5 Boundary Value Problems for Impulsive NIFDE

5.1 Introduction and Motivations

The theory of impulsive differential equations of integer order has found extensive application in realistic mathematical modeling of a wide variety of practical situations and has emerged as an important area of investigation in recent years. See [76, 77, 100, 148, 186, 215, 240], as well as [124, 157, 158, 251], and references therein.

Very recently, antiperiodic boundary value problems (BVPs) of fractional differential equations have received considerable attention because they occur in the mathematical modeling of a variety of physical processes. See for example [45, 53, 54, 52, 55, 88, 98, 125, 248, 249, 250, 259].

In this chapter, we establish existence, uniqueness, and stability results for some classes of BVPs for impulsive nonlinear implicit fractional differential equations (NIFDEs). Next, we present other results of existence and uniqueness for BVPs for NIFDEs with impulses in Banach spaces.

5.2 Existence Results for Impulsive NIFDEs

5.2.1 Introduction

In this section, we establish existence, uniqueness, and stability results of solutions for the BVP for NIFDEs with impulse and Caputo fractional derivatives:

$$ {}^cD^{\alpha}_{t_k} y(t) = f(t, y, {}^cD^{\alpha}_{t_k} y(t)) , \quad \text{for each, } t \in (t_k, t_{k+1}],\ k = 0, \dots, m,\ 0 < \alpha \le 1 , \tag{5.1} $$

$$ \Delta y|_{t=t_k} = I_k(y(t_k^-)) , \quad k = 1, \dots, m , \tag{5.2} $$

$$ ay(0) + by(T) = c , \tag{5.3} $$

where ${}^cD^{\alpha}_{t_k}$ is the Caputo fractional derivative, $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function, $I_k : \mathbb{R} \to \mathbb{R}$, and a, b, c are real constants, with $a+b \neq 0$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, and $y(t_k^+) = \lim_{h \to 0^+} y(t_k + h)$ and $y(t_k^-) = \lim_{h \to 0^-} y(t_k + h)$ represent the right and left limits of $y(t)$ at $t = t_k$, respectively.

The arguments are based upon the Banach contraction principle and Schaefer's fixed point theorem.

5.2.2 Existence of Solutions

Consider the Banach space

$$ PC(J, \mathbb{R}) = \{y : J \to \mathbb{R} : y \in C((t_k, t_{k+1}], \mathbb{R}),\ k = 0, \dots, m \\ \text{and there exist } y(t_k^-) \text{ and } y(t_k^+),\ k = 1, \dots, m \text{ with } y(t_k^-) = y(t_k)\} , $$

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with the norm

$$\|y\|_{PC} = \sup_{t \in J} |y(t)| .$$

Definition 5.1. A function $y \in PC(J, \mathbb{R})$ whose α-derivative exists on J_k is said to be a solution of (5.1)–(5.3) if y satisfies the equation ${}^cD^{\alpha}_{t_k} y(t) = f(t, y(t), {}^cD^{\alpha}_{t_k} y(t))$ on J_k and satisfies the conditions

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)) , \quad k = 1, \dots, m ,$$
$$ay(0) + by(T) = c .$$

To prove the existence of solutions to (5.1)–(5.3), we need the following auxiliary lemma.

Lemma 5.2. *Let $0 < \alpha \le 1$, and let $\sigma \colon J \to \mathbb{R}$ be continuous. A function y is a solution of the fractional integral equation*

$$y(t) = \begin{cases} \dfrac{-1}{a+b}\left[b\sum_{i=1}^{m} I_i(y(t_i^-)) + \dfrac{b}{\Gamma(\alpha)} \sum_{i=1}^{m} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1}\sigma(s)ds \right. \\ \left. + \dfrac{b}{\Gamma(\alpha)} \int_{t_m}^{T} (T-s)^{\alpha-1}\sigma(s)ds - c \right] + \dfrac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\sigma(s)ds \quad \text{if } t \in [0, t_1] \\ \dfrac{-1}{a+b}\left[b\sum_{i=1}^{m} I_i(y(t_i^-)) + \dfrac{b}{\Gamma(\alpha)} \sum_{i=1}^{m} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1}\sigma(s)ds \right. \\ \left. + \dfrac{b}{\Gamma(\alpha)} \int_{t_m}^{T} (T-s)^{\alpha-1}\sigma(s)ds - c \right] + \sum_{i=1}^{k} I_i(y(t_i^-)) + \dfrac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1}\sigma(s)ds \\ + \dfrac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1}\sigma(s)ds \quad \text{if } t \in (t_k, t_{k+1}] , \end{cases} \tag{5.4}$$

where $k = 1, \dots, m$ if and only if y is a solution of the fractional BVP

$${}^cD^{\alpha} y(t) = \sigma(t) , \quad t \in J_k , \tag{5.5}$$
$$\Delta y|_{t=t_k} = I_k(y(t_k^-)) , \quad k = 1, \dots, m , \tag{5.6}$$
$$ay(0) + by(T) = c . \tag{5.7}$$

Proof. Assume that y satisfies (5.5)–(5.7). If $t \in [0, t_1]$, then

$${}^cD^{\alpha} y(t) = \sigma(t) .$$

Lemma 1.9 implies

$$y(t) = c_0 + I^{\alpha}\sigma(t) = c_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\sigma(s)ds$$

for $c_0 \in \mathbb{R}$. If $t \in (t_1, t_2]$, then Lemma 1.9 implies

$$\begin{aligned}
y(t) &= y(t_1^+) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t} (t-s)^{\alpha-1} \sigma(s) ds \\
&= \Delta y|_{t=t_1} + y(t_1^-) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t} (t-s)^{\alpha-1} \sigma(s) ds \\
&= I_1(y(t_1^-)) + \left[c_0 + \frac{1}{\Gamma(\alpha)} \int_{0}^{t_1} (t_1-s)^{\alpha-1} \sigma(s) ds \right] \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t} (t-s)^{\alpha-1} \sigma(s) ds . \\
&= c_0 + I_1(y(t_1^-)) + \frac{1}{\Gamma(\alpha)} \int_{0}^{t_1} (t_1-s)^{\alpha-1} \sigma(s) ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t} (t-s)^{\alpha-1} \sigma(s) ds .
\end{aligned}$$

If $t \in (t_2, t_3]$, then from Lemma 1.9 we get

$$\begin{aligned}
y(t) &= y(t_2^+) + \frac{1}{\Gamma(\alpha)} \int_{t_2}^{t} (t-s)^{\alpha-1} \sigma(s) ds \\
&= \Delta y|_{t=t_2} + y(t_2^-) + \frac{1}{\Gamma(\alpha)} \int_{t_2}^{t} (t-s)^{\alpha-1} \sigma(s) ds \\
&= I_2(y(t_2^-)) + \left[c_0 + I_1(y(t_1^-)) + \frac{1}{\Gamma(\alpha)} \int_{0}^{t_1} (t_1-s)^{\alpha-1} \sigma(s) ds \right. \\
&\quad \left. + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} \sigma(s) ds \right] + \frac{1}{\Gamma(\alpha)} \int_{t_2}^{t} (t-s)^{\alpha-1} \sigma(s) ds . \\
&= c_0 + [I_1(y(t_1^-)) + I_2(y(t_2^-))] + \left[\frac{1}{\Gamma(\alpha)} \int_{0}^{t_1} (t_1-s)^{\alpha-1} \sigma(s) ds \right. \\
&\quad \left. + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} \sigma(s) ds \right] + \frac{1}{\Gamma(\alpha)} \int_{t_2}^{t} (t-s)^{\alpha-1} \sigma(s) ds .
\end{aligned}$$

Repeating the process in this way, the solution $y(t)$ for $t \in (t_k, t_{k+1}]$, where $k = 1, \dots, m$, can be written

$$y(t) = c_0 + \sum_{i=1}^{k} I_i(y(t_i^-)) + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1}\sigma(s)ds + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1}\sigma(s)ds\,.$$

Applying the boundary conditions $ay(0) + by(T) = c$, we get

$$c = c_0(a+b) + b\sum_{i=1}^{m} I_i(y(t_i^-)) + \frac{b}{\Gamma(\alpha)} \sum_{i=1}^{m} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1}\sigma(s)ds + \frac{b}{\Gamma(\alpha)} \int_{t_m}^{T} (T-s)^{\alpha-1}\sigma(s)ds\,.$$

Then

$$c_0 = \frac{-1}{a+b}\left[b\sum_{i=1}^{m} I_i(y(t_i^-)) + \frac{b}{\Gamma(\alpha)} \sum_{i=1}^{m} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1}\sigma(s)ds + \frac{b}{\Gamma(\alpha)} \int_{t_m}^{T} (T-s)^{\alpha-1}\sigma(s)ds - c \right].$$

Thus, if $t \in (t_k, t_{k+1}]$, where $k = 1, \dots, m$, then

$$y(t) = \frac{-1}{a+b}\left[b\sum_{i=1}^{m} I_i(y(t_i^-)) + \frac{b}{\Gamma(\alpha)} \sum_{i=1}^{m} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1}\sigma(s)ds + \frac{b}{\Gamma(\alpha)} \int_{t_m}^{T} (T-s)^{\alpha-1}\sigma(s)ds - c \right] + \sum_{i=1}^{k} I_i(y(t_i^-)) + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1}\sigma(s)ds + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1}\sigma(s)ds\,.$$

Conversely, assume that y satisfies impulsive fractional integral equation (5.4). If $t \in [0, t_1]$, then $ay(0) + by(T) = c$. Using the fact that ${}^cD^\alpha$ is the left inverse of I^α, we get

$${}^cD^\alpha y(t) = \sigma(t) \quad \text{for each} \quad t \in [0, t_1]\,.$$

If $t \in (t_k, t_{k+1}]$, $k = 1, \dots, m$, and using the fact that ${}^cD^\alpha C = 0$, where C is a constant, we get

$${}^cD^\alpha y(t) = \sigma(t)\,, \qquad \text{for each} \quad t \in (t_k, t_{k+1}]\,.$$

Also, we can easily show that

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)) , \quad k = 1, \ldots, m . \qquad \square$$

We are now in a position to state and prove our existence result for problem (5.1)–(5.3) based on Banach's fixed point.

Theorem 5.3. *Make the following assumptions:*
(5.3.1) *The function* $f: J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is continuous.*
(5.3.2) *There exist constants* $K > 0$ *and* $0 < L < 1$ *such that*

$$|f(t, u, v) - f(t, \bar{u}, \bar{v})| \le K|u - \bar{u}| + L|v - \bar{v}|$$

for any $u, v, \bar{u}, \bar{v} \in \mathbb{R}$ *and* $t \in J$.
(5.3.3) *There exists a constant* $\widetilde{l} > 0$ *such that*

$$|I_k(u) - I_k(\overline{u})| \le \widetilde{l}|u - \overline{u}|$$

for each $u, \overline{u} \in \mathbb{R}$ *and* $k = 1, \ldots, m$.
If

$$\left(\frac{|b|}{|a+b|} + 1\right)\left[m\widetilde{l} + \frac{(m+1)KT^\alpha}{(1-L)\Gamma(\alpha+1)}\right] < 1 , \tag{5.8}$$

then there exists a unique solution for BVP (5.1)–(5.3) *on* J.

Proof. Transform problem (5.1)–(5.3) into a fixed point problem. Consider the operator $N: PC(J, \mathbb{R}) \to PC(J, \mathbb{R})$ defined by

$$\begin{aligned} N(y)(t) = \frac{-1}{a+b}&\left[b\sum_{i=1}^{m} I_i(y(t_i^-)) + \frac{b}{\Gamma(\alpha)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}(t_i - s)^{\alpha-1}g(s)ds\right. \\ &\left.+ \frac{b}{\Gamma(\alpha)}\int_{t_m}^{T}(T-s)^{\alpha-1}g(s)ds - c\right] + \frac{1}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k - s)^{\alpha-1}g(s)ds \\ &+ \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}g(s)ds + \sum_{0<t_k<t} I_k(y(t_k^-)) , \end{aligned} \tag{5.9}$$

where $g \in C(J, \mathbb{R})$ is such that

$$g(t) = f(t, y(t), g(t)) .$$

Clearly, the fixed points of operator N are solutions of problem (5.1)–(5.3).

Let $u, w \in PC(J, \mathbb{R})$. Then for $t \in J$ we have

$$
\begin{aligned}
|N(u)(t) - N(w)(t)| \leq \frac{|b|}{|a+b|} & \left[\sum_{i=1}^{m} |I_i(u(t_i^-)) - I_i(w(t_i^-))| \right. \\
& + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{m} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} |g(s) - h(s)| ds \\
& \left. + \frac{1}{\Gamma(\alpha)} \int_{t_m}^{T} (T - s)^{\alpha-1} |g(s) - h(s)| ds \right] \\
& + \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} |g(s) - h(s)| ds \\
& + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t - s)^{\alpha-1} |g(s) - h(s)| ds \\
& + \sum_{0<t_k<t} |I_k(u(t_k^-)) - I_k(w(t_k^-))| ,
\end{aligned}
$$

where $g, h \in C(J, \mathbb{R})$ is such that

$$
g(t) = f(t, u(t), g(t)) ,
$$

and

$$
h(t) = f(t, w(t), h(t)) .
$$

By (5.3.2) we have

$$
\begin{aligned}
|g(t) - h(t)| &= |f(t, u(t), g(t)) - f(t, w(t), h(t))| \\
&\leq K|u(t) - w(t)| + L|g(t) - h(t)| .
\end{aligned}
$$

Then

$$
|g(t) - h(t)| \leq \frac{K}{1-L} |u(t) - w(t)| .
$$

Therefore, for each $t \in J$

$$
\begin{aligned}
|N(u)(t) - N(w)(t)| \leq \frac{|b|}{|a+b|} & \left[\sum_{k=1}^{m} \tilde{l} |u(t_k^-) - w(t_k^-)| \right. \\
& + \frac{K}{(1-L)\Gamma(\alpha)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} |u(s) - w(s)| ds \\
& \left. + \frac{K}{(1-L)\Gamma(\alpha)} \int_{t_m}^{T} (T - s)^{\alpha-1} |u(s) - w(s)| ds \right]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{K}{(1-L)\Gamma(\alpha)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}|u(s)-w(s)|ds \\
&+\frac{K}{(1-L)\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}|u(s)-w(s)|ds \\
&+\sum_{k=1}^{m}\widetilde{l}|u(t_k^-)-w(t_k^-)| \, . \\
\le &\left(\frac{|b|}{|a+b|}+1\right)\left[m\widetilde{l}+\frac{mKT^\alpha}{(1-L)\Gamma(\alpha+1)}\right. \\
&\left.+\frac{KT^\alpha}{(1-L)\Gamma(\alpha+1)}\right]\|u-w\|_{PC} \, .
\end{aligned}
$$

Thus,

$$
\|N(u)-N(w)\|_{PC}\le\left(\frac{|b|}{|a+b|}+1\right)\left[m\widetilde{l}+\frac{(m+1)KT^\alpha}{(1-L)\Gamma(\alpha+1)}\right]\|u-w\|_{PC} \, .
$$

By (5.8), operator N is a contraction. Hence, by Banach's contraction principle, N has a unique fixed point that is the unique solution of problem (5.1)–(5.3). □

Our second result is based on Schaefer's fixed point theorem.

Theorem 5.4. *Assume* (5.3.1) *and* (5.3.2) *hold and*
(5.4.1) *There exist* $p, q, r \in C(J, \mathbb{R}_+)$ *with* $r^* = \sup_{t\in J} r(t) < 1$ *such that*

$$
|f(t,u,w)|\le p(t)+q(t)|u|+r(t)|w| \quad \text{for } t\in J \text{ and } u,w\in\mathbb{R}\, ;
$$

(5.4.2) *The functions* $I_k:\mathbb{R}\to\mathbb{R}$ *are continuous, and there exist constants* M^*, $N^*>0$ *such that*

$$
|I_k(u)|\le M^*|u|+N^* \quad \text{for each } u\in\mathbb{R},\ k=1,\dots,m\, .
$$

If

$$
\left(\frac{|b|}{|a+b|}+1\right)\left(mM^*+\frac{(m+1)q^*T^\alpha}{(1-r^*)\Gamma(\alpha+1)}\right)<1\, , \tag{5.10}
$$

then BVP (5.1)–(5.3) *has at least one solution on* J.

Proof. Consider operator N defined in (5.9). We will use Schaefer's fixed point theorem to prove that N has a fixed point. The proof will be given in several steps.

Step 1: N is continuous. Let $\{u_n\}$ be a sequence such that $u_n \to u$ in $PC(J, \mathbb{R})$. Then for each $t \in J$

$$
\begin{aligned}
|N(u_n)(t) - N(u)(t)| \le \frac{|b|}{|a+b|}\Bigg[& \sum_{i=1}^{m} |I_k(u_n(t_k^-)) - I_k(u(t_k^-))| \\
& + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{m} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} |g_n(s) - g(s)| ds \\
& + \frac{1}{\Gamma(\alpha)} \int_{t_m}^{T} (T-s)^{\alpha-1} |g_n(s) - g(s)| ds \Bigg] \\
& + \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} |g_n(s) - g(s)| ds \\
& + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} |g_n(s) - g(s)| ds \\
& + \sum_{0<t_k<t} |I_k(u_n(t_k^-)) - I_k(u(t_k^-))| \,,
\end{aligned} \tag{5.11}
$$

where $g_n, g \in C(J, \mathbb{R})$ such that

$$g_n(t) = f(t, u_n(t), g_n(t))$$

and

$$g(t) = f(t, u(t), g(t)) \,.$$

By (5.3.2) we have

$$
\begin{aligned}
|g_n(t) - g(t)| &= |f(t, u_n(t), g_n(t)) - f(t, u(t), g(t))| \\
&\le K|u_n(t) - u(t)| + L|g_n(t) - g(t)| \,.
\end{aligned}
$$

Then

$$|g_n(t) - g(t)| \le \frac{K}{1-L} |u_n(t) - u(t)| \,.$$

Since $u_n \to u$, we get $g_n(t) \to g(t)$ as $n \to \infty$ for each $t \in J$. Let $\eta > 0$ be such that, for each $t \in J$, we have $|g_n(t)| \le \eta$ and $|g(t)| \le \eta$. Then we have

$$
\begin{aligned}
(t-s)^{\alpha-1} |g_n(s) - g(s)| &\le (t-s)^{\alpha-1} [|g_n(s)| + |g(s)|] \\
&\le 2\eta (t-s)^{\alpha-1}
\end{aligned}
$$

and

$$
\begin{aligned}
(t_k-s)^{\alpha-1} |g_n(s) - g(s)| &\le (t_k-s)^{\alpha-1} [|g_n(s)| + |g(s)|] \\
&\le 2\eta (t_k-s)^{\alpha-1} \,.
\end{aligned}
$$

For each $t \in J$ the functions $s \to 2\eta(t-s)^{\alpha-1}$ and $s \to 2\eta(t_k - s)^{\alpha-1}$ are integrable on $[0, t]$, then the Lebesgue dominated convergence theorem and (5.11) imply that

$$|N(u_n)(t) - N(u)(t)| \to 0 \ \text{ as } \ n \to \infty .$$

Hence

$$\|N(u_n) - N(u)\|_{PC} \to 0 \ \text{ as } \ n \to \infty .$$

Consequently, N is continuous.

Step 2: N maps bounded sets to bounded sets in $PC(J, \mathbb{R})$. Indeed, it is enough to show that for any $\eta^* > 0$ there exists a positive constant ℓ such that for each $u \in B_{\eta^*} = \{u \in PC(J, \mathbb{R}) : \|u\|_{PC} \le \eta^*\}$ we have $\|N(u)\|_{PC} \le \ell$. For each $t \in J$ we have

$$\begin{aligned} N(u)(t) = \frac{-1}{a+b} \Bigg[& b \sum_{i=1}^{m} I_i(u(t_i^-)) + \frac{b}{\Gamma(\alpha)} \sum_{i=1}^{m} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} g(s) ds \\ & + \frac{b}{\Gamma(\alpha)} \int_{t_m}^{T} (T-s)^{\alpha-1} g(s) ds - c \Bigg] + \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} g(s) ds \\ & + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} g(s) ds + \sum_{0<t_k<t} I_k(u(t_k^-)) , \end{aligned} \tag{5.12}$$

where $g \in C(J, \mathbb{R})$ is such that

$$g(t) = f(t, u(t), g(t)) .$$

By (5.4.1), for each $t \in J$ we have

$$\begin{aligned} |g(t)| &= |f(t, u(t), g(t))| \\ &\le p(t) + q(t)|u(t)| + r(t)|g(t)| \\ &\le p(t) + q(t)\eta^* + r(t)|g(t)| \\ &\le p^* + q^*\eta^* + r^*|g(t)| , \end{aligned}$$

where $p^* = \sup_{t\in J} p(t)$, and $q^* = \sup_{t\in J} q(t)$.

Then

$$|g(t)| \le \frac{p^* + q^*\eta^*}{1 - r^*} := M .$$

Thus, (5.12) implies

$$\begin{aligned} |N(u)(t)| \le & \frac{|b|}{|a+b|} \left[m(M^*|u| + N^*) + \frac{mMT^\alpha}{\Gamma(\alpha+1)} + \frac{MT^\alpha}{\Gamma(\alpha+1)} \right] \\ & + \frac{|c|}{|a+b|} + \frac{mMT^\alpha}{\Gamma(\alpha+1)} + \frac{MT^\alpha}{\Gamma(\alpha+1)} + m(M^*|u| + N^*) \\ \le & \left(\frac{|b|}{|a+b|} + 1 \right) \left[m(M^*|u| + N^*) + \frac{(m+1)MT^\alpha}{\Gamma(\alpha+1)} \right] + \frac{|c|}{|a+b|} . \end{aligned}$$

Then

$$\|N(u)\|_{PC} \le \left(\frac{|b|}{|a+b|}+1\right)\left[m(M^*\eta^*+N^*)+\frac{(m+1)MT^\alpha}{\Gamma(\alpha+1)}\right]+\frac{|c|}{|a+b|} := \ell\,.$$

Step 3: N maps bounded sets to equicontinuous sets of $PC(J,\mathbb{R})$. Let $\tau_1,\tau_2\in J$, $\tau_1<\tau_2$, B_{η^*} be a bounded set of $PC(J,\mathbb{R})$ as in Step 2, and let $u\in B_{\eta^*}$. Then

$$\begin{aligned}
&|N(u)(\tau_2)-N(u)(\tau_1)|\\
&\le \frac{1}{\Gamma(\alpha)}\int_0^{\tau_1}|(\tau_2-s)^{\alpha-1}-(\tau_1-s)^{\alpha-1}\|g(s)|ds\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2}|(\tau_2-s)^{\alpha-1}\|g(s)|ds+\sum_{0<t_k<\tau_2-\tau_1}|I_k(u(t_k^-))|\\
&\le \frac{M}{\Gamma(\alpha+1)}[2(\tau_2-\tau_1)^\alpha+(\tau_2^\alpha-\tau_1^\alpha)]+(\tau_2-\tau_1)(M^*|u|+N^*)\\
&\le \frac{M}{\Gamma(\alpha+1)}[2(\tau_2-\tau_1)^\alpha+(\tau_2^\alpha-\tau_1^\alpha)]+(\tau_2-\tau_1)(M^*\eta^*+N^*)\,.
\end{aligned}$$

As $\tau_1\to\tau_2$, the right-hand side of the preceding inequality tends to zero. As a consequence of Steps 1–3, together with the Ascoli–Arzelà theorem, we can conclude that $N\colon PC(J,\mathbb{R})\to PC(J,\mathbb{R})$ is completely continuous.

Step 4: A priori bounds. Now it remains to show that the set

$$E=\{u\in PC(J,\mathbb{R})\colon u=\lambda N(u)\ \text{ for some }\ 0<\lambda<1\}$$

is bounded. Let $u\in E$; then $u=\lambda N(u)$ for some $0<\lambda<1$. Thus, for each $t\in J$ we have

$$\begin{aligned}
u(t)=\frac{-1}{a+b}&\left[b\lambda\sum_{i=1}^m I_i(u(t_i^-))+\frac{b\lambda}{\Gamma(\alpha)}\sum_{i=1}^m\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}g(s)ds\right.\\
&\left.+\frac{b\lambda}{\Gamma(\alpha)}\int_{t_m}^{T}(T-s)^{\alpha-1}g(s)ds-c\lambda\right]+\frac{\lambda}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\alpha-1}g(s)ds\\
&+\frac{\lambda}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}g(s)ds+\lambda\sum_{0<t_k<t}I_k(u(t_k^-))\,.
\end{aligned}\tag{5.13}$$

From (5.4.1), for each $t\in J$ we have

$$\begin{aligned}
|g(t)|&=|f(t,u(t),g(t))|\\
&\le p(t)+q(t)|u(t)|+r(t)|g(t)|\\
&\le p^*+q^*|u(t)|+r^*|g(t)|\,.
\end{aligned}$$

Thus,

$$\begin{aligned}
|g(t)|&\le\frac{1}{1-r^*}(p^*+q^*|u(t)|)\\
&\le\frac{1}{1-r^*}(p^*+q^*\|u\|_{PC})\,.
\end{aligned}$$

This implies, by (5.13) and (5.4.2), that for each $t \in J$ we have

$$|u(t)| \leq \frac{|b|}{|a+b|}\left[m(M^*\|u\|_{PC} + N^*) + \frac{mT^\alpha(p^* + q^*\|u\|_{PC})}{(1-r^*)\Gamma(\alpha+1)} + \frac{T^\alpha(p^* + q^*\|u\|_{PC})}{(1-r^*)\Gamma(\alpha+1)}\right] + \frac{|c|}{|a+b|} + \frac{mT^\alpha(p^* + q^*\|u\|_{PC})}{(1-r^*)\Gamma(\alpha+1)} + \frac{T^\alpha(p^* + q^*\|u\|_{PC})}{(1-r^*)\Gamma(\alpha+1)} + m(M^*\|u(t)\|_{PC} + N^*) .$$

Then

$$\begin{aligned}\|u\|_{PC} &\leq \left(\frac{|b|}{|a+b|} + 1\right)\left[m(M^*\|u(t)\|_{PC} + N^*) + \frac{(m+1)\,(p^* + q^*\|u\|_{PC})\,T^\alpha}{(1-r^*)\Gamma(\alpha+1)}\right] \\ &\quad + \frac{|c|}{|a+b|} \\ &\leq \left(\frac{|b|}{|a+b|} + 1\right)\left(mN^* + \frac{(m+1)p^*T^\alpha}{(1-r^*)\Gamma(\alpha+1)}\right) + \frac{|c|}{|a+b|} \\ &\quad + \left(\frac{|b|}{|a+b|} + 1\right)\left(mM^* + \frac{(m+1)q^*T^\alpha}{(1-r^*)\Gamma(\alpha+1)}\right)\|u\|_{PC} .\end{aligned}$$

Thus,

$$\left[1 - \left(\frac{|b|}{|a+b|} + 1\right)\left(mM^* + \frac{(m+1)q^*T^\alpha}{(1-r^*)\Gamma(\alpha+1)}\right)\right]\|u\|_{PC} \leq \left(\frac{|b|}{|a+b|} + 1\right)\left[\frac{|c|}{|a+b|} + mN^* + \frac{(m+1)p^*T^\alpha}{(1-r^*)\Gamma(\alpha+1)}\right] .$$

Finally, by (5.10) we have

$$\|u\|_{PC} \leq \frac{\left(\frac{|b|}{|a+b|} + 1\right)\left[mN^* + \frac{(m+1)p^*T^\alpha}{(1-r^*)\Gamma(\alpha+1)} + \frac{|c|}{|a+b|}\right]}{\left[1 - \left(\frac{|b|}{|a+b|} + 1\right)\left(mM^* + \frac{(m+1)q^*T^\alpha}{(1-r^*)\Gamma(\alpha+1)}\right)\right]} := R .$$

This shows that set E is bounded. As a consequence of Schaefer's fixed point theorem, we deduce that N has a fixed point that is a solution of problem (5.1)–(5.3). □

5.2.3 Ulam–Hyers Rassias stability

Here we adopt the concepts in Wang et al. [252] and introduce Ulam's type stability concepts for problem (5.1)–(5.2).

Let $z \in PC^1(J, \mathbb{R})$, $\epsilon > 0$, $\psi > 0$, and $\varphi \in PC(J, \mathbb{R}_+)$ is nondecreasing. We consider the set of inequalities

$$\begin{cases} |{}^cD^\alpha z(t) - f(t, z(t), {}^cD^\alpha z(t))| \leq \epsilon , & t \in (t_k, t_{k+1}],\ k = 1, \ldots, m \\ |\Delta z(t_k) - I_k(z(t_k^-))| \leq \epsilon , & k = 1, \ldots, m ; \end{cases} \tag{5.14}$$

the set of inequalities

$$\begin{cases} |{}^cD^\alpha z(t) - f(t, z(t), {}^cD^\alpha z(t))| \leq \varphi(t) , & t \in (t_k, t_{k+1}],\ k = 1, \ldots, m \\ |\Delta z(t_k) - I_k(z(t_k^-))| \leq \psi , & k = 1, \ldots, m ; \end{cases} \tag{5.15}$$

and the set of inequalities

$$\begin{cases} |{}^cD^\alpha z(t) - f(t, z(t), {}^cD^\alpha z(t))| \le \epsilon\varphi(t)\,, & t \in (t_k, t_{k+1}],\ k = 1, \dots, m \\ |\Delta z(t_k) - I_k(z(t_k^-))| \le \epsilon\psi\,, & k = 1, \dots, m\,. \end{cases} \tag{5.16}$$

Definition 5.5. Problem (5.1)–(5.2) is Ulam–Hyers stable if there exists a real number $c_{f,m} > 0$ such that for each $\epsilon > 0$ and for each solution $z \in PC^1(J, \mathbb{R})$ of inequality (5.14) there exists a solution $y \in PC^1(J, \mathbb{R})$ of problem (5.1)–(5.2), with

$$|z(t) - y(t)| \le c_{f,m}\epsilon\,, \quad t \in J\,.$$

Definition 5.6. Problem (5.1)–(5.2) is generalized Ulam–Hyers stable if there exists $\theta_{f,m} \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\theta_{f,m}(0) = 0$ such that for each solution $z \in PC^1(J, \mathbb{R})$ of inequality (5.14) there exists a solution $y \in PC^1(J, \mathbb{R})$ of problem (5.1)–(5.2), with

$$|z(t) - y(t)| \le \theta_{f,m}(\epsilon)\,, \quad t \in J\,.$$

Definition 5.7. Problem (5.1)–(5.2) is Ulam–Hyers–Rassias stable with respect to (φ, ψ) if there exists $c_{f,m,\varphi} > 0$ such that for each $\epsilon > 0$ and for each solution $z \in PC^1(J, \mathbb{R})$ of inequality (5.16) there exists a solution $y \in PC^1(J, \mathbb{R})$ of problem (5.1)–(5.2), with

$$|z(t) - y(t)| \le c_{f,m,\varphi}\epsilon(\varphi(t) + \psi)\,, \quad t \in J\,.$$

Definition 5.8. Problem (5.1)–(5.2) is generalized Ulam–Hyers–Rassias stable with respect to (φ, ψ) if there exists $c_{f,m,\varphi} > 0$ such that for each solution $z \in PC^1(J, \mathbb{R})$ of inequality (5.15) there exists a solution $y \in PC^1(J, \mathbb{R})$ of problem (5.1)–(5.2), with

$$|z(t) - y(t)| \le c_{f,m,\varphi}(\varphi(t) + \psi)\,, \quad t \in J\,.$$

Remark 5.9. It is clear that (i) Definition 5.5 implies Definition 5.6, (ii) Definition 5.7 implies Definition 5.8, and (iii) Definition 5.7 for $\varphi(t) = \psi = 1$ implies Definition 5.5.

Remark 5.10. A function $z \in PC^1(J, \mathbb{R})$ is a solution of inequality (5.16) if and only if there is $\sigma \in PC(J, \mathbb{R})$ and a sequence σ_k, $k = 1, \dots, m$ (which depend on z) such that
(i) $|\sigma(t)| \le \epsilon\varphi(t),\ t \in (t_k, t_{k+1}]$, $k = 1, \dots, m$ and $|\sigma_k| \le \epsilon\psi$, $k = 1, \dots, m$;
(ii) ${}^cD^\alpha z(t) = f(t, z(t), {}^cD^\alpha z(t)) + \sigma(t),\ t \in (t_k, t_{k+1}]$, $k = 1, \dots, m$;
(iii) $\Delta z(t_k) = I_k(z(t_k^-)) + \sigma_k,\ k = 1, \dots, m$.

One could make similar remarks for inequalities 5.15 and 5.14. Now we state the Ulam–Hyers–Rassias stability result.

Theorem 5.11. *Assume* (5.3.1)–(5.3.3) *and* (5.8) *hold and*
(5.11.1) *there exists a nondecreasing function* $\varphi \in PC(J, \mathbb{R}_+)$ *and there exists* $\lambda_\varphi > 0$ *such that for any* $t \in J$,

$$I^\alpha\varphi(t) \le \lambda_\varphi\varphi(t)\,.$$

Then problem (5.1)–(5.2) *is Ulam–Hyers–Rassias stable with respect to* (φ, ψ).

Proof. Let $z \in PC^1(J, \mathbb{R})$ be a solution of inequality (5.16). Denote by y the unique solution of the BVP

$$\begin{cases} {}^cD^{\alpha}_{t_k} y(t) = f(t, y(t), {}^cD^{\alpha}_{t_k} y(t)), & t \in (t_k, t_{k+1}], k = 1, \dots, m\,; \\ \Delta y(t_k) = I_k(y(t_k^-)), & k = 1, \dots, m\,; \\ ay(0) + by(T) = c\,; \\ y(0) = z(0)\,. \end{cases}$$

Using Lemma 5.2, we obtain for each $t \in (t_k, t_{k+1}]$

$$y(t) = y(0) + \sum_{i=1}^{k} I_i(y(t_i^-)) + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} g(s)ds$$
$$+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t - s)^{\alpha-1} g(s)ds\,, \quad t \in (t_k, t_{k+1}]\,,$$

where $g \in C(J, \mathbb{R})$ is such that

$$g(t) = f(t, y(t), g(t))\,.$$

Since z is a solution of inequality (5.16) and by Remark 5.10, we have

$$\begin{cases} {}^cD^{\alpha}_{t_k} z(t) = f(t, z(t), {}^cD^{\alpha}_{t_k} z(t)) + \sigma(t), & t \in (t_k, t_{k+1}], k = 1, \dots, m\,; \\ \Delta z(t_k) = I_k(z(t_k^-)) + \sigma_k, & k = 1, \dots, m\,. \end{cases} \tag{5.17}$$

Clearly, the solution of (5.17) is given by

$$z(t) = z(0) + \sum_{i=1}^{k} I_i(z(t_i^-)) + \sum_{i=1}^{k} \sigma_i + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} h(s)ds$$
$$+ \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} \sigma(s)ds + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t - s)^{\alpha-1} h(s)ds$$
$$+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t - s)^{\alpha-1} \sigma(s)ds\,, \quad t \in (t_k, t_{k+1}]\,,$$

where $h \in C(J, \mathbb{R})$ is such that

$$h(t) = f(t, z(t), h(t))\,.$$

Hence, for each $t \in (t_k, t_{k+1}]$, it follows that

$$\begin{aligned}|z(t) - y(t)| \le \sum_{i=1}^{k} |\sigma_i| + \sum_{i=1}^{k} |I_i(z(t_i^-)) - I_i(y(t_i^-))| \\ + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} |h(s) - g(s)| ds \\ + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} |\sigma(s)| ds \\ + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} |h(s) - g(s)| ds \\ + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} |\sigma(s)| .\end{aligned}$$

Thus,

$$\begin{aligned}|z(t) - y(t)| \le m\epsilon\psi + (m+1)\epsilon\lambda_\varphi \varphi(t) + \sum_{i=1}^{k} \tilde{l} |z(t_i^-) - y(t_i^-)| \\ + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} |h(s) - g(s)| ds \\ + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} |h(s) - g(s)| ds .\end{aligned}$$

By (5.3.2) we have

$$\begin{aligned}|h(t) - g(t)| &= |f(t, z(t), h(t)) - f(t, y(t), g(t))| \\ &\le K|z(t) - y(t)| + L|g(t) - h(t)| .\end{aligned}$$

Then

$$|h(t) - g(t)| \le \frac{K}{1-L} |z(t) - y(t)| .$$

Therefore, for each $t \in J$

$$\begin{aligned}|z(t) - y(t)| \le m\epsilon\psi + (m+1)\epsilon\lambda_\varphi \varphi(t) + \sum_{i=1}^{k} \tilde{l} |z(t_i^-) - y(t_i^-)| \\ + \frac{K}{(1-L)\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} |z(s) - y(s)| ds \\ + \frac{K}{(1-L)\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} |z(s) - y(s)| ds .\end{aligned}$$

Thus,

$$|z(t)-y(t)| \le \sum_{i=1}^{k} \tilde{l}|z(t_i^-)-y(t_i^-)| + \epsilon(\psi+\varphi(t))(m+(m+1)\lambda_\varphi) + \frac{K(m+1)}{(1-L)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}|z(s)-y(s)|ds\,.$$

Applying Lemma 1.53, we get

$$\begin{aligned}|z(t)-y(t)| &\le \epsilon(\psi+\varphi(t))(m+(m+1)\lambda_\varphi) \\ &\quad\times\left[\prod_{0<t_k<t}(1+\tilde{l})\exp\left(\int_0^t \frac{K(m+1)}{(1-L)\Gamma(\alpha)}(t-s)^{\alpha-1}ds\right)\right] \\ &\le c_\varphi\epsilon(\psi+\varphi(t))\,,\end{aligned}$$

where

$$\begin{aligned}c_\varphi &= (m+(m+1)\lambda_\varphi)\left[\prod_{k=1}^{m}(1+\tilde{l})\exp\left(\frac{K(m+1)T^\alpha}{(1-L)\Gamma(\alpha+1)}\right)\right] \\ &= (m+(m+1)\lambda_\varphi)\left[(1+\tilde{l})\exp\left(\frac{K(m+1)T^\alpha}{(1-L)\Gamma(\alpha+1)}\right)\right]^m\,.\end{aligned}$$

Thus, problem (5.1)–(5.2) is Ulam–Hyers–Rassias stable with respect to (φ, ψ). The proof is complete. □

Next we present the Ulam–Hyers stability result.

Theorem 5.12. *Assume* (5.3.1)–(5.3.3) *and* (5.8) *hold. Then problem* (5.1)–(5.2) *is Ulam–Hyers stable.*

Proof. Let $z \in PC^1(J, \mathbb{R})$ be a solution of inequality (5.14). Denote by y the unique solution of the BVP

$$\begin{cases} {}^cD^\alpha_{t_k}y(t) = f(t, y(t), {}^cD^\alpha_{t_k}y(t)), & t \in (t_k, t_{k+1}],\, k = 1, \dots, m\,; \\ \Delta y(t_k) = I_k(y(t_k^-)), & k = 1, \dots, m\,; \\ ay(0) + by(T) = c\,; \\ y(0) = z(0)\,. \end{cases}$$

From the proof of Theorem 5.11 we get the inequality

$$|z(t)-y(t)| \le \sum_{i=1}^{k} \tilde{l}|(z(t_i^-))-(y(t_i^-))| + m\epsilon + \frac{T^\alpha\epsilon(m+1)}{\Gamma(\alpha+1)} + \frac{K(m+1)}{(1-L)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}|z(s)-y(s)|ds\,.$$

Applying Lemma 1.53, we get

$$
\begin{aligned}
|z(t) - y(t)| &\le \epsilon \left(\frac{m\Gamma(\alpha+1) + T^{\alpha}(m+1)}{\Gamma(\alpha+1)} \right) \\
&\quad \times \left[\prod_{0<t_k<t} (1+\tilde{l}) \exp\left(\int_0^t \frac{K(m+1)}{(1-L)\Gamma(\alpha)} (t-s)^{\alpha-1} ds \right) \right] \\
&\le c_{\varphi}\epsilon \,,
\end{aligned}
$$

where

$$
\begin{aligned}
c_{\varphi} &= \left(\frac{m\Gamma(\alpha+1) + T^{\alpha}(m+1)}{\Gamma(\alpha+1)} \right) \left[\prod_{k=1}^{m} (1+\tilde{l}) \exp\left(\frac{K(m+1)T^{\alpha}}{(1-L)\Gamma(\alpha+1)} \right) \right] \\
&= \left(\frac{m\Gamma(\alpha+1) + T^{\alpha}(m+1)}{\Gamma(\alpha+1)} \right) \left[(1+\tilde{l}) \exp\left(\frac{K(m+1)T^{\alpha}}{(1-L)\Gamma(\alpha+1)} \right) \right]^{m} .
\end{aligned}
$$

Moreover, if we set $\gamma(\epsilon) = c\epsilon; \gamma(0) = 0$, then problem (5.1)–(5.2) is generalized Ulam–Hyers stable. □

5.2.4 Examples

Example 1. Consider the impulsive BVP

$$
{}^{c}D^{\frac{1}{2}}_{t_k} y(t) = \frac{1}{5e^{t+2}\left(1 + |y(t)| + |{}^{c}D^{\frac{1}{2}}_{t_k} y(t)|\right)} \,, \quad \text{for each, } t \in J_0 \cup J_1 \,, \tag{5.18}
$$

$$
\Delta y|_{t=\frac{1}{2}} = \frac{\left|y\left(\frac{1}{2}^{-}\right)\right|}{10 + \left|y\left(\frac{1}{2}^{-}\right)\right|} \,, \tag{5.19}
$$

$$
2y(0) - y(1) = 3 \,, \tag{5.20}
$$

where $J_0 = [0, \frac{1}{2}]$, $J_1 = (\frac{1}{2}, 1]$, $t_0 = 0$, and $t_1 = \frac{1}{2}$.

Set

$$
f(t,u,v) = \frac{1}{5e^{t+2}(1+|u|+|v|)} \,, \quad t \in [0,1], \; u,v \in \mathbb{R} \,.
$$

Clearly, the function f is jointly continuous.

For each $u, v, \bar{u}, \bar{v} \in \mathbb{R}$ and $t \in [0,1]$

$$
|f(t,u,v) - f(t,\bar{u},\bar{v})| \le \frac{1}{5e^2}(|u-\bar{u}| + |v-\bar{v}|) \,.
$$

Hence, condition (5.3.2) is satisfied by $K = L = \frac{1}{5e^2}$. Let

$$
I_1(u) = \frac{u}{10+u} \,, \quad u \in [0,\infty) \,.
$$

Let $u, v \in [0, \infty)$. Then we have

$$|I_1(u) - I_1(v)| = \left|\frac{u}{10+u} - \frac{v}{10+v}\right| = \frac{10|u-v|}{(10+u)(10+v)} \le \frac{1}{10}|u-v| .$$

Thus, the condition

$$\left(\frac{|b|}{|a+b|}+1\right)\left[m\tilde{l} + \frac{(m+1)KT^{\alpha}}{(1-L)\Gamma(\alpha+1)}\right] = 2\left[\frac{1}{10} + \frac{\frac{2}{5e^2}}{\left(1-\frac{1}{5e^2}\right)\Gamma\left(\frac{3}{2}\right)}\right]$$
$$= 2\left[\frac{4}{(5e^2-1)\sqrt{\pi}} + \frac{1}{10}\right] < 1$$

is satisfied by $T = 1$, $a = 2$, $b = -1$, $c = 3$, $m = 1$, and $\tilde{l} = \frac{1}{10}$. It follows from Theorem 5.3 that problem (5.18)–(5.20) has a unique solution on J.

Set, for any $t \in [0, 1]$, $\varphi(t) = t$, $\psi = 1$.

Since

$$I^{\frac{1}{2}}\varphi(t) = \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^t (t-s)^{\frac{1}{2}-1} s ds \le \frac{2t}{\sqrt{\pi}} ,$$

(5.11.1) is satisfied by $\lambda_{\varphi} = \frac{2}{\sqrt{\pi}}$. Thus, problem (5.18)–(5.19) is Ulam–Hyers–Rassias stable with respect to (φ, ψ).

Example 2. Consider the impulsive antiperiodic problem

$$^{c}D_{t_k}^{\frac{1}{2}}y(t) = \frac{2+|y(t)|+|^{c}D_{t_k}^{\frac{1}{2}}y(t)|}{108e^{t+3}\left(1+|y(t)|+|^{c}D_{t_k}^{\frac{1}{2}}y(t)|\right)}, \quad \text{for each, } t \in J_0 \cup J_1 , \tag{5.21}$$

$$\Delta y|_{t=\frac{1}{3}} = \frac{|y\left(\frac{1}{3}^{-}\right)|}{6+|y\left(\frac{1}{3}^{-}\right)|} , \tag{5.22}$$

$$y(0) = -y(1) , \tag{5.23}$$

where $J_0 = [0, \frac{1}{3}]$, $J_1 = (\frac{1}{3}, 1]$, $t_0 = 0$, and $t_1 = \frac{1}{3}$. Set

$$f(t, u, v) = \frac{2+|u|+|v|}{108e^{t+3}(1+|u|+|v|)} , \quad t \in [0, 1],\ u, v \in \mathbb{R} .$$

Clearly, the function f is jointly continuous.

For any $u, v, \bar{u}, \bar{v} \in \mathbb{R}$ and $t \in [0, 1]$

$$|f(t, u, v) - f(t, \bar{u}, \bar{v})| \le \frac{1}{108e^3}(|u-\bar{u}| + |v-\bar{v}|) .$$

Hence, condition (5.3.2) is satisfied by $K = L = \frac{1}{108e^3}$.

For each $t \in [0, 1]$ we have

$$|f(t, u, v)| \le \frac{1}{108e^{t+3}}(2 + |u| + |v|) .$$

Thus, condition (5.4.1) is satisfied by $p(t) = \frac{1}{54e^{t+3}}$ and $q(t) = r(t) = \frac{1}{108e^{t+3}}$. Let

$$I_1(u) = \frac{u}{6+u}\,, \qquad u \in [0, \infty)\,.$$

For each $u \in [0, \infty)$ we have

$$|I_1(u)| \le \frac{1}{6}u + 1\,.$$

Thus, condition (5.4.2) is satisfied by $M^* = \frac{1}{6}$ and $N^* = 1$. Thus, the condition

$$\left(\frac{|b|}{|a+b|} + 1\right)\left(mM^* + \frac{(m+1)q^* T^\alpha}{(1-r^*)\Gamma(\alpha+1)}\right) = \frac{3}{2}\left(\frac{1}{6} + \frac{4}{(108e^3 - 1)\sqrt{\pi}}\right) < 1$$

is satisfied by $T = 1$, $a = 1$, $b = 1$, $c = 0$, $m = 1$, and $q^*(t) = r^*(t) = \frac{1}{108e^3}$. It follows from Theorem 5.4 that problem (5.21)–(5.23) has at least one solution on J.

5.3 Existence Results for Impulsive NIFDE in Banach Space

5.3.1 Introduction

The purpose of this section is to establish existence and uniqueness results to the BVPs for NIFDEs

$${}^cD^\nu_{t_k} y(t) = f(t, y, {}^cD^\nu_{t_k} y(t))\,, \quad \text{for each,} \quad t \in (t_k, t_{k+1}],\ k = 0, \ldots, m,\ 0 < \nu \le 1\,, \tag{5.24}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-))\,, \qquad k = 1, \ldots, m\,, \tag{5.25}$$

$$ay(0) + by(T) = c\,, \tag{5.26}$$

where ${}^cD^\nu_{t_k}$ is the Caputo fractional derivative, $(E, \|\cdot\|)$ is a real Banach space, $f : J \times E \times E \to E$ is a given function, $I_k : E \to E$, a, b are real constants with $a + b \neq 0$ and $c \in E$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, and $y(t_k^+) = \lim_{h\to 0^+} y(t_k + h)$ and $y(t_k^-) = \lim_{h \to 0^-} y(t_k + h)$ represent the right and left limits of $y(t)$ at $t = t_k$.

In this section, two results are discussed; the first is based on Darbo's fixed point theorem combined with the technique of measures of noncompactness, the second is based on Mönch's fixed point theorem. Finally, two examples are given to demonstrate the application of our main results.

5.3.2 Existence of Solutions

Definition 5.13. A function $y \in PC(J, E)$ whose ν-derivative exists on J_k is said to be a solution of (5.24)–(5.26) if y satisfies the equation ${}^cD^\nu_{t_k} y(t) = f(t, y(t), {}^cD^\nu_{t_k} y(t))$ on J_k

and satisfies the conditions

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \dots, m,$$
$$ay(0) + by(T) = c.$$

To prove the existence of solutions to (5.24)–(5.26), we need the following auxiliary lemma.

Lemma 5.14. *Let $0 < \nu \leq 1$, and let $\sigma \colon J \to E$ be continuous. A function y is a solution of the fractional integral equation*

$$y(t) = \begin{cases} \dfrac{-1}{a+b}\left[b\sum_{i=1}^{m} I_i(y(t_i^-)) + \dfrac{b}{\Gamma(\nu)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}(t_i - s)^{\nu-1}\sigma(s)ds \right. \\ \left. + \dfrac{b}{\Gamma(\nu)}\int_{t_m}^{T}(T-s)^{\nu-1}\sigma(s)ds - c\right] + \dfrac{1}{\Gamma(\nu)}\int_0^t (t-s)^{\nu-1}\sigma(s)ds, & \text{if } t \in [0, t_1], \\ \dfrac{-1}{a+b}\left[b\sum_{i=1}^{m} I_i(y(t_i^-)) + \dfrac{b}{\Gamma(\nu)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}(t_i - s)^{\nu-1}\sigma(s)ds \right. \\ \left. + \dfrac{b}{\Gamma(\nu)}\int_{t_m}^{T}(T-s)^{\nu-1}\sigma(s)ds - c\right] + \sum_{i=1}^{k} I_i(y(t_i^-)) + \dfrac{1}{\Gamma(\nu)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\nu-1}\sigma(s)ds \\ + \dfrac{1}{\Gamma(\nu)}\int_{t_k}^{t}(t-s)^{\nu-1}\sigma(s)ds, & \text{if } t \in (t_k, t_{k+1}], \end{cases} \tag{5.27}$$

where $k = 1, \dots, m$, if and only if y is a solution of the fractional BVP

$$^cD^\nu y(t) = \sigma(t), \quad t \in J_k,$$
$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \dots, m,$$
$$ay(0) + by(T) = c.$$

We list the following conditions:
(5.15.1) The function $f \colon J \times E \times E \to E$ is continuous.
(5.15.2) There exist constants $K > 0$ and $0 < L < 1$ such that

$$\|f(t, u, v) - f(t, \bar{u}, \bar{v})\| \leq K\|u - \bar{u}\| + L\|v - \bar{v}\|$$

for any $u, \bar{u}, v, \bar{v} \in E$ and $t \in J$.
(5.15.3) There exists a constant $\widetilde{l} > 0$ such that

$$\|I_k(u) - I_k(\overline{u})\| \leq \widetilde{l}\|u - \overline{u}\|$$

for each $u, \overline{u} \in E$ and $k = 1, \dots, m$.

We are now in a position to state and prove our existence result for problem (5.24)–(5.26) based on the concept of measures of noncompactness and Darbo's fixed point theorem.

Remark 5.15 ([66]). Conditions (5.15.2) and (5.15.3) are respectively equivalent to the inequalities

$$\alpha\left(f(t, B_1, B_2)\right) \le K\alpha(B_1) + L\alpha(B_2)$$
$$\alpha(I_k(B_1)) \le \tilde{l}\alpha(B_1)\ ,$$

for any bounded sets $B_1, B_2 \subseteq E$, for each $t \in J$ and $k = 1, \dots, m$.

Theorem 5.16. *Assume* (5.15.1)–(5.15.3) *hold. If*

$$\left(\frac{|b|}{|a+b|} + 1\right)\left(m\tilde{l} + \frac{(m+1)KT^{\nu}}{(1-L)\Gamma(\nu+1)}\right) < 1\ , \tag{5.28}$$

then BVP (5.24)–(5.26) *has at least one solution on J.*

Proof. Transform problem (5.24)–(5.26) into a fixed point problem. Consider the operator $N\colon PC(J,E) \to PC(J,E)$ defined by

$$\begin{aligned} N(y)(t) = \frac{-1}{a+b}&\left[b\sum_{i=1}^{m} I_i(y(t_i^-)) + \frac{b}{\Gamma(\nu)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}(t_i-s)^{\nu-1}g(s)ds \right. \\ &\left. + \frac{b}{\Gamma(\nu)}\int_{t_m}^{T}(T-s)^{\nu-1}g(s)ds - c\right] + \frac{1}{\Gamma(\nu)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\nu-1}g(s)ds \\ &+ \frac{1}{\Gamma(\nu)}\int_{t_k}^{t}(t-s)^{\nu-1}g(s)ds + \sum_{0<t_k<t} I_k(y(t_k^-))\ , \end{aligned} \tag{5.29}$$

where $g \in C(J,E)$ is such that

$$g(t) = f(t, y(t), g(t))\ .$$

Clearly, the fixed points of operator N are solutions of problem (5.24)–(5.26).

We will show that N satisfies the assumptions of *Darbo's* fixed point theorem. The proof will be given in several claims.

Claim 1: N is continuous. Let $\{u_n\}$ be a sequence such that $u_n \to u$ in $PC(J,E)$. Then, for each $t \in J$,

$$\begin{aligned} \|N(u_n)(t) - N(u)(t)\| \le \frac{|b|}{|a+b|}&\left[\sum_{i=1}^{m}\|I_k(u_n(t_k^-)) - I_k(u(t_k^-))\|\right. \\ &+ \frac{1}{\Gamma(\nu)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}(t_i-s)^{\nu-1}\|g_n(s)-g(s)\|ds \\ &\left. + \frac{1}{\Gamma(\nu)}\int_{t_m}^{T}(T-s)^{\nu-1}\|g_n(s)-g(s)\|ds\right] \end{aligned}$$

$$+\frac{1}{\Gamma(\nu)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\nu-1}\|g_n(s)-g(s)\|ds$$
$$+\frac{1}{\Gamma(\nu)}\int_{t_k}^{t}(t-s)^{\nu-1}\|g_n(s)-g(s)\|ds$$
$$+\sum_{0<t_k<t}\|I_k(u_n(t_k^-))-I_k(u(t_k^-))\|\,, \tag{5.30}$$

where $g_n, g \in C(J, E)$ such that

$$g_n(t) = f(t, u_n(t), g_n(t))$$

and

$$g(t) = f(t, u(t), g(t))\,.$$

By (P2) we have

$$\begin{aligned}\|g_n(t)-g(t)\| &= \|f(t,u_n(t),g_n(t)) - f(t,u(t),g(t))\|\\ &\le K\|u_n(t)-u(t)\| + L\|g_n(t)-g(t)\|\,.\end{aligned}$$

Then

$$\|g_n(t)-g(t)\| \le \frac{K}{1-L}\|u_n(t)-u(t)\|\,.$$

Since $u_n \to u$, we get $g_n(t) \to g(t)$ as $n \to \infty$ for each $t \in J$. Let $\eta > 0$ be such that, for each $t \in J$, we have $\|g_n(t)\| \le \eta$ and $\|g(t)\| \le \eta$. Then we have

$$\begin{aligned}(t-s)^{\nu-1}\|g_n(s)-g(s)\| &\le (t-s)^{\nu-1}[\|g_n(s)\| + \|g(s)\|]\\ &\le 2\eta(t-s)^{\nu-1}\end{aligned}$$

and

$$\begin{aligned}(t_k-s)^{\nu-1}\|g_n(s)-g(s)\| &\le (t_k-s)^{\nu-1}[\|g_n(s)\| + \|g(s)\|]\\ &\le 2\eta(t_k-s)^{\nu-1}\,.\end{aligned}$$

For each $t \in J$ the functions $s \to 2\eta(t-s)^{\nu-1}$ and $s \to 2\eta(t_k-s)^{\nu-1}$ are integrable on $[0, t]$; then the Lebesgue dominated convergence theorem and (5.30) imply that

$$\|N(u_n)(t) - N(u)(t)\| \to 0 \;\text{ as }\; n \to \infty\,.$$

Hence,

$$\|N(u_n) - N(u)\|_{PC} \to 0 \;\text{ as }\; n \to \infty\,.$$

Consequently, N is continuous.

Let R be the constant such that

$$R \geq \frac{\|c\|\Gamma(\nu+1)(1-L) + (|b|+|a+b|)\,[mc_1\Gamma(\nu+1)(1-L)+(m+1)T^{\nu}f^*]}{|a+b|\Gamma(\nu+1)(1-L) - (|b|+|a+b|)\left[m\widetilde{l}\Gamma(\nu+1)(1-L)+(m+1)T^{\nu}K\right]}, \quad (5.31)$$

where $c_1 = \sup_{v\in E}\|I(v)\|$ and $f^* = \sup_{t\in J}\|f(t,0,0)\|$.

Define

$$D_R = \{u \in PC(J,E) : \|u\|_{PC} \leq R\}\,.$$

It is clear that D_R is a bounded, closed, and convex subset of $PC(J,E)$.

Claim 2: $N(D_R) \subset D_R$. Let $u \in D_R$; we show that $Nu \in D_R$. For each $t \in J$ we have

$$\begin{aligned}\|N(y)(t)\| \leq{}& \frac{\|c\|}{|a+b|} + \frac{|b|}{|a+b|}\left[\sum_{i=1}^{m}\|I_i(y(t_i^-))\| + \frac{1}{\Gamma(\nu)}\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}(t_i-s)^{\nu-1}\|g(s)\|ds\right.\\ &\left.+\frac{1}{\Gamma(\nu)}\int_{t_m}^{T}(T-s)^{\nu-1}\|g(s)\|ds\right] + \frac{1}{\Gamma(\nu)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{\nu-1}\|g(s)\|ds\\ &+\frac{1}{\Gamma(\nu)}\int_{t_k}^{t}(t-s)^{\nu-1}\|g(s)\|ds + \sum_{0<t_k<t}\|I_k(y(t_k^-))\|\,. \end{aligned} \quad (5.32)$$

By (P2) we have for each $t \in J$

$$\begin{aligned}\|g(t)\| &\leq \|f(t,u(t),g(t)) - f(t,0,0)\| + \|f(t,0,0)\|\\ &\leq K\|u(t)\| + L\|g(t)\| + f^*\\ &\leq K\|u(t)\|_{PC} + L\|g(t)\| + f^*\\ &\leq KR + L\|g(t)\| + f^*\,.\end{aligned}$$

Then

$$\|g(t)\| \leq \frac{f^*+KR}{1-L} := M\,.$$

Thus, (5.31), (5.32), and (5.15.3) imply that

$$\begin{aligned}\|Nu(t)\| \leq{}& \frac{\|c\|}{|a+b|} + \left(\frac{|b|}{|a+b|}+1\right)\left(\sum_{i=1}^{m}\|I_i(y(t_i^-)) - I_i(0)\| + \sum_{i=1}^{m}\|I_i(0)\|\right)\\ &+\left(\frac{|b|}{|a+b|}+1\right)\frac{(m+1)T^{\nu}M}{\Gamma(\nu+1)}\\ \leq{}& \frac{\|c\|}{|a+b|} + \left(\frac{|b|}{|a+b|}+1\right)\left[m(\widetilde{l}R + c_1) + \frac{(m+1)T^{\nu}M}{\Gamma(\nu+1)}\right]\\ \leq{}& R\,.\end{aligned}$$

Thus, for each $t \in J$ we have $\|Nu(t)\| \leq R$. This implies that $\|Nu\|_{PC} \leq R$. Consequently,

$$N(D_R) \subset D_R\,.$$

Claim 3: $N(D_R)$ is bounded and equicontinuous. By Claim 2 we have $N(D_R) = \{N(u): u \in D_R\} \subset D_R$. Thus, for each $u \in D_R$ we have $\|N(u)\|_{PC} \leq R$. Thus, $N(D_R)$ is bounded. Let $t_1, t_2 \in (0, T]$, $t_1 < t_2$, and let $u \in D_R$. Then

$$\begin{aligned}
&\|N(u)(t_2) - N(u)(t_1)\| \\
&\leq \frac{1}{\Gamma(\nu)} \int_0^{t_1} |(t_2 - s)^{\nu-1} - (t_1 - s)^{\nu-1}| \|g(s)\| ds + \frac{1}{\Gamma(\nu)} \int_{t_1}^{t_2} |(t_2 - s)^{\nu-1}| \|g(s)\| ds \\
&\quad + \sum_{0<t_k<t_2-t_1} \|I_k(u(t_k^-)) - I_k(0)\| + \sum_{0<t_k<t_2-t_1} \|I_k(0)\| \\
&\leq \frac{M}{\Gamma(\nu+1)} [2(t_2 - t_1)^\nu + (t_2^\nu - t_1^\nu)] + (t_2 - t_1)(\tilde{l}\|u(t_k^-)\| + c_1) \\
&\leq \frac{M}{\Gamma(\nu+1)} [2(t_2 - t_1)^\nu + (t_2^\nu - t_1^\nu)] + (t_2 - t_1)(\tilde{l}\|u\|_{PC} + c_1) \\
&\leq \frac{M}{\Gamma(\nu+1)} [2(t_2 - t_1)^\nu + (t_2^\nu - t_1^\nu)] + (t_2 - t_1)(\tilde{l}R + c_1) .
\end{aligned}$$

As $t_1 \to t_2$, the right-hand side of the preceding inequality tends to zero.

Claim 4: The operator $N: D_R \to D_R$ is a strict set contraction. Let $V \subset D_R$ and $t \in J$; then we have

$$\begin{aligned}
\alpha(N(V)(t)) &= \alpha((Ny)(t), y \in V) \\
&\leq \frac{|b|}{|a+b|} \left[\sum_{i=1}^{m} \{\alpha(I_i(y(t_i^-))), y \in V\} \right. \\
&\quad + \frac{1}{\Gamma(\nu)} \sum_{i=1}^{m} \left\{ \int_{t_{i-1}}^{t_i} (t_i - s)^{\nu-1} \alpha(g(s)) ds, y \in V \right\} \\
&\quad \left. + \frac{1}{\Gamma(\nu)} \left\{ \int_{t_m}^{T} (T - s)^{\nu-1} \alpha(g(s)) ds, y \in V \right\} \right] \\
&\quad + \frac{1}{\Gamma(\nu)} \sum_{0<t_k<t} \left\{ \int_{t_{k-1}}^{t_k} (t_k - s)^{\nu-1} \alpha(g(s)) ds, y \in V \right\} \\
&\quad + \frac{1}{\Gamma(\nu)} \left\{ \int_{t_k}^{t} (t - s)^{\nu-1} \alpha(g(s)) ds, y \in V \right\} \\
&\quad + \sum_{0<t_k<t} \{\alpha(I_k(y(t_k^-))), y \in V\} .
\end{aligned}$$

Then Remark 5.15 and Lemma 1.32 imply that, for each $s \in J$,

$$\begin{aligned}
\alpha(\{g(s), y \in V\}) &= \alpha(\{f(s, y(s), g(s)), y \in V\}) \\
&\leq K\alpha(\{y(s), y \in V\}) + L\alpha(\{g(s), y \in V\}) .
\end{aligned}$$

Thus,

$$\alpha(\{g(s), y \in V\}) \leq \frac{K}{1-L}\alpha\{y(s), y \in V\}.$$

On the other hand, for each $t \in J$ and $k = 1, \dots, m$ we have

$$\sum_{0<t_k<t} \alpha\left(\{I_k(y(t_k^-)), y \in V\}\right) \leq m\tilde{l}\alpha(\{y(t), y \in V\}).$$

Then

$$\begin{aligned}
\alpha(N(V)(t)) \leq \frac{|b|}{|a+b|}\Bigg[& m\tilde{l}\alpha(\{y(t), y \in V\}) \\
& + \frac{mK}{\Gamma(\nu)(1-L)}\left\{\int_0^t (t-s)^{\nu-1}\{\alpha(y(s))\}ds, y \in V\right\} \\
& + \frac{K}{\Gamma(\nu)(1-L)}\left\{\int_0^T (T-s)^{\nu-1}\{\alpha(y(s))\}ds, y \in V\right\}\Bigg] \\
& + \frac{mK}{\Gamma(\nu)(1-L)}\left\{\int_0^t (t-s)^{\nu-1}\{\alpha(y(s))\}ds, y \in V\right\} \\
& + \frac{K}{\Gamma(\nu)(1-L)}\left\{\int_0^t (t-s)^{\nu-1}\{\alpha(y(s))\}ds, y \in V\right\} \\
& + m\tilde{l}\alpha(\{y(t), y \in V\}) \\
\leq & \left(\frac{|b|}{|a+b|}+1\right)\left(m\tilde{l} + \frac{(m+1)KT^{\nu}}{(1-L)\Gamma(\nu+1)}\right)\alpha_c(V).
\end{aligned}$$

Therefore,

$$\alpha_c(NV) \leq \left(\frac{|b|}{|a+b|}+1\right)\left(m\tilde{l} + \frac{(m+1)KT^{\nu}}{(1-L)\Gamma(\nu+1)}\right)\alpha_c(V).$$

Thus, operator N is a set contraction. As a consequence of Theorem 1.45, the operator N has a fixed point that is a solution of problem (5.24)–(5.26). □

Our next existence result for problem (5.24)–(5.26) is based on the concept of measures of noncompactness and *Mönch's* fixed point theorem.

Theorem 5.17. *Assume* (5.15.1)–(5.15.3) *and* (5.28) *hold. If*

$$m\tilde{l} < 1,$$

then BVP (5.24)–(5.26) *has at least one solution.*

Proof. Consider operator N defined in (5.29). We will show that N satisfies the assumptions of *Mönch's* fixed point theorem. We know that $N\colon D_R \to D_R$ is bounded and continuous, and we need to prove that the implication

$$[V = \overline{\operatorname{conv}} N(V) \;\text{ or }\; V = N(V) \cup \{0\}] \;\text{ implies }\; \alpha(V) = 0$$

holds for every subset V of D_R. Now let V be a subset of D_R such that $V \subset \overline{\operatorname{conv}}(N(V) \cup \{0\})$. V is bounded and equicontinuous, and therefore the function $t \to v(t) = \alpha(V(t))$ is continuous on $[0, T]$.

Using Lemma 5.14, we can write for each $t \in J$ and $k = 0, \ldots, m$

$$N(y(t)) = y(0) + \sum_{i=1}^{k} I_i(y(t_i^-)) + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} g(s)ds$$
$$+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} g(s)ds \,,$$

where $g \in C(J, \mathbb{R})$ is such that

$$g(t) = f(t, y(t), g(t)) \,.$$

By Remark 5.15, Lemma 1.33, and the properties of the measure α, for each $t \in J$ we have

$$\begin{aligned}
v(t) &\le \alpha(N(V)(t) \cup \{0\}) \\
&\le \alpha(N(V)(t)) \\
&\le \alpha\{(Ny)(t), y \in V\} \\
&\le \alpha(y(0)) + \frac{mK}{\Gamma(\nu)(1-L)} \left\{ \int_0^t (t-s)^{\nu-1} \{\alpha(y(s))\} ds, y \in V \right\} \\
&\quad + \frac{K}{\Gamma(\nu)(1-L)} \left\{ \int_0^t (t-s)^{\nu-1} \{\alpha(y(s))\} ds, y \in V \right\} \\
&\quad + m\tilde{l}\alpha(\{y(t), y \in V\}) \\
&\le m\tilde{l}\alpha(\{y(t), y \in V\}) + \frac{(m+1)K}{(1-L)\Gamma(\nu)} \left\{ \int_0^t (t-s)^{\nu-1} \{\alpha(y(s))\} ds, y \in V \right\} \\
&= m\tilde{l} v(t) + \frac{(m+1)K}{(1-L)\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} v(s) ds \,.
\end{aligned}$$

Then

$$v(t) \le \frac{(m+1)K}{(1-m\tilde{l})(1-L)\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} v(s) ds \,.$$

Lemma 1.52 implies that $v(t) = 0$ for each $t \in J$. Then $V(t)$ is relatively compact in E. In view of the Ascoli–Arzelà theorem, V is relatively compact in D_R. Applying now Theorem 1.46 we conclude that N has a fixed point $y \in D_R$. Hence, N has a fixed point that is a solution of problem (5.24)–(5.26). □

Remark 5.18. Our results for BVP (5.24)–(5.26) are appropriate for the following problems:
- Initial value problem: $a = 1, b = 0, c = 0$.
- Terminal value problem: $a = 0, b = 1, c$ arbitrary.
- Antiperiodic problem: $a = 1, b = 1, c = 0$.

However, our results are not applicable for the periodic problem, i.e., for $a = 1, b = -1, c = 0$.

5.3.3 Examples

Example 1. Consider the infinite system

$$ {}^cD_{t_k}^{\frac{1}{2}} y_n(t) = \frac{e^{-t}}{(11+e^t)}\left[\frac{y_n(t)}{1+y_n(t)} - \frac{{}^cD_{t_k}^{\frac{1}{2}} y_n(t)}{1+{}^cD_{t_k}^{\frac{1}{2}} y_n(t)}\right] \quad \text{for each } t \in J_0 \cup J_1 \,, \tag{5.33} $$

$$ \Delta y_n|_{t=\frac{1}{2}} = \frac{y_n\left(\frac{1}{2}^-\right)}{10+y_n\left(\frac{1}{2}^-\right)} \,, \tag{5.34} $$

$$ 2y_n(0) - y_n(1) = 3 \,, \tag{5.35} $$

where $J_0 = [0, \frac{1}{2}]$, $J_1 = (\frac{1}{2}, 1]$, $t_0 = 0$, and $t_1 = \frac{1}{2}$.

Set

$$ E = l^1 = \left\{ y = (y_1, y_2, \ldots, y_n, \ldots)\,, \quad \sum_{n=1}^{\infty} |y_n| < \infty \right\} \,, $$

$$ f = (f_1, f_2, \ldots, f_n, \ldots)\,, $$

such that

$$ f(t,u,v) = \frac{e^{-t}}{(11+e^t)}\left[\frac{u}{1+u} - \frac{v}{1+v}\right] \,, \quad t \in [0,1],\ u, v \in E \,. $$

Clearly, the function f is jointly continuous and E is a Banach space with the norm $\|y\| = \sum_{n=1}^{\infty} |y_n|$.

For any $u, \bar{u}, v, \bar{v} \in E$, and $t \in [0,1]$

$$ \|f(t,u,v) - f(t,\bar{u},\bar{v})\| \le \frac{1}{12}(\|u-\bar{u}\| + \|v-\bar{v}\|) \,. $$

Hence, condition (5.15.2) is satisfied by $K = L = \dfrac{1}{12}$. Let

$$ I_1(u) = \frac{u}{10+u} \,, \quad u \in E \,. $$

Let $u, v \in E$. Then we have

$$\|I_1(u) - I_1(v)\| = \left\| \frac{u}{10+u} - \frac{v}{10+v} \right\| \le \frac{1}{10}\|u-v\| .$$

Hence, condition (5.15.3) is satisfied by $\widetilde{l} = \frac{1}{10}$. The conditions

$$\left(\frac{|b|}{|a+b|}+1\right)\left(m\widetilde{l} + \frac{(m+1)KT^{\nu}}{(1-L)\Gamma(\nu+1)}\right) = \frac{1}{10} + \frac{\frac{2}{12}}{\left(1-\frac{1}{12}\right)\Gamma\left(\frac{3}{2}\right)}$$
$$= \frac{8}{11\sqrt{\pi}} + \frac{1}{5} < 1$$

are satisfied by $T = m = 1$, $a = 2$, $b = -1$, and $\nu = \frac{1}{2}$.

It follows from Theorem 5.16 that problem (5.33)–(5.35) has at least one solution on J.

Example 2. Consider the impulsive problem

$$^{c}D^{\frac{1}{2}}_{t_k} y_n(t) = \frac{2 + \|y_n(t)\| + \|^{c}D^{\frac{1}{2}}_{t_k} y_n(t)\|}{108 e^{t+3}\left(1 + \|y_n(t)\| + \|^{c}D^{\frac{1}{2}}_{t_k} y_n(t)\|\right)} , \quad \text{for each, } t \in J_0 \cup J_1 , \tag{5.36}$$

$$\Delta y_n|_{t=\frac{1}{3}} = \frac{\|y_n\left(\frac{1}{3}^-\right)\|}{6 + \|y_n\left(\frac{1}{3}^-\right)\|} , \tag{5.37}$$

$$y_n(0) = -y_n(1) , \tag{5.38}$$

where $J_0 = [0, \frac{1}{3}]$, $J_1 = (\frac{1}{3}, 1]$, $t_0 = 0$, and $t_1 = \frac{1}{3}$. Set

$$E = l^1 = \{y = (y_1, y_2, \ldots, y_n, \ldots) , \quad \sum_{n=1}^{\infty} |y_n| < \infty\} ,$$

$$f = (f_1, f_2, \ldots, f_n, \ldots) ,$$

such that

$$f(t,u,v) = \frac{2 + \|u\| + \|v\|}{108 e^{t+3}(1 + \|u\| + \|v\|)} , \quad t \in [0,1],\ u, v \in E .$$

Clearly, the function f is jointly continuous.

E is a Banach space with the norm $\|y\| = \sum_{n=1}^{\infty} |y_n|$.

For any $u, \bar{u}, v, \bar{v} \in E$ and $t \in [0,1]$

$$\|f(t,u,v) - f(t,\bar{u},\bar{v})\| \le \frac{1}{108e^3}(\|u-\bar{u}\| + \|v-\bar{v}\|) .$$

Hence, condition (5.15.2) is satisfied by $K = L = \frac{1}{108e^3}$. Let

$$I_1(u) = \frac{\|u\|}{6 + \|u\|} , \quad u \in E .$$

Let $u, v \in E$. Then we have

$$\|I_1(u) - I_1(v)\| = \left\| \frac{u}{6+u} - \frac{v}{6+v} \right\| \le \frac{1}{6} \|u - v\| .$$

Hence, condition (5.15.3) is satisfied by $\widetilde{l} = \frac{1}{6}$.
The condition

$$\left(\frac{|b|}{|a+b|} + 1 \right) \left(m\widetilde{l} + \frac{(m+1)KT^{\nu}}{(1-L)\Gamma(\nu+1)} \right) = \frac{3}{2} \left(\frac{1}{6} + \frac{\frac{2}{12}}{\left(1 - \frac{1}{12}\right) \Gamma\left(\frac{3}{2}\right)} \right)$$
$$= \frac{6}{11\sqrt{\pi}} + \frac{1}{4} < 1$$

is satisfied by $T = m = 1$, $a = 1$, $b = 1$, and $\nu = \frac{1}{2}$. Also, we have

$$m\widetilde{l} = \frac{1}{6} < 1 .$$

It follows from Theorem 5.17 that problem (5.36)–(5.38) has at least one solution on J.

5.4 Notes and Remarks

The results of Chapter 5 are taken from Benchohra et al. [92]. Other results may be found in [57, 125, 249, 250].