3 Impulsive Nonlinear Implicit Fractional Differential Equations

3.1 Introduction

Impulsive fractional differential equations are a very important class of fractional differential equations because many phenomena from physics, chemistry, engineering, and biology, for example, can be represented by impulsive fractional differential equations.

Impulsive differential equations describes processes subject to abrupt changes in their states. They have received much attention in the literature and we refer the reader to the books [23, 35, 76, 77, 100, 148, 186, 215, 240], the papers [17, 24, 39, 96, 106, 124, 157, 158, 251], and the references therein.
In this chapter, we establish uniqueness and some Ulam stability and results for several classes of nonlinear implicit fractional differential equations (NIFDEs) with finite delay and fixed time impulses.

3.2 Existence and Stability Results for Impulsive NIFDEs with Finite Delay

3.2.1 Introduction

In this section, we consider the problem of nonlinear implicit fractional differential equations with finite delay and impulses,

$$^cD^{\alpha}_{t_k}y(t) = f(t, y_t, {}^c D^{\alpha}_{t_k}y(t)) , \quad \text{for each, } t \in (t_k, t_{k+1}],\ k = 0, \ldots, m,\ 0 < \alpha \le 1 , \tag{3.1}$$

$$\Delta y|_{t_k} = I_k(y_{t_k^-}) , \quad k = 1, \ldots, m , \tag{3.2}$$

$$y(t) = \varphi(t) , \quad t \in [-r, 0],\ r > 0 , \tag{3.3}$$

where $^cD^{\alpha}_{t_k}$ is the Caputo fractional derivative, $f : J \times PC([-r, 0], \mathbb{R}) \times \mathbb{R} \to \mathbb{R}$ is a given function, $I_k : PC([-r, 0], \mathbb{R}) \to \mathbb{R}$, and $\varphi \in PC([-r, 0], \mathbb{R})$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$. For each function y_t defined on $[-r, T]$ and for any $t \in J$, we denote by y_t the element of $PC([-r, 0], \mathbb{R})$ defined by

$$y_t(\theta) = y(t + \theta) , \quad \theta \in [-r, 0] . \tag{3.4}$$

The arguments are based on the Banach contraction principle and Schaefer's fixed point theorem; here we also present two examples to show the applicability of our results.

https://doi.org/10.1515/9783110553819-003

3.2.2 Existence of Solutions

Let $J_0 = [t_0, t_1]$ and $J_k = (t_k, t_{k+1}]$, $k = 1, \dots, m$. Consider the set of functions

$$PC([-r,0],\mathbb{R}) = \{y\colon [-r,0] \to \mathbb{R}\colon y \in C((\tau_k, \tau_{k+1}],\mathbb{R}),\quad k = 0,\dots,m,\\ \text{and there exist } y(\tau_k^-) \text{ and } y(\tau_k^+),\quad k = 1,\dots,m,\ \text{with } y(\tau_k^-) = y(\tau_k)\}\,.$$

$PC([-r,0],\mathbb{R})$ is a Banach space with the norm

$$\|y\|_{PC} = \sup_{t\in[-r,0]} |y(t)|\,.$$

Let

$$PC([0,T],\mathbb{R}) = \{y\colon [0,T] \to \mathbb{R} | y \in C((t_k, t_{k+1}],\mathbb{R}),\quad k = 1,\dots,m,\\ \text{and there exist } y(t_k^-) \text{ and } y(t_k^+),\quad k = 1,\dots,m,\ \text{with } y(t_k^-) = y(t_k)\}\,.$$

$PC([0,T],\mathbb{R})$ is a Banach space with the norm

$$\|y\|_{C} = \sup_{t\in[0,T]} |y(t)|\,.$$

Notice that

$$\Omega = \{y\colon [-r,T] \to \mathbb{R}\colon y|_{[-r,0]} \in PC([-r,0],\mathbb{R}) \text{ and } y|_{[0,T]} \in PC([0,T],\mathbb{R})\}$$

is a Banach space with the norm

$$\|y\|_{\Omega} = \sup_{t\in[-r,T]} |y(t)|\,.$$

Definition 3.1. A function $y \in \Omega$ whose α-derivative exists on J_k is said to be a solution of (3.1)–(3.3) if y satisfies the equation ${}^cD^{\alpha}_{t_k}y(t) = f(t, y_t, {}^cD^{\alpha}_{t_k}y(t))$ on J_k and satisfies the conditions

$$\Delta y|_{t=t_k} = I_k(y_{t_k^-})\,,\quad k = 1,\dots,m\,,$$
$$y(t) = \varphi(t)\,,\quad t \in [-r,0]\,.$$

To prove the existence of solutions to (3.1)–(3.3), we need the following auxiliary lemma.

Lemma 3.2. *Let $0 < \alpha \le 1$, and let $\sigma\colon J \to \mathbb{R}$ be continuous. A function y is a solution of the fractional integral equation*

$$y(t) = \begin{cases} \varphi(0) + \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1}\sigma(s)ds, & \text{if } t \in [0,t_1]\,,\\ \varphi(0) + \displaystyle\sum_{i=1}^{k} I_i(y_{t_i^-}) + \dfrac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\sigma(s)ds & \\ +\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_{t_k}^{t}(t-s)^{\alpha-1}\sigma(s)ds, & \text{if } t \in (t_k, t_{k+1}]\,,\\ \varphi(t), & t \in [-r,0]\,, \end{cases} \tag{3.5}$$

where $k = 1, \ldots, m$, if and only if y is a solution of the fractional problem

$$^cD^\alpha y(t) = \sigma(t)\,, \quad t \in J_k\,, \tag{3.6}$$

$$\Delta y|_{t=t_k} = I_k(y_{t_k^-})\,, \quad k = 1, \ldots, m\,, \tag{3.7}$$

$$y(t) = \varphi(t)\,, \quad t \in [-r, 0]\,. \tag{3.8}$$

Proof. Assume that y satisfies (3.6)–(3.8). If $t \in [0, t_1]$, then

$$^cD^\alpha y(t) = \sigma(t)\,.$$

Lemma 1.9 implies

$$y(t) = \varphi(0) + I^\alpha \sigma(t) = \varphi(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\sigma(s)ds\,.$$

If $t \in (t_1, t_2]$, then Lemma 1.9 implies

$$\begin{aligned} y(t) &= y(t_1^+) + \frac{1}{\Gamma(\alpha)}\int_{t_1}^t (t-s)^{\alpha-1}\sigma(s)ds \\ &= \Delta y|_{t=t_1} + y(t_1^-) + \frac{1}{\Gamma(\alpha)}\int_{t_1}^t (t-s)^{\alpha-1}\sigma(s)ds \\ &= I_1(y_{t_1^-}) + \left[\varphi(0) + \frac{1}{\Gamma(\alpha)}\int_0^{t_1} (t_1-s)^{\alpha-1}\sigma(s)ds\right] \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_{t_1}^t (t-s)^{\alpha-1}\sigma(s)ds\,. \\ &= \varphi(0) + I_1(y_{t_1^-}) + \frac{1}{\Gamma(\alpha)}\int_0^{t_1} (t_1-s)^{\alpha-1}\sigma(s)ds \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_{t_1}^t (t-s)^{\alpha-1}\sigma(s)ds\,. \end{aligned}$$

If $t \in (t_2, t_3]$, then from Lemma 1.9 we get

$$\begin{aligned} y(t) &= y(t_2^+) + \frac{1}{\Gamma(\alpha)} \int_{t_2}^{t} (t-s)^{\alpha-1}\sigma(s)ds \\ &= \Delta y|_{t=t_2} + y(t_2^-) + \frac{1}{\Gamma(\alpha)} \int_{t_2}^{t} (t-s)^{\alpha-1}\sigma(s)ds \\ &= I_2(y_{t_2^-}) + \left[\varphi(0) + I_1(y_{t_1^-}) + \frac{1}{\Gamma(\alpha)} \int_{0}^{t_1} (t_1-s)^{\alpha-1}\sigma(s)ds \right. \\ &\quad \left. + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1}\sigma(s)ds \right] + \frac{1}{\Gamma(\alpha)} \int_{t_2}^{t} (t-s)^{\alpha-1}\sigma(s)ds \,. \\ &= \varphi(0) + \left[I_1(y_{t_1^-}) + I_2(y_{t_2^-}) \right] + \left[\frac{1}{\Gamma(\alpha)} \int_{0}^{t_1} (t_1-s)^{\alpha-1}\sigma(s)ds \right. \\ &\quad \left. + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1}\sigma(s)ds \right] + \frac{1}{\Gamma(\alpha)} \int_{t_2}^{t} (t-s)^{\alpha-1}\sigma(s)ds \,. \end{aligned}$$

Repeating the process in this way, the solution $y(t)$ for $t \in (t_k, t_{k+1}]$, where $k = 1, \ldots, m$, can be written

$$\begin{aligned} y(t) = \varphi(0) &+ \sum_{i=1}^{k} I_i(y_{t_i^-}) + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i-s)^{\alpha-1}\sigma(s)ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1}\sigma(s)ds \,. \end{aligned}$$

Conversely, assume that y satisfies the impulsive fractional integral equation (3.5). If $t \in [0, t_1]$, then $y(0) = \varphi(0)$. Using the fact that ${}^cD^\alpha$ is the left inverse of I^α, we obtain

$$ {}^cD^\alpha y(t) = \sigma(t) \,, \quad \text{for each } t \in [0, t_1] \,. $$

If $t \in (t_k, t_{k+1}]$, $k = 1, \ldots, m$, using the fact that ${}^cD^\alpha C = 0$, where C is a constant, we have

$$ {}^cD^\alpha y(t) = \sigma(t) \,, \quad \text{for each } t \in (t_k, t_{k+1}] \,. $$

Also, we can easily show that

$$ \Delta y|_{t=t_k} = I_k(y_{t_k^-}) \,, \quad k = 1, \ldots, m \,. \qquad \square $$

We are now in a position to state and prove our existence result for problem (3.1)–(3.3) based on Banach's fixed point.

Theorem 3.3. *Make the following assumptions:*
(3.3.1) *The function* $f: J \times PC([-r, 0], \mathbb{R}) \times \mathbb{R} \to \mathbb{R}$ *is continuous.*
(3.3.2) *There exist constants* $K > 0$ *and* $0 < L < 1$ *such that*

$$|f(t, u, v) - f(t, \bar{u}, \bar{v})| \leq K\|u - \bar{u}\|_{PC} + L|v - \bar{v}|$$

for any $u, \bar{u} \in PC([-r, 0], \mathbb{R}),\ v, \bar{v} \in \mathbb{R}$ *and* $t \in J$.
(3.3.3) *There exists a constant* $\widetilde{l} > 0$ *such that*

$$|I_k(u) - I_k(\overline{u})| \leq \widetilde{l}\|u - \overline{u}\|_{PC}$$

for each $u, \overline{u} \in PC([-r, 0], \mathbb{R})$ *and* $k = 1, \ldots, m$.
If

$$m\widetilde{l} + \frac{(m+1)KT^\alpha}{(1-L)\Gamma(\alpha+1)} < 1\,, \tag{3.9}$$

then there exists a unique solution for problem (3.1)–(3.3) *on* J.

Proof. Transform problem (3.1)–(3.3) into a fixed point problem. Consider the operator $N: \Omega \to \Omega$ defined by

$$(Ny)(t) = \begin{cases} \varphi(0) + \sum\limits_{0<t_k<t} I_k(y_{t_i^-}) + \frac{1}{\Gamma(\alpha)} \sum\limits_{0<t_k<t} \int\limits_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} g(s)ds \\ + \frac{1}{\Gamma(\alpha)} \int\limits_{t_k}^{t} (t-s)^{\alpha-1}\sigma(s)ds, & t \in [0, T]\,, \\ \varphi(t), & t \in [-r, 0]\,, \end{cases} \tag{3.10}$$

where $g \in C(J, \mathbb{R})$ is such that

$$g(t) = f(t, y_t, g(t))\,.$$

Clearly, the fixed points of operator N are solutions of problem (3.1)–(3.3).
Let $u, w \in \Omega$. If $t \in [-r, 0]$, then

$$|(Nu)(t) - (Nw)(t)| = 0\,.$$

For $t \in J$ we have

$$\begin{aligned} |N(u)(t) - N(w)(t)| \leq & \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int\limits_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1}|g(s) - h(s)|ds \\ & + \frac{1}{\Gamma(\alpha)} \int\limits_{t_k}^{t} (t-s)^{\alpha-1}|g(s) - h(s)|ds \\ & + \sum_{0<t_k<t} |I_k(u_{t_k^-}) - I_k(w_{t_k^-})|\,, \end{aligned}$$

where $g, h \in C(J, \mathbb{R})$ are given by

$$g(t) = f(t, u_t, g(t)) \,,$$

and

$$h(t) = f(t, w_t, h(t)) \,.$$

By (3.3.2) we have

$$\begin{aligned}|g(t) - h(t)| &= |f(t, u_t, g(t)) - f(t, w_t, h(t))| \\ &\le K\|u_t - w_t\|_{PC} + L|g(t) - h(t)| \,.\end{aligned}$$

Hence,

$$|g(t) - h(t)| \le \frac{K}{1-L}\|u_t - w_t\|_{PC} \,.$$

Therefore, for each $t \in J$

$$\begin{aligned}|N(u)(t) - N(w)(t)| \le{}& \frac{K}{(1-L)\Gamma(\alpha)}\sum_{k=1}^{m}\int_{t_{k-1}}^{t_k}(t_k - s)^{\alpha-1}\|u_s - w_s\|_{PC}ds \\ &+ \frac{K}{(1-L)\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}\|u_s - w_s\|_{PC}ds \\ &+ \sum_{k=1}^{m}\widetilde{l}\|u_{t_k^-} - w_{t_k^-}\|_{PC} \,. \\ \le{}& \left[m\widetilde{l} + \frac{mKT^\alpha}{(1-L)\Gamma(\alpha+1)}\right. \\ &\left.+\frac{KT^\alpha}{(1-L)\Gamma(\alpha+1)}\right]\|u-w\|_\Omega \,.\end{aligned}$$

Thus,

$$\|N(u) - N(w)\|_\Omega \le \left[m\widetilde{l} + \frac{(m+1)KT^\alpha}{(1-L)\Gamma(\alpha+1)}\right]\|u-w\|_\Omega \,.$$

By (3.9), operator N is a contraction. Hence, by Banach's contraction principle, N has a unique fixed point that is the unique solution of problem (3.1)–(3.3). □

Our second result is based on Schaefer's fixed point theorem.

Theorem 3.4. *In addition to* (3.3.1), (3.3.2) *assumes that:*
(3.4.1) *There exist* $p, q, r \in C(J, \mathbb{R}_+)$ *with* $r^* = \sup_{t\in J} r(t) < 1$ *such that*

$$|f(t, u, w)| \le p(t) + q(t)\|u\|_{PC} + r(t)|w| \quad \textit{for } t \in J,\ u \in PC([-r, 0], \mathbb{R}) \textit{ and } w \in \mathbb{R} \,.$$

(3.4.2) *The functions* $I_k\colon PC([-r, 0], \mathbb{R}) \to \mathbb{R}$ *are continuous, and there exist constants* $M^*, N^* > 0$, *with* $mM^* < 1$, *such that*

$$|I_k(u)| \le M^*\|u\|_{PC} + N^* \quad \textit{for each } u \in PC([-r, 0], \mathbb{R}),\ k = 1, \ldots, m \,.$$

Then problem (3.1)–(3.3) *has at least one solution.*

Proof. Let operator N be defined as in (3.10). We will use Schaefer's fixed point theorem to prove that N has a fixed point. The proof will be given in several steps.

Step 1: N is continuous. Let $\{u_n\}$ be a sequence such that $u_n \to u$ in Ω. If $t \in [-r, 0]$, then

$$|(Nu_n)(t) - (Nu)(t)| = 0 \, .$$

For $t \in J$ we have

$$\begin{aligned} |(Nu_n)(t) - (Nu)(t)| \leq & \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} |g_n(s) - g(s)| ds \\ & + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} |g_n(s) - g(s)| ds \\ & + \sum_{0<t_k<t} |I_k(u_{nt_k^-}) - I_k(u_{t_k^-})| \, , \end{aligned} \tag{3.11}$$

where $g_n, g \in C(J, \mathbb{R})$ are given by

$$g_n(t) = f(t, u_{nt}, g_n(t))$$

and

$$g(t) = f(t, u_t, g(t)) \, .$$

From (3.3.2) we have

$$\begin{aligned} |g_n(t) - g(t)| &= |f(t, u_{nt}, g_n(t)) - f(t, u_t, g(t))| \\ &\leq K\|u_{nt} - u_t\|_{PC} + L|g_n(t) - g(t)| \, . \end{aligned}$$

Then

$$|g_n(t) - g(t)| \leq \frac{K}{1-L} \|u_{nt} - u_t\|_{PC} \, .$$

Since $u_n \to u$, $g_n(t) \to g(t)$ as $n \to \infty$ for each $t \in J$. Let $\eta > 0$ be such that for each $t \in J$ we have $|g_n(t)| \leq \eta$ and $|g(t)| \leq \eta$. Then

$$\begin{aligned} (t-s)^{\alpha-1} |g_n(s) - g(s)| &\leq (t-s)^{\alpha-1} [|g_n(s)| + |g(s)|] \\ &\leq 2\eta (t-s)^{\alpha-1} \end{aligned}$$

and

$$\begin{aligned} (t_k - s)^{\alpha-1} |g_n(s) - g(s)| &\leq (t_k - s)^{\alpha-1} [|g_n(s)| + |g(s)|] \\ &\leq 2\eta (t_k - s)^{\alpha-1} \, . \end{aligned}$$

For each $t \in J$ the functions $s \to 2\eta(t-s)^{\alpha-1}$ and $s \to 2\eta(t_k - s)^{\alpha-1}$ are integrable on $[0, t]$, so by the Lebesgue dominated convergence theorem and (3.11),

$$|(Nu_n)(t) - (Nu)(t)| \to 0 \quad \text{as} \quad n \to \infty \, .$$

Hence,

$$\|(Nu_n) - (Nu)\|_\Omega \to 0 \text{ as } n \to \infty .$$

Consequently, N is continuous.

Step 2: F maps bounded sets to bounded sets in Ω. It is enough to show that for any $\eta^* > 0$ there exists a positive constant ℓ such that for each $u \in B_{\eta^*} = \{u \in \Omega : \|u\|_\Omega \le \eta^*\}$ we have $\|N(u)\|_\Omega \le \ell$. For each $t \in J$ we have

$$(Nu)(t) = \varphi(0) + \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} g(s)ds$$
$$+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} g(s)ds + \sum_{0<t_k<t} I_k(u_{t_k^-}) , \tag{3.12}$$

where $g \in C(J, \mathbb{R})$ is given by

$$g(t) = f(t, u_t, g(t)) .$$

By (3.4.1), for each $t \in J$ we have

$$\begin{aligned}|g(t)| &= |f(t, u_t, g(t))| \\ &\le p(t) + q(t)\|u_t\|_{PC} + r(t)|g(t)| \\ &\le p(t) + q(t)\|u\|_\Omega + r(t)|g(t)| \\ &\le p(t) + q(t)\eta^* + r(t)|g(t)| \\ &\le p^* + q^*\eta^* + r^*|g(t)| ,\end{aligned}$$

where $p^* = \sup_{t\in J} p(t)$ and $q^* = \sup_{t\in J} q(t)$.
Then

$$|g(t)| \le \frac{p^* + q^*\eta^*}{1 - r^*} := M .$$

Thus, (3.12) implies

$$\begin{aligned}|N(u)(t)| &\le |\varphi(0)| + \frac{mMT^\alpha}{\Gamma(\alpha+1)} + \frac{MT^\alpha}{\Gamma(\alpha+1)} + m(M^*\|u_{t_k^-}\|_{PC} + N^*) \\ &\le |\varphi(0)| + \frac{(m+1)MT^\alpha}{\Gamma(\alpha+1)} + m(M^*\|u\|_\Omega + N^*) \\ &\le |\varphi(0)| + \frac{(m+1)MT^\alpha}{\Gamma(\alpha+1)} + m(M^*\eta^* + N^*) := R .\end{aligned}$$

If $t \in [-r, 0]$, then

$$|N(u)(t)| \le \|\varphi\|_{PC} ,$$

so

$$\|N(u)\|_\Omega \le \max\left\{R, \|\varphi\|_{PC}\right\} := \ell .$$

Step 3: F maps bounded sets to equicontinuous sets of Ω. Let $t_1, t_2 \in (0, T]$, $t_1 < t_2$, B_{η^*} be a bounded set of Ω as in Step 2, and let $u \in B_{\eta^*}$. Then

$$
\begin{aligned}
&|N(u)(t_2) - N(u)(t_1)| \\
&\le \frac{1}{\Gamma(\alpha)} \int_0^{t_1} |(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}||g(s)|ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} |(t_2 - s)^{\alpha-1}||g(s)|ds + \sum_{0<t_k<t_2-t_1} |I_k(u_{t_k^-})| \\
&\le \frac{M}{\Gamma(\alpha+1)}[2(t_2 - t_1)^\alpha + (t_2^\alpha - t_1^\alpha)] + (t_2 - t_1)(M^*\|u_{t_k^-}\|_{PC} + N^*) \\
&\le \frac{M}{\Gamma(\alpha+1)}[2(t_2 - t_1)^\alpha + (t_2^\alpha - t_1^\alpha)] + (t_2 - t_1)(M^*\|u\|_\Omega + N^*) \\
&\le \frac{M}{\Gamma(\alpha+1)}[2(t_2 - t_1)^\alpha + (t_2^\alpha - t_1^\alpha)] + (t_2 - t_1)(M^*\eta^* + N^*) .
\end{aligned}
$$

As $t_1 \to t_2$, the right-hand side of the preceding inequality tends to zero. As a consequence of Steps 1–3, together with the Ascoli–Arzelà theorem, we can conclude that $N : \Omega \to \Omega$ is completely continuous.

Step 4: A priori bounds. Now it remains to show that the set

$$E = \{u \in \Omega : u = \lambda N(u) \text{ for some } 0 < \lambda < 1\}$$

is bounded. Let $u \in E$; then $u = \lambda N(u)$ for some $0 < \lambda < 1$. Thus, for each $t \in J$ we have

$$
\begin{aligned}
u(t) = \lambda\varphi(0) &+ \frac{\lambda}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} g(s)ds \\
&+ \frac{\lambda}{\Gamma(\alpha)} \int_{t_k}^{t} (t - s)^{\alpha-1} g(s)ds + \lambda \sum_{0<t_k<t} I_k(u_{t_k^-}) ,
\end{aligned}
\tag{3.13}
$$

and by (3.4.1), for each $t \in J$ we have

$$
\begin{aligned}
|g(t)| &= |f(t, u_t, g(t))| \\
&\le p(t) + q(t)\|u_t\|_{PC} + r(t)|g(t)| \\
&\le p^* + q^*\|u_t\|_{PC} + r^*|g(t)| .
\end{aligned}
$$

Thus,

$$|g(t)| \le \frac{1}{1 - r^*}(p^* + q^*\|u_t\|_{PC}) .$$

Now (3.13) and (3.4.2) imply that for each $t \in J$ we have

$$
\begin{aligned}
|u(t)| \le |\varphi(0)| &+ \frac{1}{(1-r^*)\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1}(p^* + q^*\|u_s\|_{PC})ds \\
&+ \frac{1}{(1-r^*)\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1}(p^* + q^*\|u_s\|_{PC})ds \\
&+ m(M^*\|u_{t_k^-}\|_{PC} + N^*) .
\end{aligned}
$$

Consider the function ζ defined by

$$\zeta(t) = \sup\{|u(s)| : \ -r \le s \le t\}, 0 \le t \le T .$$

Then there exists $t^* \in [-r, T]$ such that $\zeta(t) = |u(t^*)|$. If $t \in [0, T]$, then, by the previous inequality, for $t \in J$ we have

$$
\begin{aligned}
\zeta(t) \le |\varphi(0)| &+ \frac{1}{(1-r^*)\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1}(p^* + q^*\zeta(s))ds \\
&+ \frac{1}{(1-r^*)\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1}(p^* + q^*\zeta(s))ds \\
&+ mM^*\zeta(t) + mN^* .
\end{aligned}
$$

Thus, for $t \in J$

$$
\begin{aligned}
\zeta(t) \le &\frac{|\varphi(0)| + mN^*}{1 - mM^*} + \frac{1}{(1-mM^*)(1-r^*)\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1}(p^* + q^*\zeta(s))ds \\
&+ \frac{1}{(1-mM^*)(1-r^*)\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1}(p^* + q^*\zeta(s))ds \\
\le &\frac{|\varphi(0)| + mN^*}{1 - mM^*} + \frac{(m+1)p^*T^\alpha}{(1-mM^*)(1-r^*)\Gamma(\alpha+1)} \\
&+ \frac{(m+1)q^*}{(1-mM^*)(1-r^*)\Gamma(\alpha)} \int_{0}^{t} (t-s)^{\alpha-1}\zeta(s)ds .
\end{aligned}
$$

Applying Lemma 1.52, we get

$$
\begin{aligned}
\zeta(t) \le &\left[\frac{|\varphi(0)| + mN^*}{1 - mM^*} + \frac{(m+1)p^*T^\alpha}{(1-mM^*)(1-r^*)\Gamma(\alpha+1)}\right] \\
&\times \left[1 + \frac{\delta(m+1)q^*T^\alpha}{(1-mM^*)(1-r^*)\Gamma(\alpha+1)}\right] := A ,
\end{aligned}
$$

where $\delta = \delta(\alpha)$ is a constant. If $t^* \in [-r, 0]$, then $\zeta(t) = \|\varphi\|_{PC}$. Thus, for any $t \in J$ and $\|u\|_\Omega \le \zeta(t)$ we have

$$\|u\|_\Omega \le \max\{\|\varphi\|_{PC}, A\} .$$

This shows that the set E is bounded. As a consequence of Schaefer's fixed point theorem, we deduce that N has a fixed point that is a solution of problem (3.1)–(3.3). □

3.2.3 Ulam–Hyers–Rassias stability

Here we adopt the concepts in Wang et al. [252] and introduce Ulam's type stability concepts for problem (3.1)–(3.2).

Let $z \in \Omega$, $\epsilon > 0$, $\psi > 0$, and $\omega \in PC(J, \mathbb{R}_+)$ be nondecreasing. We consider the set of inequalities

$$\begin{cases} |{}^cD^\alpha z(t) - f(t, z_t, {}^cD^\alpha z(t))| \le \epsilon, & t \in (t_k, t_{k+1}],\ k = 1, \dots, m, \\ |\Delta y|_{t_k} - I_k(y_{t_k^-})| \le \epsilon, & k = 1, \dots, m; \end{cases} \tag{3.14}$$

the set of inequalities

$$\begin{cases} |{}^cD^\alpha z(t) - f(t, z_t, {}^cD^\alpha z(t))| \le \omega(t), & t \in (t_k, t_{k+1}],\ k = 1, \dots, m, \\ |\Delta y|_{t_k} - I_k(y_{t_k^-})| \le \psi, & k = 1, \dots, m; \end{cases} \tag{3.15}$$

and the set of inequalities

$$\begin{cases} |{}^cD^\alpha z(t) - f(t, z_t, {}^cD^\alpha z(t))| \le \epsilon\omega(t), & t \in (t_k, t_{k+1}],\ k = 1, \dots, m, \\ |\Delta y|_{t_k} - I_k(y_{t_k^-})| \le \epsilon\psi, & k = 1, \dots, m. \end{cases} \tag{3.16}$$

Definition 3.5. Problem (3.1)–(3.2) is Ulam–Hyers stable if there exists a real number $c_{f,m} > 0$ such that for each $\epsilon > 0$ and for each solution $z \in \Omega$ of inequality (3.14) there exists a solution $y \in \Omega$ of problem (3.1)–(3.2), with

$$|z(t) - y(t)| \le c_{f,m}\epsilon , \quad t \in J .$$

Definition 3.6. Problem (3.1)–(3.2) is generalized Ulam–Hyers stable if there exists $\theta_{f,m} \in C(\mathbb{R}_+, \mathbb{R}_+)$ with $\theta_{f,m}(0) = 0$ such that for each solution $z \in \Omega$ of inequality (3.14) there exists a solution $y \in \Omega$ of problem (3.1)–(3.2), with

$$|z(t) - y(t)| \le \theta_{f,m}(\epsilon) , \quad t \in J .$$

Definition 3.7. Problem (3.1)–(3.2) is Ulam–Hyers–Rassias stable with respect to (ω, ψ) if there exists $c_{f,m,\omega} > 0$ such that for each $\epsilon > 0$ and for each solution $z \in \Omega$ of inequality (3.16) there exists a solution $y \in \Omega$ of problem (3.1)–(3.2), with

$$|z(t) - y(t)| \le c_{f,m,\omega}\epsilon(\omega(t) + \psi) , \quad t \in J .$$

Definition 3.8. Problem (3.1)–(3.2) is generalized Ulam–Hyers–Rassias stable with respect to (ω, ψ) if there exists $c_{f,m,\omega} > 0$ such that for each solution $z \in \Omega$ of inequality (3.15) there exists a solution $y \in \Omega$ of problem (3.1)–(3.2), with

$$|z(t) - y(t)| \le c_{f,m,\omega}(\omega(t) + \psi) , \quad t \in J .$$

Remark 3.9. It is clear that (i) Definition 3.5 implies Definition 3.6, (ii) Definition 3.7 implies Definition 3.8, and (iii) Definition 3.7 for $\omega(t) = \psi = 1$ implies Definition 3.5.

Remark 3.10. A function $z \in \Omega$ is a solution of inequality (3.16) if and only if there is $\sigma \in PC(J, \mathbb{R})$ and a sequence σ_k, $k = 1, \dots, m$ (which depend on z) such that
(i) $|\sigma(t)| \le \epsilon\omega(t),\ t \in (t_k, t_{k+1}],\ k = 1, \dots, m$ and $|\sigma_k| \le \epsilon\psi$, $k = 1, \dots, m$;
(ii) ${}^cD^\alpha z(t) = f(t, z_t, {}^cD^\alpha z(t)) + \sigma(t),\ t \in (t_k, t_{k+1}], k = 1, \dots, m$;
(iii) $\Delta z|_{t_k} = I_k(z_{t_k^-}) + \sigma_k,\ k = 1, \dots, m$.

Similar remarks hold for inequalities 3.15 and 3.14.

Now we state the following Ulam–Hyers–Rassias stable result.

Theorem 3.11. *Assume* (3.3.1)–(3.3.3) *and* (3.9) *hold and*
(3.11.1) *there exists a nondecreasing function* $\omega \in PC(J, \mathbb{R}_+)$, *and there exists* $\lambda_\omega > 0$ *such that, for any* $t \in J$,

$$I^\alpha \omega(t) \le \lambda_\omega \omega(t) .$$

Then problem (3.1)–(3.2) *is Ulam–Hyers–Rassias stable with respect to* (ω, ψ).

Proof. Let $z \in \Omega$ be a solution of inequality (3.16). Denote by y the unique solution of the problem

$$\begin{cases} {}^cD^\alpha_{t_k} y(t) = f(t, y_t, {}^cD^\alpha_{t_k} y(t)), & t \in (t_k, t_{k+1}], k = 1, \dots, m , \\ \Delta y|_{t=t_k} = I_k(y_{t_k^-}), & k = 1, \dots, m , \\ y(t) = z(t) = \varphi(t), & t \in [-r, 0] . \end{cases}$$

Using Lemma 3.2, we obtain for each $t \in (t_k, t_{k+1}]$

$$y(t) = \varphi(0) + \sum_{i=1}^{k} I_i(y_{t_i^-}) + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} g(s) ds$$
$$+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t - s)^{\alpha-1} g(s) ds ,$$

where $g \in C(J, \mathbb{R})$ is given by

$$g(t) = f(t, y_t, g(t)) .$$

Since z is a solution of inequality (3.16), by Remark 3.10 we have

$$\begin{cases} {}^cD^\alpha_{t_k} z(t) = f(t, z_t, {}^cD^\alpha_{t_k} z(t)) + \sigma(t), & t \in (t_k, t_{k+1}], k = 1, \dots, m , \\ \Delta z|_{t=t_k} = I_k(z_{t_k^-}) + \sigma_k, & k = 1, \dots, m . \end{cases} \quad (3.17)$$

Clearly, the solution of (3.17) is given by

$$z(t) = \varphi(0) + \sum_{i=1}^{k} I_i(z_{t_i^-}) + \sum_{i=1}^{k} \sigma_i + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} h(s)ds$$
$$+ \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} \sigma(s)ds + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t - s)^{\alpha-1} h(s)ds$$
$$+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t - s)^{\alpha-1} \sigma(s)ds , \quad t \in (t_k, t_{k+1}] ,$$

where $h \in C(J, \mathbb{R})$ is given by

$$h(t) = f(t, z_t, h(t)) .$$

Hence, for each $t \in (t_k, t_{k+1}]$, it follows that

$$|z(t) - y(t)| \leq \sum_{i=1}^{k} |\sigma_i| + \sum_{i=1}^{k} |I_i(z_{t_i^-}) - I_i(y_{t_i^-})|$$
$$+ \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} |h(s) - g(s)|ds$$
$$+ \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} |\sigma(s)|ds$$
$$+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t - s)^{\alpha-1} |h(s) - g(s)|ds$$
$$+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t - s)^{\alpha-1} |\sigma(s)| .$$

Thus,

$$|z(t) - y(t)| \leq m\epsilon\psi + (m+1)\epsilon\lambda_\omega \omega(t) + \sum_{i=1}^{k} \tilde{l} \|z_{t_i^-} - y_{t_i^-}\|_{PC}$$
$$+ \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} |h(s) - g(s)|ds$$
$$+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t - s)^{\alpha-1} |h(s) - g(s)|ds .$$

By (33.3.2) we have

$$|h(t) - g(t)| = |f(t, z_t, h(t)) - f(t, y_t, g(t))| \\ \leq K\|z_t - y_t\|_{PC} + L|g(t) - h(t)| .$$

Then

$$|h(t) - g(t)| \leq \frac{K}{1-L}\|z_t - y_t\|_{PC} .$$

Therefore, for each $t \in J$

$$|z(t) - y(t)| \leq m\epsilon\psi + (m+1)\epsilon\lambda_\omega\omega(t) + \sum_{i=1}^{k}\tilde{l}\|z_{t_i^-} - y_{t_i^-}\|_{PC} \\ + \frac{K}{(1-L)\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i - s)^{\alpha-1}\|z_s - y_s\|_{PC}ds \\ + \frac{K}{(1-L)\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}\|z_s - y_s\|_{PC}ds .$$

Thus,

$$|z(t) - y(t)| \leq \sum_{0<t_i<t}\tilde{l}\|z_{t_i^-} - y_{t_i^-}\|_{PC} + \epsilon(\psi + \omega(t))(m + (m+1)\lambda_\omega) \\ + \frac{K(m+1)}{(1-L)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|z_s - y_s\|_{PC}ds .$$

We consider the function ζ_1 defined by

$$\zeta_1(t) = \sup\{\|z(s) - y(s)\| : \ -r \leq s \leq t\}, 0 \leq t \leq T ;$$

then there exists $t^* \in [-r, T]$ such that $\zeta_1(t) = \|z(t^*) - y(t^*)\|$. If $t^* \in [-r, 0]$, then $\zeta_1(t) = 0$. If $t^* \in [0, T]$, then by the previous inequality we have

$$\zeta_1(t) \leq \sum_{0<t_i<t}\tilde{l}\zeta_1(t_i^-) + \epsilon(\psi + \omega(t))(m + (m+1)\lambda_\omega) \\ + \frac{K(m+1)}{(1-L)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\zeta_1(s)ds .$$

Applying Lemma 1.53, we get

$$\zeta_1(t) \leq \epsilon(\psi + \omega(t))(m + (m+1)\lambda_\omega) \\ \times \left[\prod_{0<t_i<t}(1+\tilde{l})\exp\left(\int_0^t \frac{K(m+1)}{(1-L)\Gamma(\alpha)}(t-s)^{\alpha-1}ds\right)\right] \\ \leq c_\omega\epsilon(\psi + \omega(t)) ,$$

where

$$c_\omega = (m + (m+1)\lambda_\omega)\left[\prod_{i=1}^{m}(1+\tilde{l})\exp\left(\frac{K(m+1)T^\alpha}{(1-L)\Gamma(\alpha+1)}\right)\right]$$
$$= (m + (m+1)\lambda_\omega)\left[(1+\tilde{l})\exp\left(\frac{K(m+1)T^\alpha}{(1-L)\Gamma(\alpha+1)}\right)\right]^m .$$

Thus, problem (3.1)–(3.2) is Ulam–Hyers–Rassias stable with respect to (ω, ψ). □

Next, we present the following Ulam–Hyers stability result.

Theorem 3.12. *Assume* (3.3.1)–(3.3.3) *and* (3.9) *hold. Then problem* (3.1)–(3.2) *is Ulam–Hyers stable.*

Proof. Let $z \in \Omega$ be a solution of inequality (3.14). Denote by y the unique solution of the problem

$$\begin{cases} {}^cD^\alpha_{t_k} y(t) = f(t, y_t, {}^cD^\alpha_{t_k} y(t)), & t \in (t_k, t_{k+1}], k = 1, \dots, m ; \\ \Delta y|_{t=t_k} = I_k(y_{t_k^-}), & k = 1, \dots, m ; \\ y(t) = z(t) = \varphi(t), & t \in [-r, 0] . \end{cases}$$

From the proof of Theorem 3.11 we get the inequality

$$\zeta_1(t) \le \sum_{0<t_i<t} \tilde{l}\zeta_1(t_i^-) + m\epsilon + \frac{T^\alpha \epsilon(m+1)}{\Gamma(\alpha+1)}$$
$$+ \frac{K(m+1)}{(1-L)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\zeta_1(s)ds .$$

An application of Lemma 1.53 gives

$$\zeta_1(t) \le \epsilon\left(\frac{m\Gamma(\alpha+1) + T^\alpha(m+1)}{\Gamma(\alpha+1)}\right)$$
$$\times\left[\prod_{0<t_i<t}(1+\tilde{l})\exp\left(\int_0^t \frac{K(m+1)}{(1-L)\Gamma(\alpha)}(t-s)^{\alpha-1}ds\right)\right]$$
$$\le c_\omega \epsilon ,$$

where

$$c_\omega = \left(\frac{m\Gamma(\alpha+1) + T^\alpha(m+1)}{\Gamma(\alpha+1)}\right)\left[\prod_{i=1}^{m}(1+\tilde{l})\exp\left(\frac{K(m+1)T^\alpha}{(1-L)\Gamma(\alpha+1)}\right)\right]$$
$$= \left(\frac{m\Gamma(\alpha+1) + T^\alpha(m+1)}{\Gamma(\alpha+1)}\right)\left[(1+\tilde{l})\exp\left(\frac{K(m+1)T^\alpha}{(1-L)\Gamma(\alpha+1)}\right)\right]^m .$$

Moreover, if we set $\gamma(\epsilon) = c_\omega\epsilon$, $\gamma(0) = 0$, then problem (3.1)–(3.2) is generalized Ulam–Hyers stable. □

3.2.4 Examples

Example 1. Consider the impulsive problem

$$ {}^cD^{\frac{1}{2}}_{t_k}y(t) = \frac{e^{-t}}{(11+e^t)}\left[\frac{y_t}{1+y_t} - \frac{{}^cD^{\frac{1}{2}}_{t_k}y(t)}{1+{}^cD^{\frac{1}{2}}_{t_k}y(t)}\right] \quad \text{for } t \in J_0 \cup J_1 , \tag{3.18} $$

$$ \Delta y|_{t=\frac{1}{2}} = \frac{y\left(\frac{1}{2}^-\right)}{10+y\left(\frac{1}{2}^-\right)} , \tag{3.19} $$

$$ y(t) = \varphi(t) , \quad t \in [-r, 0], \; r > 0 , \tag{3.20} $$

where $\varphi \in PC([-r, 0], \mathbb{R})$, $J_0 = [0, \frac{1}{2}]$, $J_1 = (\frac{1}{2}, 1]$, $t_0 = 0$, and $t_1 = \frac{1}{2}$.

Set

$$ f(t, u, v) = \frac{e^{-t}}{(11+e^t)}\left[\frac{u}{1+u} - \frac{v}{1+v}\right] , \quad t \in [0,1], \; u \in PC([-r,0],\mathbb{R}) \text{ and } v \in \mathbb{R} . $$

Clearly, the function f is jointly continuous.

For each $u, \bar{u} \in PC([-r, 0], \mathbb{R})$, $v, \bar{v} \in \mathbb{R}$, and $t \in [0, 1]$:

$$ \begin{aligned} \|f(t,u,v) - f(t,\bar{u},\bar{v})\| &\le \frac{e^{-t}}{(11+e^t)}(\|u-\bar{u}\|_{PC} + \|v-\bar{v}\|) \\ &\le \frac{1}{12}\|u-\bar{u}\|_{PC} + \frac{1}{12}\|v-\bar{v}\| . \end{aligned} $$

Hence condition (3.3.2) is satisfied by $K = L = \frac{1}{12}$. Let

$$ I_1(u) = \frac{u}{10+u} , \quad u \in PC([-r,0],\mathbb{R}) , $$

and let $u, v \in PC([-r, 0], \mathbb{R})$. Then we have

$$ |I_1(u) - I_1(v)| = \left|\frac{u}{10+u} - \frac{v}{10+v}\right| \le \frac{1}{10}\|u-v\|_{PC} . $$

Thus the condition

$$ \begin{aligned} m\tilde{l} + \frac{(m+1)KT^\alpha}{(1-L)\Gamma(\alpha+1)} &= \left[\frac{1}{10} + \frac{\frac{2}{12}}{\left(1-\frac{1}{12}\right)\Gamma\left(\frac{3}{2}\right)}\right] \\ &= \frac{4}{11\sqrt{\pi}} + \frac{1}{10} < 1 \end{aligned} $$

is satisfied by $T = 1$, $m = 1$, and $\tilde{l} = \frac{1}{10}$. It follows from Theorem 3.3 that problem (2.35)–(2.37) has a unique solution on J.

For any $t \in [0, 1]$, take $\omega(t) = t$ and $\psi = 1$.

Since

$$ I^{\frac{1}{2}}\omega(t) = \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^t (t-s)^{\frac{1}{2}-1} s\,ds \le \frac{2t}{\sqrt{\pi}} , $$

condition (3.11.1) is satisfied with $\lambda_\omega = \frac{2}{\sqrt{\pi}}$. It follows that problem (3.18)–(3.19) is Ulam–Hyers–Rassias stable with respect to (ω, ψ).

Example 2. Consider the impulsive problem

$$^cD^{\frac{1}{2}}_{t_k}y(t) = \frac{2 + |y_t| + |^cD^{\frac{1}{2}}_{t_k}y(t)|}{108e^{t+3}\left(1 + |y_t| + |^cD^{\frac{1}{2}}_{t_k}y(t)|\right)}, \quad \text{for each, } t \in J_0 \cup J_1, \tag{3.21}$$

$$\Delta y|_{t=\frac{1}{3}} = \frac{|y\left(\frac{1}{3}^-\right)|}{6 + |y\left(\frac{1}{3}^-\right)|}, \tag{3.22}$$

$$y(t) = \varphi(t), \quad t \in [-r, 0], \ r > 0, \tag{3.23}$$

where $\varphi \in PC([-r, 0], \mathbb{R})$ $J_0 = [0, \frac{1}{3}]$, $J_1 = (\frac{1}{3}, 1]$, $t_0 = 0$, and $t_1 = \frac{1}{3}$. Set

$$f(t, u, v) = \frac{2 + |u| + |v|}{108e^{t+3}(1 + |u| + |v|)}, \quad t \in [0, 1], \ u \in PC([-r, 0], \mathbb{R}), v \in \mathbb{R}.$$

Clearly, the function f is jointly continuous. For any $u, \bar{u} \in PC([-r, 0], \mathbb{R})$, $v, \bar{v} \in \mathbb{R}$ and $t \in [0, 1]$

$$|f(t, u, v) - f(t, \bar{u}, \bar{v})| \le \frac{1}{108e^3}(\|u - \bar{u}\|_{PC} + |v - \bar{v}|).$$

Hence condition (3.3.2) is satisfied by $K = L = \frac{1}{108e^3}$. For each $t \in [0, 1]$ we have

$$|f(t, u, v)| \le \frac{1}{108e^{t+3}}(2 + \|u\|_{PC} + |v|).$$

Thus condition (3.4.1) is satisfied by $p(t) = \frac{1}{54e^{t+3}}$ and $q(t) = r(t) = \frac{1}{108e^{t+3}}$. Let

$$I_1(u) = \frac{|u|}{6 + |u|}, \quad u \in PC([-r, 0], \mathbb{R}).$$

For each $u \in PC([-r, 0], \mathbb{R})$ we have

$$|I_1(u)| \le \frac{1}{6}\|u\|_{PC} + 1.$$

Thus, condition (3.4.2) is satisfied by $M^* = \frac{1}{6}$ and $N^* = 1$. It follows from Theorem 3.4 that problem (3.21)–(3.23) has at least one solution on J.

3.3 Existence Results for Impulsive NIFDE with Finite Delay in Banach Space

3.3.1 Introduction

The purpose of this section is to establish existence and uniqueness results for implicit fractional differential equations with finite delay and impulses

$$ {}^cD^{\nu}_{t_k} y(t) = f(t, y_t, {}^c D^{\nu}_{t_k} y(t)) , \quad \text{for each, } t \in (t_k, t_{k+1}],\ k = 0, \dots, m,\ 0 < \nu \le 1 , \tag{3.24} $$

$$ \Delta y|_{t=t_k} = I_k(y_{t_k^-}) , \quad k = 1, \dots, m , \tag{3.25} $$

$$ y(t) = \varphi(t) , \quad t \in [-r, 0],\ r > 0 , \tag{3.26} $$

where ${}^cD^{\nu}_{t_k}$ is the Caputo fractional derivative, $(E, \|\cdot\|)$ is a real Banach space, $f: J \times PC([-r, 0], E) \times E \to E$ is a given function, $I_k: PC([-r, 0], E) \to E$, $\varphi \in PC([-r, 0], E)$, and $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$. For each function y_t defined on $[-r, T]$ and for any $t \in J$ we denote by y_t the element of $PC([-r, 0], E)$ defined by

$$ y_t(\theta) = y(t + \theta) , \quad \theta \in [-r, 0] . \tag{3.27} $$

Here, $y_t(.)$ represents the history of the state from time $t - r$ up to time t. We have $\Delta y|_{t_k} = y(t_k^+) - y(t_k^-)$, where $y(t_k^+) = \lim_{h\to 0^+} y(t_k + h)$ and $y(t_k^-) = \lim_{h\to 0^-} y(t_k + h)$ represent the right and left limits of y_t at $t = t_k$, respectively.

In this section, two results are discussed: the first is based on Darbo's fixed point theorem combined with the technique of measures of noncompactness; the second uses Mönch's fixed point theorem. Two examples are given to demonstrate the application of our main results.

3.3.2 Existence of Solutions

Consider the set of functions

$$ \begin{aligned} PC([-r, 0], E) = \{y: [-r, 0] \to E: y \in C((\tau_k, \tau_{k+1}], E),\ \ k = 1, \dots, m,\\ \text{and there exist } y(\tau_k^-) \text{ and } y(\tau_k^+),\ \ k = 1, \dots, m \text{ with } y(\tau_k^-) = y(\tau_k)\} . \end{aligned} $$

Let $PC([-r, 0], E)$ be the Banach space with the norm

$$ \|y\|_{PC} = \sup_{t\in[-r,0]} \|y(t)\| . $$

Also, we take

$$ \begin{aligned} PC([0, T], E) = \{y: [0, T] \to E: y \in C((t_k, t_{k+1}], E),\ \ k = 1, \dots, m,\\ \text{and there exist } y(t_k^-) \text{ and } y(t_k^+),\ \ k = 1, \dots, m \text{ with } y(t_k^-) = y(t_k)\} \end{aligned} $$

and $PC([0,T],E)$ to be the Banach space with the norm

$$\|y\|_C = \sup_{t\in[0,T]} \|y(t)\| .$$

Let

$$\Omega = \{y\colon [-r,T] \to E\colon y|_{[-r,0]} \in PC([-r,0],E) \text{ and } y|_{[0,T]} \in PC([0,T],E)\} ,$$

and note that Ω is a Banach space with the norm

$$\|y\|_\Omega = \sup_{t\in[-r,T]} \|y(t)\| .$$

Let us define what we mean by a solution of problem (3.24)–(3.26).

Definition 3.13. A function $y \in \Omega$ whose ν-derivative exists on J_k is said to be a solution of (3.24)–(3.26) if y satisfies the equation ${}^cD^\nu_{t_k} y(t) = f(t, y_t, {}^cD^\nu_{t_k} y(t))$ on J_k and satisfies the conditions

$$\begin{aligned} \Delta y|_{t=t_k} &= I_k(y_{t_k^-}) , \quad k = 1,\dots,m ,\\ y(t) &= \varphi(t) , \quad t \in [-r,0] . \end{aligned}$$

To prove the existence of solutions to (3.24)–(3.26), we need the following auxiliary lemma.

Lemma 3.14. *Let $0 < \nu \le 1$, and let $\sigma\colon J \to E$ be continuous. A function y is a solution of the fractional integral equation*

$$y(t) = \begin{cases} \varphi(0) + \dfrac{1}{\Gamma(\nu)} \displaystyle\int_0^t (t-s)^{\nu-1}\sigma(s)ds, & \text{if } t \in [0,t_1] ,\\ \varphi(0) + \displaystyle\sum_{i=1}^k I_i(y_{t_i^-}) + \dfrac{1}{\Gamma(\nu)} \displaystyle\sum_{i=1}^k \int_{t_{i-1}}^{t_i} (t_i-s)^{\nu-1}\sigma(s)ds & \\ +\dfrac{1}{\Gamma(\nu)} \displaystyle\int_{t_k}^t (t-s)^{\nu-1}\sigma(s)ds, & \text{if } t \in (t_k,t_{k+1}] ,\\ \varphi(t), & t \in [-r,0] , \end{cases} \tag{3.28}$$

where $k = 1,\dots,m$, if and only if y is a solution of the fractional problem

$$\begin{aligned} {}^cD^\nu y(t) &= \sigma(t), & t \in J_k ,\\ \Delta y|_{t=t_k} &= I_k(y_{t_k^-}), & k = 1,\dots,m ,\\ y(t) &= \varphi(t), & t \in [-r,0] . \end{aligned}$$

Let us introduce the following conditions:
(3.14.1) The function $f: J \times PC([-r,0],E) \times E \to E$ is continuous.
(3.14.2) There exist constants $K > 0$ and $0 < L < 1$ such that

$$\|f(t,u,v) - f(t,\bar{u},\bar{v})\| \le K\|u-\bar{u}\|_{PC} + L\|v-\bar{v}\|$$

for any $u, \bar{u} \in PC([-r,0],E)$, $v, \bar{v} \in E$ and $t \in J$.
(3.14.3) There exists a constant $\widetilde{l} > 0$ such that

$$\|I_k(u) - I_k(\overline{u})\| \le \widetilde{l}\|u - \overline{u}\|_{PC}$$

for each $u, \overline{u} \in PC([-r,0],E)$ and $k = 1, \ldots, m$.

We are now in a position to state and prove our existence result for problem (3.24)–(3.26) based on the concept of measures of noncompactness and Dafixth's fixed point theorem.

Remark 3.15 ([66]). Conditions (3.14.2) and (3.14.3) are respectively equivalent to the inequalities

$$\alpha\left(f(t,B_1,B_2)\right) \le K\alpha(B_1) + L\alpha(B_2)$$

and

$$\alpha(I_k(B_1)) \le \widetilde{l}\alpha(B_1)$$

for any bounded sets $B_1 \subseteq PC([-r,0],E)$, $B_2 \subseteq E$, for each $t \in J$ and $k = 1, \ldots, m$.

Theorem 3.16. *Assume* (3.14.1)–(3.14.3). *If*

$$m\widetilde{l} + \frac{(m+1)KT^{\nu}}{(1-L)\Gamma(\nu+1)} < 1\,, \tag{3.29}$$

then IVP (3.24)–(3.26) *has at least one solution on* J.

Proof. Transform problem (3.24)–(3.26) into a fixed point problem. Consider the operator $N\colon \Omega \to \Omega$ defined by

$$(Ny)(t) = \begin{cases} \varphi(0) + \sum\limits_{0<t_k<t} I_k(y_{t_k^-}) + \dfrac{1}{\Gamma(\nu)} \sum\limits_{0<t_k<t} \int\limits_{t_{k-1}}^{t_k} (t_k - s)^{\nu-1} g(s)ds & \\ + \dfrac{1}{\Gamma(\nu)} \int\limits_{t_k}^{t} (t-s)^{\nu-1} g(s)ds, & t \in [0,T]\,, \\ \varphi(t), & t \in [-r,0]\,, \end{cases} \tag{3.30}$$

where $g \in C(J,E)$ is such that

$$g(t) = f(t, y_t, g(t))\,.$$

Clearly, the fixed points of operator N are solutions of problem (3.24)–(3.26).

We will show that N satisfies the assumptions of Darbo's fixed point theorem. The proof will be given in several claims.

Claim 1: N is continuous. Let $\{u_n\}$ be a sequence such that $u_n \to u$ in Ω. If $t \in [-r, 0]$, then

$$\|N(u_n)(t) - N(u)(t)\| = 0\,.$$

For $t \in J$ we have

$$\begin{aligned}\|N(u_n)(t) - N(u)(t)\| \le \frac{1}{\Gamma(\nu)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\nu-1} \|g_n(s) - g(s)\| ds \\ + \frac{1}{\Gamma(\nu)} \int_{t_k}^{t} (t-s)^{\nu-1} \|g_n(s) - g(s)\| ds \\ + \sum_{0<t_k<t} \|I_k(u_{nt_k^-}) - I_k(u_{t_k^-})\|\,,\end{aligned} \tag{3.31}$$

where $g_n, g \in C(J, E)$ are given by

$$g_n(t) = f(t, u_{nt}, g_n(t))$$

and

$$g(t) = f(t, u_t, g(t))\,.$$

By (3.14.2) we have

$$\begin{aligned}\|g_n(t) - g(t)\| &= \|f(t, u_{nt}, g_n(t)) - f(t, u_t, g(t))\| \\ &\le K\|u_{nt} - u_t\|_{PC} + L\|g_n(t) - g(t)\|\,.\end{aligned}$$

Then

$$\|g_n(t) - g(t)\| \le \frac{K}{1-L} \|u_{nt} - u_t\|_{PC}\,.$$

Since $u_n \to u$, we get $g_n(t) \to g(t)$ as $n \to \infty$ for each $t \in J$. Let $\eta > 0$ be such that for each $t \in J$ we have $\|g_n(t)\| \le \eta$ and $\|g(t)\| \le \eta$. Then we have

$$\begin{aligned}(t-s)^{\nu-1}\|g_n(s) - g(s)\| &\le (t-s)^{\nu-1}[\|g_n(s)\| + \|g(s)\|] \\ &\le 2\eta(t-s)^{\nu-1}\end{aligned}$$

and

$$\begin{aligned}(t_k-s)^{\nu-1}\|g_n(s) - g(s)\| &\le (t_k-s)^{\nu-1}[\|g_n(s)\| + \|g(s)\|] \\ &\le 2\eta(t_k-s)^{\nu-1}\,.\end{aligned}$$

For each $t \in J$ the functions $s \to 2\eta(t-s)^{\nu-1}$ and $s \to 2\eta(t_k - s)^{\nu-1}$ are integrable on $[0, t]$. The Lebesgue dominated convergence theorem and (3.31) imply

$$\|N(u_n)(t) - N(u)(t)\| \to 0 \text{ as } n \to \infty\,.$$

Hence,

$$\|N(u_n) - N(u)\|_\Omega \to 0 \text{ as } n \to \infty .$$

Consequently, N is continuous.

Let R be a constant such that

$$R \geq \max\left\{ \frac{(\|\varphi(0)\| + mc_1)\Gamma(\nu+1)(1-L) + (m+1)T^\nu f^*}{\Gamma(\nu+1)(1-L) - \left[(m+1)T^\nu K + m\tilde{l}\Gamma(\nu+1)(1-L)\right]}, \|\varphi\|_{PC} \right\} , \tag{3.32}$$

where

$$c_1 = \max_{1\leq k\leq m} \{\sup\{\|I_k(v)\|, v \in PC([-r,0],E)\}\}$$

and

$$f^* = \sup_{t\in J} \|f(t,0,0)\| .$$

Define

$$D_R = \{u \in \Omega : \|u\|_\Omega \leq R\} .$$

It is clear that D_R is a bounded, closed, and convex subset of Ω.

Claim 2: $N(D_R) \subset D_R$. Let $u \in D_R$; we show that $Nu \in D_R$. If $t \in [-r,0]$, then

$$\|N(u)(t)\| \leq \|\varphi\|_{PC} \leq R .$$

If $t \in J$, then we have

$$\begin{aligned} \|N(u)(t)\| \leq \|\varphi(0)\| &+ \frac{1}{\Gamma(\nu)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\nu-1} \|g(s)\| ds \\ &+ \frac{1}{\Gamma(\nu)} \int_{t_k}^{t} (t-s)^{\nu-1} \|g(s)\| ds + \sum_{0<t_k<t} \|I_k(u_{t_k^-})\| . \end{aligned} \tag{3.33}$$

By (3.14.2), for each $t \in J$ we have

$$\begin{aligned} \|g(t)\| &\leq \|f(t,u_t,g(t)) - f(t,0,0)\| + \|f(t,0,0)\| \\ &\leq K\|u_t\|_{PC} + L\|g(t)\| + f^* \\ &\leq K\|u\|_\Omega + L\|g(t)\| + f^* \\ &\leq KR + L\|g(t)\| + f^* . \end{aligned}$$

Then

$$\|g(t)\| \leq \frac{f^* + KR}{1-L} := M .$$

Thus, (3.32), (3.33), and (3.14.3) imply that

$$\begin{aligned}
\|Nu(t)\| &\leq \|\varphi(0)\| + \frac{mMT^\nu}{\Gamma(\nu+1)} + \frac{MT^\nu}{\Gamma(\nu+1)} + \sum_{k=1}^{m} \|I_k(u_{t_k^-}) - I_k(0)\| + \sum_{k=1}^{m} \|I_k(0)\| \\
&\leq \|\varphi(0)\| + \frac{(m+1)MT^\nu}{\Gamma(\nu+1)} + m\widetilde{l}\|u_{t_k^-}\|_{PC} + mc_1 \\
&\leq \|\varphi(0)\| + \frac{(m+1)MT^\nu}{\Gamma(\nu+1)} + m\widetilde{l}\|u\|_\Omega + mc_1 \\
&\leq \|\varphi(0)\| + \frac{(m+1)MT^\nu}{\Gamma(\nu+1)} + m\widetilde{l}R + mc_1 \\
&\leq R\,.
\end{aligned}$$

Thus, for each $t \in [-r, T]$ we have $\|Nu(t)\| \leq R$. This implies that $\|Nu\|_\Omega \leq R$. Consequently,

$$N(D_R) \subset D_R\,.$$

Claim 3: $N(D_R)$ is bounded and equicontinuous. By Claim 2 we have $N(D_R) = \{N(u)\colon u \in D_R\} \subset D_R$. Thus, for each $u \in D_R$ we have $\|N(u)\|_\Omega \leq R$. Hence, $N(D_R)$ is bounded. Let $t_1, t_2 \in (0, T]$, $t_1 < t_2$, and let $u \in D_R$. Then

$$\begin{aligned}
&\|N(u)(t_2) - N(u)(t_1)\| \\
&\leq \frac{1}{\Gamma(\nu)} \int_0^{t_1} |(t_2-s)^{\nu-1} - (t_1-s)^{\nu-1}| \|g(s)\| ds \\
&\quad + \frac{1}{\Gamma(\nu)} \int_{t_1}^{t_2} |(t_2-s)^{\nu-1}| \|g(s)\| ds + \sum_{0<t_k<t_2-t_1} \|I_k(u_{t_k^-}) - I_k(0)\| + \sum_{0<t_k<t_2-t_1} \|I_k(0)\| \\
&\leq \frac{M}{\Gamma(\nu+1)} [2(t_2-t_1)^\nu + (t_2^\nu - t_1^\nu)] + (t_2-t_1)(\widetilde{l}\|u_{t_k^-}\|_{PC} + c_1) \\
&\leq \frac{M}{\Gamma(\nu+1)} [2(t_2-t_1)^\nu + (t_2^\nu - t_1^\nu)] + (t_2-t_1)(\widetilde{l}\|u\|_\Omega + c_1) \\
&\leq \frac{M}{\Gamma(\nu+1)} [2(t_2-t_1)^\nu + (t_2^\nu - t_1^\nu)] + (t_2-t_1)(\widetilde{l}R + c_1)\,.
\end{aligned}$$

As $t_1 \to t_2$, the right-hand side of the preceding inequality tends to zero.

Claim 4: The operator $N\colon D_R \to D_R$ is a strict set contraction. Let $V \subset D_R$. If $t \in [-r, 0]$, then

$$\begin{aligned}
\alpha(N(V)(t)) &= \alpha(N(y)(t), y \in V) \\
&= \alpha(\varphi(t), y \in V) \\
&= 0\,.
\end{aligned}$$

If $t \in J$, then we have

$$\begin{aligned}\alpha(N(V)(t)) &= \alpha((Ny)(t), y \in V)\\ &\leq \sum_{0<t_k<t} \left\{\alpha(I_k(y_{t_k^-})), y \in V\right\} + \frac{1}{\Gamma(\nu)} \sum_{0<t_k<t} \left\{\int_{t_{k-1}}^{t_k} (t_k - s)^{\nu-1} \alpha(g(s))ds, y \in V\right\}\\ &\quad + \frac{1}{\Gamma(\nu)} \left\{\int_{t_k}^{t} (t-s)^{\nu-1} \alpha(g(s))ds, y \in V\right\} .\end{aligned}$$

Then Remark 3.15 and Lemma 1.32 imply that for each $s \in J$

$$\begin{aligned}\alpha(\{g(s), y \in V\}) &= \alpha(\{f(s, y(s), g(s)), y \in V\})\\ &\leq K\alpha(\{y(s), y \in V\}) + L\alpha(\{g(s), y \in V\}) .\end{aligned}$$

Thus,

$$\alpha\left(\{g(s), y \in V\}\right) \leq \frac{K}{1-L} \alpha\{y(s), y \in V\} .$$

On the other hand, for each $t \in J$ and $k = 1, \ldots, m$ we have

$$\sum_{0<t_k<t} \alpha\left(\left\{I_k(y_{t_k^-}), y \in V\right\}\right) \leq m\tilde{l}\alpha(\{y(t), y \in V\}) .$$

Then

$$\begin{aligned}\alpha(N(V)(t)) &\leq m\tilde{l}\alpha(\{y(t), y \in V\}) + \frac{mK}{(1-L)\Gamma(\nu)} \left\{\int_0^t (t-s)^{\nu-1} \{\alpha(y(s))\} ds, y \in V\right\}\\ &\quad + \frac{K}{(1-L)\Gamma(\nu)} \left\{\int_0^t (t-s)^{\nu-1} \{\alpha(y(s))\} ds, y \in V\right\}\\ &\leq m\tilde{l}\alpha_c(V) + \left[\frac{mKT^\nu}{(1-L)\Gamma(\nu+1)} + \frac{KT^\nu}{(1-L)\Gamma(\nu+1)}\right] \alpha_c(V)\\ &= \left[m\tilde{l} + \frac{(m+1)KT^\nu}{(1-L)\Gamma(\nu+1)}\right] \alpha_c(V) .\end{aligned}$$

Therefore,

$$\alpha_c(NV) \leq \left[m\tilde{l} + \frac{(m+1)KT^\nu}{(1-L)\Gamma(\nu+1)}\right] \alpha_c(V) .$$

Thus, by (3.29), operator N is a set contraction. As a consequence of Theorem 1.45, we deduce that N has a fixed point that is a solution of problem (3.24)–(3.26). □

Our next existence result for problem (3.24)–(3.26) is based on the concept of measures of noncompactness and Mönch's fixed point theorem.

Theorem 3.17. *Assume* (3.14.1)–(3.14.4) *and* (3.29) *hold. If*

$$m\widetilde{l} < 1\,,$$

then IVP (3.24)–(3.26) *has at least one solution.*

Proof. Consider the operator N defined in (3.30). We will show that N satisfies the assumptions of Mönch's fixed point theorem. We know that $N\colon D_R \to D_R$ is bounded and continuous, and we need to prove that the implication

$$[V = \overline{\text{conv}}N(V) \text{ or } V = N(V) \cup \{0\}] \text{ implies } \alpha(V) = 0$$

holds for every subset V of D_R. Now let V be a subset of D_R such that $V \subset \overline{\text{conv}}(N(V) \cup \{0\})$; V is bounded and equicontinuous, and therefore the function $t \to v(t) = \alpha(V(t))$ is continuous on $[-r, T]$. By Remark 3.15, Lemma 1.33, and the properties of the measure α, for each $t \in J$ we have

$$\begin{aligned} v(t) &\le \alpha(N(V)(t) \cup \{0\}) \\ &\le \alpha(N(V)(t)) \\ &\le \alpha\{(Ny)(t), y \in V\} \\ &\le m\widetilde{l}\alpha(\{y(t), y \in V\}) + \frac{(m+1)K}{(1-L)\Gamma(\nu)}\left\{\int_0^t (t-s)^{\nu-1}\{\alpha(y(s))\}ds, y \in V\right\} \\ &= m\widetilde{l}v(t) + \frac{(m+1)K}{(1-L)\Gamma(\nu)}\int_0^t (t-s)^{\nu-1}v(s)ds\,. \end{aligned}$$

Then

$$v(t) \le \frac{(m+1)K}{(1-m\widetilde{l})(1-L)\Gamma(\nu)}\int_0^t (t-s)^{\nu-1}v(s)ds\,.$$

Lemma 1.52 implies that $v(t) = 0$ for each $t \in J$.

For $t \in [-r, 0]$ we have $v(t) = \alpha(\varphi(t)) = 0$, so $V(t)$ is relatively compact in E. In view of the Ascoli–Arzelà theorem, V is relatively compact in D_R. Applying Theorem 1.46, we conclude that N has a fixed point $y \in D_R$. Hence, N has a fixed point that is a solution of problem (3.24)–(3.26). □

3.3.3 Examples

Example 1. Consider the infinite system

$$ {}^cD^{\frac{1}{2}}_{t_k} y_n(t) = \frac{e^{-t}}{(11+e^t)}\left[\frac{y_n(t)}{1+y_n(t)} - \frac{{}^cD^{\frac{1}{2}}_{t_k} y_n(t)}{1+{}^cD^{\frac{1}{2}}_{t_k} y_n(t)}\right], \quad \text{for each, } t \in J_0 \cup J_1 , \quad (3.34)$$

$$ \Delta y_n|_{t=\frac{1}{2}} = \frac{y_n\left(\frac{1}{2}^-\right)}{10 + y_n\left(\frac{1}{2}^-\right)} , \quad (3.35)$$

$$ y_n(t) = \varphi(t) , \quad t \in [-r, 0],\ r > 0 , \quad (3.36)$$

where $\varphi \in PC([-r,0],E)$, $J_0 = [0, \frac{1}{2}]$, $J_1 = (\frac{1}{2}, 1]$, $t_0 = 0$, and $t_1 = \frac{1}{2}$.

Set

$$ E = l^1 = \{y = (y_1, y_2, \ldots, y_n, \ldots), \sum_{n=1}^{\infty} |y_n| < \infty\} ,$$

and

$$ f(t,u,v) = \frac{e^{-t}}{(11+e^t)}\left[\frac{u}{1+u} - \frac{v}{1+v}\right], \quad t \in [0,1],\ u \in PC([-r,0],E),\ \text{and } v \in E .$$

Clearly, the function f is jointly continuous; now E is a Banach space with the norm $\|y\| = \sum_{n=1}^{\infty} |y_n|$. For any $u, \bar{u} \in PC([-r,0],E)$, $v, \bar{v} \in E$ and $t \in [0,1]$

$$ \|f(t,u,v) - f(t,\bar{u},\bar{v})\| \le \frac{1}{12}(\|u-\bar{u}\|_{PC} + \|v-\bar{v}\|) .$$

Hence, condition (3.14.2) is satisfied by $K = L = \frac{1}{12}$.

Let

$$ I_1(u) = \frac{u}{10+u} , \quad u \in PC([-r,0],E)$$

and take $u, v \in PC([-r,0],E)$. Then we have

$$ \|I_1(u) - I_1(v)\| = \left\|\frac{u}{10+u} - \frac{v}{10+v}\right\| \le \frac{1}{10}\|u-v\|_{PC} .$$

Hence, condition (3.14.3) is satisfied by $\widetilde{l} = \frac{1}{10}$.

The conditions

$$ m\widetilde{l} + \frac{(m+1)KT^{\nu}}{(1-L)\Gamma(\nu+1)} = \left[\frac{1}{10} + \frac{\frac{2}{12}}{\left(1-\frac{1}{12}\right)\Gamma\left(\frac{3}{2}\right)}\right] = \frac{4}{11\sqrt{\pi}} + \frac{1}{10} < 1$$

are satisfied by $T = m = 1$ and $\nu = \frac{1}{2}$. It follows from Theorem 3.16 that problem (3.34)–(3.36) has a at least one solution on J.

Example 2. Consider the impulsive problem

$$^cD^{\frac{1}{2}}_{t_k}y_n(t) = \frac{2 + \|y_n(t)\| + \|^cD^{\frac{1}{2}}_{t_k}y_n(t)\|}{108e^{t+3}\left(1 + \|y_n(t)\| + \|^cD^{\frac{1}{2}}_{t_k}y_n(t)\|\right)}\,, \quad \text{for each, } t \in J_0 \cup J_1\,, \tag{3.37}$$

$$\Delta y_n|_{t=\frac{1}{3}} = \frac{\|y_n\left(\frac{1}{3}^-\right)\|}{6 + \|y_n\left(\frac{1}{3}^-\right)\|}\,, \tag{3.38}$$

$$y_n(t) = \varphi(t)\,, \quad t \in [-r, 0],\ r > 0\,, \tag{3.39}$$

where $\varphi \in PC([-r, 0], E)$, $J_0 = [0, \frac{1}{3}]$, $J_1 = (\frac{1}{3}, 1]$, $t_0 = 0$, and $t_1 = \frac{1}{3}$. Set

$$E = l^1 = \{y = (y_1, y_2, \dots, y_n, \dots),\ \sum_{n=1}^{\infty} |y_n| < \infty\}\,,$$

and

$$f(t, u, v) = \frac{2 + \|u\| + \|v\|}{108e^{t+3}(1 + \|u\| + \|v\|)}\,, \quad t \in [0, 1],\ u \in PC([-r, 0], E),\ v \in E\,.$$

Clearly, the function f is jointly continuous. Now E is a Banach space with the norm $\|y\| = \sum_{n=1}^{\infty} |y_n|$. For any $u, \bar{u} \in PC([-r, 0], E)$, $v, \bar{v} \in E$, and $t \in [0, 1]$,

$$\|f(t, u, v) - f(t, \bar{u}, \bar{v})\| \le \frac{1}{108e^3}(\|u - \bar{u}\|_{PC} + \|v - \bar{v}\|)\,.$$

Hence, condition (3.14.2) is satisfied by $K = L = \frac{1}{108e^3}$.

Let

$$I_1(u) = \frac{\|u\|}{6 + \|u\|}\,, \quad u \in PC([-r, 0], E)\,,$$

and take $u, v \in PC([-r, 0], E)$. Then we have

$$\|I_1(u) - I_1(v)\| = \left\|\frac{u}{6 + u} - \frac{v}{6 + v}\right\| \le \frac{1}{6}\|u - v\|_{PC}\,.$$

Hence, condition (3.14.3) is satisfied by $\widetilde{l} = \frac{1}{6}$.

The condition

$$m\widetilde{l} + \frac{(m + 1)KT^v}{(1 - L)\Gamma(v + 1)} = \left[\frac{1}{6} + \frac{\frac{2}{12}}{\left(1 - \frac{1}{12}\right)\Gamma\left(\frac{3}{2}\right)}\right] = \frac{4}{11\sqrt{\pi}} + \frac{1}{6} < 1$$

is satisfied by $T = m = 1$ and $v = \frac{1}{2}$. We also have

$$m\widetilde{l} = \frac{1}{6} < 1\,.$$

It follows from Theorem 3.17 that problem (3.37)–(3.39) has at least one solution on J.

3.4 Existence and Stability Results for Perturbed Impulsive NIFDE with Finite Delay

3.4.1 Introduction

In this section, we establish existence, uniqueness, and stability results for the nonlinear implicit perturbed fractional differential equation with finite delay and impulses

$$^cD^{\alpha}_{t_k}y(t) = f(t, y_t, {}^cD^{\alpha}_{t_k}y(t)) + \phi(t, y_t)\,, \quad \text{for } t \in (t_k, t_{k+1}],\ k = 0, \dots, m,\ 0 < \alpha \le 1\,, \tag{3.40}$$

$$\Delta y|_{t_k} = I_k(y_{t_k^-})\,, \quad k = 1, \dots, m\,, \tag{3.41}$$

$$y(t) = \varphi(t)\,, \quad t \in [-r, 0],\ r > 0\,, \tag{3.42}$$

where $J_0 = [t_0, t_1], J_k = (t_k, t_{k+1}];\ k = 1, \dots, m,\ {}^cD^{\alpha}_{t_k}$ is the Caputo fractional derivative, $f: J \times PC([-r, 0], \mathbb{R}) \times \mathbb{R} \to \mathbb{R}$ and $\phi: J \times PC([-r, 0], \mathbb{R}) \to \mathbb{R}$ are given functions, $I_k: PC([-r, 0], \mathbb{R}) \to \mathbb{R}$, $\varphi \in PC([-r, 0], \mathbb{R})$, and $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$.

The arguments are based on Banach's contraction principle and Schaefer's fixed point theorem. Finally, we present two examples to show the applicability of our results.

3.4.2 Existence of Solutions

Consider the Banach space

$$PC([-r, 0], \mathbb{R}) = \{y: [-r, 0] \to \mathbb{R}: y \in C((\tau_k, \tau_{k+1}], \mathbb{R}),\ \ k = 1, \dots, m, \text{ and there exist } y(\tau_k^-) \text{ and } y(\tau_k^+),\ \ k = 1, \dots, m \text{ with } y(\tau_k^-) = y(\tau_k)\}\,,$$

with the norm

$$\|y\|_{PC} = \sup_{t \in [-r,0]} |y(t)|\,;$$

$$PC([0, T], \mathbb{R}) = \{y: [0, T] \to \mathbb{R}: y \in C((t_k, t_{k+1}], \mathbb{R}),\ \ k = 1, \dots, m, \text{ and there exist } y(t_k^-) \text{ and } y(t_k^+),\ \ k = 1, \dots, m \text{ with } y(t_k^-) = y(t_k)\}\,,$$

with the norm

$$\|y\|_C = \sup_{t \in [0,T]} |y(t)|\,;$$

and

$$\Omega = \{y: [-r, T] \to \mathbb{R}: y|_{[-r,0]} \in PC([-r, 0], \mathbb{R}) \text{ and } y|_{[0,T]} \in PC([0, T], \mathbb{R})\}\,,$$

with the norm

$$\|y\|_{\Omega} = \sup_{t \in [-r,T]} |y(t)|\,.$$

Definition 3.18. A function $y \in \Omega$ whose α-derivative exists on J_k is said to be a solution of (3.40)–(3.42) if y satisfies the equation ${}^cD^{\alpha}_{t_k}y(t) = f(t, y_t, {}^cD^{\alpha}_{t_k}y(t)) + \phi(t, y_t)$ on J_k and satisfies the conditions

$$\Delta y|_{t=t_k} = I_k(y_{t_k^-}), k = 1, \dots, m ,$$
$$y(t) = \varphi(t) , \quad t \in [-r, 0] .$$

To prove the existence of solutions to (3.40)–(3.42), we need the following auxiliary lemma.

Lemma 3.19. *Let $0 < \alpha \le 1$, and let $\sigma : J \to \mathbb{R}$ be continuous. A function y is a solution of the fractional integral equation*

$$y(t) = \begin{cases} \varphi(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\sigma(s)ds, & \text{if } t \in [0, t_1] , \\ \varphi(0) + \sum_{i=1}^{k} I_i(y_{t_i^-}) + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1}\sigma(s)ds , & \\ + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1}\sigma(s)ds, & \text{if } t \in (t_k, t_{k+1}] , \\ \varphi(t), & t \in [-r, 0] , \end{cases} \tag{3.43}$$

where $k = 1, \dots, m$, if and only if y is a solution of the fractional problem

$${}^cD^{\alpha}y(t) = \sigma(t) , \quad t \in J_k , \tag{3.44}$$
$$\Delta y|_{t=t_k} = I_k(y_{t_k^-}) , \quad k = 1, \dots, m , \tag{3.45}$$
$$y(t) = \varphi(t) , \quad t \in [-r, 0] . \tag{3.46}$$

Proof. Assume that y satisfies (3.44)–(3.46). If $t \in [0, t_1]$, then

$${}^cD^{\alpha}y(t) = \sigma(t) .$$

Lemma 1.9 implies

$$y(t) = \varphi(0) + I^{\alpha}\sigma(t) = \varphi(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\sigma(s)ds .$$

If $t \in (t_1, t_2]$, then Lemma 1.9 implies

$$
\begin{aligned}
y(t) &= y(t_1^+) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t} (t-s)^{\alpha-1}\sigma(s)ds \\
&= \Delta y|_{t=t_1} + y(t_1^-) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t} (t-s)^{\alpha-1}\sigma(s)ds \\
&= I_1(y_{t_1^-}) + \left[\varphi(0) + \frac{1}{\Gamma(\alpha)} \int_{0}^{t_1} (t_1-s)^{\alpha-1}\sigma(s)ds\right] \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t} (t-s)^{\alpha-1}\sigma(s)ds\,. \\
&= \varphi(0) + I_1(y_{t_1^-}) + \frac{1}{\Gamma(\alpha)} \int_{0}^{t_1} (t_1-s)^{\alpha-1}\sigma(s)ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t} (t-s)^{\alpha-1}\sigma(s)ds\,.
\end{aligned}
$$

If $t \in (t_2, t_3]$, then from Lemma 1.9 we get

$$
\begin{aligned}
y(t) &= y(t_2^+) + \frac{1}{\Gamma(\alpha)} \int_{t_2}^{t} (t-s)^{\alpha-1}\sigma(s)ds \\
&= \Delta y|_{t=t_2} + y(t_2^-) + \frac{1}{\Gamma(\alpha)} \int_{t_2}^{t} (t-s)^{\alpha-1}\sigma(s)ds \\
&= I_2(y_{t_2^-}) + \left[\varphi(0) + I_1(y_{t_1^-}) + \frac{1}{\Gamma(\alpha)} \int_{0}^{t_1} (t_1-s)^{\alpha-1}\sigma(s)ds\right. \\
&\quad \left. + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1}\sigma(s)ds\right] + \frac{1}{\Gamma(\alpha)} \int_{t_2}^{t} (t-s)^{\alpha-1}\sigma(s)ds\,. \\
&= \varphi(0) + \left[I_1(y_{t_1^-}) + I_2(y_{t_2^-})\right] + \left[\frac{1}{\Gamma(\alpha)} \int_{0}^{t_1} (t_1-s)^{\alpha-1}\sigma(s)ds\right. \\
&\quad \left. + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1}\sigma(s)ds\right] + \frac{1}{\Gamma(\alpha)} \int_{t_2}^{t} (t-s)^{\alpha-1}\sigma(s)ds\,.
\end{aligned}
$$

Repeating the process in this way, the solution $y(t)$ for $t \in (t_k, t_{k+1}]$, where $k = 1, \dots, m$, can be written

$$y(t) = \varphi(0) + \sum_{i=1}^{k} I_i(y_{t_i^-}) + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1}\sigma(s)ds$$
$$+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1}\sigma(s)ds \,.$$

Conversely, assume that y satisfies the impulsive fractional integral equation (3.43). If $t \in [0, t_1]$, then $y(0) = \varphi(0)$. Using the fact that ${}^cD^\alpha$ is the left inverse of I^α, we get

$${}^cD^\alpha y(t) = \sigma(t)\,, \quad \text{for each } t \in [0, t_1]\,.$$

If $t \in (t_k, t_{k+1}]$, $k = 1, \dots, m$ and using the fact that ${}^cD^\alpha C = 0$, where C is a constant, we get

$${}^cD^\alpha y(t) = \sigma(t)\,, \quad \text{for each } t \in (t_k, t_{k+1}]\,.$$

Also, we can easily show that

$$\Delta y|_{t=t_k} = I_k(y_{t_k^-})\,, \quad k = 1, \dots, m\,. \qquad \square$$

We are now in a position to state and prove our existence result for problem (3.40)–(3.42) based on Banach's fixed point.

Theorem 3.20. *Make the following assumptions:*

(3.20.1) *The functions* $f: J \times PC([-r, 0], \mathbb{R}) \times \mathbb{R} \to \mathbb{R}$ *and* $\phi: J \times PC([-r, 0], \mathbb{R}) \to \mathbb{R}$ *are continuous.*

(3.20.2) *There exist constants* $K > 0$, $\overline{K} > 0$ *and* $0 < L < 1$ *such that*

$$|f(t, u, v) - f(t, \bar{u}, \bar{v})| \le K\|u - \bar{u}\|_{PC} + L|v - \bar{v}|$$

and

$$|\phi(t, u) - \phi(t, \bar{u})| \le \overline{K}\|u - \bar{u}\|_{PC}$$

for any $u, \bar{u} \in PC([-r, 0], \mathbb{R})$, $v, \bar{v} \in \mathbb{R}$, *and* $t \in J$.

(3.20.3) *There exists a constant* $\widetilde{l} > 0$ *such that*

$$|I_k(u) - I_k(\overline{u})| \le \widetilde{l}\|u - \overline{u}\|_{PC}$$

for each $u, \overline{u} \in PC([-r, 0], \mathbb{R})$ *and* $k = 1, \dots, m$.

If

$$m\widetilde{l} + \frac{(m+1)(K + \overline{K})T^\alpha}{(1-L)\Gamma(\alpha+1)} < 1\,, \tag{3.47}$$

then there exists a unique solution for problem (3.40)–(3.42) *on* J.

Proof. Transform problem (3.40)–(3.42) into a fixed point problem. Consider the operator $N\colon \Omega \to \Omega$ defined by

$$(Ny)(t) = \begin{cases} \varphi(0) + \sum_{0<t_k<t} I_k(y_{t_k^-}) + \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} g(s)ds \\ + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} g(s)ds, & t \in [0, T]\,, \\ \varphi(t), & t \in [-r, 0]\,, \end{cases} \tag{3.48}$$

where $g \in C(J, \mathbb{R})$ is given by

$$g(t) = f(t, y_t, g(t)) + \phi(t, y_t)\,.$$

Clearly, the fixed points of operator N are solutions of problem (3.40)–(3.42).

Let $u, w \in \Omega$. If $t \in [-r, 0]$, then

$$|(Nu)(t) - (Nw)(t)| = 0\,.$$

For $t \in J$ we have

$$\begin{aligned} |(Nu)(t) - (Nw)(t)| \le{}& \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} |g(s) - h(s)| ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} |g(s) - h(s)| ds \\ &+ \sum_{0<t_k<t} |I_k(u_{t_k^-}) - I_k(w_{t_k^-})|\,, \end{aligned}$$

where $g, h \in C(J, \mathbb{R})$ are given by

$$g(t) = f(t, u_t, g(t)) + \phi(t, u_t)\,,$$

and

$$h(t) = f(t, w_t, h(t)) + \phi(t, w_t)\,.$$

By (3.20.2) we have

$$\begin{aligned} |g(t) - h(t)| &\le |f(t, u_t, g(t)) - f(t, w_t, h(t))| + |\phi(t, u_t) - \phi(t, w_t)| \\ &\le K\|u_t - w_t\|_{PC} + L|g(t) - h(t)| + \overline{K}\|u_t - w_t\|_{PC}\,. \end{aligned}$$

Then

$$|g(t) - h(t)| \le \frac{K + \overline{K}}{1 - L} \|u_t - w_t\|_{PC}\,.$$

Therefore, for each $t \in J$

$$
\begin{aligned}
|(Nu)(t) - (Nw)(t)| &\le \frac{K+\overline{K}}{(1-L)\Gamma(\alpha)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} \|u_s - w_s\|_{PC} ds \\
&\quad + \frac{K+\overline{K}}{(1-L)\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} \|u_s - w_s\|_{PC} ds \\
&\quad + \sum_{k=1}^{m} \widetilde{l} \|u_{t_k^-} - w_{t_k^-}\|_{PC} . \\
&\le \left[m\widetilde{l} + \frac{m(K+\overline{K})T^\alpha}{(1-L)\Gamma(\alpha+1)} \right. \\
&\quad \left. + \frac{(K+\overline{K})T^\alpha}{(1-L)\Gamma(\alpha+1)} \right] \|u - w\|_\Omega .
\end{aligned}
$$

Thus,

$$
\|(Nu) - (Nw)\|_\Omega \le \left[m\widetilde{l} + \frac{(m+1)(K+\overline{K})T^\alpha}{(1-L)\Gamma(\alpha+1)} \right] \|u - w\|_\Omega .
$$

By (3.47), operator N is a contraction. Hence, by Banach's contraction principle, N has a unique fixed point that is the unique solution of (3.40)–(3.42). □

Our second result is based on Schaefer's fixed point theorem.

Theorem 3.21. *Assume* (3.20.1) *and* (3.20.2) *hold and*

(3.21.1) *There exist* $p, q, r \in C(J, \mathbb{R}_+)$ *with* $r^* = \sup_{t\in J} r(t) < 1$ *such that*

$$
|f(t, u, w)| \le p(t) + q(t)\|u\|_{PC} + r(t)|w| \quad \textit{for } t \in J,\ u \in PC([-r, 0], \mathbb{R}) \textit{ and } w \in \mathbb{R} .
$$

(3.21.2) *The functions* $I_k : PC([-r, 0], \mathbb{R}) \to \mathbb{R}$ *are continuous and there exist constants* $M^*, N^* > 0$, *with* $mM^* < 1$, *such that*

$$
|I_k(u)| \le M^*\|u\|_{PC} + N^* \quad \textit{for each } u \in PC([-r, 0], \mathbb{R}),\ k = 1, \dots, m .
$$

Then problem (3.40)–(3.42) *has at least one solution.*

Proof. Consider operator N defined in (3.48). We will use Schaefer's fixed point theorem to prove that N has a fixed point. The proof will be given in several steps.

Step 1: N is continuous. Let $\{u_n\}$ be a sequence such that $u_n \to u$ in Ω. If $t \in [-r, 0]$, then

$$
|(Nu_n)(t) - (Nu)(t)| = 0 .
$$

For $t \in J$ we have

$$|(Nu_n)(t) - (Nu)(t)| \leq \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} |g_n(s) - g(s)| ds$$
$$+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t - s)^{\alpha-1} |g_n(s) - g(s)| ds$$
$$+ \sum_{0<t_k<t} \left| I_k\left(u_n\left(t_k^-\right)\right) - I_k\left(u\left(t_k^-\right)\right)\right| , \tag{3.49}$$

where $g_n, g \in C(J, \mathbb{R})$ are given by

$$g_n(t) = f(t, u_{nt}, g_n(t)) + \phi(t, u_{nt})$$

and

$$g(t) = f(t, u_t, g(t)) + \phi(t, u_t) .$$

By (3.21.1) we have

$$|g_n(t) - g(t)| \leq |f(t, u_{nt}, g_n(t)) - f(t, u_t, g(t))| + |\phi(t, u_{nt}) - \phi(t, u_t)|$$
$$\leq K\|u_{nt} - u_t\|_{PC} + L|g_n(t) - g(t)| + \overline{K}\|u_{nt} - u_t\|_{PC} .$$

Then

$$|g_n(t) - g(t)| \leq \frac{K + \overline{K}}{1 - L} \|u_{nt} - u_t\|_{PC} .$$

Since $u_n \to u$, we get $g_n(t) \to g(t)$ as $n \to \infty$ for each $t \in J$. Let $\eta > 0$ be such that for each $t \in J$ we have $|g_n(t)| \leq \eta$ and $|g(t)| \leq \eta$. Then

$$(t - s)^{\alpha-1} |g_n(s) - g(s)| \leq (t - s)^{\alpha-1} [|g_n(s)| + |g(s)|]$$
$$\leq 2\eta(t - s)^{\alpha-1}$$

and

$$(t_k - s)^{\alpha-1} |g_n(s) - g(s)| \leq (t_k - s)^{\alpha-1} [|g_n(s)| + |g(s)|]$$
$$\leq 2\eta(t_k - s)^{\alpha-1} .$$

For each $t \in J$ the functions $s \to 2\eta(t - s)^{\alpha-1}$ and $s \to 2\eta(t_k - s)^{\alpha-1}$ are integrable on $[0, t]$; the Lebesgue dominated convergence theorem and (3.49) imply that

$$|(Nu_n)(t) - (Nu)(t)| \to 0 \quad \text{as} \quad n \to \infty .$$

Hence

$$\|(Nu_n) - (Nu)\|_{\Omega} \to 0 \quad \text{as} \quad n \to \infty ,$$

and so N is continuous.

Step 2: N maps bounded sets to bounded sets in Ω. Indeed, it is enough to show that for any $\eta^* > 0$ there exists a positive constant ℓ such that for each $u \in B_{\eta^*} = \{u \in \Omega : \|u\|_\Omega \le \eta^*\}$ we have $\|N(u)\|_\Omega \le \ell$. For each $t \in J$ we have

$$(Nu)(t) = \varphi(0) + \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} g(s)ds + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} g(s)ds + \sum_{0<t_k<t} I_k(u_{t_k^-}) , \tag{3.50}$$

where $g \in C(J, \mathbb{R})$ is given by

$$g(t) = f(t, u_t, g(t)) + \phi(t, y_t) .$$

By (3.21.2), for each $t \in J$ we have

$$\begin{aligned} |g(t)| &\le |f(t, u_t, g(t))| + |\phi(t, y_t)| \\ &\le p(t) + q(t)\|u_t\|_{PC} + r(t)|g(t)| + |\phi(t, y_t) - \phi(t, 0)| + |\phi(t, 0)| \\ &\le p(t) + q(t)\|u_t\|_{PC} + r(t)|g(t)| + \overline{K}\|u_t\|_{PC} + |\phi(t, 0)| \\ &\le p(t) + q(t)\|u\|_\Omega + r(t)|g(t)| + \overline{K}\|u\|_\Omega + |\phi(t, 0)| \\ &\le p(t) + (q(t) + \overline{K})\eta^* + r(t)|g(t)| + |\phi(t, 0)| \\ &\le p^* + (q^* + \overline{K})\eta^* + r^*|g(t)| + \phi^* , \end{aligned}$$

where $p^* = \sup_{t\in J} p(t)$, $q^* = \sup_{t\in J} q(t)$, and $\phi^* = \sup_{t\in J} |g(t, 0)|$.
Then

$$|g(t)| \le \frac{p^* + (q^* + \overline{K})\eta^* + \phi^*}{1 - r^*} := M .$$

Thus, (3.50) implies

$$\begin{aligned} |(Nu)(t)| &\le |\varphi(0)| + \frac{mMT^\alpha}{\Gamma(\alpha+1)} + \frac{MT^\alpha}{\Gamma(\alpha+1)} + m(M^*\|u_{t_k^-}\|_{PC} + N^*) \\ &\le |\varphi(0)| + \frac{(m+1)MT^\alpha}{\Gamma(\alpha+1)} + m(M^*\|u\|_\Omega + N^*) \\ &\le |\varphi(0)| + \frac{(m+1)MT^\alpha}{\Gamma(\alpha+1)} + m(M^*\eta^* + N^*) := R . \end{aligned}$$

If $t \in [-r, 0]$, then

$$|(Nu)(t)| \le \|\varphi\|_{PC} ,$$

so

$$\|N(u)\|_\Omega \le \max \{R, \|\varphi\|_{PC}\} := \ell .$$

Step 3: N maps bounded sets to equicontinuous sets of Ω.
Let $t_1, t_2 \in (0, T]$, $t_1 < t_2$, let B_{η^*} be a bounded set of Ω as in Step 2, and let $u \in B_{\eta^*}$.

Then

$$
\begin{aligned}
&|(Nu)(t_2) - (Nu)(t_1)| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^{t_1} |(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}\| g(s)| ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} |(t_2 - s)^{\alpha-1}\| g(s)| ds + \sum_{0<t_k<t_2-t_1} |I_k(u_{t_k^-})| \\
&\leq \frac{M}{\Gamma(\alpha+1)} [2(t_2 - t_1)^\alpha + (t_2^\alpha - t_1^\alpha)] + (t_2 - t_1)(M^* \|u_{t_k^-}\|_{PC} + N^*) \\
&\leq \frac{M}{\Gamma(\alpha+1)} [2(t_2 - t_1)^\alpha + (t_2^\alpha - t_1^\alpha)] + (t_2 - t_1)(M^* \|u\|_\Omega + N^*) \\
&\leq \frac{M}{\Gamma(\alpha+1)} [2(t_2 - t_1)^\alpha + (t_2^\alpha - t_1^\alpha)] + (t_2 - t_1)(M^* \eta^* + N^*) .
\end{aligned}
$$

As $t_1 \to t_2$, the right-hand side of the preceding inequality tends to zero. As a consequence of Steps 1–3, together with the Ascoli–Arzelà theorem, we can conclude that $N: \Omega \to \Omega$ is completely continuous.

Step 4: A priori bounds. Now it remains to show that the set

$$E = \{u \in \Omega : u = \lambda N(u) \quad \text{for some } 0 < \lambda < 1\}$$

is bounded. Let $u \in E$; then $u = \lambda N(u)$ for some $0 < \lambda < 1$. Thus, for each $t \in J$ we have

$$
\begin{aligned}
u(t) = \lambda\varphi(0) &+ \frac{\lambda}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} g(s) ds \\
&+ \frac{\lambda}{\Gamma(\alpha)} \int_{t_k}^{t} (t - s)^{\alpha-1} g(s) ds + \lambda \sum_{0<t_k<t} I_k(u_{t_k^-}) . \qquad (3.51)
\end{aligned}
$$

And, by (3.21.1), for each $t \in J$ we have

$$
\begin{aligned}
|g(t)| &\leq |f(t, u_t, g(t))| + |\phi(t, y_t)| \\
&\leq p(t) + q(t)\|u_t\|_{PC} + r(t)|g(t)| + |\phi(t, y_t) - \phi(t, 0)| + |\phi(t, 0)| \\
&\leq p(t) + q(t)\|u_t\|_{PC} + r(t)|g(t)| + \overline{K}\|u_t\|_{PC} + |\phi(t, 0)| \\
&\leq p^* + (q^* + \overline{K})\|u_t\|_{PC} + r^*|g(t)| + \phi^* .
\end{aligned}
$$

Thus,

$$|g(t)| \leq \frac{1}{1 - r^*}(p^* + (q^* + \overline{K})\|u_t\|_{PC} + \phi^*) .$$

This implies, by (3.51) and (3.21.2), that for each $t \in J$ we have

$$|u(t)| \leq |\varphi(0)| + \frac{1}{(1-r^*)\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1}(p^* + \phi^* + (q^* + \overline{K})\|u_s\|_{PC})ds$$
$$+ \frac{1}{(1-r^*)\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1}(p^* + \phi^* + (q^* + \overline{K})\|u_s\|_{PC})ds$$
$$+ m(M^*\|u_t\|_{PC} + N^*) .$$

Consider the function v defined by

$$v(t) = \sup\{|u(s)| : \ -r \leq s \leq t\} , \quad 0 \leq t \leq T ;$$

there exists $t^* \in [-r, T]$ such that $v(t) = |u(t^*)|$. If $t \in [0, T]$, then by the previous inequality, for $t \in J$ we have

$$v(t) \leq |\varphi(0)| + \frac{1}{(1-r^*)\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1}(p^* + \phi^* + (q^* + \overline{K})v(s))ds$$
$$+ \frac{1}{(1-r^*)\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1}(p^* + \phi^* + (q^* + \overline{K})v(s))ds$$
$$+ mM^*v(t) + mN^* .$$

Thus,

$$v(t) \leq \frac{1}{(1-mM^*)(1-r^*)\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1}(p^* + \phi^* + (q^* + \overline{K})v(s))ds$$
$$+ \frac{|\varphi(0)| + mN^*}{1-mM^*} + \frac{1}{(1-mM^*)(1-r^*)\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1}(p^* + \phi^* + (q^* + \overline{K})v(s))ds$$
$$\leq \frac{|\varphi(0)| + mN^*}{1-mM^*} + \frac{(m+1)(p^* + \phi^*)T^\alpha}{(1-mM^*)(1-r^*)\Gamma(\alpha+1)}$$
$$+ \frac{(m+1)(q^* + \overline{K})}{(1-mM^*)(1-r^*)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}v(s)ds .$$

Applying Lemma 1.52, we get

$$v(t) \leq \left[\frac{|\varphi(0)| + mN^*}{1-mM^*} + \frac{(m+1)(p^* + \phi^*)T^\alpha}{(1-mM^*)(1-r^*)\Gamma(\alpha+1)}\right]$$
$$\times \left[1 + \frac{\delta(m+1)(q^* + \overline{K})T^\alpha}{(1-mM^*)(1-r^*)\Gamma(\alpha+1)}\right] := A ,$$

where $\delta = \delta(\alpha)$ is a constant. If $t^* \in [-r, 0]$, then $v(t) = \|\varphi\|_{PC}$; thus, for any $t \in J$, $\|u\|_\Omega \leq v(t)$, and we have

$$\|u\|_\Omega \leq \max\{\|\varphi\|_{PC}, A\} .$$

This shows that the set E is bounded. As a consequence of Schaefer's fixed point theorem, we deduce that N has a fixed point that is a solution of problem (3.40)–(3.42). □

3.4.3 Ulam–Hyers Stability Results

Let $z \in PC(J, \mathbb{R})$, $\epsilon > 0$, $\psi > 0$, and $\omega \in PC(J, \mathbb{R}_+)$ is nondecreasing. We consider the sets of inequalities

$$\begin{cases} |{}^cD^\alpha z(t) - f(t, z_t, {}^cD^\alpha z(t)) - \phi(t, z_t)| \leq \epsilon, & t \in (t_k, t_{k+1}],\ k = 1, \dots, m , \\ |\Delta z|_{t_k} - I_k(z_{t_k^-})| \leq \epsilon, & k = 1, \dots, m , \end{cases} \tag{3.52}$$

$$\begin{cases} |{}^cD^\alpha z(t) - f(t, z_t, {}^cD^\alpha z(t)) - \phi(t, z_t)| \leq \omega(t), & t \in (t_k, t_{k+1}],\ k = 1, \dots, m , \\ |\Delta z|_{t_k} - I_k(z_{t_k^-})| \leq \psi, & k = 1, \dots, m , \end{cases} \tag{3.53}$$

and

$$\begin{cases} |{}^cD^\alpha z(t) - f(t, z_t, {}^cD^\alpha z(t)) - \phi(t, z_t)| \leq \epsilon\omega(t), & t \in (t_k, t_{k+1}],\ k = 1, \dots, m , \\ |\Delta z|_{t_k} - I_k(z_{t_k^-})| \leq \epsilon\psi, & k = 1, \dots, m . \end{cases} \tag{3.54}$$

Definition 3.22. Problem (3.40)–(3.41) is Ulam–Hyers stable if there exists a real number $c_{f,m} > 0$ such that, for each $\epsilon > 0$ and for each solution $z \in PC(J, \mathbb{R})$ of inequality (3.52), there exists a solution $y \in \Omega$ of problem (3.40)–(3.41), with

$$|z(t) - y(t)| \leq c_{f,m}\epsilon , \quad t \in J .$$

Definition 3.23. Problem (3.40)–(3.41) is generalized Ulam–Hyers stable if there exists $\theta_{f,m} \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\theta_{f,m}(0) = 0$ such that, for each solution $z \in PC(J, \mathbb{R})$ of inequality (3.52), there exists a solution $y \in \Omega$ of problem (3.40)–(3.41), with

$$|z(t) - y(t)| \leq \theta_{f,m}(\epsilon) , \quad t \in J .$$

Definition 3.24. Problem (3.40)–(3.41) is Ulam–Hyers–Rassias stable with respect to (ω, ψ) if there exists $c_{f,m,\omega} > 0$ such that, for each $\epsilon > 0$ and for each solution $z \in PC(J, \mathbb{R})$ of inequality (3.54), there exists a solution $y \in \Omega$ of problem (3.40)–(3.41), with

$$|z(t) - y(t)| \leq c_{f,m,\omega}\epsilon(\omega(t) + \psi) , \quad t \in J .$$

Definition 3.25. Problem (3.40)–(3.41) is generalized Ulam–Hyers–Rassias stable with respect to (ω, ψ) if there exists $c_{f,m,\omega} > 0$ such that, for each solution $z \in PC(J, \mathbb{R})$ of inequality (3.53), there exists a solution $y \in \Omega$ of problem (3.40)–(3.41), with

$$|z(t) - y(t)| \leq c_{f,m,\omega}(\omega(t) + \psi) , \quad t \in J .$$

Remark 3.26. It is clear that (i) Definition 3.22 implies Definition 3.23, (ii) Definition 3.24 implies Definition 3.25, and (iii) Definition 3.24 for $\omega(t) = \psi = 1$ implies Definition 3.22.

Remark 3.27. A function $z \in PC(J, \mathbb{R})$ is a solution of inequality (3.54) if and only if there is $\sigma \in PC(J, \mathbb{R})$ and a sequence σ_k, $k = 1, \dots, m$ (which depend on z) such that
(i) $|\sigma(t)| \le \epsilon\omega(t)$, $t \in (t_k, t_{k+1}]$, $k = 1, \dots, m$ and $|\sigma_k| \le \epsilon\psi$, $k = 1, \dots, m$;
(ii) ${}^cD^\alpha z(t) = f(t, z_t, {}^cD^\alpha z(t)) + \phi(t, z_t) + \sigma(t),\ t \in (t_k, t_{k+1}]$, $k = 1, \dots, m$;
(iii) $\Delta z|_{t_k} = I_k(z_{t_k^-}) + \sigma_k,\ k = 1, \dots, m.$

One can have similar remarks for inequalities 3.53 and 3.52.

Now we state the following Ulam–Hyers–Rassias stability results.

Theorem 3.28. *Assume* (3.20.1)–(3.20.3) *and* (3.47) *hold and*
(3.28.1) *there exists a nondecreasing function* $\omega \in PC(J, \mathbb{R}_+)$ *and there exists* $\lambda_\omega > 0$ *such that for any* $t \in J$*:*

$$I^\alpha \omega(t) \le \lambda_\omega \omega(t)\,.$$

Then problem (3.40)–(3.41) *is Ulam–Hyers–Rassias stable with respect to* (ω, ψ).

Proof. Let $z \in \Omega$ be a solution of inequality (3.54). Denote by y the unique solution of the problem

$$\begin{cases} {}^cD^\alpha_{t_k} y(t) = f(t, y_t, {}^cD^\alpha_{t_k} y(t)) + \phi(t, y_t), & t \in (t_k, t_{k+1}], k = 1, \dots, m\,, \\ \Delta y|_{t=t_k} = I_k(y_{t_k^-}), & k = 1, \dots, m\,, \\ y(t) = z(t) = \varphi(t), & t \in [-r, 0]\,. \end{cases}$$

Using Lemma 3.19, we obtain for each $t \in (t_k, t_{k+1}]$

$$y(t) = \varphi(0) + \sum_{i=1}^{k} I_i(y_{t_i^-}) + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} g(s)ds$$
$$+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t - s)^{\alpha-1} g(s)ds\,, \quad t \in (t_k, t_{k+1}]\,,$$

where $g \in C(J, \mathbb{R})$ is such that

$$g(t) = f(t, y_t, g(t)) + \phi(t, y_t)\,.$$

Since z is a solution of inequality (3.54), by Remark 3.27 we have

$$\begin{cases} {}^cD^\alpha_{t_k} z(t) = f(t, z_t, {}^cD^\alpha_{t_k} z(t)) + \phi(t, z_t) + \sigma(t), & t \in (t_k, t_{k+1}],\ k = 1, \dots, m\,; \\ \Delta z|_{t=t_k} = I_k(z_{t_k^-}) + \sigma_k, & k = 1, \dots, m\,. \end{cases} \tag{3.55}$$

Clearly, the solution of (3.55) is given by

$$\begin{aligned} z(t) = \varphi(0) + \sum_{i=1}^{k} I_i(z_{t_i^-}) + \sum_{i=1}^{k} \sigma_i + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} h(s) ds \\ + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} \sigma(s) ds + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} h(s) ds \\ + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} \sigma(s) ds , \quad t \in (t_k, t_{k+1}] , \end{aligned}$$

where $h \in C(J, \mathbb{R})$ is given by

$$h(t) = f(t, z_t, h(t)) + \phi(t, z_t) .$$

Hence, for each $t \in (t_k, t_{k+1}]$ it follows that

$$\begin{aligned} |z(t) - y(t)| \le \sum_{i=1}^{k} |\sigma_i| + \sum_{i=1}^{k} |I_i(z_{t_i^-}) - I_i(y_{t_i^-})| \\ + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} |h(s) - g(s)| ds \\ + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} |\sigma(s)| ds \\ + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} |h(s) - g(s)| ds \\ + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} |\sigma(s)| . \end{aligned}$$

Thus,

$$\begin{aligned} |z(t) - y(t)| \le m\epsilon\psi + (m+1)\epsilon\lambda_\omega \omega(t) + \sum_{i=1}^{k} \widetilde{l} \|z_{t_i^-} - y_{t_i^-}\|_{PC} \\ + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} |h(s) - g(s)| ds \\ + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} |h(s) - g(s)| ds . \end{aligned}$$

By (3.20.2) we have

$$\begin{aligned}|h(t)-g(t)| &\le |f(t,z_t,h(t))-f(t,y_t,g(t))|+|\phi(t,z_t)-\phi(t,y_t)|\\ &\le K\|z_t-y_t\|_{PC}+L|g(t)-h(t)|+\overline{K}\|z_t-y_t\|_{PC}\,.\end{aligned}$$

Then

$$|h(t)-g(t)|\le \frac{K+\overline{K}}{1-L}\|z_t-y_t\|_{PC}\,.$$

Therefore, for each $t\in J$

$$\begin{aligned}|z(t)-y(t)| &\le m\epsilon\psi+(m+1)\epsilon\lambda_\omega\omega(t)+\sum_{i=1}^{k}\widetilde{l}\|z_{t_i^-}-y_{t_i^-}\|_{PC}\\ &+\frac{K+\overline{K}}{(1-L)\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{\alpha-1}\|z_s-y_s\|_{PC}ds\\ &+\frac{K+\overline{K}}{(1-L)\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}\|z_s-y_s\|_{PC}ds\,.\end{aligned}$$

Thus,

$$\begin{aligned}|z(t)-y(t)| &\le \sum_{0<t_i<t}\widetilde{l}\|z_{t_i^-}-y_{t_i^-}\|_{PC}+\epsilon(\psi+\omega(t))(m+(m+1)\lambda_\omega)\\ &+\frac{(K+\overline{K})(m+1)}{(1-L)\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\|z_s-y_s\|_{PC}ds\,.\end{aligned}$$

We consider the function v_1 defined by

$$v_1(t)=\sup\{\|z(s)-y(s)\|:\ -r\le s\le t\}\,,\quad 0\le t\le T\,.$$

Then there exists $t^*\in[-r,T]$ such that $v_1(t)=\|z(t^*)-y(t^*)\|$. If $t^*\in[-r,0]$, then $v_1(t)=0$. If $t^*\in[0,T]$, then, by the previous inequality, we have

$$\begin{aligned}v_1(t) &\le \sum_{0<t_i<t}\widetilde{l}v_1(t_i^-)+\epsilon(\psi+\omega(t))(m+(m+1)\lambda_\omega)\\ &+\frac{(K+\overline{K})(m+1)}{(1-L)\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}v_1(s)ds\,.\end{aligned}$$

Applying Lemma 1.53, we get

$$\begin{aligned}v_1(t) &\le \epsilon(\psi+\omega(t))(m+(m+1)\lambda_\omega)\\ &\quad\times\left[\prod_{0<t_i<t}(1+\widetilde{l})\exp\left(\int_0^t\frac{(K+\overline{K})(m+1)}{(1-L)\Gamma(\alpha)}(t-s)^{\alpha-1}ds\right)\right]\\ &\le c_\omega\epsilon(\psi+\omega(t))\,,\end{aligned}$$

where

$$c_\omega = (m + (m+1)\lambda_\omega)\left[\prod_{i=1}^{m}(1+\widetilde{l})\exp\left(\frac{(K+\overline{K})(m+1)T^\alpha}{(1-L)\Gamma(\alpha+1)}\right)\right]$$
$$= (m + (m+1)\lambda_\omega)\left[(1+\widetilde{l})\exp\left(\frac{(K+\overline{K})(m+1)T^\alpha}{(1-L)\Gamma(\alpha+1)}\right)\right]^m .$$

Thus, problem (3.40)–(3.41) is Ulam–Hyers–Rassias stable with respect to (ω, ψ). □

Next we present the following Ulam–Hyers stability result.

Theorem 3.29. *Assume* (3.20.1)–(3.20.3) *and* (3.47) *hold. Then problem* (3.40)–(3.41) *is Ulam–Hyers stable.*

Proof. Let $z \in \Omega$ be a solution of inequality (3.52). Denote by y the unique solution of the problem

$$\begin{cases} {}^cD^\alpha_{t_k} y(t) = f(t, y_t, {}^cD^\alpha_{t_k} y(t)) + \phi(t, y_t), & t \in (t_k, t_{k+1}],\ k = 1, \dots, m\,, \\ \Delta y|_{t=t_k} = I_k(y_{t_k^-}), & k = 1, \dots, m\,, \\ y(t) = z(t) = \varphi(t), & t \in [-r, 0]\,. \end{cases}$$

From the proof of Theorem 3.28 we get the inequality

$$v_1(t) \le \sum_{0<t_i<t} \widetilde{l} v_1(t_i^-) + m\epsilon + \frac{T^\alpha \epsilon(m+1)}{\Gamma(\alpha+1)} + \frac{(K+\overline{K})(m+1)}{(1-L)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v_1(s)\,ds\,.$$

Applying Lemma 1.53, we obtain

$$v_1(t) \le \epsilon\left(\frac{m\Gamma(\alpha+1) + T^\alpha(m+1)}{\Gamma(\alpha+1)}\right) \times \left[\prod_{0<t_i<t}(1+\widetilde{l})\exp\left(\int_0^t \frac{(K+\overline{K})(m+1)}{(1-L)\Gamma(\alpha)}(t-s)^{\alpha-1}ds\right)\right]$$
$$\le c_\omega \epsilon\,,$$

where

$$c_\omega = \left(\frac{m\Gamma(\alpha+1) + T^\alpha(m+1)}{\Gamma(\alpha+1)}\right)\left[\prod_{i=1}^{m}(1+\widetilde{l})\exp\left(\frac{(K+\overline{K})(m+1)T^\alpha}{(1-L)\Gamma(\alpha+1)}\right)\right]$$
$$= \left(\frac{m\Gamma(\alpha+1) + T^\alpha(m+1)}{\Gamma(\alpha+1)}\right)\left[(1+\widetilde{l})\exp\left(\frac{(K+\overline{K})(m+1)T^\alpha}{(1-L)\Gamma(\alpha+1)}\right)\right]^m .$$

Moreover, if we set $\gamma(\epsilon) = c_\omega\epsilon; \gamma(0) = 0$, then problem (3.40)–(3.41) is generalized Ulam–Hyers stable. □

3.4.4 Examples

Example 1. Consider the impulsive problem

$$^{c}D_{t_k}^{\frac{1}{2}}y(t) = \frac{e^{-t}}{(11+e^t)}\left[\frac{|y_t|}{1+|y_t|} - \frac{|^{c}D_{t_k}^{\frac{1}{2}}y(t)|}{1+|^{c}D_{t_k}^{\frac{1}{2}}y(t)|}\right] + \frac{|y_t|}{6(1+|y_t|)}\,, \quad \text{for each, } t \in J_0 \cup J_1\,, \tag{3.56}$$

$$\Delta y|_{t=\frac{1}{2}} = \frac{|y\left(\frac{1}{2}^-\right)|}{10+|y\left(\frac{1}{2}^-\right)|}\,, \tag{3.57}$$

$$y(t) = \varphi(t)\,, \quad t \in [-r, 0],\; r > 0\,, \tag{3.58}$$

where $\varphi \in PC([-r, 0], \mathbb{R})$, $J_0 = [0, \frac{1}{2}]$, $J_1 = (\frac{1}{2}, 1]$, $t_0 = 0$, and $t_1 = \frac{1}{2}$. Set

$$f(t, u, v) = \frac{e^{-t}}{(11+e^t)}\left[\frac{|u|}{1+|u|} - \frac{|v|}{1+|v|}\right]$$

and

$$\phi(t, u) = \frac{|u|}{6(1+|u|)}$$

for any $t \in [0, 1]$, $u \in PC([-r, 0], \mathbb{R})$, and $v \in \mathbb{R}$. Clearly, the functions f, ϕ are jointly continuous. For each $u, \bar{u} \in PC([-r, 0], \mathbb{R})$, $v, \bar{v} \in \mathbb{R}$, and $t \in [0, 1]$,

$$\begin{aligned}\|f(t, u, v) - f(t, \bar{u}, \bar{v})\| &\le \frac{e^{-t}}{(11+e^t)}(\|u-\bar{u}\|_{PC} + \|v-\bar{v}\|)\\ &\le \frac{1}{12}\|u-\bar{u}\|_{PC} + \frac{1}{12}\|v-\bar{v}\|\end{aligned}$$

and

$$|\phi(t, u) - \phi(t, \bar{u})| \le \frac{1}{6}\|u-\bar{u}\|_{PC}\,.$$

Hence, condition (3.20.2) is satisfied by $K = L = \frac{1}{12}$, $\overline{K} = \frac{1}{6}$.

Let

$$I_1(u) = \frac{|u|}{10+|u|}\,, \quad u \in PC\left([-r, 0], \mathbb{R}\right)\,,$$

and $u, v \in PC([-r, 0], \mathbb{R})$. Then we have

$$|I_1(u) - I_1(v)| = \left|\frac{|u|}{10+|u|} - \frac{|v|}{10+|v|}\right| \le \frac{1}{10}\|u-v\|_{PC}\,.$$

Thus,

$$\begin{aligned}m\widetilde{l} + \frac{(m+1)(K+\overline{K})T^{\alpha}}{(1-L)\Gamma(\alpha+1)} &= \frac{1}{10} + \frac{\frac{3}{6}}{\left(1-\frac{1}{12}\right)\Gamma\left(\frac{3}{2}\right)}\\ &= \frac{12}{11\sqrt{\pi}} + \frac{1}{10} < 1\,.\end{aligned}$$

It follows from Theorem 3.20 that problem (3.56)–(3.58) has a unique solution on J.

Set, for any $t \in [0,1]$, $\omega(t) = t$, $\psi = 1$. Since

$$I^{\frac{1}{2}}\omega(t) = \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^t (t-s)^{\frac{1}{2}-1} s\,ds \le \frac{2t}{\sqrt{\pi}},$$

condition (3.28.1) is satisfied by $\lambda_\omega = \frac{2}{\sqrt{\pi}}$. It follows that problem (3.56)–(3.57) is Ulam–Hyers–Rassias stable with respect to (ω, ψ).

Example 2. Consider the impulsive problem

$${}^cD_{t_k}^{\frac{1}{2}}y(t) = \frac{2 + |y_t| + |{}^cD_{t_k}^{\frac{1}{2}}y(t)|}{108e^{t+3}\left(1 + |y_t| + |{}^cD_{t_k}^{\frac{1}{2}}y(t)|\right)} + \frac{e^{-t}|y_t|}{(3+e^t)(1+|y_t|)} \quad \text{for each} \quad t \in J_0 \cup J_1, \tag{3.59}$$

$$\Delta y|_{t=\frac{1}{3}} = \frac{|y\left(\frac{1}{3}^-\right)|}{6 + |y\left(\frac{1}{3}^-\right)|}, \tag{3.60}$$

$$y(t) = \varphi(t), \quad t \in [-r, 0],\ r > 0, \tag{3.61}$$

where $\varphi \in PC([-r,0],\mathbb{R})$ $J_0 = [0, \frac{1}{3}]$, $J_1 = (\frac{1}{3}, 1]$, $t_0 = 0$, and $t_1 = \frac{1}{3}$. Set

$$f(t,u,v) = \frac{2 + |u| + |v|}{108e^{t+3}(1 + |u| + |v|)}$$

and

$$\phi(t,u) = \frac{e^{-t}|u|}{(3+e^t)(1+|u|)}$$

for any $t \in [0,1]$, $u \in PC([-r,0],\mathbb{R})$, $v \in \mathbb{R}$. Clearly, the functions f, ϕ are jointly continuous.

For any $u, \bar{u} \in PC([-r,0],\mathbb{R})$, $v, \bar{v} \in \mathbb{R}$, and $t \in [0,1]$,

$$|f(t,u,v) - f(t,\bar{u},\bar{v})| \le \frac{1}{108e^3}(\|u - \bar{u}\|_{PC} + |v - \bar{v}|)$$

and

$$|\phi(t,u) - \phi(t,\bar{u})| \le \frac{1}{4}\|u - \bar{u}\|_{PC}.$$

Hence, condition (3.20.2) is satisfied by $K = L = \frac{1}{108e^3}$, $\overline{K} = \frac{1}{4}$.

For each $t \in [0,1]$ we have

$$|f(t,u,v)| \le \frac{1}{108e^{t+3}}(2 + \|u\|_{PC} + |v|).$$

Thus, condition (3.21.1) is satisfied by $p(t) = \frac{1}{54e^{t+3}}$ and $q(t) = r(t) = \frac{1}{108e^{t+3}}$. Let

$$I_1(u) = \frac{|u|}{6 + |u|}, \quad u \in PC([-r,0],\mathbb{R}).$$

For each $u \in PC([-r, 0], \mathbb{R})$ we have

$$|I_1(u)| \leq \frac{1}{6}\|u\|_{PC} + 1 .$$

Since $mM^* < 1$, condition (3.21.2) is satisfied. It follows from Theorem 3.21 that problem (3.59)–(3.61) has at least one solution on J.

3.5 Existence and Stability Results for Neutral Impulsive NIFDE with Finite Delay

3.5.1 Introduction

The purpose of this section is to establish some existence, uniqueness, and stability results for the following implicit neutral differential equations of fractional order with finite delay and impulses:

$$^cD^{\alpha}_{t_k}[y(t) - \phi(t, y_t)] = f(t, y_t, {}^c D^{\alpha}_{t_k} y(t)) , \qquad \text{for each } t \in (t_k, t_{k+1}] , \quad k = 0, \dots, m, 0 < \alpha \leq 1 , \tag{3.62}$$

$$\Delta y|_{t_k} = I_k(y_{t_k^-}) \qquad k = 1, \dots, m , \tag{3.63}$$

$$y(t) = \varphi(t) , \qquad t \in [-r, 0],\ r > 0 , \tag{3.64}$$

where $f\colon J \times PC([-r, 0], \mathbb{R}) \times \mathbb{R} \to \mathbb{R}$ and $\phi\colon J \times PC([-r, 0], \mathbb{R}) \to \mathbb{R}$ are given functions with $\phi(0, \varphi) = 0$, $I_k\colon PC([-r, 0], \mathbb{R}) \to \mathbb{R}$ and $\varphi \in PC([-r, 0], \mathbb{R})$, $0 = t_0 < t_1 < \dots < t_m < t_{m+1} = T$, and $PC([-r, 0], \mathbb{R})$ is a space to be specified later. Here, $\Delta y|_{t_k} = y(t_k^+) - y(t_k^-)$, where $y(t_k^+) = \lim_{h\to 0^+} y(t_k + h)$ and $y(t_k^-) = \lim_{h\to 0^-} y(t_k + h)$ represent the right and left limits of y_t at $t = t_k$, respectively.

The arguments are based upon the Banach contraction principle and Schaefer's fixed point theorem. An example is included to show the applicability of our results.

3.5.2 Existence of Solutions

Consider the Banach space

$$PC([-r, 0], \mathbb{R}) = \{y\colon [-r, 0] \to \mathbb{R}\colon y \in C((\tau_k, \tau_{k+1}], \mathbb{R}),\ k=1, \dots, l, \text{ and there exist } y(\tau_k^-) \text{ and } y(\tau_k^+),\ \ k=1, \dots, l \text{ with } y(\tau_k^-)=y(\tau_k)\} ,$$

with the norm

$$\|y\|_{PC} = \sup_{t\in[-r,0]} |y(t)| .$$

Take

$$PC([0, T], \mathbb{R}) = \{y\colon [0, T] \to \mathbb{R}\colon y \in C((t_k, t_{k+1}], \mathbb{R}),\ k = 1, \dots, m, \text{ and there exist } y(t_k^-) \text{ and } y(t_k^+),\ \ k = 1, \dots, m \text{ with } y(t_k^-) = y(t_k)\}$$

to be our Banach space with the norm

$$\|y\|_C = \sup_{t\in[0,T]} |y(t)| .$$

Also,

$$\Omega = \{y\colon [-r,T] \to \mathbb{R}\colon y|_{[-r,0]} \in PC([-r,0],\mathbb{R}) \text{ and } y|_{[0,T]} \in PC([0,T],\mathbb{R})\}$$

is a Banach space with the norm

$$\|y\|_\Omega = \sup_{t\in[-r,T]} |y(t)| .$$

Definition 3.30. A function $y \in \Omega$ whose α-derivative exists on J_k is said to be a solution of (3.62)–(3.64) if y satisfies the equation ${}^cD^\alpha_{t_k}(y(t) - \phi(t, y_t)) = f(t, y_t, {}^cD^\alpha_{t_k} y(t))$ on J_k and satisfies the conditions

$$\Delta y|_{t=t_k} = I_k(y_{t_k^-}) , \quad k = 1,\dots,m ,$$
$$y(t) = \varphi(t) , \quad t \in [-r,0] .$$

To prove the existence of solutions to (3.62)–(3.64), we need the following auxiliary lemma.

Lemma 3.31. *Let $0 < \alpha \le 1$, and let $\sigma\colon J \to \mathbb{R}$ be continuous. A function y is a solution of the fractional integral equation*

$$y(t) = \begin{cases} \varphi(0) + \phi(t,y_t) + \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1}\sigma(s)ds, & \text{if } t \in [0,t_1] , \\ \varphi(0) + \phi(t,y_t) + \displaystyle\sum_{i=1}^k I_i(y_{t_i^-}) + \dfrac{1}{\Gamma(\alpha)}\sum_{i=1}^k \int_{t_{i-1}}^{t_i} (t_i-s)^{\alpha-1}\sigma(s)ds & \\ + \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_{t_k}^t (t-s)^{\alpha-1}\sigma(s)ds, & \text{if } t \in (t_k,t_{k+1}] , \\ \varphi(t), & t \in [-r,0] , \end{cases} \tag{3.65}$$

where $k = 1,\dots,m$, if and only if y is a solution of the following fractional problem:

$${}^cD^\alpha(y(t) - \phi(t,y_t)) = \sigma(t) , \quad t \in J_k , \tag{3.66}$$
$$\Delta y|_{t=t_k} = I_k(y_{t_k^-}) , \quad k = 1,\dots,m , \tag{3.67}$$
$$y(t) = \varphi(t) , \quad t \in [-r,0] . \tag{3.68}$$

Proof. Assume that y satisfies (3.66)–(3.68). If $t \in [0,t_1]$, then

$${}^cD^\alpha(y(t) - \phi(t,y_t)) = \sigma(t) .$$

Lemma 1.9 implies

$$y(t) - \phi(t, y_t) = \varphi(0) + I^\alpha \sigma(t) = \varphi(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sigma(s) ds \, .$$

If $t \in (t_1, t_2]$, then Lemma 1.9 implies

$$\begin{aligned} y(t) - \phi(t, y_t) &= y(t_1^+) - \phi(t_1, y_{t_1}) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} \sigma(s) ds \\ &= \Delta y|_{t=t_1} + y(t_1^-) - \phi(t_1, y_{t_1}) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} \sigma(s) ds \\ &= I_1(y_{t_1^-}) + \left[\varphi(0) + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - s)^{\alpha-1} \sigma(s) ds \right] \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} \sigma(s) ds \, . \\ &= \varphi(0) + I_1(y_{t_1^-}) + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - s)^{\alpha-1} \sigma(s) ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} \sigma(s) ds \, . \end{aligned}$$

If $t \in (t_2, t_3]$, then from Lemma 1.9 we get

$$\begin{aligned} y(t) - \phi(t, y_t) &= y(t_2^+) - \phi(t_2, y_{t_2}) + \frac{1}{\Gamma(\alpha)} \int_{t_2}^t (t-s)^{\alpha-1} \sigma(s) ds \\ &= \Delta y|_{t=t_2} + y(t_2^-) - \phi(t_2, y_{t_2}) + \frac{1}{\Gamma(\alpha)} \int_{t_2}^t (t-s)^{\alpha-1} \sigma(s) ds \\ &= I_2(y_{t_2^-}) + \left[\varphi(0) + I_1(y_{t_1^-}) + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - s)^{\alpha-1} \sigma(s) ds \right. \\ &\quad \left. + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \sigma(s) ds \right] + \frac{1}{\Gamma(\alpha)} \int_{t_2}^t (t-s)^{\alpha-1} \sigma(s) ds \end{aligned}$$

$$= \varphi(0) + \left[I_1(y_{t_1^-}) + I_2(y_{t_2^-})\right] + \left[\frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1 - s)^{\alpha-1}\sigma(s)ds + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2 - s)^{\alpha-1}\sigma(s)ds\right] + \frac{1}{\Gamma(\alpha)}\int_{t_2}^{t}(t - s)^{\alpha-1}\sigma(s)ds\,.$$

Repeating the process in this way, the solution $y(t)$ for $t \in (t_k, t_{k+1}]$, $k = 1, \dots, m$ can be written

$$y(t) = \varphi(0) + \phi(t, y_t) + \sum_{i=1}^{k} I_i(y_{t_i^-}) + \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i - s)^{\alpha-1}\sigma(s)ds + \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t - s)^{\alpha-1}\sigma(s)ds\,.$$

Conversely, assume that y satisfies impulsive fractional integral equation (3.65). If $t \in [0, t_1]$, then $y(0) = \varphi(0)$. Using the fact that ${}^cD^\alpha$ is the left inverse of I^α, we get

$${}^cD^\alpha(y(t) - \phi(t, y_t)) = \sigma(t)\,, \quad \text{for each } t \in [0, t_1]\,.$$

If $t \in (t_k, t_{k+1}]$, $k = 1, \dots, m$, and using the fact that ${}^cD^\alpha C = 0$, where C is a constant, we get

$${}^cD^\alpha(y(t) - \phi(t, y_t)) = \sigma(t) \quad \text{for each } t \in (t_k, t_{k+1}]\,.$$

Also, we can easily show that

$$\Delta y|_{t=t_k} = I_k(y_{t_k^-})\,, \quad k = 1, \dots, m\,. \qquad \square$$

We are now in a position to state and prove our existence result for problem (3.62)–(3.64) based on Banach's fixed point.

Theorem 3.32. *Make the following assumptions:*
(3.32.1) *The function $f : J \times PC([-r, 0], \mathbb{R}) \times \mathbb{R} \to \mathbb{R}$ is continuous.*
(3.32.2) *There exist constants $K > 0$, $\overline{L} > 0$ and $0 < L < 1$ such that*

$$|f(t, u, v) - f(t, \bar{u}, \bar{v})| \le K\|u - \bar{u}\|_{PC} + L|v - \bar{v}|$$

and

$$|\phi(t, u) - \phi(t, \bar{u})| \le \overline{L}\|u - \bar{u}\|_{PC}$$

for any $u, \bar{u} \in PC([-r, 0], \mathbb{R})$, $v, \bar{v} \in \mathbb{R}$, and $t \in J$.
(3.32.3) *There exists a constant $\tilde{l} > 0$ such that*

$$|I_k(u) - I_k(\overline{u})| \le \tilde{l}\|u - \overline{u}\|_{PC}$$

for each $u, \overline{u} \in PC([-r, 0], \mathbb{R})$ and $k = 1, \dots, m$.

If

$$m\widetilde{l} + \overline{L} + \frac{(m+1)KT^{\alpha}}{(1-L)\Gamma(\alpha+1)} < 1\,, \tag{3.69}$$

then there exists a unique solution to problem (3.62)–(3.64) *on* J.

Proof. Consider the operator $N\colon \Omega \to \Omega$ defined by

$$(Ny)(t) = \begin{cases} \varphi(0) + \phi(t, y_t) + \sum_{0<t_k<t} I_k(y_{t_k^-}) + \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} g(s)ds \\ + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} g(s)ds, & t \in [0, T]\,, \\ \varphi(t), & t \in [-r, 0]\,, \end{cases} \tag{3.70}$$

where $g \in C(J, \mathbb{R})$ is such that

$$g(t) = f(t, y_t, g(t))\,.$$

Clearly, the fixed points of operator N are solutions of problem (3.62)–(3.64). Let $u, w \in \Omega$. If $t \in [-r, 0]$, then

$$|(Nu)(t) - (Nw)(t)| = 0\,.$$

For $t \in J$ we have

$$\begin{aligned} |(Nu)(t) - (Nw)(t)| \le{} & \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} |g(s) - h(s)|ds \\ & + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} |g(s) - h(s)|ds + |\phi(t, u_t) - \phi(t, w_t)| \\ & + \sum_{0<t_k<t} |I_k(u_{t_k^-}) - I_k(w_{t_k^-})|\,, \end{aligned}$$

where $g, h \in C(J, \mathbb{R})$ are given by

$$g(t) = f(t, u_t, g(t))$$

and

$$h(t) = f(t, w_t, h(t))\,.$$

By (3.32.2) we have

$$\begin{aligned} |g(t) - h(t)| &= |f(t, u_t, g(t)) - f(t, w_t, h(t))| \\ &\le K\|u_t - w_t\|_{PC} + L|g(t) - h(t)|\,. \end{aligned}$$

Then

$$|g(t) - h(t)| \le \frac{K}{1-L} \|u_t - w_t\|_{PC}\,.$$

Therefore, for each $t \in J$

$$
\begin{aligned}
|(Nu)(t) - (Nw)(t)| \le{}& \frac{K}{(1-L)\Gamma(\alpha)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} \|u_s - w_s\|_{PC} ds \\
&+ \frac{K}{(1-L)\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} \|u_s - w_s\|_{PC} ds \\
&+ \sum_{k=1}^{m} \widetilde{l} \|u_{t_k^-} - w_{t_k^-}\|_{PC} + \overline{L} \|u_t - w_t\|_{PC} \\
\le{}& \left[m\widetilde{l} + \overline{L} + \frac{mKT^{\alpha}}{(1-L)\Gamma(\alpha+1)} \right. \\
&\left. + \frac{KT^{\alpha}}{(1-L)\Gamma(\alpha+1)} \right] \|u - w\|_{\Omega} .
\end{aligned}
$$

Thus,

$$
\|N(u) - N(w)\|_{\Omega} \le \left[m\widetilde{l} + \overline{L} + \frac{(m+1)KT^{\alpha}}{(1-L)\Gamma(\alpha+1)} \right] \|u - w\|_{\Omega} .
$$

By (3.69), operator N is a contraction. Hence, by Banach's contraction principle, N has a unique fixed point that is the unique solution of problem (3.62)–(3.64). □

Our second result is based on Schaefer's fixed point theorem.

Theorem 3.33. *Assume* (3.32.1) *and* (3.32.2) *hold and*

(3.33.1) *There exist* $p, q, r \in C(J, \mathbb{R}_+)$ *with* $r^* = \sup_{t \in J} r(t) < 1$ *such that*

$$
|f(t, u, w)| \le p(t) + q(t)\|u\|_{PC} + r(t)|w| \quad \text{for } t \in J,\ u \in PC([-r, 0], \mathbb{R}) \text{ and } w \in \mathbb{R} ;
$$

(3.33.2) *The functions* $I_k : PC([-r, 0], \mathbb{R}) \to \mathbb{R}$ *are continuous and there exist constants* $M^*, N^* > 0$ *such that*

$$
|I_k(u)| \le M^* \|u\|_{PC} + N^* \quad \text{for each } u \in PC([-r, 0], \mathbb{R}),\ k = 1, \dots, m ;
$$

(3.33.3) *The function* ϕ *is completely continuous, and for each bounded set* B_{η^*} *in* Ω *the set* $\{t \to \phi(t, y_t) : y \in B_{\eta^*}\}$ *is equicontinuous in* $PC(J, \mathbb{R})$ *and there exist two constants* $d_1 > 0$, $d_2 > 0$ *with* $mM^* + d_1 < 1$ *such that*

$$
|\phi(t, u)| \le d_1 \|u\|_{PC} + d_2 , \quad t \in J,\ u \in PC([-r, 0], \mathbb{R}) .
$$

Then problem (3.62)–(3.64) *has at least one solution.*

Proof. We consider the operator $N_1 : \Omega \to \Omega$ defined by

$$
N_1 y(t) = \begin{cases}
\varphi(0) + \sum\limits_{0<t_k<t} I_k(y_{t_k^-}) + \dfrac{1}{\Gamma(\alpha)} \sum\limits_{0<t_k<t} \displaystyle\int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} g(s) ds \\
+ \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{t_k}^{t} (t-s)^{\alpha-1} g(s) ds, & t \in [0, T] , \\
\varphi(t), & t \in [-r, 0] .
\end{cases}
$$

Operator N defined in (3.70) can be written

$$(Ny)(t) = \phi(t, y_t) + N_1 y(t) , \quad \text{for each } t \in J .$$

We will use Schaefer's fixed point theorem to prove that N has a fixed point. So we must show that N is completely continuous. Since ϕ is completely continuous by (3.33.3), we will show that N_1 is completely continuous. The proof will be given in several steps.

Step 1: N_1 is continuous. Let $\{u_n\}$ be a sequence such that $u_n \to u$ in Ω. If $t \in [-r, 0]$, then

$$|N_1(u_n)(t) - N_1(u)(t)| = 0 .$$

For $t \in J$ we have

$$\begin{aligned}|N_1(u_n)(t) - N_1(u)(t)| &\le \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} |g_n(s) - g(s)| ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t - s)^{\alpha-1} |g_n(s) - g(s)| ds \\ &\quad + \sum_{0<t_k<t} |I_k(u_{nt_k^-}) - I_k(u_{t_k^-})| \\ &\le \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} |g_n(s) - g(s)| ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t - s)^{\alpha-1} |g_n(s) - g(s)| ds \\ &\quad + \sum_{0<t_k<t} \widetilde{l} \|u_{nt_k^-} - u_{t_k^-}\|_{PC} ,\end{aligned}$$

and so

$$\begin{aligned}|N_1(u_n)(t) - N_1(u)(t)| &\le \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} |g_n(s) - g(s)| ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t - s)^{\alpha-1} |g_n(s) - g(s)| ds \\ &\quad + m\widetilde{l} \|u_n - u\|_{\Omega} ,\end{aligned} \tag{3.71}$$

where $g_n, g \in C(J, \mathbb{R})$ are given by

$$g_n(t) = f(t, u_{nt}, g_n(t)) ,$$

and

$$g(t) = f(t, u_t, g(t)) .$$

By (3.32.2) we have

$$\begin{aligned}|g_n(t) - g(t)| &= |f(t, u_{nt}, g_n(t)) - f(t, u_t, g(t))| \\ &\le K\|u_{nt} - u_t\|_{PC} + L|g_n(t) - g(t)| \,.\end{aligned}$$

Then

$$|g_n(t) - g(t)| \le \frac{K}{1-L}\|u_{nt} - u_t\|_{PC} \,.$$

Since $u_n \to u$, we get $g_n(t) \to g(t)$ as $n \to \infty$ for each $t \in J$. Let $\eta > 0$ be such that for each $t \in J$ we have $|g_n(t)| \le \eta$ and $|g(t)| \le \eta$. Then we have

$$\begin{aligned}(t-s)^{\alpha-1}|g_n(s) - g(s)| &\le (t-s)^{\alpha-1}[|g_n(s)| + |g(s)|] \\ &\le 2\eta(t-s)^{\alpha-1}\end{aligned}$$

and

$$\begin{aligned}(t_k-s)^{\alpha-1}|g_n(s) - g(s)| &\le (t_k-s)^{\alpha-1}[|g_n(s)| + |g(s)|] \\ &\le 2\eta(t_k-s)^{\alpha-1} \,.\end{aligned}$$

For each $t \in J$ the functions $s \to 2\eta(t-s)^{\alpha-1}$ and $s \to 2\eta(t_k-s)^{\alpha-1}$ are integrable on $[0, t]$, then the Lebesgue dominated convergence theorem and (3.71) imply that

$$|N_1(u_n)(t) - N_1(u)(t)| \to 0 \quad \text{as } n \to \infty \,.$$

Hence,

$$\|N_1(u_n) - N_1(u)\|_\Omega \to 0 \quad \text{as } n \to \infty \,.$$

Consequently, N_1 is continuous.

Step 2: N_1 maps bounded sets to bounded sets in Ω. Indeed, it is enough to show that for any $\eta^* > 0$ there exists a positive constant ℓ such that for each $u \in B_{\eta^*} = \{u \in \Omega : \|u\|_\Omega \le \eta^*\}$ we have $\|N_1(u)\|_\Omega \le \ell$. For each $t \in J$ we have

$$\begin{aligned}N_1(u)(t) = \varphi(0) &+ \frac{1}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} g(s)ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} g(s)ds \,, \\ &+ \sum_{0<t_k<t} I_k(u_{t_k^-}) \,,\end{aligned} \tag{3.72}$$

where $g \in C(J, \mathbb{R})$ is such that

$$g(t) = f(t, u_t, g(t)) \,.$$

By (3.33.1), for each $t \in J$ we have

$$\begin{aligned}|g(t)| &= |f(t, u_t, g(t))| \\ &\le p(t) + q(t)\|u_t\|_{PC} + r(t)|g(t)| \\ &\le p(t) + q(t)\|u\|_{\Omega} + r(t)|g(t)| \\ &\le p(t) + q(t)\eta^* + r(t)|g(t)| \\ &\le p^* + q^*\eta^* + r^*|g(t)| ,\end{aligned}$$

where $p^* = \sup_{t\in J} p(t)$, and $q^* = \sup_{t\in J} q(t)$.

Then

$$|g(t)| \le \frac{p^* + q^*\eta^*}{1 - r^*} := M .$$

Thus, (3.72) implies

$$\begin{aligned}|N_1(u)(t)| &\le |\varphi(0)| + \frac{mMT^\alpha}{\Gamma(\alpha+1)} + \frac{MT^\alpha}{\Gamma(\alpha+1)} + \sum_{k=1}^{m}\left(M^*\|u_{t_k^-}\|_{PC} + N^*\right) \\ &\le |\varphi(0)| + \frac{(m+1)MT^\alpha}{\Gamma(\alpha+1)} + m\left(M^*\|u_{t_k^-}\|_{\Omega} + N^*\right) \\ &\le |\varphi(0)| + \frac{(m+1)MT^\alpha}{\Gamma(\alpha+1)} + m\left(M^*\eta^* + N^*\right) := R .\end{aligned}$$

If $t \in [-r, 0]$, then

$$|N_1(u)(t)| \le \|\varphi\|_{PC} ,$$

so

$$\|N_1(u)\|_{\Omega} \le \max\left\{R, \|\varphi\|_{PC}\right\} := \ell .$$

Step 3: N_1 maps bounded sets to equicontinuous sets of Ω. Let $\tau_1, \tau_2 \in (0, T]$, $\tau_1 < \tau_2$, B_{η^*} be a bounded set of Ω as in Step 2, and let $u \in B_{\eta^*}$. Then

$$\begin{aligned}&|N_1(u)(\tau_2) - N_1(u)(\tau_1)| \\ &\le \frac{1}{\Gamma(\alpha)}\int_0^{\tau_1} |(\tau_2 - s)^{\alpha-1} - (\tau_1 - s)^{\alpha-1}||g(s)|ds \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2} |(\tau_2 - s)^{\alpha-1}||g(s)|ds + \sum_{0<t_k<\tau_2-\tau_1} |I_k(u_{t_k^-})| \\ &\le \frac{M}{\Gamma(\alpha+1)}[2(\tau_2-\tau_1)^\alpha + (\tau_2^\alpha - \tau_1^\alpha)] + (\tau_2 - \tau_1)\left(M^*\|u_{t_k^-}\|_{\Omega} + N^*\right) \\ &\le \frac{M}{\Gamma(\alpha+1)}[2(\tau_2-\tau_1)^\alpha + (\tau_2^\alpha - \tau_1^\alpha)] + (\tau_2 - \tau_1)\left(M^*\eta^* + N^*\right) .\end{aligned}$$

As $\tau_1 \to \tau_2$, the right-hand side of the preceding inequality tends to zero. As a consequence of Steps 1–3, together with the Ascoli–Arzelà theorem, we can conclude that $N_1 : \Omega \to \Omega$ is completely continuous.

Step 4: A priori bounds. Now it remains to show that the set

$$E = \{u \in \Omega : u = \lambda N(u) \text{ for some } 0 < \lambda < 1\}$$

is bounded. Let $u \in E$. Then $u = \lambda N(u)$ for some $0 < \lambda < 1$. Thus, for each $t \in J$ we have

$$u(t) = \lambda\varphi(0) + \lambda\phi(t, y_t) + \frac{\lambda}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} g(s)ds + \frac{\lambda}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} g(s)ds + \lambda \sum_{0<t_k<t} I_k(u_{t_k^-}) . \tag{3.73}$$

From (3.33.1), for each $t \in J$ we have

$$\begin{aligned} |g(t)| &= |f(t, u_t, g(t))| \\ &\le p(t) + q(t)\|u_t\|_{PC} + r(t)|g(t)| \\ &\le p^* + q^*\|u_t\|_{PC} + r^*|g(t)| . \end{aligned}$$

Thus,

$$|g(t)| \le \frac{1}{1-r^*}(p^* + q^*\|u_t\|_{PC}) .$$

This implies, by (3.73), (3.33.2), and (3.33.3), that for each $t \in J$ we have

$$\begin{aligned} |u(t)| \le{}& |\varphi(0)| + d_1\|u_t\|_{PC} + d_2 \\ &+ \frac{1}{(1-r^*)\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1}(p^* + q^*\|u_s\|_{PC})ds \\ &+ \frac{1}{(1-r^*)\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1}(p^* + q^*\|u_s\|_{PC})ds \\ &+ m\left(M^*\|u_{t_k^-}\|_{PC} + N^*\right) . \end{aligned}$$

Consider the function v defined by

$$v(t) = \sup\{|u(s)| : -r \le s \le t\} , \quad 0 \le t \le T .$$

Then there exists $t^* \in [-r, T]$ such that $v(t) = |u(t^*)|$. If $t \in [0, T]$, then, by the previous inequality, for $t \in J$ we have

$$\begin{aligned} v(t) \le{}& |\varphi(0)| + \frac{1}{(1-r^*)\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1}(p^* + q^* v(s))ds \\ &+ \frac{1}{(1-r^*)\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1}(p^* + q^* v(s))ds \\ &+ (mM^* + d_1)v(t) + (mN^* + d_2) . \end{aligned}$$

Thus,

$$\begin{aligned}
v(t) \leq\; & \frac{1}{(1-(mM^*+d_1))(1-r^*)\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1}(p^* + q^* v(s))ds \\
& + \frac{|\varphi(0)| + mN^* + d_2}{1-(mM^*+d_1)} \\
& + \frac{1}{(1-(mM^*+d_1))(1-r^*)\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1}(p^* + q^* v(s))ds \\
\leq\; & \frac{|\varphi(0)| + mN^* + d_2}{1-(mM^*+d_1)} + \frac{(m+1)p^* T^\alpha}{(1-(mM^*+d_1))(1-r^*)\Gamma(\alpha+1)} \\
& + \frac{(m+1)q^*}{(1-(mM^*+d_1))(1-r^*)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s)ds\,.
\end{aligned}$$

Applying Lemma 1.52, we get

$$\begin{aligned}
v(t) \leq\; & \left[\frac{|\varphi(0)| + mN^* + d_2}{1-(mM^*+d_1)} + \frac{(m+1)p^* T^\alpha}{(1-(mM^*+d_1))(1-r^*)\Gamma(\alpha+1)}\right] \\
& \times \left[1 + \frac{\delta(m+1)q^* T^\alpha}{(1-(mM^*+d_1))(1-r^*)\Gamma(\alpha+1)}\right] := A\,,
\end{aligned}$$

where $\delta = \delta(\alpha)$ is a constant. If $t^* \in [-r, 0]$, then $v(t) = \|\varphi\|_{PC}$; thus, for any $t \in J$, $\|u\|_\Omega \leq v(t)$ we have

$$\|u\|_\Omega \leq \max\{\|\varphi\|_{PC}, A\}\,.$$

This shows that set E is bounded. As a consequence of Schaefer's fixed point theorem, we deduce that N has a fixed point that is a solution of the problem (3.62)–(3.64). □

3.5.3 Ulam–Hyers Stability Results

Here we adopt the concepts in Wang et al. [252] and introduce Ulam's type stability concepts for problem (3.62)–(3.63).

Let $z \in PC(J, \mathbb{R})$, $\epsilon > 0$, $\psi > 0$, and let $\omega \in PC(J, \mathbb{R}_+)$ be nondecreasing. We consider the sets of inequalities

$$\begin{cases} |{}^cD^\alpha(z(t) - \phi(t, z_t)) - f(t, z_t, {}^cD^\alpha z(t))| \leq \epsilon, & t \in (t_k, t_{k+1}], k = 1, \dots, m\,, \\ |\Delta z|_{t=t_k} - I_k(z_{t_k^-})| \leq \epsilon, & k = 1, \dots, m\,, \end{cases} \tag{3.74}$$

$$\begin{cases} |{}^cD^\alpha(z(t) - \phi(t, z_t)) - f(t, z_t, {}^cD^\alpha z(t))| \leq \omega(t), & t \in (t_k, t_{k+1}], k = 1, \dots, m\,, \\ |\Delta z|_{t=t_k} - I_k(z_{t_k^-})| \leq \psi, & k = 1, \dots, m\,, \end{cases} \tag{3.75}$$

and

$$\begin{cases} |^cD^\alpha(z(t) - \phi(t, z_t)) - f(t, z_t, {}^cD^\alpha z(t))| \le \epsilon\omega(t), & t \in (t_k, t_{k+1}], k = 1, \dots, m\,, \\ |\Delta z|_{t=t_k} - I_k(z_{t_k^-})| \le \epsilon\psi, & k = 1, \dots, m\,. \end{cases} \tag{3.76}$$

Remark 3.34. A function $z \in PC(J, \mathbb{R})$ is a solution of inequality (3.76) if and only if there is $\sigma \in PC(J, \mathbb{R})$ and a sequence σ_k, $k = 1, \dots, m$ (which depend on z) such that
(i) $|\sigma(t)| \le \epsilon\omega(t),\ t \in (t_k, t_{k+1}],\ k = 1, \dots, m$ and $|\sigma_k| \le \epsilon\psi$, $k = 1, \dots, m$;
(ii) ${}^cD^\alpha(z(t) - \phi(t, z_t)) = f(t, z_t, {}^cD^\alpha z(t)) + \sigma(t),\ t \in (t_k, t_{k+1}], k = 1, \dots, m$;
(iii) $\Delta z|_{t_k} = I_k(z_{t_k^-}) + \sigma_k,\ k = 1, \dots, m.$

One can provide remarks for inequalities 3.75 and 3.74.

Theorem 3.35. *Assume* (3.32.1)–(3.32.3) *and* (3.69) *hold and*
(3.35.1) *there exists a nondecreasing function* $\omega \in PC(J, \mathbb{R}_+)$, *and there exists* $\lambda_\omega > 0$ *such that for any* $t \in J$

$$I^\alpha\omega(t) \le \lambda_\omega\omega(t)\,.$$

If $\overline{L} < 1$, *then problem* (3.62)–(3.63) *is Ulam–Hyers–Rassias stable with respect to* (ω, ψ).

Proof. Let $z \in \Omega$ be a solution of inequality (3.76). Denote by y the unique solution of the problem

$$\begin{cases} {}^cD^\alpha_{t_k}[y(t) - \phi(t, y_t)] = f(t, y_t, {}^cD^\alpha_{t_k} y(t)), & t \in (t_k, t_{k+1}],\ k = 1, \dots, m\,, \\ \Delta y|_{t=t_k} = I_k(y_{t_k^-}), & k = 1, \dots, m\,, \\ y(t) = z(t) = \varphi(t), & t \in [-r, 0]\,. \end{cases}$$

Using Lemma 3.31, for each $t \in (t_k, t_{k+1}]$ we obtain

$$y(t) = \varphi(0) + \phi(t, y_t) + \sum_{i=1}^{k} I_i(y_{t_i^-}) + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} g(s)ds$$
$$+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t - s)^{\alpha-1} g(s)ds\,, \quad t \in (t_k, t_{k+1}]\,,$$

where $g \in C(J, \mathbb{R})$ is given by

$$g(t) = f(t, y_t, g(t))\,.$$

Since z is a solution of inequality (3.76), by Remark 3.34 we have

$$\begin{cases} {}^cD^\alpha_{t_k}[z(t) - \phi(t, z_t)] = f(t, z_t, {}^cD^\alpha_{t_k} z(t)) + \sigma(t), & t \in (t_k, t_{k+1}], k = 1, \dots, m\,, \\ \Delta z|_{t=t_k} = I_k(z_{t_k^-}) + \sigma_k, & k = 1, \dots, m\,. \end{cases} \tag{3.77}$$

Clearly, the solution of (3.77) is given by

$$z(t) = \varphi(0) + \phi(t, z_t) + \sum_{i=1}^{k} I_i(z_{t_i^-}) + \sum_{i=1}^{k} \sigma_i + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} h(s)ds$$
$$+ \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} \sigma(s)ds + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t - s)^{\alpha-1} h(s)ds$$
$$+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t - s)^{\alpha-1} \sigma(s)ds , \quad t \in (t_k, t_{k+1}] ,$$

where $h \in C(J, \mathbb{R})$ is given by

$$h(t) = f(t, z_t, h(t)) .$$

Hence, for each $t \in (t_k, t_{k+1}]$ it follows that

$$|z(t) - y(t)| \le \sum_{i=1}^{k} |\sigma_i| + |\phi(t, z_t) - \phi(t, y_t)| + \sum_{i=1}^{k} |I_i(z_{t_i^-}) - I_i(y_{t_i^-})|$$
$$+ \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} |\sigma(s)|ds$$
$$+ \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} |h(s) - g(s)|ds$$
$$+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t - s)^{\alpha-1} |h(s) - g(s)|ds$$
$$+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t - s)^{\alpha-1} |\sigma(s)| .$$

Thus,

$$|z(t) - y(t)| \le m\epsilon\psi + (m+1)\epsilon\lambda_\omega \omega(t) + \overline{L}\|z_t - y_t\|_{PC} + \sum_{i=1}^{k} \widetilde{l}\|z_{t_i^-} - y_{t_i^-}\|_{PC}$$
$$+ \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1} |h(s) - g(s)|ds$$
$$+ \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t - s)^{\alpha-1} |h(s) - g(s)|ds .$$

By (3.32.2) we have

$$|h(t) - g(t)| = |f(t, z_t, h(t)) - f(t, y_t, g(t))| \\ \leq K\|z_t - y_t\|_{PC} + L|g(t) - h(t)| .$$

Then

$$|h(t) - g(t)| \leq \frac{K}{1-L}\|z_t - y_t\|_{PC} .$$

Therefore, for each $t \in J$

$$|z(t) - y(t)| \leq m\epsilon\psi + (m+1)\epsilon\lambda_\omega \omega(t) + \overline{L}\|z_t - y_t\|_{PC} + \sum_{i=1}^{k} \widetilde{l}\|z_{t_i^-} - y_{t_i^-}\|_{PC} \\ + \frac{K}{(1-L)\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i - s)^{\alpha-1}\|z_s - y_s\|_{PC} ds \\ + \frac{K}{(1-L)\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1}\|z_s - y_s\|_{PC} ds .$$

Thus,

$$|z(t) - y(t)| \leq \sum_{0<t_i<t} \widetilde{l}\|z_{t_i^-} - y_{t_i^-}\|_{PC} + \epsilon(\psi + \omega(t))(m + (m+1)\lambda_\omega) \\ + \overline{L}\|z_t - y_t\|_{PC} + \frac{K(m+1)}{(1-L)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\|z_s - y_s\|_{PC} ds .$$

We consider the function v_1 defined by

$$v_1(t) = \sup\{\|z(s) - y(s)\| : \, -r \leq s \leq t\} , \quad 0 \leq t \leq T .$$

Then there exists $t^* \in [-r, T]$ such that $v_1(t) = \|z(t^*) - y(t^*)\|$. If $t^* \in [-r, 0]$, then $v_1(t) = 0$. If $t^* \in [0, T]$, then, by the previous inequality, we have

$$v_1(t) \leq \sum_{0<t_i<t} \frac{\widetilde{l}}{1-\overline{L}} v_1(t_i^-) + \frac{\epsilon(\psi + \omega(t))(m + (m+1)\lambda_\omega)}{1-\overline{L}} \\ + \frac{K(m+1)}{(1-\overline{L})(1-L)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v_1(s) ds .$$

Applying Lemma 1.53, we get

$$v_1(t) \leq \frac{\epsilon(\psi + \omega(t))(m + (m+1)\lambda_\omega)}{1-\overline{L}} \\ \times \left[\prod_{0<t_i<t} \left(1 + \frac{\widetilde{l}}{1-\overline{L}}\right) \exp\left(\int_0^t \frac{K(m+1)}{(1-\overline{L})(1-L)\Gamma(\alpha)} (t-s)^{\alpha-1} ds \right) \right] \\ \leq c_\omega \epsilon(\psi + \omega(t)) ,$$

where

$$c_\omega = \frac{(m+(m+1)\lambda_\omega)}{1-\overline{L}}\left[\prod_{i=1}^{m}\left(1+\frac{\widetilde{l}}{1-\overline{L}}\right)\exp\left(\frac{K(m+1)T^\alpha}{(1-\overline{L})(1-L)\Gamma(\alpha+1)}\right)\right]$$
$$= \frac{(m+(m+1)\lambda_\omega)}{1-\overline{L}}\left[\left(1+\frac{\widetilde{l}}{1-\overline{L}}\right)\exp\left(\frac{K(m+1)T^\alpha}{(1-\overline{L})(1-L)\Gamma(\alpha+1)}\right)\right]^m .$$

Thus, problem (3.62)–(3.63) is Ulam–Hyers–Rassias stable with respect to (ω, ψ). □

Next we present the following Ulam–Hyers stability result.

Theorem 3.36. *Assume* (3.32.1)–(3.32.3) *and* (3.69) *hold. If* $\overline{L} < 1$, *then problem* (3.62)–(3.63) *is Ulam–Hyers stable.*

Proof. Let $z \in \Omega$ be a solution of inequality (3.74). Denote by y the unique solution of the problem

$$\begin{cases} {}^cD_{t_k}^\alpha[y(t)-\phi(t,y_t)] = f(t,y_t,{}^cD_{t_k}^\alpha y(t)), & t\in(t_k,t_{k+1}],\ k=1,\dots,m\,,\\ \Delta y|_{t=t_k} = I_k(y_{t_k^-}), & k=1,\dots,m\,,\\ y(t)=z(t)=\varphi(t), & t\in[-r,0]\,. \end{cases}$$

From the proof of Theorem 3.35 we get the inequality

$$v_1(t) \le \sum_{0<t_i<t}\frac{\widetilde{l}}{1-\overline{L}}v_1(t_i^-) + \frac{m\epsilon}{1-\overline{L}} + \frac{T^\alpha\epsilon(m+1)}{(1-\overline{L})\Gamma(\alpha+1)}$$
$$+ \frac{K(m+1)}{(1-\overline{L})(1-L)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}v_1(s)ds\,.$$

Applying Lemma 1.53, we get

$$v_1(t) \le \epsilon\left(\frac{m\Gamma(\alpha+1)+T^\alpha(m+1)}{(1-\overline{L})\Gamma(\alpha+1)}\right)$$
$$\times\left[\prod_{0<t_i<t}\left(1+\frac{\widetilde{l}}{1-\overline{L}}\right)\exp\left(\int_0^t\frac{K(m+1)}{(1-\overline{L})(1-L)\Gamma(\alpha)}(t-s)^{\alpha-1}ds\right)\right]$$
$$\le c\epsilon\,,$$

where

$$c = \left(\frac{m\Gamma(\alpha+1)+T^\alpha(m+1)}{(1-\overline{L})\Gamma(\alpha+1)}\right)\left[\prod_{i=1}^{m}\left(1+\frac{\widetilde{l}}{1-\overline{L}}\right)\exp\left(\frac{K(m+1)T^\alpha}{(1-\overline{L})(1-L)\Gamma(\alpha+1)}\right)\right]$$
$$= \left(\frac{m\Gamma(\alpha+1)+T^\alpha(m+1)}{(1-\overline{L})\Gamma(\alpha+1)}\right)\left[\left(1+\frac{\widetilde{l}}{1-\overline{L}}\right)\exp\left(\frac{K(m+1)T^\alpha}{(1-\overline{L})(1-L)\Gamma(\alpha+1)}\right)\right]^m .$$

Moreover, if we set $\gamma(\epsilon) = c\epsilon; \gamma(0) = 0$, then problem (3.62)–(3.63) is generalized Ulam–Hyers stable. □

3.5.4 An Example

Consider the impulsive problem, for each $t \in J_0 \cup J_1$,

$$ {}^cD^{\frac{1}{2}}_{t_k}\left[y(t) - \frac{te^{-t}|y_t|}{(9+e^t)(1+|y_t|)}\right] = \frac{e^{-t}}{(11+e^t)}\left[\frac{|y_t|}{1+|y_t|} - \frac{|{}^cD^{\frac{1}{2}}_{t_k}y(t)|}{1+|{}^cD^{\frac{1}{2}}_{t_k}y(t)|}\right], \tag{3.78} $$

$$ \Delta y|_{t=\frac{1}{2}} = \frac{|y\left(\frac{1}{2}^-\right)|}{10+|y\left(\frac{1}{2}^-\right)|}, \tag{3.79} $$

$$ y(t) = \varphi(t), \quad t \in [-r, 0], \ r > 0, \tag{3.80} $$

where $\varphi \in PC([-r, 0], \mathbb{R})$, $J_0 = [0, \frac{1}{2}]$, $J_1 = (\frac{1}{2}, 1]$, $t_0 = 0$, and $t_1 = \frac{1}{2}$.

For $t \in [0, 1]$, $u \in PC([-r, 0], \mathbb{R})$, and $v \in \mathbb{R}$, set

$$ f(t, u, v) = \frac{e^{-t}}{(11+e^t)}\left[\frac{|u|}{1+|u|} - \frac{|v|}{1+|v|}\right] $$

and

$$ \phi(t, u) = \frac{te^{-t}|u|}{(9+e^t)(1+|u|)}. $$

Notice that $\phi(0, \varphi) = 0$ for any $\varphi \in PC([-r, 0], \mathbb{R})$. Clearly, the function f is jointly continuous. For each $u, \bar{u} \in PC([-r, 0], \mathbb{R})$, $v, \bar{v} \in \mathbb{R}$ and $t \in [0, 1]$, and we have

$$ \begin{aligned} \|f(t, u, v) - f(t, \bar{u}, \bar{v})\| &\le \frac{e^{-t}}{(11+e^t)}(\|u-\bar{u}\|_{PC} + \|v-\bar{v}\|) \\ &\le \frac{1}{12}\|u-\bar{u}\|_{PC} + \frac{1}{12}\|v-\bar{v}\| \end{aligned} $$

and

$$ \|\phi(t, u) - \phi(t, \bar{u})\| \le \frac{1}{10}\|u-\bar{u}\|_{PC}. $$

Hence, condition (3.32.2) is satisfied by $K = L = \frac{1}{12}$, $\overline{L} = \frac{1}{10}$.

Let

$$ I_1(u) = \frac{|u|}{10+|u|}, \quad u \in PC([-r, 0], \mathbb{R}). $$

For each $u, v \in PC([-r, 0], \mathbb{R})$ we have

$$ |I_1(u) - I_1(v)| = \left|\frac{|u|}{10+|u|} - \frac{|v|}{10+|v|}\right| \le \frac{1}{10}\|u-v\|_{PC}. $$

Thus condition

$$ \begin{aligned} m\tilde{l} + \overline{L} + \frac{(m+1)KT^\alpha}{(1-L)\Gamma(\alpha+1)} &= \frac{2}{10} + \frac{\frac{1}{6}}{\left(1-\frac{1}{12}\right)\Gamma\left(\frac{3}{2}\right)} \\ &= \frac{4}{11\sqrt{\pi}} + \frac{2}{10} < 1 \end{aligned} $$

is satisfied. From Theorem 3.32, problem (3.78)–(3.80) has a unique solution on J.

Set, for any $t \in [0, 1]$, $\omega(t) = t$ and $\psi = 1$. Since

$$I^{\frac{1}{2}}\,\omega(t) = \frac{1}{\Gamma\left(\frac{1}{2}\right)} \int_0^t (t-s)^{\frac{1}{2}-1} s ds \le \frac{2t}{\sqrt{\pi}}\,,$$

(3.35.1) is satisfied by $\lambda_\omega = \frac{2}{\sqrt{\pi}}$. Since $\overline{L} < 1$, it follows that problem (3.78)–(3.79) is Ulam–Hyers–Rassias stable with respect to (ω, ψ).

3.6 Notes and Remarks

The results of Chapter 3 are taken from Benchohra et al. [90, 92]. Other results may be found in [17, 14, 41, 53, 57, 106, 124, 158].