6 Integrable Solutions for Implicit Fractional Differential Equations

6.1 Introduction

This chapter deals with the existence of integrable solutions for initial value problems (IVPs) for implicit fractional differential equations. We present results based on Schauder's fixed point theorem and the Banach contraction . Then we present other results with infinite delay. In the literature devoted to equations with finite delay, the state space is usually the space of all continuous functions on $[-r, 0]$, $r > 0$ and $\alpha = 1$ endowed with the uniform norm topology; see the book by Hale and Lunel [155]. When the delay is infinite, the selection of the state $\mathcal{B}$ (i.e., phase space) plays an important role in the study of both qualitative and quantitative theory for functional differential equations. A usual choice is a seminormed space satisfying suitable axioms introduced by Hale and Kato [154]. For a detailed discussion on this topic we refer the reader to the book by Hino et al. [162].

6.2 Integrable Solutions for NIFDE

6.2.1 Introduction

In this section we deal with the existence of integrable solutions for initial value problems (IVPs) for the fractional order implicit differential equation

$$^cD^\alpha y(t) = f(t, y(t), {}^cD^\alpha y(t)) , \quad t \in J = [0, T] \tag{6.1}$$

$$y(0) = y_0 , \tag{6.2}$$

where $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function, $y_0 \in \mathbb{R}$.

6.2.2 Existence of solutions

Definition 6.1. A function $y \in L^1(J, \mathbb{R})$ is said to be a solution of IVP (6.1)–(6.2) if y satisfies (6.1) and (6.2).

For the existence of solutions to problem (6.1)–(6.2), we need the following auxiliary lemma.

Lemma 6.2. *The solution of IVP* (6.1)–(6.2) *can be expressed by the integral equation*

$$y(t) = y_0 + I_0^\alpha x(t) , \tag{6.3}$$

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where $x \in L^1(J, \mathbb{R})$ *satisfies*

$$x(t) = f\left(t, y_0 + I_0^\alpha x(t), x(t)\right) . \tag{6.4}$$

Proof. Let ${}^cD^\alpha y(t) = x(t)$. Then

$$x(t) = f(t, y(t), x(t)) \tag{6.5}$$

and

$$\begin{aligned} y(t) &= y(0) + I^\alpha x(t) \qquad (6.6)\\ &= y(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s) ds . \end{aligned}$$

□

Let us introduce the following conditions:

(6.2.1) $f : J \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is measurable in $t \in J$ for any $(u_1, u_2) \in \mathbb{R}^2$ and continuous in $(u_1, u_2) \in \mathbb{R}^2$ for almost all $t \in J$.

(6.2.2) There exist a positive function $a \in L^1(J, \mathbb{R})$ and constants, $b_i > 0, i = 1, 2$, such that

$$|f(t, u_1, u_2)| \le |a(t)| + b_1|u_1| + b_2|u_2|, \forall (t, u_1, u_2) \in J \times \mathbb{R}^2 .$$

Our first result is based on Schauder's fixed point theorem.

Theorem 6.3. *Assume* (6.2.1) *and* (6.2.2) *hold. If*

$$\frac{b_1 T^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{b_2 T^\alpha}{\Gamma(\alpha+1)} < 1 , \tag{6.7}$$

then IVP (6.1)–(6.2) *has at least one solution* $y \in L^1(J, \mathbb{R})$.

Proof. Transform problem (6.1)–(6.2) into a fixed point problem. Consider the operator

$$H : L^1(J, \mathbb{R}) \longrightarrow L^1(J, \mathbb{R})$$

defined by

$$(Hx)(t) = y_0 + I^\alpha x(t) , \tag{6.8}$$

where

$$x(t) = f\left(t, y_0 + I^\alpha x(t), x(t)\right) .$$

The operator H is well defined, indeed, for each $x \in L^1(J, \mathbb{R})$, and from conditions (6.2.1) and (6.2.2) we obtain

$$\begin{aligned}
\|Hx\|_{L_1} &= \int_0^T |Hx(t)|dt \\
&= \int_0^T |y_0 + I^\alpha x(t)|dt \\
&\leq T|y_0| + \int_0^T \left(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |x(s)|ds \right) dt \\
&\leq T|y_0| + \int_0^T \left(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f\left(s, y_0 + I^\alpha x(s), x(s)\right)|ds \right) dt \\
&\leq T|y_0| + \int_0^T \left(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |a(s) + b_1(y_0 + I^\alpha x(s)) + b_2(x(s)|ds \right) dt \\
&\leq T|y_0| + \frac{T^\alpha}{\Gamma(\alpha+1)}\|a\|_{L_1} + \frac{b_1|y_0|T^{\alpha+1}}{\Gamma(\alpha+1)} + \frac{b_2 T^\alpha}{\Gamma(\alpha+1)}\|x\|_{L_1} \\
&\quad + b_1 \int_0^T \left(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} I^\alpha |x(s)|ds \right) dt \\
&\leq T|y_0| + \frac{T^\alpha}{\Gamma(\alpha+1)}\|a\|_{L_1} + \frac{b_1|y_0|T^{\alpha+1}}{\Gamma(\alpha+1)} + \frac{b_2 T^\alpha}{\Gamma(\alpha+1)}\|x\|_{L_1} \\
&\quad + \frac{b_1 T^{2\alpha}}{\Gamma(2\alpha+1)}\|x\|_{L_1} < +\infty \,. \qquad (6.9)
\end{aligned}$$

Let

$$r = \frac{T|y_0| + \left(\frac{T^\alpha \|a\|_{L_1} + b_1|y_0|T^{\alpha+1}}{\Gamma(\alpha+1)}\right)}{1 - \left(\frac{b_1 T^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{b_2 T^\alpha}{\Gamma(\alpha+1)}\right)} \,,$$

and consider the set

$$B_r = \{x \in L^1(J, \mathbb{R}) : \|x\|_{L_1} \leq r.\} \,.$$

Clearly B_r is nonempty, bounded, convex, and closed.

Now we will show that $HB_r \subset B_r$, indeed, for each $x \in B_r$, and from (6.7) and (6.9) we get

$$\begin{aligned}
\|Hx\|_{L_1} &\leq T|y_0| + \left(\frac{T^\alpha \|a\|_{L_1} + b_1|y_0|T^{\alpha+1}}{\Gamma(\alpha+1)}\right) \\
&\quad + \left(\frac{b_1 T^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{b_2 T^\alpha}{\Gamma(\alpha+1)}\right)\|x\|_{L_1} \\
&\leq r \,.
\end{aligned}$$

Then $HB_r \subset B_r$. Assumption (6.2.1) implies that H is continuous. Now we will show that H is compact, that is, HB_r is relatively compact. Clearly HB_r is bounded in $L^1(J, \mathbb{R})$, i.e., condition (i) of the Kolmogorov compactness criterion is satisfied. It remains to show $(Hx)_h \longrightarrow (Hx)$ in $L^1(J, \mathbb{R})$ for each $x \in B_r$.

Let $x \in B_r$; then we have

$$
\begin{aligned}
&\|(Hx)_h - (Hx)\|_{L^1} \\
&= \int_0^T |(Hx)_h(t) - (Hx)(t)|dt \\
&= \int_0^T \left| \frac{1}{h} \int_t^{t+h} (Hx)(s)ds - (Hx)(t) \right| dt \\
&\le \int_0^T \left(\frac{1}{h} \int_t^{t+h} |(Hx)(s) - (Hx)(t)|ds \right) dt \\
&\le \int_0^T \left(\frac{1}{h} \int_t^{t+h} |I^\alpha x(s) - I^\alpha x(t)|ds \right) dt \\
&\le \int_0^T \frac{1}{h} \int_t^{t+h} |I^\alpha f\left(s, y_0 + I^\alpha x(s), x(s)\right) - I^\alpha f\left(t, y_0 + I^\alpha x(t), x(t)\right)| dsdt \,.
\end{aligned}
$$

Since $x \in B_r \subset L^1(J, \mathbb{R})$, condition (6.2.2) implies that $f \in L^1(J, \mathbb{R})$. From Proposition 1.10 (v) it follows that $I^\alpha f \in L^1(J, \mathbb{R})$; thus, we have

$$
\frac{1}{h} \int_t^{t+h} |I^\alpha f\left(s, y_0 + I^\alpha x(s), x(s)\right) - I^\alpha f\left(t, y_0 + I^\alpha x(t), x(t)\right)| ds \longrightarrow 0 \text{ as } h \longrightarrow 0,\ t \in J \,.
$$

Hence,

$$
(Hx)_h \longrightarrow (Hx) \ \text{ uniformly as } \ h \longrightarrow 0 \,.
$$

Then by the Kolmogorov compactness criterion, HB_r is relatively compact. As a consequence of Schauder's fixed point theorem, IVP (6.1)–(6.2) has at least one solution in B_r. □

The following result is based on the Banach contraction principle.

Theorem 6.4. *Assume* (6.2.1) *holds and*

(6.4.1) *There exist constants* $k_1, k_2 > 0$ *such that*

$$
|f(t, x_1, y_1) - f(t, x_2, y_2)| \le k_1|x_1 - x_2| + k_2|y_1 - y_2| \,, \quad t \in J,\ x_1, x_2, y_1, y_2 \in \mathbb{R} \,.
$$

If

$$
\frac{k_1 T^{2\alpha}}{\Gamma(2\alpha + 1)} + \frac{k_2 T^\alpha}{\Gamma(\alpha + 1)} < 1 \,, \tag{6.10}
$$

then IVP (6.1)–(6.2) *has a unique solution* $y \in L^1(J, \mathbb{R})$.

Proof. We will use the Banach contraction principle to prove that H defined by (6.8) has a fixed point. Let $x, y \in L^1(J, \mathbb{R})$ and $t \in J$. Then we have

$$
\begin{aligned}
|(Hx)(t) - (Hy)(t)| &= |I^\alpha \left[f(t, y_0 + I^\alpha x(t), x(t)) - f(t, y_0 + I^\alpha y(t), y(t))\right]| \\
&\le k_1 I^{2\alpha}|x(t) - y(t)| + k_2 I^\alpha |x(t) - y(t)| \\
&\le \frac{k_1}{\Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1}|x(s) - y(s)| ds \\
&\quad + \frac{k_2}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}|x(s) - y(s)| ds \, .
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\|(Hx) - (Hy)\|_{L_1} &\le \frac{k_1 T^{2\alpha}}{\Gamma(2\alpha+1)} \|x - y\|_{L_1} + \frac{k_2 T^\alpha}{\Gamma(\alpha+1)} \|x - y\|_{L_1} \\
&\le \left(\frac{k_1 T^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{k_2 T^\alpha}{\Gamma(\alpha+1)} \right) \|x - y\|_{L_1} \, .
\end{aligned}
$$

Consequently, by (6.10) H is a contraction. As a consequence of the Banach contraction principle, we deduce that H has a fixed point that is a solution of problem (6.1)–(6.2). □

6.2.3 Example

Let us consider the fractional IVP

$$
{}^cD^\alpha y(t) = \frac{e^{-t}}{(e^t + 8)(1 + |y(t)| + |{}^cD^\alpha y(t)|)} \, , \quad t \in J := [0, 1], \ \alpha \in (0, 1] \, , \tag{6.11}
$$

$$
y(0) = 1 \, . \tag{6.12}
$$

Set

$$
f(t, y, z) = \frac{e^{-t}}{(e^t + 8)(1 + y + z)} \, , \quad (t, y, z) \in J \times [0, +\infty) \times [0, +\infty) \, .
$$

Let $y, z \in [0, +\infty)$ and $t \in J$. Then we have

$$
\begin{aligned}
|f(t, y_1, z_1) - f(t, y_2, z_2)| &= \left| \frac{e^{-t}}{e^t + 8} \left(\frac{1}{1 + y_1 + z_1} - \frac{1}{1 + y_2 + z_2} \right) \right| \\
&\le \frac{e^{-t}(|y_1 - y_2| + |z_1 - z_2|)}{(e^t + 8)(1 + y_1 + z_1)(1 + y_2 + z_2)} \\
&\le \frac{e^{-t}}{(e^t + 8)}(|y_1 - y_2| + |z_1 - z_2|) \\
&\le \frac{1}{9}|y_1 - y_2| + \frac{1}{9}|z_1 - z_2| \, .
\end{aligned}
$$

Hence, condition (6.4.1) holds with $k_1 = k_2 = \frac{1}{9}$. Condition (6.10) is satisfied with $T = 1$. Indeed,

$$\frac{k_1 T^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{k_2 T^{\alpha}}{\Gamma(\alpha+1)} = \frac{1}{9\Gamma(2\alpha+1)} + \frac{1}{9\Gamma(\alpha+1)} < 1 . \tag{6.13}$$

Then, by Theorem 6.4, problem (6.11)–(6.12) has a unique integrable solution on $[0, 1]$.

6.3 L^1-Solutions for NIFDEs with Nonlocal Conditions

6.3.1 Introduction

In this section, we deal with the existence of solutions of the nonlocal problem for fractional order implicit differential equation

$$^cD^{\alpha}y(t) = f(t, y(t), {}^cD^{\alpha}y(t)) , \quad a.e, \ t \in J =: (0, T] , \tag{6.14}$$

$$\sum_{k=1}^{m} a_k y(t_k) = y_0 , \tag{6.15}$$

where $f : J\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is a given function, $y_0 \in \mathbb{R}$, $a_k \in \mathbb{R}$, and $0 < t_1 < t_2 < \dots , t_m < T$, $k = 1, 2, \dots , m$.

Fractional differential equations with nonlocal conditions are discussed in [50] and references therein. Nonlocal conditions were initiated by Byszewski [118] when he proved the existence and uniqueness of mild and classical solutions of nonlocal Cauchy problems. As remarked by Byszewski ([116, 117]), the nonlocal condition can be more useful than the standard initial condition for describing some physical phenomena.

6.3.2 Existence of solutions

Definition 6.5. A function $y \in L^1([0, T], \mathbb{R})$ is said to be a solution of IVP (6.14)–(6.15) if y satisfies (6.14) and (6.15).

Set

$$a = \frac{1}{\sum_{k=1}^{m} a_k} .$$

For the existence of solutions for nonlocal problem (6.14)–(6.15), we need the following auxiliary lemma.

Lemma 6.6. *Assume that* $\sum_{k=1}^{m} a_k \neq 0$*; then nonlocal problem* (6.14)–(6.15) *is equivalent to the integral equation*

$$y(t) = ay_0 - a\sum_{k=1}^{m} a_k \int_0^{t_k} \frac{(t_k - s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds + \int_0^{t} \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds , \tag{6.16}$$

where x is the solution of the functional integral equation

$$x(t) = f\left(t, ay_0 - a\sum_{k=1}^{m} a_k \int_0^{t_k} \frac{(t_k - s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds) + \int_0^{t} \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds, x(t) \right). \quad (6.17)$$

Proof. Let ${}^cD^\alpha y(t) = x(t))$ in equation (6.14); then

$$x(t) = f(t, y(t), x(t)) \quad (6.18)$$

and

$$\begin{aligned} y(t) &= y(0) + I^\alpha x(t) \\ &= y(0) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds . \end{aligned} \quad (6.19)$$

Let $t = t_k$ in (6.19); we obtain

$$y(t_k) = y(0) + \int_0^{t_k} \frac{(t_k - s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds$$

and

$$\sum_{k=1}^{m} a_k y(t_k) = \sum_{k=1}^{m} a_k y(0) + \sum_{k=1}^{m} a_k \int_0^{t_k} \frac{(t_k - s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds . \quad (6.20)$$

Substituting (6.15) into (6.20), we get

$$y_0 = \sum_{k=1}^{m} a_k y(0) + \sum_{k=1}^{m} a_k \int_0^{t_k} \frac{(t_k - s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds$$

and

$$y(0) = a\left(y_0 - \sum_{k=1}^{m} a_k \int_0^{t_k} \frac{(t_k - s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds \right). \quad (6.21)$$

Substituting (6.21) into (6.18) and (6.19), we obtain (6.16) and (6.17).

To complete the proof, we prove that equation (6.16) satisfies nonlocal problem (6.14)–(6.15). Differentiating (6.16), we get

$${}^cD^\alpha y(t) = x(t) = f(t, y(t), {}^cD^\alpha y(t)) .$$

Letting $t = t_k$ in (6.16), we obtain

$$\begin{aligned} y(t_k) &= ay_0 - a\sum_{k=1}^{m} a_k \int_0^{t_k} \frac{(t_k - s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds) + \int_0^{t_k} \frac{(t_k - s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds \\ &= ay_0 + \left(1 - a\sum_{k=1}^{m} a_k\right) \int_0^{t_k} \frac{(t_k - s)^{\alpha-1}}{\Gamma(\alpha)} x(s)ds . \end{aligned}$$

Then

$$\sum_{k=1}^{m} a_k y(t_k) = \sum_{k=1}^{m} a_k a y_0 + \sum_{k=1}^{m} a_k \left(1 - a \sum_{k=1}^{m} a_k\right) \int_0^{t_k} \frac{(t_k - s)^{\alpha-1}}{\Gamma(\alpha)} x(s) ds = y_0 . \qquad \square$$

Let us introduce the following conditions:

(6.6.1) $f\colon [0,T] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is measurable in $t \in [0,T]$ for any $(u_1, u_2) \in \mathbb{R}^2$ and continuous in $(u_1, u_2) \in \mathbb{R}^2$ for almost all $t \in [0,T]$.

(6.6.2) There exist a positive function $a \in L^1[0,T]$ and constants, $b_i > 0$, $i = 1, 2$, such that

$$|f(t, u_1, u_2)| \le |a(t)| + b_1|u_1| + b_2|u_2| , \quad \forall (t, u_1, u_2) \in [0,T] \times \mathbb{R}^2 .$$

Our first result is based on Schauder's fixed point theorem.

Theorem 6.7. *Assume* (6.6.1)–(6.6.2). *If*

$$\frac{2b_1 T^\alpha}{\Gamma(\alpha+1)} + b_2 < 1 , \tag{6.22}$$

then IVP (6.14)–(6.15) *has at least one solution* $y \in L^1([0,T], \mathbb{R})$.

Proof. Transform nonlocal problem (6.14)–(6.15) into a fixed point problem. Consider the operator

$$H\colon L^1([0,T], \mathbb{R}) \longrightarrow L^1([0,T], \mathbb{R})$$

defined by

$$(Hx)(t) = f\left(t, a y_0 - a \sum_{k=1}^{m} a_k \int_0^{t_k} \frac{(t_k - s)^{\alpha-1}}{\Gamma(\alpha)} x(s) ds) + \int_0^{t} \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} x(s) ds, x(t)\right) . \tag{6.23}$$

Let

$$r = \frac{Tab_1|y_0| + \|a\|_{L_1}}{1 - \left(\frac{2b_1 T^\alpha}{\Gamma(\alpha+1)} + b_2\right)} ,$$

and consider the set

$$B_r = \{x \in L^1([0,T], \mathbb{R})\colon \|x\|_{L_1} \le r\} .$$

Clearly B_r is nonempty, bounded, convex, and closed.

Now we will show that $HB_r \subset B_r$, indeed, for each $x \in B_r$, and from (6.22) and (6.23) we get

$$\begin{aligned}
\|Hx\|_{L_1} &= \int_0^T |Hx(t)|dt \\
&= \int_0^T \left| f\left(t, ay_0 - a\sum_{k=1}^m a_k \int_0^{t_k} \frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)}x(s)ds) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}x(s)ds, x(t)\right)\right| dt \\
&\le \int_0^T \left[|a(t)| + b_1|ay_0 - a\sum_{k=1}^m a_k I^\alpha x(t)|_{t=t_k} + I^\alpha x(t)| + b_2|x(t)| \right] dt \\
&\le Tab_1|y_0| + \|a\|_{L_1} + \frac{b_1 a \sum_{k=1}^m a_k t_k^\alpha}{\Gamma(\alpha+1)}\|x\|_{L_1} + \frac{b_1 T^\alpha}{\Gamma(\alpha+1)}\|x\|_{L_1} + b_2\|x\|_{L_1} \\
&\le Tab_1|y_0| + \|a\|_{L_1} + \left(\frac{2b_1T^\alpha}{\Gamma(\alpha+1)} + b_2\right)\|x\|_{L_1} \\
&\le r\,.
\end{aligned}$$

Then $HB_r \subset B_r$. Assumption (6.6.1) implies that H is continuous. Now we will show that H is compact, that is, HB_r is relatively compact. Clearly HB_r is bounded in $L^1([0,T],\mathbb{R})$, i.e., condition (i) of the Kolmogorov compactness criterion is satisfied. It remains to show $(Hx)_h \longrightarrow (Hx)$ in $L^1([0,T],\mathbb{R})$ for each $x \in B_r$.

Let $x \in B_r$; then we have

$$\begin{aligned}
&\|(Hx)_h - (Hx)\|_{L^1} \\
&= \int_0^T |(Hx)_h(t) - (Hx)(t)|dt \\
&= \int_0^T \left| \frac{1}{h}\int_t^{t+h} (Hx)(s)ds - (Hx)(t) \right| dt \\
&\le \int_0^T \left(\frac{1}{h}\int_t^{t+h} |(Hx)(s) - (Hx)(t)|ds \right) dt \\
&\le \int_0^T \frac{1}{h}\int_t^{t+h} |f\left(t, ay_0 - a\sum_{k=1}^m a_k \int_0^{s_k} \frac{(s_k-\tau)^{\alpha-1}}{\Gamma(\alpha)}x(\tau)d\tau) + \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}x(\tau)d\tau, x(s)\right) \\
&\quad - f\left(t, ay_0 - a\sum_{k=1}^m a_k \int_0^{t_k} \frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)}x(s)ds) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}x(s)ds, x(t)\right)|dsdt\,.
\end{aligned}$$

Since $x \in B_r \subset L^1([0,T],\mathbb{R})$, condition (6.6.2) implies that $f \in L^1([0,T],\mathbb{R})$. Thus,

$$\frac{1}{h}\int_t^{t+h}\left|f\left(t, ay_0 - a\sum_{k=1}^m a_k \int_0^{s_k}\frac{(s_k-\tau)^{\alpha-1}}{\Gamma(\alpha)}x(\tau)d\tau + \int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}x(\tau)d\tau, x(s)\right)\right.$$
$$\left.-f\left(t, ay_0 - a\sum_{k=1}^m a_k \int_0^{t_k}\frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)}x(s)ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}x(s)ds, x(t)\right)\right| ds \to 0$$

as $h \to 0$.

Hence,

$$(Hx)_h \to (Hx) \text{ uniformly as } h \to 0 .$$

Then, by the Kolmogorov compactness criterion, HB_r is relatively compact. As a consequence of Schauder's fixed point theorem, nonlocal problem (6.14)–(6.15) has at least one solution in B_r. □

The following result is based on the Banach contraction principle.

Theorem 6.8. *Assume* (6.6.1) *holds and*
(6.8.1) *there exist constants* $k_1, k_2 > 0$ *such that*

$$|f(t,x_1,y_1) - f(t,x_2,y_2)| \le k_1|x_1-x_2| + k_2|y_1-y_2|, \quad t \in [0,T],\ x_1,x_2,y_1,y_2 \in \mathbb{R}.$$

If

$$\frac{2k_1 T^\alpha}{\Gamma(\alpha+1)} + k_2 < 1 , \tag{6.24}$$

then IVP (6.14)–(6.15) *has a unique solution* $y \in L^1([0,T],\mathbb{R})$.

Proof. We will use the Banach contraction principle to prove that H defined by (6.23) has a fixed point. Let $x, y \in L^1([0,T],\mathbb{R})$, and $t \in [0,T]$. Then we have

$$\begin{aligned}
&|(Hx)(t) - (Hy)(t)| \\
&= \left|f\left(t, ay_0 - a\sum_{k=1}^m a_k I^\alpha x(t)|_{t=t_k} + I^\alpha x(t), x(t)\right)\right. \\
&\quad \left.-f\left(t, ay_0 - a\sum_{k=1}^m a_k I^\alpha y(t)|_{t=t_k} + I^\alpha y(t), y(t)\right)\right| \\
&\le k_1 a \sum_{k=1}^m a_k \int_0^{t_k}\frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)}|x(s)-y(s)|ds \\
&\quad + k_1\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|x(s)-y(s)|ds + k_2|x-y| .
\end{aligned}$$

Thus,

$$\begin{aligned}\|(Hx)-(Hy)\|_{L_1} &\le \frac{k_1 t_k^\alpha a \sum_{k=1}^m a_k}{\Gamma(\alpha+1)} \int_0^T |x(t)-y(t)|dt + \frac{k_1 T^\alpha}{\Gamma(\alpha+1)} \int_0^T |x(t)-y(t)|dt \\ &\quad + k_2 \int_0^T |x(t)-y(t)|dt \\ &\le \frac{2k_1 T^\alpha}{\Gamma(\alpha+1)} \|x-y\|_{L_1} + k_2 \|x-y\|_{L_1} \\ &\le \left(\frac{2k_1 T^\alpha}{\Gamma(\alpha+1)} + k_2 \right) \|x-y\|_{L_1} .\end{aligned}$$

Consequently, by (6.24), H is a contraction. As a consequence of the Banach contraction principle, we deduce that H has a fixed point that is a solution of nonlocal problem (6.14)–(6.15). □

6.3.3 Example

Let us consider the fractional nonlocal problem

$$ {}^cD^\alpha y(t) = \frac{1}{(e^t+5)(1+|y(t)|+|{}^cD^\alpha y(t)|)}, \quad t \in J := [0,1],\ \alpha \in (0,1] , \tag{6.25}$$

$$\sum_{k=1}^m a_k y(t_k) = 1 , \tag{6.26}$$

where $a_k \in \mathbb{R}$, $0 < t_1 < t_2 < \cdots < 1$.

Set

$$f(t,y,z) = \frac{1}{(e^t+5)(1+y+z)}, \quad (t,y,z) \in J \times [0,+\infty) \times [0,+\infty) .$$

Let $y, z \in [0,+\infty)$ and $t \in J$. Then we have

$$\begin{aligned}|f(t,y_1,z_1)-f(t,y_2,z_2)| &= \left| \frac{1}{e^t+5} \left(\frac{1}{1+y_1+z_1} - \frac{1}{1+y_2+z_2} \right) \right| \\ &\le \frac{|y_1-y_2|+|z_1-z_2|}{(e^t+5)(1+y_1+z_1)(1+y_2+z_2)} \\ &\le \frac{1}{(e^t+5)} (|y_1-y_2|+|z_1-z_2|) \\ &\le \frac{1}{6}|y_1-y_2| + \frac{1}{6}|z_1-z_2| .\end{aligned}$$

Hence condition (6.8.1) holds with $k_1 = k_2 = \frac{1}{6}$. Condition (6.24) is satisfied. Indeed,

$$\frac{2k_1}{\Gamma(\alpha+1)} + k_2 = \frac{1}{3\Gamma(\alpha+1)} + \frac{1}{6} < 1 . \tag{6.27}$$

Then, by Theorem 3.2, nonlocal problem (6.25)–(6.26) has a unique integrable solution on $[0,1]$.

6.4 Integrable Solutions for NIFDEs with Infinite Delay

6.4.1 Introduction

In this section we deal with the existence of solutions for IVPs for implicit fractional order functional differential equations with infinite delay of the form

$$^cD^\alpha y(t) = f(t, y_t, {}^cD^\alpha y_t)\,, \quad t \in J := [0, b] \tag{6.28}$$

$$y(t) = \phi(t)\,, \quad t \in (-\infty, 0]\,, \tag{6.29}$$

where $f : J \times \mathcal{B} \times \mathcal{B} \to \mathbb{R}$ is a given function.

6.4.2 Existence of solutions

Set

$$\Omega = \{y : (-\infty, b] \to \mathbb{R} : y|_{(-\infty,0]} \in \mathcal{B} \text{ and } y|_J \in L^1(J)\}\,.$$

Definition 6.9. A function $y \in \Omega$ is said to be a solution of IVP (6.28)–(6.29) if y satisfies (6.28) and (6.29).

For the existence of solutions to problem (6.28)–(6.29), we need the following auxiliary lemma.

Lemma 6.10. *The solution to IVP* (6.28)–(6.29) *can be expressed by*

$$y(t) = \phi(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}x(s)ds, \qquad t \in J\,, \tag{6.30}$$

$$y(t) = \phi(t), \qquad t \in (-\infty, 0]\,, \tag{6.31}$$

where x is the solution of the functional integral equation

$$x(t) = f\left(t, \phi(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}x_s ds, x_t\right)\,. \tag{6.32}$$

Proof. Let y be a solution of (6.30)–(6.31); for $t \in J$ and $t \in (-\infty, 0]$ we have (6.28) and (6.29), respectively. □

To present the main result, let us introduce the following conditions:

(6.10.1) $f : J \times \mathcal{B}^2 \longrightarrow \mathbb{R}$ is measurable in $t \in J$ for any $(u_1, u_2) \in \mathcal{B}^2$ and continuous in $(u_1, u_2) \in \mathcal{B}^2$ for almost all $t \in J$.

(6.10.2) There exist constants $k_1, k_2 > 0$ such that

$$|f(t, x_1, y_1) - f(t, x_2, y_2)| \le k_1\|x_1 - x_2\|_{\mathcal{B}} + k_2\|y_1 - y_2\|_{\mathcal{B}}\,,$$

for $t \in J$ and every $x_1, x_2, y_1, y_2 \in \mathcal{B}$.

Our first existence result for IVP (6.28)–(6.29) is based on the Banach contraction principle. Set

$$K_b = \sup\{|K(t)| : t \in J\} .$$

Theorem 6.11. *Assume* (6.10.1)–(6.10.2). *If*

$$\frac{k_1 K_b b^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{k_2 K_b b^{\alpha}}{\Gamma(\alpha+1)} < 1 , \tag{6.33}$$

then IVP (6.28)–(6.29) *has a unique solution on the interval* $(-\infty, b]$.

Proof. Transform problem (6.28)–(6.29) into a fixed point problem. Consider the operator $N : \Omega \longrightarrow \Omega$ defined by

$$(Ny)(t) = \begin{cases} \phi(t), & t \in (-\infty, 0] \\ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, I^\alpha y_s, y_s) ds, & t \in J . \end{cases}$$

We use the Banach contraction principle to prove that N has a fixed point.
Let $x(.) : (-\infty, b] \to \mathbb{R}$ be the function defined by

$$x(t) = \begin{cases} 0, & \text{if } t \in J \\ \phi(t), & \text{if } t \in (-\infty, 0] . \end{cases}$$

Then $x_0 = \phi$. For each $z \in L^1(J, \mathbb{R})$, with $z(0) = 0$, we denote by $\overline{z}$ the function defined by

$$\overline{z}(t) = \begin{cases} z(t), & \text{if } t \in J \\ 0, & \text{if } t \in (-\infty, 0] . \end{cases}$$

If $y(.)$ satisfies the integral equation

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, I^\alpha y_s, y_s) ds ,$$

we can decompose $y(.)$ as $y(t) = \overline{z}(t) + x(t)$, $0 \le t \le b$, then $y_t = \overline{z}_t + x_t$, for every $0 \le t \le b$, and the function $z(.)$ satisfies

$$z(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, I^\alpha(\overline{z}_s + x_s), \overline{z}_s + x_s) ds .$$

Set

$$L_0 = \{z \in L^1(J, \mathbb{R}) : z_0 = 0\} ,$$

and let $\|\cdot\|_b$ be the seminorm in L_0 defined by

$$\|z\|_b = \|z_0\|_{\mathcal{B}} + \int_0^b |z(t)| dt = \int_0^b |z(t)| dt , \quad z \in L_0 .$$

Then L_0 is a Banach space with norm $\|\cdot\|_b$. Let the operator $P\colon L_0 \to L_0$ be defined by

$$(Pz)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, I^\alpha(\overline{z}_s + x_s), \overline{z}_s + x_s)ds\,, \quad t \in J\,. \tag{6.34}$$

The operator N having a fixed point is equivalent to P having a fixed point, and so we turn to proving that P has a fixed point. We will show that $P\colon L_0 \to L_0$ is a contraction map. Indeed, consider $z, z^* \in L_0$. Then for each $t \in J$ we have

$$\begin{aligned}
&|(Pz)(t) - (Pz^*)(t)| \\
&\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, I^\alpha(\overline{z}_s + x_s), \overline{z}_s + x_s) - f(s, I^\alpha(\overline{z}^*_s + x_s), \overline{z}^*_s + x_s)|ds \\
&\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [k_1\|I^\alpha(\overline{z}_s - \overline{z}^*_s)\|_{\mathcal{B}} + k_2\|\overline{z}_s - \overline{z}^*_s\|_{\mathcal{B}}]ds \\
&\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} K_b\left[k_1\|I^\alpha(z(s) - z^*(s))\| + k_2\|z(s) - z^*(s)\|\right] ds \\
&\le \left(\frac{k_1 K_b b^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{k_2 b^\alpha}{\Gamma(\alpha+1)}\right)\|z - z^*\|_b\,.
\end{aligned}$$

Therefore,

$$\|P(z) - P(z^*)\|_b \le \left(\frac{k_1 K_b b^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{k_2 K_b b^\alpha}{\Gamma(\alpha+1)}\right)\|z - z^*\|_b\,.$$

Consequently, by (6.33), P is a contraction. The Banach contraction principle implies that P has a unique fixed point that is the unique solution of problem (6.28)–(6.29). □

The next result is based on Schauder's fixed point theorem.

Theorem 6.12. *Assume that* (6.10.1) *holds and*

(6.12.1) *There exist a positive function $a \in L^1(J)$ and constants, $q_i > 0$, $i = 1, 2$, such that*

$$|f(t, u_1, u_2)| \le |a(t)| + q_1\|u_1\|_{\mathcal{B}} + q_2\|u_2\|_{\mathcal{B}}\,, \quad \forall (t, u_1, u_2) \in J \times \mathbb{R}^2\,.$$

If

$$K_b\left(\frac{q_1 b^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{q_2 b^\alpha}{\Gamma(\alpha+1)}\right) < 1\,, \tag{6.35}$$

then IVP (6.28)–(6.29) *has at least one solution $y \in L^1(J, \mathbb{R})$.*

Proof. Let $P\colon L_0 \to L_0$ be defined as in (6.34), and

$$r = \frac{\frac{b^\alpha\|a\|_{L^1}}{\Gamma(\alpha+1)} + M_b\|\phi\|_{\mathcal{B}}\left(\frac{q_1 b^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{q_2 b^\alpha}{\Gamma(\alpha+1)}\right)}{1 - K_b\left(\frac{q_1 b^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{q_2 b^\alpha}{\Gamma(\alpha+1)}\right)}\,,$$

where $M_b = \sup\{|M(t)|: t \in J\}$, and consider the set

$$B_r := \{z \in L_0, \|z\|_b \leq r\} .$$

Clearly B_r is nonempty, bounded, convex, and closed. We will show that operator P satisfies the conditions of Schauder's fixed point theorem. The proof will be given in three steps.

Step 1: *P is continuous.* Let z_n be a sequence such that $z_n \to z$ in L_0. Then

$$\begin{aligned}|(Pz_n)(t) - (Pz)(t)| \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, I^\alpha(\overline{z}_{n_s} + x_s), \overline{z}_{n_s} + x_s) \\ - f(s, I^\alpha(\overline{z}_s + x_s), \overline{z}_s + x_s)|ds .\end{aligned}$$

Since f is a continuous function, we have

$$\begin{aligned}&\|P(z_n) - P(z)\|_b \\ &\leq \frac{b^\alpha}{\Gamma(\alpha+1)} \|f(., I^\alpha(\overline{z}_{n_{(.)}} + x_{(.)}, \overline{z}_{n_{(.)}}) + x_{(.)}) - f(., I^\alpha(\overline{z}_{(.)} + x_{(.)}), \overline{z}_{(.)} + x_{(.)})\|_{L_1} \to 0 \\ &\text{as } n \to \infty .\end{aligned}$$

Step 2: *P maps B_r to itself.* Let $z \in B_r$. Since f is a continuous function, we have for each $t \in [0, b]$

$$\begin{aligned}|(Pz)(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, I^\alpha(\overline{z}_s + x_s), \overline{z}_s + x_s)|ds \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [a(t)| + q_1 \|I^\alpha(\overline{z}_s + x_s)\|_{\mathcal{B}} + q_2 \|\overline{z}_s + x_s\|_{\mathcal{B}}]ds \\ &\leq \frac{b^\alpha \|a\|_{L_1}}{\Gamma(\alpha+1)} + \left(\frac{q_1 b^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{q_2 b^\alpha}{\Gamma(\alpha+1)}\right)(K_b r + M_b \|\phi\|_{\mathcal{B}}) ,\end{aligned}$$

where

$$\|\overline{z}_s + x_s\|_{\mathcal{B}} \leq \|\overline{z}_s\|_{\mathcal{B}} + \|x_s\|_{\mathcal{B}} .$$

Hence, $\|P(z)\|_{L_1} \leq r$. Then $P(B_r) \subset B_r$.

Step 3: *P is compact.* We will show that $P(B_r)$ is relatively compact. Clearly $P(B_r)$ is bounded in L_0, i.e., condition (i) of the Kolmogorov compactness criterion is satisfied. It remains to show $(Pz)_h \longrightarrow P(z)$, in L_0 for each $z \in B_r$.

Let $z \in B_r$; then we have

$$\begin{aligned}
&\|P(z)_h - P(z)\|_{L^1}\\
&= \int_0^b |(Pz)_h(t) - (Pz)(t)|dt\\
&= \int_0^b \left| \frac{1}{h} \int_t^{t+h} (Pz)(s)ds - (Pz)(t) \right| dt\\
&\le \int_0^b \left(\frac{1}{h} \int_t^{t+h} |(Pz)(s) - (Pz)(t)|ds \right) dt\\
&\le \int_0^b \frac{1}{h} \int_t^{t+h} |I^\alpha f\left(s, \overline{z}_s + x_s), \overline{z}_s + x_s\right) - I^\alpha f\left(t, I^\alpha(\overline{z}_t + x_t), \overline{z}_t + x_t\right)|dsdt\,.
\end{aligned}$$

Since $z \in B_r \subset L_0$, condition (6.12.1) implies that $f \in L_0$. From Proposition 1.10 it follows that $I^\alpha f \in L^1(J, \mathbb{R})$; then we have

$$\frac{1}{h} \int_t^{t+h} |I^\alpha f\left(\overline{z}_s + x_s), \overline{z}_s + x_s\right) - I^\alpha f\left(t, I^\alpha(\overline{z}_t + x_t), \overline{z}_t + x_t\right)|ds \longrightarrow 0 \text{ as } h \longrightarrow 0,\ \ t \in J\,.$$

Hence,

$$(Pz)_h \longrightarrow (Pz) \text{ uniformly as } h \longrightarrow 0\,.$$

Then by the Kolmogorov compactness criterion, PB_r is relatively compact. As a consequence of Schauder's fixed point theorem, IVP (6.28)–(6.29) has at least one solution in B_r. □

6.4.3 Example

In this section we give an example to illustrate the usefulness of our main results. Let us consider the fractional initial value problem

$$^cD^\alpha y(t) = \frac{ce^{-\gamma t+t}}{(e^t + e^{-t})(1 + \|y_t\| + \|^cD^\alpha y_t\|)}\,, \quad t \in J := [0, b],\ \alpha \in (0, 1]\,, \tag{6.36}$$

$$y(t) = \phi(t)\,, \quad t \in (-\infty, 0], \tag{6.37}$$

where $c > 1$ is fixed. Let γ be a positive real constant and

$$B_\gamma = \{y \in L^1(-\infty, 0] : \lim_{\theta \to -\infty} e^{\gamma\theta} y(\theta),\ \text{exists in } \mathbb{R}\}\,.$$

The norm of B_γ is given by

$$\|y\|_\gamma = \int_{-\infty}^0 e^{\gamma\theta}|y(\theta)|d\theta\,.$$

Let $y\colon (-\infty, b] \to \mathbb{R}$ be such that $y_0 \in B_\gamma$. Then

$$\begin{aligned}\lim_{\theta\to-\infty} e^{\gamma\theta} y_t(\theta) &= \lim_{\theta\to-\infty} e^{\gamma\theta} y(t+\theta)\\ &= \lim_{\theta\to-\infty} e^{\gamma(\theta-t)} y(\theta)\\ &= e^{\gamma t} \lim_{\theta\to-\infty} e^{\gamma\theta} y_0(\theta) < \infty\,.\end{aligned}$$

Hence, $y_t \in B_\gamma$. Finally, we prove that

$$\|y_t\|_\gamma \le K(t)\int_0^t |y(s)|ds + M(t)\|y_0\|_\gamma\,,$$

where $K = M = 1$ and $H = 1$. We have

$$|y_t(\theta)| = |y(t+\theta)|\,.$$

If $\theta + t \le 0$, we get

$$|y_t(\theta)| \le \int_{-\infty}^0 |y(s)|ds\,.$$

Then for $t + \theta \ge 0$ we have

$$|y_t(\theta)| \le \int_0^t |y(s)|ds\,.$$

Thus, for all $t + \theta \in J$ we get

$$|y_t(\theta)| \le \int_{-\infty}^0 |y(s)|ds + \int_0^t |y(s)|ds\,.$$

Then

$$\|y_t\|_\gamma \le \|y_0\|_\gamma + \int_0^t |y(s)|ds\,.$$

It is clear that $(B_\gamma, \|\cdot\|)$ is a Banach space. We can conclude that B_γ is a phase space. Set

$$f(t,y,z) = \frac{e^{-\gamma t+t}}{c(e^t+e^{-t})(1+y+z)}\,, \quad (t,x,z) \in J \times B_\gamma \times B_\gamma\,.$$

For $t \in J$, $y_1, y_2, z_1, z_2 \in B_\gamma$ we have

$$\begin{aligned}|f(t,y_1,z_1) - f(t,y_2,z_2)| &= \frac{e^{-\gamma t+t}}{c(e^t+e^{-t})}\left|\frac{1}{1+y_1+z_1} - \frac{1}{1+y_2+z_2}\right|\\ &= \frac{e^{-\gamma t+t}(|y_1-y_2| + |z_1-z_2|)}{c(e^t+e^{-t})(1+y_1+z_1)(1+y_2+z_2)}\end{aligned}$$

$$\leq \frac{e^{-\gamma t} \times e^{t}(|y_1 - y_2| + |z_1 - z_2|)}{c(e^t + e^{-t})}$$
$$\leq \frac{e^{-\gamma t}(\|y_1 - y_2\|_\gamma + \|z_1 - z_2\|_\gamma)}{c}$$
$$\leq \frac{1}{c}\|y_1 - y_2\|_\gamma + \frac{1}{c}\|z_1 - z_2\|_\gamma .$$

Hence condition (6.10.2) holds. We choose b such that $\frac{K_b b^{2\alpha}}{c\Gamma(2\alpha+1)} + \frac{K_b b^{\alpha}}{c\Gamma(\alpha+1)} < 1$. Since $K_b = 1$, then

$$\frac{b^{2\alpha}}{c\Gamma(2\alpha+1)} + \frac{b^{\alpha}}{c\Gamma(\alpha+1)} < 1 .$$

Then, by Theorem 6.11, problem (6.36)–(6.37) has a unique integrable solution on $[-\infty, b]$.

6.5 An Existence Result of Integrable Solutions for NIFDEs

6.5.1 Introduction

This section deals with the existence of integrable solutions for an IVP for the implicit fractional order differential equation

$$^{c}D^{\alpha}y(t) = f(t, y(t), {}^{c}D^{\alpha}y(t)) , \quad t \in J = [0, T] , \tag{6.38}$$
$$y(0) = y_0 , \tag{6.39}$$

where $f : J \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is a given function. We will use the technique of measures of noncompactness, which is often used in several branches of nonlinear analysis. In particular, that technique turns out to be a very useful tool in existence for several types of integral equations; details can be found in Akhmerov et al. [58] and Banaś et al. [81, 83].

The principal goal here is to prove the existence of integral solutions to problem (6.38)–(6.39) using Darbo's fixed point theorem.

6.5.2 Existence of solutions

Let us start by defining what we mean by a solution to problem (6.38)–(6.39).

Definition 6.13. A function $y \in L^1(J, \mathbb{R})$ is said to be a solution to IVP (6.38)–(6.39) if y satisfies the equation $^{c}D^{\alpha}y(t) = f(t, y(t), {}^{c}D^{\alpha}y(t))$ on J and the condition $y(0) = y_0$.

For the existence of solutions to problem (6.38)–(6.39), we need the following auxiliary lemma.

Lemma 6.14. *The solution to IVP* (6.38)–(6.39) *can be expressed by the integral equation*

$$y(t) = y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s)ds \,, \tag{6.40}$$

where x is the solution of the functional integral equation

$$x(t) = f\left(t, y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s)ds, x(t) \right) . \tag{6.41}$$

Let us introduce the following conditions:

(6.14.1) $f : J \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfies the Carathéodory conditions.

(6.14.2) There exist a positive function $a \in L^1(J)$ and two constants, $q_1, q_2 > 0$, such that

$$|f(t, u_1, u_2)| \le |a(t)| + q_1|u_1| + q_2|u_2| \,, \quad \forall (t, u_1, u_2) \in J \times \mathbb{R} \times \mathbb{R} \,.$$

(6.14.3) We first consider two real numbers, $0 < |\rho| < \delta$. There exists a positive valued function $L_f(\cdot)$ that is continuous in a neighborhood of 0 with $L_f(0) = 0$ and two constants $k_1, k_2 > 0$ such that

$$\begin{aligned}&|f(t+\rho, x_1, y_1) - f(t, x_2, y_2)| \le L_f(\rho) + k_1|x_1 - x_2| + k_2|y_1 - y_2| \,,\\ &t \in [0, T],\ x_i, y_i \in \mathbb{R},\ i = 1, 2 \,.\end{aligned}$$

In this section, we study the existence of a solution to problem (6.38)–(6.39) by using the concept of measure of noncompactness in $L^1(J)$.

Theorem 6.15. *Assume* (6.14.1)–(6.14.3). *If*

$$\frac{k_1 T^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{k_2 T^{\alpha}}{\Gamma(\alpha+1)} < 1 \,, \tag{6.42}$$

then IVP (6.38)–(6.39) *has at least one solution $y \in L^1(J, \mathbb{R})$.*

Proof. Consider the operator $N : L^1(J, \mathbb{R}) \to L^1(J, \mathbb{R})$ defined by

$$(Nx)(t) = y_0 + I^{\alpha} x(t) \,, \tag{6.43}$$

where $x(t) = f(t, y_0 + I^{\alpha}x(t), x(t))$. Clearly, the fixed points of the operator N are solutions to problem (6.38)–(6.39). Let

$$r = \frac{T|y_0| + \left(\frac{T^{\alpha}\|a\|_{L^1} + q_1|y_0|T^{\alpha+1}}{\Gamma(\alpha+1)} \right)}{1 - \left(\frac{q_1 T^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{q_2 T^{\alpha}}{\Gamma(\alpha+1)} \right)} \,,$$

where

$$\frac{q_1 T^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{q_2 T^{\alpha}}{\Gamma(\alpha+1)} < 1 \,,$$

and consider the set

$$B_r = \{x \in L^1(J, \mathbb{R}) : \|x\|_{L^1} \le r,\ r > 0\} .$$

Clearly, the subset B_r is closed, bounded, and convex. We will show that N satisfies the assumptions of Darbo's fixed point theorem. The proof will be given in three steps.

Step 1. N is continuous. Let x_n be a sequence such that $x_n \to x$ in B_r. Then for each $t \in J$,

$$\begin{aligned}
&\|N(x_n) - N(x)\|_{L^1} \\
&= \|I^\alpha x_n(t) - I^\alpha x(t)\|_{L^1} \\
&= \left\| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (x_n(s) - x(s))\, ds \right\|_{L^1} \\
&\le \int_0^T \left(\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, y_0 + I^\alpha x_n(s), x(s)) - f(s, y_0 + I^\alpha x(s), x(s))| ds \right) dt .
\end{aligned}$$

Since f is of Carathéodory type, then by the Lebesgue dominated convergence theorem, we have

$$\|N(x_n) - N(x)\|_{L^1} \to 0 \ \text{ as } \ n \to \infty .$$

Step 2. N maps B_r to itself. Let x be an arbitrary element in B_r. Then from (6.14.1)–(6.14.2) we obtain

$$\begin{aligned}
\|Nx\|_{L^1} &= \int_0^T |Nx(t)| dt \\
&= \int_0^T |y_0 + I^\alpha x(t)| dt \\
&\le T|y_0| + \int_0^T \left(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |x(s)| ds \right) dt \\
&\le T|y_0| + \int_0^T \left(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f\left(s, y_0 + I^\alpha x(s), x(s)\right)| ds \right) dt \\
&\le T|y_0| + \int_0^T \left(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |a(s) + q_1 \left(y_0 + I^\alpha x(s)\right) + q_2(x(s))| ds \right) dt \\
&\le T|y_0| + \frac{T^\alpha}{\Gamma(\alpha+1)} \|a\|_{L^1} + \frac{q_1 |y_0| T^{\alpha+1}}{\Gamma(\alpha+1)} + \frac{q_2 T^\alpha}{\Gamma(\alpha+1)} \|x\|_{L^1} \\
&\quad + q_1 \int_0^T \left(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} I^\alpha |x(s)| ds \right) dt \\
&\le T|y_0| + \frac{T^\alpha}{\Gamma(\alpha+1)} \|a\|_{L^1} + \frac{q_1 |y_0| T^{\alpha+1}}{\Gamma(\alpha+1)} + \frac{q_2 T^\alpha}{\Gamma(\alpha+1)} \|x\|_{L^1} + \frac{q_1 T^{2\alpha}}{\Gamma(2\alpha+1)} \|x\|_{L^1} \\
&\le T|y_0| + \frac{T^\alpha \|a\|_{L^1} + q_1 |y_0| T^{\alpha+1}}{\Gamma(\alpha+1)} + \frac{q_1 T^{2\alpha} r}{\Gamma(2\alpha+1)} + \frac{q_2 T^\alpha r}{\Gamma(\alpha+1)} \le r .
\end{aligned}$$

Then $\|Nx\|_{L^1} \le r$. Thus, the operator N maps B_r to itself.

Step 3. N is a contraction, i.e., $\mu(NX) \le k\mu(X)$, $k \in [0,1)$. Now let us fix a nonempty subset X of B_r. We first consider two real numbers, $0 < |\rho| < \delta$, and an arbitrary fixed $x \in X$. By (6.14.3) we have

$$
\begin{aligned}
&\|Nx(t+\rho) - Nx(t)\|_{L^1}\\
&= \int_0^T |(Nx)(t+\rho) - (Nx)(t)|dt\\
&= \int_0^T |I^\alpha x(t+\rho) - I^\alpha x(t)|dt\\
&= \int_0^T |I^\alpha (x(t+\rho) - x(t))|dt\\
&= \int_0^T \left|I^\alpha \left(f(t+\rho, y_0 + I^\alpha x(t+\rho), x(t+\rho)) - f(t, y_0 + I^\alpha x(t), x(t))\right)\right| dt\\
&\le \int_0^T \left(\left|I^\alpha L_f(\rho)\right| + k_1 \left|I^{2\alpha}(x(t+\rho) - x(t))\right| + k_2 \left|I^\alpha(x(t+\rho) - x(t))\right|\right) dt\\
&\le \frac{T^\alpha}{\Gamma(\alpha+1)} \int_0^T |L_f(\rho)|dt + \frac{k_1 T^{2\alpha}}{\Gamma(2\alpha+1)} \int_0^T |x(t+\rho) - x(t)|dt\\
&\quad + \frac{k_2 T^\alpha}{\Gamma(\alpha+1)} \int_0^T |x(t+\rho) - x(t)|dt\,.
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
\|Nx(.+\rho) - Nx(.)\|_{L^1} \le\ & \frac{T^{\alpha+1}}{\Gamma(\alpha+1)} L_f(\rho)\\
& + \left(\frac{k_1 T^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{k_2 T^\alpha}{\Gamma(\alpha+1)}\right) \|x(.+\rho) - x(.)\|_{L^1}\,.
\end{aligned}
$$

Taking into account that

$$\lim_{\delta\to 0} \sup_{|\rho|\le\delta} L_f(\rho) = 0\,,$$

we get

$$\mu(NX) \le \left(\frac{k_1 T^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{k_2 T^\alpha}{\Gamma(\alpha+1)}\right)\mu(X)\,.$$

Here, $\mu(.)$ is the measure of noncompactness in $L^1[0,T]$ by (4.3). This means that the operator N is a contraction with respect to μ. Since

$$\frac{k_1 T^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{k_2 T^\alpha}{\Gamma(\alpha+1)} < 1\,,$$

by applying Darbo's fixed point theorem, we conclude that IVP (6.38)–(6.39) has at least one solution belonging to the set $B_r \subset L^1(J, \mathbb{R})$. □

6.5.3 Example

In this section we present an example to illustrate the usefulness of our main results. Let us consider the fractional initial value problem

$$^cD^\alpha y(t) = \frac{t(1 + |y(t)| + |^cD^\alpha y(t)|)}{(t+5)}, \quad t \in J := [0,1],\ \alpha \in (0,1], \tag{6.44}$$

$$y(0) = y_0 . \tag{6.45}$$

Set

$$f(t, y, z) = \frac{t(1 + y + z)}{(t+5)}, \quad (t, y, z) \in J \times \mathbb{R}_+ \times \mathbb{R}_+ .$$

Clearly, the function f satisfies the Carathéodory conditions. For $y, z \in \mathbb{R}_+$ and $t \in J$, we have

$$\begin{aligned} |f(t,y,z)| &= \left| \frac{t}{t+5} + \frac{ty}{t+5} + \frac{tz}{t+5} \right| \\ &\le \left| \frac{t}{t+5} \right| + \left| \frac{ty}{t+5} \right| + \left| \frac{tz}{t+5} \right| \\ &\le \left| \frac{t}{t+5} \right| + \frac{1}{6}|y| + \frac{1}{6}|z| . \end{aligned}$$

We first show that $a \in L^1[0,1]$, where $a(t) = \frac{t}{t+5}$; indeed, $a(t)$ is a measurable function and

$$\begin{aligned} \int_0^1 a(t)dt &= \int_0^1 \frac{t}{t+5} dt \\ &= [t - 5 \ln|t+5|)]_0^1 \\ &= 1 - 5 \ln 6 + 5 \ln 5 < \infty . \end{aligned}$$

Then $a \in L^1[0,1]$. Hence, (6.14.2) holds with $a(t) = \frac{t}{t+5}$ and $q_1 = q_2 = \frac{1}{6}$.

Moreover, for each $t \in [0,1]$, $x_i, y_i \in \mathbb{R}_+$, $i = 1, 2$, we have

$$\begin{aligned} &|f(t+\rho, y_1, z_1) - f(t, y_2, z_2)| \\ &= \left| \frac{t+\rho}{t+\rho+5} - \frac{t}{t+5} + \frac{(t+\rho)y_1}{t+\rho+5} - \frac{ty_2}{t+5} + \frac{(t+\rho)z_1}{t+\rho+5} - \frac{tz_2}{t+5} \right| \\ &\le \frac{\rho}{\rho+5} + \left| \frac{(t+\rho)y_1}{t+\rho+5} - \frac{ty_2}{t+5} \right| + \left| \frac{(t+\rho)z_1}{t+\rho+5} - \frac{tz_2}{t+5} \right| \\ &\le L_f(\rho) + \left| \frac{(t+\rho)y_1}{t+\rho+5} - \frac{ty_2}{t+5} \right| + \left| \frac{(t+\rho)z_1}{t+\rho+5} - \frac{tz_2}{t+5} \right| , \end{aligned}$$

where $L_f(\rho) = \frac{\rho}{\rho+5}$.

As $\delta \to 0$, we have

$$|f(t+\rho, y_1, z_1) - f(t, y_2, z_2)| \leq L_f(\rho) + \frac{1}{6}|y_1 - y_2| + \frac{1}{6}|z_1 - z_2| .$$

Then (6.14.3) holds with $L_f(\rho) = \frac{\rho}{\rho+5}$ and $k_1 = k_2 = \frac{1}{6}$. Condition (6.42) is satisfied for appropriate values of $\alpha \in (0, 1]$ with $T = 1$. Indeed,

$$\frac{k_1}{\Gamma(2\alpha+1)} + \frac{k_2}{\Gamma(\alpha+1)} < 1 \Leftrightarrow \frac{1}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(2\alpha+1)} < 6 . \tag{6.46}$$

Then, by Theorem 6.15, problem (6.44)–(6.45) has at least one solution on $[0, 1]$ for values of α satisfying condition (6.46). For example,
If $\alpha = \frac{1}{2}$, then $\Gamma(\alpha+1) = \Gamma(\frac{3}{2}) \simeq 0.88$ and $\Gamma(2\alpha+1) = \Gamma(2) = 1$ and

$$\frac{k_1}{\Gamma(2\alpha+1)} + \frac{k_2}{\Gamma(\alpha+1)} = \frac{1}{6} + \frac{\frac{1}{6}}{0.88} \simeq 0.35659 < 1 .$$

6.6 Notes and Remarks

The results of Chapter 6 are taken from Benchohra et al. [107, 108]. Other results may be found in [179, 181].