10 Ulam Stabilities for Random Hadamard Fractional Integral Equations

10.1 Introduction

Let β_E be the σ-algebra of Borel subsets of E. A mapping $v\colon \Omega \to E$ is said to be measurable if for any $B \in \beta_E$ one has

$$v^{-1}(B) = \{w \in \Omega\colon v(w) \in B\} \subset \mathcal{A}\,.$$

To define integrals of sample paths of a random process, it is necessary to define a jointly measurable map.

Definition 10.1. A mapping $T\colon \Omega \times E \to E$ is called jointly measurable if for any $B \in \beta_E$ one has

$$T^{-1}(B) = \{(w, v) \in \Omega \times E\colon T(w, v) \in B\} \subset \mathcal{A} \times \beta_E\,,$$

where $\mathcal{A} \times \beta_E$ is the direct product of the σ-algebras $\mathcal{A}$ and β_E defined in Ω and E, respectively.

Lemma 10.2 ([136]). *Let $T\colon \Omega \times E \to E$ be a mapping such that $T(., v)$ is measurable for all $v \in E$, and $T(w, .)$ is continuous for all $w \in \Omega$. Then the map $(w, v) \mapsto T(w, v)$ is jointly measurable.*

Definition 10.3 ([156]). A function $f\colon J \times E \times \Omega \to E$ is called random Carathéodory if the following conditions hold.
(i) The map $(x, y, w) \to f(x, y, u, w)$ is jointly measurable for all $u \in E$.
(ii) The map $u \to f(x, y, u, w)$ is continuous for almost all $(x, y) \in J$ and $w \in \Omega$.

Let $T\colon \Omega \times E \to E$ be a mapping. Then T is called a random operator if $T(w, u)$ is measurable in w for all $u \in E$ and it is expressed as $T(w)u = T(w, u)$. In this case we also say that $T(w)$ is a random operator on E. A random operator $T(w)$ on E is called continuous (resp. compact, totally bounded, and completely continuous) if $T(w, u)$ is continuous (resp. compact, totally bounded, and completely continuous) in u for all $w \in \Omega$. The details of completely continuous random operators in Banach spaces and their properties appear in Itoh [169].

Definition 10.4 ([140]). Let $\mathcal{P}(Y)$ be the family of all nonempty subsets of Y and C a mapping from Ω to $\mathcal{P}(Y)$. A mapping $T\colon \{(w, y)\colon w \in \Omega, y \in C(w)\} \to Y$ is called a random operator with stochastic domain C if C is measurable (i.e., for all closed $A \subset Y$, $\{w \in \Omega, C(w) \cap A \neq \emptyset\}$ is measurable) and for all open $D \subset Y$ and all $y \in Y$, $\{w \in \Omega\colon y \in C(w), T(w, y) \in D\}$ is measurable. T will be called continuous if every $T(w)$ is continuous. For a random operator T, a mapping $y\colon \Omega \to Y$ is called a random (stochastic) fixed point of T if for P-almost all $w \in \Omega$, $y(w) \in C(w)$ and $T(w)y(w) = y(w)$ and for all open $D \subset Y$, $\{w \in \Omega\colon y(w) \in D\}$ is measurable.

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Let $\emptyset \neq \Lambda \subset BC$, let $G\colon \Lambda \to \Lambda$, and consider the solutions of the random equation

$$G(w)u(t,x) = u(t,x,w)\,; \quad w \in \Omega\,. \tag{10.1}$$

Inspired by the definition of the attractivity of solutions of integral equations (e.g., [36]), we introduce the following concept of attractivity of solutions for random equation (10.1).

Definition 10.5. Solutions of random equation (10.1) are locally attractive if there exists a ball $B(u_0,\eta)$ in the space BC such that, for arbitrary random solutions $v = v(t,x,w)$ and $z = z(t,x,w)$ of equations (10.1) belonging to $B(u_0,\eta) \cap \Lambda$, we have that, for each $x \in [0,b]$ and $w \in \Omega$,

$$\lim_{t\to\infty}(v(t,x,w) - z(t,x,w)) = 0\,. \tag{10.2}$$

When the limit (10.2) is uniform with respect to $B(u_0,\eta) \cap \Lambda$, solutions of equation (10.1) are said to be uniformly locally attractive (or, equivalently, that solutions of (10.1) are locally asymptotically stable).

Definition 10.6. The solution $v = v(t,x,w)$ of random equation (10.1) is said to be globally attractive if (10.2) holds for each solution $z = z(t,x,w)$ of (10.1). If condition (10.2) is satisfied uniformly with respect to the set Λ, solutions of equation (10.1) are said to be globally asymptotically stable (or uniformly globally attractive).

In the sequel, we employ the following random fixed point theorem.

Theorem 10.7 (Itoh [169]). *Let X be a nonempty, closed, convex, bounded subset of a Banach space E, and let $N\colon \Omega \times X \to X$ be a compact and continuous random operator. Then the random equation $N(w)u = u$ has a random solution.*

10.2 Partial Hadamard Fractional Integral Equations with Random Effects

10.2.1 Introduction

This section deals with some existence results and Ulam stabilities for a class of random partial functional partial integral equations via Hadamard's fractional integral by applying random fixed point theorem with a stochastic domain.

This section deals with the existence of the Ulam stability of solutions to the Hadamard partial fractional integral equation of the form

$$\begin{aligned} u(x,y,w) &= \mu(x,y,w) \\ &\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left(\log\frac{x}{s}\right)^{r_1-1} \left(\log\frac{y}{t}\right)^{r_2-1} \frac{f(s,t,u(s,t,w),w)}{st}\,dt\,ds\,; \\ &\quad \text{if } (x,y) \in J,\ w \in \Omega\,, \end{aligned} \tag{10.3}$$

where $J := [1, a] \times [1, b]$, $a, b > 1$, $r_1, r_2 > 0$, $(\Omega, \mathcal{A})$ is a measurable space, and $\mu \colon J \times \Omega \to \mathbb{R}$ and $f \colon J \times \mathbb{R} \times \Omega \to \mathbb{R}$ are given continuous functions.

10.2.2 Existence and Ulam Stabilities Results

In this section, we discuss the existence of solutions, and we present conditions for the Ulam stability for the Hadamard integral equation (10.3).

Lemma 10.8 ([113]). *If Y is a bounded subset of Banach space X, then for each $\epsilon > 0$ there is a sequence $\{y_k\}_{k=1}^{\infty} \subset Y$ such that*

$$\alpha(Y) \le 2\alpha(\{y_k\}_{k=1}^{\infty}) + \epsilon .$$

Lemma 10.9 ([202, 261]). *() If $\{u_k\}_{k=1}^{\infty} \subset L^1(J)$ is uniformly integrable, then $\alpha(\{u_k\}_{k=1}^{\infty})$ is measurable and for each $(x, y) \in J$*

$$\alpha\left(\left\{\int_0^x \int_0^y u_k(s, t)dtds\right\}_{k=1}^{\infty}\right) \le 2\int_0^x \int_0^y \alpha(\{u_k(s, t)\}_{k=1}^{\infty})dtds .$$

Lemma 10.10 ([195]). *Let F be a closed and convex subset of a real Banach space, and let $G \colon F \to F$ be a continuous operator and $G(F)$ be bounded. If there exists a constant $k \in [0, 1)$ such that for each bounded subset $B \subset F$,*

$$\alpha(G(B)) \le k\alpha(B) ,$$

then G has a fixed point in F.

The following conditions will be used in the sequel.

(10.4.1) The function $w \mapsto \mu(x, y, w)$ is measurable and bounded for a.e. $(x, y) \in J$.

(10.4.2) The function f is random Carathéodory on $J \times \mathbb{R} \times \Omega$.

(10.4.3) There exist functions $p_1, p_2 \colon J \times \Omega \to [0, \infty)$ with $p_i(w) \in C(J, \mathbb{R}_+)$; $i = 1, 2$ such that for each $w \in \Omega$

$$|f(x, y, u, w)| \le p_1(x, y, w) + \frac{p_2(x, y, w)}{1 + |u(x, y)|}|u(x, y, w)|$$

for all $u \in \mathbb{R}$ and a.e. $(x, y) \in J$.

(10.4.4) There exists a function $q \colon J \times \Omega \to [0, \infty)$ with $q(w) \in L^{\infty}(J, [0, \infty))$ for each $w \in \Omega$ such that for any bounded $B \subset \mathbb{R}$

$$\alpha(f(x, y, B, w)) \le q(x, y, w)\alpha(B) , \quad \text{for a.e. } (x, y) \in J .$$

(10.4.5) There exists a random function $R \colon \Omega \to (0, \infty)$ such that

$$R(w) \ge \mu^*(w) + \frac{(p_1^*(w) + p_2^*(w))(\log a)^{r_1}(\log b)^{r_2}}{\Gamma(1 + r_1)\Gamma(1 + r_2)} ,$$

where

$$\mu^*(w) = \sup_{(x,y)\in J} |\mu(x,y,w)| , \quad p_i^*(w) = \sup_{(x,y)\in J} \operatorname{ess} p_i(x,y,w); \quad i = 1, 2 .$$

(10.4.6) There exist $q_1, q_2 : J \times \Omega \to [0, \infty)$, with $q_i(., w) \in L^\infty(J, [0, \infty))$, $i = 1, 2$, such that for each $w \in \Omega$ and a.e. $(x, y) \in J$ we have

$$p_i(x,y,w) \le q_i(x,y,w,w)\Phi(x,y,w) .$$

(10.4.7) $\Phi(w) \in L^1(J, [0, \infty))$ for all $w \in \Omega$, and there exists $\lambda_\Phi > 0$ such that for each $(x, y) \in J$ we have

$$({}^H I_\sigma^r \Phi)(x,y,w) \le \lambda_\Phi \Phi(x,y,w) ,$$

$$q^* = \sup_{(x,y,w)\in J\times\Omega} \operatorname{ess} \; q(x,y,w) .$$

Theorem 10.11. *Assume* (10.4.1)–(10.4.5). *If*

$$\ell := \frac{4q^*(\log a)^{r_1}(\log b)^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} < 1 ,$$

then integral equation (10.3) *has a random solution defined on* J. *Furthermore, if conditions* (10.4.6) *and* (10.4.7) *hold, then the random equation* (10.3) *is generalized Ulam–Hyers–Rassias stable.*

Proof. From conditions (10.4.2) and (10.4.3), for each $w \in \Omega$ and almost all $(x, y) \in J$, we have that $f(x, y, u(x, y, w), w)$ is in L^1. Since the function f is continuous, the indefinite integral is continuous for all $w \in \Omega$ and almost all $(x, y) \in J$. Again, as the map μ is continuous for all $w \in \Omega$ and the indefinite integral is continuous on J, $N(w)$ defines a mapping $N : \Omega \times C \to C$. Hence, u is a solution for integral equation (10.3) if and only if $u = (N(w))u$.

We will show that the operator N satisfies all conditions of Lemma 10.10. The proof will be given in several steps.

Step 1: $N(w)$ is a random operator with a stochastic domain on C. Since $f(x, y, u, w)$ is random Carathéodory, the map $w \to f(x, y, u, w)$ is measurable. Similarly, the product $(\log \frac{x}{s})^{r_1-1}(\log \frac{y}{t})^{r_2-1}\frac{f(s,t,u(s,t,w),w)}{st}$ of a continuous and measurable function is again measurable. Further, the integral is a limit of a finite sum of measurable functions; therefore, the map

$$w \mapsto \mu(x,y,w) + \int_0^x\int_0^y \left(\log\frac{x}{s}\right)^{r_1-1}\left(\log\frac{y}{t}\right)^{r_2-1}\frac{f(s,t,u(s,t,w),w)}{st\Gamma(r_1)\Gamma(r_2)}dtds$$

is measurable. As a result, N is a random operator on $\Omega \times C$ to C.

Let $W : \Omega \to \mathcal{P}(C)$ be defined by

$$W(w) = \{u \in C : \|u\|_C \le R(w)\} ,$$

with $W(w)$ bounded, closed, convex, and solid for all $w \in \Omega$. Then W is measurable by Lemma [[140], Lemma 17]. Let $w \in \Omega$ be fixed; then from (10.4.4), for any $u \in w(w)$, we get

$$
\begin{aligned}
&|(N(w)u)(x,y)| \\
&\le |\mu(x,y,w)| + \int_0^x\int_0^y \left|\log\frac{x}{s}\right|^{r_1-1}\left|\log\frac{y}{t}\right|^{r_2-1}\frac{|f(s,t,u(s,t,w),w)|}{st\Gamma(r_1)\Gamma(r_2)}dtds \\
&\le |\mu(x,y,w)| + \int_0^x\int_0^y \left|\log\frac{x}{s}\right|^{r_1-1}\left|\log\frac{y}{t}\right|^{r_2-1}\frac{|p_1(s,t,w)+p_2(s,t,w)|}{\Gamma(r_1)\Gamma(r_2)}dtds \\
&\le \mu^*(w) + \frac{(p_1^*(w)+p_2^*(w))(\log a)^{r_1}(\log b)^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} \\
&\le R(w)\,.
\end{aligned}
$$

Therefore, N is a random operator with stochastic domain W and $N(w)\colon W(w) \to N(w)$. Furthermore, $N(w)$ maps bounded sets to bounded sets in C.

Step 2: $N(w)$ is continuous. Let $\{u_n\}$ be a sequence such that $u_n \to u$ in $\mathcal{C}$. Then, for each $(x,y) \in J$ and $w \in \Omega$, we have

$$
\begin{aligned}
|(N(w)u_n)(x,y) - (N(w)u)(x,y)| \le& \int_0^x\int_0^y \left|\log\frac{x}{s}\right|^{r_1-1}\left|\log\frac{y}{t}\right|^{r_2-1} \\
&\times \frac{|f(s,t,u_n(s,t,w),w) - f(s,t,u(s,t,w),w)|}{\Gamma(r_1)\Gamma(r_2)}dtds\,.
\end{aligned}
$$

Using the Lebesgue dominated convergence theorem, we get

$$
\|N(w)u_n - N(w)u\|_C \to 0 \ \text{ as } \ n \to \infty\,.
$$

As a consequence of Steps 1 and 2, we can conclude that $N(w)\colon W(w) \to N(w)$ is a continuous random operator with stochastic domain W, and $N(w)(W(w))$ is bounded.

Step 3: For each bounded subset B of $W(w)$ we have $\alpha(N(w)B) \le \ell\alpha(B)$. Let $w \in \Omega$ be fixed. From Lemmas 10.8 and 10.9, for any $B \subset W$ and any $\epsilon > 0$ there exists a sequence $\{u_n\}_{n=0}^{\infty} \subset B$, such that for all $(x,y) \in J$ we have

$$
\begin{aligned}
&\alpha((N(w)B)(x,y)) \\
&= \alpha\left(\left\{\mu(x,y) + \int_1^x\int_1^y \left(\log\frac{x}{s}\right)^{r_1-1}\left(\log\frac{y}{t}\right)^{r_2-1}\frac{f(s,t,u(s,t,w),w)}{st\Gamma(r_1)\Gamma(r_2)}dtds;\ u \in B\right\}\right) \\
&\le 2\alpha\left(\left\{\int_1^x\int_1^y \left(\log\frac{x}{s}\right)^{r_1-1}\left(\log\frac{y}{t}\right)^{r_2-1}\frac{f(s,t,u_n(s,t,w),w)}{st\Gamma(r_1)\Gamma(r_2)}dtds\right\}_{n=1}^{\infty}\right) + \epsilon \\
&\le 4\int_1^x\int_1^y \alpha\left(\left\{\left(\log\frac{x}{s}\right)^{r_1-1}\left(\log\frac{y}{t}\right)^{r_2-1}\frac{f(s,t,u(s,t,w),w)}{st\Gamma(r_1)\Gamma(r_2)}dtds\right\}_{n=1}^{\infty}\right)dtds + \epsilon
\end{aligned}
$$

$$
\begin{aligned}
&\le 4\int_1^x\int_1^y\left(\log\frac{x}{s}\right)^{r_1-1}\left(\log\frac{y}{t}\right)^{r_2-1}\frac{1}{\Gamma(r_1)\Gamma(r_2)}\alpha\left(\{f(s,t,u_n(s,t,w),w)\}_{n=1}^{\infty}\right)dtds+\epsilon\\
&\le 4\int_1^x\int_1^y\left(\log\frac{x}{s}\right)^{r_1-1}\left(\log\frac{y}{t}\right)^{r_2-1}\frac{1}{\Gamma(r_1)\Gamma(r_2)}q(s,t,w)\alpha\left(\{u_n(s,t,w)\}_{n=1}^{\infty}\right)dtds+\epsilon\\
&\le\left(4\int_1^x\int_1^y\left(\log\frac{x}{s}\right)^{r_1-1}\left(\log\frac{y}{t}\right)^{r_2-1}\frac{1}{\Gamma(r_1)\Gamma(r_2)}q(s,t,w)dsdt\right)\alpha\left(\{u_n\}_{n=1}^{\infty}\right)+\epsilon\\
&\le\left(4\int_1^x\int_1^y\left(\log\frac{x}{s}\right)^{r_1-1}\left(\log\frac{y}{t}\right)^{r_2-1}\frac{1}{\Gamma(r_1)\Gamma(r_2)}q(s,t,w)dtds\right)\alpha(B)+\epsilon\\
&\le\frac{4q^*(\log a)^{r_1}(\log b)^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}\alpha(B)+\epsilon\\
&=\ell\alpha(B)+\epsilon\,.
\end{aligned}
$$

Since $\epsilon > 0$ is arbitrary,

$$\alpha(N(B)) \le \ell\alpha(B)\,.$$

Hence, from Lemma 10.10 it follows that for each $w \in \Omega$, N has at least one fixed point in W. Since $\bigcap_{w\in\Omega} intW(w) \neq \emptyset$, the measurable selector of $intW$ exists. From Lemma 10.10, the operator N has a stochastic fixed point, i.e., integral equation (10.3) has at least one random solution on C.

Step 4: Generalized Ulam–Hyers–Rassias stability. Set

$$q_i^* = \sup_{(x,y,w)\in J\times\Omega}\operatorname{ess}\, q_i(x,y,w)\,;\quad i=1,2\,.$$

Let $u\colon \Omega \to C$ be a solution of inequality (9.8). By Theorem 10.11, there exists v, which is a solution of random equation (10.3). Hence,

$$
\begin{aligned}
v(x,y,w) &= \mu(x,y,w)\\
&\quad+\int_1^x\int_1^y\left(\log\frac{x}{s}\right)^{r_1-1}\left(\log\frac{y}{t}\right)^{r_2-1}\frac{f(s,t,v(s,t,w),w)}{st\Gamma(r_1)\Gamma(r_2)}dtds\,;\quad (x,y)\in J,\ w\in\Omega\,.
\end{aligned}
$$

From conditions (10.4.6) and (10.4.7), for each $(x, y) \in J$ and $w \in \Omega$, we have

$$
\begin{aligned}
&|u(x,y,w)-v(x,y,w)| \le |u(x,y,w)-N(w)(u)|+|N(w)(u)-N(w)(v)|\\
&\le \Phi(x,y,w)+\int_1^x\int_1^y\left|\log\frac{x}{s}\right|^{r_1-1}\left|\log\frac{y}{t}\right|^{r_2-1}\frac{|f(s,t,u(s,t,w))-f(s,t,v(s,t,w))|}{\Gamma(r_1)\Gamma(r_2)}dtds\\
&\le \Phi(x,y,w)+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^x\int_1^y\left|\log\frac{x}{s}\right|^{r_1-1}\left|\log\frac{y}{t}\right|^{r_2-1}\\
&\quad\times\left(2q_1^*+\frac{q_2^*|u(s,t,w)|}{1+|u|}+\frac{q_2^*|v(s,t,w)|}{1+|v|}\right)\frac{\Phi(s,t,w)}{st}dtds
\end{aligned}
$$

$$\begin{aligned}
&\leq \Phi(x,y,w) + 2(q_1^* + q_2^*)({}^H I_\sigma^r \Phi)(x,y,w) \\
&\leq [1 + 2(q_1^* + q_2^*)\lambda_\phi]\Phi(x,y,w) \\
&:= c_{N,\Phi}\Phi(x,y,w)\,.
\end{aligned}$$

Hence, random equation (10.3) is generalized Ulam–Hyers–Rassias stable. □

10.2.3 An Example

Let $E = \mathbb{R}$, $\Omega = (-\infty, 0)$ be equipped with the usual σ-algebra consisting of Lebesgue measurable subsets of $(-\infty, 0)$. Given a measurable function $u\colon \Omega \to C([1,e]\times[1,e])$, consider the partial random Hadamard integral equation

$$\begin{aligned}
u(x,y,w) &= \mu(x,y,w) \\
&\quad + \int_1^x \int_1^y \left(\log\frac{x}{s}\right)^{r_1-1}\left(\log\frac{y}{t}\right)^{r_2-1}\frac{f(s,t,u(s,t,w),w)}{st\Gamma(r_1)\Gamma(r_2)}\,dtds
\end{aligned} \tag{10.4}$$

for $(x,y) \in [1,e]\times[1,e]$, $w \in \Omega$, where

$$r_1, r_2 > 0\,, \quad \mu(x,y,w) = x\sin w + y^2\cos w;\quad (x,y)\in[1,e]\times[1,e],$$

and

$$f(x,y,u(x,y)) = \frac{w^2xy^2}{(1 + w^2 + u(x,y,w)|)e^{x+y+3}}, \quad (x,y)\in[1,e]\times[1,e],\ w\in\Omega\,.$$

The function $w \mapsto \mu(x,y,w) = x\sin w + y^2\cos w$ is measurable and bounded, with

$$|\mu(x,y,w)| \leq e + e^2\,;$$

hence, condition (10.4.1) is satisfied.

The map $(x,y,w) \mapsto f(x,y,u,w)$ is jointly continuous for all $u \in \mathbb{R}$, so jointly measurable for all $u\in\mathbb{R}$. Also, the map $u \mapsto f(x,y,u,w)$ is continuous for all $(x,y) \in [1,e]\times[1,e]$ and $w\in\Omega$. So the function f is Carathéodory on $[1,e]\times[1,e]\times\mathbb{R}\times\Omega$.

For each $u\in\mathbb{R}$, $(x,y)\in[1,e]\times[1,e]$ and $w\in\Omega$ we have

$$|f(x,y,u,w)| \leq w^2xy^2\left(1 + \frac{1}{e^3}|u|\right)\,.$$

Hence, condition (10.4.3) is satisfied by $p_1^* = e^3$ and $p_1(x,y,w) = p_2^* = 1$. The condition $\ell < 1$ holds with $a = b = e$ and $q^* = \frac{1}{e^3}$. Indeed, for each $r_1, r_2 > 0$ we get

$$\begin{aligned}
\ell &= \frac{4q^*(\log a)^{r_1}(\log b)^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} \\
&\leq \frac{4}{e^3\Gamma(1+r_1)\Gamma(1+r_2)} \\
&< 1\,.
\end{aligned}$$

Condition (10.4.6) is satisfied by

$$\Phi(x, y, w) = w^2 w^2 x y^2, \quad \text{and} \quad \lambda_\Phi = \frac{1}{\Gamma(1+r_1)\Gamma(1+r_2)} .$$

Indeed, for each $(x, y) \in [1, e] \times [1, e]$ we get

$$\begin{aligned}({}^H I_\sigma^r \Phi)(x, y, w) &\le \frac{w^2 e^3}{\Gamma(1+r_1)\Gamma(1+r_2)} \\ &= \lambda_\Phi \Phi(x, y, w) .\end{aligned}$$

Finally, we can see that condition (10.4.7) is satisfied by $q_1(x, y, w) = 1$ and $q_2(x, y, w) = \frac{1}{e^3}$. Consequently, Theorem 10.11 implies that the Hadamard integral equation (10.4) has a solution defined on $[1, e] \times [1, e]$, and (10.4) is generalized Ulam–Hyers–Rassias stable.

10.3 Global Stability Results for Volterra–Hadamard Random Partial Fractional Integral Equations

10.3.1 Introduction

This section deals with the existence and stability of random solutions of a class of functional partial integral equations of Hadamard fractional order with random effects in Banach spaces.

The initial value problems of ordinary random differential equations have been studied in the literature on bounded as well as unbounded intervals. See, for example, Burton and Furumochi [114], Zielinski et al. [265], and the references therein.

In [8, 32], Abbas et al. studied existence and stability results for some classes of nonlinear differential and integral equations of fractional order. This section deals with the existence and the asymptotic behavior of random solutions to the nonlinear quadratic Volterra random partial integral equation of Hadamard fractional order

$$\begin{aligned}u(t, x, w) = f(t, x, u(t, x, w), w) &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^t \int_1^x \left(\log \frac{t}{s}\right)^{r_1 - 1} \left(\log \frac{x}{\xi}\right)^{r_2 - 1} \\ &\times g(t, x, s, \xi, u(s, \xi, w), w) \frac{d\xi ds}{s\xi} , \quad (t, x) \in J := [1, \infty) \times [1, b], \ w \in \Omega ,\end{aligned} \tag{10.5}$$

where $b > 1$, $r_1, r_2 \in (0, \infty)$, $\alpha, \beta, \gamma \colon [1, \infty) \to [1, \infty)$, $(\Omega, \mathcal{A})$ is a measurable space, $f \colon J \times \mathbb{R} \times \Omega \to \mathbb{R}$ and $g \colon J_1 \times \mathbb{R} \times \Omega \to \mathbb{R}$ are given continuous functions, and $J_1 = \{(t, x, s, \xi) \colon 1 \le s \le t, 1 \le \xi \le s \le b\}$. Our existence results are based on Itoh's random fixed point theorem. Also, we obtain some results about the global asymptotic stability of random solutions of the integral equation in question. Finally, we present an example illustrating the applicability of the imposed conditions.

10.3.2 Existence of Random Solutions and Global Stability Results

In this section, we are concerned with the existence and the asymptotic stability of random solutions for the Hadamard partial integral equation (10.5). The following conditions will be used in the sequel:

(10.5.1) The functions f and g are random Carathéeodory.

(10.5.2) There exist a constant $M, L > 0$ with $M < L$ and a nondecreasing function $\psi_1 : [0, \infty) \to (0, \infty)$ such that

$$|f(t, x, u, w) - f(t, x, v, w)| \le \frac{M|u - v|}{(1 + t)(L + |u - v|)}$$

and

$$|f(t_1, x_1, u, w) - f(t_2, x_2, u, w)| \le (|t_1 - t_2| + |x_1 - x_2|)\psi_1(|u|)$$

for each $(t, x), (t_1, x_1), (t_2, x_2) \in J$, $u, v \in \mathbb{R}$ and $w \in \Omega$.

(10.5.3) The function $t \to f(t, x, 0, 0, w)$ is bounded on $J \times \Omega$ with

$$f^* = \sup_{(t,x,w)\in J\times\Omega} f(t, x, 0, 0, w)$$

and

$$\lim_{t\to\infty} |f(t, x, 0, 0, w)| = 0 ; \quad x \in [1, b], \ w \in \Omega .$$

(10.5.4) There exist continuous measurable functions $\varphi : J \times \Omega \to \mathbb{R}_+$, $p : J_1 \times \Omega \to \mathbb{R}_+$ and a nondecreasing function $\psi_2 : [0, \infty) \to (0, \infty)$ such that

$$|g(t_1, x_1, s, \xi, u, w) - g(t_2, x_2, s, \xi, u, w)| \le \varphi(s, \xi, w)(|x_1 - x_2| + |y_1 - y_2|)\psi_2(|u|)$$

and

$$|g(t, x, s, \xi, u, w)| \le \frac{p(t, x, s, \xi, w)}{1 + t + |u|}$$

for each $(t, x), (s, t), (t_1, x_1), (t_2, x_2) \in J$, $u \in \mathbb{R}$, and $w \in \Omega$. Moreover, assume that

$$\lim_{t\to\infty} \int_1^t \int_1^x \left|\log \frac{t}{s}\right|^{r_1-1} \left|\log \frac{x}{\xi}\right|^{r_2-1} p(t, x, s, \xi, w)d\xi ds = 0 ; \quad x \in [1, b] .$$

Theorem 10.12. *Assume* (10.5.1)–(10.5.4)*, then integral equation* (10.5) *has at least one random solution in the space BC. Moreover, the random solutions of* (10.5) *are globally asymptotically stable.*

Proof. Set $d^* := \sup_{(t,x,w)\in J\times\Omega} d(t, x, w)$, where

$$d(t, x, w) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^t \int_1^x \left|\log \frac{t}{s}\right|^{r_1-1} \left|\log \frac{x}{\xi}\right|^{r_2-1} p(t, x, s, \xi, w)d\xi ds .$$

From condition (10.5.4) we infer that d^* is finite. Define a mapping $N\colon \Omega \times BC \to BC$ such that

$$N(w)u(t,x) = f(t,x,u(t,x,w),w) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^t \int_1^x \left(\log \frac{t}{s}\right)^{r_1-1} \left(\log \frac{x}{\xi}\right)^{r_2-1} \times g(t,x,s,\xi,u(s,\xi,w),w) \frac{d\xi ds}{s\xi}\,, \quad (t,x) \in J,\ w \in \Omega\,. \tag{10.6}$$

The maps f and g are continuous for all $w \in \Omega$. Again, as the indefinite integral is continuous on J, $N(w)$ defines a mapping $N\colon \Omega \times BC \to BC$. Then u is a solution for integral equation (10.5) if and only if $u = N(w)u$.

Next we show that the function $N(w)u \in BC$ for any $u \in BC$ and each $w \in \Omega$. By considering the conditions of this theorem, for each $(t,x) \in J$ and $w \in \Omega$ we have

$$\begin{aligned}
|(Nw)u(t,x)| &\le |f(t,x,u(t,x,w),w) - f(t,x,0,w)| + |f(t,x,0,w)| \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^t \int_1^x \left|\log \frac{t}{s}\right|^{r_1-1} \left|\log \frac{x}{\xi}\right|^{r_2-1} \\
&\quad \times |g(t,x,s,\xi,u(s,\xi,w),w)| \frac{d\xi ds}{s\xi} \\
&\le \frac{M|u(t,x,w)|}{(1+t)(L+|u(t,x,w)|)} + |f(t,x,0,w)| \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^t \int_1^x \left|\log \frac{t}{s}\right|^{r_1-1} \left|\log \frac{x}{\xi}\right|^{r_2-1} \\
&\quad \times \frac{p(t,x,s,\xi)}{1+\alpha(t)+|u(s,\xi))|+|u(\gamma(s),\xi))|} \frac{d\xi ds}{s\xi} \\
&\le M + f^* + d^*\,.
\end{aligned}$$

Hence, $N(w)u \in BC$, and $N(w)$ transforms the ball $B_\eta := B(0,\eta)$ into itself, where $\eta = M + f^* + d^*$. We will show that $N\colon \Omega \times B_\eta \to B_\eta$ satisfies the assumptions of Theorem 10.7. The proof will be given in several steps.

Step 1: $N(w)$ is a random operator on $\Omega \times B_\eta$ into B_η. Since $f(t,x,u,w)$ is random Carathéodory, the map $w \to f(t,x,u,w)$ is measurable in view of Lemma 10.2. Similarly, the product $(\log \frac{t}{s})^{r_1-1}(\log \frac{x}{\xi})^{r_2-1} g(t,x,s,\xi,u(s,\xi,w),w)$ of a continuous and a measurable function is again measurable. Further, the integral is a limit of a finite sum of measurable functions; therefore, the map

$$w \mapsto N(w)u(t,x,w)$$

is measurable. As a result, $N(w)$ is a random operator on $\Omega \times B_\eta$ into B_η.

Step 2: $N(w)$ is continuous. Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence such that $u_n \to u$ in B_η. Then, for each $(t,x) \in J$ and $w \in \Omega$ we have

$$\begin{aligned}
&|N(w)u_n(t,x) - N(w)u(t,x)| \le |f(t,x,u_n(t,x,w),w) - f(t,x,u(t,x,w),w)| \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^t \int_1^x \left|\log \frac{t}{s}\right|^{r_1-1} \left|\log \frac{x}{\xi}\right|^{r_2-1} \\
&\quad \times \sup_{(s,\xi)\in J} |g(t,x,s,\xi,u_n(s,\xi,w),w) - g(t,x,s,\xi,u(s,\xi,w),w)| \frac{d\xi ds}{s\xi} \\
&\le \frac{M}{L}\|u_n - u\|_{BC} \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^t \int_1^x \left|\log \frac{t}{s}\right|^{r_1-1} \left|\log \frac{x}{\xi}\right|^{r_2-1} \\
&\quad \times \|g(t,x,.,.,u_n(.,.,w),w) - g(t,x,.,.,u(.,.,w),w)\|_{BC} d\xi ds\,. \qquad (10.7)
\end{aligned}$$

Case 1. If $(t,x) \in [1,T] \times [1,b]$, $T > 1$, then, since $u_n \to u$ as $n \to \infty$ and f, g are continuous, (10.7) gives

$$\|N(w)u_n - N(w)u\|_{BC} \to 0 \text{ as } n \to \infty\,.$$

Case 2. If $(t,x) \in (T,\infty) \times [1,b]$, $T > 1$, then from (10.5.4) and (10.7) for each $(t,x) \in J$ we have

$$\begin{aligned}
&|N(w)u_n(t,x) - N(w)u(t,x)| \le \frac{M}{L}\|u_n - u\|_{BC} \\
&\quad + \frac{2}{\Gamma(r_1)\Gamma(r_2)} \int_1^t \int_1^x \left|\log \frac{\beta(t)}{s}\right|^{r_1-1} \left|\log \frac{x}{\xi}\right|^{r_2-1} \frac{p(t,x,s,\xi)}{s\xi} d\xi ds \\
&\le \frac{M}{L}\|u_n - u\|_{BC} + 2d(t,x)\,.
\end{aligned}$$

Thus, we get

$$|N(w)u_n(t,x) - N(w)u(t,x)| \le \frac{M}{L}\|u_n - u\|_{BC} + 2d(t,x,w)\,. \qquad (10.8)$$

Since $u_n \to u$ as $n \to \infty$ and $t \to \infty$, (10.8) gives

$$\|N(w)u_n - N(w)u\|_{BC} \to 0 \text{ as } n \to \infty\,.$$

Step 3: $N(w)(B_\eta)$ is uniformly bounded. This is clear since $N(w)(B_\eta) \subset B_\eta$; $w \in \Omega$ and B_η is bounded.

Step 4: $N(B_\eta)$ is equicontinuous on every compact subset $[1,a] \times [1,b]$ of J, $a > 1$. Let $w \in \Omega$, $(t_1,x_1), (t_2,x_2) \in [1,a] \times [1,b]$, $t_1 < t_2$, $x_1 < x_2$, and let $u \in B_\eta$. Then we have

$$
\begin{aligned}
&|N(w)u(t_2,x_2) - N(w)u(t_1,x_1)| \\
&\le |f(t_2,x_2,u(t_2,x_2,w),w) - f(t_2,x_2,u(t_1,x_1,w),w)| \\
&\quad + |f(t_2,x_2,u(t_1,x_1,w),w) - f(t_1,x_1,u(t_1,x_1,w),w)| \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^{t_2}\int_1^{x_2} \left|\log\frac{t_2}{s}\right|^{r_1-1} \left|\log\frac{x_2}{\xi}\right|^{r_2-1} \\
&\qquad \times |g(t_2,x_2,s,\xi,u(s,\xi,w),w) - g(t_1,x_1,s,\xi,u(s,\xi,w),w)|d\xi ds \\
&\quad + \left|\frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^{t_2}\int_1^{x_2} \left(\log\frac{t_2}{s}\right)^{r_1-1} \left(\log\frac{x_2}{\xi}\right)^{r_2-1}\right. \\
&\qquad \times g(t_1,x_1,s,\xi,u(s,\xi,w),w)d\xi ds \\
&\quad - \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^{t_1}\int_1^{x_1} \left(\log\frac{t_2}{s}\right)^{r_1-1} \left(\log\frac{x_2}{\xi}\right)^{r_2-1} \\
&\qquad \left.\times g(t_1,x_1,s,\xi,u(s,\xi,w),w)d\xi ds\right| \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^{t_1}\int_1^{x_1} \left|\left(\log\frac{t_2}{s}\right)^{r_1-1} \left(\log\frac{x_2}{\xi}\right)^{r_2-1}\right. \\
&\quad \left. - \left(\log\frac{t_1}{s}\right)^{r_1-1} \left(\log\frac{x_1}{\xi}\right)^{r_2-1}\right| |g(t_1,x_1,s,\xi,u(s,\xi,w),w)|d\xi ds\,.
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
&|N(w)u(t_2,x_2) - N(w)u(t_1,x_1)| \\
&\le \frac{M}{L}(|u(t_2,x_2,w) - u(t_1,x_1,w)| + |u(t_2,x_2,w) - u(t_1,x_1,w)|) \\
&\quad + (|t_2 - t_1| + |x_2 - x_1|)\psi_1(\|u\|_{BC}) \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^{t_2}\int_1^{x_2} \left|\log\frac{t_2}{s}\right|^{r_1-1} \left|\log\frac{x_2}{\xi}\right|^{r_2-1} \\
&\qquad \times \varphi(t,x,s,\xi,w)(|t_2 - t_1| + |x_2 - x_1|)\psi_2(\|u\|_{BC})d\xi ds \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{t_1}^{t_2}\int_1^{x_2} \left|\log\frac{t_2}{s}\right|^{r_1-1} \left|\log\frac{x_2}{\xi}\right|^{r_2-1} \\
&\qquad \times |g(t_1,x_1,s,\xi,u(s,\xi,w),w)|d\xi ds
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^{t_2}\int_{x_1}^{x_2}\left|\log\frac{t_2}{s}\right|^{r_1-1}\left|\log\frac{x_2}{\xi}\right|^{r_2-1}\\
&\quad\times|g(t_1,x_1,s,\xi,u(s,\xi,w),w)|d\xi ds\\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{t_1}^{t_2}\int_{x_1}^{x_2}\left|\log\frac{t_2}{s}\right|^{r_1-1}\left|\log\frac{x_2}{\xi}\right|^{r_2-1}\\
&\quad\times|g(t_1,x_1,s,\xi,u(s,\xi,w),w)|d\xi ds\\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^{t_1}\int_1^{x_1}\left|\left(\log\frac{t_2}{s}\right)^{r_1-1}\left(\log\frac{x_2}{\xi}\right)^{r_2-1}\right.\\
&\left.-\left(\log\frac{\beta(t_1)}{s}\right)^{r_1-1}\left(\log\frac{x_1}{\xi}\right)^{r_2-1}\right||g(t_1,x_1,s,\xi,u(s,\xi,w),w)|d\xi ds\,.
\end{aligned}
$$

Hence, we get

$$
\begin{aligned}
&|N(w)u(t_2,x_2)-N(w)u(t_1,x_1)|\\
&\le\frac{M}{L}(|u(t_2,x_2,w)-u(t_1,x_1,w)|+|u(t_2,x_2,w)-u(t_1,x_1,w)|)\\
&\quad+(|t_2-t_1|+|x_2-x_1|)\psi_1(\eta)\\
&\quad+\frac{(|t_2-t_1|+|x_2-x_1|)\psi_2(\eta)}{\Gamma(r_1)\Gamma(r_2)}\\
&\qquad\times\int_1^{t_2}\int_1^{x_2}\left|\log\frac{t_2}{s}\right|^{r_1-1}\left|\log\frac{x_2}{\xi}\right|^{r_2-1}\varphi(s,\xi)d\xi ds\\
&\quad+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{\beta(t_1)}^{\beta(t_2)}\int_1^{x_2}\left|\log\frac{\beta(t_2)}{s}\right|^{r_1-1}\left|\log\frac{x_2}{\xi}\right|^{r_2-1}p(t_1,x_1,s,\xi)|d\xi ds\\
&\quad+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^{t_2}\int_{x_1}^{x_2}\left|\log\frac{t_2}{s}\right|^{r_1-1}\left|\log\frac{x_2}{\xi}\right|^{r_2-1}p(t_1,x_1,s,\xi)|d\xi ds\\
&\quad+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{t_1}^{t_2}\int_{x_1}^{x_2}\left|\log\frac{t_2}{s}\right|^{r_1-1}\left|\log\frac{x_2}{\xi}\right|^{r_2-1}p(t_1,x_1,s,\xi)|d\xi ds\\
&\quad+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^{t_1}\int_1^{x_1}\left|\left(\log\frac{t_2}{s}\right)^{r_1-1}\left(\log\frac{x_2}{\xi}\right)^{r_2-1}\right.\\
&\quad\left.-\left(\log\frac{t_1}{s}\right)^{r_1-1}\left(\log\frac{x_1}{\xi}\right)^{r_2-1}\right|p(t_1,x_1,s,\xi,w)|d\xi ds\,.
\end{aligned}
$$

From the continuity of φ, p and as $t_1\to t_2$ and $x_1\to x_2$, the right-hand side of the preceding inequality tends to zero.

Step 5: $N(w)(B_\eta)$ is equiconvergent. Let $(t,x) \in J$, $w \in \Omega$ and $u \in B_\eta$. Then we have

$$\begin{aligned}
|N(w)u(t,x)| &\le |f(t,x,u(t,x,w),w) - f(t,x,0,w) + f(t,x,0,w)| \\
&\quad + \left| \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^t \int_1^x \left(\log\frac{t}{s}\right)^{r_1-1} \left(\log\frac{x}{\xi}\right)^{r_2-1} \right. \\
&\qquad \left. \times\, g(t,x,s,\xi,u(s,\xi,w),w)\frac{d\xi ds}{s\xi} \right| \\
&\le \frac{M|u(t,x,w)|}{(1+t)(L+|u(t,x,w)|)} + |f(t,x,0,w)| \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^t \int_1^x \left(\log\frac{t}{s}\right)^{r_1-1} \left(\log\frac{x}{\xi}\right)^{r_2-1} \\
&\qquad \times \frac{p(t,x,s,\xi,w)}{1+t+|u(s,\xi,w)|} d\xi ds \\
&\le \frac{M}{1+t} + |f(t,x,0,w)| \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)(1+t)} \int_1^t \int_1^x \left(\log\frac{\beta(t)}{s}\right)^{r_1-1} \left(\log\frac{x}{\xi}\right)^{r_2-1} p(t,x,s,\xi,w)d\xi ds \\
&\le \frac{M}{1+t} + |f(t,x,0,w)| + \frac{d^*}{1+t} \,.
\end{aligned}$$

Thus, for each $x \in [1,b]$ we get

$$|N(w)u(t,x)| \to 0, \quad \text{as } \ t \to +\infty \,.$$

Hence,

$$|N(w)u(t,x) - N(w)u(+\infty,x)| \to 0, \quad \text{as } \ t \to +\infty \,.$$

As a consequence of Steps 1–5, together with Lemma 1.57, we can conclude that $N\colon \Omega \times B_\eta \to B_\eta$ is continuous and compact. From an application of Theorem 10.7 we deduce that the operator equation $N(w)u = u$ has a random solution. This further implies that random integral equation (10.5) has a random solution.

Step 6: The uniform global attractivity. Let us assume that u_0 is a solution of integral equation (7.1) with the conditions of this theorem. Consider the ball $B(u_0, \eta^*)$ with $\eta^* = \frac{LM^*}{L-M}$, where

$$\begin{aligned}
M^* := \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sup_{(t,x,w)\in J\times\Omega} &\left\{ \int_1^t \int_1^x \left(\log\frac{t}{s}\right)^{r_1-1} \left(\log\frac{x}{\xi}\right)^{r_2-1} \right. \\
&\times |g(t,x,s,\xi,u(s,\xi,w),w) \\
&\left. - g(t,x,s,\xi,u_0(s,\xi,w),w)| d\xi ds;\ u \in BC \right\} .
\end{aligned}$$

Taking $w \in \Omega$ and $u \in B(u_0, \eta^*)$, we have

$$
\begin{aligned}
&|N(w)u(t,x) - u_0(t,x,w)| = |N(w)u(t,x) - N(w)u_0(t,x)| \\
&\le |f(t,x,u(t,x,w),w) - f(t,x,u_0(t,x,w),w)| \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^t \int_1^x \left(\log \frac{t}{s}\right)^{r_1-1} \left(\log \frac{x}{\xi}\right)^{r_2-1} \\
&\quad \times |g(t,x,s,\xi,u(s,\xi,w),w) - g(t,x,s,\xi,u_0(s,\xi,w),w)| \frac{d\xi ds}{s\xi} \\
&\le \frac{M}{L}\|u - u_0\|_{BC} + M^* \\
&\le \frac{M}{L}\eta^* + M^* = \eta^* .
\end{aligned}
$$

Thus, we observe that $N(w)$ is a continuous function such that $N(w)(B(u_0,\eta^*)) \subset B(u_0,\eta^*)$. Moreover, if u is a solution of integral equation (10.5), then

$$
\begin{aligned}
&|u(t,x,w) - u_0(t,x,w)| = |N(w)u(t,x) - N(w)u_0(t,x)| \\
&\le |f(t,x,u(t,x,w),w) - f(t,x,u_0(t,x,w),w)| \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^t \int_1^x \left(\log \frac{t}{s}\right)^{r_1-1} \left(\log \frac{x}{\xi}\right)^{r_2-1} \\
&\quad \times |g(t,x,s,\xi,u(s,\xi,w),w) - g(t,x,s,\xi,u_0(s,\xi,w),w)| d\xi ds .
\end{aligned}
$$

Thus,

$$
\begin{aligned}
|u(t,x,w) - u_0(t,x,w)| &\le \frac{M}{L}|u(t,x,w) - u_0(t,x,w)| \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^t \int_1^x \left(\log \frac{t}{s}\right)^{r_1-1} \left(\log \frac{x}{\xi}\right)^{r_2-1} p(t,x,s,\xi,w) d\xi ds . \qquad (10.9)
\end{aligned}
$$

Using (10.9), we get

$$
\begin{aligned}
\lim_{t\to\infty} |u(t,x,w) - u_0(t,x,w)| \le \lim_{t\to\infty} \frac{L}{\Gamma(r_1)\Gamma(r_2)(L-M)} \int_1^t \int_1^x \left(\log \frac{t}{s}\right)^{r_1-1} \left(\log \frac{x}{\xi}\right)^{r_2-1} \\
\times p(t,x,s,\xi,w) d\xi ds = 0 .
\end{aligned}
$$

Consequently, all random solutions of integral equation (7.1) are globally asymptotically stable. □

10.3.3 An Example

Let $\Omega = (-\infty, 0)$ be equipped with the usual σ-algebra consisting of Lebesgue measurable subsets of $(-\infty, 0)$. Given a measurable function $u: \Omega \to AC([1,\infty) \times [1,e])$, consider the partial Hadamard random fractional integral equation

$$u(t,x,w) = \frac{tx}{10(1+t+t^2+t^3+tw^2+w^2)}(1+\sin(u(t,x,w)))$$
$$+ \frac{1}{\Gamma^2(q)} \int_1^t \int_1^x \left(\log \frac{t}{s}\right)^{q-1} \left(\log \frac{x}{\xi}\right)^{q-1} \frac{\ln(1+2x(s\xi)^{-1}|u(s,\xi)|)}{(1+t+|u(s,\xi)|)^2(1+x^2+t^4)} d\xi ds\,;$$
$$(t,x) \in [1,\infty) \times [1,e],\ w \in \Omega\,, \tag{10.10}$$

where $r_1 = r_2 = q > 0$,

$$f(t,x,u,w) = \frac{tx(1+\sin(u))}{10(1+t)(1+w^2+t^2)}$$

for $(t,x) \in J\ w \in \Omega$ and $u \in \mathbb{R}$ and

$$g(t,x,s,\xi,u,w) = \frac{\ln(1+x(s\xi)^{-1}|u|)}{(1+t+|u|)^2(1+x^2+t^4)}$$

for $(t,x,s,\xi) \in J_1\ w \in \Omega$ and $u \in \mathbb{R}$.

We can easily check that the assumptions of Theorem 10.12 are satisfied. In fact, clearly, the maps $(t,x,w) \mapsto f(t,x,u,w)$ and $(t,x,w) \mapsto g(t,x,s,\xi,u,w)$ are jointly continuous for all $u \in \mathbb{R}$ and, thus, jointly measurable for all $u \in \mathbb{R}$. Also, the maps $u \mapsto f(t,x,u,w)$ and $u \mapsto g(t,x,s,\xi,u,w)$ are continuous for all $(t,x) \in J$ and $w \in \Omega$. Thus, the functions f and g are Carathéodory; then condition (10.5.1) is satisfied. The function f is continuous and satisfies (10.5.2), where $M = \frac{1}{10}$, $L = 1$. Also, f satisfies (10.5.3), with $f^* = \frac{e}{10}$. Next, let us note that the function g satisfies (10.5.4), where $p(t,x,s,\xi) = \frac{x(s\xi)^{-1}}{1+x^2+t^4}$. Also,

$$\lim_{t\to\infty} p(t,x) \int_1^t \int_1^x \left|\log \frac{t}{s}\right|^{q-1} \left|\log \frac{x}{\xi}\right|^{q-1} p(t,x,s,\xi) d\xi ds$$
$$= \lim_{t\to\infty} \frac{x}{1+x^2+t^4} \int_1^t \int_1^x \left|\log \frac{t}{s}\right|^{q-1} \left|\log \frac{x}{\xi}\right|^{q-1} \frac{d\xi ds}{s\xi}$$
$$= \lim_{t\to\infty} \frac{9x(\log t)^q}{1+x^2+t^4} = 0\,.$$

Hence, by Theorem 10.12, integral equation (10.10) has a random solution defined on $[1,\infty) \times [1,e]$, and the random solutions of this integral equation are globally asymptotically stable.

10.4 Multidelay Hadamard Fractional Integral Equations in Fréchet Spaces with Random Effects

10.4.1 Introduction

In this section, we present some results concerning the existence and Ulam stabilities of random solutions for some functional integral equations of Hadamard fractional order and random effects in Fréchet spaces.

Recently, some interesting results on the existence and Ulam stabilities of the solutions of some classes of differential equations were obtained by Abbas et al. [5, 24, 25, 28]. This section deals with the existence and Ulam stabilities of random solutions of the problem of Hadamard fractional integral equations

$$\begin{aligned} u(t,x,w) &= \mu(t,x,w) + f(t,x,({}^HI_\sigma^r u)(t,x,w), u(t,x,w), w) \\ &\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^t \int_1^x \left(\log\frac{t}{s}\right)^{r_1-1} \left(\log\frac{x}{y}\right)^{r_2-1} \\ &\quad \times g(t,x,s,y,u(s-\tau_1,y-\xi_1,w),\dots,u(s-\tau_m,y-\xi_m,w),w)\frac{dyds}{sy}\,, \\ &\qquad \text{if } (t,x) \in J := [1,+\infty)\times[1,b],\; w\in\Omega\,, \end{aligned} \tag{10.11}$$

$$u(t,x,w) = \Phi(t,x,w)\,, \quad \text{if } (t,x)\in\tilde{J} := [-T,\infty)\times[-\xi,b]\backslash(1,\infty)\times(1,b],\; w\in\Omega\,, \tag{10.12}$$

where $b > 1$, $\sigma = (1,1)$, $r = (r_1,r_2)$, $r_1, r_2 \in (0,\infty)$, ${}^HI_\sigma^r$ is the Hadamard integral of order r, $\tau_i, \xi_i \geq -1$; $i = 1\dots,m$, $T = \max_{i=1\dots,m}\{\tau_i\}$, $\xi = \max_{i=1\dots,m}\{\xi_i\}$, $(\Omega,\mathcal{A})$ is a measurable space, $\mu\colon J\times\Omega\to\mathbb{R}$, $f\colon J\times\mathbb{R}\times\mathbb{R}\times\Omega\to\mathbb{R}$, $g\colon J'\times\mathbb{R}\times\Omega\to\mathbb{R}$ are given continuous functions, and $J' = \{(t,x,s,y)\colon 1\leq s\leq t, 1\leq y\leq x\leq b\}$.

Our investigations are conducted in Fréchet spaces with the application of a stochastic fixed point theorem of Goudarzi for the existence of solutions of problem(10.11)–(10.12), and we prove that all solutions are generalized Ulam–Hyers–Rassias stable.

10.4.2 Existence of Random Solutions and Ulam stabilities results

Let us start by defining what we mean by a random solution of problem (10.11)–(10.12).

Definition 10.13. A function $u \in C$ is said to be a random solution of (10.11)–(10.12) if u satisfies equation (10.11) on J and (10.12) in $\tilde{J}$.

Now we are concerned with the existence and uniform global attractivity of random solutions for problem (10.11)–(10.12). Set

$$J_p := [1,p]\times[1,b]\,, \quad J_p' = \{(t,x,s,y)\colon 1\leq s\leq t\leq p, 1\leq y\leq x\leq b\};\;\; p\in\mathbb{N}\backslash\{0,1\}\,.$$

The following conditions will be used in the sequel:

(10.7.1) The functions $w \mapsto \mu(t,x,w)$ and $w \mapsto \Phi(t,x,w)$ are measurable for a.e. $(t,x) \in J_p$ or $(t,x) \in \tilde{J}$, respectively, and the functions f and g are random Carathéeodory.

(10.7.2) There exist continuous measurable functions $l, k\colon J_p \times \Omega \to \mathbb{R}_+$ such that

$$|f(t,x,u_1,v_1,w) - f(t,x,u_2,v_2,w)| \le l(t,x,w)|u_1 - u_2| + k(t,x,w)|v_1 - v_2|$$

for each $(t,x) \in J_p$, $u_1, u_2, v_1, v_2 \in \mathbb{R}$, and $w \in \Omega$. Moreover, assume that the function $(u,v) \mapsto f(t,x,u,v,w)$ satisfies

$$f(t,x,\lambda u,\lambda v,w) = \lambda f(t,x,u,v,w)\,; \quad \text{for } \lambda \in (0,1), (t,x) \in J_p, \text{and } w \in \Omega\,.$$

(10.7.3) There exist continuous measurable functions $P_i\colon J_p' \times \Omega \to \mathbb{R}_+$, $i = 1,\dots,m$, such that

$$|g(t,x,s,y,u_1,\dots,u_m,w)| \le \sum_{i=1}^{m} P_i(t,x,s,y,w)|u_i|$$

for $(t,x,s,y) \in J_p'$, $u_i \in \mathbb{R}$, and $w \in \Omega$. Moreover, assume that the function $(u_1,\dots,u_m) \mapsto g(t,x,u_1,\dots,u_m,w)$ satisfies

$$g(t,x,\lambda u_1,\dots,\lambda u_m,w) = \lambda g(t,x,u_1,\dots,u_m,w)\,; \quad \text{for } \lambda \in (0,1), (t,x) \in J_p, \text{ and } w \in \Omega\,.$$

(10.7.4) There exist $Q_i\colon J_p \times \Omega \to [0,\infty)$, $i = 1,\dots,m$, with $Q_i(.,w) \in L^\infty(J_p,[0,\infty))$, $i = 1,\dots,m$, such that for each $w \in \Omega$ and a.e. $(t,x) \in J_p$ we have

$$P_i(t,x,s,y,w) \le \varphi(t,x,w)Q_i(s,y,w)\,, \quad i = 1\dots,m\,.$$

For any $p \in \mathbb{N}\backslash\{0,1\}$ set

$$\Phi^* = \sup_{(t,x,w)\in\tilde{J}\times\Omega} |\Phi(t,x,w)|\,, \quad \mu_p = \sup_{(t,x,w)\in J_p\times\Omega} |\mu(t,x,w)|\,,$$

$$f_p = \sup_{(t,x,w)\in J_p\times\Omega} |f(t,x,0,0,w)|\,,$$

$$k_p = \sup_{(t,x,w)\in J_p\times\Omega} k(t,x,w)\,, \quad l_p = \sup_{(t,x,w)\in J_p\times\Omega} l(t,x,w)\,,$$

$$P_{ip} = \sup_{(t,x,w)\in J_p\times\Omega} \int_1^t \int_1^x \left|\log\frac{t}{s}\right|^{r_1-1} \left|\log\frac{x}{y}\right|^{r_2-1} \frac{P_i(t,x,s,y,w)}{\Gamma(r_1)\Gamma(r_2)}\,dyds\,, \quad P_p = \sum_{i=1}^{m} P_{ip}\,.$$

Theorem 10.14. *Assume* (10.7.1)–(10.7.3). *If*

$$\ell_p := P_p + k_p + \frac{l_p(\log p)^{r_1}(\log b)^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} < 1\,, \tag{10.13}$$

then problem (10.11)–(10.12) *has at least one random solution in space C. Furthermore, if condition (10.7.4) holds, then problem* (10.11)–(10.12) *is generalized Ulam–Hyers–Rassias stable.*

Proof. Let $N\colon \Omega \times C \to C$ be the mapping defined by

$$N(w)u(t,x) = \begin{cases} \Phi(t,x,w), & (t,x) \in \tilde{J}\,, \\ \mu(t,x,w) + f(t,x,({}^H I_\sigma^r u)(t,x,w), u(t,x,w), w) & \\ +\dfrac{1}{\Gamma(r_1)\Gamma(r_2)} \displaystyle\int_1^t \int_1^x \left(\log \frac{t}{s}\right)^{r_1-1} \left(\log \frac{x}{y}\right)^{r_2-1} & w \in \Omega\,, \\ \times\, g(t,x,s,y,u(s-\tau_1, y-\xi_1, w), \dots, & \\ u(s-\tau_m, y-\xi_m, w), w)\dfrac{dyds}{sy}, & (t,x) \in J\,. \end{cases} \tag{10.14}$$

The maps Φ, μ, f, and g are continuous for all $w \in \Omega$. Again, as the indefinite integral is continuous on J, $N(w)$ defines a mapping $N\colon \Omega \times C \to C$. Then u is a random solution of problem (10.11)–(10.12) if and only if $u = N(w)u$.
For each $p \in \mathbb{N}\backslash\{0,1\}$ and any $w \in \Omega$ we can show that $N(w)$ transforms the ball $B_\eta := \{u \in C\colon \|u\|_p \le \eta_p\}$ into itself, where $\eta_p := \max\{\Phi^*, \eta_p'\}$, with

$$\eta_p' \ge \frac{\mu_p + f_p}{1 - \ell_p}\,.$$

Indeed, for any $w \in \Omega$ and each $u \in C$ and $(t,x) \in \tilde{J}$ we have

$$|N(w)u(t,x)| \le |\Phi(t,x,w)| \le \Phi^*\,,$$

and for any $w \in \Omega$ and each $u \in C$ and $(t,x) \in J_p$ we have

$$\begin{aligned} |N(w)u(t,x)| &\le |\mu(t,x,w)| + |f(t,x,({}^H I_\sigma^r u)(t,x,w), u(t,x,w), w)| \\ &\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^t \int_1^x \left|\log \frac{t}{s}\right|^{r_1-1} \left|\log \frac{x}{y}\right|^{r_2-1} \\ &\quad \times |g(t,x,s,y,u(s-\tau_1, y-\xi_1, w), \dots, u(s-\tau_m, y-\xi_m, w), w)|dyds \\ &\le |\mu(t,x,w)| + |f(t,x,0,0,w)| \\ &\quad + l(t,x,w)|({}^H I_\sigma^r u)(t,x,w)| + k(t,x,w)|u(t,x,w)| \\ &\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^t \int_1^x \left|\log \frac{t}{s}\right|^{r_1-1} \left|\log \frac{x}{y}\right|^{r_2-1} \\ &\quad \times \sum_{i=1}^m P_i(t,x,s,y)|u(s-\tau_i, y-\xi_i)|dyds \\ &\le \mu(t,x,w)| + |f(t,x,0,0,w)| + \eta_p' l(t,x,w)|{}^H I_\sigma^r 1| + \eta_p' k(t,x,w) \\ &\quad + \frac{\eta_p'}{\Gamma(r_1)\Gamma(r_2)} \sum_{i=1}^m \int_1^t \int_1^x \left|\log \frac{t}{s}\right|^{r_1-1} \left|\log \frac{x}{y}\right|^{r_2-1} P_i(t,x,s,y)dyds \end{aligned}$$

$$\begin{aligned}
&\le \mu_p + f_p + \frac{\eta'_p l_p (\log p)^{r_1} (\log b)^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} + \eta'_p k_p + \eta'_p \sum_{i=1}^{m} P_{ip} \\
&\le \mu_p + f_p + \eta'_p \left(P_p + k_p + \frac{l_p (\log p)^{r_1} (\log b)^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} \right) \\
&= \mu_p + f_p + \eta'_p \ell_p \\
&\le \eta'_p \,.
\end{aligned}$$

Thus,

$$\|N(u)\|_p \le \eta_p \,.$$

Hence, $N(w)$ transforms the ball B_η into itself. We will show that $N\colon \Omega \times B_\eta \to B_\eta$ satisfies the assumptions of [147, Theorem 3.1]. The proof will be given in three steps.

Step 1. $N(w)$ is a random operator on $\Omega \times B_\eta$ into B_η. Since $f(t,x,u,v,w)$ is random Carathéodory, the map $w \to f(t,x,u,v,w)$ is measurable in view of Lemma 10.2. Similarly, the product $(\log \frac{t}{s})^{r_1-1}(\log \frac{x}{\xi})^{r_2-1} g(t,x,u_1,\dots,u_m,w)$ of a continuous and a measurable function is again measurable. Further, the integral is a limit of a finite sum of measurable functions; therefore, the map

$$w \mapsto N(w)u(t,x,w)$$

is measurable. As a result, $N(w)$ is a random operator on $\Omega \times B_\eta$ into B_η.

Step 2. $N(w)$ is continuous. Let $\{u_n\}$ be a sequence such that $u_n \to u$ in B_η. Then for each $(x,y) \in J_p$ and $w \in \Omega$ we have

$$\begin{aligned}
&|(N(w)u_n)(x,y) - (N(w)u)(x,y)| \\
&\le |f(t,x,({}^H I^r_\sigma u)(t,x,w), u_n(t,x,w), w) - f(t,x,({}^H I^r_\sigma u)(t,x,w), u(t,x,w), w)| \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^t \int_1^x \left|\log \frac{t}{s}\right|^{r_1-1} \left|\log \frac{x}{y}\right|^{r_2-1} \\
&\qquad \times |g(t,x,s,y,u_n(s-\tau_1,y-\xi_1,w),\dots,u_n(s-\tau_m,y-\xi_m,w),w) \\
&\quad - g(t,x,s,y,u(s-\tau_1,y-\xi_1,w),\dots,u(s-\tau_m,y-\xi_m,w),w)| \frac{dyds}{sy} \\
&\le l(t,x,w) {}^H I^r_\sigma |u_n(t,x,w) - u(t,x,w)| + k(t,x,w)|u_n(t,x,w) - u(t,x,w)| \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^t \int_1^x \left|\log \frac{t}{s}\right|^{r_1-1} \left|\log \frac{x}{y}\right|^{r_2-1} \\
&\qquad \times |g(t,x,s,y,u_n(s-\tau_1,y-\xi_1,w),\dots,u_n(s-\tau_m,y-\xi_m,w),w) \\
&\quad - g(t,x,s,y,u(s-\tau_1,y-\xi_1,w),\dots,u(s-\tau_m,y-\xi_m,w),w)| dyds \,.
\end{aligned}$$

From the continuity of g and ${}^H I^r_\sigma$ and using the Lebesgue dominated convergence theorem, we get

$$\|N(w)u_n - N(w)u\|_p \to 0 \ \text{ as } \ n \to \infty \,.$$

Step 3. $N(w)$ is affine. For each $u, v \in B_\eta$, $(t,x) \in J_p^*$ and any $\lambda \in (0,1)$ and $w \in \Omega$ we have

$$\begin{aligned}
N(w)(\lambda u + (1-\lambda)v) &= \mu(t,x,w) + \lambda f(t,x,({}^H I_\sigma^r u)(t,x,w), u(t,x,w), w) \\
&\quad + \frac{\lambda}{\Gamma(r_1)\Gamma(r_2)} \int_1^t \int_1^x \left(\log \frac{t}{s}\right)^{r_1-1} \left(\log \frac{x}{y}\right) r_2 - 1 \\
&\quad \times g(t,x,s,y,u(s-\tau_1, y-\xi_1, w), \dots, u(s-\tau_m, y-\xi_m, w), w) dyds \\
&\quad + (1-\lambda) f(t,x,({}^H I_\sigma^r u)(t,x,w), u(t,x,w), w) \\
&\quad + \frac{1-\lambda}{\Gamma(r_1)\Gamma(r_2)} \int_1^t \int_1^x \left(\log \frac{t}{s}\right)^{r_1-1} \left(\log \frac{x}{y}\right) r_2 - 1 \\
&\quad \times g(t,x,s,y,u(s-\tau_1, y-\xi_1, w), \dots, u(s-\tau_m, y-\xi_m, w), w) dyds \\
&\quad = \lambda N(w)(u) + (1-\lambda) N(w)(v) .
\end{aligned}$$

Hence, $N(w)$ is affine.

As a consequence of Steps 1–3, together with [147, Theorem 3.1], we deduce that N has a fixed point v that is a random solution of problem (10.11)–(10.12).

Step 4. Generalized Ulam–Hyers–Rassias stability. Set

$$Q_{ip} = \sup_{(s,y,w) \in J_p \times \Omega} \operatorname{ess} Q_i(s,y,w), \quad Q_p = \sum_{i=1}^m Q_{ip} .$$

Let $u\colon \Omega \to B_\eta$ be a solution of the inequality

$$\|u(t,x,w) - (N(w)u)(t,x)\|_p \le \varphi(t,x,w), \quad \text{for a.e. } (t,x) \in J_p^*, \; w \in \Omega, \tag{10.15}$$

and v a random solution of problem (10.11)–(10.12). Then $\|u\|_p \le \eta$, $\|v\|_p \le \eta$, and

$$v(t,x,w) = \begin{cases} \Phi(t,x,w), & (t,x) \in \tilde{J}, \\ \mu(t,x,w) + f(t,x,({}^H I_\sigma^r v)(t,x,w), v(t,x,w), w) & \\ + \dfrac{1}{\Gamma(r_1)\Gamma(r_2)} \displaystyle\int_1^t \int_1^x \left(\log \frac{t}{s}\right)^{r_1-1} \left(\log \frac{x}{y}\right)^{r_2-1} & w \in \Omega, \\ \times g(t,x,s,y,v(s-\tau_1, y-\xi_1, w), \dots, & \\ v(s-\tau_m, y-\xi_m, w), w) \dfrac{dyds}{sy}; & (t,x) \in J. \end{cases}$$

For each $(t,x) \in \tilde{J}$ and any $w \in \Omega$ we have

$$\begin{aligned}
|u(t,x,w) - v(x,y,w)| &\le |u(t,x,w) - N(w)(u(t,x,w))| \\
&\quad + |N(w)(u(t,x,w)) - N(w)(v(t,x,w))| \\
&\le \varphi(x,y,w) .
\end{aligned}$$

Next, from condition (10.7.4), for each $(t,x) \in J_p$ and any $w \in \Omega$ we have

$$
\begin{aligned}
|u(t,x,w) - v(x,y,w)| &\le |u(t,x,w) - N(w)(u(t,x,w))| \\
&\quad + |N(w)(u(t,x,w)) - N(w)(v(t,x,w))| \\
&\le \varphi(x,y,w) + |f(t,x,({}^H I_\sigma^r u)(t,x,w), u(t,x,w)) - f(t,x,({}^H I_\sigma^r v)(t,x,w), v(t,x,w))| \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^t \int_1^x \left|\log \frac{t}{s}\right|^{r_1-1} \left|\log \frac{x}{y}\right|^{r_2-1} \\
&\quad \times |g(t,x,s,y,u(s-\tau_1,y-\xi_1,w),\dots,u(s-\tau_m,y-\xi_m,w),w) \\
&\quad - g(t,x,s,y,v(s-\tau_1,y-\xi_1,w),\dots,v(s-\tau_m,y-\xi_m,w),w)| \frac{dyds}{sy} \\
&\le \varphi(x,y,w) + l(t,x,w)|({}^H I_\sigma^r u)(t,x,w) - ({}^H I_\sigma^r v)(t,x,w)| \\
&\quad + k(t,x,w)|u(t,x,w) - v(t,x,w)| \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^t \int_1^x \left|\log \frac{t}{s}\right|^{r_1-1} \left|\log \frac{x}{y}\right|^{r_2-1} \varphi(t,x,w) \\
&\quad \times \left(\sum_{i=1}^m Q_i(s,y)(|u(s-\tau_i,y-\xi_i,w)| + |v(s-\tau_i,y-\xi_i,w)|) \right) dyds \\
&\le \varphi(x,y,w) + \ell_p |u(t,x,w) - v(t,x,w)| \\
&\quad + \frac{2\eta\varphi(t,x,w)}{\Gamma(r_1)\Gamma(r_2)} \int_1^t \int_1^x \left|\log \frac{t}{s}\right|^{r_1-1} \left|\log \frac{x}{y}\right|^{r_2-1} \left(\sum_{i=1}^m Q_{ip} \right) dyds \\
&\le \varphi(t,x,w) + \ell_p |u(t,x,w) - v(t,x,w)| \\
&\quad + \frac{2\eta Q_p \varphi(t,x,w)}{\Gamma(r_1)\Gamma(r_2)} \int_1^t \int_1^x \left|\log \frac{t}{s}\right|^{r_1-1} \left|\log \frac{x}{y}\right|^{r_2-1} dyds\,.
\end{aligned}
$$

Thus, for each $(t,x) \in J_p$ and any $w \in \Omega$ we obtain

$$
\begin{aligned}
|u(t,x,w) - v(x,y,w)| &\le \frac{\varphi(t,x,w)}{1-\ell_p} \left(1 + \frac{2\eta Q_p}{\Gamma(r_1)\Gamma(r_2)} \int_1^t \int_1^x \left|\log \frac{t}{s}\right|^{r_1-1} \left|\log \frac{x}{y}\right|^{r_2-1} dyds \right) \\
&\le \frac{1}{1-\ell_p} \left(1 + \frac{2\eta Q_p (\log p)^{r_1} (\log b)^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} \right) \varphi(t,x,w) \\
&:= c'_{N,\varphi} \varphi(t,x,w)\,.
\end{aligned}
$$

Hence, for each $(t,x) \in J_p^*$ and any $w \in \Omega$ we get

$$
|u(t,x,w) - v(x,y,w)| \le c_{N,\varphi} \varphi(x,y,w)\,,
$$

where $c_{N,\varphi} := \max\{1, c'_{N,\varphi}\}$. Consequently, random problem (10.11)–(10.12) is generalized Ulam–Hyers–Rassias stable. □

10.4.3 An Example

Let $\Omega = (-\infty, 0)$ be equipped with the usual σ-algebra consisting of Lebesgue measurable subsets of $(-\infty, 0)$. Given a measurable function $u\colon \Omega \to C([-1,\infty) \times [-2, e])$, consider the problem of Hadamard fractional order integral equations

$$\begin{aligned} u(t,x,w) &= \frac{xe^{3-2t}}{(1+w^2)(1+t+x^2)} + \frac{xc_pe^{-t-2}}{1+w^2}(e^{2p}|({}^HI^r_\sigma u)(t,x,w)| + e^p|u(t,x,w)|) \\ &\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^t\int_1^x \left(\log\frac{t}{s}\right)^{r_1-1}\left(\log\frac{x}{y}\right)^{r_2-1} \\ &\quad \times g(t,x,s,y,u(s-1,y-2,w),u(s-\frac{1}{2},y-\frac{2}{5},w),w) \\ &\quad \times \frac{1}{sy}dyds\,, \quad \text{if } (t,x) \in J := [1,+\infty) \times [1,e],\ w \in \Omega\,, \end{aligned} \tag{10.16}$$

$$u(t,x,w) = \frac{2}{(1+w^2)(2+t^2)(2+x^2)}\,, \quad \text{if } (t,x) \in \tilde{J},\ w \in \Omega\,, \tag{10.17}$$

where

$$\tilde{J} := [-1,\infty) \times [-2,e]\backslash(1,\infty) \times (1,e]\,, \quad r = (r_1,r_2) \in (0,\infty) \times (0,\infty)\,,$$

$$c_p = \frac{e^{-2}}{p^{\frac{-3}{4}}e + e^{-2+p} + \frac{e^{-2+2p}p^{r_1}}{\Gamma(1+r_1)\Gamma(1+r_2)}}\,, \quad p \in \mathbb{N}\backslash\{0,1\}\,,$$

$$g(t,x,s,y,u_1,u_2,w) = \frac{xc_ps^{\frac{-3}{4}}(|u_1|+|u_2|)\sin\sqrt{t}\sin s}{(1+w^2)(1+x^2+t^2)}\,, \quad \text{if } (t,x,s,y) \in J'\,,$$

and $u_1, u_2 \in \mathbb{R}$,

and

$$J' = \{(t,x,s,y)\colon 1 \le s \le t \text{ and } 1 \le x \le y \le e\}\,.$$

Set

$$\mu(t,x,w) = \frac{xe^{3-2t}}{(1+w^2)(1+t+x^2)}\,,$$

$$f(t,x,u,v,w) = \frac{xc_pe^{-t-2}}{1+w^2}(e^{2p}|u| + e^p|v|)\,; \quad p \in \mathbb{N}\backslash\{0,1\}\,.$$

We have $\mu_p = e^2$. The function f is continuous and satisfies (10.7.2), with

$$l(t,x,w) = \frac{xc_pe^{-t-2+2p}}{c}1+w^2\,, \qquad k(t,x,w) = \frac{xc_pe^{-t-2+p}}{1+w^2}\,,$$
$$l_p = c_pe^{-2+2p}, \qquad l_p = c_pe^{-2+p}\,.$$

Also, the function g is continuous and satisfies (10.7.3), with

$$P_1(t,x,s,y,w) = P_2(t,x,s,y,w) = \frac{xc_ps^{\frac{-3}{4}}\sin\sqrt{t}\sin s}{(1+w^2)(1+x^2+t^2)}\,; \quad (t,x,s,y) \in J'\,,$$

$$P_p = c_pp^{\frac{-3}{4}}e\,.$$

Condition (10.7.4) is satisfied by

$$Q_1(s, y, w) = Q_2(s, y, w) = \frac{c_p s^{\frac{-3}{4}} \sin s}{1 + w^2}, \quad \text{and} \quad \varphi(t, x, w) = \frac{x \sin \sqrt{t}}{1 + x^2 + t^2} .$$

Condition (10.13) holds, with $b = e$. Indeed, for each $p \in \mathbb{N} \backslash \{0, 1\}$ we get

$$P_p + k_p + \frac{l_p (\log p)^{r_1} (\log b)^{r_2}}{\Gamma(1 + r_1)\Gamma(1 + r_2)} = c_p \left(p^{\frac{-3}{4}} e + e^{-2+p} + \frac{e^{-2+2p} p^{r_1}}{\Gamma(1 + r_1)\Gamma(1 + r_2)} \right) = e^{-2} < 1 .$$

Hence, by Theorem 10.14, problem (10.16)–(10.17) has a random solution defined on $[-1, +\infty) \times [-2, e]$ and is generalized Ulam–Hyers–Rassias stable.

10.5 Notes and Remarks

The results of Chapter 10 are taken from Abbas et al. [8, 7, 6, 27]. Other results may be found in [5, 7, 21, 36, 40].