Introduction

Fractional calculus is a generalization of differentiation and integration to the arbitrary (noninteger) order fundamental operator D^{α}_{a+}, where α, a, $\in \mathbb{R}$. The concept of fractional differential and integral equations has a long history. One may wonder what meaning may be ascribed to the derivative of a fractional order, that is, $\frac{d^n y}{dx^n}$, where n is a fraction. In fact, the French mathematician l'Hôpital himself considered this very possibility in a correspondence with Leibniz. In 1695, in a letter to l'Hôpital, Leibniz raised the following question: *Can the meaning of derivatives with integer order be generalized to derivatives with noninteger orders?* l'Hôpital was somewhat curious about that question and replied with another question to Leibniz: "*What if the order is* $\frac{1}{2}$?" In a letter dated September 30, Leibniz replied: "$d^{\frac{1}{2}}x$ *would be equal to* $x\sqrt{dx : x}$. *This is an apparent paradox from which, one day, useful consequences will be drawn.*" Thus, September 30, 1695, marks the exact date of birth of the *fractional calculus*! Therefore, the fractional calculus has its origin in the works of Leibniz, l'Hôpital (1695), Bernoulli (1697), Euler (1730), and Lagrange (1772). Some years later, Laplace (1812), Fourier (1822), Abel (1823), Liouville (1832), Riemann (1847), Grünwald (1867), Letnikov (1868), Nekrasov (1888), Hadamard (1892), Heaviside (1892), Hardy (1915), Weyl (1917), Riesz (1922), P. Levy (1923), Davis (1924), Kober (1940), Zygmund (1945), Kuttner (1953), J. L. Lions (1959), Liverman (1964), and others developed the basic concept of fractional calculus.

Several approaches to fractional derivatives exist, for example, Riemann–Liouville (RL), Hadamard, Grunwald–Letnikov (GL), Weyl, and Caputo. The Caputo fractional derivative is well suited to the physical interpretation of initial conditions and boundary conditions. We refer readers, for example, to the books [35, 63, 78, 137, 181, 187, 200, 209, 210, 219, 239], the articles [46, 43, 47, 73, 72, 85, 91, 89, 94, 97, 101, 102, 103, 104, 180, 241], and references therein.

In 1783, Leonhard Euler made his first comments on fractional order derivatives. He worked on progressions of numbers and introduced for the first time the generalization of factorials to the *gamma* function. A little more than 50 years after the death of Leibniz, Lagrange, in 1772, indirectly contributed to the development of exponent laws for differential operators of integer order, which can be transferred to arbitrary order under certain conditions. In 1812, Laplace provided the first detailed definition of a fractional derivative. Laplace states that a fractional derivative can be defined for functions with representation by an integral; in modern notation it can be written as $\int y(t)t^{-x}dt$. A few years later, Lacroix worked on generalizing the integer order derivative of the function $y(t) = t^m$ to fractional order, where m is some natural number. In modern notation, the integer order nth derivative derived by Lacroix can be given as

$$\frac{d^n y}{dt^n} = \frac{m!}{(m-n)!}t^{m-n} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)}t^{m-n}, \quad m > n\,,$$

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where Γ is *Euler's gamma function* defined by

$$\Gamma(\varsigma) = \int_0^\infty t^{\varsigma-1} e^{-t} dt, \quad \varsigma > 0 .$$

Thus, replacing n with $\frac{1}{2}$ and letting $m = 1$, one obtains the derivative of order $\frac{1}{2}$ of the function y:

$$\frac{d^{\frac{1}{2}} y}{dt^{\frac{1}{2}}} = \frac{\Gamma(2)}{\Gamma\left(\frac{3}{2}\right)} t^{\frac{1}{2}} = \frac{2}{\sqrt{\pi}} \sqrt{t} .$$

Euler's gamma function (or Euler's integral of the second kind) has the same importance in fractional order calculus, and it is basically given by the integral

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt .$$

The exponential provides the convergence of this integral at ∞. The convergence at zero obviously occurs for all complex z from the right half of the complex plane ($\mathrm{Re}(z) > 0$).

This function is a generalization of a factorial in the following form:

$$\Gamma(n) = (n-1)! .$$

Other generalizations for values in the left half of the complex plane can be obtained in the following way. If we replace e^{-t} by the well-known limit

$$e^{-t} = \lim_{n\to\infty} \left(1 - \frac{t}{n}\right)^n$$

and then use n-times integration by parts, we obtain the following limit definition of the gamma function:

$$\Gamma(z) = \lim_{n\to\infty} \frac{n! n^z}{z(z+1)\ldots(z+n)} .$$

Therefore, historically the first discussion of a derivative of fractional order appeared in a calculus written by Lacroix in 1819.

It was Liouville who engaged in the first major study of fractional calculus. Liouville's first definition of a derivative of arbitrary order v involved an infinite series. Here, the series must be convergent for some v. Liouville's second definition succeeded in giving a fractional derivative of x^{-a} whenever both x and a are positive. Based on the definite integral related to Euler's gamma integral, the integral formula can be calculated for x^{-a}. Note that in the integral

$$\int_0^\infty u^{a-1} e^{-xu} du ,$$

if we make change the variables $t = xu$, then

$$\int_0^\infty u^{a-1} e^{-xu} du = \int_0^\infty \left(\frac{t}{x}\right)^{a-1} e^{-t} \frac{1}{x} dt = \frac{1}{x^a} \int_0^\infty t^{a-1} e^{-t} dt \, .$$

Thus,

$$\int_0^\infty u^{a-1} e^{-xu} du = \frac{1}{x^a} \int_0^\infty t^{a-1} e^{-t} dt \, .$$

Through the gamma function we obtain the integral formula

$$x^{-a} = \frac{1}{\Gamma(a)} \int_0^\infty u^{a-1} e^{-xu} du \, .$$

Consequently, by assuming that $\frac{d^\nu}{dx^\nu} e^{ax} = a^\nu e^{ax}$, for any $\nu > 0$,

$$\frac{d^\nu}{dx^\nu} x^{-a} = \frac{\Gamma(a+\nu)}{\Gamma(a)} x^{-a-\nu} = (-1)^\nu \frac{\Gamma(a+\nu)}{\Gamma(a)} x^{-a-\nu} \, .$$

In 1884, Laurent published what is now recognized as the definitive paper on the foundations of fractional calculus. Using Cauchy's integral formula for complex valued analytical functions, and a simple change of notation to employ a positive ν rather than a negative ν, will now yield Laurent's definition of integration of arbitrary order

$${}_{x_0}D_x^\alpha h(x) = \frac{1}{\Gamma(\nu)} \int_{x_0}^x (x-t)^{\nu-1} h(t) dt \, .$$

The Riemann–Liouville differential operator of fractional calculus of order α is defined as

$$(D_{a+}^\alpha f)(t) := \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \left(\dfrac{d}{dt}\right)^n \displaystyle\int_a^t (t-s)^{n-\alpha-1} f(s) ds, & \text{if } n-1 < \alpha < n \, , \\ \left(\dfrac{d}{dt}\right)^n f(t), & \text{if } \alpha = n \, , \end{cases}$$

where $\alpha, a, t \in \mathbb{R}$, $t > a$, $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of the real number α, and Γ is the gamma function.

The Grünwald–Letnikov differential operator of fractional calculus of order α is defined as

$$(D_{a+}^\alpha f)(t) := \lim_{h \to 0} h^{-\alpha} \sum_{j=0}^{[\frac{t-a}{h}]} (-1)^j \tbinom{\alpha}{j} f(t - jh) \, .$$

Binomial coefficients with alternating signs for positive values of n are defined as

$$\binom{n}{j} = \frac{n(n-1)(n-2)\ldots(n-j+1)}{j!} = \frac{n!}{j!(n-j)!} \, .$$

For binomial coefficient calculations, we can use the relation between Euler's gamma function and factorials given by

$$\binom{\alpha}{j} = \frac{\alpha!}{j!(\alpha-j)!} = \frac{\Gamma(\alpha)}{\Gamma(j+1)\Gamma(\alpha-j+1)} .$$

The Grünwald–Letnikov definition of differintegral starts from classical definitions of derivatives and integrals based on infinitesimal division and limit. The disadvantages of this approach are its technical difficulty in computations and in proofs and with the large restrictions on functions (see [262]).

The Caputo (1967) differential operator of fractional calculus of order α is defined as

$$({}^{c}D_{a+}^{\alpha}f)(t) := \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_{a}^{t} (t-s)^{n-\alpha-1} f^{(n)}(s)ds, & \text{if } n-1 < \alpha < n\,, \\ \left(\dfrac{d}{dt}\right)^{n} f(t), & \text{if } \alpha = n\,, \end{cases}$$

where $\alpha, a, t \in \mathbb{R}$, $t > a$, and $n = [\alpha] + 1$. This operator was introduced in 1967 by the Italian mathematician Caputo.

This consideration is based on the fact that for a wide class of functions, the three best known definitions (GL, RL, and Caputo) are equivalent under some conditions (see ([160]). Unfortunately, fractional calculus still lacks a geometric interpretation of integration or differentiation of arbitrary order. We refer readers, for example, to books such as [23, 36, 35, 78, 161, 181, 187, 200, 209, 219, 239], the articles [46, 43, 47, 73, 72, 85, 89, 94, 97, 101, 102, 103, 180, 241], and the references therein.

In June 1974, Ross organized the "*First Conference on Fractional Calculus and Its Applications*" at the University of New Haven and edited its proceedings [227]. Subsequently, in 1974, Spanier published the first monograph devoted to *Fractional Calculus* [209]. Integrals and derivatives of noninteger order and fractional integrodifferential equations have found many applications in recent studies in theoretical physics, mechanics, and applied mathematics. There is a remarkably comprehensive encyclopedic-type monograph by Samko, Kilbas, and Marichev that was published in Russian in 1987 and in English in 1993 [239] (for more details see [197]). Works devoted largely to fractional differential and integral equations include the book by Miller and Ross (1993) [200], Podlubny (1999) [219], Kilbas et al. (2006) [181], Diethelm (2010) [137], Mainardi (2010) [198], Ortigueira (2011) [210], Abbas et al. (2012, 2015) [23, 36, 35], Baleanu et al. (2012) [78], Zhou (2014, 2016) [263, 264], Almeida et al. (2015) [59], Sabatier et al. (2015) [237], Povstenko (2015) [222, 221], Umarov (2015) [246], Cattani et al. (2016) [122], Goodrich and Peterson (2016) [144], and Uchaikin and Sibatov (2016) [243].

Since the second half of the twentieth century, the study of fractional differential and integral equations has made great strides (Oldham and Spanier 1974, Samko et al. 1993, Miller and Ross 1993, Kiryakova 1994, Gorenflo and Mainardi 1997, Podlubny 1999, Kilbas et al. 2006). Thanks to these advances, fractional differentiation has been applied in many areas: Electrical engineering (modeling of motors, modeling of transformers, skin effect), electronics, telecommunications (phase locking loops), electromagnetism (modeling of complex dielectric materials), electrochemistry (modeling of batteries, fuel cells, and ultracapacitors), thermal engineering (modeling and identification of thermal systems), mechanics, mechatronics (vibration insulation, suspension), rheology (behavior identification of materials, viscoelastic properties), automatic control (fractional order PID, robust control, system identification, observation and control of fractional systems), robotics (modeling, path tracking, path planning, obstacle avoidance), signal processing (filtering, restoration, reconstruction, analysis of fractal noise), image processing (fractal environment modeling, pattern recognition, edge detection), biology, biophysics (electrical conductance of biological systems, fractional modeling of neurons, muscle modeling, lung modeling), physics (analysis and modeling of diffusion phenomenon), and economics (analysis of stock exchange signals). In these applications, fractional differentiation is often used to model phenomena that exhibit nonstandard dynamical behaviors with long memory or with hereditary effects. We will now present a brief survey of applications of fractional calculus in science and engineering.

The Tautochrone Problem. This example was studied for the first time by Abel in the early nineteenth century. It was one of the basic problems where the framework of the fractional calculus was used, although it is not essentially necessary.

Signal and Image Processing. In the last decade, the use of fractional calculus in *signal processing* has increased tremendously. In signal processing, the fractional operators are used in the design of differentiators and integrators of fractional order, fractional order differentiator *FIR* (finite impulse response), *infinite impulse response* (IIR)-type digital fractional order differentiator, a new *IIR*-type digital fractional order differentiator (*DFOD*), and for modeling speech signals. The fractional calculus allows for edge detection, enhances the quality of images, and has interesting possibilities in various image enhancement applications such as image restoration, image denoising, and texture enhancement. It is used, in particular, in satellite image classification and astronomical image processing.

Electromagnetic Theory. The use of fractional calculus in electromagnetic theory has emerged in the last two decades. In 1998, Engheta [139] introduced the concept of fractional curl operators, and this concept was extended by Naqvi and Abbas [205]. Engheta's work gave birth to a new field of research in electromagnetics, namely, *fractional paradigms in electromagnetic theory*. Nowadays fractional calculus is widely used in electromagnetics to explore new results; for example, Faryad and Naqvi [141] have used fractional calculus for the analysis of a rectangular waveguide.

Control Engineering. In industrial environments, robots must execute their tasks quickly and precisely, minimizing production time, and the robustness of control systems is becoming imperative these days. This requires flexible robots working in large workspaces, which means that they are influenced by nonlinear and fractional order dynamic effects.

Biological Population Models. The problems of the diffusion of biological populations occur nonlinearly, and fractional order differential equations are appearing more and more frequently in various research areas.

Reaction–Diffusion Equations. Fractional equations can be used to describe some physical phenomena more accurately than classical integer order differential equations. Reaction–diffusion equations play an important role in dynamical systems in mathematics, physics, chemistry, bioinformatics, finance, and other research areas. There has been a wide variety of analytical and numerical methods proposed for fractional equations [196, 258], for example, finite difference methods [127], finite element methods, the Adomian decomposition method [226], and spectral techniques [194]. Interest in fractional reaction–diffusion equations has increased.

In recent years, there has been a significant development in the theory of fractional differential and integral equations. It was brought about by its applications in the modeling of many phenomena in various fields of science and engineering, such as acoustics, control theory, chaos and fractals, signal processing, porous media, electrochemistry, viscoelasticity, rheology, polymer physics, optics, economics, astrophysics, chaotic dynamics, statistical physics, thermodynamics, proteins, biosciences, and bioengineering. Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. See, for example, [24, 25, 45, 55, 88, 98, 79, 80, 125, 159, 161, 197, 216, 238, 242, 248].

Fractional differential equations with nonlocal conditions have been discussed in [44, 50, 138, 152, 126, 206, 207] and the references therein. Nonlocal conditions were initiated by Byszewski [118] when he proved the existence and uniqueness of mild and classical solutions of nonlocal Cauchy problems (*C.P.* for short). As remarked by Byszewski [116, 117], nonlocal conditions can be more useful than the standard initial conditions to describe some physical phenomena.

Two measures are most important. The Kuratowski measure of noncompactness $\alpha(B)$ of a bounded set B in a metric space is defined as the infimum of numbers $r > 0$ such that B can be covered with a finite number of sets of diameter smaller than r. The Hausdorff measure of noncompactness $\chi(B)$ is defined as the infimum of numbers $r > 0$ such that B can be covered with a finite number of balls of radii smaller than r. Several authors have studied measures of noncompactness in Banach spaces. See, for example, books such as [58, 81, 71], the articles [62, 83, 84, 93, 103, 105, 163, 202], and the references therein.

Recently, considerable attention has been paid to the existence of solutions of boundary value problems (BVPs) and boundary conditions for implicit fractional differential equations and integral equations with Caputo fractional derivatives. See, for example, [47, 51, 57, 56, 74, 94, 95, 97, 103, 164, 177, 188, 190, 191, 189, 241, 260] and the references therein.

Functional implicit differential and integral equations involving Caputo fractional derivatives were analyzed recently by many authors; see, for instance, [20, 17, 15, 14, 26, 34, 43, 90, 91, 92, 102, 104, 107, 108] and the references therein.

Ordinary and partial fractional differential and integral equations are one of the useful mathematical tools in both pure and applied analysis. There has been a significant development in ordinary and partial fractional integral equations in recent years; see the monographs of Abbas et al. [23, 36, 35], Appell et al. [67], Banaś and Mursaleen [82], Miller and Ross [200], Podlubny [219], and the papers by Abbas et al.[8, 7, 5], Banaś et al. [81, 83], and the references therein.

During the last 10 years, impulsive differential equations and impulsive differential inclusions with different conditions have been intensely studied by many mathematicians. The concept of differential equations with impulses were introduced by Milman and Myshkis in 1960 [201]. This subject was, thereafter, extensively investigated. Impulsive differential equations have become more important in recent years in some mathematical models of real-world phenomena, especially in biological or medical domains and in control theory; see, for example, the monograph of Graef et al. [148], Lakshmikantham et al. [186], Perestyuk et al. [215], and Samoilenko and Perestyuk [240]; several articles have also been published, for example, see [48, 57, 76, 90, 87, 86, 100, 105, 106, 157, 251, 252, 250, 254] and the references therein.

In the theory of functional differential and integral equations, there is a special kind of data dependency: Ulam, Hyers, Aoki, and Rassias [234]. The stability of functional equations was originally raised by Ulam in 1940 in a talk given at the University of Wisconsin. The problem posed by Ulam was the following: Under what conditions does there exist an additive mapping near an approximately additive mapping? (For more details see [244, 245]). The first answer to Ulam's question was given by Hyers in 1941 in the case of Banach spaces in [166]. Thereafter, this type of stability has been known as the Ulam–Hyers stability. The Hyers theorem was generalized by Aoki [65] for additive mappings and by Rassias [224] for linear mappings by considering an unbounded Cauchy difference. In 1978, Rassias [224] provided a remarkable generalization of the Ulam–Hyers stability of mappings by considering variables. A generalization of the Rassias theorem was obtained by Gavruta [142]. The concept of stability for a functional equation arises when we replace the functional equation by an inequality that acts as a perturbation of the equation. Thus, the stability question of functional equations is how the solutions of the inequality differ from those of the given functional equation. Considerable attention has been devoted to the study of the Ulam–Hyers and Ulam–Hyers–Rassias stability of all kinds of functional equations; one may consult the monographs [167, 172]. Bota–Boriceanu and Petrusel [112], Petru et al. [217], and Rus [234] discussed the Ulam–Hyers stability for operator equations and inclusions. Ulam stability for fractional differential equations with Caputo derivatives are proposed by Wang et al. [253]. More details from a historical point of view and recent developments of such stabilities are reported in the monographs [128, 167, 172, 174, 223, 225] and the papers [6, 24, 61, 92, 168, 170, 171, 173, 176, 182, 234, 252, 253, 254].

In this book we are interested in the existence and stability of solutions to initial and BVPs for functional differential and integral equations and inclusions that involve Caputo's fractional derivative and Hadamard's fractional integral. The book is arranged and organized as follows.

In Chapter 1, we introduce notations, definitions, and some preliminary notions. In *Section 1.1*, we give some concepts from the theory of Banach spaces; in *Section 1.2* we recall some basic definitions and facts on the theory of fractional calculus. *Section 1.3* recalls some properties of set-valued maps. In *Section 1.4*, we give some properties of the measure of noncompactness. *Section 1.5* presents definitions and examples concerning the phase space. *Section 1.6* is devoted to fixed point theory; here we give the main theorems that will be used in the following chapters. In *Section 1.7*, we give other auxiliary lemmas.

In Chapter 2, we will be concerned with the existence and stability of solutions for some classes of nonlinear implicit fractional differential equations (NIFDEs). In *Section 2.2*, we prove some results concerning the existence and stability of solutions for a system of NIFDEs, and *Section 2.3* is concerned with the existence and stability results for NIFDE with nonlocal conditions. In *Section 2.4*, we present other existence results for NIFDE in Banach spaces. *Section 2.5* is devoted to the existence and stability results for perturbed NIFDEs with finite delay. In *Section 2.6*, we establish a sufficient condition for the existence and stability of solutions of a system of neutral NIFDEs with finite delay.

In Chapter 3, we will be concerned with the existence and stability of solutions for some classes of impulsive NIFDEs. In *Section 3.2*, we establish some existence and stability results for impulsive NIFDE with finite delay. *Section 3.3* is devoted to the existence and stability results for impulsive NIFDEs with finite delay in Banach space. In *Section 3.4*, we prove existence and stability results for perturbed impulsive NIFDEs with finite delay. The last section is devoted to proving other existence and stability results for neutral impulsive NIFDEs with finite delay.

In Chapter 4, we prove sufficient conditions for the existence and stability of solutions for some classes of BVPs for NIFDEs. In *Section 4.2*, we establish some existence and stability results for BVP for NIFDEs with $0 < \alpha \leq 1$. In *Section 4.3*, we prove results for BVPs for NIFDEs with $1 < \alpha \leq 2$. In *Section 4.4*, we give some stability results for BVPs for NIFDEs. *Section 4.5* is devoted to other stability results for BVPs for NIFDEs in Banach spaces. In the last section, we prove the existence of L^1-solutions of BVPs for NIFDEs with local and nonlocal conditions.

In Chapter 5, we prove some existence and stability results for some classes of BVPs for impulsive NIFDEs. In *Section 5.2*, we give some existence and stability results for impulsive NIFDEs. *Section 5.3* is devoted to other existence results for impulsive NIFDEs in Banach spaces.

In Chapter 6, we shall give results about the integrable solutions for implicit fractional differential equations. In *Section 6.2*, we give some existence results for integrable solutions of NIFDEs. *Section 6.3* is devoted to L^1-solutions of NIFDEs with nonlocal conditions. In *Section 6.4*, we give some existence results for integrable solutions for NIFDEs with infinite delay. *Section 6.4* is devoted to other existence results for integrable solutions for NIFDEs.

In Chapter 7, we shall prove some existence results for some classes of partial Hadamard fractional integral equations and inclusions. In *Section 7.2*, we give some existence results for a class of functional partial Hadamard fractional integral equations. *Section 7.3* is devoted to existence results for Fredholm-type Hadamard fractional integral equations. In *Section 7.4*, we use the upper and lower solutions method for partial Hadamard fractional integral equations and inclusions.

In Chapter 8, we shall present results on the stability of solutions for partial Hadamard fractional integral equations and inclusions. In *Section 8.2*, we give some Ulam stability results for partial Hadamard fractional integral equations. *Section 8.3* is devoted to some global stability results for Volterra-type partial Hadamard fractional integral equations. In *Section 8.4*, we prove some Ulam stability results for Hadamard fractional integral equations in Frèchet spaces. In *Section 8.5* we present some Ulam stability results for Hadamard partial fractional integral inclusions via Picard operators.

In Chapter 9, we present results on the stability of solutions for Hadamard–Stieltjes fractional integral equations. In *Section 9.2*, we prove results on the stability of solutions for Hadamard–Stieltjes fractional integral equations. *Section 9.3* is devoted to global stability results for Volterra-type fractional Hadamard–Stieltjes partial integral equations. In *Section 9.4*, we prove some Ulam stability results for a class of Volterra-type nonlinear multidelay Hadamard–Stieltjes fractional integral equations.

In Chapter 10, we prove some results on the Ulam stability for random Hadamard fractional integral equations. In *Section 10.2*, we present results on the stability of solutions for partial Hadamard fractional integral equations with random effects. *Section 10.3* is devoted to global stability results for Volterra–Hadamard random partial fractional integral equations. In *Section 10.4*, we prove the existence and Ulam stability for multidelay Hadamard fractional integral equations in Frèchet spaces with random effects.

Keywords and phrases: Differential and integral equations, implicit differential equation, fractional order, left-sided mixed Riemann–Liouville integral, Riemann–Liouville and Caputo fractional order derivatives, Hadamard fractional integral, solution, upper and lower solutions, boundary value problem, initial value problem, nonlocal conditions, contraction, existence, uniqueness, Banach space, ARéchet space, phase space, impulse, finite delay, infinite delay, fixed point, attractivity, Ulam–Hyers–Rassias stability.