2 Nonlinear Implicit Fractional Differential Equations

2.1 Introduction

Many techniques have been developed for studying the existence and uniqueness of solutions of initial value problems (IVPs) for fractional differential equations. Several authors have tried to develop techniques that depend on the Darbo or Mönch fixed point theorem with the Hausdorff or Kuratowski measure of noncompactness. The notion of the measure of noncompactness has been defined in many ways. In 1930, Kuratowski [185] defined the measure of noncompactness, $\alpha(A)$, of a bounded subset A of a metric space (X, d), and in 1955, Darbo [132] introduced a new type of fixed point theorem for noncompactness maps.

Recently, fractional differential equations have been studied by Abbas et al. [35, 43], Baleanu et al. [78, 80], Diethelm [137], Kilbas and Marzan [180], Srivastava et al. [181], Lakshmikantham et al. [187], and Samko et al. [239]. The purpose of this chapter is to establish existence and uniqueness results for some classes of implicit fractional differential equations by using fixed point theory (Banach contraction principle, Schauder's fixed point theorem, the nonlinear alternative of a Leray–Schauder type). Two other results are discussed; the first is based on Darbo's fixed point theorem combined with the technique of measures of noncompactness, the second is based on Mönch's fixed point theorem. Some examples are included to show the applicability of our results.

2.2 Existence and Stability Results for NIFDE

2.2.1 Introduction and Motivations

Recently, some mathematicians have considered boundary value problems (BVPs) for fractional differential equations depending on the fractional derivative. In [89], Benchohra et al. studied the problem involving Caputo's derivative

$$\begin{aligned} &{}^cD^{\alpha}u(t) = f(t, u(t), {}^cD^{\alpha-1}u(t)) , \quad \text{for each } t \in J := [0, \infty),\ 1 < \alpha \le 2 , \\ &u(0) = u_0,\ u \text{ is bounded on } J . \end{aligned}$$

In [203], Murad and Hadid, by means of Schauder fixed point theorem and the Banach contraction principle, considered the BVP for the fractional differential equation

$$\begin{aligned} &D^{\alpha}y(t) = f(t, y(t), D^{\beta}y(t)) , \quad t \in (0, 1),\ 1 < \alpha \le 2,\ 0 < \beta < 1,\ 0 < \gamma \le 1 , \\ &y(0) = 0,\ y(1) = I_0^{\gamma}y(s) , \end{aligned}$$

where D^{α} is the Riemann–Liouville fractional derivative and $f\colon [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

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In [150], Lakoud and Khaldi studied the following BVP for fractional integral boundary conditions:

$$
\begin{aligned}
&{}^cD^q y(t) = f(t, y(t), {}^cD^p y(t)) , \quad t \in (0,1),\ 1 < q \le 2,\ 0 < p < 1 ,\\
&y(0) = 0,\ y'(1) = \alpha I_0^p y(1) ,
\end{aligned}
$$

where ${}^cD^\alpha$ is the Caputo fractional derivative and $f\colon [0,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a continuous function.
The purpose of this section is to establish existence and uniqueness results for the implicit fractional order differential equation

$$ {}^cD^\alpha y(t) = f(t, y(t), {}^cD^\alpha y(t)) , \quad \text{for each } t \in J = [0, T],\ T > 0,\ 0 < \alpha \le 1 , \tag{2.1} $$

$$ y(0) = y_0 , \tag{2.2} $$

where ${}^cD^\alpha$ is the Caputo fractional derivative, $f\colon J\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a given function, and $y_0 \in \mathbb{R}$.

2.2.2 Existence of Solutions

Let us define what we mean by a solution of problem (2.1)–(2.2).

Definition 2.1. A function $u \in C^1(J, \mathbb{R})$ is said to be a solution of problem (2.1)–(2.2) if u satisfies equation (2.1) and conditions (2.2) on J.

For the existence of solutions for problem (2.1)–(2.2), we need the following auxiliary lemma.

Lemma 2.2. *Let a function $f(t,u,v)\colon J\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be continuous. Then the problem* (2.1)–(2.2) *is equivalent to the problem*

$$ y(t) = y_0 + I^\alpha g(t) , \tag{2.3} $$

where $g \in C(J, \mathbb{R})$ satisfies the functional equation

$$ g(t) = f(t, y_0 + I^\alpha g(t), g(t)) . $$

Proof. If ${}^cD^\alpha y(t) = g(t)$, then $I^\alpha\, {}^cD^\alpha y(t) = I^\alpha g(t)$. Thus, we obtain $y(t) = y_0 + I^\alpha g(t)$. □

We are now in a position to state and prove our existence result for problem (2.1)–(2.2) based on Banach's fixed point theorem.

Theorem 2.3. *Make the following assumptions:*
(2.3.1) *The function $f\colon J\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is continuous.*

(2.3.2) *There exist constants* $K > 0$ *and* $0 < L < 1$ *such that*

$$|f(t, u, v) - f(t, \bar{u}, \bar{v})| \leq K|u - \bar{u}| + L|v - \bar{v}|$$

for any $u, v, \bar{u}, \bar{v} \in \mathbb{R}$ *and* $t \in J$.

If

$$L + \frac{KT^\alpha}{\Gamma(\alpha + 1)} < 1\,, \tag{2.4}$$

then there exists a unique solution for IVP (2.1)–(2.2) *on* J.

Proof. The proof will be given in several steps. Transform problem (2.1)–(2.2) into a fixed point problem. Define the operator $N\colon C(J, \mathbb{R}) \to C(J, \mathbb{R})$ by

$$(Ny)(t) = y_0 + I^\alpha g(t)\,, \tag{2.5}$$

where $g \in C(J, \mathbb{R})$ satisfies the functional equation

$$g(t) = f(t, y(t), g(t))\,.$$

Clearly, the fixed points of operator N are solutions of problem (2.1)–(2.2). Let $u, w \in C(J, \mathbb{R})$. Then for $t \in J$ we have

$$(Nu)(t) - (Nw)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1}(g(s) - h(s))ds\,,$$

where $g, h \in C(J, \mathbb{R})$ are given by

$$g(t) = f(t, u(t), g(t))\,,$$
$$h(t) = f(t, w(t), h(t))\,.$$

Then for $t \in J$

$$|(Nu)(t) - (Nw)(t)| \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1}|g(s) - h(s)|ds\,. \tag{2.6}$$

By (2.3.2) we have

$$\begin{aligned}|g(t) - h(t)| &= |f(t, u(t), g(t)) - f(t, w(t), h(t))|\\ &\leq K|u(t) - w(t)| + L|g(t) - h(t)|\,.\end{aligned}$$

Thus,

$$|g(t) - h(t)| \leq \frac{K}{1 - L}|u(t) - w(t)|\,.$$

By (2.6) we have

$$\begin{aligned}|(Nu)(t) - (Nw)(t)| &\leq \frac{K}{(1 - L)\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1}|u(s) - w(s)|ds\\ &\leq \frac{KT^\alpha}{(1 - L)\Gamma(\alpha + 1)}\|u - w\|_\infty\,.\end{aligned}$$

Then

$$\|Nu - Nw\|_\infty \le \frac{KT^\alpha}{(1-L)\Gamma(\alpha+1)}\|u-w\|_\infty .$$

By (2.4), operator N is a contraction. Hence, by Banach's contraction principle, N has a unique fixed point that is the unique solution of problem (2.1)–(2.2). □

Our next existence result is based on Schauder's fixed point theorem.

Theorem 2.4. *Assume* (2.3.1) *and* (2.3.2) *hold and*
(2.4.1) *There exist* $p, q, r \in C(J, \mathbb{R}_+)$ *with* $r^* = \sup_{t\in J} r(t) < 1$ *such that*

$$|f(t,u,w)| \le p(t) + q(t)|u| + r(t)|w| \quad \text{for } t \in J, \text{ and } u, w \in \mathbb{R} .$$

If

$$\frac{q^* T^\alpha}{(1-r^*)\Gamma(\alpha+1)} < 1 , \tag{2.7}$$

where $p^* = \sup_{t\in J} p(t)$, *and* $q^* = \sup_{t\in J} q(t)$, *then the IVP* (2.1)–(2.2) *has at least one solution.*

Proof. Consider operator N defined in (2.5). We will show that N satisfies the assumptions of Schauder's fixed point theorem. The proof will be given in several steps.

Claim 1: N is continuous. Let $\{u_n\}$ be a sequence such that $u_n \to u$ in $C(J,\mathbb{R})$. Then for each $t \in J$

$$|N(u_n)(t) - N(u)(t)| \le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|g_n(s) - g(s)|ds , \tag{2.8}$$

where $g_n, g \in C(J,\mathbb{R})$ satisfy

$$g_n(t) = f(t, u_n(t), g_n(t))$$

and

$$g(t) = f(t, u(t), g(t)) .$$

By (2.3.2) we have

$$\begin{aligned}|g_n(t) - g(t)| &= |f(t,u_n(t),g_n(t)) - f(t,u(t),g(t))| \\ &\le K|u_n(t) - u(t)| + L|g_n(t) - g(t)| .\end{aligned}$$

Then

$$|g_n(t) - g(t)| \le \frac{K}{1-L}|u_n(t) - u(t)| .$$

Since $u_n \to u$, we have $g_n(t) \to g(t)$ as $n \to \infty$ for each $t \in J$. Let $\eta > 0$ be such that, for each $t \in J$, we have $|g_n(t)| \le \eta$ and $|g(t)| \le \eta$. Then

$$\begin{aligned}(t-s)^{\alpha-1}|g_n(s) - g(s)| &\le (t-s)^{\alpha-1}[|g_n(s)| + |g(s)|] \\ &\le 2\eta(t-s)^{\alpha-1} .\end{aligned}$$

For each $t \in J$ the function $s \to 2\eta(t-s)^{\alpha-1}$ is integrable on $[0, t]$. By the Lebesgue dominated convergence theorem and (2.8),

$$|N(u_n)(t) - N(u)(t)| \to 0 \quad \text{as } n \to \infty .$$

Hence,

$$\|N(u_n) - N(u)\|_\infty \to 0 \quad \text{as } n \to \infty .$$

Consequently, N is continuous.

Let

$$R \geq \frac{M|y_0| + p^* T^\alpha}{M - q^* T^\alpha} ,$$

where $M := (1 - r^*)\Gamma(\alpha + 1)$, and define

$$D_R = \{u \in C(J, \mathbb{R}) : \|u\|_\infty \leq R\} .$$

It is clear that D_R is a bounded, closed, and convex subset of $C(J, \mathbb{R})$.

Claim 2: $N(D_R) \subset D_R$. Let $u \in D_R$; we will show that $Nu \in D_R$. For each $t \in J$ we have

$$|Nu(t)| \leq |y_0| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |g(s)| ds . \tag{2.9}$$

By (2.4.1) we have for each $t \in J$

$$\begin{aligned} |g(t)| &= |f(t, u(t), g(t))| \\ &\leq p(t) + q(t)|u(t)| + r(t)|g(t)| \\ &\leq p(t) + q(t)R + r(t)|g(t)| \\ &\leq p^* + q^* R + r^* |g(t)| . \end{aligned}$$

Then

$$|g(t)| \leq \frac{p^* + q^* R}{1 - r^*} := \overline{M} .$$

Thus, (2.9) implies that

$$\begin{aligned} |Nu(t)| &\leq |y_0| + \frac{p^* T^\alpha}{(1 - r^*)\Gamma(\alpha + 1)} + \frac{q^* R T^\alpha}{M} \\ &\leq |y_0| + \frac{p^* T^\alpha}{M} + \frac{q^* R T^\alpha}{M} \\ &\leq R . \end{aligned}$$

Hence, $N(D_R) \subset D_R$.

Claim 3: $N(D_R)$ is relatively compact. Let $t_1, t_2 \in J$, $t_1 < t_2$, and let $u \in D_R$. Then

$$\begin{aligned}|N(u)(t_2) - N(u)(t_1)| &= \left|\frac{1}{\Gamma(\alpha)}\int_0^{t_1}[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}]g(s)ds\right.\\ &\quad\left.+\frac{1}{\Gamma(\alpha)}\int_{t1}^{t_2}[(t_2-s)^{\alpha-1}g(s)ds\right|\\ &\le \frac{\overline{M}}{\Gamma(\alpha+1)}(t_2^\alpha - t_1^\alpha + 2(t_2-t_1)^\alpha)\,.\end{aligned}$$

As $t_1 \to t_2$, the right-hand side of the preceding inequality tends to zero.

As a consequence of Claims 1–3, together with the Ascoli–Arzelà theorem, we conclude that $N\colon C(J,\mathbb{R}) \to C(J,\mathbb{R})$ is continuous and compact. As a consequence of Schauder's fixed point theorem [149], we deduce that N has a fixed point that is a solution of problem (2.1)–(2.2). □

Our next existence result is based on the nonlinear alternative of the Leray–Schauder type.

Theorem 2.5. *Assume* (2.3.1), (2.3.2), *and* (2.4.1) *hold. Then IVP* (2.1)–(2.2) *has at least one solution.*

Proof. Consider operator N defined in (2.5). We will show that N satisfies the assumptions of the Leray–Schauder fixed point theorem. The proof will be given in several claims.

Claim 1: Clearly N is continuous.

Claim 2: N maps bounded sets to bounded sets in $C(J,\mathbb{R})$. Indeed, it is enough to show that for any $\rho > 0$ there exist a positive constant ℓ such that for each $u \in B_\rho = \{u \in C(J,\mathbb{R})\colon \|u\|_\infty \le \rho\}$ we have $\|N(u)\|_\infty \le \ell$.

For $u \in B_\rho$ we have, for each $t \in J$,

$$|Nu(t)| \le |y_0| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|g(t)|ds\,. \tag{2.10}$$

By (2.4.1), for each $t \in J$, we have

$$\begin{aligned}|g(t)| &= |f(t,u(t),g(t))|\\ &\le p(t) + q(t)|u(t)| + r(t)|g(t)|\\ &\le p(t) + q(t)\rho + r(t)|g(t)|\\ &\le p^* + q^*\rho + r^*|g(t)|\,.\end{aligned}$$

Then

$$|g(t)| \le \frac{p^* + q^*\rho}{1-r^*} := M^*\,.$$

Thus, (2.10) implies that

$$|Nu(t)| \le |y_0| + \frac{M^* T^\alpha}{\Gamma(\alpha+1)} \,.$$

Hence,

$$\|Nu\|_\infty \le |y_0| + \frac{M^* T^\alpha}{\Gamma(\alpha+1)} := l \,.$$

Claim 3: Clearly, N maps bounded sets to equicontinuous sets of $C(J,\mathbb{R})$. We conclude that $N\colon C(J,\mathbb{R}) \longrightarrow C(J,\mathbb{R})$ is continuous and completely continuous.

Claim 4: A priori bounds. We now show there exists an open set $U \subseteq C(J,\mathbb{R})$, with $u \ne \lambda N(u)$, for $\lambda \in (0,1)$ and $u \in \partial U$. Let $u \in C(J,\mathbb{R})$ and $u = \lambda N(u)$ for some $0 < \lambda < 1$. Thus, for each $t \in J$ we have

$$u(t) = \lambda y_0 + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)ds \,.$$

This implies by (2.3.2) that for each $t \in J$ we have

$$|u(t)| \le |y_0| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |g(s)|ds \,. \tag{2.11}$$

From (2.4.1) we have for each $t \in J$

$$\begin{aligned} |g(t)| &= |f(t,u(t),g(t))| \\ &\le p(t) + q(t)|u(t)| + r(t)|g(t)| \\ &\le p^* + q^*|u(t)| + r^*|g(t)| \,. \end{aligned}$$

Thus,

$$|g(t)| \le \frac{1}{1-r^*}(p^* + q^*|u(t)|) \,.$$

Hence,

$$|u(t)| \le |y_0| + \frac{p^* T^\alpha}{(1-r^*)\Gamma(\alpha+1)} + \frac{q^*}{(1-r^*)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |u(s)|ds \,.$$

Then Lemma 1.52 implies that for each $t \in J$

$$|u(t)| \le \left(|y_0| + \frac{p^* T^\alpha}{(1-r^*)\Gamma(\alpha+1)} \right) \left(1 + \frac{K q^* T^\alpha}{(1-r^*)\Gamma(\alpha+1)} \right) .$$

Thus,

$$\|u\|_\infty \le \left(|y_0| + \frac{p^* T^\alpha}{(1-r^*)\Gamma(\alpha+1)} \right) \left(1 + \frac{K q^* T^\alpha}{(1-r^*)\Gamma(\alpha+1)} \right) := \overline{M} \,. \tag{2.12}$$

Let

$$U = \{u \in C(J,\mathbb{R})\colon \|u\|_\infty < \overline{M} + 1\} \,.$$

By our choice of U, there is no $u \in \partial U$ such that $u = \lambda N(u)$ for $\lambda \in (0,1)$. As a consequence from Leray–Schauder's theorem we deduce that N has a fixed point u in $\overline{U}$ that is a solution of problem (2.1)–(2.2). □

2.2.3 Examples

Example 1. Consider the Cauchy problem

$$^cD^{\frac{1}{2}}y(t) = \frac{1}{2e^{t+1}\left(1 + |y(t)| + |^cD^{\frac{1}{2}}y(t)|\right)}, \quad \text{for each } t \in [0,1], \tag{2.13}$$

$$y(0) = 1. \tag{2.14}$$

Set

$$f(t,u,v) = \frac{1}{2e^{t+1}(1+|u|+|v|)}, \quad t \in [0,1],\ u,v \in \mathbb{R}.$$

Clearly, the function f is jointly continuous.

For any $u, v, \bar{u}, \bar{v} \in \mathbb{R}$ and $t \in [0,1]$

$$|f(t,u,v) - f(t,\bar{u},\bar{v})| \le \frac{1}{2e}(|u-\bar{u}| + |v-\bar{v}|).$$

Hence, condition (2.3.2) is satisfied by $K = L = \frac{1}{2e}$.

It remains to show that condition (2.4) is satisfied. Indeed, we have

$$\frac{KT^\alpha}{(1-L)\Gamma(\alpha+1)} = \frac{1}{(2e-1)\Gamma\left(\frac{3}{2}\right)} < 1.$$

It follows from Theorem 2.3 that problem (2.13)–(2.14) has a unique solution.

Example 2. Consider the Cauchy problem

$$^cD^{\frac{1}{2}}y(t) = \frac{\left(2 + |y(t)| + |^cD^{\frac{1}{2}}y(t)|\right)}{2e^{t+1}\left(1 + |y(t)| + |^cD^{\frac{1}{2}}y(t)|\right)}, \quad \text{for each } t \in [0,1], \tag{2.15}$$

$$y(0) = 1. \tag{2.16}$$

Set

$$f(t,u,v) = \frac{(2+|u|+|v|)}{2e^{t+1}(1+|u|+|v|)}, \quad t \in [0,1],\ u,v \in \mathbb{R}.$$

Clearly, function f is jointly continuous.

For any $u, v, \bar{u}, \bar{v} \in \mathbb{R}$ and $t \in [0,1]$

$$|f(t,u,v) - f(t,\bar{u},\bar{v})| \le \frac{1}{2e}(|u-\bar{u}| + |v-\bar{v}|).$$

Hence, condition (2.3.2) is satisfied by $K = L = \frac{1}{2e}$. Also, we have

$$|f(t,u,v)| \le \frac{1}{2e^{t+1}}(2+|u|+|v|).$$

Thus, condition (2.4.1) is satisfied by $p(t) = \frac{1}{e^{t+1}}$ and $q(t) = r(t) = \frac{1}{2e^{t+1}}$.

Also,

$$\frac{q^* T^\alpha}{(1-r^*)\Gamma(\alpha+1)} = \frac{1}{(2e-1)\Gamma\left(\frac{3}{2}\right)} < 1$$

holds with $T = 1$, $\alpha = \frac{1}{2}$, and $q^* = r^* = \frac{1}{2e}$. It follows from Theorem 2.4 that problem (2.15)–(2.16) has at least one solution.

2.3 NIFDE with Nonlocal Conditions

2.3.1 Introduction and Motivations

The purpose of this section is to establish the existence, uniqueness, and uniform stability of solutions of the implicit fractional-order differential equation with nonlocal condition:

$$^cD^\alpha y(t) = f(t, y(t), {}^cD^\alpha y(t)) , \quad \text{for each } t \in J = [0, T],\ T > 0,\ 0 < \alpha \le 1 , \tag{2.17}$$

$$y(0) + \varphi(y) = y_0 , \tag{2.18}$$

where $^cD^\alpha$ is the Caputo fractional derivative, $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function, $\varphi : C(J, \mathbb{R}) \to \mathbb{R}$ is a continuous function, and $y_0 \in \mathbb{R}$.

2.3.2 Existence of Solutions

Let us define what we mean by a solution of problem (2.17)–(2.18).

Definition 2.6. A function $u \in C^1(J, \mathbb{R})$ is said to be a solution of problem (2.17)–(2.18) if u satisfies equation (2.17) on J and conditions (2.18).

For the existence of solutions for problem (2.17)–(2.18), we need the following auxiliary lemma.

Lemma 2.7. *Let $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Then problem* (2.17)–(2.18) *is equivalent to the problem*

$$y(t) = y_0 - \varphi(y) + I^\alpha g(t) , \tag{2.19}$$

where $g \in C(J, \mathbb{R})$ satisfies the functional equation

$$g(t) = f(t, y_0 - \varphi(y) + I^\alpha g(t), g(t)) .$$

Proof. If $^cD^\alpha y(t) = g(t)$, then $I^\alpha\, {}^cD^\alpha y(t) = I^\alpha g(t)$. Thus, we obtain $y(t) = y_0 - \varphi(y) + I^\alpha g(t)$. □

We are now in a position to state and prove our existence result for problem (2.17)–(2.18) based on Banach's fixed point theorem.

Theorem 2.8. *Make the following assumptions:*
(2.8.1)*The function $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous.*
(2.8.2)*There exist constants $K > 0$ and $0 < L < 1$ such that*

$$|f(t, u, v) - f(t, \bar{u}, \bar{v})| \le K|u - \bar{u}| + L|v - \bar{v}| \quad \text{for any } u, v, \bar{u}, \bar{v} \in \mathbb{R}, t \in J .$$

(2.8.3)*There exists a constant $0 < \gamma < 1$ such that*

$$|\varphi(u) - \varphi(\bar{u})| \le \gamma|u - \bar{u}| \quad \text{for any } u, \bar{u} \in C(J, \mathbb{R}) .$$

If

$$C := \gamma + \frac{KT^{\alpha}}{(1-L)\Gamma(\alpha+1)} < 1\,, \tag{2.20}$$

then there exists a unique solution for problem (2.17)–(2.18) *on* J*.*

Proof. The proof will be given in several steps. Transform problem (2.17)–(2.18) into a fixed point problem. Define the operator $N\colon C(J,\mathbb{R}) \to C(J,\mathbb{R})$ by

$$N(y)(t) = y_0 - \varphi(y) + I^{\alpha} g(t)\,, \tag{2.21}$$

where $g \in C(J,\mathbb{R})$ satisfies the functional equation

$$g(t) = f(t, y(t), g(t))\,.$$

Clearly, the fixed points of operator N are solutions of problem (2.17)–(2.18). Let $u, w \in C(J,\mathbb{R})$. Then for $t \in J$ we have

$$\begin{aligned}(Nu)(t) - (Nw)(t) &= \varphi(w) - \varphi(u)\\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(g(s)-h(s))ds\,,\end{aligned}$$

where $g, h \in C(J,\mathbb{R})$ are given by

$$\begin{aligned}g(t) &= f(t, u(t), g(t))\,,\\ h(t) &= f(t, w(t), h(t))\,.\end{aligned}$$

Then, for $t \in J$,

$$\begin{aligned}|(Nu)(t) - (Nw)(t)| &\le |\varphi(u) - \varphi(w)|\\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|g(s)-h(s)|ds\,. \end{aligned} \tag{2.22}$$

By (2.8.2) we have

$$\begin{aligned}|g(t) - h(t)| &= |f(t, u(t), g(t)) - f(t, w(t), h(t))|\\ &\le K|u(t) - w(t)| + L|g(t) - h(t)|\,.\end{aligned}$$

Thus,

$$|g(t) - h(t)| \le \frac{K}{1-L}|u(t) - w(t)|\,.$$

From (2.22) and (2.8.3) we have

$$\begin{aligned}|(Nu)(t)-(Nw)(t)| &\le \gamma|u(t)-w(t)| \\ &\quad + \frac{K}{(1-L)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|u(s)-w(s)|ds \\ &\le \gamma\|u-w\|_\infty \\ &\quad + \sup_{0\le t\le T}|u(t)-w(t)|\frac{K}{(1-L)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds \\ &\le \gamma\|u-w\|_\infty + \frac{KT^\alpha}{(1-L)\Gamma(\alpha+1)}\|u-w\|_\infty .\end{aligned}$$

Then

$$\|Nu-Nw\|_\infty \le \left[\gamma + \frac{KT^\alpha}{(1-L)\Gamma(\alpha+1)}\right]\|u-w\|_\infty .$$

By (2.20), operator N is a contraction. Hence, by Banach's contraction principle, N has a unique fixed point that is the unique solution of problem (2.17)–(2.18). □

The second result is based on Krasnosel'skii's fixed point theorem.

Let $\widetilde{M} := \frac{T^\alpha}{(1-L)\Gamma(\alpha+1)}$, $a := |\varphi(0)|$, and $f^* := \sup_{0\le t\le T}|f(t,0,0)|$.

Theorem 2.9. *Assume that* (2.8.1)–(2.8.3) *hold. If*

$$\gamma + \widetilde{M}K < 1 , \tag{2.23}$$

then problem (2.17)–(2.18) *has at least one solution.*

Proof. Consider operator N to be defined as in (2.21). We have

$$N(y)(t) = y_0 - \varphi(y) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}g(s)ds ,$$

where $g \in C(J,\mathbb{R})$ satisfies the functional equation

$$g(t) = f(t, y(t), g(t)) .$$

Let

$$R \ge \frac{|y_0| + a + \widetilde{M}f^*}{1-\gamma-\widetilde{M}K} ,$$

and define

$$D_R = \{u \in C(J,\mathbb{R}) : \|u\|_\infty \le R\} .$$

It is clear that D_R is a bounded, closed, and convex subset of $C(J,\mathbb{R})$. Define on D_R operators P and Q by

$$P(u)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}g(s)ds , \tag{2.24}$$

where $g \in C(J, \mathbb{R})$ satisfies the functional equation

$$g(t) = f(t, u(t), g(t))$$

and

$$Q(v)(t) = y_0 - \varphi(v)\,. \tag{2.25}$$

Claim 1: For any $u, v \in D_R$, $Pu + Qv \in D_R$. For any $u, v \in D_R$ and $t \in J$ we have

$$|P(u)(t) + Q(v)(t)| \leq |y_0| + |\varphi(v)| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |g(s)| ds\,. \tag{2.26}$$

By (2.8.2) we have for each $t \in J$

$$\begin{aligned} |g(t)| &= |f(t, u(t), g(t))| \\ &\leq |f(t, u(t), g(t)) - f(t, 0, 0)| + |f(t, 0, 0)|) \\ &\leq K|u(t)| + L|g(t)| + \sup_{0\leq t\leq T} |f(t, 0, 0)| \\ &\leq KR + L|g(t)| + f^*\,. \end{aligned}$$

Then

$$(1 - L)|g(t)| \leq KR + f^*\,.$$

Thus,

$$|g(t)| \leq \frac{KR + f^*}{1 - L}\,. \tag{2.27}$$

From (2.8.3) we have

$$\begin{aligned} |\varphi(v)| &\leq |\varphi(v) - \varphi(0)| + |\varphi(0)| \\ &\leq \gamma|v| + a \\ &\leq \gamma R + a\,. \end{aligned}$$

Then, by (2.26), we get

$$\begin{aligned} |P(u)(t) + Q(v)(t)| &\leq |y_0| + (\gamma R + a) + \frac{KR + f^*}{(1 - L)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds \\ &\leq |y_0| + (\gamma R + a) + \frac{(KR + f^*)T^\alpha}{(1 - L)\Gamma(\alpha + 1)} \\ &= |y_0| + \gamma R + a + \widetilde{M}(KR + f^*) \\ &\leq R\,. \end{aligned}$$

Thus, $Pu + Qv \in D_R$.

Claim 2: Q is a contraction mapping on D_R. For any $v_1, v_2 \in D_R$, by (2.8.3) we have

$$\begin{aligned} |Q(v_2) - Q(v_1)| &\leq |\varphi(v_2) - \varphi(v_1)| \\ &\leq \gamma|v_2 - v_1|\,. \end{aligned}$$

Thus,

$$\|Q(v_2) - Q(v_1)\|_\infty \le \gamma \|v_2 - v_1\|_\infty ,$$

and so Q is a contraction mapping.

Claim 3: P is continuous. Let $\{u_n\}$ be a sequence such that $u_n \to u$ in $C(J, \mathbb{R})$. Then for each $t \in J$

$$|P(u_n)(t) - P(u)(t)| \le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |g_n(s) - g(s)| ds , \tag{2.28}$$

where $g_n, g \in C(J, \mathbb{R})$ satisfy

$$g_n(t) = f(t, u_n(t), g_n(t))$$

and

$$g(t) = f(t, u(t), g(t)) .$$

By (2.8.2) we have

$$\begin{aligned} |g_n(t) - g(t)| &= |f(t, u_n(t), g_n(t)) - f(t, u(t), g(t))| \\ &\le K|u_n(t) - u(t)| + L|g_n(t) - g(t)| , \end{aligned}$$

so

$$|g_n(t) - g(t)| \le \frac{K}{1-L} |u_n(t) - u(t)| .$$

Since $u_n \to u$, we get $g_n(t) \to g(t)$ as $n \to \infty$ for each $t \in J$. Let $\eta > 0$ be such that, for each $t \in J$, we have $|g_n(t)| \le \eta$ and $|g(t)| \le \eta$; then we have

$$\begin{aligned} (t-s)^{\alpha-1} |g_n(s) - g(s)| &\le (t-s)^{\alpha-1} [|g_n(s)| + |g(s)|] \\ &\le 2\eta (t-s)^{\alpha-1} . \end{aligned}$$

For each $t \in J$ the function $s \to 2\eta(t-s)^{\alpha-1}$ is integrable on $[0, t]$; then the Lebesgue dominated convergence theorem and (2.28) imply that

$$|(Pu_n)(t) - (Pu)(t)| \to 0 \quad \text{as } n \to \infty .$$

Hence,

$$\|P(u_n) - P(u)\|_\infty \to 0 \quad \text{as } n \to \infty .$$

Consequently, P is continuous.

Claim 4: P is compact. Let $\{u_n\}$ be a sequence on D_R. Then, for each $t \in J$, we have

$$|P(u_n)(t)| \le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |g_n(s)| ds , \tag{2.29}$$

where $g_n \in C(J, \mathbb{R})$ is given by

$$g_n(t) = f(t, u_n(t), g_n(t)) .$$

By (2.8.2) we have for each $t \in J$

$$
\begin{aligned}
|g_n(t)| &= |f(t, u_n(t), g(t))| \\
&\leq |f(t, u_n(t), g_n(t)) - f(t, 0, 0)| + |f(t, 0, 0)|) \\
&\leq K|u_n(t)| + L|g_n(t)| + \sup_{0 \leq t \leq T} |f(t, 0, 0)| \\
&\leq KR + L|g_n(t)| + f^* .
\end{aligned}
$$

Then

$$
(1 - L)|g_n(t)| \leq KR + f^* ,
$$

and so

$$
|g_n(t)| \leq \frac{KR + f^*}{1 - L} . \tag{2.30}
$$

Thus, (2.29) implies

$$
\begin{aligned}
|P(u_n)(t)| &\leq \frac{KR + f^*}{(1 - L)\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} ds \\
&\leq \frac{(KR + f^*)T^\alpha}{(1 - L)\Gamma(\alpha + 1)} \\
&\leq \widetilde{M}(KR + f^*) ,
\end{aligned}
$$

and we see that $\{u_n\}$ is uniformly bounded.

Now we prove that $\{P(u_n)\}$ is equicontinuous. Let $t_1, t_2 \in J$, $t_1 < t_2$, and let $u \in D_R$. Then

$$
\begin{aligned}
|(Pu)(t_2) - (Pu)(t_1)| &= \left| \frac{1}{\Gamma(\alpha)} \int_0^{t_1} [(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}] g(s) ds \right. \\
&\quad \left. + \frac{1}{\Gamma(\alpha)} \int_{t1}^{t_2} (t_2 - s)^{\alpha-1} g(s) ds \right| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^{t_1} |[(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}]||g(s)| ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{t1}^{t_2} (t_2 - s)^{\alpha-1} |g(s)| ds \\
&\leq \frac{KR + f^*}{(1 - L)\Gamma(\alpha + 1)} (t_2^\alpha - t_1^\alpha + 2(t_2 - t_1)^\alpha) .
\end{aligned}
$$

As $t_1 \to t_2$, the right-hand side of the preceding inequality tends to zero. As a consequence of Claims 1–4, together with the Ascoli–Arzelà theorem, we conclude that $N\colon C(J, \mathbb{R}) \to C(J, \mathbb{R})$ is continuous and compact. As a consequence of Krasnosel'skii's fixed point theorem, we deduce that N has a fixed point that is a solution of problem (2.17)–(2.18). □

2.3.3 Stability Results

Here we consider the uniform stability of the solutions of problem (2.17)–(2.18) and adopt the definitions in [138].

Definition 2.10. The solution of equation (2.17) is uniformly stable if for any $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that for any two solutions $y(t)$ and $\tilde{y}(t)$ corresponding to the initial conditions (2.18) and $\tilde{y}(0) = \tilde{y}_0 - \varphi(\tilde{y})$, respectively, with $|y_0 - \tilde{y}_0| \le \delta$, one has $\|y - \tilde{y}\|_\infty \le \epsilon$.

Theorem 2.11. *Assume* (2.8.1)–(2.8.3) *and* (2.20) *hold. Then the solutions of the Cauchy problem* (2.17)–(2.18) *are uniformly stable.*

Proof. Let y be a solution of

$$y(t) = y_0 - \varphi(y) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)ds \,, \tag{2.31}$$

where $g \in C(J, \mathbb{R})$ satisfies the functional equation

$$g(t) = f(t, y(t), g(t)) \,,$$

and let $\tilde{y}$ be a solution of equation (2.31) such that

$$\tilde{y}(0) = \tilde{y}_0 - \varphi(\tilde{y}) \,.$$

Then we have

$$\tilde{y}(t) = \tilde{y}_0 - \varphi(\tilde{y}) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)ds \,, \tag{2.32}$$

where $h \in C(J, \mathbb{R})$ satisfies the functional equation

$$h(t) = f(t, \tilde{y}(t), h(t)) \,.$$

By (2.31) and (2.32) we have

$$\begin{aligned} |y(t) - \tilde{y}(t)| &\le |y_0 - \tilde{y}_0| + |\varphi(y) - \varphi(\tilde{y})| \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |g(s) - h(s)| ds \,, \end{aligned} \tag{2.33}$$

and by (2.8.2) we have

$$\begin{aligned} |g(t) - h(t)| &= |f(t, y(t), g(t)) - f(t, \tilde{y}(t), h(t))| \\ &\le K|y(t) - \tilde{y}(t)| + L|g(t) - h(t)| \,, \end{aligned}$$

so

$$|g(t) - h(t)| \le \frac{K}{1-L} |y(t) - \tilde{y}(t)| \,.$$

Thus, (2.33) and (2.8.3) imply that

$$\begin{aligned}
|y(t)-\tilde{y}(t)| &\le |y_0-\tilde{y}_0| + \gamma|y(t)-\tilde{y}(t)| \\
&\quad + \frac{K}{(1-L)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|y(s)-\tilde{y}(s)|ds \\
&\le |y_0-\tilde{y}_0| + \gamma\|y-\tilde{y}\|_\infty \\
&\quad + \sup_{0\le t\le T}|y(t)-\tilde{y}(t)|\frac{K}{(1-L)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds \\
&\le |y_0-\tilde{y}_0| + \gamma\|y-\tilde{y}\|_\infty \\
&\quad + \|y-\tilde{y}\|_\infty\frac{KT^\alpha}{(1-L)\Gamma(\alpha+1)}\,.
\end{aligned}$$

Then

$$\left[1-\gamma-\frac{KT^\alpha}{(1-L)\Gamma(\alpha+1)}\right]\|y-\tilde{y}\|_\infty \le |y_0-\tilde{y}_0|\,.$$

This implies that

$$\|y-\tilde{y}\|_\infty \le (1-C)^{-1}|y_0-\tilde{y}_0|\,. \tag{2.34}$$

For $\epsilon > 0$ it suffices to make $(1-C)^{-1}|y_0-\tilde{y}_0| \le \epsilon$. This suggests that we choose $\delta = (1-C)\epsilon$. Therefore, if $|y_0-\tilde{y}_0| \le \delta(\epsilon)$, then $\|y-\tilde{y}\|_\infty \le \epsilon$. This implies that the solution y is uniformly stable. □

2.3.4 An Example

Consider the problem with nonlocal conditions

$$^cD^{\frac{1}{2}}y(t) = \frac{1}{2e^{t+1}\left(1+|y(t)|+|^cD^{\frac{1}{2}}y(t)|\right)}\,, \quad \text{for each } t\in[0,1]\,, \tag{2.35}$$

$$y(0)+\varphi(y) = 1\,, \tag{2.36}$$

where

$$\varphi(y) = \frac{|y|}{10+|y|}\,. \tag{2.37}$$

Set

$$f(t,u,v) = \frac{1}{2e^{t+1}(1+|u|+|v|)}\,, \quad t\in[0,1],\; u,v\in\mathbb{R}\,.$$

Clearly, the function f is jointly continuous. For any $u, v, \bar{u}, \bar{v} \in \mathbb{R}$ and $t \in [0,1]$

$$|f(t,u,v)-f(t,\bar{u},\bar{v})| \le \frac{1}{2e}(|u-\bar{u}|+|v-\bar{v}|)\,.$$

Hence, condition (2.8.2) is satisfied by $K = L = \frac{1}{2e}$. Let

$$\varphi(u) = \frac{u}{10+u}\,, \qquad u \in [0,\infty)\,,$$

and take $u, v \in [0,\infty)$. Then we have

$$\begin{aligned}|\varphi(u) - \varphi(v)| &= \left|\frac{u}{10+u} - \frac{v}{10+v}\right| = \frac{10|u-v|}{(10+u)(10+v)} \\ &\le \frac{1}{10}|u-v|\,.\end{aligned}$$

Thus the condition

$$C = \gamma + \frac{KT^{\alpha}}{(1-L)\Gamma(\alpha+1)} < 1$$

is satisfied by $T = 1$, $\gamma = \frac{1}{10}$, and $\alpha = \frac{1}{2}$. It follows from Theorems 2.9 and 2.11 that problem (2.35)–(2.37) is a unique uniformly stable solution on J.

2.4 Existence Results for NIFDE in Banach Space

2.4.1 Introduction and Motivations

Recently, fractional differential equations have been studied by Abbas et al. [35, 43], Baleanu et al. [78, 80], Diethelm [137], Kilbas and Marzan [180], Srivastava et al. [181], Lakshmikantham et al. [187], and Samko et al. [239]. More recently, some mathematicians have considered BVPs and boundary conditions for implicit fractional differential equations.

In [164], Hu and Wang investigated the existence of solutions of nonlinear fractional differential equations with integral boundary conditions

$$\begin{aligned}&D^{\alpha}u(t) = f(t, u(t), D^{\beta}u(t))\,, \quad t \in (0,1),\ 1 < \alpha \le 2,\ 0 < \beta < 1\,, \\ &u(0) = u_0,\ u(1) = \int_0^1 g(s)u(s)ds\,,\end{aligned}$$

where D^{α} is the Riemann–Liouville fractional derivative, $f\colon [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, and g is an integrable function.

In [241], by means of Schauder's fixed point theorem, Su and Liu studied the existence of solutions of nonlinear fractional BVPs involving Caputo's derivative

$$\begin{aligned}&{}^cD^{\alpha}u(t) = f(t, u(t), {}^cD^{\beta}u(t))\,, \quad \text{for each } t \in (0,1),\ 1 < \alpha \le 2,\ 0 < \beta \le 1\,, \\ &u(0) = u'(1) = 0,\ \text{or } u'(1) = u(1) = 0,\ \text{or } u(0) = u(1) = 0\,,\end{aligned}$$

where $f\colon [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given continuous function.

The purpose of this section is to establish existence and uniqueness results for the implicit fractional differential equation

$$^cD^\nu y(t) = f(t, y(t), {}^cD^\nu y(t)), \quad \text{for each } t \in J := [0, T],\ T > 0,\ 0 < \nu \le 1, \tag{2.38}$$

with the initial condition

$$y(0) = y_0, \tag{2.39}$$

where $^cD^\nu$ is the Caputo fractional derivative, $(E, \|\cdot\|)$ is a real Banach space, $f: J \times E \times E \to E$ is a continuous function, and $y_0 \in E$.

2.4.2 Existence of Solutions

Let $(E; \|\cdot\|)$ be a Banach space and $t \in J$. For a given set V of functions $v: J \to E$, let us use the notation

$$V(t) = \{v(t), v \in V\}, \quad t \in J$$

and

$$V(J) = \{v(t) : v \in V, t \in J\}.$$

Next we define what we mean by a solution of problem (2.38)–(2.39).

Definition 2.12. A function $u \in C^1(J, E)$ is said to be a solution of problem (2.38)–(2.39) if u satisfies equation (2.38) and condition (2.39) on J.

For the existence of solutions of problem (2.38)–(2.39), we need the following auxiliary lemma.

Lemma 2.13. *Suppose that the function $f(t, u, v): J \times E \times E \to E$ is continuous; then problem* (2.38)–(2.39) *is equivalent to the problem*

$$y(t) = y_0 + I^\nu g(t), \tag{2.40}$$

where $g \in C(J, E)$ satisfies the functional equation

$$g(t) = f(t, y_0 + I^\nu g(t), g(t)).$$

Proof. If $^cD^\nu y(t) = g(t)$, then $I^\nu\, {}^cD^\nu y(t) = I^\nu g(t)$. Thus, we obtain $y(t) = y_0 + I^\nu g(t)$. □

We list the following conditions:
(2.13.1) The function $f: J \times E \times E \to E$ is continuous.
(2.13.2) There exist constants $K > 0$ and $0 < L < 1$, such that

$$\|f(t, u, v) - f(t, \bar{u}, \bar{v})\| \le K\|u - \bar{u}\| + L\|v - \bar{v}\|$$

for any $u, v, \bar{u}, \bar{v} \in E$ and $t \in J$.

(2.13.3) There exist $p, q, r \in C(J, \mathbb{R}_+)$, with $r^* = \sup_{t \in J} r(t) < 1$, such that

$$\|f(t, u, w)\| \leq p(t) + q(t)|u| + r(t)|w| \quad \text{for } t \in J \text{ and } u, w \in \mathbb{R} .$$

Remark 2.14 ([66]). If

$$\|f(t, u, v) - f(t, \bar{u}, \bar{v})\| \leq K\|u - \bar{u}\| + L\|v - \bar{v}\|$$

for any $u, v, \bar{u}, \bar{v} \in E$ and $t \in J$, then
(2.14.1)

$$\alpha(f(t, B_1, B_2)) \leq K\alpha(B_1) + L\alpha(B_2)$$

for each $t \in J$ and bounded sets $B_1, B_2 \subseteq E$.

We are now in a position to state and prove our existence result for problem (2.38)–(2.39) based on the concept of measures of noncompactness and Darbo's fixed point theorem.

Theorem 2.15. *Assume* (2.13.1)–(2.13.3). *If*

$$\frac{KT^{\nu}}{(1 - L)\Gamma(\nu + 1)} < 1 , \tag{2.41}$$

then IVP (2.38)–(2.39) *has at least one solution on* J.

Proof. Transform problem (2.38)–(2.39) into a fixed point problem. Define the operator $N\colon C(J, E) \to C(J, E)$ by

$$(Ny)(t) = y_0 + I^{\nu} g(t) , \tag{2.42}$$

where $g \in C(J, E)$ satisfies the functional equation

$$g(t) = f(t, y(t), g(t)) .$$

Clearly, the fixed points of operator N are solutions of problem (2.38)–(2.39). We will show that N satisfies the assumptions of Darbo's fixed point theorem. The proof will be given in several steps.

Claim 1: N is continuous. Let $u, w \in C(J, E)$, and let $\{u_n\}$ be a sequence such that $u_n \to u$ in $C(J, E)$. Then for each $t \in J$

$$\|N(u_n)(t) - N(u)(t)\| \leq \frac{1}{\Gamma(\nu)} \int_0^t (t - s)^{\nu - 1} \|g_n(s) - g(s)\| ds , \tag{2.43}$$

where $g_n, g \in C(J, E)$ such that

$$g_n(t) = f(t, u_n(t), g_n(t))$$

and

$$g(t) = f(t, u(t), g(t)) .$$

By (2.13.2), for each $t \in J$, we have

$$\begin{aligned}\|g_n(t) - g(t)\| &= \|f(t, u_n(t), g_n(t)) - f(t, u(t), g(t))\| \\ &\le K\|u_n(t) - u(t)\| + L\|g_n(t) - g(t)\| .\end{aligned}$$

Then

$$\|g_n(t) - g(t)\| \le \frac{K}{1-L}\|u_n(t) - u(t)\| .$$

Since $u_n \to u$, we get $g_n(t) \to g(t)$ as $n \to \infty$ for each $t \in J$.

Let a positive constant $\eta > 0$ be such that, for each $t \in J$, we have $\|g_n(t)\| \le \eta$ and $\|g(t)\| \le \eta$. Then we have

$$\begin{aligned}(t-s)^{\nu-1}\|g_n(s) - g(s)\| &\le (t-s)^{\nu-1}[\|g_n(s)\| + \|g(s)\|] \\ &\le 2\eta(t-s)^{\nu-1} .\end{aligned}$$

For each $t \in J$, the function $s \to 2\eta(t-s)^{\nu-1}$ is integrable on $[0, t]$, so by the Lebesgue dominated convergence theorem and (7.2),

$$\|N(u_n)(t) - N(u)(t)\| \to 0 \; as \; n \to \infty .$$

Then

$$\|N(u_n) - N(u)\|_\infty \to 0 \; as \; n \to \infty .$$

Consequently, N is continuous.

Let

$$R \ge \frac{M|y_0| + p^* T^\alpha}{M - q^* T^\alpha} , \tag{2.44}$$

where $M := (1 - r^*)\Gamma(\alpha + 1)$, $p^* = \sup_{t\in J} p(t)$, and $q^* = \sup_{t\in J} q(t)$.
Define

$$D_R = \{u \in C(J, E) : \|u\|_\infty \le R\} .$$

It is clear that D_R is a bounded, closed, and convex subset of $C(J, E)$.

Claim 2: $N(D_R) \subset D_R$. Let $u \in D_R$. We will show that $Nu \in D_R$. We have for each $t \in J$

$$\|Nu(t)\| \le \|y_0\| + \frac{1}{\Gamma(\nu)}\int_0^t (t-s)^{\nu-1}\|g(t)\| ds . \tag{2.45}$$

By (2.13.3) we have

$$\begin{aligned}\|g(t)\| &= \|f(t, u(t), g(t))\| \\ &\le p(t) + q(t)\|u(t)\| + r(t)\|g(t)\| \\ &\le p(t) + q(t)R + r(t)|g(t)| \\ &\le p^* + q^* R + r^*\|g(t)\| .\end{aligned}$$

Then for each $t \in J$

$$\|g(t)\| \leq \frac{p^* + q^* R}{1 - r^*} .$$

Thus, (2.44) and (2.45) imply that

$$\begin{aligned}
\|Nu(t)\| &\leq \|y_0\| + \frac{p^* T^\nu}{(1 - r^*)\Gamma(\nu + 1)} + \frac{q^* R T^\nu}{(1 - r^*)\Gamma(\nu + 1)} \\
&\leq \|y_0\| + \frac{p^* T^\nu}{M} + \frac{q^* R T^\nu}{M} \\
&\leq R .
\end{aligned}$$

Consequently,

$$N(D_R) \subset D_R .$$

Claim 3: $N(D_R)$ is bounded and equicontinuous. By Claim 2 we have $N(D_R) = \{N(u) : u \in D_R\} \subset D_R$. Then for each $u \in D_R$ we have $\|N(u)\|_\infty \leq R$. Thus, $N(D_R)$ is bounded. Let $t_1, t_2 \in J$, $t_1 < t_2$, and let $u \in D_R$. Then

$$\begin{aligned}
|(Nu)(t_2) - (Nu)(t_1)| &= \left| \frac{1}{\Gamma(\nu)} \int_0^{t_1} [(t_2 - s)^{\nu-1} - (t_1 - s)^{\nu-1}] g(s) ds \right. \\
&\quad \left. + \frac{1}{\Gamma(\nu)} \int_{t1}^{t_2} [(t_2 - s)^{\nu-1} g(s) ds \right| \\
&\leq \frac{M}{\Gamma(\nu + 1)} (t_2^\nu - t_1^\nu + 2(t_2 - t_1)^\nu) .
\end{aligned}$$

As $t_1 \to t_2$, the right-hand side of the preceding inequality tends to zero, so $N(D_R)$ is equicontinuous.

Claim 4: The operator $N : B_R \to B_R$ is a strict set contraction. Let $V \subset B_R$ and $t \in J$; then we have

$$\begin{aligned}
\alpha(N(V)(t)) &= \alpha(\{(Ny)(t), y \in V\}) \\
&\leq \frac{1}{\Gamma(\nu)} \left\{ \int_0^t (t - s)^{\nu-1} \alpha(g(s)) ds : y \in V \right\} .
\end{aligned}$$

By (2.14.1), Remark 2.14, and Lemma 1.32, for each $s \in J$,

$$\begin{aligned}
\alpha(\{g(s) : y \in V\}) &= \alpha(\{f(s, y(s), g(s)) : y \in V\}) \\
&\leq K\alpha(\{y(s), y \in V\}) + L\alpha(\{g(s) : y \in V\}) .
\end{aligned}$$

Thus,

$$\alpha\,(\{g(s) : y \in V\}) \leq \frac{K}{1 - L} \alpha\{y(s) : y \in V\} .$$

Then

$$\begin{aligned}\alpha(N(V)(t)) &\leq \frac{K}{(1-L)\Gamma(\nu)}\int_0^t (t-s)^{\nu-1}\{\alpha(y(s))ds : y \in V\}\\ &\leq \frac{K\alpha_c(V)}{(1-L)\Gamma(\nu)}\int_0^t (t-s)^{\nu-1}ds\\ &\leq \frac{KT^{\nu}}{(1-L)\Gamma(\nu+1)}\alpha_c(V)\,.\end{aligned}$$

Therefore,

$$\alpha_c(NV) \leq \frac{KT^{\nu}}{(1-L)\Gamma(\nu+1)}\alpha_c(V)\,.$$

So, by (2.41), operator N is a set contraction. As a consequence of Theorem 1.45, we deduce that N has a fixed point that is a solution of problem (2.38)–(2.39). □

Our next existence result for problem (2.38)–(2.39) is based on the concept of measures of noncompactness and Mönch's fixed point theorem.

Theorem 2.16. *Assume* (2.13.1)–(2.13.3). *Then IVP* (2.38)–(2.39) *has at least one solution.*

Proof. Consider operator N defined in (2.42). We will show that N satisfies the assumptions of Mönch's fixed point theorem. We know that $N : B_R \to B_R$ is bounded and continuous; we need to prove that the implication

$$[V = \overline{\text{conv}}N(V) \text{ or } V = N(V) \cup \{0\}] \Rightarrow \alpha(V) = 0$$

holds for every subset V of B_R.

Now let V be a subset of B_R such that $V \subset \overline{\text{conv}}(N(V) \cup \{0\})$; now V is bounded and equicontinuous, and therefore the function $t \to v(t) = \alpha(V(t))$ is continuous on J. By (2.14.1), Lemma 1.33, and the properties of the measure α, we have for each $t \in J$

$$\begin{aligned}v(t) &\leq \alpha(N(V)(t) \cup \{0\})\\ &\leq \alpha(N(V)(t))\\ &\leq \alpha\{(Ny)(t) : y \in V\}\\ &\leq \frac{K}{(1-L)\Gamma(\nu)}\int_0^t (t-s)^{\nu-1}\alpha\left(\{(y(s) : y \in V\}\right)ds\\ &\leq \frac{K}{(1-L)\Gamma(\nu)}\int_0^t (t-s)^{\nu-1}v(s)ds\,.\end{aligned}$$

Lemma 1.52 implies that $v(t) = 0$ for each $t \in J$, and so $V(t)$ is relatively compact in E. In view of the Ascoli–Arzelà theorem, V is relatively compact in B_R. Applying Theorem 1.46 we conclude that N has a fixed point $y \in B_R$. Hence, N has a fixed point that is a solution of the problem (2.38)–(2.39). □

2.4.3 An Example

Consider the infinite system

$$ {}^cD^{\frac{1}{2}}y_n(t) = \frac{(3 + \|y_n(t)\| + \|{}^cD^{\frac{1}{2}}y_n(t)\|)}{3e^{t+2}(1 + \|y_n(t)\| + \|{}^cD^{\frac{1}{2}}y_n(t)\|)} \quad \text{for each } t \in [0,1] , \tag{2.46} $$

$$ y_n(0) = 1 . \tag{2.47} $$

Set

$$ E = l^1 = \{y = (y_1, y_2, \dots, y_n, \dots), \sum_{n=1}^{\infty} |y_n| < \infty\} $$

and

$$ f(t, u, v) = \frac{(3 + \|u\| + \|v\|)}{3e^{t+2}(1 + \|u\| + \|v\|)} , \quad t \in [0,1],\ u, v \in E , $$

where E is a Banach space with the norm $\|y\| = \sum_{n=1}^{\infty} |y_n|$. Clearly, the function f is jointly continuous. For any $u, v, \bar{u}, \bar{v} \in E$ and $t \in [0, 1]$

$$ \|f(t, u, v) - f(t, \bar{u}, \bar{v})\| \le \frac{2}{3e^2}(\|u - \bar{u}\| + \|v - \bar{v}\|) . $$

Hence condition (2.13.2) is satisfied by $K = L = \frac{2}{3e^2}$. Also,

$$ \|f(t, u, v)\| \le \frac{1}{3e^{t+2}}(3 + \|u\| + \|v\|) . $$

Thus, conditions (2.13.3) and (2.14.1) are satisfied by $p(t) = \frac{1}{e^{t+2}}$, and $q(t) = r(t) = \frac{1}{3e^{t+2}}$. Theorem 2.16 implies that problem (2.46)–(2.47) has at least one solution on J.

2.5 Existence and Stability Results for Perturbed NIFDE with Finite Delay

2.5.1 Introduction

In this section, we establish existence, uniqueness, and stability results for the perturbed functional differential equations of fractional order with finite delay

$$ {}^cD^{\alpha}y(t) = f(t, y_t, {}^cD^{\alpha}y(t)) + g(t, y_t) , \quad t \in J = [0, T],\ T > 0,\ 0 < \alpha \le 1 , \tag{2.48} $$

$$ y(t) = \varphi(t) , \quad t \in [-r, 0],\ r > 0 , \tag{2.49} $$

where $f: J \times C([-r, 0], \mathbb{R}) \times \mathbb{R} \to \mathbb{R}$ and $g: J \times C([-r, 0], \mathbb{R}) \to \mathbb{R}$ are two given functions and $\varphi \in C([-r, 0], \mathbb{R})$. The arguments are based upon the Banach contraction principle and a fixed point theorem of Burton and Kirk.

2.5.2 Existence of Solutions

Set

$$Q = \{y : [-r, T] \to \mathbb{R} : y|_{[-r,0]} \in C([-r, 0], \mathbb{R}) \text{ and } y|_{[0,T]} \in C([0, T], \mathbb{R})\} ;$$

then Q is a Banach space with the norm

$$\|y\|_Q = \sup_{t\in[-r,T]} |y(t)| .$$

Definition 2.17. A function $y \in Q$ is called a solution of problem (2.48)–(2.49) if it satisfies equation (2.48) on J and condition (2.49) on $[-r, 0]$.

Lemma 2.18. *Let $0 < \alpha \leq 1$ and $h : [0, T] \to \mathbb{R}$ be a continuous function. The linear problem*

$$^cD^\alpha y(t) = h(t) , \quad t \in J ,$$
$$y(t) = \varphi(t) , \quad t \in [-r, 0] ,$$

has a unique solution given by

$$y(t) = \begin{cases} \varphi(0) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} h(s)ds, & t \in J \\ \varphi(t), & t \in [-r, 0] . \end{cases}$$

Lemma 2.19. *Let $f(t, u, v) : J \times C([-r, 0], \mathbb{R}) \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Problem* (2.48)–(2.49) *is equivalent to the problem*

$$y(t) = \begin{cases} \varphi(0) + I^\alpha K_y(t), & t \in J , \\ \varphi(t), & t \in [-r, 0] , \end{cases} \tag{2.50}$$

where $K_y \in C(J, \mathbb{R})$ satisfies the functional equation

$$K_y(t) = f(t, y_t, K_y(t)) + g(t, y_t) .$$

Proof. Let y be a solution of problem (2.50); we want to show that y is a solution of (2.48)–(2.49). We have

$$y(t) = \begin{cases} \varphi(0) + I^\alpha K_y(t), & t \in J \\ \varphi(t), & t \in [-r, 0] \end{cases}$$

for $t \in [-r, 0]$, so $y(t) = \varphi(t)$, and we see that condition (2.49) is satisfied.
On the other hand, for $t \in J$ we have

$$^cD^\alpha y(t) = K_y(t) = f(t, y_t, K_y(t)) + g(t, y_t) ,$$

so

$$^cD^\alpha y(t) = f(t, y_t, {}^cD^\alpha y(t)) + g(t, y_t) .$$

Then y is a solution of problem (2.48)–(2.49). □

Lemma 2.20. *Assume*
(2.20.1)$f: J \times C([-r,0],\mathbb{R}) \times \mathbb{R} \to \mathbb{R}$ *is a continuous function.*
(2.20.2)*There exist* $K > 0$ *and* $0 < \overline{K} < 1$ *such that*

$$|f(t,u,v) - f(t,\bar{u},\bar{v})| \le K\,\|u-\bar{u}\|_C + \overline{K}|v-\bar{v}|$$

for any $u,\bar{u} \in C([-r,0],\mathbb{R})$, $v,\bar{v} \in \mathbb{R}$ *and* $t \in J$.
(2.20.3)*There exists* $L > 0$ *such that*

$$|g(t,u) - g(t,v)| \le L\|u-v\|_C$$

for any $u, v \in C([-r,0],\mathbb{R})$ *and* $t \in J$.
If

$$\frac{(K+L)T^\alpha}{\left(1-\overline{K}\right)\Gamma(\alpha+1)} < 1\,, \tag{2.51}$$

then problem (2.48)–(2.49) *has a unique solution.*

Proof. Consider that the operator $N\colon Q \to Q$ is defined by

$$(Ny)(t) = \begin{cases} \varphi(0) + I^\alpha K_y(t), & t \in J \\ \varphi(t), & t \in [-r,0]\,. \end{cases} \tag{2.52}$$

From Lemma 2.19 it is clear that the fixed points of N are the solutions of problem (2.48)–(2.49). Let $y,\tilde{y} \in Q$. If $t \in [-r,0]$, then

$$\|Ny(t) - N\tilde{y}(t)\| = 0\,,$$

and for $t \in J$

$$\|Ny(t) - N\tilde{y}(t)\| = \|I^\alpha K_y(t) - I^\alpha K_{\tilde{y}}(t)\| \le I^\alpha\|K_y(t) - K_{\tilde{y}}(t)\|\,. \tag{2.53}$$

For any $t \in J$ we have

$$\begin{aligned}\|K_y(t) - K_{\tilde{y}}(t)\| &\le \|f(t,y_t,K_y(t)) - f(t,\tilde{y}_t,K_{\tilde{y}}(t))\| \\ &\quad + \|g(t,y_t) - g(t,\tilde{y}_t)\| \\ &\le K\,\|y_t - \tilde{y}_t\|_C + \overline{K}\|K_y(t) - K_{\tilde{y}}(t)\| \\ &\quad + L\,\|y_t - \tilde{y}_t\|_C\,.\end{aligned}$$

Thus,

$$\|K_y(t) - K_{\tilde{y}}(t)\| \le \frac{K+L}{1-\overline{K}}\,\|y_t - \tilde{y}_t\|_C\,. \tag{2.54}$$

From (2.54) and (2.53) we find that

$$\begin{aligned}\|Ny(t) - N\tilde{y}(t)\| &\le \frac{K+L}{\left(1-\overline{K}\right)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\,\|y_s - \tilde{y}_s\|_C\,ds \\ &\le \frac{(K+L)T^\alpha}{\left(1-\overline{K}\right)\Gamma(\alpha+1)}\,\|y-\tilde{y}\|_Q\,.\end{aligned}$$

Thus,

$$\|Ny - N\tilde{y}\|_Q \le \frac{(K+L)T^\alpha}{\left(1-\overline{K}\right)\Gamma(\alpha+1)}\,\|y-\tilde{y}\|_Q \ .$$

From (2.51) it follows that N has a unique fixed point that is the unique solution of problem (2.48)–(2.49). □

Our next existence result is based on the fixed point theorem of Burton and Kirk.

Lemma 2.21. *We consider the operators $F, G\colon Q \to Q$ defined by*

$$F(y)(t) = \begin{cases} \varphi(0) + \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1} f(s, y_s, K_y(s))ds, & t \in J\ , \\ \varphi(t), & t \in [-r, 0]\ , \end{cases}$$

$$G(y)(t) = \begin{cases} \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1} g(s, y_s)ds, & t \in J\ , \\ 0, & t \in [-r, 0]\ , \end{cases}$$

where $K_y \in C(J, \mathbb{R})$ satisfies the functional equation

$$K_y(t) = f(t, y_t, K_y(t)) + g(t, y_t)\ .$$

To find solutions of (2.48)–(2.49), we must find solutions of the equation

$$y(t) = F(y)(t) + G(y)(t)\ , \quad \text{for each } t \in [-r, T]\ .$$

Remark 2.22. If y is a fixed point of the operator $F + G$, then y is a solution of problem (2.48)–(2.49).

Theorem 2.23. *Assume* (2.20.1) *and* (2.20.3) *hold and*
(2.23.1)*There exist $p, q, r \in C(J, \mathbb{R}_+)$ with $r^* = \sup_{t\in J} r(t) < 1$ such that*

$$|f(t, u, w)| \le p(t) + q(t)\|u\|_C + r(t)|w|$$

for $t \in J$, $w \in \mathbb{R}$, and $u \in C([-r, 0], \mathbb{R})$.
If

$$\frac{LT^\alpha}{\Gamma(\alpha+1)} < 1\ , \tag{2.55}$$

then IVP (2.48)–(2.49) *has at least one solution.*

Proof. We must show that operators F and G satisfy all conditions of Theorem 1.43. By (2.20.1), (2.20.3), (2.23.1), and the choices of F and G, we show that F is completely continuous and G is a contraction. The proof will be given in several claims.

Claim 1: F is continuous. Let $\{y_n\}_{n\ge 0}$ be a sequence such that $y_n \to y$ in Q; then

$$\|F(y_n)(t) - F(y)(t)\|_Q \le \frac{T^\alpha}{\Gamma(\alpha+1)}\left\|f(.,y_{n.},K_{y_n}(.)) - f(.,y_.,K_y(.))\right\|_Q \xrightarrow[n\to\infty]{} 0$$

because f is continuous.

Claim 2: F maps bounded sets to bounded sets in Q. Let $\gamma^* > 0$, $B_{\gamma^*} = \{y \in Q : \|y\|_Q \le \gamma^*\}$; we need to show that for each $\gamma^* > 0$ there exists $l > 0$ such that for $y \in B_{\gamma^*}$, $\|Fy\|_Q \le l$. By (2.23.1) we have for any $t \in J$

$$\begin{aligned}\|F(y)(t)\| &\le |\varphi(0)| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(s, y_s, K_y(s))\|ds \\ &\le |\varphi(0)| + \frac{T^\alpha(p^* + q^*\gamma^*)}{\Gamma(\alpha+1)} + \frac{r^*}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|K_y(s)\|ds\,, \end{aligned} \tag{2.56}$$

where $p^* = \sup_{t\in J} p(t)$, and $q^* = \sup_{t\in J} q(t)$. On the other hand, we have for any $t \in J$

$$\begin{aligned}\|K_y(t)\| &\le \|f(t, y_t, K_y(t))\| + \|g(t, y_t) - g(t, 0)\| + \|g(t, 0)\| \\ &\le p^* + q^*\gamma^* + r^*\|K_y(t)\| + L\gamma^* + g^*\,,\end{aligned}$$

where $g^* = \sup_{s\in J}\|g(s, 0)\|$. Thus,

$$\|K_y(t)\| \le \frac{1}{1-r^*}(p^* + q^*\gamma^* + L\gamma^* + g^*) := M\,. \tag{2.57}$$

Combining (2.57) and (2.56), we find that

$$\|F(y)(t)\| \le |\varphi(0)| + \frac{T^\alpha(p^* + q^*\gamma^*)}{\Gamma(\alpha+1)} + \frac{r^*T^\alpha}{\Gamma(\alpha+1)}M := d\,.$$

If $t \in [-r, 0]$, then

$$\|F(y)(t)\| \le \|\varphi\|_C\,,$$

so

$$\|Fy\|_Q \le \max\{\|\varphi\|_C\,, d\} = l\,.$$

Claim 3: F maps bounded sets to equicontinuous sets of Q. Let $t_1, t_2 \in (0, T]$, $t_1 < t_2$, B_{γ^*} be a bounded set of Q, which is as defined previously (Claim 2), and let $y \in B_{\gamma^*}$. Then

$$\begin{aligned}\|F(y)(t_2) - F(y)(t_1)\| &\le \|\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\right]f(s, y_s, K_y(s))ds\| \\ &\quad + \|\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}f(s, y_s, K_y(s))ds\| \\ &\le \frac{(p^* + q^*\gamma^* + r^*M)}{\Gamma(\alpha)}\int_0^{t_1}\left[(t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}\right]ds \\ &\quad + \frac{(p^* + q^*\gamma^* + r^*M)}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds\end{aligned}$$

$$\leq \frac{(p^* + q^* \gamma^* + r^* M)}{\Gamma(\alpha+1)} \left[2(t_2 - t_1)^\alpha - t_1^\alpha - t_2^\alpha\right]$$
$$\leq \frac{2\,(p^* + q^* \gamma^* + r^* M)}{\Gamma(\alpha+1)} (t_2 - t_1)^\alpha \,.$$

As $t_1 \to t_2$, the right-hand side of the preceding inequality tends to zero. Consequently, Theorem 1.54 allows us to conclude that the operator $F\colon Q \to Q$ is relatively compact. Hence, operator F is completely continuous.

Claim 4: G is a contraction. Let $y, \tilde{y} \in Q$; then for every $t \in J$

$$\begin{aligned}
\|G(y)(t) - G(\tilde{y})(t)\| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|g(s, y_s) - g(s, \tilde{y}_s)\| ds \\
&\leq \frac{L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|y_s - \tilde{y}_s\|_C \, ds \\
&\leq \frac{T^\alpha L}{\Gamma(\alpha+1)} \|y - \tilde{y}\|_\infty \,.
\end{aligned}$$

Thus,

$$\|G(y) - G(\tilde{y})\|_Q \leq \frac{T^\alpha L}{\Gamma(\alpha+1)} \|y - \tilde{y}\|_Q \,.$$

By (2.55) G is a contraction.

Claim 5: A priori bounds. We will show that the set

$$\Omega = \left\{ y \in Q\colon y = \lambda F(y) + \lambda G\left(\frac{y}{\lambda}\right) \quad \text{for } \lambda \in (0,1) \right\}$$

is bounded. In fact, let $y \in \Omega$; then $y = \lambda F(y) + \lambda G(\frac{y}{\lambda})$ for some $0 < \lambda < 1$. Then for each $t \in J$ we have

$$y(t) = \lambda \left[|\varphi(0)| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y_s, K_y(s)) ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g\left(s, \frac{y_s}{\lambda}\right) ds \right] .$$

By (2.20.3) and (2.23.1), for every $t \in J$ we have

$$\begin{aligned}
\|y(t)\| &\leq |\varphi(0)| + \frac{q^*}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|y_s\|_C \, ds + \frac{T^\alpha (p^* + r^* M)}{\Gamma(\alpha+1)} \\
&\quad + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left\|g\left(s, \frac{y_s}{\lambda}\right) - g(s,0)\right\| ds \\
&\quad + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|g(s,0)\| ds \\
&\leq |\varphi(0)| + \frac{T^\alpha}{\Gamma(\alpha+1)} (p^* + r^* M + g^*) + \frac{(q^* + L)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|y_s\|_C \, ds \,.
\end{aligned}$$

We consider the function γ defined by

$$\gamma(t) = \sup\{\|y(s)\| : \ -r \le s \le t\} \ , \quad 0 \le t \le T \ .$$

There exists $t^* \in [-r, T]$ such that $\gamma(t) = \|y(t^*)\|$.

If $t^* \in [0, T]$, then, by the previous inequality, for $t \in J$ we have

$$\gamma(t) \le |\varphi(0)| + \frac{T^\alpha}{\Gamma(\alpha+1)}(p^* + r^*M + g^*) + \frac{(q^* + L)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\gamma(s)ds \ .$$

Applying Lemma 1.52, we obtain

$$\gamma(t) \le \left[|\varphi(0)| + \frac{T^\alpha}{\Gamma(\alpha+1)}(p^* + r^*M + g^*)\right]\left[1 + \frac{\overline{\delta}(q^* + L)T^\alpha}{\Gamma(\alpha+1)}\right] := R \ ,$$

where $\overline{\delta} = \overline{\delta}(\alpha)$ is a constant. Thus for any $t \in J$, $\|y\|_\infty \le \gamma(t) \le R$. If $t^* \in [-r, 0]$, then $\gamma(t) = \|\varphi\|_C$. Therefore,

$$\|y\|_Q \le \max\{\|\varphi\|_C, R\} := A \ .$$

Thus, the set Ω is bounded. Therefore, problem (2.48)–(2.49) has at least one solution. □

2.5.3 Ulam–Hyers Stability Results

For the implicit fractional order differential equation (2.48), we adopt the definition in Rus [224] for Ulam–Hyers stability, generalized Ulam–Hyers stability, Ulam–Hyers–Rassias stability, and generalized Ulam–Hyers–Rassias stability.

Definition 2.24. Equation (2.48) is Ulam–Hyers stable if there exists a real number $c_f > 0$ such that for each $\epsilon > 0$ and for each solution $z \in C^1(J, \mathbb{R})$ of the inequality

$$\|{}^cD^\alpha z(t) - f(t, z_t, {}^cD^\alpha z(t)) - g(t, z_t)\| \le \epsilon \ , \quad t \in J \ ,$$

there exists a solution $y \in C^1(J, \mathbb{R})$ of equation (2.48), with

$$\|z(t) - y(t)\| \le c_f\epsilon \ , \quad t \in J \ .$$

Definition 2.25. Equation (2.48) is generalized Ulam–Hyers stable if there exists $\psi_f \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\psi_f(0) = 0$, such that for each solution $z \in C^1(J, \mathbb{R})$ of the inequality

$$\|{}^cD^\alpha z(t) - f(t, z_t, {}^cD^\alpha z(t)) - g(t, z_t)\| \le \epsilon \ , \quad t \in J \ ,$$

there exists a solution $y \in C^1(J, \mathbb{R})$ of equation (2.48), with

$$\|z(t) - y(t)\| \le \psi_f(\epsilon) \ , \quad t \in J \ .$$

Definition 2.26. Equation (2.48) is Ulam–Hyers–Rassias stable with respect to $\phi \in C(J, \mathbb{R}_+)$ if there exists a real number $c_f > 0$ such that for each $\epsilon > 0$ and for each solution $z \in C^1(J, \mathbb{R})$ of the inequality

$$\|{}^cD^\alpha z(t) - f(t, z_t, {}^cD^\alpha z(t)) - g(t, z_t)\| \leq \epsilon\phi(t)\,, \quad t \in J\,,$$

there exists a solution $y \in C^1(J, \mathbb{R})$ of equation (2.48), with

$$\|z(t) - y(t)\| \leq c_f \epsilon\phi(t), \quad t \in J\,.$$

Definition 2.27. Equation (2.48) is generalized Ulam–Hyers–Rassias stable with respect to $\phi \in C(J, \mathbb{R}_+)$ if there exists a real number $c_{f,\phi} > 0$ such that for each solution $z \in C^1(J, \mathbb{R})$ of the inequality

$$\|{}^cD^\alpha z(t) - f(t, z_t, {}^cD^\alpha z(t)) - g(t, z_t)\| \leq \phi(t)\,, \quad t \in J\,,$$

there exists a solution $y \in C^1(J, \mathbb{R})$ of equation (2.48), with

$$\|z(t) - y(t)\| \leq c_{f,\phi}\phi(t), \quad t \in J\,.$$

Remark 2.28. A function $z \in C^1(J, \mathbb{R})$ is a solution of the inequality

$$\|{}^cD^\alpha z(t) - f(t, z_t, {}^cD^\alpha z(t)) - g(t, z_t)\| \leq \epsilon\,, \quad t \in J\,,$$

if and only if there exists a function $h \in C(J, \mathbb{R})$ (which depends on y) such that
(i) $\|h(t)\| \leq \epsilon$, $t \in J$,
(ii) ${}^cD^\alpha z(t) = f(t, z_t, {}^cD^\alpha z(t)) + g(t, z_t) + h(t)$, $t \in J$.

Remark 2.29. Clearly:
(i) Definition 2.24 $\Rightarrow$ Definition 2.25.
(ii) Definition 2.26 $\Rightarrow$ Definition 2.27.

Remark 2.30. A solution of the implicit differential equation

$$\|{}^cD^\alpha z(t) - f(t, z_t, {}^cD^\alpha z(t)) - g(t, z_t)\| \leq \epsilon\,, \quad t \in J\,,$$

with fractional order is called a fractional ϵ-solution of the implicit fractional differential equation (2.48).

Theorem 2.31. *Assume* (2.20.1)–(2.20.3) *and* (2.51) *hold. Then problem* (2.48)–(2.49) *is Ulam–Hyers stable.*

Proof. Let $\epsilon > 0$ and $z \in Q$ be a function such that

$$\|{}^cD^\alpha z(t) - f(t, z_t, {}^cD^\alpha z(t)) - g(t, z_t)\| \leq \epsilon \quad \text{for each } t \in J\,.$$

This inequality is equivalent to

$$\|{}^cD^\alpha z(t) - K_z(t)\| \leq \epsilon\,. \tag{2.58}$$

Let $y \in Q$ be the unique solution of the problem

$$\begin{cases} {}^cD^\alpha y(t) = f(t, y_t, {}^cD^\alpha y(t)) + g(t, y_t), & t \in J\,, \\ z(t) = y(t) = \varphi(t), & t \in [-r, 0]\,. \end{cases}$$

Integrating inequality (2.58), we obtain

$$\|z(t) - I^\alpha K_z(t)\| \le \frac{\epsilon T^\alpha}{\Gamma(\alpha+1)}\,.$$

We consider the function γ_2 defined by

$$\gamma_2(t) = \sup\{\|z(s) - y(s)\| : \; -r \le s \le t\}\,, \quad 0 \le t \le T\,.$$

Then there exists $t^* \in [-r, T]$ such that $\gamma_2(t) = \|z(t^*) - y(t^*)\|$. If $t^* \in [-r, 0]$, then $\gamma_2(t) = 0$. If $t^* \in [0, T]$, then

$$\begin{aligned} \gamma_2(t) &\le \|z(t) - I^\alpha K_z(t)\| + I^\alpha\|K_z(t) - K_y(t)\| \\ &\le \frac{\epsilon T^\alpha}{\Gamma(\alpha+1)} + I^\alpha\|K_z(t) - K_y(t)\|\,. \end{aligned} \tag{2.59}$$

On the other hand, we have

$$\begin{aligned} \|K_z(t) - K_y(t)\| &\le \|f(t, z_t, K_z(t)) - f(t, y_t, K_y(t)\| \\ &\quad + \|g(t, z_t) - g(t, y_t)\| \\ &\le (K + L)\gamma_2(t) + \overline{K}\|K_z(t) - K_y(t)\|\,, \end{aligned}$$

so

$$\|K_z(t) - K_y(t)\| \le \frac{K+L}{1-\overline{K}}\gamma_2(t)\,. \tag{2.60}$$

Substituting (2.60) in inequality (2.59), we get

$$\gamma_2(t) \le \frac{\epsilon T^\alpha}{\Gamma(\alpha+1)} + \frac{K+L}{\left(1-\overline{K}\right)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\gamma_2(s)ds\,,$$

and by Gronwall's lemma

$$\gamma_2(t) \le \frac{\epsilon T^\alpha}{\Gamma(\alpha+1)}\left[1 + \frac{(K+L)T^\alpha\sigma_1}{(1-\overline{K})\Gamma(\alpha+1)}\right] := c\epsilon\,,$$

where $\sigma_1 = \sigma_1(\alpha)$ is a constant. □

If we set $\psi(\epsilon) = c\psi$; $\psi(0) = 0$, then problem (2.48)–(2.49) is generalized Ulam–Hyers stable.

Theorem 2.32. *Assume* (2.20.1)–(2.20.3) *and* (2.51) *hold and (2.32.1) there exists an increasing function* $\phi \in C(J, \mathbb{R}_+)$ *and there exists* $\lambda_\phi > 0$ *such that for any* $t \in J$

$$I^\alpha \phi(t) \le \lambda_\phi \phi(t) .$$

Then problem (2.48)–(2.49) *is Ulam–Hyers–Rassias stable.*

Proof. Let $z \in Q$ be a solution of the inequality

$$\|{}^c D^\alpha z(t) - f(t, z_t, {}^c D^\alpha z(t)) - g(t, z_t)\| \le \epsilon\phi(t) , \quad t \in J,\ \epsilon > 0 .$$

This inequality is equivalent to

$$\|{}^c D^\alpha z(t) - K_z(t)\| \le \epsilon\phi(t) . \tag{2.61}$$

Let $y \in Q$ be the unique solution of the Cauchy problem

$$\begin{cases} {}^c D^\alpha y(t) = f(t, y_t, {}^c D^\alpha y(t)) + g(t, y_t), & t \in J , \\ z(t) = y(t) = \varphi(t), & t \in [-r, 0] . \end{cases}$$

Integrating (2.61), we obtain for any $t \in J$

$$\|z(t) - I^\alpha K_z(t)\| \le \epsilon I^\alpha \phi(t) \le \epsilon\lambda_\phi \phi(t) .$$

Using the function y_2 defined in the proof of Theorem 2.31, we see that if $t^* \in [-r, 0]$, then $y_2(t) = 0$, and if $t^* \in [0, T]$, then we have

$$\begin{aligned} y_2(t) &\le \|z(t) - I^\alpha K_z(t)\| + I^\alpha \|K_z(t) - K_y(t)\| \\ &\le \epsilon\lambda_\phi \phi(t) + I^\alpha \|K_z(t) - K_y(t)\| . \end{aligned} \tag{2.62}$$

It follows that

$$\|K_z(t) - K_y(t)\| \le \frac{K + L}{1 - \overline{K}} y_2(t) . \tag{2.63}$$

Substituting into (2.63) in the inequality (2.62), we obtain

$$y_2(t) \le \epsilon\lambda_\phi \phi(t) + \frac{K + L}{\left(1 - \overline{K}\right) \Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} y_2(s) ds ,$$

and by Gronwall's lemma we get

$$\begin{aligned} y_2(t) &\le \epsilon\lambda_\phi \phi(t) \left[1 + \frac{(K + L)T^\alpha \sigma_2}{(1 - \overline{K})\Gamma(\alpha + 1)} \right] \\ &\le \left[\lambda_\phi \left(1 + \frac{(K + L)T^\alpha \sigma_2}{(1 - \overline{K})\Gamma(\alpha + 1)} \right) \right] \epsilon\phi(t) = c\epsilon\phi(t) , \end{aligned}$$

where $\sigma_2 = \sigma_2(\alpha)$ is a constant. Thus problem (2.48)–(2.49) is Ulam–Hyers–Rassias stable. □

2.5.4 An Example

Consider the problem of the perturbed differential equation of fractional order

$$^cD^{\frac{1}{2}}y(t) = \frac{2 + \|y_t\|_C + \|^cD^{\frac{1}{2}}y(t)\|}{12e^{t+9}\left(1 + \|y_t\|_C + \|^cD^{\frac{1}{2}}y(t)\|\right)} + \frac{e^{-t}\|y_t\|_C}{3\,(1 + \|y_t\|_C)}\,, \quad t \in [0, 1]\,, \tag{2.64}$$

$$y(t) = \varphi(t)\,, \quad t \in [-r, 0]\,, \tag{2.65}$$

where $\varphi \in C([-r, 0], \mathbb{R})$. Set

$$f(t, u, v) = \frac{2 + u + v}{12e^{t+9}(1 + u + v)}\,, \quad t \in [0, 1],\ u, v \in [0, +\infty) \times [0, +\infty)\,.$$

It is clear that f is jointly continuous.

For each $u \in C([-r, 0], \mathbb{R})$, $v \in \mathbb{R}$, and $t \in [0, 1]$,

$$f(t, u, v) \le \frac{1}{12e^{t+9}}\,(2 + \|u\|_C + \|v\|)\,.$$

Hence, condition (2.23.1) is satisfied by $p(t) = \frac{1}{6e^{t+9}}$, $r(t) = q(t) = \frac{1}{12e^{t+9}}$, and $r^* = \frac{1}{12e^9} < 1$. Set

$$g(t, w) = \frac{e^{-t}w}{3(1 + w)}, \, t \in [0, 1]\,, \quad w \in [0, +\infty)\,.$$

It is clear that g is continuous; moreover, we have for any $u, v \in C([-r, 0], \mathbb{R})$ and $t \in J$

$$\|g(t, u) - g(t, v)\| \le \frac{1}{3}\|u - v\|_C\,.$$

Thus, (2.20.3) is satisfied, and we have

$$\frac{T^\alpha L}{\Gamma(\alpha + 1)} = \frac{(1)^{\frac{1}{2}} \times \frac{1}{3}}{\Gamma\left(\frac{3}{2}\right)} = \frac{1}{3\Gamma\left(\frac{3}{2}\right)} = \frac{1}{3 \times \frac{1}{2}\Gamma\left(\frac{1}{2}\right)} = \frac{2}{3\sqrt{\pi}} < 1\,.$$

Hence, (2.20.1), (2.20.3), (2.23.1), and (2.55) are satisfied, so by Theorem 2.23, problem (2.64)–(2.65) has at least one solution.

2.6 Existence and Stability Results for Neutral NIFDE with Finite Delay

2.6.1 Introduction

In this section, we establish existence, uniqueness, and stability results for the nonlinear implicit neutral fractional differential equation with finite delay

$$^cD^\alpha\,[y(t) - g(t, y_t)] = f(t, y_t, {}^cD^\alpha y(t))\,, \quad t \in J = [0, T],\ T > 0,\ 0 < \alpha \le 1\,, \tag{2.66}$$

$$y(t) = \varphi(t),\ t \in [-r, 0],\ r > 0\,, \tag{2.67}$$

where $f: J \times C([-r, 0], \mathbb{R}) \times \mathbb{R} \to \mathbb{R}$ and $g: J \times C([-r, 0], \mathbb{R})$ are given functions such that $g(0, \varphi) = 0$ and $\varphi \in C([-r, 0], \mathbb{R})$.

Two examples are given to show the applicability of our results.

2.6.2 Existence of Solutions

Set

$$\Omega = \{y: [-r, T] \to \mathbb{R}: y|_{[-r,0]} \in C([-r, 0], \mathbb{R}) \text{ and } y|_{[0,T]} \in C([0, T], \mathbb{R})\} .$$

Note that Ω is a Banach space with the norm

$$\|y\|_\Omega = \sup_{t\in[-r,T]} |y(t)| .$$

Definition 2.33. A function $y \in \Omega$ is called a solution of problem (2.66)–(2.67) if it satisfies equation (2.66) on J and condition (2.67) on $[-r, 0]$.

Lemma 2.34. *Let $0 < \alpha \le 1$ and $h: [0, T] \to \mathbb{R}$ be a continuous function. Then the linear problem*

$$\begin{aligned} &{}^cD^\alpha [y(t) - g(t, y_t)] = h(t) , \quad t \in J , \\ &y(t) = \varphi(t) , \quad t \in [-r, 0] , \end{aligned}$$

has a unique solution given by

$$y(t) = \begin{cases} \varphi(0) + g(t, y_t) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} h(s) ds, & t \in J , \\ \varphi(t), & t \in [-r, 0] . \end{cases}$$

Lemma 2.35. *Let $f(t, u, v): J \times C([-r, 0], \mathbb{R}) \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Then problem* (2.66)–(2.67) *is equivalent to the problem*

$$y(t) = \begin{cases} \varphi(0) + I^\alpha K_y(t), & t \in J , \\ \varphi(t), & t \in [-r, 0] , \end{cases} \tag{2.68}$$

where $K_y \in C(J, \mathbb{R})$ satisfies the functional equation

$$K_y(t) = f(t, y_t, K_y(t)) + {}^cD^\alpha g(t, y_t) .$$

Proof. Let y be a solution of problem (2.68); we need to show that y is a solution of (2.66)–(2.67). We have

$$y(t) = \begin{cases} \varphi(0) + I^\alpha K_y(t), & t \in J , \\ \varphi(t), & t \in [-r, 0] . \end{cases}$$

For $t \in [-r, 0]$ we have $y(t) = \varphi(t)$, so condition (2.67) is satisfied.
On the other hand, for $t \in J$ we have

$$^cD^\alpha y(t) = K_y(t) = f(t, y_t, K_y(t)) + {}^cD^\alpha g(t, y_t) .$$

So

$$^cD^\alpha \left[y(t) - g(t, y_t)\right] = f(t, y_t, {}^cD^\alpha y(t)) .$$

Then y is a solution of problem (2.66)–(2.67). □

Lemma 2.36. *Assume*

(2.36.1) $f : J \times C([-r, 0], \mathbb{R}) \times \mathbb{R} \to \mathbb{R}$ *is a continuous function.*

(2.36.2) *There exist* $K > 0$ *and* $0 < \overline{K} < 1$ *such that*

$$|f(t, u, v) - f(t, \bar{u}, \bar{v})| \leq K \|u - \bar{u}\|_C + \overline{K}|v - \bar{v}|$$

for any $u, \bar{u} \in C([-r, 0], \mathbb{R})$, $v, \bar{v} \in \mathbb{R}$ *and* $t \in J$.

(2.36.3) *There exists* $L > 0$ *such that*

$$|g(t, u) - g(t, v)| \leq L\|u - v\|_C$$

for any $u, v \in C([-r, 0], \mathbb{R})$ *and* $t \in J$. *If*

$$\frac{KT^\alpha}{\left(1 - \overline{K}\right)\Gamma(\alpha + 1)} + \frac{L}{\left(1 - \overline{K}\right)} < 1 , \tag{2.69}$$

then problem (2.66)–(2.67) *has a unique solution.*

Proof. Consider the operator $N : \Omega \to \Omega$ defined by

$$(Ny)(t) = \begin{cases} \varphi(0) + I^\alpha K_y(t), & t \in J \\ \varphi(t), & t \in [-r, 0] . \end{cases} \tag{2.70}$$

By Lemma 2.35, it is clear that the fixed points of N are the solutions of problem (2.66)–(2.67).

Let $y, \tilde{y} \in \Omega$. If $t \in [-r, 0]$, then

$$\|(Ny)(t) - (N\tilde{y})(t)\| = 0 .$$

For $t \in J$ we have

$$\|(Ny)(t) - (N\tilde{y})(t)\| = \|I^\alpha K_y(t) - I^\alpha K_{\tilde{y}}(t)\| \leq I^\alpha \|K_y(t) - K_{\tilde{y}}(t)\| . \tag{2.71}$$

For any $t \in J$

$$\begin{aligned} \|K_y(t) - K_{\tilde{y}}(t)\| &\leq \|f(t, y_t, K_y(t)) - f(t, \tilde{y}_t, K_{\tilde{y}}(t))\| \\ &\quad + {}^cD^\alpha \|g(t, y_t) - g(t, \tilde{y}_t)\| \\ &\leq K \|y_t - \tilde{y}_t\|_C + \overline{K}\|K_y(t) - K_{\tilde{y}}(t)\| \\ &\quad + {}^cD^\alpha \|g(t, y_t) - g(t, \tilde{y}_t)\| . \end{aligned}$$

Thus,

$$\|K_y(t) - K_{\tilde{y}}(t)\| \le \frac{K}{1-\overline{K}} \|y_t - \tilde{y}_t\|_C + \left(\frac{1}{1-\overline{K}}\right) {}^cD^\alpha \|g(t, y_t) - g(t, \tilde{y}_t)\| \,. \tag{2.72}$$

Substituting into (2.72) in the inequality (2.71) we have

$$\begin{aligned}
\|Ny(t) - N\tilde{y}(t)\| &\le \frac{K}{\left(1-\overline{K}\right)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|y_s - \tilde{y}_s\|_C \, ds \\
&\quad + \frac{1}{1-\overline{K}} I^\alpha \, {}^cD^\alpha \|g(t, y_t) - g(t, \tilde{y}_t)\| \\
&\le \frac{KT^\alpha}{\left(1-\overline{K}\right)\Gamma(\alpha+1)} \|y - \tilde{y}\|_\Omega \\
&\quad + \frac{1}{1-\overline{K}} \left(\|g(t, y_t) - g(t, \tilde{y}_t)\| + \|g(0, y_0) - g(0, \tilde{y}_0)\|\right) \\
&\le \frac{KT^\alpha}{\left(1-\overline{K}\right)\Gamma(\alpha+1)} \|y - \tilde{y}\|_\Omega + \frac{L}{1-\overline{K}} \|y_t - \tilde{y}_t\|_C \\
&\le \left[\frac{KT^\alpha}{\left(1-\overline{K}\right)\Gamma(\alpha+1)} + \frac{L}{1-\overline{K}}\right] \|y - \tilde{y}\|_\Omega \,.
\end{aligned}$$

Thus,

$$\|Ny - N\tilde{y}\|_\Omega \le \left[\frac{KT^\alpha}{\left(1-\overline{K}\right)\Gamma(\alpha+1)} + \frac{L}{\left(1-\overline{K}\right)}\right] \|y - \tilde{y}\|_\Omega \,.$$

From (2.69) it follows that N has a unique fixed point that is the unique solution of problem (2.66)–(2.67). □

2.6.3 Ulam–Hyers Stability Results

A solution of the implicit differential equation

$$\|{}^cD^\alpha z(t) - f(t, z_t, {}^cD^\alpha z(t)) - {}^cD^\alpha g(t, z_t)\| \le \epsilon \,, \quad t \in J \,,$$

with fractional order is called a fractional ϵ-solution of implicit fractional differential equation (2.66).

Theorem 2.37. *Assume* (2.36.1)–(2.36.3) *and* (2.69) *hold. If*

$$\overline{K} + L < 1 \,, \tag{2.73}$$

then problem (2.66)–(2.67) *is Ulam–Hyers stable.*

Proof. Let $\epsilon > 0$ and $z \in \Omega$ be a function such that

$$\|{}^cD^\alpha z(t) - f(t, z_t, {}^cD^\alpha z(t)) - {}^cD^\alpha g(t, z_t)\| \le \epsilon \quad \text{for each } t \in J \,.$$

This inequality is equivalent to

$$\|{}^cD^\alpha z(t) - K_z(t)\| \le \epsilon \,. \tag{2.74}$$

Let $y \in \Omega$ be the unique solution of the problem

$$\begin{cases} {}^cD^\alpha\,[y(t) - g(t, y_t)] = f(t, y_t, {}^cD^\alpha y(t)), & t \in J \\ z(t) = y(t) = \varphi(t), & t \in [-r, 0]\,. \end{cases}$$

Integrating the inequality (2.74), we obtain

$$\|z(t) - I^\alpha K_z(t)\| \le \frac{\epsilon T^\alpha}{\Gamma(\alpha+1)}\,.$$

We consider the function y_1 defined by

$$y_1(t) = \sup\,\{\|z(s) - y(s)\| :\; -r \le s \le t\}\;, \quad 0 \le t \le T\,.$$

Then there exists $t^* \in [-r, T]$ such that $y_1(t) = \|z(t^*) - y(t^*)\|$. If $t^* \in [-r, 0]$, then $y_1(t) = 0$. If $t^* \in [0, T]$, then

$$\begin{aligned} y_1(t) &\le \|z(t) - I^\alpha K_z(t)\| + I^\alpha\|K_z(t) - K_y(t)\| \\ &\le \frac{\epsilon T^\alpha}{\Gamma(\alpha+1)} + I^\alpha\|K_z(t) - K_y(t)\|\,. \end{aligned} \tag{2.75}$$

On the other hand, we have

$$\begin{aligned} \|K_z(t) - K_y(t)\| &\le \|f(t, z_t, K_z(t)) - f(t, y_t, K_y(t)\| \\ &\quad + {}^cD^\alpha\|g(t, z_t) - g(t, y_t)\| \\ &\le K y_1(t) + \overline{K}\|K_z(t) - K_y(t)\| \\ &\quad + {}^cD^\alpha\|g(t, z_t) - g(t, y_t)\|\,, \end{aligned}$$

so

$$\|K_z(t) - K_y(t)\| \le \frac{K}{1-\overline{K}} y_1(t) + \frac{1}{1-\overline{K}}\; {}^cD^\alpha\|g(t, z_t) - g(t, y_t)\|\,. \tag{2.76}$$

From (2.76) and (2.75) we obtain

$$\begin{aligned} y_1(t) &\le \frac{\epsilon T^\alpha}{\Gamma\,(\alpha+1)} + \frac{K}{\left(1-\overline{K}\right)\Gamma\,(\alpha)} \int_0^t (t-s)^{\alpha-1} y_1(s)ds \\ &\quad + \frac{1}{1-\overline{K}}\|g(t, z_t) - g(t, y_t)\| \\ &\le \frac{\epsilon T^\alpha}{\Gamma(\alpha+1)} + \frac{K}{\left(1-\overline{K}\right)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y_1(s)ds \\ &\quad + \frac{L}{1-\overline{K}} y_1(t)\,. \end{aligned}$$

Then

$$y_1(t) \le \frac{\epsilon T^\alpha \left(1-\overline{K}\right)}{\left[1-\left(\overline{K}+L\right)\right]\Gamma(\alpha+1)} + \frac{K}{\left[1-\left(\overline{K}+L\right)\right]\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y_1(s)ds \,.$$

By Gronwall's lemma

$$y_1(t) \le \frac{\epsilon T^\alpha \left(1-\overline{K}\right)}{\left[1-\left(\overline{K}+L\right)\right]\Gamma(\alpha+1)} \left[1 + \frac{KT^\alpha \sigma_1}{\left[1-\left(\overline{K}_1+L\right)\right]\Gamma(\alpha+1)}\right] := c\epsilon \,,$$

where $\sigma_1 = \sigma_1(\alpha)$ is a constant. This completes the proof of the theorem. Moreover, if we set $\psi(\epsilon) = c\psi$; $\psi(0) = 0$, then problem (2.66)–(2.67) is generalized Ulam–Hyers stable. □

Theorem 2.38. *Assume* (2.36.1)–(2.36.3), (2.69), *and* (2.73) *hold and*

(2.45.1) *there exists an increasing function* $\phi \in C(J, \mathbb{R}_+)$, *and there exists* $\lambda_\phi > 0$ *such that for any* $t \in J$

$$I^\alpha \phi(t) \le \lambda_\phi \phi(t) \,.$$

Then problem (2.66)–(2.67) *is Ulam–Hyers–Rassias stable.*

Proof. Let $z \in \Omega$ be a solution of the inequality

$$\|{}^cD^\alpha z(t) - f(t, z_t, {}^cD^\alpha z(t)) - {}^cD^\alpha g(t, z_t)\| \le \epsilon\phi(t) \,, \quad t \in J,\ \epsilon > 0 \,.$$

This inequality is equivalent to

$$\|{}^cD^\alpha z(t) - K_z(t)\| \le \epsilon\phi(t) \,. \tag{2.77}$$

Let $y \in \Omega$ be the unique solution of the Cauchy problem

$$\begin{cases} {}^cD^\alpha \left[y(t) - g(t, y_t)\right] = f(t, y_t, {}^cD^\alpha y(t)), & t \in J \,, \\ z(t) = y(t) = \varphi(t), & t \in [-r, 0] \,. \end{cases}$$

Integrating (2.77), we obtain for any $t \in J$

$$\|z(t) - I^\alpha K_z(t)\| \le \epsilon I^\alpha \phi(t) \le \epsilon\lambda_\phi \phi(t) \,.$$

Using the function y_1 defined in the proof of Theorem 2.37, we have that if $t^* \in [-r, 0]$, then $y_1(t) = 0$. If $t^* \in [0, T]$, then we have

$$\begin{aligned} y_1(t) &\le \|z(t) - I^\alpha K_z(t)\| + I^\alpha \|K_z(t) - K_y(t)\| \\ &\le \epsilon\lambda_\phi\phi(t) + I^\alpha\|K_z(t) - K_y(t)\| \,. \end{aligned} \tag{2.78}$$

Thus,

$$\|K_z(t) - K_y(t)\| \le \frac{K}{1-\overline{K}} y_1(t) + \frac{1}{1-\overline{K}}\ {}^cD^\alpha \|g(t, z_t) - g(t, y_t)\| \,. \tag{2.79}$$

Substituting into (2.79) in (2.78), we obtain

$$\begin{aligned} y_1(t) &\le \epsilon\lambda_\phi\phi(t) + \frac{K}{\left(1-\overline{K}\right)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y_1(s)ds \\ &\quad + \frac{1}{1-\overline{K}}\|g(t,z_t)-g(t,y_t)\| \\ &\le \epsilon\lambda_\phi\phi(t) + \frac{K}{\left(1-\overline{K}\right)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y_1(s)ds + \frac{L}{1-\overline{K}}y_1(t)\,, \end{aligned}$$

so

$$y_1(t) \le \frac{\left(1-\overline{K}\right)\epsilon\lambda_\phi\phi(t)}{1-\left(\overline{K}+L\right)} + \frac{K}{\left[1-\left(\overline{K}+L\right)\right]\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y_1(s)ds\,.$$

By Gronwall's lemma we get

$$\begin{aligned} y_1(t) &\le \frac{\left(1-\overline{K}\right)\epsilon\lambda_\phi\phi(t)}{1-\left(\overline{K}+L\right)}\left[1+\frac{KT^\alpha\sigma_2}{\left[1-\left(\overline{K}+L\right)\right]\Gamma(\alpha+1)}\right] \\ &\le \left[\frac{\left(1-\overline{K}\right)\lambda_\phi}{1-\left(\overline{K}+L\right)}\left(1+\frac{KT^\alpha\sigma_2}{\left[1-\left(\overline{K}+L\right)\right]\Gamma(\alpha+1)}\right)\right]\epsilon\phi(t) = c\epsilon\phi(t)\,, \end{aligned}$$

where $\sigma_2 = \sigma_2(\alpha)$ is a constant. Then problem (2.66)–(2.67) is Ulam–Hyers–Rassias stable. □

2.6.4 Examples

Example 1. Consider the neutral fractional differential equation

$$^cD^{\frac{1}{2}}\left[y(t) - \frac{te^{-t}\|y_t\|_C}{(9+e^t)(1+\|y_t\|_C)}\right] = \frac{2+\|y_t\|_C+|^cD^{\frac{1}{2}}y(t)|}{12e^{t+9}\left(1+\|y_t\|_C+|^cD^{\frac{1}{2}}y(t)|\right)}\,, \quad t\in[0,1]\,, \tag{2.80}$$

$$y(t) = \varphi(t); \quad t\in[-r,0]\,, \quad r>0\,, \tag{2.81}$$

where $\varphi \in C([-r,0],\mathbb{R})$. Set

$$g(t,w) = \frac{te^{-t}w}{(9+e^t)(1+w)}\,, \quad (t,w)\in[0,1]\times[0,+\infty)$$

and

$$f(t,u,v) = \frac{2+u+v}{12e^{t+9}(1+u+v)}\,, \quad (t,u,v)\in[0,1]\times[0,+\infty)\times[0,+\infty)\,.$$

Observe that $g(0, w) = 0$ for any $w \in [0, +\infty)$. Clearly, the function f is continuous. Hence, (2.36.1) is satisfied. We have

$$|f(t, u, v) - f(t, \bar{u}, \bar{v})| \leq \frac{1}{12e^9} (\|u - \bar{u}\|_C + \|v - \bar{v}\|)$$
$$|g(t, u) - g(t, \bar{u})| \leq \frac{1}{10} \|u - \bar{u}\|_C$$

for any $u, \bar{u} \in C([-r, 0], \mathbb{R})$, $v, \bar{v} \in \mathbb{R}$ and $t \in [0, 1]$. Hence, conditions (2.36.2) and (2.36.3) are satisfied by $K = \overline{K} = \frac{1}{12e^9}$ and $L = \frac{1}{10}$.
The condition

$$\frac{KT^\alpha}{(1+\overline{K})\Gamma(\alpha+1)} + \frac{L}{(1-\overline{K})} = \frac{20 + 12e^9\sqrt{\pi}}{10\sqrt{\pi}(12e^9 - 1)} < 1$$

is satisfied by $T = 1$, $\alpha = \frac{1}{2}$.

By Lemma 2.36, problem (2.80)–(2.81) admits a unique solution.

Since
$$\overline{K} + L = \frac{10 + 12e^9}{120e^9} < 1 ,$$
by Theorem 2.37, problem (2.80)–(2.81) is Ulam–Hyers stable.

Example 2. Consider the neutral fractional differential equation

$$^cD^{\frac{1}{2}} \left[y(t) - \frac{t}{5e^{t+2}\,(1 + \|y_t\|_C)} \right] = \frac{e^{-t}}{7 + e^t} \left[\frac{\|y_t\|_C}{1 + \|y_t\|_C} - \frac{|^cD^{\frac{1}{2}} y(t)|}{1 + |^cD^{\frac{1}{2}} y(t)|} \right], \quad t \in [0, 1] , \tag{2.82}$$

$$y(t) = \varphi(t) , \quad t \in [-r, 0] , \quad r > 0 , \tag{2.83}$$

where $\varphi \in C([-r, 0], \mathbb{R})$.
Set
$$g(t, w) = \frac{t}{5e^{t+2}(1 + w)} , \quad (t, w) \in [0, 1] \times [0, +\infty) ,$$
and
$$f(t, u, v) = \frac{e^{-t}}{(7 + e^t)} \left(\frac{u}{1+u} - \frac{v}{1+v} \right) , \quad (t, u, v) \in [0, 1] \times [0, +\infty) \times [0, +\infty) .$$
Observe that $g(0, w) = 0$ for any $w \in [0, +\infty)$. Clearly, the function f is continuous, so (2.36.1) is satisfied.

$$|f(t, u, v) - f(t, \bar{u}, \bar{v})| \leq \frac{1}{8} \|u - \bar{u}\|_C + \frac{1}{8} \|v - \bar{v}\|$$
$$|g(t, u) - g(t, \bar{u})| \leq \frac{1}{5e^2} \|u - \bar{u}\|_C$$

for any $u, \bar{u} \in C([-r, 0], \mathbb{R}),\ v, \bar{v} \in \mathbb{R}$ *and* $t \in [0, 1]$.
Hence, conditions (2.36.2) and (2.36.3) are satisfied by $K = \overline{K} = \frac{1}{8}$ and $L = \frac{1}{5e^2}$.
We have
$$\frac{KT^\alpha}{(1+\overline{K})\Gamma(\alpha+1)} + \frac{L}{(1-\overline{K})} = \frac{10e^2 + 8\sqrt{\pi}}{35e^2\sqrt{\pi}} < 1 ,$$

so by Lemma 2.36, problem (2.82)–(2.83) admits a unique solution.
Since

$$\overline{K} + L = \frac{5e^2 + 8}{40e^2} < 1 ,$$

by Theorem 2.37, problem (2.82)–(2.83) is Ulam–Hyers stable.

2.7 Notes and Remarks

The results of Chapter 2 are taken from Benchohra et al. [91, 102, 104]. Other results may be found in [20, 15, 26, 34, 43, 94, 248, 253].