1 Preliminary Background

In this chapter, we introduce notations, definitions, and preliminary facts that will be used in the remainder of the book. Some notations and definitions from fractional calculus, definitions and properties of measures of noncompactness, and fixed point theorems are presented.

1.1 Notations and Definitions

Let $C(J, \mathbb{R})$ be the Banach space of all continuous functions from $J := [0, T];\ T > 0$ to $\mathbb{R}$ with the usual norm

$$\|y\| = \sup_{t\in J} |y(t)| ,$$

and let $L^1(J, \mathbb{R})$ denote the Banach space of functions $: J \to \mathbb{R}$ that are Lebesgue integrable with the norm

$$\|y\|_{L_1} = \int_0^T |y(t)|dt .$$

Definition 1.1 ([131]). A map $f: J \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is said to be L^1-Carathéodory if

(i) the map $t \longmapsto f(t, x, y)$ is measurable for each $(x, y) \in \mathbb{R} \times \mathbb{R}$,

(ii) the map $(x, y) \longmapsto f(t, x, y)$ is continuous for almost all $t \in J$,

(iii) for each $q > 0$ there exists $\varphi_q \in L^1(J, \mathbb{R})$ such that

$$|f(t, x, y)| \le \varphi_q(t)$$

for all $|x| \le q,\ |y| \le q$ and for a.e. $t \in J$.

The map f is said to be of Carathéodory if it satisfies just (i) and (ii).

Definition 1.2. An operator $T: E \longrightarrow E$ is called compact if the image of each bounded set $B \subset E$ is relatively compact, i.e., $\overline{T(B)}$ is compact. T is called a completely continuous operator if it is continuous and compact.

Theorem 1.3 (Kolmogorov compactness criterion [133]).
Let $\Omega \subseteq L^p(J, \mathbb{R})$ and $1 \le p \le \infty$. If

(i) *Ω is bounded in $L^p(J, \mathbb{R})$ and*

(ii) *$u_h \longrightarrow u$ as $h \longrightarrow 0$ uniformly with respect to $u \in \Omega$,*

then Ω is relatively compact in $L^p(J, \mathbb{R})$, where

$$u_h(t) = \frac{1}{h} \int_t^{t+h} u(s)ds .$$

https://doi.org/10.1515/9783110553819-001

1.2 Fractional Calculus

Definition 1.4 ([35, 181, 219]). The Riemann–Liouville fractional (arbitrary) order integral of the function $h \in L^1([a, b], \mathbb{R}_+)$ of order $\alpha \in \mathbb{R}_+$ is defined by

$$I_a^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} h(s) ds \,,$$

where $\Gamma(.)$ is the Euler gamma function. If $a = 0$, we write $I^\alpha h(t) = h(t) * \varphi_\alpha(t)$, where $\varphi_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $t > 0$, $\varphi_\alpha(t) = r$ for $t \leq 0$, and $\varphi_\alpha \to \delta(t)$ as $\alpha \to 0$, where δ is the delta function.

Definition 1.5 ([35, 181, 219]). The Riemann–Liouville fractional derivative of order $\alpha > 0$ of function $h \in L^1([a, b], \mathbb{R}_+)$ is given by

$$(D_{a+}^\alpha h)(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1} h(s) ds \,.$$

Here $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α. If $\alpha \in (0, T]$, then

$$(D_{a+}^\alpha h)(t) = \frac{d}{dt} I_{a+}^{1-\alpha} h(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{ds} \int_a^t (t-s)^{-\alpha} h(s) ds \,.$$

Definition 1.6 ([35, 181]). The Caputo fractional derivative of order $\alpha > 0$ of a function $h \in L^1([a, b], \mathbb{R}_+)$ is given by

$$({}^c D_{a+}^\alpha h)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} h^{(n)}(s) ds \,,$$

where $n = [\alpha] + 1$. If $\alpha \in (0, 1]$, then

$$({}^c D_{a+}^\alpha h)(t) = I_{a+}^{1-\alpha} \frac{d}{dt} h(t) = \int_a^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} \frac{d}{ds} h(s) ds \,.$$

The following properties are some of the main ones of fractional derivatives and integrals.

Lemma 1.7 ([200]). *Let $\alpha > 0$ and $n = [\alpha] + 1$. Then*

$$I^\alpha({}^c D^\alpha f(t)) = f(t) - \sum_{k=0}^{n-1} \frac{f^k(0)}{k!} t_k \,.$$

Lemma 1.8 ([181]). *Let $\alpha > 0$; then the differential equation*

$${}^c D^\alpha h(t) = 0$$

has the solution

$$h(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}, \quad c_i \in \mathbb{R},\ i = 0, 1, 2, \ldots, n-1,\ n = [\alpha] + 1 \,.$$

Lemma 1.9 ([181]). *Let $\alpha > 0$; then*

$$I^{\alpha\,c}D^{\alpha}h(t) = h(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1}t^{n-1} ,$$

for arbitrary $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$, $n = [\alpha] + 1$.

Proposition 1.10 ([181]). *Let $\alpha, \beta > 0$. Then we have*

(1) *$I^{\alpha}\colon\ L^1(J, \mathbb{R}) \to L^1(J, \mathbb{R})$, and if $f \in L^1(J, \mathbb{R})$, then*

$$I^{\alpha}I^{\beta}f(t) = I^{\beta}I^{\alpha}f(t) = I^{\alpha+\beta}f(t) .$$

(2) *If $f \in L^p(J, \mathbb{R})$, $1 \le p \le +\infty$, then $\|I^{\alpha}f\|_{L^p} \le \frac{T^{\alpha}}{\Gamma(\alpha+1)}\|f\|_{L^p}$.*

(3) *The fractional integration operator I^{α} is linear.*

(4) *The fractional order integral operator I^{α} maps $L^1(J, \mathbb{R})$ to itself continuously.*

(5) *If $\alpha = n \in \mathbb{N}$, then I_0^{α} is the n-fold integration.*

(6) *The Caputo and Riemann–Liouville fractional derivatives are linear.*

(7) *The Caputo fractional derivative of a constant is equal to zero.*

Now we recall some definitions and properties of Hadamard fractional integration and differentiation. We refer to [153, 181] for a more detailed analysis.

Definition 1.11 ([153, 181]). The Hadamard fractional integral of order $q > 0$ for a function $g \in L^1([1, a], \mathbb{R})$ is defined as

$$({}^{H}I_1^{r}g)(x) = \frac{1}{\Gamma(q)} \int_1^x \left(\ln \frac{x}{s}\right)^{q-1} \frac{g(s)}{s}\,ds$$

provided the integral exists.

Analogous to the Riemann–Liouville fractional calculus, the Hadamard fractional derivative is defined in terms of the Hadamard fractional integral in the following way. Set

$$\delta = x\frac{d}{dx}, \quad q > 0, \quad n = [q] + 1 ,$$

where $[q]$ is the integer part of q, and

$$AC_{\delta}^{n} := \{u\colon [1, a] \to \mathbb{R}\colon \delta^{n-1}[u(x)] \in AC[1, a]\} .$$

Definition 1.12 ([153, 181]). The Hadamard fractional derivative of order q applied to the function $w \in AC_{\delta}^{n}$ is defined as

$$({}^{H}D_1^{q}w)(x) = \delta^{n}({}^{H}I_1^{n-q}w)(x) .$$

It has been proved (e.g., Kilbas [[178], Theorem 4.8]) that in the space $L^1([1, a], \mathbb{R})$, the Hadamard fractional derivative is the left-inverse operator to the Hadamard fractional integral, i.e.,

$$({}^{H}D_1^{q})({}^{H}I_1^{q}w)(x) = w(x) .$$

Analogously to the Caputo partial fractional integral and derivative [36, 35], we can define the Hadamard partial fractional integral and derivative. Also, the Hadamard partial fractional derivative is defined in terms of the Hadamard partial fractional integral.

Definition 1.13. Let $r_1, r_2 \geq 0$, $\sigma = (1,1)$, and $r = (r_1, r_2)$. For $w \in L^1(J, \mathbb{R})$, define the Hadamard partial fractional integral of order r by the expression

$$({}^H I_\sigma^r w)(x,y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_1^x \int_1^y \left(\ln \frac{x}{s}\right)^{r_1-1} \left(\ln \frac{y}{t}\right)^{r_2-1} \frac{w(s,t)}{st} dt ds \,.$$

By $1-r$ we mean $(1-r_1, 1-r_2) \in (0,1] \times (0,1]$. Denote by $D_{xy}^2 := \frac{\partial^2}{\partial x \partial y}$ the mixed second-order partial derivative.

Definition 1.14. Let $r = (r_1, r_2) \in (0,1] \times (0,1]$ and $u \in L^1(J)$. Define the Hadamard fractional order derivative of order r of u by the expression

$${}^H D_\sigma^r u(x,y) = D_{xy}^2 [xy D_{xy}^2 ({}^H I_\sigma^{1-r} u)](x,y) \,.$$

Definition 1.15. Let $\alpha \in (0, \infty)$ and $u \in L^1(J)$. The partial Hadamard integral of order α of $u(x,y)$ with respect to x is defined by

$${}^H I_{1,x}^\alpha u(x,y) = \frac{1}{\Gamma(\alpha)} \int_1^x \left(\ln \frac{x}{s}\right)^{\alpha-1} \frac{u(s,y)}{s} ds \quad \text{for almost all } x \in [1,a] \text{ and all } y \in [1,b].$$

Analogously, we define the integral

$${}^H I_{1,y}^\alpha u(x,y) = \frac{1}{\Gamma(\alpha)} \int_1^y \left(\ln \frac{y}{s}\right)^{\alpha-1} \frac{u(x,s)}{s} ds \quad \text{for all } x \in [1,a] \text{ and almost all } y \in [1,b].$$

Definition 1.16. Let $\alpha \in (0,1]$ and $u \in L^1(J)$. The Hadamard fractional derivative of order α of $u(x,y)$ with respect to x is defined by

$${}^H D_{1,x}^\alpha u(x,y) = \frac{\partial}{\partial x} \left[x \frac{\partial}{\partial x} ({}^H I_{1,x}^{1-\alpha} u) \right](x,y) \quad \text{for almost all } x \in [1,a] \text{ and all } y \in [1,b] \,.$$

Analogously, we define the derivative of order α of $u(x,y)$ with respect to y by

$${}^H D_{1,y}^\alpha u(x,y) = \frac{\partial}{\partial y} \left[y \frac{\partial}{\partial y} ({}^H I_{1,y}^{1-\alpha} u) \right](x,y) \quad \text{for all } x \in [1,a] \text{ and almost all } y \in [1,b] \,.$$

1.3 Multivalued Analysis

Let $(X, \|\cdot\|)$ be a Banach space and K be a subset of X. We use the notation

$$\begin{aligned}
\mathcal{P}(X) &= \{K \subset X\colon K \neq \emptyset\}\,,\\
\mathcal{P}_{cl}(X) &= \{K \subset \mathcal{P}(X)\colon K \text{ is closed}\}\,,\\
\mathcal{P}_{b}(X) &= \{K \subset \mathcal{P}(X)\colon K \text{ is bounded}\}\,,\\
\mathcal{P}_{cv}(X) &= \{K \subset \mathcal{P}(X)\colon K \text{ is convex}\}\,,\\
\mathcal{P}_{cp}(X) &= \{K \subset \mathcal{P}(X)\colon K \text{ is compact}\}\,,\\
\mathcal{P}_{cv,cp}(X) &= \mathcal{P}_{cv}(X) \cap \mathcal{P}_{cp}(X)\,.
\end{aligned}$$

Let $A, B \in \mathcal{P}(X)$. Consider $H_d\colon \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}_+ \cup \{\infty\}$ the Hausdorff distance between A and B given by

$$H_d(A, B) = \max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b)\}\,,$$

where $d(A, b) = \inf_{a\in A} d(a, b)$ and $d(a, B) = \inf_{b\in B} d(a, b)$. As usual, $d(x, \emptyset) = +\infty$.

Then $(\mathcal{P}_{b,cl}(X), H_d)$ is a metric space and $(\mathcal{P}_{cl}(X), H_d)$ is a generalized (complete) metric space [184].

Definition 1.17. A multivalued operator $N\colon X \to \mathcal{P}_{cl}(X)$ is called:
(a) γ-Lipschitz if there exists $\gamma > 0$ such that

$$H_d(N(x), N(y)) \le \gamma d(x, y), \quad \text{for all } x, y \in X\,;$$

(b) a contraction if it is γ-Lipschitz with $\gamma < 1$.

Definition 1.18. A multivalued map $F\colon J \to \mathcal{P}_{cl}(X)$ is said to be measurable if, for each $y \in X$, the function

$$t \longmapsto d(y, F(t)) = \inf\{d(x, z)\colon z \in F(t)\}$$

is measurable.

Definition 1.19. The selection set of a multivalued map $G\colon J \to \mathcal{P}(X)$ is defined by

$$S_G = \{u \in L^1(J)\colon u(t) \in G(t), \ \text{ a.e. } t \in J\}\,.$$

For each $u \in C$, the set $S_{F\circ u}$ known as the set of selectors from F is defined by

$$S_{F\circ u} = \{v \in L^1(J)\colon v(t) \in F(t, u(t)), \ \text{ a.e. } t \in J\}\,.$$

Definition 1.20. Let X and Y be metric spaces. A set-valued map F from X to Y is characterized by its graph $Gr(F)$, the subset of the product space $X \times Y$ defined by

$$Gr(F) := \{(x, y) \in X \times Y\colon y \in F(x)\}\,.$$

Definition 1.21. Let $(X, \|\cdot\|)$ be a Banach space. A multivalued map $F\colon X \to \mathcal{P}(X)$ is convex (closed) if $F(X)$ is convex (closed) for all $x \in X$.

The map F is bounded on bounded sets if $F(\mathcal{B}) = \cup_{x\in\mathcal{B}} F(x)$ is bounded in X for all $\mathcal{B} \in \mathcal{P}_b(X)$, i.e., $\sup_{x\in\mathcal{B}}\{\sup\{|y|\colon y \in F(x)\}\} < \infty$.

Definition 1.22. A multivalued map F is called upper semicontinuous (u.s.c.) on X if for each $x_0 \in X$ the set $F(x_0)$ is a nonempty, closed subset of X and for each open set U of X containing $F(x_0)$ there exists an open neighborhood V of x_0 such that $F(V) \subset U$. A set-valued map F is said to be u.s.c. if it is so at every point $x_0 \in X$. F is said to be completely continuous if $F(\mathcal{B})$ is relatively compact for every $\mathcal{B} \in \mathcal{P}_b(X)$.

If the multivalued map F is completely continuous with nonempty compact values, then F is u.s.c. if and only if F has closed graph (i.e., $x_n \to x_*,\ y_n \to y_*,\ y_n \in G(x_n)$ imply $y_* \in F(x_*)$).
The map F has a fixed point if there exists $x \in X$ such that $x \in Gx$. The set of fixed points of the multivalued operator G will be denoted by $FixG$.

Definition 1.23. A measurable multivalued function $F\colon J \to \mathcal{P}_{b,cl}(X)$ is said to be integrably bounded if there exists a function $g \in L^1(\mathbb{R}_+)$ such that $|f| \le g(t)$ for almost all $t \in J$ for all $f \in F(t)$.

Lemma 1.24 ([165]). *Let G be a completely continuous multivalued map with nonempty compact values. Then G is u.s.c. if and only if G has a closed graph (i.e., $u_n \to u$, $w_n \to w$, $w_n \in G(u_n)$ imply $w \in G(u)$).*

Lemma 1.25 ([192]). *Let X be a Banach space. Let $F\colon J \times X \longrightarrow \mathcal{P}_{cp,cv}(X)$ be an L^1-Carathéodory multivalued map, and let Λ be a linear continuous mapping from $L^1(J, X)$ to $C(J, X)$. Then the operator*

$$\begin{aligned} \Lambda \circ S_{F\circ u}\colon\ C(J, X) &\longrightarrow \mathcal{P}_{cp,cv}(C(J, X))\,, \\ w &\longmapsto (\Lambda \circ S_{F\circ u})(w) := (\Lambda S_{F\circ u})(w) \end{aligned}$$

is a closed graph operator in $C(J, X) \times C(J, X)$.

Proposition 1.26 ([165]). *Let $F\colon X \to Y$ be an u.s.c. map with closed values. Then $Gr(F)$ is closed.*

Definition 1.27. A multivalued map $F\colon J \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be L^1-Carathéodory if

(i) $t \to F(t, x, y)$ is measurable for each $x, y \in \mathbb{R}$;
(ii) $x \to F(t, x, y)$ is u.s.c. for almost all $t \in J$;
(iii) for each $q > 0$ there exists $\varphi_q \in L^1(J, \mathbb{R}_+)$ such that

$$\|F(t, x, y)\|_{\mathcal{P}} = \sup\{|f|\colon f \in F(t, x, y)\} \le \varphi_q(t)$$
$$\text{for all } |x| \le q,\ |y| \le q \text{ and for a.e. } t \in J\,.$$

The multivalued map F is said to be Carathéodory if it satisfies (i) and (ii).

Lemma 1.28 ([145]). *Let X be a separable metric space. Then every measurable multivalued map $F\colon X \to \mathcal{P}_{cl}(X)$ has a measurable selection.*

For more details on multivalued maps and the proof of the known results cited in this section, we refer interested reader to the books of Aubin and Cellina [68], Deimling [134], Gorniewicz [145], and Hu and Papageorgiou [165].

1.4 Measure of Noncompactness

We will define the Kuratowski (1896–1980) and Hausdorff (1868–1942) measures of noncompactness ($\mathcal{MNC}$ for short) and give their basic properties. Let us recall some fundamental facts of the notion of measure of noncompactness in a Banach space.

Let (X, d) be a complete metric space and $\mathcal{P}_{bd}(X)$ be the family of all bounded subsets of X. Analogously denote by $\mathcal{P}_{rcp}(X)$ the family of all relatively compact and nonempty subsets of X. Recall that $B \subset X$ is said to be bounded if B is contained in some ball. If $B \subset \mathcal{P}_{bd}(X)$ is not relatively compact, (precompact) then there exists an $\epsilon > 0$ such that B cannot be covered by a finite number of ϵ-baIls, and it is then also impossible to cover B by finitely many sets of diameter $< \epsilon$. Recall that the diameter of B is given by

$$\operatorname{diam}(B) := \begin{cases} \sup\limits_{(x,y)\in B^2} d(x,y), & \text{if } B \neq \phi\,, \\ 0, & \text{if } B = \phi\,. \end{cases}$$

Definition 1.29 ([183]). Let (X, d) be a complete metric space and $\mathcal{P}_{bd}(X)$ be the family of bounded subsets of X. For every $B \in \mathcal{P}_{bd}(X)$, we define the Kuratowski measure of noncompactness $\alpha(B)$ of the set B as the infimum of the numbers d such that B admits a finite covering by sets of diameter smaller than d.

Remark 1.30. It is clear that $0 \leq \alpha(B) \leq \operatorname{diam}(B) < +\infty$ for each nonempty bounded subset B of X and that $\operatorname{diam}(B) = 0$ if and only if B is an empty set or consists of exactly one point.

Definition 1.31 ([81]). Let X be a Banach space and $\mathcal{P}_{bd}(X)$ be the family of bounded subsets of X. For every $B \in \mathcal{P}_{bd}(X)$, the Kuratowski measure of noncompactness is the map $\alpha\colon \mathcal{P}_{bd}(X) \to [0, +\infty]$ defined by

$$\alpha(B) = \inf\{r > 0\colon\ B \subseteq \cup_{i=1}^{n} B_i \text{ and } \operatorname{diam}(B_i) < r\}\,.$$

The Kuratowski measure of noncompactness satisfies the following properties:

Proposition 1.32 ([81, 83, 183]). *Let X be a Banach space. Then for all bounded subsets A, B of X the following assertions hold:*

1. *$\alpha(B) = 0$ implies $\overline{B}$ is compact (B is relatively compact), where B denotes the closure of B.*

2. $\alpha(\phi) = 0$.
3. $\alpha(B) = \alpha(\overline{B}) = \alpha(\operatorname{conv} B)$, *where* $\operatorname{conv} B$ *is the convex hull of* B.
4. *monotonicity:* $A \subset B$ *implies* $\alpha(A) \leq \alpha(B)$.
5. *algebraic semi-additivity:* $\alpha(A+B) \leq \alpha(A)+\alpha(B)$, *where* $A+B = \{x+y : x \in A; y \in B\}$.
6. *semihomogeneity:* $\alpha(\lambda B) = |\lambda|\alpha(B), \quad \lambda \in \mathbb{R}$, *where* $\lambda(B) = \{\lambda x : x \in B\}$.
7. *semi-additivity:* $\alpha(A \cup B) = \max\{\alpha(A), \alpha(B)\}$.
8. *semi-additivity:* $\alpha(A \cap B) = \min\{\alpha(A), \alpha(B)\}$.
9. *invariance under translations:* $\alpha(B + x_0) = \alpha(B)$ *for any* $x_0 \in X$.

Lemma 1.33 ([151]). *If* $V \subset C(J, E)$ *is a bounded and equicontinuous set, then*
(i) the function $t \to \alpha(V(t))$ *is continuous on* J *and*

$$\alpha_c(V) = \sup_{0 \leq t \leq T} \alpha(V(t)) ;$$

(ii) $\alpha\left(\int_0^T x(s)ds : x \in V\right) \leq \int_0^T \alpha(V(s))ds,$

where

$$V(s) = \{x(s) : x \in V\}, \quad s \in J .$$

The following definition of measure of noncompactness appeared in Banaś and Goebel [81].

Definition 1.34. A function $\mu : \mathcal{P}_{bd}(X) \longrightarrow [0, \infty)$ will be called a measure of noncompactness if it satisfies the following conditions:
1. $\operatorname{Ker} \mu(A) = \{A \in \mathcal{P}_{bd}(X) : \mu(A) = 0\}$ is nonempty and $\operatorname{Ker} \mu(A) \subset \mathcal{P}_{rcp}(X)$.
2. $A \subset B$ implies $\mu(A) \leq \mu(B)$.
3. $\mu(\overline{A}) = \mu(A)$.
4. $\mu(\operatorname{conv} A) = \mu(A)$.
5. $\mu(\lambda A + (1 - \lambda)B) \leq \lambda\mu(A) + (1 - \lambda)\mu(B)$ for $\lambda \in [0, 1]$.
6. If $(A_n)_{n \in \geq 1}$ is a sequence of closed sets in $\mathcal{P}_{bd}(X)$ such that

$$X_{n+1} \subset A_n \ (n = 1, 2, \dots)$$

   and

$$\lim_{n \to +\infty} \mu(A_n) = 0 ,$$

   then the intersection set $A_\infty = \bigcap_{n=1}^{\infty} A_n$ is nonempty.

Remark 1.35. The family $\operatorname{Ker} \mu$ described in 1 is said to be the kernel of the measure of noncompactness μ. Observe that the intersection set A_∞ in 6 is a member of the family $\operatorname{Ker} \mu$. Since $\mu(A_\infty) \leq \mu(A_n)$ for any n, we infer that $\mu(A_\infty) = 0$. This yields that $\mu(A_\infty) \in \operatorname{Ker} \mu$. This simple observation will be essential in our further investigations.

Moreover, we introduce the notion of a measure of noncompactness in $L^1(J)$. We let $\mathcal{P}_{bd}(J)$ be the family of all bounded subsets of $L^1(J)$. Analogously, denote by $\mathcal{P}_{rcp}(J)$

the family of all relatively compact and nonempty subsets of $L^1(J)$. In particular, the measure of noncompactness in $L^1(J)$ is defined as follows. Let X be a fixed nonempty and bounded subset of $L^1(J)$. For $x \in X$, set

$$\mu(X) = \lim_{\delta \to 0} \left\{ \sup \left\{ \sup \left(\int_0^T |x(t+h) - x(t)| dt \right), \ |h| \le \delta \right\}, \quad x \in X \right\}. \tag{1.1}$$

It can be easily shown that μ is a measure of noncompactness in $L^1(J)$ [81]. For more details on the measure of noncompactness and the proof of the known results cited in this section, we refer the reader to Akhmerov et al. [58] and Banaś et al. [81, 83].

1.5 Phase Spaces

In this section, we assume that the state space $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is a seminormed linear space of functions mapping $(-\infty, 0]$ to $\mathbb{R}$ and satisfying the following fundamental axioms introduced by Hale and Kato in [154].

(A_1) If $y: (-\infty, b] \to \mathbb{R}$ and $y_0 \in \mathcal{B}$, then for every $t \in J$ the following conditions hold:
- (i) $y_t \in \mathcal{B}$.
- (ii) $\|y_t\|_{\mathcal{B}} \le K(t) \int_0^t |y(s)| ds + M(t)\|y_0\|_{\mathcal{B}}$.
- (iii) $|y(t)| \le H\|y_t\|_{\mathcal{B}}$, where $H \ge 0$ is a constant, $K: J \to [0, \infty)$ is continuous, $M: [0, \infty) \to [0, \infty)$ is locally bounded, and H, K, M are independent of $y(\cdot)$.

(A_2) For the function $y(\cdot)$ in **(A_1)**, y_t is a $\mathcal{B}$-valued continuous function on J.

(A_3) The space $\mathcal{B}$ is complete.

Use the notation $K_b = \sup\{K(t): t \in J\}$ and $M_b = \sup\{M(t): t \in J\}$.

Remark 1.36. 1. (A_1)(ii) is equivalent to $|\phi(0)| \le H\|\phi\|_{\mathcal{B}}$ for every $\phi \in \mathcal{B}$.
2. Since $\|\cdot\|_{\mathcal{B}}$ is a seminorm, two elements $\phi, \psi \in \mathcal{B}$ can satisfy $\|\phi - \psi\|_{\mathcal{B}} = 0$ without necessarily $\phi(\theta) = \psi(\theta)$ for all $\theta \le 0$.
3. From the equivalence in the first remark, we can see that for all $\phi, \psi \in \mathcal{B}$ such that $\|\phi - \psi\|_{\mathcal{B}} = 0$. We necessarily have that $\phi(0) = \psi(0)$.

We now present some examples of phase spaces. For other details see, for instance, the book by Hino et al. [162].

1.5.1 Examples of Phase Spaces

Example 1.37. *Let us define the following spaces:*
BC the space of bounded and continuous functions defined from $(-\infty, 0] \to E$*;*
BUC the space of bounded and uniformly continuous functions defined from $(-\infty, 0] \to E$*;*

$C^\infty := \{\phi \in BC\colon\ \lim_{\theta\to-\infty}\phi(\theta)$ *exist in* $E\}$*;*
$C^0 := \{\phi \in BC\colon\ \lim_{\theta\to-\infty}\phi(\theta) = 0\}$, *endowed with the uniform norm*

$$\|\phi\| = \sup\{|\phi(\theta)|\colon\ \theta \le 0\}\ .$$

We have that the spaces BUC, C^∞ *and* C^0 *satisfy conditions (A1)–(A3). However,* BC *satisfies (A1) and (A3), but (A2) is not satisfied.*

Example 1.38. *Let* g *be a positive continuous function on* $(-\infty, 0]$*. We define:*
$C_g := \{\phi \in C((-\infty, 0]), E)\colon\ \frac{\phi(\theta)}{g(\theta)}$ *is bounded on* $(-\infty, 0]\}$,
$C_g^0 := \{\phi \in C_g\colon\ \lim_{\theta\to-\infty}\frac{\phi(\theta)}{g(\theta)} = 0\}$ *endowed with the uniform norm*

$$\|\phi\| = \sup\left\{\frac{|\phi(\theta)|}{g(\theta)}\colon\ \theta \le 0\right\}\ .$$

Then we have that the spaces C_g *and* C_g^0 *satisfy condition (A3). We consider the following condition on the function* g*:*
(g_1) *For all* $a > 0$, $\sup_{0\le t\le a}\sup\{\frac{\phi(t+\theta)}{g(\theta)}\colon\ -\infty < \theta \le -t\}$.
Then C_g *and* C_g^0 *satisfy conditions (A1) and (A2) if* (g_1) *holds.*

Example 1.39. *The space* C_γ *for any real positive constant* γ *is defined by*

$$C_\gamma := \{\phi \in C((-\infty, 0]), E)\colon\ \lim_{\theta\to-\infty} e^{\gamma\theta}\phi(\theta) \text{ exist in } E$$

endowed with the norm

$$\|\phi\| = \sup\{e^{\gamma\theta}|\phi(\theta)|\colon\ \theta \le 0\}\ .$$

Then in the space C_γ *axioms (A1)–(A3) are satisfied.*

1.6 Some Fixed Point Theorems

In this section, we give the main fixed point theorems that will be used in subsequent chapters.

Definition 1.40 ([60]). Let (M, d) be a metric space. The map $T\colon M \longrightarrow M$ is said to be Lipschitzian if there exists a constant $k > 0$ (called a Lipschitz constant) such that

$$d(T(x), T(y)) \le kd(x, y) \quad \text{for all } x, y \in M\ .$$

A Lipschitzian mapping with a Lipschitz constant $k < 1$ is called a contraction.

Theorem 1.41 (Banach's fixed point theorem [149]). *Let* C *be a nonempty closed subset of a Banach space* X*. Then any contraction mapping* T *of* C *to itself has a unique fixed point.*

Theorem 1.42 (Schauder fixed point theorem [149]). *Let* E *be a Banach space,* Q *a convex subset of* E*, and* $T\colon Q \longrightarrow Q$ *a compact and continuous map. Then* T *has at least one fixed point in* Q*.*

Theorem 1.43 (Burton and Kirk fixed point theorem [115]). *Let X be a Banach space and $A, B\colon X \to X$ two operators satisfying*
(i) *A is a contraction,*
(ii) *B is completely continuous.*
Then either
- *The operator equation $y = A(y) + B(y)$ admits a solution or*
- *the set $\Omega = \{u \in X\colon u = \lambda A(\frac{u}{\lambda}) + \lambda B(u)\}$ is unbounded for $\lambda \in [0, 1]$.*

In the next definition, we will consider a special class of continuous and bounded operators.

Definition 1.44. Let $T\colon M \subset E \longrightarrow E$ be a bounded operator from a Banach space E to itself. The operator T is called a k-set contraction if there is a number $k \geq 0$ such that

$$\mu(T(A)) \leq k\mu(A)$$

for all bounded sets A in M. The bounded operator T is called condensing if $\mu(T(A)) < \mu(A)$ for all bounded sets A in M with $\mu(M) > 0$.

Obviously, every k-set contraction for $0 \leq k < 1$ is condensing. Every compact map T is a k-set contraction with $k = 0$.

Theorem 1.45 (Darbo fixed point theorem [81]). *Let M be a nonempty, bounded, convex, and closed subset of a Banach space E and $T\colon M \longrightarrow M$ a continuous operator satisfying $\mu(TA) \leq k\mu(A)$ for any nonempty subset A of M and for some constant $k \in [0, 1)$. Then T has at least one fixed point in M.*

Theorem 1.46 (Mönch's fixed point theorem [49, 202]). *Let D be a bounded, closed, and convex subset of a Banach space such that $0 \in D$, α the Kuratowski measure of noncompactness, and N a continuous mapping of D to itself. If the implication $[V = \overline{\text{conv}} N(V)$ or $V = N(V) \cup \{0\}]$ implies $\alpha(V) = 0$ holds for every subset V of D, then N has a fixed point.*

For more details, see [49, 64, 145, 149, 183, 257].

Theorem 1.47 (Nonlinear alternative to Leray–Schauder type [149]). *Let X be a Banach space and C a nonempty convex subset of X. Let U be a nonempty open subset of C, with $0 \in U$ and $T\colon \overline{U} \to C$ a continuous and compact operator.*
Then, either
(a) T has fixed points or
(b) there exist $u \in \partial U$ and $\lambda \in (0, 1)$ with $u = \lambda T(u)$.

Theorem 1.48 (Martelli's fixed point theorem [199]). *Let X be a Banach space and $N\colon X \to \mathcal{P}_{cl,cv}(X)$ an u.s.c. and condensing map. If the set $\Omega := \{u \in X\colon \lambda u \in N(u)$ for some $\lambda > 1\}$ is bounded, then N has a fixed point.*

Theorem 1.49 ([70]). *Let $(X, \|\cdot\|_n)$ be a Fréchet space, and let $A, B\colon X \to X$ be two operators such that*
(a) A is a compact operator;
(b) B is a contraction operator with respect to a family of seminorms $\{\|\cdot\|_n\}$;
(c) the set $\{x \in X\colon x = \lambda A(x) + \lambda B(\frac{x}{\lambda}),\quad \lambda \in (0,1)\}$ is bounded.
Then the operator equation $A(u) + B(u) = u$ has a solution in X.

Next, we state two multivalued fixed point theorems.

Lemma 1.50 (Bohnenblust–Karlin 1950 [111]). *Let X be a Banach space and $K \in \mathcal{P}_{cl,cv}(X)$, and suppose that the operator $G\colon K \to \mathcal{P}_{cl,cv}(K)$ is u.s.c. and the set $G(K)$ is relatively compact in X. Then G has a fixed point in K.*

Lemma 1.51 (Covitz–Nadler [130]). *Let (X, d) be a complete metric space. If $N\colon X \to \mathcal{P}_{cl}(X)$ is a contraction, then $FixN \neq \phi$.*

1.7 Auxiliary Lemmas

We state the following generalization of Gronwall's lemma for a singular kernel.

Lemma 1.52 ([256]). *Let $v\colon [0,T] \to [0,+\infty)$ be a real function and $w(\cdot)$ a nonnegative, locally integrable function on $[0,T]$. Assume that there exist constants $a > 0$ and $0 < \alpha < 1$ such that*

$$v(t) \leq w(t) + a\int_0^t (t-s)^{-\alpha} v(s)ds\,.$$

Then there exists a constant $K = K(\alpha)$ such that

$$v(t) \leq w(t) + Ka\int_0^t (t-s)^{-\alpha} w(s)ds \quad \text{for every } t \in [0,T]\,.$$

Bainov and Hristova [75] introduced the following integral inequality of the Gronwall type for piecewise continuous functions that can be used in the sequel.

Lemma 1.53. *Let, for $t \geq t_0 \geq 0$, the following inequality hold:*

$$x(t) \leq a(t) + \int_{t_0}^t g(t,s)x(s)ds + \sum_{t_0<t_k<t} \beta_k(t)x(t_k)\,,$$

where $\beta_k(t)(k \in \mathbb{N})$ are nondecreasing functions for $t \geq t_0$, $a \in PC([t_0,\infty), \mathbb{R}_+)$, a is nondecreasing, and $g(t,s)$ is a continuous nonnegative function for $t, s \geq t_0$ and nondecreasing with respect to t for any fixed $s \geq t_0$. Then, for $t \geq t_0$,

$$x(t) \leq a(t) \prod_{t_0<t_k<t} (1 + \beta_k(t)) \exp\left(\int_{t_0}^t g(t,s)ds\right)\,.$$

Lemma 1.54 (Ascoli–Arzelà, [155]). *Let $A \subset C(J, \mathbb{R})$; A is relatively compact (i.e., $\overline{A}$ is compact) if*
1. *A is uniformly bounded, i.e., there exists $M > 0$ such that*
$$\|f(x)\| < M \quad \text{for every } f \in A \text{ and } x \in J\,;$$
2. *A is equicontinuous, i.e., for every $\epsilon > 0$ there exists $\delta > 0$ such that for each $x, \overline{x} \in J$, $\|x - \overline{x}\| \le \delta$ implies $\|f(x) - f(\overline{x})\| \le \epsilon$ for every $f \in A$.*

Set $J_0 := \{(x, y, s) \colon 0 \le s \le x \le a, y \in [0, b]\}$, $J_1 := \{(x, y, s, t) \colon 0 \le s \le x \le a, 0 \le t \le y \le b\}$, $D_1 := \frac{\partial}{\partial x}$, $D_2 := \frac{\partial}{\partial y}$, and $D_1 D_2 := \frac{\partial^2}{\partial x \partial y}$.

In the sequel we will make use of the following variant of the inequality for two independent variables due to Pachpatte.

Lemma 1.55 ([211]). *Let $w \in C(J, \mathbb{R}_+)$, $p, D_1 p \in C(J_0, \mathbb{R}_+)$, $q, D_1 q, D_2 q, D_1 D_2 q \in C(J_1, \mathbb{R}_+)$, and $c > 0$ a constant. If*
$$w(x, y) \le c + \int_0^x p(x, y, s) w(s, y) ds + \int_0^x \int_0^y q(x, y, s, t) w(s, t) dt ds$$
for $(x, y) \in [0, a] \times [0, b]$, then
$$w(x, y) \le c A(x, y) \exp\left(\int_0^x \int_0^y B(s, t) dt ds \right),$$
where
$$A(x, y) = \exp(Q(x, y))\,,$$
$$Q(x, y) = \int_0^x \left[p(s, y, s) + \int_0^s D_1 p(s, y, \xi) d\xi \right] ds\,,$$
and
$$B(x, y) = q(x, y, x, y) A(x, y) + \int_0^x D_1 q(x, y, s, y) A(s, y) ds$$
$$+ \int_0^y D_2 q(x, y, x, t) A(x, t) dt + \int_0^x \int_0^y D_1 D_2 q(x, y, s, t) A(s, t) dt ds\,.$$

From the preceding lemma and with $p \equiv 0$, we get the following lemma.

Lemma 1.56. *Let $w \in C(J, \mathbb{R}_+)$, $q, D_1 q, D_2 q, D_1 D_2 q \in C(J_1, \mathbb{R}_+)$, and let $c > 0$ be a constant. If*
$$a w(x, y) \le c + \int_1^x \int_1^y q(x, y, s, t) w(s, t) dt ds$$

for $(x, y) \in J$, *then*

$$w(x,y) \leq c \exp\left(\int_1^x \int_1^y B(s,t)\,dt\,ds \right),$$

where

$$B(x,y) = q(x,y,x,y) + \int_1^x D_1 q(x,y,s,y)\,ds + \int_1^y D_2 q(x,y,x,t)\,dt + \int_1^x \int_1^y D_1 D_2 q(x,y,s,t)\,dt\,ds\,.$$

Lemma 1.57 ([129]). *Let $D \subset BC$. Then D is relatively compact in BC if the following conditions hold:*

(a) D is uniformly bounded in BC,

(b) The functions belonging to D are almost equicontinuous on $[1,\infty) \times [1,b]$, i.e., equicontinuous on every compact of J.

(c) The functions from D are equiconvergent, that is, given $\epsilon > 0$ and $x \in [1,b]$, there is a corresponding $T(\epsilon, x) > 0$ such that $|u(t,x) - \lim_{t\to\infty} u(t,x)| < \epsilon$ for any $t \geq T(\epsilon, x)$ and $u \in D$.