4 Boundary Value Problems for Nonlinear Implicit Fractional Differential Equations

4.1 Introduction

In this chapter, we establish the existence and uniqueness of solutions to some boundary value problem (BVPs) for implicit fractional differential equations with $0 < \alpha \leq 1$ and $1 < \alpha \leq 2$. Then we consider the stability of solutions of other classes of BVP of implicit fractional differential equations with local and nonlocal conditions in Banach spaces.

4.2 BVP for NIFDE with $0 < \alpha \leq 1$

4.2.1 Introduction and Motivations

The purpose of this section is to establish existence and uniqueness results of solutions for a class of boundary value problem (BVP) for the implicit fractional order differential equation

$$^cD^\alpha y(t) = f(t, y(t), {}^c D^\alpha y(t)) , \quad \text{for each } t \in J = [0, T], T > 0,\ 0 < \alpha \leq 1 , \tag{4.1}$$

$$ay(0) + by(T) = c , \tag{4.2}$$

where $f: J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function and a, b, and c are real constants, with $a + b \neq 0$.

We present three results for problem (4.1)–(4.2). The first one is based on the Banach contraction principle, the second one on Schauder's fixed point theorem, and the last one on the nonlinear alternative of Leray–Schauder type.

4.2.2 Existence of Solutions

Let us define what we mean by a solution of problem (4.1)–(4.2).

Definition 4.1. A function $u \in C(J)$ is said to be a solution of problem (4.1)–(4.2) if u satisfies equation (4.1) and conditions (4.2) on J.

For the existence of solutions to problem (4.1)–(4.2), we need the following auxiliary lemma.

Lemma 4.2 ([95]). *Let $0 < \alpha \leq 1$ and $g: J \to \mathbb{R}$ be continuous. A function y is a solution of the implicit fractional boundary value problem*

$$^cD^\alpha y(t) = g(t) , \quad \textit{for each } t \in J,\ 0 < \alpha \leq 1 ,$$

$$ay(0) + b(T) = c ,$$

https://doi.org/10.1515/9783110553819-004

where a, b, and c are real constants with $a + b \neq 0$, if and only if y is a solution of the fractional integral equation

$$y(t) = \frac{c}{a+b} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)ds - \frac{b}{(a+b)\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} g(s)ds \,.$$

Theorem 4.3. *Make the following assumptions:*
(4.3.1) *The function $f: J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous.*
(4.3.2) *There exist constants $K > 0$ and $0 < L < 1$ such that*

$$|f(t,u,v) - f(t,\bar{u},\bar{v})| \leq K|u-\bar{u}| + L|v-\bar{v}|$$

for any $u, v, \bar{u}, \bar{v} \in \mathbb{R}$ and $t \in J$.

If

$$\frac{KT^\alpha}{(1-L)\Gamma(\alpha+1)} \left(1 + \frac{|b|}{|a+b|}\right) < 1 \,, \tag{4.3}$$

then there exists a unique solution for BVP (4.1)–(4.2) on J.

Proof. Define the operator $N: C(J,\mathbb{R}) \to C(J,\mathbb{R})$ by

$$N(y)(t) = \frac{c}{a+b} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)ds - \frac{b}{(a+b)\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} g(s)ds \,, \tag{4.4}$$

where $g \in C(J,\mathbb{R})$ satisfies the functional equation

$$g(t) = f(t, y(t), g(t)) \,.$$

Clearly, the fixed points of the operator N are solutions of problem (4.1)–(4.2). Let $u, w \in C(J,\mathbb{R})$. Then for $t \in J$ we have

$$(Nu)(t) - (Nw)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}(g(s) - h(s))ds - \frac{b}{(a+b)\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1}(g(s) - h(s))ds \,,$$

where $g, h \in C(J,\mathbb{R})$ is such that

$$g(t) = f(t, u(t), g(t)) \,,$$

and

$$h(t) = f(t, w(t), h(t)) .$$

Then, for $t \in J$,

$$|(Nu)(t) - (Nw)(t)| \le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |g(s) - h(s)| ds + \frac{|b|}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} |g(s) - h(s)| ds . \tag{4.5}$$

By (4.3.2) we have

$$\begin{aligned} |g(t) - h(t)| &= |f(t, u(t), g(t)) - f(t, w(t), h(t))| \\ &\le K|u(t) - w(t)| + L|g(t) - h(t)| . \end{aligned}$$

Thus,

$$|g(t) - h(t)| \le \frac{K}{1-L} |u(t) - w(t)| .$$

By (4.5), for $t \in J$ we have

$$\begin{aligned} |(Nu)(t) - (Nw)(t)| &\le \frac{K}{(1-L)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |u(s) - w(s)| ds \\ &\quad + \frac{|b|K}{|a+b|(1-L)\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} |u(s) - w(s)| ds \\ &\le \frac{KT^\alpha}{(1-L)\Gamma(\alpha+1)} \left(1 + \frac{|b|}{|a+b|}\right) \|u - w\|_\infty . \end{aligned}$$

Then

$$\|Nu - Nw\|_\infty \le \frac{KT^\alpha}{(1-L)\Gamma(\alpha+1)} \left(1 + \frac{|b|}{|a+b|}\right) \|u - w\|_\infty .$$

By (4.3), the operator N is a contraction. Hence, by Banach's contraction principle, N has a unique fixed point that is the unique solution of problem (4.1)–(4.2). □

Our next existence result is based on Schauder's fixed point theorem.

Theorem 4.4. *Assume* (4.3.1) *and* (4.3.2) *hold and*
(4.4.1) *There exist $p, q, r \in C(J, \mathbb{R}_+)$ with $r^* = \sup_{t \in J} r(t) < 1$ such that*

$$|f(t, u, w)| \le p(t) + q(t)|u| + r(t)|w| \text{ for } t \in J, \text{ and } u, w \in \mathbb{R} .$$

If

$$q^* M \left(1 + \frac{|b|}{|a+b|}\right) < 1 , \tag{4.6}$$

where $q^* = \sup_{t \in J} q(t)$, and $M = \frac{T^\alpha}{(1-r^*)\Gamma(\alpha+1)}$, then BVP (4.1)–(4.2) *has at least one solution.*

Proof. We will show that the operator N defined in (4.4) satisfies the assumptions of Schauder's fixed point theorem. The proof will be given in several steps.

Claim 1: N is continuous. Let $\{u_n\}$ be a sequence such that $u_n \to u$ in $C(J, \mathbb{R})$. Then for each $t \in J$

$$|N(u_n)(t) - N(u)(t)| \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |g_n(s) - g(s)| ds + \frac{|b|}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} |g_n(s) - g(s)| ds\,, \tag{4.7}$$

where $g_n, g \in C(J, \mathbb{R})$ such that

$$g_n(t) = f(t, u_n(t), g_n(t))$$

and

$$g(t) = f(t, u(t), g(t))\,.$$

By (4.3.2) we have

$$\begin{aligned} |g_n(t) - g(t)| &= |f(t, u_n(t), g_n(t)) - f(t, u(t), g(t))| \\ &\leq K|u_n(t) - u(t)| + L|g_n(t) - g(t)|\,. \end{aligned}$$

Then

$$|g_n(t) - g(t)| \leq \frac{K}{1-L} |u_n(t) - u(t)|\,.$$

Since $u_n \to u$, we get $g_n(t) \to g(t)$ as $n \to \infty$ for each $t \in J$. Let $\eta > 0$ be such that, for each $t \in J$, we have $|g_n(t)| \leq \eta$ and $|g(t)| \leq \eta$; then we have

$$\begin{aligned} (t-s)^{\alpha-1} |g_n(s) - g(s)| &\leq (t-s)^{\alpha-1} [|g_n(s)| + |g(s)|] \\ &\leq 2\eta (t-s)^{\alpha-1}\,. \end{aligned}$$

For each $t \in J$, the function $s \to 2\eta(t-s)^{\alpha-1}$ is integrable on $[0, t]$; then the Lebesgue dominated convergence theorem and (4.7) imply that

$$|N(u_n)(t) - N(u)(t)| \to 0 \;\text{ as }\; n \to \infty\,.$$

Hence,

$$\|N(u_n) - N(u)\|_\infty \to 0 \;\text{ as }\; n \to \infty\,.$$

Consequently, N is continuous.

Let $p^* = \sup_{t \in J} p(t)$ and

$$R \geq \frac{\frac{|c|}{|a+b|} + \left(1 + \frac{|b|}{|a+b|}\right) p^* M}{1 - \left(1 + \frac{|b|}{|a+b|}\right) q^* M}\,,$$

and define

$$D_R = \{u \in C(J, \mathbb{R}) : \|u\|_\infty \leq R\} .$$

It is clear that D_R is a bounded, closed, and convex subset of $C(J, \mathbb{R})$.

Claim 2: $N(D_R) \subset D_R$. Let $u \in D_R$; we show that $Nu \in D_R$. For each $t \in J$ we have

$$|Nu(t)| \leq \frac{|c|}{|a+b|} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |g(s)| ds + \frac{|b|}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} |g(s)| ds . \tag{4.8}$$

By (4.4.1), for each $t \in J$ we have

$$\begin{aligned} |g(t)| &= |f(t, u(t), g(t))| \\ &\leq p(t) + q(t)|u(t)| + r(t)|g(t)| \\ &\leq p(t) + q(t)R + r(t)|g(t)| \\ &\leq p^* + q^* R + r^* |g(t)| . \end{aligned}$$

Then

$$|g(t)| \leq \frac{p^* + q^* R}{1 - r^*} := M_1 .$$

Thus, (4.8) implies that

$$\begin{aligned} |Nu(t)| &\leq \frac{|c|}{|a+b|} + \frac{(p^* + q^* R)T^\alpha}{(1-r^*)\Gamma(\alpha+1)} + \frac{|b|(p^* + q^* R)T^\alpha}{|a+b|(1-r^*)\Gamma(\alpha+1)} \\ &\leq \frac{|c|}{|a+b|} + (p^* + q^* R)M + \frac{|b|(p^* + q^* R)M}{|a+b|} \\ &\leq \frac{|c|}{|a+b|} + p^* M \left(1 + \frac{|b|}{|a+b|}\right) + q^* M \left(1 + \frac{|b|}{|a+b|}\right) R \\ &\leq R . \end{aligned}$$

Then $N(D_R) \subset D_R$.

Claim 3: $N(D_R)$ is relatively compact. Let $t_1, t_2 \in J$, $t_1 < t_2$, and let $u \in D_R$. Then

$$\begin{aligned} |N(u)(t_2) - N(u)(t_1)| &= \left| \frac{1}{\Gamma(\alpha)} \int_0^{t_1} [(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}] g(s) ds \right. \\ &\quad \left. + \frac{1}{\Gamma(\alpha)} \int_{t1}^{t_2} [(t_2 - s)^{\alpha-1} g(s) ds \right| \\ &\leq \frac{M_1}{\Gamma(\alpha+1)} (t_2^\alpha - t_1^\alpha + 2(t_2 - t_1)^\alpha) . \end{aligned}$$

As $t_1 \to t_2$, the right-hand side of the preceding inequality tends to zero.

As a consequence of Claims 1–3, together with the Ascoli–Arzelà theorem, we conclude that $N\colon C(J,\mathbb{R}) \to C(J,\mathbb{R})$ is continuous and compact. As a consequence of Schauder's fixed point theorem [149], we deduce that N has a fixed point that is a solution of problem (4.1)–(4.2). □

Our next existence result is based on a nonlinear alternative of the Leray–Schauder type.

Theorem 4.5. *Assume* (4.3.1), (4.3.2), (4.4.1), *and* (4.6) *hold. Then BVP* (4.1)–(4.2) *has at least one solution.*

Proof. Consider the operator N defined in (4.4). We will show that N satisfies the assumptions of the Leray–Schauder fixed point theorem. The proof will be given in several claims.

Claim 1: Clearly N is continuous.

Claim 2: N maps bounded sets to bounded sets in $C(J,\mathbb{R})$. Indeed, it is enough to show that for any $\rho > 0$ there exist a positive constant ℓ such that for each $u \in B_\rho = \{u \in C(J,\mathbb{R})\colon \|u\|_\infty \le \rho\}$ we have $\|N(u)\|_\infty \le \ell$.

For $u \in B_\rho$, we have, for each $t \in J$,

$$|(Nu)(t)| \le \frac{|c|}{|a+b|} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|g(s)|ds + \frac{|b|}{|a+b|\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}|g(s)|ds\,. \tag{4.9}$$

By (4.4.1), for each $t \in J$ we have

$$\begin{aligned}|g(t)| &= |f(t,u(t),g(t))|\\ &\le p(t) + q(t)|u(t)| + r(t)|g(t)|\\ &\le p(t) + q(t)\rho + r(t)|g(t)|\\ &\le p^* + q^*\rho + r^*|g(t)|\,.\end{aligned}$$

Then

$$|g(t)| \le \frac{p^* + q^*\rho}{1-r^*} := M^*\,.$$

Thus, (4.9) implies that

$$|(Nu)(t)| \le \frac{|c|}{|a+b|} + \frac{M^*T^\alpha}{\Gamma(\alpha+1)} + \frac{|b|M^*T^\alpha}{|a+b|\Gamma(\alpha+1)}\,.$$

Consequently,

$$\|N(u)\|_\infty \le \frac{|c|}{|a+b|} + \frac{M^*T^\alpha}{\Gamma(\alpha+1)} + \frac{|b|M^*T^\alpha}{|a+b|\Gamma(\alpha+1)} := l\,.$$

Claim 3: Clearly, N maps bounded sets to equicontinuous sets of $C(J, \mathbb{R})$.

We conclude that $N: C(J, \mathbb{R}) \longrightarrow C(J, \mathbb{R})$ is continuous and completely continuous.

Claim 4: A priori bounds. We now show that there exists an open set $U \subseteq C(J, \mathbb{R})$, with $u \neq \lambda N(u)$, for $\lambda \in (0, 1)$ and $u \in \partial U$. Let $u \in C(J, \mathbb{R})$ and $u = \lambda N(u)$ for some $0 < \lambda < 1$. Thus, for each $t \in J$ we have

$$u(t) = \lambda \frac{c}{a+b} + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)ds + \frac{\lambda b}{(a+b)\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} |g(s)|ds .$$

This implies by (4.3.2) that for each $t \in J$ we have

$$\begin{aligned} |u(t)| \le{}& \frac{|c|}{|a+b|} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)ds \\ &+ \frac{|b|}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} |g(s)|ds . \end{aligned} \tag{4.10}$$

Additionally, by (4.4.1), for each $t \in J$ we have

$$\begin{aligned} |g(t)| &= |f(t, u(t), g(t))| \\ &\le p(t) + q(t)|u(t)| + r(t)|g(t)| \\ &\le p^* + q^*|u(t)| + r^*|g(t)| . \end{aligned}$$

Thus,

$$|g(t)| \le \frac{1}{1-r^*}(p^* + q^*|u(t)|) .$$

Hence,

$$\begin{aligned} |u(t)| \le{}& \frac{|c|}{|a+b|} + \frac{p^* T^\alpha}{(1-r^*)\Gamma(\alpha+1)} \left(1 + \frac{|b|}{|a+b|}\right) \\ &+ \frac{q^*}{(1-r^*)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |u(s)|ds \\ &+ \frac{|b|q^*}{(1-r^*)|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} |u(s)|ds \\ \le{}& \frac{|c|}{|a+b|} + \frac{p^* T^\alpha}{(1-r^*)\Gamma(\alpha+1)} \left(1 + \frac{|b|}{|a+b|}\right) \\ &+ \frac{q^*\|u\|_\infty}{(1-r^*)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds \\ &+ \frac{|b|q^*\|u\|_\infty}{(1-r^*)|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} ds \end{aligned}$$

$$\leq \frac{|c|}{|a+b|} + \frac{p^* T^\alpha}{(1-r^*)\Gamma(\alpha+1)}\left(1 + \frac{|b|}{|a+b|}\right) + \frac{q^* T^\alpha}{(1-r^*)\Gamma(\alpha+1)}\left(1 + \frac{|b|}{|a+b|}\right)\|u\|_\infty .$$

Then for each $t \in J$ we have

$$\|u\|_\infty \leq \frac{|c|}{|a+b|} + \frac{p^* T^\alpha}{(1-r^*)\Gamma(\alpha+1)}\left(1 + \frac{|b|}{|a+b|}\right) + \frac{q^* T^\alpha}{(1-r^*)\Gamma(\alpha+1)}\left(1 + \frac{|b|}{|a+b|}\right)\|u\|_\infty .$$

Thus, for each $t \in J$,

$$\|u\|_\infty \left[1 - \left(1 + \frac{|b|}{|a+b|}\right) q^* M\right] \leq \frac{|c|}{|a+b|} + \left(1 + \frac{|b|}{|a+b|}\right) p^* M .$$

Consequently,

$$\|u\|_\infty \leq \frac{\frac{|c|}{|a+b|} + \left(1 + \frac{|b|}{|a+b|}\right) p^* M}{1 - \left(1 + \frac{|b|}{|a+b|}\right) q^* M} := \overline{M} . \tag{4.11}$$

Let

$$U = \{u \in C(J, \mathbb{R}) : \|u\|_\infty < \overline{M} + 1\} .$$

By our choice of U, there is no $u \in \partial U$ such that $u = \lambda N(u)$ for $\lambda \in (0,1)$. As a consequence of Leray–Schauder's theorem ([149]), we deduce that N has a fixed point u in $\overline{U}$ that is a solution of problem (4.1)–(4.2). □

4.2.3 Examples

Example 1. Consider the BVP

$$^cD^{\frac{1}{2}} y(t) = \frac{1}{10e^{t+2}\left(1 + |y(t)| + |^cD^{\frac{1}{2}} y(t)|\right)} , \quad \text{for each } t \in [0,1] , \tag{4.12}$$

$$y(0) + y(1) = 0 . \tag{4.13}$$

Set

$$f(t,u,v) = \frac{1}{10e^{t+2}(1 + |u| + |v|)} , \quad t \in [0,1],\ u, v \in \mathbb{R} .$$

Clearly, the function f is jointly continuous.

For any $u, v, \bar{u}, \bar{v} \in \mathbb{R}$ and $t \in [0,1]$,

$$|f(t,u,v) - f(t,\bar{u},\bar{v})| \leq \frac{1}{10e^2}(|u - \bar{u}| + |v - \bar{v}|) .$$

Hence condition (4.3.2) is satisfied by $K = L = \frac{1}{10e^2}$. Thus, the condition

$$\frac{KT^\alpha}{(1-L)\Gamma(\alpha+1)}\left(1 + \frac{|b|}{|a+b|}\right) = \frac{3}{2(10e^2 - 1)\Gamma\left(\frac{3}{2}\right)} = \frac{3}{(10e^2-1)\sqrt{\pi}} < 1$$

is satisfied by $a = b = T = 1$, $c = 0$, and $\alpha = \frac{1}{2}$. It follows from Theorem 4.3 that problem (4.12)–(4.13) as a unique solution on J.

Example 2. Consider the BVP

$$^cD^{\frac{1}{2}}y(t) = \frac{\left(2 + |y(t)| + |^cD^{\frac{1}{2}}y(t)|\right)}{12e^{t+9}\left(1 + |y(t)| + |^cD^{\frac{1}{2}}y(t)|\right)}\,, \quad \text{for each } t \in [0,1]\,, \tag{4.14}$$

$$\frac{1}{2}y(0) + \frac{1}{2}y(1) = 1\,. \tag{4.15}$$

Set

$$f(t,u,v) = \frac{(2 + |u| + |v|)}{12e^{t+9}(1 + |u| + |v|)}\,, \quad t \in [0,1],\ u, v \in \mathbb{R}\,.$$

Clearly, the function f is jointly continuous. For any $u, v, \bar{u}, \bar{v} \in \mathbb{R}$ and $t \in [0,1]$

$$|f(t,u,v) - f(t,\bar{u},\bar{v})| \le \frac{1}{12e^9}(|u - \bar{u}| + |v - \bar{v}|)\,.$$

Hence, condition (4.3.2) is satisfied by $K = L = \frac{1}{12e^9}$. Also, we have

$$|f(t,u,v)| \le \frac{1}{12e^{t+9}}(2 + |u| + |v|)\,.$$

Thus, condition (4.34.1) is satisfied by $p(t) = \frac{1}{6e^{t+9}}$ and $q(t) = r(t) = \frac{1}{12e^{t+9}}$. The condition

$$q^*M\left(1 + \frac{|b|}{|a+b|}\right) = \frac{3}{2(12e^9 - 1)\Gamma\left(\frac{3}{2}\right)} = \frac{3}{(12e^9 - 1)\sqrt{\pi}} < 1$$

is satisfied by $a = b = \frac{1}{2}$, $c = T = 1$, $\alpha = \frac{1}{2}$, and $q^* = r^* = \frac{1}{12e^9}$.

It follows from Theorem 4.4 that problem (4.14)–(4.15) has at least one solution on J.

4.3 BVP for NIFDE with $1 < \alpha \le 2$

4.3.1 Introduction and Motivations

The purpose of this section is to establish existence and uniqueness results to the following implicit fractional order differential equation:

$$^cD^{\alpha}y(t) = f(t, y(t), {}^cD^{\alpha}y(t))\,, \quad \text{for each } t \in J = [0,T],\ T > 0,\ 1 < \alpha \le 2\,, \tag{4.16}$$

$$y(0) = y_0\,, \quad y(T) = y_1\,, \tag{4.17}$$

where $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function and $y_0, y_1 \in \mathbb{R}$.

We present three results for problem (4.16)–(4.17). The first one is based on the Banach contraction principle, the second one on Schauder's fixed point theorem, and the last one on the nonlinear alternative of the Leray–Schauder type.

4.3.2 Existence of Solutions

Let us define what we mean by a solution of problem (4.16)–(4.17).

Definition 4.6. A function $u \in C^1(J)$ is said to be a solution of problem (4.16)–(4.17) if u satisfies equation (4.16) on J and conditions (4.17).

4.3.2.1 Preparatory Lemmas

For the existence of solutions of problem (4.16)–(4.17), we need the following auxiliary lemma.

Lemma 4.7. *Let* $1 < \alpha \le 2$ *and* $g: J \to \mathbb{R}$ *be continuous. A function* y *is a solution of the fractional BVP*

$$\begin{aligned} &{}^cD^\alpha y(t) = f(t, y(t), {}^cD^\alpha y(t)), \quad \textit{for each, } t \in J,\ 1 < \alpha \le 2, \\ &y(0) = y_0, \quad y(T) = y_1, \end{aligned}$$

if and only if y *is a solution of the fractional integral equation*

$$y(t) = l(t) + \int_0^T G(t,s) f\left(s, l(s) + \int_0^T G(t,\tau) g(\tau) d\tau, g(s) \right) ds, \tag{4.18}$$

where

$$\begin{aligned} l(t) &= \left(1 - \frac{t}{T}\right) y_0 + \frac{t}{T} y_1 = y_0 + \frac{(y_1 - y_0)}{T} t, \\ {}^cD^\alpha y(t) &= g(t), \end{aligned} \tag{4.19}$$

and

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} (t-s)^{\alpha-1} - \frac{t}{T}(T-s)^{\alpha-1} & \textit{if } 0 \le s \le t, \\ -\frac{t}{T}(T-s)^{\alpha-1} & \textit{if } t \le s \le T. \end{cases} \tag{4.20}$$

Proof. By Lemma 1.9 we reduce (4.16)–(4.17) to the equation

$$y(t) = I^\alpha g(t) + c_0 + c_1 t = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s) ds + c_0 + c_1 t$$

for some constants c_0, and $c_1 \in \mathbb{R}$. Conditions (4.17) give

$$c_0 = y_0, \quad c_1 = \frac{1}{T} y_T - \frac{1}{T} y_0 - \frac{1}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} g(s) ds.$$

Then the solution of (4.16)–(4.17) is given by

$$
\begin{aligned}
y(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)ds - \frac{t}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} g(s)ds \\
&\quad + \left(1 - \frac{t}{T}\right) y_0 + \frac{t}{T} y_1 \\
&= \frac{1}{\Gamma(\alpha)} \Bigg[\int_0^t [(t-s)^{\alpha-1} - \frac{t}{T}(T-s)^{\alpha-1}] g(s)ds \\
&\quad - \frac{t}{T} \int_t^T (T-s)^{\alpha-1} g(s)ds \Bigg] + \left(1 - \frac{t}{T}\right) y_0 + \frac{t}{T} y_1 .
\end{aligned}
$$

Hence, we get (4.18). Inversely, if y satisfies (4.18), then equations (4.16)–(4.17) hold. □

Remark 4.8. From the expression of $G(t, s)$, it is obvious that $G(t, s)$ is continuous on $[0, T] \times [0, T]$. Use the notation

$$
G^* := \sup\{|G(t,s)|, \ (t,s) \in J \times J\} .
$$

We are now in a position to state and prove our existence result for problem (4.16)–(4.17) based on Banach's fixed point.

Theorem 4.9. *Make the following assumptions:*
(4.9.1) *The function $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous.*
(4.9.2) *There exist constants $K > 0$ and $0 < L < 1$ such that*

$$
|f(t,u,v) - f(t,\bar{u},\bar{v})| \leq K|u - \bar{u}| + L|v - \bar{v}|
$$

for any $u, v, \bar{u}, \bar{v} \in \mathbb{R}$ and $t \in J$.

If

$$
\frac{KTG^*}{1-L} < 1 , \tag{4.21}
$$

then there exists a unique solution for BVP (4.16)–(4.17).

Proof. The proof will be given in several steps. Transform problem (4.16)–(4.17) into a fixed point problem. Define the operator $N : C(J, \mathbb{R}) \to C(J, \mathbb{R})$ by

$$
N(y)(t) = l(t) + \int_0^T G(t,s)k(s)ds , \tag{4.22}
$$

where $k \in C(J)$ satisfies the implicit functional equation

$$
k(t) = f(t, y(t), k(t)) ,
$$

and l and G are the functions defined by (4.19) and (4.20), respectively.

Clearly, the fixed points of the operator N are solutions of problem (4.16)–(4.17). Let $u, w \in C(J)$. Then for $t \in J$ we have

$$(Nu)(t) - (Nw)(t) = \int_0^T G(t,s)(g(s) - h(s))ds ,$$

where $g, h \in C(J)$ are such that

$$g(t) = f(t, u(t), g(t))$$

and

$$h(t) = f(t, w(t), h(t)) .$$

Then, for $t \in J$,

$$|(Nu)(t) - (Nw)(t)| \le \int_0^T |G(t,s)||g(s) - h(s)|ds . \tag{4.23}$$

By (4.9.2) we have

$$\begin{aligned} |g(t) - h(t)| &= |f(t, u(t), g(t)) - f(t, w(t), h(t))| \\ &\le K|u(t) - w(t)| + L|g(t) - h(t)| . \end{aligned}$$

Thus,

$$|g(t) - h(t)| \le \frac{K}{1-L}|u(t) - w(t)| .$$

By (4.23) we have

$$\begin{aligned} |(Nu)(t) - (Nw)(t)| &\le \frac{K}{(1-L)} \int_0^T |G(t,s)||u(s) - w(s)|ds \\ &\le \frac{KTG^*}{1-L}\|u - w\|_\infty . \end{aligned}$$

Then

$$\|Nu - Nw\|_\infty \le \frac{KTG^*}{1-L}\|u - w\|_\infty .$$

By (4.21), the operator N is a contraction. Hence, by Banach's contraction principle, N has a unique fixed point that is the unique solution of problem (4.16)–(4.17). □

Our next existence result is based on Schauder's fixed point theorem.

Theorem 4.10. *Assume* (4.9.1) *and* (4.9.2) *hold and*
(4.10.1) *There exist* $p, q, r \in C(J, \mathbb{R}_+)$ *with* $r^* = \sup_{t \in J} r(t) < 1$ *such that*

$$|f(t, u, w)| \le p(t) + q(t)|u| + r(t)|w| \quad \text{for } t \in J, \text{ and } u, w \in \mathbb{R} .$$

If

$$\frac{q^* T G^*}{1 - r^*} < 1 , \tag{4.24}$$

where $q^* = \sup_{t\in J} q(t)$, *then BVP* (4.16)–(4.17) *has at least one solution.*

Proof. Consider the operator N defined in (4.22). We will show that N satisfies the assumptions of Schauder's fixed point theorem. The proof will be given in several steps.

Claim 1: N is continuous. Let $\{u_n\}$ be a sequence such that $u_n \to u$ in $C(J)$. Then for each $t \in J$

$$|N(u_n)(t) - N(u)(t)| \le \int_0^T |G(t,s)||g_n(s) - g(s)|ds , \tag{4.25}$$

where $g_n, g \in C(J)$ such that

$$g_n(t) = f(t, u_n(t), g_n(t))$$

and

$$g(t) = f(t, u(t), g(t)) .$$

By (4.9.2), we have

$$\begin{aligned}|g_n(t) - g(t)| &= |f(t, u_n(t), g_n(t)) - f(t, u(t), g(t))| \\ &\le K|u_n(t) - u(t)| + L|g_n(t) - g(t)| .\end{aligned}$$

Then

$$|g_n(t) - g(t)| \le \frac{K}{1-L}|u_n(t) - u(t)| .$$

Since $u_n \to u$, we get $g_n(t) \to g(t)$ as $n \to \infty$ for each $t \in J$. Let $\eta > 0$ be such that, for each $t \in J$, we have $|g_n(t)| \le \eta$ and $|g(t)| \le \eta$. Then we have

$$\begin{aligned}|G(t,s)||g_n(s) - g(s)| &\le |G(t,s)|[|g_n(s)| + |g(s)|] \\ &\le 2\eta|G(t,s)| .\end{aligned}$$

For each $t \in J$ the function $s \to 2\eta|G(t,s)|$ is integrable on J. Then the Lebesgue dominated convergence theorem and (4.25) imply that

$$|N(u_n)(t) - N(u)(t)| \to 0 \ \text{ as } \ n \to \infty .$$

Hence,

$$\|N(u_n) - N(u)\|_\infty \to 0 \ \text{ as } \ n \to \infty .$$

Consequently, N is continuous.

Let

$$R \ge \frac{(2|y_0| + |y_1|)(1 - r^*) + G^* T p^*}{M} ,$$

where $M := 1 - r^* - G^* T q^*$ and $p^* = \sup_{t\in J} p(t)$. Define

$$D_R = \{u \in C(J) : \|u\|_\infty \le R\} .$$

It is clear that D_R is a bounded, closed, and convex subset of $C(J)$.

Claim 2: $N(D_R) \subset D_R$. Let $u \in D_R$; we show that $Nu \in D_R$. For each $t \in J$ we have

$$\begin{aligned}|Nu(t)| &\leq |l(t)| + \int_0^T |G(t,s)||g(s)|ds \\ &\leq |y_0| + |y_1 - y_0| + G^* \int_0^T |g(s)|ds \\ &\leq 2|y_0| + |y_1| + G^* \int_0^T |g(s)|ds\,, \end{aligned} \tag{4.26}$$

where $g(t) = f(t, u(t), g(t))$.

From (4.10.1), for each $t \in J$ we have

$$\begin{aligned}|g(t)| &= |f(t, u(t), g(t))| \\ &\leq p(t) + q(t)|u(t)| + r(t)|g(t)| \\ &\leq p(t) + q(t)R + r(t)|g(t)| \\ &\leq p^* + q^*R + r^*|g(t)|\,. \end{aligned}$$

Then

$$|g(t)| \leq \frac{p^* + q^*R}{1 - r^*}\,.$$

Thus, (4.26) implies that, for each $t \in J$,

$$\begin{aligned}|Nu(t)| &\leq 2|y_0| + |y_1| + \frac{p^* + q^*R}{1 - r^*}G^*T \\ &\leq R\,. \end{aligned}$$

Then $N(D_R) \subset D_R$.

Claim 3: $N(D_R)$ is relatively compact. Let $t_1, t_2 \in J$, $t_1 < t_2$, and let $u \in D_R$. Then

$$\begin{aligned}|N(u)(t_2) - N(u)(t_1)| &= \left| l(t_2) - l(t_1) + \int_0^T [G(t_2,s) - G(t_1,s)]g(s)ds \right| \\ &= \left| \frac{(y_1 - y_0)}{T}(t_2 - t_1) + \int_0^T [G(t_2,s) - G(t_1,s)]g(s)ds \right| \\ &\leq \left| \frac{(y_1 - y_0)}{T}(t_2 - t_1) \right| + \frac{p^* + q^*R}{1 - r^*} \left| \int_0^T [G(t_2,s) - G(t_1,s)]ds \right|\,. \end{aligned}$$

As $t_1 \to t_2$, the right-hand side of the preceding inequality tends to zero.

As a consequence of Claims 1–3, together with the Ascoli–Arzelà theorem, we conclude that $N\colon C(J) \to C(J)$ is continuous and compact. As a consequence of Schauder's fixed point theorem, we deduce that N has a fixed point that is a solution of problem (4.16)–(4.17). □

Our next existence result is based on a nonlinear alternative of the Leray–Schauder type.

Theorem 4.11. *Assume* (4.9.1), (4.9.2), (4.10.1), *and* (4.24) *hold. Then the initial value problem (IVP)* (4.16)–(4.17) *has at least one solution.*

Proof. Consider the operator N defined in (4.22). We will show that N satisfies the assumptions of the Leray–Schauder fixed point theorem. The proof will be given in several claims.

Claim 1: Clearly N is continuous.

Claim 2: N maps bounded sets to bounded sets in $C(J)$. Indeed, it is enough to show that for any $\rho > 0$ there exist a positive constant ℓ such that for each $u \in B_\rho = \{u \in C(J, \mathbb{R}) : \|u\|_\infty \le \rho\}$ we have $\|N(u)\|_\infty \le \ell$.

For $u \in B_\rho$ we have, for each $t \in J$,

$$\begin{aligned}|Nu(t)| &\le |l(t)| + \int_0^T |G(t,s)\|g(s)|ds \\ &\le |y_0| + |y_1 - y_0| + G^* \int_0^T |g(s)|ds .\end{aligned}$$

Then

$$|Nu(t)| \le 2|y_0| + |y_1| + G^* \int_0^T |g(t)|ds . \tag{4.27}$$

By (4.10.1), for each $t \in J$ we have

$$\begin{aligned}|g(t)| &= |f(t, u(t), g(t))| \\ &\le p(t) + q(t)|u(t)| + r(t)|g(t)| \\ &\le p(t) + q(t)\rho + r(t)|g(t)| \\ &\le p^* + q^*\rho + r^*|g(t)| .\end{aligned}$$

Then

$$|g(t)| \le \frac{p^* + q^*\rho}{1 - r^*} := M^* .$$

Thus, (4.27) implies that

$$|Nu(t)| \le 2|y_0| + |y_1| + G^* M^* T .$$

Thus,

$$\|Nu\|_\infty \le 2|y_0| + |y_1| + G^* M^* T := l .$$

Claim 3: Clearly, N maps bounded sets to equicontinuous sets of $C(J)$.

We conclude that $N : C(J) \longrightarrow C(J)$ is continuous and completely continuous.

Claim 4: A priori bounds. We now show there exists an open set $U \subseteq C(J)$ with $u \neq \lambda N(u)$ for $\lambda \in (0,1)$ and $u \in \partial U$. Let $u \in C(J)$ and $u = \lambda N(u)$ for some $0 < \lambda < 1$. Thus, for each $t \in J$ we have

$$u(t) = \lambda l(t) + \lambda \int_0^T G(t,s)g(s)ds \, .$$

This implies by (3.9.2) that, for each $t \in J$, we have

$$|u(t)| \leq 2|y_0| + |y_1| + \int_0^T |G(t,s)||g(s)|ds \, . \tag{4.28}$$

By (4.10.1), for each $t \in J$, we have

$$\begin{aligned} |g(t)| &= |f(t,u(t),g(t))| \\ &\leq p(t) + q(t)|u(t)| + r(t)|g(t)| \\ &\leq p^* + q^*|u(t)| + r^*|g(t)| \, . \end{aligned}$$

Thus,

$$|g(t)| \leq \frac{1}{1-r^*}(p^* + q^*|u(t)|) \, .$$

Hence,

$$\begin{aligned} |u(t)| &\leq \left(2|y_0| + |y_1| + \frac{p^*TG^*}{1-r^*}\right) + \frac{q^*G^*}{1-r^*}\int_0^T |u(s)|ds \\ &\leq \left(2|y_0| + |y_1| + \frac{p^*TG^*}{1-r^*}\right) + \frac{q^*TG^*}{1-r^*}\|u\|_\infty \, . \end{aligned}$$

Then

$$\|u\|_\infty \leq \left(2|y_0| + |y_1| + \frac{p^*TG^*}{1-r^*}\right) + \frac{q^*TG^*}{1-r^*}\|u\|_\infty \, .$$

Thus,

$$\|u\|_\infty \leq \frac{M_1}{1 - \dfrac{q^*TG^*}{1-r^*}} := \overline{M} \, ,$$

where $M_1 = 2|y_0| + |y_1| + \dfrac{p^*TG^*}{1-r^*}$.

Let

$$U = \{u \in C(J) : \|u\|_\infty < \overline{M} + 1\} \, .$$

By our choice of U, there is no $u \in \partial U$ such that $u = \lambda N(u)$ for $\lambda \in (0,1)$. As a consequence of Leray–Schauder's theorem, we deduce that N has a fixed point u in $\overline{U}$ that is a solution of (4.16)–(4.17). □

4.3.3 Examples

Example 1. Consider the BVP

$$ {}^cD^{\frac{3}{2}}y(t) = \frac{1}{3e^{t+2}\left(1 + |y(t)| + |{}^cD^{\frac{3}{2}}y(t)|\right)} , \quad \text{for each } t \in [0,1] , \tag{4.29} $$

$$ y(0) = 1 , \quad y(1) = 2 . \tag{4.30} $$

Set

$$ f(t,u,v) = \frac{1}{3e^{t+2}(1 + |u| + |v|)} , \quad t \in [0,1],\ u, v \in \mathbb{R} . $$

Clearly, the function f is jointly continuous.

For any $u, v, \bar{u}, \bar{v} \in \mathbb{R}$ and $t \in [0,1]$

$$ |f(t,u,v) - f(t,\bar{u},\bar{v})| \leq \frac{1}{3e^2}(|u - \bar{u}| + |v - \bar{v}|) . $$

Hence, condition (4.9.2) is satisfied by $K = \frac{1}{3e^2}$ and $L = \frac{1}{3e^2} < 1$.

From (4.20) the function G is given by

$$ G(t,s) = \frac{1}{\Gamma\left(\frac{3}{2}\right)} \begin{cases} (t-s)^{\frac{1}{2}} - t(1-s)^{\frac{1}{2}} & \text{if } 0 \leq s \leq t \\ -t(1-s)^{\frac{1}{2}} & \text{if } t \leq s \leq 1 . \end{cases} $$

Clearly, $G^* < \frac{2}{\Gamma(\frac{3}{2})}$. Thus, condition

$$ \frac{KTG^*}{1-L} < 1 $$

is satisfied by $T = 1$ and $\alpha = \frac{3}{2}$. It follows from Theorem 4.9 that problem (4.29)–(4.30) has a unique solution on J.

Example 2. Consider the BVP

$$ {}^cD^{\frac{3}{2}}y(t) = \frac{\left(6 + |y(t)| + |{}^cD^{\frac{3}{2}}y(t)|\right)}{10e^{t+1}\left(1 + |y(t)| + |{}^cD^{\frac{3}{2}}y(t)|\right)} , \quad \text{for each } t \in [0,1] , \tag{4.31} $$

$$ y(0) = 1 , \quad y(1) = 2 . \tag{4.32} $$

Set

$$ f(t,u,v) = \frac{6 + |u| + |v|}{10e^{t+1}(1 + |u| + |v|)} , \quad t \in [0,1],\ u, v \in \mathbb{R} . $$

Clearly, the function f is jointly continuous.

For each $u, v, \bar{u}, \bar{v} \in \mathbb{R}$ and $t \in [0,1]$,

$$ |f(t,u,v) - f(t,\bar{u},\bar{v})| \leq \frac{1}{2e}(|u - \bar{u}| + |v - \bar{v}|) . $$

Hence, condition (4.9.2) is satisfied by $K = L = \frac{1}{2e}$. Also, for each $u, v, \in \mathbb{R}$ and $t \in [0,1]$ we have

$$ |f(t,u,v)| \leq \frac{1}{10e^{t+1}}(6 + |u| + |v|) . $$

Thus, condition (4.10.1) is satisfied by $p(t) = \frac{3}{5e^{t+1}}$ and $q(t) = r(t) = \frac{1}{10e^{t+1}}$. Clearly, $p^* = \frac{3}{5e}$, $q^* = \frac{1}{10e}$, and $r^* = \frac{1}{10e} < 1$.

From (4.20) the function G is given by

$$G(t,s) = \frac{1}{\Gamma\left(\frac{3}{2}\right)} \begin{cases} (t-s)^{\frac{1}{2}} - t(1-s)^{\frac{1}{2}} & \text{if } 0 \le s \le t \\ -t(1-s)^{\frac{1}{2}} & \text{if } t \le s \le 1 \,. \end{cases}$$

Clearly, $G^* < \frac{2}{\Gamma(\frac{3}{2})}$. Thus, condition

$$\frac{q^* T G^*}{1 - r^*} < 1$$

is satisfied by $T = 1$ and $\alpha = \frac{3}{2}$. It follows from Theorems 4.10 and 4.11 that problem (4.31)–(4.32) has at least one solution on J.

4.4 Stability Results for BVP for NIFDE

4.4.1 Introduction and Motivations

In this section, we establish some existence, uniqueness, and stability results for the implicit fractional order differential equations

$$^cD^\alpha y(t) = f(t, y(t), {}^cD^\alpha y(t)) \quad \text{for each } t \in J = [0,T],\ T > 0,\ 0 < \alpha \le 1\,, \tag{4.33}$$

$$ay(0) + by(T) = c\,, \tag{4.34}$$

where $f\colon J \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function, and a, b, c are real constants, with $a + b \neq 0$ and

$$^cD^\alpha y(t) = f(t, y(t), {}^cD^\alpha y(t))\,, \quad \text{for each } t \in J = [0,T],\ T > 0,\ 0 < \alpha \le 1\,, \tag{4.35}$$

$$y(0) + g(y) = y_0\,, \tag{4.36}$$

where $f\colon J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function, $g\colon C(J,\mathbb{R}) \to \mathbb{R}$ is a continuous function, and $y_0 \in \mathbb{R}$.

4.4.2 Existence of solutions

Let us define what we mean by a solution of problem (4.33)–(4.34) and (4.35)–(4.36).

Definition 4.12. A function $u \in C^1(J,\mathbb{R})$ is said to be a solution of problem (4.33)–(4.34) if u satisfies equation (4.33) and conditions (4.34) on J, and a function $y \in C^1(J,\mathbb{R})$ is called a solution of problem (4.35)–(4.36) if y satisfies equation (4.35) and conditions (4.36) on J.

For the existence of solutions to problems (4.33)–(4.34) and (4.35)–(4.36), we need the following auxiliary lemma.

Lemma 4.13. *Let* $0 < \alpha \le 1$, *and let* $h: [0, T] \longrightarrow \mathbb{R}$ *be a continuous function. The linear problem*

$$^cD^\alpha y(t) = h(t)\,, \qquad t \in J\,, \tag{4.37}$$

$$ay(0) + by(T) = c \tag{4.38}$$

has a unique solution given by

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)ds - \frac{1}{a+b}\left[\frac{b}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} h(s)ds - c\right]. \tag{4.39}$$

Proof. By the integration of formula (4.37) we obtain

$$y(t) = y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)ds\,. \tag{4.40}$$

We use condition (4.38) to compute the constant y_0, so we have

$$ay(0) = ay_0 \text{ and by } y(T) = by_0 + \frac{b}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} h(s)ds\,;$$

then $ay(0) + by(T) = c$. Since

$$y_0 = \frac{-1}{(a+b)}\left[\frac{b}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} h(s)ds - c\right],$$

we can use this in (4.40) to obtain (4.39). □

Lemma 4.14. *Let* $f(t, u, v): J \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ *be a continuous function; then problem* (4.33)–(4.34) *is equivalent to the problem*

$$y(t) = \tilde{A} + I^\alpha g(t)\,, \tag{4.41}$$

where $g \in C(J, \mathbb{R})$ *satisfies the functional equation*

$$g(t) = f(t, \tilde{A} + I^\alpha g(t), g(t))$$

and

$$\tilde{A} = \frac{1}{a+b}\left[c - \frac{b}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} g(s)ds\right].$$

Proof. Let y be a solution of (4.41). We will show that y is a solution of (4.33)–(4.34). We have

$$y(t) = \tilde{A} + I^{\alpha} g(t) .$$

Thus, $y(0) = \tilde{A}$ and $y(T) = \tilde{A} + \frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} g(s) ds$, so

$$\begin{aligned} ay(0) + by(T) &= \frac{-ab}{(a+b)\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} g(s) ds \\ &\quad + \frac{ac}{a+b} - \frac{b^2}{(a+b)\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} g(s) ds \\ &\quad + \frac{bc}{a+b} + \frac{b}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} g(s) ds . \\ &= c . \end{aligned}$$

Then

$$ay(0) + by(T) = c .$$

On the other hand, we have

$$\begin{aligned} {}^cD^{\alpha} y(t) &= {}^cD^{\alpha}(\tilde{A} + I^{\alpha} g(t)) = g(t) \\ &= f(t, y(t), {}^c D^{\alpha} y(t)) . \end{aligned}$$

Thus, y is a solution of problem (4.33)–(4.34). □

Lemma 4.15. *Let $0 < \alpha \leq 1$, and let $h: [0, T] \longrightarrow \mathbb{R}$ be a continuous function. The linear problem*

$$\begin{aligned} {}^cD^{\alpha} y(t) &= h(t) , \quad t \in J \\ y(0) + g(y) &= y_0 \end{aligned}$$

has a unique solution given by

$$y(t) = y_0 - g(y) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds .$$

Lemma 4.16. *Let $f: J \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function; then problem* (4.35)–(4.36) *is equivalent to the problem*

$$y(t) = y_0 - g(y) + I^{\alpha} K_y(t) ,$$

where $K_y(t) = f(t, y(t), K_y(t))$.

Theorem 4.17. *Make the following assumption:*
(4.17.1) *There exist two constants $K > 0$ and $0 < L < 1$ such that*

$$\|f(t,u,v) - f(t,\bar{u},\bar{v})\| \le K\|u-\bar{u}\| + L\|v-\bar{v}\| \quad \text{for each } t \in J \text{ and } u,\bar{u},v,\bar{v} \in \mathbb{R}\,.$$

If

$$\frac{KT^{\alpha}}{(1-L)\Gamma(\alpha+1)}\left(1+\frac{\|b\|}{\|a+b\|}\right) < 1\,, \tag{4.42}$$

then problem (4.33)–(4.34) *has a unique solution.*

Proof. Let N be the operator defined by

$$N\colon C(J,\mathbb{R}) \longrightarrow C(J,\mathbb{R})$$

$$Ny(t) = \tilde{A}_y + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g_y(s)ds\,,$$

where

$$g_y(t) = f(t, \tilde{A}_y + I^{\alpha}g_y(t), g_y(t))$$

and

$$\tilde{A}_y = \frac{1}{a+b}\left[c - \frac{b}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} g_y(s)ds\right]\,.$$

By Lemma 4.14, it is clear that the fixed points of N are solutions of (4.33)–(4.34). Let $y_1, y_2 \in C(J,\mathbb{R})$ and $t \in J$; then we have

$$\begin{aligned}\|Ny_1(t) - Ny_2(t)\| \le{}& \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|g_{y_1}(s) - g_{y_2}(s)\|ds \\ &+ \frac{\|b\|}{\|a+b\|\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\|g_{y_1}(s) - g_{y_2}(s)\|ds\end{aligned} \tag{4.43}$$

and

$$\begin{aligned}\|g_{y_1}(t) - g_{y_2}(t)\| &= \|f(t, y_1(t), {}^cD^{\alpha}y_1(t)) - f(t, y_2(t), {}^cD^{\alpha}y_2(t))\| \\ &\le K\|y_1(t) - y_2(t)\| + L\|g_{y_1}(t) - g_{y_2}(t)\|\,.\end{aligned}$$

Then

$$\|g_{y_1}(t) - g_{y_2}(t)\| \le \frac{K}{1-L}\|y_1(t) - y_2(t)\|\,. \tag{4.44}$$

By replacing (4.44) in inequality (4.43), we obtain

$$\begin{aligned}\|Ny_1(t) - Ny_2(t)\| &\le \frac{K}{(1-L)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|y_1(s) - y_2(s)\|ds \\ &\quad + \frac{\|b\|K}{(1-L)\|a+b\|\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\|y_1(s) - y_2(s)\|ds \\ &\le \frac{KT^\alpha}{(1-L)\Gamma(\alpha+1)}\|y_1 - y_2\|_\infty \\ &\quad + \frac{\|b\|KT^\alpha}{(1-L)\|a+b\|\Gamma(\alpha+1)}\|y_1 - y_2\|_\infty .\end{aligned}$$

Then

$$\|Ny_1 - Ny_2\|_\infty \le \left[\frac{KT^\alpha}{(1-L)\Gamma(\alpha+1)}\left(1 + \frac{\|b\|}{\|a+b\|}\right)\right]\|y_1 - y_2\|_\infty .$$

From (4.42), the operator N has a unique fixed point that is the unique solution. □

Theorem 4.18. *Make the following assumption:*
(4.18.1) *There exist* $K > 0$, $0 < \overline{K} < 1$, *and* $0 < L < 1$ *such that*

$$\|f(t,u,v) - f(t,\overline{u},\overline{v})\| \le K\|u - \overline{u}\| + \overline{K}\|v - \overline{v}\| \quad \text{for any } u, \overline{u}, v, \overline{v} \in \mathbb{R}$$

and

$$\|g(y) - g(\overline{y})\| \le L\|y - \overline{y}\| \quad \text{for any } y, \overline{y} \in C(J,\mathbb{R}) .$$

If

$$L + \frac{KT^\alpha}{(1-\overline{K})\Gamma(\alpha+1)} < 1 , \tag{4.45}$$

then BVP (4.35)–(4.36) *has a unique solution on* J.

Proof. Let

$$N\colon C(J,\mathbb{R}) \to C(J,\mathbb{R})$$

$$Ny(t) = y_0 - g(y) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}K_y(s)ds ,$$

where

$$K_y(t) = f(t, y(t), K_y(t)) .$$

By Lemma 4.16, it is easy to see that the fixed points of N are the solutions of problem (4.35)–(4.36). Let $y_1, y_2 \in C(J,\mathbb{R})$; for any $t \in J$ we have

$$\|Ny_1(t) - Ny_2(t)\| \le \|g(y_1) - g(y_2)\| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|K_{y_1}(s) - K_{y_2}(s).\|ds .$$

Then

$$\|Ny_1(t) - Ny_2(t)\| \le L\|y_1(t) - y_2(t)\| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|K_{y_1}(s) - K_{y_2}(s)\|ds \,. \tag{4.46}$$

On the other hand, for every $t \in J$ we have

$$\begin{aligned}\|K_{y_1}(t) - K_{y_2}(t)\| &= \|f(t, y_1(t), K_{y_1}(t)) - f(t, y_2(t), K_{y_2}(t))\| \\ &\le K\|y_1(t) - y_2(t)\| + \overline{K}\|K_{y_1}(t) - K_{y_2}(t)\| \,.\end{aligned}$$

Thus,

$$\|K_{y_1}(t) - K_{y_2}(t)\| \le \frac{K}{1-\overline{K}}\|y_1(t) - y_2(t)\| \,. \tag{4.47}$$

Replacing (4.47) in inequality (4.46), we obtain

$$\begin{aligned}\|Ny_1(t) - Ny_2(t)\| &\le L\|y_1(t) - y_2(t)\| \\ &\quad + \frac{K}{\left(1-\overline{K}\right)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|y_1(s) - y_2(s)\| \\ &\le \left[L + \frac{KT^\alpha}{\left(1-\overline{K}\right)\Gamma(\alpha+1)}\right]\|y_1 - y_2\|_\infty \,.\end{aligned}$$

Then

$$\|Ny_1 - Ny_2\|_\infty \le \left[L + \frac{KT^\alpha}{\left(1-\overline{K}\right)\Gamma(\alpha+1)}\right]\|y_1 - y_2\|_\infty \,.$$

Thus, N is a contraction. Hence, the operator N has a unique fixed point that is the unique solution of problem (4.35)–(4.36). □

4.4.3 Ulam–Hyers–Rassias stability

Definition 4.19. A solution of the implicit differential inequality

$$\|{}^cD^\alpha z(t) - f(t, z(t), {}^cD^\alpha z(t))\| \le \epsilon \,, \quad t \in J \,,$$

with fractional order is called a fractional ϵ-solution of the implicit fractional differential equation (4.33).

Theorem 4.20. *Assume* (4.17.1) *and* (4.42) *hold; then problem* (4.33)–(4.34) *is Ulam–Hyers stable.*

Proof. Let $\epsilon > 0$ and $z \in C^1(J, \mathbb{R})$ be a function that satisfies the inequality

$$\|^c D^\alpha z(t) - f(t, z(t), {}^c D^\alpha z(t))\| \le \epsilon \text{ for any } t \in J\,, \tag{4.48}$$

and let $y \in C(J, \mathbb{R})$ be the unique solution of the Cauchy problem

$$\begin{cases} {}^c D^\alpha y(t) = f(t, y(t), {}^c D^\alpha y(t)); & t \in J;\ 0 < \alpha \le 1 \\ y(0) = z(0)\,, \quad y(T) = z(T)\,. \end{cases}$$

Using Lemma 4.14, we obtain

$$y(t) = \tilde{A}_y + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g_y(s)ds\,.$$

On the other hand, if $y(T) = z(T)$ and $y(0) = z(0)$, then $\tilde{A}_y = \tilde{A}_z$. Indeed,

$$\|\tilde{A}_y - \tilde{A}_z\| \le \frac{\|b\|}{\|a+b\|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \|g_y(s) - g_z(s)\| ds\,,$$

and by inequality (4.44) we find

$$\begin{aligned} \|\tilde{A}_y - \tilde{A}_z\| &\le \frac{\|b\|K}{(1-L)\|a+b\|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \|y(s) - z(s)\| ds \\ &= \frac{\|b\|K}{(1-L)\|a+b\|} I^\alpha \|y(T) - z(T)\| = 0\,. \end{aligned}$$

Thus,

$$\tilde{A}_y = \tilde{A}_z\,.$$

Hence, we have

$$y(t) = \tilde{A}_z + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g_y(s)ds\,.$$

By integration of inequality (4.48), we obtain

$$\|z(t) - \tilde{A}_z - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g_z(s)ds\| \le \frac{\epsilon t^\alpha}{\Gamma(\alpha+1)} \le \frac{\epsilon T^\alpha}{\Gamma(\alpha+1)}\,,$$

with

$$g_z(t) = f(t, \tilde{A}_z + I^\alpha g_z(t), g_z(t))\,.$$

We have for any $t \in J$

$$\begin{aligned}\|z(t) - y(t)\| &= \left| z(t) - \tilde{A}_z - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g_z(s) ds \right. \\ &\quad \left. + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(g_z(s) - g_y(s) \right) ds \right| \\ &\le \|z(t) - \tilde{A}_z - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g_z(s) ds\| \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|g_z(s) - g_y(s)\| ds \,.\end{aligned}$$

Using (4.44), we obtain

$$\|z(t) - y(t)\| \le \frac{\epsilon T^\alpha}{\Gamma(\alpha+1)} + \frac{K}{(1-L)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|z(s) - y(s)\| ds \,,$$

and by Gronwall's lemma we get

$$\|z(t) - y(t)\| \le \frac{\epsilon T^\alpha}{\Gamma(\alpha+1)} \left[1 + \frac{\gamma K T^\alpha}{(1-L)\Gamma(\alpha+1)} \right] := c\epsilon \,,$$

where $\gamma = \gamma(\alpha)$ is a constant. Moreover, if we set $\psi(\epsilon) = c\epsilon$, $\psi(0) = 0$, then problem (4.33)–(4.34) is generalized Ulam–Hyers stable. □

Theorem 4.21. *Assume* (4.17.1) *and* (4.42) *hold and*

(4.27.1) *there exists an increasing function* $\varphi \in C(J, \mathbb{R}_+)$, *and there exists* $\lambda_\varphi > 0$ *such that for any* $t \in J$,

$$I^\alpha \varphi(t) \le \lambda_\varphi \varphi(t) \,.$$

Then problem (4.33)–(4.34) *is Ulam–Hyers–Rassias stable.*

Proof. Let $z \in C^1(J, \mathbb{R})$ be a solution of the inequality

$$\|{}^cD^\alpha z(t) - f(t, z(t), {}^cD^\alpha z(t))\| \le \epsilon\varphi(t) \,, \quad t \in J, \ \epsilon > 0 \,, \tag{4.49}$$

and let $y \in C(J, \mathbb{R})$ be the unique solution of the Cauchy problem

$$\begin{cases} {}^cD^\alpha y(t) = f(t, y(t), {}^cD^\alpha y(t)); & t \in J; \ 0 < \alpha \le 1 \\ y(0) = z(0) \,, \quad y(T) = z(T) \,. \end{cases}$$

It follows from the proof of the previous theorem that

$$y(t) = \tilde{A}_z + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g_y(s) ds \,.$$

By integration of (4.49), we obtain

$$\|z(t) - \tilde{A}_z - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g_z(s)ds\| \le \frac{\epsilon}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\varphi(s)ds$$
$$\le \epsilon\lambda_\varphi \varphi(t) .$$

On the other hand, we have

$$\begin{aligned}\|z(t) - y(t)\| &= \left| z(t) - \tilde{A}_z - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g_z(s)ds \right.\\ &\quad \left. + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left(g_z(s) - g_y(s)\right)ds \right|\\ &\le \|z(t) - \tilde{A}_z - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g_z(s)ds\|\\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|g_z(s) - g_y(s)\|ds .\end{aligned}$$

Using (4.44), we have

$$\|z(t) - y(t)\| \le \epsilon\lambda_\varphi\varphi(t) + \frac{K}{(1-L)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|z(s) - y(s)\|ds .$$

By applying Gronwall's lemma, we get that for any $t \in J$:

$$\|z(t) - y(t)\| \le \epsilon\lambda_\varphi\varphi(t) + \frac{\gamma_1\epsilon K\lambda_\varphi}{(1-L)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\varphi(s)ds ,$$

where $\gamma_1 = \gamma_1(\alpha)$ is constant, and by (2.27.2) we have

$$\|z(t) - y(t)\| \le \epsilon\lambda_\varphi\varphi(t) + \frac{\gamma_1\epsilon K\lambda_\varphi^2\varphi(t)}{(1-L)} = \left(1 + \frac{\gamma_1 K\lambda_\varphi}{(1-L)}\right)\epsilon\lambda_\varphi\varphi(t) .$$

Then for any $t \in J$

$$\|z(t) - y(t)\| \le \left[\left(1 + \frac{\gamma_1 K\lambda_\varphi}{1-L}\right)\lambda_\varphi\right]\epsilon\varphi(t) = c\epsilon\varphi(t) .$$ □

Theorem 4.22. *Assume* (4.27.1) *and* (4.45) *hold; then problem* (4.35)–(4.36) *is Ulam–Hyers stable.*

Proof. Let $\epsilon > 0$, and let $z \in C^1(J, \mathbb{R})$, satisfying the inequality

$$\|{}^cD^\alpha z(t) - f\left(t, z(t), {}^cD^\alpha z(t)\right)\| \le \epsilon \quad \text{for every } t \in J , \tag{4.50}$$

and let $y \in C(J, \mathbb{R})$ be the unique solution of the Cauchy problem

$$\begin{cases} {}^cD^\alpha y(t) = f\left(t, y(t), {}^cD^\alpha y(t)\right), & t \in J,\ 0 < \alpha \le 1 \\ z(0) + g(y) = y_0 . \end{cases}$$

Thus,

$$y(t) = y_0 - g(y) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} K_y(s)ds ,$$

where $K_y(t) = f(t, y(t), K_y(t))$. By integration of inequality (4.50), we find

$$\|z(t) - y_0 + g(z) - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} K_z(s)ds\| \le \frac{\epsilon T^\alpha}{\Gamma(\alpha+1)} ,$$

where $K_z(t) = f(t, z(t), K_z(t))$. For every $t \in J$ we have

$$\begin{aligned} \|z(t) - y(t)\| &\le \|z(t) - y_0 + g(z) - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} K_z(s)ds\| \\ &\quad + \|g(y) - g(z) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(K_z(s) - K_y(s)\right) ds\| \\ &\le \frac{\epsilon T^\alpha}{\Gamma(\alpha+1)} + \|g(z) - g(y)\| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|K_z(s) - K_y(s)\| ds . \end{aligned}$$

Using (4.47), we obtain

$$\|z(t) - y(t)\| \le \frac{\epsilon T^\alpha}{\Gamma(\alpha+1)} + L\|z(t) - y(t)\| + \frac{K}{\left(1-\overline{K}\right)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|z(s) - y(s)\| ds .$$

Thus,

$$\|z(t) - y(t)\| \le \frac{\epsilon T^\alpha}{(1-L)\Gamma(\alpha+1)} + \frac{K}{(1-L)\left(1-\overline{K}\right)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|z(s) - y(s)\| ds .$$

Using Gronwall's lemma, for every $t \in J$ we obtain

$$\|z(t) - y(t)\| \le \frac{\epsilon T^\alpha}{(1-L)\Gamma(\alpha+1)} \left[1 + \frac{\gamma K T^\alpha}{(1-L)\left(1-\overline{K}\right)\Gamma(\alpha+1)}\right] := c\epsilon ,$$

where $\gamma = \gamma(\alpha)$ is a constant, so problem (4.35)–(4.36) is Ulam–Hyers stable. If we set $\psi(\epsilon) = c\epsilon$; $\psi(0) = 0$, then problem (4.35)–(4.36) is generalized Ulam–Hyers stable . □

Theorem 4.23. *Assume that* (4.27.1) *and inequality* (4.45) *and*

(4.23.1) *there exists an increasing function* $\varphi \in C(J, \mathbb{R}_+)$, *and there exists* $\lambda_\varphi > 0$ *such that for any* $t \in J$

$$I^\alpha \varphi(t) \le \lambda_\varphi \varphi(t)$$

are satisfied;

then problem (4.35)–(4.36) *is Ulam–Hyers–Rassias stable.*

4.4.4 Examples

Example 1. Consider the BVP

$$ {}^cD^{\frac{1}{2}}y(t) = \frac{1}{10e^{t+2}\left(1 + |y(t)| + |{}^cD^{\frac{1}{2}}y(t)|\right)} \quad \text{for each } t \in [0,1]\,, \tag{4.51}$$

$$ y(0) + y(1) = 0\,. \tag{4.52}$$

Set

$$ f(t,u,v) = \frac{1}{10e^{t+2}(1 + |u| + |v|)}\,, \quad t \in [0,1],\ u,v \in \mathbb{R}\,. $$

Clearly, the function f is jointly continuous.

For any $u, v, \bar{u}, \bar{v} \in \mathbb{R}$ and $t \in [0,1]$

$$ |f(t,u,v) - f(t,\bar{u},\bar{v})| \le \frac{1}{10e^2}(|u-\bar{u}| + |v-\bar{v}|)\,. $$

Hence, condition (4.27.1) is satisfied by $K = L = \frac{1}{10e^2}$.
Thus, condition

$$ \frac{KT^\alpha}{(1-L)\Gamma(\alpha+1)}\left(1 + \frac{|b|}{|a+b|}\right) = \frac{3}{2(10e^2-1)\Gamma\left(\frac{3}{2}\right)} = \frac{3}{(10e^2-1)\sqrt{\pi}} < 1 $$

is satisfied by $a = b = T = 1$, $c = 0$, and $\alpha = \frac{1}{2}$. From Theorem 4.17, problem (4.51)–(4.52) has a unique solution on J, and Theorem 4.20 implies that problem (4.51)–(4.52) is Ulam–Hyers stable.

Example 2. Consider the BVP

$$ {}^cD^{\frac{1}{2}}y(t) = \frac{e^{-t}}{(9+e^t)}\left[\frac{\|y(t)\|}{1+\|y(t)\|} - \frac{\|{}^cD^{\frac{1}{2}}y(t)\|}{1+\|{}^cD^{\frac{1}{2}}y(t)\|}\right]\,, \quad t \in J = [0,1]\,, \tag{4.53}$$

$$ y(0) + \sum_{i=1}^{n} c_i y(t_i) = 1\,, \tag{4.54}$$

where $0 < t_1 < t_2 < \cdots < t_n < 1$ and $c_i = 1, \ldots, n$ are positive constants, with

$$ \sum_{i=1}^{n} c_i < \frac{1}{3}\,. $$

Set

$$ f(t,u,v) = \frac{e^{-t}}{(9+e^t)}\left[\frac{u}{1+u} - \frac{v}{1+v}\right]\,, \quad t \in [0,1],\ u,v \in [0,+\infty)\,. $$

Clearly, the function f is continuous. For each $u, \bar{u}, v, \bar{v} \in \mathbb{R}$ and $t \in [0,1]$

$$ \begin{aligned} \|f(t,u,v) - f(t,\bar{u},\bar{v})\| &\le \frac{e^{-t}}{(9+e^t)}(\|u-\bar{u}\| + \|v-\bar{v}\|) \\ &\le \frac{1}{10}\|u-\bar{u}\| + \frac{1}{10}\|v-\bar{v}\|\,. \end{aligned} $$

On the other hand, we have

$$\begin{aligned}\|g(u) - g(\bar{u})\| &= \|\sum_{i=1}^{n} c_i u - \sum_{i=1}^{n} c_i \bar{u}\| \\ &\le \sum_{i=1}^{n} c_i \|u - \bar{u}\| \\ &< \frac{1}{3}\|u - \bar{u}\| .\end{aligned}$$

Hence, condition (4.27.1) is satisfied by $K = \overline{K} = \frac{1}{10}$ and $L = \frac{1}{3}$. We have

$$L + \frac{KT^\alpha}{\left(1 - \overline{K}\right)\Gamma(\alpha+1)} = \frac{1}{3} + \frac{1}{9\Gamma\left(\frac{3}{2}\right)} = \frac{9\sqrt{\pi}+6}{27\sqrt{\pi}} < 1 .$$

From Theorem 4.18, problem (4.53)–(4.54) has a unique solution on J, and Theorem 4.22 implies that problem (4.53)–(4.54) is Ulam–Hyers stable.

4.5 BVP for NIFDE in Banach Space

4.5.1 Introduction and Motivations

Recently, fractional differential equations have been studied by Abbas et al. [35, 43], Baleanu et al. [78, 80], Diethelm [137], Kilbas and Marzan [180], Srivastava et al. [181], Lakshmikantham et al. [187], and Samko et al. [239]. More recently, some mathematicians have considered BVPs and boundary conditions for implicit fractional differential equations.

In [164], Hu and Wang investigated the existence of a solutions to a nonlinear fractional differential equation with an integral boundary condition:

$$\begin{aligned}&D^\alpha u(t) = f(t, u(t), D^\beta u(t)) , \quad t \in (0,1),\ 1 < \alpha \le 2,\ 0 < \beta < 1 , \\ &u(0) = u_0 , \quad u(1) = \int_0^1 g(s)u(s)ds ,\end{aligned}$$

where $f\colon [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function and g is an integrable function. In [241], by means of Schauder's fixed point theorem, Su and Liu studied the existence of nonlinear fractional BVPs involving Caputo's derivative:

$$\begin{aligned}&{}^cD^\alpha u(t) = f(t, u(t), {}^cD^\beta u(t)) , \quad \text{for each } t \in (0,1),\ 1 < \alpha \le 2,\ 0 < \beta \le 1 , \\ &u(0) = u'(1) = 0, \quad \text{or} \quad u'(1) = u(1) = 0, \quad \text{or} \quad u(0) = u(1) = 0 ,\end{aligned}$$

where $f\colon [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

Many techniques have been developed for studying the existence and uniqueness of solutions of initial and BVPs for fractional differential equations. Several authors tried to develop a technique that depends on the Darbo or the Mönch fixed point theorem with the Hausdorff or Kuratowski measure of noncompactness. The notion of the measure of noncompactness was defined in many ways. In 1930, Kuratowski [185] defined the measure of noncompactness, $\alpha(A)$, of a bounded subset A of a metric space (X, d), and in 1955 Darbo [132] introduced a new type of fixed point theorem for set contractions.

The purpose of this section is to establish existence and uniqueness results for problems of implicit fractional differential equations in Banach space:

$$^{c}D^{\nu}y(t) = f(t, y(t), {}^{c}D^{\nu}y(t)), \quad \text{for each, } t \in J := [0, T],\ T > 0,\ 0 < \nu \leq 1\,,$$
$$ay(0) + by(T) = c\,,$$

where $(E, \|\cdot\|)$ is a real Banach space, $f: J \times E \times E \to E$ is a given function, and a, b are real, with $a + b \neq 0$, $c \in E$, and

$$^{c}D^{\nu}y(t) = f(t, y(t), {}^{c}D^{\nu}y(t)), \quad \text{for every } t \in J := [0, T], T > 0,\ 0 < \nu \leq 1\,,$$
$$y(0) + g(y) = y_0\,,$$

where $f: J \times E \times E \to E$ is a given function, $g: C(J, E) \to E$ is a continuous function, and $y_0 \in E$. The results of this section are based on Darbo's fixed point theorem combined with the technique of measures of noncompactness and on Mönch's fixed point theorem.

4.5.2 Existence Results for BVPs in Banach Space

The purpose of this section is to establish sufficient conditions for the existence of solutions to the problem of implicit fractional differential equations with a Caputo fractional derivative:

$$^{c}D^{\nu}y(t) = f(t, y(t), {}^{c}D^{\nu}y(t)), \quad \text{for each, } t \in J := [0, T],\ T > 0,\ 0 < \nu \leq 1\,, \tag{4.55}$$
$$ay(0) + by(T) = c\,, \tag{4.56}$$

where $f: J \times E \times E \to E$ is a given function and a, b are real, with $a + b \neq 0$ and $c \in E$.

For a given set V of functions $v: J \to E$ let us use the notation

$$V(t) = \{v(t), v \in V\}\,, \quad t \in J$$

and

$$V(J) = \{v(t): v \in V,\ t \in J\}\,.$$

Let us define what we mean by a solution of problem (4.55)–(4.56).

Definition 4.24. A function $y \in C^1(J, E)$ is said to be a solution of problem (4.55)–(4.56) if y satisfies equation (4.55) on J and conditions (4.56).

For the existence of solutions of problem (4.55)–(4.56), we need the following auxiliary lemma.

Lemma 4.25 ([79]). *Let $0 < \nu \leq 1$, and let $h\colon [0,T] \longrightarrow E$ be a continuous function. The linear problem*

$$
\begin{aligned}
&{}^{c}D^{\nu}y(t) = h(t)\,, \quad t \in J\,,\\
&ay(0) + by(T) = c\,,
\end{aligned}
$$

has a unique solution given by

$$
\begin{aligned}
y(t) = {}& \frac{1}{\Gamma(\nu)}\int_0^t (t-s)^{\nu-1}h(s)ds \\
& - \frac{1}{a+b}\left[\frac{b}{\Gamma(\nu)}\int_0^T (T-s)^{\nu-1}h(s)ds - c\right].
\end{aligned}
$$

Lemma 4.26. *Let $f(t,u,v)\colon J \times E \times E \longrightarrow E$ be a continuous function; then problem* (4.55)–(4.56) *is equivalent to the problem*

$$y(t) = \tilde{A} + I^{\nu}g(t)\,, \tag{4.57}$$

where $g \in C(J,E)$ satisfies the functional equation

$$g(t) = f(t, \tilde{A} + I^{\nu}g(t), g(t))$$

and

$$\tilde{A} = \frac{1}{a+b}\left[c - \frac{b}{\Gamma(\nu)}\int_0^T (T-s)^{\nu-1}g(s)ds\right].$$

Proof. Let y be a solution of (4.57). We will show that y is a solution of (4.55)–(4.56). We have

$$y(t) = \tilde{A} + I^{\nu}g(t)\,.$$

Thus, $y(0) = \tilde{A}$ and $y(T) = \tilde{A} + \frac{1}{\Gamma(\nu)}\int_0^T (T-s)^{\nu-1}g(s)ds$, so

$$
\begin{aligned}
ay(0) + by(T) = {}& \frac{-ab}{(a+b)\Gamma(\nu)}\int_0^T (T-s)^{\alpha-1}g(s)ds \\
& + \frac{ac}{a+b} - \frac{b^2}{(a+b)\Gamma(\nu)}\int_0^T (T-s)^{\nu-1}g(s)ds \\
& + \frac{bc}{a+b} + \frac{b}{\Gamma(\nu)}\int_0^T (T-s)^{\nu-1}g(s)ds\,. \\
= {}& c\,.
\end{aligned}
$$

Then

$$ay(0) + by(T) = c \,.$$

On the other hand, we have

$$\begin{aligned} {}^cD^\nu y(t) &= {}^cD^\nu(\tilde{A} + I^\nu g(t)) = g(t) \\ &= f(t, y(t), {}^cD^\nu y(t)) \,. \end{aligned}$$

Thus, y is a solution of problem (4.55)–(4.56). □

Let us list the conditions:
(4.33.1) The function $f: J \times E \times E \to E$ is continuous.
(4.33.2) There exist constants $K > 0$ and $0 < L < 1$ such that

$$\|f(t, u, v) - f(t, \bar{u}, \bar{v})\| \le K\|u - \bar{u}\| + L\|v - \bar{v}\|$$

for any $u, v, \bar{u}, \bar{v} \in E$, and $t \in J$.

We are now in a position to state and prove our existence result for problem (4.55)–(4.56) based on the concept of measures of noncompactness and Darbo's fixed point theorem.

Remark 4.27 ([66]). Condition (4.33.2) is equivalent to the inequality

$$\alpha\,(f(t, B_1, B_2)) \le K\alpha(B_1) + L\alpha(B_2)$$

for any bounded sets $B_1, B_2 \subseteq E$ and for each $t \in J$.

Theorem 4.28. *Assume that* (4.33.1) *and* (4.33.2) *hold. If*

$$\frac{(|b| + |a + b|)T^\nu K}{|a + b|\Gamma(\nu + 1)(1 - L)} < 1 \,, \tag{4.58}$$

and

$$\frac{KT^\nu}{(1 - L)\Gamma(\nu + 1)} < 1 \,, \tag{4.59}$$

then IVP (4.55)–(4.56) *has at least one solution on* J.

Proof. Transform problem (4.55)–(4.56) into a fixed point problem. Define the operator $N: C(J, E) \to C(J, E)$ by

$$N(y)(t) = \tilde{A} + I^\nu g(t) \,, \tag{4.60}$$

where $g \in C(J, E)$ satisfies the functional equation

$$g(t) = f(t, y(t), g(t))$$

and

$$\tilde{A} = \frac{1}{a + b}\left[c - \frac{b}{\Gamma(\nu)}\int_0^T (T - s)^{\nu-1} g(s)ds\right] \,.$$

Clearly, the fixed points of the operator N are solutions of problem (4.55)–(4.56). We will show that N satisfies the assumptions of Darbo's fixed point theorem. The proof will be given in several claims.

Claim 1: N is continuous. Let $\{u_n\}$ be a sequence such that $u_n \to u$ in $C(J, E)$. Then for each $t \in J$

$$\|N(u_n)(t) - N(u)(t)\| \leq \frac{|b|}{|a+b|\Gamma(\nu)} \int_0^T (T-s)^{\nu-1} \|g_n(s) - g(s)\| ds + \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} \|g_n(s) - g(s)\| ds \,, \tag{4.61}$$

where $g_n, g \in C(J, E)$ such that

$$g_n(t) = f(t, u_n(t), g_n(t))$$

and

$$g(t) = f(t, u(t), g(t)) \,.$$

By (4.33.2), for each $t \in J$ we have

$$\begin{aligned} \|g_n(t) - g(t)\| &= \|f(t, u_n(t), g_n(t)) - f(t, u(t), g(t))\| \\ &\leq K\|u_n(t) - u(t)\| + L\|g_n(t) - g(t)\| \,. \end{aligned}$$

Then

$$\|g_n(t) - g(t)\| \leq \frac{K}{1-L} \|u_n(t) - u(t)\| \,.$$

Since $u_n \to u$, we get $g_n(t) \to g(t)$, as $n \to \infty$ for each $t \in J$.

Let $\eta > 0$ be such that, for each $t \in J$, we have $\|g_n(t)\| \leq \eta$ and $\|g(t)\| \leq \eta$. Then we have

$$\begin{aligned} (t-s)^{\nu-1}\|g_n(s) - g(s)\| &\leq (t-s)^{\nu-1}[\|g_n(s)\| + \|g(s)\|] \\ &\leq 2\eta(t-s)^{\nu-1} \,. \end{aligned}$$

For each $t \in J$, the function $s \to 2\eta(t-s)^{\nu-1}$ is integrable on $[0, t]$; then the Lebesgue dominated convergence theorem and (4.61) imply that

$$\|N(u_n)(t) - N(u)(t)\| \to 0 \ \text{ as } \ n \to \infty \,.$$

Thus,

$$\|N(u_n) - N(u)\|_\infty \to 0 \ \text{ as } \ n \to \infty \,.$$

Hence, N is continuous.

Let R be a constant such that

$$R \geq \frac{\|c\|\Gamma(\nu+1)(1-L) + (|b| + |a+b|)T^\nu f^*}{|a+b|\Gamma(\nu+1)(1-L) - (|b| + |a+b|)T^\nu K}, \quad \text{where } f^* = \sup_{t\in J} \|f(t, 0, 0)\| \,. \tag{4.62}$$

Define

$$D_R = \{u \in C(J, E) : \|u\|_\infty \le R\} .$$

It is clear that D_R is a bounded, closed, and convex subset of $C(J, E)$.

Claim 2: $N(D_R) \subset D_R$. Let $u \in D_R$; we show that $Nu \in D_R$. For each $t \in J$ we have

$$\|Nu(t)\| \le \frac{\|c\|}{|a+b|} + \frac{|b|}{|a+b|\Gamma(\nu)} \int_0^T (T-s)^{\nu-1} \|g(s)\| ds + \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} \|g(s)\| ds . \tag{4.63}$$

By (4.33.2), for each $t \in J$ we have

$$\begin{aligned} \|g(t)\| &= \|f(t, u(t), g(t)) - f(t, 0, 0) + f(t, 0, 0)\| \\ &\le \|f(t, u(t), g(t)) - f(t, 0, 0)\| + \|f(t, 0, 0)\| \\ &\le K\|u(t)\| + L\|g(t)\| + f^* \\ &\le KR + L\|g(t)\| + f^* . \end{aligned}$$

Then

$$\|g(t)\| \le \frac{f^* + KR}{1-L} := M .$$

Thus, (4.62) and (4.63) imply that

$$\begin{aligned} \|Nu(t)\| &\le \frac{\|c\|}{|a+b|} + \left[\frac{|b|}{|a+b|} + 1\right] \frac{T^\nu}{\Gamma(\nu+1)} \left(\frac{f^* + KR}{1-L}\right) \\ &\le \frac{\|c\|}{|a+b|} + \frac{(|b| + |a+b|)T^\nu f^*}{|a+b|\Gamma(\nu+1)(1-L)} \\ &\quad + \frac{(|b| + |a+b|)T^\nu KR}{|a+b|\Gamma(\nu+1)(1-L)} \\ &\le R . \end{aligned}$$

Consequently,

$$N(D_R) \subset D_R .$$

Claim 3: $N(D_R)$ is bounded and equicontinuous. By Claim 2 we have $N(D_R) = \{N(u) : u \in D_R\} \subset D_R$. Thus, for each $u \in D_R$ we have $\|N(u)\|_\infty \le R$. Thus, $N(D_R)$ is bounded. Let $t_1, t_2 \in J$, $t_1 < t_2$, and let $u \in D_R$. Then

$$\begin{aligned} \|N(u)(t_2) - N(u)(t_1)\| &= \left\| \frac{1}{\Gamma(\nu)} \int_0^{t_1} [(t_2-s)^{\nu-1} - (t_1-s)^{\nu-1}] g(s) ds \right. \\ &\quad \left. + \frac{1}{\Gamma(\nu)} \int_{t1}^{t_2} (t_2-s)^{\nu-1} g(s) ds \right\| \\ &\le \frac{M}{\Gamma(\nu+1)} (t_2^\nu - t_1^\nu + 2(t_2-t_1)^\nu) . \end{aligned}$$

As $t_1 \to t_2$, the right-hand side of the preceding inequality tends to zero.

Claim 4: The operator $N: D_R \to D_R$ is a strict set contraction. Let $V \subset D_R$ and $t \in J$; then we have

$$\begin{aligned}\alpha(N(V)(t)) &= \alpha((Ny)(t), y \in V) \\ &\le \frac{1}{\Gamma(\nu)} \left\{ \int_0^t (t-s)^{\nu-1} \alpha(g(s)) ds, y \in V \right\} .\end{aligned}$$

Then Remark 4.27 implies that, for each $s \in J$,

$$\begin{aligned}\alpha(\{g(s), y \in V\}) &= \alpha(\{f(s, y(s), g(s)), y \in V\}) \\ &\le K\alpha(\{y(s), y \in V\}) + L\alpha(\{g(s), y \in V\}) .\end{aligned}$$

Thus,

$$\alpha\left(\{g(s), y \in V\}\right) \le \frac{K}{1-L} \alpha\{y(s), y \in V\} .$$

Then

$$\begin{aligned}\alpha(N(V)(t)) &\le \frac{K}{(1-L)\Gamma(\nu)} \left\{ \int_0^t (t-s)^{\nu-1} \{\alpha(y(s))\} ds, y \in V \right\} \\ &\le \frac{K\alpha_c(V)}{(1-L)\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} ds \\ &\le \frac{KT^\nu}{(1-L)\Gamma(\nu+1)} \alpha_c(V) .\end{aligned}$$

Therefore,

$$\alpha_c(NV) \le \frac{KT^\nu}{(1-L)\Gamma(\nu+1)} \alpha_c(V) .$$

So, by (4.59), the operator N is a set contraction. As a consequence of Theorem 1.45, we deduce that N has a fixed point that is a solution of problem (4.55)–(4.56). □

Our next existence result for problem (4.55)–(4.56) is based on the concept of measures of noncompactness and Mönch's fixed point theorem.

Theorem 4.29. *Assume* (4.33.1), (4.33.2), *and* (4.58) *hold. Then IVP* (4.55)–(4.56) *has at least one solution.*

Proof. Consider the operator N defined in (4.60). We will show that N satisfies the assumptions of Mönch's fixed point theorem. We know that $N: D_R \to D_R$ is bounded and continuous, and we need to prove that the implication

$$[V = \overline{\text{conv}} N(V) \ \text{ or } V = N(V) \cup \{0\}] \ \text{ implies } \ \alpha(V) = 0$$

holds for every subset V of D_R. Now let V be a subset of D_R such that $V \subset \overline{\text{conv}}(N(V) \cup \{0\})$. V is bounded and equicontinuous, and therefore the function $t \to v(t) = \alpha(V(t))$ is continuous on J. By Remark 4.27, Lemma 1.33, and the properties of the measure α we have for each $t \in J$

$$\begin{aligned} v(t) &\le \alpha(N(V)(t) \cup \{0\}) \\ &\le \alpha(N(V)(t)) \\ &\le \alpha\{(Ny)(t), y \in V\} \\ &\le \frac{K}{(1-L)\Gamma(v)} \int_0^t (t-s)^{v-1}\{\alpha(y(s))ds, y \in V\} \\ &\le \frac{K}{(1-L)\Gamma(v)} \int_0^t (t-s)^{v-1} v(s)ds \, . \end{aligned}$$

Lemma 1.52 implies that $v(t) = 0$ for each $t \in J$, $V(t)$ is relatively compact in E. In view of the Ascoli–Arzelà theorem, V is relatively compact in D_R. Applying now Theorem 1.46, we conclude that N has a fixed point $y \in D_R$. Hence, N has a fixed point that is a solution of problem (4.55)–(4.56). □

4.5.3 Existence Results for Nonlocal BVP in Banach Space

The purpose of this section is to establish sufficient conditions for the existence of solutions to the BVP for implicit fractional differential equations with a Caputo fractional derivative:

$$^cD^v y(t) = f(t, y(t), {}^cD^v y(t)) \, , \quad \text{for every } t \in J := [0, T], T > 0, \; 0 < v \le 1 \, , \tag{4.64}$$

$$y(0) + g(y) = y_0 \, , \tag{4.65}$$

where $f : J \times E \times E \to E$ is a given function, $g : C(J, E) \to E$ is a continuous function, and $y_0 \in E$. Finally, an example is given to demonstrate the application of our main results.

Let $(E; \|\cdot\|)$ be a Banach space, and $t \in J$. We denote by $C(J, E)$ the space of E valued continuous functions on J with the usual supremum norm

$$\|y\|_\infty = \sup\{\|y(t)\| : t \in J\}$$

for any $y \in C(J, E)$.

Definition 4.30. A function $y \in C^1(J, E)$ is called a solution of problem (4.64)–(4.65) if it satisfies equation (4.64) on J and condition (4.65).

Lemma 4.31. *Let $0 < \nu \le 1$, and let $h: [0, T] \longrightarrow E$ be a continuous function. The linear problem*

$$\begin{aligned} {}^cD^\nu y(t) &= h(t)\,, \quad t \in J\,, \\ y(0) + g(y) &= y_0 \end{aligned}$$

has a unique solution given by

$$y(t) = y_0 - g(y) + \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} h(s)ds\,.$$

Lemma 4.32. *Let $f: J \times E \times E \longrightarrow E$ be a continuous function; then problem* (4.64)–(4.65) *is equivalent to the problem*

$$y(t) = y_0 - g(y) + I^\nu H(t)$$

where $H(t) = f(t, y(t), H(t))$.

Introduce the following condition:
(4.39.1) There exists $0 < \overline{K}$ such that

$$\|g(u) - g(\overline{u})\| \le \overline{K}\|u - \overline{u}\| \quad \text{for any } u, \overline{u} \in C(J, E)\,.$$

Remark 4.33 ([66]). Condition (4.39.1) is equivalent to the inequality

$$\alpha(g(B)) \le \overline{K}\alpha(B)$$

for any bounded sets $B \subseteq E$.

Theorem 4.34. *Assume* (4.33.11), (4.33.2), *and* (4.39.1). *If*

$$\overline{K} + \frac{KT^\nu}{(1-L)\Gamma(\nu+1)} < 1\,, \tag{4.66}$$

then IVP (4.64)–(4.65) *has at least one solution on J.*

Theorem 4.35. *Assume* (4.33.11), (4.33.2), (4.39.1), *and* (4.66) *hold. If $\overline{K} < 1$, then IVP* (4.64)–(4.65) *has at least one solution.*

4.5.4 Examples

Example 1. Consider the infinite system

$${}^cD^{\frac{1}{2}} y_n(t) = \frac{\left(3 + \|y_n(t)\| + \|{}^cD^{\frac{1}{2}} y_n(t)\|\right)}{3e^{t+2}\left(1 + \|y_n(t)\| + \|{}^cD^{\frac{1}{2}} y_n(t)\|\right)} \quad \text{for each } t \in [0, 1]\,, \tag{4.67}$$

$$y_n(0) + y_n(1) = 0\,. \tag{4.68}$$

Set

$$E = l^1 = \{y = (y_1, y_2, \dots, y_n, \dots), \sum_{n=1}^{\infty} |y_n| < \infty\},$$

and

$$f(t, u, v) = \frac{(3 + \|u\| + \|v\|)}{3e^{t+2}(1 + \|u\| + \|v\|)}, \quad t \in [0, 1],\ u, v \in E.$$

E is a Banach space with the norm $\|y\| = \sum_{n=1}^{\infty} |y_n|$.

Clearly, the function f is jointly continuous. For any $u, v, \bar{u}, \bar{v} \in E$ and $t \in [0, 1]$

$$\|f(t, u, v) - f(t, \bar{u}, \bar{v})\| \leq \frac{1}{3e^2}(\|u - \bar{u}\| + \|v - \bar{v}\|).$$

Hence, condition (4.33.2) is satisfied by $K = L = \frac{1}{3e^2}$. The conditions

$$\frac{(|b| + |a + b|)T^\nu K}{|a + b|\Gamma(\nu + 1)(1 - L)} = \frac{1}{\sqrt{\pi}\left(e^2 - \frac{1}{3}\right)} < 1$$

and

$$\frac{KT^\nu}{(1 - L)\Gamma(\nu + 1)} = \frac{2}{(3e^2 - 1)\sqrt{\pi}} < 1$$

are satisfied by $a = b = T = 1, c = 0$, and $\nu = \frac{1}{2}$. From Theorem 4.28, problem (4.67)–(4.68) has at least one solution on J.

Example 2. Consider the BVP

$$^cD^{\frac{1}{2}} y_n(t) = \frac{e^{-t}}{(9 + e^t)}\left[1 + \frac{\|y_n(t)\|}{1 + \|y_n(t)\|} - \frac{\|^cD^{\frac{1}{2}} y_n(t)\|}{1 + \|^cD^{\frac{1}{2}} y_n(t)\|}\right], \quad t \in J = [0, 1], \tag{4.69}$$

$$y_n(0) + \sum_{i=1}^{m} c_i y_n(t_i) = 1, \tag{4.70}$$

where $0 < t_1 < t_2 < \cdots < t_m < 1$ and $c_i = 1, \dots, m$ are positive constants, with

$$\sum_{i=1}^{m} c_i < \frac{1}{3}.$$

Set

$$E = l^1 = \{y = (y_1, y_2, \dots, y_n, \dots), \quad \sum_{n=1}^{\infty} |y_n| < \infty\},$$

and

$$f(t, u, v) = \frac{e^{-t}}{(9 + e^t)}\left[1 + \frac{\|u\|}{1 + \|u\|} - \frac{\|v\|}{1 + \|v\|}\right], \quad t \in [0, 1],\ u, v \in E.$$

E is a Banach space with the norm $\|y\| = \sum_{n=1}^{\infty} |y_n|$.

Clearly, the function f is continuous. For each $u, \bar{u}, v, \bar{v} \in E$ and $t \in [0, 1]$

$$\begin{aligned}\|f(t, u, v) - f(t, \bar{u}, \bar{v})\| &\leq \frac{e^{-t}}{9 + e^t}(\|u - \bar{u}\| + \|v - \bar{v}\|) \\ &\leq \frac{1}{10}\|u - \bar{u}\| + \frac{1}{10}\|v - \bar{v}\|.\end{aligned}$$

Hence, condition (4.33.2) is satisfied by $K = L = \frac{1}{10}$.
On the other hand, we have for any $u, \bar{u} \in E$

$$\|g(u) - g(\bar{u})\| \leq \frac{1}{3}\|u - \bar{u}\| .$$

Hence, condition (4.39.1) is satisfied by $\overline{K} = \frac{1}{3}$. Also, the condition

$$\overline{K} + \frac{KT^{\nu}}{(1-L)\Gamma(\nu+1)} = \frac{9\sqrt{\pi}+6}{27\sqrt{\pi}} < 1$$

is satisfied by $T = 1$ and $\nu = \frac{1}{2}$. It follows from Theorem 4.35 that problem (4.69)–(4.70) has at least one solution on J.

4.6 L^1-Solutions of BVP for NIFDE

4.6.1 Introduction and Motivations

More recently, considerable attention has been paid to the existence of solutions of BVPs and boundary conditions for implicit fractional differential equations and integral equations with a Caputo fractional derivative. See, for example, [47, 53, 94, 164, 260] and references therein.

In [203], Murad and Hadid, by means of Schauder's fixed-point theorem and the Banach contraction principle, considered the BVP for the fractional differential equation

$$\begin{aligned} &D^{\alpha}y(t) = f(t, y(t), D^{\beta}y(t)) , \quad t \in J := (0,1),\ 1 < \alpha \leq 2,\ 0 < \beta < 1,\ 0 < \gamma \leq 1 , \\ &y(0) = 0 , \quad y(1) = I_0^{\gamma}y(s) , \end{aligned}$$

where $f\colon [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, and D^{α} is the Riemann–Liouville fractional derivative.

In [150], Guezane-Lakoud and Khaldi studied the BVP of the fractional integral boundary conditions

$$\begin{aligned} &{}^cD^{q}y(t) = f(t, y(t), {}^cD^{p}y(t)) , \quad t \in J := (0,1),\ 1 < q \leq 2,\ 0 < p < 1 , \\ &y(0) = 0 , \quad y'(1) = \alpha I_0^{p}y(1) , \end{aligned}$$

where $f\colon [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, and D^{α} is the Caputo fractional derivative.

In [241], by means of Schauder's fixed-point theorem, Su and Liu studied the existence of nonlinear fractional BVPs involving Caputo's derivative:

$$\begin{aligned} &{}^cD^{\alpha}u(t) = f(t, u(t), {}^cD^{\beta}u(t)) , \quad t \in J := (0,1),\ 1 < \alpha \leq 2,\ 0 < \beta < 1 , \\ &u(0) = 0 = u'(1) = 0 \ \text{ or } \ u'(1) = u(1) = 0 \ \text{ or } \ u(0) = u(1) = 0 , \end{aligned}$$

where $f\colon [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

In [103], Benchohra and Lazreg studied the existence of continuous solutions of problem (4.71)–(4.72) and the implicit fractional order differential equation

$$^{c}D^{\alpha}y(t) = f(t, y(t), {}^{c}D^{\alpha}y(t)) , \quad t \in J := [0, T],\ 0 < \alpha \le 1 ,$$

with boundary condition

$$ay(0) = y_0 + By(T) = c ,$$

where $f: J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function, $^{c}D^{\alpha}$ is the Caputo fractional derivative, and a, b, c are real constants, with $a + b \neq 0$.

The purpose of this section is to establish existence and uniqueness of integrable solutions to BVPs for the fractional order implicit differential equation

$$^{c}D^{\alpha}y(t) = f(t, y(t), {}^{c}D^{\alpha}y(t)) , \quad t \in J := [0, T],\ 1 < \alpha \le 2 , \tag{4.71}$$

$$y(0) = y_0 , \quad y(T) = y_T , \tag{4.72}$$

where $f: J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function, $y_0, y_T \in \mathbb{R}$.

4.6.2 Existence of solutions

Definition 4.36. A function $y \in L^1(J, \mathbb{R})$ is said to be a solution of BVP (4.71)–(4.72) if y satisfies (4.71) and (4.72).

For the existence of solutions to problem (4.71)–(4.72), we need the following auxiliary lemma.

Lemma 4.37. *Let $1 < \alpha \le 2$ and let $x \in L^1(J, \mathbb{R})$. The BVP (4.71)–(4.72) is equivalent to the integral equation*

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_0^T G(t, s)x(s)ds + y_0 + \frac{(y_T - y_0)t}{T} , \tag{4.73}$$

where x is the solution of the functional integral equation

$$x(t) = f\left(t, \frac{1}{\Gamma(\alpha)} \int_0^T G(t, s)x(s)ds + y_0 + \frac{(y_T - y_0)t}{T}, x(t) \right) , \tag{4.74}$$

and $G(t, s)$ is the Green's function defined by

$$G(t, s) := \begin{cases} (t-s)^{\alpha-1} - \frac{t(T-s)^{\alpha-1}}{T}, & 0 \le s \le t \le T , \\ \frac{-t(T-s)^{\alpha-1}}{T}, & 0 \le t \le s \le T . \end{cases} \tag{4.75}$$

Proof. Let $^{c}D^{\alpha}y(t) = x(t)$ in equation (4.71); then

$$x(t) = f(t, y(t), x(t)) , \tag{4.76}$$

and Lemma 1.9 implies that

$$y(t) = c_0 + c_1 t + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s) ds \,.$$

From (4.72), a simple calculation gives

$$c_0 = y_0$$

and

$$c_1 = -\frac{1}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} x(s) ds + \frac{(y_T - y_0)}{T} \,.$$

Hence, we get equation (4.73).

Inversely, we prove that equation (4.73) satisfies BVP (4.71)–(4.72). Differentiating (4.73), we get

$${}^c D^\alpha y(t) = x(t) = f(t, y(t), {}^c D^\alpha y(t)) \,.$$

By (4.73) and (4.75) we have

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s) ds - \frac{t}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} x(s) ds + y_0 + \frac{(y_T - y_0)t}{T} \,. \quad (4.77)$$

A simple calculation gives $y(0) = y_0$ and $y(T) = y_T$. □

Let us introduce the following conditions:

(4.44.1) $f\colon [0,T] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is measurable in $t \in [0,T]$, for any $(u_1, u_2) \in \mathbb{R}^2$ and continuous in $(u_1, u_2) \in \mathbb{R}^2$ for almost all $t \in [0,T]$.

(4.44.2) There exist a positive function $a \in L^1[0,T]$ and constants $b_i > 0$, $i = 1, 2$, such that

$$|f(t, u_1, u_2)| \le |a(t)| + b_1|u_1| + b_2|u_2|, \forall (t, u_1, u_2) \in [0,T] \times \mathbb{R}^2 \,.$$

Our first result is based on Schauder's fixed point theorem.

Theorem 4.38. *Assume* (4.44.1) *and* (4.44.2) *hold. If*

$$\frac{b_1 G_0 T}{\Gamma(\alpha)} + b_2 < 1 \,, \quad (4.78)$$

then BVP (4.71)–(4.72) *has at least one solution* $y \in L^1(J, \mathbb{R})$.

Proof. Transform problem (4.71)–(4.72) into a fixed point problem. Consider the operator

$$H\colon L^1(J, \mathbb{R}) \longrightarrow L^1(J, \mathbb{R})$$

defined by

$$(Hx)(t) = f\left(t, \frac{1}{\Gamma(\alpha)}\int_0^T G(t,s)x(s)ds + y_0 + \frac{(y_T - y_0)t}{T}, x(t)\right), \tag{4.79}$$

where G is given by 4.75. Let

$$G_0 := \max[|G(t,s)|, (t,s) \in J \times J]$$

and

$$r = \frac{b_1(|y_0| + |y_T|)T + \|a\|_{L_1}}{1 - \left(\frac{b_1 G_0 T}{\Gamma(\alpha)} + b_2\right)} .$$

Consider the set

$$B_r = \{x \in L^1([0,T], \mathbb{R}) : \|x\|_{L_1} \le r\} .$$

Clearly, B_r is nonempty, bounded, convex, and closed.

We will now show that $HB_r \subset B_r$; indeed, for each $x \in B_r$, from conditions (4.44.2) and (4.78) we get

$$\begin{aligned}
\|Hx\|_{L_1} &= \int_0^T |Hx(t)|dt \\
&= \int_0^T \left| f\left(t, \frac{1}{\Gamma(\alpha)}\int_0^T G(t,s)x(s)ds + y_0 + \frac{(y_T - y_0)t}{T}, x(t)\right)\right| dt \\
&\le \int_0^T \left[|a(t)| + b_1 \left| \frac{1}{\Gamma(\alpha)}\int_0^T G(t,s)x(s)ds - \left(\frac{t}{T} - 1\right) y_0 + \frac{t}{T} y_T \right| + b_2|x(t)| \right] dt \\
&\le \|a\|_{L_1} + \frac{b_1 G_0 T}{\Gamma(\alpha)}\|x\|_{L_1} + b_1(|y_0| + |y_T|)T + b_2\|x\|_{L_1} \\
&\le b_1(|y_0| + |y_T|)T + \|a\|_{L_1} + \left(\frac{b_1 G_0 T}{\Gamma(\alpha)} + b_2\right) r \\
&\le r .
\end{aligned}$$

Then $HB_r \subset B_r$. Assumption (4.44.1) implies that H is continuous. We will now show that H is compact, that is, HB_r is relatively compact. Clearly, HB_r is bounded in $L^1(J, \mathbb{R})$, i.e., condition (i) of Kolmogorov's compactness criterion is satisfied. It remains to show that $(Hx)_h \longrightarrow (Hx)$ in $L^1(J, \mathbb{R})$ for each $x \in B_r$.

Let $x \in B_r$; then we have

$$\begin{aligned}
&\|(Hx)_h - (Hx)\|_{L^1}\\
&= \int_0^T |(Hx)_h(t) - (Hx)(t)|dt\\
&= \int_0^T \left| \frac{1}{h} \int_t^{t+h} (Hx)(s)ds - (Hx)(t) \right| dt\\
&\le \int_0^T \left(\frac{1}{h} \int_t^{t+h} |(Hx)(s) - (Hx)(t)|ds \right) dt\\
&\le \int_0^T \left(\frac{1}{h} \int_t^{t+h} \left| f\left(s, \frac{1}{\Gamma(\alpha)} \int_0^T G(s,\tau)x(\tau)d\tau + y_0 + \frac{(y_T - y_0)s}{T}, x(s) \right) \right.\right.\\
&\qquad \left.\left. -f\left(t, \frac{1}{\Gamma(\alpha)} \int_0^T G(t,s)x(s)ds + y_0 + \frac{(y_T - y_0)t}{T}, x(t) \right) \right| ds \right) dt\,.
\end{aligned}$$

Since $x \in B_r \subset L^1(J, \mathbb{R})$, condition (4.44.2) implies that $f \in L^1(J, \mathbb{R})$. Thus, we have

$$\begin{aligned}
&\frac{1}{h} \int_t^{t+h} \left| f\left(s, \frac{1}{\Gamma(\alpha)} \int_0^T G(s,\tau)x(\tau)d\tau + y_0 + \frac{(y_T - y_0)s}{T}, x(s) \right) \right.\\
&\quad \left. -f\left(t, \frac{1}{\Gamma(\alpha)} \int_0^T G(t,s)x(s)ds + y_0 + \frac{(y_T - y_0)t}{T}, x(t) \right) \right| ds \longrightarrow 0\,, \quad \text{as } h \longrightarrow 0,\ t \in J\,.
\end{aligned}$$

Hence,

$$(Hx)_h \longrightarrow (Hx) \ \text{ uniformly as } \ h \longrightarrow 0\,.$$

Then by Kolmogorov's compactness criterion, HB_r is relatively compact. As a consequence of Schauder's fixed point theorem, BVP (4.71)–(4.72) has at least one solution in B_r. □

The next result is based on the Banach contraction principle.

Theorem 4.39. *Assume* (4.44.1) *holds and*
(4.46.1) *There exist constants* $k_1, k_2 > 0$ *such that*

$$|f(t,x_1,y_1) - f(t,x_2,y_2)| \le k_1|x_1 - x_2| + k_2|y_1 - y_2|, \quad t \in [0,T],\ x_1, x_2, y_1, y_2 \in \mathbb{R}.$$

If

$$\frac{k_1 T G_0}{\Gamma(\alpha)} + k_2 < 1\,, \tag{4.80}$$

then BVP (4.71)–(4.72) *has a unique solution* $y \in L^1([0,T], \mathbb{R})$.

Proof. We will use the Banach contraction principle to prove that H defined by (4.79) has a fixed point. Let $x, y \in L^1(J, \mathbb{R})$, and $t \in J$. Then we have

$$
\begin{aligned}
|(Hx)(t) - (Hy)(t)| &= \left| f\left(t, \frac{1}{\Gamma(\alpha)} \int_0^T G(t,s)x(s)ds + y_0 + \frac{(y_T - y_0)t}{T}, x(t) \right) \right. \\
&\quad \left. - f\left(t, \frac{1}{\Gamma(\alpha)} \int_0^T G(t,s)y(s)ds + y_0 + \frac{(y_T - y_0)t}{T}, y(t) \right) \right| . \\
&\le \frac{k_1}{\Gamma(\alpha)} \int_0^T |G(t,s)(x(s) - y(s))|ds + k_2|x(t) - y(t)| \\
&\le \frac{k_1 G_0}{\Gamma(\alpha)} \int_0^T |x(s) - y(s)|ds + k_2|x(t) - y(t)| .
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\|(Hx) - (Hy)\|_{L_1} &\le \frac{k_1 T G_0}{\Gamma(\alpha)} \|x - y\|_{L_1} + k_2 \int_0^T |x(t) - y(t)|dt \\
&\le \frac{k_1 T G_0}{\Gamma(\alpha)} \|x - y\|_{L_1} + k_2 \|x - y\|_{L_1} \\
&\le \left(\frac{k_1 T G_0}{\Gamma(\alpha)} + k_2 \right) \|x - y\|_{L_1} .
\end{aligned}
$$

Consequently, by (4.80), H is a contraction. As a consequence of the Banach contraction principle, the operator H has a fixed point that is a solution of problem (4.71)–(4.72). □

4.6.3 Nonlocal problem

This section is devoted to some existence and uniqueness results for the class of the nonlocal problem

$$ {}^cD^\alpha y(t) = f(t, y(t), {}^cD^\alpha y(t)) , \quad t \in J := [0, T], \ 1 < \alpha \le 2 , \tag{4.81} $$

$$ y(0) = g(y) , \quad y(T) = y_T , \tag{4.82} $$

where $g\colon L^1(J, \mathbb{R}) \to \mathbb{R}$ a continuous function. The nonlocal condition can be applied in physics with better effect than the classical initial condition $y(0) = y_0$. For example, $g(y)$ may be given by

$$ g(y) = \sum_{i=1}^{p} c_i y(t_i) , $$

where c_i, $i = 1, 2, \ldots, p$ are given constants and $0 < \cdots < t_p < T$. Nonlocal conditions were initiated by Byszewski [117] when he proved the existence and uniqueness of mild and classical solutions of nonlocal Cauchy problems. As remarked by Byszewski [117, 118], the nonlocal condition can be more useful than the standard initial condition to describe some physical phenomena.

Let us introduce the following set of conditions on the function g.

(4.46.2) There exists a constant $\tilde{k} > 0$ such that

$$|g(y) - g(\tilde{y}| \le \tilde{k}|y - \tilde{y}| \quad \text{for each } y, \tilde{y} \in L^1(J, \mathbb{R}) .$$

Theorem 4.40. *Assume* (4.44.1), (4.46.1), *and* (4.46.2) *hold. If*

$$\frac{2k_1 T^\alpha}{\Gamma(\alpha+1)} + k_1\tilde{k} + k_2 < 1 , \tag{4.83}$$

then BVP (4.81)–(4.82) *has a unique solution* $y \in L^1(J, \mathbb{R})$.

Transform problem (4.81)–(4.82) into a fixed point problem. Consider the operator

$$\tilde{H} : L^1(J, \mathbb{R}) \longrightarrow L^1(J, \mathbb{R})$$

defined by

$$\begin{aligned}(\tilde{H}x)(t) &= f\Bigg(t, \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}x(s)ds - \frac{t}{T\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}x(s)ds \\ &\quad - \left(\frac{t}{T} - 1\right)g(y) + \frac{t}{T}y_T, x(t)\Bigg) .\end{aligned} \tag{4.84}$$

Proof. We will use the Banach contraction principle to prove that $\tilde{H}$ defined by (4.84) has a fixed point. Let $x, y \in L^1(J, \mathbb{R})$, and $t \in J$. Then we have

$$\begin{aligned}&|(\tilde{H}x)(t) - (\tilde{H}y)(t)| \\ &= \Bigg| f\Bigg(t, \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}x(s)ds - \frac{t}{T\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}x(s)ds \\ &\quad - \left(\frac{t}{T} - 1\right)g(x) + \frac{t}{T}y_T, x(t)\Bigg) \\ &\quad - f\Bigg(t, \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds - \frac{t}{T\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}y(s)ds \\ &\quad - \left(\frac{t}{T} - 1\right)g(y) + \frac{t}{T}y_T, y(t)\Bigg)\Bigg| \\ &\le \frac{k_1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|(x(s) - y(s))|ds + \frac{k_1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}|(x(s) - y(s))|ds \\ &\quad + k_1|g(x) - g(y)| + k_2|x(t) - y(t)| .\end{aligned}$$

Thus,

$$\begin{aligned}\|(\tilde{H}x)-(\tilde{H}y)\|_{L_1} &\le \frac{k_1\|x-y\|_{L_1}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds + \frac{k_1\|x-y\|_{L_1}}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}ds \\ &\quad + k_1\tilde{k}\|x-y\|_{L_1} + k_2\|x-y\|_{L_1} \\ &\le \frac{2k_1T^\alpha}{\Gamma(\alpha+1)}\|x-y\|_{L_1} + k_1\tilde{k}\|x-y\|_{L_1} + k_2\|x-y\|_{L_1} \\ &\le \left(\frac{2k_1T^\alpha}{\Gamma(\alpha+1)} + k_1\tilde{k} + k_2\right)\|x-y\|_{L_1}\ .\end{aligned}$$

Consequently, by (4.83), $\tilde{H}$ is a contraction. As a consequence of the Banach contraction principle, we deduce that $\tilde{H}$ has a fixed point that is a solution of problem (4.81)–(4.82). □

4.6.4 Examples

Example 1. Let us consider the BVP

$$^cD^\alpha y(t) = \frac{e^{-t}}{(e^t+6)(1+|y(t)|+|^cD^\alpha y(t)|)}\ , \quad t \in J := [0,1],\ 1 < \alpha \le 2\ , \tag{4.85}$$

$$y(0) = 1\ , \quad y(1) = 2\ . \tag{4.86}$$

Set

$$f(t,y,z) = \frac{e^{-t}}{(e^t+6)(1+y+z)}\ , \quad (t,y,z) \in J \times [0,+\infty) \times [0,+\infty)\ .$$

Let $y, z \in [0,+\infty)$ and $t \in J$. Then we have

$$\begin{aligned}|f(t,y_1,z_1)-f(t,y_2,z_2)| &= \left|\frac{e^{-t}}{e^t+6}\left(\frac{1}{1+y_1+z_1} - \frac{1}{1+y_2+z_2}\right)\right| \\ &\le \frac{e^{-t}(|y_1-y_2|+|z_1-z_2|)}{(e^t+6)(1+y_1+z_1)(1+y_2+z_2)} \\ &\le \frac{e^{-t}}{(e^t+6)}(|y_1-y_2|+|z_1-z_2|) \\ &\le \frac{1}{7}|y_1-y_2| + \frac{1}{7}|z_1-z_2|\ .\end{aligned}$$

Hence, condition (4.46.1) holds, with $k_1 = k_2 = \frac{1}{7}$. Condition (4.80) is satisfied by $T = 1$. Indeed,

$$\frac{k_1TG_0}{\Gamma(\alpha)} + k_2 = \frac{G_0}{7\Gamma(\alpha)} + \frac{1}{7} < 1\ . \tag{4.87}$$

Then, by Theorem 4.39, problem (4.85)–(4.86) has a unique integrable solution on $[0,1]$ for values of α satisfying condition (4.87).

Example 2. Let us consider the nonlocal BVP

$$ {}^cD^\alpha y(t) = \frac{e^{-t}}{(e^t+9)(1+|y(t)|+|{}^cD^\alpha y(t)|)}\,, \quad t \in J := [0,1],\ 1<\alpha\le 2\,, \tag{4.88}$$

$$ y(0) = \sum_{i=1}^n c_i y(t_i)\,, \quad y(1) = 0\,, \tag{4.89}$$

where $0 < \cdots < t_n < 1$, c_i, $i = 1, 2, \ldots, n$, are given positive constants with $\sum_{i=1}^n c_i < \frac{4}{5}$. Set

$$ f(t,y,z) = \frac{e^{-t}}{(e^t+9)(1+y+z)}\,, \quad (t,y,z) \in J\times[0,+\infty)\times[0,+\infty)\,,$$

and

$$ g(y) = \sum_{i=1}^n c_i y(t_i)\,.$$

Let $y, z \in [0,+\infty)$ and $t \in J$. Then we have

$$\begin{aligned}
|f(t,y_1,z_1)-f(t,y_2,z_2)| &= \left|\frac{e^{-t}}{e^t+9}\left(\frac{1}{1+y_1+z_1}-\frac{1}{1+y_2+z_2}\right)\right| \\
&\le \frac{e^{-t}(|y_1-y_2|+|z_1-z_2|)}{(e^t+9)(1+y_1+z_1)(1+y_2+z_2)} \\
&\le \frac{e^{-t}}{(e^t+9)}(|y_1-y_2|+|z_1-z_2|) \\
&\le \frac{1}{10}|y_1-y_2| + \frac{1}{10}|z_1-z_2|\,.
\end{aligned}$$

Hence, condition (4.46.1) holds, with $k_1 = k_2 = \frac{1}{10}$. Also, we have

$$ |g(x)-g(y)| \le \sum_{i=1}^n c_i|x-y|\,.$$

Hence, (4.46.2) is satisfied by $\tilde{k} = \sum_{i=1}^n c_i$. Condition (4.83) is satisfied by $T = 1$. Indeed,

$$ \frac{2k_1T^\alpha}{\Gamma(\alpha+1)} + k_1\tilde{k} + k_2 = \frac{1}{5\Gamma(\alpha+1)} + \frac{1}{10}\sum_{i=1}^n c_i + \frac{1}{10} < 1 \Longleftrightarrow \Gamma(\alpha+1) > \frac{10}{41}\,. \tag{4.90}$$

Then by Theorem 4.40, problem (4.88)–(4.89) has a unique integrable solution on $[0,1]$ for values of α satisfying condition (4.90).

4.7 Notes and Remarks

The results of Chapter 4 are taken from Benchohra et al. [91, 103, 109]. Other results may be found in [95, 97, 202].