9 Hadamard–Stieltjes Fractional Integral Equations

9.1 Introduction

If u is a real function defined on the interval $[a, b]$, then the symbol $\bigvee_a^b u$ denotes the variation of u on $[a, b]$. We say that u is of bounded variation on the interval $[a, b]$ whenever $\bigvee_a^b u$ is finite. If $w\colon [a, b] \times [c, b] \to \mathbb{R}$, then the symbol $\bigvee_{t=p}^q w(t, s)$ indicates the variation of the function $t \to w(t, s)$ on the interval $[p, q] \subset [a, b]$, where s is arbitrarily fixed in $[c, d]$. In the same way we define $\bigvee_{s=p}^q w(t, s)$. For the properties of functions of bounded variation we refer the reader to [204].

If u and φ are two real functions defined on the interval $[a, b]$, then under some conditions (see [204]) we can define the Stieltjes integral (in the Riemann–Stieltjes sense)

$$\int_a^b u(t)d\varphi(t)$$

of the function u with respect to φ. In this case we say that u is Stieltjes integrable on $[a, b]$ with respect to φ. Several conditions are known guaranteeing Stieltjes integrability [204]. One of the most frequently used requires that u be continuous and φ be of bounded variation on $[a, b]$.

If u is Stieltjes integrable on the interval $[a, b]$ with respect to a function φ of bounded variation, then

$$\left|\int_a^b u(t)d\varphi(t)\right| \le \int_a^b |u(t)|d\left(\bigvee_a^t \varphi\right) .$$

If u and v are Stieltjes integrable functions on the interval $[a, b]$ with respect to a nondecreasing function φ such that $u(t) \le v(t)$ for $t \in [a, b]$. Then

$$\int_a^b u(t)d\varphi(t) \le \int_a^b v(t)d\varphi(t) .$$

In the sequel we consider Stieltjes integrals of the form

$$\int_a^b u(t)d_s g(t, s)$$

and Hadamard–Stieltjes integrals of fractional order

$$\frac{1}{\Gamma(r)} \int_1^t \left(\log \frac{t}{s}\right)^{q-1} u(s)d_s g(t, s) ,$$

where $g\colon [1, \infty) \times [1, \infty) \to \mathbb{R}$, $q \in (0, \infty)$, and the symbol d_s indicates the integration with respect to s.

https://doi.org/10.1515/9783110553819-009

Definition 9.1. Let $r_1, r_2 \geq 0$, $\sigma = (1,1)$ and $r = (r_1, r_2)$. For $w \in L^1(J, \mathbb{R})$ define the Hadamard–Stieltjes partial fractional integral of order r by the expression

$$({}^{HS}I^r_\sigma w)(x,y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left(\log\frac{x}{s}\right)^{r_1-1} \left(\log\frac{y}{t}\right)^{r_2-1} \frac{w(s,t)}{st} d_t g_2(y,t) d_s g_1(x,s) ,$$

where $g_1, g_2 \colon [1,\infty) \times [1,\infty) \to \mathbb{R}$.

9.2 Existence and Stability of Solutions for Hadamard–Stieltjes Fractional Integral Equations

9.2.1 Introduction

We give some existence results and Ulam stability results for a class of Hadamard–Stieltjes integral equations. We present two results, the first one an existence result based on Schauder's fixed point theorem, and the second one about the generalized Ulam–Hyers–Rassias stability.

This section deals with the existence of the Ulam stability of solutions to the Hadamard–Stieltjes fractional integral equation

$$\begin{aligned} u(x,y) &= \mu(x,y) \\ &+ \int_1^x \int_1^y \left(\log\frac{x}{s}\right)^{r_1-1} \left(\log\frac{y}{t}\right)^{r_2-1} \frac{f(s,t,u(s,t))}{st\Gamma(r_1)\Gamma(r_2)} d_t g_2(y,t) d_s g_1(x,s) ; \quad \text{if } (x,y) \in J , \end{aligned} \tag{9.1}$$

where $J := [1,a] \times [1,b]$, $a, b > 1$, $r_1, r_2 > 0$, and $\mu \colon J \to \mathbb{R}$, $f \colon J \times \mathbb{R} \to \mathbb{R}$, $g_1 \colon [1,a]^2 \to \mathbb{R}$, and $g_2 \colon [1,b]^2 \to \mathbb{R}$ are given continuous functions.

Our investigations are conducted with an application of Schauder's fixed point theorem for the existence of solutions of integral equation (9.1). Also, we obtain some results on the generalized Ulam–Hyers–Rassias stability of solutions of (9.1). Finally, we present an example illustrating the applicability of the imposed conditions.

9.2.2 Existence and Ulam Stabilities Results

In this section, we discuss the existence of solutions and present conditions for the Ulam stability for the Hadamard integral equation (7.1).

The following conditions will be used in the sequel:

(9.2.1) There exist functions $p_1, p_2 \in C(J, \mathbb{R}_+)$ such that for any $u \in \mathbb{R}$ and $(x,y) \in J$

$$|f(x,y,u)| \leq p_1(x,y) + \frac{p_2(x,y)}{1 + |u(x,y)|} |u(x,y)| ,$$

with

$$p_i^* = \sup_{(x,y)\in J} \sup_{(s,t)\in[1,x]\times[1,y]} \left|\log\frac{x}{s}\right|^{r_1-1} \left|\log\frac{y}{t}\right|^{r_2-1} \frac{p_i(s,t)}{st\Gamma(r_1)\Gamma(r_2)}\,; \quad i = 1,2\,.$$

(9.2.2) For all $x_1, x_2 \in [1,a]$ such that $x_1 < x_2$ the function $s \mapsto g(x_2,s) - g(x_1,s)$ is nondecreasing on $[1,a]$. Also, for all $y_1, y_2 \in [1,b]$ such that $y_1 < y_2$ the function $s \mapsto g(y_2,t) - g(y_1,t)$ is nondecreasing on $[1,b]$.

(9.2.3) The functions $s \mapsto g_1(0,s)$ and $t \mapsto g_2(0,t)$ are nondecreasing on $[1,a]$ and $[1,b]$, respectively.

(9.2.4) The functions $s \mapsto g_1(x,s)$ and $x \mapsto g_1(x,s)$ are continuous on $[1,a]$ for each fixed $x \in [1,a]$ and $s \in [1,a]$, respectively. Also, the functions $t \mapsto g_2(y,t)$ and $y \mapsto g_2(y,t)$ are continuous on $[1,b]$ for each fixed $y \in [1,b]$ or $t \in [1,b]$, respectively.

(9.2.5) There exists $\lambda_\Phi > 0$ such that for each $(x,y) \in J$ we have

$$({}^{HS}I_\sigma^r \Phi)(x,y) \le \lambda_\Phi \Phi(x,y)\,.$$

Set

$$g^* = \sup_{(x,y)\in J} \bigvee_{k_2=1}^{y} g_2(y,k_2) \bigvee_{k_1=1}^{x} g_1(x,k_1)\,.$$

Theorem 9.2. *Assume* (9.2.1)–(9.2.4)*; then integral equation* (9.1) *has a solution defined on J.*

Proof. Let $\rho > 0$ be a constant such that

$$\rho > \|\mu\|_\infty + g^*(p_1^* + p_2^*)\,.$$

We will use Schauder's fixed point theorem [149] to prove that the operator $N\colon C \to C$ defined by

$$(Nu)(x,y) = \mu(x,y) + \int_1^x \int_1^y \left(\log\frac{x}{s}\right)^{r_1-1} \left(\log\frac{y}{t}\right)^{r_2-1} \frac{f(s,t,u(s,t))}{st\Gamma(r_1)\Gamma(r_2)} d_t g_2(y,t) d_s g_1(x,s) \tag{9.2}$$

has a fixed point. The proof will be given in four steps.

Step 1: N transforms the ball $B_\rho := \{u \in \mathcal{C}\colon \|u\|_C \le \rho\}$ into itself. For any $u \in B_\rho$ and each $(x,y) \in J$ we have

$$\begin{aligned}|(Nu)(x,y)| \le |\mu(x,y)| &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left|\log\frac{x}{s}\right|^{r_1-1} \left|\log\frac{y}{t}\right|^{r_2-1} \\ &\times \frac{p_1(s,t)}{st} |d_t g_2(y,t) d_s g_1(x,s)| \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left|\log\frac{x}{s}\right|^{r_1-1} \left|\log\frac{y}{t}\right|^{r_2-1}\end{aligned}$$

$$\times \frac{p_2(s,t)|u(s,t)|}{st(1+|u(s,t)|)}|d_t g_2(y,t) d_s g_1(x,s)|$$
$$\le \|\mu\|_C + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left|\log \frac{x}{s}\right|^{r_1-1} \left|\log \frac{y}{t}\right|^{r_2-1}$$
$$\times \frac{p_1(s,t) + p_2(s,t)\rho}{st} d_t \bigvee_{k_2=1}^{t} g_2(y,k_2) d_s \bigvee_{k_1=1}^{s} g_1(x,k_1)$$
$$\le \|\mu\|_C + (p_1^* + p_2^*) \int_1^x \int_1^y d_t \bigvee_{k_2=1}^{t} g_2(y,k_2) d_s \bigvee_{k_1=1}^{s} g_1(x,k_1)$$
$$\le \|\mu\|_C + g^*(p_1^* + p_2^*)$$
$$\le \rho\,.$$

Thus, $\|(Nu)\|_C \le \rho$. This implies that N transforms the ball B_ρ into itself.

Step 2: $N\colon B_\rho \to B_\rho$ is continuous. Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence such that $u_n \to u$ in B_ρ. Then

$$|(Nu_n)(x,y) - (Nu)(x,y)| \le \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left|\log \frac{x}{s}\right|^{r_1-1} \left|\log \frac{y}{t}\right|^{r_2-1}$$
$$\times \frac{|f(s,t,u_n(s,t)) - f(s,t,u(s,t))|}{st} d_t g_2(y,t) d_s g_1(x,s)$$
$$\le \frac{\sup_{(s,t)\in J} |f(s,t,u_n(s,t)) - f(s,t,u(s,t))|}{\Gamma(r_1)\Gamma(r_2)}$$
$$\times \int_1^x \int_1^y \left|\log \frac{x}{s}\right|^{r_1-1} \left|\log \frac{y}{t}\right|^{r_2-1} d_t \bigvee_{k_2=1}^{t} g_2(y,k_2) d_s \bigvee_{k_1=1}^{s} g_1(x,k_1)$$
$$\le g^* \|f(.,.,u_n(.,.)) - f(.,.,u(.,.))\|_C\,.$$

From Lebesgue's dominated convergence theorem and the continuity of the function f we get

$$|(Nu_n)(x,y) - (Nu)(x,y)| \to 0 \quad \text{as} \quad n \to \infty\,.$$

Step 3: *$N(B_\rho)$ is bounded.* This is clear since $N(B_\rho) \subset B_\rho$ and B_ρ is bounded.

Step 4: *$N(B_\rho)$ is equicontinuous.* Let $(x_1,y_1), (x_2,y_2) \in J$, $x_1 < x_2$, $y_1 < y_2$. Then

$$|(Nu)(x_2,y_2) - (Nu)(x_1,y_1)| \le |\mu(x_1,y_1) - \mu(x_2,y_2)|$$
$$+ \left| \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^{x_1} \int_1^{y_1} \left|\log \frac{x_2}{s}\right|^{r_1-1} \left|\log \frac{y_2}{t}\right|^{r_2-1} \times \frac{f(s,t,u(s,t))}{st} d_t g_2(y_2,t) d_s g_1(x_2,s) \right.$$
$$\left. - \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^{x_1} \int_1^{y_1} \left|\log \frac{x_1}{s}\right|^{r_1-1} \left|\log \frac{y_1}{t}\right|^{r_2-1} \times \frac{f(s,t,u(s,t))}{st} d_t g_2(y_1,t) d_s g_1(x_1,s) \right|$$

$$+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_1}^{x_2}\int_{y_1}^{y_2}\left|\log\frac{x_2}{s}\right|^{r_1-1}\left|\log\frac{y_2}{t}\right|^{r_2-1}\frac{|f(s,t,u(s,t))|}{st}|d_t g_2(y_2,t)d_s g_1(x_2,s)|$$
$$+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{1}^{x_1}\int_{y_1}^{y_2}\left|\log\frac{x_2}{s}\right|^{r_1-1}\left|\log\frac{y_2}{t}\right|^{r_2-1}\frac{|f(s,t,u(s,t))|}{st}|d_t g_2(y_2,t)d_s g_1(x_2,s)|$$
$$+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_1}^{x_2}\int_{1}^{y_1}\left|\log\frac{x_2}{s}\right|^{r_1-1}\left|\log\frac{y_2}{t}\right|^{r_2-1}\frac{|f(s,t,u(s,t))|}{st}|d_t g_2(y_2,t)d_s g_1(x_2,s)|\,.$$

Thus, we obtain

$$\begin{aligned}|(Nu)(x_2,y_2)-(Nu)(x_1,y_1)| &\le |\mu(x_1,y_1)-\mu(x_2,y_2)|\\
&+(p_1^*+p_2^*)\int_1^{x_1}\int_1^{y_1}\Bigg|d_t\bigvee_{k_2=1}^{t}g_2(y_2,k_2)d_s\bigvee_{k_1=1}^{s}g_1(x_2,k_1)\\
&-d_t\bigvee_{k_2=1}^{t}g_2(y_1,k_2)d_s\bigvee_{k_1=1}^{s}g_1(x_1,k_1)\Bigg|\\
&+(p_1^*+p_2^*)\int_{x_1}^{x_2}\int_{y_1}^{y_2}d_t\bigvee_{k_2=1}^{t}g_2(y_2,k_2)d_s\bigvee_{k_1=1}^{s}g_1(x_2,k_1)\\
&+(p_1^*+p_2^*)\int_{1}^{x_1}\int_{y_1}^{y_2}d_t\bigvee_{k_2=1}^{t}g_2(y_2,k_2)d_s\bigvee_{k_1=1}^{s}g_1(x_2,k_1)\\
&+(p_1^*+p_2^*)\int_{x_1}^{x_2}\int_{1}^{y_1}d_t\bigvee_{k_2=1}^{t}g_2(y_2,k_2)d_s\bigvee_{k_1=1}^{s}g_1(x_2,k_1)\,.\end{aligned}$$

Hence, we get

$$\begin{aligned}&|(Nu)(x_2,y_2)-(Nu)(x_1,y_1)| \le |\mu(x_1,y_1)-\mu(x_2,y_2)|\\
&+(p_1^*+p_2^*)\left|\bigvee_{k_2=1}^{y_1}g_2(y_2,k_2)\bigvee_{k_1=1}^{x_1}g_1(x_2,k_1)-\bigvee_{k_2=1}^{y_1}g_2(y_1,k_2)\bigvee_{k_1=1}^{x_1}g_1(x_1,k_1)\right|\\
&+(p_1^*+p_2^*)\bigvee_{k_2=y_1}^{y_2}g_2(y_2,k_2)\bigvee_{k_1=x_1}^{x_2}g_1(x_2,k_1)\\
&+(p_1^*+p_2^*)\bigvee_{k_2=y_1}^{y_2}g_2(y_2,k_2)\bigvee_{k_1=1}^{x_2}g_1(x_2,k_1)\\
&+(p_1^*+p_2^*)\bigvee_{k_2=1}^{y_2}g_2(y_2,k_2)\bigvee_{k_1=x_1}^{x_2}g_1(x_2,k_1)\,.\end{aligned}$$

As $x_1 \to x_2$ and $y_1 \to y_2$, the right-hand side of the preceding inequality tends to zero.

As a consequence of Steps 1–4, together with the Ascoli–Arzelà theorem, we can conclude that N is continuous and compact. From an application of Schauder's theorem [149] we deduce that N has a fixed point u that is a solution of integral equation (9.1). □

Now we are concerned with the stability of solutions for integral equation (9.1). Let $\epsilon > 0$ and $\Phi\colon J \to [0, \infty)$ be a continuous function. We consider the inequalities

$$|u(x, y) - (Nu)(x, y)| \le \epsilon; \quad (x, y) \in J , \tag{9.3}$$

$$|u(x, y) - (Nu)(x, y)| \le \Phi(x, y); \quad (x, y) \in J , \tag{9.4}$$

$$|u(x, y) - (Nu)(x, y)| \le \epsilon\Phi(x, y); \quad (x, y) \in J . \tag{9.5}$$

Theorem 9.3. *Assume* (9.2.1)–(9.2.5). *If there exist* $q_i \in C(J, \mathbb{R}_+)$; $i = 1, 2$ *such that for each* $(x, y) \in J$ *we have*

$$p_i(x, y) \le q_i(x, y)\Phi(x, y) ,$$

then integral equation (9.1) *is generalized Ulam–Hyers–Rassias stable.*

Proof. Let u be a solution of inequality (9.4). By Theorem 9.2, there exists v that is a solution of integral equation (9.1). Hence,

$$v(x, y) = \mu(x, y) + \int_1^x \int_1^y \left(\log \frac{x}{s}\right)^{r_1-1} \left(\log \frac{y}{t}\right)^{r_2-1} \frac{f(s, t, v(s, t))}{st\Gamma(r_1)\Gamma(r_2)} d_t g_2(y, t) d_s g_1(x, s) .$$

By inequality (9.4) for each $(x, y) \in J$ we have

$$\left| u(x, y) - \mu(x, y) - \int_1^x \int_1^y \left(\log \frac{x}{s}\right)^{r_1-1} \left(\log \frac{y}{t}\right)^{r_2-1} \right.$$
$$\left. \times \frac{f(s, t, u(s, t))}{st\Gamma(r_1)\Gamma(r_2)} d_t g_2(y, t) d_s g_1(x, s) \right| \le \Phi(x, y) .$$

Set

$$q_i^* = \sup_{(x,y)\in J} q_i(x, y) ; \quad i = 1, 2 .$$

For each $(x, y) \in J$ we have

$$|u(x, y) - v(x, y)|$$
$$\le \left| u(x, y) - \mu(x, y) - \int_1^x \int_1^y \left(\log \frac{x}{s}\right)^{r_1-1} \left(\log \frac{y}{t}\right)^{r_2-1} \right.$$
$$\left. \times \frac{f(s, t, u(s, t))}{st\Gamma(r_1)\Gamma(r_2)} d_t g_2(y, t) d_s g_1(x, s) \right|$$
$$+ \int_1^x \int_1^y \left|\log \frac{x}{s}\right|^{r_1-1} \left|\log \frac{y}{t}\right|^{r_2-1}$$
$$\times \frac{|f(s, t, u(s, t)) - f(s, t, v(s, t))|}{st\Gamma(r_1)\Gamma(r_2)} d_t g_2(y, t) d_s g_1(x, s)$$
$$\le \Phi(x, y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left|\log \frac{x}{s}\right|^{r_1-1} \left|\log \frac{y}{t}\right|^{r_2-1}$$
$$\times \left(2q_1^* + \frac{q_2^*|u(s, t)|}{1 + |u|} + \frac{q_2^*|v(s, t)|}{1 + |v|}\right) \frac{\Phi(s, t)}{st} d_t g_2(y, t) d_s g_1(x, s)$$

$$\le \Phi(x,y) + 2(q_1^* + q_2^*)({}^{HS}I_\sigma^r \Phi)(x,y)$$
$$\le [1 + 2(q_1^* + q_2^*)\lambda_\phi]\Phi(x,y)$$
$$:= c_{N,\Phi}\Phi(x,y)\,.$$

Hence, integral equation (9.1) is generalized Ulam–Hyers–Rassias stable. □

9.2.3 An Example

As an application of our results we consider the Hadamard–Stieltjes integral equation

$$u(x,y) = \mu(x,y) + \int_1^x \int_1^y \left(\log \frac{x}{s}\right)^{r_1-1} \left(\log \frac{y}{t}\right)^{r_2-1} \frac{f(s,t,u(s,t))}{st\Gamma(r_1)\Gamma(r_2)} d_t g_2(y,t) d_s g_1(x,s)\,;$$
$$(x,y) \in [1,e] \times [1,e]\,, \tag{9.6}$$

where

$$r_1, r_2 > 0, \qquad \mu(x,y) = x + y^2; \qquad (x,y) \in [1,e] \times [1,e],$$
$$g_1(x,s) = s, \qquad g_2(y,t) = t, \qquad s,t \in [1,e]\,,$$

and

$$f(x,y,u(x,y)) = xy^2\left(e^{-7} + \frac{u(x,y)}{e^{x+y+5}}\right), \quad (x,y) \in [1,e] \times [1,e]\,.$$

Condition (9.2.1) is satisfied with $p_1(x,y) = xy^2e^{-7}$ and $p_2^* = \frac{xy^2}{e^{x+y+5}}$. We can see that the functions g_1 and g_2 satisfy (9.2.2)–(9.2.4). Consequently, Theorem 9.2 implies that Hadamard integral equation (9.6) has a solution defined on $[1,e] \times [1,e]$.
Condition (9.2.5) is satisfied by

$$\Phi(x,y) = e^3, \text{ and } \lambda_\Phi = \frac{1}{\Gamma(1+r_1)\Gamma(1+r_2)}\,.$$

Indeed, for each $(x,y) \in [1,e] \times [1,e]$ we get

$$({}^{HS}I_\sigma^r \Phi)(x,y) \le \frac{e^3}{\Gamma(1+r_1)\Gamma(1+r_2)}$$
$$= \lambda_\Phi \Phi(x,y)\,.$$

Consequently, Theorem 9.3 implies that equation (9.6) is generalized Ulam–Hyers–Rassias stable.

9.3 Global Stability Results for Volterra-Type Fractional Hadamard–Stieltjes Partial Integral Equations

9.3.1 Introduction

This section deals with the existence and global stability of solutions of a new class of Volterra partial integral equations of Hadamard–Stieltjes fractional order.

Recently, Abbas et al. [47, 33, 38] studied some existence and stability results for some classes of nonlinear quadratic Volterra integral equations of Riemann–Liouville fractional order. This section deals with the global existence and stability of solutions to the nonlinear quadratic Volterra partial integral equation of Hadamard fractional order

$$u(t,x) = f(t,x,u(t,x),u(\alpha(t),x)) + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^{\beta(t)}\int_1^{x}\left(\log\frac{\beta(t)}{s}\right)^{r_1-1}\left(\log\frac{x}{\xi}\right)^{r_2-1}$$
$$\times h(t,x,s,\xi,u(s,\xi),u(\gamma(s),\xi))\frac{d_\xi g_2(x,\xi)d_s g_1(t,s)}{s\xi}\,, \quad (t,x)\in J\,, \tag{9.7}$$

where $J := [1,\infty)\times[1,b]$, $b>1$, $r_1,r_2\in(0,\infty)$, $\alpha,\beta,\gamma\colon[1,\infty)\to[1,\infty)$, $f\colon J\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, $g_1\colon\triangle_1\to\mathbb{R}$, $g_2\colon\triangle_2\to\mathbb{R}$, and $h\colon J_1\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are given continuous functions, $\triangle_1=\{(t,s)\colon 1\le s\le t\}$, $\triangle_2=\{(x,\xi)\colon 1\le\xi\le x\le b\}$, $J_1=\{(t,x,s,\xi)\colon (t,s)\in\triangle_1$, and $(x,\xi)\in\triangle_2\}$.

In this section we provide some existence and asymptotic stability of such a new class of fractional integral equations. Finally, we present an example illustrating the applicability of the imposed conditions.

9.3.2 Existence and Asymptotic Stability Results

In this section, we are concerned with the existence and asymptotic stability of solutions to the Hadamard partial integral equation (9.7).

Let us introduce the following conditions:

(9.4.1) The function $\alpha\colon[1,\infty)\to[1,\infty)$ satisfies $\lim_{t\to\infty}\alpha(t)=\infty$.

(9.4.2) There exist constants $M, L>0$ and a nondecreasing function $\psi_1\colon[0,\infty)\to(0,\infty)$ such that $M<\frac{L}{2}$,

$$|f(t,x,u_1,v_1)-f(t,x,u_2,v_2)|\le\frac{M(|u_1-u_2|+|v_1-v_2|)}{(1+\alpha(t))(L+|u_1-u_2|+|v_1-v_2|)}\,,$$

and

$$|f(t_1,x_1,u,v)-f(t_2,x_2,u,v)|\le(|t_1-t_2|+|x_1-x_2|)\psi_1(|u|+|v|)$$

for each $(t,x),(t_1,x_1),(t_2,x_2)\in J$ and $u,v,u_1,v_1,u_2,v_2\in\mathbb{R}$.

(9.4.3) The function $t \to f(t, x, 0, 0)$ is bounded on J with

$$f^* = \sup_{(t,x)\in[1,\infty)\times[1,b]} f(t, x, 0, 0)$$

and

$$\lim_{t\to\infty} |f(t, x, 0, 0)| = 0; \; x \in [1, b] .$$

(9.4.4) There exist continuous functions $\varphi\colon J \to \mathbb{R}_+$, $p\colon J_1 \to \mathbb{R}_+$ and a nondecreasing function $\psi_2\colon [0,\infty) \to (0,\infty)$ such that

$$|h(t_1, x_1, s, \xi, u, v) - h(t_2, x_2, s, \xi, u, v)| \le \varphi(s, \xi)(|t_1 - t_2| + |x_1 - x_2|)\psi_2(|u| + |v|)$$

and

$$|h(t, x, s, \xi, u, v)| \le \frac{p(t, x, s, \xi)}{1 + \alpha(t) + |u| + |v|}$$

for each $(t, s), (t_1, s), (t_2, s) \in \triangle_1$, $(x, \xi), (x_1, \xi), (x_2, \xi) \in \triangle_2$ and $u, v \in \mathbb{R}$. Moreover, assume that

$$\lim_{t\to\infty} \int_1^{\beta(t)} \int_1^{x} \left|\log \frac{\beta(t)}{s}\right|^{r_1-1} \left|\log \frac{x}{\xi}\right|^{r_2-1} p(t, x, s, \xi) d_\xi \bigvee_{k_2=1}^{\xi} g_2(x, k_2) d_s \bigvee_{k_1=1}^{s} g_1(t, k_1) = 0$$

for each $x \in [1, b]$.

(9.4.5) The functions $s \mapsto g_1(t, s)$ and $\xi \mapsto g_2(x, \xi)$ have bounded variations for each fixed $t \in [1,\infty)$ or $x \in [1, b]$, respectively. Moreover, the functions $s \mapsto g_1(1, s)$ and $\xi \mapsto g_2(1, \xi)$ are nondecreasing on $[1,\infty)$ or $[1, b]$, respectively,

(9.4.6) For each $(t, s), (t_1, s), (t_2, s) \in \triangle_1$, $(x, \xi), (x_1, \xi), (x_2, \xi) \in \triangle_2$ we have

$$\left|\bigvee_{k_2=1}^{x_2} g_2(x_2, k_2) \bigvee_{k_1=1}^{t_2} g_1(t_2, k_1) - \bigvee_{k_2=1}^{x_1} g_2(x_1, k_2) d_s \bigvee_{k_1=1}^{t_1} g_1(t_1, k_1)\right| \to 0$$

as $t_1 \to t_2$ and $x_1 \to x_2$.

(9.4.7) $g_1(t, 1) = g_2(x, 1) = 0$ for any $t \in [1,\infty)$ and any $x \in [1, b]$.

Theorem 9.4. *Assume* (9.4.1)–(9.4.7). *Then integral equation* (9.7) *has at least one solution in the space BC. Moreover, solutions of equation* (9.7) *are globally asymptotically stable.*

Proof. Set $d^* := \sup_{(t,x)\in J} d(t, x)$, where

$$d(t, x) = \int_1^{\beta(t)} \int_1^{x} \left|\log \frac{\beta(t)}{s}\right|^{r_1-1} \left|\log \frac{x}{\xi}\right|^{r_2-1} \frac{p(t, x, s, \xi)}{\Gamma(r_1)\Gamma(r_2)} d_\xi \bigvee_{k_2=1}^{\xi} g_2(x, k_2) d_s \bigvee_{k_1=1}^{s} g_1(t, k_1) .$$

From condition (9.4.4), we infer that d^* is finite. Let us define the operator N such that, for any $u \in BC$,

$$(Nu)(t, x) = f(t, x, u(t, x), u(\alpha(t), x)) + \int_1^{\beta(t)} \int_1^{x} \left(\log \frac{\beta(t)}{s}\right)^{r_1-1} \left(\log \frac{x}{\xi}\right)^{r_2-1}$$
$$\times h(t, x, s, \xi, u(s, \xi), u(\gamma(s), \xi)) \frac{d_\xi g_2(x, \xi) d_s g_1(t, s)}{s\xi\Gamma(r_1)\Gamma(r_2)} , \quad (t, x) \in J . \quad (9.8)$$

By considering the conditions of this theorem, we infer that $N(u)$ is continuous on J. Now we prove that $N(u) \in BC$ for any $u \in BC$. For arbitrarily fixed $(t, x) \in J$ we have

$$\begin{aligned}
|(Nu)(t,x)| &\le |f(t,x,u(t,x),u(\alpha(t),x)) - f(t,x,0,0)| + |f(t,x,0,0)| \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^{\beta(t)} \int_1^{x} \left|\log\frac{\beta(t)}{s}\right|^{r_1-1} \left|\log\frac{x}{\xi}\right|^{r_2-1} \\
&\quad \times |h(t,x,s,\xi,u(s,\xi),u(\gamma(s),\xi))| \frac{|d_\xi g_2(x,\xi) d_s g_1(t,s)|}{s\xi} \\
&\le \frac{M(|u(t,x)| + |u(\alpha(t),x)|)}{(1+\alpha(t))(L + |u(t,x)| + |u(\alpha(t),x)|)} + |f(t,x,0,0)| \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^{\beta(t)} \int_1^{x} \left|\log\frac{\beta(t)}{s}\right|^{r_1-1} \left|\log\frac{x}{\xi}\right|^{r_2-1} \\
&\quad \times \frac{p(t,x,s,\xi)}{1+\alpha(t) + |u(s,\xi))| + |u(\gamma(s),\xi))|} d_\xi \bigvee_{k_2=1}^{\xi} g_2(x,k_2) d_s \bigvee_{k_1=1}^{s} g_1(t,k_1) \\
&\le \frac{M(|u(t,x)| + |u(\alpha(t),x)|)}{|u(t,x)| + |u(\alpha(t),x)|} + f^* + d^* .
\end{aligned}$$

Thus,

$$\|N(u)\|_{BC} \le M + f^* + d^* . \tag{9.9}$$

Hence, $N(u) \in BC$. Equation (9.9) yields that N transforms the ball $B_\eta := B(0,\eta)$ into itself, where $\eta = M + f^* + d^*$. We will show that $N\colon B_\eta \to B_\eta$ satisfies the conditions of Theorem 1.42. The proof will be given in several steps and cases.

Step 1: N is continuous. Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence such that $u_n \to u$ in B_η. Then for each $(t,x) \in J$ we have

$$\begin{aligned}
|(Nu_n)(t,x) - (Nu)(t,x)| &\le |f(t,x,u_n(t,x),u_n(\alpha(t),x)) - f(t,x,u(t,x),u(\alpha(t),x))| \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^{\beta(t)} \int_1^{x} \left|\log\frac{\beta(t)}{s}\right|^{r_1-1} \left|\log\frac{x}{\xi}\right|^{r_2-1} \\
&\quad \times \sup_{(s,\xi)\in J} |h(t,x,s,\xi,u_n(s,\xi),u_n(\gamma(s),\xi)) - h(t,x,s,\xi,u(s,\xi),u(\gamma(s),\xi))| \\
&\quad \times \frac{|d_\xi g_2(x,\xi) d_s g_1(t,s)|}{s\xi} \\
&\le \frac{2M}{L}\|u_n - u\|_{BC} \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^{\beta(t)} \int_1^{x} \left|\log\frac{\beta(t)}{s}\right|^{r_1-1} \left|\log\frac{x}{\xi}\right|^{r_2-1} \\
&\quad \times \|h(t,x,.,.,u_n(.,.),u_n(\gamma(.),.)) - h(t,x,.,.,u(.,.),u(\gamma(.),.))\|_{BC} \\
&\quad \times d_\xi \bigvee_{k_2=1}^{\xi} g_2(x,k_2) d_s \bigvee_{k_1=1}^{s} g_1(t,k_1) .
\end{aligned} \tag{9.10}$$

Case 1. If $(t,x) \in [1,T] \times [1,b]$, $T > 1$, then, since $u_n \to u$ as $n \to \infty$ and h, γ are continuous, (9.10) gives

$$\|N(u_n) - N(u)\|_{BC} \to 0 \;\text{ as }\; n \to \infty \,.$$

Case 2. If $(t,x) \in (T,\infty) \times [1,b]$, $T > 1$, then from (9.4.4) and (9.10), for each $(t,x) \in J$, we have

$$\begin{aligned}|(Nu_n)(t,x) - (Nu)(t,x)| &\le \frac{2M}{L}\|u_n - u\|_{BC} \\ &\quad + \frac{2}{\Gamma(r_1)\Gamma(r_2)} \int_1^{\beta(t)}\int_1^{x} \left|\log\frac{\beta(t)}{s}\right|^{r_1-1} \left|\log\frac{x}{\xi}\right|^{r_2-1} \\ &\quad \times p(t,x,s,\xi) d_\xi \bigvee_{k_2=1}^{\xi} g_2(x,k_2) d_s \bigvee_{k_1=1}^{s} g_1(t,k_1)\,. \\ &\le \frac{2M}{L}\|u_n - u\|_{BC} + 2d(t,x)\,.\end{aligned}$$

Thus, we get

$$|(Nu_n)(t,x) - (Nu)(t,x)| \le \frac{2M}{L}\|u_n - u\|_{BC} + 2d(t,x)\,. \tag{9.11}$$

Since $u_n \to u$ as $n \to \infty$ and $t \to \infty$, then (6.6) gives

$$\|N(u_n) - N(u)\|_{BC} \to 0 \;\text{ as }\; n \to \infty \,.$$

Step 2: *$N(B_\eta)$ is uniformly bounded.* This is clear since $N(B_\eta) \subset B_\eta$ and B_η is bounded.

Step 3: $N(B_\eta)$ is equicontinuous on every compact subset $[1,a] \times [1,b]$ of J, $a > 0$. Let $(t_1,x_1), (t_2,x_2) \in [1,a] \times [1,b]$, $t_1 < t_2$, $x_1 < x_2$, and let $u \in B_\eta$. Also, without loss of generality, suppose that $\beta(t_1) \le \beta(t_2)$. Then we have

$$\begin{aligned}&|(Nu)(t_2,x_2) - (Nu)(t_1,x_1)| \\ &\quad \le |f(t_2,x_2,u(t_2,x_2),u(\alpha(t_2),x_2)) - f(t_2,x_2,u(t_1,x_1),u(\alpha(t_1),x_1))| \\ &\qquad + |f(t_2,x_2,u(t_1,x_1),u(\alpha(t_1),x_1)) - f(t_1,x_1,u(t_1,x_1),u(\alpha(t_1),x_1))| \\ &\qquad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^{\beta(t_2)}\int_1^{x_2} \left|\log\frac{\beta(t_2)}{s}\right|^{r_1-1} \left|\log\frac{x_2}{\xi}\right|^{r_2-1} \\ &\qquad \times |h(t_2,x_2,s,\xi,u(s,\xi),u(\gamma(s),\xi)) - h(t_1,x_1,s,\xi,u(s,\xi),u(\gamma(s),\xi))| \\ &\qquad \times |d_\xi g_2(x_2,\xi) d_s g_1(t_2,s)| \\ &\qquad + \left| \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^{\beta(t_2)}\int_1^{x_2} \left(\log\frac{\beta(t_2)}{s}\right)^{r_1-1} \left(\log\frac{x_2}{\xi}\right)^{r_2-1} \right. \\ &\qquad \times h(t_1,x_1,s,\xi,u(s,\xi),u(\gamma(s),\xi)) d_\xi g_2(x_2,\xi) d_s g_1(t_2,s)\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^{\beta(t_1)} \int_1^{x_1} \left(\log \frac{\beta(t_2)}{s}\right)^{r_1-1} \left(\log \frac{x_2}{\xi}\right)^{r_2-1} \\
&\times h(t_1, x_1, s, \xi, u(s,\xi), u(\gamma(s),\xi)) d_\xi g_2(x_2,\xi) d_s g_1(t_2, s)\Big| \\
&+ \left| \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^{\beta(t_1)} \int_1^{x_1} \left(\log \frac{\beta(t_2)}{s}\right)^{r_1-1} \left(\log \frac{x_2}{\xi}\right)^{r_2-1} \right. \\
&\times h(t_1, x_1, s, \xi, u(s,\xi), u(\gamma(s),\xi)) (d_\xi g_2(x_2,\xi) d_s g_1(t_2, s) - d_\xi g_2(x_1,\xi) d_s g_1(t_1, s))\Big| \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^{\beta(t_1)} \int_1^{x_1} \left| \left(\log \frac{\beta(t_2)}{s}\right)^{r_1-1} \left(\log \frac{x_2}{\xi}\right)^{r_2-1} \right. \\
&\left. - \left(\log \frac{\beta(t_1)}{s}\right)^{r_1-1} \left(\log \frac{x_1}{\xi}\right)^{r_2-1} \right| |h(t_1, x_1, s, \xi, u(s,\xi), u(\gamma(s),\xi))| \\
&\times |d_\xi g_2(x_1,\xi) d_s g_1(t_1, s)| .
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
&|(Nu)(t_2,x_2) - (Nu)(t_1,x_1)| \\
&\quad \le \frac{M}{L}(|u(t_2,x_2) - u(t_1,x_1)| + |u(\alpha(t_2),x_2) - u(\alpha(t_1),x_1)|) \\
&\quad + (|t_2 - t_1| + |x_2 - x_1|)\psi_1(2\|u\|_{BC}) \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^{\beta(t_2)} \int_1^{x_2} \left|\log \frac{\beta(t_2)}{s}\right|^{r_1-1} \left|\log \frac{x_2}{\xi}\right|^{r_2-1} \\
&\quad \times \varphi(s,\xi)(|t_2 - t_1| + |x_2 - x_1|)\psi_2(2\|u\|_{BC}) d_\xi \bigvee_{k_2=1}^{\xi} g_2(x_2,k_2) d_s \bigvee_{k_1=1}^{s} g_1(t_2,k_1) \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{\beta(t_1)}^{\beta(t_2)} \int_1^{x_2} \left|\log \frac{\beta(t_2)}{s}\right|^{r_1-1} \left|\log \frac{x_2}{\xi}\right|^{r_2-1} \\
&\quad \times |h(t_1, x_1, s, \xi, u(s,\xi), u(\gamma(s),\xi))| d_\xi \bigvee_{k_2=1}^{\xi} g_2(x_2,k_2) d_s \bigvee_{k_1=1}^{s} g_1(t_2,k_1) \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^{\beta(t_2)} \int_{x_1}^{x_2} \left|\log \frac{\beta(t_2)}{s}\right|^{r_1-1} \left|\log \frac{x_2}{\xi}\right|^{r_2-1} \\
&\quad \times |h(t_1, x_1, s, \xi, u(s,\xi), u(\gamma(s),\xi))| d_\xi \bigvee_{k_2=1}^{\xi} g_2(x_2,k_2) d_s \bigvee_{k_1=1}^{s} g_1(t_2,k_1) \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{\beta(t_1)}^{\beta(t_2)} \int_{x_1}^{x_2} \left|\log \frac{\beta(t_2)}{s}\right|^{r_1-1} \left|\log \frac{x_2}{\xi}\right|^{r_2-1}
\end{aligned}
$$

$$\times |h(t_1, x_1, s, \xi, u(s,\xi), u(\gamma(s),\xi))| d_\xi \bigvee_{k_2=1}^{\xi} g_2(x_2, k_2) d_s \bigvee_{k_1=1}^{s} g_1(t_2, k_1)$$

$$+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^{\beta(t_1)} \int_1^{x_1} \left|\log \frac{\beta(t_2)}{s}\right|^{r_1-1} \left|\log \frac{x_2}{\xi}\right|^{r_2-1}$$

$$\times |h(t_1, x_1, s, \xi, u(s,\xi), u(\gamma(s),\xi))| |d_\xi g_2(x_2,\xi) d_s g_1(t_2, s) - d_\xi g_2(x_1,\xi) d_s g_1(t_1, s))|$$

$$+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^{\beta(t_1)} \int_1^{x_1} \left| \left(\log \frac{\beta(t_2)}{s}\right)^{r_1-1} \left(\log \frac{x_2}{\xi}\right)^{r_2-1} \right.$$

$$\left. - \left(\log \frac{\beta(t_1)}{s}\right)^{r_1-1} \left(\log \frac{x_1}{\xi}\right)^{r_2-1} \right| |h(t_1, x_1, s, \xi, u(s,\xi), u(\gamma(s),\xi))|$$

$$\times d_\xi \bigvee_{k_2=1}^{\xi} g_2(x_1, k_2) d_s \bigvee_{k_1=1}^{s} g_1(t_1, k_1) .$$

Hence, we get

$$\begin{aligned}
&|(Nu)(t_2, x_2) - (Nu)(t_1, x_1)| \\
&\le \frac{M}{L}(|u(t_2, x_2) - u(t_1, x_1)| + |u(\alpha(t_2), x_2) - u(\alpha(t_1), x_1)|) \\
&\quad + (|t_2 - t_1| + |x_2 - x_1|)\psi_1(2\eta) \\
&\quad + \frac{(|t_2 - t_1| + |x_2 - x_1|)\psi_2(2\eta)}{\Gamma(r_1)\Gamma(r_2)} \\
&\quad \times \int_1^{\beta(t_2)} \int_1^{x_2} \left|\log \frac{\beta(t_2)}{s}\right|^{r_1-1} \left|\log \frac{x_2}{\xi}\right|^{r_2-1} \varphi(s,\xi) d_\xi \bigvee_{k_2=1}^{\xi} g_2(x_2, k_2) d_s \bigvee_{k_1=1}^{s} g_1(t_2, k_1) \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{\beta(t_1)}^{\beta(t_2)} \int_1^{x_2} \left|\log \frac{\beta(t_2)}{s}\right|^{r_1-1} \left|\log \frac{x_2}{\xi}\right|^{r_2-1} \\
&\quad \times p(t_1, x_1, s, \xi) d_\xi \bigvee_{k_2=1}^{\xi} g_2(x_2, k_2) d_s \bigvee_{k_1=1}^{s} g_1(t_2, k_1) \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^{\beta(t_2)} \int_{x_1}^{x_2} \left|\log \frac{\beta(t_2)}{s}\right|^{r_1-1} \left|\log \frac{x_2}{\xi}\right|^{r_2-1} \\
&\quad \times p(t_1, x_1, s, \xi) d_\xi \bigvee_{k_2=1}^{\xi} g_2(x_2, k_2) d_s \bigvee_{k_1=1}^{s} g_1(t_2, k_1) \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{\beta(t_1)}^{\beta(t_2)} \int_{x_1}^{x_2} \left|\log \frac{\beta(t_2)}{s}\right|^{r_1-1} \left|\log \frac{x_2}{\xi}\right|^{r_2-1} \\
&\quad \times p(t_1, x_1, s, \xi) d_\xi \bigvee_{k_2=1}^{\xi} g_2(x_2, k_2) d_s \bigvee_{k_1=1}^{s} g_1(t_2, k_1)
\end{aligned}$$

$$+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^{\beta(t_1)}\int_1^{x_1}\left|\log\frac{\beta(t_2)}{s}\right|^{r_1-1}\left|\log\frac{x_2}{\xi}\right|^{r_2-1}p(t_1,x_1,s,\xi)$$
$$\times|d_\xi\bigvee_{k_2=1}^{\xi}g_2(x_2,k_2)d_s\bigvee_{k_1=1}^{s}g_1(t_2,k_1)-d_\xi\bigvee_{k_2=1}^{\xi}g_2(x_1,k_2)d_s\bigvee_{k_1=1}^{s}g_1(t_1,k_1))|$$
$$+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^{\beta(t_1)}\int_1^{x_1}\left|\left(\log\frac{\beta(t_2)}{s}\right)^{r_1-1}\left(\log\frac{x_2}{\xi}\right)^{r_2-1}\right.$$
$$\left.-\left(\log\frac{\beta(t_1)}{s}\right)^{r_1-1}\left(\log\frac{x_1}{\xi}\right)^{r_2-1}\right|p(t_1,x_1,s,\xi)d_\xi\bigvee_{k_2=1}^{\xi}g_2(x_1,k_2)d_s\bigvee_{k_1=1}^{s}g_1(t_1,k_1)\,.$$

From the continuity of $\alpha, \beta, \varphi, p$ and as $t_1 \to t_2$ and $x_1 \to x_2$, the right-hand side of the preceding inequality tends to zero.

Step 4: $N(B_\eta)$ is equiconvergent. Let $(t,x) \in J$ and $u \in B_\eta$; then we have

$$\begin{aligned}|u(t,x)| &\le |f(t,x,u(t,x),u(\alpha(t),x)) - f(t,x,0,0) + f(t,x,0,0)|\\
&\quad+\left|\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^{\beta(t)}\int_1^{x}\left(\log\frac{\beta(t)}{s}\right)^{r_1-1}\left(\log\frac{x}{\xi}\right)^{r_2-1}\right.\\
&\qquad\left.\times h(t,x,s,\xi,u(s,\xi),u(\gamma(s),\xi))\frac{d_\xi g_2(x,\xi)d_s g_1(t,s)}{s\xi}\right|\\
&\le \frac{M(|u(t,x)|+|u(\alpha(t),x)|)}{(1+\alpha(t))(L+|u(t,x)|+|u(\alpha(t),x)|)} + |f(t,x,0,0)|\\
&\quad+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^{\beta(t)}\int_1^{x}\left(\log\frac{\beta(t)}{s}\right)^{r_1-1}\left(\log\frac{x}{\xi}\right)^{r_2-1}\\
&\qquad\times\frac{p(t,x,s,\xi)}{1+\alpha(t)+|u(s,\xi)|+|u(\gamma(s),\xi)|}d_\xi g_2(x,\xi)d_s g_1(t,s)\\
&\le \frac{M}{1+\alpha(t)} + |f(t,x,0,0)|\\
&\quad+\frac{1}{\Gamma(r_1)\Gamma(r_2)(1+\alpha(t))}\int_1^{\beta(t)}\int_1^{x}\left(\log\frac{\beta(t)}{s}\right)^{r_1-1}\left(\log\frac{x}{\xi}\right)^{r_2-1}\\
&\qquad\times p(t,x,s,\xi)d_\xi\bigvee_{k_2=1}^{\xi}g_2(x,k_2)d_s\bigvee_{k_1=1}^{s}g_1(t,k_1)\\
&\le \frac{M}{1+\alpha(t)} + |f(t,x,0,0)| + \frac{d^*}{1+\alpha(t)}\,.\end{aligned}$$

Thus, for each $x \in [1,b]$ we get

$$|u(t,x)| \to 0, \quad \text{as } t \to +\infty\,.$$

Hence,

$$|u(t,x) - u(+\infty,x)| \to 0, \quad \text{as } t \to +\infty\,.$$

As a consequence of Steps 1–4, together with Lemma 1.57, we can conclude that $N\colon B_\eta \to B_\eta$ is continuous and compact. From an application of Theorem 1.42, we deduce that N has a fixed point u that is a solution of the Hadamard integral equation (9.7).

Step 5: Uniform global attractivity. Let us assume that u_0 is a solution of integral equation (9.7) with the conditions of this theorem. Consider the ball $B(u_0, \eta)$ with $\eta^* = \frac{LM^*}{L-2M}$, where

$$M^* := \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sup_{(t,x)\in J} \left\{ \int_1^{\beta(t)} \int_1^{x} \left|\log \frac{\beta(t)}{s}\right|^{r_1-1} \left|\log \frac{x}{\xi}\right|^{r_2-1} \right.$$
$$\times |h(t,x,s,\xi,u(s,\xi),u(\gamma(s),\xi)) - h(t,x,s,\xi,u_0(s,\xi),u_0(\gamma(s),\xi))|$$
$$\left. \times d_\xi \bigvee_{k_2=1}^{\xi} g_2(x,k_2) d_s \bigvee_{k_1=1}^{s} g_1(t,k_1);\ u \in BC \right\} .$$

Taking $u \in B(u_0, \eta^*)$, we then have

$$\begin{aligned}
&|(Nu)(t,x) - u_0(t,x)| = |(Nu)(t,x) - (Nu_0)(t,x)| \\
&\le |f(t,x,u(t,x),u(\alpha(t),x)) - f(t,x,u_0(t,x),u_0(\alpha(t),x))| \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^{\beta(t)} \int_1^{x} \left|\log \frac{\beta(t)}{s}\right|^{r_1-1} \left|\log \frac{x}{\xi}\right|^{r_2-1} \\
&\quad\quad \times |h(t,x,s,\xi,u(s,\xi),u(\gamma(s),\xi)) - h(t,x,s,\xi,u_0(s,\xi),u_0(\gamma(s),\xi))| \\
&\quad\quad \times d_\xi \bigvee_{k_2=1}^{\xi} g_2(x,k_2) d_s \bigvee_{k_1=1}^{s} g_1(t,k_1) \\
&\le \frac{2M}{L} \|u - u_0\|_{BC} + M^* \\
&\le \frac{2M}{L} \eta^* + M^* = \eta^* .
\end{aligned}$$

Thus we observe that N is a continuous function such that $N(B(u_0, \eta^*)) \subset B(u_0, \eta^*)$. Moreover, if u is a solution of equation (7.1), then

$$\begin{aligned}
&|u(t,x) - u_0(t,x)| = |(Nu)(t,x) - (Nu_0)(t,x)| \\
&\le |f(t,x,u(t,x),u(\alpha(t),x)) - f(t,x,u_0(t,x),u_0(\alpha(t),x))| \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^{\beta(t)} \int_1^{x} \left|\log \frac{\beta(t)}{s}\right|^{r_1-1} \left|\log \frac{x}{\xi}\right|^{r_2-1} \\
&\quad\quad \times |h(t,x,s,\xi,u(s,\xi),u(\gamma(s),\xi)) - h(t,x,s,\xi,u_0(s,\xi),u_0(\gamma(s),\xi))| \\
&\quad\quad \times d_\xi \bigvee_{k_2=1}^{\xi} g_2(x,k_2) d_s \bigvee_{k_1=1}^{s} g_1(t,k_1) .
\end{aligned}$$

Thus,

$$|u(t,x)-u_0(t,x)| \le \frac{M}{L}(|u(t,x)-u_0(t,x)| + |u(\alpha(t),x)-u_0(\alpha(t),x)|)$$
$$+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_2^{\beta(t)} \int_1^x \left|\log\frac{\beta(t)}{s}\right|^{r_1-1} \left|\log\frac{x}{\xi}\right|^{r_2-1}$$
$$\times p(t,x,s,\xi) d_\xi \bigvee_{k_2=1}^{\xi} g_2(x,k_2) d_s \bigvee_{k_1=1}^{s} g_1(t,k_1)\,. \tag{9.12}$$

By using (9.12) and the fact that $\alpha(t) \to \infty$ as $t \to \infty$, we get

$$\lim_{t\to\infty} |u(t,x)-u_0(t,x)| \le \lim_{t\to\infty} \frac{2L}{\Gamma(r_1)\Gamma(r_2)(L-2M)} \int_1^{\beta(t)} \int_1^x \left|\log\frac{\beta(t)}{s}\right|^{r_1-1} \left|\log\frac{x}{\xi}\right|^{r_2-1}$$
$$\times p(t,x,s,\xi) d_\xi \bigvee_{k_2=1}^{\xi} g_2(x,k_2) d_s \bigvee_{k_1=1}^{s} g_1(t,k_1) = 0\,.$$

Consequently, all solutions of the Hadamard–Volterra–Stieltjes integral equation (7.1) are globally asymptotically stable. □

9.3.3 An Example

As an application of our results we consider the partial Hadamard Volterra–Stieltjes integral equation of fractional order

$$u(t,x) = \frac{tx}{10(1+t+t^2+t^3)}(1+2\sin(u(t,x))) + \frac{1}{\Gamma^2\left(\frac{1}{3}\right)} \int_1^t \int_1^x \left(\log\frac{t}{s}\right)^{\frac{-2}{3}} \left(\log\frac{x}{\xi}\right)^{\frac{-2}{3}}$$
$$\times \frac{\ln(1+2x(s\xi)^{-1}|u(s,\xi)|)}{(1+t+2|u(s,\xi)|)^2(1+x^2+t^4)} d_\xi g_2(x,\xi) d_s g_1(t,s)\,, \tag{9.13}$$
$$(t,x) \in [1,\infty) \times [1,e]\,,$$

where $r_1 = r_2 = \frac{1}{3}$, $\alpha(t) = \beta(t) = \gamma(t) = t$, $g_1(t,s) = s$, $g_2(x,\xi) = \xi$; $s, \xi \in [1,e]$,

$$f(t,x,u,v) = \frac{tx(1+\sin(u)+\sin(v))}{10(1+t)(1+t^2)}\,,$$

and

$$h(t,x,s,\xi,u,v) = \frac{\ln(1+x(s\xi)^{-1}(|u|+|v|))}{(1+t+|u|+|v|)^2(1+x^2+t^4)}$$

for $(t,x), (s,\xi) \in [1,\infty) \times [1,e]$, and $u, v \in \mathbb{R}$.

We can easily check that the conditions of Theorem 9.4 are satisfied. In fact, we have that the function f is continuous and satisfies (9.4.2), where $M = \frac{1}{10}$, $L = 1$. Also, f

satisfies (9.4.3), with $f^* = \frac{e}{10}$. Next, let us notice that the function h satisfies (9.4.4), where $p(t, x, s, \xi) = \frac{1}{s\xi(1+x^2+t^4)}$. Also,

$$\begin{aligned}
&\lim_{t\to\infty} \int_1^t \int_1^x \left|\log \frac{t}{s}\right|^{\frac{-2}{3}} \left|\log \frac{x}{\xi}\right|^{\frac{-2}{3}} p(t, x, s, \xi) d_\xi \bigvee_{k_2=1}^{\xi} g_2(x, k_2) d_s \bigvee_{k_1=1}^{s} g_1(t, k_1) \\
&= \lim_{t\to\infty} \frac{x}{1 + x^2 + t^4} \int_1^t \int_1^x \left|\log \frac{t}{s}\right|^{\frac{-2}{3}} \left|\log \frac{x}{\xi}\right|^{\frac{-2}{3}} \frac{1}{s\xi} d_\xi \bigvee_{k_2=1}^{\xi} g_2(x, k_2) d_s \bigvee_{k_1=1}^{s} g_1(t, k_1) \\
&= \lim_{t\to\infty} \frac{x}{1 + x^2 + t^4} \int_1^t \int_1^x \left|\log \frac{t}{s}\right|^{\frac{-2}{3}} \left|\log \frac{x}{\xi}\right|^{\frac{-2}{3}} \frac{d\xi ds}{s\xi} \\
&= \lim_{t\to\infty} \frac{9x(\log t)^{\frac{1}{3}}}{1 + x^2 + t^4} = 0 .
\end{aligned}$$

Hence, by Theorem 9.4, the Volterra–Stieltjes equation (9.13) has a solution defined on $[1, \infty) \times [1, e]$, and solutions of this equation are globally asymptotically stable.

9.4 Volterra-Type Nonlinear Multidelay Hadamard–Stieltjes Fractional Integral Equations

9.4.1 Introduction

This section deals with the existence and global attractivity of solutions for Volterra–Stieltjes quadratic delay integral equations of Hadamard fractional order.

In [24, 25, 32, 30], the authors studied the existence and stability of solutions for some integral equations. Motivated by the aforementioned papers, in this section we establish some sufficient conditions for the existence and attractivity of solutions of the class of Volterra-type delay fractional order Hadamard–Stieltjes quadratic integral equations,

$$\begin{aligned}
u(t, x) = \mu(t, x) + \frac{f(t, x, u(t, x))}{\Gamma(r_1)\Gamma(r_2)} \int_1^t \int_1^x \left(\log \frac{t}{s}\right)^{r_1-1} \left(\log \frac{x}{y}\right)^{r_2-1} \\
\times h(t, x, s, y, u(s - \tau_1, y - \xi_1), \ldots, u(s - \tau_m, y - \xi_m)) \\
\times \frac{1}{sy} d_y g_2(x, y) d_s g_1(t, s) ; \quad \text{if } (t, x) \in J := [1, +\infty) \times [1, b] , \qquad (9.14)
\end{aligned}$$

$$u(t, x) = \Phi(t, x) \quad \text{if } (t, x) \in \tilde{J} := [-T, \infty) \times [-\xi, b] \backslash (1, \infty) \times (1, b] , \qquad (9.15)$$

where $b > 1$, $r_1, r_2 \in (0, \infty)$, $\tau_i, \xi_i \geq -1$; $i = 1 \ldots, m$, $T = \max_{i=1\ldots,m} \{\tau_i\}$, $\xi = \max_{i=1\ldots,m} \{\xi_i\}$, $\mu : J \to \mathbb{R}$, $f : J \times \mathbb{R} \to \mathbb{R}$, $g_1 : \triangle_1 \to \mathbb{R}$, $g_2 : \triangle_2 \to \mathbb{R}$, $h : J' \times \mathbb{R}^m \to \mathbb{R}$ are given continuous functions, $\triangle_1 = \{(t, s) : 1 \leq s \leq t\}$, $\triangle_2 = \{(x, y) : 1 \leq y \leq x \leq b\}$,

$J_1 = \{(t, x, s, y): (t, s) \in \triangle_1$ and $(x, y) \in \triangle_2\}$, and $\Phi: \tilde{J} \to \mathbb{R}$ is continuous with $\mu(t, 1) = \Phi(t, 1)$ for each $t \in [1, +\infty)$ and $\mu(1, x) = \Phi(1, x)$ for each $x \in [1, b]$.

We use the Schauder fixed point theorem for the existence of solutions of problem (9.14)–(9.15), and we prove that all solutions are uniformly globally attractive.

Let $\emptyset \neq \Omega \subset BC$, let $G: \Omega \to \Omega$, and consider the solutions of the equation

$$(Gu)(t, x) = u(t, x)\,. \tag{9.16}$$

Definition 9.5 ([35]). The solutions of equation (9.16) are locally attractive if there exists a ball $B(u_0, \eta)$ in the space BC such that, for arbitrary solutions $v = v(t, x)$ and $w = w(t, x)$ of equations (9.16) belonging to $B(u_0, \eta) \bigcap \Omega$, we have that, for each $x \in [1, b]$,

$$\lim_{t\to\infty} (v(t, x) - w(t, x)) = 0\,. \tag{9.17}$$

When the limit (9.17) is uniform with respect to $B(u_0, \eta)$, the solutions of equation (9.16) are said to be uniformly locally attractive (or, equivalently, that solutions of (9.16) are locally asymptotically stable).

Definition 9.6 ([35]). The solution $v = v(t, x)$ of equation (9.16) is said to be globally attractive if (9.17) holds for each solution $w = w(t, x)$ of (9.16). If condition (9.17) is satisfied uniformly with respect to the set Ω, then solutions of equation (9.16) are said to be globally asymptotically stable (or uniformly globally attractive).

9.4.2 Existence and Attractivity Results

Let us start in this section by defining what we mean by a solution of problem (9.14)–(9.15).

Definition 9.7. By a solution of problem (9.14)–(9.15) we mean every function $u \in BC$ such that u satisfies equation (9.14) on J and equation (9.15) on $\tilde{J}$.

The following conditions will be used in the sequel.

(9.9.1) The functions μ and Φ are in BC. Moreover, assume that

$$\lim_{t\to\infty} \mu(t, x) = 0\,; \quad x \in [1, b]\,.$$

(9.9.2) There exist a positive function $P \in BC$ and a nondecreasing function $\psi_1: [0, \infty) \to (0, \infty)$ such that

$$|f(t, x, u) - f(t, x, v)| \le P(t, x)|u - v|\,; \quad (t, x) \in J,\ u, v \in \mathbb{R}$$

and

$$|f(t_1, x_1, u) - f(t_2, x_2, u)| \le (|t_1 - t_2| + |x_1 - x_2|)\psi_1(|u|); \quad (t_1, x_1), (t_2, x_2) \in J,\ u \in \mathbb{R}.$$

(9.9.3) The function $t \to f(t, x, 0)$ is bounded on J, and $\lim_{t\to\infty} f(t, x, 0) = 0;\ x \in [1, b]$.

(9.9.4) The functions $s \mapsto g_1(t, s)$ and $y \mapsto g_2(x, y)$ have bounded variations for each fixed $t \in [1, \infty)$ or $x \in [1, b]$, respectively. Moreover, the functions $s \mapsto g_1(1, s)$ and $y \mapsto g_2(1, y)$ are nondecreasing on $[1, \infty)$ or $[1, b]$, respectively.

(9.9.5) For each $(t_1, s), (t_2, s) \in \triangle_1, (x_1, y_1), (x_2, y_2) \in \triangle_2$ we have

$$\left| \bigvee_{k_2=1}^{x_1} g_2(x_2, k_2) \bigvee_{k_1=1}^{t_1} g_1(t_2, k_1) - \bigvee_{k_2=1}^{x_1} g_2(x_1, k_2) \bigvee_{k_1=1}^{t_1} g_1(t_1, k_1)) \right| \to 0$$

as $t_1 \to t_2$ and $x_1 \to x_2$.

(9.9.6) $g_1(t, 1) = g_2(x, 1) = 0$ for any $t \in [1, \infty)$ and any $x \in [1, b]$.

(9.9.7) There exists a continuous function $Q\colon J' \to \mathbb{R}_+$ and a nondecreasing function $\psi_2\colon [0, \infty) \to (0, \infty)$ such that

$$|h(t, x, s, y, u_1, \dots, u_m)| \le \frac{Q(t, x, s, y)}{1 + \sum_{i=1}^{m} |u_i|}; \quad (t, x, s, y) \in J',\ u_i \in \mathbb{R},\ i = 1 \dots m,$$

and

$$\begin{aligned} &|h(t_1, x_1, s, y, u_1, \dots, u_m) - h(t_2, x_2, s, y, u_1, \dots, u_m)| \\ &\le \varphi(s, y)(|t_1 - t_2| + |x_1 - x_2|)\psi_2\left(\sum_{i=1}^{m} |u_i|\right); \\ &(t_1, s), (t_2, s) \in \triangle_1,\ (x_1, y), (x_2, y) \in \triangle_2,\ u_i \in \mathbb{R},\ i = 1 \dots m. \end{aligned}$$

Moreover, assume that

$$\lim_{t\to\infty} \int_1^t \left|\log \frac{t}{s}\right|^{r_1-1} \frac{Q(t, x, s, y)}{s} d_s G_1(t, s) = 0$$

for each $(x, y) \in \triangle_2$, where $G_1(t, s) = \bigvee_{k_1=1}^{s} g_1(t, k_1)$.

Remark 9.8. Set

$$\mu^* := \sup_{(t,x)\in J} \mu(t, x), \quad \Phi^* := \sup_{(t,x)\in \tilde{J}} \Phi(t, x), \quad f^* := \sup_{(t,x)\in J} f(t, x, 0), \quad p^* := \sup_{(t,x)\in J} P(t, x)$$

and

$$q^* := \sup_{(t,x)\in J} \int_1^t \int_1^x \left|\log \frac{t}{s}\right|^{r_1-1} \left|\log \frac{x}{y}\right|^{r_2-1} \frac{Q(t, x, s, y)}{sy\Gamma(r_1)\Gamma(r_2)} d_y G_2(x, y) d_s G_1(t, s),$$

where $G_2(x, y) = \bigvee_{k_2=1}^{y} g_2(x, k_2)$. From our conditions we infer that μ^*, Φ^*, f^*, p^*, and q^* are finite.

Now we prove the following theorem concerning the existence and attractivity of solutions of problem (9.14)–(9.15).

Theorem 9.9. *Assume* (9.9.1)–(9.9.7). *If*

$$p^* q^* < 1\,, \tag{9.18}$$

then problem (9.14)–(9.15) *has at least one solution in the space BC. Moreover, the solutions of problem* (9.14)–(9.15) *are uniformly globally attractive.*

Proof. Define the operator N such that, for any $u \in BC$,

$$(Nu)(t,x) = \begin{cases} \Phi(t,x), & (t,x) \in \tilde{J}, \\ \mu(t,x) + \dfrac{f(t,x,u(t,x))}{\Gamma(r_1)\Gamma(r_2)} \displaystyle\int_1^t \int_1^x \left(\log \frac{t}{s}\right)^{r_1-1} \left(\log \frac{x}{y}\right)^{r_2-1} & \\ \times\, h(t,x,s,y,u(s-\tau_1,y-\xi_1),\ldots,u(s-\tau_m,y-\xi_m)) & \\ \times \dfrac{1}{sy} d_y g_2(x,y) d_s g_1(t,s), & (t,x) \in J. \end{cases} \tag{9.19}$$

The operator N maps BC to BC. Indeed, the map $N(u)$ is continuous on J for any $u \in BC$, and for each $(t,x) \in J$ we have

$$\begin{aligned} |(Nu)(t,x)| &\le |\mu(t,x)| + \frac{|f(t,x,0)|}{\Gamma(r_1)\Gamma(r_2)} \int_1^t \int_1^x \left|\log \frac{t}{s}\right|^{r_1-1} \left|\log \frac{x}{y}\right|^{r_2-1} \\ &\quad \times |h(t,x,s,y,u(s-\tau_1,y-\xi_1),\ldots,u(s-\tau_m,y-\xi_m))| \frac{d_y g_2(x,y) d_s g_1(t,s)}{sy} \\ &\quad + \frac{|f(t,x,u(t,x)) - f(t,x,0)|}{\Gamma(r_1)\Gamma(r_2)} \int_1^t \int_1^x \left|\log \frac{t}{s}\right|^{r_1-1} \left|\log \frac{x}{y}\right|^{r_2-1} \\ &\quad \times |h(t,x,s,y,u(s-\tau_1,y-\xi_1),\ldots,u(s-\tau_m,y-\xi_m))| d_y G_2(x,y) d_s G_1(t,s) \\ &\le \mu^* + \frac{f^*}{\Gamma(r_1)\Gamma(r_2)} \int_1^t \int_1^x \left|\log \frac{t}{s}\right|^{r_1-1} \left|\log \frac{x}{y}\right|^{r_2-1} \\ &\quad \times \frac{Q(t,x,s,y)}{1 + \sum_{i=1}^m |u_i(s-\tau_i,y-\xi_i)|} d_y G_2(x,y) d_s G_1(t,s) \\ &\quad + \frac{P(t,x)|u(t,x)|}{\Gamma(r_1)\Gamma(r_2)} \int_1^t \int_1^x \left|\log \frac{t}{s}\right|^{r_1-1} \left|\log \frac{x}{y}\right|^{r_2-1} \\ &\quad \times Q(t,x,s,y) d_y G_2(x,y) d_s G_1(t,s) \\ &\le \mu^* + \frac{f^*}{\Gamma(r_1)\Gamma(r_2)} \int_1^t \int_1^x \left|\log \frac{t}{s}\right|^{r_1-1} \left|\log \frac{x}{y}\right|^{r_2-1} \\ &\quad \times Q(t,x,s,y) d_y G_2(x,y) d_s G_1(t,s) \end{aligned}$$

$$+ \frac{p^*\|u\|_{BC}}{\Gamma(r_1)\Gamma(r_2)} \int_1^t \int_1^x \left|\log \frac{t}{s}\right|^{r_1-1} \left|\log \frac{x}{y}\right|^{r_2-1}$$
$$\times Q(t,x,s,y) d_y G_2(x,y) d_s G_1(t,s)$$
$$\le \mu^* + q^*(f^* + p^*\|u\|_{BC}) .$$

Thus, for each $(t,x) \in J$ we have

$$|(Nu)(t,x)| \le \mu^* + q^*(f^* + p^*\|u\|_{BC}) . \tag{9.20}$$

Also, for $(t,x) \in \tilde{J}$ we have

$$|(Nu)(t,x)| = |\Phi(t,x)| \le \Phi^* .$$

Thus,

$$\|Nu\|_{BC} \le \max\{\Phi^*, \mu^* + q^*(f^* + p^*\|u\|_{BC})\} .$$

Hence, $N(u) \in BC$. This proves that the operator N maps BC to itself.

The issue of finding the solutions of problem (9.14)–(9.15) is reduced to finding the solutions of the operator equation $N(u) = u$. We can show that N transforms the ball $B_\eta := B(0,\eta)$ into itself, where $\eta = \max\{\Phi^*, \eta^*\}$ and $\eta^* > \frac{\mu^*+q^*f^*}{1-p^*q^*}$. We will show that $N\colon B_\eta \to B_\eta$ satisfies the conditions of Theorem 1.42. The proof will be given in several steps and cases.

Step 1: N is continuous. Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence such that $u_n \to u$ in B_η. Then for each $(t,x) \in [-T,\infty) \times [-\xi, b]$ we have

$$|(Nu_n)(t,x) - (Nu)(t,x)|$$
$$\le |f(t,x,u_n(t,x)) - f(t,x,u(t,x))| \int_1^t \int_1^x \left|\log \frac{t}{s}\right|^{r_1-1} \left|\log \frac{x}{y}\right|^{r_2-1}$$
$$\times |h(t,x,s,y,u_n(s-\tau_1, y-\xi_1), \ldots, u_n(s-\tau_m, y-\xi_m))| \frac{d_y g_2(x,y) d_s g_1(t,s)}{sy\Gamma(r_1)\Gamma(r_2)}$$
$$+ |f(t,x,u(t,x))| \int_1^t \int_1^x \left|\log \frac{t}{s}\right|^{r_1-1} \left|\log \frac{x}{y}\right|^{r_2-1}$$
$$\times |h(t,x,s,y,u_n(s-\tau_1, y-\xi_1), \ldots, u_n(s-\tau_m, y-\xi_m))$$
$$- h(t,x,s,y,u(s-\tau_1, y-\xi_1), \ldots, u(s-\tau_m, y-\xi_m))| \frac{d_y g_2(x,y) d_s g_1(t,s)}{sy\Gamma(r_1)\Gamma(r_2)}$$
$$\le p^*q^*|u_n(t,x) - u(t,x)| + \frac{f^* + p^*\|u\|_{BC}}{\Gamma(r_1)\Gamma(r_2)} \int_1^t \int_1^x \left|\log \frac{t}{s}\right|^{r_1-1} \left|\log \frac{x}{y}\right|^{r_2-1}$$
$$\times |h(t,x,s,y,u_n(s-\tau_1, y-\xi_1), \ldots, u_n(s-\tau_m, y-\xi_m))$$
$$- h(t,x,s,y,u(s-\tau_1, y-\xi_1), \ldots, u(s-\tau_m, y-\xi_m))| d_y G_2(x,y) d_s G_1(t,s)$$

$$\leq p^*q^*|u_n(t,x)-u(t,x)| + \frac{f^*+p^*\eta}{\Gamma(r_1)\Gamma(r_2)} \int_1^t \int_1^x \left|\log \frac{t}{s}\right|^{r_1-1} \left|\log \frac{x}{y}\right|^{r_2-1}$$
$$\times |h(t,x,s,y,u_n(s-\tau_1,y-\xi_1),\dots,u_n(s-\tau_m,y-\xi_m))$$
$$-h(t,x,s,y,u(s-\tau_1,y-\xi_1),\dots,u(s-\tau_m,y-\xi_m))|d_yG_2(x,y)d_sG_1(t,s)\,. \tag{9.21}$$

Case 1. If $(t,x) \in \tilde{J} \cup [1,a] \times [1,b]$, $a > 1$, then, since $u_n \to u$ as $n \to \infty$ and h is continuous, (9.21) gives

$$\|N(u_n) - N(u)\|_{BC} \to 0 \text{ as } n \to \infty\,.$$

Case 2. If $(t,x) \in (a,\infty) \times [1,b]$, $a > 1$, then from (9.9.6) and (9.21) we get

$$|(Nu_n)(t,x)-(Nu)(t,x)|$$
$$\leq p^*q^*|u_n(t,x)-u(t,x)| + \frac{f^*+p^*\eta}{\Gamma(r_1)\Gamma(r_2)} \int_1^t \int_1^x \left|\log \frac{t}{s}\right|^{r_1-1} \left|\log \frac{x}{y}\right|^{r_2-1}$$
$$\times 2Q(t,x,s,y)d_yG_2(x,y)d_sG_1(t,s)$$
$$\leq p^*q^*|u_n(t,x)-u(t,x)| + \frac{2(f^*+p^*\eta)}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \left|\log \frac{x}{y}\right|^{r_2-1} \left[\int_1^t \left|\log \frac{t}{s}\right|^{r_1-1}\right.$$
$$\left.\times\; Q(t,x,s,y)d_sG_1(t,s)\right] d_yG_2(x,y)\,. \tag{9.22}$$

Thus, (9.22) gives

$$\|N(u_n) - N(u)\|_{BC} \to 0 \text{ as } n \to \infty\,.$$

Step 2: $N(B_\eta)$ is uniformly bounded. This is clear since $N(B_\eta) \subset B_\eta$ and B_η is bounded.

Step 3: $N(B_\eta)$ is equicontinuous on every compact subset $[-T,a] \times [-\xi,b]$ of $[-T,\infty) \times [-\xi,b]$; $a > 1$. Let $(t_1,x_1),(t_2,x_2) \in [1,a] \times [1,b]$, $t_1 < t_2$, $x_1 < x_2$, and let $u \in B_\eta$. Then we have

$$|(Nu)(t_2,x_2)-(Nu)(t_1,x_1)| \leq |\mu(t_2,x_2)-\mu(t_1,x_1)|$$
$$+\frac{|f(t_2,x_2,u(t_2,x_2))-f(t_1,x_1,u(t_1,x_1))|}{\Gamma(r_1)\Gamma(r_2)} \int_1^{t_2} \int_1^{x_2} \left|\log \frac{t_2}{s}\right|^{r_1-1} \left|\log \frac{x_2}{y}\right|^{r_2-1}$$
$$\times |h(t_2,x_2,s,y,u(s-\tau_1,y-\xi_1),\dots,u(s-\tau_m,y-\xi_m))||d_yg_2(x_2,y)d_sg_1(t_2,s)|$$
$$+\frac{|f(t_1,x_1,u(t_1,x_1))|}{\Gamma(r_1)\Gamma(r_2)} \int_1^{t_2} \int_1^{x_2} \left|\log \frac{t_2}{s}\right|^{r_1-1} \left|\log \frac{x_2}{y}\right|^{r_2-1}$$
$$\times |h(t_2,x_2,s,y,u(s-\tau_1,y-\xi_1),\dots,u(s-\tau_m,y-\xi_m))$$

$$
\begin{aligned}
&- h(t_1, x_1, s, y, u(s-\tau_1, y-\xi_1), \dots, u(s-\tau_m, y-\xi_m))| \\
&\quad \times |d_\xi g_2(x_2, \xi) d_s g_1(t_2, s)| \\
&+ \left| \frac{|f(t_1, x_1, u(t_1, x_1))|}{\Gamma(r_1)\Gamma(r_2)} \int_1^{t_2}\int_1^{x_2} \left|\log\frac{t_2}{s}\right|^{r_1-1} \left|\log\frac{x_2}{y}\right|^{r_2-1} \right. \\
&\quad \times h(t_1, x_1, s, y, u(s-\tau_1, y-\xi_1), \dots, u(s-\tau_m, y-\xi_m)) d_\xi g_2(x_2, \xi) d_s g_1(t_2, s) \\
&- \frac{|f(t_1, x_1, u(t_1, x_1))|}{\Gamma(r_1)\Gamma(r_2)} \int_1^{t_1}\int_1^{x_1} \left(\log\frac{t_2}{s}\right)^{r_1-1} \left(\log\frac{x_2}{\xi}\right)^{r_2-1} \\
&\quad \left. \times h(t_1, x_1, s, y, u(s-\tau_1, y-\xi_1), \dots, u(s-\tau_m, y-\xi_m)) d_y g_2(x_2, y) d_s g_1(t_2, s) \right| \\
&+ \left| \frac{|f(t_1, x_1, u(t_1, x_1))|}{\Gamma(r_1)\Gamma(r_2)} \int_1^{t_1}\int_1^{x_1} \left(\log\frac{t_2}{s}\right)^{r_1-1} \left(\log\frac{x_2}{y}\right)^{r_2-1} \right. \\
&\quad \times h(t_1, x_1, s, y, u(s-\tau_1, y-\xi_1), \dots, u(s-\tau_m, y-\xi_m)) \\
&\quad \left. \times (d_y g_2(x_2, y) d_s g_1(t_2, s) - d_y g_2(x_1, y) d_s g_1(t_1, s)) \right| \\
&+ \frac{|f(t_1, x_1, u(t_1, x_1))|}{\Gamma(r_1)\Gamma(r_2)} \int_1^{t_1}\int_1^{x_1} \left| \left(\log\frac{t_2}{s}\right)^{r_1-1} \left(\log\frac{x_2}{y}\right)^{r_2-1} - \left(\log\frac{t_1}{s}\right)^{r_1-1} \left(\log\frac{x_1}{y}\right)^{r_2-1} \right| \\
&\quad \times |h(t_1, x_1, s, y, u(s-\tau_1, y-\xi_1), \dots, u(s-\tau_m, y-\xi_m))| |d_y g_2(x_1, y) d_s g_1(t_1, s)| .
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
&|(Nu)(t_2, x_2) - (Nu)(t_1, x_1)| \le |\mu(t_2, x_2) - \mu(t_1, x_1)| \\
&+ \frac{(|t_1 - t_2| + |x_1 - x_2|)\psi_1(|u|)}{\Gamma(r_1)\Gamma(r_2)} \int_1^{t_2}\int_1^{x_2} \left|\log\frac{t_2}{s}\right|^{r_1-1} \left|\log\frac{x_2}{y}\right|^{r_2-1} \\
&\times Q(t_2, x_2, s, y) |d_y g_2(x_2, y) d_s g_1(t_2, s)| \\
&+ \frac{|P(t_1, x_1)| u| + |f(t_1, x_1, 0)|}{\Gamma(r_1)\Gamma(r_2)} \int_1^{t_2}\int_1^{x_2} \left|\log\frac{t_2}{s}\right|^{r_1-1} \left|\log\frac{x_2}{y}\right|^{r_2-1} \\
&\times \varphi(s, y)(|t_1 - t_2| + |x_1 - x_2|)\psi_2\left(\sum_{i=1}^{m} |u|\right) |d_\xi g_2(x_2, \xi) d_s g_1(t_2, s)| \\
&+ \frac{|P(t_1, x_1)| u| + |f(t_1, x_1, 0)|}{\Gamma(r_1)\Gamma(r_2)} \int_1^{t_2}\int_{x_1}^{x_2} \left|\log\frac{t_2}{s}\right|^{r_1-1} \left|\log\frac{x_2}{y}\right|^{r_2-1} \\
&\times Q(t_1, x_1, s, y) |d_\xi g_2(x_2, \xi) d_s g_1(t_2, s)| \\
&+ \frac{|P(t_1, x_1)| u| + |f(t_1, x_1, 0)|}{\Gamma(r_1)\Gamma(r_2)} \int_{t_1}^{t_2}\int_1^{x_2} \left|\log\frac{t_2}{s}\right|^{r_1-1} \left|\log\frac{x_2}{y}\right|^{r_2-1} \\
&\times Q(t_1, x_1, s, y) |d_y g_2(x_2, y) d_s g_1(t_2, s)|
\end{aligned}
$$

$$+\frac{|P(t_1,x_1)|u|+|f(t_1,x_1,0)|}{\Gamma(r_1)\Gamma(r_2)}\int_{t_1}^{t_2}\int_{x_1}^{x_2}\left|\log\frac{t_2}{s}\right|^{r_1-1}\left|\log\frac{x_2}{y}\right|^{r_2-1}$$

$$\times Q(t_1,x_1,s,y)|d_y g_2(x_2,y)d_s g_1(t_2,s)|$$

$$+\frac{|P(t_1,x_1)|u|+|f(t_1,x_1,0)|}{\Gamma(r_1)\Gamma(r_2)}\int_{1}^{t_1}\int_{1}^{x_1}\left(\log\frac{t_2}{s}\right)^{r_1-1}\left(\log\frac{x_2}{y}\right)^{r_2-1}$$

$$\times Q(t_1,x_1,s,y)|d_y g_2(x_2,y)d_s g_1(t_2,s)-d_y g_2(x_1,y)d_s g_1(t_1,s)|$$

$$+\frac{|P(t_1,x_1)|u|+|f(t_1,x_1,0)|}{\Gamma(r_1)\Gamma(r_2)}\int_{1}^{t_1}\int_{1}^{x_1}Q(t_1,x_1,s,y)$$

$$\times\left|\left(\log\frac{t_2}{s}\right)^{r_1-1}\left(\log\frac{x_2}{y}\right)^{r_2-1}-\left(\log\frac{t_1}{s}\right)^{r_1-1}\left(\log\frac{x_1}{y}\right)^{r_2-1}\right|$$

$$\times|d_y g_2(x_1,y)d_s g_1(t_1,s)|\,.$$

Hence, we get

$$|(Nu)(t_2,x_2)-(Nu)(t_1,x_1)|\le|\mu(t_2,x_2)-\mu(t_1,x_1)|$$

$$+\frac{(|t_1-t_2|+|x_1-x_2|)\psi_1(\eta)}{\Gamma(r_1)\Gamma(r_2)}\int_{1}^{t_2}\int_{1}^{x_2}\left|\log\frac{t_2}{s}\right|^{r_1-1}\left|\log\frac{x_2}{y}\right|^{r_2-1}$$

$$\times Q(t_2,x_2,s,y)d_y G_2(x_2,y)d_s G_1(t_2,s)$$

$$+\frac{(p^*\eta+f^*)(|t_1-t_2|+|x_1-x_2|)\psi_2(m\eta)}{\Gamma(r_1)\Gamma(r_2)}\int_{1}^{t_2}\int_{1}^{x_2}\left|\log\frac{t_2}{s}\right|^{r_1-1}\left|\log\frac{x_2}{y}\right|^{r_2-1}$$

$$\times\varphi(s,y)d_y G_2(x_2,y)d_s G_1(t_2,s)$$

$$+\frac{p^*\eta+f^*}{\Gamma(r_1)\Gamma(r_2)}\int_{1}^{t_2}\int_{x_1}^{x_2}\left|\log\frac{t_2}{s}\right|^{r_1-1}\left|\log\frac{x_2}{y}\right|^{r_2-1}Q(t_1,x_1,s,y)d_y G_2(x_2,y)d_s G_1(t_2,s)$$

$$+\frac{p^*\eta+f^*}{\Gamma(r_1)\Gamma(r_2)}\int_{t_1}^{t_2}\int_{1}^{x_2}\left|\log\frac{t_2}{s}\right|^{r_1-1}\left|\log\frac{x_2}{y}\right|^{r_2-1}Q(t_1,x_1,s,y)d_y G_2(x_2,y)d_s G_1(t_2,s)$$

$$+\frac{p^*\eta+f^*}{\Gamma(r_1)\Gamma(r_2)}\int_{t_1}^{t_2}\int_{x_1}^{x_2}\left|\log\frac{t_2}{s}\right|^{r_1-1}\left|\log\frac{x_2}{y}\right|^{r_2-1}Q(t_1,x_1,s,y)|d_y G_2(x_2,y)d_s G_1(t_2,s)$$

$$+\frac{p^*\eta+f^*}{\Gamma(r_1)\Gamma(r_2)}\sup_{(s,y)\in[1,t_1]\times[1,x_1]}\left[\left(\log\frac{t_2}{s}\right)^{r_1-1}\left(\log\frac{x_2}{y}\right)^{r_2-1}Q(t_1,x_1,s,y)\right]$$

$$\times\left|\bigvee_{k_2=1}^{x_1}g_2(x_2,k_2)\bigvee_{k_1=1}^{t_1}g_1(t_2,k_1)-\bigvee_{k_2=1}^{x_1}g_2(x_1,k_2)d_s\bigvee_{k_1=1}^{t_1}g_1(t_1,k_1)\right|$$

$$+\frac{p^*\eta+f^*}{\Gamma(r_1)\Gamma(r_2)}\int_1^{t_1}\int_1^{x_1} Q(t_1,x_1,s,y)$$
$$\times\left|\left(\log\frac{t_2}{s}\right)^{r_1-1}\left(\log\frac{x_2}{y}\right)^{r_2-1}-\left(\log\frac{t_1}{s}\right)^{r_1-1}\left(\log\frac{x_1}{y}\right)^{r_2-1}\right|$$
$$\times d_yG_2(x_1,y)d_sG_1(t_1,s)\,.$$

From the continuity of μ, Q, g_1, g_2 and as $t_1 \to t_2$ and $x_1 \to x_2$, the right-hand side of the preceding inequality tends to zero. The equicontinuity for the cases $t_1 < t_2 < 0$, $x_1 < x_2 < 0$ and $t_1 \le 0 \le t_2$, $x_1 \le 0 \le x_2$ is obvious.

Step 4: $N(B_\eta)$ is equiconvergent. Let $(t,x) \in J$ and $u \in B_\eta$; then we have

$$|(Nu)(t,x)| \le |\mu(t,x)| + \frac{|f(t,x,0)|}{\Gamma(r_1)\Gamma(r_2)}\int_1^{t}\int_1^{x}\left|\log\frac{t}{s}\right|^{r_1-1}\left|\log\frac{x}{y}\right|^{r_2-1}$$
$$\times|h(t,x,s,y,u(s-\tau_1,y-\xi_1),\ldots,u(s-\tau_m,y-\xi_m))||d_yg_2(x,y)d_sg_1(t,s)|$$
$$+\frac{|f(t,x,u(t,x))-f(t,x,0)|}{\Gamma(r_1)\Gamma(r_2)}\int_1^{t}\int_1^{x}\left|\log\frac{t}{s}\right|^{r_1-1}\left|\log\frac{x}{y}\right|^{r_2-1}$$
$$\times|h(t,x,s,y,u(s-\tau_1,y-\xi_1),\ldots,u(s-\tau_m,y-\xi_m))||d_yg_2(x,y)d_sg_1(t,s)|$$
$$\le|\mu(t,x)|+\frac{f^*}{\Gamma(r_1)\Gamma(r_2)}\int_1^{t}\int_1^{x}\left|\log\frac{t}{s}\right|^{r_1-1}\left|\log\frac{x}{y}\right|^{r_2-1}$$
$$\times\frac{Q(t,x,s,y)}{1+\sum_{i=1}^m|u_i(s-\tau_i,y-\xi_i)|}d_yG_2(x,y)d_sG_1(t,s)$$
$$+\frac{P(t,x)|u(t,x)|}{\Gamma(r_1)\Gamma(r_2)}\int_1^{t}\int_1^{x}\left|\log\frac{t}{s}\right|^{r_1-1}\left|\log\frac{x}{y}\right|^{r_2-1}$$
$$\times Q(t,x,s,y)d_yG_2(x,y)d_sG_1(t,s)$$
$$\le|\mu(t,x)|+\frac{f^*+p^*\eta}{\Gamma(r_1)\Gamma(r_2)}\int_1^{t}\int_1^{x}\left|\log\frac{t}{s}\right|^{r_1-1}\left|\log\frac{x}{y}\right|^{r_2-1}$$
$$\times Q(t,x,s,y)d_yG_2(x,y)d_sG_1(t,s)\,.$$

Thus, for each $x \in [0,b]$ we have

$$|(Nu)(t,x)| \to 0, \quad \text{as } t \to +\infty\,.$$

Hence, for each $x \in [0,b]$ we get

$$|(Nu)(t,x)-(Nu)(+\infty,x)| \to 0, \quad \text{as } t\to+\infty\,.$$

As a consequence of Steps 1–4, together with Lemma 1.57, we can conclude that $N\colon B_\eta \to B_\eta$ is continuous and compact. From an application of Schauder's theorem, we deduce that N has a fixed point u that is a solution of problem (9.14)–(9.15).

Step 5: Uniform global attractivity of solutions. Let us assume that u and v are two solutions of problem (9.14)–(9.15). Then for each $(t, x) \in J$ we have

$$\begin{aligned}
|u(t,x) - v(t,x)| &= |(Nu)(t,x) - (Nv)(t,x)| \\
&\le p^* q^* |u(t,x) - v(t,x)| + \frac{f^* + p^* \|u\|_{BC}}{\Gamma(r_1)\Gamma(r_2)} \int_1^t \int_1^x \left|\log \frac{t}{s}\right|^{r_1-1} \left|\log \frac{x}{y}\right|^{r_2-1} \\
&\quad \times 2Q(t,x,s,y) d_y G_2(x,y) d_s G_1(t,s)\, . \\
&\le p^* q^* |u(t,x) - v(t,x)| + \frac{2(f^* + p^* \|u\|_{BC})}{\Gamma(r_1)\Gamma(r_2)} \\
&\quad \times \int_1^x \left|\log \frac{x}{y}\right|^{r_2-1} \left[\int_1^t \left|\log \frac{t}{s}\right|^{r_1-1} Q(t,x,s,y) d_s G(t,s) \right] d_y G_2(x,y)\, .
\end{aligned}$$

Thus, for each $(t, x) \in J$ we get

$$\begin{aligned}
|u(t,x) - v(t,x)| \le{} & \frac{2(f^* + p^* \|u\|_{BC})}{(1 - p^* q^*)\Gamma(r_1)\Gamma(r_2)} \\
& \times \int_1^x \left|\log \frac{x}{y}\right|^{r_2-1} \left[\int_1^t \left|\log \frac{t}{s}\right|^{r_1-1} Q(t,x,s,y) d_s G(t,s) \right] d_y G_2(x,y)\, .
\end{aligned} \tag{9.23}$$

Using (9.23), we deduce that for each $x \in [0, b]$ we get

$$\lim_{t\to\infty} |u(t,x) - v(t,x)| = 0\, .$$

Consequently, all solutions of problem (9.14)–(9.15) are uniformly globally attractive. □

9.4.3 An Example

As an application and to illustrate our results, we consider the problem of fractional order Volterra–Stieltjes quadratic multidelay Hadamard integral equations

$$\begin{aligned}
u(t,x) = \mu(t,x) &+ \frac{f(t,x,u(t,x))}{\Gamma(r_1)\Gamma(r_2)} \int_1^t \int_1^x \left(\log \frac{t}{s}\right)^{r_1-1} \left(\log \frac{x}{y}\right)^{r_2-1} \\
&\times h\left(t,x,s,y,u(s-1,y-2), u\left(s - \frac{1}{2}, y - \frac{2}{5}\right)\right) \\
&\times \frac{1}{sy} d_y g_2(x,y) d_s g_1(t,s) \quad \text{if } (t,x) \in J := [1,+\infty) \times [1,e]\, ,
\end{aligned} \tag{9.24}$$

$$u(t,x) = \frac{2}{(2+t^2)(2+x^2)} \quad \text{if } (t,x) \in \tilde{J} := [-1,\infty) \times [-2,e] \backslash (1,\infty) \times (1,e]\, , \tag{9.25}$$

where

$$r_1 = \frac{1}{4}, \quad r_2 = \frac{1}{2}, \quad \mu(t,x) = \frac{1}{2+t^2+x^2}; \ (t,x) \in J,$$
$$f(t,x,u) = 1 + \frac{e^{x-t}|u|}{1+|u|}; \ (t,x) \in J, \ u \in \mathbb{R},$$
$$g_1(t,s) = s; \ (t,s) \in [1,+\infty)^2, \quad g_2(x,y) = y; \ (x,y) \in [1,e]^2,$$
$$h(t,x,s,y,u_1,u_2) = \frac{cxs^{\frac{-3}{4}}|u|\sin\sqrt{t}\sin s}{(1+t^2+y^2)(2+|u_1|+|u_2|)}; \ (t,x,s,y) \in J', \ u_1,u_2 \in \mathbb{R},$$
$$c = \frac{\pi}{16e\Gamma\left(\frac{1}{4}\right)} \text{ and } J' = \{(t,x,s,y) : 1 \le s \le t, \ 1 \le y \le x \le e\}.$$

Set

$$\tau_1 = 1, \tau_2 = \frac{1}{2}, \quad \xi_1 = 2, \quad \xi_2 = \frac{2}{5} \text{ and } \Phi(t,x) = \frac{2}{(2+t^2)(2+x^2)}; \ (t,x) \in \tilde{J}.$$

Then $T = e$ and $\xi = 2$.
First, we can see that for each $x \in [1,e]$ we have $\lim_{t\to+\infty} \frac{1}{2+t^2+x^2} = 0$. Then (9.9.1) is satisfied by $\mu^* = \frac{1}{4}$, and $\Phi^* = \frac{2}{9}$.
Next, the function f is continuous, $f^* = 1$, and

$$|f(t,x,u) - f(t,x,v)| \le e^{x-t}|u-v|; \quad (t,x) \in J, \ u,v \in \mathbb{R}.$$

Then (9.9.2) is satisfied by $P(t,x) = e^{x-t}$; $(t,x) \in J$, and then $p^* = e^2$. Also, we can easily see that the function g satisfies conditions (9.9.3)–(9.9.5). Also, the function h satisfies condition (9.9.6). Indeed, h is continuous and

$$|h(t,x,s,y,u_1,u_2)| \le \frac{Q(t,x,s,y)}{1+|u_1|+|u_2|}, \quad (t,x,s,y) \in J', \ u_1,u_2 \in \mathbb{R},$$

where

$$Q(t,x,s,y) = \frac{cxs^{\frac{-3}{4}}\sin\sqrt{t}\sin s}{1+t^2+y^2}, \quad (t,x,s,y) \in J'.$$

Then

$$\begin{aligned}
\left|\int_1^t \left|\log\frac{t}{s}\right|^{r_1-1} \frac{Q(t,x,s,y)}{s} d_s g_1(t,s)\right| &\le \int_1^t \left|\log\frac{t}{s}\right|^{\frac{-3}{4}} cxs^{\frac{-7}{4}} |\sin\sqrt{t}\sin s| d_s G_1(t,s) \\
&\le cx|\sin\sqrt{t}| \int_1^t \left|\log\frac{t}{s}\right|^{\frac{-3}{4}} s^{\frac{-7}{4}} ds \\
&\le \frac{cx\Gamma^2\left(\frac{1}{4}\right)}{\sqrt{\pi}} \left|\frac{\sin\sqrt{t}}{\sqrt{\log t}}\right| \\
&\le \frac{cx\Gamma^2\left(\frac{1}{4}\right)}{\sqrt{\pi\log t}} \longrightarrow 0 \ \text{as} \ t \to \infty
\end{aligned}$$

and

$$q^* := \sup_{(t,x)\in J} \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^t \int_1^x \left|\log \frac{t}{s}\right|^{r_1-1} \left|\log \frac{x}{y}\right|^{r_2-1} \frac{Q(t,x,s,y)}{sy} dyds$$
$$\leq \sup_{(t,x)\in J} \frac{1}{\Gamma\left(\frac{3}{2}\right)} \frac{cx\Gamma\left(\frac{1}{4}\right)}{\sqrt{\pi}} \left|\frac{\sin\sqrt{t}}{\sqrt{\log t}}\right| \leq \frac{2ec\Gamma\left(\frac{1}{4}\right)}{\pi} = \frac{1}{8}.$$

Finally, we can see that $p^*q^* \leq \frac{e^2}{8} < 1$. Consequently, by Theorem 9.9, problem (7.4)–(9.25) has at least one solution in the space $BC([-1,\infty) \times [-2,e])$, and all solutions of (9.24)–(9.25) are uniformly globally attractive.

9.5 Notes and Remarks

The results of Chapter 9 are taken from Abbas et al. [4, 3]. Other results may be found in [19, 18, 16, 30, 33, 38].