12}\left(du^1\delta u^2 + du^2\delta u^1\right) + b_{22}du^2\delta u^2 = 0,$$

or

$$b_{\alpha\beta}du^\alpha\delta u^\beta = 0, \tag{8.37}$$

then these are called conjugate directions at P. Thus, the necessary and sufficient condition that the direction $\frac{\delta u^1}{\delta u^2}$ be conjugate to the direction $\frac{du^1}{du^2}$ is

$$b_{11}\frac{du^1}{du^2}\ \frac{\delta u^1}{\delta u^2} + b_{12}\left(\frac{du^1}{du^2} + \frac{\delta u^1}{\delta u^2}\right) + b_{22} = 0 \tag{8.38}$$

and the symmetry of the relation shows that the property is a reciprocal one. Moreover, Eq. (8.38) is linear in each of the ratios $du^1 : du^2$ and $\delta u^1 : \delta u^2$; so that to a given direction there is one and only one conjugate direction.

EXAMPLE 8.3.10 *Find a necessary condition so that the two directions determined by* $p_{\alpha\beta} du^\alpha du^\beta = 0;\ p_{\alpha\beta} = p_{\beta\alpha}$ *are conjugate.*

Solution: Given that, $p_{\alpha\beta} du^\alpha du^\beta = 0$, i.e.

$$p_{11}(du^1)^2 + 2p_{12} du^1 du^2 + p_{22}(du^2)^2 = 0.$$

Let (du^1, du^2) and $(\delta u^1, \delta u^2)$ be the solutions, then

$$\frac{du^1}{du^2} + \frac{\delta u^1}{\delta u^2} = -2\frac{p_{12}}{p_{11}} \text{ and } \frac{du^1}{du^2} \cdot \frac{\delta u^1}{\delta u^2} = \frac{p_{22}}{p_{11}}.$$

From Eq. (8.37), we get

$$b_{11}\frac{p_{22}}{p_{11}} + b_{12}\left(-2\frac{p_{12}}{p_{11}}\right) + b_{22} = 0$$

or

$$b_{11}p_{22} - 2b_{12}p_{12} + b_{22}p_{11} = 0$$

or

$$b^{\alpha\beta} p_{\alpha\beta} = 0,$$

where $b^{\alpha\beta}$ are given by

$$b^{\alpha\beta} = \frac{\text{cofactor of } b_{\alpha\beta} \text{ in } |b_{\alpha\beta}|}{|b_{\alpha\beta}|}.$$

Theorem 8.3.3 *Parametric curves have conjugate directions if and only if* $b_{12} = 0$.

Proof: Curve conjugate to u^1 curve is given by Eq. (8.37) as

$$b_{11} du^1 \delta u^1 + b_{12} du^1 \delta u^2 = 0; \text{ as } du^2 = 0.$$

Since the conjugate of u^1 curve is the u^2 curve, $\delta u^1 = 0$ but $\delta u^2 \neq 0$. Therefore, $b_{12} = 0$ which is the necessary condition.

Conversely, if $b_{12} = 0$, then the curve conjugate of u^1curve is given by

$$b_{11}\delta u^1 = 0 \Rightarrow \delta u^1 = 0,$$

which shows that it is the u^2 curve. Hence, the principal directions at a point of the surface are conjugate directions. Thus the null lines on a minimal surface form a conjugate system.

Deduction 8.3.4 Let the lines of curvature be taken as parametric curves, so that $a_{12} = 0$ and $b_{12} = 0$. The directions $\frac{du^1}{du^2}$ and $\frac{\delta u^1}{\delta u^2}$ are inclined to the curve $u^2 =$ constant at angles θ, θ', such that

$$\tan\theta = \sqrt{\frac{a_{22}}{a_{11}}}\frac{du^1}{du^2};\ \ \tan\theta' = \sqrt{\frac{a_{22}}{a_{11}}}\frac{\delta u^1}{\delta u^2}.$$

The condition that two directions be conjugate may be expressed

$$b_{11}\frac{du^1}{du^2}\ \frac{\delta u^1}{\delta u^2} + b_{12}\left(\frac{du^1}{du^2} + \frac{\delta u^1}{\delta u^2}\right) + b_{22} = 0$$

or

$$\tan\theta\tan\theta' = -\frac{b_{11}}{a_{11}}\frac{a_{22}}{b_{22}} = -\frac{\beta}{\alpha},\ \text{as } b_{12} = 0,$$

that is to say, provided they are parallel to conjugate diameters of the indicatrix.

EXAMPLE 8.3.11 *Show that the parametric curves are conjugate on the surface of revolution*

$$x^1 = u^1\cos u^2;\ x^2 = u^1\sin u^2;\ x^3 = f(u^1)$$

with $f(u^1)$ of class C^2.

Solution: The parametric representation of the right helicoid is given by

$$x^1 = u^1\cos u^2;\ x^2 = u^1\sin u^2;\ x^3 = f(u^1)$$

i.e.

$$\mathbf{r} = (x^1, x^2, x^3) = \left(u^1\cos u^2; u^1\sin u^2; f(u^1)\right).$$

The symmetric covariant tensors $b_{\alpha\beta}$ for the second fundamental form are given by

$$b_{11} = \frac{f_{11}}{\sqrt{1+f_1^2}}, b_{22} = \frac{u^1 f_1}{\sqrt{1+f_1^2}}\ \text{and}\ b_{12} = 0 = b_{21}.$$

where $f_1 = \frac{\partial f}{\partial u^1}$ and $f_{11} = \frac{\partial^2 f}{\partial u^1 \partial u^1}$. Since $b_{12} = 0$, the parametric curves on the given surface are conjugate.

8.3.3 Asymptotic Directions

The asymptotic directions at a point on the surface are the self-conjugate directions; and the asymptotic line is a curve whose direction at every point is self-conjugate.

Consequently, in Eq. (8.38) connecting conjugate directions we put $\frac{\delta u^1}{\delta u^2}$ equal to $\frac{du^1}{du^2}$, the directions on the surface given by

$$b_{11}(du^1)^2 + 2b_{12}du^1du^2 + b_{22}(du^2)^2 = 0$$

or

$$b_{\alpha\beta}du^\alpha du^\beta = b_{\alpha\beta}\lambda^\alpha\lambda^\beta = 0 \tag{8.39}$$

and are called the *asymptotic directions.* The curves whose tangents are asymptotic directions are called an *asymptotic line.* Thus, there are two asymptotic directions at a point. The asymptotic directions are

(i) real and distinct, when $b_{12}^2 - b_{11}b_{22}$ is positive, i.e. $\chi < 0$;
(ii) imaginary, when $\chi > 0$;
(iii) identical, when $\chi = 0$.

Now, the normal curvature $\chi_{(n)}$ is given in Eq. (8.14). In the case (iii) the surface is a developable, and the single asymptotic line through a point is a generator. Since for an asymptote direction Eq. (8.39) holds, it follows that, for an asymptotic direction the normal curvature is zero, i.e.

$$\chi_{(n)} = 0 \Rightarrow b_{\alpha\beta}du^\alpha du^\beta = 0$$

for the asymptotic directions. These directions are, therefore, the directions of the asymptotes of the indicatrix. They are at right angles when the indicatrix is a rectangular hyperbola, i.e. when the principal curvature are equal and opposite. Thus, the asymptotic lines are orthogonal when the surface is a minimal surface. The osculating plane at any point of an asymptotic line is the tangent plane to the surface.

Deduction 8.3.5 Let (du^1, du^2) be the direction of asymptotic line at P then we know

$$b_{\alpha\beta}du^\alpha du^\beta = b_{11}(du^1)^2 + 2b_{12}du^1du^2 + b_{22}(du^2)^2 = 0.$$

Now, the normal curvature $\chi_{(n)}$ is given by

$$\chi_{(n)} = \frac{b_{11}(du^1)^2 + 2b_{12}du^1du^2 + b_{22}(du^2)^2}{a_{11}(du^1)^2 + 2a_{12}du^1du^2 + a_{22}(du^2)^2} = 0.$$

If χ_a and χ_b are principal curvatures at P, then, by Euler's theorem

$$\chi_a \cos^2\theta + \chi_b \sin^2\theta = \chi_{(n)} = 0. \tag{8.40}$$

Now, if χ be the normal curvature in a direction perpendicular to asymptotic line then from Eq. (8.40), we get

$$\chi = \chi_a \cos^2\left(\frac{\pi}{2}+\theta\right) + \chi_b \sin^2\left(\frac{\pi}{2}+\theta\right)$$

or

$$\chi = \chi_a \sin^2\theta + \chi_b \cos^2\theta. \tag{8.41}$$

Adding Eqs. (8.40) and (8.41), we get

$$\chi = \chi_a + \chi_b = 2\left(\frac{\chi_a+\chi_b}{2}\right)$$
$$= \text{ twice of mean normal curvature at } P.$$

Thus, the normal curvature in a direction perpendicular to an asymptotic line is twice the mean normal curvature.

EXAMPLE 8.3.12 *Show that a straight line on a surface is an asymptotic line.*

Solution: We know, the normal curvature $\chi_{(n)}$ is given by

$$\chi_{(n)} = \chi \cos\theta,$$

where χ is the curvature of the curve on the surface and θ is the angle between the principal normal and the surface normal. Since for a straight line $\chi = 0$, so,

$$\chi_{(n)} = \frac{b_{\alpha\beta}du^\alpha du^\beta}{a_{\alpha\beta}du^\alpha du^\beta} = 0$$
$$\Rightarrow b_{\alpha\beta}du^\alpha du^\beta = b_{\alpha\beta}\lambda^\alpha\lambda^\beta = 0.$$

Hence, any straight line on a surface is an asymptotic line. From this result, it follows that at each point of an asymptotic curve $\mathcal{C}$, at which $\kappa > 0$, the principal normal of $\mathcal{C}$ lies in the tangent plane of the surface. Thus the generating straight lines of a cylinder or as cone are asymptotic curves on the surface. From this we obtain the following important property of asymptotic curves.

At any point of an asymptotic curve $\mathcal{C}$ for which the curvature $\kappa > 0$, the binormal of $\mathcal{C}$ and the normal to the surface coincide. Consequently, at any point of $\mathcal{C}$, the osculating plane of $\mathcal{C}$ and the tangent plane to the surface then coincide.

Theorem 8.3.4 *The parametric curves are asymptotic lines if and only if*

$$b_{11} = b_{22} = 0.$$

Proof: First, let the parametric curves u^1 = constant and u^2 = constant be asymptotic lines. Then from Eq. (8.39), we get

$$b_{\alpha\beta} du^\alpha du^\beta = b_{\alpha\beta} \lambda^\alpha \lambda^\beta = 0.$$

For u^1 curve, i.e. $du^2 = 0$, this equation reduces to

$$b_{\alpha\beta} \lambda^\alpha_{(1)} \lambda^\beta_{(1)} = b_{\alpha\beta} \frac{1}{\sqrt{a_{11}}} \delta^\alpha_{(1)} \frac{1}{\sqrt{a_{11}}} \delta^\beta_{(1)} = 0$$

$$\Rightarrow \frac{b_{11}}{a_{11}} = 0 \Rightarrow b_{11} = 0.$$

Similarly, for u^2 curve, i.e. $du^1 = 0$; i.e.

$$b_{\alpha\beta} \lambda^\alpha_{(2)} \lambda^\beta_{(2)} = b_{\alpha\beta} \frac{1}{\sqrt{a_{22}}} \delta^\alpha_{(2)} \frac{1}{\sqrt{a_{22}}} \delta^\beta_{(2)} = 0$$

$$\Rightarrow \frac{b_{22}}{a_{22}} = 0 \Rightarrow b_{22} = 0.$$

Conversely, let $b_{11} = b_{22} = 0$, then the differential Eq. (8.39) determining the asymptotic lines is

$$b_{12} du^1 du^2 = 0 \Rightarrow du^1 du^2 = 0; \quad \text{as } b_{12} \neq 0,$$

i.e. the curves are parametric. Of course, co-ordinates of this type can be introduced on a surface S if and only if at any point of S there are two (different real) asymptotic directions.

EXAMPLE 8.3.13 *Prove that the parametric curves on a surface are asymptotic lines if and only if $b_{11} = b_{22} = 0$ and show that*

$$\kappa = -\frac{b_{12}^2}{a}; \quad H = -\frac{a_{12} b_{12}}{a}; \quad \frac{a_{12}}{b_{12}} = \frac{H}{\kappa}.$$

Solution: We know, parametric curves are asymptotic lines if and only if $b_{11} = b_{22} = 0$. Now, the Gaussian curvature κ is given by

$$\kappa = \frac{b}{a} = \frac{b_{11} b_{22} - b_{12}^2}{a_{11} a_{22} - a_{12}^2} = -\frac{b_{12}^2}{a_{11} a_{22} - a_{12}^2} = -\frac{b_{12}^2}{a}.$$

Since $b_{11} = b_{22} = 0$, therefore, the mean curvature H is given by

$$H = \frac{1}{2} a^{\alpha\beta} b_{\alpha\beta} = \frac{1}{2} \left[a^{11} b_{11} + a^{22} b_{22} + 2a^{12} b_{12} \right]$$

$$= a^{12} b_{12} = -\frac{a_{12} b_{12}}{a}.$$

These are the values of the mean curvature H and the Gaussian curvature κ, when parametric curves are asymptotic lines. Thus, the ratio is given by

$$\frac{H}{\kappa} = -\frac{a_{12}b_{12}}{a} \times \left(-\frac{a}{b_{12}^2}\right) = \frac{a_{12}}{b_{12}}.$$

EXAMPLE 8.3.14 *Show that the parametric curves are asymptotic lines to the surfaces*

$$x^1 = u^1 \cos u^2;\ x^2 = u^1 \sin u^2;\ x^3 = cu^2.$$

Prove that the asymptotic lines consist of generators and the curves of intersection with coaxial right circular cylinders.

Solution: The parametric curves are asymptotic lines if $b_{11} = 0$, $b_{22} = 0$ and $b_{12} \neq 0$. For the given surface

$$x^1 = u^1 \cos u^2;\quad x^2 = u^1 \sin u^2;\quad x^3 = cu^2$$

or

$$\mathbf{r} = (x^1, x^2, x^3) = \left(u^1 \cos u^2; u^1 \sin u^2; cu^2\right)$$

the symmetric covariant tensors $b_{\alpha\beta}$ for the second fundamental form are given by

$$b_{11} = 0, b_{22} = 0 \text{ and } b_{12} = -\frac{c}{\sqrt{c^2 + (u^1)^2}} \neq 0.$$

The differential equation giving the asymptotic lines is

$$b_{\alpha\beta}du^\alpha du^\beta = b_{11}(du^1)^2 + 2b_{12}du^1du^2 + b_{22}(du^2)^2 = 0$$

or

$$-\frac{c}{\sqrt{c^2 + (u^1)^2}}du^1du^2 = 0$$

or

$$u^1 = \text{ constant and } u^2 = \text{ constant},$$

i.e. the parametric curves. Now, the curves $u^2 =$ constant are generators, whereas the curves $u^1 =$ constant are the curves of intersection of the given helicoid with the circular cylinders $(x^1)^2 + (x^2)^2 = (u^1)^2$. The axis of these coaxial cylinders is x^3 axis which is also the axis of the given helicoid.

EXAMPLE 8.3.15 *Determine the asymptotic curves of a cylinder of revolution.*

Solution: Let the parametric equation of the surface $\mathcal{S}$ is of the form

$$\mathbf{r} = (x^1, x^2, x^3) = (a\cos u^1, a\sin u^1, u^2).$$

We know any straight line on a surface S of class $r \geq 2$ is an asymptotic curve. Thus the generating straight lines of a cylinder (or a cone) $\mathcal{S}$ are asymptotic curves on these surfaces. We will prove that those curves are the only asymptotic curves of $\mathcal{S}$. The second fundamental magnitudes are given by

$$b_{12} = b_{22} = 0 \text{ and } b_{11} = -1 \neq 0$$

Thus Eq. (8.39) takes the form

$$\begin{aligned} & b_{11}(du^1)^2 + b_{22}(du^2)^2 + 2b_{12}du^1du^2 = 0 \\ \Rightarrow\ & b_{11}(du^1)^2 = 0;\ \text{where } b_{11} = -1 \neq 0 \\ \Rightarrow\ & du^1 = 0,\ \text{that is, } u^1 = \text{constant}. \end{aligned}$$

Since this is the only solution of that equation the above straight lines are the only asymptotic curves on the cylinder.

EXAMPLE 8.3.16 *Show that the two directions given by* $h_{\alpha\beta}du^\alpha du^\beta = 0$, *are orthogonal if and only if* $a^{\alpha\beta}h_{\alpha\beta} = 0$.

Solution: The given equation $h_{\alpha\beta}du^\alpha du^\beta = 0$ can be written in the form

$$h_{11}(du^1)^2 + h_{12}du^1du^2 + h_{21}du^2du^1 + h_{22}(du^2)^2 = 0$$

or

$$h_{11}\left(\frac{du^1}{du^2}\right)^2 + (h_{12} + h_{21})\frac{du^1}{du^2} + h_{22} = 0.$$

Let $\frac{du^1}{du^2}$ and $\frac{\delta u^1}{\delta u^2}$ be the roots of the quadratic equation. Then

$$\frac{du^1}{du^2} + \frac{\delta u^1}{\delta u^2} = -\frac{h_{12} + h_{21}}{h_{11}} \text{ and } \frac{du^1}{du^2}\frac{\delta u^1}{\delta u^2} = \frac{h_{22}}{h_{11}}.$$

The two directions (du^1, du^2) and $(\delta u^1, \delta u^2)$ will be perpendicular if and only if

$$a_{11}du^1\delta u^1 + a_{12}du^1\delta u^2 + a_{21}du^2\delta u^1 + a_{22}du^2\delta u^2 = 0$$

or

$$a_{11}\frac{du^1}{du^2}\frac{\delta u^1}{\delta u^2} + a_{12}\left[\frac{du^1}{du^2} + \frac{\delta u^1}{\delta u^2}\right] + a_{22} = 0$$

or

$$a_{11}\frac{h_{22}}{h_{11}} - a_{12}\left[\frac{h_{12}+h_{21}}{h_{11}}\right] + a_{22} = 0$$

or

$$a_{11}h_{22} - a_{12}(h_{12}+h_{21}) + a_{22}h_{11} = 0$$

or

$$a\left[a^{22}h_{22} + a^{12}h_{12} + a^{21}h_{21} + a^{11}h_{11}\right] = 0$$

or

$$a^{\alpha\beta}h_{\alpha\beta} = 0; \quad \text{as } a \neq 0.$$

EXAMPLE 8.3.17 *The asymptotic lines are at right angles if and only if $H = 0$.*

Solution: The asymptotic lines are given by the equation

$$b_{\alpha\beta}du^{\alpha}du^{\beta} = b_{\alpha\beta}\lambda^{\alpha}\lambda^{\beta} = 0.$$

Thus, the asymptotic lines are at right angles if and only if

$$a^{\alpha\beta}b_{\alpha\beta} = 0 \Rightarrow H = \frac{1}{2}a^{\alpha\beta}b_{\alpha\beta} = 0,$$

where H is the mean curvature. Thus, the asymptotic lines are right angles if and only if the surface is a minimal surface. Hence the asymptotic lines form an orthogonal system, bisecting the angles between the lines of curvature.

EXAMPLE 8.3.18 *Show that on a sphere there is no real asymptotic line.*

Solution: The parametric representation of the sphere of radius a is given by

$$x^1 = a\sin u^1 \cos u^2, \quad x^2 = a\sin u^1 \sin u^2, x^3 = a\cos u^2$$

or

$$\mathbf{r} = (x^1, x^2, x^3) = \left(a\sin u^1 \cos u^2, a\sin u^1 \sin u^2, a\cos u^2\right).$$

For the given surface, the symmetric covariant second order tensors $a_{\alpha\beta}$ for the first fundamental form are given by

$$a_{11} = a^2; \quad a_{22} = a^2\sin^2 u^1; \quad a_{12} = 0 = a_{21}$$

and the symmetric covariant second order tensors $b_{\alpha\beta}$ for the second fundamental form are given by

$$b_{11} = -a; \quad b_{22} = -a\sin^2 u^1; \quad b_{12} = 0 = b_{21}.$$

Since $a_{12} = 0 = b_{12}$, this given parametric curve has the line of curvature. In fact,

$$\frac{b_{11}}{a_{11}} = \frac{b_{12}}{a_{12}} = \frac{b_{22}}{a_{22}} = -\frac{1}{a},$$

i.e. $b_{\alpha\beta}$ are proportional to $a_{\alpha\beta}$. Now, the asymptotic lines are given by

$$b_{\alpha\beta}du^\alpha du^\beta = b_{11}(du^1)^2 + 2b_{12}du^1du^2 + b_{22}(du^2)^2 = 0$$

or

$$-a(du^1)^2 - a\sin^2 u^1(du^2)^2 = 0$$

or

$$(du^1)^2 + \sin^2 u^1(du^2)^2 = 0.$$

Hence, there is no real asymptote.

Theorem 8.3.5 *The torsion of an asymptotic line equals to* $\pm\sqrt{-\kappa}$, *where* κ *is the Gaussian curvature of the surface.*

Proof: From Eq. (8.6), we have, for an asymptotic curve,

$$\chi\mu^i = b_{\alpha\beta}\lambda^\alpha\lambda^\beta\xi^i + \chi_g\zeta^i, = \chi_g\zeta^i; \quad \text{as } b_{\alpha\beta}\lambda^\alpha\lambda^\beta = 0, \tag{8.42}$$

where μ^i is the principal normal and ζ^i is the space vector giving the same direction on the surface as ξ^i; the unit normal to the surface $\mathcal{S}$. Therefore,

$$(\chi\mu^i)(\chi\mu^i) = (\chi_g\zeta^i)(\chi_g\zeta^i)$$

or

$$\chi^2 = \chi_h^2 \Rightarrow \chi = \pm\chi_h \tag{8.43}$$

from which we conclude that the curvature and the geodesic curvature of an asymptotic line are equal in magnitude. Also,

$$\chi\mu^i = \chi_g\zeta^i \Rightarrow \mu^i = \pm\zeta^i. \tag{8.44}$$

We now derive the well known formula for torsion of an asymptotic line. From Eq. (8.44), it follows that the principal normal μ^i of an asymptotic line is tangent to the surface and hence the binormal of an asymptotic line coincides with the surface normal ξ^i, i.e.

$$\gamma^i = \pm\xi^i. \tag{8.45}$$

Taking intrinsic derivative of both the sides of Eq. (8.45), we get

$$\frac{\delta\gamma^i}{\delta s} = \frac{\delta\xi^i}{\delta s} \Rightarrow \tau\mu^i = \pm\xi^i_{,\alpha}\frac{du^\alpha}{ds} = \pm\xi^i_{,\alpha}\lambda^\alpha.$$

Therefore,

$$\begin{aligned}\tau^2 &= g_{ij}(\tau\mu^i)(\tau\mu^j) = g_{ij}\xi^i_{,\alpha}\xi^j_{,\beta}\lambda^\alpha\lambda^\beta \\ &= c_{\alpha\beta}\lambda^\alpha\lambda^\beta; \quad \text{as } c_{\alpha\beta} = g_{ij}\xi^i_{,\alpha}\xi^j_{,\beta}.\end{aligned}$$

Using the relation

$$c_{\alpha\beta} - 2Hb_{\alpha\beta} + \kappa a_{\alpha\beta} = 0,$$

where H is the mean curvature and κ is the Gaussian curvature, we get

$$c_{\alpha\beta}\lambda^\alpha\lambda^\beta - 2Hb_{\alpha\beta}\lambda^\alpha\lambda^\beta + \kappa a_{\alpha\beta}\lambda^\alpha\lambda^\beta = 0$$

or

$$\tau^2 - 0 + \kappa^2 = 0 \Rightarrow \tau = \pm\sqrt{-\kappa}.$$

Therefore, the torsions of the two asymptotic lines through a point are equal in magnitude and opposite in sign; and the square of either is negative of the specific/Gaussian curvature of the surface. This theorem is formulated by *Beltrami Enneper.*

EXAMPLE 8.3.19 *Find the asymptotic lines on the surface of revolution*

$$x^1 = u^1\cos u^2, \quad x^2 = u^1\sin u^2, x^3 = f(u^1).$$

Find also the values of their torsions.

Solution: For the given surface

$$x^1 = u^1\cos u^2, \ x^2 = u^1\sin u^2, x^3 = f(u^1)$$

or

$$\mathbf{r} = (x^1, x^2, x^3) = \left(u^1\cos u^2, \quad u^1\sin u^2, f(u^1)\right)$$

the symmetric covariant tensors $b_{\alpha\beta}$ for the second fundamental form are given by

$$b_{11} = \frac{u^1 f_{11}}{\sqrt{1+f_1^2}}, b_{12} = 0 = b_{21} \text{ and } b_{22} = \frac{(u^1)^2 f_1}{\sqrt{1+f_1^2}}.$$

The differential equation giving the asymptotic lines is

$$b_{11}(du^1)^2 + 2b_{12}du^1du^2 + b_{22}(du^2)^2 = 0$$

or

$$f_{11}(du^1)^2 + u^1 f_1 (du^2)^2 = 0.$$

Using *Beltrami Enneper's formula* the torsion of an asymptotic line equals to $\pm\sqrt{-\kappa}$, where κ is the Gaussian curvature of the surface, as

$$\tau = \pm\sqrt{-\kappa} = \pm\frac{\sqrt{-(u^1)^3 f_{11} f_1}}{(u^1)^2(1+f_1^2)}.$$

8.4 Conformal Mapping

Isometric mappings of a surface $\mathcal{S}$ on another surface $\mathcal{S}^*$ have been already mentioned. It is often convenient to consider mappings which are more general than isometrices- for example, it is useful to map parts of the earth's surface onto a flat atlas. Certain of these maps are particular cases of conformal mappings.

A surface $\mathcal{S}$ is said to be conformally mapped on a surface $\mathcal{S}^*$ if there is a differentiable homomorphism of $\mathcal{S}$ on $\mathcal{S}^*$ such that the angle between any two curves at an arbitrary point P on $\mathcal{S}$ is equal to the angle between the corresponding curves on $\mathcal{S}^*$.

An isometric mappings preserves both distances and angles, whereas a conformal mapping just preserves angles. Let the fundamental forms of $\mathcal{S}$, $\mathcal{S}^*$ be respectively

$$ds^2 = a_{11}du^2 + 2a_{12}dudv + a_{22}dv^2$$
$$ds^{*2} = a^*_{11}du^2 + 2a^*_{12}dudv + a^{*2}_{12}dv^2$$

the correspondence being such that corresponding points P, P^* have the same parametric values. Then if $\frac{ds^*}{ds}$ has the same value for all directions at the given point, we must have

$$\frac{a^*_{11}}{a_{11}} = \frac{a^*_{12}}{a_{12}} = \frac{a^*_{22}}{a_{22}} = \frac{ds^{*2}}{ds^2} = \rho^2, \tag{8.46}$$

where ρ is function of u and v or a constant. Equation (8.46) is evidently a necessary and sufficient condition for a differentiable homomorphism to be conformal. Therefore

$$ds^* = \rho ds.$$

The quantity ρ may be called the linear magnification. When $\rho = 1$, for all points of the surface, $ds^* = ds$. The conformal representation is then said to be isometric, and the surfaces are said to be applicable. In this case, corresponding elements of the two surfaces are congruent. Thus every isometric mapping is conformal.

EXAMPLE 8.4.1 *Prove that every point on a surface has a neighbourhood which can be mapped conformably on some neighbourhood of any other surface.*

Solution: Let S be the given surface with metric

$$ds^2 = a_{11}du^2 + 2a_{12}dudv + a_{22}dv^2$$

in some co-ordinate domain. At any point P there are two (imaginary) directions such that $ds^2 = 0$. These are called isotropic directions at P, and since $a = a_{11}a_{22} - a_{12}^2 \neq 0$ if follows that these directions are always distinct.

When curves along these directions are chosen as parametric curves the metric assumes the form $ds^2 = \lambda dudv$. The change of parameters

$$u = U + iV, v = U - iV,$$

where U and V are real, leads to a metric of the form

$$ds^2 = \Delta^2[dU^2 + dV^2].$$

If this is compared with the metric of the plane

$$ds^2 = du^{*2} + dv^{*2},$$

it is rapidly seen that the mapping $u^* = U, v^* = V$ gives a conformal mapping of a region of the given surface on a region of the plane. Hence the result follows.

EXAMPLE 8.4.2 *Show that null lines on a surface correspond to null lines in the conformal representation.*

Solution: We have deduce that $ds^{*2} = r^2ds^2$, if ds^2 vanishes along a curve $\mathcal{S}, ds^{*2}$ will vanish along the corresponding curve on $\mathcal{S}^*$. Conversely, let the null lines be taken as parametric curves. Then

$$a_{11} = a_{22} = 0 \quad \text{and} \quad a_{11}^* = a_{22}^* = 0$$

$$\Rightarrow \frac{ds^{*2}}{ds^2} = \frac{2a_{12}^*dudv}{2a_{12}dudv} = \frac{a_{12}^*}{a_{12}}.$$

Since the $\dfrac{ds^*}{ds}$ has the same value for all arcs through a given point, the representation is conformal. Thus, if null lines on $\mathcal{S}$ corresponds to null lines on $\mathcal{S}^*$, the representation is conformal.

8.5 Spherical Image

We shall now consider briefly the spherical representation of a surface, in which each point or configuration on the surface has its representation on a unit sphere, whose centre may be taken as origin. If $\boldsymbol{\xi}$ is the unit normal at the point P on the surface S, the point Q whose position vector is $\boldsymbol{\xi}$ is said to correspond to P, or the image of P. Clearly, Q lies on the unit sphere; and if P moves in any curve on the surface, Q moves in the corresponding curve on the sphere. The mapping thus defined is called the Gaussian spherical mapping of the surface $\mathcal{S}$. The set of all image points is called the spherical image of $\mathcal{S}$. Since the position vector $\overline{r}$ of Q is given by $\overline{\mathbf{r}} = \boldsymbol{\xi}$, it follows that

$$\frac{\partial \overline{\mathbf{r}}}{\partial u} = \frac{\partial : \boldsymbol{\xi}}{\partial u} = \frac{1}{a^2}\left[(a_{12}b_{12} - a_{22}b_{11})\frac{\partial \mathbf{r}}{\partial v} + (a_{12}b_{11} - a_{11}b_{12})\frac{\partial \mathbf{r}}{\partial v}\right]$$

$$\frac{\partial \overline{\mathbf{r}}}{\partial v} = \frac{\partial \boldsymbol{\xi}}{\partial v} = \frac{1}{a^2}\left[(a_{12}b_{22} - a_{22}b_{11})\frac{\partial \mathbf{r}}{\partial u} + (a_{12}b_{12} - a_{11}b_{12})\frac{\partial \mathbf{r}}{\partial v}\right].$$

Consequently, if $a^*_{11}, a^*_{12}, a^*_{22}$ denote the fundamental magnitudes of the first order for the spherical image

$$\left.\begin{aligned} a^*_{11} &= \frac{1}{a^2}\left[a_{11}b^2_{12} - 2a_{12}b_{11}b_{12} + a_{22}b_{11}\right], \\ a^*_{12} &= \frac{1}{a^2}\left[a_{11}b_{12}b_{22} - a_{12}b^2_{12} - a_{12}b_{11}b_{22} + a_{22}b_{11}b_{12}\right], \\ a^*_{22} &= \frac{1}{a^2}\left[a_{11}b^2_{22} - 2a_{12}b_{12}b_{22} + a_{22}b^2_{12}\right], \end{aligned}\right\} \tag{8.47}$$

or, in terms of the first and second curvatures

$$a^*_{11} = Hb_{11} - ka_{11};\quad a^*_{12} = Hb_{12} - ka_{12};\quad a^*_{22} = Hb_{22} - ka_{22} \tag{8.48}$$

In virtue of Eq. (8.48) we may write the square of the linear element of the image.

$$ds^{*2} = H(b_{11}du^2 + 2b_{12}dudv + b_{22}dv^2) - k(a_{11}du^2 + 2a_{12}dudv + a_{22}dv^2),$$

or, if $\chi_{(n)}$ is the normal curvature of the given surface in the direction of the arc-length

$$ds^{*2} = \left[\chi_{(n)}H - k\right]ds^2.$$

If then $\chi_{(1)}$ and $\chi_{(2)}$ are the principal curvatures of the surface, we may write this in virtue of Euler's theorem

$$\begin{aligned} ds^{*2} &= \left[\left\{\chi_{(1)} + \chi_{(2)}\right\}\left\{\chi_{(1)}\cos^2\theta + \chi_{(2)}\sin^2\theta\right\} - \chi_{(1)}\chi_{(2)}\right]ds^2 \\ &= \left[\chi_{(1)}\cos^2\theta + \chi^2_{(2)}\sin^2\theta\right]ds^2. \end{aligned} \tag{8.49}$$

It is clear from either of these formulae that the value of the quotient $\frac{ds^*}{ds}$ depends upon the direction of the arc element. The fundamental magnitudes for the spherical representation, as given by Eq. (8.48), become in the case of a minimal surface

$$a_{11}^* = -\kappa a_{11}; \;\; a_{12}^* = -\kappa a_{12}; \;\; a_{22}^* = -\, \kappa a_{22}. \tag{8.50}$$

From these relations several interesting general properties may be deduced.

EXAMPLE 8.5.1 *Show that the spherical image is not a conformal representation in general.*

Solution: The spherical image be conformal, however, if $\chi_{(1)} = \pm\chi_{(2)}$. When $\chi_{(1)} = -\chi_{(2)}$ at all points, the Gaussian curvature becomes

$$H = \frac{1}{2}\left[\chi_{(1)} + \chi_{(2)}\right] = 0$$

i.e. H vanishes indentically and the surface is a minimal surface. Thus, for a minimal surface

$$ds^{*2} = -\kappa ds^2.$$

Thus $\dfrac{ds^*}{ds}$ is independent of the direction of the arc-element through the point, and the representation is conformal. The magnification has the value $\sqrt{-\kappa}$, the second curvature being essentially negative for a real minimal surface. Thus, the spherical representation of a minimal surface is conformal. Therefore, in general, the spherical image is not a conformal representation.

Moreover it follows from Eq. (8.49) that the turning values of $\dfrac{ds^*}{ds}$ are given by $\cos\theta = 0$ and $\sin\theta = 0$. Thus the greatest and least values of the magnification at a point are numerically equal to the principal curvatures. From this it follows that if the spherical image of a surface is a conformal representation, either the surface is minimal, or else its principal curvatures are equal at each point.

EXAMPLE 8.5.2 *Prove that null lines on a minimal surface become both null lines and asymptotic lines in the spherical representation.*

Solution: Let the null lines be taken as parametric curves, then $a_{11} = 0 = a_{22}$. Therefore

$$Hb_{11} - \kappa a_{11} = 0 \quad \text{and} \quad Hb_{22} - \kappa a_{22} = 0.$$

Thus the parametric curves in the spherical image are null lines. Again considering the second order magnitudes for the sphere, we have

$$b_{11} = 0 = b_{12}$$

and therefore, the parametric curves in the spherical image are also asymptotic lines. Conversely, let the null lines are parametric curves, then $a_{11} = 0 = a_{22}$; and since the parametric curves are also null lines in the spherical image

$$Hb_{11} - ka_{11} = 0; Hb_{22} - ka_{22} = 0$$
$$\Rightarrow Hb_{11} = 0 \quad \text{and} \quad Hb_{22} = 0.$$

Consequently, either $H = 0$ and the surface is minimal, or else $b_{11} = 0 = b_{22}$. In the later case, it follows that

$$H = \frac{2b_{12}}{a_{12}} \quad \text{and} \quad k = \frac{b_{12}^2}{a_{12}^2} \quad \Rightarrow \quad H^2 - 4\kappa = 0,$$

which is the condition that the principal curvatures should be equal. Thus, if the null lines in the spherical representation, either the surface is minimal, or else its principal curvatures are equal.

EXAMPLE 8.5.3 *Show that the isometric lines on a minimal surface are also isometric in the spherical representation.*

Solution: If the isometric lines are taken as parametric curves, we have

$$a_{12} = 0 \quad \text{and} \quad \frac{a_{11}}{a_{22}} = \frac{U(u)}{V(v)},$$

where U is a function of u only and V is a function of v only. Therefore, it follows that

$$\frac{H}{\kappa} = \frac{a_{12}}{b_{12}}; \quad \frac{a_{12}}{a_{22}} = \frac{U(u)}{V(v)},$$

showing that the parametric curves in the spherical image are also isomorphic. Hence the theorem. This result is obvious from the fact that the spherical image on a minimal surface is conformal.

In particular, the spherical image of the lines of curvature on a minimal surface are isometric curves, for the lines of curvature on a minimal surface have been shown to be isometric.

EXAMPLE 8.5.4 *Show that the lines of curvature on a surface are orthogonal in their spherical representation.*

Solution: The lines of curvature on a surface are orthogonal in their spherical representation. For if they are taken as parametric curves, we have $a_{12} = 0 = b_{12}$, hence $a_{12}^* = 0$ which proves the statement. Further if $a_{12} = 0$ and $a_{12}^* = 0$ we must also have $b_{12} = 0$ unless the Gaussian curvature H vanishes identically. Thus, if the surface is not a minimal surface, the lines of curvature are the only orthogonal system whose spherical image is orthogonal.

Moreover, by Rodrigue's formula along a line of curvature dr is parallel to $d\xi$ and therefore also to $d\bar{r}$. Hence the tangent to a line of curvature is parallel to the tangent to its spherical image at the corresponding point.

Conversely if dr is parallel to $d\bar{r}$ it is parallel to $d\xi$. The vectors $\xi, \xi + d\xi$, dr are therefore coplanar, and the line is a line of curvature. Thus if the relation holds for a curve on a surface it must be a line of curvature.

EXAMPLE 8.5.5 *Prove that if two directions are conjugate at a point on a given surface, each is perpendicular to the spherical image of the other at the corresponding point.*

Solution: If dr and ds be two infinitesimal displacements on a given surface, and $d\xi$ the change in the unit normal due to the former, the directions of the displacements will be conjugate provided

$$d\xi \cdot ds = 0.$$

Conversely this relation holds if the directions are conjugate. But $d\xi = d\bar{r}$, where $d\bar{r}$ is the spherical image of dr. Consequently

$$d\bar{r} \cdot ds = 0.$$

Thus if two directions are conjugate at a point on a given surface, each is perpendicular to the spherical image of the other at the corresponding point. It follows that the inclination of two conjugate directions are equal, or supplementary, to that of their spherical representations.

Further, an asymptotic line is self-conjugate. Hence an asymptotic line on a surface is perpendicular to its spherical image at the corresponding point.

8.6 Surface of Revolution

A surface of revolution may be generated by the rotation of a plane curve about an axis in its plane. Let $\mathcal{C}: x^3 = f(x^1)$ be a curve drawn on the plane $x^2 = 0$, where x^i are Cartesian co-ordinates of a point. Let $P(x^1, 0, x^3)$ be a point on the curve, then P traces a circle as the curve $\mathcal{C}$ is rotated about the x^3 axis. The surface generated by the curve is called the *surface of revolution* and its implicit equation is

$$x^3 = \phi\left(\sqrt{(x^1)^2 + (x^2)^2}\right). \tag{8.51}$$

To obtain the parametric equation of the surface, let the generating curve be given by

$$x^1 = f_1(u^1), \;\; x^2 = 0, \;\; x^3 = f_2(u^2).$$

Let u^2 be the angle of rotation about the x^3 axis measured from the x^1 axis, then

$$x^1 = f_1(u^1)\cos u^2, x^2 = f_1(u^1)\sin u^2,\ x^3 = f_2(u^1).$$

This is the parametric representation of the surface of revolution obtained by rotating a curve about the x^3 axis. The curve C is called the *generator of the curve.*

Let the axis of rotation be x^3 and u^1 denote perpendicular distance from it, the co-ordinates of a point on the surface may be expressed

$$x^1 = u^1\cos u^2; x^2 = u^1\sin u^2;\quad x^3 = f(u^1),$$

the longitude u^2 being the inclination of the axial plane through the given point to the x^3x^1 plane. The parametric curves u^2 = constant are the *meridian lines*, or intersections of the surface by the axial planes; the curves u^1 = constant are the *parallels*, or intersection of the surface by planes perpendicular to the axis.

EXAMPLE 8.6.1 *Let $\mathcal{S}$ is a surface of revolution*

$$x^1 = u^1\cos u^2; x^2 = u^1\sin u^2;\quad x^3 = f(u^1);\quad f(u^1)\in C^2.$$

(a) Find the surface of revolution that has a minimal surface.

(b) Find the surface of revolution that has a constant negative Gaussian curvature.

(c) Find the complete integral of the equations of geodesics.

Solution: With u^1, u^2 as parameters, the parametric representation of the right helicoid is given by

$$x^1 = u^1\cos u^2; x^2 = u^1\sin u^2;\quad x^3 = f(u^1)$$

i.e.

$$\mathbf{r} = (x^1, x^2, x^3) = \left(u^1\cos u^2; u^1\sin u^2; f(u^1)\right).$$

Here we have to show that the catenoid is the only minimal surface of revolution. The first order magnitudes are given by

$$a_{11} = 1 + f_1^2;\ a_{22} = (u^1)^2 \text{ and } a_{12} = 0 = a_{21},$$

$$\Rightarrow a = |a_{\alpha\beta}| = \begin{vmatrix} 1+f_1^2 & 0 \\ 0 & (u^1)^2 \end{vmatrix} = (u^1)^2(1+f_1^2),$$

where $f_1 = \frac{\partial f}{\partial u^1}$. Since $a_{12} = 0$, it follows that the parallels cut the meridians orthogonally. The second order magnitudes are given by

$$b_{11} = \frac{f_{11}}{\sqrt{1+f_1^2}}, b_{22} = \frac{u^1 f_1}{\sqrt{1+f_1^2}} \text{ and } b_{12} = 0 = b_{21},$$

$$\Rightarrow b = |b_{\alpha\beta}| = \begin{vmatrix} \dfrac{f_{11}}{\sqrt{1+f_1^2}} & 0 \\ 0 & \dfrac{u^1 f_1}{\sqrt{1+f_1^2}} \end{vmatrix} = \frac{u^1 f_1 f_{11}}{1+f_1^2},$$

where $f_1 = \frac{\partial f}{\partial u^1}$ and $f_{11} = \frac{\partial^2 f}{\partial u^1 \partial u^1}$. Since $a_{12} = 0$ and $b_{12} = 0$, the *parametric curves are the lines of curvature.* The equation for the principal curvatures reduces to

$$u^1(1+f_1^2)^2\chi_\rho^2 - \sqrt{1+f_1^2}\left[u^1 f_{11} + f_1(1+f_1^2)\right]\chi_\rho + f_1 f_{11} = 0.$$

This is a quadratic equation in χ_ρ which gives the two values $\chi_{(1)}$ and $\chi_{(2)}$ given by

$$\chi_{(1)} = \frac{f_{11}}{(1+f_1^2)^{3/2}};\quad \chi_{(2)} = \frac{f_1}{u^1\sqrt{1+f_1^2}}.$$

The first of those is the curvature of the generating curve. The second is the reciprocal of the length of the normal intercepted between the curve and the axis of rotation.

(a) Now, the mean curvature H is given by

$$H = \frac{a_{11}b_{22} - 2a_{12}b_{12} + a_{22}b_{11}}{a} = \frac{f_1(1+f_1^2) + u^1 f_{11}}{u^1(1+f_1^2)^{3/2}}.$$

The surface will be a minimal surface if $H = 0$, i.e. if

$$f_1(1+f_1^2) + u^1 f_{11} = 0$$

or

$$F(1+F^2) + u^1 F_1 = 0;\quad F = f_1 = \frac{\partial f}{\partial u^1}$$

or

$$\frac{du^1}{u^1} + \frac{dF}{F(1+F^2)} = 0$$

or

$$\log u^1 + \log F - \frac{1}{2}\log(1+F^2) = \log c;\ \text{integrating}$$

or

$$\frac{u^1 F}{\sqrt{1+f_1^2}} = c \Rightarrow F = \frac{\partial f}{\partial u^1} = \pm\frac{c}{\sqrt{(u^1)^2 - c^2}}.$$

Take the positive sign (the other case is exactly similar) and integrating we get

$$f + c_1 = \cosh^{-1}\left(\frac{u^1}{c}\right) \Rightarrow u^1 = c\cosh(x^3 + c_1).$$

Hence, the required surface is the surface of revolution obtained by revolving the catenary $u^1 = c\cosh(x^3 + c_1)$ about x^3 axis. Thus the only minimal surface of revolution is that formed by the revolution of a catenary about its directrix.

(b) Now, the Gaussian curvature κ is given by

$$\kappa = \frac{b}{a} = \frac{f_1 f_{11}}{(u^1)^2(1+f_1^2)}.$$

Given that, the surface of revolution that has a constant negative Gaussian curvature, say $\kappa = -\frac{1}{c^2}$. Therefore,

$$\frac{f_1 f_{11}}{(u^1)^2(1+f_1^2)} = -\frac{1}{c^2}$$

or

$$\frac{dF}{F^2} = -\frac{2u^1 du^1}{c^2}; \quad F = 1 + f_1^2 = 1 + \left(\frac{\partial f}{\partial u^1}\right)^2$$

or

$$\frac{1}{F} = \frac{1}{1+f_1^2} = \frac{(u^1)^2}{c^2}; \quad \text{integrating}$$

or

$$1 + f_1^2 = \frac{c^2}{(u^1)^2} \Rightarrow f_1 = \pm\frac{\sqrt{c^2-(u^1)^2}}{u^1},$$

where we choose the constant of integration equal to 0. Take the positive sign (the other case is exactly similar) and integrating we get

$$f = x^3 = \sqrt{c^2-(u^1)^2} + \frac{c}{2}\log\frac{c-\sqrt{c^2-(u^1)^2}}{c+\sqrt{c^2-(u^1)^2}}.$$

Hence, the required surface is the surface of revolution obtained by revolving the tractrix

$$x^3 = \sqrt{c^2-(u^1)^2} + \frac{c}{2}\log\frac{c-\sqrt{c^2-(u^1)^2}}{c+\sqrt{c^2-(u^1)^2}}.$$

about x^3 axis, which is the axis of tractrix.

(c) The second equation for geodesics is of the form

$$\frac{d^2u^2}{ds^2} + \frac{2}{u^1}\frac{du^1}{ds}\frac{du^2}{ds} = 0.$$

On multiplication by u^2 this equation becomes exact, and has for its integral

$$(u^1)^2\frac{du^2}{ds} = h; \quad h = \text{constant}.$$

If θ is the angle at which the geodesics cuts the meridian, we may write $u^1 \sin\theta = h$, a theorem due to Clairant. This is a first integral of the equation of geodesics, involving one orbitrary constant h.

To obtain the complete integral we observe that, for any arc on the surface,

$$ds^2 = (1+f_1^2)(du^1)^2 + (u^1)^2(du^2)^2$$

and therefore, for the arc of a geodesic

$$(u^1)^4(du^2)^2 = h^2(1+f_1^2)(du^1)^2 + h^2(u^1)^2(du^2)^2$$

or

$$du^2 = \pm\frac{h}{u^1}\sqrt{\frac{1+f_1^2}{(u^1)^2-h^2}}du^1$$

or

$$u^2 = c \pm h\int\frac{1}{u}\sqrt{\frac{1+f_1^2}{(u^1)^2-h^2}}du^1$$

involving the two arbitrary constants c and h, is the complete integral of the equation of geodesics on a surface of revolution.

8.7 Exercises

1. Find the principal curvatures of the surface $\mathcal{S}$ defined by

$$x^1 = u^1; x^2 = u^2;\quad x^3 = f(u^1, u^2).$$

2. Show that the principal curvatures are $-1, -1$ for the surface

$$(x^1)^2 + (x^2)^2 + (x^3)^2 = 1.$$

3. Find the equation for the principal curvatures, and the differential equation of the lines of curvature, for the surfaces

   (i) $2z = \dfrac{x^2}{a} + \dfrac{y^2}{b}$, (ii) $3z = ax^3 + by^3$, (iii) $z = c\tan^{-1}\frac{y}{x}$.

4. Examine the curvature, and find the lines of curvature, on the surface $xyz = abc$.
5. Find the principal direction and principal curvatures on the surface

$$x^1 = a(u^1+u^2);\quad x^2 = b(u^1-u^2);\quad x^3 = u^1u^2.$$

   Show also that on the surface the parametric curves are asymptotic.

6. Find equations for the principal radii, the lines of curvature, and the first and second curvatures of the following surfaces:

   (i) the conoid $x^1 = u^1\cos u^2;\ x^2 = u^1\sin u^2;\ x^3 = f(u^1)$;
   (ii) the catenoid $x^1 = u^1\cos u^2;\ x^2 = u^1\sin u^2;\ x^3 = c\log\left(u^1 + \sqrt{(u^1)^2-c^2}\right)$;
   (iii) the cylindroid $x^3((x^1)^2+(x^2)^2) = 2mx^1x^2$; find also the lines of curvature and the principal curvatures;
   (iv) the surface $2z = ax^2 + 2hxy + by^2$;

(v) the surface

$$x^1 = 3u^1(1 + (u^2)^2) - (u^1)^3;$$
$$x^2 = 3u^2(1 + (u^1)^2) - (u^2)^3;$$
$$x^3 = 3\left((u^1)^2 - (u^2)^2\right);$$

Also show that the asymptotic lines are $u \pm v =$ constant.

(vi) the surface

$$\frac{x}{a} = \frac{1+uv}{u+v};\ \frac{y}{b} = \frac{u-v}{u+v};\ \frac{z}{c} = \frac{1-uv}{u+v};$$

(vii) the surface $xyz = a^3$.

7. Find the principal curvatures and the lines of curvature on the surface $z^2(x^2 + y^2) = c^4$.
8. Prove that along a line of curvature of a conoid, one principal radius varies as the cube of the other.
9. Prove that if a plane cuts a surface everywhere at the same angle, the section is a line of curvature on the surface.
10. Prove that the only developable surface which have isometric lines of curvature are either conical or cylindrical.
11. Prove that the co-ordinate surfaces of every triply orthogonal curvilinear co-ordinate system in E^3 intersect along the lines of curvature of co-ordinate surfaces.

    Hints: Consider the surface $x^3 =$ constant and take $x^1 = u^1, x^2 = u^2$ as surface co-ordinates on it. Show that along the co-ordinate lines $u^1 =$ constant, $u^2 =$ constant, $b_{12} = 0$ if $a_{12} = 0$.
12. Find the components of the curvature tensors $R_{\alpha\beta\gamma\delta}$ and $R^{\alpha}_{\beta\gamma\delta}$ on $x^3 = f(x^1, x^2)$.
13. Prove that the Gaussian curvature k depends only on the coefficients of the first fundamental form (and their derivatives) but not on the second fundamental form.
14. Given an ellipsoid of revolution, whose surface is given by

    $$x^1 = a\cos u^1 \sin u^2;\quad x^2 = a\sin u^1 \sin u^2;\quad x^3 = c\cos u^2,$$

    where a and c are constants satisfying $a^2 > c^2$ and $\left(x^1, x^2, x^3\right)$ are orthogonal Cartesian co-ordinates. Show that

    $$a_{11} = a^2\sin^2 u^2;\ a_{12} = 0;\quad a_{22} = a^2\cos^2 u^2 + c^2\sin^2 u^2,$$

    $$b_{11} = \frac{ac\sin^2 u^2}{\sqrt{a^2\cos^2 u^2 + c^2\sin^2 u^2}};\ b_{12} = 0;\ b_{22} = \frac{ac}{\sqrt{a^2\cos^2 u^2 + c^2\sin^2 u^2}}$$

and the total curvature K is given by

$$K = \chi_1\chi_2 = \frac{c^2}{(a^2\cos^2 u^2 + c^2\sin^2 u^2)^2}.$$

Discuss the lines of curvature on this surface.

15. Show that the lines of curvatures for the surface given by

$$(x^1)^2 + (x^2)^2 + (x^3)^2 = 1$$

are the parametric curves.

16. Show that the lines of curvatures for the surface given by

$$\mathbf{r} = (u\cos v, u\sin v, cv)$$

are given by $v = \pm\sinh^{-1}\frac{u}{c} +$ constant. Also, show that on the surface the parametric curves are asymptotic lines.

17. Show that a system of confocal ellipses and hyperbolas are isometric lines in the plane.

18. Show that the umbilics of the surface

$$\left(\frac{x^1}{a}\right)^{2/3} + \left(\frac{x^2}{b}\right)^{2/3} + \left(\frac{x^3}{c}\right)^{2/3} = 1$$

lie on a sphere of radius $abc/(ab+bc+ca)$ whose centre is the origin.

19. If $a > b > c$, the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ has umbilici at the points

$$y = 0,\ x^2 = \frac{a^2(a^2-b^2)}{a^2-c^2};\ z^2 = \frac{c^2(b^2-c^2)}{a^2-c^2}.$$

20. Show that the surface $4a^2z^2 = (x^2-2a^2)(y^2-2a^2)$ has a line of umbilics lying on the sphere $x^2+y^2+z^2 = 4a^2$.

21. Prove that, at any point of the surface, the sum of the radii of normal curvature in conjugate directions is constant.

22. Prove that the product of the radii of normal curvature in conjugate directions is a minimum for the lines of curvature.

23. Prove that the normal curvature in a direction perpendicular to an asymptotic line is twice the mean curvature.

24. Show that on the paraboloid $2z = \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}$ the asymptotic lines are $\dfrac{x}{a} \pm \dfrac{y}{b} =$ constant.

25. Find the asymptotic lines on the surface

$$x = a(1+\cos u)\cot v;\ y = a(1+\cos u);\ z = \frac{a\cos u}{\sin v}.$$

26. Prove that the asymptotic lines of the right conoid

$$x^1 = u^1\cos u^2; x^2 = u^1\sin u^2;\ x^3 = f(u^2)$$

are given by $u^2 =$ constant, $\frac{\partial f}{\partial u^2} = (u^1)^2 c$. Hence, find the asymptotic lines of the cylindroid

$$x^1 = u^1\cos u^2;\ x^2 = u^1\sin u^2;\ x^3 = m\sin 2u^2.$$

Determine $f(u^2)$ so that on this conoid the parametric curves may be isometric lines.

27. Prove that the asymptotic lines of the catenoid $u^1 = c\cosh\frac{x^3}{c}$ lie on the cylinders

$$2u^1 = c\left(ae^{u^2} + \frac{1}{a}e^{-u^2}\right), \text{where } a \text{ is arbitrary.}$$

Hints: Consider the parametric representation of the curve as

$$x^1 = u^1\cos u^2; x^2 = u^1\sin u^2;\ \ x^3 = f(u^1) = c\cosh^{-1}\frac{u^1}{c}.$$

28. Find the asymptotic lines on the surface $z = y\sin x$.

29. Prove that the asymptotic lines on the surface of revolution

$$x^1 = u^1\cos u^2; x^2 = u^1\sin u^2;\ x^3 = f(u^1)$$

are given by $f_{11}d(u^1)^2 + u^1 f_1 d(u^2)^2 = 0$. Write down the value of the torsions. Further if $u^1 = a\sin\phi$ and $f(u^1) = a(\log\tan\frac{\phi}{2} + \cos\phi)$ the asymptotic lines are given by $du^2 = \pm\dfrac{d\phi}{\sin\phi}$.

30. Find the asymptotic lines and torsions on the surface whose equation is

$$\mathbf{r} = (x^1, x^2, x^3) = \left(x^1;\ x^2;\ f(x^1, x^2)\right).$$

Hence, show that

(i) if $f(x^1, x^2) = x^2\sin x^1$, then the asymptotic line is $(x^2)^2\cos x^1 = c_1$.

(ii) if $f(x^1, x^2) = \dfrac{1}{2}\left[\dfrac{(x^1)^2}{a^2} - \dfrac{(x^2)^2}{b^2}\right]$, then the asymptotic line is $\dfrac{x^1}{a} \pm \dfrac{x^2}{b} = c_1$.

31. Prove that the generating straight lines of a cylinder or a cone are asymptotic curves on those surfaces.
32. Prove that the asymptotic lines on a minimal surface form an orthogonal system.
33. Prove that the indicatrix at a point of a surface $z = f(x, y)$ is a rectangular hyperbola if

$$(1 + p^2)t - 2pqs + (1 + q^2)r = 0.$$

Also, show that the asymptotic lines and torsions are

$$rdx^2 + 2sdxdy + tdy^2 = 0;\ \tau = \pm \frac{\sqrt{s^2 - rt}}{1 + p^2 + q^2}.$$

34. Prove that the indicatrix at every point of a helicoid $z = c \tan^{-1} \frac{y}{x}$ is a rectangular hyperbola.
35. Show that the parametric curves are conjugate on the surface

$$x^3 = f(x^1) + f(x^2),$$

where x^1 and x^2 are parameters.

36. If a curve is a geodesic on the surface, prove that it is either a straight line or its principal normal is orthogonal to the surface at every point and conversely.
37. Prove that the normal curvature in a direction perpendicular to an asymptotic line is twice the mean normal curvature.
38. Prove that the sum of the normal curvatures in two directions at right angles is constant, and equal to the sum of the principal curvatures.
39. Prove that, at each non-umbilical point of a surface there exist two mutually orthogonal directions for which the normal curvature attains its extreme values.
40. Show that the helicoid

$$x^1 = u^1 \cos u^2;\ x^2 = u^1 \sin u^2;\ x^3 = au^2$$

is a surface of negative curvature.

41. If $\mathcal{S}$ is a surface of revolution

$$x^1 = u^1 \cos u^2;\ x^2 = u^1 \sin u^2;\ x^3 = f(u^1)$$

with $f(u^1)$ of class C^2, show that the points on a surface of revolution $\mathcal{S}$ for which $f'f'' > 0$ are elliptic; those for which $f'f'' < 0$ are hyperbolic; and if $f'' = 0$, then $\mathcal{S}$ is a cone.

42. Find the differential equation of the asymptotic curves of a surface of revolution

$$x^1 = u^1 \cos u^2;\ x^2 = u^1 \sin u^2;\ x^3 = f(u^1)$$

and determine the location of elliptic, parabolic, and hyperbolic points on this surface.

43. Prove that the co-ordinate curves of any allowable co-ordinate system on a surface S coincide with the lines of curvature if and only if for this system $a_{12} = 0 = b_{12}$ at any point where those co-ordinates are allowable.

44. Prove that at elliptic points the Gaussian curvature is positive, at parabolic ones it vanishes, and at hyperbolic ones it is negative.

45. Prove that if every point of a surface S which has a representation of class ≥ 3 is an umbilic, S is a plane or a sphere.

46. Show that on a cylinder $x^1 = c\cos u^1;\ x^2 = c\sin u^1;\ x^3 = u^2$ one distinct real asymptotic direction is given by u^1 curve.

47. Prove that the osculating plane to an asymptotic line which is not a straight line coincides with the tangent plane.

48. Prove that if every point of the surface $\mathcal{S}$ is parabolic, then $\mathcal{S}$ is developable.

49. Prove that any point of the following ellipsoid is elliptic

$$x^1 = a\cos u^1 \cos u^2;\ x^2 = b\sin u^1 \cos u^2;\ x^3 = c\sin u^2.$$

50. Prove that any point of a cylinder or of a cone is parabolic.

51. If $\mathcal{S}$ is a surface of revolution

$$x^1 = u^1 \cos u^2; x^2 = u^1 \sin u^2;\ x^3 = f(u^1)$$

with $f(u^1)$ of class C^2, show that a parallel surface $\overline{\mathcal{S}}$ is also a surface of revolution.

52. Show that the meridians and parallels on a sphere form an isometric system, and determine the isometric parameters.

53. On the surface formed by the revolution of a parabola about its directrix, one principal curvature is double the other.

54. Find the surface of revolution for which

$$ds^2 = du^2 + (a^2 - u^2)dv^2.$$

55. If the surface of revolution is a minimal surface,

$$u\frac{d^2f}{du^2} + \frac{df}{du}\left[1 + \left(\frac{df}{du}\right)^2\right] = 0.$$

Hence, show that the only real minimal surface of revolution is that formed by the revolution of a catenary about its directrix.

56. Find the equations of the helicoid generated by a circle of radius a, whose plane passes through the axis; and determine the lines of curvature on the surface.
57. Prove that the null lines are conjugate if and only if the surface is minimal.
58. Show that the only minimal surface of the type $z = f(x) + F(y)$ is the surface $az = \log\cos(ax) - \log\cos(ay)$.
59. Show that the surface $\sin az = \sinh ax \sinh ay$ is minimal.
60. Show that the surfaces

$$x^1 = u^1 \cos u^2, \;\; x^2 = u^1 \sin u^2, \;\; x^3 = cu^2$$

and

$$x^1 = u^1 \cos u^2, \;\; x^2 = u^1 \sin u^2, \;\; x^3 = c \cosh^{-1} \frac{u^1}{c}$$

are applicable.

61. Find the first and second curvatures for the spherical image.
62. Prove that the osculating planes of a line of curvature and of its spherical image at corresponding points are parallel.
63. Show that the lines of curvature of a surface of revolution remains isometric in their spherical representation.
64. Show that the spherical images of the asymptotic lines on a minimal surface, as well as the asymptotic lines themselves, are an isometric system.
65. Show that the helicoids

$$x^1 = u^1 \cos u^2, \;\; x^2 = u^1 \sin u^2, \;\; x^3 = cu^2 + c \int \sqrt{\frac{(u^1)^2 + c^2}{(u^1)^2 - c^2}} \frac{du^1}{u^1}$$

is a minimal surface.

Chapter 9

Classical Mechanics

Differential geometry is the building block in different branches of physics. Classical mechanics was originated with the work of Galileo and was developed extensively by Newton. It deals with the motion of particles in a fixed frame of reference (rectangular co-ordinate system). The basic premise of Newtonian mechanics is the concept of absolute time measurement between two reference frames at constant velocity relative to each other. Within those frames, other co-ordinate systems may be used so long as the metric remains Euclidean. This means that some of the theory of tensors can be brought to bear on this study.

9.1 Newton's Laws of Motion

Newton's laws of motion are the basis of the development of mechanics. Newton's laws of motion are stated in the following form:

(i) Everybody continues to be in its state of rest or of uniform motion in a straight line unless it is compelled to change that state by external forces acting on it (Law of inertia).

(ii) The time rate of change of momentum of the particle is proportional to the external force and is in the direction of the force (Law of causality).

(iii) To every action there is always an equal and opposite reaction. This law prescribes the general nature of forces of reaction in relation to the forces of action (Law of reciprocity).

Here, we will derive the equation of motion of a particle. Let the equation of path $\mathcal{C}$ of the particle in E^3 be

$$\mathcal{C}\colon x^i = x^i(t) \tag{9.1}$$

and the curve $\mathcal{C}$, the trajectory of the particle. Let at time t, particle is at $P\{x^i(t)\}$. If v^i be the component of velocity of the moving particle then $v^i = \frac{dx^i}{dt}$, and if a^i be the component of acceleration of moving particle then

$$a^i = \frac{\delta v^i}{\delta t} = \frac{dv^i}{dt} + \left\{ {i \atop j\ \ k} \right\} v^j \frac{dx^k}{dt}$$

or

$$a^i = \frac{d^2x^i}{dt^2} + \left\{ \begin{matrix} i \\ j \quad k \end{matrix} \right\} \frac{dx^j}{dt}\frac{dx^k}{dt}, \tag{9.2}$$

where $\frac{\delta v^i}{\delta t}$ is the intrinsic derivative. If m be the mass of a particle then by Newton's second law of motion

$$F^i = m\frac{\delta v^i}{\delta t} = ma^i = m\left[\frac{d^2x^i}{dt^2} + \left\{ \begin{matrix} i \\ j \quad k \end{matrix} \right\} \frac{dx^j}{dt}\frac{dx^k}{dt}\right]. \tag{9.3}$$

Deduction 9.1.1 If there is no external force, then

$$\frac{dv^i}{dt} + \left\{ \begin{matrix} i \\ j \quad k \end{matrix} \right\} v^j v^k = 0$$

or

$$\frac{\partial v^i}{\partial x^p}u^p + \left\{ \begin{matrix} i \\ j \quad k \end{matrix} \right\} v^j v^k = 0$$

or

$$\left[\frac{\partial v^i}{\partial x^p} + \left\{ \begin{matrix} i \\ j \quad p \end{matrix} \right\} v^p\right] v^p = 0$$

or

$$v^i_{,p}u^p = 0 \Rightarrow \frac{\delta v^i}{\delta t} = 0$$

Show that the acceleration is zero, which is the Newton's first law. Thus, Newton's first law can be derived from the second law.

Deduction 9.1.2 If there is external force, then

$$\frac{\delta}{\delta t}\left(\frac{dx^i}{dt}\right) = 0 \Rightarrow \frac{\delta}{\delta t}\left(m\frac{dx^i}{dt}\right) = 0 \Rightarrow \frac{\delta p^i}{\delta t} = 0$$

$$\Rightarrow \frac{\delta p_i}{\delta t} = 0 \Rightarrow p_{i,j}\frac{dx^j}{dt} = 0 \Rightarrow p^j p_{i,j} = 0$$

$$\Rightarrow p^j\left[p_{i,j} - \left\{ \begin{matrix} \alpha \\ i \quad j \end{matrix} \right\} p_\alpha\right] = 0$$

$$\Rightarrow m\frac{\delta p_i}{\delta x^j}\frac{\delta x^j}{\delta t} - m\left\{ \begin{matrix} \alpha \\ i \quad j \end{matrix} \right\} p_\alpha p^j = 0$$

$$\Rightarrow m\frac{dp_i}{dt} = g^{\alpha\rho}\left[\alpha j, \rho\right] p_\alpha p^j$$

or

$$m\frac{dp_i}{dt} = \frac{1}{2}(g^{\alpha\rho}p_\alpha)\left(g_{j\rho,i} + g_{i\rho,j} - g_{ij,\rho}\right)p^j = \frac{1}{2}g_{j\rho,i}p^\rho p^j.$$

Therefore, if all the metric coefficients are independent of the x^i co-ordinate then p_i is constant along the trajectory.

Deduction 9.1.3 If there is external force, then from Eq. (9.2), we get

$$\frac{d^2x^i}{dt^2} + \left\{ \begin{matrix} i \\ j \quad k \end{matrix} \right\} \frac{dx^j}{dt}\frac{dx^k}{dt} = 0$$

or

$$2g_{i\alpha}\frac{d^2x^i}{dt^2}\frac{dx^\alpha}{dt} + 2\left(g_{i\alpha}\left\{ \begin{matrix} i \\ j \quad k \end{matrix} \right\}\right)\frac{dx^j}{dt}\frac{dx^k}{dt}\frac{dx^\alpha}{dt} = 0$$

or

$$2g_{i\alpha}\frac{dx^\alpha}{dt}\frac{d^2x^i}{dt^2} + (g_{k\alpha,j} + g_{j\alpha,k} - g_{jk,\alpha})\frac{dx^j}{dt}\frac{dx^k}{dt}\frac{dx^\alpha}{dt} = 0$$

or

$$2g_{i\alpha}\frac{dx^\alpha}{dt}\frac{d^2x^i}{dt^2} + g_{k\alpha,j}\frac{dx^j}{dt}\frac{dx^k}{dt}\quad\frac{dx^\alpha}{dt} = 0$$

or

$$\frac{d}{dt}\left(g_{i\alpha}\frac{dx^\alpha}{dt}\frac{dx^i}{dt}\right) = 0 \quad\Rightarrow\quad g_{i\alpha}\frac{dx^\alpha}{dt}\frac{dx^i}{dt} = \text{ constant},$$

this shows that the tangent vector $\frac{dx^i}{dt}$ is parallely transported along the geodesic and its magnitude is constant.

9.1.1 Work Done

We define the element of work done by the force $\mathbf{F}$ in producing a displacement $d\mathbf{r}$ by invariant

$$dW = \mathbf{F} \cdot d\mathbf{r}. \tag{9.4}$$

Since the components of $\mathbf{F}$ and $d\mathbf{r}$ are F^i and dx^i, respectively, so,

$$dW = g_{ij}F^i dx^i = F_j dx^j; \tag{9.5}$$

where $F_j = g_{ij}F^i$. The work is done in displacing a particle along the trajectory $\mathcal{C}$, joining a pair of points P_1 and P_2 is given by

$$\begin{aligned} W &= \int_{P_1}^{P_2} F_i dx^i = \int_{P_1}^{P_2} mg_{ij}\frac{\delta v^i}{\delta t}dx^j \\ &= \int_{P_1}^{P_2} mg_{ij}\frac{\delta v^i}{\delta t}\frac{dx^j}{dt}dt = \int_{P_1}^{P_2} mg_{ij}\frac{\delta v^i}{\delta t}v^j dt. \end{aligned}$$

Since $g_{ij}v^iv^j$ is an invariant, then

$$\frac{\delta}{\delta t}\left(g_{ij}v^iv^j\right) = \frac{d}{dt}\left(g_{ij}v^iv^j\right)$$

or

$$\frac{d}{dt}\left(g_{ij}v^iv^j\right) = 2g_{ij}\frac{\delta v^i}{\delta t}v^j$$

or

$$g_{ij}\frac{\delta v^i}{\delta t}v^j = \frac{1}{2}\frac{d}{dt}\left(g_{ij}v^iv^j\right).$$

The work is done in displacing a particle along the trajectory $\mathcal{C}$, joining a pair of points P_1 and P_2 is given by

$$W = \int_{P_1}^{P_2} \frac{m}{2}\frac{d}{dt}\left(g_{ij}v^iv^j\right)\, dt = \frac{m}{2}\left[g_{ij}v^iv^j\right]_{P_1}^{P_2}.$$

Let T denote the kinetic energy at P on the trajectory $\mathcal{C}$, then

$$T = \frac{1}{2}mg_{ij}v^iv^j = \frac{1}{2}mv^2. \tag{9.6}$$

Let T_1 and T_2 be the kinetic energies of the particle at P_1 and P_2, respectively, then

$$W = \frac{m}{2}\left[g_{ij}v^iv^j\right]_{P_1}^{P_2} = T_2 - T_1.$$

Let the force field F_i be such that the integral $W = \int_{P_1}^{P_2} F_i dx^i$ is independent of the path. In this case, $F_i dx^i$ is an exact differential and we can write

$$F_i = -\frac{\partial V}{\partial x^i}, \tag{9.7}$$

where V is a function of co-ordinates x^i, known as the force potential or potential energy. This force field is called *conservative force field.*

Theorem 9.1.1 *A necessary and sufficient condition that a force field F_i, defined in a simply connected region, be conservative is that $F_{i,j} = F_{j,i}$.*

Proof: First, let the force field F_i be conservative then $F_i = -\frac{\partial V}{\partial x^i}$. Now,

$$\begin{aligned} F_{i,j} &= \frac{\partial F_i}{\partial x^j} - \left\{ \begin{matrix} k \\ i\ \ j \end{matrix} \right\} F_k = \frac{\partial}{\partial x^j}\left(-\frac{\partial V}{\partial x^i}\right) - \left\{ \begin{matrix} k \\ i\ \ j \end{matrix} \right\} F_k \\ &= -\frac{\partial^2 V}{\partial x^j \partial x^i} - \left\{ \begin{matrix} k \\ i\ \ j \end{matrix} \right\} F_k. \end{aligned}$$

Similarly,

$$F_{j,i} = -\frac{\partial^2 V}{\partial x^i \ \partial x^j} - \left\{ \begin{matrix} k \\ j \ \ i \end{matrix} \right\} F_k.$$

From the above two relations of $F_{i,j}$ and $F_{j,i}$ it follows that

$$\begin{aligned} F_{i,j} - F_{j,i} &= -\frac{\partial^2 V}{\partial x^j \ \partial x^i} - \left\{ \begin{matrix} k \\ i \ \ j \end{matrix} \right\} F_k + \frac{\partial^2 V}{\partial x^i \ \partial x^j} + \left\{ \begin{matrix} k \\ j \ \ i \end{matrix} \right\} F_k. \\ &= -\frac{\partial^2 V}{\partial x^i \partial x^j} - \left\{ \begin{matrix} k \\ i \ \ j \end{matrix} \right\} F_k + \frac{\partial^2 V}{\partial x^i \ \partial x^j} + \left\{ \begin{matrix} k \\ i \ \ j \end{matrix} \right\} F_k = 0. \end{aligned}$$

Therefore, if the force field F_i be conservative then $F_{i,j} = F_{j,i}$. Conversely, let the relation $F_{i,j} = F_{j,i}$ holds, then

$$\frac{\partial F_i}{\partial x^j} - \left\{ \begin{matrix} k \\ i \ \ j \end{matrix} \right\} F_k = \frac{\partial F_j}{\partial x^i} - \left\{ \begin{matrix} k \\ j \ \ i \end{matrix} \right\} F_k$$

or

$$\frac{\partial F_i}{\partial x^j} = \frac{\partial F_j}{\partial x^i}; \quad \text{as } \left\{ \begin{matrix} k \\ i \ \ j \end{matrix} \right\} \text{ is symmetric.}$$

So if we take $F_i = -\frac{\partial V}{\partial x^i}$ then the relation $F_{i,j} = F_{j,i}$ holds. Hence, F_i is conservative.

9.1.2 Lagrange's Equation of Motion

Consider a particle moving on the curve be $\mathcal{C}: q^i = q^i(t)$ and the curve $\mathcal{C}$ the trajectory of the particle. Let at time t, particle is at $P\{q^i(t)\}$. The kinetic energy $T = \frac{1}{2}mv^2$ can be written as

$$T = \frac{1}{2} m g_{ij} \dot{q}^i \dot{q}^j = \frac{1}{2} m g_{jk} \dot{q}^j \dot{q}^k; \quad \text{as } \dot{q}^i = v^i. \tag{9.8}$$

Differentiating Eq. (9.8) with respect to q^i, we get

$$\frac{\partial T}{\partial q^i} = \frac{1}{2} m \frac{\partial g_{jk}}{\partial q^i} \dot{q}^j \dot{q}^k. \tag{9.9}$$

Differentiating Eq. (9.8) with respect to $\dot{q}^i$, we get

$$\begin{aligned} \frac{\partial T}{\partial \dot{q}^i} &= \frac{1}{2} m \ g_{jk} \left[\frac{\partial \dot{q}^j}{\partial \dot{q}^i} \dot{q}^k + \dot{q}^j \frac{\partial \dot{q}^k}{\partial \dot{q}^i} \right] = \frac{1}{2} m \ g_{jk} \left(\delta^j_i \dot{q}^k + \dot{q}^j \delta^k_i \right) \\ &= \frac{1}{2} m \left(g_{jk} \delta^j_i \dot{q}^k + \dot{q}^j g_{jk} \delta^k_i \right) = \frac{1}{2} m \left(g_{ik} \dot{q}^k + g_{ji} \dot{q}^j \right) \\ &= \frac{1}{2} m \left(g_{ij} \dot{q}^j + g_{ij} \dot{q}^j \right) = m g_{ij} \dot{q}^j = m g_{ik} \dot{q}^k; \quad \text{as } g_{ij} = g_{ji}. \end{aligned} \tag{9.10}$$

Differentiating Eq. (9.10) with respect to t, we get

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}^i}\right) = m\frac{d}{dt}\left(g_{ik}\dot{q}^k\right) = m\left(\frac{d}{dt}g_{ik}\dot{q}^k + g_{ik}\ddot{q}^k\right)$$
$$= m\left(\frac{\partial g_{ik}}{\partial q^i}\frac{dq^j}{dt}\dot{q}^k + g_{ik}\ddot{q}^k\right). \tag{9.11}$$

Therefore, from Eqs. (9.9) and (9.11), we get

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}^i}\right) - \frac{\partial T}{\partial q^i} &= m\frac{\partial g_{ik}}{\partial q^i}\frac{dq^j}{dt}\dot{q}^k + mg_{ik}\ddot{q}^k - \frac{1}{2}m\frac{\partial g_{jk}}{\partial q^i}\dot{q}^j\dot{q}^k \\
&= mg_{ik}\ddot{q}^k + \frac{1}{2}m\frac{\partial g_{ik}}{\partial q^j}\dot{q}^j\dot{q}^k + \frac{1}{2}m\frac{\partial g_{ik}}{\partial q^j}\dot{q}^j\dot{q}^k - \frac{1}{2}m\frac{\partial g_{jk}}{\partial q^i}\dot{q}^j\dot{q}^k \\
&= mg_{ik}\ddot{q}^k + \frac{1}{2}m\frac{\partial g_{ik}}{\partial q^j}\dot{q}^j\dot{q}^k + \frac{1}{2}m\frac{\partial g_{ij}}{\partial q^k}\dot{q}^k\dot{q}^j - \frac{1}{2}m\frac{\partial g_{jk}}{\partial q^i}\ \dot{q}^j\dot{q}^k \\
&= mg_{ik}\ddot{q}^k + \frac{1}{2}m\left(\frac{\partial g_{ik}}{\partial q^j} + \frac{\partial g_{ij}}{\partial q^k} - \frac{\partial g_{jk}}{\partial q^i}\right)\ \dot{q}^j\dot{q}^k \\
&= mg_{ik}\ddot{q}^k + m\,[jk,i]\ \dot{q}^j\dot{q}^k \\
&= mg_{ik}\ddot{q}^k + mg^{il}g_{il}\,[jk,i]\ \dot{q}^j\dot{q}^k \\
&= mg_{il}\ddot{q}^l + mg^{il}g_{il}\,[jk,i]\ \dot{q}^j\dot{q}^k \\
&= mg_{il}\left(\ddot{q}^l + g^{il}\,[jk,i]\ \dot{q}^j\dot{q}^k\right) \\
&= mg_{il}\left(\ddot{q}^l + \left\{\begin{matrix} l \\ j\quad k\end{matrix}\right\}\dot{q}^j\dot{q}^k\right);\ \text{as } g^{il}\,[jk,i] = \left\{\begin{matrix} l \\ j\quad k\end{matrix}\right\} \\
&= mg_{il}a^l;\quad \text{as } a^l = \ddot{q}^l + \left\{\begin{matrix} l \\ j\quad k\end{matrix}\right\}\dot{q}^j\dot{q}^k,
\end{aligned}$$

where a^i is the component of acceleration. If $F_i = ma_i$ are the components of force field, then

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}^i}\right) - \frac{\partial T}{\partial q^i} = ma_i = F_i, \tag{9.12}$$

which is known as *Lagrange's equation of motion.* For a conservative system, forces $\mathbf{F}_i$ are derivable from scalar potential function V, of position only but not generalised velocities, i.e.

$$\mathbf{F}_i = -\boldsymbol{\nabla}_i V = -\frac{\partial V}{\partial \mathbf{r}_i}.$$

$$\Rightarrow Q_j = \sum_i \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} = -\sum_i \frac{\partial V}{\partial \mathbf{r}_i} \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} = -\frac{\partial V}{\partial q_j}.$$

Hence, Eq. (9.12), we get

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = -\frac{\partial V}{\partial q_j}.$$

or

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial}{\partial q_j}(T - V) = 0$$

or

$$\frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_j}(T - V)\right] - \frac{\partial}{\partial q_j}(T - V) = 0$$

since V is not a function of $\dot{q}_j$

or

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_j}\right] - \frac{\partial L}{\partial q_j} = 0, \tag{9.13}$$

where $L = T - V$ is known as Lagrangian for conservative system. Equation (9.13) is known as Lagrange's equation of motion for conservative, holonomic system.

9.1.3 Hamilton's Canonical Equations

Consider a conservative holonomic dynamical system with n degrees of freedom and the integral

$$J = \int_{t_1}^{t_2} L(q, \dot{q})dt, \tag{9.14}$$

where $L = T - V$ is the kinetic potential. Using variational principle J will be extremum if it satisfies the Euler's equation, i.e. it consists of the n simultaneous second order ordinary differential equations in the form

$$\frac{dL_{\dot{q}^i}}{dt} - L_{q^i} = 0;\ i = 1, 2, \ldots, n \tag{9.15}$$

by using the subscript notation for partial derivatives of $L(q, \dot{q})$. The function

$$L(q, \dot{q}) = T(q, \dot{q}) - V(q)$$

depends on n generalised co-ordinates q^i and n generalised velocities $\dot{q}^i$. Instead of the variables $\dot{q}^i$ we can introduce a set of n new variables p_i defined by the relations

$$p_i = L_{\dot{q}^i}(q, \dot{q});\ i = 1, 2, \ldots, n, \tag{9.16}$$

where we suppose that this system Eq. (9.16) is solvable for the $\dot{q}^i$ in terms of p_i and q^i as $\left|\frac{\partial L_{\dot{q}^i}}{\partial \dot{q}^j}\right| \neq 0$. Let us construct a function $H(p, q)$ of the independent variables p and q as

$$H(q, p, t) = p_i \dot{q}^i - L(q, \dot{q}, t), \tag{9.17}$$

which is related to the Lagrangian L. Differentiating Eq. (9.17) with respect to q^j, we get

$$H_{q^j} = \frac{\partial \dot{q}^i}{\partial q^j} p_j - L_{q^j} - L_{\dot{q}^i} \frac{\partial \dot{q}^i}{\partial q^j} = -L_{q^j};\ \text{using Eq. (9.16).} \tag{9.18}$$

Similarly, differentiating Eq. (9.17) with respect to p_j, we get

$$H_{p_j} = \dot{q}^j + \frac{\partial \dot{q}^i}{\partial p_j} p_i - L_{\dot{q}^i} \frac{\partial \dot{q}^i}{\partial p^j} = \dot{q}^j;\ \text{using Eq. (9.16).} \tag{9.19}$$

Since the Lagrange's equation of motion [Eq. (9.15)] gives $\frac{dL_{\dot{q}^i}}{dt} = L_{q^i}$, from formulas [Eqs. (9.16) and (9.18)] we obtain a set of n first order equations

$$\frac{dp_i}{dt} = -H_{q^i};\ i = 1, 2, \ldots, n, \tag{9.20}$$

which together with n equations [Eq. (9.19)]

$$\frac{dq^i}{dt} = H_{p^i};\ i = 1, 2, \cdots, n, \tag{9.21}$$

constitute the system of $2n$ first order *Hamilton's canonical equations.* The function H defined in Eq. (9.17) is called *Hamiltonian* or *Hamilton's function.* Normally, the Hamiltonian function for each problem must be constructed via Lagrangian formulation. Thus, H consists of two quantities:

(i) $p_i \dot{q}^i$ is the part that is homogeneous in $\dot{q}^i$ in the second degree and

(ii) L is a part of Lagrangian independent of generalised velocities.

The quantities q_k and p_k form a set of independent variables in Hamiltonian formalism, whereas q_k and $\dot{q}_k$ form the independent set in Lagrangian formulation.

The *Hamilton's function* $H(p, q)$ has an important physical meaning. The kinetic energy $T = \frac{1}{2}mv^2$ can be written as

$$T = \frac{1}{2}a_{ij}\dot{q}^i\dot{q}^j \Rightarrow \frac{\partial T}{\partial \dot{q}^i} = a_{ij}\dot{q}^j$$

or

$$\dot{q}^i \frac{\partial T}{\partial \dot{q}^i} = a_{ij}\dot{q}^i\dot{q}^j = 2T.$$

Since $L = T - V$ and V is a function of the q^i alone, we can rewrite Eq. (9.17) as

$$\begin{aligned} H(p, q, t) &= \dot{q}^i \frac{\partial L}{\partial \dot{q}^i} - L(q, p, t) = \dot{q}^i \frac{\partial L}{\partial \dot{q}^i} - T + V \\ &= T + V. \end{aligned}$$

Thus, H is the total energy of the system. The variables

$$p_i = \frac{\partial L}{\partial \dot{q}^i} = a_{ij}\dot{q}^j$$

are called *generalised momenta.* The square of the magnitude of the vector p_i is

$$p^2 = a^{ij}p_ip_j = a^{ij}a_{ik}a_{jl}\dot{q}^k\dot{q}^l = a_{kl}\dot{q}^k\dot{q}^l = 2T.$$

EXAMPLE 9.1.1 For the planetary problem, the kinetic energy T and potential energy V are, respectively, given by

$$T = \frac{1}{2}m[\dot{r}^2 + r^2\dot{\theta}^2]; \ V = -\frac{\mu}{r}.$$

$$L = T - V = \frac{1}{2}m[\dot{r}^2 + r^2\dot{\theta}^2] + \frac{\mu}{r}.$$

The conjugate momenta are given by $p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}$; $p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$. Thus, the Hamiltonian H is given by

$$H(r, \theta, p_r, p_\theta) = T + V = \frac{1}{2}m[\dot{r}^2 + r^2\dot{\theta}^2] - \frac{\mu}{r} = \frac{1}{2m}[p_r^2 + \frac{p_\theta^2}{r^2}] - \frac{\mu}{r}.$$

The Hamilton's canonical equation of motion gives

$$\begin{aligned} \dot{r} &= \frac{\partial H}{\partial p_r} = \frac{p_r}{m}; \ \dot{p}_r = -\frac{\partial H}{\partial r} = \frac{p_\theta^2}{mr^3} - \frac{\mu}{r^2} \\ \dot{\theta} &= \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2}; \ \dot{p}_\theta = 0. \end{aligned}$$

Now

$$\dot{p}_\theta = 0 \Rightarrow \frac{d}{dt}(mr^2\dot{\theta}) = 0 \Rightarrow mr^2\dot{\theta} = \text{ constant } = l(\text{ say })$$

$$\Rightarrow m\ddot{r} = \frac{l^2}{mr^3} - \frac{\mu}{r^2}.$$

I have discussed the nature of the solution in my book of Mechanics.

9.1.4 Hamilton's Principle

Hamilton's principle states that, the motion of a conservative, holonomic dynamical system from time t_1 to t_2 takes place along such a path (among of all possible paths) that the line integral of the Lagrangian of the system between the limits t_1 and t_2 is extremum (stationary) for the path (consistent with constraints, if any) of the motion. Mathematically

$$J = \int_{t_1}^{t_2} L(q, \dot{q}, t)dt = \text{ extreme;}$$

or

$$\delta J = \delta \int_{t_1}^{t_2} L(q, \dot{q}, t)dt = 0,$$

where δ is variation symbol, that does not include time variation. Unlike all other techniques, this one does not start with a differential equation, rather it starts with an integral which is then optimised against some possible variations of the path. This principle helps to distinguish the actual path of the system from the infinite number of neighboring possible paths. For the deduction of the principle, the following two conditions must be satisfied by the system:

(i) All paths are traversed in the same time, i.e. at time t_1, the system must be at point P and at time t_2, it must be at Q, i.e. δt must be zero at the end points P and Q.

(ii) For all paths whether dynamical or varied, have the same termini. Since the points P and Q are fixed in configuration space, the co-ordinate variation $\delta \mathbf{r}$ must be zero at the end points P and Q.

If the particle is in motion under the influence of the force $\mathbf{F}$, and our problem is to determine the trajectory in a three-dimensional manifold E^3 as

$$\mathcal{C}: x^i = x^i(t);\ i = 1, 2, 3, t_1 \le t \le t_2,$$

where t denotes the time. The kinetic energy T of the particle is given by

$$T(q^i, \dot{q}^i) = \frac{1}{2}m\ \ g_{jk}\dot{q}^j\dot{q}^k \text{ as } \dot{q}^i = v^i.$$

Let us consider another curve $\mathcal{C}'$, joining t_1 and t_2 close to be $\mathcal{C}$ is

$$\mathcal{C}' : \overline{x}^i(\varepsilon, t) = x^i(t) + \delta x^i(t) \tag{9.22}$$

with $\delta x^i(t) = \varepsilon \xi^i(t)$ and $\xi^i(t_1) = \xi^i(t_2) = 0$, belonging to the h neighbourhood of $\mathcal{C}$. If δT be the variation in T, then

$$\delta T = \frac{\partial T}{\partial \dot{x}^i}\delta \dot{x}^i + \frac{\partial T}{\partial x^i}\delta x^i.$$

Now,

$$\begin{aligned}
&\int_{t_1}^{t_2} \left[(\delta T + F_i)\,\delta x^i\right] dt \\
&= \int_{t_1}^{t_2} \left[\frac{\partial T}{\partial \dot{x}^i}\,\delta \dot{x}^i + \frac{\partial T}{\partial x^i}\delta x^i + F_i \delta x^i\right] dt \\
&= \int_{t_1}^{t_2} \frac{\partial T}{\partial \dot{x}^i}\delta \dot{x}^i dt + \int_{t_1}^{t_2} \frac{\partial T}{\partial x^i}\delta x^i dt + \int_{t_1}^{t_2} F_i \delta x^i dt \\
&= \left[\frac{\partial T}{\partial \dot{x}^i}\delta x^i\right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}^i}\right)\delta x^i dt + \int_{t_1}^{t_2} \frac{\partial T}{\partial x^i}\delta x^i dt + \int_{t_1}^{t_2} F_i \delta x^i dt \\
&= \int_{t_1}^{t_2} \frac{\partial T}{\partial x^i}\delta x^i dt - \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}^i}\right)\delta x^i dt + \int_{t_1}^{t_2} F_i \delta x^i dt \\
&\qquad \text{as } \delta x^i(t_1) = 0 \text{ and } \delta x^i(t_2) = 0;\ \text{i.e. } \left[\frac{\partial T}{\partial \dot{x}^i}\delta x^i\right]_{t_1}^{t_2} = 0 \\
&= \int_{t_1}^{t_2} \left[\frac{\partial T}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}^i}\right) + F_i\right]\,\delta x^i dt. \qquad (9.23)
\end{aligned}$$

Since the particle satisfies the Lagrange's equation of motion, so

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}^i}\right) - \frac{\partial T}{\partial x^i} = F_i.$$

Therefore, Eq. (9.23) reduces to

$$\int_{t_1}^{t_2} \left[(\delta T + F_i)\,\delta x^i\right] dt = \int_{t_1}^{t_2} [-F_i + F_i]\delta x^i dt$$

or

$$\int_{t_1}^{t_2} \left[(\delta T + F_i)\,\delta x^i\right] dt = 0.$$

Theorem 9.1.2 (Integral of energy): *The motion of a particle in a conservative field of force is such that the sum of its kinetic and potential energies is a constant.*

Proof: Consider a particle moving on the curve

$$\mathcal{C}: x^i = x^i(t); t_1 \leq t \leq t_2,$$

where t denotes the time. The kinetic energy T of the particle is given by

$$T(q^i, \dot{q}^i) = \frac{1}{2}m\ g_{ij}\dot{q}^i\dot{q}^j = \frac{1}{2}m\ g_{ij}v^iv^j.$$

As T is invariant, taking intrinsic derivative with respect to t, we get

$$\begin{aligned}\frac{dT}{dt} = \frac{\delta T}{\delta t} &= \frac{\delta}{\delta t}\left(\frac{1}{2}m\ g_{ij}v^iv^j\right)\\ &= \frac{1}{2}m\ g_{ij}\left(\frac{\delta v^i}{\delta t}v^j + v^i\frac{\delta v^j}{\delta t}\right)\\ &= \frac{1}{2}m\ \left(g_{ij}\frac{\delta v^i}{\delta t}v^j + g_{ij}v^i\frac{\delta v^j}{\delta t}\right)\\ &= \frac{1}{2}m\ \left(g_{ij}\frac{\delta v^i}{\delta t}v^j + g_{ji}v^j\frac{\delta v^i}{\delta t}\right)\\ &= \frac{1}{2}m2g_{ij}\frac{\delta v^i}{\delta t}v^j = mg_{ij}\frac{\delta v^i}{\delta t}v^j;\ \text{ as } g_{ij} = g_{ji}.\end{aligned}$$

Therefore, the intrinsic derivative of T is given by

$$\begin{aligned}\frac{dT}{dt} &= mg_{ij}\ \frac{\delta v^j}{\delta t}v^i = mg_{ij}a^jv^j;\ \text{ as } a^j = \frac{\delta v^i}{\delta t}\\ &= ma_iv^i = F_iv^i;\ \text{since } g_{ij}a^j = a_i\end{aligned}$$

as $F_i = ma_i$ is a component of force field. But given F_i is conservative, then $F_i = -\frac{\partial V}{\partial x^i}$, where V is potential energy. Therefore,

$$\frac{dT}{dt} = -\frac{\partial V}{\partial x^i}\ \ v^i = -\frac{\partial V}{\partial x^i}\frac{dx^i}{dt} = -\frac{dV}{dt}$$

or

$$\begin{aligned}\frac{dT}{dt} + \frac{dV}{dt} = 0 &\Rightarrow \frac{d}{dt}(T + V) = 0\\ &\Rightarrow T + V = h = \text{ constant.}\end{aligned}$$

9.1.5 Principle of Least Action

The principle of least action is of fundamental importance in classical mechanics. An important variational principle associated with Hamiltonian formulation is the principle of least action. To obtain the principle of least action we restrict our further considerations by three important qualifications:

(i) Only systems are considered for which L, and so H, are not explicit functions of time, and consequently H is conserved.

(ii) The variation is such that H is conserved on the varied path as well as on the actual path.

(iii) The varied paths are constrained requiring that $\triangle q_k$ vanish at the endpoints but not $\triangle t$.

Let us consider the integral,

$$A = \int_{P_1}^{P_2} mv\, ds, \tag{9.24}$$

evaluated over the path $\mathcal{C}:\ x^i = x^i(t); t_1 \leq t \leq t_2$, where $\mathcal{C}$ is the trajectory of the particle of mass m moving in a conservative field of force. We suppose that neither the kinetic energy T nor the potential energy V is a function of time. In the three-dimensional space with curvilinear co-ordinates, integral Eq. (9.24) can be written as

$$\begin{aligned} A &= \int_{P_1}^{P_2} mg_{ij}\frac{dx^i}{dt}dx^j = \int_{t(P_1)}^{t(P_2)} mg_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}dt \\ &= \int_{t(P_1)}^{t(P_2)} 2T dt;\ \text{as } T = \frac{1}{2}mg_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}dt. \end{aligned}$$

This integral has a physical meaning only when evaluated over the trajectory $\mathcal{C}$, but its value can be computed along any varied path joining the points P_1 and P_2. Let us consider a particular set of admissible paths $\mathcal{C}'$ along which the function $T + V$, for each value of parameter t, has the same constant value h. The integral A in Eq. (9.24) is called the *action integral.*

The *principle of least action* states that "of all curves $\mathcal{C}'$ passing through P_1 and P_2 in the neighbourhood of the trajectory $\mathcal{C}$, which are traversed at a rate such that, for each $\mathcal{C}'$, for every value of t, $T + V = h$, that one for which the action integral A is stationary is the trajectory of the particle".

9.2 Exercises

1. Show that the covariant components of the acceleration vector in a spherical co-ordinate with

$$ds^2 = (dr)^2 + r^2(d\theta)^2 + r^2 \sin^2\theta(d\phi)^2.$$

Hence, derive the Lagrange's equation of motion.

2. Use Lagrangian equations to show that, if a particle is not subjected to the action of forces, then its trajectory is given by $y^i = a^i t + b^i$, where the a^i and b^i are constants and y^i are orthogonal Cartesian co-ordinates.
3. Prove that if a particle moves so that its velocity is constant in magnitude then its acceleration vector is either orthogonal to the velocity or it is zero.
4. Find, with the aid of Lagrange equations, the trajectory of a particle moving in a uniform gravitational field.
5. Find the dynamical equations in spherical co-ordinates with

$$ds^2 = (dr)^2 + r^2(d\theta)^2 + r^2 \sin^2\theta(d\phi)^2.$$

6. Find the dynamical equations in cylindrical co-ordinates with

$$ds^2 = (dr)^2 + r^2(d\theta)^2 + (dz)^2.$$

7. Let a particle of mass m be constrained to move on the surface of a sphere of radius a. Relate the orthogonal Cartesian co-ordinate y^i to the surface co-ordinates u^i by the formula

$$y^1 = a\sin u^1 \cos u^2, y^2 = a\sin u^1 \sin u^2, y^3 = a\cos u^1.$$

Show that Eq. (9.12) yield

$$\ddot{u}^1 - (\dot{u}^2)^2 \sin u^1 \cos u^2 = \frac{F_1}{ma^2}$$
$$\ddot{u}^2 \sin^2 u^1 + 2\dot{u}^1\dot{u}^2 \sin u^1 \cos u^1 = \frac{F_2}{ma^2}.$$

Solve these equations for the case when $F^i = 0$, and show that the trajectory is an arc of a great circle and the speed is constant.

8. A particle of mass m oscillates about the lowest point of a smooth surface $z = \frac{1}{2}(ax^2 + 2hxy + by^2)$, where the co-ordinates are orthogonal Cartesian and the z axis is directed vertically up. We suppose that the vertical component of the velocity is small, so that $T = \frac{1}{2}(\dot{x}^2 + \dot{y}^2)$ and $V = mgz$. Obtain equations of motion, determine the solutions.

9. If a particle of mass m is constrained to move on a smooth surface, show that the system of Hamilton's equations are

$$\frac{du^i}{dt} = \frac{\partial H}{\partial p^i}; \quad \frac{dp^i}{dt} = -\frac{\partial H}{\partial u^i}; i = 1, 2$$

with $p_i = ma_{ij}\dot{u}^j$ and $H = \frac{1}{2m}a^{ij}p_ip_j + V$.

Chapter 10

Relativistic Mechanics

Differential geometry was used to great advantage by Einstein in his development of relativity. The formulation of the fundamental laws of classical mechanics is based on the hypothesis that physical phenomena take place in a three-dimensional Euclidean space. It is also assumed that these phenomena can be ordered in the one-dimensional continuum of time t. The time variable t is regarded to be independent not only the space variable x^i but also of the possible motion of the space reference systems. The mass m of the body is likewise supposed to be independent of the motion of reference systems, and in particular, it is invariant with respect to a group of Galilean transformations of co-ordinates.

10.1 Event Space

It is first necessary to give the concepts of time and space. Thus, each event (atomic collision, flash of lighting, etc.) is assigned four co-ordinates (x, y, z, t), where t is the time (in seconds) of the event and (x, y, z) is the location (in metres) of the event in ordinary rectangular co-ordinates. Such co-ordinates are called *space-time* co-ordinates.

An event space in an R^4 whose points are events, co-ordinated by $(x^i) = (x^1, x^2, x^3, x^4)$ where $x^4 = ct$ is the temporal co-ordinate, and $\left(x^1, x^2, x^3\right) = (x, y, z)$ the rectangular positional co-ordinates of an event. Two events E_1 and E_2 are identical if $x_1^i = x_2^i$, for all i. Thus,

(i) Compositional: $x_1^i = x_2^i;\ i = 1, 2, 3$
(ii) Simultaneous : $x_1^4 = x_2^4$.

The spatial distance between E_1 and E_2 is the number

$$d = \sqrt{(\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)^2},$$

where $\Delta x^i = x_2^i - x_1^i;\quad i = 1, 2, 3$.

10.1.1 Inertial Reference Frames

The general settings for Einstein's special theory of relativity consists of two (or more) observers $O, O,' \cdots$ moving at constant velocities relative to each other, which is set up of space-time co-ordinates $(x^i), (\overline{x}^i), \cdots$ to record events and make calculations for experiments they conduct. Such co-ordinate systems in uniform relative motion are called *inertial frames* provided by Newton's first law in each system. All the systems are assumed to have a common origin at some instant, which is taken as $t = \bar{t} = \cdots = 0$.

Let us consider two inertial frames of reference S and S' having their respective axes parallel Figure (10.1). For simplicity, let the inertial frame S' be moving with

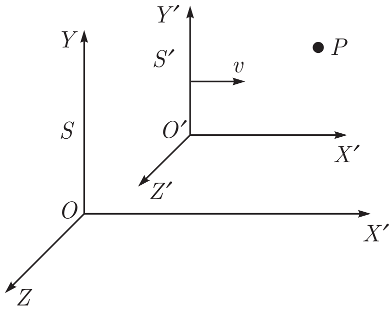

Figure 10.1: Galilean transformation.

a relative constant velocity v along the x direction relative to the frame S. Let us suppose from the instant $(t = \bar{t} = 0)$, the origin O and O' of the frames S and S' coincide with each other. Suppose that as measured by the observer of the frame S, a particle which is moving in space, is at a point P at any time t, whose co-ordinates are (x^1, x^2, x^3) with respect to the origin of the frame S. Let the observer of frames S' measures the co-ordinates of P as $\left(\overline{x}^1, \overline{x}^2, \overline{x}^3\right)$ at any time t. According to the Newton's concept of time the time flows uniformly without any reference to anything external, i.e. there is no effect of the motion of the frame S' on the flow of time. Therefore, $\bar{t} = t$. The co-ordinates are related to each other by the transformation equations

$$\overline{x}^1 = x^1 - vt;\ \overline{x}^2 = x^2;\ \overline{x}^3 = x^3 \text{ and } \bar{t} = t. \tag{10.1}$$

These set of equations [Eq. (10.1)] are called *Galilean Transformation* from stationary frame S to moving from S'. To the observer of frame S', the frame S appears to move with the same speed, but along negative x-axis i.e. with velocity $-v$. Thus, we have the following set of equations:

$$\overline{x}^1 = x^1 + vt;\ \overline{x}^2 = x^2;\ \overline{x}^3 = x^3 \text{ and } \bar{t} = t. \tag{10.2}$$

These set of equations [Eq. (10.2)] are called *inverse Galilean Transformation.* If the measurement of a physical quantity does not undergo a change under Galilean Transformation or its measurement is independent of relative motion of the observer, it is called a *Galilean Invariance.*

10.1.2 Invariance of Space

Let us consider, two inertial frames of reference S and S', where the frame S' is moving with a constant velocity v along x direction. Let a rod AB be placed in the frame S' parallel to the direction of x axis. For the observer of the frame S', the rod AB is at rest. Let the distances of the two ends A and B of the rod from the origin of frame S' be $\overline{x}_1$ and $\overline{x}_2$, respectively. Then length of the rod as measured by the observer of the frame S' is

$$L_0 = x_2' - x_1',$$

where the suffix '0' tells that the observer of the frame S' measures the length of the rod at rest. For the observer of frame S, the rod is in motion. Hence, to measure the length correctly, the observer of the frame S measures the distance x_1 and x_2 of the ends A and B of the rod from the origin simultaneously. From Galilean transformations, we have

$$x_1' = x_1 - vt; \;\; x_2' = x_2 - vt$$
$$\Rightarrow L_0 = (x_2 - vt) - (x_1 - vt) = x_2 - x_1 = L,$$

where L is the length of the rod as measured by the observer of the frame S. Therefore, when the length is measured in two inertial frames moving with uniform velocity relative to each other, the length interval remains unchanged.

10.1.3 Invariance of Time Interval

Suppose that an event, say a flash of light, occurs in frame S' at time t_1' and then again at time t_2'. Then as observed by an observer of the frame S', the time interval between the two flashes is given by

$$\tau_0 = t_2' - t_1'.$$

Here, the suffix zero tells that the event of flashing of light occurs at rest with respect to the observer of the frame S'. Suppose that the observer of frame S records the time of occurrence of the two flashes as t_1 and t_2. According to Galilean transformations, $t_1' = t_1$ and $t_2' = t_2$. Thus,

$$\tau_0 = t_2' - t_1' = t_2 - t_1 = \tau,$$

where $\tau = t_2 - t_1$ is the time interval between the flashes as recorded by the observer of the frame S. Thus, $\tau_0 = \tau$. Hence, the time interval does not change, when it is measured in two inertial frames moving with uniform velocity relative to each other.

10.1.4 Invariance of Velocity

Here, we will discuss how the velocity of a particle in one inertial frame of reference are measured by the observer of another frame moving with a uniform velocity relative to each other. The Galilean transformation from frame S to S' are given by

$$\begin{aligned} & x' = x - vt, y' = y, z' = z, t' = t \\ \Rightarrow & \frac{dx'}{dt} = \frac{dx}{dt} - v; \frac{dy'}{dt'} = \frac{dy}{dt}; \frac{dz'}{dt'} = \frac{dz}{dt} \\ \Rightarrow & u'_x = u_x - v; u'_y = u_y; u'_z = u_z \\ & \mathbf{u}' = \mathbf{u} - \mathbf{v} \end{aligned} \tag{10.3}$$

where $\mathbf{u} = (u_x, u_y, u_z)$ is the velocity of the particle with respect to the observer of the frame S and $\mathbf{u}' = (u'_x, u'_y, u'_z)$ is of the frame S'. The set of transformation equations [Eq. (10.3)] is the classical velocity addition formula in Newtonian mechanics. Also, from equations [Eq. (10.3)], we conclude that velocity of a particle is not a Galilean invariant. Also, the inverse Galilean velocity transformations are

$$u_x = u'_x + v; \quad u_y = u'_y; \quad u_z = u'_z.$$

10.1.5 Invariance of Acceleration

Differentiating the transformation set of equations [Eq. (10.3)], with respect to time t, we get

$$\begin{aligned} & \frac{du'_x}{dt} = \frac{du_x}{dt}; \frac{du'_y}{dt} = \frac{du_y}{dt}; \frac{du'_z}{dt} = \frac{du_z}{dt} \\ \Rightarrow & \frac{du'_x}{dt'} = \frac{du_x}{dt}; \frac{du'_y}{dt'} = \frac{du_y}{dt}; \frac{du'_z}{dt'} = \frac{du_z}{dt} \text{ as } t = t'. \end{aligned}$$

Since the components of the acceleration of the particle measured in the two inertial frames are independent of the uniform relative velocity of the two frames, acceleration of the particle is invariant under Galilean transformation.

10.1.6 Invariance of Newton's Law

In fact, Newton's first law of motion, whether a given frame of reference is an inertial frame of reference or not. Since Newton's first law of motion always holds good in an inertial frame of reference, it will also hold in two inertial frames of reference moving with uniform velocity relative to each other. We know that, the measurement of acceleration of a particle is independent of the uniform relative motion of the two inertial frames, i.e. $\mathbf{a}' = \mathbf{a}$. Let m be the mass of the particle. In Newtonian mechanics, mass is a constant quantity and is independent of the motion of the observer. Multiplying both sides of the above equation with m, we have

$$m\mathbf{a}' = m\mathbf{a} \Rightarrow \mathbf{F}' = \mathbf{F},$$

where $\mathbf{F}' = m\mathbf{a}'$ and $\mathbf{F} = m\mathbf{a}$. Thus, the observers of the two inertial frames of reference moving with uniform velocity with respect to each other measure the force on the particle to be the same. Therefore, equations of motion ($\mathbf{F} = m\mathbf{a}$) of a particle is invariant under Galilean transformation. In other words, Newton's second law of motion is Galilean invariant. Since measurement of force is not affected by the uniform relative motion between the two inertial frames of reference, Newton's third law of motion $\mathbf{F}_{12} = -\mathbf{F}_{21}$ must also hold in the two inertial frames moving with uniform velocity relative to each other. Hence, Newton's laws of motion are invariant under Galilean transformations.

10.1.7 Light Cone and Relativistic Length

A flash of light at position $(0, 0, 0)$ at time $t = 0$ sends out an expanding spherical wave front, with equation $x^2 + y^2 + z^2 = c^2t^2$, or

$$-(x^1)^2 - (x^2)^2 - (x^3)^2 + (x^4)^2 = 0. \tag{10.4}$$

Equation (10.4) is the equation of the light cone in event space, relative to the inertial frame (x^i). Figure 10.2 shows the projection of the light cone onto hyperplane $x^3 = 0$. In any other inertial frame (x^i), the equation of the light cone is exactly the same (since all observers measure the velocity of light as c)

$$-(\overline{x}^1)^2 - (\overline{x}^2)^2 - (\overline{x}^3)^2 + (\overline{x}^4)^2 = 0. \tag{10.5}$$

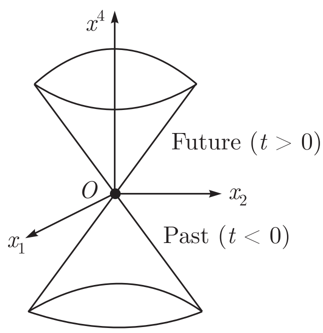

Figure 10.2: Light cone.

As one of the four components of the four radius vector is imaginary, the square of its magnitude may be positive, negative or zero.

For an arbitrary event $E(\mathbf{x})$, the quantity the left hand expression of Eq. (10.4) may be positive, negative or zero. The *relativistic distance* from $E(\mathbf{x})$ to the origin $E_0(\mathbf{0})$ is the real number $s \geq 0$, such that,

$$s^2 = -(x^1)^2 - (x^2)^2 - (x^3)^2 + (x^4)^2. \tag{10.6}$$

More generally, the *length of interval* or *relativistic distance* between $E_1(\mathbf{x_1})$ and $E_2(\mathbf{x_2})$ is the unique real number $\Delta s \geq 0$, such that,

$$(\Delta s)^2 = -(\Delta x^1)^2 - (\Delta x^2)^2 - (\Delta x^3)^2 + (\Delta x^4)^2, \tag{10.7}$$

where $\Delta x^i = x_2^i - x_1^i$ for $i = 0, 1, 2, 3$. The chief significance of this length concept is relative distance is an invariant across all inertial frames. The interval between $E_1(\mathbf{x_1})$ and $E_2(\mathbf{x_2})$ is

(i) *spacelike* if: $(\Delta\overline{x}^1)^2 + (\Delta\overline{x}^2)^2 + (\Delta\overline{x}^3)^2 > (\Delta\overline{x}^4)^2$
(ii) *lightlike* if: $(\Delta\overline{x}^1)^2 + (\Delta\overline{x}^2)^2 + (\Delta\overline{x}^3)^2 = (\Delta\overline{x}^4)^2$
(iii) *timelike* if: $(\Delta\overline{x}^1)^2 + (\Delta\overline{x}^2)^2 + (\Delta\overline{x}^3)^2 < (\Delta\overline{x}^4)^2$.

Also, the categorisation is independent of the particular inertial frame.

10.2 Postulates of Relativity

The special theory of relativity is based on the following two postulates:

(i) **Principle of relativity and invariance of uniform motion:** All the basic laws of physics which can be expressed in the form of equations have the same form in all the inertial frames of reference moving with an uniform velocity with respect to one another.

(ii) **Invariance of light speed:** The measured speed of light in free space has always the same value c for all the observers irrespective of the relative motion of the source of light and the observer.

The first postulate bears the fact that, there is no existence of an universal frame. If we reformulate the different laws of physics for observers in a relative constant motion with respect to one another, we obtain other set of transformations in place of Galilean Transformation, under which the laws would be invariant.

The second postulate requires that the bijective transformation

$$\mathcal{T}: \quad \overline{x}^i = F^i\left(x^1, x^2, x^3, x^4\right);\; x^4 = ct \tag{10.8}$$

be such as to map straight lines into straight lines. Consequently, each F^i must be a linear function. Since $F^i(0,0,0,0) = (0,0,0,0)$, constants a^i_j exist such that

$$T: \quad x^i = a^i_j x^j. \tag{10.9}$$

These postulates of the special theory of relativity do not appear so radical at first sight, but lead to very revolutionary ideas. The concepts of the invariance of mass, length interval and time interval and their mutual independence are neither invariant nor independent of one another in special theory of relativity.

The name special theory of relativity comes from the fact that this theory permits the independence of the physical laws of those co-ordinate systems which are moving with constant velocity relative to one another. Later, Einstein propounded his general theory of relativity which allows for the indepence of the physical laws of all co-ordinate systems, having any general relative motion.

These two postulates of special theory of relativity look to be very simple, but they have revolutionized the physics with far reaching consequences. First we deduce transformation equations, connecting any two inertial systems moving with constant relative velocity. The transformation should be such that they are applicable to both Newtonian mechanics and electromagnetism. Such transformations were deduced by Einstein in 1905 and are known as Lorentz transformations because Lorentz deduced them first in his theory of electromagnetism.

10.3 Lorentz Transformation

In classical physics, we are concerned with invariance of physical quantities under Gallilean transformations, according to which all physical phenomena appear to be the same to all observers, who are stationary relative to each other.

In 1904, H.A. Lorentz derived a set of transformations for space and time co-ordinates of an event in two inertial frames moving with a uniform velocity relative to each other. However, Einstein obtained these transformations quite independently on the basis of the two postulates of the special theory of relativity. These equations are known as *Lorentz–Einstein transformations* or simply *Lorentz transformations.* This transformation should confirm to the following general requirements:

(i) The transformation must be linear, i.e. any single event in one inertial frame must transform to a single event in another frame, with a single set of co-ordinates.
(ii) The postulate of the equivalence of all interval frames requires that the direct and inverse transformations should be symmetrical with respect to each other.
(iii) In case, there are no relative motion between the two frames of reference, Lorentz transformations should be reduce to identity transformations.

(iv) In case, the velocity of moving frame is very small as compared to the velocity of light, Lorentz transformation should reduce to Galilean transformations.

(v) The relativistic law of addition of velocities as derived from the Lorentz transformations should leave the velocity of light as invariant.

The invariance of the equation of the light cone (in consequence of postulate 2) may be expressed as

$$g_{ij}x^i x^j = 0 = g_{ij}\overline{x}^i\overline{x}^j, \tag{10.10}$$

where $g_{11} = g_{22} = g_{33} = -1, g_{44} = 1$ and $g_{ij} = 0$ for $i \neq j$. Now, using Eq. (10.9), from Eq. (10.10), we get

$$\begin{aligned} g_{ij}\overline{x}^i\overline{x}^j &= g_{ij}\left(a^i_r x^r\right)\left(a^i_s x^s\right) \\ &= g_{rs}a^r_i a^s_j x^i x^j = 0 \text{ whenever } g_{ij}x^i x^j = 0. \end{aligned} \tag{10.11}$$

Now, apply Examples 1.7.12 to 10.11, with $g_{rs}a^r_i a^s_j = c_{ij}$, where $C = (c_{ij}) = A^T G A$ is symmetric, we obtain

$$g_{rs}a^r_i a^s_j = \lambda g_{ij}, \text{ or } A^T G A = \lambda G. \tag{10.12}$$

Since $G^2 = I$, multiplication of Eq. (10.12) by $\lambda^{-1}G$ gives,

$$\begin{aligned} &\left\{G(\lambda^{-1}A^T)G\right\} A = I \\ &\Rightarrow A^{-1} = \frac{1}{\lambda}GA^T G = \begin{bmatrix} -\dfrac{a^1_1}{\lambda} & -\dfrac{a^2_1}{\lambda} & -\dfrac{a^3_1}{\lambda} & \dfrac{a^4_1}{\lambda} \\ \dfrac{a^1_2}{\lambda} & \dfrac{a^2_2}{\lambda} & \dfrac{a^3_2}{\lambda} & -\dfrac{a^4_2}{\lambda} \\ \dfrac{a^1_3}{\lambda} & \dfrac{a^2_3}{\lambda} & \dfrac{a^3_3}{\lambda} & -\dfrac{a^4_3}{\lambda} \\ \dfrac{a^1_4}{\lambda} & \dfrac{a^2_4}{\lambda} & \dfrac{a^3_4}{\lambda} & -\dfrac{a^4_4}{\lambda} \end{bmatrix} \\ &= B = (b^i_j)_{4\times 4}, \text{ say,} \end{aligned} \tag{10.13}$$

In particular, $b^4_1 = \frac{a^4_1}{\lambda}$. Now, since observers O and $\overline{O}$ are receding from each other at constant velocity v and are using identical measuring devices, it is clear that each views the other in the same way. It follows that:

$$a^4_1 = b^4_1 \quad \text{and} \quad \lambda = \frac{a^4_1}{b^4_1} = 1.$$

Hence, Eq. (10.12) can be written as

$$A^T G A = G \text{ or, } g_{ij} a^i_r a^j_s = g_{rs}. \tag{10.14}$$

Equation (10.14) can also be written as

$$-(a^1_j)^2 - (a^2_j)^2 - (a^3_j)^2 + (a^4_j)^2 = -1;\ j = 1, 2, 3$$
$$-(a^1_4)^2 - (a^2_4)^2 - (a^3_4)^2 + (a^4_4)^2 = 1$$
$$-a^1_i a^1_j - a^2_i a^2_j - a^3_i a^3_j + a^4_i a^4_j = 0; \quad \text{for, } i \neq j.$$

Now, from the relation $A^T G A = G$, we get

$$(Ax)^T G(Ax) = x^T (A^T G A) x = x^T G x = q. \tag{10.15}$$

Therefore, the $g_{ij} x^i x^j = 0$ is invariant due to invariance of $g_{ij} x^i x^j = q$, for every value of q, so that $A^T G A = G$ is a criterion for the quadratic form $x^T G x$ to be invariant. Any 4×4 matrix (or corresponding linear transformation) that preserves the quadratic form $x^T G x$ is called *Lorentz.* If the terms $\overline{g}_{ij} \equiv g_{ij}$ are defined for the $(\overline{x}^i)$ system, then Eq. (10.14) becomes

$$g_{rs} = \overline{g}_{ij} a^i_r a^j_s;$$

which makes g_{ij} a covariant tensor of the second order under Lorentz transformations of co-ordinates. Therefore, the metric for R^4 is chosen as

$$ds^2 = g_{ij} dx^i dx^j = -(dx^1)^2 - (dx^2)^2 - (dx^3)^2 + (dx^4)^2. \tag{10.16}$$

Let us consider two inertial frames of reference S and $\overline{S}$ having their respective axes parallel to each other. The frame $\overline{S}$ is moving with a relative constant velocity v along x^1-direction. According to Galilean transformation, the velocity transformations are

$$u_{\overline{x}^1} = u_{x^1} - v;\ \ u_{\overline{x}^2} = u_{x^2};\ \ u_{\overline{x}^3} = x^3,$$

which violate the postulates of the special theory of relativity. The first postulates of the special theory of relativity implies that a uniform linear motion is one frame of reference must appear as the same in the other inertial frame also and likewise a linear relationship among (x^1, x^2, x^3) and t must go over a linear relationship among $(\overline{x}^1, \overline{x}^2, \overline{x}^3)$ and $\overline{t}$. Let us assume that x^1 and $\overline{x}^1$ may be connected by the relation

$$\overline{x}^1 = a x^1 + b x^4, \tag{10.17}$$

where a and b are constants. Further as the motion of the frame $\overline{S}$ is only along x^1 direction, we have $\overline{x}^2 = x^2$ and $\overline{x}^3 = x^3$. In the special theory of relativity, space and time are independent and they form a space time continumn. Therefore, time co-ordinates t and t' may be assumed to be connected by the relation

$$\overline{x}^4 = px^1 + qx^4, \tag{10.18}$$

where p and q are some constants. To determine the constants a, b, p, q we use the second postulate of special theory of relativity. If according to the observer of the frame S, the spherical wavefront passes through the point $P(x^1, x^2, x^3)$ in space at time t, then

$$(x^1)^2 + (x^2)^2 + (x^3)^2 = (x^4)^2. \tag{10.19}$$

Let the point P in the frame S be observed as $P'(y^1, y^2, y^3)$ at a time $\bar{t}$ in frame $\overline{S}$. Since according to second postulate of the special theory of relativity, the speed of the light is same in all the inertial frames, i.e.

$$(\overline{x}^1)^2 + (\overline{x}^2)^2 + (\overline{x}^3)^2 = (\overline{x}^4)^2. \tag{10.20}$$

Using Eq. (10.14), we get

$$q^2 - b^2 = 1; \;\; p^2 - a^2 = -1; \;\; pq - ab = 0.$$

By considering the co-ordinates which O and $\overline{O}$ would assign to each other's origin, we find that

$$b = -\left(\frac{v}{c}\right) q \equiv -\beta q \text{ and } q = a,$$

where $\beta = \frac{v}{c}$. Noting $q > 0$, we get

$$q = \left(1 - \beta^2\right)^{-1/2} = a \text{ and } p = -\beta\left(1 - \beta^2\right)^{-1/2} = b.$$

Therefore, the co-ordinate transformation takes on the simplest form

$$\overline{x}^1 = \frac{1}{\sqrt{1-\beta^2}}x^1 - \frac{\beta}{\sqrt{1-\beta^2}}x^4 = \frac{1}{\sqrt{1-\beta^2}}\left(x^1 - \beta x^4\right),$$

$$\overline{x}^4 = -\frac{\beta}{\sqrt{1-\beta^2}}x^1 + \frac{1}{\sqrt{1-\beta^2}}x^4 = \frac{1}{\sqrt{1-\beta^2}}\left(x^4 - \beta x^1\right),$$

where the Lorentz matrix A is given by

$$A = \begin{bmatrix} -\frac{\beta}{\sqrt{1-\beta^2}} & \frac{1}{\sqrt{1-\beta^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{1}{\sqrt{1-\beta^2}} & -\frac{\beta}{\sqrt{1-\beta^2}} & 0 & 0 \end{bmatrix} = \begin{bmatrix} b & a & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a & b & 0 & 0 \end{bmatrix},$$

where $a^2 - b^2 = 1$, will be termed simple Lorentz. The relative velocity in the physical situation modelled by A is recovered as $\beta = -\frac{b}{a}$. Hence, putting these values in Eqs. (10.17) and (10.18), we obtain

$$\overline{x}^1 = \frac{x^1 - vt}{\sqrt{1-(v^2/c^2)}};\ \ \overline{x}^2 = x^2;\ \ \overline{x}^3 = x^3;\ \ \overline{t} = \frac{t-(vx^1)/c^2}{\sqrt{1-(v^2/c^2)}}. \tag{10.21}$$

These set of equations are called *Lorentz transformations.* The inverse of the simple Lorentz matrix A is given by

$$A^{-1} = \begin{bmatrix} -b & a & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a & -b & 0 & 0 \end{bmatrix}.$$

Thus, one can obtain x^1, x^2, x^3 and t in terms of $\overline{x}^1, \overline{x}^2, \overline{x}^3$ and $\overline{t}$ as

$$x^1 = \frac{\overline{x}^1 + v\overline{t}}{\sqrt{1-(v^2/c^2)}};\ \ x^2 = \overline{x}^2;\ \ x^3 = \overline{x}^3;\ \ t = \frac{\overline{t}+(v\overline{x}^1)/c^2}{\sqrt{1-(v^2/c^2)}}, \tag{10.22}$$

which is called *inverse Lorentz transformations.* The direct and inverse Lorentz transformations are symmetrical with respect to each other. Now,

(i) When $v \to 0$, the Lorentz transformation relations [Eq. (10.21)] reduces to $\overline{x}^1 = x^1, \overline{x}^2 = x^2, \overline{x}^3 = x^3$ and $\overline{t} = t$, which are termed as identity transformations. Thus, in the limit, when $v \to 0$, the Lorentz transformation reduces to identity transformations.

(ii) When $v \ll c$, the factor $\frac{v^2}{c^2}$ can be neglected as compared to 1 and $\frac{vx^1}{c^2}$ in comparison to t. So, when $v \ll c$, the Lorentz transformation [Eq. (10.21)] reduces to $\overline{x}^1 = x^1 - vt$, $\overline{x}^2 = x^2, \overline{x}^3 = x^3$ and $\overline{t} = t$, which is the Galilean transformations. Thus, the Lorentz transformation equations reduce to the Galilean transformations when relative velocity v is very small in comparison with velocity c of light.

(iii) Note that, Lorentz transformation put an upper limit on the velocity, a moving frame of reference (or an object) can possess. It follows that, in case $v = c$, or $v > c$, the Lorentz transformation gives unphysical results. Hence, the relative velocity of an inertial frame with respect to another must always be less than the velocity of light.

Result 10.3.1 The particular set of equations [Eq. (10.21)] leaves the quadratic form

$$d\sigma^2 = (dx^4)^2 - dx^i dx^i \tag{10.23}$$

invariant, which plays an important role in Minkowski space. If instead of cartesian variables x^i we had chosen curvilinear co-ordinates u^i, related to Cartesian co-ordinates x^i by formula (2.42)

$$T: \quad x^i = x^i\left(u^1, u^2, u^3\right),$$

then the form [Eq. (10.23)] would be

$$d\sigma^2 = c^2 dt^2 - g_{ij} du^i du^j; \text{ where } g_{ij} = \frac{\partial x^k}{\partial u^i}\frac{\partial x^k}{\partial u^j}. \tag{10.24}$$

The determinant of coefficients of this form [Eq. (10.24)] has the value $-c^2 g$.

Result 10.3.2 The forgoing formulas can be cast in a symmetric form by setting $t = u^4$, then Eq. (10.24) becomes

$$d\sigma^2 = a_{\alpha\beta} du^\alpha du^\beta; \quad \alpha, \beta = 1, 2, 3, 4. \tag{10.25}$$

where

$$a_{ij} = -g_{ij}; \quad i, j = 1, 2, 3; \quad a_{i4} = 0, a_{44} = c^2, \text{ and } a = -c^2 g.$$

Now, we shall discuss some physical implication of the simple Lorentz transformations.

10.3.1 Length Contraction

Suppose that a rod AB is placed along the axis of $\overline{x}^1$ in the moving frame of reference $\overline{S}$, whose respective axes are parallel to the reference frame S and which is moving with a constant velocity v along x^1 direction relative to frame S. A frame in which the object AB is at rest is known as proper frame and the length of the rod in such a frame is called *the proper length.* Let y^1 and $\overline{x}^2$ be the distances of the ends A and B of the rod from the origin O' of the moving reference frame $\overline{S}$, if L_0 be the proper length then $L_0 = \overline{x}^2 - \overline{x}^1$.

To an observer of frame S, the rod AB is moving with a constant velocity v and its length L (say) as measured by the observer of frame S is called *non-proper or moving length* of the rod. The process of measuring the length of moving object requires that the distances of the two ends from the origin must be measured simultaneously. Thus, if at any instant t, the distances of the ends A and B of the rod AB from the origin O of the frame S are x^1 and x^2, respectively, then $L = x^2 - x^1$.

Now, let us find length L of the rod in system S relative to which the rod is in motion with velocity v. The direct Lorentz transformation equations give

$$\overline{x}^1 = \frac{x^1 - vt}{\sqrt{1-(v^2/c^2)}}; \quad \overline{x}^2 = \frac{x^2 - vt}{\sqrt{1-(v^2/c^2)}}$$

$$\Rightarrow L_0 = \overline{x}^2 - \overline{x}^1 = \frac{(x^2 - x^1) - v(t^2 - t^1)}{\sqrt{1 - v^2/c^2}}$$

$$= \frac{L}{\sqrt{1 - v^2/c^2}} - \frac{v}{\sqrt{1 - v^2/c^2}}(t^2 - t^1).$$

Here, t^2 and t^1 are the times at which the end co-ordinates x^2 and x^1 of the rod are measured. Since the measured should be simultaneous in frame S for determining the length of the rod, we have

$$L_0 = \frac{L}{\sqrt{1 - v^2/c^2}} \Rightarrow L = L_0\sqrt{1 - v^2/c^2}.$$

If follows that $L < L_0$, i.e. the length of the rod in motion with respect to an observer appears to the observer to be shorter than when it is at rest with respect to him/her, and the length appears to contract by a factor $\sqrt{1 - v^2/c^2}$. Hence, it is clear that, the length of an object is a maximum in the frame of reference in which it is stationary. This phenomenon is known as *Lorentz–Fitzgerald contraction.*

10.3.2 Time Dilation

Let us suppose that a light signal is emitted in space from $P(x', 0, 0)$ in frame $\overline{S}$ (which is moving with a constant velocity v along x-direction with respect to the frame S) at time t_1' and again at time t_2' from the same point. Thus, the proper time τ_0 is the time interval between the two light signals and so $\tau_0 = t_2' - t_1'$.

Suppose that t_1 and t_2 are two instants recorded by the observer in the frame S, then the non-proper time τ is the time interval between the two signals and is given by $\tau = t_2 - t_1$. The observer in frame S, however, measures these instants given by the Lorentz transformation equations give

$$t_1 = \frac{t_1' + \frac{x'v}{c^2}}{\sqrt{1 - v^2/c^2}}; \quad t_2 = \frac{t_2' + \frac{x'v}{c^2}}{\sqrt{1 - v^2/c^2}}$$

$$\Rightarrow \tau = \frac{t_2' - t_1'}{\sqrt{1 - v^2/c^2}} = \frac{\tau_0}{\sqrt{1 - v^2/c^2}}.$$

Since $\sqrt{1 - v^2/c^2}$ is a fraction, it follows that $\tau > \tau_0$, i.e. the time interval between two light signals as recorded by the clock of the frame $\overline{S}$. Since the event of emitting of light signal takes place in frame $\overline{S}$ and it appears to the observer of frame S in motion, the time interval between the two light signals is recorded by the clock of frame S in motion. In other words, to an observer in motion relative to the clock, the time intervals appear to lengthened. The phenomenon of kinematical effect of relativity of time is called *time dilation*. Now, we are able to observe the following phenomena:

(i) The time dilation effect is expected only, when the relative velocity of frame S' is comparable to the velocity of light. In case $v \ll c$, i.e. $\frac{v^2}{c^2} \ll 1$, then $\tau = \tau_0$.

(ii) The opinion of the observer will also reciprocal in nature, like length contraction effect,

$$\tau = \frac{\tau_0}{\sqrt{1 - v^2/c^2}} = \frac{\tau_0}{\sqrt{1 - (-v)^2/c^2}}.$$

Hence, to the observer of the frame S', the clock of frame S will appear to run slower.

(iii) Time does not run backward for any observer. The sequence of the events in a series of events is never altered for any observer. Since the velocity available for communication is always less than or equal to c, the intervals of the time between any two events may be different.

(iv) No observer can see an event before it takes place.

10.3.3 Relativistic Velocity and Acceleration

Let us consider a body that moves with respect to both inertial frames of reference S and $\overline{S}$ having their respective axes parallel and the frame $\overline{S}$ is moving with a constant velocity v along x direction relative to frame S. In the inertial frame $x^i = \left(x^1, x^2, x^3\right)$, let a particle describe the class C^2 curve

$$\Gamma: \quad (x^i) = (\mathbf{r}(t), ct) = \left(x^1(t), x^2(t), x^3(t), ct\right).$$

Let $P\left(x^1, x^2, x^3\right)$ be the position of the particle at time t with respect to the observer of the frame S. Then the velocity is given by the classical formula as

$$(v^i) = \left(\frac{dx^i}{dt}\right) = (\mathbf{v}, c)\,, \text{ where, } \mathbf{v} = \frac{d\mathbf{r}}{dt}$$

and

$$\hat{v} = |\mathbf{v}| = \sqrt{(v^1)^2 + (v^2)^2 + (v^3)^2}.$$

At the instant, let $P\left(\overline{x}^1, \overline{x}^2, \overline{x}^3\right)$ be the position of the particle at time $\overline{t}$. An observer of the frame of reference $\overline{S}$ makes the following measurements of velocity $\overline{\mathbf{v}} = \left(\overline{v}^1, \overline{v}^2, \overline{v}^3\right)$ of the body measured by him/her as

$$(\overline{v}^i) = \left(\frac{d\overline{x}^i}{d\overline{t}}\right) = (\overline{\mathbf{v}}, c)\,, \hat{v}' = |\overline{\mathbf{v}}| = \sqrt{(\overline{v}^1)^2 + (\overline{v}^2)^2 + (\overline{v}^3)^2}.$$

The acceleration is given by the classical formula as

$$(a^i) = \left(\frac{d^2x^i}{dt^2}\right) = (\mathbf{a}, 0)\,, \text{ where } \mathbf{a} = \frac{d\mathbf{v}}{dt}$$

and $\hat{a} = |\mathbf{a}|$. As defined, neither the velocity nor the acceleration is a tensor under Lorentz transformation [Eq. (10.21)]. Now, we find the Lorentz transformations of velocity and acceleration, that define how $\overline{S}$ tracks the motion of the particle in S's frame. To simplify notation, let $\gamma \equiv (1-\beta^2)^{-1/2}$, then the transformation [Eq. (10.21)] becomes

$$\overline{x}^1 = \gamma(x^1 - \beta ct); \;\; \overline{x}^2 = x^2; \;\; \overline{x}^3 = x^3; \;\; c\bar{t} = \gamma(ct - \beta x^1). \tag{10.26}$$

Differentiate the last equation with respect to $\bar{t}$, we get

$$c = \left(c - \beta v^1\right)\frac{dt}{d\bar{t}} \Rightarrow \frac{dt}{d\bar{t}} = \frac{1}{\gamma\left(1 - \frac{vv^1}{c^2}\right)}; \;\; \beta = \frac{v}{c}.$$

Now, differentiate the first three equations of Eq. (10.26), we get,

$$\overline{v}^1 = \frac{d\overline{x}^1}{d\bar{t}} = \gamma(v^1 - \beta c)\frac{dt}{d\bar{t}} = \frac{\gamma(v^1 - \beta c)}{\gamma\left(1 - \frac{vv^1}{c^2}\right)} = \frac{v^1 - v}{1 - \frac{vv^1}{c^2}}$$

$$\overline{v}^2 = \frac{d\overline{x}^2}{d\bar{t}} = \frac{v^2}{\gamma\left(1 - \frac{vv^1}{c^2}\right)} = \frac{v^2\sqrt{1-\beta^2}}{1 - \frac{vv^1}{c^2}}$$

$$\overline{v}^3 = \frac{d\overline{x}^3}{d\bar{t}} = \frac{v^3}{\gamma\left(1 - \frac{vv^1}{c^2}\right)} = \frac{v^3\sqrt{1-\beta^2}}{1 - \frac{vv^1}{c^2}}.$$

These equations constitute the relativistic transformation law equations of addition of velocities. These equations transform the velocity components of a particle in unprimed frame S to the velocity components in primed frame $\overline{S}$ moving with a constant velocity v with respect to the reference frame S. The inverse transformation laws are

$$v^1 = \frac{\overline{v}^1 + v}{1 + \frac{v}{c^2}\overline{v}^1}; \;\; v^2 = \frac{\overline{v}^2\sqrt{1 - v^2/c^2}}{1 + \frac{v}{c^2}\overline{v}^1}; \;\; v^3 = \frac{\overline{v}^3\sqrt{1 - v^2/c^2}}{1 - \frac{v}{c^2}\overline{v}^1}. \tag{10.27}$$

By differentiation of the velocity components just found the acceleration components as

$$\overline{a}^1 = \frac{d\overline{v}^1}{dt}\frac{dt}{d\bar{t}} = \frac{(a^1 - 0)\left(1 - \dfrac{v^1 v}{c^2}\right) - (v^1 - v)\left(0 - \dfrac{a^1 v}{c^2}\right)}{\left(1 - \dfrac{vv^1}{c^2}\right)^2}\frac{dt}{d\bar{t}}$$

$$= \frac{a^1 - \dfrac{a^1 v^1 v}{c^2} + \dfrac{v^1 a^1 v}{c^2} - \dfrac{a^1 v \cdot v}{c^2}}{\gamma \left(1 - \dfrac{vv^1}{c^2}\right)^3} = \frac{a^1(1-\beta^2)^{3/2}}{\left(1 - \dfrac{vv^1}{c^2}\right)^3}$$

$$\overline{a}^2 = \frac{d\overline{v}^2}{dt}\frac{dt}{d\overline{t}} = \frac{a^2\left(1 - \dfrac{v^1 v}{c^2}\right) - v^2\left(0 - \dfrac{a^1 v}{c^2}\right)}{\left(1 - \dfrac{vv^1}{c^2}\right)^2} \frac{1-\beta^2}{\left(1 - \dfrac{v^1 v}{c^2}\right)}$$

$$= \frac{a^2 + \left(a^1 v^2 - v^1 a^2\right)\dfrac{v}{c^2}}{\left(1 - \dfrac{v^1}{c^2}\right)^3}(1-\beta^2).$$

$$\overline{a}^3 = \frac{d\overline{v}^3}{dt}\frac{dt}{d\overline{t}} = \frac{a^3 + \left(a^1 v^3 - v^1 a^3\right)\dfrac{v}{c^2}}{\left(1 - \dfrac{v^1}{c^2}\right)^3}(1-\beta^2).$$

Following are some points about the relativistic law of addition of velocities:

(i) In case, the reference frame S' moves with a velocity very small as compared to the velocity of light, then both $\frac{v^2}{c^2}$ and $\frac{vv^1}{c^2}$ can be neglected as compared to 1. Hence, the direct velocity transformations reduces to

$$\overline{v}^1 = v^1 - v; \;\; \overline{v}^2 = v^2; \;\; \overline{v}^3 = v^3$$

i.e. the relativistic law of addition of velocities reduces to Galilean law of addition of velocities.

(ii) In case the motion of the particle is confined to x direction, i.e. $v^1 = u, v^2 = v^3 = 0$, then the direct velocity transformation reduces to

$$\overline{v}^1 = \frac{u - v}{1 - \frac{uv}{c^2}}; \;\; \overline{v}^2 = \overline{v}^3 = 0.$$

(iii) Let $v^1 = c, v^2 = v^3 = 0$, i.e. the ray of light signal is emitted with velocity c along x direction in frame S. Applying the direct velocity transformation, we get

$$\overline{v}^1 = \frac{c - v}{1 - \frac{vc}{c^2}} = c; \;\; \overline{v}^2 = \overline{v}^3 = 0.$$

Thus, we find that the two inertial frames moving relative to each other, the velocity of light remains the same, which is consistent with the second postulate of the theory of relativity. From this we also conclude that c is the highest limit to the velocity that can be acquired by material bodies.

EXAMPLE 10.3.1 *Verify that the following matrix is Lorentz:*

$$\begin{pmatrix} \sqrt{2} & 0 & 0 & \sqrt{3} \\ \dfrac{\sqrt{6}}{2} & \dfrac{1}{2} & \dfrac{1}{2} & 1 \\ \dfrac{\sqrt{6}}{2} & -\dfrac{1}{2} & -\dfrac{1}{2} & 1 \\ 0 & -\dfrac{\sqrt{2}}{2} & \dfrac{\sqrt{2}}{2} & 0 \end{pmatrix}.$$

Solution: We verify directly condition Eq. (10.14) which can be written as

$$\begin{aligned} &-(a_1^1)^2 - (a_1^2)^2 - (a_1^3)^2 + (a_1^4)^2 \\ &\quad = -(\sqrt{2})^2 - \left(\frac{\sqrt{6}}{2}\right)^2 - \left(\frac{\sqrt{6}}{2}\right)^2 + (0)^2 = -1 \\ &-(a_2^1)^2 - (a_2^2)^2 - (a_2^3)^2 + (a_2^4)^2 \\ &\quad = -\left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 - \left(\pm\frac{\sqrt{2}}{2}\right)^2 + (0)^2 = -1. \end{aligned}$$

Similar calculations give

$$\begin{aligned} &-(a_4^1)^2 - (a_4^2)^2 - (a_4^3)^2 + (a_4^4)^2 = 1 \\ &-a_i^1 a_j^1 - a_i^2 a_j^2 - a_i^3 a_j^3 + a_i^4 a_j^4 = 0; \quad \text{for, } i \neq j. \end{aligned}$$

Therefore, the given matrix is Lorentz.

10.4 Minkowski Space

Consider a point P whose space co-ordinates relative to some reference frame S are $\left(x^1, x^2, x^3\right)$. Let the velocity of P, relative to this frame at that instant t, be $\mathbf{v}$. We shall introduce a Galilean reference frame $\overline{S}$ moving with the point P so that, at each instant t, the point P is at rest relative to the system $\overline{S}$. We shall call the system $\overline{S}$ a *local or proper co-ordinate system.*

Obviously the choice of local co-ordinate systems is not unique, since the definition just laid down merely requires that the velocity of the local frame be the same as that of the particle. This implies that the estimates of time (measured by the clocks carried in two different local co-ordinate frames) are the same. Hence, the transformation from one local system $\overline{S}$ to another $\overline{S}'$ has the form

$$\overline{x}'^i = \overline{x}'^i\left(\overline{x}^1, \overline{x}^2, \overline{x}^3\right); \quad \bar{t}' = \bar{t}.$$

The interval $d\sigma$ is defined by the formula

$$d\sigma^2 = a_{\alpha\beta}dx^\alpha dx^\beta = c^2dt^2 - g_{ij}dx^idx^j,$$

or

$$\left(\frac{d\sigma}{dt}\right)^2 = c^2 - g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} = c^2 - \hat{v}^2, \tag{10.28}$$

where v is the magnitude of the velocity $\mathbf{v}$ of the point P relative to the X co-ordinate frame. If a local co-ordinate system $\overline{X}$ is introduced at P, then, relative to $\overline{X}$, $\hat{v} = 0$ and then $\frac{d\sigma}{dt} = c$ in the local system. Using the particular set of equations [Eq. (10.21)], we see that

$$c^2dt^2 - dx^idx^i = c^2d\bar{t}^2 - d\overline{x}^id\overline{x}^i. \tag{10.29}$$

This invariance of this equation with respect to Lorentz transformations [Eq. (10.21)] suggests that the *Minkowski space or space time continuum* defined by the metric

$$d\sigma^2 = c^2(dx^4)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2, \tag{10.30}$$

where we have written $x^4 = t$, is appropriate for the geometrical discussion of special relativity. We denote the line-element of this four-dimensional space by $d\sigma$ (not by ds) in order to emphasise that $d\sigma$ is not the physical distance between two neighbouring points. A comparison of the metric given by Eq. (10.30) in the Minkowski space with Eq. (2.1) shows that the metric tensor of the space with the chosen co-ordinate system is

$$(g_{ij}) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

or

$$g_{11} = g_{22} = g_{33} = -1; g_{44} = 1 \text{ and } g_{ij} = 0 \text{ for } i \neq j. \tag{10.31}$$

The Minkowski space is flat and its signature, which equals the excess of the number of positive terms over the number of the negative terms in its metric is -2. It is well known that there is no real transformation of co-ordinates which will reduce [Eq. (10.30)] to the metric of a four-dimensional Euclidean space, whose signature is

4. Thus, the geometry of Minkowski space differs in many respects from Euclidean geometry; for example, there exist real null curves and real null geodesics.

The Lorentz transformation [Eq. (10.21)] for x^1 and t co-ordinates are

$$\overline{x}^1 = \frac{x^1 - vt}{\sqrt{1-(v^2/c^2)}}; \quad \overline{t} = \frac{t - (vx^1)/c^2}{\sqrt{1-(v^2/c^2)}},$$

which can be written in the form,

or

$$\overline{x}^1 = \frac{x^1 - (v/c).ct}{\sqrt{1-(v^2/c^2)}}; \quad c\overline{t} = \frac{ct - (v/c)x^1}{\sqrt{1-(v^2/c^2)}}$$

or

$$\overline{x}^1 = \frac{x^1 - \beta\omega}{\sqrt{1-\beta^2}}; \quad \overline{\omega} = \frac{\overline{\omega} - \beta x^1}{\sqrt{1-\beta^2}}; \quad \omega = ct. \tag{10.32}$$

This four-dimensional world which is a linking together of space and time is called the *four-dimensional space-time continuum.* Any four-dimensional space involving time in one of the axes is referred to as *four space* or world space. The four space with *ict* as the fourth co-ordinate is referred to as the *Minkowski four space.* Thus, events are defined by four space time co-ordinates and represented by points called *points.*

Minkowski referred to space-time as *the world* and a point in space-time as the *world point.* The motion of a particle in space-time can be represented by a curve called *world line*, which gives the loci of space-time points corresponding to the motion. Since the homogeneous second degree equation

$$(dx^4)^2 - (dx^1)^2 + (dx^2)^2 + (dx^3)^2 = 0$$

represents a cone, so, the world line of a particle must lie in its light cones. The quantity $d\sigma$ defined in Eq. (10.30) is an invariant under Lorentz transformation. The quantity $d\sigma$ is called the *element of proper time or world time* in the Minkowski four space. As one of the four components of the four radius vector is imaginary, the square magnitude may be positive, negative or zero. The four radius vector is called *space like interval*, if $(d\sigma)^2 < 0$. Also, if $(d\sigma)^2 > 0$, then it is called *time like interval.* If $(d\sigma)^2 = 0$, then the corresponding interval is called a *light-like interval.*

10.4.1 Minkowski Velocity and Acceleration

The velocity v of a particle, which is at the point x^α has the components u^α by the formula,

$$u^\alpha = \frac{dx^\alpha}{d\sigma}; \quad \alpha = 1, 2, 3, 4 \tag{10.33}$$

referred to the system S, known as *Minkowski velocity vector*. It follows from Eq. (10.28) that

$$\frac{dt}{d\sigma} = \frac{1}{c}\left[1 - \frac{\hat{v}^2}{c^2}\right]^{-1/2} = \frac{1}{\sqrt{c^2 - \hat{v}^2}}. \tag{10.34}$$

Let v^α be the nonrelativistic components of velocity. Using Eq. (10.24), the velocity components can be written as

$$u^\alpha = \frac{dx^\alpha}{(c^2dt^2 - g_{ij}dx^i dx^j)^{1/2}} = \frac{dx^\alpha}{dt} \cdot \frac{1}{\sqrt{c^2 - \hat{v}^2}} = \frac{v^\alpha}{\sqrt{c^2 - \hat{v}^2}}.$$

For the local co-ordinate system, $v^1 = v^2 = v^3 = 0, x^4 = t$ so the components of velocity in a local co-ordinate system are $\left(0, 0, 0, \frac{1}{c}\right)$. The *Minkowski acceleration vector* f^α is given by

$$f^\alpha \equiv \frac{du^\alpha}{d\sigma} = \frac{d^2x^\alpha}{d\sigma^2}; \quad \alpha = 1, 2, 3, 4. \tag{10.35}$$

Thus, the acceleration components can be written as

$$\begin{aligned} f^\alpha &= \frac{du^\alpha}{dt}\frac{dt}{d\sigma} = \frac{d}{dt}\left(\frac{v^\alpha}{\sqrt{c^2 - \hat{v}^2}}\right)\frac{dt}{d\sigma} \\ &= \frac{a^\alpha\sqrt{c^2 - \hat{v}^2} - v^\alpha\frac{1}{2}\left(c^2 - \hat{v}^2\right)^{-1/2}\left(-2a^1v^1 - 2a^2v^2 - 2a^3v^3\right)}{c^2 - \hat{v}^2}\frac{dt}{d\sigma} \\ &= \frac{a^\alpha}{(c^2 - \hat{v}^2)} + \frac{(\mathbf{a}\cdot\mathbf{v})v^\alpha}{(c^2 - \hat{v}^2)^2}. \end{aligned}$$

Using the intrinsic differentiation, the *Minkowski acceleration vector* f^α by the formula

$$f^\alpha = \frac{\delta u^\alpha}{\delta\sigma} = \frac{d^2x^\alpha}{d\sigma^2} + \left\{\begin{matrix} & \alpha & \\ \beta & & \gamma \end{matrix}\right\}\frac{dx^\beta}{d\sigma}\frac{dx^\gamma}{d\sigma}; \quad \alpha, \beta, \gamma = 1, 2, 3, 4. \tag{10.36}$$

If our local reference frame $\overline{X}$ is Cartesian so that

$$d\sigma^2 = c^2d\bar{t}^2 - d\overline{y}^i d\overline{y}^i,$$

the components $\overline{f}^\alpha$ of the Minkowski acceleration relative to it are

$$\overline{f}^\alpha = \frac{d^2\overline{y}^\alpha}{d\sigma^2} = \frac{d}{d\bar{t}}\left(\frac{d\overline{y}^\alpha}{d\sigma}\right)\frac{d\bar{t}}{d\sigma} = \frac{1}{c}\frac{d}{d\bar{t}}\left(\frac{d\overline{y}^\alpha}{d\sigma}\right) = \frac{1}{c^2}\frac{d^2\overline{y}^\alpha}{d\bar{t}^2},$$

so that,

$$\overline{f}^i = \frac{1}{c^2}\frac{d^2\overline{y}^i}{d\bar{t}^2}; \quad i = 1, 2, 3 \text{ and } \overline{f}^4 = 0; \text{ as } \overline{y}^4 = \bar{t}.$$

10.4.2 Minkowski Momentum

The four-dimensional *Minkowski momentum vector* is defined by

$$p^\alpha = m_0 c \frac{dx^\alpha}{d\sigma}; \text{ where, } m_0 = \text{ the constant.} \tag{10.37}$$

The special theory identifies the fourth component $m_0 c\frac{dx^4}{d\sigma}$ with mass m of the moving particle. In virtue of Eq. (10.34), we have

$$m = m_0 c \frac{dt}{d\sigma} = m_0 \left[1 - \frac{v^2}{c^2}\right]^{-1/2} = \frac{m_0}{\sqrt{1-\beta^2}}. \tag{10.38}$$

The constant m_0 is the mass when $v = 0$ so, it is called the *rest mass* of the particle. The mass m, which clearly increases with the velocity, is called the *relativistic mass* of the particle. The components

$$m_0 c \frac{dx^\alpha}{d\sigma} = m_0 c \frac{dx^\alpha}{dt}\frac{dt}{d\sigma} = m\frac{dx^\alpha}{dt}$$

and are evident generalisations of the *Newtonian momentum vector.*

10.4.3 Minkowski Four Force Vector

In Newtonian mechanics, the generalised equation of motion is given by

$$\mathbf{F}_k = \frac{d}{dt}(m\mathbf{v}_k)$$

which is not invariant under Lorentz transformation. Its relativistic generalisation should be a four-vector equation, in special part of which reduce to the above equation in the limit as $\beta \to 0$. Now, we have the following assumptions:

(i) Since time t is not Lorentz invariant, it should be replaced by proper time τ.

(ii) The mass m can be taken as an invariant property of the particle.

(iii) In place of the velocity v_i, world velocity v_μ should be substituted.

(iv) Force F_i should be replaced by some four vector, called the *Minkowski force.*

In the Minkowski space, a quantity is called a four-vector it has four components, each of which can be transformed by using the Lorentz transformation equations.

We define the four-dimensional *Minkowski force vector* F^α by

$$F^\alpha = m_0 c^2 \frac{d^2 x^\alpha}{d\sigma^2} = c^2 \frac{d}{d\sigma}\left(m_0 \frac{dx^\alpha}{d\sigma}\right).$$

Consider the formula suggested by Newton's second law,

$$F^\alpha = \frac{\delta}{\delta\sigma}(m_0 u^\alpha); \quad \alpha = 1, 2, 3, 4,$$

where $u^\alpha = \frac{dx^\alpha}{d\sigma}$ is the Minkowski velocity and m_0 is a constant whose significance will appear presently. Now,

$$\begin{aligned} F^\alpha &= \frac{\delta}{\delta t}(m_0 u^\alpha)\frac{dt}{d\sigma} = \frac{1}{\sqrt{c^2 - v^2}}\frac{\delta}{\delta t}\left(m_0 \frac{dx^\alpha}{d\sigma}\right) \\ &= \frac{1}{\sqrt{c^2 - v^2}}\frac{\delta}{\delta t}\left(\frac{m_0}{\sqrt{c^2 - v^2}}\frac{dx^\alpha}{dt}\right) \\ &= \frac{1}{c^2\sqrt{1 - \beta^2}}\frac{\delta}{\delta t}\left(\frac{m_0}{\sqrt{1 - \beta^2}}\frac{dx^\alpha}{dt}\right) \end{aligned}$$

or

$$\sqrt{1 - \beta^2} F^\alpha = \frac{1}{c^2}\frac{\delta}{\delta t}\left(m \frac{dx^\alpha}{dt}\right). \tag{10.39}$$

The Newtonian force vector is $X^\alpha = \frac{\delta}{\delta t}\left(m\frac{dx^\alpha}{dt}\right)$ so, Eq. (10.39) can be written as

$$F^\alpha = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} X^\alpha.$$

In the local co-ordinate system $\overline{Y}$, since $\beta = 0$ and $m = m_0$, we have

$$\overline{F}^\alpha = \frac{m_0}{c^2}\frac{d^2\overline{y}^\alpha}{d\overline{t}^2} = m_0 \overline{f}^\alpha. \tag{10.40}$$

This is the form of the Newton's second law used in classical mechanics. We see that the invariant m_0 is the mass of the particle P referred to a local reference frame. It is called the *rest* or *proper mass* of the particle. The quantity m is called the *relativistic mass*, or simply mass. Since Eq. (10.40) is a tensor equation, we can write the force equation as $F^\alpha = m_0 f^\alpha$, which is valid in all Galilean reference frames. We shall rewrite Eq. (10.39) in the form

$$\mathcal{F}^\alpha = \frac{\delta}{\delta t}\left(\frac{m_0 v^2}{\sqrt{1 - \beta^2}}\right), \tag{10.41}$$

where $v^\alpha = \frac{dx^\alpha}{dt}$, and $\mathcal{F}^\alpha = c^2\sqrt{1 - \beta^2}F^\alpha$, and shall take it as the equation of motion of a particle in the restricted theory of relativity.

10.4.4 Mass-energy Relation

Let us consider a body of rest mass m_0 moving with velocity v. For simplicity in writing we suppose that the co-ordinates x^i used in this section are rectangular Cartesian; and we recall that the work done by the force $F_i; i = 1, 2, 3$; in producing a displacement dx^i is equal to the change in the kinetic energy. Therefore, classical theory gives

$$\begin{aligned} T - T_0 &= \int_{v_0}^{v} mv dv = \int_{v_0}^{v} m\frac{dx^i}{dt} d\left(\frac{dx^i}{dt}\right) \\ &= \int_{t_0}^{t} m\frac{dx^i}{dt}\frac{d^2x^i}{dt^2}dt = \int_{P_0}^{P} m\frac{d^2x^i}{dt^2}dx^i = \int_{P_0}^{P} F_i dx^i. \end{aligned}$$

If we take as our definition of kinetic energy in the restricted theory of relativity the expression

$$\begin{aligned} T &= \int_{P_0}^{P} \mathcal{F}_i dx^i = \int_{P_0}^{P} \frac{\delta}{\delta t}\left(\frac{m_0 v^i}{1-\beta^2}\right) \quad dx^i \\ &= m_0 \int_{t_0}^{t} \left[\frac{d}{dt}\left(\frac{1}{1-\beta^2}\right) v^i \frac{dx^i}{dt} + \frac{dv^i}{dt}\frac{1}{1-\beta^2}\frac{dx^i}{dt}\right] dt \end{aligned}$$

where

$$\begin{aligned} \beta^2 &= \frac{v^2}{c^2} = \frac{v^i v^i}{c^2}; v^i = \frac{dx^i}{dt} \text{ and } v^i\frac{dx^i}{dt} = \beta^2 c^2, \frac{v^i}{c^2}\frac{dv^i}{dt} = \beta\dot{\beta} \\ &= m_0 \int_{t_0}^{t} \left[\frac{d}{dt}\left(\frac{1}{1-\beta^2}\right) \beta^2 c^2 + c^2\beta\dot{\beta}\frac{1}{1-\beta^2}\right] dt \\ &= m_0 \int_{t_0}^{t} \left[\frac{\beta\dot{\beta}}{(1-\beta^2)^{3/2}} \quad \beta^2 c^2 + c^2\beta\dot{\beta}\frac{1}{1-\beta^2}\right] dt \\ &= m_0 c^2 \int_{t_0}^{t} \frac{\beta\dot{\beta}}{(1-\beta^2)^{3/2}} dt = m_0 c^2 \int_{P_0}^{P} \frac{\beta d\beta}{(1-\beta^2)^{3/2}} \\ &= m_0 c^2 \int_{P_0}^{P} d\left[\frac{1}{(1-\beta^2)^{1/2}}\right] \end{aligned}$$

or

$$T = \frac{m_0 c^2}{\sqrt{1-\beta^2}} + \text{ constant.}$$

If we wish to have $T = 0$ when $\beta = \frac{v}{c} = 0$, the constant of integration is $-m_0c^2$, so that

$$T = \left[\frac{m_0}{\sqrt{1-\beta^2}} - m_0\right] c^2 = (m - m_0)c^2. \tag{10.42}$$

Thus, the KE of a moving body is equal to the product of the increase in the mass with square of the speed of light. Since the body possesses mass m_0 even when at rest, it may, therefore, be assumed that the rest mass of the body is due to an internal store of energy $E_0 = m_0c^2$. The quantity E_0 is called the *rest mass energy* or the *intrinsic energy* of the body. Now, Eq. (10.42) can be written as

$$mc^2 = E = T + m_0c^2 = T + E_0,$$

where $E = mc^2$ = total energy of the body; m_0c^2 = the intrinsic energy and T = the KE. This famous relation is known as *Einstein's mass-energy relation* and it shows the equivalence of mass and energy. The expression $E_0 = m_0c^2$ shows that mass is yet another form of energy. Since mass and energy are related to each other, we have to consider a principle of conservation of mass and energy. Mass can be created or destroyed, provided that an equivalent amount of energy vanishes or is being created and vice versa. Now, when $v \ll c$ (non-relativistic approximation), we have

$$\begin{aligned} T &= mc^2 - m_0c^2 = \frac{m_0c^2}{\sqrt{1-v^2/c^2}} - m_0c^2 \\ &= m_0c^2[(1-\frac{v^2}{c^2})^{-\frac{1}{2}} - 1] = m_0c^2[1 + \frac{1}{2}\frac{v^2}{c^2} - \frac{3}{8}\frac{v^4}{c^4} - \cdots] \approx \frac{1}{2}m_0c^2, \end{aligned}$$

which is the expression for KE of the particle in Newtonian mechanics. Now,

$$F^4 = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \frac{dm}{dt} = \frac{1}{c^2}\left(1 - \frac{v^2}{c^2}\right)^{-1/2} \frac{dE}{dt}.$$

The motion of a particle which moves under the action of some force system can be represented in Minkowski space by a curve, called the *world line* of the particle. If no forces act on the particle, we see from Eq. (10.39) that $\frac{d^2x^\alpha}{d\sigma^2} = 0$. Thus, the world-line of a free particle is a geodesic of the Minkowski space.

The velocity of a light ray is the constant c, and so we see from Eq. (10.23) that for such a ray $d\sigma = 0$. Accordingly the world line of a light ray is a null geodesic of the Minkowski space.

10.4.5 Differential Operator

We now consider the differential operator in the four-dimensional space time. By the chain rule of partial differentiation, we have

$$\frac{\partial}{\partial \overline{x}^\alpha} \equiv \frac{\partial x^i}{\partial \overline{x}^\alpha}\frac{\partial}{\partial x^i}. \tag{10.43}$$

A comparison of Eq. (10.43) with Eq. (1.43) shows that $\frac{\partial}{\partial x^i}$ is a *covariant vector operator*. The components of the operator are

$$\frac{\partial}{\partial x^i} \equiv \left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}, \frac{\partial}{\partial x^4}\right) \equiv \left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}, \frac{1}{c}\frac{\partial}{\partial t}\right). \tag{10.44}$$

Noting that, the first three components just make the Galilean gradient operator ∇, we can write

$$\partial_i \equiv \frac{\partial}{\partial x^i} \equiv \left(\nabla, \frac{\partial}{\partial x^4}\right). \tag{10.45}$$

Since $g^{ij} = g_{ij}$, we can write the *contravariant vector operator*

$$\partial^i \equiv \frac{\partial}{\partial x_i} \equiv \left(-\nabla, \frac{\partial}{\partial x^4}\right); \quad x_i = g_{ij}x^j, \tag{10.46}$$

so that $x_i = -x^i$ for $i = 1, 2, 3$ and $x_4 = x^4$. Let $\left(A^1, A^2, A^3, A^4\right)$ be any four vector with respect to the frame S, so that, A^1, A^2, A^3 are the components of a *Galilean vector* $\mathbf{A}$ and A^4 is a *Galilean scalar.* If $\left(A^1, A^2, A^3, A^4\right)$ are contravariant components of A^i, its covariant components A_i are given by Eq. (2.10), by lowering the index. Using the metric tensor of Eq. (10.31), we see that

$$A_1 = -A^1, A_2 = -A^2, \quad A_3 = -A^3, \quad A_4 = A^4. \tag{10.47}$$

Noting that, A^1, A^2, A^3 are the components of a *Galilean vector* $\mathbf{A}$ and A^4 is a *Galilean scalar*, we can write the contravariant and covariant components of a four vector precisely as

$$A^i = \left(\mathbf{A}, A^4\right), \quad A_i = \left(-\mathbf{A}, A^4\right). \tag{10.48}$$

10.5 Maxwell Equations

The classical theory of electrodynamics, according to Lorentz, is specified by the electric potential ϕ, which is a scalar and the magnetic potential A_i which is a vector. The electric field strength vector E_i and the magnetic field strength vector H_i are derived from those potentials by the equations

$$E_i = -\text{grad}\ \phi - \frac{1}{c}\frac{\partial A_i}{\partial t} \tag{10.49}$$

$$H_i = \text{curl}\ A_i. \tag{10.50}$$

We shall denote the components of $\mathbf{E}$ and $\mathbf{H}$ by E_x, E_y, E_z, not by E^1, E^2, E^3, etc., as $\mathbf{E}$ and $\mathbf{H}$ are not parts of a four vector. We can write the x components of Eqs. (10.49) and (10.50), by using Eq. (10.46) as

$$E_x = -\left(-\partial^1 A^4 + \partial^4 A^1\right); \quad H_x = -\left(-\partial^2 A^3 - \partial^3 A^2\right). \tag{10.51}$$

This form of E_x and H_x suggests that we define a second rank tensor,

$$F^{ij} = \partial^i A^j - \partial^j A^i. \tag{10.52}$$

This shows that F^{ij} is an antisymmetric tensor, so that the diagonal elements of F^{ij} vanish. The tensor F^{ij} then takes the form

$$F^{ij} = \begin{pmatrix} -E_x & -E_y & -E_z & 0 \\ 0 & -H_z & H_y & E_x \\ H_z & 0 & -H_x & E_y \\ -H_y & H_x & 0 & E_z \end{pmatrix} \tag{10.53}$$

and is known as the *electromagnetic field strength tensor*. Equation (10.53) gives F^{ij} explicitly in terms of **E** and **H**, while Eq. (10.52) gives F^{ij} in terms of the scalar and the vector potentials Φ and **A**. We can obtain the covariant field strength tensor

$$F_{ij} = g_{ik} g_{jl} F^{kl}. \tag{10.54}$$

Remembering that g_{ij} is given by Eq. (10.31) and is diagonal, we have

$$\begin{aligned} F_{ij} &= g_{ii} g_{jj} F^{ij}; \quad \text{no summation} \\ &= \begin{pmatrix} E_x & E_y & E_z & 0 \\ 0 & -H_z & H_y & -E_x \\ H_z & 0 & -H_x & -E_y \\ -H_y & H_x & 0 & -E_z \end{pmatrix}. \end{aligned} \tag{10.55}$$

We also often dual field strength tensor $\mathcal{F}^{ij}$ defined by

$$\mathcal{F}^{ij} = \frac{1}{2} \varepsilon^{ijkl} F_{kl} = \begin{pmatrix} -H_x & -H_y & -H_z & 0 \\ 0 & -E_z & E_y & H_x \\ -E_z & 0 & E_x & H_y \\ E_y & -E_x & 0 & H_z \end{pmatrix}, \tag{10.56}$$

where ε^{ijkl} is a fully antisymmetric tensor of rank 4. Using electrostatic units, the four Maxwell field equations for electromagnetic field in terms of **E** and **H** are

$$\nabla \cdot \mathbf{E} = 4\pi\varrho; \qquad \text{div } E_i = 4\pi\varrho \tag{10.57}$$

$$\nabla \cdot \mathbf{H} = 0; \qquad \text{div } H_i = 0 \tag{10.58}$$

$$\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{H}}{\partial t} = \vec{0}; \qquad \text{curl } E_i + \frac{1}{c}\frac{\partial H_i}{\partial t} = 0 \tag{10.59}$$

$$\nabla \times \mathbf{H} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi}{c}\vec{J}; \quad \text{curl } H_i - \frac{1}{c}\frac{\partial E_i}{\partial t} = \frac{4\pi}{c} J_i, \tag{10.60}$$

where J_i is the current density vector and ϱ is the charge density. The first and last of the above equations are inhomogeneous equations and contain the components of the four vector J_i on the right-hand side. The middle two equations are homogeneous equations. In Minkowski space, with the metric Eq. (10.30), let us form the four-dimensional potential vector Φ_α and the four-dimensional current density vector J^α defined, respectively, by

$$\Phi_\alpha \equiv (A_1, A_2, A_3, -c\phi); \quad J^\alpha = (J_1, J_2, J_3, \varrho)$$

with respect to a particular co-ordinate system. Using Eq. (10.53) and covariant differential operator Eq. (10.45), Eq. (10.57) can be written as

$$\partial_1 F^{14} + \partial_2 F^{24} + \partial_3 F^{34} = \frac{4\pi}{c} J_4$$

or

$$\partial_i F^{i4} = \frac{4\pi}{c} J_4, \tag{10.61}$$

where we have used the fact $F^{44} = 0$. The x component of Eq. (10.60) can be written as

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} - \frac{1}{c}\frac{\partial E_x}{\partial t} = \frac{4\pi}{c} J_1.$$

Using Eq. (10.53) for F^{ij} and Eq. (10.45) for covariant differential operator, the above expression can be written in the form

$$\partial_2 F^{21} + \partial_3 F^{31} + \partial_4 F^{41} = \frac{4\pi}{c} J_1$$

or

$$\partial_i F^{i1} = \frac{4\pi}{c} J_1$$

or

$$\partial_i F^{ij} = \frac{4\pi}{c} J_j; \quad j = 1, 2, 3, 4. \tag{10.62}$$

Combining to the homogeneous equations, we see that Eq. (10.58) can be written as

$$\frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} = 0.$$

In terms of the tensors F^{ij} or $\mathcal{F}^{ij}$, this equation can be written as

$$\partial_1 F^{32} + \partial_2 F^{13} + \partial_3 F^{21} = 0 \tag{10.63}$$

and

$$\partial_1 \mathcal{F}^{14} + \partial_2 \mathcal{F}^{24} + \partial_3 \mathcal{F}^{34} = 0; \quad \text{i.e. } \partial_i \mathcal{F}^{i4} = 0. \tag{10.64}$$

The x component of Eq. (10.59) is

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} + \frac{1}{c}\frac{\partial H_x}{\partial t} = 0.$$

In terms of Field tensors, this equation takes the form

$$\partial_2 F^{34} + \partial_3 F^{42} + \partial_4 F^{32} = 0 \tag{10.65}$$

or

$$\partial_i \mathcal{F}^{i1} = 0.$$

The generalisations in terms of dual field-strength tensor is immediately obvious and gives

$$\partial_i \mathcal{F}^{ij} = 0, \tag{10.66}$$

which is equivalent to Eqs. (10.58) and (10.59). Thus, the Maxwell's Eqs. (10.57), (10.58), (10.59) and (10.60) can be written in the covariant form as

$$\partial_i F^{ij} = \frac{4\pi}{c} J_j; \quad j = 1, 2, 3, 4$$
$$\partial_i \mathcal{F}^{ij} = 0.$$

While in Eq. (10.63), the indices 1, 2, 3 appear in a cyclic manner in the three terms, we see that 4, 2, 3 in Eq. (10.65) do not appear in a cyclic manner, due to the third term. To rectify this, we use the contravariant differentiation operator rather than covariant operator, so that Eq. (10.65) becomes

$$\partial^2 F^{34} + \partial^3 F^{42} + \partial^4 F^{23} = 0,$$

where the indices 4, 2, 3 appear in a cyclic manner in the three terms. This does not affect Eq. (10.63) in which all the terms get multiplied by -1, so that Eq. (10.63) becomes

$$\partial^1 F^{23} + \partial^2 F^{31} + \partial^3 F^{12} = 0.$$

These above two equations suggest the generalisation

$$\partial^i F^{jk} + \partial^j F^{ki} + \partial^k F^{ij} = 0, \tag{10.67}$$

where i, j, k take any of the possible values 1, 2, 3, 4. We have accordingly written Maxwell's equations in tensor form in Minkowski space. Thus, they are invariant under the Lorentz group of transformations.

10.6 Generalised Relativity

We now turn to the *general theory of relativity* which was developed by Einstein in order to discuss gravitation. He postulated the principle of covariance, which asserts that the laws of physics must be independent of the space-time co-ordinates. This swept away the privileged role of the Lorentz transformation. As a result, Minkowski space was replaced by the metric coefficients of the four-dimensional relativity manifold by $g_{ij}\left(x^1, x^2, x^3, x^4\right)$, with the general metric

$$d\sigma^2 = g_{ij}dx^i dx^j; \quad i,j = 1,2,3,4. \tag{10.68}$$

In the special instance of the restricted theory the form Eq. (10.68) can be reduced by a suitable transformation to the canonical form

$$d\sigma^2 = c^2(dt)^2 - dy^i dy^i. \tag{10.69}$$

Einstein also introduced the principle of equivalence, which in essence states that the fundamental tensor g_{ij} can be chosen to account for the presence of the gravitational field. That is, g_{ij} depends on the distribution of matter and energy in physical sense.

Matter and energy can be specified by the energy momentum tensor T^{ij} which in the special theory satisfies the equation

$$T^{ij}_{,j} = F^j; \;\; F^j = \text{ external force.}$$

The only forces, namely those due to gravitation, are however already taken into account by the choice of the fundamental tensor g_{ij}. We therefore, ignore F^j and, in accordance with the principle of covariance, the energy momentum tensor must now satisfy

$$T^{ij}_{,i} = 0; \text{ equivalently, } T^i_{.j,i} = 0, \tag{10.70}$$

where $T^i_{.j} = g_{jk}T^{ik}$ is the mixed energy momentum tensor. The problem now is to determine $T^i_{.j}$ as a function of the g_{ij} and their derivatives up to the second order, bearing in mind that $T^i_{.j,i} = 0$.

The Riemannian curvature tensor R^i_{jkl}, associated with the manifold of restricted theory, vanishes, and the rectilinear geodesics of the manifold correspond to the trajectories of particles in absence of a gravitational field. Consequently, if the manifold with the metric Eq. (10.68) is to account for non-rectilinear trajectories, the Riemannian curvature tensor must not vanish.

We recall from Eq. (4.45) that Einstein tensor defined by

$$\Gamma^i_j = g^{ik}R_{jk} - \frac{1}{2}R\delta^i_j = R^i_j - \frac{1}{2}\delta^i_j R = 0, \tag{10.71}$$

as we assume, with Einstein, that the field of large gravitating mass is such that the potential functions g_{ij} satisfy in vacuum Eq. (10.71). The equations of motion require $T^{i}_{.j,i} = 0$, but very remarkably $\Gamma^{i}_{.j,i} = 0$, is an identity in Riemannian geometry. If we contract Γ^{i}_{j}, we get,

$$R - \frac{1}{2}4R = 0 \Rightarrow R = 0 \quad \text{i.e. } R_{ij} \equiv R^{\alpha}_{ij\alpha} = 0, \tag{10.72}$$

where R_{ij} the Ricci tensor. The left-hand side of Eq. (10.72) can also be written from Eq. (4.10) as

$$R_{ij} = \frac{\partial^2 \log \sqrt{|g|}}{\partial x^j \partial x^i} - \frac{\partial}{\partial x^\alpha}\left\{ \begin{matrix} \alpha \\ i \;\; j \end{matrix} \right\} + \left\{ \begin{matrix} \alpha \\ \beta \;\; j \end{matrix} \right\}\left\{ \begin{matrix} \beta \\ \alpha \;\; i \end{matrix} \right\} - \left\{ \begin{matrix} \beta \\ i \;\; j \end{matrix} \right\} \frac{\partial \log \sqrt{|g|}}{\partial x^\beta} \tag{10.73}$$

These equations include the flat manifold of restricted theory and admit the case for which the components of the curvature tensor do not vanish. It is obvious from the forgoing that the system of ten nonlinear partial differential equations $R_{ij} = 0$ for the ten unknown functions g_{ij} is extremely complicated. The general solution of this system is not known, and one is obliged to seek particular solutions, essentially by trial, and use Newtonian mechanics as a guide in selecting sensible forms for the coefficients g_{ij}. Once a set of g_{ij}s satisfying Eq. (10.72) is found, we can form the equations of geodesics

$$\frac{d^2x^i}{d\sigma^2} + \left\{ \begin{matrix} i \\ j \;\; k \end{matrix} \right\} \frac{dx^j}{d\sigma}\frac{dx^k}{d\sigma} = 0, \tag{10.74}$$

agrees to the first order of small quantities with the corresponding situations in Newtonian theory, all is well.

In the special theory, the world lines of free particles and of light rays are, respectively, the geodesics and the null geodesics of Minkowski space. The principle of equivalence demands that all particles be regarded as free particles when gravitation is the only force under consideration. Then it follows from the principle of covariance that the world-line of a particle under the action of gravitational forces is a geodesic of the V_4 with the metric [Eq. (10.68)]. Similarly the world line of a light ray is a null geodesic.

10.6.1 Spherically Symmetric Static Field

Generally relativity discusses several important problems in which the co-ordinate system r, θ, ϕ and t is such that the metric in the presence of a spherically symmetrical static gravitational field,

$$d\sigma^2 = c^2 f_1(r)(dt)^2 - f_2(r)(dr)^2 - r^2(d\theta)^2 - r^2 \sin^2\theta(d\phi)^2, \tag{10.75}$$

where f_1 and f_2 are unknown functions of r, each reducing to unity, when r is increased indefinitely. The coefficients of $(dr)^2$ and $(dt)^2$ have been selected as exponentials in order of ensure that the signature of $d\sigma^2$ is -2. Therefore, for the purpose of calculating f_1 and f_2 it is convenient to set,

$$f_1(r) = e^{\mu(r)} = e^{\mu}; \quad f_2(r) = e^{\lambda(r)} = e^{\lambda},$$

where λ and μ are the functions of r. Since effects of the gravitational field diminish as $r \to \infty$, the function λ and μ must be tend to zero when r increases infinitely. Thus, the spherically symmetric metric [Eq. (10.75)] takes the form

$$d\sigma^2 = -e^{\lambda}(dr)^2 - r^2(d\theta)^2 - r^2 \sin^2\theta (d\phi)^2 + c^2 e^{\mu}(dt)^2. \tag{10.76}$$

Let us write, $x^1 = r, x^2 = \theta, x^3 = \phi$ and $x^4 = ct$, then the components of the fundamental tensor are

$$g_{11} = e^{\lambda}; \; g_{22} = -r^2; \; g_{33} = -r^2 \sin^2\theta; \; g_{44} = e^{\mu}$$

and

$$g_{ij} = 0, \text{ for } i \neq j.$$

Therefore,

$$g = g_{11}g_{22}g_{33}g_{44} = -r^4 e^{\lambda+\mu} \sin^2\theta.$$

The covariant tensor g^{ij} is given by the matrix

$$(g^{ij}) = \begin{pmatrix} -e^{-\lambda} & 0 & 0 & 0 \\ 0 & -\dfrac{1}{r^2} & 0 & 0 \\ 0 & 0 & -\dfrac{1}{r^2\sin^2\theta} & 0 \\ 0 & 0 & 0 & e^{-\mu} \end{pmatrix}.$$

The non-vanishing Christoffel symbols of second kind are given by

$$\left\{ {1 \atop 1\ 1} \right\} = \frac{1}{2}\lambda'; \quad \left\{ {2 \atop 1\ 2} \right\} = \frac{1}{r}; \quad \left\{ {3 \atop 1\ 3} \right\} = \frac{1}{r};$$

$$\left\{ {4 \atop 1\ 4} \right\} = \frac{1}{2}\mu'; \quad \left\{ {1 \atop 2\ 2} \right\} = -re^{-\lambda}; \quad \left\{ {3 \atop 2\ 3} \right\} = \cot\theta;$$

$$\left\{ {1 \atop 3\ 3} \right\} = -r\sin^2\theta e^{-\lambda}; \quad \left\{ {2 \atop 3\ 3} \right\} = -\sin\theta\cos\theta; \quad \left\{ {1 \atop 4\ 4} \right\} = \frac{1}{2}e^{\mu-\lambda}\mu';$$

where a dash denotes differentiation with respect to r. We now calculate the components of the Ricci tensor by means of Eq. (10.73) and obtain after tedious but simple calculations of the following set of differential equations:

$$R_{11} = -\frac{1}{r}\lambda' - \frac{1}{4}\lambda'\mu' + \frac{1}{2}\mu'' + \frac{1}{4}(\mu')^2 = 0 \tag{10.77}$$

$$R_{22} = -1 + e^{-\lambda}\left[1 - \frac{1}{2}r\lambda' + \frac{1}{2}r\mu'\right] = 0 = \text{cosec}^2\theta R_{33} \tag{10.78}$$

$$R_{44} = e^{-\lambda+\mu}\left[\frac{1}{4}\lambda'\mu' - \frac{1}{2}\mu'' - \frac{1}{r}\mu' - \frac{1}{4}(\mu')^2\right] = 0 \tag{10.79}$$

and

$$R_{ij} = 0, \text{ for } i \neq j.$$

These lead the curvature invariant

$$R = \frac{2}{r^2} + e^{-\lambda}\left[-\frac{2}{r^2} + \frac{2}{r}\lambda' + \frac{1}{2}\lambda'\mu' - \mu'' - \frac{2}{r}\mu' - \frac{1}{2}(\mu')^2\right].$$

Therefore, the necessary and sufficient condition that a space with a spherically symmetric metric be an Einstein space is

$$\frac{2}{r^2} + e^{-\lambda}\left[-\frac{2}{r^2} + \frac{2}{r}\lambda' + \frac{1}{2}\lambda'\mu' - \mu'' - \frac{2}{r}\mu' - \frac{1}{2}(\mu')^2\right] = 0.$$

The components of the Einstein tensor for the spherically symmetric metric Eq. (10.76) are given by

$$\Gamma^1_{.1} = -\frac{1}{r^2} + e^{-\lambda}\left[\frac{1}{r^2} + \frac{1}{r}\mu'\right]$$

$$\Gamma^2_{.2} = \Gamma^3_{.3} = e^{-\lambda}\left[-\frac{1}{2r}\lambda' - \frac{1}{4}\lambda'\mu' + \frac{1}{2}\mu'' + \frac{1}{2r}\mu' + \frac{1}{4}(\mu')^2\right]$$

$$\Gamma^4_{.4} = -\frac{1}{r^2} + e^{-\lambda}\left[\frac{1}{r^2} - \frac{1}{r}\lambda'\right]; \quad \Gamma^i_{.j} = 0; \quad \text{for } i \neq j.$$

We now seek the spherically symmetric metric Eq. (10.76) consistent with the existence of one gravitating point particle situated at the origin, and surrounded by empty space. When the origin itself is excluded from our discussion, the energy momentum tensor $T^i_{,j}$ is zero at all points, so that $\Gamma^i_{,j} = 0$. Now, from Eqs. (10.77) and (10.79), we get

$$\lambda' = -\mu' \Rightarrow \lambda = -\mu + \text{ constant}.$$

Since as $r \to \infty$; $\lambda, \mu \to 0$, so $\lambda(r) = -\mu(r)$. Thus, Eq. (10.78) becomes

$$e^{\mu}\left(1 + r\mu'\right) = 1$$
$$\Rightarrow \gamma + \gamma\mu' = 1; \quad e^{\mu} = \gamma$$
$$\Rightarrow \gamma = 1 - \frac{2m}{c^2 r} \equiv e^{\mu},$$

where $2m$ is a constant of integration. We shall identify m physically with the rest mass of the graviting particle. Noting that $e^{\mu} = e^{-\lambda} = \gamma$, the metric Eq. (10.76) is given by

$$\begin{aligned} d\sigma^2 &= -\gamma^{-1}(dr)^2 - r^2(d\theta)^2 - r^2\sin^2\theta(d\phi)^2 + \gamma(dt)^2, \qquad (10.80)\\ &= -\left(1 - \frac{2m}{c^2 r}\right)^{-1}(dr)^2 - r^2(d\theta)^2 - r^2\sin^2\theta(d\phi)^2 + c^2\left(1 - \frac{2m}{c^2 r}\right)(dt)^2, \end{aligned}$$

where $\gamma = 1 - \frac{2m}{c^2 r}$ and is known as *Schwarzschild metric.* If the constant of integration $2m$ vanishes, $\gamma = 1$, and the resulting manifold is the flat manifold of restricted theory. For $m \neq 0$, the manifold is a curve. The solution obtained is of interest because it is only static solution of our equations satisfying specified boundary conditions at infinity.

10.6.2 Planetary Motion

Let us investigate the motion of a planet in the gravitational field of the sun. The sun will be selected as a gravitating particle and the planet as a free particle whose mass is so small that it does not affect the metric, and whose world line is then a geodesic in the V_4 with the Schwarzschild metric [Eq. (10.80)], given by

$$\begin{aligned} -\left(1 - \frac{2m}{c^2 r}\right)^{-1}\left(\frac{dr}{d\sigma}\right)^2 - r^2\left(\frac{d\theta}{d\sigma}\right)^2 - r^2\sin^2\theta\left(\frac{d\phi}{d\sigma}\right)^2 \\ + c^2\left(1 - \frac{2m}{c^2 r}\right)\left(\frac{dt}{d\sigma}\right)^2 = 1. \qquad (10.81) \end{aligned}$$

The trajectory of the particle is a geodesic, so we have to solve the set of four equations

$$\frac{d^2x^i}{d\sigma^2} + \left\{\begin{matrix} i \\ j \quad k \end{matrix}\right\}\frac{dx^j}{d\sigma}\frac{dx^k}{d\sigma} = 0,$$

where, $x^1 = r, x^2 = \theta, x^3 = \phi$ and $x^4 = ct$. We shall omit one of these equations, in practice the most formidable one involving $\frac{d^2r}{d\sigma^2}$. Thus, we no longer require the

Christoffel symbols of the type $\left\{ \begin{matrix} 1 \\ i \;\; j \end{matrix} \right\}$. The remaining non-vanishing symbols of the second kind are

$$\left\{ \begin{matrix} 2 \\ 1 \;\; 2 \end{matrix} \right\} = \frac{1}{r} = \left\{ \begin{matrix} 3 \\ 1 \;\; 3 \end{matrix} \right\}; \; \left\{ \begin{matrix} 3 \\ 2 \;\; 3 \end{matrix} \right\} = \cot\theta;$$

$$\left\{ \begin{matrix} 2 \\ 3 \;\; 3 \end{matrix} \right\} = -\sin\theta\cos\theta; \; \left\{ \begin{matrix} 4 \\ 1 \;\; 4 \end{matrix} \right\} = \frac{2m}{c^2r^2}\left(1 - \frac{2m}{c^2r}\right)^{-1}.$$

Thus, the equations of the geodesic are

$$\frac{d^2\theta}{d\sigma^2} + \frac{2}{r}\frac{dr}{d\sigma}\frac{d\theta}{d\sigma} - \sin\theta\cos\theta\left(\frac{d\phi}{d\sigma}\right)^2 = 0 \tag{10.82}$$

$$\frac{d^2\phi}{d\sigma^2} + \frac{2}{r}\frac{dr}{d\sigma}\frac{d\phi}{d\sigma} + 2\cot\theta\frac{d\phi}{d\sigma}\frac{d\theta}{d\sigma} = 0 \tag{10.83}$$

$$\frac{d^2t}{d\sigma^2} + \frac{2m}{c^2r^2}\left(1 - \frac{2m}{c^2r}\right)^{-1}\frac{dr}{d\sigma}\frac{dt}{d\sigma} = 0. \tag{10.84}$$

We may assume that the planet moves initially in the plane $\theta = \frac{\pi}{2}$. That is, $\frac{d\theta}{d\sigma}$ and $\cos\theta$ are both initially zero. Then Eq. (10.82) tells us that $\frac{d^2\theta}{d\sigma^2}$ is also zero. Repeated differentiation of this equation shows that $\frac{d^i\theta}{d\sigma^i}$ vanishes at $t = 0$ for all i. Hence, $\theta = \frac{\pi}{2}$ permanently, Eqs. (10.81), (10.83), (10.84) simplify to

$$-\left(1 - \frac{2m}{c^2r}\right)^{-1}\left(\frac{dr}{d\sigma}\right)^2 - r^2\left(\frac{d\phi}{d\sigma}\right)^2 + c^2\left(1 - \frac{2m}{c^2r}\right)\left(\frac{dt}{d\sigma}\right)^2 = 1 \tag{10.85}$$

$$\frac{d^2\phi}{d\sigma^2} + \frac{2}{r}\frac{dr}{d\sigma}\frac{d\phi}{d\sigma} = 0 \tag{10.86}$$

$$\frac{d^2t}{d\sigma^2} + \frac{2m}{c^2r^2}\left(1 - \frac{2m}{c^2r}\right)^{-1}\frac{dr}{d\sigma}\frac{dt}{d\sigma} = 0. \tag{10.87}$$

We can immediately integrate Eqs. (10.86) and (10.87) and the results are

$$r^2\frac{d\phi}{d\sigma} = h; \quad \left(1 - \frac{2m}{c^2r}\right)\frac{dt}{d\sigma} = k, \tag{10.88}$$

where h and k are constants. On eliminating t and σ from Eqs. (10.85) and (10.88) we obtain,

$$-\frac{1}{r^4}\left(\frac{dr}{d\phi}\right)^2 - \frac{1}{r^2}\left(1 - \frac{2m}{c^2r}\right) + \frac{c^2k^2}{h^2} = \frac{1}{h^2}\left(1 - \frac{2m}{c^2r}\right)$$

or

$$\frac{d^2u}{d\phi^2} + u = \frac{m}{c^2h^2}(1 + 3h^2u^2); \quad u = \frac{1}{r}, \tag{10.89}$$

which is the relativistic equation for the orbit of a planet. For the planets of our solar system, the term $\frac{m}{c^2h^2}$ is much larger than $\frac{3mu^2}{c^2}$ and this justifies us in attempting to obtain a solution of this equation by the method of perturbations. But when we neglect this latter term, we obtain Newton's equation for the motion of a planet. Accordingly, we neglect the small term $\frac{3mu^2}{c^2}$ and obtain for our first approximation u_1 the Newtonian equation

$$\frac{d^2u_1}{d\phi^2} + u_1 = \frac{m}{c^2h^2} \Rightarrow u_1 = \frac{m}{c^2h^2}[1 + e\cos(\phi - \omega)], \tag{10.90}$$

where e is the eccentricity of the elliptic orbit and ω is the longitude of the perihelion. Putting Eq. (10.90) in the right-hand side of Eq. (10.89), we get

$$\begin{aligned}\frac{d^2u}{d\phi^2} + u &= \frac{m}{c^2h^2}(1 + 3h^2u_1^2) \\ &= \frac{m}{c^2h^2} + \frac{6m^3}{c^2h^4}e\cos(\phi - \omega) + \frac{3m^3e^2}{2c^2h^4}[1 + \cos 2(\phi - \omega)] + \frac{3m^3}{c^2h^4}. \end{aligned} \tag{10.91}$$

Since planetary orbits are nearly circular, the contribution of the perturbation term containing e^2 will be negligible. Also, the term $\frac{3m^3}{c^2h^4}$ will not have significant effect on the shape of the orbit, but the second term, containing $\cos(\phi - \omega)$, may have a pronounced cumulative effect on the displacement of the perihelion. Thus, Eq. (10.91) becomes

$$\frac{d^2u}{d\phi^2} + u = \frac{m}{c^2h^2} + \frac{6m^3}{c^2h^4}e\cos(\phi - \omega).$$

The solution of this linear equation is clearly made up of the solution u_1 and the solution of

$$\frac{d^2u}{d\phi^2} + u = \frac{6m^3}{c^2h^4}e\cos(\phi - \omega).$$

A second approximation u_2 to the solution can then be obtained in the form

$$u_2 = \frac{m}{c^2h^2}\left[1 + e\cos(\phi - \omega) + \frac{3m^2}{c^4h^2}e\phi\sin(\phi - \omega)\right].$$

Let us introduce, $\Delta\omega = \frac{3m^2\phi}{c^4h^2}$ and note that

$$\cos(\phi - \omega) + \Delta\omega\phi\sin(\phi - \omega) = \sqrt{1 + (\Delta\omega)^2} \quad \cos(\phi - \omega - \alpha),$$

where $\alpha = \tan^{-1}\Delta\omega \approx \Delta\omega$, the approximation u_2 can be written in the form

$$u_2 \approx \frac{m}{c^2h^2}\left[1 + e\cos(\phi - \omega - \Delta\omega)\right]. \tag{10.92}$$

This means that the major axis of the elliptic orbit is slowly rotating about its focus (the sun). The increase of $\Delta\omega$ corresponding to a complete revolution $\phi = 2\pi$ is thus $\frac{3m^2}{c^4h^2}2\pi$ rad. Equation (10.92) represents a closed orbit, only approximately elliptical in shape, because $\Delta\omega$ is a function of ϕ. Since $u = \frac{1}{r}$, we have

$$\frac{c^2h^2/m}{r} = 1 + e\cos(\phi - \omega - \Delta\omega),$$

where the semi latus rectum $l = \frac{c^2h^2}{m}$.

10.6.3 Einstein Universe

Einstein was led by cosmological considerations to consider the universe with the metric

$$d\sigma^2 = -\left(1 - \frac{r^2}{\mathcal{R}^2}\right)^{-1}(dr)^2 - r^2(d\theta)^2 - r^2\sin^2\theta(d\phi)^2 + c^2(dt)^2, \qquad (10.93)$$

where $\mathcal{R}$ is a constant. Thus metric is spherically symmetrical with $e^{-\lambda} = \left(1 - \frac{r^2}{\mathcal{R}^2}\right)$ and $\mu = 0$. The non-vanishing Christoffel symbols of second kind are given by

$$\begin{Bmatrix} 1 \\ 1\ 1 \end{Bmatrix} = \frac{r}{\mathcal{R}^2}\left(1 - \frac{r^2}{\mathcal{R}^2}\right)^{-1}; \begin{Bmatrix} 2 \\ 1\ 2 \end{Bmatrix} = \frac{1}{r}; \begin{Bmatrix} 3 \\ 1\ 3 \end{Bmatrix} = \frac{1}{r};$$
$$\begin{Bmatrix} 1 \\ 2\ 2 \end{Bmatrix} = -r\left(1 - \frac{r^2}{\mathcal{R}^2}\right); \begin{Bmatrix} 3 \\ 2\ 3 \end{Bmatrix} = \cot\theta;$$
$$\begin{Bmatrix} 1 \\ 3\ 3 \end{Bmatrix} = -r\sin^2\theta\left(1 - \frac{r^2}{\mathcal{R}^2}\right); \begin{Bmatrix} 2 \\ 3\ 3 \end{Bmatrix} = -\sin\theta\cos\theta.$$

Let us investigate the path of a ray of light in Einstein's universe. The path must be a null geodesic and so its equations are given by three of the four equations of Eq. (4.64) taken together with the equation

$$\frac{d^2\theta}{du^2} + \frac{2}{r}\frac{dr}{du}\frac{d\theta}{du} - \sin\theta\cos\theta\left(\frac{d\phi}{du}\right)^2 = 0 \qquad (10.94)$$
$$\frac{d^2\phi}{du^2} + \frac{2}{r}\frac{dr}{du}\frac{d\phi}{du} + 2\cot\theta\frac{d\phi}{du}\frac{d\theta}{du} = 0; \quad \frac{d^2t}{du^2} = 0$$
$$-\left(1 - \frac{r^2}{\mathcal{R}^2}\right)^{-1}\left(\frac{dr}{du}\right)^2 - r^2\left(\frac{d\theta}{du}\right)^2 - r^2\sin^2\theta\left(\frac{d\phi}{du}\right)^2 + c^2\left(\frac{dt}{du}\right)^2 = 0,$$

where u is some parameter. Again, following the argument as in Section 10.5, Eq. (10.94) tells us that we can take θ to have the permanent value $\frac{\pi}{2}$. With this choice, the equations reduces to

$$\frac{d^2\phi}{du^2} + \frac{2}{r}\frac{dr}{du}\frac{d\phi}{du} = 0; \quad \frac{d^2t}{du^2} = 0 \tag{10.95}$$

$$-\left(1 - \frac{r^2}{\mathcal{R}^2}\right)^{-1}\left(\frac{dr}{du}\right)^2 - r^2\left(\frac{d\phi}{du}\right)^2 + c^2\left(\frac{dt}{du}\right)^2 = 0. \tag{10.96}$$

We integrate Eq. (10.95) and obtain

$$r^2\frac{d\phi}{du} = h; \quad \frac{dt}{du} = k \Rightarrow \frac{dt}{d\phi} = \frac{k}{h}r^2, \tag{10.97}$$

where h and k are constants, and then eliminate t and u from these equations and Eq. (10.96). The result is

$$\left(\frac{dr}{d\phi}\right)^2 = r^2\left(1 - \frac{r^2}{\mathcal{R}^2}\right)\left(\frac{c^2k^2}{h^2}r^2 - 1\right). \tag{10.98}$$

The solution of Eq. (10.98) is

$$\frac{1}{r^2} = \frac{1}{\mathcal{R}^2}\cos^2(\phi - \xi) + \frac{c^2k^2}{h^2}\sin^2(\phi - \xi), \tag{10.99}$$

where ξ is a constant. We immediately see that r regains its initial value when ϕ is increased by π and that r is never infinite for any value of ϕ. Thus, all the null geodesics of the Einstein's universe, that is the light rays, are closed curves. Hence, the time taken for a light ray to make a complete circuit is given by, from Eq. (10.97), as

$$T = \frac{k}{h}\int_0^{2\pi}\left[\frac{1}{\mathcal{R}^2}\cos^2(\phi - \xi) + \frac{c^2k^2}{h^2}\sin^2(\phi - \xi)\right]^{-1} d\phi.$$

Owing to the periodicity of ϕ, we have

$$T = \frac{k}{h}\int_0^{2\pi}\left[\frac{1}{\mathcal{R}^2}\cos^2\phi + \frac{c^2k^2}{h^2}\sin^2\phi\right]^{-1} d\phi$$

$$= \frac{4k}{h}\int_0^{\pi/2}\left[\frac{1}{\mathcal{R}^2}\cos^2\phi + \frac{c^2k^2}{h^2}\sin^2\phi\right]^{-1} d\phi = \frac{2\pi\mathcal{R}}{c}.$$

Other cosmological considerations suggested to DeSitter that the universe could be described by the spherically symmetry metric

$$d\sigma^2 = -\left(1 - \frac{r^2}{\mathcal{R}^2}\right)^{-1}(dr)^2 - r^2(d\theta)^2 - r^2\sin^2\theta(d\phi)^2 + c^2\left(1 - \frac{r^2}{\mathcal{R}^2}\right)(dt)^2, \tag{10.100}$$

where the constant $\mathcal{R}$ has not the same value as the corresponding constant of the Einstein universe. In this case the geodesic is given by

$$\frac{1}{r} = a \quad \cos(\phi - \xi),$$

where ξ is a constant. These trajectories correspond to straight lines and arc not closed, since r becomes infinite when $\phi - \xi = \pi/2$.

10.7 Exercises

1. Verify that the following matrices are Lorentz and calculate the velocity between the two observers

$$\text{(i)} \begin{pmatrix} \sqrt{2} & 0 & 0 & \sqrt{3} \\ \frac{\sqrt{6}}{2} & \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{\sqrt{6}}{2} & -\frac{1}{2} & -\frac{1}{2} & 1 \\ 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \end{pmatrix}; \quad \text{(ii)} \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & -\frac{1}{2} & \frac{5}{4} \\ -\frac{5}{6} & -\frac{5}{12} & \frac{5}{6} & -\frac{3}{4} \\ \frac{2}{3} & \frac{2}{15} & \frac{11}{15} & 0 \\ -\frac{1}{3} & \frac{14}{15} & \frac{2}{15} & 0 \end{pmatrix}.$$

2. A relativistic transformation of the space time co-ordinates of two inertial systems whose relative motion is parallel to x^1 axis is

$$\overline{x}^1 = \gamma_1(x^1 - \beta_1 x^4), \overline{x}^2 = x^2, \overline{x}^3 = x^3, \overline{x}^4 = \gamma_1(x^4 - \beta_1 x^1),$$

where $\beta_1 = \frac{\gamma_1}{c}, \gamma_1 = (1 - \beta_1^2)^{-1/2}$ and γ_1 is the relative velocity between the two frames. If we express the co-ordinate four vectors as column vectors $\mathbf{x} = \{x^1, x^2, x^3, x^4\}), \overline{\mathbf{x}} = \{\overline{x}^1, \overline{x}^2, \overline{x}^3, \overline{x}^4\})$, write this transformation through a matrix equation in the form $\overline{\mathbf{x}} = A_1\mathbf{x}$ and determine A_1.

3. Show that the expression $dx^2 + dy^2 + dz^2 - c^2dt^2$ is invariant under Lorentz transformations.

4. Prove that the four-dimensional volume element $dxdydzdt$ is invariant under Lorentz transformation.

5. Show that, under the Lorentz transformation, the wave equation for the propagation of the electromagnetic potential $\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = 0$, where $\triangle^2 = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$ is the D'Alembert operator remains invariant.

6. Find the relativistic expression for KE of a particle, whose rest mass is m_0, moving with velocity $\mathbf{v}$. Obtain the relation $E^2 = p^2c^2 + m_0^2c^4$. Show also that, in the limit of low velocities the usual expression for KE can be obtained from the relativistic expression of it.

7. What should be the speed of a particle of rest mass m_0 in order that its relativistic momentum is m_0c. Show that its total relativistic energy is $\sqrt{2}m_0c^2$.
8. A rod of proper length L rests in the xy plane inclined at an angle θ to the x axis. What does the observer moving with speed v along the x axis. Find its length and the angle of inclination.
9. Show that the metric of the Minkowski space in spherical polar co-ordinates (r, θ, ϕ) can be written as

$$d\sigma^2 = c^2dt^2 - dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\phi^2.$$

10. Consider the transformation

$$x^1 = r\sin\theta\cos\phi\sinh\psi; \quad x^2 = r\sin\theta\sin\phi\sinh\psi;$$
$$x^3 = r\cos\theta\sinh\psi; \quad x^4 = r\cosh\psi$$

form the co-ordinates $\left(x^1, x^2, x^3, x^4\right)$ to (r, θ, ϕ, ψ) in Minkowski space.

(i) Obtain the metric in terms of (r, θ, ϕ, ψ).
(ii) Obtain the inverse transformation giving (r, θ, ϕ, ψ) in terms of (x^1, x^2, x^3, x^4).

11. Find the necessary and sufficient conditions that a space with a spherically symmetric metric be an Einstein space.
12. Show that a space with Schwartzschild's metric is an Einstein space, but not a space of constant curvature.
13. Show that the Einstein universe is neither an Einstein space nor a space of constant curvature.
14. Prove that the curvature invariant of Einstein's universe is $R = \frac{6}{\mathcal{R}^2}$.
15. Show that the DeSitter universe is an Einstein space with constant curvature $\frac{12}{\mathcal{R}^2}$.

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