7 Partial Hadamard Fractional Integral Equations and Inclusions

7.1 Introduction

The fractional calculus represents a powerful tool in applied mathematics for studying many problems in various fields of science and engineering, with many breakthrough results found in mathematical physics, finance, hydrology, biophysics, thermodynamics, control theory, statistical mechanics, astrophysics, cosmology, and bioengineering [242]. There have been significant developments in ordinary and partial fractional differential and integral equations in recent years; see the monographs of Abbas et al. [35, 35], Kilbas et al. [181], and Miller and Ross [200], the papers of Abbas et al. [24, 25, 43], Vityuk et al. [247], and the references therein.

In [119], Butzer et al. investigated the properties of the Hadamard fractional integral and derivative. In [120], they obtained the Mellin transform of the Hadamard fractional integral and differential operators, and in [220], Pooseh et al. obtained expansion formulas of the Hadamard operators in terms of integer order derivatives. Many other interesting properties of those operators and others are summarized in [239] and the references therein.

This chapter deals with the existence and uniqueness of solutions to several classes of Hadamard partial fractional integral equations. We present results based on Banach's contraction principle and others on the nonlinear alternative of Leray–Schauder type. This chapter initiates the study of Hadamard integral equations of two independent variables.

7.2 Functional Partial Hadamard Fractional Integral Equations

7.2.1 Introduction

This section deals with the existence and uniqueness of solutions to the Hadamard partial fractional integral equation of the form

$$\begin{aligned} u(x,y) &= \mu(x,y) \\ &\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left(\log\frac{x}{s}\right)^{r_1-1} \left(\log\frac{y}{t}\right)^{r_2-1} \frac{f(s,t,u(s,t))}{st}\,dt\,ds\,; \quad \text{if } (x,y)\in J\,, \end{aligned} \tag{7.1}$$

where $J := [1,a]\times[1,b]$, $a,b > 1$, $r_1, r_2 > 0$, $\mu\colon J \to \mathbb{R}$, $f\colon J\times\mathbb{R} \to \mathbb{R}$ are given continuous functions.

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We present two results for integral equation (7.1). The first one is based on Banach's contraction principle and the second one on the nonlinear alternative of Leray–Schauder type. This section initiates the study of Hadamard integral equations of two independent variables.

7.2.2 Main Results

Definition 7.1. A function $u \in C$ is said to be a solution of (7.1) if u satisfies equation (7.1) on J.

Further, we present conditions for the existence and uniqueness of a solution to equation (7.1).

Theorem 7.2. *Make the following assumption:*
(7.2.1) *For any $u, v \in C$ and $(x, y) \in J$, there exists $k > 0$ such that*

$$|f(x, y, u) - f(x, y, v)| \leq k\|u - v\|_C .$$

If

$$L := \frac{k(\log a)^{r_1}(\log b)^{r_2}}{\Gamma(1 + r_1)\Gamma(1 + r_2)} < 1 , \tag{7.2}$$

then there exists a unique solution of equation (7.1) *on J.*

Proof. Transform integral equation (7.1) into a fixed point equation. Consider the operator $N\colon C \to C$ defined by

$$(Nu)(x, y) = \mu(x, y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left(\log \frac{x}{s}\right)^{r_1-1} \left(\log \frac{y}{t}\right)^{r_2-1} \frac{f(s, t, u(s, t))}{st} dtds . \tag{7.3}$$

Let $v, w \in C$. Then for $(x, y) \in J$ we have

$$\begin{aligned} |(Nv)(x, y) - (Nw)(x, y)| &\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left|\log \frac{x}{s}\right|^{r_1-1} \left|\log \frac{y}{t}\right|^{r_2-1} \\ &\quad \times \frac{|f(s, t, u(s, t)) - f(s, t, v(s, t))|}{st} dtds \\ &\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left|\log \frac{x}{s}\right|^{r_1-1} \left|\log \frac{y}{t}\right|^{r_2-1} \\ &\quad \times \frac{k\|u - v\|_C}{st} dtds \\ &\leq \frac{k(\log a)^{r_1}(\log b)^{r_2}}{\Gamma(1 + r_1)\Gamma(1 + r_2)} \|v - w\|_C . \end{aligned}$$

Consequently,

$$\|N(v) - N(w)\|_C \le L\|v - w\|_C \,.$$

From (7.2), N is a contraction, so N has a unique fixed point by Banach's contraction principle. □

Theorem 7.3. *Make the following assumption:*
(7.3.1) *There exist functions* $p_1, p_2 \in C(J, \mathbb{R}_+)$ *such that*

$$|f(x,y,u)| \le p_1(x,y) + p_2(x,y)|u(x,y)| \quad \text{for any } u \in \mathbb{R} \text{ and } (x,y) \in J \,.$$

Then integral equation (7.1) *has at least one solution defined on* J.

Proof. Consider the operator N defined in (7.3). We will show that the operator N is continuous and completely continuous.

Step 1. N is continuous. Let $\{u_n\}$ be a sequence such that $u_n \to u$ in C. Let $\eta > 0$ be such that $\|u_n\|_C \le \eta$. Then

$$\begin{aligned} |(Nu_n)(x,y) - (Nu)(x,y)| &\le \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left|\log\frac{x}{s}\right|^{r_1-1} \left|\log\frac{y}{t}\right|^{r_2-1} \\ &\quad \times \frac{|f(s,t,u_n(s,t)) - f(s,t,u(s,t))|}{st}\,dtds \\ &\le \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left|\log\frac{x}{s}\right|^{r_1-1} \left|\log\frac{y}{t}\right|^{r_2-1} \\ &\quad \times \frac{\sup_{(s,t)\in J}|f(s,t,u_n(s,t)) - f(s,t,u(s,t))|}{st}\,dtds \\ &\le \frac{(\log a)^{r_1}(\log b)^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} \|f(.,.,u_n(.,.)) - f(.,.,u(.,.))\|_C \,. \end{aligned}$$

From Lebesgue's dominated convergence theorem and the continuity of the function f we get

$$|(Nu_n)(x,y) - (Nu)(x,y)| \to 0 \quad \text{as} \quad n \to \infty \,.$$

Step 2. N maps bounded sets to bounded sets in C. Indeed, it is enough to show that for any $\eta^* > 0$ there exists a positive constant $\tilde{\ell}$ such that, for each $u \in B_{\eta^*} = \{u \in C : \|u\|_C \le \eta^*\}$, we have $\|N(u)\|_C \le \tilde{\ell}$. Set

$$p_i^* = \sup_{(x,y)\in J} p_i(x,y)\,; \quad i = 1, 2 \,.$$

From (7.3.1), for each $(x, y) \in J$ we have

$$\begin{aligned}|(Nu)(x,y)| &\le |\mu(x,y)| + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left|\log \frac{x}{s}\right|^{r_1-1} \left|\log \frac{y}{t}\right|^{r_2-1} \\ &\quad \times \frac{p_1(s,t) + p_2(s,t)\|u\|_C}{st} dtds \\ &\le \|\mu\|_\infty + \frac{(\log a)^{r_1} (\log b)^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} (p_1^* + p_2^* \eta^*) \\ &:= \tilde{\ell}\,.\end{aligned}$$

Hence,

$$\|N(u)\|_C \le \tilde{\ell}\,.$$

Step 3: N maps bounded sets to equicontinuous sets in C. Let $(x_1, y_1), (x_2, y_2) \in (1, a] \times (1, b]$, $x_1 < x_2$, $y_1 < y_2$, B_{η^*} be a bounded set of C as in Step 2, and let $u \in B_{\eta^*}$. Then

$$\begin{aligned}&|(Nu)(x_2,y_2) - (Nu)(x_1,y_1)| \le |\mu(x_1,y_1) - \mu(x_2,y_2)| \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^{x_1} \int_1^{y_1} \left[\left|\log \frac{x_2}{s}\right|^{r_1-1} \left|\log \frac{y_2}{t}\right|^{r_2-1} - \left|\log \frac{x_1}{s}\right|^{r_1-1} \left|\log \frac{y_1}{t}\right|^{r_2-1}\right] \\ &\times \frac{|f(s,t,u(s,t))|}{st} dtds \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^{x_2} \int_{y_1}^{y_2} \left|\log \frac{x_2}{s}\right|^{r_1-1} \left|\log \frac{y_2}{t}\right|^{r_2-1} \frac{|f(s,t,u(s,t))|}{st} dtds \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^{x_1} \int_{y_1}^{y_2} \left|\log \frac{x_2}{s}\right|^{r_1-1} \left|\log \frac{y_2}{t}\right|^{r_2-1} \frac{|f(s,t,u(s,t))|}{st} dtds \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^{x_2} \int_1^{y_1} \left|\log \frac{x_2}{s}\right|^{r_1-1} \left|\log \frac{y_2}{t}\right|^{r_2-1} \frac{|f(s,t,u(s,t))|}{st} dtds\,.\end{aligned}$$

Thus,

$$\begin{aligned}&|(Nu)(x_2,y_2) - (Nu)(x_1,y_1)| \le |\mu(x_1,y_1) - \mu(x_2,y_2)| \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^{x_1} \int_1^{y_1} \left[\left|\log \frac{x_2}{s}\right|^{r_1-1} \left|\log \frac{y_2}{t}\right|^{r_2-1} - \left|\log \frac{x_1}{s}\right|^{r_1-1} \left|\log \frac{y_1}{t}\right|^{r_2-1}\right] \\ &\times \frac{p_1^* + p_2^*\eta^*}{st} dtds \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^{x_2} \int_{y_1}^{y_2} \left|\log \frac{x_2}{s}\right|^{r_1-1} \left|\log \frac{y_2}{t}\right|^{r_2-1} \frac{p_1^* + p_2^*\eta^*}{st} dtds\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^{x_1}\int_{y_1}^{y_2}\left|\log\frac{x_2}{s}\right|^{r_1-1}\left|\log\frac{y_2}{t}\right|^{r_2-1}\frac{p_1^*+p_2^*\eta^*}{st}dtds\\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_1}^{x_2}\int_1^{y_1}\left|\log\frac{x_2}{s}\right|^{r_1-1}\left|\log\frac{y_2}{t}\right|^{r_2-1}\frac{p_1^*+p_2^*\eta^*}{st}dtds\\
\le{}&\frac{p_1^*+p_2^*\eta^*}{\Gamma(1+r_1)\Gamma(1+r_2)}\\
&\times[2(\log y_2)^{r_2}(\log x_2-\log x_1)^{r_1}+2(\log x_2)^{r_1}(\log y_2-\log y_1)^{r_2}\\
&+(\log x_1)^{r_1}(\log y_1)^{r_2}-(\log x_2)^{r_1}(\log y_2)^{r_2}\\
&-2(\log x_2-\log x_1)^{r_1}(\log y_2-\log y_1)^{r_2}]\,.
\end{aligned}
$$

As $x_1 \to x_2$ and $y_1 \to y_2$, the right-hand side of the preceding inequality tends to zero.

As a consequence of Steps 1–3, together with the Ascoli–Arzelà theorem, we can conclude that N is continuous and completely continuous.

Step 4. A priori bounds. We now show that there exists an open set $U \subseteq C$ with $u \neq \lambda N(u)$ for $\lambda \in (0,1)$ and $u \in \partial U$. Let $u \in C$ be such that $u = \lambda N(u)$ for some $0 < \lambda < 1$. Thus, for each $(x,y) \in J$,

$$
u(x,y)=\lambda\mu(x,y)+\frac{\lambda}{\Gamma(r_1)\Gamma(r_2)}\int_1^x\int_1^y\left(\log\frac{x}{s}\right)^{r_1-1}\left(\log\frac{y}{t}\right)^{r_2-1}\frac{f(s,t,u(s,t))}{st}dtds\,.
$$

This implies that for each $(x,y) \in J$ we have

$$
\begin{aligned}
|u(x,y)|\le{}&|\mu(x,y)|+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^x\int_1^y\left|\log\frac{x}{s}\right|^{r_1-1}\left|\log\frac{y}{t}\right|^{r_2-1}\\
&\times\frac{p_1(s,t)+p_2(s,t)|u(s,t)|}{st}dtds\\
\le{}&\|\mu\|_\infty+\frac{p_1^*(\log a)^{r_1}(\log b)^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}\\
&+\frac{p_2^*}{\Gamma(r_1)\Gamma(r_2)}\int_1^x\int_1^y\left|\log\frac{x}{s}\right|^{r_1-1}\left|\log\frac{y}{t}\right|^{r_2-1}\frac{|u(s,t)|}{st}dtds\,.
\end{aligned}
$$

Thus, for each $(x,y) \in J$ we get

$$
\begin{aligned}
|u(x,y)|\le{}&\|\mu\|_\infty+\frac{p_1^*(\log a)^{r_1}(\log b)^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}\\
&+\frac{p_2^*}{\Gamma(r_1)\Gamma(r_2)}\int_1^x\int_1^y\left|\log\frac{x}{s}\right|^{r_1-1}\left|\log\frac{y}{t}\right|^{r_2-1}\frac{|u(s,t)|}{st}dtds\\
\le{}&c+\int_1^x\int_1^y q(x,y,s,t)|u(s,t)|\,,
\end{aligned}
$$

where

$$c := \|\mu\|_\infty + \frac{p_1^*(\log a)^{r_1}(\log b)^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}$$

and

$$q(x,y,s,t) := \frac{p_2^*}{st\Gamma(r_1)\Gamma(r_2)}\left|\log\frac{x}{s}\right|^{r_1-1}\left|\log\frac{y}{t}\right|^{r_2-1}.$$

From Lemma 1.56 we obtain

$$|u(x,y)| \le c\exp\left(\int_1^x\int_1^y B(s,t)dtds\right),$$

where

$$\begin{aligned}B(x,y) &= q(x,y,x,y) + \int_1^x D_1q(x,y,s,y)ds \\ &\quad + \int_1^y D_2q(x,y,x,t)dt + \int_1^x\int_1^y D_1D_2q(x,y,s,t)dtds \\ &\le \frac{p_2^*}{xy\Gamma(r_1)\Gamma(r_2)}(\log x)^{r_1-1}(\log y)^{r_2-1}.\end{aligned}$$

Hence,

$$\begin{aligned}|u(x,y)| &\le c\exp\left(\int_1^x\int_1^y \frac{p_2^*}{st\Gamma(r_1)\Gamma(r_2)}(\log s)^{r_1-1}(\log t)^{r_2-1}dtds\right) \\ &\le c\exp\left(\frac{p_2^*(\log a)^{r_1}(\log b)^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}\right) \\ &:= R.\end{aligned}$$

Set

$$U = \{u \in C : \|u\|_\infty < R+1\}.$$

By our choice of U, there is no $u \in \partial U$ such that $u = \lambda N(u)$ for $\lambda \in (0,1)$. As a consequence of the nonlinear alternative of the Leray–Schauder type [149], we deduce that N has a fixed point u in $\overline{U}$ that is a solution of our equation (7.1). □

7.2.3 An Example

Consider a partial Hadamard integral equation of the form

$$\begin{aligned}u(x,y) &= \mu(x,y) \\ &\quad + \int_1^x\int_1^y\left(\log\frac{x}{s}\right)^{r_1-1}\left(\log\frac{y}{t}\right)^{r_2-1}\frac{f(s,t,u(s,t))}{st\Gamma(r_1)\Gamma(r_2)}dtds; \quad (x,y)\in[1,e]\times[1,e],\end{aligned} \tag{7.4}$$

where

$$r_1, r_2 > 0\,, \quad \mu(x, y) = x + y^2; \quad (x, y) \in [1, e] \times [1, e]$$

and

$$f(x, y, u(x, y)) = \frac{cu(x, y)}{e^{x+y+2}}\,; \quad (x, y) \in [1, e] \times [1, e]\,,$$

with

$$c := \frac{e^4}{2}\Gamma(1 + r_1)\Gamma(1 + r_2)\,.$$

For each $u, \overline{u} \in \mathbb{R}$ and $(x, y) \in [1, e] \times [1, e]$ we have

$$|f(x, y, u(x, y)) - f(x, y, \overline{u}(x, y))| \le \frac{c}{e^4}\|u - \overline{u}\|_C\,.$$

Hence, condition (7.2.1) is satisfied by $k = \frac{c}{e^4}$. Condition (7.2) holds with $a = b = e$. Indeed,

$$\frac{k(\log a)^{r_1}(\log b)^{r_2}}{\Gamma(1 + r_1)\Gamma(1 + r_2)} = \frac{c}{e^4\Gamma(1 + r_1)\Gamma(1 + r_2)} = \frac{1}{2} < 1\,.$$

Consequently, Theorem 7.2 implies that integral equation (7.4) has a unique solution defined on $[1, e] \times [1, e]$.

7.3 Fredholm-Type Hadamard Fractional Integral Equations

7.3.1 Introduction

The qualitative properties and structure of the set of solutions of the Darboux problem for hyperbolic partial integer order differential equations have been studied by many authors, for instance, [43, 123, 146]. In [110], Bica et al. initiated the study of the Fredholm integral equation

$$x(t) = f(t) + \int_0^a g(t, s, x(s), x'(s))ds \tag{7.5}$$

in a Banach space setting. In [213], Pachpatte studied the qualitative behavior of solutions of equation (7.5) and its further generalization. Inspired by the results in [110, 212, 213], Pachpatte in [214] studied the Fredholm-type integral equation

$$u(x, y) = f(x, y) + \int_0^a\int_0^b g(x, y, s, t, u(s, t), D_1u(s, t), D_2u(s, t))dtds\,, \tag{7.6}$$

where u is an unknown function. Recently, in [24], Abbas and Benchohra studied some uniqueness results for the Fredholm-type Riemann–Liouville integral equation

$$u(x, y) = \mu(x, y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^a\int_0^b (a - s)^{r_1-1}(b - t)^{r_2-1} \times f(x, y, s, t, u(s, t), ({}^cD_\theta^r u)(s, t))dtds\,; \quad \text{if } (x, y) \in J := [0, a] \times [0, b]\,, \tag{7.7}$$

where $a, b \in (0, \infty)$, $\theta = (0, 0)$, ${}^cD^r_\theta$ is the standard Caputo's fractional derivative of order $r = (r_1, r_2) \in (0, 1] \times (0, 1]$, $\mu: J \to \mathbb{R}^n$, and $f: J \times J \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ are given continuous functions.

This section deals with the existence and uniqueness of solutions to the Fredholm-type Hadamard partial integral equation

$$u(x, y) = \mu(x, y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^a \int_1^b \left(\ln \frac{a}{s}\right)^{r_1-1} \left(\ln \frac{b}{t}\right)^{r_2-1} \times \frac{f(x, y, s, t, u(s, t), ({}^HD^r_\sigma u)(s, t))}{st} dtds\,; \quad \text{if } (x, y) \in J := [1, a] \times [1, b]\,, \tag{7.8}$$

where $a, b \in (1, \infty)$, $\sigma = (1, 1)$, ${}^HD^r_\theta$ is the standard Hadamard fractional derivative of order $r = (r_1, r_2) \in (0, 1] \times (0, 1]$, $\mu: J \to \mathbb{R}^n$, and $f: J \times J \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ are given continuous functions.

7.3.2 Main Results

Define the space $E = E(J, \mathbb{R}^n)$ by

$$E := \{w \in C(J):\ {}^HD^r_\sigma w \text{ exists and } {}^HD^r_\sigma w \in C(J)\}\,.$$

For $w \in E$, use the notation

$$\|w(x, y)\|_1 = \|w(x, y)\| + \|{}^HD^r_\sigma w(x, y)\|\,.$$

In the space E we define the norm

$$\|w\|_E = \sup_{(x,y)\in J} \|w(x, y)\|_1\,.$$

Lemma 7.4. *$(E, \|\cdot\|_E)$ is a Banach space.*

Proof. Let $\{u_n\}_{n=0}^\infty$ be a Cauchy sequence in the space $(E, \|\cdot\|_E)$. Then

$$\forall \epsilon > 0, \exists N > 0 \text{ such that for all } n, m > N \text{ we have } \|u_n - u_m\|_E < \epsilon\,.$$

Thus, $\{u_n(x, y)\}_{n=0}^\infty$ and $\{({}^HD^r_\sigma u_n)(x, y)\}_{n=0}^\infty$ are Cauchy sequences in $\mathbb{R}^n$. Then $\{u_n(x, y)\}_{n=0}^\infty$ converges to some $u(x, y)$ in $\mathbb{R}^n$, and $\{{}^HD^r_\sigma u_n\}_{n=0}^\infty$ converges uniformly to some $v(x, y) \in E$. Next, we need to prove that $u \in E$ and $v = {}^HD^r_\sigma u$. According to the uniform convergence of $\{({}^HD^r_\sigma u_n)(x, y)\}_{n=0}^\infty$ and the dominated convergence theorem, we obtain

$$v(x, y) = \lim_{n\to\infty} ({}^HD^r_\sigma u_n)(x, y)\,.$$

Thus, $\{{}^HD^r_\sigma u_n\}_{n=0}^\infty$ converges uniformly to ${}^HD^r_\sigma u$ in E. Hence, $u \in E$ and

$$v(x, y) = ({}^HD^r_\sigma u)(x, y)\,.$$ □

Definition 7.5. By a solution to equation (7.8), we mean every function $w \in E$ such that w satisfies (7.8) on J.

Next we present conditions for the existence of solutions of integral equation (7.8).

Theorem 7.6. *Make the following assumptions:*

(7.6.1) *There exist* $0 < r_3 < \min\{r_1, r_2\}$, *functions* $\rho_1 : J \times J \to \mathbb{R}^+$, $\varphi : J \to \mathbb{R}^+$, *with* $\rho_1(x, y, \cdot, \cdot), \varphi \in L^{\frac{1}{r_3}}(J)$, *and a nondecreasing function* $\psi : [0, \infty) \to (0, \infty)$ *such that*

$$\|f(x, y, s, t, u, v)\| \le \rho_1(x, y, s, t)(\|u\| + \|v\|) \tag{7.9}$$

and

$$\begin{aligned}&\|f(x_1, y_1, s, t, u, v) - f(x_2, y_2, s, t, u, v)\| \\ &\le \varphi(s, t)(|x_1 - x_2| + |y_1 - y_2|)\psi(\|u\| + \|v\|) \\ &\text{for each } (x, y), (s, t), (x_1, y_1), (x_2, y_2) \in J \text{ and } u, v \in \mathbb{R}^n .\end{aligned} \tag{7.10}$$

(7.6.2) *There exist nonnegative constants* α, β_1, β_2 *such that for* $(x, y) \in J$ *we have*

$$\begin{cases} \|\mu(x, y)\|_1 & \le \alpha , \\ \int_1^a \int_1^b \rho_1^{\frac{1}{r_3}}(x, y, s, t)dtds & \le \beta_1^{\frac{1}{r_3}} , \\ \int_1^a \int_1^b \rho_2^{\frac{1}{r_3}}(x, y, s, t)dtds & \le \beta_2^{\frac{1}{r_3}} , \end{cases} \tag{7.11}$$

where

$$\rho_2(x, y, \cdot, \cdot) \in L^{\frac{1}{r_3}}(J) \text{ and } \rho_2(x, y, s, t) = ({}^H D_\sigma^r \rho_1)(x, y, s, t) .$$

If

$$\ell := \frac{(\beta_1 + \beta_2)(\ln a)^{(\omega_1+1)(1-r_3)}(\ln b)^{(\omega_2+1)(1-r_3)}}{(\omega_1 + 1)^{(1-r_3)}(\omega_2 + 1)^{(1-r_3)}\Gamma(r_1)\Gamma(r_2)} < 1 , \tag{7.12}$$

where $\omega_1 = \frac{r_1-1}{1-r_3}$, $\omega_2 = \frac{r_2-1}{1-r_3}$, *then the Fredholm–Hadamard integral equation* (7.8) *has at least one solution on* J.

Remark 7.7. It is clear that condition (7.9) implies

$$\|({}^H D_\sigma^r f)(x, y, s, t, u, v)\| \le \rho_2(x, y, s, t)(\|u\| + \|v\|) . \tag{7.13}$$

Proof. Let $u \in E$, and define the operator $N : E \to E$ by

$$\begin{aligned}(Nu)(x, y) =& \mu(x, y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^a \int_1^b \left(\ln \frac{a}{s}\right)^{r_1-1} \left(\ln \frac{b}{t}\right)^{r_2-1} \\ &\times \frac{f(x, y, s, t, u(s, t), ({}^H D_\sigma^r u)(s, t))}{st} dtds .\end{aligned} \tag{7.14}$$

Differentiating both sides of (7.14) by applying the Hadamard fractional derivative, we get

$$ {}^H D_\sigma^r (Nu)(x,y) = {}^H D_\sigma^r \mu(x,y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^a \int_1^b \left(\ln \frac{a}{s}\right)^{r_1-1} \left(\ln \frac{b}{t}\right)^{r_2-1} \times \frac{{}^H D_\sigma^r f(x,y,s,t,u(s,t),({}^H D_\sigma^r u)(s,t))}{st} dtds \,. \tag{7.15} $$

Set

$$ M = \frac{\alpha}{1-\ell} \quad \text{and} \quad D = \{u \in E : \|u\|_E \le M\} \,. $$

Clearly, D is a closed convex subset of E. Now we show that N maps D to itself. Evidently, $N(u)$, ${}^H D_\theta^r (Nu)$ are continuous on J. From (7.11) and using (7.6.1) and (7.6.2), for each $(x,y) \in J$ we have

$$ \begin{aligned} \|(Nu)(x,y)\|_1 \le\ & \|\mu(x,y)\|_1 \\ & + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^a \int_1^b \left|\ln \frac{a}{s}\right|^{r_1-1} \left|\ln \frac{b}{t}\right|^{r_2-1} \\ & \times \left\| \frac{f(x,y,s,t,u(s,t),({}^H D_\sigma^r u)(s,t))}{st} \right\| dtds \\ & + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^a \int_1^b \left(\ln \frac{a}{s}\right)^{r_1-1} \left(\ln \frac{b}{t}\right)^{r_2-1} \\ & \times \left\| \frac{{}^H D_\sigma^r f(x,y,s,t,u(s,t),({}^H D_\sigma^r u)(s,t))}{st} \right\| dtds \\ \le\ & \|\mu(x,y)\|_1 \\ & + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \left(\int_1^a \int_1^b \frac{1}{st} \left|\ln \frac{a}{s}\right|^{\frac{r_1-1}{1-r_3}} \left|\ln \frac{b}{t}\right|^{\frac{r_2-1}{1-r_3}} dtds \right)^{1-r_3} \\ & \times \left(\int_1^a \int_1^b \|f(x,y,s,t,u(s,t),({}^H D_\sigma^r u)(s,t))\|^{\frac{1}{r_3}} dtds \right)^{r_3} \\ & + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \left(\int_1^a \int_1^b \frac{1}{st} \left|\ln \frac{a}{s}\right|^{\frac{r_1-1}{1-r_3}} \left|\ln \frac{b}{t}\right|^{\frac{r_2-1}{1-r_3}} dtds \right)^{1-r_3} \\ & \times \left(\int_1^a \int_1^b \|{}^H D_\sigma^r f(x,y,s,t,u(s,t),({}^H D_\sigma^r u)(s,t))\|^{\frac{1}{r_3}} dtds \right)^{r_3} . \end{aligned} $$

Then for each $(x, y) \in J$ we obtain

$$\|(Nu)(x,y)\|_1 \le \|\mu(x,y)\|_1 + \frac{(\ln a)^{(\omega_1+1)(1-r_3)}(\ln b)^{(\omega_2+1)(1-r_3)}}{(\omega_1+1)^{(1-r_3)}(\omega_2+1)^{(1-r_3)}\Gamma(r_1)\Gamma(r_2)}$$
$$\times\left[\left(\int_1^a\int_1^b \rho_1^{\frac{1}{r_3}}(x,y,s,t)\|u(s,t)\|_1^{\frac{1}{r_3}}dtds\right)^{r_3} + \left(\int_1^a\int_1^b \rho_2^{\frac{1}{r_3}}(x,y,s,t)\|u(s,t)\|_1^{\frac{1}{r_3}}dtds\right)^{r_3}\right]$$
$$\le \alpha + \frac{(\ln a)^{(\omega_1+1)(1-r_3)}(\ln b)^{(\omega_2+1)(1-r_3)}}{(\omega_1+1)^{(1-r_3)}(\omega_2+1)^{(1-r_3)}\Gamma(r_1)\Gamma(r_2)}$$
$$\times\left[\|u\|_E\left(\int_1^a\int_1^b \rho_1^{\frac{1}{r_3}}(x,y,s,t)dtds\right)^{r_3} + \|u|_E\left(\int_0^a\int_0^b \rho_2^{\frac{1}{r_3}}(x,y,s,t)dtds\right)^{r_3}\right]$$
$$\le \alpha + \frac{(M\beta_1 + M\beta_2)(\ln a)^{(\omega_1+1)(1-r_3)}(\ln b)^{(\omega_2+1)(1-r_3)}}{(\omega_1+1)^{(1-r_3)}(\omega_2+1)^{(1-r_3)}\Gamma(r_1)\Gamma(r_2)}$$
$$= \alpha + M\ell\,.$$

From (7.12) and the definition of M we get

$$\|N(u)\|_E \le M\,.$$

Hence, $N(u) \in D$. This proves that the operator N maps D to itself. Next we verify that the operator N satisfies the assumptions of Schauder's fixed point theorem. The proof will be given in several steps.

Step 1. N is continuous. Let $\{u_n\}$ be a sequence such that $u_n \to u$ in D. Then

$$\|(Nu_n)(x,y) - (Nu)(x,y)\|_1 \le \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^a\int_1^b \frac{1}{st}\left|\ln\frac{a}{s}\right|^{r_1-1}\left|\ln\frac{b}{t}\right|^{r_2-1}$$
$$\times \|f(x,y,s,t,u_n(s,t),({}^HD_\sigma^r u_n)(s,t))$$
$$- f(x,y,s,t,u(s,t),({}^HD_\sigma^r u)(s,t))\|dtds$$
$$+ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^a\int_1^b \frac{1}{st}\left|\ln\frac{a}{s}\right|^{r_1-1}\left|\ln\frac{b}{t}\right|^{r_2-1}$$
$$\times \|{}^HD_\sigma^r f(x,y,s,t,u_n(s,t),({}^HD_\sigma^r u_n)(s,t))$$
$$- {}^HD_\sigma^r f(x,y,s,t,u(s,t),({}^HD_\sigma^r u)(s,t))\|dtds\,.$$

Thus,

$$\begin{aligned}\|(Nu_n)(x,y)-(Nu)(x,y)\|_1 \le{}& \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^a\int_1^b \frac{1}{st}\left|\ln\frac{a}{s}\right|^{r_1-1}\left|\ln\frac{b}{t}\right|^{r_2-1}\\
&\times \sup_{(s,t)\in J}\|f\left(x,y,s,t,u_n(s,t),({}^HD^r_\sigma u_n)(s,t)\right)\\
&\quad -f\left(x,y,s,t,u(s,t),({}^HD^r_\sigma u)(s,t)\right)\|dtds\\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^a\int_1^b \frac{1}{st}\left|\ln\frac{a}{s}\right|^{r_1-1}\left|\ln\frac{b}{t}\right|^{r_2-1}\\
&\times \sup_{(s,t)\in J}\|{}^HD^r_\sigma f\left(x,y,s,t,u_n(s,t),({}^HD^r_\sigma u_n)(s,t)\right)\\
&\quad -{}^HD^r_\sigma f\left(x,y,s,t,u(s,t),({}^HD^r_\sigma u)(s,t)\right)\|dtds\\
\le{}& \frac{(\ln a)^{r_1}(\ln b)^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}\Big(\|f\left(x,y,\cdot,\cdot,u_n(\cdot,\cdot),({}^HD^r_\sigma u_n)(\cdot,\cdot)\right)\\
&\quad -f\left(x,y,\cdot,\cdot,u(\cdot,\cdot),({}^HD^r_\sigma u)(\cdot,\cdot)\right)\|\\
&+\|{}^HD^r_\sigma f\left(x,y,\cdot,\cdot,u_n(\cdot,\cdot),({}^HD^r_\sigma u_n)(\cdot,\cdot)\right)\\
&\quad -{}^HD^r_\sigma f\left(x,y,\cdot,\cdot,u(\cdot,\cdot),({}^HD^r_\sigma u)(\cdot,\cdot)\right)\|\Big)\\
\le{}& \frac{(\ln a)^{r_1}(\ln b)^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}\|f\left(x,y,\cdot,\cdot,u_n(\cdot,\cdot),({}^HD^r_\sigma u_n)(\cdot,\cdot)\right)\\
&-f\left(x,y,\cdot,\cdot,u(\cdot,\cdot),({}^HD^r_\sigma u)(\cdot,\cdot)\right)\|_1 .\end{aligned}$$

Hence, from Lebesgue's dominated convergence theorem and the continuity of the function f we get

$$\|(N(u_n)-N(u)\|_E \to 0 \text{ as } n\to\infty .$$

Step 2. $N(D)$ is bounded. This is clear since $N(D)\subset D$ and D is bounded.

Step 3. $N(D)$ is equicontinuous. Let $(x_1,y_1),(x_2,y_2)\in(1,a]\times(1,b]$, $x_1<x_2$, $y_1<y_2$, and let $u\in D$. Then

$$\begin{aligned}\|(Nu)(x_2,y_2)-(Nu)(x_1,y_1)\|_1 \le{}& \|\mu(x_1,y_1)-\mu(x_2,y_2)\|_1\\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^a\int_1^b \frac{1}{st}\left|\ln\frac{a}{s}\right|^{r_1-1}\left|\ln\frac{b}{t}\right|^{r_2-1}\\
&\times\|f(x_2,y_2,s,t,u(s,t),({}^HD^r_\sigma u)(s,t))\\
&\quad -f(x_1,y_1,s,t,u(s,t),({}^HD^r_\sigma u)(s,t))\|dtds ,\\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^a\int_1^b \frac{1}{st}\left|\ln\frac{a}{s}\right|^{r_1-1}\left|\ln\frac{b}{t}\right|^{r_2-1}\\
&\times\|{}^HD^r_\sigma f(x_2,y_2,s,t,u(s,t),({}^HD^r_\sigma u)(s,t))\\
&\quad -{}^HD^r_\sigma f(x_1,y_1,s,t,u(s,t),({}^HD^r_\sigma u)(s,t))\|dtds .\end{aligned}$$

Thus,

$$\begin{aligned}
\|(Nu)(x_2,y_2)-(Nu)(x_1,y_1)\|_1 \le{}& \|\mu(x_1,y_1)-\mu(x_2,y_2)\|_1 \\
&+\frac{(\ln a)^{(\omega_1+1)(1-r_3)}(\ln b)^{(\omega_2+1)(1-r_3)}}{(\omega_1+1)^{(1-r_3)}(\omega_2+1)^{(1-r_3)}\Gamma(r_1)\Gamma(r_2)} \\
&\times\Bigg[\Bigg(\int_1^a\int_1^b \|f(x_2,y_2,s,t,u(s,t),({}^H D_\sigma^r u)(s,t)) \\
&-f(x_1,y_1,s,t,u(s,t),({}^H D_\sigma^r u)(s,t))\|^{\frac{1}{r_3}}\,dtds\Bigg)^{r_3} \\
&+\Bigg(\int_1^a\int_1^b \|{}^H D_\sigma^r f(x_2,y_2,s,t,u(s,t),({}^H D_\sigma^r u)(s,t)) \\
&-{}^H D_\sigma^r f(x_1,y_1,s,t,u(s,t),({}^H D_\sigma^r u)(s,t))\|^{\frac{1}{r_3}}\,dtds\Bigg)^{r_3}\Bigg].
\end{aligned}$$

Hence,

$$\begin{aligned}
\|(Nu)(x_2,y_2)-(Nu)(x_1,y_1)\|_1 \le{}& \|\mu(x_1,y_1)-\mu(x_2,y_2)\|_1 \\
&+\frac{(\ln a)^{(\omega_1+1)(1-r_3)}(\ln b)^{(\omega_2+1)(1-r_3)}}{(\omega_1+1)^{(1-r_3)}(\omega_2+1)^{(1-r_3)}\Gamma(r_1)\Gamma(r_2)} \\
&\times(|x_1-x_2|+|y_1-y_2|)\psi(\|u\|_1) \\
&\times\Bigg[\Bigg(\int_1^a\int_1^b \|\varphi(s,t)\|^{\frac{1}{r_3}}\,dtds\Bigg)^{r_3} \\
&\quad+\Bigg(\int_1^a\int_1^b \|({}^H D_\sigma^r \varphi)(s,t)\|^{\frac{1}{r_3}}\,dtds\Bigg)^{r_3}\Bigg] \\
\le{}& \|\mu(x_1,y_1)-\mu(x_2,y_2)\|_1 \\
&+\frac{(\ln a)^{(\omega_1+1)(1-r_3)}(\ln b)^{(\omega_2+1)(1-r_3)}}{(\omega_1+1)^{(1-r_3)}(\omega_2+1)^{(1-r_3)}\Gamma(r_1)\Gamma(r_2)} \\
&\times\left[(\|\varphi\|_{L^{\frac{1}{r_3}}})^{r_3}+\left(\|{}^H D_\sigma^r\varphi\|_{L^{\frac{1}{r_3}}}\right)^{r_3}\right]\psi(M) \\
&\times(|x_1-x_2|+|y_1-y_2|)\,.
\end{aligned}$$

As $x_1 \to x_2$ and $y_1 \to y_2$, the right-hand side of the preceding inequality tends to zero.

As a consequence of Steps 1–3, together with the Ascoli–Arzelà theorem, we can conclude that N is continuous and completely continuous. From an application of Schauder's theorem [149], we deduce that N has a fixed point u that is a solution of integral equation (7.8). □

Now we define the Banach space

$$X := \{w \in C(J) : \ {}^H D^{r_1}_{1,x} w, {}^H D^{r_2}_{1,y} w \ \text{exist and} \ {}^H D^{r_1}_{1,x} w, {}^H D^{r_2}_{1,y} w \in C(J)\} ,$$

with the norm

$$\|w\|_X = \sup_{(x,y)\in J} \|w(x,y)\|_1 ,$$

where

$$\|w(x,y)\|_1 = \|w(x,y)\| + \|{}^H D^{r_1}_{1,x} w(x,y)\| + \|{}^H D^{r_2}_{1,y} w(x,y)\| .$$

Corollary 7.8. *Consider the Fredholm-type Hadamard integral equation*

$$\begin{aligned} u(x,y) = \mu(x,y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^a \int_1^b \left(\ln \frac{x}{s}\right)^{r_1-1} \left(\ln \frac{y}{t}\right)^{r_2-1} \\ \times f(x,y,s,t,u(s,t),({}^H D^{r_1}_{1,s} u)(s,t),({}^H D^{r_2}_{1,t} u)(s,t)) dt ds ; \qquad (7.16) \\ \text{if } (x,y) \in J := [1,a] \times [1,b] . \end{aligned}$$

Make the following assumptions:

(7.8.1) *There exist* $0 < r_3 < \min\{r_1, r_2\}$, *functions* $\rho_1 : J \times J \to \mathbb{R}^+$, $\varphi : J \to \mathbb{R}^+$, *with* $\rho_1(x,y,\cdot,\cdot), \varphi \in L^{\frac{1}{r_3}}(J)$, *and a nondecreasing function* $\psi : [0,\infty) \to (0,\infty)$ *such that*

$$\|f(x,y,s,t,u,v,w)\| \le \rho_1(x,y,s,t)(\|u\| + \|v\| + \|w\|) \qquad (7.17)$$

and

$$\begin{aligned} &\|f(x_1,y_1,s,t,u,v,w) - f(x_2,y_2,s,t,u,v,w)\| \\ &\le \varphi(s,t)(|x_1 - x_2| + |y_1 - y_2|)\psi(\|u\| + \|v\| + \|w\|) \qquad (7.18) \end{aligned}$$

for each $(x,y), (s,t), (x_1,y_1), (x_2,y_2) \in J$ *and* $u, v, w \in \mathbb{R}^n$.

(7.8.2) *There exist nonnegative constants* $\alpha, \beta_1, \beta_2, \beta_3$ *such that for* $(x,y) \in J$ *we have*

$$\begin{cases} \|\mu(x,y)\|_1 & \le \alpha , \\ \int_1^a \int_1^b \rho_1^{\frac{1}{r_3}}(x,y,s,t) dt ds & \le \beta_1^{\frac{1}{r_3}} , \\ \int_1^a \int_1^b \rho_2^{\frac{1}{r_3}}(x,y,s,t) dt ds & \le \beta_2^{\frac{1}{r_3}} , \\ \int_1^a \int_1^b \rho_3^{\frac{1}{r_3}}(x,y,s,t) dt ds & \le \beta_3^{\frac{1}{r_3}} , \end{cases} \qquad (7.19)$$

where

$$\rho_2(x,y,s,t) = ({}^H D^{r}_{1,x} \rho_1)(x,y,s,t), \ \text{and} \ \rho_3(x,y,s,t) = ({}^H D^{r}_{1,y} \rho_1)(x,y,s,t) .$$

If

$$\frac{(\beta_1 + \beta_2 + \beta_3) a^{(\omega_1+1)(1-r_3)} b^{(\omega_2+1)(1-r_3)}}{(\omega_1 + 1)^{(1-r_3)} (\omega_2 + 1)^{(1-r_3)} \Gamma(r_1)\Gamma(r_2)} < 1 , \qquad (7.20)$$

where $\omega_1 = \frac{r_1-1}{1-r_3}$, $\omega_2 = \frac{r_2-1}{1-r_3}$, *then equation* (7.16) *has at least one solution on* J *in* X.

7.3.3 An Example

As an application of our results we consider the Fredholm partial Hadamard integral equation

$$u(x,y) = \mu(x,y) + \int_1^e \int_1^e (1-\ln s)^{r_1-1}(1-\ln t)^{r_2-1} \frac{f(x,y,s,t,u(s,t),{}^H(D_\sigma^r u)(s,t))}{st\Gamma(r_1)\Gamma(r_2)} dtds \tag{7.21}$$

for $(x,y) \in [1,e]\times[1,e]$, where $r_1, r_2 > 0$, $\mu(x,y) = x + y^2$; $(x,y) \in [1,e]\times[1,e]$, and

$$f(x,y,s,t,u(x,y),v(x,y)) = c(x+y)st^2 \frac{u(x,y)+v(x,y)}{e^{x+y+5}}, \quad (x,y) \in [1,e]\times[1,e],$$

with

$$c := \frac{(\omega_1+1)^{(1-r_3)}(\omega_2+1)^{(1-r_3)}\Gamma(r_1)\Gamma(r_2)}{\frac{2e^{-3}}{(\Gamma(1+r_1))^{r_3}(\Gamma(1+r_2))^{r_3}}\left(1+\frac{1}{\Gamma(1-r_1)\Gamma(1-r_2)}\right)},$$

$$0 < r_3 < \min\{r_1, r_2\}, \quad \omega_1 = \frac{r_1-1}{1-r_3}, \quad \text{and } \omega_2 = \frac{r_2-1}{1-r_3}.$$

For each $u, v \in \mathbb{R}$ and $(x,y) \in [1,e]\times[1,e]$ we have

$$|f(x,y,u,v)| \le 2ce^{-3}(|u|+|v|),$$

and for each $(x,y), (s,t), (x_1,y_1), (x_2,y_2) \in [1,e]\times[1,e]$, and $u, v \in \mathbb{R}$ we have

$$|f(x_1,y_1,s,t,u,v) - f(x_2,y_2,s,t,u,v)| \le 2ce^{-3}(|x_1-x_2|+|y_1-y_2|)(|u|+|v|).$$

Hence, condition (7.6.1) is satisfied by

$$\rho_1 = ce^{-3}, \quad \rho_2 = \frac{ce^{-3}}{\Gamma(1-r_1)\Gamma(1-r_2)}, \quad \varphi(s,t) = 2ce^{-3}, \quad \psi(x) = 1.$$

Also, (7.6.2) is satisfied by

$$\alpha = (e+e^2)\left(1+\frac{1}{\Gamma(1-r_1)\Gamma(1-r_2)}\right), \quad \beta_1 = c\frac{e^{-3}}{(\Gamma(1+r_1))^{r_3}(\Gamma(1+r_2))^{r_3}},$$

and

$$\beta_2 = c\frac{e^{-3}}{\Gamma(1-r_1)\Gamma(1-r_2)\Gamma(1+r_1))^{r_3}(\Gamma(1+r_2))^{r_3}}.$$

Condition (7.12) holds with $a = b = e$. Indeed,

$$\begin{aligned} \ell &= \frac{(\beta_1+\beta_2)(\ln a)^{(\omega_1+1)(1-r_3)}(\ln b)^{(\omega_2+1)(1-r_3)}}{(\omega_1+1)^{(1-r_3)}(\omega_2+1)^{(1-r_3)}\Gamma(r_1)\Gamma(r_2)} \\ &= \frac{c\frac{e^{-3}}{(\Gamma(1+r_1))^{r_3}(\Gamma(1+r_2))^{r_3}}\left(1+\frac{1}{\Gamma(1-r_1)\Gamma(1-r_2)}\right)}{(\omega_1+1)^{(1-r_3)}(\omega_2+1)^{(1-r_3)}\Gamma(r_1)\Gamma(r_2)} \\ &= \frac{1}{2} < 1. \end{aligned}$$

Consequently, Theorem 7.6 implies the Fredholm–Hadamard integral equation (7.21) has at least one solution on $[1,e]\times[1,e]$.

7.4 Upper and Lower Solutions Method for Partial Hadamard Fractional Integral Equations and Inclusions

7.4.1 Introduction

In this section, we use the upper and lower solutions method combined with Schauder's fixed point theorem and a fixed point theorem for condensing multivalued maps to Martelli to investigate the existence of solutions for some classes of partial Hadamard fractional integral equations and inclusions.

The method of upper and lower solutions has been successfully applied to study the existence of solutions for ordinary and partial differential equations and inclusions. See the monographs by Benchohra et al. [101], the papers of Abbas et al. [25, 20, 17, 15, 29], Pachpatte [211], and the references therein.

In this section, we use the method of upper and lower solutions for the existence of solutions to the Hadamard partial fractional integral equation

$$\begin{aligned} u(x,y) &= \mu(x,y) \\ &\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left(\log \frac{x}{s}\right)^{r_1-1} \left(\log \frac{y}{t}\right)^{r_2-1} \frac{f(s,t,u(s,t))}{st} dt ds \,; \quad \text{if } (x,y) \in J \,, \end{aligned} \tag{7.22}$$

where $J := [1,a] \times [1,b]$, $a, b > 1$, $r_1, r_2 > 0$, $\mu: J \to \mathbb{R}$, $f: J \times \mathbb{R} \to \mathbb{R}$ are given continuous functions. Next we discuss the existence of solutions to the Hadamard partial fractional integral inclusion

$$u(x,y) - \mu(x,y) \in ({}^H I_\sigma^r F)(x,y,u(x,y)); \ (x,y) \in J \,, \tag{7.23}$$

where $\sigma = (1,1)$, $F: J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a compact-valued multivalued map, ${}^H I_\sigma^r F$ is the definite Hadamard integral for the set-valued function F of order $r = (r_1, r_2) \in (0,\infty) \times (0,\infty)$, and $\mu: J \to \mathbb{R}$ is a given continuous function; additionally, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$.

This section initiates the application of the upper and lower solutions method to these new classes of problems.

7.4.2 Existence Results for Partial Hadamard Fractional Integral Equations

Let us start by defining what we mean by a solution of integral equation (7.1).

Definition 7.9. A function $u \in C$ is said to be a solution of (7.1) if u satisfies equation (7.1) on J.

Definition 7.10. A function $z \in C$ is said to be a lower solution of integral equation (7.1) if z satisfies

$$u(x,y) \le \mu(x,y) + \int_1^x\int_1^y \left(\log\frac{x}{s}\right)^{r_1-1}\left(\log\frac{y}{t}\right)^{r_2-1}\frac{f(s,t,u(s,t))}{st\Gamma(r_1)\Gamma(r_2)}dtds\,; \quad (x,y)\in J\,.$$

The function z is said to be an upper solution of (7.22) if the reverse inequality holds.

Further, we present our main result for equation (7.1).

Theorem 7.11. *Assume*
(7.11.1) *There exist* v *and* $w \in C$*, lower and upper solutions to equation* (7.1) *such that* $v \le w$.
Then integral equation (7.22) *has at least one solution* u *such that*

$$v(x,y) \le u(x,y) \le w(x,y) \;\; \textit{for all} \;\; (x,y)\in J\,.$$

Proof. Consider the modified integral equation

$$u(x,y) = \mu(x,y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^x\int_1^y \left(\log\frac{x}{s}\right)^{r_1-1}\left(\log\frac{y}{t}\right)^{r_2-1}\frac{g(s,t,u(s,t))}{st}dtds\,, \quad (7.24)$$

where

$$g(x,y,u(x,y)) = f(x,y,h(x,y,u(x,y)))\,,$$
$$h(x,y,u(x,y)) = \max\{v(x,y), \min\{u(x,y), w(x,y)\}\}$$

for each $(x,y)\in J$.

A solution of (7.24) is a fixed point of the operator $N\colon C \to C$ defined by

$$(Nu)(x,y) = \mu(x,y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^x\int_1^y \left(\log\frac{x}{s}\right)^{r_1-1}\left(\log\frac{y}{t}\right)^{r_2-1}\frac{g(s,t,u(s,t))}{st}dtds\,.$$

Notice that g is a continuous function, and from (7.11.1) there exists $M > 0$ such that

$$|g(x,y,u)| \le M\,, \quad \text{for each } (x,y)\in J, \text{ and } u\in\mathbb{R}\,. \quad (7.25)$$

Set

$$\eta = \|\mu\|_C + \frac{M(\log a)^{r_1}(\log b)^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}$$

and

$$D = \{u \in C\colon \|u\|_C \le \eta\}\,.$$

Clearly, D is a closed convex subset of C and N maps D to itself. We will show that N satisfies the assumptions of Theorem 1.42. The proof will be given in several steps.

Step 1. N is continuous. Let $\{u_n\}$ be a sequence such that $u_n \to u$ in D. Then

$$\begin{aligned}|(Nu_n)(x,y)-(Nu)(x,y)| &\le \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^x\int_1^y \left|\log\frac{x}{s}\right|^{r_1-1}\left|\log\frac{y}{t}\right|^{r_2-1} \\ &\quad\times\frac{|g(s,t,u_n(s,t))-g(s,t,u(s,t))|}{st}dtds \\ &\le \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^x\int_1^y \left|\log\frac{x}{s}\right|^{r_1-1}\left|\log\frac{y}{t}\right|^{r_2-1} \\ &\quad\times\frac{\sup_{(s,t)\in J}|g(s,t,u_n(s,t))-g(s,t,u(s,t))|}{st}dtds \\ &\le \frac{(\log a)^{r_1}(\log b)^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}\|g(\cdot,\cdot,u_n(\cdot,\cdot))-g(\cdot,\cdot,u(\cdot,\cdot))\|_C\,.\end{aligned}$$

From Lebesgue's dominated convergence theorem and the continuity of the function g we get

$$|(Nu_n)(x,y)-(Nu)(x,y)| \to 0 \ \text{ as } \ n\to\infty\,.$$

Step 2. $N(D)$ is bounded. This is clear since $N(D)\subset D$ and D is bounded.

Step 3. $N(D)$ is equicontinuous. Let $(x_1,y_1),(x_2,y_2)\in(1,a]\times(1,b]$, $x_1<x_2$, $y_1<y_2$, and let $u\in D$. Then

$$\begin{aligned}&|(Nu)(x_2,y_2)-(Nu)(x_1,y_1)| \le |\mu(x_1,y_1)-\mu(x_2,y_2)| \\ &+\int_1^{x_1}\int_1^{y_1}\left|\left(\log\frac{x_2}{s}\right)^{r_1-1}\left(\log\frac{y_2}{t}\right)^{r_2-1}-\left(\log\frac{x_1}{s}\right)^{r_1-1}\left(\log\frac{y_1}{t}\right)^{r_2-1}\right| \\ &\times\frac{|g(s,t,u(s,t))|}{st\Gamma(r_1)\Gamma(r_2)}dtds \\ &+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_1}^{x_2}\int_{y_1}^{y_2}\left|\log\frac{x_2}{s}\right|^{r_1-1}\left|\log\frac{y_2}{t}\right|^{r_2-1}\frac{|g(s,t,u(s,t))|}{st}dtds \\ &+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^{x_1}\int_{y_1}^{y_2}\left|\log\frac{x_2}{s}\right|^{r_1-1}\left|\log\frac{y_2}{t}\right|^{r_2-1}\frac{|g(s,t,u(s,t))|}{st}dtds \\ &+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_1}^{x_2}\int_1^{y_1}\left|\log\frac{x_2}{s}\right|^{r_1-1}\left|\log\frac{y_2}{t}\right|^{r_2-1}\frac{|g(s,t,u(s,t))|}{st}dtds\,.\end{aligned}$$

Thus,

$$\begin{aligned}&|(Nu)(x_2,y_2)-(Nu)(x_1,y_1)| \le |\mu(x_1,y_1)-\mu(x_2,y_2)| \\ &+\int_1^{x_1}\int_1^{y_1}\left|\left(\log\frac{x_2}{s}\right)^{r_1-1}\left(\log\frac{y_2}{t}\right)^{r_2-1}-\left(\log\frac{x_1}{s}\right)^{r_1-1}\left(\log\frac{y_1}{t}\right)^{r_2-1}\right| \\ &\times\frac{M}{st\Gamma(r_1)\Gamma(r_2)}dtds\end{aligned}$$

$$+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_1}^{x_2}\int_{y_1}^{y_2}\left|\log\frac{x_2}{s}\right|^{r_1-1}\left|\log\frac{y_2}{t}\right|^{r_2-1}\frac{M}{st}dtds$$
$$+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{1}^{x_1}\int_{y_1}^{y_2}\left|\log\frac{x_2}{s}\right|^{r_1-1}\left|\log\frac{y_2}{t}\right|^{r_2-1}\frac{M}{st}dtds$$
$$+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_1}^{x_2}\int_{1}^{y_1}\left|\log\frac{x_2}{s}\right|^{r_1-1}\left|\log\frac{y_2}{t}\right|^{r_2-1}\frac{M}{st}dtds$$
$$\leq |\mu(x_1,y_1)-\mu(x_2,y_2)|$$
$$+\frac{M}{\Gamma(1+r_1)\Gamma(1+r_2)}$$
$$\times\left[2(\log y_2)^{r_2}(\log x_2-\log x_1)^{r_1}+2(\log x_2)^{r_1}(\log y_2-\log y_1)^{r_2}\right.$$
$$+(\log x_1)^{r_1}(\log y_1)^{r_2}-(\log x_2)^{r_1}(\log y_2)^{r_2}$$
$$\left.-2(\log x_2-\log x_1)^{r_1}(\log y_2-\log y_1)^{r_2}\right].$$

As $x_1 \to x_2$ and $y_1 \to y_2$, the right-hand side of the preceding inequality tends to zero.

As a consequence of Steps 1–3, together with the Ascoli–Arzelà theorem, we can conclude that N is continuous and completely continuous. From an application of Theorem 1.42 we deduce that N has a fixed point u that is a solution of equation (8.11).

Step 4. The solution u of (7.24) *satisfies*

$$v(x,y)\leq u(x,y)\leq w(x,y)\ \text{ for all }\ (x,y)\in J\,.$$

Let u be the preceding solution of (7.24). We prove that

$$u(x,y)\leq w(x,y)\ \text{ for all }\ (x,y)\in J\,.$$

Assume that $u-w$ attains a positive maximum on J at $(\overline{x},\overline{y})\in J$; then

$$(u-w)(\overline{x},\overline{y})=\max\{u(x,y)-w(x,y):\ (x,y)\in J\}>0\,.$$

We distinguish the following cases.

Case 1. If $(\overline{x},\overline{y})\in(1,a)\times[1,b]$, then there exists $(x^*,y^*)\in(1,a)\times[1,b]$ such that

$$\begin{aligned}&[u(x,y^*)-w(x,y^*)]+[u(x^*,y)-w(x^*,y)]-[u(x^*,y^*)-w(x^*,y^*)]\leq 0\,;\\&\text{for all }(x,y)\in([x^*,\overline{x}]\times\{y^*\})\cup(\{x^*\}\times[y^*,b])\,,\end{aligned}\tag{7.26}$$

and

$$u(x,y)-w(x,y)>0;\ \text{ for all }\ (x,y)\in(x^*,\overline{x}]\times(y^*,b]\,.\tag{7.27}$$

By the definition of h we have

$$u(x,y)=\mu(x,y)+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_1^x\int_1^y\left(\log\frac{x}{s}\right)^{r_1-1}\left(\log\frac{y}{t}\right)^{r_2-1}\frac{g(s,t,u(s,t))}{st}dtds\tag{7.28}$$

for all $(x, y) \in [x^*, \overline{x}] \times [y^*, b]$, where

$$g(x, y, u(x, y)) = f(x, y, w(x, y)), \quad (x, y) \in [x^*, \overline{x}] \times [y^*, b].$$

Thus equation (7.28) gives

$$\begin{aligned} &u(x, y) + u(x^*, y^*) - u(x, y^*) - u(x^*, y) \\ &= \int_{x^*}^{x}\int_{y^*}^{y} \left(\log \frac{x}{s}\right)^{r_1-1} \left(\log \frac{y}{t}\right)^{r_2-1} \frac{g(s, t, u(s, t))}{st\Gamma(r_1)\Gamma(r_2)} dtds. \end{aligned}$$

Using the fact that w is an upper solution of (7.1) we get

$$u(x, y) + u(x^*, y^*) - u(x, y^*) - u(x^*, y) \le w(x, y) + w(x^*, y^*) - w(x, y^*) - w(x^*, y).$$

Then

$$[u(x, y)-w(x, y)] \le [u(x, y^*)-w(x, y^*)]+[u(x^*, y)-w(x^*, y)]-[u(x^*, y^*)-w(x^*, y^*)]. \tag{7.29}$$

Thus, from (7.26), (7.27), and (7.29) we obtain the contradiction

$$\begin{aligned} &0 < [u(x, y) - w(x, y)] \le [u(x, y^*) - w(x, y^*)] \\ &+ [u(x^*, y) - w(x^*, y)] - [u(x^*, y^*) - w(x^*, y^*)] \le 0 \quad \text{for all } (x, y) \in [x^*, \overline{x}] \times [y^*, b]. \end{aligned}$$

Case 2. If $\overline{x} = 1$, then

$$w(1, \overline{y}) < u(1, \overline{y}) \le w(1, \overline{y}),$$

which is a contradiction. Thus,

$$u(x, y) \le w(x, y) \quad \text{for all } (x, y) \in J.$$

Analogously, we can prove that

$$u(x, y) \ge v(x, y), \quad \text{for all } (x, y) \in J.$$

This shows that integral equation (7.24) has a solution u that satisfies $v \le u \le w$, which is a solution of (7.22). □

7.4.3 Existence Results for Partial Hadamard Fractional Integral Inclusions

Definition 7.12. A function $z \in C$ is said to be a lower solution of (7.23) if there exists a function $f \in S_{F\circ z}$ such that z satisfies

$$z(x, y) \le \mu(x, y) + \int_{1}^{x}\int_{1}^{y} \left(\log \frac{x}{s}\right)^{r_1-1} \left(\log \frac{y}{t}\right)^{r_2-1} \frac{f(s, t)}{st\Gamma(r_1)\Gamma(r_2)} dtds; \quad (x, y) \in J.$$

The function z is said to be an upper solution of (7.23) if the reverse inequality holds.

Theorem 7.13. *Make the following assumptions:*
(7.13.1) *The multifunction $F\colon J \times \mathbb{R} \longrightarrow \mathcal{P}_{cp,cv}(\mathbb{R})$ is L^1-Carathéodory.*
(7.13.2) *There exist v and $w \in C$, lower and upper solutions for the integral inclusion* (7.23)*, such that $v \le w$.*
Then the Hadamard integral inclusion (7.23) *has at least one solution u such that*

$$v(x,y) \le u(x,y) \le w(x,y) \ \text{for all} \ (x,y) \in J .$$

Remark 7.14. Solutions of inclusion (7.23) are solutions of the Hadamard integral inclusion

$$u(x,y) \in \left\{\mu(x,y) + ({}^H I_\sigma^r f)(x,y) \colon\ f \in S_{F\circ u}\right\} ; \quad (x,y) \in J .$$

Proof. Consider the modified integral inclusion

$$u(x,y) - \mu(x,y) \in ({}^H I_\sigma^r F)(x,y,(gu)(x,y)) ; \quad (x,y) \in J , \tag{7.30}$$

where $g\colon C \longrightarrow C$ is the truncation operator defined by

$$(gu)(x,y) = \begin{cases} v(x,y); & u(x,y) < v(x,y), \\ u(x,y); & v(x,y) \le u(x,y) \le w(x,y), \\ w(x,y); & w(x,y) < u(x,y). \end{cases}$$

A solution of (7.30) is a fixed point of the operator $N\colon C \to \mathcal{P}(C)$ defined by

$$(Nu)(x,y) = \left\{ h \in C\colon \begin{cases} h(x,y) = \mu(x,y) \\ + \int_1^x \int_1^y \left(\log \frac{x}{s}\right)^{r_1-1} \left(\log \frac{y}{t}\right)^{r_2-1} \frac{f(s,t)}{st\Gamma(r_1)\Gamma(r_2)} dt ds;\ (x,y) \in J , \end{cases} \right\}$$

where

$$\begin{aligned} f \in \tilde{S}^1_{F\circ g(u)} &= \{f \in S^1_{F\circ g(u)} \colon f(x,y) \ge f_1(x,y) \text{ on } A_1 \text{ and } f(x,y) \le f_2(x,y) \text{ on } A_2\} , \\ A_1 &= \{(x,y) \in J\colon u(x,y) < v(x,y) \le w(x,y)\} , \\ A_2 &= \{(x,y) \in J\colon u(x,y) \le w(x,y) < u(x,y)\} , \end{aligned}$$

and

$$S^1_{F\circ g(u)} = \{f \in L^1(J)\colon f(x,y) \in F(t,x,(gu)(x,y));\ \text{for } (x,y) \in J\} .$$

Remark 7.15. (A) For each $u \in C$ the set $\tilde{S}_{F\circ g(u)}$ is nonempty. In fact, (7.13.1) implies the existence of $f_3 \in S_{F\circ g(u)}$, so we set

$$f = f_1\chi_{A_1} + f_2\chi_{A_2} + f_3\chi_{A_3} ,$$

where χ_{A_i} is a characteristic function of A_i; $i = 1, 2, 3$ and

$$A_3 = \{(x,y) \in J\colon v(x,y) \le u(x,y) \le w(x,y)\} .$$

Then, by decomposability, $f \in \tilde{S}_{F\circ g(u)}$.

(B) By the definition of g, it is clear that $F(.,.,(gu)(.,.))$ is an L^1-Carathéodory multivalued map with compact convex values, and there exists $\phi \in C(J, \mathbb{R}_+)$ such that

$$\|F(t,x,(gu)(x,y))\|_{\mathcal{P}} \le \phi(x,y)\,; \quad \text{for each } u \in \mathbb{R} \text{ and } (x,y) \in J\,.$$

Set

$$\phi^* := \sup_{(x,y)\in J} \phi(x,y)\,.$$

From Remark 7.14 and the fact that $g(u) = u$ for all $v \le u \le w$, the problem of finding the solutions of integral inclusion (7.23) is reduced to finding the solutions of the operator inclusion $u \in N(u)$. We will show that N is a completely continuous multivalued map, u.s.c. with convex closed values. The proof will be given in several steps.

Step 1: $N(u)$ is convex for each $u \in C$. Indeed, if h_1, h_2 belong to $N(u)$, then there exist $f_1, f_2 \in \tilde{S}^1_{F\circ g(u)}$ such that for each $(x,y) \in J$ we have

$$h_i(x,y) = \mu(x,y) + \int_1^x\int_1^y \left(\log\frac{x}{s}\right)^{r_1-1}\left(\log\frac{y}{t}\right)^{r_2-1}\frac{f_i(s,t)}{st\Gamma(r_1)\Gamma(r_2)}dtds\,; \quad i=1,2\,.$$

Let $0 \le \xi \le 1$. Then for each $(x,y) \in J$ we have

$$(\xi h_1 + (1-\xi)h_2)(x,y) = \mu(x,y) + \int_1^x\int_1^y \left(\log\frac{x}{s}\right)^{r_1-1}\left(\log\frac{y}{t}\right)^{r_2-1}\frac{((\xi f_1 + (1-\xi)f_2))(s,t)}{st\Gamma(r_1)\Gamma(r_2)}dtds\,.$$

Since $\tilde{S}^1_{F\circ g(u)}$ is convex (because F has convex values), we have

$$\xi h_1 + (1-\xi)h_2 \in N(u)\,.$$

Step 2: N sends bounded sets of C to bounded sets. Indeed, we can prove that $N(C)$ is bounded. It is enough to show that there exists a positive constant ℓ such that for each $h \in N(u)$, $u \in C$ one has $\|h\|_C \le \ell$.
If $h \in N(u)$, then there exists $f \in \tilde{S}^1_{F\circ g(u)}$ such that for each $(x,y) \in J$ we have

$$h(x,y) = \mu(x,y) + \int_1^x\int_1^y \left(\log\frac{x}{s}\right)^{r_1-1}\left(\log\frac{y}{t}\right)^{r_2-1}\frac{f(s,t)}{st\Gamma(r_1)\Gamma(r_2)}dtds\,.$$

Then we get

$$|h(x,y)| \le |\mu(x,y)| + \int_1^x\int_1^y \left|\log\frac{x}{s}\right|^{r_1-1}\left|\log\frac{y}{t}\right|^{r_2-1}\frac{\phi(s,t)}{st\Gamma(r_1)\Gamma(r_2)}dtds\,.$$

Thus, we obtain

$$\begin{aligned}|h(x,y)| &\le \|\mu\|_C \\ &\quad + \int_{\phi^*}^{x}\int_{1}^{y} \left|\log\frac{x}{s}\right|^{r_1-1}\left|\log\frac{y}{t}\right|^{r_2-1}\frac{\phi^*}{st\Gamma(r_1)\Gamma(r_2)}dtds \\ &\le \|\mu\|_C + \frac{(\log a)^{r_1}(\log b)^{r_2}\phi^*}{\Gamma(1+r_1)\Gamma(1+r_2)} := \ell\,.\end{aligned}$$

Hence,

$$\|h\|_C \le \ell\,.$$

Step 3: *N sends bounded sets of C to equicontinuous sets.* Let $(x_1, y_1), (x_2, y_2) \in J$, $x_1 < x_2, y_1 < y_2$ and $B_\rho = \{u \in C\colon \|u\|_C \le \rho\}$ be a bounded set of C. For each $u \in B_\rho$ and $h \in N(u)$ there exists $f \in \tilde{S}^1_{F\circ g(u)}$ such that for each $(x, y) \in J$ we get

$$\begin{aligned}|h(x_2,y_2) - h(x_1,y_1)| &\le |\mu(x_1,y_1) - \mu(x_2,y_2)| \\ &\quad + \int_{1}^{x_1}\int_{1}^{y_1}\left|\left(\log\frac{x_2}{s}\right)^{r_1-1}\left(\log\frac{y_2}{t}\right)^{r_2-1} - \left(\log\frac{x_1}{s}\right)^{r_1-1}\left(\log\frac{y_1}{t}\right)^{r_2-1}\right| \\ &\quad \times \frac{|f(s,t)|}{st\Gamma(r_1)\Gamma(r_2)}dtds \\ &\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_1}^{x_2}\int_{y_1}^{y_2}\left|\log\frac{x_2}{s}\right|^{r_1-1}\left|\log\frac{y_2}{t}\right|^{r_2-1}\frac{|f(s,t)|}{st}dtds \\ &\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{1}^{x_1}\int_{y_1}^{y_2}\left|\log\frac{x_2}{s}\right|^{r_1-1}\left|\log\frac{y_2}{t}\right|^{r_2-1}\frac{|f(s,t)|}{st}dtds \\ &\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_1}^{x_2}\int_{1}^{y_1}\left|\log\frac{x_2}{s}\right|^{r_1-1}\left|\log\frac{y_2}{t}\right|^{r_2-1}\frac{|f(s,t)|}{st}dtds\,.\end{aligned}$$

Hence,

$$\begin{aligned}|h(x_2,y_2) - h(x_1,y_1)| &\le |\mu(x_1,y_1) - \mu(x_2,y_2)| \\ &\quad + \frac{\phi^*}{\Gamma(1+r_1)\Gamma(1+r_2)} \\ &\quad \times \big[2(\log y_2)^{r_2}(\log x_2 - \log x_1)^{r_1} + 2(\log x_2)^{r_1}(\log y_2 - \log y_1)^{r_2} \\ &\quad + (\log x_1)^{r_1}(\log y_1)^{r_2} - (\log x_2)^{r_1}(\log y_2)^{r_2} \\ &\quad -2(\log x_2 - \log x_1)^{r_1}(\log y_2 - \log y_1)^{r_2}\big]\,.\end{aligned}$$

As $x_1 \to x_2$ and $y_1 \to y_2$, the right-hand side of the preceding inequality tends to zero.

As a consequence of Steps 1–3, together with the Ascoli–Arzelà theorem, we can conclude that N is completely continuous and, therefore, a condensing multivalued map.

Step 4: N has a closed graph. Let $u_n \to u_*$, $h_n \in N(u_n)$, and $h_n \to h_*$. We need to show that $h_* \in N(u_*)$.
$h_n \in N(u_n)$ means that there exists $f_n \in \tilde{S}^1_{F\circ g(u_n)}$ such that for each $(x, y) \in J$ we have

$$h_n(x,y) = \mu(x,y) + \int_1^x\int_1^y \left(\log\frac{x}{s}\right)^{r_1-1}\left(\log\frac{y}{t}\right)^{r_2-1}\frac{f_n(s,t)}{st\Gamma(r_1)\Gamma(r_2)}dtds\,.$$

We must show that there exists $f_* \in \tilde{S}^1_{F\circ g(u_*)}$ such that, for each $(x, y) \in J$,

$$h_*(x,y) = \mu(x,y) + \int_1^x\int_1^y \left(\log\frac{x}{s}\right)^{r_1-1}\left(\log\frac{y}{t}\right)^{r_2-1}\frac{f_*(s,t)}{st\Gamma(r_1)\Gamma(r_2)}dtds\,.$$

Now we consider the linear continuous operator

$$\Lambda : L^1(J) \longrightarrow C(J)\,,\qquad f \longmapsto \Lambda f$$

defined by

$$(\Lambda f)(x,y) = \mu(x,y) + \int_1^x\int_1^y \left(\log\frac{x}{s}\right)^{r_1-1}\left(\log\frac{y}{t}\right)^{r_2-1}\frac{f(s,t)}{st\Gamma(r_1)\Gamma(r_2)}dtds\,.$$

Remark 7.16. Remark 7.15 (B) implies that the operator Λ is well defined.

From Lemma 1.25 it follows that $\Lambda \circ \tilde{S}^1_F$ is a closed graph operator. Clearly we have

$$|h_n(x,y) - h_*(x,y)| = \mu(x,y) + \int_1^x\int_1^y \left(\log\frac{x}{s}\right)^{r_1-1}\left(\log\frac{y}{t}\right)^{r_2-1}\frac{|f_n(s,t)-f_*(s,t)|}{st\Gamma(r_1)\Gamma(r_2)}dtds \to 0$$

as $n \to \infty$.

Moreover, from the definition of Λ we have

$$|h_n(x,y) - h_*(x,y)| \in \Lambda(\tilde{S}^1_{F\circ g(u_n)})\,;\quad \text{if } (x,y) \in J\,.$$

Since $u_n \to u_*$, it follows from Lemma 1.25 that for some $f_* \in \Lambda(\tilde{S}^1_{F\circ g(u_*)})$ we have

$$h_*(x,y) = \mu(x,y) + \int_1^x\int_1^y \left(\log\frac{x}{s}\right)^{r_1-1}\left(\log\frac{y}{t}\right)^{r_2-1}\frac{f_*(s,t)}{st\Gamma(r_1)\Gamma(r_2)}dtds\,.$$

From Lemma 1.24 we can conclude that N is u.s.c.

Step 5: The set $\Omega = \{u \in C : \lambda u \in N(u)$ for some $\lambda > 1\}$ is bounded. Let $u \in \Omega$. Then there exists $f \in \Lambda(\tilde{S}^1_{F\circ g(u)})$ such that

$$\lambda u(x,y) = \mu(x,y) + \int_1^x \int_1^y \left(\log \frac{x}{s}\right)^{r_1-1} \left(\log \frac{y}{t}\right)^{r_2-1} \frac{f(s,t)}{st\Gamma(r_1)\Gamma(r_2)} dtds\,.$$

As in Step 2, this implies that for each $(x,y) \in J$ we have

$$\|u\|_C \le \frac{\ell}{\lambda} < \ell\,.$$

This shows that Ω is bounded. As a consequence of Theorem 1.48, we deduce that N has a fixed point that is a solution of (7.30) on J.

Step 6: The solution u of (7.30) *satisfies*

$$v(x,y) \le u(x,y) \le w(x,y)\,; \quad \text{for all } (x,y) \in J\,.$$

First we prove that

$$u(x,y) \le w(x,y)\,; \quad \text{for all } (x,y) \in J\,.$$

Assume that $u - w$ attains a positive maximum on J at $(\overline{x}, \overline{y}) \in J$; then

$$(u-w)(\overline{x}, \overline{y}) = \max\{u(x,y) - w(x,y);\ (x,y) \in J\} > 0\,.$$

We distinguish the following cases.

Case 1. If $(\overline{x}, \overline{y}) \in (1,a) \times [1,b]$, then there exists $(x^*, y^*) \in (1,a) \times [1,b]$ such that

$$\begin{aligned} &[u(x,y^*) - w(x,y^*)] + [u(x^*,y) - w(x^*,y)] \\ &- [u(x^*,y^*) - w(x^*,y^*)] \le 0; \quad \text{for all } (x,y) \in ([x^*,\overline{x}] \times \{y^*\}) \cup (\{x^*\} \times [y^*,b]) \end{aligned} \tag{7.31}$$

and

$$u(x,y) - w(x,y) > 0\,; \quad \text{for all } (x,y) \in (x^*,\overline{x}] \times (y^*,b]\,. \tag{7.32}$$

For all $(x,y) \in [x^*,\overline{x}] \times [y^*,b]$ we have

$$u(x,y) = \mu(x,y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left(\log \frac{x}{s}\right)^{r_1-1} \left(\log \frac{y}{t}\right)^{r_2-1} \frac{f(s,t)}{st} dtds\,, \tag{7.33}$$

where $f \in S_{F\circ u}$. Thus, equation (7.33) gives

$$\begin{aligned} &u(x,y) + u(x^*,y^*) - u(x,y^*) - u(x^*,y) \\ &= \int_{x^*}^x \int_{y^*}^y \left(\log \frac{x}{s}\right)^{r_1-1} \left(\log \frac{y}{t}\right)^{r_2-1} \frac{f(s,t)}{st\Gamma(r_1)\Gamma(r_2)} dtds\,. \end{aligned} \tag{7.34}$$

From (7.34) and using the fact that w is an upper solution of (7.23) we get

$$u(x,y)+u(x^*,y^*)-u(x,y^*)-u(x^*,y)\le w(x,y)+w(x^*,y^*)-w(x,y^*)-w(x^*,y)\,.$$

Then

$$[u(x,y)-w(x,y)]\le[u(x,y^*)-w(x,y^*)]+[u(x^*,y)-w(x^*,y)]-[u(x^*,y^*)-w(x^*,y^*)]\,. \tag{7.35}$$

Thus, from (7.31), (7.32), and (7.35) we obtain the contradiction

$$0<[u(x,y)-w(x,y)]\le[u(x,y^*)-w(x,y^*)]$$
$$+[u(x^*,y)-w(x^*,y)]-[u(x^*,y^*)-w(x^*,y^*)]\le 0;\quad\text{for all } (x,y)\in[x^*,\overline{x}]\times[y^*,b]\,.$$

Case 2. If $\overline{x}=1$, then

$$w(1,\overline{y})<u(1,\overline{y})\le w(1,\overline{y})\,,$$

which is a contradiction. Thus,

$$u(x,y)\le w(x,y);\quad\text{for all}\quad (x,y)\in J\,.$$

Analogously, we can prove that

$$u(x,y)\ge v(x,y)\quad\text{for all}\quad (x,y)\in J\,.$$

This shows that problem (7.30) has a solution u that satisfies $v\le u\le w$, which is a solution of integral inclusion (7.23). □

7.5 Notes and Remarks

The results of Chapter 7 are taken from Abbas et al. [1, 12, 32]. Other results may be found in [29, 42, 119, 120].