# 8 Stability Results for Partial Hadamard Fractional Integral Equations and Inclusions

## 8.1 Introduction

This chapter deals with some existence and Ulam stability results for several classes of partial integral equations via Hadamard's fractional integral by applying some fixed point theorems.

## 8.2 Ulam Stabilities for Partial Hadamard Fractional Integral Equations

### 8.2.1 Introduction

This section deals with the existence the Ulam stability of solutions to the Hadamard partial fractional integral equation

$$u(x,y) = \mu(x,y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left(\log \frac{x}{s}\right)^{r_1-1} \left(\log \frac{y}{t}\right)^{r_2-1} \frac{f(s,t,u(s,t))}{st} dtds\,; \quad \text{if} \quad (x,y) \in J\,, \tag{8.1}$$

where $J := [1,a] \times [1,b]$, $a, b > 1$, $r_1, r_2 > 0$, $\mu : J \to \mathbb{R}$, $f : J \times \mathbb{R} \to \mathbb{R}$ are given continuous functions.

We present two results for integral equation (8.1). The first one is based on Banach's contraction principle and the second one on the nonlinear alternative of the Leray–Schauder type.

### 8.2.2 Existence and Ulam Stabilities Results

In this section, we discuss the existence of solutions and present conditions for the Ulam stability for the Hadamard integral equation (8.1).

The following conditions will be used in the sequel.

(8.1.1) There exist functions $p_1, p_2 \in C(J, \mathbb{R}_+)$ such that for any $u \in \mathbb{R}$ and $(x,y) \in J$,

$$|f(x,y,u)| \le p_1(x,y) + \frac{p_2(x,y)}{1+|u(x,y)|}|u(x,y)|\,,$$

with

$$p_i^* = \sup_{(x,y)\in J} p_i(x,y)\,; \quad i = 1,2\,.$$

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(8.1.2) There exists $\lambda_\Phi > 0$ such that for each $(x, y) \in J$ we have

$$({}^H I_\sigma^r \Phi)(x, y) \leq \lambda_\Phi \Phi(x, y) .$$

**Theorem 8.1.** *Assume* (8.1.1). *If*

$$\frac{(\log a)^{r_1}(\log b)^{r_2}}{\Gamma(1 + r_1)\Gamma(1 + r_2)} p_2^* < 1 , \tag{8.2}$$

*then integral equation* (8.1) *has a solution defined on J.*

*Proof.* Let $\rho > 0$ be a constant such that

$$\rho > \frac{\|\mu\|_\infty + \frac{(\log a)^{r_1}(\log b)^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} p_1^*}{1 - \frac{(\log a)^{r_1}(\log b)^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} p_2^*} .$$

We use Schauder's fixed point theorem [149] to prove that the operator $N: C \to C$ defined by

$$(Nu)(x, y) = \mu(x, y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left(\log \frac{x}{s}\right)^{r_1 - 1} \left(\log \frac{y}{t}\right)^{r_2 - 1} \frac{f(s, t, u(s, t))}{st} dtds \tag{8.3}$$

has a fixed point. The proof will be given in four steps.

*Step 1: N transforms the ball $B_\rho := \{u \in \mathcal{C}: \|u\|_C \leq \rho\}$ into itself.* For any $u \in B_\rho$ and each $(x, y) \in J$ we have

$$\begin{aligned} |(Nu)(x, y)| &\leq |\mu(x, y)| + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left|\log \frac{x}{s}\right|^{r_1 - 1} \left|\log \frac{y}{t}\right|^{r_2 - 1} \\ &\quad \times \frac{p_1(s, t) + p_2(s, t)\|u\|_C}{st} dtds \\ &\leq \|\mu\|_\infty + \frac{(\log a)^{r_1}(\log b)^{r_2}}{\Gamma(1 + r_1)\Gamma(1 + r_2)} (p_1^* + p_2^* \rho) . \end{aligned}$$

Thus, by (8.2) and the definition of $\rho$ we get $\|(Nu)\|_C \leq \rho$. This implies that $N$ transforms the ball $B_\rho$ into itself.

*Step 2: $N: B_\rho \to B_\rho$ is continuous.*
Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence such that $u_n \to u$ in $B_\rho$. Then

$$\begin{aligned} |(Nu_n)(x, y) - (Nu)(x, y)| &\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left|\log \frac{x}{s}\right|^{r_1 - 1} \left|\log \frac{y}{t}\right|^{r_2 - 1} \\ &\quad \times \frac{|f(s, t, u_n(s, t)) - f(s, t, u(s, t))|}{st} dtds \end{aligned}$$

$$\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left|\log \frac{x}{s}\right|^{r_1-1} \left|\log \frac{y}{t}\right|^{r_2-1}$$
$$\times \frac{\sup_{(s,t)\in J} |f(s,t,u_n(s,t)) - f(s,t,u(s,t))|}{st} dtds$$
$$\leq \frac{(\log a)^{r_1}(\log b)^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} \|f(\cdot,\cdot,u_n(\cdot,\cdot)) - f(\cdot,\cdot,u(\cdot,\cdot))\|_C .$$

From Lebesgue's dominated convergence theorem and the continuity of the function $f$ we get

$$|(Nu_n)(x,y) - (Nu)(x,y)| \to 0 \text{ as } n \to \infty .$$

*Step 3*: *$N(B_\rho)$ is bounded.* This is clear since $N(B_\rho) \subset B_\rho$ and $B_\rho$ is bounded.

*Step 4*: *$N(B_\rho)$ is equicontinuous.*

Let $(x_1,y_1),(x_2,y_2) \in J$, $x_1 < x_2$, $y_1 < y_2$. Then

$$|(Nu)(x_2,y_2) - (Nu)(x_1,y_1)| \leq |\mu(x_1,y_1) - \mu(x_2,y_2)|$$
$$+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^{x_1} \int_1^{y_1} \left[\left|\log \frac{x_2}{s}\right|^{r_1-1} \left|\log \frac{y_2}{t}\right|^{r_2-1} - \left|\log \frac{x_1}{s}\right|^{r_1-1} \left|\log \frac{y_1}{t}\right|^{r_2-1}\right]$$
$$\times \frac{|f(s,t,u(s,t))|}{st} dtds$$
$$+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^{x_2} \int_{y_1}^{y_2} \left|\log \frac{x_2}{s}\right|^{r_1-1} \left|\log \frac{y_2}{t}\right|^{r_2-1} \frac{|f(s,t,u(s,t))|}{st} dtds$$
$$+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^{x_1} \int_{y_1}^{y_2} \left|\log \frac{x_2}{s}\right|^{r_1-1} \left|\log \frac{y_2}{t}\right|^{r_2-1} \frac{|f(s,t,u(s,t))|}{st} dtds$$
$$+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^{x_2} \int_1^{y_1} \left|\log \frac{x_2}{s}\right|^{r_1-1} \left|\log \frac{y_2}{t}\right|^{r_2-1} \frac{|f(s,t,u(s,t))|}{st} dtds .$$

Thus,

$$|(Nu)(x_2,y_2) - (Nu)(x_1,y_1)| \leq |\mu(x_1,y_1) - \mu(x_2,y_2)|$$
$$+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^{x_1} \int_1^{y_1} \left[\left|\log \frac{x_2}{s}\right|^{r_1-1} \left|\log \frac{y_2}{t}\right|^{r_2-1} - \left|\log \frac{x_1}{s}\right|^{r_1-1} \left|\log \frac{y_1}{t}\right|^{r_2-1}\right]$$
$$\times \frac{p_1^* + p_2^*\rho}{st} dtds$$
$$+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^{x_2} \int_{y_1}^{y_2} \left|\log \frac{x_2}{s}\right|^{r_1-1} \left|\log \frac{y_2}{t}\right|^{r_2-1} \frac{p_1^* + p_2^*\rho}{st} dtds$$
$$+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^{x_1} \int_{y_1}^{y_2} \left|\log \frac{x_2}{s}\right|^{r_1-1} \left|\log \frac{y_2}{t}\right|^{r_2-1} \frac{p_1^* + p_2^*\rho}{st} dtds$$

$$+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_1}^{x_2}\int_{1}^{y_1}\left|\log\frac{x_2}{s}\right|^{r_1-1}\left|\log\frac{y_2}{t}\right|^{r_2-1}\frac{p_1^*+p_2^*\rho}{st}dtds$$
$$\leq\frac{p_1^*+p_2^*\rho}{\Gamma(1+r_1)\Gamma(1+r_2)}$$
$$\times[2(\log y_2)^{r_2}(\log x_2-\log x_1)^{r_1}+2(\log x_2)^{r_1}(\log y_2-\log y_1)^{r_2}$$
$$+(\log x_1)^{r_1}(\log y_1)^{r_2}-(\log x_2)^{r_1}(\log y_2)^{r_2}-2(\log x_2-\log x_1)^{r_1}(\log y_2-\log y_1)^{r_2}]\,.$$

As $x_1 \to x_2$ and $y_1 \to y_2$, the right-hand side of the preceding inequality tends to zero.

As a consequence of Steps 1–4, together with the Ascoli–Arzelà theorem, we can conclude that $N$ is continuous and compact. From an application of Schauder's theorem [149], we deduce that $N$ has a fixed point $u$ that is a solution of integral equation (8.1). □

Now we are concerned with the stability of solutions for integral equation (8.1).

Recall $N: C \to C$ as defined in 8.3. Let $\epsilon > 0$, and let $\Phi: J \to [0, \infty)$ be a continuous function. We consider the inequalities

$$|u(x,y)-(Nu)(x,y)|\leq\epsilon\,;\quad (x,y)\in J\,, \tag{8.4}$$
$$|u(x,y)-(Nu)(x,y)|\leq\Phi(x,y)\,;\quad (x,y)\in J\,, \tag{8.5}$$
$$|u(x,y)-(Nu)(x,y)|\leq\epsilon\Phi(x,y)\,;\quad (x,y)\in J\,. \tag{8.6}$$

**Definition 8.2** ([35, 233]). Equation (8.1) is Ulam–Hyers stable if there exists a real number $c_N > 0$ such that for each $\epsilon > 0$ and for each solution $u \in C$ of inequality (8.4) there exists a solution $v \in C$ of equation (8.1) with

$$|u(x,y)-v(x,y)|\leq\epsilon c_N\,;\quad (x,y)\in J\,.$$

**Definition 8.3** ([35, 233]). Equation (8.1) is generalized Ulam–Hyers stable if there exists $c_N: C([0,\infty),[0,\infty))$, with $c_N(0) = 0$, such that for each $\epsilon > 0$ and for each solution $u \in \mathcal{C}$ of (8.4) there exists a solution $v \in C$ of equation (8.1) with

$$|u(x,y)-v(x,y)|\leq c_N(\epsilon)\,;\quad (x,y)\in J\,.$$

**Definition 8.4** ([35, 233]). Equation (8.1) is Ulam–Hyers–Rassias stable with respect to $\Phi$ if there exists a real number $c_{N,\Phi} > 0$ such that for each $\epsilon > 0$ and for each solution $u \in C$ of (8.6) there exists a solution $v \in C$ of equation (8.1) with

$$|u(x,y)-v(x,y)|\leq\epsilon c_{N,\Phi}\Phi(x,y)\,;\quad (x,y)\in J\,.$$

**Definition 8.5** ([35, 233]). Equation (8.1) is generalized Ulam–Hyers–Rassias stable with respect to $\Phi$ if there exists a real number $c_{N,\Phi} > 0$ such that for each solution $u \in C$ of (8.5) there exists a solution $v \in \mathcal{C}$ of equation (8.1) with $|u(x,y) - v(x,y)| \leq c_{N,\Phi}\Phi(x,y)$; $(x,y) \in J$.

**Remark 8.6.** It is clear that (i) Definition 8.2 implies Definition 8.3, (ii) Definition 8.4 implies Definition 8.5, and (iii) Definition 8.4 for $\Phi(.,.) = 1$ implies Definition 8.2.

One could make similar remarks for inequalities (8.4) and (8.6).

**Theorem 8.7.** *Assume* (8.1.1), (8.1.2), *and* (8.2) *hold. Furthermore, suppose that there exist* $q_i \in C(J, \mathbb{R}_+)$, $i = 1, 2$, *such that for each* $(x, y) \in J$ *we have*

$$p_i(x, y) \leq q_i(x, y)\Phi(x, y) .$$

*Then integral equation* (8.1) *is generalized Ulam–Hyers–Rassias stable.*

*Proof.* Let $u$ be a solution of inequality (8.5). By Theorem 8.1 there exists $v$ that is a solution of integral equation (8.1). Hence,

$$v(x, y) = \mu(x, y) + \int_1^x \int_1^y \left(\log \frac{x}{s}\right)^{r_1-1} \left(\log \frac{y}{t}\right)^{r_2-1} \frac{f(s, t, v(s, t))}{st\Gamma(r_1)\Gamma(r_2)} dtds .$$

By inequality (8.5), for each $(x, y) \in J$ we have

$$\left| u(x, y) - \mu(x, y) - \int_1^x \int_1^y \left(\log \frac{x}{s}\right)^{r_1-1} \left(\log \frac{y}{t}\right)^{r_2-1} \frac{f(s, t, u(s, t))}{st\Gamma(r_1)\Gamma(r_2)} dtds \right| \leq \Phi(x, y) .$$

Set

$$q_i^* = \sup_{(x,y)\in J} q_i(x, y) ; \quad i = 1, 2 .$$

For each $(x, y) \in J$ we have

$$\begin{aligned}
|u(x, y) - v(x, y)| &\leq \left| u(x, y) - \mu(x, y) - \int_1^x \int_1^y \left(\log \frac{x}{s}\right)^{r_1-1} \left(\log \frac{y}{t}\right)^{r_2-1} \frac{f(s, t, u(s, t))}{st\Gamma(r_1)\Gamma(r_2)} dtds \right| \\
&\quad + \int_1^x \int_1^y \left|\log \frac{x}{s}\right|^{r_1-1} \left|\log \frac{y}{t}\right|^{r_2-1} \frac{|f(s, t, u(s, t)) - f(s, t, v(s, t))|}{st\Gamma(r_1)\Gamma(r_2)} dtds \\
&\leq \Phi(x, y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left|\log \frac{x}{s}\right|^{r_1-1} \left|\log \frac{y}{t}\right|^{r_2-1} \\
&\quad \times \left( 2q_1^* + \frac{q_2^*|u(s, t)|}{1 + |u|} + \frac{q_2^*|v(s, t)|}{1 + |v|} \right) \frac{\Phi(s, t)}{st} dtds \\
&\leq \Phi(x, y) + 2(q_1^* + q_2^*)({}^H I_\sigma^r \Phi)(x, y) \\
&\leq [1 + 2(q_1^* + q_2^*)\lambda_\phi]\Phi(x, y) \\
&:= c_{N,\Phi}\Phi(x, y) .
\end{aligned}$$

Hence, integral equation (8.1) is generalized Ulam–Hyers–Rassias stable. □

### 8.2.3 An Example

As an application of our results we consider the partial Hadamard integral equation

$$u(x,y) = \mu(x,y) + \int_1^x \int_1^y \left(\log\frac{x}{s}\right)^{r_1-1}\left(\log\frac{y}{t}\right)^{r_2-1}\frac{f(s,t,u(s,t))}{st\Gamma(r_1)\Gamma(r_2)}dtds\,, \quad (x,y)\in[1,e]\times[1,e]\,, \tag{8.7}$$

where

$$r_1, r_2 > 0\,, \quad \mu(x,y) = x + y^2\,, \quad (x,y)\in[1,e]\times[1,e]\,,$$

and

$$f(x,y,u(x,y)) = cxy^2\left(e^{-4} + \frac{u(x,y)}{e^{x+y+5}}\right)\,, \quad (x,y)\in[1,e]\times[1,e]\,,$$

with

$$c := \frac{e^4}{2}\Gamma(1+r_1)\Gamma(1+r_2)\,.$$

For each $u \in \mathbb{R}$ and $(x,y)\in[1,e]\times[1,e]$ we have

$$|f(x,y,u(x,y))| \le ce^{-4}(1+|u|)\,.$$

Hence, condition (8.1.1) is satisfied by $p_1^* = p_2^* = ce^{-4}$. Condition (8.2) holds with $a = b = e$. Indeed,

$$\frac{(\log a)^{r_1}(\log b)^{r_2}p_2^*}{\Gamma(1+r_1)\Gamma(1+r_2)} = \frac{c}{e^4\Gamma(1+r_1)\Gamma(1+r_2)} = \frac{1}{2} < 1\,.$$

Consequently, Theorem 8.1 implies that Hadamard integral equation (8.7) has a solution defined on $[1,e]\times[1,e]$. Also, condition (8.1.2) is satisfied by

$$\Phi(x,y) = e^3, \text{ and } \lambda_\Phi = \frac{1}{\Gamma(1+r_1)\Gamma(1+r_2)}\,.$$

Indeed, for each $(x,y)\in[1,e]\times[1,e]$ we get

$$\begin{aligned}({}^H I_\sigma^r \Phi)(x,y) &\le \frac{e^3}{\Gamma(1+r_1)\Gamma(1+r_2)} \\ &= \lambda_\Phi \Phi(x,y)\,.\end{aligned}$$

Consequently, Theorem 8.7 implies that equation (8.7) is generalized Ulam–Hyers–Rassias stable.

## 8.3 Global Stability Results for Volterra-Type Partial Hadamard Fractional Integral Equations

### 8.3.1 Introduction

In [47], Abbas et al. studied some existence and stability results for the nonlinear quadratic Volterra integral equation of Riemann–Liouville fractional order

$$u(t,x) = f(t,x,u(t,x),u(\alpha(t),x)) + \frac{1}{\Gamma(r)}\int_0^{\beta(t)} (\beta(t)-s)^{r-1} \times g(t,x,s,u(s,x),u(\gamma(s),x))ds\,, \quad (t,x) \in \mathbb{R}_+ \times [0,b]\,, \tag{8.8}$$

where $b > 0$, $\mathbb{R}_+ = [0,\infty)$, $r \in (0,\infty)$, $\alpha, \beta, \gamma\colon \mathbb{R}_+ \to \mathbb{R}_+$, and $f\colon \mathbb{R}_+ \times [0,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R}_+ \times [0,b] \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions.

This section deals with the global existence and stability of solutions to the nonlinear quadratic Volterra partial integral equation of Hadamard fractional order

$$u(t,x) = f(t,x,u(t,x),u(\alpha(t),x)) + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^{\beta(t)}\int_1^{x} \left(\log\frac{\beta(t)}{s}\right)^{r_1-1}\left(\log\frac{x}{\xi}\right)^{r_2-1} \times g(t,x,s,\xi,u(s,\xi),u(\gamma(s),\xi))\frac{d\xi ds}{s\xi}\,, \quad (t,x) \in J := [1,\infty) \times [1,b]\,, \tag{8.9}$$

where $b > 1$, $r_1, r_2 \in (0,\infty)$, $\alpha, \beta, \gamma\colon [1,\infty) \to [1,\infty)$, and $f\colon J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $g\colon J \times J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions. Our existence results are based upon Schauder's fixed point theorem. Also, we obtain some results about the local asymptotic stability of solutions of the equation in question. Finally, we present an example illustrating the applicability of the imposed conditions.

### 8.3.2 Existence and Global Stability Results

In this section, we are concerned with the existence and the asymptotic stability of solutions for Hadamard partial integral equation (8.9).

In the sequel, we will use the following conditions.

(8.3.1) The function $\alpha\colon [1,\infty) \to [1,\infty)$ satisfies $\lim_{t\to\infty} \alpha(t) = \infty$.

(8.3.2) There exist constants $M, L > 0$ and a nondecreasing function $\psi_1\colon [0,\infty) \to (0,\infty)$ such that $M < \frac{L}{2}$,

$$|f(t,x,u_1,v_1) - f(t,x,u_2,v_2)| \le \frac{M(|u_1-u_2| + |v_1-v_2|)}{(1+\alpha(t))(L + |u_1-u_2| + |v_1-v_2|)}\,,$$

and

$$|f(t_1,x_1,u,v) - f(t_2,x_2,u,v)| \le (|t_1-t_2| + |x_1-x_2|)\psi_1(|u|+|v|)$$

for each $(t,x), (t_1,x_1), (t_2,x_2) \in J$ and $u, v, u_1, v_1, u_2, v_2 \in \mathbb{R}$.

(8.3.3) The function $t \to f(t, x, 0, 0)$ is bounded on $J$ with

$$f^* = \sup_{(t,x)\in[1,\infty)\times[1,b]} f(t, x, 0, 0)$$

and

$$\lim_{t\to\infty} |f(t, x, 0, 0)| = 0\,; \quad x \in [1, b]\,.$$

(8.3.4) There exist continuous functions $p, q, \varphi : J \to \mathbb{R}_+$ and a nondecreasing function $\psi_2 : [0, \infty) \to (0, \infty)$ such that

$$|g(t_1, x_1, s, \xi, u, v) - g(t_2, x_2, s, \xi, u, v)| \le \varphi(s, \xi)(|t_1 - t_2| + |x_1 - x_2|)\psi_2(|u| + |v|)$$

and

$$|g(t, x, s, \xi, u, v)| \le \frac{p(t, x)q(s, \xi)}{1 + \alpha(t) + |u| + |v|}$$

for each $(t, x), (s, \xi), (t_1, x_1), (t_2, x_2) \in J$ and $u, v \in \mathbb{R}$. Moreover, assume that

$$\lim_{t\to\infty} p(t, x) \int_1^{\beta(t)} \int_1^{x} \left|\log \frac{\beta(t)}{s}\right|^{r_1-1} \left|\log \frac{x}{\xi}\right|^{r_2-1} q(s, \xi)d\xi ds = 0\,.$$

**Theorem 8.8.** *Assume* (8.3.1)–(8.3.4). *Then integral equation* (7.1) *has at least one solution in the space BC. Moreover, solutions of equation* (7.1) *are globally asymptotically stable.*

*Proof.* Set $d^* := \sup_{(t,x)\in J} d(t, x)$, where

$$d(t, x) = \frac{p(t, x)}{\Gamma(r_1)\Gamma(r_2)} \int_1^{\beta(t)} \int_1^{x} \left|\log \frac{\beta(t)}{s}\right|^{r_1-1} \left|\log \frac{x}{\xi}\right|^{r_2-1} q(s, \xi)d\xi ds\,.$$

From condition (8.3.4) we infer that $d^*$ is finite. Let us define the operator $N$ such that, for any $u \in BC$,

$$(Nu)(t, x) = f(t, x, u(t, x), u(\alpha(t), x)) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^{\beta(t)} \int_1^{x} \left(\log \frac{\beta(t)}{s}\right)^{r_1-1} \left(\log \frac{x}{\xi}\right)^{r_2-1}$$
$$\times g(t, x, s, \xi, u(s, \xi), u(\gamma(s), \xi))\frac{d\xi ds}{s\xi}\,, \quad (t, x) \in J\,. \tag{8.10}$$

By considering the conditions of this theorem, we infer that $N(u)$ is continuous on $J$. Now we prove that $N(u) \in BC$ for any $u \in BC$. For arbitrarily fixed $(t, x) \in J$ we have

$$|(Nu)(t, x)| \le |f(t, x, u(t, x), u(\alpha(t), x)) - f(t, x, 0, 0)| + |f(t, x, 0, 0)|$$
$$+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^{\beta(t)} \int_1^{x} \left|\log \frac{\beta(t)}{s}\right|^{r_1-1} \left|\log \frac{x}{\xi}\right|^{r_2-1}$$
$$\times |g(t, x, s, \xi, u(s, \xi), u(\gamma(s), \xi))|\frac{d\xi ds}{s\xi}|$$

$$
\begin{aligned}
&\leq \frac{M(|u(t,x)|+|u(\alpha(t),x)|)}{(1+\alpha(t))(L+|u(t,x)|+|u(\alpha(t),x)|)} + |f(t,x,0,0)| \\
&\quad + \frac{p(t,x)}{\Gamma(r_1)\Gamma(r_2)} \int_1^{\beta(t)} \int_1^{x} \left|\log \frac{\beta(t)}{s}\right|^{r_1-1} \left|\log \frac{x}{\xi}\right|^{r_2-1} \\
&\quad \times \frac{q(s,\xi)}{1+\alpha(t)+|u(s,\xi))|+|u(\gamma(s),\xi))|} \frac{d\xi ds}{s\xi} \\
&\leq \frac{M(|u(t,x)|+|u(\alpha(t),x)|)}{|u(t,x)|+|u(\alpha(t),x)|} + f^* + d^* .
\end{aligned}
$$

Thus,

$$
\|N(u)\|_{BC} \leq M + f^* + d^* . \tag{8.11}
$$

Hence, $N(u) \in BC$. Equation (8.11) yields that $N$ transforms the ball $B_\eta := B(0,\eta)$ into itself, where $\eta = M + f^* + d^*$. We will show that $N\colon B_\eta \to B_\eta$ satisfies the assumptions of Schauder's fixed point theorem [149]. The proof will be given in several steps and cases.

*Step 1: N is continuous.* Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence such that $u_n \to u$ in $B_\eta$. Then for each $(t,x) \in J$ we have

$$
\begin{aligned}
|(Nu_n)(t,x) - (Nu)(t,x)| &\leq |f(t,x,u_n(t,x),u_n(\alpha(t),x)) - f(t,x,u(t,x),u(\alpha(t),x))| \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^{\beta(t)} \int_1^{x} \left|\log \frac{\beta(t)}{s}\right|^{r_1-1} \left|\log \frac{x}{\xi}\right|^{r_2-1} \\
&\quad \times \sup_{(s,\xi)\in J} |g(t,x,s,\xi,u_n(s,\xi),u_n(\gamma(s),\xi)) \\
&\qquad - g(t,x,s,\xi,u(s,\xi),u(\gamma(s),\xi))| \frac{d\xi ds}{s\xi} \\
&\leq \frac{2M}{L}\|u_n - u\|_{BC} \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^{\beta(t)} \int_1^{x} \left|\log \frac{\beta(t)}{s}\right|^{r_1-1} \left|\log \frac{x}{\xi}\right|^{r_2-1} \\
&\quad \times \|g(t,x,.,.,u_n(.,.),u_n(\gamma(.),.)) \\
&\qquad - g(t,x,.,.,u(.,.),u(\gamma(.),.))\|_{BC} d\xi ds .
\end{aligned} \tag{8.12}
$$

*Case 1.* If $(t,x) \in [1,T] \times [1,b]$, $T > 1$, then, since $u_n \to u$ as $n \to \infty$ and $g, \gamma$ are continuous, then (8.12) gives

$$
\|N(u_n) - N(u)\|_{BC} \to 0 \ \text{ as } \ n \to \infty .
$$

*Case 2.* If $(t,x) \in (T,\infty)\times[1,b]$, $T>1$, then from (8.3.4) and (8.12), for each $(t,x)\in J$, we have

$$\begin{aligned}|(Nu_n)(t,x)-(Nu)(t,x)| &\le \frac{2M}{L}\|u_n-u\|_{BC} \\ &\quad + \frac{2p(t,x)}{\Gamma(r_1)\Gamma(r_2)}\int_1^{\beta(t)}\int_1^{x}\left|\log\frac{\beta(t)}{s}\right|^{r_1-1}\left|\log\frac{x}{\xi}\right|^{r_2-1}\frac{q(s,\xi)}{s\xi}d\xi ds \\ &\le \frac{2M}{L}\|u_n-u\|_{BC}+d(t,x)\,.\end{aligned}$$

Thus, we get

$$|(Nu_n)(t,x)-(Nu)(t,x)| \le \frac{2M}{L}\|u_n-u\|_{BC}+d(t,x)\,. \tag{8.13}$$

Since $u_n\to u$ as $n\to\infty$ and $t\to\infty$, then (8.13) gives

$$\|N(u_n)-N(u)\|_{BC}\to 0 \quad\text{as}\quad n\to\infty\,.$$

*Step 2*: *$N(B_\eta)$ is uniformly bounded.* This is clear since $N(B_\eta)\subset B_\eta$ and $B_\eta$ is bounded.

*Step 3: $N(B_\eta)$ is equicontinuous on every compact subset* $[1,a]\times[1,b]$ *of* $J$, $a>0$. Let $(t_1,x_1),(t_2,x_2)\in[1,a]\times[1,b]$, $t_1<t_2$, $x_1<x_2$, and let $u\in B_\eta$. Also, without loss of generality, suppose that $\beta(t_1)\le\beta(t_2)$. Then we have

$$\begin{aligned}&|(Nu)(t_2,x_2)-(Nu)(t_1,x_1)| \\ &\le |f(t_2,x_2,u(t_2,x_2),u(\alpha(t_2),x_2))-f(t_2,x_2,u(t_1,x_1),u(\alpha(t_1),x_1))| \\ &\quad + |f(t_2,x_2,u(t_1,x_1),u(\alpha(t_1),x_1))-f(t_1,x_1,u(t_1,x_1),u(\alpha(t_1),x_1))| \\ &\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^{\beta(t_2)}\int_1^{x_2}\left|\log\frac{\beta(t_2)}{s}\right|^{r_1-1}\left|\log\frac{x_2}{\xi}\right|^{r_2-1} \\ &\qquad \times|g(t_2,x_2,s,\xi,u(s,\xi),u(\gamma(s),\xi))-g(t_1,x_1,s,\xi,u(s,\xi),u(\gamma(s),\xi))|d\xi ds \\ &\quad + \left|\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^{\beta(t_2)}\int_1^{x_2}\left(\log\frac{\beta(t_2)}{s}\right)^{r_1-1}\left(\log\frac{x_2}{\xi}\right)^{r_2-1}\right. \\ &\qquad \times g(t_1,x_1,s,\xi,u(s,\xi),u(\gamma(s),\xi))d\xi ds \\ &\quad - \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^{\beta(t_1)}\int_1^{x_1}\left(\log\frac{\beta(t_2)}{s}\right)^{r_1-1}\left(\log\frac{x_2}{\xi}\right)^{r_2-1} \\ &\qquad \left.\times g(t_1,x_1,s,\xi,u(s,\xi),u(\gamma(s),\xi))d\xi ds\right| \\ &\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^{\beta(t_1)}\int_1^{x_1}\left|\left(\log\frac{\beta(t_2)}{s}\right)^{r_1-1}\left(\log\frac{x_2}{\xi}\right)^{r_2-1}\right. \\ &\quad \left.-\left(\log\frac{\beta(t_1)}{s}\right)^{r_1-1}\left(\log\frac{x_1}{\xi}\right)^{r_2-1}\right||g(t_1,x_1,s,\xi,u(s,\xi),u(\gamma(s),\xi))|d\xi ds\,.\end{aligned}$$

Thus, we obtain

$$
\begin{aligned}
&|(Nu)(t_2, x_2) - (Nu)(t_1, x_1)| \\
&\leq \frac{M}{L}(|u(t_2, x_2) - u(t_1, x_1)| + |u(\alpha(t_2), x_2) - u(\alpha(t_1), x_1)|) \\
&\quad + (|t_2 - t_1| + |x_2 - x_1|)\psi_1(2\|u\|_{BC}) \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^{\beta(t_2)} \int_1^{x_2} \left|\log \frac{\beta(t_2)}{s}\right|^{r_1-1} \left|\log \frac{x_2}{\xi}\right|^{r_2-1} \\
&\qquad \times \varphi(s,\xi)(|t_2 - t_1| + |x_2 - x_1|)\psi_2(2\|u\|_{BC}) d\xi ds \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{\beta(t_1)}^{\beta(t_2)} \int_1^{x_2} \left|\log \frac{\beta(t_2)}{s}\right|^{r_1-1} \left|\log \frac{x_2}{\xi}\right|^{r_2-1} \\
&\qquad \times |g(t_1, x_1, s, \xi, u(s,\xi), u(\gamma(s),\xi))| d\xi ds \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^{\beta(t_2)} \int_{x_1}^{x_2} \left|\log \frac{\beta(t_2)}{s}\right|^{r_1-1} \left|\log \frac{x_2}{\xi}\right|^{r_2-1} \\
&\qquad \times |g(t_1, x_1, s, \xi, u(s,\xi), u(\gamma(s),\xi))| d\xi ds \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{\beta(t_1)}^{\beta(t_2)} \int_{x_1}^{x_2} \left|\log \frac{\beta(t_2)}{s}\right|^{r_1-1} \left|\log \frac{x_2}{\xi}\right|^{r_2-1} \\
&\qquad \times |g(t_1, x_1, s, \xi, u(s,\xi), u(\gamma(s),\xi))| d\xi ds \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^{\beta(t_1)} \int_1^{x_1} \left|\left(\log \frac{\beta(t_2)}{s}\right)^{r_1-1} \left(\log \frac{x_2}{\xi}\right)^{r_2-1}\right. \\
&\quad - \left.\left(\log \frac{\beta(t_1)}{s}\right)^{r_1-1} \left(\log \frac{x_1}{\xi}\right)^{r_2-1}\right| |g(t_1, x_1, s, \xi, u(s,\xi), u(\gamma(s),\xi))| d\xi ds \, .
\end{aligned}
$$

Hence, we get

$$
\begin{aligned}
&|(Nu)(t_2, x_2) - (Nu)(t_1, x_1)| \\
&\leq \frac{M}{L}(|u(t_2, x_2) - u(t_1, x_1)| + |u(\alpha(t_2), x_2) - u(\alpha(t_1), x_1)|) \\
&\quad + (|t_2 - t_1| + |x_2 - x_1|)\psi_1(2\eta) \\
&\quad + \frac{(|t_2 - t_1| + |x_2 - x_1|)\psi_2(2\eta)}{\Gamma(r_1)\Gamma(r_2)} \\
&\qquad \times \int_1^{\beta(t_2)} \int_1^{x_2} \left|\log \frac{\beta(t_2)}{s}\right|^{r_1-1} \left|\log \frac{x_2}{\xi}\right|^{r_2-1} \varphi(s,\xi) d\xi ds \\
&\quad + \frac{p(t_1, x_1)}{\Gamma(r_1)\Gamma(r_2)} \int_{\beta(t_1)}^{\beta(t_2)} \int_1^{x_2} \left|\log \frac{\beta(t_2)}{s}\right|^{r_1-1} \left|\log \frac{x_2}{\xi}\right|^{r_2-1} q(s,\xi)| d\xi ds
\end{aligned}
$$

$$+\frac{p(t_1,x_1)}{\Gamma(r_1)\Gamma(r_2)}\int_1^{\beta(t_2)}\int_{x_1}^{x_2}\left|\log\frac{\beta(t_2)}{s}\right|^{r_1-1}\left|\log\frac{x_2}{\xi}\right|^{r_2-1}q(s,\xi)|d\xi ds$$

$$+\frac{p(t_1,x_1)}{\Gamma(r_1)\Gamma(r_2)}\int_{\beta(t_1)}^{\beta(t_2)}\int_{x_1}^{x_2}\left|\log\frac{\beta(t_2)}{s}\right|^{r_1-1}\left|\log\frac{x_2}{\xi}\right|^{r_2-1}q(s,\xi)|d\xi ds$$

$$+\frac{p(t_1,x_1)}{\Gamma(r_1)\Gamma(r_2)}\int_1^{\beta(t_1)}\int_1^{x_1}\left|\left(\log\frac{\beta(t_2)}{s}\right)^{r_1-1}\left(\log\frac{x_2}{\xi}\right)^{r_2-1}\right.$$
$$\left.-\left(\log\frac{\beta(t_1)}{s}\right)^{r_1-1}\left(\log\frac{x_1}{\xi}\right)^{r_2-1}\right|q(s,\xi)|d\xi ds\,.$$

From the continuity of $\alpha,\beta,f,g$ and as $t_1\to t_2$ and $x_1\to x_2$, the right-hand side of the preceding inequality tends to zero.

*Step 4: $N(B_\eta)$ is equiconvergent.* Let $(t,x)\in J$ and $u\in B_\eta$; then we have

$$\begin{aligned}
|u(t,x)| &\le |f(t,x,u(t,x),u(\alpha(t),x))-f(t,x,0,0)+f(t,x,0,0)|\\
&\quad+\left|\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^{\beta(t)}\int_1^{x}\left(\log\frac{\beta(t)}{s}\right)^{r_1-1}\left(\log\frac{x}{\xi}\right)^{r_2-1}\right.\\
&\quad\left.\times g(t,x,s,\xi,u(s,\xi),u(\gamma(s),\xi))\frac{d\xi ds}{s\xi}\right|\\
&\le\frac{M(|u(t,x)|+|u(\alpha(t),x)|)}{(1+\alpha(t))(L+|u(t,x)|+|u(\alpha(t),x)|)}+|f(t,x,0,0)|\\
&\quad+\frac{p(t,x)}{\Gamma(r_1)\Gamma(r_2)}\int_1^{\beta(t)}\int_1^{x}\left(\log\frac{\beta(t)}{s}\right)^{r_1-1}\left(\log\frac{x}{\xi}\right)^{r_2-1}\\
&\quad\times\frac{q(s,\xi)}{1+\alpha(t)+|u(s,\xi)|+|u(\gamma(s),\xi)|}d\xi ds\\
&\le\frac{M}{1+\alpha(t)}+|f(t,x,0,0)|\\
&\quad+\frac{p(t,x)}{\Gamma(r_1)\Gamma(r_2)(1+\alpha(t))}\int_1^{\beta(t)}\int_1^{x}\left(\log\frac{\beta(t)}{s}\right)^{r_1-1}\left(\log\frac{x}{\xi}\right)^{r_2-1}q(s,\xi)d\xi ds\\
&\le\frac{M}{1+\alpha(t)}+|f(t,x,0,0)|+\frac{d^*}{1+\alpha(t)}\,.
\end{aligned}$$

Thus, for each $x\in[1,b]$ we get

$$|u(t,x)|\to 0,\quad\text{as } t\to+\infty\,.$$

Hence,

$$|u(t,x)-u(+\infty,x)|\to 0,\quad\text{as } t\to+\infty\,.$$

As a consequence of Steps 1–4, together with Lemma 1.57, we can conclude that $N: B_\eta \to B_\eta$ is continuous and compact. From an application of Schauder's fixed point theorem [149] we deduce that $N$ has a fixed point $u$ that is a solution of Hadamard integral equation (8.9).

*Step 5: the uniform global attractivity.* Let us assume that $u_0$ is a solution of integral equation (8.9) with the conditions of this theorem. Consider the ball $B(u_0, \eta)$ with $\eta^* = \frac{LM^*}{L-2M}$, where

$$M^* := \frac{1}{\Gamma(r_1)\Gamma(r_2)} \sup_{(t,x)\in J} \left\{ \int_1^{\beta(t)} \int_1^x \left(\log \frac{\beta(t)}{s}\right)^{r_1-1} \left(\log \frac{x}{\xi}\right)^{r_2-1} \times |g(t,x,s,\xi,u(s,\xi),u(\gamma(s),\xi)) - g(t,x,s,\xi,u_0(s,\xi),u_0(\gamma(s),\xi))| d\xi ds;\ u \in BC \right\}.$$

Taking $u \in B(u_0, \eta^*)$, we then have

$$\begin{aligned} |(Nu)(t,x) - u_0(t,x)| &= |(Nu)(t,x) - (Nu_0)(t,x)| \\ &\le |f(t,x,u(t,x),u(\alpha(t),x)) - f(t,x,u_0(t,x),u_0(\alpha(t),x))| \\ &\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^{\beta(t)} \int_1^x \left(\log \frac{\beta(t)}{s}\right)^{r_1-1} \left(\log \frac{x}{\xi}\right)^{r_2-1} \\ &\quad \times |g(t,x,s,\xi,u(s,\xi),u(\gamma(s),\xi)) \\ &\quad - g(t,x,s,\xi,u_0(s,\xi),u_0(\gamma(s),\xi))| \frac{d\xi ds}{s\xi} \\ &\le \frac{2M}{L\|}\|u - u_0\|_{BC} + M^* \\ &\le \frac{2M}{L}\eta^* + M^* = \eta^* . \end{aligned}$$

Thus, we observe that $N$ is a continuous function such that $N(B(u_0, \eta^*)) \subset B(u_0, \eta^*)$. Moreover, if $u$ is a solution of equation (8.9), then

$$\begin{aligned} |u(t,x) - u_0(t,x)| &= |(Nu)(t,x) - (Nu_0)(t,x)| \\ &\le |f(t,x,u(t,x),u(\alpha(t),x)) - f(t,x,u_0(t,x),u_0(\alpha(t),x))| \\ &\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^{\beta(t)} \int_1^x \left(\log \frac{\beta(t)}{s}\right)^{r_1-1} \left(\log \frac{x}{\xi}\right)^{r_2-1} \\ &\quad \times |g(t,x,s,\xi,u(s,\xi),u(\gamma(s),\xi)) \\ &\quad - g(t,x,s,\xi,u_0(s,\xi),u_0(\gamma(s),\xi))| d\xi ds . \end{aligned}$$

Thus,

$$|u(t,x)-u_0(t,x)| \le \frac{M}{L}(|u(t,x)-u_0(t,x)|+|u(\alpha(t),x)-u_0(\alpha(t),x)|) + \frac{p(t,x)}{\Gamma(r_1)\Gamma(r_2)}\int_1^{\beta(t)}\int_1^{x}\left(\log\frac{\beta(t)}{s}\right)^{r_1-1}\left(\log\frac{x}{\xi}\right)^{r_2-1} q(s,\xi)d\xi ds\,. \tag{8.14}$$

By using (8.14) and the fact that $\alpha(t)\to\infty$ as $t\to\infty$, we get

$$\lim_{t\to\infty}|u(t,x)-u_0(t,x)| \le \lim_{t\to\infty}\frac{L.p(t,x)}{\Gamma(r_1)\Gamma(r_2)(L-2M)}\int_1^{\beta(t)}\int_1^{x}\left(\log\frac{\beta(t)}{s}\right)^{r_1-1}\left(\log\frac{x}{\xi}\right)^{r_2-1} \times q(s,\xi)d\xi ds = 0\,.$$

Consequently, all solutions of integral equation (7.1) are globally asymptotically stable. □

### 8.3.3 An Example

As an application of our results we consider the partial Hadamard integral equation of fractional order

$$u(t,x) = \frac{tx}{10(1+t+t^2+t^3)}(1+2\sin(u(t,x))) + \frac{1}{\Gamma^2\left(\frac{1}{3}\right)}\int_1^{t}\int_1^{x}\left(\log\frac{t}{s}\right)^{\frac{-2}{3}}\left(\log\frac{x}{\xi}\right)^{\frac{-2}{3}} \times \frac{\ln\left(1+2x(s\xi)^{-1}|u(s,\xi)|\right)}{(1+t+2|u(s,\xi)|)^2(1+x^2+t^4)}d\xi ds\,;\quad (t,x)\in[1,\infty)\times[1,e]\,, \tag{8.15}$$

where $r_1=r_2=\frac{1}{3}$, $\alpha(t)=\beta(t)=\gamma(t)=t$,

$$f(t,x,u,v) = \frac{tx(1+\sin(u)+\sin(v))}{10(1+t)(1+t^2)}\,,$$

and

$$g(t,x,s,\xi,u,v) = \frac{\ln(1+x(s\xi)^{-1}(|u|+|v|))}{(1+t+|u|+|v|)^2(1+x^2+t^4)}$$

for $(t,x),(s,\xi)\in[1,\infty)\times[1,e]$, and $u,v\in\mathbb{R}$.

We can easily check that the conditions of Theorem 8.8 are satisfied. In fact, we have that the function $f$ is continuous and satisfies (8.3.2), where $M=\frac{1}{10}$, $L=1$. Also, $f$ satisfies (8.3.3), with $f^*=\frac{e}{10}$. Next, let us note that the function $g$ satisfies (8.3.4), where $p(t,x)=\frac{1}{1+x^2+t^4}$ and $q(s,\xi)=(s\xi)^{-1}$.

Additionally,

$$\lim_{t\to\infty} p(t,x)\int_1^t\int_1^x \left|\log\frac{t}{s}\right|^{\frac{-2}{3}}\left|\log\frac{x}{\xi}\right|^{\frac{-2}{3}} q(s,\xi)d\xi ds$$

$$= \lim_{t\to\infty}\frac{x}{1+x^2+t^4}\int_1^t\int_1^x \left|\log\frac{t}{s}\right|^{\frac{-2}{3}}\left|\log\frac{x}{\xi}\right|^{\frac{-2}{3}} \frac{d\xi ds}{s\xi}$$

$$= \lim_{t\to\infty}\frac{9x(\log t)^{\frac{1}{3}}}{1+x^2+t^4} = 0\,.$$

Hence by Theorem 8.8, equation (8.15) has a solution defined on $[1,\infty)\times[1,e]$, and solutions of this equation are globally asymptotically stable.

## 8.4 Ulam Stabilities for Hadamard Fractional Integral Equations in Fréchet Spaces

### 8.4.1 Introduction

In this section, we present some results concerning the existence and Ulam stabilities of solutions for some functional integral equations of Hadamard fractional order. We use an extension of the Burton–Kirk fixed point theorem in Fréchet spaces.

Recently some interesting results on the existence and Ulam stabilities of the solutions of some classes of differential equations were obtained by Abbas et al. [5, 24, 25, 28]. This section deals with the existence and Ulam stabilities of solutions of the following Hadamard fractional integral equations:

$$\begin{aligned} u(t,x) &= \mu(t,x) + f(t,x,({}^H I_\sigma^r u)(t,x), u(t,x)) \\ &\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^t\int_1^x \left(\log\frac{t}{s}\right)^{r_1-1}\left(\log\frac{x}{y}\right)^{r_2-1} \\ &\quad \times g(t,x,s,y,u(s,y))\frac{dyds}{sy}\,, \quad (t,x)\in J := [1,+\infty)\times[1,b]\,, \end{aligned} \tag{8.16}$$

where $b > 1$, $\sigma = (1,1)$, $r = (r_1, r_2)$, $r_1, r_2 \in (0,\infty)$, $\mu\colon J \to \mathbb{R}$, $f\colon J\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and $g\colon J'\times\mathbb{R}\to\mathbb{R}$ are given continuous functions, and $J' = \{(t,x,s,y)\in J^2\colon s\le t, y\le x\}$. Our investigations are conducted in Fréchet spaces with an application of the fixed point theorem of Burton–Kirk to the existence of solutions of integral equation (8.16), and we prove that all solutions are generalized Ulam–Hyers–Rassias stable.

### 8.4.2 Existence and Ulam Stabilities Results

Here we are concerned with the existence and the Ulam stability of solutions for integral equation (8.16). Set

$$J'_p = \{(t,x,s,y)\colon 1 \le s \le t \le p, 1 \le y \le x \le b\}\,; \quad p \in \mathbb{N}\backslash\{0,1\}\,.$$

The following conditions will be used in the sequel:

(8.4.1) There exist continuous functions $l, k\colon J_p \to \mathbb{R}_+$ such that

$$|f(t,x,u_1,v_1) - f(t,x,u_2,v_2)| \le \frac{l(t,x)|u_1-u_2| + k(t,x)|v_1-v_2|}{1+|u_1-u_2|+|v_1-v_2|}$$

for each $(t,x) \in J_p$ and each $u_1, u_2, v_1, v_2 \in \mathbb{R}$.

(8.4.2) There exist continuous functions $P, Q, \varphi\colon J'_p \to \mathbb{R}_+$ and a nondecreasing function $\psi\colon [0,\infty) \to (0,\infty)$ such that

$$|g(t,x,s,y,u)| \le \frac{P(t,x,s,y) + Q(t,x,s,y)|u|}{1+|u|}$$

for $(t,x,s,y) \in J'$, $u \in \mathbb{R}$, and

$$|g(t_1,x_1,s,y,u) - g(t_2,x_2,s,y,u)| \le \varphi(s,y)(|t_1-t_2| + |x_1-x_2|)$$
$$\times \psi(|u|)\,; \quad (t_1,x_1,s,y),(t_2,x_2,s,y) \in J'_p, u \in \mathbb{R}\,.$$

(8.4.3) There exist continuous functions $P_1, Q_1\colon J_p \to [0,\infty)$ such that for each $(t,s),(t,x) \in J_p$ we have

$$P(t,x,s,y,w) \le \phi(t,x)P_1(s,y), \text{ and } Q(t,x,s,y,w) \le \phi(t,x)Q_1(s,y)\,.$$

**Theorem 8.9.** *Assume* (8.4.1) *and* (8.4.2). *If*

$$\ell := k_p + \frac{l_p(\log p)^{r_1}(\log b)^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} < 1\,, \tag{8.17}$$

*where*

$$k_p = \sup_{(t,x)\in J_p} k(t,x),\ l_p = \sup_{(t,x)\in J_p} l(t,x);\ p \in \mathbb{N}\backslash\{0,1\}\,,$$

*then Hadamard integral equation* (8.16) *has at least one solution in the space* $C$. *Furthermore, if condition* (8.4.3) *holds, then equation* (8.16) *is generalized Ulam–Hyers–Rassias stable.*

*Proof.* Let us define the operators $A, B\colon C \to C$ defined by

$$(Au)(t,x) = \int_1^t\int_1^x \left(\log\frac{t}{s}\right)^{r_1-1}\left(\log\frac{x}{y}\right)^{r_2-1}\frac{g(t,x,s,y,u(s,y))}{sy\Gamma(r_1)\Gamma(r_2)}dyds\,; \quad (t,x) \in J\,, \tag{8.18}$$

$$(Bu)(t,x) = \mu(t,x) + f(t,x,(^H I^r_\sigma u)(t,x),u(t,x))\,; \quad (t,x) \in J\,. \tag{8.19}$$

We will show that operators $A$ and $B$ satisfy all the conditions of Theorem 1.43. The proof will be given in several steps.

*Step 1. $A$ is compact.* To this end, we must prove that $A$ is continuous and it transforms every bounded set into a relatively compact set. Let $M \subset C$ be a bounded set of $C$. The proof will be given in several claims.

*Claim 1. $A$ is continuous.* Let $\{u_n\}_{n\in\mathbb{N}\setminus\{0,1\}}$ be a sequence in $M$ such that $u_n \to u$ in $M$. Then, for each $(t,x) \in J_p;\ p \in \mathbb{N}\setminus\{0,1\}$, we have

$$\begin{aligned}
&|(Au_n)(t,x) - (Au)(t,x)| \\
&\le \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^t \int_1^x \left|\log \frac{t}{s}\right|^{r_1-1} \left|\log \frac{x}{y}\right|^{r_2-1} \\
&\quad \times |g(t,x,s,y,u_n(s,y)) - g(t,x,s,y,u(s,y))| dyds \\
&\le \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^t \int_1^x \left|\log \frac{t}{s}\right|^{r_1-1} \left|\log \frac{x}{y}\right|^{r_2-1} \\
&\quad \times |g(t,x,s,y,u_n(s,y)) - g(t,x,s,y,u(s,y))| dyds\,. 
\end{aligned} \tag{8.20}$$

Since $u_n \to u$ as $n \to \infty$ and $g$ is continuous, (8.20) gives

$$\|A(u_n) - A(u)\|_p \to 0 \ \text{ as } \ n \to \infty\,.$$

*Claim 2. $A$ maps bounded sets to bounded sets in $C$.* For arbitrarily fixed $(t,x) \in J_p$ and $u \in M$, we have

$$\begin{aligned}
|(Au)(t,x)| &\le \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^t \int_1^x \left|\log \frac{t}{s}\right|^{r_1-1} \left|\log \frac{x}{y}\right|^{r_2-1} \\
&\quad \times |g(t,x,s,y,u(s,y))| dyds \\
&\le \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^t \int_1^x \left|\log \frac{t}{s}\right|^{r_1-1} \left|\log \frac{x}{y}\right|^{r_2-1} \\
&\quad \times \frac{P(t,x,s,y) + Q(t,x,s,y)|u(s,y)|}{1+|u(s,y)|} dyds \\
&\le \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^t \int_1^x \left|\log \frac{t}{s}\right|^{r_1-1} \left|\log \frac{x}{y}\right|^{r_2-1} \\
&\quad \times (P(t,x,s,y) + Q(t,x,s,y)) dyds \\
&\le P_p + Q_p\,,
\end{aligned}$$

where

$$P_p = \sup_{(t,x)\in J_p} \int_1^t \int_1^x \left|\log \frac{t}{s}\right|^{r_1-1} \left|\log \frac{x}{y}\right|^{r_2-1} \frac{P(t,x,s,y)}{\Gamma(r_1)\Gamma(r_2)} dyds$$

and

$$Q_p = \sup_{(t,x)\in J_p} \int_1^t \int_1^x \left|\log \frac{t}{s}\right|^{r_1-1} \left|\log \frac{x}{y}\right|^{r_2-1} \frac{Q(t,x,s,y)}{\Gamma(r_1)\Gamma(r_2)} dyds .$$

Thus,

$$\|A(u)\|_p \le P_p + Q_p := \ell'_p .$$

*Claim 3. A maps bounded sets to equicontinuous sets in C.* Let $(t_1, x_1), (t_2, x_2) \in J_p$, $t_1 < t_2, x_1 < x_2$, and let $u \in M$; thus, we have

$$\begin{aligned}
&|(Au)(t_2,x_2) - (Au)(t_1,x_1)| \\
&\le \frac{1}{\Gamma(r_1)\Gamma(r_2)} \Bigg| \int_1^{t_2}\int_1^{x_2} \left|\log \frac{t_2}{s}\right|^{r_1-1} \left|\log \frac{x_2}{y}\right|^{r_2-1} \\
&\quad \times [g(t_2,x_2,s,y,u(s,y)) - g(t_1,x_1,s,y,u(s,y))]dyds \Bigg| \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \Bigg| \int_1^{t_2}\int_1^{x_2} \left|\log \frac{t_2}{s}\right|^{r_1-1} \left|\log \frac{x_2}{y}\right|^{r_2-1} g(t_1,x_1,s,y,u(s,y))dyds \\
&\quad - \int_1^{t_2}\int_1^{x_2} \left|\log \frac{t_1}{s}\right|^{r_1-1} \left|\log \frac{x_1}{y}\right|^{r_2-1} g(t_1,x_1,s,y,u(s,y))dyds \Bigg| \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \Bigg| \int_1^{t_2}\int_1^{x_2} \left|\log \frac{t_1}{s}\right|^{r_1-1} \left|\log \frac{x_1}{y}\right|^{r_2-1} g(t_1,x_1,s,y,u(s,y))dyds \\
&\quad - \int_1^{t_1}\int_1^{x_1} \left|\log \frac{t_1}{s}\right|^{r_1-1} \left|\log \frac{x_1}{y}\right|^{r_2-1} g(t_1,x_1,s,y,u(s,y))dyds \Bigg| .
\end{aligned}$$

Thus,

$$\begin{aligned}
&|(Au)(t_2,x_2) - (Au)(t_1,x_1)| \\
&\le \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^{t_2}\int_1^{x_2} \left|\log \frac{t_2}{s}\right|^{r_1-1} \left|\log \frac{x_2}{y}\right|^{r_2-1} \\
&\quad \times |g(t_2,x_2,s,y,u(s,y)) - g(t_1,x_1,s,y,u(s,y))|\, dyds \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^{t_1}\int_1^{x_1} \left| \left(\log \frac{t_2}{s}\right)^{r_1-1} \left(\log \frac{x_2}{y}\right)^{r_2-1} - \left(\log \frac{t_1}{s}\right)^{r_1-1} \left(\log \frac{x_1}{y}\right)^{r_2-1} \right| \\
&\quad \times |g(t_1,x_1,s,y,u(s,y))|\, dyds \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^{t_1}\int_{x_1}^{x_2} \left|\log \frac{t_2}{s}\right|^{r_1-1} \left|\log \frac{x_2}{y}\right|^{r_2-1} |g(t_1,x_1,s,y,u(s,y))|dyds
\end{aligned}$$

$$+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{t_1}^{t_2}\int_{1}^{x_1}\left|\log\frac{t_2}{s}\right|^{r_1-1}\left|\log\frac{x_2}{y}\right|^{r_2-1}|g(t_1,x_1,s,y,u(s,y))|dyds$$

$$+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{t_1}^{t_2}\int_{x_1}^{x_2}\left|\log\frac{t_2}{s}\right|^{r_1-1}\left|\log\frac{x_2}{y}\right|^{r_2-1}|g(t_1,x_1,s,y,u(s,y))|dyds\,.$$

Hence,

$$\begin{aligned}
&|(Au)(t_2,x_2)-(Au)(t_1,x_1)|\\
&\le\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{1}^{t_2}\int_{1}^{x_2}\left|\log\frac{t_2}{s}\right|^{r_1-1}\left|\log\frac{x_2}{y}\right|^{r_2-1}\\
&\quad\times\varphi(s,y)(|t_1-t_2|+|x_1-x_2|)\psi(\ell_p)dyds\\
&\quad+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{1}^{t_1}\int_{1}^{x_1}\left|\left(\log\frac{t_2}{s}\right)^{r_1-1}\left(\log\frac{x_2}{y}\right)^{r_2-1}-\left(\log\frac{t_1}{s}\right)^{r_1-1}\left(\log\frac{x_1}{y}\right)^{r_2-1}\right|\\
&\quad\times(P(t_1,x_1,s,y)+Q(t_1,x_1,s,y))dyds\\
&\quad+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{t_2}^{t_2}\int_{1}^{x_2}\left|\log\frac{t_2}{s}\right|^{r_1-1}\left|\log\frac{x_2}{y}\right|^{r_2-1}(P(t_1,x_1,s,y)+Q(t_1,x_1,s,y))dyds\\
&\quad+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{1}^{t_1}\int_{x_1}^{x_2}\left|\log\frac{t_2}{s}\right|^{r_1-1}\left|\log\frac{x_2}{y}\right|^{r_2-1}(P(t_1,x_1,s,y)+Q(t_1,x_1,s,y))dyds\\
&\quad+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{t_1}^{t_2}\int_{x_1}^{x_2}\left|\log\frac{t_2}{s}\right|^{r_1-1}\left|\log\frac{x_2}{y}\right|^{r_2-1}(P(t_1,x_1,s,y)+Q(t_1,x_1,s,y))dyds\,.
\end{aligned}$$

From the continuity of functions $P$, $Q$, $\varphi$ and as $t_1\longrightarrow t_2$ and $x_1\longrightarrow x_2$, the right-hand side of the preceding inequality tends to zero. As a consequence of Claims 1–3 and from the Ascoli–Arzelà theorem, we can conclude that $A$ is continuous and compact.

*Step 2. B is a contraction.* Consider $v, w\in C$. Then, by (8.4.1), for any $p\in\mathbb{N}\backslash\{0,1\}$ and each $(t,x)\in J_P$, we have

$$\begin{aligned}
|(Bv)(t,x)-(Bw)(t,x)|&\le l(t,x)|^H I_\sigma^r(v-w)(t,x)|+k(t,x)|(v-w)(t,x)|\\
&\le\left(k(t,x)+\frac{l(t,x)(\log p)^{r_1}(\log b)^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}\right)|v-w|\,.
\end{aligned}$$

Thus,

$$\|(B(v)-B(w)\|_p\le\left(k_p+\frac{l_p(\log p)^{r_1}(\log b)^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}\right)\|v-w\|_p\,.$$

By (8.17) we conclude that $B$ is a contraction.

*Step 3.* The set $\mathcal{E} := \{u \in C(J) : u = \lambda A(u) + \lambda B(\frac{u}{\lambda}), \lambda \in (0,1)\}$ is bounded. Let $u \in C$ such that $u = \lambda A(u) + \lambda B(\frac{u}{\lambda})$ for some $\lambda \in (0,1)$. Then for any $p \in \mathbb{N}\backslash\{0,1\}$ and each $(t,x) \in J_p$ we have

$$\begin{aligned}|u(t,x)| &\le \lambda|A(u)| + \lambda\left|B\left(\frac{u}{\lambda}\right)\right| \\ &\le |\mu(t,x)| + |f(t,x,0,0)| + k(t,x) + l(t,x) \\ &\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^t\int_1^x \left|\log\frac{t}{s}\right|^{r_1-1}\left|\log\frac{x}{y}\right|^{r_2-1} \\ &\quad \times \frac{P(t,x,s,y)+Q(t,x,s,y)}{sy}dyds \\ &\le \mu_p + f_p + k_p + l_p + P_p + Q_p\,,\end{aligned}$$

where

$$\mu_p = \sup_{(t,x)\in[1,p]\times[1,b]} \mu(t,x)\,, \quad f_p = \sup_{(t,x)\in[1,p]\times[1,b]} |f(t,x,0,0)|;\; p \in \mathbb{N}\backslash\{0,1\}\,.$$

Thus,

$$\|u\|_p \le \mu_p + f_p + k_p + l_p + P_p + Q_p =: \ell_p^*\,.$$

Hence, the set $\mathcal{E}$ is bounded.

As a consequence of Steps 1–3 and from an application of Theorem 1.43, we deduce that $N$ has a fixed point $u$ that is a solution of integral equation (8.16).

*Step 4. The generalized Ulam–Hyers–Rassias stability.* Set

$$P_{1p} = \sup_{(s,y)\in J_p} P_1(s,y), \quad \text{and} \quad Q_{1p} = \sup_{(s,y)\in J_p} Q_1(s,y)\,.$$

Let $u$ be a solution of inequality (8.18) and $v$ be a solution of equation (8.16). Then

$$\begin{aligned}v(t,x) &= \mu(t,x) + f(t,x,{}^H I_\sigma^r v(t,x), v(t,x)) \\ &\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^t\int_1^x \left(\log\frac{t}{s}\right)^{r_1-1}\left(\log\frac{x}{y}\right)^{r_2-1} \\ &\quad \times g(t,x,s,y,v(s,y))\frac{dyds}{sy}\,, \quad (t,x) \in J := [1,+\infty)\times[1,b]\,.\end{aligned}$$

From inequality (8.18) and condition (8.4.3), for each $(t,x) \in J_p$, we have

$$
\begin{aligned}
|u(t,x) - v(x,y)| &\le |u(t,x) - (Nu)(t,x)| + |(Nu)(t,x) - (Nv)(t,x)| \\
&\le \phi(x,y) + |f(t,x,({}^H I_\sigma^r u)(t,x,), u(t,x)) - f(t,x,{}^H I_\sigma^r v(t,x), v(t,x))| \\
&\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^t \int_1^x \left|\log \frac{t}{s}\right|^{r_1-1} \left|\log \frac{x}{y}\right|^{r_2-1} \\
&\qquad \times |g(t,x,s,y,u(s,y)) - g(t,x,s,y,v(s,y))| \frac{dyds}{sy} \\
&\le \phi(x,y) + l(t,x)|({}^H I_\sigma^r u)(t,x) - {}^H I_\sigma^r v(t,x)| + k(t,x)|u(t,x) - v(t,x)| \\
&\quad + \frac{2\phi(t,x)}{\Gamma(r_1)\Gamma(r_2)} \int_1^t \int_1^x \left|\log \frac{t}{s}\right|^{r_1-1} \left|\log \frac{x}{y}\right|^{r_2-1} \phi(t,x) \\
&\qquad \times (P_1(s,y) + Q_1(s,t))dyds \\
&\le \phi(x,y) + \ell_p|u(t,x) - v(t,x)| \\
&\quad + \frac{2\phi(t,x)}{\Gamma(r_1)\Gamma(r_2)} \int_1^t \int_1^x \left|\log \frac{t}{s}\right|^{r_1-1} \left|\log \frac{x}{y}\right|^{r_2-1} (P_{1p} + Q_{1p})dyds \\
&\le \phi(t,x) + \ell_p|u(t,x) - v(t,x)| \\
&\quad + \frac{2(P_{1p} + Q_{1p})\phi(t,x)}{\Gamma(r_1)\Gamma(r_2)} \int_1^t \int_1^x \left|\log \frac{t}{s}\right|^{r_1-1} \left|\log \frac{x}{y}\right|^{r_2-1} dyds\,.
\end{aligned}
$$

Thus, for each $(t,x) \in J_p$ we obtain

$$
\begin{aligned}
|u(t,x) - v(x,y)| &\le \frac{\phi(t,x)}{1-\ell_p}\left(1 + \frac{2(P_{1p} + Q_{1p})}{\Gamma(r_1)\Gamma(r_2)} \int_1^t \int_1^x \left|\log \frac{t}{s}\right|^{r_1-1} \left|\log \frac{x}{y}\right|^{r_2-1} dyds\right) \\
&\le \frac{1}{1-\ell_p}\left(1 + \frac{2(P_{1p} + Q_{1p})(\log p)^{r_1}(\log b)^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}\right)\phi(t,x) \\
&:= c_{N,\phi}\phi(t,x)\,.
\end{aligned}
$$

Hence, for each $(t,x) \in J_p$ we get

$$
|u(t,x) - v(x,y)| \le c_{N,\phi}\phi(x,y)\,.
$$

Consequently, equation (8.16) is generalized Ulam–Hyers–Rassias stable. □

### 8.4.3 An Example

Consider the Hadamard fractional order integral equation

$$u(t,x) = \frac{xe^{3-2t}}{1+t+x^2} + \frac{xe^{-t-2}}{c_p(1+e^{-2p}|({}^H I_\sigma^r u)(t,x)| + e^{-p}|u(t,x)|)}$$
$$+ \int_1^t\int_1^x \left(\log\frac{t}{s}\right)^{r_1-1}\left(\log\frac{x}{y}\right)^{r_2-1}\frac{g(t,x,s,y,u(s,y))}{\Gamma(r_1)\Gamma(r_2)}dyds\,,$$
$$(t,x)\in[1,+\infty)\times[1,e]\,, \tag{8.21}$$

where $c_p = e^{-p} + \frac{e^{-2p}p^{r_1}}{\Gamma(1+r_1)\Gamma(1+r_2)}$; $p \in \mathbb{N}\backslash\{0,1\}$, $r=(r_1,r_2)\in(0,\infty)\times(0,\infty)$,

$$g(t,x,s,y,u) = \frac{xs^{\frac{-3}{4}}(1+|u|)\sin\sqrt{t}\sin s}{(1+x^2+t^2)(1+|u|)} \quad \text{if } (t,x,s,y)\in J' \text{ and } u\in\mathbb{R}\,,$$

and

$$J' = \{(t,x,s,y)\colon 1\le s\le t \text{ and } 1\le x\le y\le e\}\,.$$

Set

$$\mu(t,x) = \frac{xe^{3-2t}}{1+t+x^2},\; f(t,x,u,v) = \frac{xe^{-t-2}}{c_p(1+e^{-2p}|u|+e^{-p}|v|)}\,;\quad p\in\mathbb{N}\backslash\{0,1\}\,.$$

The function $f$ is continuous and satisfies (8.4.1), with $k(t,x) = \frac{xe^{-t-2-p}}{c_p}$, $l(t,x) = \frac{xe^{-t-2-2p}}{c_p}$, $k_p = \frac{e^{-2-p}}{c_p}$, and $l_p = \frac{e^{-2-2p}}{c_p}$. Additionally, the function $g$ is continuous and satisfies (8.4.2), with

$$P(t,x,s,y) = Q(t,x,s,y) = \frac{xs^{\frac{-3}{4}}\sin\sqrt{t}\sin s}{1+x^2+t^2}\,;\quad (t,x,s,y)\in J'\,.$$

Further, the function $g$ is continuous and satisfies (8.4.3), with

$$P_1(s,y) = Q_(s,y) = s^{\frac{-3}{4}}\sin s\,,\quad P_{1p} = Q_{1p} = p^{\frac{-3}{4}}$$

and

$$\phi(t,x) = \frac{x\sin\sqrt{t}}{1+x^2+t^2}\,.$$

Finally, we will show that condition (8.17) holds with $b = e$. Indeed, for each $p\in\mathbb{N}\backslash\{0,1\}$ we get

$$k_p + \frac{l_p(\log p)^{r_1}(\log b)^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} = \frac{1}{c_p}\left(e^{-2-p} + \frac{e^{-2-2p}p^{r_1}}{\Gamma(1+r_1)\Gamma(1+r_2)}\right) = e^{-2} < 1\,.$$

Hence, by Theorem 8.9, equation (8.21) has a solution defined on $[1,+\infty)\times[1,e]$ and (8.21) is generalized Ulam–Hyers–Rassias stable.

## 8.5 Ulam Stability Results for Hadamard Partial Fractional Integral Inclusions via Picard Operators

### 8.5.1 Introduction

In this section, using weakly Picard operators theory, we investigate some existence results and Ulam-type stability concepts of fixed point inclusions due to Rus for a class of partial Hadamard fractional integral inclusions.

In [47, 37, 39], Abbas et al. studied some Ulam stabilities for functional fractional partial differential and integral inclusions via Picard operators. In this section, we discuss the Ulam–Hyers and the Ulam–Hyers–Rassias stability for the new class of fractional partial integral inclusions

$$u(x,y) - \mu(x,y) \in ({}^{H}I_{\sigma}^{r}F)(x,y,u(x,y)) ; \quad (x,y) \in J := [1,a] \times [1,b] , \tag{8.22}$$

where $a, b > 1, \sigma = (1,1), F : J \times E \to \mathcal{P}(E)$ is a set-valued function with nonempty values in a (real or complex) separable Banach space $E, \mathcal{P}(E)$ is the family of all nonempty subsets of $E$, and $\mu : J \to E$ is a given continuous function.

### 8.5.2 Picard Operators Theory

In what follows we will give some basic definitions and results on Picard operators [228, 229]. Let $(X, d)$ be a metric space and $A : X \to X$ an operator. We denote by $F_A$ the set of the fixed points of $A$. We denote by $A^0 := 1_X,\ A^1 := A, \ldots, A^{n+1} := A^n \circ A;\ n \in \mathbb{N}$ the iterate operators of the operator $A$.

**Definition 8.10.** The operator $A : X \to X$ is a Picard operator (PO) if there exists $x^* \in X$ such that
(i) $F_A = \{x^*\}$,
(ii) The sequence $(A^n(x_0))_{n\in\mathbb{N}}$ converges to $x^*$ for all $x_0 \in X$.

**Definition 8.11.** The operator $A : X \to X$ is a weakly Picard operator (WPO) if the sequence $(A^n(x))_{n\in\mathbb{N}}$ converges for all $x \in X$ and its limit (which may depend on $x$) is a fixed point of $A$.

**Definition 8.12.** If $A$ is a WPO, then we consider the operator $A^\infty$ defined by

$$A^\infty : X \to X ; \quad A^\infty(x) = \lim_{n\to\infty} A^n(x) .$$

**Remark 8.13.** It is clear that $A^\infty(X) = F_A$.

**Definition 8.14.** Let $A$ be a WPO and $c > 0$. The operator $A$ is a c-weakly Picard operator if

$$d(x, A^\infty(x)) \le c\, d(x, A(x)) ; \quad x \in X .$$

In the multivalued case we have the following concepts (see [218, 235]).

**Definition 8.15.** Let $(X, d)$ be a metric space and $F\colon X \to \mathcal{P}_{cl}(X)$ a multivalued operator. By definition, $F$ is a multivalued weakly Picard operator (MWPO) if for each $u \in X$ and each $v \in F(x)$ there exists a sequence $(u_n)_{n\in\mathbb{N}}$ such that
(i) $u_0 = u, u_1 = v$,
(ii) $u_{n+1} \in F(u_n)$ for each $n \in \mathbb{N}$,
(iii) the sequence $(u_n)_{n\in\mathbb{N}}$ is convergent and its limit is a fixed point of $F$.

**Remark 8.16.** A sequence $(u_n)_{n\in\mathbb{N}}$ satisfying conditions (i) and (ii) in the preceding definition is called a sequence of successive approximations of $F$ starting from $(x, y) \in \text{Graph}(F)$.

If $F\colon X \to \mathcal{P}_{cl}(X)$ is a MWPO, then we define $F_1\colon \text{Graph}(F) \to \mathcal{P}(Fix(F))$ by the formula $F_1(x, y) := \{u \in \text{Fix}(F)$: there exists a sequence of successive approximations of $F$ starting from $(x, y)$ that converges to $u\}$.

**Definition 8.17.** Let $(X, d)$ be a metric space, and let $\Psi\colon [0,\infty) \to [0,\infty)$ be an increasing function that is continuous at 0 and $\Psi(0) = 0$. Then $F\colon X \to \mathcal{P}_{cl}(X)$ is said to be a multivalued $\Psi$-weakly Picard operator ($\Psi$-MWPO) if it is a MWPO and there exists a selection $A^{\infty}\colon \text{Graph}(F) \to \text{Fix}(F)$ of $F^{\infty}$ such that

$$d(u, A^{\infty}(u, v)) \leq \Psi(d(u, v))\,; \quad \text{for all } (u, v) \in \text{Graph}(F)\,.$$

If there exists $c > 0$ such that $\Psi(t) = ct$ for each $t \in [0,\infty)$, then $F$ is called a multivalued c-weakly Picard operator ($c$-MWPO).

Let us recall the notion of comparison.

**Definition 8.18.** A function $\varphi\colon [0,\infty) \to [0,\infty)$ is said to be a comparison function (see [228]) if it is increasing and $\varphi^n \to 0$ as $n \to \infty$.

As a consequence, we have $\varphi(t) < t$ for each $t > 0$, $\varphi(0) = 0$, and $\varphi$ is continuous at 0.

**Definition 8.19.** A function $\varphi\colon [0,\infty) \to [0,\infty)$ is said to be a strict comparison function (see [228]) if it is strictly increasing and $\sum_{n=1}^{\infty} \varphi^n(t) < \infty$ for each $t > 0$.

**Example 8.20.** *The mappings* $\varphi_1, \varphi_2\colon [0,\infty) \to [0,\infty)$ *given by* $\varphi_1(t) = ct$, $c \in [0, 1)$, *and* $\varphi_2(t) = \frac{t}{1+t}$, $t \in [0,\infty)$, *are strict comparison functions.*

**Definition 8.21.** A multivalued operator $N\colon X \to \mathcal{P}_{cl}(X)$ is called
(a) $\gamma$-Lipschitz if and only if there exists $\gamma \geq 0$ such that

$$H_d(N(u), N(v)) \leq \gamma d(u, v) \quad \text{for each } u, v \in X\,,$$

(b) a multivalued $\gamma$-contraction if and only if it is $\gamma$-Lipschitz with $\gamma \in [0, 1)$,
(c) a multivalued $\varphi$-contraction if and only if there exists a strict comparison function $\varphi\colon [0,\infty) \to [0,\infty)$ such that

$$H_d(N(u), N(v)) \leq \varphi(d(u, v)) \quad \text{for each } u, v \in X\,.$$

The following result, a generalization of the Covitz–Nadler fixed point principle (see [130]), is known in the literature as Węgrzyk's fixed point theorem.

**Lemma 8.22** ([255]). *Let $(X, d)$ be a complete metric space. If $A: X \to \mathcal{P}_{cl}(X)$ is a $\varphi$-contraction, then $Fix(A)$ is nonempty, and for any $u_0 \in X$ there exists a sequence of successive approximations of $A$ starting from $u_0$, which converges to a fixed point of $A$.*

The next result is known as Węgrzyk's theorem.

**Lemma 8.23** ([255]). *Let $(X, d)$ be a Banach space. If an operator $A: X \to \mathcal{P}_{cl}(X)$ is a $\varphi$-contraction, then $A$ is a MWPO.*

Now we present an important characterization lemma from the point of view of Ulam–Hyers stability.

**Lemma 8.24** ([217]). *Let $(X, d)$ be a metric space. If $A: X \to \mathcal{P}_{cp}(X)$ is a $\Psi$-MWPO, then the fixed point inclusion $u \in A(u)$ is generalized Ulam–Hyers stable. In particular, if $A$ is c-MWPO, then the fixed point inclusion $u \in A(u)$ is Ulam–Hyers stable.*

Another Ulam–Hyers stability result, more efficient for applications, was proved in [193].

**Theorem 8.25** ([193]). *Let $(X, d)$ be a complete metric space and $A: X \to \mathcal{P}_{cp}(X)$ a multivalued $\varphi$-contraction. Then:*
(i) Existence of fixed point: *$A$ is a MWPO;*
(ii) Ulam–Hyers stability for fixed point inclusion: *If additionally $\varphi(ct) \le c\varphi(t)$ for every $t \in [0, \infty)$ (where $c > 1$) and $t = 0$ is a point of uniform convergence for the series $\sum_{n=1}^{\infty} \varphi^n(t)$, then $A$ is a $\Psi$-MWPO, with $\Psi(t) := t + \sum_{n=1}^{\infty} \varphi^n(t)$, for each $t \in [0, \infty)$;*
(iii) Data dependence of fixed point set: *Let $S: X \to \mathcal{P}_{cl}(X)$ be a multivalued $\varphi$-contraction and $\eta > 0$ be such that $H_d(S(x), A(x)) \le \eta$ for each $x \in X$. Suppose that $\varphi(ct) \le c\varphi(t)$ for every $t \in [0, \infty)$ (where $c > 1$) and $t = 0$ is a point of uniform convergence for the series $\sum_{n=1}^{\infty} \varphi^n(t)$. Then $H_d(Fix(S), Fix(F)) \le \Psi(\eta)$.*

### 8.5.3 Existence and Ulam Stability Results

In this section, we present conditions for the existence and Ulam stability of Hadamard integral inclusion (8.22).

**Theorem 8.26.** *Make the following assumptions:*
(8.21.1) *$(x, y) \longmapsto F(x, y, u)$ is jointly measurable for each $u \in E$.*
(8.21.2) *$u \longmapsto F(x, y, u)$ is lower semicontinuous for almost all $(x, y) \in J$.*
(8.21.3) *There exist $p \in L^{\infty}(J, [0, \infty))$ and a strict comparison function $\varphi: [0, \infty) \to [0, \infty)$ such that for each $(x, y) \in J$ and each $u, v \in E$ we have*

$$H_d(F(x, y, u), F(x, y, \overline{u})) \le p(x, y)\varphi(\|u - \overline{u}\|_E) \tag{8.23}$$

*and*

$$\frac{(\log a)^{r_1}(\log b)^{r_2}\|p\|_{L^\infty}}{\Gamma(1+r_1)\Gamma(1+r_2)} \leq 1 . \tag{8.24}$$

(8.21.4) *There exists an integrable function* $q\colon [1,b] \to [0,\infty)$ *such that for each* $x \in [1,a]$ *and* $u \in E$ *we have* $F(x,y,u) \subset q(y)B(0,1)$*, a.e.* $y \in [1,b]$*, where* $B(0,1) = \{u \in E\colon \|u\|_E < 1\}$.

*Then we have that:*

(a) *The integral inclusion* (7.1) *has at least one solution and N is a MWPO.*

(b) *If additionally* $\varphi(ct) \leq c\varphi(t)$ *for every* $t \in [0,\infty)$ *(where* $c > 1$*) and* $t = 0$ *is a point of uniform convergence for the series* $\sum_{n=1}^{\infty}\varphi^n(t)$*, then integral inclusion* (7.1) *is generalized Ulam–Hyers stable, and N is a* $\Psi$*-MWPO, with the function* $\Psi$ *defined by* $\Psi(t) := t + \sum_{n=1}^{\infty}\varphi^n(t)$*, for each* $t \in [0,\infty)$*. Moreover, in this case the continuous data dependence of the solution set of integral inclusion* (8.23) *holds.*

**Remark 8.27.** For each $u \in \mathcal{C}$, the set $S_{F\circ u}$ is nonempty since, by (8.21.1), $F$ has a measurable selection (see [121] Theorem III.6).

*Proof.* The proof will be given in two steps.

*Step 1.* $N(u) \in P_{cp}(\mathcal{C})$ *for each* $u \in \mathcal{C}$. From the continuity of $\mu$ and Theorem 2 in Rybiński [236] we have that for each $u \in \mathcal{C}$ there exists $f \in S_{F\circ u}$, for all $(x,y) \in J$, such that $f(x,y)$ is integrable with respect to $y$ and continuous with respect to $x$. Then the function $v(x,y) = \mu(x,y) + {}^H I_\sigma^r f(x,y)$ has the property $v \in N(u)$. Moreover, from (8.21.1) and (8.21.4), via Theorem 8.6.3. in Aubin and Frankowska [69], we get that $N(u)$ is a compact set for each $u \in \mathcal{C}$.

*Step 2.* $H_d(N(u), N(\overline{u})) \leq \varphi(\|u-\overline{u}\|_\infty)$ *for each* $u, \overline{u} \in \mathcal{C}$. Let $u, \overline{u} \in \mathcal{C}$ and $h \in N(u)$. Then there exists $f(x,y) \in F(x,y,u(x,y))$ such that for each $(x,y) \in J$ we have

$$h(x,y) = \mu(x,y) + {}^H I_\sigma^r f(x,y) .$$

From (8.21.3) it follows that

$$H_d(F(x,y,u(x,y)), F(x,y,\overline{u}(x,y))) \leq p(x,y)\varphi(\|u(x,y)-\overline{u}(x,y)\|_E) .$$

Hence, there exists $w(x,y) \in F(x,y,\overline{u}(x,y)$ such that

$$\|f(x,y) - w(x,y)\|_E \leq p(x,y)\varphi(\|u(x,y)-\overline{u}(x,y)\|_E) ; \quad (x,y) \in J .$$

Consider $U\colon J \to \mathcal{P}(E)$ given by

$$U(x,y) = \{w \in E\colon \|f(x,y) - w(x,y)\|_E \leq p(x,y)\varphi(\|u(x,y)-\overline{u}(x,y)\|_E)\} .$$

Since the multivalued operator $u(x,y) = U(x,y) \cap F(x,y,\overline{u}(x,y))$ is measurable (see Proposition III.4 in [121]), there exists a function $\overline{f}(x,y)$ that is a measurable selection for $u$. Thus, $\overline{f}(x,y) \in F(x,y,\overline{u}(x,y))$, and for each $(x,y) \in J$,

$$\|f(x,y) - \overline{f}(x,y)\|_E \leq p(x,y)\varphi(\|u(x,y)-\overline{u}(x,y)\|_E) .$$

Let us define for each $(x, y) \in J$

$$\overline{h}(x, y) = \mu(x, y) + {}^H I_\sigma^r \overline{f}(x, y) .$$

Then for each $(x, y) \in J$ we have

$$\begin{aligned}
\|h(x, y) - \overline{h}(x, y)\|_E &\leq {}^H I_\sigma^r \|f(x, y) - \overline{f}(x, y)\|_E \\
&\leq {}^H I_\sigma^r (p(x, y)\varphi(\|u(x, y) - \overline{u}(x, y)\|_E)) \\
&\leq \|p\|_{L^\infty} \varphi(\|u - \overline{u}\|_\infty) \left( \int_1^x \int_1^y \frac{\left|\log \frac{x}{s}\right|^{r_1 - 1} \left|\log \frac{y}{t}\right|^{r_2 - 1}}{st\Gamma(r_1)\Gamma(r_2)} dtds \right) \\
&\leq \frac{(\log a)^{r_1} (\log b)^{r_2} \|p\|_{L^\infty}}{\Gamma(1 + r_1)\Gamma(1 + r_2)} \varphi(\|u - \overline{u}\|_\infty) .
\end{aligned}$$

Thus, by (10.2), we get

$$\|h - \overline{h}\|_\infty \leq \varphi(\|u - \overline{u}\|_\infty) .$$

By an analogous relation, obtained by interchanging the roles of $u$ and $\overline{u}$, it follows that

$$H_d(N(u), N(\overline{u})) \leq \varphi(\|u - \overline{u}\|_\infty) .$$

Hence, $N$ is a $\varphi$-contraction.

(a) By Lemma 8.22, $N$ has a fixed point that is a solution of inclusion (7.1) on $J$, and by [Theorem 8.25 (i)], $N$ is a MWPO.

(b) We will prove that the fixed point inclusion problem (7.1) is generalized Ulam–Hyers stable. Indeed, let $\epsilon > 0$ and $v \in \mathcal{C}$ for which there exists $u \in \mathcal{C}$ such that

$$u(x, y) \in \mu(x, y) + ({}^H I_\sigma^r F)(x, y, v(x, y)) , \quad \text{if } (x, y) \in J ,$$

and

$$\|u - v\|_\infty \leq \epsilon .$$

Then $H_d(v, N(v)) \leq \epsilon$. Moreover, by the preceding proof we have that $N$ is a multivalued $\varphi$-contraction, and using [Theorem 8.25 (i)-(ii)], we obtain that $N$ is a $\Psi$-MWPO. Then, by Lemma 8.24, we obtain that the fixed point problem $u \in N(u)$ is generalized Ulam–Hyers stable. Thus, integral inclusion (8.22) is generalized Ulam–Hyers stable.

Concerning the conclusion of the theorem, we apply [Theorem 8.25 (iii)]. □

### 8.5.4 An Example

Let

$$E = l^1 = \left\{ w = (w_1, w_2, \ldots, w_n, \ldots) : \sum_{n=1}^{\infty} |w_n| < \infty \right\}$$

be a Banach space with norm

$$\|w\|_E = \sum_{n=1}^{\infty} |w_n| \,,$$

and consider the partial functional fractional order integral inclusion of the form

$$u(x, y) \in \mu(x, y) + ({}^H I_\sigma^r F)(x, y, u(x, y)) \,, \quad \text{a.e. } (x, y) \in [1, e] \times [1, e] \,, \tag{8.25}$$

where $r = (r_1, r_2), r_1, r_2 \in (0, \infty)$,

$$u = (u_1, u_2, \dots, u_n, \dots) \,, \quad \mu(x, y) = (x + e^{-y}, 0, \dots, 0, \dots) \,,$$

and

$$\begin{aligned} &F(x, y, u(x, y)) \\ &\quad = \{v \in C([1, e] \times [1, e], \mathbb{R}) \colon \|f_1(x, y, u(x, y))\|_E \le \|v\|_E \le \|f_2(x, y, u(x, y))\|_E\} \,, \end{aligned}$$

$(x, y) \in [1, e] \times [1, e]$, where $f_1, f_2 \colon [1, e] \times [1, e] \times E \to E$,

$$f_k = (f_{k,1}, f_{k,2}, \dots, f_{k,n}, \dots) \,; \quad k \in \{1, 2\}, \ n \in \mathbb{N} \,,$$

$$f_{1,n}(x, y, u_n(x, y)) = \frac{xy^2 u_n}{(1 + \|u_n\|_E) e^{10+x+y}} \,, \quad n \in \mathbb{N} \,,$$

and

$$f_{2,n}(x, y, u_n(x, y)) = \frac{xy^2 u_n}{e^{10+x+y}} \,; \quad n \in \mathbb{N} \,.$$

We assume that $F$ is closed and convex valued. We can see that the solutions of the inclusion (7.4) are solutions of the fixed point inclusion $u \in A(u)$, where $A \colon C([1, e] \times [1, e], \mathbb{R}) \to \mathcal{P}(C([1, e] \times [1, e], \mathbb{R}))$ is the multifunction operator defined by

$$(Au)(x, y) = \left\{\mu(x, y) + ({}^H I_\sigma^r f)(x, y); \ f \in S_{F \circ u}\right\} \,; \quad (x, y) \in [1, e] \times [1, e] \,.$$

For each $(x, y) \in [1, e] \times [1, e]$ and all $z_1, z_2 \in E$ we have

$$\|f_2(x, y, z_2) - f_1(x, y, z_1)\|_E \le xy^2 e^{-10-x-y} \|z_2 - z_1\|_E \,.$$

Thus, conditions (8.21.1)–(8.21.3) are satisfied by $p(x, y) = xy^2 e^{-10-x-y}$. Condition (10.2) holds with $a = b = e$. Indeed, $\|p\|_{L^\infty} = e^{-9}$, $\Gamma(1+r_i) > \frac{1}{2}$; $i = 1, 2$. A simple computation shows that

$$\zeta := \frac{(\log a)^{r_1} (\log b)^{r_2} \|p\|_{L^\infty}}{\Gamma(1 + r_1) \Gamma(1 + r_2)} < 4e^{-9} < 1 \,.$$

Condition (8.21.4) is satisfied by $q(y) = \frac{y^2 e^{-10-y}}{\|F\|_{\mathcal{P}}}$; $y \in [1, e]$, where

$$\|F\|_{\mathcal{P}} = \sup\{\|f\|_{\mathcal{C}} : f \in S_{F \circ u}\} \,; \quad \text{for all } u \in \mathcal{C} \,.$$

Consequently, by Theorem 8.26, we draw the following conclusions:

(a) Integral inclusion (8.25) has least one solution and $A$ is a (MWPO).

(b) The function $\varphi\colon [0,\infty) \to [0,\infty)$ defined by $\varphi(t) = \zeta t$ satisfies $\varphi(\zeta t) \le \zeta\varphi(t)$ for every $t \in [0,\infty)$ and $t = 0$ is a point of uniform convergence for the series $\sum_{n=1}^{\infty}(\zeta t)^n$. Then the integral inclusion (7.4) is generalized Ulam–Hyers stable, and $A$ is a $\Psi$-MWPO, with the function $\Psi$ defined by $\Psi(t) := t + (1-\zeta t)^{-1}$ for each $t \in [0,\zeta^{-1})$. Moreover, the continuous data dependence of the solution set of integral inclusion (8.23) holds.

## 8.6 Notes and Remarks

The results of Chapter 8 are taken from Abbas et al. [2, 10, 9, 11]. Other results may be found in [24, 25, 22, 31, 153].