Topological Fixed Point Theory of Multivalued Mappings

Topological Fixed Point Theory and Its Applications

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Topological Fixed Point Theory of Multivalued Mappings

Second edition

by

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Dedicated to the memory
of my Parents

TABLE OF CONTENTS

PREFACE

This book is an attempt to give a systematic presentation of results and methods which concern the fixed point theory of multivalued mappings and some of its applications. In selecting the material we have restricted ourselves to studying topological methods in the fixed point theory of multivalued mappings and applications, mainly to differential inclusions.

Thus in Chapter III the approximation (on the graph) method in fixed point theory of multivalued mappings is presented. Chapter IV is devoted to the homological methods and contains more general results, e.g. the Lefschetz Fixed Point Theorem, the fixed point index and the topological degree theory. In Chapter V applications to some special problems in fixed point theory are formulated. Then in the last chapter a direct applications to differential inclusions are presented. Note that Chapters I and II have an auxiliary character, and only results connected with the Banach Contraction Principle (see Chapter II) are strictly related to topological methods in the fixed point theory. In the last section of our book (see Section 75) we give a bibliographical guide and also signal some further results which are not contained in our monograph.

The author thanks several colleagues and my wife Maria who read and commented on the manuscript. These include J. Andres, A. Buraczewski, G. Gabor, A. Górka, M. Górniewicz, S. Park and A. Wieczorek.

The author wish to express his gratitude to P. Konstanty for preparing the electronic version of this monograph.

Lech Górniewicz

Toruń, June 1998

PREFACE TO THE SECOND EDITION

During the last decade a rapid development of multivalued methods can be observed in many branches of mathematics such as:

- topological and metric fixed point theory,
- multivalued nonlinear analysis,
- ordinary and patrial (deterministic and stochastic) differential inclusions,
- convex analysis,
- game theory,
- mathematical economics

(comp. [AnGo-M], [Au-M], [BFGJ-M], [BGM-M], [Bot1-M], [Bot2-M], [CV-M], [Cwi-M], [De1-M], [DMNZ-M], [Fry-M], [KOZ-M], [Kr2-M], [Mik2-M], [HuPa-M], [Me-M], [PeM-M], [Ski-M]).

Our monograph mainly concetrates on the topological fixed point theory of multivalued mappings. The second edition differs from the first one. Firstly, a completely new Chapter VII (Sections 76–85) is added. In this chapter new results obtained mainly in the last six years are presented. Chapters I–VI are in principal the same as in the first edition, but many changes and improvements are made. Moreover, in the references, there are added all new positions connected with topological fixed point theory for multivalued mappings, which appeared during last six years.

We believe that the second edition of our monograph is adequate to the current scientific status of the topological fixed point theory of multivalued mappings.

I am indebted to my young colleague R. Skiba who read the first edition and suggested many important improvements.

The author also wishes to express his gratitude to J. Szelatyńska and M. Czerniak who prepared the electronic version of the text.

Lech Górniewicz

Toruń, December 2005

CHAPTER I

BACKGROUND IN TOPOLOGY

In this chapter, we present a concise review of the requisite mathematical background. First we recall fundamental facts from geometric topology, later we discuss the part of homology theory related to the Vietoris mapping theorem and, finally, necessary information about the Lefschetz number.

Our main references for these topics are: [Bo-M], [Go1-M], [DG-M], [HW-M], [ES-M], [SP-M].

All topological spaces are assumed to be metric and all mappings are assumed to be continuous; $\mathbb{R}^n$ stands for the n-dimensional Euclidean space; by a Banach (normed) space we shall always understand a real Banach (normed) space and all mappings are assumed to be continuous.

1. Extension and embedding properties

We start this section with the following.

(1.1) Definition. We shall say that a space X *possesses an extension property* (written $X \in \mathrm{ES}$) provided that for every space Y, every closed $B \subset Y$, and every map $f\colon B \to X$, there exists an extension $\widetilde{f}\colon Y \to X$ of f onto Y, i.e. $\widetilde{f}(x) = f(x)$, for each $x \in B$; similarly, X *possesses a neighbourhood extension property* (written $X \in \mathrm{NES}$) provided that for every space Y every closed $B \subset Y$ and every $f\colon B \to X$ there exists an open neighbourhood U of B in Y and an extension $\widetilde{f}\colon U \to X$ of f onto U.

Of course, every ES-space is NES. Before we formulate more properties of these spaces we need the notion of a retract. Recall that a subset $A \subset X$ is called *the retract of* X if there exists a retraction $r\colon X \to A$, i.e. $r(x) = x$, for every $x \in A$. Observe that A is a retract of X if and only if the identity map id_A over A possesses a continuous extension onto X. It is also easy to see that if A is a retract of X then A is a closed subset of X. Similarly, we shall say that A is a *neighbourhood retract* of X if there exists an open subset $U \subset X$ such that $A \subset U$ and A is a retract of U.

Below we collect some simple but important properties of ES and NES-spaces.

(1.2) PROPERTIES.

(1.2.1) *If* X *is homeomorphic to* Z *and* $X \in \mathrm{ES}$ $(X \in \mathrm{NES})$, *then* $Z \in \mathrm{ES}$ $(Z \in \mathrm{NES})$;

(1.2.2) *If* $X \in \mathrm{ES}$ $(X \in \mathrm{NES})$ *and* A *is a retract of* X, *then* $A \in \mathrm{ES}$ $(A \in \mathrm{NES})$;

(1.2.3) *If* $X \in \mathrm{NES}$ *and* V *is an open subset of* X, *then* $V \in \mathrm{NES}$;

(1.2.4) *If* $X_1, \dots, X_n \in \mathrm{ES}$ $(X_1, \dots, X_n \in \mathrm{NES})$, *then the Cartesian product* $X = X_1 \times \dots \times X_k$ *of* $X_1, \dots, X_k$ *is an* ES-*space* (NES-*space*).

The proof of (1.2) is self-evident and therefore, is left to the reader. Note, that property (1.2.4) for ES-spaces can be formulated for arbitrarily many (even infinitely many) $X_1, \dots, X_k$ provided its Cartesian product is defined (we are considering metric spaces only!).

In a normed space E we shall understand by the convex hull, $\mathrm{conv}(A)$, of a subset $A \subset E$ the set of all points $y \in E$ of the form:

$$y = \sum_{i=1}^{n} t_i a_i,$$

where a_i are in A, and the coefficients t_i are greater or equal to zero $(t_i \geq 0)$ and their sum is equal to 1, $i = 1, \dots, n$. It is easy to see that $\mathrm{conv}(A)$ is equal to the intersection of all the convex subsets of E which contain A.

It is well known that the theorem of Tietze asserts that each real continuous function defined on a closed subset of a metric space X can be extended onto X. The generalization of this theorem proved by J. Dugundji (cf. [DG-M]) shows that for the range space we can take any normed space (in fact, even locally convex space) as well. More precisely, we have the following:

(1.3) THEOREM (Dugundji Extension Theorem). *If* E *is a normed space, then* $E \in \mathrm{ES}$. *Moreover, for every closed subset* B *of a metric space* Y *and for every map* $f\colon B \to E$ *there exists a continuous extension* $\widetilde{f}\colon Y \to E$ *such that:*

(1.3.1) $$\widetilde{f}(Y) \subset \mathrm{conv}(f(B)).$$

PROOF. Let $d\colon Y \times Y \to \mathbb{R}_+ = [0, +\infty)$ be a metric for Y. Cover $Y \setminus B$ by the balls

$$\left\{ B\left(x, \frac{1}{2}\,\mathrm{dist}(x, B)\right) \,\middle|\, x \in Y \setminus B \right\},$$

where

$$\mathrm{dist}(A, B) = \inf\{d(x, y) \mid x \in A \text{ and } y \in B\},$$
$$B(x, r) = \{y \in Y \mid d(x, y) < r\}.$$

By using the theorem of Stone, this cover has a locally finite refinement $\{V_\lambda \mid \lambda \in \Lambda\}$. For each V_λ choose a $B(v_\lambda, (1/2)\operatorname{dist}(v_\lambda, B)) \supset V_\lambda$, then choose $b_\lambda \in B$ such that $d(v_\lambda, b_\lambda) \leq 2 \operatorname{dist}(v_\lambda, B)$. Then we have

$$\operatorname{dist}(v_\lambda, B) \leq d(v_\lambda, v) + \operatorname{dist}(v, B) \leq \frac{1}{2} \operatorname{dist}(v_\lambda, B) + \operatorname{dist}(v, B),$$

so we get:

$$\text{(i)} \qquad \operatorname{dist}(v_\lambda, B) \leq 2 \operatorname{dist}(v, B), \quad \text{for each } v \in V_\lambda.$$

Moreover, we have:

$$\begin{aligned} d(b, b_\lambda) &\leq d(b, v) + d(v, v_\lambda) + d(v_\lambda, b_\lambda) \\ &\leq d(b, v) + \frac{1}{2} \operatorname{dist}(v_\lambda, B) + 2 \operatorname{dist}(v_\lambda, B) \\ &\leq d(b, v) + \operatorname{dist}(v, B) + 4 \operatorname{dist}(v, B). \end{aligned}$$

Therefore we obtain:

$$\text{(ii)} \qquad d(b, b_\lambda) \leq 6 d(b, v) \quad \text{for every } b \in B \text{ and } v \in V_\lambda.$$

Now, we consider a partition of unity ([1]) $\{\kappa_\lambda\}_{\lambda \in \Lambda}$ subordinated to the cover $\{V_\lambda\}_{\lambda \in \Lambda}$ and using points b_λ we are able to define the extension: $\widetilde{f}\colon Y \to E$ by letting:

$$\widetilde{f}(y) = \begin{cases} f(y) & \text{if } y \in B, \\ \displaystyle\sum_{\lambda \in \Lambda} \kappa_\lambda(y) f(b_\lambda) & \text{if } y \in Y \setminus B. \end{cases}$$

The function $\widetilde{f}$ is evidently continuous in every point $y \in Y \setminus B$, so only its continuity at the points of B needs to be proved. Let $b \in B$, and we let $f(b) \in W$ be an open set. Since E is normed and f is continuous on B, there is a convex C and $\delta > 0$ such that $f(B \cap B(b, \delta)) \subset C \subset W$. We are going to show that $\widetilde{f}(B(b, \delta/6)) \subset C \subset W$ which will prove the continuity of $\widetilde{f}$ at $b \in B$. Let y be any point of $B(b, \delta/6) \setminus B$; it belongs to only finitely many sets $V_{\lambda_1}, \ldots, V_{\lambda_n}$. Then $d(y, b) < \delta/6$ so, since $y \in V_{\lambda_i}$, we have $d(b, b_{\lambda_i}) < \delta$ by (ii). Therefore all the $b_{\lambda_i} \in B \cap B(a, \delta)$, consequently all $f(b_{\lambda_i}) \in C$ and because $\widetilde{f}(y) = \sum_{i=1}^n \kappa_{\lambda_i}(y) \cdot f(b_{\lambda_i})$ is a convex combination of points in C, we conclude that $\widetilde{f}(y) \in C$. Thus $\widetilde{f}(B(a, \delta/6)) \subset W$ and $\widetilde{f}$ is continuous at b. The fact that $\widetilde{f}(X) \subset \operatorname{conv}(f(B))$ is evident, the proof is complete. □

As an immediate consequence of the above theorem we get:

([1]) For example we can take:

$$\kappa_\lambda(y) = \frac{\operatorname{dist}(y, Y \setminus V_\lambda)}{\sum_{\mu \in \Lambda} \operatorname{dist}(y, Y \setminus V_\mu)}, \quad y \in Y.$$

(1.4) COROLLARY. *Let C be a convex subset of a normed space E. Then $C \in$ ES.*

For the proof of (1.4) we use (1.3.1).

From (1.2) and (1.4) it follows that the class of ES-spaces is quite large. Now, we obtain the following corollary from (1.3).

(1.5) COROLLARY. *Let $S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$ be the unit sphere in $\mathbb{R}^{n+1}$. Then $S^n \in$ NES.*

PROOF. Let B be an arbitrary closed subset of a space Y and let $f\colon B \to S^n$ be a continuous mapping. Denote by $i\colon S^n \to \mathbb{R}^{n+1}$, $i(x) = x$, the inclusion map. We let:

$$\overline{f}\colon B \to \mathbb{R}^{n+1}, \quad \overline{f} = i \circ f.$$

Since $\mathbb{R}^{n+1} \in$ ES, there exists an extension $f_1\colon Y \to \mathbb{R}^{n+1}$ of $\overline{f}$ onto Y. We define:

$$U = f_1^{-1}(\mathbb{R}^{n+1} \setminus \{0\}).$$

Then U is an open subset of Y containing B. Finally, we define a map $\widetilde{f}\colon U \to S^{n+1}$, by putting

$$\widetilde{f}(y) = \frac{f_1(y)}{\|f_1(y)\|}, \quad \text{for } y \in U.$$

Then $\widetilde{f}$ is a continuous extension of $f\colon B \to S^n$ over U and the proof is completed.□

Now, we are going to express extension spaces (neighbourhood extension spaces) in terms of absolute retracts (absolute neighbourhood retracts).

Before we do it we shall prove an important embedding theorem.

(1.6) THEOREM (Arens–Eells Embedding Theorem). *Let X be a space. Then there exists a normed space E and an isometry $\Theta\colon X \to E$ on X into E such that $\Theta(X)$ is a closed subset of E.*

PROOF. Denote by d the metric in X. Let Σ be the set of all finite subsets of X. Taking Σ with the discrete topology, let $C(\Sigma)$ be the Banach space of all bounded (of course continuous) real-valued functions on Σ equipped with the sup norm. We first embed X isometrically into $C(\Sigma)$.

Choose a point $p \in X$ and for $x \in X$ let $f_x\colon \Sigma \to \mathbb{R}$ be the function

$$f_x(\xi) = \operatorname{dist}(x, \xi) - \operatorname{dist}(p, \xi).$$

Then $f_x \in C(\Sigma)$ because $|f_x(\xi)| = |\operatorname{dist}(x, \xi) - \operatorname{dist}(p, \xi)| \leq d(x, p)$ shows it is bounded.

The map $\Theta: X \to C(\Sigma)$ given by $\Theta(x) = f_x$ is an isometric embedding because

$$\|f_x - f_z\| = \sup_{\xi} |\operatorname{dist}(x,\xi) - \operatorname{dist}(z,\xi)| \le d(x,z)$$

and the sup is attained at $\xi = \{z\} \in \Sigma$.

Thus $\|\Theta(x) - \Theta(z)\| = d(x,z)$ and Θ is isometry. Now, we observe:

(i) $f_p(\xi) = 0$ for every $\xi \in \Sigma$,

(ii) for each $x \in X$, we have $f_x(\xi) = 0$ whenever $\xi \supset \{x,p\}$.

In particular $\Theta(X)$ contains the origin of $C(\Sigma)$.

Let E be the linear span of $\Theta(X)$ in $C(\Sigma)$; clearly E is a normed space which need not to be closed in $C(\Sigma)$; but $\Theta(Y) \subset E$ and we shall now show that $\Theta(Y)$ is closed in E.

Let $g \in E \setminus \Theta(X)$. Then $g = \sum_{i=1}^n \alpha_i f_{x_i}$ for suitable real α_i and $x_i \in X$ for $i = 1, \ldots, n$. To show that $\Theta(X)$ is closed in E, it is sufficient to get $B(g,\delta) \cap \Theta(X) = \emptyset$ for some $\delta > 0$. Let $\delta > 0$ be smaller than

$$\min\left\{\frac{1}{2}\|g\|, \frac{1}{2}\|g - f_{x_1}\|, \ldots, \frac{1}{2}\|g - f_{x_n}\|\right\}$$

and assume $\|g - f_x\| < \delta$ for some $f_x \in \Theta(X)$.

Then for this f_x we would have $\|f_x - f_{x_i}\| \ge \delta$ and $\|f_x\| = \|f_x - f_p\| \ge \delta$; therefore, because Θ is an isometry, $d(x,x_i) \ge \delta$ and $d(x,p) \ge \delta$. But $\|g - f_x\| \ge |g(\xi) - f_x(\xi)|$ for every $\xi \in \Sigma$. In particular for $\xi = \{x_1, \ldots, x_n, p\}$ we have $g(\xi) = 0$ by (ii). So

$$\|g - f_x\| \ge |f_x(\xi)| = \operatorname{dist}(x; \{x_1, \ldots, x_n, p\})$$

contradicting the assumption that $\|g - f_x\| < \delta$. Thus $B(g,\delta) \cap \Theta(X) = \emptyset$ and the proof is complete. □

Now, following K. Borsuk ([Bo-M]) we introduce the notion of absolute retracts (AR-spaces) and the notion of absolute neighbourhood retracts (ANR-spaces).

It is useful to use the notion of an r-map. A mapping $r: Z \to T$ is called *an r-map* provided there exists a map $s: T \to Z$ such that $r \circ s = \mathrm{id}_T$, i.e. $(r \circ s)(t) = t$ for every $t \in T$.

We shall also use the notion of an embedding. Namely, by an embedding of a space X into Y we shall understand any homeomorphism $h: X \to Y$ from X to Y such that $h(X)$ is a closed subset of Y.

Now, we are able to formulate the following:

(1.7) Definition. We shall say that $X \in \mathrm{AR}$ ($X \in \mathrm{ANR}$) if and only if for any space Y and for any embedding $h: X \to Y$ the set $h(X)$ is a retract of Y ($h(X)$ a neighbourhood retract of Y).

In view of (1.6), we obtain:

(1.8) Proposition.

(1.8.1) $X \in \mathrm{AR}$ *if and only if* X *is an* r*-image of some normed space* E*;*

(1.8.2) $X \in \mathrm{ANR}$ *if and only if* X *is an* r*-image of some open subset* U *of a normed space* E*.*

Proof. For the proof of (1.8.1) it is sufficient to show that, if there exists an r-map $r\colon E \to X$ from a normed space onto X, then $X \in \mathrm{AR}$. Let Y be an arbitrary space and $h\colon X \to Y$ be an embedding. We have to prove that $h(X)$ is a retract of Y. Let us denote $h(X)$ by B. So B is a closed subset of Y. We define $f\colon h(X) \to E$ by putting:

$$f = s \circ h^{-1} \quad \text{where } r \circ s = \mathrm{id}_X.$$

Since $E \in \mathrm{ES}$ we have the extension $\widetilde{f}\colon Y \to E$ of f onto Y. Then the map $\varrho\colon Y \to h(X)$ given by $\varrho = r \circ \widetilde{f}$ is the needed retraction and the proof of (1.8.1) is complete. The proof of (1.8.2) is strictly analogous and therefore we leave it to the reader. □

Now, we shall prove the main result of this section.

(1.9) Theorem.

(1.9.1) $$X \in \mathrm{ES} \Leftrightarrow X \in \mathrm{AR},$$

(1.9.2) $$X \in \mathrm{NES} \Leftrightarrow X \in \mathrm{ANR}.$$

Proof. Since the proof of (1.9.2) is analogous to the proof of (1.9.1), we will restrict our considerations to the proof of (1.9.1) only.

First, assume that $X \in \mathrm{ES}$. To prove that $X \in \mathrm{AR}$ let $h\colon X \to Y$ be an embedding. We let $B = h(X)$. Then B is a closed subset of Y. We consider the map $f\colon B \to X$ defined by $f = h^{-1}$. Since $X \in \mathrm{ES}$ there exists an extension $\widetilde{f}\colon Y \to X$ of f onto Y. Then the map $r\colon Y \to h(X)$ defined as: $r = h \circ \widetilde{f}$ is a retraction from Y onto $h(X)$ what proves that $X \in \mathrm{AR}$.

Now, assume that $X \in \mathrm{AR}$. We would like to prove that $X \in \mathrm{ES}$. Let B be a closed subset of Y and let $f\colon B \to X$ be a mapping. For the proof it is sufficient to define the extension $\widetilde{f}\colon B \to X$ of f onto Y.

Since $X \in \mathrm{AR}$, in view of (1.8.1) there exists a normed space E and an r-map $r\colon E \to X$ (i.e. there exists $s\colon X \to E$ such that $r \circ s = \mathrm{id}_X$). We define $f_1\colon B \to E$ by the formula: $f_1 = s \circ f$

Since $E \in \mathrm{ES}$ we obtain an extension $\widetilde{f_1}\colon Y \to E$ of f_1 onto Y. Then the map $\widetilde{f}\colon Y \to X$ given by $\widetilde{f} = r \circ f_1$ is an extension of f onto Y and the proof of (1.8.1) is complete. □

We suggest the reader prove (1.8.2).

In view of (1.9) we see that all properties of ES and NES spaces remain valid for AR and ANR spaces respectively. In particular, r-image of AR-space (ANR-space) is AR space (ANR-space) again.

The following theorem gives a relation between the AR and ANR properties of two sets, of their union and of their common part.

(1.10) THEOREM. *Suppose that the space X is the union of two closed subsets X_1 and X_2 and let $X_0 = X_1 \cap X_2$. Then:*

(1.10.1) *If $X_0, X_1, X_2 \in \mathrm{AR}$, then $X \in \mathrm{AR}$,*
(1.10.2) *If $X_0, X_1, X_2 \in \mathrm{ANR}$, then $X \in \mathrm{ANR}$,*
(1.10.3) *If $X, X_0 \in \mathrm{AR}$, then $X_1, X_2 \in \mathrm{AR}$,*
(1.10.4) *If $X, X_0 \in \mathrm{ANR}$, then $X_1, X_2 \in \mathrm{ANR}$.*

PROOF. In order to prove (1.10.1) it is sufficient to show that if X is a closed subset of a space Z and $X_0, X_1, X_2 \in \mathrm{AR}$, then X is a retract of Z. Let us set

$$\begin{aligned} Z_0 &= \{z \in Z \mid \operatorname{dist}(z, X_1) = \operatorname{dist}(z, X_2)\}, \\ Z_1 &= \{z \in Z \mid \operatorname{dist}(z, X_1) < \operatorname{dist}(z, X_2)\}, \\ Z_2 &= \{z \in Z \mid \operatorname{dist}(z, X_1) > \operatorname{dist}(z, X_2)\}. \end{aligned}$$

It is evident that $Z = Z_0 \cup Z_1 \cup Z_2$, the set $X_0 \subset Z_0$ is closed in Z_0 and $X_i \cap Z_0 = X_0$ for $i = 1, 2$. Hence there exists a retraction $r_0 \colon Z_0 \to X_0$. Moreover, the set $X_i \cup Z_0$ is closed in $Z_i \cup Z_0$ for $i = 1, 2$ and we infer from (1.9) that the map $r_i \colon X_i \cup Z_0 \to X_i$, $i = 1, 2$, given by the formula:

$$r_i(z) = \begin{cases} z & \text{for every } z \in X_i, \\ r_0(z) & \text{for every } z \in Z_0, \end{cases}$$

has a continuous extension $f_i \colon Z_i \cup Z_0 \to X_i$ over $Z_i \cup Z_0$. It is sufficient to set $r(z) = f_i(z)$ for $z \in Z_i \cup Z_0$, $i = 1, 2$, to obtain a retraction $r \colon Z \to X$.

Passing to (1.10.2), we need to show that if X is a closed subset of space Z and if $X_0, X_1, X_2 \in \mathrm{ANR}$, then there exists in Z an open neighbourhood U of the set X such that X is a retract of U. Consider the sets Z_0, Z_1, Z_2 defined in the proof of (1.10.1). Then X_0 is a closed subset of Z_0 and hence there is a neighbourhood W_0 of X_0 in Z_0 and a retraction $r_0 \colon W_0 \to X_0$. Setting

$$r_i(z) = \begin{cases} r_0(z) & \text{for every } z \in W_0, \\ z & \text{for every } z \in X_i, \end{cases}$$

we obtain a retraction r_i of the set $X_i \cup W_0$ (which is closed in $Z_i \cup Z_0$) onto the set X_i, $i = 1, 2$. Since $X_i \in \mathrm{ANR}$, we infer by (1.9) that there exists a continuous extension r_i' of r_i onto a neighbourhood V_i of $X_i \cup W_0$ in $Z_0 \cup Z_i$ with r_i having values in X_i.

It is clear that V_i contains a closed neighbourhood U_i of X_i in the space $Z_0 \cup Z_i$ such that $U_i \cap Z_0 \subset W_0$. Then the formula:

$$r(z) = r_i'(z) \quad \text{for } z \in U_i,\ i = 1, 2$$

defines a retraction r of the set $U = U_1 \cup U_2$, which is an open neighbourhood of X in Z, onto the set X. Thus the proof of (1.10.2) is complete.

In order to prove (1.10.3) let us observe that the condition $X_0 \in \mathrm{AR}$ implies that there exists a retraction $r_i\colon X_i \to X_0$, $i = 1, 2$. If we set

$$r(x) = \begin{cases} x & \text{for } x \in X_1, \\ r_2(x) & \text{for } x \in X_2, \end{cases}$$

then we obtain a retraction $r\colon X \to X_1$. Since $X \in \mathrm{AR}$ we have from (1.9) and (1.2.2) that $X_1 \in \mathrm{AR}$. Similar reasoning shows that $X_2 \in \mathrm{AR}$.

In order to prove (1.10.4) let us observe that $X_0 \in \mathrm{ANR}$ implies that there exists an open neighbourhood U_0 of X_0 in X such that, for $i = 1, 2$ there exists a retraction $r_i\colon X_i \cap U_0 \to X_0$. Setting

$$r(x) = \begin{cases} x & \text{for } x \in X_1, \\ r_2(x) & \text{for } x \in X_2 \cap U_0, \end{cases}$$

we obtain a retraction $r\colon U_0 \cup X_1 \to X_1$. Since $U_0 \cup X_1$ is an open neighbourhood of X_1 in X, and $X \in \mathrm{ANR}$ it follows by (1.9) and (1.2.1) that $X_1 \in \mathrm{ANR}$. A similar argument shows that $X_2 \in \mathrm{ANR}$ and the proof of theorem (1.10) is complete. □

Let us remark that an important application of (1.10) is to show that every polyhedron is an absolute neighbourhood retract.

First, we are going to recall the notion of a polyhedron.

A Hilbert space l_2 consists of all (real) sequences $x = \{x_k\}$ for which the series $\sum_{k=1}^{\infty} x_k^2$ converges. Then x_k is called the k-th coordinate of $x = \{x_k\}$. The space l_2 becomes a metric space, if we define the distance $d(x, y)$ between points $x = \{x_k\}$, $y = \{y_k\}$ of l_2 by the formula

$$d(x, y) = \sqrt{\sum_{k=1}^{\infty} (x_k - y_k)^2}.$$

The subset of l_2 consisting of all points $x = \{x_k\}$ with $0 \le x_k \le 1/k$, for $k = 1, 2, \ldots$ is denoted by K^{ω} and is called the *Hilbert cube*. Let us recall that K^{ω} is a compact subset of l_2. Of course l_2 possess also the linear structure over the field of real numbers $\mathbb{R}$. Namely, if $x = \{x_k\},\ y = \{y_k\} \in l_2$ and $t, s \in \mathbb{R}$ then we have:

$$t \cdot x + s \cdot y = \{tx_k + sy_k\}.$$

moreover, l_2 is a Banach space with the norm defined as follows:

$$\|x\| = \sqrt{\sum_{k=1}^{\infty} x_k^2}, \quad \text{for } x = \{x_k\} \in l_2.$$

Since $\{x_k\} \in K^\omega$ and $\{y_k\} \in K^\omega$ implies that $\{tx_k + (1-t)y_k\} \in K^\omega$, i.e. the Hilbert cube K^ω is a convex subset of l_2 too.

Euclidean n-dimensional space $\mathbb{R}^n$ consists of all points $\{x_k\}$ of l_2 such that $x_k = 0$, for each $k > n$. It follows, in particular, that $\mathbb{R}^n \subset \mathbb{R}^m$ for every $n < m$.

A system $x^i = \{x_k^i\}$, $i = 0, \dots, m$ of points of l_2 is said to be *affine independent* provided that the linear combination

$$t_0x^0 + t_1x^1 + \ldots + t_mx^m \quad \text{where } t_0 + \ldots + t_m = 1$$

is equal to 0 only if all coefficients t_i vanish. Then the set $\sigma^m = (x^0, \dots, x^m)$ consisting of all points x of l_2 of the form

$$x = s_0x^0 + \ldots + s_mx^m, \quad \text{where } s_i \geq 0 \text{ and } s_0 + \ldots + s_m = 1$$

is called an *m-dimensional geometric simplex.*

In what follows we shall denote by Δ^m the m-dimensional standard simplex, we let:

$$\Delta^m = (e^0, \dots, e^m),$$

where $e^0 = (0, 0, 0 \ldots), e^1 = (1, 0, \ldots), \dots, e^m = (0, \dots, 0, \underbrace{1}_{m\text{-th}}, 0 \ldots)$.

We leave to the reader to prove that:

$$\Delta^m = \operatorname{conv}(\{e^0, \dots, e^m\}).$$

In view of the Dugundji extension theorem, we see that $\Delta^m \in \mathrm{AR}$, $m = 0, 1, \ldots$ A subset $A \subset l_2$ is a *geometric polyhedron* if it is the union of a finite number of geometric simplexes. A metric space X is called a *polyhedron* if there exists a geometric polyhedron A such that X is homeomorphic to A. We have:

(1.11) Corollary.

(1.11.1) $K^\omega \in$ AR,

(1.11.2) *If X is a polyhedron, then $X \in$* ANR.

Proof. Observe that (1.11.1) follows from the fact that K^ω is a convex set (cf. (1.4) and (1.9)). For the proof of (1.11.2) assume that X is homeomorphic to $A \subset l_2$, where A is a geometric polyhedron. In view of (1.9) and (1.2.1) it is sufficient to show that $A \in$ ANR.

We can assume that $A = \bigcup\{\sigma_i,\ i = 1, \dots, l\}$, where σ_i are geometric simplexes.

If $l = 1$ or $l = 2$ then our corollary follows immediately from (1.10) because $\sigma_i \in$ AR. So by using (1.10) we can prove (1.11.2) by induction. □

It is well known that every compact metric space can be embedded into K^ω. The above result was proved by Urysohn (see [DG1-M]). We shall end this section by stating the characterisation of compact AR and ANR-spaces.

Namely, if we take Urysohn Theorem in the place of the Arens–Eells Embedding Theorem then analogously to the proof of (1.8) we can prove the following.

(1.12) Theorem. *Compact* AR*-spaces are precisely the r-images of the Hilbert cube K^ω. Compact* ANR*-spaces coincide with r-images of open subsets of the Hilbert cube K^ω.*

We shall also make use of the following. Let E be a normed space and $K, U \subset E$ be such that K is compact, U is open in E and $K \subset U$.

(1.13) Proposition ([Gi]). *There exists a compact* ANR*-space A such that*

$$K \subset A \subset U.$$

Note that for E to be a Banach space Proposition (1.13) one can get directly from (1.10) and Mazur's convexification theorem saying that the closed convex ball of a compact set in a Banach space is compact, too. In the general case when E is an arbitrary normed space the proof was given by J. Girolo ([Gi]). Note that a version of (1.13) we shall prove in Section 3 (cf. Lemma (3.5)).

2. Homotopical properties of spaces

The notion of homotopy plays an important role in geometric topology. In what follows by $[0, 1]$ we shall denote the closed unit interval in $\mathbb{R}$.

(2.1) Definition. Consider two maps $f, g\colon X \to Y$. We shall say that f is *homotopic to g* (written $f \sim g$), if there exists a mapping $h\colon X \times [0, 1] \to Y$ such that:

$$h(x, 0) = f(x) \quad \text{and} \quad h(x, 1) = g(x) \quad \text{for every } x \in X.$$

In what follows the mapping h is called the homotopy joining f and g.

The notion of homotopy can be reinterpreted in terms of the extension property. Namely, let us consider a closed subset $(X \times \{0\}) \cup (X \times \{1\})$ of $X \times [0,1]$ and a map $\overline{f}: X \times \{0\} \cup X \times \{1\} \to Y$ defined as follows:

$$\overline{f}(x,t) = \begin{cases} f(x) & \text{for } t = 0, \\ g(x) & \text{for } t = 1. \end{cases}$$

One can say that $f \sim g$ if and only if $\overline{f}$ possesses an extension $\widetilde{f}$ over $X \times [0,1]$.

Then, of course, $\widetilde{f}$ is a homotopy joining f and g. From (1.10) we infer:

(2.2) Proposition. *If $Y \in \mathrm{AR}$, then any two mappings $f, g: X \to Y$ are homotopic.*

For given X and Y we shall denote by $C(X,Y)$ the set of all (continuous) mappings from X to Y. We have:

(2.3) Proposition. *The relation "$\sim$" is an equivalence relation in $C(X,Y)$.*

Proof. In order to prove that for every $f: X \to Y$ we have $f \sim f$, it is sufficient to consider the homotopy $h: X \times [0,1] \to Y$ defined by the formula:

$$h(x,t) = f(x) \quad \text{for every } x \in X \text{ and } t \in [0,1].$$

Assume that $f \sim g$. We want to prove that $g \sim f$. Let h be a homotopy joining f and g, then the map $\widetilde{h}: X \times [0,1] \to Y$ given by:

$$\widetilde{h}(x,t) = h(x, 1-t) \quad \text{for every } x \in X \text{ and } t \in [0,1]$$

is a homotopy joining g and f.

Finally, assume that $f \sim g$ and $g \sim g_1$. We have to prove that $f \sim g_1$.

In order to do that assume that h_1 is a homotopy joining f and g and h_2 is a homotopy joining g and g_1. We let $h: X \times [0,1] \to Y$ by putting:

$$h(x,t) = \begin{cases} h_1(x, 2t) & \text{for } x \in X \text{ and } 0 \le t \le 1/2, \\ h_2(x, 2t-1) & \text{for } x \in X \text{ and } 1/2 < t \le 1. \end{cases}$$

It is easy to see that h is a homotopy joining f with g_1 and the proof of Proposition (2.3) is completed. □

We define:

$$[X,Y] = C(X,Y)|_{\sim}.$$

Then $[X,Y]$ is called the set of all homotopy classes under the homotopical equivalence.

(2.4) Corollary. *If* $Y \in \mathrm{AR}$*, then* $[X, Y]$ *is a singleton.*

Corollary (2.4) immediately follows from (2.2).

(2.5) Definition. Two spaces X and Y are said to be *homotopically equivalent* (written $X \sim Y$) provided. There are two maps $f\colon X \to Y$ and $g\colon Y \to X$ such that:

$$g \circ f \sim \mathrm{id}_X, \tag{2.5.1}$$

$$f \circ g \sim \mathrm{id}_Y. \tag{2.5.2}$$

Of course, if two spaces X and Y are homeomorphic then they are homotopically equivalent.

The following notion is especially important in our considerations.

(2.6) Definition. A space X is called *contractible* provided it is equivalent to the one-point space $\{p\}$, i.e. $X \sim \{p\}$.

One can easily see that the space X is contractible if and only if there exists a point $x_0 \in X$ such that:

$$\mathrm{id}_X \sim g,$$

where $g\colon X \to X$ is defined by $g(x) = x_0$ for every $x \in X$. Moreover, the above consideration does not depend on the choice of the point $x_0 \in X$ because every two one-point spaces are homeomorphic and hence homotopically equivalent.

From this we deduce:

(2.7) Proposition. *If* $X \in \mathrm{AR}$*, then* X *is a contractible space.*

Note, that the converse to (2.7) is not true. Namely, consider so called the comb space $C \subset \mathbb{R}^2$, i.e.

$$C = \left\{ (x, y) \in \mathbb{R}^2 \;\middle|\; \left(x = 0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \text{ and } 0 \le y \le 1 \right) \right. \\ \left. \text{or } (x \in [0,1] \text{ and } y = 0) \right\}.$$

Evidently, C is a contractible space but id_C can not be extended over $\mathbb{R}^2$, so C is not an AR-space, the respective homotopy can defined as follows:

$$h((x, y), t) = \begin{cases} (x, (1-t)y) & \text{for } y \neq 0, \\ ((1-t)x, 0) & \text{for } y = 0. \end{cases}$$

As we already know ANR-spaces need not be contractible (compare S^n or a non contractible polyhedron). We are going to explain what type of contractibility possess ANR-spaces.

A space X is said to be *locally contractible at a point* $x_0 \in X$ provided for each $\varepsilon > 0$, there exists $\delta > 0$ ($\delta < \varepsilon$) and a homotopy $h\colon [0,1] \times B(x_0, \delta) \to B(x_0, \varepsilon)$ such that:

$$h(x,0) = x_0 \quad \text{and} \quad h(x,1) = x, \quad \text{for every } x \in B(x_0, \delta);$$

in other words the ball $B(x_0, \delta)$ is contractible in $B(x_0, \varepsilon)$.

It is evident that the local contractibility at a point x_0 implies local arcwise connectivity at this point. A space X is said to be locally contractible if it is locally contractible at each of its points. For the sake of brevity we shall write $X \in \mathrm{LC}$ if X is a locally contractible space. We see that every open subset of a locally contractible space is itself locally contractible.

Now, let us observe that open subsets of normed spaces are locally contractible because the open balls are convex. On the other hand it is easy to see that every r-image of a locally contractible space is locally contractible. Summing up the above we obtain:

(2.8) Proposition. *If* $X \in \mathrm{ANR}$, *then* $X \in \mathrm{LC}$.

Observe that the comb space $C \subset \mathbb{R}^2$ is not locally contractible, so $C \notin \mathrm{ANR}$. The example of an LC-space which is not an ANR-space is not trivial. We recommend Chapter V, Sections 10 and 11 in [Bo-M] for details.

In the case of compact metric spaces it is useful to consider uniformly locally contractible spaces (ULC-spaces). Namely, a compact metric space (A, d) is said to be a ULC-space provided for every $\varepsilon > 0$ there is $\delta > 0$ and a map

$$g\colon [0,1] \times \{(a,b) \in A \times A \mid d(a,b) < \delta\} \to A$$

such that

$$g(0,a,b) = a, \quad g(1,a,b) = b, \quad g(t,a,a) = a$$

and

$$\mathrm{diam}\{g(a,b,t) \mid t \in [0,1]\} < \varepsilon,$$

where

$$\mathrm{diam}\{g(a,b,t) \mid t \in [0,1]\} \\ = \sup\{d(c,d) \mid c = g(a,b,t_1),\ d = g(a,b,t_2),\ t_1, t_2 \in [0,1]\}.$$

A compact space (A, d) is called k-ULC, $k \geq 1$, provided for every $\varepsilon > 0$ there exists $\delta > 0$ such that any map $\overline{g}\colon S^k \to A$, with $\mathrm{diam}(f(S^k)) < \delta$, is homotopic to a constant map by a homotopy $h\colon S^k \times [0,1] \to A$ such that $\mathrm{diam}(h(S^k \times [0,1])) < \varepsilon$.

Of course every ULC-space is a LC-space. In view of (2.8) and compactness of A we conclude:

(2.9) PROPOSITION. *If A is a compact* ANR-*space, then A is a* ULC-*space.*

PROOF. In view of (1.12) we can assume that A is a neighbourhood retract of the Hilbert cube K^ω. Let U be an open subset of K^ω containing A and let $r\colon U \to A$ be a retraction. Let $\eta > 0$ denote the distance from A to $K^\omega \setminus V$, where V is an open neighbourhood of A in K^ω such that the closure $\overline{V}$ of V in K^ω is contained in U and

$$\eta = \operatorname{dist}(A, K^\omega \setminus V) = \inf\{\|x-y\| \mid x \in A \text{ and } y \in K^\omega \setminus V\}.$$

By uniform continuity of r on V there exists $\delta > 0$, $\delta < \eta$, such that $y, z \in V$ and $\|z - y\| < \delta$ then $\|r(y) - r(z)\| < \varepsilon$. For $x, x' \in A$ with $\|x - x'\| < \delta$ define:

$$g(x, x', t) = r((1-t)x + tx')$$

and the Proposition (2.9) is proved. □

In fact, it is possible to show the following result:

(2.10) THEOREM. *Every compact* ANR-*space A is homotopically equivalent to some polyhedron.*

Theorem (2.10) was proved by West (see [Bo-M]). The proof is quite difficult and we are not able to present it here.

It is known that every compact metric space can be represented as an intersection of a decreasing sequence of compact ANRs. Below we shall characterize the compact metric spaces which can be written as the intersection of a decreasing sequence of compact ARs.

(2.11) DEFINITION. A compact nonempty space is called *an R_δ set* provided there exists a decreasing sequence $\{A_n\}$ of compact absolute retracts such that:

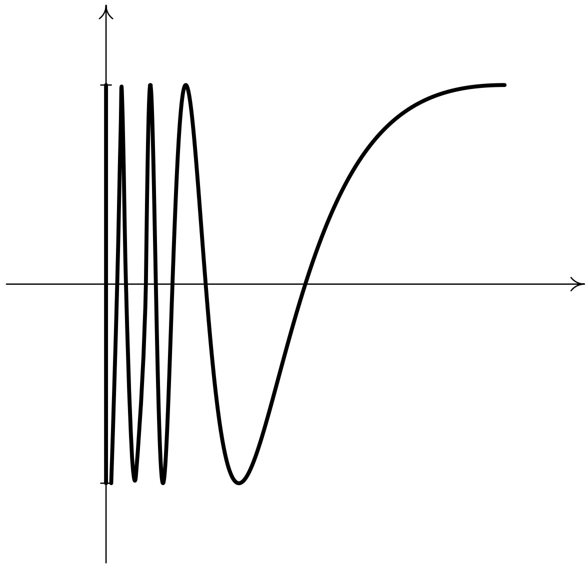
$$A = \bigcap_{n \geq 1} A_n.$$

Note that any intersection of a decreasing sequence of R_δ-sets is R_δ. Observe that A is not AR-space and even ANR-space in general. More, A need not be contractible in general.

(2.12) EXAMPLE. We shall construct an R_δ-space which is not contractible. Let $f\colon (0, (1/\pi)] \to \mathbb{R}$ be a function defined as follows:

$$f(x) = \sin\frac{1}{x}.$$

Let $B = \{(x,y) \in \mathbb{R}^2 \mid y = f(x),\ x \in (0, (1/\pi)\}$, $C = \{(x,y) \in \mathbb{R}^2 \mid x = 0 \text{ and } -1 \leq y \leq 1\}$, and $A = B \cup C$. We have:



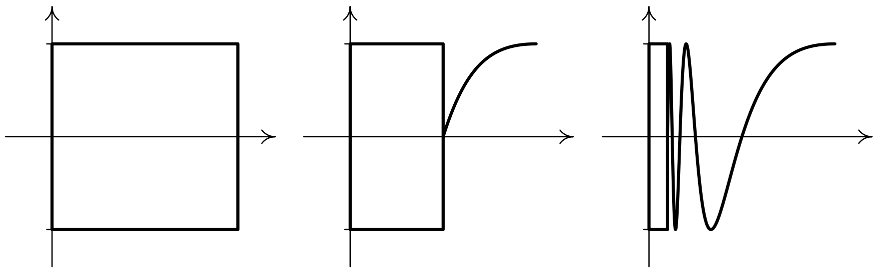
Of course A is not contractible space (in fact it is not locally contractible!).

We let:

$$A_n = \left[0, \frac{1}{n\pi}\right] \times [-1,1] \cup B_n,$$

where

$$B_n = \left\{(x,y) \in \mathbb{R}^2 \;\middle|\; \frac{1}{n\pi} \le x \le \frac{1}{\pi} \text{ and } y = f(x)\right\}, \quad \text{i.e.}$$



then $A = \bigcap_{n \ge 1} A_n$. The fact that A_n is an AR-space for every n follows, for example, from (1.10.1).

Let A be a compact subset of X. We will say that A is ∞-*proximally connected* subset of X if for every $\varepsilon > 0$ there exists $\varepsilon > \delta > 0$ such that for every $n = 0, 1, \ldots$ and for every map $g\colon \partial\Delta^{n+1} \to O_\delta(A)$ there is a map $\widetilde{g}\colon \Delta^{n+1} \to O_\varepsilon(A)$ such that $g(x) = \widetilde{g}(x)$ for every $x \in \partial\Delta^{n+1}$, where $\partial\Delta^{n+1}$ stands for the boundary of Δ^{n+1} and $O_\varepsilon(A) = \{y \in X \mid \mathrm{dist}(y, A) < \varepsilon\}$ is an ε-hull of A in X.

It is easy to see that we can replace in the above definition Δ^{n+1} by the unit ball K^{n+1} in $\mathbb{R}^{n+1}$ and $\partial\Delta^{n+1}$ by the unit sphere S^n.

First we shall formulate the following characterization theorem proved by D. M. Hyman in 1969 (see [Hy]).

(2.13) THEOREM. *Let $X \in$ ANR and $A \subset X$ be a compact nonempty subset. Then the following statements are equivalent:*

(2.13.1) *A is an R_δ-set,*
(2.13.2) *A is an intersection of a decreasing sequence $\{A_n\}$ of compact contractible spaces,*
(2.13.3) *A is ∞-proximally connected,*
(2.13.4) *for every $\varepsilon > 0$ the set A is contractible in $O_\varepsilon(A) = \{x \in X \mid \operatorname{dist}(x, A) < \varepsilon\}$.*

First observe that as an immediate consequence of (2.13.2) we obtain.

(2.14) COROLLARY. *An intersection of a decreasing sequence of R_δ-sets is again an R_δ-set.*

We shall make use from the following:

(2.15) PROPOSITION (cf. [BrGu]). *Let $\{A_n\}$ be a sequence of compact* AR*s contained in X, and let A be a subset of X such that the following conditions hold:*

(2.15.1) *$A \subset A_n$ for every n;*
(2.15.2) *A is the set-theoretic limit of the sequence $\{A_n\}$;*
(2.15.3) *for each open neighbourhood U of A in X there is a subsequence $\{A_{n_i}\}$ of $\{A_n\}$ such that $A_{n_i} \subset U$ for every n_i.*

Then A is an R_δ.

Finally, we shall come back to the Proposition (2.9). Namely, we would like to point out properties of mappings into compact ANR-spaces which are very important in the theory of fixed points.

Let $f, g: Y \to X$ be two mappings and let d be a metric in X. We shall say that f and g are ε-close (written $f \sim_\varepsilon q$) provided for every $y \in Y$ we have:

$$d(f(y), g(y)) < \varepsilon.$$

We shall prove the following

(2.16) THEOREM. *Let X be a compact* ANR*-space. Then there exists $\sigma_0 > 0$ such that for every $0 < \sigma < \sigma_0$ and for every two mappings $f, g: Y \to X$, if $f \sim_\sigma g$, then $f \sim g$* (f is homotopic to g).

PROOF. In fact, it is sufficient to put in (2.9) $\varepsilon = 1$ and then the obtained σ_0 (for $\varepsilon = 1$) is a needed number. It is convenient to consider also the non-compact case. Of course the notion of contractability has a sense for arbitrary space X not necessary compact (see Definition (2.6)). The same is true in the case of locally contractible sets. □

Because an intersection of an decreasing sequence of non-compact sets can be empty, we see that this is no longer true for R_δ sets. Below we shall formulate a non-compact version of ∞-proximally connected sets.

(2.17) DEFINITION. Let X be a space and $K \subset X$ be a closed (not necessary compact) subset of X. The set K is ∞-proximally connected in X (written $K \in \mathrm{PC}^\infty_X$) provided for every open neighbourhood U of K in X there exists an open neighbourhood $V \subset U$ of K in X such that for every $n = 0, 1, \ldots$ and for every map $g\colon \partial\Delta^{n+1} \to V$ there exists a mapping $\widetilde{g}\colon \Delta^{n+1} \to U$ such that $\widetilde{g}(x) = g(x)$ for every $x \in \partial\Delta$.

Since for any open neighbourhood W of compact $K \subset X$ there is $\varepsilon > 0$ such that $O_\varepsilon(K) \subset W$, we see that Definition (2.17) is equivalent with formulated above. Therefore we shall use the notation $K \in m\mathrm{PC}^\infty_X$ if U and V will be replaced by $O_\varepsilon(K)$ and $O_\delta(K)$.

Of course, we have:

(2.18) If K is compact, then $K \in m\mathrm{PC}^\infty_X$ if and only if $K \in \mathrm{PC}^\infty_X$.

Generally, $\mathrm{PC}^\infty_X \not\subset m\mathrm{PC}^\infty_X$ and $m\mathrm{PC}^\infty_X \not\subset \mathrm{PC}^\infty_X$.

(2.19) EXAMPLE. Consider the set $K \subset \mathbb{R}^2$ defined as follows:

$$K = \{(x,y) \in \mathbb{R}^2 \mid y = 0 \text{ and } x \geq 1\} \cup \{(x,y) \in \mathbb{R}^2 \mid x = 1 \text{ and } 0 \leq y \leq 1\} \cup \left\{(x,y) \in \mathbb{R}^2 \;\middle|\; x \geq 1 \text{ and } y = \frac{1}{x}\right\}.$$

Then the set K is homeomorphic to $\mathbb{R}$, hence $K \subset \mathrm{PC}^\infty_{\mathbb{R}^2}$. Moreover, $K \notin m\mathrm{PC}^\infty_{\mathbb{R}^2}$ since for every $\varepsilon > 0$ the set $O_\varepsilon(K)$ is homotopically equivalent to S^1.

(2.20) EXAMPLE. Consider $K \subset \mathbb{R}^2$ defined as follows:

$$K = \{(x,y) \in \mathbb{R}^2 \mid x = 0 \text{ and } y \geq 1\} \cup \left\{(x,y) \in \mathbb{R}^2 \;\middle|\; x > 0 \text{ and } y = \frac{1}{x}\right\}.$$

Then for every $\varepsilon > 0$ the set $O_\varepsilon(K)$ is contractible and hence $K \in m\mathrm{PC}^\infty_{\mathbb{R}^2}$. Since there is an open neighbourhood U of K in $\mathbb{R}^2$ such that $U = U_1 \cup U_2$ and $U_1 \cap U_2 = \emptyset$ we conclude that $K \notin \mathrm{PC}^\infty_{\mathbb{R}^2}$.

Observe that Theorem (2.13) is not true for non-compact K, but still we are able to prove the following:

(2.21) THEOREM. *Let K be a closed contractible subset of X and $X \in$ ANR. Then $X \in \mathrm{PC}^{\infty}_{X}$.*

Theorem (2.21) is a consequence of the following homotopy extension property.

(2.22) PROPERTY (Homotopy Extension Property). *Let $Y \in$ ANR, X be an arbitrary space and $K \subset X$ be a closed subset. Assume that $f, g\colon X \to Y$ are such that there is a homotopy $h\colon K \times [0,1] \to Y$ with $h(x,0) = f(x)$, $h(x,1) = g(x)$ for every $x \in K$. Then there exists a homotopy $H\colon X \times [0,1] \to Y$ such that $H(x,0) = f(x)$, for $x \in X$ and $H(x,t) = h(x,t)$, for $x \in K$ and $t \in [0,1]$.*

PROOF. Let $Z = X \times \{0\} \cup K \times [0,1]$ and $Z' = X \times [0,1]$. First we claim that for any open neighbourhood V of Z in Z' there is a map $\eta\colon Z' \to V$ such that $\eta(z) = z$ for every $z \in Z$.

To prove our claim let us assign to every $x \in K$ the segment $L_x = \{x\} \times [0,1]$. Since L_x is compact and V is an open neighbourhood of Z, then there exists an open neighbourhood U_x of X in K such that $U_x \times [0,1] \subset V$. Then the set

$$U = \bigcup_{x \in K} U_x$$

is an open neighbourhood of K in X and hence $U \times [0,1]$ is open in Z'. Now we consider the Urysohn function $\alpha\colon X \to [0,1]$ which takes the value 0 on $X \setminus U$ and 1 on K. If we set

$$\eta(x,t) = (x, \alpha(x)t) \quad \text{for } (x,t) \in Z'$$

then we obtain the desired map.

Now, consider the map $\overline{f}\colon Z \to Y$ defined by the conditions:

$$\overline{f}(x,0) = f(x) \quad \text{for } x \in X \quad \text{and} \quad \overline{f}(x,t) = h(x,t) \quad \text{for } x \in K \text{ and } t \in [0,1].$$

Since $Y \in$ ANR and Z is closed in Z', there exists a continuous extension $\widetilde{f}$ of $\overline{f}$ to a neighbourhood V of Z in Z' which has values in Y. Finally we define $H\colon Z' = X \times [0,1] \to Y$ by putting

$$H(x,t) = \widetilde{f}(\eta(x,t)) \quad \text{for } (x,1) \in X \times [0,1]$$

and we obtain the required homotopy. □

In what follows we shall need the following special version of (2.10).

(2.23) PROPOSITION ([Br1-M]). *Let X be a compact ANR and $\varepsilon > 0$. Then there exists a compact polyhedron P_ε and two maps $r_\varepsilon\colon P_\varepsilon \to X$, $s_\varepsilon\colon X \to P_\varepsilon$ and a homotopy $h_\varepsilon\colon X \times [0,1] \to X$ such that $h_\varepsilon(x,0) = r_\varepsilon(s_\varepsilon(x))$, $h_\varepsilon(x,1) = x$ and $\operatorname{diam}(h(\{x\} \times [0,1]) < \varepsilon$ for every $x \in X$. In such a case we say that P_ε ε-dominates X.*

3. Approximative and proximative retracts

First we generalize the notion of compact ANRs onto the case of compact approximate ANRs (written AANR). It is important to remember that for AANRs the global fixed point theorems, like Schauder or more generally the Lefschetz fixed point theorem still hold true. Since AANRs are not locally contractible the fixed point index for maps of AANRs is not possible.

Let A be a subset of X and let d be a metric in X. A mapping $r_\varepsilon\colon X \to A$ is said to be an ε-retraction, $\varepsilon > 0$, if for every $x \in A$ we have $d(x, r_\varepsilon(x)) < \varepsilon$.

Note that the retraction $r\colon X \to A$ is an ε-retraction for every $\varepsilon > 0$.

A subset $A \subset X$ is called an *approximative retract* of X provided for every $\varepsilon > 0$ there exists an ε-retraction $r_\varepsilon\colon X \to A$; A is called an *approximative neighbourhood retract* of X provided there exists an open neighbourhood U of A in X such that A is an approximative retract of U.

(3.1) Definition. A compact space X is called an *absolute approximative retract* (written $X \in$ AAR) provided that for every embedding $h\colon X \to Y$ the set $h(X)$ is approximative retract of Y; X is called an *absolute approximative neighbourhood retract* (written $X \in$ AANR) provided that for every embedding $h\colon X \to Y$ the set $h(X)$ is approximative neighbourhood retract of Y.

(3.2) Proposition. *For compact spaces we have:*

(3.2.1) $$\mathrm{AR} \subset A\mathrm{AR},$$

(3.2.2) $$\mathrm{ANR} \subset \mathrm{AANR}.$$

We prove the following:

(3.4) Theorem. *Assume that $X \in$ AANR. Then there exists a compact ANR-space Y such that X is homeomorphic to an approximative retract of Y.*

For the proof of (3.4) we need the following lemma:

(3.5) Lemma. *Let A be a compact subset of the Hilbert cube K^ω and let U be an open neighbourhood of A in K^ω. Then there exists a compact ANR-space K such that:*

$$A \subset K \subset U \subset K^\omega.$$

Proof. First observe that because K^ω is a convex subset of the space l_2 so every open ball in K^ω is convex. We cover A by a finite number of open balls $B(x_1, r_1), \ldots, B(x_k, r_k)$ in K^ω such that

$$\overline{B(x_1, r_1)} \cup \ldots \cup \overline{B(x_k, r_k)} \subset U \subset K^\omega.$$

Since every ball $\overline{B(x_i, r_i)}$ is a compact convex set in AR-space, from theorem (1.10) we deduce that the set $K = \overline{B(x_1, r_1)} \cup \ldots \cup \overline{B(x_k, r_k)}$ is the needed compact ANR-space and the proof of (3.5) is completed. □

Proof of Theorem (3.4). Without loss of generality we can assume that $X \subset K^\omega$. By definition we can find an open neighbourhood U of X in K^ω such that X is an approximative retract of U. So by applying Lemma (3.5) we obtain the needed compact ANR-space Y, and the proof is completed. □

(3.6) Remark. By using the Mazur theorem, which says that the convex closed hull of a compact subset of a Banach space is again compact, one can show very easily, that Lemma (3.5) remains true for subsets of Banach spaces.

Instead of the notion of approximative retracts we shall need also the notion of proximative retracts.

Let A be a closed subset of the euclidean space $\mathbb{R}^n$ and let U be an open neighbourhood of A in $\mathbb{R}^n$. A mapping $r: U \to A$ is called a proximative retraction (or metric projection) provided the following condition holds true:

$$\|r(x) - x\| = \operatorname{dist}(x, A) \quad \text{for every } x \in U. \tag{3.7}$$

Evidently every proximative retraction is a retraction map but not conversely.

(3.8) Definition. A compact subset $A \subset \mathbb{R}^n$ is called a *proximative neighbourhood retract* (written $A \in$ PANR) provided there exists an open neighbourhood U of A in $\mathbb{R}^n$ and a proximative retraction $r: U \to A$.

Of course we have:

(3.9) Proposition. *For compact spaces we have* PANR $\subset$ ANR.

Let us remark that any compact convex subset $A \subset \mathbb{R}^n$ is a PANR-space; then as U we can take $\mathbb{R}^n$. Below we shall list important properties of PANR-spaces.

(3.10) Proposition. *Let $A \in$ PANR. Then there exists an $\varepsilon > 0$ such that $\overline{O_\varepsilon(A)} \in$ PANR.*

Proof. Because $A \subset \mathbb{R}^n$ is a proximative absolute neighbourhood retract there exists an open neighbourhood U of A in $\mathbb{R}^n$ and a proximative retraction $r: U \to A$. Since A is compact, there exists an $\varepsilon > 0$ such that $O_{2\varepsilon}(A) \subset U$.

Now, for the proof we define a proximative retraction: $s: O_{2\varepsilon}(A) \to \overline{O_\varepsilon(A)}$ by putting:

$$s(x) = \begin{cases} x & \text{if } x \in O_\varepsilon(A), \\ r(x) + \varepsilon \cdot \dfrac{x - r(x)}{\|x - r(x)\|} & \text{if } x \in O_{2\varepsilon}(A) \setminus O_\varepsilon(A). \end{cases}$$

Observe that if $x \in \partial(\overline{O_\varepsilon(A)})$, then:

$$s(x) = r(x) + \varepsilon \cdot \frac{x - r(x)}{\|x - r(x)\|} = r(x) + x - r(x) = x,$$

where $\partial(\overline{O_\varepsilon(A)})$ denotes the boundary of $O_\varepsilon(A)$ in $\mathbb{R}^n$.

On the other hand, if $x \in O_{2\varepsilon}(A) \setminus O_\varepsilon(A)$, then we have:

$$\begin{aligned} \|x - s(x)\| &= \left\| x - r(x) - \varepsilon \frac{x - r(x)}{\|x - r(x)\|} \right\| = \|x - r(x)\| - \varepsilon \\ &= \operatorname{dist}(x, A) - \varepsilon \leq \operatorname{dist}(x, \overline{O_\varepsilon(A)}). \end{aligned}$$

So s is a continuous map and hence it is a proximative retraction. □

Now observe, that the definition of PANRs can be reformulated as follows:

(3.11) PROPOSITION. *$A \in$ PANR if and only if there exists an open neighbourhood of A in $\mathbb{R}^n$ such that the following condition is satisfied:*

(3.11.1) *for every $y \in U$ there exists exactly one point $x = x(y) \in A$ such that:*

$$\|y - x\| = \operatorname{dist}(y, A).$$

(3.12) PROPOSITION. *If A is a compact C^2-manifold with or without the boundary, then $A \in$ PANR.*

If A is a compact C^2-manifold without boundary then taking a tubular neighbourhood of A in $\mathbb{R}^n$ we are able to obtain that $A \in$ PANR. If the boundary ∂M of M is not empty the situation is more difficult because we have two tubular neighbourhoods (for M and ∂M) but still we can obtain that $A \in$ PANR. For details we recommend [BiGP].

Now, we shall show an example of a compact C^1-manifold A such that $A \notin$ PANR.

(3.13) EXAMPLE. Consider the function $f\colon [0,1] \to \mathbb{R}^n$ given by

$$f(x) = \begin{cases} |x|^{-(1/2)} \cdot \sin(1/x) & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$

An elementary calculation shows that the function $h\colon [0,1] \to \mathbb{R}$ given by

$$h(x) = \begin{cases} \displaystyle\int_0^x f(t)\,dt & \text{for } x \neq 0, \\ 0 & \text{for } x = 0, \end{cases}$$

is differentiable on $(0,1)$ and $h'(x) = f(x)$ for every $x \in (0,1)$.

Finally, we set

$$g(x) = \int_0^x h(t)\,dt.$$

Obviously, we have $g''(x) = f(x)$, for every $x \in (0,1)$.

We let

$$A = \Gamma_g = \{(x, g(x)) \mid x \in [0,1]\} \subset \mathbb{R}^2.$$

Then A is a C^1-manifold but $A \notin$ PANR. Indeed, let

$$r(x) = \|g''(x)\|^{-1}(1 + g'(x))^{3/2}$$

be the radius of curvature of A at $(x, g(x))$. We have:

$$\liminf_{x \to 0} r(x) = 0.$$

Thus one can show that $A \notin$ PANR.

4. Hyperspaces of metric spaces

Let (X,d) be a metric space. Let $B(X)$ and $C(X)$ denote the family of all nonempty closed bounded and nonempty compact, respectively, subsets of X. Evidently, we have $C(X) \subset B(X)$. Given $A, B \in B(X)$ let:

$$d_H(A,B) = \inf\{\varepsilon > 0 \mid A \subset O_\varepsilon(B) \text{ and } B \subset O_\varepsilon(A)\}. \tag{4.1}$$

Observe that

$$d_H(A,B) = \max\Big\{\sup_{a \in A} \operatorname{dist}(a,B), \sup_{b \in B} \operatorname{dist}(A,b)\Big\}. \tag{4.2}$$

Note that (4.2) can be rewritten as follows:

$$d_H(A,B) = \inf\{\varepsilon > 0 \mid A \subset O_\varepsilon(B) \text{ and } B \subset O_\varepsilon(A)\}. \tag{4.2.1}$$

Formula (4.2.1) is more geometrical than (4.2).

(4.3) Proposition. *The function $d_H\colon B(X) \times B(X) \to \mathbb{R}_+ = [0,+\infty)$ is a metric on $B(X)$.*

Proof. It is clear that $d_H(A,B) \geq 0$ and $d_H(A,B) = 0$ if and only if $A = B$. Furthermore, for every $A, B \in B(X)$ we have $d_H(A,B) = d_H(B,A)$. Let $A, B, C \in B(X)$ be fixed. For every $x \in A$ and $y \in B$ one has

$$\operatorname{dist}(x,C) \leq d(x,y) + \operatorname{dist}(y,C) \leq d(x,y) + d_H(B,C).$$

Hence it follows that $\operatorname{dist}(x, C) \leq d_H(A, B) + d_H(B, C)$ for every $x \in A$. Therefore

$$\sup_{x \in A} \operatorname{dist}(x, C) \leq d_H(A, B) + d_H(B, C).$$

In a similar way we obtain

$$\sup_{z \in C} \operatorname{dist}(z, A) \leq d_H(A, B) + d_H(B, C).$$

Consequently we obtain $d_H(A, C) \leq d_H(A, B) + d_H(B, C)$ and the proof is completed. □

The metric d_H defined on $B(X)$ is called the *Hausdorff distance* or Hausdorff metric in $B(X)$.

We have the following theorem:

(4.4) THEOREM. $(B(X), d_H)$ *is a complete metric space whenever* (X, d) *is complete.*

PROOF. Let $\{A_n\}$ be a Cauchy sequence in $B(X)$. We shall prove first that the set A defined as follows:

$$A = \bigcap_{n=1}^{\infty} \operatorname{cl}\left(\bigcup_{m=n}^{\infty} A_m\right)$$

is nonempty, bounded and $\lim_n A_n = A$.

Let $\varepsilon > 0$ and N be the set of all natural numbers. For each $k \in N$ there exists n_k such that $n, m \geq n_k$ implies $d_H(A_n, A_m) < 2^{-k} \cdot \varepsilon$. Let $\{n_k\}$ be a strictly increasing sequence of elements of N chosen for $k = 0, 1, \ldots$ Let $x_0 \in A_{n_0}$. Suppose we have chosen $x_0, \ldots, x_k$ with properties $x_i \in A_{n_i}$, $d(x_i, x_{i+1}) < 2^{-i}\varepsilon$, for $i = 0, \ldots, k-1$. Then x_{k+1} is chosen in $A_{n_{k+1}}$ so as to satisfy $d(x_k, x_{k+1}) < 2^{-k} \cdot \varepsilon$. Observe that such x_{k+1} exists because $\operatorname{dist}(x_k, A_{n_{k+1}}) \leq d_H(A_{n_k}, A_{n_{k+1}}) < 2^{-k} \cdot \varepsilon$. It is easy to see that $\{x_k\}$ is a Cauchy sequence in X. Then there exists $x \in X$ such that

$$\lim_k x_k = x.$$

We have, of course $x \in A$ and, furthermore, $d(x_0, x) \leq 2\varepsilon$. Therefore, for every $\widetilde{n}_0 \geq n_0$ and $x_0 \in A_{\tilde{n}_0}$, there exists a point $x \in A$ such that $d(x_0, x) \leq 2\varepsilon$. Hence

$$\sup_{x \in A_{\tilde{n}_0}} \operatorname{dist}(x, A) \leq 2\varepsilon, \quad \text{for } \widetilde{n}_0 \geq n_0.$$

Now, we will show that $\sup_{x \in A} \operatorname{dist}(x, A) \to 0$, as $n \to \infty$ which together with the above will prove that $d_H(A_n, A) \to 0$ as $n \to \infty$. Let $\widetilde{n}$ be such that $m, n \geq \widetilde{n}$

implies $d_H(A_n, A_m) \le \varepsilon$. Let $x \in A$. Then $x \in \operatorname{cl}(\bigcup_{m=n}^{\infty} A_m)$. Therefore there exists $n_0 \ge \widetilde{n}$ and $y \in A_{n_0}$ such that $d(x, y) \le \varepsilon$. For each $m \ge \widetilde{n}$ we have

$$\operatorname{dist}(x, A_m) \le \operatorname{dist}(x, A_{n_0}) + \sup_{y \in A_{n_0}} \operatorname{dist}(y, A) \le 2\varepsilon.$$

Hence $\sup_{y \in A} \operatorname{dist}(y, A_m) \le 2\varepsilon$ and the proof is completed. □

Note that the topology in $C(X)$ derived from the Hausdorff distance d_H is not determined by the metric topology of (X, d). Two topologically equivalent metrics d and d' may lead to very different topologies on $C(X)$ by the Hausdorff distance procedure. It follows from the example given below.

(4.5) Example. Let $X = \mathbb{R}_+ = [0, +\infty)$, $d(x, y) = |x/(1+x) - y/(1+y)|$ and $d'(x, y) = \min\{1, |x - y|\}$. The metrics d and d' define the same topology on $\mathbb{R}_+$ but the topologies of the Hausdorff distance on $C(\mathbb{R}_+)$ are different, i.e. the set N of natural numbers belongs to the closure of the set of all finite subsets of N in the first space but not in the second.

The example (4.5) shows us that $C(X)$ is not a closed subset of $(B(X), d_H)$ in general. However the following result holds true.

(4.6) Proposition. *If (X, d) is a complete space, then $C(X)$ is a closed subset of the metric space $(B(X), d_H)$.*

Proof. Assume that $\{A_n\} \subset C(X)$ and $\lim_n A_n = A$, where $A \in B(X)$. We have to prove that $A \in C(X)$. Let $\{x_n\}$ be a sequence of points in A. Let $\varepsilon_n = d_H(A_n, A)$. Then $\lim_n \varepsilon_n = 0$. It implies that for every j there exists a point $x_{n,j} \in A_n$ such that $d(x_j, x_{n,j}) \le \varepsilon_n$. We can assume, without loss of generality, that $\lim_j x_{n,j} = u_n \in A_n$.

Now, we claim that $\{u_n\}$ is a Cauchy sequence, because $d(x_{l,m}, x_{s,m}) \le \varepsilon_l + \varepsilon_s$ so we obtain $d(u_l, u_s) \le \varepsilon_l + \varepsilon_s$. Since (X, d) is a complete space we can assume that $\lim_n u_n = u$.

Now, it suffices to prove that the sequence $\{x_n\}$ contains a subsequence $\{x_{n_j}\}$ such that $\lim x_{n_j} = u$.

Assume to the contrary that there exists $\delta > 0$ such that $d(x_n, u) \ge \delta$ for every n. But $\lim_n u_n = u$, so we have $d(u_m, u) < \delta/3$ for sufficiently large m. Because $\lim_m x_{l,m} = u_l$ there then exists n_i as large as needed such that $d(x_{i,n_i}, u_i) < \delta/3$. Then we obtain $d(x_{n_i}, u) < \delta$ but this is a contradiction and the proof is completed. □

Now, we would like to point out that more appropriate than d_H metric in $C(X)$ is the metric d_C, called the (Borsuk) metric of continuity. We let:

(4.7) $$d_C(A, B) = \inf\{\varepsilon > 0 \mid \text{exists } f\colon A \to B \text{ and } g\colon B \to A \text{ such that } d(x, f(x)) \le \varepsilon \text{ and } d(y, g(y)) \le \varepsilon \text{ for every } x \in A \text{ and } y \in B\},$$

where $A, B \in C(X)$ and f, g stand for continuous functions. Evidently, we have:

$$d_H(A,B) \le d_C(A,B) \quad \text{for every } A, B \in C(X).$$

For a given Banach space E we shall consider the family $\mathcal{B}(E)$ defined as follows:

$$\mathcal{B}(E) = \{A \subset E \mid A \text{ is a bounded subset of } E\}. \tag{4.8}$$

Of course we have $C(E) \subset B(E) \subset \mathcal{B}(E)$.

Note that the Hausdorff distance d_H can be extended onto $\mathcal{B}(E)$ but it is no longer a metric. In fact, it is easy to see that $d_H(A, \operatorname{cl} A) = 0$.

We shall define the *measure of noncompactness* on $\mathcal{B}(E)$. We shall say that a subset $A \subset E$ is relatively compact provided the set $\operatorname{cl} A$ is compact.

(4.9) DEFINITION. Let E be a Banach space and $\mathcal{B}(E)$ the family of all bounded subsets of E. Then the function: $\alpha\colon \mathcal{B}(E) \to \mathbb{R}_+$ defined by:

$$\alpha(A) = \inf\{\varepsilon > 0 \mid A \text{ admits a finite cover by sets of diameter } \le \varepsilon\}$$

is called the (Kuratowski) *measure of noncompactness*, the α-MNC for short.

Another function $\beta\colon \mathcal{B}(E) \to \mathbb{R}_+$ defined by:

$$\beta(A) = \inf\{r > 0 \mid A \text{ can be covered by finitely many balls of radius } r\}$$

is called the (Hausdorff) *measure of noncompactness*.

Definition (4.9) is very useful since α and β have interesting properties, some of which are listed in the following

(4.10) PROPOSITION. *Let E be a Banach space with $\dim E = +\infty$ and $\gamma\colon \mathcal{B}(E) \to \mathbb{R}_+$ be either α or β. Then:*

(4.10.1) $\gamma(A) = 0$ *if and only if A is relatively compact,*

(4.10.2) $\gamma(\lambda A) = |\lambda|\gamma(A)$ *and* $\gamma(A_1 + A_2) \le \gamma(A_1) + \gamma(A_2)$, *for every $\lambda \in \mathbb{R}$ and $A, A_1, A_2 \in \mathcal{B}(E)$,*

(4.10.3) $A_1 \subset A_2$ *implies* $\gamma(A_1) \le \gamma(A_2)$,

(4.10.4) $\gamma(A_1 \cup A_2) = \max\{\gamma(A_1), \gamma(A_2)\}$,

(4.10.5) $\gamma(A) = \gamma(\operatorname{conv}(A))$,

(4.10.6) *the function $\gamma\colon \mathcal{B}(E) \to \mathbb{R}_+$ is continuous (with respect to the metric d_H on $\mathcal{B}(E)$).*

PROOF. You will have no difficulty in checking (4.10.1)–(4.10.4) and (4.10.6) by means of Definition (4.9).

Concerning (4.10.5), we only have to show that $\gamma(\mathrm{conv}(A)) \leq \gamma(A)$, since $A \subset \mathrm{conv}(A)$ and therefore $\gamma(A) \leq \gamma(\mathrm{conv}(A))$. Let $\mu > \gamma(A)$ and $A \subset \bigcup_{i=1}^m M_i$ with $\delta(M_i) \leq \mu$ if $\gamma = \alpha$ and $M_i = B(x_i, \mu)$ if $\gamma = \beta$. Since $\delta(\mathrm{conv}(\mu_i)) \leq \mu$ and $B(x_i, \mu)$ are convex, we may assume that the M_i are convex. Since

$$\begin{aligned}\mathrm{conv}(A) &\subset \mathrm{conv}\left[M_1 \cup \mathrm{conv}\left(\bigcup_{i=2}^m M_i\right)\right] \\ &\subset \mathrm{conv}\left[M_1 \cup \mathrm{conv}\left[M_2 \cup \mathrm{conv}\left(\bigcup_{i=3}^m M_i\right)\right]\right] \subset \dots,\end{aligned}$$

it suffices to show that

$$\gamma(\mathrm{conv}(C_1 \cup C_2)) \leq \max\{\gamma(C_1), \gamma(C_2)\} \quad \text{for convex } C_1 \text{ and } C_2.$$

Now, we have

$$\mathrm{conv}(C_1 \cup C_2) \subset \bigcup_{0 \leq \lambda \leq 1} [\lambda C_1 + (1-\lambda) C_2],$$

and since $C_1 - C_2$ is bounded there exists an $r > 0$ such that $\|x\| \leq r$ for all $x \in (C_1 - C_2)$.

Finally, given $\varepsilon > 0$, we find $\lambda_1, \dots, \lambda_p$ such that

$$[0,1] \subset \bigcup_{i=1}^p \left(\lambda_i - \frac{\varepsilon}{r}, \lambda_i + \frac{\varepsilon}{r}\right)$$

and therefore

$$\mathrm{conv}(C_1 \cup C_2) \subset \bigcup_{i=1}^p [\lambda_i C_1 + (1-\lambda_i) C_2 + \mathrm{cl}\, B(0, \varepsilon)].$$

Hence, (4.10.2)–(4.10.4) and the obvious estimate $\gamma(\mathrm{cl}\, B(0,\varepsilon)) \leq 2\varepsilon$ imply

$$\gamma(\mathrm{conv}(C_1 \cup C_2)) \leq \max\{\gamma(C_1), \gamma(C_2)\} + 2\varepsilon,$$

for every $\varepsilon > 0$. Consequently the proof is completed. □

Now, let us state the following obvious observation.

(4.11) REMARK. For every $A \in \mathcal{B}(E)$ we have $\beta(A) \leq \alpha(A) \leq 2\beta(A)$.

We shall end this section by considering two examples and by formulating a generalization of the Cantor theorem.

(4.12) EXAMPLE. Assume that $\dim E = +\infty$. Now, let us complete the measures of a ball $B(x_0, r) = \{x_0\} + r \cdot B(0,1)$. Evidently,

$$\gamma(B(x_0, r)) = r\gamma(\operatorname{cl} B(0,1)) = r\gamma(S),$$

where $S = \delta B(0,1) = \{x \in E \mid \|x\| = 1\}$.

Furthermore, $\alpha(S) \leq 2$ and $\beta(S) \leq 1$. Suppose $\alpha(S) < 2$. Then $S = \bigcup_{i=1}^{n} M_i$ with the closed sets M_i and $\delta(M_i) < 2$. Let E^n be an n-dimensional subspace of E. Then

$$S \cap E^n = \bigcup_{i=1}^{n} M_i \cap E^n$$

and in view of the Lusternik–Schnirelman–Borsuk theorem (see [De3-M, p. 22] or [DG-M, p. 43]) there exists i such that the set $M_i \cap E^n$ contains a pair of antipodal points, x and $-x$. Hence $\delta(M_i) \geq 2$ for this i, a contradiction. Thus $\alpha(S) = 2$ and

$$1 = \frac{\alpha(S)}{2} \leq \beta(S) \leq 1,$$

i.e. we have $\alpha(B(x_0, r)) = 2r$ and $\beta(B(x_0, r)) = r$ provided $\dim E = +\infty$.

(4.13) EXAMPLE. Let $r\colon E \to \operatorname{cl} B(0,1)$ be the retraction map defined as follows:

$$r(x) = \begin{cases} x & \text{if } \|x\| \leq 1, \\ \dfrac{x}{\|x\|} & \text{if } \|x\| > 1. \end{cases}$$

Let $A \in \mathcal{B}(E)$. Since $r(A) \subset \operatorname{conv}(A \cup \{0\})$, we obtain $\gamma(r(A)) \leq \gamma(A)$. In other words we can say that r is a nonexpansive map with respect to the Kuratowski or Hausdorff measure of noncompactness.

Finally, note that the following version of the Cantor theorem holds true.

(4.14) THEOREM. *If $\gamma = \alpha$ or $\gamma = \beta$ and $\{A_n\}$ is a decreasing sequence of closed nonempty subsets in $B(E)$ such that* $\lim_n \gamma(A_n) = 0$. *Then $A = \bigcap_{n=1}^{\infty} A_n$ is a nonempty and compact subset of E.*

5. The Čech homology (cohomology) functor

By a *pair* of spaces (X, X_0) we understand a pair consisting of a metric space X and of its subset X_0. A pair of the form $(X, \emptyset)$ will be identified with the space X. Let (X, X_0), (Y, Y_0) be two pairs; if $X \subset Y$ and $X_0 \subset Y_0$ then the pair (X, X_0) is a subpair of (Y, Y_0) and we indicate this by writing $(X, X_0) \subset (Y, Y_0)$. A pair (X, X_0) is called *compact* provided X is a compact space and X_0 is a closed subset of X. By a map $f\colon (X, X_0) \to (Y, Y_0)$ we understand a continuous map $f\colon X \to Y$ satisfying the condition $f(X_0) \subset Y_0$.

The category of all pairs and maps will be denoted by $\mathcal{E}$. By $\widetilde{\mathcal{E}}$ will be denoted the subcategory of $\mathcal{E}$ consisting of all compact pairs and maps of such pairs. For maps of pairs we can consider also the notion of homotopy. Namely, two maps $f, g\colon (X, X_0) \to (Y, Y_0)$ are said to be *homotopic* (written $f \sim g$) provided that there is a map $h\colon (X \times [0,1], X_0 \times [0,1]) \to (Y, Y_0)$ such that $h(x,0) = f(x)$ and $h(x,1) = g(x)$ for every $x \in X$. Let us observe that if (X, X_0) is a pair in $\widetilde{\mathcal{E}}$, then $(X \times [0,1], X_0 \times [0,1])$ is in $\widetilde{\mathcal{E}}$ too.

Below we recall some basic facts concerning the Čech homology (cohomology) functor.

For details we recommend [Do-M], [ES-M], [Sp-M].

By H_* (H^*) we denote the *Čech homology* (*cohomology*) *functor* with the coefficients in the field of rational numbers Q (or in a group G if necessary) from the category $\widetilde{\mathcal{E}}$ ($\mathcal{E}$) to the category $\mathcal{A}$ of graded vector spaces over Q and linear maps of degree zero.

Thus, for a pair (X, X_0),

$$H_*(X, X_0) = \{H_q(X, X_0)\}, \quad (H^*(X, X_0) = \{H^q(X, X_0)\}),$$

is a graded vector space and, for $f\colon (X, X_0) \to (Y, Y_0)$ we have $H_*(f)$ ($H^*(f)$) to be the induced linear map:

$$H_*(f) = f_* = \{f_{*q}\}\colon H_*(X, X_0) \to H_*(Y, Y_0),$$
$$(H^*(f) = f^* = \{f^{*q}\}\colon H^*(Y, Y_0) \to H^*(X, X_0)),$$

where $f_{*q}\colon H_q(X, X_0) \to H_q(Y, Y_0)$ ($f^{*q}\colon H^q(Y, Y_0) \to H^q(X, X_0)$).

We have assumed as well known that the functor H_* (H^*) satisfies all of the Eilenberg–Steenrod axioms for homology (cohomology). Recall (cf. [ES-M]) that the Čech homology functor can be defined also on the category $\mathcal{E}$ but then it satisfies all of the Eilenberg–Steenrod axioms except that of exactness. Note that in Section 7 we shall define on $\mathcal{E}$ the Čech homology functor with compact carrieres which is more useful for our considerations.

By $\mathrm{Hom}_Q\colon \mathcal{A} \to \mathcal{A}$ we denote the contravariant functor which to a graded vector space $E = \{E_q\}$ assigns the conjugate graded space $\mathrm{Hom}_Q(E) = \{\mathrm{Hom}(E_q, Q)\}$ and to a linear map $L\colon E_1 \to E_2$ between graded spaces assigns the conjugate map $\mathrm{Hom}_Q(L)\colon \mathrm{Hom}_Q(E_2) \to \mathrm{Hom}_Q(E_1)$ given by the formula:

$$\mathrm{Hom}_Q(L)(u) = u \circ L \quad \text{for every } u \in \mathrm{Hom}_Q(E_2).$$

Moreover, by $\otimes\colon \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ we shall denote the tensor product functor of two variables which assigns to two graded vector spaces $E = \{E_q\}$ and $F = \{F_q\}$ the graded vector space $G = \{G_q\}$, where $G = E \otimes F = \{G_q\}$ and

$$G_q = \bigoplus_{i+j=q} E_i \otimes E_j$$

is the direct sum of the respective tensor products.

For two linear maps $L\colon E_1 \to E_2$, $T\colon F_1 \to F_2$ the tensor product $L \otimes T\colon E_1 \otimes F_1 \to E_2 \otimes F_2$ is defined in the natural way (see [Sp-M]).

We now formulate the Duality Theorem between the Čech homology and cohomology (for the proof see [HW-M]).

(5.1) THEOREM. *On the category $\widetilde{\mathcal{E}}$ the functors H_* and $\operatorname{Hom}_Q \circ H^*$ are naturally isomorphic; in other words there are linear isomorphisms*

$$\eta_X\colon H_*(X, X_0) \to \operatorname{Hom}_Q(H^*(X, X_0))$$

for every pair (X, X_0) in $\widetilde{\mathcal{E}}$ such that, for every map $f\colon (X, X_0) \to (Y, Y_0)$ in $\widetilde{\mathcal{E}}$, the following diagram is commutative

$$\begin{array}{ccc} H_*(X, X_0) & \xrightarrow[\sim]{\eta_X} & \operatorname{Hom}_Q(H^*(X, X_0)) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle \operatorname{Hom}_Q(f^*)} \\ H_*(Y, Y_0) & \xrightarrow[\sim]{\eta_Y} & \operatorname{Hom}_Q(H^*(Y, Y_0)) \end{array}$$

A graded vector space $E = \{E_q\}$ in $\mathcal{A}$ is said to be of finite type provided:

(i) $\dim E_q < \infty$, for all q and

(ii) $E_q = 0$, for almost all q.

The following fact is well known from the first course of linear algebra.

(5.2) PROPOSITION. *If E is a graded vector space of finite type, then the graded vector space $\operatorname{Hom}_Q(E)$ is isomorphic to E; in particular it is also of finite type.*

We need the following:

(5.3) DEFINITION. A pair (X, X_0) in $\widetilde{\mathcal{E}}$ is of finite type with respect to H_* (H^*) provided the graded vector space $H_*(X, X_0)$ ($H^*(X, X_0)$) is of finite type.

From (5.1) and (5.2) immediately follows:

(5.4) PROPOSITION. *A pair (X, X_0) in $\widetilde{\mathcal{E}}$ is of finite type with respect to H_* if and only if it is of finite type with respect to H^*.*

For two pairs (X, X_0), (Y, Y_0) in $\mathcal{E}$ we define the Cartesian product $(X, X_0) \times (Y, Y_0)$ as a pair of the following form:

$$(X, X_0) \times (Y, Y_0) = (X \times Y, X \times Y_0 \cup X_0 \times Y),$$

where the product metric in $X \times Y$ is considered.

Given two maps $f\colon (X, X_0) \to (Y, Y_0)$ and $g\colon (X', X_0') \to (Y', Y_0')$ we can define the product map $f \times g\colon (X, X_0) \times (X', X_0') \to (Y, Y_0) \times (Y', Y_0')$ by letting:

$$(f \times g)(x, x') = (f(x), g(x')),$$

for every $x \in X$ and $x' \in X'$. It is easy to see that $f \times g$ is a map of pairs.

Note that for metric spaces the Čech cohomology functor and the Alexander–Spanier cohomology functor are naturally equivalent. So we can formulate the following version of the Künneth theorem for Čech cohomology functor (see [Sp-M], p. 405).

(5.5) THEOREM (Künneth Theorem). *For every two pairs* (X, X_0), (X', X_0') *in* $\widetilde{\mathcal{E}}$, *there is a linear isomorphism*

$$L\colon\ H^*((X, X_0) \times (X', X_0')) \xrightarrow{\ \sim\ } H^*(X, X_0) \otimes H^*(X', X_0')$$

such that if $f\colon (X, X_0) \to (Y, Y_0)$ *and* $g\colon (X', X_0') \to (Y', Y_0')$ *are two maps in* $\widetilde{\mathcal{E}}$, *then the following diagram commutes:*

$$\begin{array}{ccc} H^*((X, X_0) \times (X', X_0')) & \xleftarrow{(f\times g)^*} & H^*((Y, Y_0) \times (Y', Y_0')) \\ \downarrow{\scriptstyle L} & & \downarrow{\scriptstyle L} \\ H^*(X, X_0) \otimes H^*(X', X_0') & \xleftarrow[f^*\otimes g^*]{} & H^*(Y, Y_0) \otimes H^*(Y', Y_0'). \end{array}$$

Now, from the Duality Theorem and the commutativity of functors $\otimes$ and Hom_Q for graded vector spaces of finite type we obtain:

(5.6) THEOREM. *For every two pairs of finite type* (X, X_0) *and* (Y, Y_0) *in* $\widetilde{\mathcal{E}}$, *there is a linear isomorphism*

$$\overline{L}\colon\ H_*((X, X_0) \times (X', X_0')) \xrightarrow{\ \sim\ } H_*(X, X_0) \otimes H_*(X', X_0')$$

such that, if $f\colon (X, X_0) \to (Y, Y_0)$ *and* $g\colon (X', X_0') \to (Y', Y_0')$ *are two maps of pairs of finite type in* $\widetilde{\mathcal{E}}$, *then the following diagram commutes:*

$$\begin{array}{ccc} H_*((X, X_0) \times (X', X_0')) & \xrightarrow{(f\times g)_*} & H((Y, Y_0) \times (Y', Y_0')) \\ \downarrow{\scriptstyle \overline{L}} & & \downarrow{\scriptstyle \overline{L}} \\ H_*(X, X_0) \otimes H_*(X', X_0') & \xrightarrow[f_*\otimes g_*]{} & H_*(Y, Y_0) \otimes H_*(Y', Y_0') \end{array}$$

Let us recall the universal coefficients formula (see [Sp-M]):

(5.7) THEOREM (on Universal Coefficients). *For any space X the following sequence is exact:*

$$0 \to H^n(X, Z) \otimes Q \to H_n(X, Q) \to \mathrm{Tor}(H_{n-1}(X, Z), Q) \to 0,$$

for every $n \geq 1$, where $\mathrm{Tor}(G, H)$ *denotes the torsion product.*

In Chapter IV we shall need also the Mayer–Vietoris theorem (see [Sp-M]).

(5.8) DEFINITION. Let X be a space and A, B two subsets of X. Let

$$i\colon (A, A \cap B) \to (X, B) \quad \text{and} \quad j\colon (B, A \cap B) \to (X, A)$$

denote the respective inclusions. A triple (X, A, B) is called a *k-triad*, $k \geq 0$, if provided:

(5.8.1) $$X = A \cup B,$$

(5.8.2) $$j^{*l}\colon H^l(X, B) \to H^l(B, A \cap B), \quad i^{*l}\colon H^l(X, B) \to H^l(A, A \cap B)$$

are isomorphisms for every $l \geq k + 1$; a 0-triad (X, A, B) is called simply *triad*.

(5.9) THEOREM (Mayer–Vietoris Theorem). *Let (X, A, B) be a k-triad. Then the sequence is exact*

$$H^k(A \cap B) \xrightarrow{\delta} H^{k+1}(X) \xrightarrow{\alpha} H^{k+1}(A) \oplus H^{k+1}(B) \xrightarrow{\beta} H^{k+1}(A \cap B) \longrightarrow \cdots$$

in which δ, α, β are so called Mayer–Vietoris homomorphisms.

We shall end this section by expressing the Čech homology functor in terms of coverings simplicial and chain complexes.

Let X be a compact space. By $\mathrm{Cov}_f(X)$ we denote the family of all open finite coverings of X. Let $A \subset X$. The *star* of A with respect to a covering α is defined by

$$\mathrm{St}(A, \alpha) := \bigcup \{U \in \alpha \mid U \cap A \neq \emptyset\}.$$

The *k-th star* is defined inductively:

$$\mathrm{St}^k(A, \alpha) := \mathrm{St}(\mathrm{St}^{k-1}(A, \alpha), \alpha).$$

One associates with a given covering α an abstract simplicial complex $N(\alpha)$ called the *nerve* of α. The vertices of $N(\alpha)$ are the sets $A \in \alpha$. The sets $A_0, \ldots, A_n$ form a simplex in $N(\alpha)$ provided $A_0 \cap \ldots \cap A_n \neq \emptyset$.

If $\sigma = \{A_0, \ldots, A_n\}$ is an n-simplex in $N(\alpha)$ then the *support* of σ is the set $\sup \sigma := \bigcup_{i=0}^n A_i$. Let $C_*(N(\alpha))$ be the complex of oriented chains. Then we define the *support* of a chain $c \in C_*(N(\alpha))$ by $\operatorname{supp} c := \bigcup_i \operatorname{supp} \sigma_i$, where $c = \sum k_i\sigma_i$ is a nondegenerate representation of c (i.e. $k_i \neq 0$ for each i).

Let $\alpha, \beta \in \operatorname{Cov}_f(X)$ and assume that β is a refinement of α. Then there is a simplical map $i_\alpha^\beta\colon N(\beta) \to N(\alpha)$ defined on vertices as follows: because β refines α, for each vertex w_0 of $N(\beta)$ we can find a vertex v_0 of $N(\alpha)$ such that $\operatorname{supp} w_0 \subset \operatorname{supp} v_0$; we fix for any vertex w of $N(\beta)$ such a vertex v of $N(\alpha)$ and put $i_\alpha^\beta(w) := v$. Of course, i_α^β is not unique, but all such maps are continuous and therefore they induce the same homomorphism of homology groups.

The set $\operatorname{Cov} X$ is directed with the quasi-order relation:

$$\alpha \geq \beta \text{ if and only if } \alpha \text{ refines } \beta.$$

The Čech homology groups of X are defined as the inverse limit

$$\check{H}_q(X) := \varprojlim_{\operatorname{Cov}_f(X)} H_q(N(\alpha)).$$

If A is closed subset of X then every covering $\widetilde{\alpha} \in \operatorname{Cov} A$ can be obtained from a covering $\alpha \in \operatorname{Cov} X$ satisfying $\widetilde{U}_i = A \cap U_i$, where $U_i \in \alpha$. It is known that

$$\check{H}_q(A) = \varprojlim_{\widetilde{\alpha}} H_q(N(\widetilde{\alpha})).$$

Now, we would like to describe $\check{H}_q(A)$ using coverings in $\operatorname{Cov}_f(X)$ only. If B is a subset of X then by $N(\alpha)|_B$ we denote the subcomplex of $N(\alpha)$ which consists of all simplexes σ with $\operatorname{supp} \sigma \subset B$. We shall prove the following

(5.10) Proposition.

$$\check{H}_q(A) = \varprojlim_{\operatorname{Cov} X} H_q(N(\alpha)|_{\operatorname{St}(A,\alpha)}).$$

Before the proof of (5.10) we need:

(5.11) Lemma. *Let $\alpha = \{U_1, \dots, U_k\}$ be an open covering of X and let A be a closed subset of X. Then there exists a finite refinement $\beta = \{V_j\}_{j=1}^n$ of α which has the following property for each $p = 1, \dots, n$:*

(5.11.1) *If $V_1, \dots, V_p \in \beta$ are such that $V_i \cap A \neq \emptyset$ for each i and $\bigcap_{i=1}^p V_i \neq \emptyset$ then $\bigcap_{i=1}^p V_i \cap A \neq \emptyset$.*

Proof. We will adjust the given covering α in a number of steps.

Step 1. $p = 2$. Let $U_1 \cap A \neq \emptyset$, $U_2 \cap A \neq \emptyset$, $U_1 \cap U_2 \neq \emptyset$ and $U_1 \cap U_2 \cap A = \emptyset$. If $\overline{U_1 \cap U_2} \cap \partial A = \emptyset$ then we define

$$U_1^2 := U_1, \quad U_2^1 := U_2 - \overline{U_1 \cap U_2}$$

($\overline{U}$ denotes the closure of U and ∂U its boundary). If $\overline{U_1 \cap U_2} \cap \partial A \neq \emptyset$ then we consider the sets

$$C_j := \left\{ x \;\middle|\; x \in \partial A \cap U_j \text{ and } x \notin \bigcup_{i=3}^{k} U_i \right\}, \quad j = 1, 2.$$

Let W_1 be an open neighbourhood of C_1 in X such that $W_1 \subset U_1$, $\overline{W}_1 \cap \overline{U}_2 \cap A = \emptyset$. Define then

$$U_1^2 := (U_1 \cap \operatorname{Int} A) \cup W_1.$$

Let W_2 be an open neighbourhood of C_2 such that $W_2 \subset U_2$ and $\overline{W}_2 \cap \overline{U}_1^2 = \emptyset$. We put

$$U_2^1 := (U_2 \cap \operatorname{Int} A) \cup W_2.$$

Now, define

$$\begin{aligned} V_1 &:= \bigcap_{i \neq 1} U_1^i, \dots, V_k := \bigcap_{i \neq k} U_k^i, \\ V_{k+1} &:= U_1 \cap (X - A), \dots, V_{k+k} := U_k \cap (X - A). \end{aligned}$$

The covering $\alpha_1 = \{V_1, \dots, V_{2k}\}$ is a refinement of α and satisfies (5.11.1) for $p = 2$.

Step 2. $p = 3$. Assume that $V_1 \cap V_2 \cap V_3 \neq \emptyset$, $V_i = \alpha_1$, $V_i \cap A \neq \emptyset$ for $i = 1, 2, 3$, and $V_1 \cap V_2 \cap V_3 \cap A = \emptyset$. We can repeat the same trick as in the first step for sets $U_1 = V_1 \cap V_2$ and $U_2 = V_3$. If $\overline{U_1 \cap U_2} \cap \partial A = \emptyset$ then we put $V_1' := V_1$, $V_2' := V_2$ and $V_3' := V_3 - \overline{V_1 \cap V_2 \cap V_3}$. If $\overline{U_1 \cap U_2} \cap \partial A \neq \emptyset$ then $V_1' := (V_1 \cap \operatorname{Int} A) \cup W_1$, $V_2' := (V_2 \cap \operatorname{Int} A) \cup W_1$ and $V_3' := (V_3 \cap \operatorname{Int} A) \cup W_2$. The same correction is done for every triple of sets V_i, V_j, V_l $(1 \leq i < j < l \leq k)$. Taking the intersections of such V_i', one obtains a covering $\alpha_2 = \{V_1'', \dots, V_{k+1}'', \dots, V_{2k}''\}$ which is a refinement of α_1 and satisfies (5.11.1) for $p \leq 3$. After $(k-1)$ such steps we obtain the desired covering β. □

Proof of Proposition (5.10). Let Γ denote the family of all coverings which satisfy (5.11.1). Lemma (5.9) states that Γ is a cofinal subfamily in $\operatorname{Cov} X$. If $\alpha \in \Gamma$ and if we consider the induced covering $\widetilde{\alpha} \in \operatorname{Cov}(A)$ then (5.11.1) ensures that the simplical complexes $N(\widetilde{\alpha})$ and $N(\alpha)|_{\operatorname{St}(A,\alpha)}$ are simplically isomorphic. Therefore $H_*(N(\widetilde{\alpha})) = H_*(N(\alpha)|_{\operatorname{St}(A,\alpha)})$. Hence

$$\check{H}_*(A) = \varprojlim_{\Gamma} H_*(N(\widetilde{\alpha})) = \varprojlim_{\Gamma} H_*(N(\alpha)|_{\operatorname{St}(A,\alpha)}) = \varprojlim_{\operatorname{Cov} X} H_*(N(\alpha)|_{\operatorname{St}(A,\alpha)})$$

and the proof is finished. □

In the above the coefficient group was inessential. From now on we assume that the coefficient group is a field F. A compact set A is *acyclic* provided

$$\check{H}_q(A) = \begin{cases} 0 & \text{for } q > 0, \\ F & \text{for } q = 0. \end{cases}$$

Denote by $\check{H}(X)$ the reduced homology vector space of X (see [ES-M] or [Sp-M]).

(5.12) Proposition. *Let A be a closed acyclic subset of X. Then for every covering $\alpha \in \operatorname{Cov} X$ there exists a refinement $\beta \in \operatorname{Cov} X$ of α such that the homomorphism*

$$i^{\beta}_{\alpha *} \colon \widetilde{H}_*(N(\beta)|_{\mathrm{St}^2(A,\beta)}) \to \widetilde{H}_*(N(\alpha)|_{\mathrm{St}(A,\alpha)})$$

is a trivial homomorphism of vector spicas.

Proof. We recall that the coefficients are in a field F. Hence $H_*(N(\alpha))$ are finite-dimensional graded vector spaces. Since $\widetilde{H}_*(A) = 0$, by (5.10) we can find a covering $\gamma \in \operatorname{Cov} X$ such that the homomorphism

$$i^{\gamma}_{\alpha *} \colon \widetilde{H}_*(N(\gamma)|_{\mathrm{St}(A,\gamma)}) \to \widetilde{H}_*(N(\alpha)|_{\mathrm{St}(A,\alpha)})$$

is trivial. Let β be a star-refinement of γ (i.e. for each $B \in \beta$ there is $U \in \gamma$ such that $\mathrm{St}(B,\beta) \subset U$). Then

$$\mathrm{St}^2(A,\beta) \subset \mathrm{St}(A,\gamma), \quad i^{\beta}_{\gamma}(N(\beta)|_{\mathrm{St}^2(A,\beta)}) \subset N(\gamma)|_{\mathrm{St}(A,\gamma)}.$$

Therefore, equation $i^{\beta}_{\alpha *} = i^{\gamma}_{\alpha *} \circ i^{\beta}_{\gamma *}$ is trivial on $\widetilde{H}_*(N(\beta)|_{\mathrm{St}^2(A,\beta)})$ and the proof is completed. □

We shall need also some information about chain complexes as considered in (see also: [ES-M], [SeS] or [Dz1-M]).

Let (K, τ) be a finite polyhedron with a fixed triangulation τ. Its n-th barycentric subdivision is denoted by τ^n. A subset $U \subset K$ is called *polyhedral* provided there is an integer l such that τ^l induces a triangulation of the closure $\operatorname{cl} U = \overline{U}$ of U in K.

We denote by $C_*(K,\tau)$ the oriented chain with coefficients in Q. The *carrier* of $c \in C_*(K,\tau)$ (carr c) is the smallest polyhedral subset $X \subset K$ such that $c \in C_*(X,\tau)$. By $b \colon C_*(K,\tau) \to C_*(K,\tau^l)$ we denote the baricentric subdivision map which maps each chain onto its l-th barycentric subdivision. By $c_* \colon C_*(K,\tau^l) \to C_*(K,\tau)$ we denote any chain map induced by a simplical approximation of the map id_K.

6. Maps of spaces of finite type

In this section we shall formulate homological version of Theorem (2.16). Note that the result obtained below is very useful in the fixed point theory.

The aim of this section is to prove the following:

(6.1) THEOREM. *Let (X,d) be a compact metric space of a finite type. Then there exists $\varepsilon > 0$ such that for every compact space Y and for every two maps $f,g\colon Y \to X$ if $f \sim_\varepsilon g$, (i.e. $d(f(y),g(y)) < \varepsilon$, for every $y \in Y$), then $f_* = g_*$.*

(6.2) REMARK. In view of the duality theorem, (6.1) can be formulated in terms of homology or cohomology. We prefer to prove (6.1) for cohomology, i.e. we shall prove that $f^* = g^*$.

Before the proof of (6.1) we shall formulate a lemma.

(6.3) LEMMA. *Let $\beta = \{U_1,\dots,U_n\} \in \operatorname{Cov}_f(X)$ of a metric space X. Then there exists $\alpha = \{V_1,\dots,V_n\} \in \operatorname{Cov}_f(X)$ such that $\alpha \geq \beta$ and $\overline{V}_i \subset U_i$, for every $i = 1,\dots,n$, where $\overline{V}_i = \operatorname{cl} V_i$.*

PROOF. For given i we consider

$$F_i = X \setminus U_i \quad \text{and} \quad F_i' = X \setminus \bigcup_{\substack{j \neq i \\ j=1}}^{m} U_j.$$

Since $F_i \cap F_i' = \emptyset$ we can find two open sets U and V_i such that:

$$F_i \subset U, \quad F_i' \subset V_i \quad \text{and} \quad U \cap V_i = \emptyset.$$

Now, it is easy to see that $\overline{V}_i \subset U_i$ and the family $\{U_1,\dots,U_{i-1},V_i,U_{i+1},\dots,U_n\}$ is a covering of X. If we repeat the above construction for $i = 1,\dots,n$ successively we get the needed covering α. □

Now, we are able to prove Theorem (6.1).

PROOF OF THEOREM (6.1). Let $[u_{\alpha_1}],\dots,[u_{\alpha_k}]$ be a basis of $H^*(X)$, where $u_{\alpha_i} \in H^*(N(\alpha_i))$ for each $i = 1,\dots,k$. We choose a covering $\alpha = \{U_1,\dots,U_n\}$ of X such that $\alpha \geq \alpha_i$ for all $i = 1,\dots,k$. Consider simplicial maps $i_{\alpha\alpha_i}\colon N(\alpha) \to N(\alpha_i)$ for each $i = 1,\dots,k$. Then

$$v_\alpha^i = i^*_{\alpha\alpha_i}(u_{\alpha_i}) \in [u_{\alpha_i}] \quad \text{for each } i.$$

Applying Lemma (6.3) to the covering α, we obtain a covering $\beta = \{V_1,\dots,V_n\}$ such that $\overline{V}_i \subset U_i$ for each $i = 1,\dots,n$. Let $i_{\beta\alpha}\colon N(\beta) \to N(\alpha)$ be a simplicial map given by the vertex transformation $i_{\beta\alpha}(V_i) = U_i$ for each i. Then

$$w_\beta^i = i^*_{\beta\alpha}(v_\alpha^i) \in [u_{\alpha_i}] \quad \text{for each } i = 1,\dots,k.$$

Let $\varepsilon = \min_i \operatorname{dist}(\overline{V}_i, X \setminus U_i)$. We may assume without loss of generality that $U_i \neq X$ for each i. Since $\overline{V}_i \cap X \setminus U_i = \emptyset$ and $\overline{V}_i$, $X \setminus U_i$ are compact, non-empty sets, we deduce that ε is a positive real number.

Let Y be a compact space and let $f, g\colon Y \to X$ be two maps such that $d(f(y), g(y)) < \varepsilon$ for each $y \in Y$. We assert that $f^* = g^*$. Consider the coverings $\gamma = f^{-1}(\alpha)$ and $\delta = g^{-1}(\beta)$. It is easy to see that

$$g^{-1}(V_i) \subset f^{-1}(U_i) \quad \text{for each } i = 1, \ldots, n \text{ and } \delta \geq \gamma.$$

Let $i_{\delta\gamma}\colon N(\delta) \to N(\gamma)$ be a simplicial map given by the vertex transformation $i_{\delta\gamma}(g^{-1}(V_i)) = f^{-1}(U_i)$ for each $i = 1, \ldots, n$. We have the following commutative diagram:

$$\begin{array}{ccc} N(\gamma) & \xrightarrow{f_\alpha} & N(\alpha) \\ {\scriptstyle i_{\delta\gamma}}\uparrow & & \uparrow{\scriptstyle i_{\beta\alpha}} \\ N(\delta) & \xrightarrow[g_\beta]{} & N(\beta) \end{array}$$

This implies that $i^*_{\delta\gamma} f^*_\alpha(v^i_\alpha) = g^*_\beta(w^i_\beta)$ for each $i = 1, \ldots, k$ and hence we obtain $[f^*_\alpha(v^i_\alpha)] = [g^*_\beta(w^i_\beta)]$. Since $g^*([u_{\alpha_i}]) = [g^*_\beta(w^i_\beta)]$ and $f^*([u_{\alpha_i}]) = [f^*_\alpha(v^i_\alpha)]$, we find that the maps f^*, g^* are equal by properties of $H^*(X)$. Finally, from this we deduce that $f^* = g^*$ and the proof of (6.1) is completed. □

Finally, observe that if X is a compact ANR-space (6.1) follows from (2.16). Later, A. Gmurczyk proved Theorem (6.1) for $X \in$ AANR (cf. [Bo-M]). The above formulation of Theorem (6.1) is taken from [Go1-M].

In particular, note that any compact AANR-space is of a finite type.

7. The Čech homology functor with compact carriers

Let (X, X_0) be an arbitrary pair in $\mathcal{E}$. We shall denote by $\mathcal{M} = \{(A_\alpha, A_{0\alpha})\}$ the directed set of all compact pairs such that $(A_\alpha, A_{0\alpha}) \subset (X, X_0)$ for each α, with the natural quasi-order relation defined by the inclusion $\leq$ defined by the condition

$$(A_\alpha, A_{0\alpha}) \leq (A_\beta, A_{0\beta}) \quad \text{if and only if} \quad (A_\alpha, A_{0\alpha}) \subset (A_\beta, A_{0\beta}).$$

If $(A_\alpha, A_{0\alpha}) \leq (A_\beta, A_{0\beta})$, then we shall denote by $i_{\alpha\beta}\colon (A_\alpha, A_{0\alpha}) \to (A_\beta, A_{0\beta})$ the inclusion map. For each pair $(A_\alpha, A_{0\alpha})$ consider the graded vector space $H_*(A_\alpha, A_{0\alpha})$, together with the linear map $i_{\alpha\beta*}$ given for $(A_\alpha, A_{0\alpha}) \leq (A_\beta, A_{0\beta})$. Then the family $\{H_*(A_\alpha, A_{0\alpha}), i_{\alpha\beta*}\}$ is a direct system in the category $\mathcal{A}$ over $\mathcal{M}$. We define a graded vector space

$$H(X, X_0) = \varinjlim \{H_*(A_\alpha, A_{0\alpha}), i_{\alpha\beta*}\}.$$

It is easy to see that $H(X, X_0) = \{H_q(X, X_0)\}$, where

$$H_q(X, X_0) = \varinjlim \{H_q(A_\alpha, A_{0\alpha}), i_{\alpha\beta *}\}, \quad \text{for each } q.$$

Let $f\colon (X, X_0) \to (Y, Y_0)$ be a map. Consider the directed sets $\mathcal{M} = \{(A_\alpha, A_{0\alpha})\}$ and $\mathcal{N} = \{(B_\gamma, B_{0\gamma})\}$ for (X, X_0) and (Y, Y_0) respectively. We define $F\colon \mathcal{M} \to \mathcal{N}$ by the formula

$$F((A_\alpha, A_{0\alpha})) = (f(A_\alpha), f(A_{0\alpha})) \quad \text{for each } (A_\alpha, A_{0\alpha}) \in \mathcal{M}.$$

We observe that if $(A_\alpha, A_{0\alpha}) \leq (A_\beta, A_{0\beta})$ then

$$F((A_\alpha, A_{0\alpha})) \leq F((A_\beta, A_{0\beta})).$$

For each α, by $f_\alpha\colon (A_\alpha, A_{0\alpha}) \to (f(A_\alpha), f(A_{0\alpha}))$ we denote a map given by $f_\alpha(x) = f(x)$ for each $x \in A$. Then the map F and the family $\{f_{\alpha *}\}$ is a map of directed systems $\{H_*(A_\alpha, A_{0\alpha}), i_{\alpha\beta *}\}$ and $\{H_*(B_\gamma, B_{0\gamma}), i_{\delta\gamma *}\}$. We define the induced linear map $H(f)$ for f, by putting

$$H(f) = f_* = \varinjlim \{f_{*\alpha}\}.$$

Then we have $f_{*q} = \varinjlim \{f_{\alpha * q}\}$ for every q.

From the functoriality of $\varinjlim$ we deduce that $H\colon \mathcal{E} \to \mathcal{A}$ is a covariant functor. The functor H is said to be the Čech homology functor with compact carriers.

We note that if (X, X_0) is a compact pair, then the family consisting of the single pair (X, X_0) is a cofinal subset of $\mathcal{M} = \{(A_\alpha, A_{0\alpha})\}$ for (X, X_0), and hence we obtain $H_*(X, X_0) = H(X, X_0)$. Similarly, if $f\colon (X, X_0) \to (Y, Y_0)$ is a map of compact pairs, then $H_*(f) = H(f)$.

The following properties of H clearly follow from the Eilenberg–Steenrod axioms for H_* and some simple properties of $\varinjlim$.

(7.1) Property. *If $f, g\colon (X, X_0) \to (Y, Y_0)$ are homotopic maps, then the induced linear maps are equal, that is, $f_* = g_*$.*

(7.2) Property. *Let (X, X_0) be a pair in $\mathcal{E}$ and let $i\colon X_0 \to X$, $j\colon X \to (X, X_0)$ be inclusions. Then there exists a linear map*

$$\partial_q\colon H_q(X, X_0) \to H_{q-1}(X_0) \quad \textit{for each } q,$$

so that

$$\cdots \longrightarrow H_q(X_0) \xrightarrow{i_{*q}} H_q(X) \xrightarrow{j_{*q}} H_q(X, X_0) \xrightarrow{\partial_q} H_{q-1}(X_0) \longrightarrow \cdots$$

is exact.

The linear map ∂_q has the additional property of being natural in the following sense:

(7.3) PROPERTY. *Given a map* $f\colon (X, X_0) \to (Y, Y_0)$ *in* $\mathcal{E}$, *the diagram*

$$\begin{array}{ccc} H_q(X, X_0) & \xrightarrow{\delta_q} & H_{q-1}(X_0) \\ \Big\downarrow{\scriptstyle f_{*q}} & & \Big\downarrow{\scriptstyle (f_{X_0})_{*q-1}} \\ H_q(Y, Y_0) & \xrightarrow[\delta_q]{} & H_{q-1}(Y_0) \end{array}$$

commutes for all q, *where* $f_{X_0}\colon X_0 \to Y_0$ *is given by the formula* $f_{X_0}(x) = f(x)$ *for each* $x \in X_0$.

We prove the following generalization of (6.1).

(7.4) THEOREM. *Let* (X, d) *be a compact metric space of finite type. Then there exists an* $\varepsilon > 0$ *such that, for every two maps* $f, g\colon Y \to X$, *where* Y *is a metric space, the condition* $d(f(y), g(y)) < \varepsilon$ *for each* $y \in Y$ *implies* $f_* = g_*$.

PROOF. Let ε be as in (6.1). Consider two maps f, g from a metric space Y to X. Let A be a compact subset of Y and let $f_A, g_A\colon A \to X$ be given by $f_A(y) = f(y)$, $g_A(y) = g(y)$ for each $y \in A$. We observe that f_A, g_A satisfy the assumptions of (6.1). So, we have $(f_A)_* = (g_A)_*$. Since

$$f_* = \varinjlim_{A} \{(f_A)_*\} \quad \text{and} \quad g_* = \varinjlim_{A} \{(g_A)_*\},$$

we infer that $f_* = g_*$ and the proof of (7.4) is completed. □

8. Vietoris maps

Let X, Y be two spaces and let $f\colon Y \to X$ be a continuous map; f is called *closed* provided for every closed set $A \subset Y$ the set $f(A)$ is closed in X; f is called *proper* provided for every compact $K \subset X$ the set $f^{-1}(K)$ is compact. We have:

(8.1) PROPOSITION. *If* $f\colon Y \to Y$ *is a proper map, then* f *is closed.*

PROOF. Let $A \subset Y$ be a closed subset of Y. We have to prove that $f(A)$ is closed in X. Consider the sequence $\{x_n\} \subset f(A)$ such that $\lim_n x_n = x$. It is sufficient to prove that $x \in f(A)$.

In order to show it let us consider the set $K = \{x_n\} \cup \{x\}$. Then K is a compact subset of X and consequently the set $f^{-1}(K)$ is compact. For every n we choose $y_n \in A$ such that $f(y_n) = x_n$. Then $\{y_n\} \subset f^{-1}(K)$ and hence we can assume, without loss of generality, that $\lim_n y_n = y$. Since A is closed we have $y \in A$ but f is continuous so $f(y) = x \in f(A)$ and the proof is completed. □

If we consider, for example, a map $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = 3$ for every $x \in \mathbb{R}$, then f is closed but not proper, so the converse to (8.1) is not true.

As we already known a space X is closed *acyclic* provided:

(i) $H_q(X) = 0$ for all $q \geq 1$, and
(ii) $H_0(X) \approx Q$.

In other words a space X is acyclic if its homology are exactly the same as the homology of a one point space $\{p\}$. An equivalent definition of acyclic spaces is the following: a space X is acyclic if and only if the map $j\colon \{p\} \to X$, $j(p) = x_0 \in X$, induces an isomorphism $j_*\colon H_*(\{p\}) \to H_*(X)$.

Now from the homotopy axiom for the Čech homology functor we get:

(8.2) PROPOSITION. *If X is a contractible space, then X is acyclic.*

From (8.2) we get:

(8.3) COROLLARY. *If X is one of the following:*

(8.3.1) *X is a convex subset of some normed space E,*
(8.3.2) *$X \in \mathrm{AR}$,*

then X is acyclic.

Since the Čech homology functor is continuous on $\widetilde{\mathcal{E}}$ (see [ES-M]) we get:

(8.4) PROPOSITION. *If X is an R_δ-space, then X is acyclic.*

So, the notion of acyclicity is more general than earlier notions of contractibility, ARs and R_δ-sets.

Now, we shall introduce the main notion of this section.

(8.5) DEFINITION. A map $p\colon (X, X_0) \to (Y, Y_0)$ of pairs is said to be a *Vietoris map* provided the following conditions are satisfied:

(8.5.1) $p\colon X \to Y$ is proper,
(8.5.2) $p^{-1}(Y_0) = X_0$,
(8.5.3) the set $p^{-1}(y)$ is acyclic, for every $y \in Y$.

In what follows we shall reserve the symbol $p\colon (X, X_0) \Rightarrow (Y, Y_0)$ for Vietoris maps. First, note the following evident proposition:

(8.6) PROPOSITION. *Let $p\colon (X, X_0) \Rightarrow (Y, Y_0)$ and $(B, B_0) \subset (Y, Y_0)$, then the map $\widetilde{p}\colon (p^{-1}(B), p^{-1}(B_0)) \to (B, B_0)$, $\widetilde{p}(x) = p(x)$, for every $x \in p^{-1}(B)$, is a Vietoris map too.*

In 1927, L. Vietoris proved the following result:

(8.7) THEOREM. *Let X and Y be compact spaces and $p\colon X \Rightarrow Y$ be a Vietoris map, then $p_*\colon H_*(X) \xrightarrow{\sim} H_*(Y)$ is an isomorphism.*

By applying to (8.7) the exactness axiom for the Čech homology (with coefficients in Q) and the Five Lemma we obtain:

(8.8) THEOREM. *Let* $(X, X_0), (Y, Y_0)$ *be compact pairs and* $p\colon (X, X_0) \Rightarrow (Y, Y_0)$ *be a Vietoris map. Then*

$$p_*\colon H_*(X, X_0) \xrightarrow{\sim} H_*(Y, Y_0)$$

is an isomorphism.

Now, the importance of the Čech homology functor with compact carriers is evident in the following Vietoris Mapping Theorem.

(8.9) THEOREM (Vietoris Mapping Theorem). *If* $p\colon (X, X_0) \Rightarrow (Y, Y_0)$ *is a Vietoris map, then* $p_*\colon H_*(X, X_0) \xrightarrow{\sim} H_*(Y, Y_0)$ *is an isomorphism.*

PROOF. Consider $\mathcal{M} = \{(A_\alpha, A_{0\alpha})\}$ and $\mathcal{N} = \{(B_\gamma, B_{0\gamma})\}$ for (X, X_0) and (Y, Y_0), respectively. Let $\mathcal{M}_0 = \{(f^{-1}(B_\gamma), f^{-1}(B_{0\gamma})) \mid (B_\gamma, B_{0\gamma}) \in \mathcal{N}$. Since f is a proper map, we have $\mathcal{M}_0 \subset \mathcal{M}$. It is easy to see that $\mathcal{M}_0$ is a cofinal subset of $\mathcal{M}$. Therefore we may assume without loss of generality that

$$H(X, X_0) = \varinjlim_{\alpha \in \mathcal{M}_0} \{H_*(A_\alpha, A_{0\alpha}), i_{\alpha\beta *}\}.$$

Then for each $\gamma \in \mathcal{N}$ the map $f_\gamma\colon (f^{-1}(B_\gamma), f^{-1}(B_{0\gamma})) \to (B_\gamma, B_{0\gamma})$ is a Vietoris map of compact pairs. Using (8.7) we infer that

$$f_{\gamma *}\colon H_*(f^{-1}(B_\gamma), f^{-1}(B_{0\gamma})) \xrightarrow{\sim} H_*(B_\gamma, B_{0\gamma})$$

is a linear isomorphism. Consequently, the linear map $f_* = \varinjlim_{\gamma \in \mathcal{N}} \{f_{\gamma *}\}$ is an isomorphism. The proof of (8.9) is completed. □

Vietoris mappings have some nice properties. Namely, first we prove:

(8.10) PROPOSITION. *If* $p_1\colon X \Rightarrow Y$ *and* $p_2\colon Y \Rightarrow Z$ *are two Vietoris maps, then so is the composition* $p_2 \circ p_1\colon X \Rightarrow Z$.

PROOF. Evidently, the composition is a proper and onto. So for the proof it is sufficient to show that for every $z \in Z$ the set $(p_2 \circ p_1)^{-1}(z)$ is acyclic. Since p_2 is a Vietoris map we know that $p_2^{-1}(z)$ is an acyclic set. Now, let us observe that the map $p\colon p_1^{-1}(p_2^{-1}(z)) \to p_2^{-1}(z)$, $p(x) = p_1(x)$ for every $x \in p_1^{-1}(p_2^{-1}(z))$, is a Vietoris map. Consequently by applying Theorem (8.7) we deduce that the set $p_1^{-1}(p_2^{-1}(z))$ is acyclic and the proof of (8.10) is completed. □

We shall need the following auxiliary notions. Consider a diagram:

$$(*) \qquad X_1 \xrightarrow{q} Y \xleftarrow{p} X_2$$

The *fibre product* of this diagram is a map $f\colon X_1 \boxtimes_Y X_2 \to Y$, where $X_1 \boxtimes_Y X_2 = \{(x_1, x_2) \in X_1 \times X_2 \mid q(x_1) = p(x_2)\}$ and $f(x_1, x_2) = q(x_1)$ $(= p(x_2))$. By the *pull-back* of $(*)$ we mean the following diagram:

$$(**) \qquad X_1 \xleftarrow{\overline{p}} X_1 \boxtimes_Y X_2 \xrightarrow{\overline{q}} X_2$$

where $\overline{p}(x_1, x_2) = x_1$, $\overline{q}(x_1, x_2) = x_2$ for $(x_1, x_2) \in X_1 \boxtimes_Y X_2$. The following proposition is self-evident:

(8.11) PROPOSITION. *If in the diagram $(*)$ the map p is a Vietoris map, then the map $\overline{p}$ in $(**)$ is a Vietoris map too.*

We will end this section by noting that for cohomology the Vietoris Mapping Theorem can be formulated in a more general form.

Let (X, d) be a metric space and C be a subset of X. We define the relative dimension $\operatorname{reldim}_X C$ of C with respect to X by putting:

$$(8.12) \qquad \operatorname{reldim}_X C = \sup\{\dim A \mid A \subset C \text{ and } A \text{ is a closed subset of } X\}.$$

We let also $\operatorname{reldim}_X \emptyset = -\infty$.

Let $p\colon X \to Y$ be a continuous map from X onto Y. We let:

$$\begin{aligned} M^k(p) &= \{y \in Y \mid H^k(p^{-1}(y)) \neq 0\} \quad \text{for } k > 0, \\ M^k(p) &= \{y \in Y \mid H^k(p^{-1}(y)) \neq Q\} \quad \text{for } k = 0. \end{aligned}$$

Moreover, we put:

$$m^n(p) = 1 + \max_{0 \le k \le n-1} \{\operatorname{reldim}_X(M^k(p)) + k\}.$$

The following generalization of the Vietoris Mapping Theorem is owed to E. Skljarenko [Sk1].

(8.13) THEOREM. *Let $p\colon X \to Y$ be a continuous, closed and onto map. If there exists an integer n such that $m^n(p) < n$, then the induced linear map:*

$$p^{*k}\colon H^k(Y) \to H^k(X)$$

is an epimorphism for $k = m^n(p)$, an isomorphism for $m^n(p) < k < n$ and a monomorphism for $k = n$.

From (8.13) we infer:

(8.14) THEOREM. *Let $p: X \to Y$ be a continuous, closed onto map such that $p^{-1}(y)$ is acyclic (with respect to the functor H^*), for every $y \in Y$ then the induced linear map*:

$$p^*: H(Y) \xrightarrow{\sim} H(X)$$

is an isomorphism.

Assume that $\dim X < +\infty$ and $Y \in \mathrm{ANR}$ is a space such that $\pi_n(Y)$ is finitely generated for every $n \geq 1$, where as usually $\pi_n(Y)$ denote n-th homotopy group of Y. Assume further that $p: \Gamma \Rightarrow X$ is a Vietoris map. If X and Y are, additionally, compact spaces then we have:

$$p^{\#}: [X, Y] \xrightarrow{\sim} [\Gamma, Y] \tag{8.15}$$

is a bijection, where for $f: X \to Y$ we let $p^{\#}([f]) = [f \circ p]$. In the other words Vietoris mappings give us a classification of homotopy classes (cf. [Kr2-M]).

9. Homology of open subsets of Euclidean spaces

Consider the subcategory $\mathcal{E}_1 \subset \mathcal{E}$ consisting of all pairs (U, V) such that U and V are open subsets in the Euclidean space $\mathbb{R}^n$ for some n, or U is a finite polyhedron and V is an open subset of U, and all maps of such pairs.

Since the family of all pairs of finite polyhedra $\{(K, K_0)\}$ is cofinal in the family of all compact pairs $\{(A, A_0)\}$ contained in (U, V), we obtain the following:

(9.1) PROPERTY. *On the category $\mathcal{E}_1$ the functors H and $\overline{H}$ are naturally isomorphic* ($\overline{H}$ denotes the singular homology functor with coefficients in Q).

Let $A \subset U \subset \mathbb{R}^n$, where A is compact and U is open in $\mathbb{R}^n$. We identify the n-sphere $S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$ and $\mathbb{R}^n \cup \{\infty\}$. Then from the excision axiom for singular homology and (9.1) we deduce:

(9.2) PROPERTY. *The inclusion $j: (U, U \setminus A) \to (S^n, S^n \setminus A)$ induces an isomorphism*

$$j_*: H(U, U \setminus A) \xrightarrow{\sim} H(S^n, S^n \setminus A).$$

Let K be a finite polyhedron and U an open subset of $\mathbb{R}^n$ where $K \subset U$. Consider a Vietoris map $p: Y \Rightarrow U$ and a map $q: Y \to K$ from a Hausdorff space Y to K. We prove the following:

(9.3) PROPERTY. *There are isomorphisms α_1, α_2, α_3 such that the following* *diagram commutes:*

$$\begin{array}{ccc}
H((U, U \setminus K) \times U) & \xrightarrow[\alpha_1]{\sim} & H(U, U \setminus K) \otimes H(U) \\
\uparrow{\scriptstyle (\mathrm{id} \times p)_*} & & \uparrow{\scriptstyle \mathrm{id} \otimes p_*} \\
H((U, U \setminus K) \times Y) & \xrightarrow[\alpha_3]{\sim} & H(U, U \setminus K) \otimes H(Y) \\
\downarrow{\scriptstyle (\mathrm{id} \times q)_*} & & \downarrow{\scriptstyle \mathrm{id} \otimes q_*} \\
H((U, U \setminus K) \times K) & \xrightarrow[\alpha_2]{\sim} & H(U, U \setminus K) \otimes H(K)
\end{array}$$

Proof. It is easy to see that the families

$$\{(M, M_0) \times L\}, \quad \{(M, M_0) \times p^{-1}(L)\}, \quad \{(M, M_0) \times K\},$$

where M, M_0, L are finite polyhedra, are cofinal in families of all compact pairs contained in $(U, U \setminus K) \times U$, $(U, U \setminus K) \times Y$ and $(U, U \setminus K) \times K$, respectively. We observe that for every L the space $p^{-1}(L)$ is of finite type (p is a Vietoris map), so we may apply (5.6) and have the commutative diagram

$$\begin{array}{ccc}
H_*((M, M_0) \times L) & \xrightarrow{\sim} & H_*(M, M_0) \otimes H_*(L) \\
\uparrow{\scriptstyle (\mathrm{id} \times p_L)_*} & & \uparrow{\scriptstyle \mathrm{id} \otimes (p_L)_*} \\
H_*((M, M_0) \times p^{-1}(L)) & \xrightarrow{\sim} & H_*(M, M_0) \otimes H_*(p^{-1}(L)) \\
\downarrow{\scriptstyle (\mathrm{id} \times q_{p^{-1}(L)})_*} & & \downarrow{\scriptstyle \mathrm{id} \otimes (q_{p^{-1}(L)})_*} \\
H_*((M, M_0) \times K) & \xrightarrow[\sim]{} & H_*(M, M_0) \otimes H_*(K).
\end{array}$$

From the commutativity of the above diagram and the commutativity of $\varinjlim$ and $\otimes$ we simply deduce (9.3).

Consider the diagram

$$U \xleftarrow{p} Y \xrightarrow{q} K$$

where p and q are as in (9.3). With the above diagram we associate the following:

$$(U, U \setminus K) \xleftarrow{\overline{p}} (Y, Y \setminus p^{-1}(K)) \xrightarrow{\overline{q}} (\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$$

where $\overline{p}(y) = p(y)$ and $\overline{q}(y) = p(y) - q(y)$ for each $y \in Y$. We observe that $\overline{p}$ is a Vietoris map. Let $\Delta\colon (U, U \setminus K) \to (U, U \setminus K) \times U$ be a map given by $\Delta(x) = (x, x)$ and let $d\colon (U, U \setminus K) \times K \to (\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$ be given by $d(x, x') = x - x'$, for each $x \in U$ and $x' \in K$. $\square$

(9.4) Lemma. *The following diagram commutes*

$$\begin{array}{ccccc} H(U,U\setminus K) & \xrightarrow{\Delta_*} & H(U,U\setminus K)\otimes H(U) & \xrightarrow{\mathrm{id}\otimes q_* p_*^{-1}} & H(U,U\setminus K)\otimes H(K) \\ & {\scriptstyle \overline{q}_*\overline{p}_*^{-1}}\searrow & & \swarrow{\scriptstyle d_*} & \\ & & H(\mathbb{R}^n,\mathbb{R}^n\setminus\{0\}) & & \end{array}$$

Proof. Consider the diagram

$$\begin{array}{ccccc} (U,U\setminus K)\times U & \xleftarrow{\mathrm{id}\times p} & (U,U\setminus K)\times Y & \xrightarrow{\mathrm{id}\times q} & (U,U\setminus K)\times K \\ \Delta\uparrow & & \uparrow f & & \downarrow d \\ (U,U\setminus K) & \xleftarrow[\overline{p}]{} & (Y,Y\setminus p^{-1}(K)) & \xrightarrow[\overline{q}]{} & (\mathbb{R}^n,\mathbb{R}^n\setminus\{0\}) \end{array}$$

where the map f is given by $f(y)=(p(y),y)$ for each $y\in Y$. From the commutativity of the above diagram and (9.3) we obtain (9.4).

Let us fix for each n an orientation $1\in H_n(S^n)\approx Q$ of the n-sphere $S^n=\mathbb{R}^n\cup\{\infty\}$. Consider the diagram

$$S^n \xrightarrow{i} (S^n,S^n\setminus A) \xleftarrow{j} (U,U\setminus A)$$

in which A is a compact subset of U and U is open in $\mathbb{R}^n$; i,j are inclusions. From (9.2) we infer that j_* is an isomorphism. We define the fundamental class O_A of the pair (U,A) by the equality $O_A=j_{*n}^{-1}i_{*n}(1)$. □

Lemma (9.5). *Let $A\subset A_1\subset V\subset U\subset\mathbb{R}^n$, where A, A_1 are compact, U, V are open subsets of $\mathbb{R}^n$ and let $k\colon(V,V\setminus A_1)\to(U,U\setminus A)$ be the inclusion map. Then we have $k_{*n}(O_{A_1})=O_A$.*

Proof. Consider the commutative diagram

$$\begin{array}{ccccc} S^n & \xrightarrow{i} & (S^n,S^n\setminus A) & \xleftarrow{j} & (U,U\setminus A) \\ & {\scriptstyle i_1}\searrow & \uparrow k_1 & & k\uparrow \\ & & (S^n,S^n\setminus A_1) & \xleftarrow[j_1]{} & (V,V\setminus A_1) \end{array}$$

in which j_1, i_1, k_1 are inclusion maps. Applying H_n to the above diagram, we obtain (9.5). □

Now, we formulate Dold's Lemma in terms of Čech homology with compact carriers. Let $K\subset U\subset\mathbb{R}^n$, where K is a finite polyhedron and U an open subset of $\mathbb{R}^n$. We define the following maps:

$$t\colon U\times K\to K\times U,\quad t(x,x')=(x',x),\quad \text{for each } x\in U \text{ and } x'\in K,$$
$$O_K^\times\colon H(K)\to H(U,U\setminus K)\otimes H(K),\quad O_K^\times(u)=O_K\otimes u,\quad \text{for each } u\in H(K),$$
$$\times\colon Q\otimes H(U)\to H(U),\quad \times(q\otimes u)=q\cdot u,\quad \text{for each } u\in H(U),\ q\in Q.$$

(9.6) LEMMA. *The composite*

$$\begin{aligned} l = l(K,U)\colon H(K) &\xrightarrow{O_k^{\times}} H(U, U\setminus K)\otimes H(K) \\ &\xrightarrow{\Delta_*\otimes \mathrm{id}} H(U, U\setminus K)\otimes H(U)\otimes H(K) \\ &\xrightarrow{\mathrm{id}\otimes t_*} H(U, U\setminus K)\otimes H(K)\otimes H(U) \\ &\xrightarrow{d_*\otimes \mathrm{id}} Q\otimes H(U) \xrightarrow{\times} H(U) \end{aligned}$$

coincides with the linear map $i_*\colon H(K)\to H(U)$.

(9.7) REMARK. Dold's Lemma was given in terms of singular homology in [Do-M] (cf. also [Do2]). Lemma (9.6), in view of (9.1), clearly follows from the original statement of Dold's Lemma.

We recall the Alexander duality theorem (cf. [Do-M], [ES-M], [HW-M], [Sp-M]):

(9.8) THEOREM. *If A is a compact subset of the Euclidean space $\mathbb{R}^{n+1}$ then for every $k\geq 0$ the vector spaces $\overline{H}^0_{n-k}(\mathbb{R}^n\setminus A)$ and $H^k(A)$ are linearly isomorphic, where $\overline{H}^0_{n-k}$ denotes the $(n-k)$-reduced singular homology functor and H^k* (*as in Section* 6) *is the k-th Čech cohomology functor.*

(9.9) DEFINITION. A compact nonempty subset $A\subset S^n$ is called *strongly acyclic* provided the complement $S^n\setminus A$ of A is infinitely connected, i.e. it is path connected and for every $k=1,2,\ldots$ any map $f\colon S^k\to S^n\setminus A$ is homotopic to a constant map.

Another words a compact nonempty set $A\subset S^n$ is called strongly acyclic if the set $S^n\setminus A$ is path connected and for every $k\geq 1$ the k-homotopy group $\pi_k(S^n\setminus A)$ of $S^n\setminus A$ is equal to zero. In what follows such A set is called infinitely connected.

As a direct application of the Hurewicz isomorphism theorem (cf. [Sp-M]) we get:

(9.10) PROPOSITION. *If the set $A\subset S^n$ is strongly acyclic, then A is acyclic.*

Let us observe that the converse to (9.9) is false, for example there exists an embedding A of the unit interval $[0,1]$ into S^3 such that $S^3\setminus A$ is not 1-connected, i.e. $\pi_1(S^3,A)\neq 0$.

We shall make use from the following two propositions (see [Bi2]).

(9.11) PROPOSITION. *Let K be a compact subset of $\mathbb{R}^n$ with* $\dim K\leq n-3$ *and such that $\mathbb{R}^n\setminus K$ is* 1-ULC. *Then for every compact subset C of K the set $\mathbb{R}^n\setminus C$ is also* 1-ULC.

(9.12) PROPOSITION. *Let K and C be two compact subsets of $\mathbb{R}^n$, $n \geq 6$, such that $\mathbb{R}^n \setminus C$ and $\mathbb{R}^n \setminus K$ are* 1-ULC. *Let $h: K \to C$ be a homeomorphism and assume that $2 \dim K + 2 \leq n$. Then there is a homeomorphism $F: \mathbb{R}^n \to \mathbb{R}^n$ such that $F(x) = h(x)$ for every $x \in K$.*

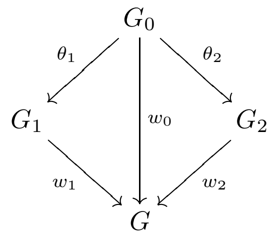
(9.13) THEOREM (Lefschetz Duality Theorem, [Sp-M]). *Let X be a compact orientable n-dimensional manifold with boundary ∂X and let $j: X \setminus \partial X \to X$ be a inclusion. Then for any $q \geq 0$ we have the following isomorphisms:*

$$H_q(X, \partial X) \xrightarrow[\sim]{\rho} H^{n-q}(X \setminus \partial X) \xleftarrow[\sim]{j*n} H^{n-q}(X)$$

where ρ is induced by the orientation of X.

We finish this section by recalling the Van Kampen theorem ([Sp-M]).

(9.14) THEOREM. *Let X_1, X_2 be two open subsets of X such that $X = X_1 \cup X_2$ and X_1, X_2, $X_0 = X_1 \cap X_2$ are path-connected. Let $x_0 \in X_0$ and $G = \pi_1(X, x_0)$, $G_i = \pi_1(X_i, x_0)$, $i = 0, 1, 2$. Assume that the following diagram is commutative:*



where θ_1, θ_2, w_0, w_1, w_2 are induced by the respective inclusions. Assume more that $w_i(G_i)$, $i = 0, 1, 2$ generate G. If H is an arbitrary group and $\psi_i: G_i \to H$, $i = 0, 1, 2$ are homomorphisms such that $\psi_0 = \psi_1 \theta_1 = \psi_2 \theta_2$ then there exists an unique homomorphism $\lambda: G \to H$ such that $\psi_i = \lambda w_i$, $i = 0, 1, 2$.

10. The (ordinary) Lefschetz number

In what follows all the vector spaces are taken over Q. Let $f: E \to E$ be an endomorphism of a finite-dimensional vector space E. If $v_1, \ldots, v_n$ is a basis for E, then we can write

$$f(v_i) = \sum_{j=1}^{n} a_{ij} v_j, \quad \text{for all } i = 1, \ldots, n.$$

The matrix $[a_{ij}]$ is called the matrix of f (with respect to the basis $v_1, \ldots, v_n$). Let $A = [a_{ij}]$ be an $(n \times n)$-matrix; then the trace of A is defined as $\sum_{i=1}^{n} a_{ii}$. If $f: E \to E$ is an endomorphism of a finite-dimensional vector space E, then the *trace* of f, written $\mathrm{tr}(f)$, is the trace of the matrix of f with respect to some basis for E. If E is a trivial vector space then, by definition, $\mathrm{tr}(f) = 0$. It is a standard result that the definition of the trace of an endomorphism is independent of the choice of the basis for E.

We recall the following two basic properties of the trace:

(10.1) PROPERTY. *Assume that in the category of finite-dimensional vector spaces the following diagram commutes*

$$\begin{array}{ccc} E' & \xrightarrow{\;f\;} & E'' \\ {\scriptstyle f'}\big\uparrow & \overset{g}{\nwarrow} & \big\uparrow{\scriptstyle f''} \\ E' & \xrightarrow[\;f\;]{} & E'' \end{array}$$

Then $\mathrm{tr}(f') = \mathrm{tr}(f'')$*; in other words* $\mathrm{tr}(gf) = \mathrm{tr}(fg)$.

(10.2) PROPERTY. *Given a commutative diagram of finite-dimensional vector spaces with exact rows*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E' & \longrightarrow & E & \longrightarrow & E'' & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle f'} & & \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle f''} & & \\ 0 & \longrightarrow & E' & \longrightarrow & E & \longrightarrow & E'' & \longrightarrow & 0 \end{array}$$

we have $\mathrm{tr}(f) = \mathrm{tr}(f') + \mathrm{tr}(f'')$.

Let $E = \{E_q\}$ be a graded vector space in $\mathcal{A}$ of finite type. If $f = \{f_q\}$ is an endomorphism of degree zero of such a graded vector space, then the (*ordinary*) *Lefschetz number* $\lambda(f)$ of f is defined by

$$\lambda(f) = \sum_q (-1)^q \,\mathrm{tr}(f_q).$$

Let E be a finite-dimensional vector space and $v_1, \ldots, v_n$ a basis for E. We define a basis $v^1, \ldots, v^n$ for $\mathrm{Hom}_Q(E)$ by putting

$$v^i(v_j) = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \neq j. \end{cases}$$

The basis $v^1, \ldots, v^n$ is called the *conjugate basis* to $v_1, \ldots, v_n$. For a vector space E and any integer q, define a linear map $\Theta_q \colon \mathrm{Hom}_Q(E) \otimes E \to \mathrm{Hom}(E, E)$ by letting

$$\Theta_q(u \otimes v)(v') = (-1)^q u(v') \cdot v \quad \text{for } u \in \mathrm{Hom}_Q(E),\ v, v' \in E,$$

and extend Θ_q to all $\mathrm{Hom}_Q(E) \otimes E$.

(10.3) LEMMA. *If the vector space E is finite-dimensional, then Θ_q is an isomorphism.*

PROOF. Let $v_1, \dots, v_n$ be a basis for vector space E and $v^1, \dots, v^n$ the conjugate basis to $v_1, \dots, v_n$. Then every element a in $\mathrm{Hom}_Q(E) \otimes E$ has the following form:

$$a = \sum_{i,j=1}^{n} a_{ij} v^i \otimes v_j.$$

If $\Theta_q(a) = 0$ then

$$\Theta_q(a)(v_k) = (-1)^q \sum_{j=1}^{n} a_{kj} v^k(v_k) \cdot v_j = (-1)^q \sum_{j=1}^{n} a_{kj} \cdot v_j = 0$$

so, $a_{kj} = 0$ for all k, j, which implies that $a = 0$. To prove Θ_q is onto, let $f \in \mathrm{Hom}(E, E)$. Then we can write

$$f(v_j) = a_{j1} v_1 + \ldots + a_{jn} v_n \quad \text{for } j = 1, \dots, n.$$

Let $a = (-1)^q \sum_{m,k=1}^{n} a_{mk} v^m \otimes v_k$. For each $j = 1, \dots, n$ we see that

$$\Theta_q(a)(v_j) = (-1)^{2q} \sum_{k=1}^{n} a_{jk} \cdot v_k = f(v_j).$$

So, f and $\Theta_q(a)$ agree on the basis for E, which implies that Θ_q is onto. The proof of (10.3) is completed. □

Define $e\colon \mathrm{Hom}_Q(E) \otimes E \to Q$ as the evaluation map

$$e(u \otimes v) = u(v) \quad \text{for } u \in \mathrm{Hom}_Q(E),\ v \in E.$$

(10.4) LEMMA. *If E is a finite-dimensional vector space and $f\colon E \to E$ is a linear map then*

$$e(\Theta_q^{-1}(f)) = (-1)^q \operatorname{tr}(f).$$

PROOF. Take a basis $v_1, \dots, v_n$ for E and write

$$f(v_j) = \sum_{k=1}^{n} a_{jk} v_k \quad \text{for } j = 1, \dots, n.$$

From the proof of (10.3) we know that

$$\Theta_q^{-1}(f) = (-1)^q \sum_{m,k=1}^{n} a_{mk}(v^m \otimes v_k),$$

so

$$e(\Theta_q^{-1}(f)) = (-1)^q \sum_{k,m=1}^{n} a_{mk} \cdot v^m(v_k) = (-1)^q \sum_k a_{kk} = (-1)^q \operatorname{tr}(f)$$

and the proof of (10.4) is completed. □

Let $E = \{E_q\}$ be a graded vector space of finite type. Define the following graded vector spaces:

(1) $E^* = \{E_q^*\}$, where $E_q^* = \operatorname{Hom}_Q(E_{-q})$,
(2) $\operatorname{Hom}(E, E) = \{(\operatorname{Hom}(E, E))_k\}$,
where $(\operatorname{Hom}(E, E))_k = \bigoplus_{-q+i=k} \operatorname{Hom}(E_q, E_i)$,
(3) $E^* \otimes E = \{(E^* \otimes E)_k\}$, where $(E^* \otimes E)_k = \bigoplus_{q+i=k} E_q^* \otimes E_i$.

Define $\Theta\colon (E^* \otimes E)_0 \to (\operatorname{Hom}(E, E))_0$ by letting

$$\Theta(u_q \otimes v_i) = \Theta_Q(u_q \otimes v_i) \quad \text{for } u_q \in \operatorname{Hom}_Q(E_q),\ v_i \in E_i,\ q = i$$

and extend Θ_q to all $(E^* \otimes E)_0$; and $e\colon (E^* \otimes E)_0 \to Q$ by letting

$$e(u_q \otimes v_i) = u_q(v_i) \quad \text{for } u_q \in \operatorname{Hom}_Q(E_q),\ v_i \in E_i,\ q = i$$

and extend e to all $(E^* \otimes E)_0$. It is immediate from Lemma (10.4) that

(10.5) THEOREM. *If $f\colon E \to E$ is a linear map of degree zero on a graded vector space of finite type E then $e(\Theta^{-1}(f)) = \lambda(f)$.*

11. The generalized Lefschetz number

Let $f\colon E \to E$ be an endomorphism of an arbitrary vector space E. Denote by $f^{(n)}\colon E \to E$ the n-th iterate of f and observe that the kernels

$$\operatorname{Ker} f \subset \operatorname{Ker} f^{(2)} \subset \ldots \subset \operatorname{Ker} f^{(n)} \subset \ldots$$

form an increasing sequence of subspaces of E. Let us now put

$$N(f) = \bigcup_n \operatorname{Ker} f^{(n)} \quad \text{and} \quad \widetilde{E} = E/N(f).$$

Clearly, f maps $N(f)$ into itself and therefore induces the endomorphism $\widetilde{f}\colon \widetilde{E} \to \widetilde{E}$ on the factor space $\widetilde{E} = E/N(f)$.

(11.1) PROPERTY. *We have $f^{-1}(N(f)) = N(f)$; consequently, the kernel of the induced map $\widetilde{f}\colon \widetilde{E} \to \widetilde{E}$ is trivial, i.e. $\widetilde{f}$ is a monomorphism.*

PROOF. If $v \in f^{-1}(N(f))$, then $f(v) \in N(f)$. This implies that for some n we have $f^{(n)}(f(v)) = 0 = f^{(n+1)}(v)$ and $v \in N(f)$. Conversely, if $v \in N(f)$, then $f^{(n)}(v) = 0$ for some n; then $f^{(n)}(f(v)) = 0$ and hence $f(v) \in N(f)$, i.e. $v \in f^{-1}(N(f))$. □

Let $f\colon E \to E$ be an endomorphism of a vector space E. Assume that $\dim \widetilde{E} < +\infty$; in this case we define the generalized trace $\operatorname{Tr}(f)$ of f by putting $\operatorname{Tr}(f) = \operatorname{tr}(\widetilde{f})$.

(11.2) PROPERTY. *Let $f: E \to E$ be an endomorphism. If $\dim E < +\infty$ then $\mathrm{Tr}(f) = \mathrm{tr}(f)$.*

PROOF. We have the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N(f) & \longrightarrow & E & \longrightarrow & E/N(f) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \overline{f}} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \widetilde{f}} & & \\ 0 & \longrightarrow & N(f) & \longrightarrow & E & \longrightarrow & E/N(f) & \longrightarrow & 0 \end{array}$$

in which $\widetilde{f}$ is induced by f. Applying (11.2), to the above diagram, we obtain

$$\mathrm{tr}(f) = \mathrm{tr}(\overline{f}) + \mathrm{tr}(\widetilde{f}) \quad \text{where } \mathrm{tr}(\widetilde{f}) = \mathrm{Tr}(f). \tag{11.2.1}$$

We prove that $\mathrm{tr}(\overline{f}) = 0$. Since $\dim E < +\infty$, we may assume that $N(f) = \mathrm{Ker}\, f^{(n)}$ for some $n \geq 1$. Now consider the commutative diagram

$$\begin{array}{ccccccccc} \mathrm{Ker}(f) & \longrightarrow & \mathrm{Ker}(f^{(2)}) & \longrightarrow \cdots \longrightarrow & \mathrm{Ker}(f^{(n-1)}) & \longrightarrow & \mathrm{Ker}(f^{(n)}) \\ {\scriptstyle 0=\overline{f}_1}\downarrow & \swarrow & \downarrow{\scriptstyle \overline{f}_2} & & {\scriptstyle \overline{f}_{n-1}}\downarrow & \swarrow & \downarrow{\scriptstyle \overline{f}_n=\overline{f}} \\ \mathrm{Ker}(f) & \longrightarrow & \mathrm{Ker}(f^{(2)}) & \longrightarrow \cdots \longrightarrow & \mathrm{Ker}(f^{(n-1)}) & \longrightarrow & \mathrm{Ker}(f^{(n)}) \end{array}$$

where the maps $\overline{f}_i, f_i$, $i = 1, \ldots, n$ are given by f (observe that if $v \in \mathrm{Ker}(f^{(i)})$, then $f(v) \in \mathrm{Ker}(f^{(i-1)})$, for every $i > 1$). Then from (11.1) we infer

$$\mathrm{tr}(\overline{f}) = \mathrm{tr}(\overline{f}_{n-1}) = \ldots = \mathrm{tr}(\overline{f}_2) = \mathrm{tr}(\overline{f}_1) = 0.$$

Finally, from (11.2.1) we obtain $\mathrm{Tr}(f) = \mathrm{tr}(\widetilde{f}) = \mathrm{tr}(f)$ and the proof of (11.2) is completed. □

Let $f = \{f_q\}$ be an endomorphism of degree zero of a graded vector space $E = \{E_q\}$. We say that f is a *Leray endomorphism* provided that the graded vector space $\widetilde{E} = \{\widetilde{E}_q\}$ is of finite type. For such an f we define the (*generalized*) *Lefschetz number* $\Lambda(f)$ of f by putting

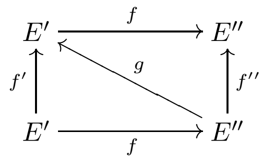
$$\Lambda(f) = \sum_q (-1)^q \,\mathrm{Tr}(f_q).$$

It is immediate from (11.2) that

(11.3) PROPERTY. *Let $f: E \to E$ be an endomorphism of degree zero. If E is a graded vector space of finite type then $\Lambda(f) = \lambda(f)$.*

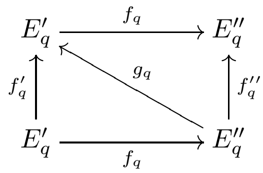
The following property of the Leray endomorphism is of importance:

(11.4) PROPERTY. *Assume that in the category $\mathcal{A}$ the following diagram commutes:*

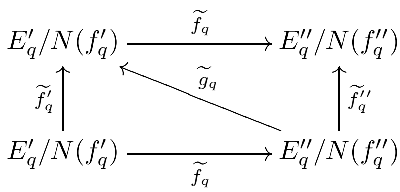


Then if either f' or f'' is a Leray endomorphism, then the other is a Leray endomorphism, and in that case $\Lambda(f') = \Lambda(f'')$.

PROOF. By assumption we have, for each q, the following commutative diagram in the category of vector spaces:

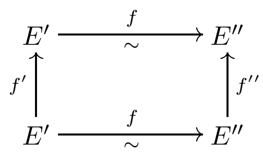


For the proof it is sufficient to show that if either $\mathrm{Tr}(f'_q)$ or $\mathrm{Tr}(f''_q)$ is defined, then so is the other trace, and in that case $\mathrm{Tr}(f'_q) = \mathrm{Tr}(f''_q)$. We observe that the commutativity of the above diagram implies that the following diagram commutes:



Since $\widetilde{f}_q$ and $\widetilde{g}_q$ are monomorphisms, the commutativity of the above diagram implies that $\dim(E'_q/N(f'_q)) < \infty$ if and only if $\dim(E''_q/N(f''_q)) < +\infty$, and hence we conclude that $\mathrm{Tr}(f'_q)$ is defined if and only if $\mathrm{Tr}(f''_q)$ is defined. Moreover, from (10.1) we deduce that $\mathrm{Tr}(f'_q) = \mathrm{Tr}(f''_q)$, if $\mathrm{Tr}(f''_q)$ is defined. The proof of (11.4) is completed. □

Assume that the following diagram



is commutative. Then we obtain the following commutative diagram:

$$\begin{array}{ccc} E' & \xrightarrow{\ \ f\ \ } & E'' \\ {\scriptstyle f'}\big\uparrow & \nwarrow{\scriptstyle f'\circ f^{-1}} & \big\uparrow{\scriptstyle f''} \\ E' & \xrightarrow[\ \ f\ \ ]{} & E'' \end{array}$$

Therefore, from (11.4) we obtain:

(11.4.1) Assume that in the category $\mathcal{A}$ the following diagram is commutative:

$$\begin{array}{ccc} E' & \xrightarrow[\sim]{\ \ f\ \ } & E'' \\ {\scriptstyle f'}\big\uparrow & & \big\uparrow{\scriptstyle f''} \\ E' & \xrightarrow[\sim]{\ \ f\ \ } & E'' \end{array}$$

and f is an isomorphism, then the conclusion of (11.4) holds true.

(11.5) Property. *Let*

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & E'_q & \longrightarrow & E_q & \longrightarrow & E''_q & \longrightarrow & E'_{q-1} & \longrightarrow & \cdots \\ & & \big\downarrow{\scriptstyle f'_q} & & \big\downarrow{\scriptstyle f_q} & & \big\downarrow{\scriptstyle f''_q} & & \big\downarrow{\scriptstyle f'_{q-1}} & & \\ \cdots & \longrightarrow & E'_q & \longrightarrow & E_q & \longrightarrow & E''_q & \longrightarrow & E'_{q-1} & \longrightarrow & \cdots \end{array}$$

be a commutative diagram of vector spaces in which the rows are exact. If two of the following endomorphisms $f = \{f_q\}$, $f' = \{f'_q\}$, $f'' = \{f''_q\}$ *are the Leray endomorphisms then so is the third, and, moreover, in that case we have:*

$$\Lambda(f'') + \Lambda(f') = \Lambda(f).$$

Proof. This immediately follows from (9.2). □

Among the above properties of the Leray endomorphisms we note also some information about weakly nilpotent endomorphisms.

(11.6) Definition. A linear map $f\colon E \to E$ of a vector space E into itself is called *weakly nilpotent* provided for every $x \in E$ there exists n_x such that $f^{n_x}(x) = 0$

Observe that if $f\colon E \to E$ is weakly nilpotent then $N(f) = E$, so, we have:

(11.7) Property. *If* $f\colon E \to E$ *is weakly nilpotent then* $\mathrm{Tr}(f)$ *is well defined and* $\mathrm{Tr}(f) = 0$.

Assume that $E = \{E_q\}$ is a graded vector space and $f = \{f_q\}\colon E \to E$ is an endomorphism. We say that f is *weakly nilpotent* if and only if f_q is weakly nilpotent for every q.

From (11.7) we deduce:

(11.8) PROPERTY. *Any weakly nilpotent endomorphism $f: E \to E$ is a Leray endomorphism and $\Lambda(f) = 0$.*

12. The coincidence problem

A natural generalization of the well known fixed point problem is the coincidence problem. Assume we have two metric spaces (X, d), (Y, d_1) and two continuous mappings $p, q: Y \to X$.

We shall say that p and q have a *coincidence* provided there exists a point $x \in X$ such that $p(x) = q(x)$. In the case when $X = Y$ and $p = \mathrm{id}_X$ is the identity map the coincidence problem for p and q reduces to the fixed point problem of q.

Observe that for arbitrary p and q usually we do not have a coincidence. Therefore in what follows we can assume that p is a Vietoris map and $q: Y \to X$ is a compact map, i.e. $\overline{q(Y)}$ is a compact subset of X.

We assume first that $X = U$ is an open subset of $\mathbb{R}^n$.

(12.1) LEMMA. *Consider the diagram*

$$U \overset{p}{\Longleftarrow} Y \overset{q}{\longrightarrow} U$$

in which p is Vietoris and q is compact. Then the set $\chi_{p,q} = \{x \in U \mid x \in q(p^{-1}(x))\}$ is compact.

PROOF. Consider a sequence $\{x_n\} \subset U$ such that $x_n \in q(p^{-1}(x_n))$ for every n.

For every n we choose $y_n \in p^{-1}(x_n)$ such that $q(y_n) = x_n$. It means that $\{x_n\} \subset \overline{q(Y)}$, and hence $\{x_n\}$ contains a convergent subsequence and the proof is completed. □

We shall now apply the Čech homology with compact carriers to the theory of Lefschetz number and establish a general coincidence theorem, which contains the classical Lefschetz Fixed Point theorem (cf. [Br1-M]) as a special case.

Let U by an open subset of the n-dimensional euclidean space $\mathbb{R}^n$. Consider the diagram:

$$U \overset{p}{\Longleftarrow} Y \overset{q}{\longrightarrow} U \tag{12.2}$$

in which p is a Vietoris map and q is a compact map. With the above diagram we associate the diagram:

$$(U, U \setminus \chi_{p,q}) \overset{\overline{p}}{\Longleftarrow} (Y, Y \setminus p^{-1}(\chi_{p,q})) \overset{\overline{q}}{\longrightarrow} (\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}), \tag{12.3}$$

where $\overline{p}(y) = p(y)$ and $\overline{q}(y) = p(y) - q(y)$ for every $y \in Y$.

Now we define the *index of coincidence* $\mathrm{I}(p, q)$ of the pair (p, q) by putting (cf. Section 9):

$$\mathrm{I}(p, q) = \overline{q}_*(\overline{p}_*)^{-1}(O_{\chi_{p,q}}) \in H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \approx Q. \tag{12.4}$$

(12.5) Proposition. *If* $\mathrm{I}(p,q) \neq 0$*, then there is a* $y \in Y$ *such that* $p(y) = q(y)$*.*

Proof. Indeed, if $p(y) \neq q(y)$ for each $y \in Y$, then $\chi_{p,q} = \emptyset$ an hence we have:

$$\mathrm{I}(p,q) = \overline{q}_*(\overline{p}_*)^{-1}(O_{\chi_{p,q}}) = \overline{q}_*(\overline{p}_*)^{-1}(O) = 0,$$

observe that then we have $H_n(U,U) = 0$. □

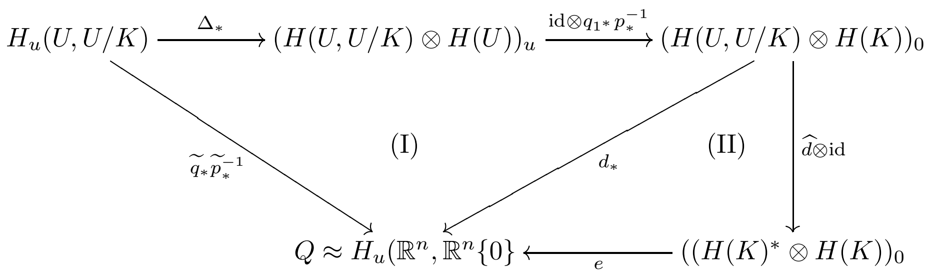
From (9.5) clearly follows:

(12.6) Proposition. *If* A *is a compact set such that* $\chi_{p,q} \subset A \subset U$*, then* $\mathrm{I}(p,q) = \widetilde{q}_*(\widetilde{p}_*)^{-1}(O_A)$*, where* $\widetilde{p}, \widetilde{q}$ *are defined by the same formulae as* $\overline{p}$ *and* $\overline{q}$ *in* (11.3).

Now we prove the following:

(12.7) Proposition. *Let* K *be a finite polyhedron such that* $q(Y) \subset K \subset U$*. Then there exists an element* $a \in (H(K))^* \otimes H(K)$ *such that* $\mathrm{I}(p,q) = e(a)$*.*

Proof. Consider the diagram



in which $q_1\colon Y \to K$ is the contraction of q to the pair (Y,K) and $\widehat{d}\colon H(U, U\setminus K) \to (H(K))^*$ is a linear map of degree $(-n)$ given by:

$$\widehat{d}(u)(v) = d_*(u \otimes v) \quad \text{for } u \in H(U, U \setminus K) \text{ and } v \in H(U \setminus K)$$

and the notations are the same as in Section 9. The subdiagram (I) commutes (cf. (9.4)).

The commutativity of (II) follows by an easy computation. We let:

$$a = (\widehat{d} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes q_{1*}p_*^{-1})(\Delta_*(O_K)).$$

Then from the commutativity of the above diagram we get $\mathrm{I}(p,q) = e(a)$ and the proof is completed. □

Now, we are able to prove the following

(12.8) THEOREM (First Coincidence Theorem). *If we have diagram* (12.2), *then $q_*p_*^{-1}$ is a Leray endomorphism and $\Lambda(q_*p_*^{-1}) \neq 0$ implies that p and q have a coincidence.*

PROOF. Since q is a compact map, there exists a finite polyhedron K such that $q(Y) \subset K \subset U$. We have the commutative diagram

$$\begin{array}{ccccc}
H(K) & & \xrightarrow{\quad i_* \quad} & & H(U) \\
{\scriptstyle q'_*}\uparrow & \nwarrow^{q_{1*}} & & & \uparrow{\scriptstyle q_*} \\
H(p^{-1}(K)) & \xrightarrow[j_*]{} & H(Y) & \xrightarrow{\text{id}} & H(Y) \\
{\scriptstyle (p'_*)^{-1}}\uparrow & & & \nwarrow_{p_*^{-1}} & \uparrow{\scriptstyle p_*^{-1}} \\
H(K) & & \xrightarrow[\quad i_* \quad]{} & & H(U)
\end{array}$$

in which i_*, j_* are linear maps induced by inclusions $i\colon K \to U$ and $j\colon p^{-1}(K) \to Y$, respectively, and q'_*, q_{1*}, p'_* are linear maps induced by the contractions of q and p, respectively. The commutativity of the above diagram and (10.3) imply

$$\Lambda(q_*p_*^{-1}) = \lambda(q'_*(p'_*)^{-1}),$$

and hence $q_*p_*^{-1}$ is a Leray endomorphism.

Assume that $\Lambda(q_*p_*^{-1}) \neq 0$. For the proof it is sufficient to show that

$$\lambda(q'_*(p'_*)^{-1}) = \mathrm{I}(p,q) \tag{12.8.1}$$

(cf. also Section 9).

Consider the following diagram:

$$\begin{array}{ccc}
H(U, U\setminus K) \otimes H(U) \otimes H(K) & \xrightarrow{\widehat{d}\otimes q_{1*}p_*^{-1}\otimes \text{id}} & (H(K))^* \otimes H(K) \otimes H(K) \\
{\scriptstyle \text{id}\otimes t_*}\downarrow & & \downarrow{\scriptstyle \text{id}\otimes t_*} \\
H(U, U\setminus K) \otimes H(K) \otimes H(U) & \xrightarrow{\widehat{d}\otimes \text{id}\otimes q_{1*}p_*^{-1}} & (H(K))^* \otimes H(K) \otimes H(K) \\
{\scriptstyle d_*\otimes \text{id}}\downarrow & & \downarrow{\scriptstyle e\otimes \text{id}} \\
H(U) \approx Q \otimes H(U) & \xrightarrow[q_{1*}p_*^{-1}]{} & Q \otimes H(K) \approx H(K)
\end{array}$$

The commutativity of the above diagram is obtained by simple calculation. Let

$$a = (\widehat{d} \otimes \text{id})(\text{id} \otimes q_{1*}p_*^{-1})\Lambda_*(O_K) \in \mathrm{Hom}_Q(H(K)) \otimes H(K).$$

Since $e(a) = \mathrm{I}(p,q)$ (see (9.4)), for the proof of (12.8.1) it is sufficient to show that

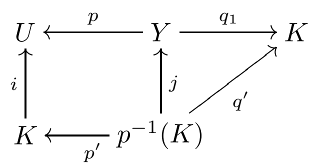
(12.8.2) $$\Theta(a) = q'_*(p'_*)^{-1}$$

(cf. Section 9).

If we follow $\Delta_*(O_K) \otimes u \in H(U, U \setminus K) \otimes H(U) \otimes H(K)$ along $\rightarrow\downarrow\downarrow$, we obtain $(\Theta(a))(u)$. If we follow it along $\downarrow\downarrow$, by Dold's Lemma (9.6) we obtain $i_*(u)$. Therefore, for the proof of (12.8.2) it is sufficient to show that

(12.8.3) $$q_{1*}p_*^{-1}i_* = q'_*(p'_*)^{-1}.$$

Consider the following commutative diagram:



Applying to the above diagram the functor H, we obtain (12.8.3) and the proof of the First Coincidence Theorem is completed. □

To generalize (12.8) we need the Schauder Approximation Theorem.

(12.9) Theorem (Schauder Approximation Theorem). *Let U be an open subset of a normed space E and let $f\colon X \to U$ be a compact map. Then for every $\varepsilon > 0$ there exists a finite dimensional subspace $E^{n(\varepsilon)}$ of E and a compact map $f_\varepsilon\colon X \to U$ such that:*

(12.9.1) $\|f(x) - f_\varepsilon(x)\| < \varepsilon$, *for every* $x \in X$,
(12.9.2) $f_\varepsilon(X) \subset E^{n(\varepsilon)}$,
(12.9.3) *the maps* $f_\varepsilon, f\colon X \to U$ *are homotopic.*

Proof. Given $\varepsilon > 0$ (we can assume to be sufficiently small) $f(X)$ is contained in the union of open balls $B(y_i, \varepsilon)$ with $B(y_i, 2\varepsilon) \subset U$, $i = 1, \dots, k$.

For every $i = 1, \dots, k$ we define $\lambda_i\colon X \to \mathbb{R}_+$, $\lambda_i(x) = \max\{0, \varepsilon - \|f(x) - y_i\|\}$ and
$$\mu_i\colon X \to [0,1], \quad \mu_i(x) = \frac{\lambda_i(x)}{\sum_{j=1}^k \lambda_j(x)}.$$

Now, we define $f_\varepsilon\colon X \to U$ by putting
$$f_\varepsilon(x) = \sum_{i=1}^k \mu_i(x) \cdot y_i.$$

Let $E^{n(\varepsilon)}$ be a subspace of E spanned by vectors $y_1, \ldots, y_n$, i.e.

$$E^{n(\varepsilon)} = \operatorname{span}\{y_1, \ldots, y_k\}.$$

Then $f_\varepsilon(X) \subset \operatorname{conv}\{y_1, \ldots, y_n\}$ so f_ε is a compact map. We have:

$$\|f(x) - f_\varepsilon(x)\| \leq \sum_{i=1}^{k} \mu_i(x)\|f(x) - y_i\| < \varepsilon.$$

Moreover, the map $h\colon X \times [0,1] \to U$,

$$h(x,t) = tf(x) + (1-t)f_\varepsilon(x)$$

is a good homotopy joining f and f_ε and the proof is completed. □

Now, we prove the following:

(12.10) Theorem (Second Coincidence Theorem). *Assume that we have a diagram:*

$$U \overset{p}{\Longleftarrow} Y \overset{q}{\longrightarrow} U,$$

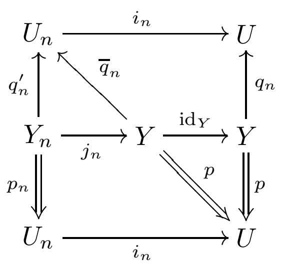
in which U is an open subset of a normed space E, p is Vietoris and q compact. Then $q_* p_*^{-1}$ is a Leray endomorphism and $\Lambda(q_* p_*^{-1}) \neq 0$ implies that p and q have a coincidence.

Proof. Since $q\colon Y \to U$ is compact, in view of the Schauder Approximation Theorem for every n we get a finite dimensional subspace $E^n \subset E$ and a compact map $q_n\colon Y \to U$ such that:

(12.10.1) $\|q(y) - q_n(y)\| < 1/n$,
(12.10.2) $q_n(Y) \subset E^n$, and
(12.10.3) $q \sim q_n$.

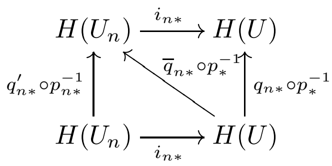
We let $U_n = U \cap E^n$.

Now, for every n, we consider the following commutative diagram:



where $q'_n(y) = q_n(y)$, $\overline{q}_n(y) = q(y)$, $p_n(y) = p(y)$, $i_n(x) = x$, $j_n(y) = y$ for respective y and x.

Consequently, its image under H is also a commutative diagram:



Now, it follows from the First Coincidence Theorem that $q'_{n*} \circ p_{n*}^{-1}$ is a Leray endomorphism. So, by the commutativity property $q_{1*}p_*^{-1}$ is a Leray endomorphism and because $q_{n*} = q_*$ (cf. (iii)) we obtain:

$$\Lambda(q'_{n*}p_{n*}^{-1}) = \Lambda(q_{n*}p_*^{-1}) = \Lambda(q_*p_*^{-1}). \tag{12.10.4}$$

Now, let us assume that $\Lambda(q_*p_*^{-1}) \neq 0$. Then, in view of (12.10.4), by the First Coincidence Theorem we deduce that

$$p(y_n) = q_n(y_n) \quad \text{for every } n.$$

Let $x_n = p(y_n) = q_n(y_n)$ for every n. We put $q(y_n) = \overline{x}_n$, $n = 1, 2, \ldots$ Since q is compact, we may assume without loss of generality that $\lim_n \overline{x}_n = x \in U$.

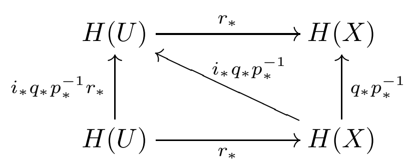
We have $\|x_n - \overline{x}_n\| = \|q_n(y_n) - q(y_n)\| < 1/n$ for every n (cf. (12.10.1)) and hence $\lim_n x_n = x$. Then $x \in q(p^{-1}(x))$ and consequently there exists $y \in p^{-1}(x)$ such that $p(y) = q(y) = x$; the proof is completed. □

(12.11) THEOREM (Coincidence Theorem for arbitrary ANRs). *Consider a diagram:*

$$X \overset{p}{\Longleftarrow} Y \overset{q}{\longrightarrow} X,$$

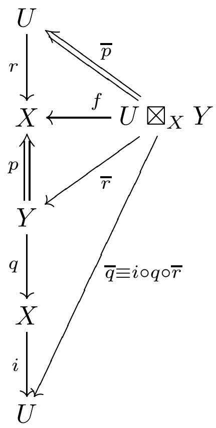
in which $X \in \mathrm{ANR}$, *p is Vietoris and q is compact. Then $q_* \circ p_*^{-1}$ is a Leray endomorphism and $\Lambda(q_* \circ p_*^{-1}) \neq 0$ implies that p and q have a coincidence.*

PROOF. Since $X \in \mathrm{ANR}$ by using the characterization theorem (see (1.8.2)) we can assume that there exists an open subset U of a normed space E such that $X \subset U$ is a retract of U. Let $r: U \to X$ be the retraction map and $i: X \to U$ the inclusion. Of course the following diagram is commutative:



By applying (12.10) the Second Coincidence Theorem we would like to deduce that $i_*q_*p_*^{-1}r_*$ is a Leray endomorphism.

Now, by considering the fibre product and pull-back construction we obtain the following commutative diagram.



where $\overline{p}(u,y) = u$, $\overline{r}(u,y) = y$, $f(u,y) = r(u) = p(y)$. Then $i_* q_* p_*^{-1} r_* = \overline{q}_* \circ \overline{p}_*^{-1}$ and moreover there is a coincidence point for p and q if and only if it is for $\overline{p}$ and $\overline{q}$. Consequently our result follows from the commutativity property of the Leray endomorphisms and the Second Coincidence Theorem, the proof is completed. □

There are many consequences of Theorem (12.11). Before we state them we need a simple observation.

(12.12) PROPERTY. *Assume we have a diagram*

$$X \overset{p}{\Longleftarrow} Y \overset{q}{\longrightarrow} X,$$

in which X is acyclic, p Vietoris and q compact. Then $q_* \circ p_*^{-1}$ is a Leray endomorphism and $\Lambda(q_* \circ p_*^{-1}) = 1$.

PROOF. In fact, from the acyclicity of X we deduce that $q_* \circ p_*^{-1} = \mathrm{id}_{H(X)}$ but

$$H_n(X) = \begin{cases} 0 & \text{for } n > 0, \\ Q & \text{for } n = 0, \end{cases}$$

so, our claim follows. □

From (12.11) and (12.12) we obtain:

(12.13) COROLLARY. *If we have the diagram:*

$$X \overset{p}{\Longleftarrow} Y \overset{q}{\longrightarrow} X,$$

in which $X \in \mathrm{AR}$, p is Vietoris and q compact, then there exists a point $y \in Y$ such that $p(y) = q(y)$.

Now, if we let $Y = X$ and $p = \mathrm{id}_X$ then from (12.11) we deduce the generalized Lefschetz fixed point theorem, proved by A. Granas in 1967 (see [Gr3]):

(12.14) Corollary. *If* $X \in \mathrm{AR}$ *and* $f: X \to X$ *is a compact map then* $f_*: H(X) \to H(X)$ *is a Leray endomorphism and* $\Lambda(f_*) \neq 0$ *implies that* f *has a fixed point.*

Finally, from (12.14) we deduce the following generalized version of the Schauder fixed point theorem:

(12.15) Corollary. *If* $X \in \mathrm{AR}$ *and* $f: X \to X$ *is a compact map then* f *has a fixed point.*

Note that the coincidence theorem can be proved in terms of AANRs too. Then theorem (7.4) is needed. We will do it in Chapter III in terms of multivalued maps.

CHAPTER II

MULTIVALUED MAPPINGS

We gather in this chapter the properties of multivalued maps (called also set-valued or multiple-valued maps) which are needed for the study of the fixed point theory and applications to nonlinear analysis. The first three sections deal with the concept of continuity. Then we consider the selection problem and the continuity of multivalued mappings with respect to the Borsuk and Hausdorff metric. Finally, we shall prove a general characterisation theorem of the set of fixed points of multivalued contractions.

13. General properties

Let X and Y be two spaces and assume that for every point $x \in X$ a nonempty closed (some time we will assume only that $\varphi(x) \neq \emptyset$) subset $\varphi(x)$ of Y is given; in this case, we say that φ is a *multivalued mapping* from X to Y and we write $\varphi\colon X \multimap Y$. More precisely a multivalued map $\varphi\colon X \multimap Y$ can be defined as a subset $\varphi \subset X \times Y$ such that the following condition is satisfied:

$$\text{for all } X \in X \text{ there exists } y \in Y \text{ such that } (x, y) \in \varphi.$$

In what follows the symbol $\varphi\colon X \to Y$ is reserved for single valued mappings, i.e. $\varphi(x)$ is a point of Y.

Let $\varphi\colon X \multimap Y$ be a multivalued map. We associate with φ the *graph* Γ_φ of φ by putting:

$$\Gamma_\varphi = \{(x, y) \in X \times Y \mid y \in \varphi(x)\}$$

and two natural projections $p_\varphi\colon \Gamma_\varphi \to X$, $q_\varphi\colon \Gamma_\varphi \to Y$ defined as follows: $p_\varphi(x, y) = x$ and $q_\varphi(x, y) = y$, for every $(x, y) \in \Gamma_\varphi$.

The point-to-set mapping $\varphi\colon X \multimap Y$ extends to a set-to-set mapping by putting:

$$\varphi(A) = \bigcup_{x \in A} \varphi(x) \quad \text{for } A \subset X,$$

then $\varphi(A)$ is called the *image* of A under φ. If $\varphi\colon X \multimap Y$ and $\psi\colon Y \multimap Z$ are two maps, then the *composition* $\psi \circ \varphi\colon X \multimap Z$ of φ and ψ is defined by:

$$(\psi \circ \varphi)(x) = \bigcup \{\psi(y) \mid y \in \varphi(x)\} \quad \text{for every } x \in X.$$

If $X \subset Y$ and $\varphi\colon X \multimap Y$, then a point $x \in X$ is called a *fixed point* of φ provided $x \in \varphi(x)$. We let:

$$\mathrm{Fix}(\varphi) = \{x \in X \mid x \in \varphi(x)\}.$$

For $\varphi\colon X \multimap Y$ and any subset $B \subset Y$ we define the small counter image $\varphi^{-1}(B)$ and the large counter image $\varphi_+^{-1}(B)$ of B under φ as follows:

$$\varphi^{-1}(B) = \{x \in X \mid \varphi(x) \subset B\}, \quad \varphi_+^{-1}(B) = \{x \in X \mid \varphi(x) \cap B \neq \emptyset\}.$$

If $\varphi\colon X \multimap Y$ and $A \subset X$ then by $\varphi|_A\colon A \multimap Y$ we will denote the restriction of φ to A if, moreover, $\varphi(A) \subset B$, then the map $\widetilde{\varphi}\colon A \multimap B$, $\widetilde{\varphi}(x) = \varphi(x)$ for every $x \in A$ is the contraction of φ to the pair (A, B).

Below, we will summarize properties of image and counterimage.

(13.1) Proposition. *Let $\varphi\colon X \multimap Y$ be a multivalued map, $A \subset X$ and $B \subset Y$, $B_j \subset Y$, $j \in J$, then we have:*

(13.1.1) $$\varphi^{-1}\big(\varphi(A)\big) \supset A,$$

(13.1.2) $$\varphi\big(\varphi^{-1}(B)\big) \subset B,$$

(13.1.3) $$X \setminus \varphi^{-1}(B) \supset \varphi^{-1}(Y \setminus B),$$

(13.1.4) $$\varphi^{-1}\Big(\bigcup_{j\in J} B_j\Big) \supset \bigcup_{j\in J} \varphi^{-1}(B_j),$$

(13.1.5) $$\varphi^{-1}\Big(\bigcap_{j\in J} B_j\Big) = \bigcap_{j\in J} \varphi^{-1}(B_j),$$

(13.1.6) $$\varphi_+^{-1}\big(\varphi(A)\big) \supset A,$$

(13.1.7) $$\varphi\big(\varphi_+^{-1}(B)\big) \supset B \cap \varphi(X),$$

(13.1.8) $$X \setminus \varphi_+^{-1}(B) = \varphi^{-1}(Y \setminus B),$$

(13.1.9) $$\varphi_+^{-1}\Big(\bigcup_{j\in J} B_j\Big) = \bigcup_{j\in J} \varphi_+^{-1}(B_j),$$

(13.1.10) $$\varphi_+^{-1}\Big(\bigcap_{j\in J} B_j\Big) \subset \bigcap_{j\in J} \varphi_+^{-1}(B_j).$$

The proof of (13.1) is straightforward and we leave it to the reader.

For given two maps $\varphi, \psi\colon X \multimap Y$ we let $\varphi \cup \psi\colon X \multimap Y$ and $\varphi \cap \psi\colon X \multimap Y$ as follows:

$$(\varphi \cup \psi)(x) = \varphi(x) \cup \psi(x) \quad \text{and} \quad (\varphi \cap \psi)(x) = \varphi(x) \cap \psi(x),$$

for every $x \in X$. Of course the map $\varphi \cap \psi$ is defined provided that $\varphi(x) \cap \psi(x) \neq \emptyset$, for every $x \in X$.

As an easy observation we obtain:

(13.2) PROPOSITION. *Let* $\varphi, \psi\colon X \multimap Y$ *be such that* $\varphi \cap \psi$ *is defined and let* $B \subset Y$ *then we have*:

(13.2.1) $$(\varphi \cup \psi)^{-1}(B) = \varphi^{-1}(B) \cap \psi^{-1}(B),$$

(13.2.2) $$(\varphi \cap \psi)^{-1}(B) \supset \varphi^{-1}(B) \cup \psi^{-1}(B),$$

(13.2.3) $$(\varphi \cup \psi)_+^{-1}(B) = \varphi_+^{-1}(B) \cup \psi_+^{-1}(B),$$

(13.2.4) $$(\varphi \cap \psi)_+^{-1}(B) \subset \varphi_+^{-1}(B) \cap \psi_+^{-1}(B).$$

If we have two maps $\varphi\colon X \multimap Y$ and $\psi\colon Y \multimap Z$, then for any $B \subset Z$ we obtain:

(13.3) PROPOSITION.

(13.3.1) $$(\psi \circ \varphi)^{-1}(B) = \varphi^{-1}\big(\psi^{-1}(B)\big),$$

(13.3.2) $$(\psi \circ \varphi)_+^{-1}(B) = \varphi_+^{-1}\big(\psi_+^{-1}(B)\big).$$

Finally, let us consider two maps $\varphi\colon X \multimap Y$ and $\psi\colon X \multimap Z$. Then we define the Cartesian product $\varphi \times \psi\colon X \multimap Y \times Z$ of φ and ψ by putting:

$$(\varphi \times \psi)(x) = \varphi(x) \times \psi(x) \quad \text{for every } x \in X.$$

As an easy observation we obtain:

(13.4) PROPOSITION. *Let* $B \subset Y$ *and* $D \subset Z$ *then we have:*

(13.4.1) $$(\varphi \times \psi)^{-1}(B \times D) = \varphi^{-1}(B) \cap \psi^{-1}(D),$$

(13.4.2) $$(\varphi \times \psi)_+^{-1}(B \times D) = \varphi_+^{-1}(B) \cap \psi_+^{-1}(D).$$

To make the notion of multivalued map more natural below we shall present a number of examples.

(13.5) EXAMPLES.
(13.5.1) Let $\varphi\colon [0,1] \multimap [0,1]$ be the map defined as follows:

$$\varphi(x) = \begin{cases} 1 & \text{for } x < 1/2, \\ \{0,1\} & \text{for } x = 1/2, \\ 0 & \text{for } x > 1/2. \end{cases}$$

(13.5.2) Let $\varphi\colon [0,1] \multimap [0,1]$ be given:

$$\varphi(x) = \begin{cases} 1 & \text{for } x < 1/2, \\ [0,1] & \text{for } x = 1/2, \\ 0 & \text{for } x > 1/2. \end{cases}$$

(13.5.3) Let $\varphi\colon [0,1] \multimap [0,1]$ be defined as $\varphi(x) = [x,1]$.
(13.5.4) We let $\varphi\colon [0,1] \multimap [0,1]$ as follows:

$$\varphi(x) = \begin{cases} [0,1/2] & \text{for } x \neq 1/2, \\ [0,1] & \text{for } x = 1/2. \end{cases}$$

(13.5.5) Let $\varphi\colon [0,1] \multimap [0,1]$ be given:

$$\varphi(x) = \begin{cases} [0,1] & \text{for } x \neq 1/2, \\ [0,1/2] & \text{for } x = 1/2. \end{cases}$$

(13.5.6) Let $\varphi\colon [0,\pi] \multimap \mathbb{R}$ be defined:

$$\varphi(x) = \begin{cases} [\mathrm{tg} x, 1 + \mathrm{tg} x] & \text{for } x \neq \pi/2, \\ \{0\} & \text{for } x = \pi/2. \end{cases}$$

(13.5.7) Let $\varphi\colon \mathbb{R}_+ = [0,+\infty) \multimap \mathbb{R}$ be defined: $\varphi(x) = [e^{-x}, 1]$.
(13.5.8) Let $\varphi\colon \mathbb{R}^2 \multimap \mathbb{R}^2$ be defined:

$$\varphi(x,y) = \{(x + z_1, y + z_2) \in \mathbb{R}^2 \mid z_1, z_2 > 0 \text{ and } z_1 \cdot z_2 = 1\}.$$

(13.5.9) Let $K^2 = \{(x,y) \in \mathbb{R}^2 \mid \|(x,y)\| \leq 1\}$, $S^1 = \{(x,y) \in \mathbb{R}^2 \mid \|(x,y)\| = 1\}$. We define a map $\varphi\colon K^2 \to K^2$ by putting:

$$\varphi(x,y) = \{(x,y) \in K^2 \mid \|(x,y)\| = \rho(x,y)\} \cup \{(x,y) \in S^1 \mid \|(x,y)\| \geq \rho(x,y)\},$$

where $\rho(x,y) = 1 - \|(x,y)\| + \|(x,y)\|^2$.

Let us make the geometrical illustrations for the above mappings.

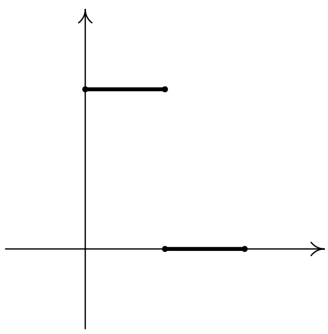


Figure 1. Graph (13.5.1)

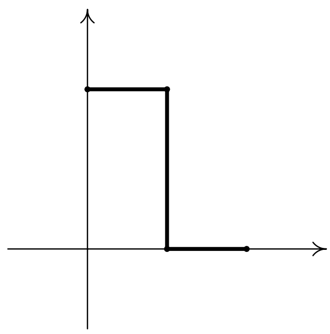


Figure 2. Graph (13.5.2)

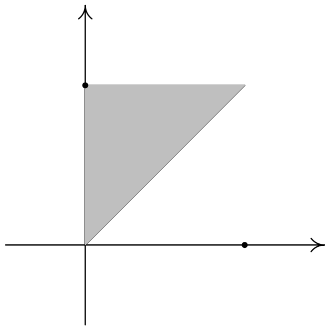


FIGURE 3. Graph (13.5.3)

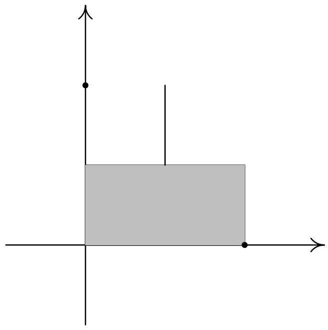


FIGURE 4. Graph (13.5.4)

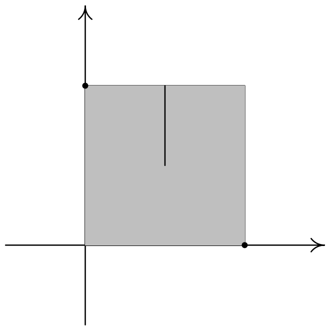


FIGURE 5. Graph (13.5.5)

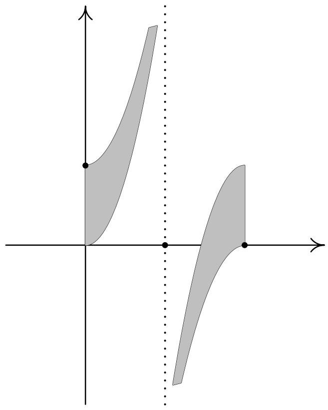


FIGURE 6. Graph (13.5.6)

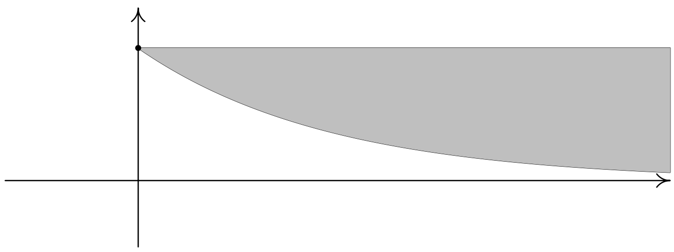


FIGURE 7. Graph (13.5.7)

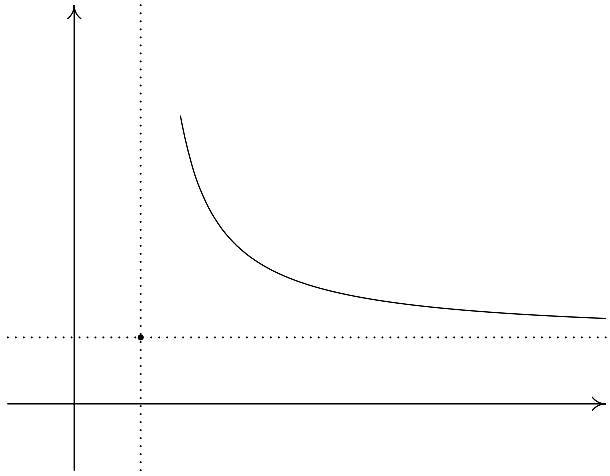


FIGURE 8. Map (13.5.7); no graph!

For every $(x, y) \in K^2$ the set $\varphi(x, y)$ is homeomorphic to S^1.

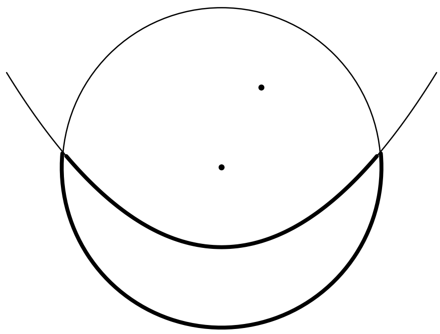


FIGURE 9. Map (13.5.9); no graph!

Let us present more general examples which give motivation for consideration of multivalued maps.

(13.6) EXAMPLE (Inverse functions). Let $f: X \to Y$ be a (singlevalued) continuous map from X onto Y. Then its inverse we can consider as a multivalued map $\varphi_f: Y \multimap X$ defined by:

$$\varphi_f(y) = f^{-1}(y) \quad \text{for } y \in Y.$$

(13.7) EXAMPLE (Implicit functions). Let $f: X \times Y \to Z$ and $g: X \to Z$ be two continuous maps such that for every $x \in X$ there exists $y \in Y$ such that $f(x, y) = g(x)$.

The implicit function (defined by f and g) is a multivalued map $\varphi: X \multimap Y$ defined as follows:

$$\varphi(x) = \{y \in Y \mid f(x, y) = g(x)\}.$$

(13.8) EXAMPLE. Let $f: X \times Y \to \mathbb{R}$ be a continuous map. Assume that there is $r > 0$ such that for every $x \in X$ there exists $y \in Y$ such that $f(x, y) \leq r$. Then we let $\varphi_r: X \multimap Y$, $\varphi_r(x) = \{y \in Y \mid f(x, y) \leq r\}$.

(13.9) EXAMPLE (Multivalued dynamical systems). Dynamical systems determined by ordinary differential equations without uniques property are multivalued maps. We will come back to this example in last chapter.

(13.10) EXAMPLE (Metric projection). Let A be a compact subset of a metric space (X, d). Then for every $x \in X$ there exists $a \in A$ such that

$$d(a, x) = \operatorname{dist}(x, A).$$

We define the metric projections $P: X \multimap A$ by putting:

$$P(x) = \{a \in A \mid d(a, x) = \operatorname{dist}(x, A)\}, \quad x \in X.$$

Note that metric retraction considered in Chapter I is a special case of metric projection.

(13.11) EXAMPLE (Control problems). Assume we have to solve the following control problem:

$$\begin{cases} x'(t) = f(t, x(t), u(t)), \\ x(0) = x_0, \end{cases} \tag{13.11.1}$$

controlled by parameters $u(t)$ (the controls), where $f: [0, a] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$.

In order to solve (13.11.1) we define a multivalued map $F: [0, a] \times \mathbb{R}^n \multimap \mathbb{R}^n$ as follows:

$$F(t, x) = \{f(t, x, u)\}_{u \in U}.$$

Then solutions of (13.11.1) are solutions of the following differential inclusions:

$$\begin{cases} x'(t) \in F(t, x(t)), \\ x(0) = x_0, \end{cases} \tag{13.11.2}$$

so any control problem (13.11.1) can be translated, in view of multivalued maps, onto problem (13.11.2).

Note that many other examples are provided by game theory, mathematical economies, convex analysis and nonlinear analysis.

Let $\varphi: X \multimap Y$ be a multivalued map and $f: X \to Y$ be a singlevalued map. We shall say that f is a selection of φ (written $f \subset \varphi$) provided $f(x) \in \varphi(x)$, for every $x \in X$.

The problem of existence of good selections for multivalued mappings is very important in the fixed point theory.

14. Upper semicontinuous mappings

The concept of upper semicontinuity is related to the notion of the small counter image of open sets. Consequently the concept of lower semicontinuity is related to the large counter image of open sets. Note that the concept of lower semicontinuity we will study in Section 15.

(14.1) Definition. A multivalued map $\varphi: X \multimap Y$ is called *upper semicontinuous* (u.s.c.) provided for every open $U \subset Y$ the set $\varphi^{-1}(U)$ is open in X.

In terms of closed sets we can formulate (14.1) as follows:

(14.2) Proposition. *A multivalued map $\varphi: X \multimap Y$ is u.s.c. if and only if for every closed set $A \subset Y$ the set $\varphi_+^{-1}(A)$ is a closed subset of X.*

The proof of (14.2) is an immediate consequence of (13.1.3) and (13.1.8).

(14.3) Example. Observe that multivalued mappings considered in examples (13.5.1)–(13.5.4), (13.5.7) and (13.5.9) are u.s.c. among all examples considered in (13.5).

(14.4) Proposition. *If $\varphi: X \multimap Y$ is u.s.c. then the graph Γ_φ is a closed subset of $X \times Y$.*

Proof. We have to prove that $X \times Y \setminus \Gamma_\varphi$, is open i.e. $y \notin \varphi(x)$. Now, we choose an open neighbourhood V_y of y in Y and $V_{\varphi(x)}$ of $\varphi(x)$ in Y such that $V_y \cap V_{\varphi(x)} = \emptyset$ (we consider metric spaces!).

Let $U_x = \varphi^{-1}(V_{\varphi(x)})$. Then U_x is an open neighbourhood of x in X. Consequently, the set $U_x \times V_y$ is an open neighbourhood of (x, y) in $X \times Y$. We observe that $U_x \times V_y \cap \Gamma_\varphi = \emptyset$. In fact, if $(x', y') \in U_x \times V_y$ then $\varphi(x') \subset V_{\varphi(x)}$ but $V_{\varphi(x)} \cap V_y = \emptyset$, hence $y' \notin V_{\varphi(x)}$ and $(x', y') \notin \Gamma_\varphi$.

The proof of (14.4) is completed. □

Observe that for example the map $f\colon \mathbb{R} \to \mathbb{R}$ defined as follows:

$$f(x) = \begin{cases} 1/x & \text{for } x \neq 0, \\ 0 & \text{for } x = 0, \end{cases}$$

has a closed graph but it is not u.s.c. i.e. continuous.

In general if $f\colon X \to Y$ is a continuous map from X onto Y, then the inverse map $\varphi_f\colon Y \multimap X$ considered in (13.6) has a closed graph but is not necessarily u.s.c. However, we have:

(14.5) PROPOSITION. *Assume $\varphi\colon X \multimap Y$ is a multivalued map such that $\varphi(X) \subset K$ and the graph Γ_φ of φ is closed, where K is a compact set. Then φ is u.s.c.*

PROOF. Assume to the contrary that φ is not u.s.c. Then there exists an open neighbourhood $V_{\varphi(x)}$ of $\varphi(x)$ in Y such that for every open neighbourhood U_x of x in X we have $\varphi(U_x) \not\subset V_{\varphi(x)}$.

We take $U_x = B(x, 1/n)$, $n = 1, 2, \ldots$ Then for every n we get a point $x_n \in B(x, 1/n)$ such that $\varphi(x_n) \not\subset V_{\varphi(x)}$. Let y_n be a point in Y such that $y_n \in \varphi(x_n)$ and $y_n \notin V_{\varphi(x)}$. Then we have $\lim_n x_n = x$ and $\{y_n\} \subset K$. Since K is compact we can assume without loss of generality that $\lim_n y_n = y \in K$. We see that $y \notin V_{\varphi(x)}$. There for every n we have $(x_n, y_n) \in \Gamma_\varphi$ and $\{(x_n, y_n)\} \to (x, y)$. So $(x, y) \in \Gamma_\varphi$ because Γ_φ is a closed subset of $X \times Y$ but it contradicts $y \notin V_{\varphi(x)}$ and the proof is completed. □

Regarding the inverse map we mentioned above we have the following.

(14.6) PROPOSITION. *If $f\colon X \to Y$ is a closed continuous map from X onto Y, then the inverse map $\varphi_f\colon Y \multimap X$* (defined in (13.6)) *is u.s.c. In fact, we have:*

$$(\varphi_f)_+^{-1}(A) = f(A)$$

for every closed subset $A \subset X$.

The proof of (14.6) is an immediate consequence of (13.2.1).

(14.7) PROPOSITION. *Assume that $\varphi, \psi\colon X \multimap Y$ are two u.s.c. mappings. Then:*

(14.7.1) *the map $\varphi \cup \psi\colon X \multimap Y$ is u.s.c.*
(14.7.2) *the map $\varphi \cap \psi\colon X \multimap Y$ is u.s.c.*

provided it is well defined.

PROOF OF (14.7.2). Let $x \in X$ and V be an open neighbourhood of $\varphi(x) \cap \psi(x)$ in Y. Then $\varphi(x) \setminus V$ and $\psi(x) \setminus V$ are closed subsets of Y such that $(\varphi(x) \setminus V) \cap$

$(\psi(x)\setminus V) = \emptyset$. Let W_1 and W_2 be open neighbourhood of $(\varphi(x)\setminus V)$ and $(\psi(x)\setminus V)$, respectively, such that $W_1 \cap W_2 = \emptyset$ (metric spaces are normal!) since φ is u.s.c. we choose an open neighbourhood U_1 of x in X such that $\varphi(U_1) \subset V \cup W_1$ and an open neighourhood U_2 of x in X such that $\psi(U_2) \subset U \cup W_2$. Let $U = U_1 \cap U_2$. Then $(\varphi \cap \psi)(U) \subset V$ and the proof is completed. □

(14.8) Proposition. *Let $\varphi\colon X \multimap Y$ and $\psi\colon X \multimap Z$ be two u.s.c. mappings. Then the map $\varphi \times \psi\colon X \multimap Y \times Z$ is u.s.c.*

Note, that (14.8) follows immediately from (13.4.1).

Until the end of this section we will restrict our considerations to multivalued maps with compact values.

(14.9) Proposition. *Let $\varphi\colon X \multimap Y$ be an u.s.c. map with compact values and let A be a compact subset of X. Then $\varphi(A)$ is compact.*

Proof. Let $\{V_t\}$ be an open covering of $\varphi(A)$. Since $\varphi(x)$ is compact for every $x \in X$, we infer that there exist a finite number of sets V_t such that $\varphi(x) \subset W_x$, where W_x is the union of the sets V_t, for every $x \in A$. This implies that the family $\{W_x\}_{x\in A}$ is an open covering of $\varphi(A)$. Let $U_x = \varphi^{-1}(W_x)$ for each $x \in A$. Then $\{U_x\}_{x\in A}$ is an open covering of A in X. Since A is compact there exists a finite subcovering $U_{x_1}, \dots, U_{x_n}$ of this covering. Consequently the sets $W_{x_1}, \dots, W_{x_n}$ cover $\varphi(A)$, and since every W_{x_i} is a finite union of sets in $\{V_t\}$ we obtain a finite subcovering $V_{t_1}, \dots, V_{t_k}$ of the covering $\{V_t\}$ and the proof is completed. □

Now, from (14.9) and (13.3.1) we obtain:

(14.10) Proposition. *If $\varphi : X \multimap Y$ and $\psi\colon Y \multimap Z$ are two u.s.c. mappings with compact values then the composition $\psi \circ \varphi : X \multimap Z$ of φ and X is an u.s.c. map with compact values.*

Finally, let us observe that the upper semicontinuity for mappings with compact values (on metric spaces) can be reformulated in the Cauchy sense as follows:

(14.11) Proposition. *Let $\varphi\colon X \multimap Y$ be a multivalued map with compact values. Then φ is u.s.c. if and only if*

(14.11.1) *for all $x \in X$ and all $\varepsilon > 0$ there exists $\delta > 0$ such that $\varphi(B(x,\delta)) \subset O_\varepsilon(\varphi(x))$.*

It is easy to see that if φ is u.s.c. then (14.11.1) holds true. Conversely, if U is an open neighbourhood of $\varphi(x)$ in Y then, in view of compactness of $\varphi(x)$, we can choose an $\varepsilon > 0$ such that $O_\varepsilon(\varphi(x)) \subset U$ and our assertion follows from (14.11.1).

Let E be a Banach space and let $\varphi\colon X \multimap E$ be an u.s.c. map with compact values. We define a map $\overline{\text{conv}}\,\varphi\colon X \multimap E$ by putting:

$$\overline{\text{conv}}\,\varphi(x) = \overline{\text{conv}}(\varphi(x)),$$

where $\overline{\text{conv}}(\varphi(x))$ denotes the closed convex hull of $\varphi(x)$ in E. Since $\varphi(x)$ is compact, in view of Mazur's Theorem (cf. Remark (3.6) or [De3-M]) the set $\overline{\text{conv}}(\varphi(x))$ is compact too. Recall that $\overline{\text{conv}}(\varphi(x))$ is the intersection of all convex closed subsets of E containing $\varphi(x)$.

We prove the following:

(14.12) PROPOSITION. *If $\varphi\colon X \multimap E$ is an u.s.c. map with compact values, then $\overline{\text{conv}}\,\varphi : X \multimap E$ is also u.s.c. with compact values.*

PROOF. In the proof we shall use (14.11). Let $\varepsilon > 0$ and let $0 < \varepsilon_1 < \varepsilon$. Assume that $x_0 \in X$. Since φ is u.s.c. there exists $\delta > 0$ such that $\varphi(B(x_0,\delta)) \subset O_{\varepsilon_1}(\varphi(x_0))$.

Consequently, $\varphi(B(x_0,\delta)) \subset O_{\varepsilon_1}(\overline{\text{conv}}\,\varphi(x_0))$. Since $O_{\varepsilon_1}(\overline{\text{conv}}\,\varphi(x_0))$ is convex we deduce that

$$\text{conv}\,\varphi(B(x_0,\delta)) \subset O_{\varepsilon_1}(\overline{\text{conv}}\,\varphi(x_0))$$

and hence

$$\overline{\text{conv}}\,\varphi(B(x_0,\delta)) \subset \text{cl}\,(O_{\varepsilon_1}(\overline{\text{conv}}\,\varphi(x_0))) \subset O_{\varepsilon}(\overline{\text{conv}}\,\varphi(x_0)),$$

where $\text{conv}\,\varphi(x) = \text{conv}(\varphi(x))$ is the convex hull of $\varphi(x)$. Therefore, we have proved (14.12). □

15. Lower semicontinuous mappings

By using the large counter image in the place of small counter image we get:

(15.1) DEFINITION. Let $\varphi\colon X \multimap Y$ be a multivalued map. If for every open $U \subset Y$ the set $\varphi_+^{-1}(U)$ is open in X then φ is called a lower semicontinuous (l.s.c.) map.

Note that for $\varphi = f\colon X \to Y$ upper semicontinuity coincide with lower semicontinuity and it means nothing more than continuity of f.

In what follows we say also that a multivalued map $\varphi\colon X \multimap Y$ is *continuous* provided it is both u.s.c. and l.s.c.

(15.2) REMARK. Note that the maps presented in: (13.5.3), (13.5.5), (13.5.7), (13.5.8) and (13.5.9) are l.s.c. The maps (13.5.3), (13.5.7) and (13.5.9) are continuous. The maps (13.5.1), (13.5.2) and (13.5.4) are u.s.c. but not l.s.c. Finally, the maps (13.5.5) and (13.5.8) are l.s.c. but not u.s.c.

In terms of the small counter image we can define the lower semicontinuity as follows.

(15.3) PROPOSITION. *A map $\varphi\colon X \multimap Y$ is l.s.c. if and only if for every closed $A \subset Y$ the set $\varphi^{-1}(A)$ is a closed subset of X.*

Proposition (15.3) is an immediate consequence of (13.1.8) and (15.1).

The following two propositions are straightforward (see (13.2.3) and (13.3.2)).

(15.4) PROPOSITION.

(15.4.1) *If $\varphi, \psi\colon X \multimap Y$ are two l.s.c. mappings, then $\varphi \cup \psi\colon X \multimap Y$ is l.s.c. too.*

(15.4.2) *If $\varphi\colon X \multimap Y$ and $\psi\colon Y \multimap Z$ are two l.s.c. maps, then the composition $\psi \circ \varphi\colon X \multimap Z$ of φ and ψ is l.s.c. too, provided for every $x \in X$ the set $\psi(\varphi(x))$ is closed.*

We would like to stress that the intersection of two l.s.c. mappings does not have to be l.s.c.

(15.5) EXAMPLE. Consider two multivalued mappings $\varphi, \psi\colon [0,\pi] \multimap \mathbb{R}^2$ defined as follows:

$$\begin{aligned} \varphi(t) &= \{(x,y) \in \mathbb{R}^2 \mid y \geq 0 \text{ and } x^2 + y^2 \leq 1\}, \quad \text{for every } t \in [0,\pi];\\ \psi(t) &= \{(x,y) \in \mathbb{R}^2 \mid x = \lambda \cos t,\ y = \lambda \sin t,\ \lambda \in [-1,1]\}. \end{aligned}$$

Then φ is a constant map and hence even continuous, ψ is l.s.c. map but $\varphi \cap \psi$ is no longer l.s.c. (to see it consider $t = 0$ or $t = \pi$).

One can prove the following:

(15.6) PROPOSITION. *Let $\varphi\colon X \multimap Y$ be an l.s.c. map $f\colon X \to Y$ and $\lambda\colon X \to (0,\infty)$ be continuous mappings. Assume further that for every $x \in X$ we have:*

$$\varphi(x) \cap B(f(x), \lambda(x)) \neq \emptyset.$$

Then the map $\psi\colon X \multimap Y$ defined by

$$\psi(x) = \operatorname{cl}(\varphi(x) \cap B(f(x), \lambda(x))$$

is a l.s.c. map.

PROOF. Let $x_0 \in X$ and V be an open set of Y such that $V \cap \psi(x_0) \neq \emptyset$. Let $y_0 \in V \cap (\varphi(x_0) \cap B(f(x_0), \lambda(x_0)))$ and let V_{y_0} be an open neighbourhood of y_0 in Y such that $V_{y_0} \subset V \cap B(f(x_0), \lambda(x_0))$.

Now, continuity of f and λ implies that there is an open neighbourhood U_{x_0} of x_0 in X such that $V_{y_0} \subset B(f(x), \lambda(x))$ for every $x \in U_{x_0}$.

Consequently, since φ is l.s.c. we choose an open neighbourhood W_{x_0} of x_0 in X such that $\varphi(x) \cap V_y \neq \emptyset$, for every $x \in W_{x_0}$. Let $U = U_{x_0} \cap W_{x_0}$. Then we get that $(\varphi(x) \cap B(f(x), \lambda(x))) \cap V_{y_0} \neq \emptyset$ implies $\mathrm{cl}(\varphi(x) \cap B(f(x), \lambda(x))) \cap V \neq \emptyset$, for every $x \in U$ and the proof is completed. □

For another formulations of (15.6) we recommend [BrGMO1-M] (see also references in [BrGMO2-M] and [BrGMO3-M]).

We shall end this section by considering important examples of multivalued maps to be tangent and normed cones.

Let M be a closed nonempty subset of a Banach space. By $T_M(x)$ we shall denote the tangent Bouligand cone at x to M and by $N_M(x)$ the normal cone at x to M, where $x \in M$. We let:

$$T_M(x) = \{y \in E \mid \liminf[t^{-1} \operatorname{dist}(x + ty, M)\ t \to 0^+] = 0\},$$
$$N_M(x) = \{y \in E \mid \text{there exists } \alpha > 0 \text{ such that } \operatorname{dist}(x + \alpha y, M) = \alpha\|y\|\}.$$

Observe that if M is convex or approximate retract contained in $\mathbb{R}^n$, then both $T_M(x)$ and $N_M(x)$ are nonempty.

Starting from now we shall assume that $M \subset \mathbb{R}^n$ is a proximate retract. We shall fix U to be an open neighbourhood of M in $\mathbb{R}^n$ and the metric retraction $r\colon U \to M$ i.e. $\|y - r(y)\| = \operatorname{dist}(y, M)$ (see (3.11)).

(15.7) PROPOSITION. *Suppose that $r\colon (M + \varepsilon B) \to M$ is a metric retraction. Then $r(x + y) = x$ for any $x \in M$ and $y \in (N_M(x) \cap \varepsilon B)$, where B is the unit open ball in $\mathbb{R}^n$.*

PROOF. Let $x \in M$. Suppose that there exists $y \in (N_M(x) \cap \varepsilon B)$ such that $r(x+y) \neq x$. Let $\alpha_0 = \sup\{\alpha > 0 \mid r(x + \alpha y) = x\}$. If the inequality $\alpha_0 > 1$ held, there would be $\alpha > 1$ such that $r(x + \alpha y) = x$. So $\mathrm{cl}\,(B(x + \alpha y, \alpha|y|) \cap M = \{x\}$. As $\mathrm{cl}\,(B(x + y, |y|) \subset \mathrm{cl}\,(B(x + \alpha y, \alpha|y|))$, we would get a contradiction to the assumption $r(x + y) \neq x$. So $\alpha_0 \leq 1$. Let z_0 be given by $z_0 = x + \alpha_0 y$. Since r is continuous, $r(z_0) = x$. Let $R = 2^{-1} \min\{\alpha_0|y|, \varepsilon - \alpha_0|y|\}$. The radial retraction of $\mathbb{R}^n \setminus B(z_0, R)$ onto $S(z_0, R)$ will be denoted by p (2). Let $f\colon S(z_0, R) \to S(z_0, R)$ be the antipodal map. As $B(z_0, R) \cap M = \emptyset$, the function q from $\mathrm{cl}\,(B(z_0, R))$ into $S(z_0, R)$ given by $q(a) = f(p(r(a)))$ is well defined. According to Brouwer's fixed point theorem there must be $z \in S(z_0, R)$ with $q(z) = z$. Since $p(r(z)) = f(z)$, the point z_0 must belong to the interval with ends z and $r(z)$. Then $r(z_0) = r(z)$. As $r(z) = x$ we have $z = f(p(x))$. Since $f(p(x)) = x + (1 + R/(\alpha_0|y|))\alpha_0 y$, it follows that $r(x + (1 + R/(\alpha_0|y|))\alpha_0 y) = x$ which is the desired contradiction. □

(15.8) PROPERTY. *The normal cone map $N_M\colon M \multimap \mathbb{R}^n$, $x \multimap N_M(x)$, has a closed graph in $M \times \mathbb{R}^n$.*

(2) $p(z_0 + b) = z_0 + R \cdot \frac{b}{|b|}$.

PROOF. Suppose that sequences $\{x_n\} \subset M$ and $y_n \in N_M(x_n)$ satisfy the following conditions: $\lim x_n = x$, $\lim y_n = y$. By (15.7), we can choose a sequence $\{\alpha_n\}$ that converges to $\alpha > 0$ and satisfies $r(x_n + \alpha_n y_n) = x_n$ for every n. Since r is continuous, it follows that $r(x + \alpha y) = x$ which completes the proof. □

Let T be a cone in a normed space E. Let E^* denote the dual to E. For $p \in E^*$ and $x \in E$ we let:

$$\langle p, x\rangle = p(x).$$

When $T \subset \mathbb{R}^n$, then $\mathbb{R}^* \simeq \mathbb{R}^n$ and $\langle p, x\rangle$ is nothing more than the scalar product in $\mathbb{R}^n$.

Now, the polar cone $T^\perp$ to T is defined as follows:

$$T^\perp = \{p \in E^* \mid \langle p, x\rangle \le 0 \;\text{ for every } x \in T\}.$$

We prove:

(15.9) LEMMA. *Let $M \subset \mathbb{R}^n$ be a proximate retract, then $T_M(x) = N_M(x)^\perp$ for any $x \in M$.*

PROOF. First we show that $T_M(x) \subset N_M(x)^\perp$. Let $y \notin N_M(x)^\perp$ Then there exists $z \in N_M(x)$ such that $\langle y, z\rangle > 0$. We can that suppose that $B(x+z, |z|) \cap M = \emptyset$. It is easy to check that

$$\lim[t^{-1} d(x + ty, \mathbb{R}^n \setminus B(x + z, |z|)) \mid t \to 0^+] > 0.$$

From this, we conclude that $y \notin T_M(x)$.

It remains to prove that $N_M(x)^\perp \subset T_M(x)$. Let $z \notin T_M(x)$. Then there exist $R > 0$ and $c \in (2^{-1}\sqrt{2}, 1)$ such that

$$\{x + y \mid 0 < |y| < R \text{ and } \langle z, y\rangle \ge c|z|\,|y|\} \cap M = \emptyset. \tag{5.9.1}$$

We let

$$C = \{x + y \mid |y| = 3^{-1}R \text{ and } \langle z, y\rangle = \sqrt{(1 - c^2)}\;|y|\;|z|\}; \quad D = \operatorname{conv} C;$$
$$P = \{x + w \mid \langle w, z\rangle = 0\};$$
$$p\colon \mathbb{R}^n \to P \text{ is the orthogonal projection;}$$

q is the inverse function of $p|_D$, where $p|_D$ denotes the restriction of the function p to the set D.

Suppose that $y, v \in \mathbb{R}^n$ satisfy the following conditions:

$$|y - v| \le |y|, \quad \langle y, z\rangle = \sqrt{(1 - c^2)}\,|z|\,|y|$$

and there exists $\alpha > 0$ such that $p(x+v) - x = -\alpha(p(x+y) - x)$. Then by an easy calculation we obtain that $\langle v, z\rangle \geq c|z|\,|v|$.

Suppose that there exists $a \in p(C)$ and $\alpha > 0$ such that $p(r(q(a))) - x = -\alpha(a - x)$. Since

$$\langle q(a) - x, z\rangle = \sqrt{(1 - c^2)}\,|q(a) - x|\,|z| \quad \text{and} \quad |q(a) - r(q(a))| \leq |q(a) - x|,$$

the point $r(q(a))$ must belong to the set $\{x + y \mid 0 < |y| < R \text{ and } \langle z, y\rangle \geq c|z|\,|y|\}$, contradicting (5.9.1). By the Birkhoff–Kellogg theorem there exists $a \in p(D)$ such that $p(r(q(a))) = x$. So, there exists $\beta \in \mathbb{R}$ such that $r(q(a)) = x + \beta z$. As $\langle q(a) - x, z\rangle > 0$ and $|q(a) - r(q(a))| \leq |q(a) - x|$ we have $\beta \geq 0$. By (5.9.1), we obtain $\beta \leq 0$. Then $(q(a) - x) \in N_M(x)$. As $\langle z, q(a) - x\rangle > 0$ we have $z \notin N_M(x)^{\perp}$, which completes the proof. □

Now, let us observe that if $N\colon M \multimap \mathbb{R}^n$ is a convex cone valued map with closed graph Γ_N, then the map $T\colon M \multimap \mathbb{R}^n$ given by $T(x) = N(x)^{\perp}$ is l.s.c. (the proof is straighforward).

So from (15.9) and (15.8) we deduce:

(15.10) PROPOSITION. *The map $T_M\colon M \multimap \mathbb{R}^n$, $x \multimap T_M(x)$, is l.s.c. with closed convex values.*

16. Michael's selection theorem

The most famous continuous selection theorem is the following result proved by Michael (see [Mi1]–[Mi3] for details).

(16.1) THEOREM. *Let X be a metric space, E a Banach space and $\varphi\colon X \multimap E$ an l.s.c. map with closed convex values. Then there exists $f\colon X \to E$, a continuous selection of φ ($f \subset \varphi$).*

PROOF. *Step 1.* Let us begin by proving the following claim: given any convex (not necessarily closed) valued l.s.c. map $\Phi\colon X \multimap E$ and every $\varepsilon > 0$, there exists a continuous $g\colon X \to E$ such that $\operatorname{dist}(g(x), \Phi(x)) \leq \varepsilon$, i.e. $g(x) \in O_\varepsilon(\Phi(x))$, for every $x \in X$.

In fact, for every $x \in X$, let $y_x \in \Phi(x)$ and let $\delta_x > 0$ be such that $B(y_x, \varepsilon) \cap \Phi(x') \neq \emptyset$, for x' in $B(x, \delta_x)$. Since X is metric, it is paracompact. Hence there exists a locally finite refinement $\{U_x\}_{x\in X}$ of $\{B(x, \delta_x)\}_{x\in X}$. Let $\{L_x\}_{x\in X}$ be a partition of unity subordinate to it. The mapping $g\colon X \to E$ defined as follows:

$$g(u) = \sum_{x\in X} L_x(u) \cdot y_x$$

is continuous since it is locally a finite sum of continuous functions. Fix $n \in X$. Whenever $L_x(u) > 0$, $n \in B(x, \delta_x)$, hence $y_x \in O_\varepsilon(\Phi(u))$. Since this latter set is convex, any convex combination of such y's belongs to it.

Step 2. Next we claim that we can define a sequence $\{f_n\}$ of continuous mappings from X to E with the following properties

$$\text{(16.1.1)} \qquad \operatorname{dist}(f_n(u), \varphi(u)) \leq \frac{1}{2^n}, \qquad n = 1, 2, \dots, \; u \in X,$$

$$\text{(16.1.2)} \qquad \|f_n(u) - f_{n-1}(u)\| \leq \frac{1}{2^{n-2}}, \quad n = 2, 3, \dots, \; u \in X.$$

For $n = 1$ it is enough to take in the Step 1, $\Phi = \varphi$ and $\varepsilon = 1/2$.

Assume we have defined mappings f_n satisfying (1.16.1) up to $n = k$. We shall define f_{k+1} satisfying (1.16.1) and (1.16.2) as follows.

Consider the set $\Phi(u) = B(f_k(u), 1/2^k) \cap \varphi(u)$. By (1.16.1) it is not empty, and it is a convex set. By (15.6) the map Φ is l.s.c. so by the claim in Step 1 there exists a continuous g such that

$$\operatorname{dist}(g(x), \Phi(x)) < \frac{1}{2^{n+1}}.$$

Set $f_{k+1}(u) = g(u)$. A portion $\operatorname{dist}(f_{k+1}(u), \varphi(u)) < 1/2^{k+1}$, proving (a). Also

$$f_{k+1}(u) \in O_{1/2^{k+1}}(\Phi(u)) \subset B\bigg(f_k(u), \frac{1}{2^k} + \frac{1}{2^{k+1}}\bigg),$$

i.e.

$$\|f_{k+1}(u) - f_n(u)\| \leq \frac{1}{2^{k-1}}$$

proving (1.16.2).

Step 3. Since the series $\sum(1/2^n)$ converges, $\{f_n\}$ is a Cauchy sequence, uniformly converging to a continuous f. Since the values of φ are closed, by (1.16.1), f is a selection from F. The proof is completed. □

Some applications of the Michael selection theorem to differential inclusions we will show in the last chapter. Now, we explain the connection between the continuous selection property and extension property. Namely, we would like to show from (16.1) the following version of the Dugundji extension theorem.

(16.2) Corollary. *If E is a Banach space, then $E \in \mathrm{ES}$.*

Proof. Let A be a closed subset of X and let $f \colon A \to E$ be a continuous map. We define $\varphi \colon X \multimap E$ as follows:

$$\varphi(x) = \begin{cases} f(x) & x \in A, \\ \overline{\operatorname{conv}}(f(A)) & x \notin A. \end{cases}$$

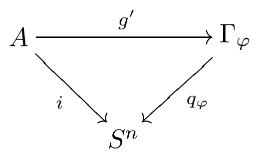
Then φ is l.s.c. with convex, closed values. So it has a selection $\widetilde{f} \subset \varphi$. It is evident that $\widetilde{f}$ is an extension of f.

We would like to stress that there are other Michael-type results concerning the existence of continuous singlevalued selections. Below we formulate a result of this type owed to McLendon and Bielawski (see [Bi-2], [Mcc2]).

Since for the proof of this result an apparatus concerning the Serre fibration theory is needed we shall sketch the proof only.

(16.3) THEOREM. *Let U be an open subset of S^n and let A be a closed subset of U. Assume further that $\varphi: U \multimap S^n$ has open finitely connected values and the graph Γ_φ of φ is an open subset of $U \times S^n$. Then any continuous selection $g: A \to S^n$ for φ (i.e. $g(x) \in \varphi(x)$ for every $x \in A$) can be extended to a continuous selection $f: U \to S^n$. Furthermore, any two such extensions are homotopic by a homotopy $h: U \times [0,1] \to S^n$ such that $h(\cdot\,, t)$ is a selection of φ for every $t \in [0,1]$.*

SKETCH OF PROOF. Assuming that the projection $p_\varphi: \Gamma_\varphi \to U$ is a Serre fibration (see [Sp-M, Chapter 7]) consider a diagram:



where $g'(x) = (x, g(x))$, $i(x) = x$ for every $x \in A$. Since $p_\varphi: \Gamma_\varphi \to X$ is a surjective Serre fibration with fibers $\varphi(x)$ which are infinitely connected, so standard obstruction theory (see [Sp-M, p. 404, Theorem 22; p. 416, Theorem 9]) gives an extension $f': U \to \Gamma_\varphi$ of g' over U, and two such extensions are homotopic. Let $f = q_\varphi \circ f'$. Then f is the extension of g required in theorem and if f_1 and f_2 are two such extensions and h is a homotopy joining f_1' with f_2' then $\overline{h} = q_\varphi \circ h$ is the needed homotopy between f_1 and f_2.

We are left with the problem of proving that $p_\varphi: \Gamma_\varphi \to U$ is a Serre fibration as a specific problem from algebraic topology. □

We will use Theorem (16.3) in Section 33. Note, that in fact Theorem (16.3) can be formulated in the following very general form (cf. [Bi-2]):

(16.4) THEOREM. *Let A be a closed subset of X such that $\dim(X \setminus A) \leq n+1$. Let Y be a locally n-connected space. Let $\varphi_0, \dots, \varphi_n: X \multimap Y$ be mappings with open infinitely connected values and open graphs. Then every continuous partial selection $g: A \to Y$ for $\varphi_0|_A$ can be extended to a continuous selection $f: X \to Y$ of φ_n.*

If in (16.4) we put $Y = S^n$, $X = U \subset S^n$ and $\varphi = \varphi_0 = \ldots = \varphi_n$ then we obtain (16.3). Observe that for A to be a singleton we can always find a partial selection. So from (16.3) (resp. (16.4)) follows the existence of a continuous selection for φ (resp. for φ_n).

17. σ-Selectionable mappings

The Michel selection theorem is not true for u.s.c. mappings (cf. (13.5.1) and (13.5.2)) but under some natural assumptions u.s.c. mappings are σ-selectionable.

(17.1) Definition. We say that a map $\varphi\colon X \multimap Y$ is *σ-selectionable*, if there exists a decreasing sequence of compact valued u.s.c. maps $\varphi_n\colon X \multimap Y$ satisfying:

(17.1.1) φ_n has a continuous selection, for all $n \geq 0$,
(17.1.2) $\varphi(x) = \bigcap_n \varphi_n(x)$, for all $x \in X$.

We prove the following:

(17.2) Theorem. *Let $\varphi\colon X \multimap E$ be an u.s.c. map with compact convex values from a metric space X to a Banach space E.*

If $\operatorname{cl}(\varphi(X))$ is a compact set, then φ is σ-selectionable. Actually, there exists a sequence of u.s.c. mappings φ_n from X to $\overline{co}(\varphi(X))$, which approximate φ in the sense that for all $x \in X$ we have:

$$(17.2.1)\qquad \begin{cases} \varphi(x) \subset \ldots \subset \varphi_{n+1}(x) \subset \varphi_n(x) \subset \ldots \subset \varphi_0(x) & \text{for all } n \geq 0, \\ \text{for all } \varepsilon > 0 \text{ there exists } n_0 = n_0(\varepsilon, x) & \\ \qquad\qquad \text{such that } \varphi_n(x) \subset \overline{0_\varepsilon(\varphi(x))} & \text{for all } n \leq n_0, \end{cases}$$

and moreover, the maps φ_n can be written in the following form:

$$(17.2.2)\qquad \varphi_n(x) = \sum_{i \in I(n)} L_i^{(n)}(x) \cdot C_i^{(n)} \quad \text{for all } x \in X,$$

where the subsets $C_i^{(n)}$ are compact and convex and where the functions $L_i^{(n)}$ form a locally Lipschitzian locally finite partition of unity.

(17.3) Remark. Note, that (17.2.1) imply that

$$\varphi(x) = \bigcap_{n \geq 0} \varphi_m(x).$$

If X is compact then the sets $I(n)$, which appear in formula (17.2.2), are finite.

Proof of (17.2). Let $K = \overline{\operatorname{conv}}\,\varphi(X)$. Then K is a compact convex subset of E. We fix $\varrho > 0$. Let us cover X with the open balls $\{B(x, \varrho)\}_{x \in X}$. Let

$\{\Omega_i^{(0)}\}_{i\in I(0)}$ be a locally finite refinement of $\{B(x,\varrho)\}_{x\in X}$. For any $i \in I(0)$ exists $x_i^{(0)} \in X$ with $\Omega_i^{(0)} \subset B(x_i^{(0)},\varrho)$. We then define for any $i \in I(0)$ the set $C_i^{(0)} = \overline{\text{conv}}\,\varphi(B(x_i^{(0)},2\varrho))$ which is a nonempty closed convex subset of K.

Now, we can associate a locally Lipschitzian partition of unity $\{L_i^{(0)}\}_{i\in I(0)}$ to open covering $\{\Omega_i^{(0)}\}_{i\in I(0)}$ (see Theorem 0.1.2 in [AuC-M]).

We define the map $\varphi_0\colon X \multimap E$ by putting:

$$\varphi_0(x) = \sum_{i\in I(0)} L_i^{(0)}(x) C_i^{(0)}.$$

Then the map $f_0\colon X \to E$ given by:

$$f_0(x) = \sum_{i\in I(0)} L_i^{(0)}(x)\cdot y_i^{(0)}$$

is a locally Lipschitzian selection of φ_0, where $y_i^{(0)} \in C_i^{(0)}$ is fixed for every $i \in I(0)$. In order to define φ_1 we do the some as before with the open covering $\{B(x,\varrho/3)\}_{x\in X}$. Thus we consider its locally finite refinement $\{\Omega_i^{(1)}\}_{i\in I(1)}$ and associated locally Lipschitzian partition of unity $\{L_i^{(1)}\}_{i\in I(1)}$.

As before we set, for all $i \in I(1)$,

$$C_i^{(1)} = \overline{\text{conv}}\,\varphi\left(B\left(x_i^{(1)}, \frac{2\varrho}{3}\right)\right) \subset K$$

and define $\varphi_1\colon X \multimap E$ by putting:

$$\varphi_1(x) = \sum_{i\in I(1)} L_i^{(1)}(x)\cdot C_i^{(1)}.$$

The map φ_1 enjoys the some properties as φ_0.

We shall now prove that $\varphi_1(x) \subset \varphi_0(x)$ for every $x \in X$. Let us fix $x \in X$. Then we define:

$$I_{(0)}^x = \{i \in I(0) \mid x \in B(x_i^{(0)},\varrho)\}, \qquad I_{(1)}^x = \left\{ i \in I(1) \,\middle|\, x \in B\left(x_i^{(1)}, \frac{\varrho}{3}\right)\right\}.$$

Let $i(0) \in I_{(0)}^x$ and $i(1) \in I_{(1)}^x$ be given. Then, if $y \in B(x_{i(1)}^{(1)}, 2\varrho/3)$ we obtain:

$$d(y, x_{i(1)}^{(1)}) < \frac{2\varrho}{3} \quad \text{with } d\big(x, x_{i(0)}^{(0)}\big) < \varrho \quad \text{and} \quad d\big(x, x_{i(1)}^{(1)}\big) < \frac{\varrho}{3}.$$

Thus we have:

$$d(y, x_{i(0)}^{(0)}) \le \frac{2\varrho}{3} + \frac{\varrho}{3} + \varrho = 2\varrho.$$

Then $B(x^{(1)}_{i(1)}, 2\varrho/3) \subset B(x^{(0)}_{i(0)}, 2\varrho)$ for all $i_{(0)} \in I^x_{(0)}$ and all $i_{(1)} \in I^x_{(1)}$. This leads to $C^{(1)}_{i(1)} \subset C^{(0)}_{i(0)}$ for such indexes. Then for all $i(1) \in I^x_{(1)}$ we have:

$$C^{(1)}_{i(1)} \subset \sum_{i \in I^x_{(0)}} L^{(0)}_i(x) C^{(0)}_i = \sum_{i \in I(0)} L^{(0)}_i C^{(0)}_i = \varphi_0(x).$$

This being true by convexity arguments, and since $\{L^{(0)}_i\}_{i\in I_{(0)}}$ is a locally finite partition of unity associated with $\{\Omega^{(0)}_i\}_{i\in I(0)}$, and thus also with $\{B(x^{(0)}_i, \varrho)\}_{i\in I(0)}$ in particular, says that $L^{(0)}_i(x) = 0$, if $i \in I^x_{(0)}$.

Consequently, for the same reasons we get:

$$\varphi_1(x) = \sum_{i\in I(1)} L^{(1)}_i(x) \cdot C^{(1)}_i = \sum_{i\in I^x_{(1)}} L^{(1)}_i(x) C^{(1)}_i \subset \varphi_0(x).$$

Moreover, it is easy to see that $\varphi(x) \subset \varphi_1(x)$ for every $x \in X$.

Now, let us define $\varrho_n = (1/3) \cdot \varrho$ for any $n = 1, 2, \ldots$. Then we can build by induction a sequence of multivalued maps $\varphi_n : X \multimap E$ each of them being u.s.c. nonempty convex compact valued and satisfying the first part of (17.2.1) (as φ_0 for $\varrho_0 = \varrho$ and φ_1 for $\varrho_1 = \varrho/3$).

So to end the proof we have to show that for every $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon, x)$ such that $\varphi_n(x) \subset O_\varepsilon(\varphi(x))$.

Let $x \in X$ be given. Since φ is u.s.c. for any $\varepsilon > 0$ there exists $\eta = \eta(\varepsilon, x)$ such that $\operatorname{cl}(y, x) \le \eta$ implies $\varphi(y) \subset O_\varepsilon(\varphi(x))$. Then there obviously exists $n_0 = n_0(\varepsilon, x)$ such that for $n \ge n_0$ we have $\varrho_n \le \eta/3$.

Let us define as before $I^x_{(n)} = \{i \in I(n) \mid x \in B(x^{(n)}_i, \varrho_n)\}$. For the same reasons as for φ_0 and φ_1 we can write:

$$\varphi_n(x) = \sum_{i\in I^x_{(n)}} L^{(n)}_i(x) \cdot C^{(n)}_i,$$

where $C^{(n)}_i = \overline{\operatorname{conv}}\, \varphi(B(x^{(n)}_i, 2\varrho) \subset K$. Then for all $y \in B(x^{(n)}_i, 2\varrho)$ with $i \in I^x_{(n)}$ we have:

$$d(y, x) \le d(y, x^{(n)}_i) + d(x^{(n)}_i, x) \le 2\varrho_n + \varrho_n = 3\varrho_n < \eta, \quad \text{if we take } n > n_0.$$

Thus for all $n \le n_0$ we have $\varphi(y) \subset O_\varepsilon(\varphi(x))$ for all $y \in B(x^{(n)}_i, 2\varrho_n)$ with $i \in I^x_{(n)}$.

But since $\overline{O_\varepsilon(\varphi(x))}$ is closed and convex we obtain: $C^{(n)}_i \subset \overline{O_\varepsilon(\varphi(x))}$ and by convexity we infer $\varphi_n(x) \subset \overline{O_\varepsilon(\varphi(x))}$ for all $n \ge n_0$. Therefore the proof of (17.2) is completed. □

(17.4) REMARKS.

(17.4.1) Note that originally Theorem (17.2) was proved in [CR-M] (cf. also [HL]).

(17.4.2) If X is a compact space then φ is automatically compact (cf. (14.9)).

(17.4.3) If $E = \mathbb{R}^n$ then any bounded u.s.c. map with convex compact values satisfies assumptions of (17.2).

(17.5) REMARK. Instead of σ-selectionable mappings we can define for example, Lipschitz σ-selectionable mappings (L-σ-selectionable) or locally Lipschitz σ-selectionable mappings (LL-σ-selectionable) if in (17.1.1) we ask φ_n has Lipschitz selection (locally Lipschitz selection) for every $n \geq 0$.

As we will see in Section 22 the following result, owed to A. Lasota and J. Yorke, (see [Go2-M]) is useful in showing that a map is LL-σ-selectionable.

(17.6) THEOREM (Lasota–Yorke Approximation Theorem). *Let E be a normed space and $f: X \to E$ be a continuous singlevalued map. Then, for each $\varepsilon > 0$ there is a locally Lipschitz* (*singlevalued*) *map $f_\varepsilon: X \to E$ such that:*

$$\|f(x) - f_\varepsilon(x)\| < \varepsilon \quad \text{for every } x \in X.$$

PROOF. Let $V_\varepsilon(x) = \{y \in X \mid \|f(y) - f(x)\| < (\varepsilon/2)\}$. Then $\alpha = \{V_\varepsilon(x)\}_{x\in X}$ is an open covering of the metric space X. Since X is paracompact there exists a locally finite subcovering $\beta = \{W_\lambda\}_{\lambda\in\Lambda}$ of α. For every $\lambda \in \Lambda$ we obtain $\mu_\lambda: X \to \mathbb{R}$ by putting

$$\mu_\lambda(x) = \begin{cases} 0 & \text{for } x \notin W_\lambda, \\ \inf\{d(x,y) \mid y \in \partial W_\lambda\} & \text{for } x \in W_\lambda. \end{cases}$$

Then μ_λ is a Lipschitz map with constant 1. Since β is locally finite we deduce that for every $\lambda \in \Lambda$ the map $\eta_\lambda: X \to [0,1]$ defined as follows

$$\eta_\lambda(x) = \frac{\mu_\lambda(x)}{\sum_{\varrho\in\Lambda} \mu_\varrho(x)}$$

is a locally Lipschitz map.

We let $f_\varepsilon: X \to E$ by putting

$$f_\varepsilon(x) = \sum_{\lambda\in\Lambda} \eta_\lambda(x) \cdot f(a_\lambda),$$

where $a_\lambda \in W_\lambda$ is an arbitrary but fixed element. Then we have:

$$\begin{aligned} \|f_\varepsilon(x) - f(x)\| &= \bigg\| \sum_{\lambda\in\Lambda} \eta_\lambda(x) \cdot f(a_\lambda) - \sum_{\lambda\in\Lambda} \eta_\lambda(x) \cdot f(x) \bigg\| \\ &\leq \sum_{\lambda\in\Lambda} \eta_\lambda(x) \|f(a_\lambda) - f(x)\| < 1 \cdot \varepsilon = \varepsilon \end{aligned}$$

and the proof is completed. □

18. Directionally continuous selections

Michael selection theorem is no true for l.s.c. mappings with arbitrary compact values. For example see (13.5.9). In 1988, A. Bressan ([Bre1], [Bre2]) observed there exists a directionally continuous selection.

In the following, a set $\Gamma \subseteq \mathbb{R}^m$ will be called a cone if Γ is a nonempty closed convex subset of $\mathbb{R}^m$ such that

(18.1.1) $$\text{if } x \in \Gamma,\ \lambda \geq 0 \text{ then } \lambda x \in \Gamma,$$

(18.1.2) $$\Gamma \cap (-\Gamma) = \{0\}.$$

We now introduce the basic concept discussed in this section.

(18.2) Definition. Let Γ be a cone in $\mathbb{R}^m$ and let Y be a metric space. A map $f\colon \mathbb{R}^m \to Y$ is Γ-continuous at a point $\overline{x} \in \mathbb{R}^m$ if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $d(f(x), f(\overline{x})) < \varepsilon$ for all $x \in B(\overline{x}, \delta) \cap (\overline{x} + \Gamma)$. We say that f is Γ-continuous on A if it is Γ-continuous at every point $\overline{x} \in A$.

In the above setting some preliminary results concerning directional continuity are now listed.

(18.3) Proposition.

(18.3.1) *f is Γ-continuous at $\overline{x}$ if and only if $\lim_{n\to\infty} f(x_n) = f(\overline{x})$ for every sequence x_n tending to $\overline{x}$ such that $(x_n - \overline{x}) \in \Gamma$ for all $n \geq 1$.*

(18.3.2) *If f is Γ-continuous at $\overline{x}$ then f is also Γ'-continuous at $\overline{x}$ for every cone $\Gamma' \subseteq \Gamma$.*

(18.3.3) *If $(f_n)_{n\geq 1}$ is a sequence of Γ-continuous functions which converges uniformly to f then f is Γ-continuous.*

All proofs are straightforward. Particularly interesting is the case where $\Gamma = \mathbb{R}^m_+$ is the cone of all points in $\mathbb{R}^m$ with non-negative coordinates (with respect to the canonical basis). Indeed, a large class of $\mathbb{R}^m_+$-continuous functions can be constructed. For every integer k define the partition $\mathcal{P}_k$ of $\mathbb{R}^m$ into half-open cubes with side length 2^{-k}:

(18.4.1) $$\mathcal{P}_k = \{Q^k_\eta \mid \eta = (\eta_1, \dots, \eta_m) \in Z^m\},$$

(18.4.2) $$Q^k_\eta = \{x = (x_1, \dots, x_m) \mid 2^{-k}(\eta_i - 1) \leq x_i < 2^{-k}\eta_i,\ i = 1, \dots, m\}.$$

In this setting, we have:

(18.5) Proposition.

(18.5.1) *Let k be any integer. If the restriction of f to each cube $Q^k_\eta \in \mathcal{P}_k$ is $\mathbb{R}^m_+$-continuous, then f is $\mathbb{R}^m_+$-continuous on $\mathbb{R}^m$.*

(18.5.2) *If, for some k, f is constant on every cube $Q^k_\eta \in \mathcal{P}_k$ then f is $\mathbb{R}^m_+$-continuous.*

PROOF. Let $\mathcal{T}^+$ be the topology on $\mathbb{R}^m$ generated by the sets

$$A_{x,\varepsilon} = \{y \in \mathbb{R}^m \mid (y - x) \in \mathbb{R}^m_+,\ \|y - x\| < \varepsilon\},$$

with $x \in \mathbb{R}^m$, $\varepsilon > 0$. Saying that a map f is $\mathbb{R}^m_+$-continuous simply means that f is continuous with respect to the topology $\mathcal{T}^+$. Since all cubes Q^k_η are closed-open sets in $\mathcal{T}^+$, assertions (18.5.1) and (18.5.2) follow. □

The next result provides a useful tool for reducing a problem concerning an arbitrary cone Γ to the special case where $\Gamma = \mathbb{R}^m_+$.

(18.6) PROPOSITION.

(18.6.1) *Let Γ be a cone in $\mathbb{R}^m$, Y a metric space. Let L be an invertible linear operator on $\mathbb{R}^m$. Then a map $f\colon \mathbb{R}^m \to Y$ is Γ-continuous at a point $\overline{x}$ if and only if the composite map $f \circ L^{-1}$ is $L(\Gamma)$-continuous at $L(\overline{x})$.*

(18.6.2) *For every cone $\Gamma \subseteq \mathbb{R}^m$, there exists an invertible linear operator ψ on $\mathbb{R}^m$ such that $\psi(\Gamma) \subseteq \mathbb{R}^m_+$.*

PROOF. To prove (18.6.1) assume that $f \circ L^{-1}$ is $L(\Gamma)$-continuous at $L(\overline{x})$. Take any sequence $x_n \to \overline{x}$ such that $(x_n - \overline{x}) \in \Gamma$ for all $n \geq 1$. Then $L(x_n) \to L(\overline{x})$ and $(L(x_n) - L(\overline{x})) \in L(\Gamma)$. Hence

$$d(f(x_n), f(\overline{x})) = d(f \circ L^{-1}(L(x_n)), f \circ L^{-1}(L(\overline{x}))) \to 0,$$

showing that f is Γ-continuous at $\overline{x}$. The converse is obtained by replacing L with L^{-1}.

To prove (18.6.2), let Γ be given and consider the positive dual cone

$$\Gamma^+ = \{y \in \mathbb{R}^m \mid \langle y, x\rangle \geq 0,\ \text{for all } x \in \Gamma\}.$$

Since Γ^+ has nonempty interior (see [De4-M]), there exists a unit vector $w_1 \in \operatorname{int}(\Gamma^+)$ and some $\varepsilon > 0$ such that

$$\langle w_1, x\rangle \geq \varepsilon\, \|x\| \quad \text{for all } x \in \Gamma. \tag{18.6.3}$$

Choose $m - 1$ unit vectors $w_2, \ldots, w_m$ such that $\{w_1, \ldots, w_m\}$ is an orthonormal basis for $\mathbb{R}^m$ and let $\{e_1, \ldots, e_m\}$ be the canonical basis. Define the invertible operators L_1, L_2 on $\mathbb{R}^m$ by setting:

$$\begin{aligned} L_1(w_1) &= w_1 + \varepsilon^{-1}(w_2 + \ldots + w_m), && \\ L_1(w_i) &= w_i && i = 2, \ldots, m, \\ L_2(w_j) &= e_j && j = 1, \ldots, m. \end{aligned}$$

The transformation $L = L_2 \circ L_1$ then satisfies our requirement. Indeed, if $u = \sum_{j=1}^{m} \lambda_j w_j \in \Gamma$, (18.4.1) implies

$$\lambda_1 \geq \varepsilon \Big(\sum_{j=1}^{m} \lambda_j^2 \Big)^{1/2} \geq \varepsilon \|\lambda_i\| \tag{18.6.4}$$

for all $i = 1, \dots, m$. Let $L_1(u) = \sum_{i=1}^{m} \mu_i w_i$. Then $\mu_1 = \lambda_1 \geq 0$ and $\mu_i = \lambda_i + \varepsilon^{-1}\lambda_i \geq 0$ for $2 \leq i \leq m$, because of (18.6.4). Therefore,

$$L(u) = \sum_{i=1}^{m} \mu_i e_i \in \mathbb{R}^m_+. \qquad \square$$

Now, we are able to prove the main result of this section:

(18.7) THEOREM. *Let $\varphi: \mathbb{R}^m \multimap Y$ be a l.s.c. map with nonempty closed values and Y be a complete* (*metric*) *space. Then for every cone $\Gamma \subset \mathbb{R}^m$ the mapping φ admits a Γ-continuous selection.*

PROOF. The proof will first be given in the special case $\Gamma = \mathbb{R}^m_+$ then extended to an arbitrary cone Γ. We begin by constructing a sequence of approximate selections $(f_n)_{n \geq 1}$ on the half-open unit cube $Q = \{(x_1, \dots, x_m) \in \mathbb{R}^m \mid 0 \leq x_i < 1,\ i = 1, \dots, m\}$. Each f_n will have the following properties:

- $(\mathrm{i})_n$ There exists an integer $h = h(n)$ such that f_n is constant on every cube $Q_\eta^{h(n)} \subseteq Q$ of the partition $\mathcal{P}_{h(n)}$, defined at (18.2.1); say, $f(x) = y_\eta^n \in Y$ for all $x \in Q_\eta^{h(n)}$,
- $(\mathrm{ii})_n$ $d(y_\eta^n, \varphi(x)) < 2^{-n}$ for all $x \in \overline{Q_\eta^{h(n)}}$,
- $(\mathrm{iii})_n$ $d(f_n(x), f_{n-1}(x)) < 2^{-n+1}$ for all $x \in Q (n \geq 2)$.

To define f_1, choose a finite set of points $a_1, \dots, a_k \in \overline{Q}$, elements $y_i \in F(a_i)$ and open neighbourhoods $V_1, \dots, V_k$ such that

$$a_i \in V_i, \tag{18.7.1}$$

$$\bigcup_{i=1}^{k} V_i \supseteq \overline{Q}, \tag{18.7.2}$$

$$d(y_i, \varphi(x)) < 2^{-1} \quad \text{for all } x \in V_i. \tag{18.7.3}$$

All this can be done because φ is lower semicontinuous on the compact set $\overline{Q}$. Let λ be a Lebesgue number for the covering $\{V_i\}$ of $\overline{Q}$ and choose an integer $h = h(1)$ so large that the closure of every cube $\overline{Q_\eta^h} \subseteq \overline{Q}$ is entirely contained in some V_i. This is certainly the case if $\sqrt{m} \cdot 2^{-h} < \lambda$. For each $Q_\eta^h \subseteq Q$, choose a V_i such that $\overline{Q_\eta^h} \subseteq V_i$ and define $f_1(x) = y_i$ for all $x \in Q_\eta^h$. Clearly $(\mathrm{i})_1$ and $(\mathrm{ii})_1$ hold.

Let now f_n be defined and satisfy $(\mathrm{i})_n$–$(\mathrm{iii})_n$. We shall construct f_{n+1} separately on each cube of the partition $\mathcal{P}_{h(n)}$. Fix $\sigma \in Z^m$ such that $Q_\sigma^{h(n)} \subseteq Q$. By $(\mathrm{i})_n$, $f_n(x) = y_\sigma^n$ is constant on $Q_\sigma^{h(n)}$. Choose a finite set of points $a_1, \dots, a_k \in \overline{Q_\sigma^{h(n)}}$ elements $y_i \in F(a_i)$ and open neighbourhoods $V_1, \dots, V_k$ such that

$$a_i \in V_i, \tag{18.7.4}$$

$$\bigcup_{i=1}^{k} V_i \supseteq \overline{Q_\sigma^{h(n)}}, \tag{18.7.5}$$

$$d(y_i, F(x)) < 2^{-n-1}, \quad \text{for all } x \in V_i, \tag{18.7.6}$$

$$d(y_i, y_\sigma^n) < 2^{-n}. \tag{18.7.7}$$

Notice that all this can be done because φ is lower semicontinuous, $\overline{Q_\sigma^{h(n)}}$ is compact and $\overline{(\mathrm{ii})_n}$ holds. Choose an integer $h(\sigma)$ so large that every closed cube $\overline{Q_\eta^{h(n)}} \subseteq \overline{Q_\sigma^{h(n)}}$ is entirely contained in some V_i. For every $Q_\eta^{h(\sigma)}$, select a V_i for which $\overline{Q_\eta^{h(\sigma)}} \subseteq V_i$ and define $f_{n+1}(x) = y_i$, for all $x \in Q_\eta^{h(\sigma)}$. Repeating this construction on every cube $Q_\sigma^{h(n)} \subseteq Q$, we obtain an approximate selection f_{n+1} defined on the whole cube Q. Setting $h(n+1) = \max\{h(\sigma) \mid \sigma \in Z^m, Q_\sigma^{h(n)} \subseteq Q\}$, conditions $(\mathrm{i})_{n+1} - (\mathrm{iii})_{n+1}$ are all satisfied.

By induction, we can now assume that a sequence $(f_n)_{n \geq 1}$ satisfying (i)–(iii) has been constructed. By (iii) and the completeness of Y, the sequence (f_n) has a uniform limit $f\colon Q \to Y$. Property (i) together with (18.5.2) and (18.3.1) imply that f is $\mathbb{R}_+^m$-continuous. Moreover, $f(x) \in F(x)$ for all $x \in Q$, because of (ii) and of the closure of $\varphi(x)$. Therefore, f is an $\mathbb{R}_+^m$-continuous selection of φ on Q. Repeating the same construction on every cube Q_η^0, $\eta \in Z^m$, and recalling (18.5.1) we obtain an $\mathbb{R}_+^m$-continuous selection of φ defined on the whole space $\mathbb{R}^m$.

Consider now the case where $\Gamma \subseteq \mathbb{R}^m$ is an arbitrary cone. Using (18.6.2), let L be an invertible linear operator such that $L(\Gamma) \subseteq \mathbb{R}_+^m$. Construct a $\mathbb{R}_+^m$-continuous selection g of the lower semicontinuous map $\varphi \circ L^{-1}$ and set $f = g \circ L$. This yields

$$f(x) = g(L(x)) \in \varphi \circ L^{-1}(L(x)) = \varphi(x)$$

for every $x \in \mathbb{R}^m$, hence f is a selection of φ. Since $L(\Gamma) \subseteq \mathbb{R}_+^m$, (18.3.2) and (18.6.1) imply that g is $L(\Gamma)$-continuous and f is Γ-continuous. The proof of (18.7) is completed. □

Theorem (18.7) has important applications to the existence results for differential inclusions with l.s.c. right hand sides.

Therefore, we shall come back to this theorem in the last chapter of our book.

Note that Theorem (18.7) is not a generalization of the Michael selection theorem. There exists the second additional result connected with Michael selection theorem. Namely, in 1983 A. Fryszkowski [Fry-1] (comp. also [FryR], [Fry-5]) proved the selection theorem for l.s.c. mappings with closed decomposable values. A generalization of Fryszkowski's theorem for separable spaces is proved by Bressan and Colombo in 1988. We do not present Fryszkowski's result because we will not apply it to differential inclusions. Moreover, let us remark that Fryszkowski's result is formulated more in terms of the measure theory rather than in topological terms.

Finally, we would like to point out that in the next section we will present Kuratowski–Ryll–Nardzewski selection theorem used in the theory of differential inclusions.

19. Measurable selections

Apart from semicontinuous multivalued mappings, multivalued measurable mappings will be great importance in the sequel. In this section we assume throughout that Y is a separable metric space, and $(\Omega, \mathcal{U}, \mu)$ is a measurable space, i.e. a set Ω equipped with σ-algebra $\mathcal{U}$ of subsets and a countably additive measure μ on $\mathcal{U}$. A typical example is when Ω is a bounded domain in the Euclidean space $\mathbb{R}^k$, equipped with the Lebesgue measure.

(19.1) Definition. *A* multivalued map $\varphi\colon \Omega \multimap Y$ with closed values is called *measurable*, if $\varphi^{-1}(V) \in \mathcal{U}$ for each open $V \subset Y$.

(19.2) Definition. *A* multivalued map $\varphi\colon \Omega \multimap Y$ with closed values is called *weakly measurable*, if $\varphi^{-1}(A) \in \mathcal{U}$ for each closed $A \subset Y$.

Another way of defining measurability is by requiring the measurability of the graph Γ_φ of φ in the product $\Omega \times Y$, equipped with the minimal σ-algebra $\mathcal{U} \quad (Y)$ generated by the sets $A \times B$ with $A \in \mathcal{U}$ and $B \in \mathcal{B}(Y)$, where $\mathcal{B}(Y)$ denotes the family of all Borel subsets of Y.

For further reference, we collect some relations between these definitions in the following

(19.3) Proposition. *Assume that $\varphi, \psi\colon \Omega \multimap Y$ are two multivalued mappings. Then the following hold true*

(19.3.1) *φ is measurable if and only if $\varphi_+^{-1}(A) \in \mathcal{U}$ for each closed $A \subset Y$,*

(19.3.2) *φ is weakly measurable if and only if $\varphi_+^{-1}(V) \in \mathcal{U}$ for each open $V \subset Y$,*

(19.3.3) *if φ is measurable then φ is also weakly measurable,*

(19.3.4) *if φ has compact values, measurability and weak measurability of φ are equivalent,*

(19.3.5) *φ is weakly measurable if and only if the distance function $f_y\colon \Omega \to \mathbb{R}$, $f_y(x) = \operatorname{dist}(y, \varphi(x))$ is measurable for all $y \in Y$,*

(19.3.6) *if φ is weakly measurable then the graph Γ_φ of φ is product measurable,*
(19.3.7) *if φ and ψ are measurable then so is $\varphi \cup \psi$,*
(19.3.8) *if φ and ψ are measurable then so is $\varphi \cap \psi$,*
(19.3.9) *if φ and ψ are measurable then so is $\varphi \times \psi$.*

The proof of (19.3) is straightfoward and therefore we left it to the reader.

Of course, the composition of two measurable multivalued mappings need not be measurable.

(19.4) EXAMPLE. Let $\Omega = [0,1]$ be equipped with the Lebesgue measure and let $f\colon \Omega \to \mathbb{R}$ be a strictly increasing Cantor function which of course is measurable. It is well known that one may find a measurable set $\mathcal{D} \subset \mathbb{R}$ such that $f^{-1}(\mathcal{D})$ is not measurable. If we define $\varphi\colon \Omega \multimap \mathbb{R}$ and $\psi\colon \mathbb{R} \multimap \mathbb{R}$ by

$$\varphi(t) = \{f(t)\} \quad \text{for } t \in \Omega, \qquad \psi(u) = \begin{cases} \{1\} & \text{if } \ u \in \mathcal{D}, \\ \{0\} & \text{if } \ u \notin \mathcal{D}, \end{cases}$$

then both φ and ψ are measurable, but $\psi \circ \varphi$ is not.

For further reference, we collect the results and counterexamples given so far on the conservation of semicontinuity or measurability properties in the following table:

φ, ψ	u.s.c.	l.s.c.	measurable
$\varphi \cup \psi$	yes	yes	yes
$\varphi \cap \psi$	yes	no	yes
$\varphi \times \psi$	yes*	yes*	yes
$\varphi \circ \psi$	yes	yes	no

* if φ and ψ have compact values

A famous relation between measurability and continuity of singlevalued functions is established by Luzin's theorem, which states, roughly speaking, that $f\colon \Omega \to Y$ is measurable if and only if f is continuous up on to subsets of Ω of arbitrarily small measure. It is not surprising that this result has an analogue for multivalued mappings (for details see [APNZ-M], [Fis-M]). Below we shall sketch Luzin's-type of multivalued results.

(19.5) DEFINITION. We will say that a multivalued map $\varphi\colon \Omega \multimap Y$ with closed values has the *Luzin property* if, given $\delta > 0$, one may find a closed subset $\Omega_\delta \subset \Omega$ such that $\mu(\Omega \setminus \Omega_\delta) \le \delta$ and the restriction $\varphi|_{\Omega_\delta}$ of φ to Ω_δ is continuous (of course we have assumed that Ω is a metric space).

We have:

(19.6) THEOREM. *A multivalued map* $\varphi\colon \Omega \multimap Y$ *is measurable if and only if* φ *has the Luzin property.*

In what follows we shall use the following Kuratowski–Ryll–Nardzewski selection theorem (see [AF-M], [AuC-M], [Ki-M], [Sr-M]).

(19.7) THEOREM. *Let* Y *be a separable complete space. Then every measurable* $\varphi\colon \Omega \multimap Y$ *has a* (*singlevalued*) *selection.*

PROOF. Without loss of generality we can change the metric of Y into an equivalent metric, preserving completeness and separability, so that Y becomes a bounded (say, with diameter M) complete metric space. Now, let us divide the proof into two steps.

Step 1. Let C be a countable dense subset of Y. Set $\varepsilon_0 = M$, $\varepsilon_i = M/2^i$. We claim that we can define a sequence of mappings $s_m\colon \Omega \to C$ such that:

(19.7.1) s_m is measurable,
(19.7.2) $s_m(x) \in O_{\varepsilon_m}(\varphi(x))$,
(19.7.3) $s_m(x) \in B(s_{m-1}(x), \varepsilon_{m-1})$, $m > 0$.

In fact, arrange the points of C into a sequence $\{c_j\}_{j=0,1,\ldots}$ and define s_0 by putting:

$$s_0(x) = c_0, \quad \text{for every } x \in \Omega.$$

Then (19.7.1) and (19.7.2) are clearly satisfied.

Assume we have defined functions s_m satisfying (19.7.1) and (19.7.2) up to $m = p - 1$, and define s_p, satisfying (19.7.1)–(19.7.3) as follows. Set

$$\begin{aligned} A_j &= \varphi_+^{-1}(B(c_j, \varepsilon_p)) \cap s_{p-1}^{-1}(B(c_j, \varepsilon_{p-1})), \\ E_0 &= A_0, \quad E_j = A_j \setminus (E_0 \cup \ldots \cup E_{j-1}). \end{aligned}$$

We claim that

$$\Omega = \bigcup_{j=0}^{\infty} E_j.$$

Of course E_j, $j = 0, 1, \ldots$ is measurable (comp. (19.3)). In fact, let $x \in \Omega$ and consider $s_{p-1}(x)$ and $\varphi(x)$. By (19.7.2) $s_{p-1}(x) \in O_{\varepsilon_{p-1}}(\varphi(x))$; by the density of C there is a c_j such that at once $s_{p-1}(x) \in B(c_j, \varepsilon_{p-1})$ and $\varphi(x) \cap B(c_j, \varepsilon_{p-1}) \neq \emptyset$, i.e. $x \in A_j$. Finally, either $x \in E$, or it is in some E_i, $i < j$. In either case $x \in \bigcup_{j=0}^{\infty} E_j$. We define $s_p\colon \Omega \to C$ by putting:

$$s_p(x) = c_j \quad \text{whenever } x \in E_j.$$

Then s_p satisfies (19.7.1)–(19.7.3). Condition (19.7.3) implies that $\{s_m(x)\}$ is a Cauchy sequence for every $x \in \Omega$.

We let $s\colon \Omega \to Y$ as follows:

$$s(x) = \lim_{m\to\infty} s_m(x), \quad x \in \Omega.$$

Since φ has closed values by (ii) we deduce that $s(x) \in \varphi(x)$ for every $x \in \Omega$.

It remains to show that s is measurable. This is equivalent to show that counter images of closed sets are measurable. Let K be a closed subset of Y. Then each set $s_m^{-1}(O_{\varepsilon_m}(K))$ is measurable. We shall complete the proof showing that $s^{-1}(K) = \bigcap s_m^{-1}(O_{\varepsilon_m}(K))$. In fact on the one hand, when $x \in s^{-1}(K)$, $s(x) \in K$ and since $d(s_m(x), s(x)) < \varepsilon_m$, $s_m \in O_{\varepsilon_m}(K)$ for every m. On the other hand, when $x \in s_m^{-1}(O_{\varepsilon_m}(K))$ for all m, $s_m(x) \in O_{\varepsilon_m}(K)$ and since $\{s_m(x)\}$ converges to $s(x)$ and K is closed we get $s(x) \in K$. The proof of Theorem (19.7) is completed.□

Now, we shall be concerned with multivalued mappings which are defined on the topological product of some measurable set with the Euclidean space $\mathbb{R}^n$. We are particularly interested in Carathéodory multivalued mappings and Scorza–Dragoni multivalued mappings. Apart from their fundamental importance in all fields of multivalued analysis, such multivalued mappings are useful in differential inclusions.

Let $\Omega = [0, a]$ be equipped with the Lebesgue measure and $Y = \mathbb{R}^n$.

(19.8) DEFINITION. *A* map $\varphi\colon [0, a] \times \mathbb{R}^n \multimap \mathbb{R}^n$ with nonempty compact values is called u-Carathéodory (resp. l-Carathéodory; resp. Carathéodory) if satisfies:

(19.8.1) $t \multimap \varphi(t, x)$ is measurable for every $x \in \mathbb{R}^n$,

(19.8.2) $x \multimap \varphi(t, x)$ is u.s.c. (resp. l.s.c.; resp. continuous) for almost all $t \in [0, a]$,

(19.8.3) $\|y\| \le \mu(t)(1 + \|x\|)$, for every $(t, x) \in [0, a] \times \mathbb{R}^n$, $y \in \varphi(t, x)$, where $\mu\colon [0, a] \to [0, +\infty)$ is an integrable function.

As before, by $\mathcal{U} \quad (\mathbb{R}^n)$ we denote the minimal σ-algebra generated by the Lebesgue measurable sets $A \in \mathcal{U}$ and the Borel subsets of $\mathbb{R}^n$, and then the term "product-measurable" means measurability with respect to $\mathcal{U} \quad (\mathbb{R}^n)$.

(19.9) PROPOSITION. *Let $\varphi\colon [0, a] \times \mathbb{R}^n \multimap \mathbb{R}^m$ be a Carathéodory multivalued map. Then φ is product-measurable.*

PROOF. Consider the countable dense subset $Q^n \subset \mathbb{R}^n$ of rationals. For closed $A \subset \mathbb{R}^n$, $a \in Q^n$ and k, the set

$$G_k(A, a) = \{t \in [0, a] \mid \varphi(t, a) \cap O_{1/k}(A) \neq \emptyset\} \times B(a, 1/k)$$

belongs to $\mathcal{U} \quad (\mathbb{R}^n)$. Since φ is l.s.c. in the second variable we have:

$$\varphi_+^{-1}(A) \subset \bigcap_{k=1}^{\infty} \bigcup_{a\in Q^n} G_k(A, a),$$

while the u.s.c. of φ implies the reverse inclusion. The proof is completed. □

The following example shows that an l-Carathéodory multivalued map needs not to be product measurable.

(19.11) EXAMPLE. Let $\varphi\colon [0,1]\times\mathbb{R} \multimap \mathbb{R}$ be defined as follows:

$$\varphi(t,u)=\begin{cases}\{0\} & \text{if } u=0,\\ [0,1] & \text{otherwise.}\end{cases}$$

Then φ is l-Carathéodory but not u-Carathéodory.

An analogous example can be constructed for u-Carathéodory mappings.

Let $\varphi\colon [0,a]\times\mathbb{R}^n \multimap \mathbb{R}^n$ be a fixed multivalued map. We are interested in the existence of Carathéodory selections, i.e. Carathéodory functions $f\colon [0,a]\times\mathbb{R}^n\to\mathbb{R}^n$ such that $f(t,u)\in\varphi(t,u)$ for almost all $t\in[0,a]$ and all $n\in\mathbb{R}^n$. It is evident that, in the case when φ is u-Carathéodory, this selection problem does not have a selection in general (the reason is exactly the same as in Michael's relation principle). For l-Carathéodory multivalued maps φ, however, this is an interesting problem.

We are now going to study this problem. We shall use the following notation:

$$C(\mathbb{R}^n,\mathbb{R}^n)=\{f\colon\mathbb{R}^n\to\mathbb{R}^n \mid f \text{ is continuous}\}.$$

We shall understand that $C(\mathbb{R}^n,\mathbb{R}^n)$ is equipped with the topology on uniform convergence on compact subsets of $\mathbb{R}^n$. In fact this topology is metrizable (see [AFG] for example). Moreover, as usually by $L_1([0,a],\mathbb{R}^n)$ we shall denote the Banach space of Lebesgue integrable functions (see [DG1-M]).

There are two ways, essentially, to deal with the above selection problem. Let $\varphi\colon [0,a]\times\mathbb{R}^n \multimap \mathbb{R}^n$ be a l-Carathéodory mapping. On the one hand, we may show that the multivalued map

$$\Phi\colon [0,a]\multimap C(\mathbb{R}^n,\mathbb{R}^n),$$
$$\Phi(t)=\{u\colon\mathbb{R}^n\to\mathbb{R}^n \mid u(x)\in\varphi(t,u(x)) \text{ and } u \text{ is continuous}\}$$

is measurable.

Then, if we assume that φ has convex values, in view of Michael selection theorem we obtain that $\Phi(t)\neq\emptyset$ for every t. Moreover, let us observe that every measurable selection of Φ will then give rise to a Carathéodory selection of φ.

On the other hand, we may show that the multivalued map:

$$\Psi\colon\mathbb{R}^n\multimap L_1([a,1],\mathbb{R}^n),$$
$$\Psi(x)=\{u\colon[a,1]\to\mathbb{R}^n \mid u(t)\in\varphi(t,u(t)),\ \text{for almost all } t\in[0,a]\}$$

is a l.s.c. mapping.

Consequently, continuous selections of Ψ will then give rise to a Carathéodory selections of φ. Hence, our problem can be solved by using Michael and Kuratowski–Ryll–Nardzewski selection theorems. Let us formulate, only for informative purposes, the following result owed to A. Cellina ([Ce1]).

(19.11) THEOREM. *Let $\varphi\colon [0,a]\times\mathbb{R}^n \multimap \mathbb{R}^n$ be a multivalued map with compact convex values. If $\varphi(\,\cdot\,,x)$ is u.s.c. for all $x\in\mathbb{R}^n$ and $\varphi(t,\,\cdot\,)$ is l.s.c. for all $t\in[0,a]$ then φ has a Carathéodory selection.*

We shall end this section by introducing mappings having Scorza–Dragoni property.

(19.12) DEFINITION. We say that a multivalued map $\varphi\colon [0,a]\times\mathbb{R}^n \multimap \mathbb{R}^n$ with closed values has the u-Scorza–Dragoni property (resp. l-Scorza–Dragoni property; resp. Scorza–Dragoni property) if, given $\delta > 0$, one may find a closed subset $A_\delta \subset [0,a]$ such that the measure $\mu([0,a]\setminus A_\delta) \le \delta$ and the restriction $\widetilde{\varphi}$ of φ to $A_\delta\times\mathbb{R}^n$ is u.s.c. (resp. l.s.c.; resp. continuous).

Let us observe that the Scorza–Dragoni property plays the same role for multivalued mappings of two variables as the Luzin property for multivalued mappings of one variable.

There is a close connection between Carathéodory multivalued mappings and multivalued mapping having the Scorza–Dragoni property.

(19.13) PROPOSITION. *Let $\varphi\colon [0,a]\times\mathbb{R}^m \multimap \mathbb{R}^n$ be a multivalued map with compact values. Then we have:*

(19.13.1) *φ is Carathéodory if and only if φ has the Scorza–Dragoni property,*
(19.13.2) *if φ has the u-Scorza–Dragoni property then φ is u-Carathéodory,*
(19.13.3) *if φ has the l-Scorza–Dragoni property then φ is l-Carathéodory,*
(19.13.4) *if φ is product-measurable l-Carathéodory then φ has the l-Scorza–Dragoni property.*

Assume further that φ satisfies the Filippov condition, i.e. for every open U, $V\subset\mathbb{R}^n$ the set $\{t\in[0,a]\mid \varphi(t,U)\subset V\}$ is Lebesgue measurable then:

(19.13.5) *φ is u-Carathéodory multivalued map if and only if φ has the u-Scorza–Dragoni property.*

Proposition (19.13) is taken from [APNZ-M]. All proofs are rather technical and need sometimes long calculations.

Therefore we shall present below only two examples showing that l-Carathéodory (u-Carathéodory) map need not have the l-Scorza–Dragoni (u-Scorza–Dragoni) property.

(19.14) Example. Let $\varphi: [0,1] \times \mathbb{R} \multimap \mathbb{R}$ be a the map defined as follows:

$$\varphi(t,u) = \begin{cases} \{0\} & \text{if } u = t \text{ and } t \in [0,1] \setminus A, \\ \{1\} & \text{if } u = t \text{ and } t \in A, \\ [0,1] & \text{otherwise,} \end{cases}$$

where A is a nonmeasurable subset of $[0,1]$. Then obviously φ is l-Carathéodory but does not have l-Scorza–Dragoni property. Moreover, φ is not product measurable.

(19.15) Example. Let $\varphi: [0,1] \times \mathbb{R} \multimap \mathbb{R}$ be defined as follows:

$$\varphi(t,u) = \begin{cases} [0,1] & \text{if } t = u \text{ and } t \in A, \\ \{0\} & \text{otherwise,} \end{cases}$$

where A is a nonmeasurable subset of $[0,1]$. It is not hard to see that φ is u-Carathéodory but does not have u-Scorza–Dragoni property.

In Section 31 we will need some additional information about product measurable functions. Until the end of this section X is a metric separable space and Ω a complete measure space. We let also that $\varphi: \Omega \times X \multimap X$ is assumed to be a product-measureable multivalued mapping with compact values.

First we shall prove:

(19.16) Proposition. *If $\varphi: \Omega \times X \multimap X$ is product-measurable then the function $f: \Omega \times X \to [0,+\infty)$ defined by the formula:*

$$f(\omega,x) = \operatorname{dist}(x, \varphi(\omega,x))$$

is also product measurable.

Proof. We have:

$$\{(\omega,x) \in \Omega \times X \mid f(\omega,x) < r\} = \{(\omega,x) \in \Omega \times X \mid \varphi(\omega,x) \cap O_r(\{x\}) \neq \emptyset\}.$$

Therefore, our assertion follows from the assumption that φ is measurable. □

(19.17) Theorem (Aumann). *If $\varphi: \Omega \multimap X$ is a multivalued map with compact values such that the graph Γ_φ of φ is measurable then φ possesses a measurable selector.*

The proof of (19.17) is not in the scope of our book. Therefore, for the proof we recommend [Hi2] (comp. also [CV-M], [Ki-M], [Ox-M]).

The following Scorza–Dragoni type result describes possible regularizations of Carathéodory maps.

(19.18) THEOREM. *Let X be a compact subset of $\mathbb{R}^n$ and $\varphi\colon [0,a]\times X \multimap \mathbb{R}^n$ a nonempty compact convex valued Carathéodory map. Then there exists an u-Scorza–Dragoni $\psi\colon [0,a]\times X \multimap \mathbb{R}^n$ with nonempty compact convex values such that:*

(19.18.1) *$\psi(t,x)\subset\varphi(t,x)$ for every $(t,x)\in[0,a]\times X$,*

(19.18.2) *if $\Delta\subset[0,a]$ is measurable, $u\colon\Delta\to\mathbb{R}^n$ and $v\colon\Delta\to X$ are measurable maps and $u(t)\in\varphi(t,v(t))$ for almost all $t\in\Delta$ then $u(t)\in\psi(t,v(t))$ for almost all $t\in\Delta$.*

Now, we prove:

(19.19) THEOREM. *Let E, E_1 be two separable Banach spaces and $\varphi\colon [a,b]\times E \multimap E_1$ an u-Scorza–Dragoni map with compact convex values then φ is σ-Ca-selectionable. The maps $\varphi_k\colon [a,b]\times E\multimap E_1$ (see Remark (17.5)) are u-Scorza–Dragoni and we have*

$$\varphi_k(t,e)\subset\Big(\bigcup_{x\in E}\varphi(t,x)\Big).$$

*Moreover, if φ is integrably bounded then φ is σ-m*LL*-selectionable.*

PROOF. Consider the family $\{B(y,r_k)\}_{y\in E}$, where $r_k=(1/3)^k$, $k=1,2,\ldots$ Using Stone's theorem for every $k=1,2,\ldots$, we get locally finite subcovering $\{U_i^k\}_{i\in I_k}$ of $\{B(y,r_k)\}_{y\in E}$. For every $i\in I_k$, $k=1,2,\ldots$, we fix the center $y_i^k\in E$ such that $U_i^k\subset B(y_i^k,r_k)$. Now, let $\eta_i^k\colon E\to[0,1]$ be a locally Lipschitz partition of unity subordinated to $\{U_i^k\}_{i\in I^k}$.

Define $\psi_i^k\colon [0,a]\multimap E$ and $f_i^k\colon [0,a]\to E$ as follows:

$$\psi_i^k(t)=\overline{\operatorname{conv}}\Big(\bigcup_{y\in B(y_i^k,2r_k)}\varphi(t,y)\Big),$$

and let f_i^k be a measurable selection of ψ_i^k which exists in view of the Kuratowski–Ryll–Nardzewski theorem.

Finally, we define $\varphi_k\colon [a,b]\times E\multimap E_1$ and $f_k\colon [a,b]\times E\to E_1$ as follows:

$$\varphi_k(t,z)=\sum_{i\in I_k}\eta_i^k(z)\cdot\psi_i^k(t),\qquad f_k(t,z)=\sum_{i\in I_k}\eta_i^k(z)\cdot f_i^k(t).$$

Then $f_k\subset\varphi_k$. Fix $t\in[a,b]$. If $\varphi(t,\cdot\,)$ is u.s.c. then $\varphi(t,z)=\bigcap_{k=1}^{\infty}\varphi_k(t,z)$ and $\varphi_{k+1}(t,z)\subset\varphi_k(t,z)$, for every $z\in E$. By the assumptions on φ the map $\varphi(t,\cdot\,)$ is u.s.c. for almost all $t\in[0,a]$, and the first part of (19.19). The second claim is an immediate consequence of the first one and theorem is proved. □

A map $\varphi\colon [a,b]\times\mathbb{R}^n\multimap\mathbb{R}^n$ is said to be *integrably bounded* if there exists an integrable function $\mu\in L^1([a,b])$ such that $\|y\|\le\mu(t)$ for every $x\in\mathbb{R}^n$, $t\in[a,b]$ and $y\in\varphi(t,x)$.

We say that φ has *linear growth* if there exists an integrable function $\mu \in L^1([a,b])$ such that

$$\|y\| \leq \mu(t)(1 + \|x\|)$$

for every $x \in \mathbb{R}^n$, $t \in [a,b]$ and $y \in \varphi(t,x)$.

20. Borsuk and Hausdorff continuity of multivalued mappings

According to Section 4 for a metric space (Y,d) we shall denote by $B(Y)$ $(C(Y))$ the family of all nonempty closed bounded (compact) subset of Y. We shall consider $B(Y)$ as a metric space with the Hausdorff metric d_H defined in Section 4 and in $C(Y)$ we shall consider moreover, the Borsuk metric of continuity defined also in Section 4. In what follows we will consider mappings of the type

$$F\colon X \to B(Y) \quad \text{or} \quad F\colon X \to C(Y).$$

Any such a map can be reinterpreted as a multivalued map $\varphi\colon X \multimap Y$ with closed bounded and nonempty values or respectively with compact nonempty values defined as follows:

$$\varphi(x) = F(x) \quad \text{for every } x \in X.$$

For simplicity we shall use only one notion F in the place of φ. We hope it will not cause any confusion.

Observe that for $F\colon X \to B(Y)$ we have notion of continuity with respect to the metric given in X and d_H in $B(Y)$ but for $F\colon X \to C(Y)$ we can speak also about the continuity of F with respect to the metric given in X and d_C in $C(Y)$.

As a first observation we get the following.

(20.1) Proposition. *If $F\colon X \to C(Y)$ is continuous with respect to d_C then F is continuous with respect to d_H.*

In fact our claim follows from the following inequality:

$$d_C(F(x), F(y)) \geq d_H(F(x), F(y)),$$

for every $x, y \in X$, which we obtained already in Section 4.

(20.2) Remark. Note that continuity with respect to d_H does not imply continuity with respect to d_C.

For example the mapping φ considered in (13.5.1) is d_H-continuous but not d_C continuous.

In what follows we shall say also that d_H-continuous maps are Hausdorff continuous and d_C-continuous maps are Borsuk continuous. We prove the following:

(20.3) THEOREM. *A mapping $F\colon X \to C(Y)$ is Hausdorff continuous if and only if it is both u.s.c. and l.s.c.*

PROOF. Assume that F is d_H continuous and let U be an open subset of Y. First we shall prove that the set $F^{-1}(U) = \{x \in X \mid F(x) \subset U\}$ is open.

Let $x_0 \in F^{-1}(U)$. Then $F(x_0) \subset U$. Since $F(x_0)$ is compact there exists $\varepsilon > 0$ such that $O_\varepsilon(F(x_0)) \subset U$. Since F is d_H-continuous we can find $\delta > 0$ such that for every $x \in B(x_0, \delta)$ we have $d_H(F(x_0), F(x)) < \varepsilon$. It implies that

$$F(x) \subset O_\varepsilon(F(x_0)) \subset U$$

so, $B(x_0, \delta) \subset F^{-1}(U)$ and $F^{-1}(U)$ is open.

Now we would like to show that the set $F_+^{-1}(U) = \{x \in U \mid F(x) \cap U \neq \emptyset\}$ is open. Let $x_0 \in F_+^{-1}(U)$. So $F(x_0) \cap U \neq 0$. Let $y_0 \in F(x_0) \cap U$. We take $\varepsilon > 0$ such that $B(y_0, \varepsilon) \subset U$. Now we find $\delta > 0$ such that for every $x \in B(x_0, \delta)$ we have:

$$d_H(F(x_0), F(x)) < \varepsilon/2.$$

We claim that $F(x) \cap B(y_0, \varepsilon) \neq \emptyset$. Assume to the contrary that $F(x) \cap B(y_0, \varepsilon) = \emptyset$. On the other hand we have $F(x_0) \subset O_{\varepsilon/2}(F(x))$. Therefore, $y_0 \in O_{\varepsilon/2}(F(x))$ and there exists $z_0 \in F(x)$ such that $d(y_0, z_0) < \varepsilon/2$. It implies $z_0 \in B(y_0, \varepsilon)$ and we obtained a contradiction.

Now assume that F is both u.s.c. and l.s.c. and let $\varepsilon > 0$, $x_0 \in X$. We let $U = O_\varepsilon(F(x_0))$. Then the sets $F^{-1}(U)$ and $F_+^{-1}(U)$ are open and $x_0 \in F^{-1}(U) \cap F_+^{-1}(U)$. Let $V = F^{-1}(U) \cap F_+^{-1}(U)$. Then V is an open neighbourhood of x_0 such that $F(x) \subset O_\varepsilon(F(x_0))$ for every $x \in V$. We are looking for $\delta > 0$ such that $B(x_0, \delta) \subset V$ and $F(x_0) \subset O_\varepsilon(F(x))$ for every $x \in B(x_0, \delta)$. To find that we cover the compact set by n open balls $B(y_i, \varepsilon)$, $i = 1, \ldots, n$. Then $F(x_0) = \bigcup_{i=1}^n B(y_i, \varepsilon) \subset O_{\varepsilon/2}(F(x_0))$ since F is l.s.c. there are open balls $B(x_0, \delta_i) \subset V$ such that

$$F(x) \cap B(y_i, \varepsilon/2) \neq \emptyset \quad \text{for every } x \in B(x_0, \delta_i).$$

Let $\delta = \min\{\delta_1, \ldots, \delta_n\}$. Then $B(x_0, \delta) \subset V$ and any $y \in F(x_0)$ belongs to $B(y_i, \varepsilon/2)$, for some i. Furthermore, we know that for any $x \in B(x_0, \delta)$, $F(x) \cap B(y_i, \varepsilon/2) \neq \emptyset$ for every $i = 1, \ldots, n$. Thus for every $x \in B(x_0, \delta)$ and $y \in F(x_0)$ there exists $i = 1, \ldots, n$ such that:

$$\operatorname{dist}(y, F(x)) \leq d(y, y_i) + \operatorname{dist}(y_i, F(x_i)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore, for every $x \in B(x_0, \delta)$ we obtain $F(x_0) \subset O_\varepsilon(F(x))$ and the proof is completed. □

Note that if F is only u.s.c. or l.s.c. then F is not Hausdorff continuous in general. For example the mappings defined in (13.5.1) or (13.5.2) are u.s.c. but not d_H-continuous. Below we present an example of l.s.c. which is not d_H-continuous.

(20.4) EXAMPLE. Let $X = Y = [0,1]$. Let $F: [0,1] \to C([0,1])$ be defined as follows:

$$F(x) = \begin{cases} [0,1] & \text{if } x \neq 0, \\ \{0\} & \text{if } x = 0. \end{cases}$$

Then F is l.s.c. with compact values but $d_H(F(0), F(x)) = 1$ for every $x \neq 0$, so F is not Hausdorff continuous.

We obtain:

(20.5) THEOREM. *Let $F: X \to B(X)$ be Hausdorff continuous, then F is l.s.c.*

The proof of (20.5) is strictly analogous to the respective part of the proof of (20.3). Note that, under assumptions of (20.5), the map F has not to be u.s.c.

(20.6) EXAMPLE. Let $X = \mathbb{R}$ be the Euclidean space of real numbers and let $Y = \mathbb{R}^2$ be equipped with the bounded metric d defined as follows:

$$d(x,y) = \frac{\|x-y\|}{1+\|x-y\|}.$$

We consider the mapping $F: \mathbb{R} \to B(\mathbb{R}^2) = 2^{\mathbb{R}^2} \setminus \{\emptyset\}$ defined as follows:

$$F(t) = \{(t,y) \mid y \in \mathbb{R}\} \quad \text{for every } t \in \mathbb{R}.$$

Then we have $d_H(F(t), F(t')) \leq 2|t-t'|$ so F is Hausdorff continuous.

Let $U = \{(x,y) \in \mathbb{R}^2 \mid |y| < 1/x \text{ or } x = 0\}$. Then U is an open subset of $\mathbb{R}^2$ but $F^{-1}(U) = \{0\}$ is not open in $\mathbb{R}$. Consequently F is not u.s.c.

21. Banach contraction principle for multivalued maps

A multivalued map $F: X \to B(X)$ is called *contraction* provided, there exists $k \in [0,1)$ such that:

$$d_H(F(x), F(y)) \leq k \cdot d(x,y) \quad \text{for every } x, y \in X.$$

We have the following result proved by H. Covitz and S. B. Nadler, Jr. ([Ki-M]).

(21.1) THEOREM. *Let (X,d) be a complete metric space and $F: X \to B(X)$ a contraction map. Then there exists $x \in X$ such that $x \in F(x)$.*

PROOF. Assume that $d_H(F(x), F(y)) \leq k \cdot d(x,y)$ for every $x, y \in X$, where $k \in [0,1)$. Let $x \in X$. We let:

$$D(x, \operatorname{dist}(x, F(x))) = \{y \in X \mid d(x,y) \leq \operatorname{dist}(x, F(x)\}.$$

Then we have:

$$D(x, \operatorname{dist}(x, F(x))) \cap F(x) \neq \emptyset,$$

so we can select $x_1 \in F(x)$ such that: $d(x, x_1) \leq \operatorname{dist}(x, F(x))$. Then for such $x_1 \in X$ select $x_2 \in F(x_1)$ such that $d(x_1, x_2) \leq \lambda \operatorname{dist}(x_1, F(x_1))$. Continuing this procedure we can find a sequence $\{x_n\} \subset X$ such that $d(x_n, x_{n+1}) \leq \operatorname{dist}(x_n, F(x_n))$. Hence it follows:

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \operatorname{dist}(x_n, F(x_n)) \leq d_H(F(x_{n-1}), F(x_n)) \\ &\leq k\, d(x_{n-1}, x_n) \leq k^n d(x, x_1) \leq k^n \operatorname{dist}(x, F(x)). \end{aligned}$$

So, it is easy to verify that $\{x_n\}$ is a Cauchy sequence. We let $u = \lim_n x_n$. Then we have $\{x_n\} \to u$ and $x_{n+1} \in F(x_n)$ for every $n = 1, 2, \ldots$ Since F is u.s.c. we deduce that the graph Γ_F of F is closed and consequently we obtain $u \in F(u)$. The proof of (21.1) is completed. □

There are many generalizations of theorem (21.1). We recommend [Ki-M] and [We-M] for details. Below we shall concentrate our considerations on the topological structure of the set of fixed points of contraction mappings. First, observe that multivalued contraction can possess not necessarily a unique fixed point.

(21.2) Example. Let $F\colon \mathbb{R} \to B(\mathbb{R})$ be a map defined as follows:

$$F(x) = [0, 1] \quad \text{for every } x \in \mathbb{R}.$$

Then F as a constant map is a contraction. Of course we have:

$$\operatorname{Fix}(F) = \{x \in \mathbb{R} \mid x \in F(x)\} = [0, 1].$$

Observe that for $A \in B(\mathbb{R})$ we can construct a multivalued contraction such that $\operatorname{Fix}(F) = A$.

Since, contrary to the singlevalued case, $\operatorname{Fix}(F)$ of a contraction F may have many elements, it is interesting to look for topological properties of it. In this framework, the following two results are well known:

(21.3) Theorem ([Ri1]). *Let E be a Banach space and let X be a nonempty convex closed subset of E. Suppose $F\colon X \to B(X)$ is a contraction with convex values. Then the set* $\operatorname{Fix}(F)$ *is an absolute retract.*

(21.4) Theorem ([BCF]). *If $X = L^1(T)$ for some measure space T and $F\colon X \to B(X)$ is a contraction with decomposable values then* $\operatorname{Fix}(F)$ *is an absolute retract.*

Below we establish a result (see [GMS] or [GM]) which unifies and extends to a larger class of multivalued contractions defined on arbitrary complete absolute retracts both Theorems (21.3) and (21.4).

Let $(T, \mathcal{F}, \mu)$ be a finite, positive, nonatomic measure space and let $(E, \|\cdot\|)$ be a Banach space. As before we denote by $L^1(T, E)$ the Banach space of all (equivalence classes) μ-measurable functions $u: T \to E$ such that the function $t \to \|u(t)\|$ is μ-integrable, equipped with the norm

$$\|u\|_{L^1(T,E)} = \int_T \|u(t)\|\, d\mu.$$

We always assume that the space $L^1(T, E)$ is separable. Now, we set

(21.5) Definition. A nonempty set $K \subset L^1(T, E)$ is said to be *decomposable*, if for every $u_1, u_2 \in K$ and every μ-measurable subset A of T, one has

$$(\chi_A \cdot u_1 + (1 - \chi_A) \cdot u_2) \in K,$$

where χ_A denotes the characteristic function of $A \subset T$.

Some basic facts about decomposable sets in $L^1(T, E)$ are collected in the following:

(21.6) Remarks.

(21.6.1) It is easily seen that every decomposable subset of $L^1(T, E)$ is contractible and, consequently, infinitely connected.

(21.6.2) Any closed decomposable subset of $L^1(T, E)$ is an absolute retract.

(21.6.3) A simple calculation shows that the open (or closed) ball unit ball of $L^1(T, E)$ is not decomposable.

For more details concerning the notion of decomposability we recommend [Fry2], [Fry3] and [Ol3].

In the proof of our main result the following proposition will play an important role.

(21.7) Proposition. *Let (X, d) be a metric space and let $\Phi: X \to 2^X$ be a Lipschitzian multifunction. Set $f(x) = d(x, \Phi(x))$ for every $x \in X$. Then the function $f: X \to [0, +\infty[$ is Lipschitzian.*

Proof. Let $L \geq 0$ be such that $d_H(\Phi(x'), \Phi(x'')) \leq Ld(x', x'')$ for all $x', x'' \in X$. Pick $x', x'' \in X$ and choose $\varepsilon > 0$. Owing to the definition of f there exists $z' \in \Phi(x')$ satisfy

$$-f(x') < -d(x', z') + \varepsilon.$$

By using the inequality $d(z', \Phi(x'')) \leq Ld(x', x'')$ we can find $z'' \in \Phi(x'')$ such that

$$d(z', z'') < Ld(x', x'') + \varepsilon.$$

Therefore,

$$\begin{aligned} f(x'') - f(x') &< d(x'', \Phi(x'')) - d(x', z') + \varepsilon \\ &\le d(x'', z'') - d(x', z') + \varepsilon < (L+1)d(x', x'') + 2\varepsilon. \end{aligned}$$

Since ε is arbitrary, we actually have

$$f(x'') - f(x') \le (L+1)d(x', x'')$$

and, interchanging x' with x'',

$$f(x') - f(x'') \le (L+1)d(x', x'').$$

This completes the proof. □

We now recall the notion of Michael family of subsets of a metric space ([GMS] Definition 1.4). In what follows, by $\mathcal{M}$ we will denote the class of all metric spaces.

(21.8) DEFINITION. Let $X \in \mathcal{M}$ and let $M(X)$ be a family of closed subsets of X, satisfying the following conditions:

(21.8.1) $X \in M(X)$, $\{x\} \in M(X)$ for all $x \in X$ and if $\{A_i\}_{i \in I}$ is any subfamily of $M(X)$ then $\bigcap_{i \in I} A_i \in M(X)$.

(21.8.2) For every $k \in \mathbb{N}$ and every $x_1, \dots, x_k \in X$, the set

$$A(x_1, x_2, \dots, x_k) = \bigcup \{A \mid A \in M(X), x_1, x_2, \dots, x_k \in A\}$$

is infinitely connected.

(21.8.3) To each $\varepsilon < 0$ there corresponds $\delta > 0$ such that for any $A \in M(X)$, any $k \in \mathbb{N}$, and any $x_1, \dots, x_k \in O_\delta(A)$, one has $A(x_1, \dots, x_k) \subseteq O_\varepsilon(A)$.

(21.8.4) $A \cap B(x, r) \in M(X)$ for all $A \in M(X)$, $x \in X$, and $r > 0$.

Then we say that $M(X)$ is a Michael family of subsets of X.

It is an easy remark (cf. [Bie-1], [GMS]) that in the Michael selection theorem the notion of convexity can be replaced by a Michael family. Namely, we obtain:

(21.9) PROPOSITION. *Let $X, Y \in \mathcal{M}$ and let $\Phi\colon X \to 2^Y$ be a lower semicontinuous multifunction. If Y is complete and there exists a Michael family $M(Y)$ of subsets of Y such that $\Phi(x) \in M(Y)$ for each $x \in X$ then, for any nonempty closed set $X_0 \subseteq X$, every continuous selection f_0 from $\Phi|_{X_0}$ admits a continuous extension f over X such that $f(x) \in \Phi(x)$ for all $x \in X$.*

The preceding result gains in interest if we realize that significant classes of sets are examples of Michael families.

(21.10) EXAMPLES.

(21.10.1) Let X be a convex subset of a normed space and let $M(X)$ be the class of all sets $A \subset X$ such that $A = \emptyset$ or A is closed and convex in X. Then $M(X)$ is a Michael family of subsets of X.

(21.10.2) Let $X \in \mathcal{M}$ and let $M(X)$ be the family of all simplicially convex closed subsets of X (in the sense of [Bie1]) or closed convex sets with respect to an abstract convex structure (see [Wie-M]) then $M(X)$ is a Michael family of subsets of X.

In (21.10.2) we only signalized some non-typical examples of Michael's families. The following definition is crucial in what follows

(21.11) DEFINITION. Let $X \in \mathcal{M}$, let $\Phi: X \to 2^X$ be a lower semicontinuous multifunction, and let $\mathcal{D} \subset \mathcal{M}$. We say that Φ has the *selection property with respect to* $\mathcal{D}$, when for any $Y \in \mathcal{D}$, any pair of continuous functions $f: Y \to X$ and $h: Y \to]0, +\infty[$ such that

$$\Psi(y) = \overline{\Phi(f(y)) \cap B(f(y), h(y))} \neq \emptyset, \quad y \in Y,$$

and for any nonempty closed set $Y_0 \subseteq Y$, every continuous selection g_0 from $\Psi|_{Y_0}$ admits a continuous extension g over Y fulfilling $g(y) \in \Psi(y)$ for all $y \in Y$. If $\mathcal{D} = \mathcal{M}$, then we say that Φ has the *selection property* (in symbols, $\Phi \in \mathrm{SP}(X)$).

The above notion has some meaningful features, as below is pointed out.

(21.12) EXAMPLE. Let $X \in \mathcal{M}$ and let $\Phi: X \to 2^X$ be a l.s.c. mapping. If X is complete and there exists a Michael family $M(X)$ of subsets of such that $\Phi(x) \in M(X)$ for all $x \in X$, then $\Phi \in \mathrm{SP}(X)$ (see (21.9)).

The above notion has some meaningful features, as below point out.

Now, we establish the following result:

(21.13) THEOREM. *Let X be a nonempty closed subset of $L^1(T, E)$ and let $\varphi: X \multimap X$ be a lower semicontinuous map, with decomposable values. Then φ has the selection property with respect to the family $\mathcal{D}$ of all separable metric spaces.*

PROOF. Throughout this proof, we write 0 to denote the zero vector of $L^1(T, E)$ with $\|\cdot\|_{L^1(T,E)}$. Pick $Y \in \mathcal{D}$ and a pair of continuous functions $f: Y \to X$, $h: Y \to]0, +\infty[$ such that $\psi(y) = \mathrm{cl}\,(\varphi(f(y)) \cap B(f(y), h(y))) \neq \emptyset$ for all $y \in Y$. If Y_0 is a nonempty closed subset of Y and g_0 denotes a continuous selection from $\psi|_{Y_0}$ then the function $k_0: Y_0 \to L^1(T, E)$ defined by

$$k_0(y) = h(y)^{-1}[g_0(y) - f(y)], \quad \text{for } y \in Y_0,$$

is a continuous selection of $\eta|_{Y_0}$, where

$$\eta(y) = \operatorname{cl}\,(h(y)^{-1}[\varphi(f(y)) - f(y)] \cap B(0,1)), \quad \text{for } y \in Y.$$

Evidently, the proof will be completed as soon as we show that k_0 admits a continuous extension k over Y, with the property $k(y) \in \eta(y)$ for every $y \in Y$. We first define

$$\xi(y) = \begin{cases} \{k_0(y)\} & \text{if } y \in Y_0, \\ h(y)^{-1}[\varphi(f(y)) - f(y)] & \text{if } y \in Y \setminus Y_0. \end{cases}$$

It is a simple matter to see that the multivalued map: $\xi: Y \to L^1(T,E)$ is lower semicontinuous and with decomposable values. Hence, by Theorem 3 of [BC1], for any $y \in Y$ and any $u \in \xi(y) \cap B(0,1)$, there exists a continuous selection $k_{y,u}: Y \to L^1(T,E)$ from ξ such that $k_{y,u}(y) = u$. Let

$$V_{y,u} = \{z \in Y \mid \|k_{y,u}(z)\|_1 < 2^{-1}(1 + \|u\|_1)\}.$$

The family of sets $\{V_{y,u} \mid y \in Y,\ u \in \xi(y) \cap B(0,1)\}$ is an open covering of the separable metric space Y, so it has a countable neighbourhood finite refinement $\{V_n \mid n \in \mathbb{N}\}$. For each $n \in \mathbb{N}$, choose $y_n \in Y$ and $u_n \in \xi(y_n) \cap B(0,1)$ such that $V_n \subseteq V_{y_n,u_n}$, and define $k_n = k_{y_n,u_n}$. Let $\{p_n\}$ be a continuous partition of unity subordinated to the covering $\{V_n\}$ and let $\{h_n\}$ be a sequence of continuous functions from Y into $[0,1]$, fulfilling the conditions $h_n(y) = 1$ on $\operatorname{supp} p_n$, $\operatorname{supp} h_n \subseteq V_n$, $n \in \mathbb{N}$. We now set, for any $y \in Y$,

$$\varphi_n(y)(t) = \|k_n(y)(t)\|, \quad \text{for } t \in [0,a] \text{ and } n \in \mathbb{N},$$

$$l(y) = \frac{1}{2}\left[1 - \sum_{n=0}^{\infty} \frac{1 + \|u_n\|_1}{2} p_n(y)\right]^{-1} \sum_{n=1}^{\infty} h_n(y).$$

Since $u_n \in B(0,1)$ and the above summations are locally finite, the function l is well defined, positive, and continuous. Therefore, there exists a continuous function $r: Y \to]0,+\infty[$ and a family $\{A_{r,\lambda} \mid r > 0,\ \lambda \in [0,1]\}$ of measurable subsets of T satisfying (comp. Lemma 2 in [BC1]):

(21.13.1) $A_{r,\lambda_1} \subseteq A_{r,\lambda_2}$ if $\lambda_1 \le \lambda_2$,

(21.13.2) $\mu(A_{r_1,\lambda_1} \Delta A_{r_2,\lambda_2}) \le |\lambda_1 - \lambda_2| + 2|r_1 - r_2|$ and $\mu(A_{r,\lambda}) = \lambda\mu(T)$,

(21.13.3) for each $y \in Y$, $\lambda \in [0,1]$, and $n \in \mathbb{N}$, if $h_n(y) = 1$ then

$$\left| \int_{A_{r(y),\lambda}} \varphi_n(y)(t)\, d\mu - \lambda \int_T \varphi_n(y)(t)\, d\mu \right| < \frac{1}{4l(y)}.$$

Finally, let us define, for $y \in Y$ and $n \in \mathbb{N}$, $\lambda_0(y) = 0$, $\lambda_n(y) = \sum_{m \le n} p_m(y)$, $\chi_{y,n} = \chi_{A_{r(y),\lambda_n(y)} \setminus A_{r(y),\lambda_{n-1}(y)}}$,

$$k(y) = \sum_{n=1}^{\infty} \chi_{y,n} \cdot k_n(y).$$

Bearing in mind condition (b), it is a simple matter to see that the function $k: Y \to L^1(T, E)$ is continuous. Furthermore, for any $y \in Y$ one has $k(y) \in \xi(y)$, because $\xi(y)$ is decomposable. Thus, to complete the proof, we only need to show that $|k(y)\|_1 < 1$ at all points of Y. Fix$(y) \in Y$ and observe that if $I(y) = \{n \in \mathbb{N} \mid p_n(y) > 0\}$ then $1 \leq \sharp I(y) \leq \sum_{n=1}^{\infty} h_n(y)$. From (21.13.1)–(21.13.3) we deduce

$$\begin{aligned}
\int_T \|k(y)(t)\| d\mu \leq \sum_{n\in I(y)} \int_{A_{r(y),\lambda_n(y)} \setminus A_{r(y),\lambda_{n-1}(y)}} \varphi_n(y)(t)\, d\mu \\
= \sum_{n\in I(y)} \bigg[\int_{A_{r(y),\lambda_n(y)}} \varphi_n(y)(t)\, d\mu - \lambda_n(y) \int_T \varphi_n(y)(t)\, d\mu \\
- \int_{A_{r(y),\lambda_{n-1}(y)}} \varphi_n(y)(t)\, d\mu + \lambda_{n-1}(y) \int_T \varphi_n(y)(t)\, d\mu + p_n(y) \int_T \varphi_n(y)(t)\, d\mu \bigg] \\
< \frac{\sharp I(y)}{2l(y)} + \sum_{n=1}^{\infty} \frac{1 + \|u_n\|_1}{2} p_n(y) \leq \frac{1}{2l(y)} \sum_{n=1}^{\infty} h_n(y) + \sum_{n=1}^{\infty} \frac{1 + \|u_n\|_1}{2} p_n(y).
\end{aligned}$$

Hence, by the definition of l, $\|k(y)\|_1 < 1$ as required. □

We are in a position now to prove our main result.

(21.14) THEOREM. *Let X be a complete absolute retract and $\Phi: X \to 2^X$ be a multivalued contraction such that $\Phi \in \mathrm{SP}(X)$. Then* Fix(Φ) *is a complete* AR-*space.*

PROOF. Since Fix(Φ) is nonempty and closed in X, we only have to show that if $Y \in \mathcal{M}$, Y^* is a nonempty closed subset of Y, and $f^*: Y^* \to \mathrm{Fix}(\Phi)$ is a continuous function then there exists a continuous extension $f: Y \to \mathrm{Fix}(\Phi)$ of f^* over Y.

Let d be the metric of X, let $L \in]0,1[$ be such that $d_H(\Phi(x'), \Phi(x'')) \leq Ld(x', x'')$ for all $x', x'' \in X$, and let $M \in]1, L^{-1}[$. The assumption $X \in AR$ yields a continuous function $f_0: Y \to X$ fulfilling $f_0(y) = f^*(y)$ in Y^*. We claim that there is a sequence $\{f_n\}$ of continuous functions from Y into X, with the following properties:

(21.14.1) $f_n|_{Y^*} = f^*$ for every $n \in \mathbb{N}$,

(21.14.2) $f_n(y) \in \Phi(f_{n-1}(y))$ for all $y \in Y$, $n \in \mathbb{N}$,

(21.14.3) $d(f_n(y), f_{n-1}(y)) \leq L^{n-1} d(f_1(y), f_0(y)) + M^{1-n}$ for every $y \in Y$, $n \in \mathbb{N}$.

To see this we proceed by induction on n. From Proposition (21.7) it follows that the function $h_0: Y \to]0, +\infty[$ defined by

$$h_0(y) = \mathrm{dist}(f_0(y), \Phi(f_0(y))) + 1 \quad \text{for } y \in Y,$$

is continuous; moreover, one clearly has $\Phi(f_0(y)) \cap B(f_0(y), h_0(y)) \neq \emptyset$ for all $y \in Y$. Bearing in mind that $\Phi \in \mathrm{SP}(X)$ we obtain a continuous function $f_1: Y \to X$

satisfying $f_1(y) = f^*(y)$ in Y^* and $f_1(y) \in \Phi(f_0(y))$ in Y. Hence, conditions (21.14.1)–(21.14.3) are true for f_1. Suppose now we have constructed p continuous functions $f_1, \ldots, f_p$ from Y into X in such a way that (21.14.1)–(21.14.3) hold whenever $n = 1, \ldots, p$. Since Φ is Lipschitzian with constant L, (21.14.2) and (2.14.3) apply if $n = p$, and $LM < 1$, for every $y \in Y$ we achieve

$$\begin{aligned} \operatorname{dist}(f_p(y), \Phi(f_p(y))) &\le d_H(\Phi(f_{p-1}(y)), \Phi(f_p(y))) \le Ld(f_{p-1}(y), f_p(y)) \\ &\le L^p d(f_1(y), f_0(y)) + LM^{1-p} < L^p d(f_1(y), f_0(y)) + M^{-p}, \end{aligned}$$

so that

$$\Phi(f_p(y)) \cap B(f_p(y), L^p\, d(f_1(y), f_0(y)) + M^{-p}) \ne \emptyset.$$

Because of the assumption $\Phi \in \mathrm{SP}(X)$, this procedure yields a continuous function $f_{p+1}\colon Y \to X$ with the properties:

$$\begin{gathered} f_{p+1}|_{Y^*} = f^*, \quad f_{p+1}(y) \in \Phi(f_p(y)) \quad \text{for every } y \in Y, \\ \operatorname{dist}(f_{p+1}(y), f_p(y)) \le L^p d(f_1(y), f_0(y)) + M^{-p} \quad \text{for all } y \in Y. \end{gathered}$$

Thus, the existence of the sequence $\{f_n\}$ is established.

We next define, for any $a > 0$, $Y_a = \{y \in Y \mid d(f_1(y), f_0(y)) < a\}$. Obviously, the family of sets $\{Y_a \mid a > 0\}$ is an open covering of Y. Moreover, due to (21.14.3) and the completeness of X, the sequence $\{f_n\}$ converges uniformly on each Y_a. Let $f\colon Y \to X$ be the pointwise limit of $\{f_n\}$. It is easy to seen that the function f is continuous. Further, due to (21.14.1) one has $f|_{Y^*} = f^*$. Finally, the range of f is a subset of $\mathrm{Fix}(\Phi)$ since, by (21.14.2), $f(y) \in \Phi(f(y))$ for all $y \in Y$. This completes the proof. □

The same arguments used to prove Theorem (21.14) actually produce the following more general result.

(21.15) THEOREM. *Let $\mathcal{D} \subseteq \mathcal{M}$, let X be a complete absolute retract, and let $\Phi\colon X \to 2^X$ be a multivalued contraction having the selection property with respect to $\mathcal{D}$. Then, for any $Y \in \mathcal{D}$ and any nonempty closed set $Y_0 \subseteq Y$, every continuous function $f_0\colon Y_0 \to \mathrm{Fix}(\Phi)$ admits a continuous extension over Y.*

Finally, note that (21.3) and (21.4) are special cases of (21.15).

CHAPTER III

APPROXIMATION METHODS IN FIXED POINT THEORY OF MULTIVALUED MAPPINGS

There are two significant sets of methods in the fixed point theory of multivalued mappings. The first are the so called homological methods, started in 1946 by S. Eilenberg and D. Montgomery ([EM]), and depend on using algebraic topology tools, e.g. homology theory, homotopy theory, etc. The second, started in 1935 by J. Von Neumann ([Neu]), are called the approximation methods.

Note that we will study homological methods in Chapter IV. In the present chapter we shall concentrate on approximation methods, which are simpler than homological methods, but they are sufficient for applications to nonlinear analysis and some other branches of mathematics.

The main idea of approximation methods is simple: one approximate on the graph a given multivalued map by a singlevalued map and, then, applying a limiting process, investigates to what extent properties of singlevalued approximations are inherited by the original map.

We recommend [Go2-M], [Kr2-M], [LR-M], [ACZ1]–[ACZ3], [Bee1]–[Bee4], [Cl1], [GGK1]–[GGK3], [MC] as bibliography for this chapter.

22. Graph-approximation

We start with two preliminary facts.

Firstly, for two metric spaces (X,d_1), (Y,d_2) in the Cartesian product $X\times Y$ we shall consider the max-metric d, i.e.

$$d((x,y),(u,v))=\max\{d_1(x,u),d_2(y,v)\}\quad \text{for } x,y\in X \text{ and } u,v\in Y.$$

Secondly, we shall use the following.

(22.1) PROPOSITION. *Let K be a compact subset of X and let $f\colon X\to Y$ be a continuous map. Then for each $\varepsilon>0$ there exists $\eta>0$ such that $d_2(f(x),f(u))<\varepsilon$, provided $d_1(u,x)<\eta$ and $x,u\in O_\eta(K)$.*

PROOF. Assume to the contrary. Then there exists $\varepsilon>0$ such that for every $n=1,2,\ldots$ there are $x_n,u_n\in O_{1/n}(K)$ such that $d_1(x_n,u_n)<1/n$,

$d_2(f(x_n), f(u_n)) \geq \varepsilon$ for every $n = 1, 2, \ldots$ Since $x_n, u_n \in O_{1/n}(K)$ we can find $\widetilde{x}_n, \widetilde{u}_n \in K$ such that:

$$d_1(x_n, \widetilde{x}_n) < \frac{1}{n} \quad \text{and} \quad d_1(u_n, \widetilde{u}_n) < \frac{1}{n}. \tag{22.1.1}$$

Then we deduce that

$$d_1(\widetilde{x}_n, \widetilde{u}_n) < \frac{3}{n}, \qquad n = 1, 2, \ldots \tag{22.1.2}$$

Now, since K is compact we can assume that sequences $\{\widetilde{x}_n\}$ and $\{\widetilde{u}_n\}$ are convergent. So, in view of (22.1.2), we have:

$$\lim_n \widetilde{x}_n = \lim_n \widetilde{u}_n = x. \tag{22.1.3}$$

Consequently, from (22.1.1) we get $\lim_n x_n = \lim_n u_n = x$. Then $\lim_n f(u_n) = \lim_n f(x_n) = f(x)$ and this contradicts the fact that:

$$d_2(f(u_n), f(x_n)) \geq \varepsilon \quad \text{for every } n.$$

The proof of (22.1) is completed. □

Assumption. Starting from now in this chapter until Section 27 all multivalued mappings are assumed to have compact values.

(22.2) Definition. Let $\varphi\colon X \multimap Y$ be a multivalued map, $Z \subset X$ and let $\varepsilon > 0$. A mapping $f\colon Z \to Y$ is an *ε-approximation* (*on the graph*) of φ if and only if $\Gamma_f \subset O_\varepsilon(\Gamma_\varphi)$.

If $Z = X$ and f is an approximation (on the graph of φ), then we write $f \in a(\varphi, \varepsilon)$.

Some important properties are summarized in the following.

(22.3) Proposition.

(22.3.1) *A mapping $f\colon Z \to Y$ is an ε-approximation of a multivalued map $\varphi\colon X \multimap Y$ if and only if $f(x) \in O_\varepsilon(\varphi(O_\varepsilon(x)))$ for each $x \in Z$, where $Z \subset X$.*

(22.3.2) *Let P be a compact space, $r\colon P \to X$ a continuous map and let $\varphi\colon X \multimap Y$ be u.s.c. Then, for each $\varrho > 0$ there exists $\varepsilon_0 > 0$ such that for any ε $(0 < \varepsilon < \varepsilon_0)$ and ε-approximation $f\colon X \to Y$ of φ, the map $f \circ r\colon P \to Y$ is a ϱ-approximation of $\varphi \circ r$.*

(22.3.3) *Let C be a compact subset of X and $\varphi\colon X \multimap X$ is an u.s.c. mapping such that $C \cap \mathrm{Fix}(\varphi) = \emptyset$. Then there exists $\varepsilon > 0$ such that, for every $f \in a(\varphi, \varepsilon)$, we have $\mathrm{Fix} f \cap C = \emptyset$.*

(22.3.4) *Let C be a compact subset of X. Then, for every $\varepsilon > 0$ there is $\delta > 0$ such that the restriction $f|_C$ of f to C is an ε-approximation of the restriction $\varphi|_C$ of $\varphi: X \multimap Y$ to C, whenever $f \in a\ (\varphi, \delta)$.*

(22.3.5) *Let X be compact and $\chi: X \times [0,1] \multimap Y$ be a multivalued map. Then, for every $t \in [0,1]$ and for every $\varepsilon > 0$ there exists $\delta > 0$ such that $h_t \in a\ (\chi_t, \varepsilon)$, whenever $h \in a\ (\chi, \delta)$, where $h_t: X \to Y$ and $\chi_t: X \multimap Y$ are defined as follows:*

$$\chi_t(x) = \chi(x,t), \quad h_t(x) = h(x,t) \quad \text{for every } x \in X \text{ and } t \in [0,1].$$

(22.3.6) *Let $\varphi: X \multimap Y$ and $g: Y \to Z$ be two mappings (with φ u.s.c. and g continuous). Then, for every $\varepsilon > 0$ there exists $\delta > 0$ such that $g \circ f \in a\ (g \circ \varphi, \varepsilon)$, whenever $f \in a\ (\varphi, \delta)$.*

(22.3.7) *Let $\varphi: X \multimap Y$ and $\psi: Z \multimap T$ be two multivalued mappings. Then, for every $\varepsilon > 0$ there exists $\delta > 0$ such that, if $f \in a\ (\varphi, \delta)$ and $g \in a\ (\psi, \delta)$, then $f \times g: X \times Z \to Y \times T$ is an ε-approximation of $\varphi \times \psi: X \times Z \multimap Y \times T$.*

PROOF. (22.3.1) Observe that $f \in a\ (\varphi, \varepsilon)$ if and only if $\Gamma_f \subset O_\varepsilon(\Gamma_\varphi)$. It is equivalent that for every $x \in Z$ there is $(\widetilde{x}, \widetilde{y}) \in \Gamma_\varphi$ such that $x \in O_\varepsilon(\widetilde{x})$ and $f(x) \in O_\varepsilon(\widetilde{y})$. Consequently we have $f(x) \in O_\varepsilon(\varphi(O_\varepsilon(x)))$ and (22.3.1) is proved.

(22.3.2) We shall prove it by contradiction. Assume that there is $\varrho > 0$ such that for every $\varepsilon > 0$ we have $f_\varepsilon \in a(\varphi, \varepsilon)$ and $f_\varepsilon \circ r \notin a(\varphi \circ r, \varrho)$. We let $\varepsilon = 1, 1/2, \ldots, 1/n, \ldots$ Let $f_n \in a(\varphi, (1/n))$. Then for every n we choose $u_n \in P$ such that:

$$f_n(r(u_n)) \notin O_\varrho(\varphi(r(O_\varrho(u_n)))), \quad n = 1, 2, \ldots \tag{22.3.8}$$

Now, we can assume that $\lim_n u_n = u$. By assumption we have $f_n(r(u_n)) \in O_{(1/n)}(\varphi(O_{(1/n)} r(u_n)))$. So, in view of (22.3.1) we have

$$(r(u_n), f_n(r(u_n))) \in O_{(1/n)}(\Gamma_\varphi).$$

So we can choose $x_n \in X$ and $y_n \in \varphi(x_n)$ such that

$$d_1(x_n, r(u_n)) < \frac{1}{n} \quad \text{and} \quad d_2(y_n, f_n(r(u_n)) < \frac{1}{n}.$$

Therefore, $\lim_n x_n = \lim_n r(u_n) = r(u)$ and $\lim_n y_n = \lim_n f_n(r(u_n)) = y$. So, $(r(u), y) \in \Gamma_\varphi$ but it contradicts (22.3.8) and the proof is completed.

(22.3.3) Assume, to the contrary, that for every $\varepsilon > 0$ there is $f_\varepsilon \in a(\varphi, \varepsilon)$ such that $\mathrm{Fix}(f_\varepsilon) \cap C \neq \emptyset$. We let $\varepsilon = 1/n$, $n = 1, 2, \ldots$ Let $f_n \in a(\varphi, (1/n))$ be such that $\mathrm{Fix} f_n \cap C \neq \emptyset$. For every $n = 1, 2, \ldots$ we choose a point $x_n \in \mathrm{Fix} f_n \cap C$.

Since C is compact we can assume that $\lim_n x_n = x$. Then $x \in C$ and we can choose $(u_n, v_n) \in \Gamma_\varphi$ such that:

$$d_1(x_n, u_n) < \frac{1}{n} \quad \text{and} \quad d_1(x_n, v_n) < \frac{1}{n}, \quad n = 1, 2, \ldots .$$

Thus $\lim_n x_n = \lim_n u_n = \lim_n v_n = x$ and since φ is u.s.c. the graph Γ_φ is closed in $X \times X$. Consequently $x \in \varphi(x)$ and we obtain $\mathrm{Fix}\varphi \cap C \neq \emptyset$ which is a contradiction.

(22.3.4) It is enough to apply (22.3.2) to the inclusion map $i: C \to X$.

(22.3.5) It follows again from (22.3.2). We take $r = i_t: X \to X \times [0,1]$ defined by $i_t(x) = (x, t)$.

(22.3.6) It is easy to prove by contradiction analogously to the proof of (22.3.2). Note that, in fact, (22.3.6) is a generalization of (22.3.2).

(22.3.7) Follows directly from the observation that for every $\varepsilon > 0$ there exists $\delta > 0$ such that $O_\delta(x) \times O_\delta(y) \subset O_\varepsilon((x, y))$. □

For given two spaces X, Y we let:

$$A_0(X, Y) = \{\varphi: X \multimap Y \mid \varphi \text{ is u.s.c. and for every } \varepsilon > 0 \text{ there is } f \in a(\varphi, \varepsilon)\}.$$

The class A_0 is adequate for obtaining global fixed point theorems, but it is not sufficient to construct the topological degree. Fortunately, we will be able to define a quite large class of multivalued maps appropriate to the topological degree theory. We will do it in the sections which follow. Now, we shall describe properties of A_0.

(22.4) THEOREM. *Let these be a map $\varphi \in A_0(X, X)$ and X be a compact* AR-*space, then* $\mathrm{Fix}(\varphi) \neq \emptyset$.

PROOF. Let $\varepsilon = 1/n$, $n = 1, 2, \ldots$ and let $f_n \in a(\varphi, 1/n)$. Then from Schauder Fixed Point Theorem (see Corollary (12.15)) we get that $f_n(x_n) = x_n$, for some $x_n \in X$. Without loss of generality we can assume that $\lim_n f_n(x_n) = \lim_n x_n = x$.

Then we can choose a sequence $(u_n, v_n) \in \Gamma_\varphi$ such that:

$$d_1(x_n, u_n) < \frac{1}{n} \quad \text{and} \quad d_1(x_n, v_n) < \frac{1}{n}, \quad n = 1, 2, \ldots$$

and hence we obtain:

$$x = \lim_n x_n = \lim_n (f_n(x_n)) = \lim_n u_n = \lim_n v_n.$$

Now because φ is u.s.c. the graf Γ_φ of φ is closed in $X \times X$ and consequently $x \in \varphi(x)$ what completes the proof. □

(22.5) THEOREM. *Let $\varphi \in A_0(X, \mathbb{R}^n)$ and $g: \mathbb{R}^n \to \mathbb{R}^m$ be a continuous map. Then the map $\psi = g \circ \varphi: X \multimap \mathbb{R}^m$ is σ-selectionable.*

PROOF. Let a mapping $\varphi_k: X \multimap \mathbb{R}^n$ be given by the formula:

$$\varphi_k(x) = \operatorname{cl}(O_{1/k}(\varphi(O_{1/k}(x)))), \quad x \in X.$$

Then, using the Lasota–Yorke Approximation Theorem (see (17.16)), we may find a locally Lipschitz map $g_k: \mathbb{R}^n \to \mathbb{R}^m$ such that:

$$\|g(z) - g_k(z)\| < \frac{1}{(2k)^2}, \quad z \in \mathbb{R}^n.$$

Next define $\psi_k: X \multimap \mathbb{R}^m$ by the formula:

$$\psi_k(x) = \operatorname{cl}(O_{1/k}(g_k(\varphi_k(x)))), \quad x \in X.$$

Then $\psi_{k+1}(x) \subset \psi_k(x)$ and

$$\psi(x) = \bigcap_{k \geq 1} \psi_k(x), \quad \text{for every } x \in X.$$

Since $\varphi \in A_0(X, \mathbb{R}^n)$ we find a $(1/3k)$-approximation h_k of φ.

Once again, by the Lasota–Yorke Approximation Theorem, take a mapping f_k which is locally Lipschitzian and:

$$\|f_k(x) - h_k(x)\| < \frac{1}{3k}, \quad x \in X, \ k = 1, 2, \ldots$$

By (22.3.1) f_k is a selection of φ_k. Consequently $w_k = g_k \circ f_k$ is a locally Lipschitz selection of ψ_k and the proof is completed. □

(22.6) REMARK. In fact we have proved that $\psi = g \circ \varphi$ is LL-σ-selectionable.

We shall end this section by proving one more approximation-type result.

(22.7) PROPOSITION. *Let E be a normed space and let $\varphi, \psi: X \multimap E$ be two multivalued mappings such that:*

(22.7.1) *ψ is l.s.c. with convex (not necessarily closed) values,*

(22.7.2) *φ is u.s.c. map with closed convex values.*

Assume further that $\varphi(x) \cap \psi(x) \neq \emptyset$ for every $x \in X$. Then for any $\delta > 0$ there exists a δ-approximation (on the graph) $f: X \to E$ of φ such that:

(22.7.3) *$B(f_\delta(t), \delta) \cap \psi(x) \neq \emptyset$, where $B(f_\delta(x), \delta)$ is the open ball with the center in $f_\delta(x)$ and radius δ.*

Proof. Let us put $U(x) = B(x, \delta/2) \cap \{x' \in X \mid \varphi(x') \subset O_{\delta/2}(\varphi(x))\}$ for every $x \in X$. Let $\alpha = \{V_i\}_{i \in J}$ be an open star-refinement of the open cover $\{U(x)\}_{x \in X}$ of X, i.e. for any $i \in J$ there exists $x = x(i) \in X$ such that $\operatorname{st}(V_i, \alpha) \subset U(x)$, where $\operatorname{st}(V_i, \alpha) = \bigcap\{V \mid V \in \alpha \text{ and } V \cap V_{\alpha_i} \neq \emptyset\}$.

For any $x \in X$, choose $z_x \in \varphi(x) \cap \psi(x)$ and consider the open cover $\mathcal{T} = \{T_V(x)\}_{x \in V}$ of X, where

$$T_V(x) = \{x' \in V \mid \varphi(x') \cap B(z_x, \delta/2) \neq \emptyset\}.$$

Let $\{\lambda_s\}_{s \in S}$ be a locally finite partition of unity subordinated to $\mathcal{T}$. Hence, for each $s \in S$ there are $V_s \in \alpha$, $x_s \in V_s$ with $\lambda_s(x') = 0$ for $x' \notin T_{V_s}(x_s)$.

The map $f\colon X \to E$ defined by

$$f(x) = \sum_{s \in S} \lambda_s(x) z_s, \quad x \in X, \text{ where } z_s = z_{x_s}$$

is clearly continuous. Moreover, for each $x \in X$ and each index s in the finite set $S(x) = \{s \in S \mid \lambda_s(x) \neq 0\}$, there exists $z'_s \in \psi(x)$ such that $\|z'_s - z_s\| < \delta$ because $x \in T_{V_s}(x_s)$. Thus by the convexity of $\psi(x)$ we obtain:

$$\Big(\sum_{s \in S(x)} \lambda_s(x) z'_s\Big) \in \psi(x)$$

and

$$\Big\|\sum_{s \in S(x)} \lambda_s(x) z'_s - f(x)\Big\| \leq \sum_{s \in S(x)} \|z'_s - z_s\| < \delta.$$

In other words, $B(f(x), \delta) \cap \psi(x) \neq \emptyset$ for every $x \in X$. On the other hand, given $x \in X$, $s \in S(x)$, it follows that $x \in T_{V_s}(x_s) \subset V_s$, where $x_s \in V_s$. Since α is a star-refinement of $\{U(x)\}_{x \in X}$, there exits $\overline{x} \in X$ such that $x, x_s \in U(\overline{x})$. Therefore $z_s \in \varphi(x_s) \subset O_\delta(\varphi(\overline{x}))$ and $\|x - \overline{x}\| < \delta$. The set $B(\varphi(\overline{x}), \delta)$ being convex, we infer that $f(x) \in O_\delta(\varphi(x))$ and the proof is competed. □

The above lemma guarantees the existence of a so-called δ-approximate selection of ψ which is also a δ-approximation of φ. As in Michael's selection theorem, assuming in a addition that the values of ψ are closed and E is a Banach space we conclude:

(22.8) Corollary. *Assume that E is a Banach space and $\varphi, \psi\colon X \multimap E$ are two multivalued mappings such that:*

(22.8.1) *ψ is l.s.c. with closed convex values,*
(22.8.2) *φ is u.s.c. with closed convex values, and*
(22.8.3) *$\varphi(x) \cap \psi(x) \neq \emptyset$ for every $x \in X$.*

Then for any $\delta > 0$ *there exists a countinuous map* $f: X \to E$ *such that:*

(22.8.4) $f \subset \psi$, *and*

(22.8.5) f *is an* δ*-approximation* (*on the graph*) *of* φ.

We will show in Chapter VI that Corollary (22.8) is useful for applications.

23. Existence of approximations

As we mention in last section to construct the topological degree theory (i.e. the local fixed point theory) more assumptions have to be made on the class of multivalued maps under consideration. Therefore, we shall define a subclass $A(X,Y)$ of the class $A_0(X,Y)$.

(23.1) Definition. We let $\varphi \in A(X,Y)$ provided $\varphi \in A_0(X,Y)$ and for each $\delta > 0$ there is an $\varepsilon_0 > 0$ such that for every ε $(0 < \varepsilon < \varepsilon_0)$, if $f, g: X \to Y$ are ε-approximations of φ, then there exists a homotopy $h: X \times [0,1] \to Y$ joining f and g such that $h_t \in a(\varphi, \delta)$ for every $t \in [0,1]$.

First, we would like to explain how large the class $A(X,Y)$ is. In order to do this we shall say that an u.s.c. map $\varphi: X \multimap Y$ is a J-mapping (write $\varphi \in J(X,Y)$) provided the set $\varphi(x)$ is ∞-proximally connected for every $x \in X$ (cf. Section 2).

Observe, that the definition of ∞-proximally connected sets can be formulated in terms of J-maps as follows:

(23.2) Proposition. *If* $\varphi \in J(X,Y)$, *then for each* $x \in X$, $\varepsilon > 0$, *there is an* $\eta = \eta(x,\varepsilon)$, $0 < \eta < \varepsilon$ *such that for any positive integer* n *and a continuous map* $f: \partial\Delta^n \to O_\eta(\varphi(x))$ *there exists a continuous map* $g: \Delta^n \to O_\varepsilon(\varphi(x))$ *such that* $g(z) = f(z)$ *for every* $z \in \partial\Delta^n$.

(23.3) Remark. Note that if $\varphi \in J(X,Y)$ and $r: Z \to X$ is a continuous map, then $\varphi \circ r \in J(Z,Y)$.

The following lemma is crucial in what follows.

(23.4) Lemma. *If* X, Y *are spaces,* X *is compact and* $\varphi: X \multimap Y$ *is a* J*-mapping, then for each* $\varepsilon > 0$, *there exists a* $\delta = \delta(\varepsilon)$, $0 < \delta < \varepsilon$, *such that for each* $x \in X$ *and a positive integer* n, *if* $g: \partial\Delta^n \to O_\delta(\varphi(O_\delta(x)))$ *is continuous, then there is a continuous map* $\widetilde{g}: \Delta^n \to O_\varepsilon(\varphi(O_\varepsilon(x)))$ *such that* $\widetilde{g}(v) = g(v)$ *for* $v \in \partial\Delta^n$.

Proof. Let $\varepsilon > 0$. By the upper semicontinuity of φ, for any $y \in X$ there is a $\mu = \mu(y)$, $0 < \mu < \eta(y,\varepsilon)/4$ (see Proposition (23.2)), such that $\varphi(O_\mu(y)) \subset O_{\eta/2}(\varphi(y))$. Let λ be the Lebesgue coefficient of the covering $\{O_\mu(y)\}_{y \in X}$ of X.

We put $\delta = \delta(\varepsilon) = \lambda/2$. For an arbitrary $x \in X$, there is $y \in X$ such that $O_\delta(x) \subset O_\mu(y)$. Obviously,

$$O_\delta(\varphi(O_\delta(x))) \subset O_\delta(\varphi(O_\mu(y))) \subset O_\delta(O_{\eta/2}(\varphi(y))) \subset O_\eta(\varphi(y))$$

because $\delta \le 2\mu < \eta/2$. Let $n \ge 1$ be an arbitrary integer and let $g\colon \partial\Delta^n \to O_\delta(\varphi(O_\delta(x)))$ be continuous. By definition of ∞-proximally connected sets, there is a $\widetilde{g}\colon \Delta^n \to O_\varepsilon(\varphi(y))$ such that $\widetilde{g}(v) = g(v)$ for $v \in \partial\Delta^n$. Since $d(x,y) < \mu(y)$, we have that $y \in O_\varepsilon(x)$, so $O_\varepsilon(\varphi(y)) \subset O_\varepsilon(\varphi(O_\varepsilon(x)))$ which completes the proof. □

(23.5) THEOREM. *Let P be a finite polyhedron and P_0 a subpolyhedron of P. Let Y be a space and $\varphi\colon P \multimap Y$ a J-mapping. For any $\varepsilon > 0$ there is $\delta > 0$ such that, if $f_0\colon P_0 \to Y$ is a continuous δ-approximation of φ, then there exists a continuous $f\colon P \to Y$ which is an ε-approximation of φ such that $f|_{P_0} = f_0$.*

PROOF. Let us fix $\varepsilon > 0$ and let $\dim P = N \ge N_0 = \dim P_0$. Let $\varepsilon_N := \varepsilon$ and assume that we have defined ε_{k+1}, $0 \le k \le N-1$. Now, we define a number $\varepsilon_k < \varepsilon_{k+1}$ such that, for any $x \in P$, any positive integer n and any continuous map $g\colon \partial\Delta^n \to O_{\varepsilon_k}(\varphi(O_{\varepsilon_k}(x)))$, there is a continuous map $\widetilde{g}\colon \Delta^n \to O_{\varepsilon_{k+1}/4}(\varphi(O_{\varepsilon_{k+1}/4}(x)))$ such that $\widetilde{g}(v) = g(v)$ for $v \in \partial\Delta^n$. The existence of such an ε_k follows from Lemma (23.4). Let $\delta = \varepsilon_0/2$ and let $f_0\colon P_0 \to Y$ be a δ-approximation of φ. Suppose that (T, T_0) is a triangulation of (P, P_0) finer than the covering $\{O_{\varepsilon_0/4}(x)\}_{x\in P}$ of P, i.e. $|T| = P$, $|T_0| = P_0$ and T_0 is a subcomplex of T. By T^k, $0 \le k \le N$, we denote the k-dimensional skeleton of T and let $P^k = |T^k|$. Similarly, T_0^k, $0 \le k \le N_0$, denotes the k-dimensional skeleton of T_0, and let $P_0^k = |T_0^k|$. It is obvious that, for $k \le N_0$, T_0^k is a subcomplex of T^k. Moreover, $P^N = P$, $P_0^{N_0} = P_0$.

We shall define a sequence $\{f^k\colon P^k \to Y\}_{k=0}^N$ of continuous mappings such that

(23.5.1) for any k, $0 \le k \le N$, $f^k\colon P^k \to Y$ is an $(\varepsilon_k/2)$-approximation of φ;
(23.5.2) for any k, $0 \le k \le N_0$, $f^k|_{P_0^k} = f_0|_{P_0^k}$ and
(23.5.3) for any k, $0 \le k \le N-1$, $f^{k+1}|_{P^k} = f^k$.

Then the mapping $f := f^N$ will satisfy the assertion of Theorem (23.5). Indeed, $f^N|_{P^{N_0}} = f^{N_0}$ and $f^N|_{P^0} = (f^N|_{P^{N_0}})|_{P_0} = f_0$. Let $P^0 = \{x_1, \ldots, x_r\}$ and let the vertices be ordered in such a manner that $x_1, \ldots, x_q \in P_0^0$, $(q \le r)$, $x_{q+1}, \ldots, x_r \notin P_0^0$. For i, $1 \le i \le q$, we put $f^0(x_i) = f_0(x_i)$ and, for i, $q+1 \le i \le r$, we put $f^0(x_i) \in \varphi(x_i)$. Obviously f^0 satisfies (23.5.1)–(23.5.3) above.

Assume that, for $k = 0, \ldots, N-1$, we have defined f^k on P^k satisfying (23.5.1)–(23.5.3). Now, it suffices to define f^{k+1} on an arbitrary $(k+1)$-dimensional simplex S from T^{k+1}. There exists $x \in P$ such that $S \subset O_{\varepsilon_0/4}(x)$.

For $x' \in \partial S$, we have

$$f^k(x') \in O_{\varepsilon_k/2}(\varphi(O_{\varepsilon_k/2}(x'))) \subset O_{\varepsilon_k}(\varphi(O_{\varepsilon_k/2}(S))) \subset O_{\varepsilon_k}(\varphi(O_{\varepsilon_k/2+\varepsilon_0/4}(x))) \subset O_{\varepsilon_k}(\varphi(O_{\varepsilon_k}(x))).$$

Thus $f^k(\partial S) \subset O_{\varepsilon_k}(\varphi(O_{\varepsilon_k}(x)))$.

(23.5.4) If $k+1 \leq N_0$ and $S \in T_0^{k+1}$, then we put $f^{k+1}|_S = f_0|_S$.

(23.5.5) If $k+1 > N_0$ or $S \notin T_0^{k+1}$, then, by the definition of ε_k, there exists $f^{k+1}\colon S \to O_{\varepsilon_{k+1}/4}(\varphi(O_{\varepsilon_{k+1}/4}(x)))$.

Let $y \in S$. If (23.5.4) holds, then

$$f^{k+1}(y) = f^0(y) \in O_{\varepsilon_0/2}(\varphi(O_{\varepsilon_0/2}(\varphi(O_{\varepsilon_0/2}(y))))) \subset O_{\varepsilon_{k+1}/2}(\varphi(O_{\varepsilon_{k+1}/2}(y))).$$

If (23.5.5) holds, then $y \in O_{\varepsilon_0/4}(x)$. Hence $x \in O_{\varepsilon_0/4}(y)$, and therefore

$$O_{\varepsilon_{k+1}/4}(x) \subset O_{\varepsilon_{k+1}/4+\varepsilon_0/4}(y) \subset O_{\varepsilon_{k+1}/2}(y).$$

Now, $f^{k+1}(y) \in O_{\varepsilon_{k+1}/4}(\varphi(O_{\varepsilon_{k+1}/4}(x))) \subset O_{\varepsilon_{k+1}/2}(\varphi(O_{\varepsilon_{k+1}/2}(y)))$. This proves that f^{k+1} satisfies (23.5.1)–(23.5.3). □

From the above theorem we immediately obtain:

(23.6) Corollary. *Let P be a finite polyhedron, Y a space and let $\varphi \in J(P,Y)$. Then for any $\varepsilon > 0$ there exists a continuous ε-approximation $f\colon P \to Y$ of φ.*

Proof. Let T be a triangulation of P and T_0 its 0-dimensional skeleton. If $P_0 = |T_0| = \{x_1, \dots, x_r\}$, then we define $f_0\colon P_0 \to Y$ by putting $f_0(x_i) \in \varphi(x_i)$ for $i = 1, \dots, r$. By Theorem (23.5), for any $\varepsilon > 0$, there is a continuous map $f\colon P \to Y$ which is an ε-approximation of φ and an extension of f_0, since f_0 is a selection of $\varphi|_{P_0}$ and, in particular, as such is a δ-approximation of φ for any $\delta > 0$. □

(23.7) Corollary. *Let P be a finite polyhedron, Y a space, and let $\varphi\colon P \multimap Y$ be a J-mapping. For any $\varepsilon > 0$, there exists a $\delta > 0$ such that if $f, g\colon P \to Y$ are δ-approximations of φ then there is a continuous mapping $h\colon P \times [0,1] \to Y$ such that $h(\cdot, 0) = f$, $h(\cdot, 1) = g$ and, for each $t \in [0,1]$, $h(\cdot, t)$ is an ε-approximation of φ.*

Proof. Let $P' = P \times [0,1]$ be a polyhedron with the canonical triangulation T'. (The triangulation T' is constructed in the following manner; to any simplex of the triangulation T of P, with vertices $\{x_0, \dots, x_r\}$, we join the family of simplices spanned by the points $x_0 \times \{0\}, \dots, x_i \times \{0\}, x_i \times \{1\}, \dots x_r \times \{1\}$, $i = 0, \dots, r$, together with their faces). We put $P_0' = P \times \{0\} \cup P \times \{1\}$. Of course, P_0' is a subpolyhedron of P'.

Assume that $\varphi' \colon P' \to Y$ is given by the formula: $\varphi'(x,t) = \varphi(x)$ for $x \in P$ and $t \in [0,1]$. Let us fix $\varepsilon > 0$. By Theorem (23.5), since φ' is J-map in view of Lemma (23.3), there exists $\delta > 0$ such that if $h_0 \colon P_0' \to Y$ is a δ-approximation of φ', then there is a continuous mapping $h \colon P' \to Y$ which is an ε-approximation of φ' and such that $h|_{P_0'} = h_0$. Now, take $f, g \colon P \to Y$ as continuous δ-approximations of φ and define $h_0 \colon P_0' \to Y$ by putting $h_0|_P \times \{0\} = f$, $h_0|_P \times \{1\} = g$. Obviously, h_0 is a δ-approximation of φ'. Then the existing map $h \colon P' \to Y$ satisfies our assertion. □

Now, we shall show that the above results may be carried over to a larger class of domains.

(23.8) Theorem. *Let X be a compact* ANR*-space, Y a space. For any $\varepsilon > 0$ and any J-mapping $\varphi \colon X \multimap Y$, there exists a continuous ε-approximation $f \colon X \to Y$ of φ.*

Proof. Using the Arens–Eells Theorem (1.6), we may assume that $X \subset U \subset E$, where $(E, \|\cdot\|)$ is a normed space, U is an open subset of E and there is a retraction $r \colon U \to X$. Take $\varepsilon > 0$. By Proposition (22.1), there is γ, $0 < \gamma < \varepsilon$, such that $O_\gamma(X) \subset U$ and for $x, z \in O_\gamma(X)$ with $\|x - z\| < \gamma$ we have $d_X(r(x), r(z)) < \varepsilon$. Take ϱ, $0 < 2\varrho < \gamma$. Then $O_{2\varrho}(X) \subset U$. Let $x_1, \ldots, x_k \in X$ be such that

$$X \subset \bigcup_{i=1}^{k} O_\varrho(x_i) = V.$$

Thus

$$X \subset V \subset \bigcup_{i=1}^{k} O_{2\varrho}(x_i) \subset U.$$

For $x \in V$, $i = 1, \ldots, k$, we put

$$\mu_i(x) = \max\{0, \varrho - \|x - x_i\|\} \quad \text{and} \quad \lambda_i(x) = \frac{\mu_i(x)}{\sum_{i=1}^{k} \mu_j(x)}.$$

Then, for $x \in V$,

$$\sum_{i=1}^{k} \lambda_i(x) = 1 \quad \text{and} \quad \lambda_i(x) \neq 0$$

if and only if $x \in O_\varrho(x_i)$.

We define $\pi \colon V \to \operatorname{span}\{x_1, \ldots, x_k\}$ by the formula

$$\pi(x) = \sum_{i=1}^{k} \lambda_i(x) x_i.$$

Hence $\|\pi(v)-v\| < \varrho$ for any $v \in V$. It is also easy to see that there exists a (finite) polyhedron P such that $\pi(V) \subset P \subset O_{2\varrho}(X) \subset U$. Let $\psi = \varphi \circ r\colon U \to Y$. By Lemma (23.3), ψ is a J-mapping. By Theorem (23.5), there exists a continuous map $g\colon P \to Y$ being a ϱ-approximation of $\psi|_P\colon P \to Y$. Let $\widetilde{f}\colon V \to Y$ be defined by the formula: $\widetilde{f}(v) := g(\pi(v))$ for $v \in V$.

We claim that $\widetilde{f}$ is a γ-approximation of ψ. To the end of proof let $v \in V$. Then $\widetilde{f}(v) = g(\pi(v)) \in O_\varrho(\psi(O_\varrho(\pi(v))))$. So, there are $z \in O_\varrho(\pi(v)) \cap P$ and $y \in \psi(z)$ such that $d_Y(\widetilde{f}(v), y) < \varrho$. But

$$\|z - v\| \leq \|z - \pi(v)\| + \|\pi(v) - v\| < 2\varrho < \gamma.$$

Hence $z \in O_\gamma(v) \cap U$, so $\widetilde{f}(v) \in O_\gamma(\psi(O_\gamma(v)))$.

Now, let $f = \widetilde{f}|_X\colon X \to Y$. We can see that f is an ε-approximation of φ. Indeed, take $x \in X$. Since $f(x) = \widetilde{f}(x) \in O_\gamma(\psi(O_\gamma(x)))$, we have $z \in O_\gamma(x)$ and $y \in \psi(z)$ such that $d_Y(f(x), y) < \gamma < \varepsilon$. Since $x, z \in O_\gamma(X)$ and $\|x - z\| < \gamma$, therefore $d_X(r(x), r(z)) < \varepsilon$. So $r(z) \in O_\varepsilon(x) \cap X$. Moreover, $y \in \psi(z) = \varphi(r(z))$. Hence $f(x) \in O_\varepsilon(\varphi(O_\varepsilon(x)))$. The proof is complete. □

Finally, we prove the following:

(23.9) Theorem. *Let X be a compact* ANR*-space, Y a space and let $\varphi \in J(X, Y)$. For any $\delta > 0$ there exists an $\varepsilon_0 > 0$ such that for ε, $0 < \varepsilon \leq \varepsilon_0$, and for arbitrary ε-approximations $f, g\colon X \to Y$ of φ, there is a continuous map $h\colon X \times [0,1] \to Y$ such that $h(\,\cdot\,, 0) = f$, $h(\,\cdot\,, 1) = g$, and $h(\,\cdot\,, t)$ is a δ-approximation of φ for all $t \in [0,1]$.*

Proof. Again, we assume that $X \subset U \subset E$, where $(E, \|\cdot\|)$ is a normed space, U is open in E, and there is a retraction $r\colon U \to X$. Let us fix $\delta > 0$. Since the proof is constructive, it will be carried out in several steps.

(23.9.1) By Proposition (23.1) we can choose γ, $0 < \gamma < \delta$ such that $O_\gamma(X) \subset U$ and for $z, z' \in O_\gamma(X)$, if $\|z - z'\| < \gamma$ then $d_X(r(z), r(z')) < \delta$.

(23.9.2) Take η, $0 < \eta < \gamma/4$, such that for $z, z' \in O_\eta(X)$, if $\|z - z'\| < \eta$ then $d_X(r(z), r(z')) < \gamma/4$. Hence $O_\eta(X) \subset U$ and, as it is easily seen, for any $z \in O_\eta(X)$, $\|z - r(z)\| < \gamma/2$.

(23.9.3) Now, take ϱ, $0 < 2\varrho < \eta$. Hence $O_{2\varrho}(X) \subset U$. Let $x_1, \ldots, x_k \in X$ be such that $X \subset V = \bigcup_{i=1}^k O_\varrho(x_i)$. As in the proof of Theorem (23.2) we construct $\pi\colon V \to \operatorname{span}\{x_1, \ldots, x_k\}$ for which there exists a (finite) polyhedron P such that $\pi(X) \subset \pi(V) \subset P \subset O_{2\varrho}(X) \subset U$. Moreover, $\|\pi(v) - v\| < \varrho$ for any $v \in V$.

(23.9.4) By Corollary (23.7), there is a $\varrho_0 > 0$ such that if $F, G\colon P \to Y$ are ϱ_0-approximations of $\psi|_P\colon P \to Y$, where $\psi = \varphi \circ r$. Then there exists a continuous map $H\colon P \times [0,1] \to Y$ such that $H(\,\cdot\,, 1) = G$ and $H(\,\cdot\,, t)$ is a ϱ-approximation of $\psi|_P$ for any $t \in [0,1]$.

(23.9.5) By (23.2.2), there exists an ε_0, $0 < \varepsilon_0 < \gamma/4$, such that for any ε, $0 < \varepsilon \le \varepsilon_0$, and $f\colon X \to Y$, we have $F = f \circ (r|_P)\colon P \to Y$ is a ϱ_0-approximation of $\psi|_P$ provided that f is an ε-approximation of φ.

(23.9.6) Take ε, $0 < \varepsilon \le \varepsilon_0$, and $f, g\colon X \to Y$ to be continuous ε-approximations of φ. By (23.9.5), $F = f \circ r|_P$, $G = g \circ r|_P$ are ϱ_0-approximations of $\psi|_P$. By (23.9.4), there is a continuous map $H\colon P \times [0,1] \to Y$ such that $H(\,\cdot\,, 0) = F$, $H(\,\cdot\,, 1) = G$ and $H(\,\cdot\,, t)$ is a ϱ-approximation of $\psi|_P$ for any $t \in [0,1]$.

(23.9.7) Consider a map $k\colon X \times [0,1] \to Y$ given by the formula $k(x,t) = H(\pi(x), t)$ for $x \in X$ and $t \in [0,1]$. We claim that $k(\,\cdot\,, t)$ is an η-approximation of ψ for any $t \in [0,1]$. Indeed, let $t \in [0,1]$ and take $x \in X$. We have

$$k(x,t) = H(\pi(x), t) \in O_\varrho(\psi(O_\varrho(\pi(x)) \cap P)).$$

Hence, there exists $z \in O_\varrho(\pi(x)) \cap P$ and $y \in \psi(z)$ such that $d_Y(k(x,t), y) < \varrho$. Since

$$\|z - x\| \le \|z - \pi(x)\| + \|\pi(x) - x\| < 2\varrho < \eta,$$

we infer that $z \in O_\eta(x) \subset U$, and that $k(x,t) \in O_\eta(\psi(O_\eta(x) \cap U))$. Moreover, observe that $k(x,0) = F(\pi(x))$, $k(x,1) = G(\pi(x))$ for each $x \in X$.

(23.9.8) Let us define $k', k''\colon X \times [0,1] \to Y$ by the formulas

$$k'(x,t) = f(r((1-t)x + t\pi(x))), \qquad k''(x,t) = g(r((1-t)\pi(x) + tx))$$

for $x \in X$, $t \in [0,1]$. This definition is correct since

$$(1-t)x + t\pi(x) = x + t(\pi(x) - x) \in O_\varrho(x) \subset U \quad \text{and} \quad (1-t)\pi(x) + tx \in U$$

for each $x \in X$, $t \in [0,1]$. Next, we define $h\colon X \times [0,1] \to Y$ by

$$h(x,t) = \begin{cases} k'(x, 3t) & \text{for } t \in [0, 1/3], \\ k(x, 3t-1) & \text{for } t \in [1/3, 2/3], \\ k''(x, 3t-2) & \text{for } t \in [2/3, 1], \end{cases}$$

for $x \in X$. Then h is continuous and $h(\,\cdot\,, 0) = f$, $h(\,\cdot\,, 1) = g$.

(23.9.9) We shall show that $h(\,\cdot\,, t)\colon X \to Y$ is a γ-approximation of ψ for any $t \in [0,1]$.

(a) Let $t \in [1/3, 2/3]$. By (23.9.7), $h(\,\cdot\,, t) = k(\,\cdot\,, 3t-1)$ is an η-approximation of ψ. Since $\eta < \gamma$, $h(\,\cdot\,, t)$ is a γ-approximation of ψ.

(b) Let $t \in [0, 1/3]$. Take $x \in X$. By the definition,

$$h(x,t) = k'(x, 3t) = f(r((1-3t)x + 3t\pi(x))) \in f \circ r(O_\varrho(x)).$$

Hence there exists $z \in O_\varrho(x) \subset O_\eta(x)$ such that $h(x,t) = f(r(z))$. We shall show now that $h(x,t) \in O_\gamma(\varphi(O_\gamma(x)))$. Since f is an ε-approximation of φ, we infer that $f(r(z)) \in O_\varepsilon(\varphi(O_\varepsilon(r(z))))$. So, there are $\widetilde{x} \in O_\varepsilon(r(z))$ and $y \in \varphi(\widetilde{x})$ such that $d_Y(f(r(z)), y) < \varepsilon$. Since $z \in O_\eta(X)$, therefore, $\|z - r(z))\| < \gamma/2$ by (23.9.2). Hence

$$\|z - \widetilde{x}\| \leq d_X(\widetilde{x}, r(z)) + \|r(z) - z\| < \varepsilon + \gamma/2 < 3\gamma/4$$

because $\varepsilon \leq \varepsilon_0 < \gamma/4$. Moreover, $\|x - z\| < \varrho < \eta/2 < \gamma/8$; thus

$$d_X(x, \widetilde{x}) \leq \|\widetilde{x} - z| + \|z - x\| < \gamma.$$

So finally, $\widetilde{x} \in O_\gamma(x)$, $y \in \varphi(\widetilde{x})$ and $h(x,t) = f(r(z)) \in O_\gamma(\varphi(O_\gamma(x)))$.

(c) Let $t \in [2/3, 1]$. By an analogous reasoning to the above we show that $h(x,t) \in O_\gamma(\varphi(O_\gamma(x)))$ for $x \in X$.

(23.9.10) By (23.9.1), proceeding in exactly the same way as in the last part of the proof of Theorem (23.8), we now show that, for any $t \in [0,1]$, $h(\cdot\,,t)$ is a δ-approximation of φ.

This completes the proof of Theorem (23.9) □

From Theorems (23.8) and (23.9), we immediately have:

(23.10) Corollary. *Let X be a compact* ANR*-space and Y a space. If $\varphi \in J(X,Y)$ then $\varphi \in A(X,Y)$.*

So, for X a compact ANR-space and Y an arbitrary space we obtain:

(23.11) Proposition. $J(X,Y) \subset A(X,Y) \subset A_0(X,Y)$.

24. Homotopy

In fixed point theory an appropriate notion of homotopy is needed. In what follows we let:

$$A(X) = A(X,X) \quad \text{and} \quad A_C(X) = \{\varphi \in A(X) \mid \mathrm{Fix}(\varphi) \cap C = \emptyset\},$$

where C is a closed subset of X.

Now we shall give the definition of homotopy in $A_C(X)$.

(24.1) Definition. Two maps $\varphi, \psi \in A_C(X)$ are called *homotopic* (in $A_C(X)$) if there exists a mapping $\chi \in A_0(X \times [0,1], X)$ such that $\chi(x,0) = \varphi(x)$, $\chi(x,1) = \psi(x)$ for each $x \in X$ and $\chi(x,t) \not\ni x$ for each $x \in C$, $t \in [0,1]$. If φ and ψ are homotopic then we write $\varphi \sim_C \psi$.

Obviously, the relation $\sim_C$ is reflexive and symmetric. To see that it a transitive, we need the following lemma.

(24.2) LEMMA. *If $\chi \in A_0(X\times[0,1],Y)$ where X is compact then for each $\varepsilon > 0$ and $t \in [0,1]$, there is $\varrho > 0$ such that if $h\colon X \times [0,1] \to Y$ is a ϱ-approximation of χ, then $h_t(\,\cdot\,,t)$ is an ε-approximation of $\chi_t = \chi(\,\cdot\,,t)$.*

PROOF. Fix$\varepsilon > 0$ and $t \in [0,1]$. Since χ is u.s.c. for any $x \in X$, there is $\delta = \delta(x)$ $(0 < \delta < \varepsilon)$ such that $\chi(O_\delta(x,t)) \subset O_{\varepsilon/2}(\chi(x,t))$.

Let $\eta(x) = \delta(x)/2$. Obviously, $X\times\{t\} \subset \bigcup\{O_{\eta(x)}(x,t) \mid x \in X\}$. Since $X\times\{t\}$ is compact, there exists $x_1,\dots,x_k \in X$ such that $X \times \{t\} \subset \bigcup\{O_{\eta(x_i)}(x_i,t) \mid i = 1,\dots,k\}$.

We put $\eta = \min\{\eta(x_i) \mid i = 1,\dots,k\}$. We claim that, for any $x \in X$ there exists i, $1 \le i \le k$, such that $O_\eta(x,t) \subset O_{\delta(x_i)}(x_i,t)$. Indeed, let $x \in X$. Then there is i, $1 \le i \le k$, such that $d(x,x_i) < \eta(x_i)$. If $(x',t') \in O_\eta(x,t)$ then $d(x',x_i) \le d(x',x) + d(x,x_i) < \eta + \eta(x_i) \le 2\eta(x_i) = \delta(x_i)$. Moreover, $|t-t'| < \eta(x_i) < \delta(x_i)$. So, we obtain $(x',t') \in O_{\delta(x_i)}(x_i,t)$.

Let $\varrho = \min\{\eta,\varepsilon/2\}$. Consider a ϱ-approximation $h\colon X\times[0,1] \to Y$ of χ. Take $x \in X$. By (22.3.1) we have $h_t(x) = h(x,t) \in O_\varrho(\chi(O_\varrho(x,t)))$. Consequently, there exists $(x',t') \in O_\varrho(x,t)$ and $y' \in \chi(x',t')$ such that $d_Y(h_t(x),y') < \varrho$.

Choose x_i, such that $O_\varrho(x,t) \subset O_{\varrho(x_i)}(x_i,t)$. Then $y' \in \chi(x',t') \subset \chi(O_\varrho(x,t)) \subset \chi(O_{\delta(x_i)}(x_i,t)) \subset O_{\varepsilon/2}(\chi(x_i,t))$. We find $y \in \chi(x_i,t)$ such that $d_Y(y,y') < \varepsilon/2$. Next, $d(x,x_i) \le \delta(x_i) < \varepsilon$. So $x \in O_\varepsilon(x)$, $y \in \chi(x_i,t)$ and $d_Y(h_t(x),y) \le d_Y(h_t(x),y') + d_Y(y,y') < \varrho + \varepsilon/2 \le \varepsilon$; hence $h_t(x) \in O_\varepsilon(\chi_t(O_\varepsilon(x)))$ and this completes the proof. □

(24.3) PROPOSITION. *If C is a closed subset of a compact space X then the relation $\sim_C$ is an equivalence.*

PROOF. Obviously $\sim_C$ is reflexive and symmetric. Let $\varphi_i \in A(X)$, $i = 1,2,3$ and $\varphi_1 \sim_C \varphi_2$, $\varphi_2 \sim_C \varphi_3$. There are $\chi_i \in A_0(X\times[0,1],X)$, $i = 1,3$ such that $\chi_1(\,\cdot\,,0) = \varphi_1$, $\chi_1(\,\cdot\,,1) = \varphi_2 = \chi_3(\,\cdot\,,0)$, and $\chi_3(\,\cdot\,,1) = \varphi_3$. Moreover, for any $t \in [0,1]$, $x \in C$, we have $\chi_i(x,t) \ne x$ $(i = 1,3)$. Let $\chi\colon X\times[0,1] \to X$ be given by the formula

$$\chi(x,t) = \begin{cases} \chi_1(x,3t) & \text{for } t \in [0,1/3],\\ \varphi_2(x) & \text{for } t \in (1/3,2/3],\\ \chi_3(x,3t-2) & \text{for } t \in (2/3,1]. \end{cases}$$

Obviously, for any $t \in [0,1]$ and $x \in C$, $x \notin \chi(x,t)$.

Take any $\delta > 0$. By Definition (24.1), since $\varphi_2 \in A_C(X)$, there is $\varepsilon_0 > 0$ such that for ε, $0 < \varepsilon \le \varepsilon_0$, any two approximations of φ_2 are homotopic to each other; moreover, the homotopy joining them is a δ-approximation of φ_2. By (24.2), there is ϱ, $0 < \varrho < \delta$, such that if $h_i\colon X\times[0,1] \to X$ is a ϱ-approximation of χ_i $(i = 1,3)$ then $h_1(\,\cdot\,,1)$, $h_3(\,\cdot\,,0)$ are ε_0-approximations of φ.

Let $h_2\colon X \times [0,1] \to X$ be a homotopy joining $h_1(\,\cdot\,,1)$ and $h_3(\,\cdot\,,0)$ such that $h_2(\,\cdot\,,t)$ is a δ-approximation of φ_2. We define $h\colon X \times [0,1] \to X$ by the formula

$$h(x,t) = \begin{cases} h_1(x,3t) & \text{for } t \in [0,1/3], \\ h_2(x,3t-1) & \text{for } t \in (1/3,2/3], \\ h_3(x,3t-2) & \text{for } t \in (2/3,1]. \end{cases}$$

It is easy to verify that h is a δ-approximation of χ, thus $\chi \in A_0(X \times [0,1], X)$ and joins φ_1 with φ_3 in the sense of the relation $\sim_C$; the proof is complete. □

We denote the homotopy class of $\varphi \in A_C(X)$ by $[\varphi]_C$ and the set of all homotopy classes by $[A_C(X)]$. Let $S_C(X)$ denote the class of all continuous (singlevalued) mappings $f\colon X \to X$ such that $\mathrm{Fix}(f) \cap C = \emptyset$.

We say that two maps in $S_C(X)$ are homotopic if there exists a singlevalued, continuous and fixed point free (on C) homotopy joining these maps. We denote the homotopy class of $f \in S_C(X)$ by $[f]_C$ and the set of all homotopy classes by $[S_C(X)]$.

Now, we are in a position to prove our first fundamental result of this section.

(24.4) THEOREM. *If C is a closed subset of a compact* ANR*-space X then there is a bijection $F\colon [A_C(X)] \to [S_C(X)]$.*

PROOF. Let $\omega = [\varphi] \in [A_C(X)]$. By (22.3.3), we have $\delta(\varphi) > 0$ such that any $\delta(\varphi)$-approximation of φ is fixed point free on C. By definition, we find $\varepsilon_0(\varphi)$, $0 < \varepsilon_0(\varphi) < \delta(\varphi)$, such that for ε, $0 < \varepsilon \le \varepsilon_0(\varphi)$, any two ε-approximations of φ are joined by a homotopy being a $\delta(\varphi)$-approximation of φ. We put

$$F(\omega) = [f]_C,$$

where $f\colon X \to X$ is an arbitrary $\varepsilon_0(\varphi)$-approximation of φ. We shall show that the definition of F is correct. Obviously $f \in S_C(X)$. It is easily seen (from the construction), what $F(\omega)$ does not depend on the choice of f. Now, let $\psi \in \omega$. Thus φ and ψ are joined in $A_C(X)$ by a homotopy $\chi \in A_0(X \times [0,1], X)$ (see Definition (24.1)).

By (22.3.3), there is $\delta(\chi) > 0$ such that for any $\delta(\chi)$-approximation $h\colon X \times [0,1] \to X$ of χ we have $h(x,t) \ne x$ for $x \in C$, $t \in [0,1]$. By (24.2), there is $\varrho < \delta(\chi)$ such that for any ϱ-approximation $h\colon X \times [0,1] \to X$ of χ, $h(\,\cdot\,,0)$ is an $\varepsilon_0(\varphi)$-approximation of φ and $h(\,\cdot\,,1)$ is an $\varepsilon_0(\psi)$-approximation of ψ. According to our definition we assigned to $[\varphi]_C$ the element $[h(\,\cdot\,,0)]_C$ and to $[\psi]_C$ the element $[h(\,\cdot\,,1)]_C$; thus our definition is correct.

Observe that since X is a compact ANR-space we have that $S_C(X) \subset A_C(X)$ and hence, F is surjective. To prove the injectivity of F it is enough to show that having $\varphi \in A_C(X)$ there is $\varepsilon > 0$ such that for any ε-approximation $f\colon X \to X$ of φ, one has $f \in [\varphi]_C$. Applying (22.3.3), we gather that there is $\delta > 0$ such that for each $x \in C$ we have $x \notin O_\delta(\varphi(O_\delta(x)))$. Take ε, $\varepsilon < \delta$ (given in (24.1) for φ and δ) and let f be an ε-approximation of φ. We put

$$\chi(x,t) = \begin{cases} \varphi(x) & \text{for } t \in [0,1/3], \\ \operatorname{cl} O_\delta(\varphi(O_\delta(x))) & \text{for } t \in (1/3,2/3], \\ f(x) & \text{for } t \in (2/3,1]. \end{cases}$$

It is easy to see that χ is an u.s.c. map. We shall show that $\chi \in A_0(X \times [0,1], X)$. Let $\eta > 0$ (we may assume without loss of generality that $\eta < \varepsilon$) and let $g\colon X \to X$ be an η-approximation of φ. By the construction there is a mapping $k\colon X \times [1/3,2/3] \to X$ such that $k(\,\cdot\,,1/3) = g$, $k(\,\cdot\,,2/3) = f$ and $k(\,\cdot\,,t)$ is a δ-approximation of φ for each $t \in [1/3,2/3]$. Define $h\colon X \times [0,1] \to X$ by the formula

$$h(x,t) = \begin{cases} g(x) & \text{for } t \in [0,1/3], \\ k(x,t) & \text{for } t \in (1/3,2/3], \\ f(x) & \text{for } t \in (2/3,1]. \end{cases}$$

We can see that h is an η-approximation of χ. It is also obvious that $x \notin \chi(x,t)$ for $x \in C$, $t \in [0,1]$; the proof is complete. □

(24.5) REMARK. Let us note that F may be defined for an arbitrary compact space and its closed subsets.

The above theorem enables one to provide a construction of the fixed point index and the Lefschetz number for maps from $A(X)$, where X is a compact ANR-space.

We will show it in the next section.

25. The fixed point index in $A(X)$

It is well known that in order to define a fixed point index for a class of multivalued mappings, some important facts from homology theory are, so far, indispensable (cf. [Br1-M], [Do-M]). Here we present a new approach to the fixed point index theory of multivalued maps, much simpler than those considered earlier. Note that for this purpose we use the approximation technique only, thus for our considerations, homology theory is superfluous. We believe that readers will find our approach more natural, convenient, and interesting from the point of view of applications in nonlinear analysis. Our presentations is connected with those presented in [CL1], [GGK1]–[GGK3] in the context of the topological degree theory.

Let U be an open subset of a compact ANR-space X and let $\varphi \in A_M(X)$. By (24.4), we obtain a map $f \in S_{\partial U}(X)$ such that $[f]_{\partial U} = F([\varphi]_{\partial U})$. We put

(25.1) $$\operatorname{ind}(X,\varphi,U) := \operatorname{ind}(X,f,U),$$

where $\operatorname{ind}(X, f, U)$ denotes the fixed point index for singlevalued f as defined in [Br1-M] or [Do-M].

The number $\operatorname{ind}(X, \varphi, U)$ is called *the fixed point index* of $\varphi\colon X \multimap X$ (from the class $A(X)$) with respect to U. In view of (24.4) this definition is correct. We define also the Lefschetz number $\lambda(\varphi)$ of $\varphi \in A(X, X)$ by putting

$$\lambda(\varphi) = \lambda(f), \tag{25.2}$$

where $f\colon X \to X$ is a singlevalued map homotopic to φ. Once again, in view of (24.4) our definition is correct.

Below there are the most important properties of the fixed point index.

(25.3) THEOREM. *Let X be a compact* ANR*-space and let U be an open subset of X. Let $\varphi, \psi \in A_{\partial U}(X)$.*

(25.3.1) (Existence) *If* $\operatorname{ind}(X, \varphi, U) \neq 0$ *then* $\operatorname{Fix}(\varphi) \neq \emptyset$.

(25.3.2) (Excision) *If* $\operatorname{Fix}(\varphi) \subset V \subset U$, *where V is open in X then* $\operatorname{ind}(X, \varphi, U) = \operatorname{ind}(X, \varphi, V)$.

(25.3.3) (Additivity) *Let U_1, U_2 be open in X and such that $U_1 \cap U_2 = \emptyset$ and* $\operatorname{Fix}(\varphi) \cap \operatorname{cl} U \setminus (U_1 \cup U_2) = \emptyset$*; then* $\operatorname{ind}(X, \varphi, U) = \operatorname{ind}(X, \varphi, U_1) + \operatorname{ind}(X, \varphi, U_2)$.

(25.3.4) (Homotopy) *If $\chi \in A_0(X \times [0,1], X)$ joins φ and ψ in $A_{\partial U}(X)$ then* $\operatorname{ind}(X, \varphi, U) = \operatorname{ind}(X, \psi, U)$.

(25.3.5) (Normalization) *If $U = X$ then* $\operatorname{ind}(X, \varphi, X) = \lambda(\varphi)$.

(25.3.6) (Contraction) *Let Y be a compact* ANR*-space such that $Y \subset X$. If $\varphi(X) \subset Y$ and $\varphi|_Y \in A_{\partial(U \cap Y)}(Y)$ then* $\operatorname{ind}(X, \varphi, U) = \operatorname{ind}(Y, \varphi|_Y, U \cap Y)$.

PROOF. (25.3.1) Assume that $\operatorname{ind}(X, \varphi, U) = m \neq 0$. For any sufficiently large n, if $f\colon X \to X$ is an n^{-1} approximation of φ, then

$$\operatorname{ind}(X, \varphi, U) = \operatorname{ind}(X, f, U) = m.$$

By the existence property for singlevalued maps, we infer that there is $x_n \in X$ such that $f(x_n) = x_n$. Hence $x_n \in O_{1/n}(\varphi(O_{1/n}(x_n)))$. Since X is compact and φ is u.s.c., we obtain x_0 such that $x_0 \in \varphi(x_0)$.

(25.3.2) follows immediately from the excision property of the classical index.

(25.3.3) is equivalent to (25.3.2).

(25.3.4) is a direct consequence of definition (25.1). (25.3.5) is obvious.

(25.3.6) Take an arbitrary $\varrho > 0$. By (22.1), there is δ, $0 < \delta < 2^{-1}\varrho$, for which there exists a retraction $r\colon O_\delta(Y) \to Y$ such that $d(r(z), z) < 2^{-1}\varrho$. Let $\varepsilon > 0$ and $\varepsilon < \delta$. By the upper semi-continuity of φ, for any $\varepsilon > 0$, $y \in Y$, there is $\eta = \eta(y, \varepsilon)$, $0 < \eta < 4^{-1}\varepsilon$ such that $\varphi(O_\eta(y)) \subset O_{\varepsilon/2}(\varphi(y))$.

Let 2λ be the Lebesgue coefficient of the covering $\{X \setminus Y, \{O_\eta(y)\}_{y\in Y}\}$ of X. Take any $y \in Y$. There is $y^\sim \in Y$ such that $O_\lambda(y) \subset O_{\eta(y^\sim)}(y^\sim)$. Then $y^\sim \in O_\varepsilon(y)$. Let $f\colon X \to X$ be a λ-approximation of φ.

By (22.3.1), $f(y) \in O_\lambda(\varphi(O_\lambda(y))) \subset O_\lambda(\varphi(O_{\eta(y^\sim)}(y^\sim))) \subset O_{\lambda+\varepsilon/2}(\varphi(y^\sim))$. Since $\lambda < 2\eta(y^\sim)$, we obtain that $\lambda + \varepsilon/2 < \varepsilon$. Thus, $f(y) \in O_\varepsilon(\varphi(O_\varepsilon(y)))$ and $f(y) \in O_\delta(Y)$. Hence, $r \circ f|_Y\colon Y \to Y$ is a ϱ-approximation of $\varphi|_Y$.

If ϱ (and, consequently, λ) is sufficiently small, then $\operatorname{ind}(X, \varphi, U) = \operatorname{ind}(X, f, U)$ and $\operatorname{ind}(Y, \varphi|_Y, U \cap Y) = \operatorname{ind}(Y, r \circ f|_Y, U \cap Y)$. By the contraction property of the ordinary index $\operatorname{ind}(Y, r \circ f|_Y, U \cap Y) = \operatorname{ind}(X, r \circ f, U)$. Since X is uniformly locally connected (ULC-space, see Section 2) then, for sufficiently small ϱ the maps $r \circ f$ and f are homotopic. So, $\operatorname{ind}(X, r \circ f, U) = \operatorname{ind}(X, f, U)$ which completes the proof. □

(25.4) Remark. Observe that we have proved that $\varphi|_Y \in A_0(Y)$ provided that $\varphi \in A_0(X)$. Moreover, it is not clear whether $\varphi|_Y \in A_{\partial(U\cap Y)}(Y)$ if $\varphi \in A_{\partial U}(X)$. That is why the condition: $\varphi|_Y \in A_{\partial(U\cap Y)}(Y)$ seems to be indispensable as an assumption (23.5.6).

Note, that from the existence and normalization properties we get:

(25.5) Theorem (Lefschetz Fixed Point Theorem). *Let X be a compact* ANR-*space and $\varphi \in A(X)$. If the Lefschetz number $\lambda(\varphi)$ of φ is different from zero, then* $\operatorname{Fix}(\varphi) \neq \emptyset$.

Since any mapping $\varphi \in A(X)$ has the Lefschetz number $\lambda(\varphi)$ equal to 1 provided $X \in \mathrm{AR}$ (see Section 12) we obtain

(25.6) Corollary. *If X is a compact* AR-*space then for any $\varphi \in A(X)$ we have* $\operatorname{Fix}(\varphi) \neq \emptyset$.

Let us note that a version of the Lefschetz fixed point theorem can be formulated also for mappings in $A_0(X, X)$.

Let X be a compact ANR and let $\varphi \in A_0(X)$. For each $\varepsilon > 0$ there exists a continuous (singlevalued) mapping $f \in a(\varphi, \varepsilon)$.

We let:

$$\Lambda_\varepsilon(\varphi) = \{\lambda(f) \mid f \in a(\varphi, \varepsilon)\}.$$

We define the *Lefschetz set* $\Lambda(\varphi)$ of $\varphi \in A_0(X, X)$ by:

$$\Lambda(\varphi) = \bigcap\{\Lambda_\varepsilon(\varphi) \mid \varepsilon > 0\}.$$

The following fact is obvious (see Section 12).

(25.7) PROPOSITION. *If X a compact* AR*-space, then the set* $\Lambda(\varphi) = \{1\}$ *is a singleton for each* $\varphi \in A_0(X)$.

By using the Lefschetz fixed point theorem for singlevalued mappings of compact ANRs analogously to the proof of (22.4) one can obtain:

(25.8) THEOREM. *Let X be a compact* ANR*-space and let* $\varphi \in A_0(X)$. *If the Lefschetz set* $\Lambda(\varphi)$ *of* φ *contains a non-zero integer then* $\mathrm{Fix}(\varphi) \neq \emptyset$.

Now, from (25.7) and (25.8) we get (22.4) as a corollary.

(25.9) COROLLARY. *If X is a compact* AR *and* $\varphi \in A_0(X)$ *then* $\mathrm{Fix}(\varphi) \neq \emptyset$.

26. Topological degree in $\mathbb{R}^n$

The fixed point index defined in the last section is unfortunately not sufficient for applications in the theory of differential inclusions, which we will consider in Chapter VI. In the next two sections therefore we will define the topological degree theory which will be adequate for the mentioned applications, and so in Chapter IV we will present a more general theory then the presented here. The aim of such approach is to give a reader who is not interested in algebraic topology or more precisely in homological methods a chance to go directly from Chapter III to Chapter VI.

In what follows we will use the following notations. Throughout this section a closed ball in $\mathbb{R}^n$ with center x and radius $r \geq 0$ is denoted by $K^n(x,r)$. Furthermore, we are putting:

$$K^n(r) = K^n(0,r), \quad S^{n-1}(r) = \partial K^n(r), \quad P^n = \mathbb{R}^n \setminus \{0\},$$

where $\partial K^n(r)$ stands for the boundary of $K^n(r)$ in $\mathbb{R}^n$.

For any ANR-space X we set

$$J(K^n(r), X) = \{F : K^n(r) \multimap X \mid F \text{ is u.s.c. which } R_\delta\text{-values}\}.$$

In view of (2.13) the above definition coincides with the one given in Section 23. Moreover, for any continuous $f\colon X \to \mathbb{R}^n$, when $X \in \mathrm{ANR}$, we put

$$J_f(K^n(r), \mathbb{R}^n) = \{\varphi : K^n(r) \multimap \mathbb{R}^n \mid \varphi = f \circ F \\ \text{for some } F \in J(K^n(r), X) \text{ and } \varphi(S^{n-1}(r)) \subset P^n\}.$$

Finally, we define

$$CJ(K^n(r), \mathbb{R}^n) = \bigcup \{J_f(K^n(r), \mathbb{R}^n) \mid f\colon X \to \mathbb{R}^n \\ \text{is continuous, with } X \in \mathrm{ANR}\}.$$

Note that from results obtained in Sections 22 and 23 immediately follows that:

$$CJ(K^n(r), \mathbb{R}^n) \subset A(K^n(r), \mathbb{R}^n).$$

We are going to show that on $CJ(K^n(r), \mathbb{R}^n)$ it is possible to define topological degree. To do so we need an appropriate notion of homotopy in $CJ(K^n(r), \mathbb{R}^n)$.

(26.1) DEFINITION. Let $\varphi_1, \varphi_2 \in CJ(K^n(r), \mathbb{R}^n)$ be two maps of the form

$$\begin{aligned} \varphi_1 &= f_1 \circ F_1, \quad K^n(r) \xrightarrow{F_1} X \xrightarrow{f_1} \mathbb{R}^n, \\ \varphi_2 &= f_2 \circ F_2, \quad K^n(r) \xrightarrow{F_2} X \xrightarrow{f_2} \mathbb{R}^n. \end{aligned}$$

We will say that φ_1 and φ_2 are *homotopic* in $CJ(K^n(r), \mathbb{R}^n)$ if there exist an u.s.c. R_δ-valued homotopy $\chi: K^n(r) \times [0,1] \multimap X$ and a continuous homotopy $h: X \times [0,1] \to \mathbb{R}^n$ satisfying:

(26.1.1) $\chi(u,0) = F_1(u)$, $\chi(u,1) = F_2(u)$ for every $u \in K^n(r)$,
(26.1.2) $h(x,0) = f_1(x)$, $h(x,1) = f_2(x)$ for every $x \in X$,
(26.1.3) for every $(u,\lambda) \in S^{n-1}(r) \times [0,1]$ and $x \in \chi(u,\lambda)$ we have $h(x,\lambda) \neq 0$,

then the map $H: K^n(r) \times [0,1] \multimap \mathbb{R}^n$ given by

$$H(u,\lambda) = h(\chi(u,\lambda),\lambda) \quad \text{for every } (u,\lambda) \in K^n(r) \times [0,1]$$

is called *homotopy* in $CJ(K^n(r), \mathbb{R}^n)$ between φ_1 and φ_2.

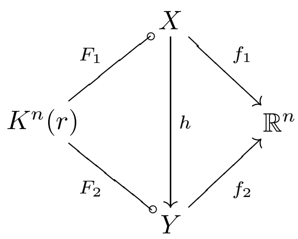
Now we are able to prove the following.

(26.2) THEOREM. *There exists a map* $\mathrm{Deg}: CJ(K^n(r), \mathbb{R}) \to \mathbb{Z}$, *called the topological degree function, satisfying the following properties:*

(26.2.1) *If $\varphi \in CJ(K^n(r), \mathbb{R}^n)$ is of the form $\varphi = f \circ F$ with F single valued and continuous then $\mathrm{Deg}(\varphi) = \deg(\varphi)$, where $\deg(\varphi)$ stands for the ordinary Brouwer degree of the single valued continuous map $\varphi: K^n(r) \to \mathbb{R}^n$.*
(26.2.2) *If $\mathrm{Deg}(\varphi) \neq 0$, where $\varphi \in CJ(K^n(r), \mathbb{R}^n)$, then there exists $u \in K^n(r)$ such that $0 \in \varphi(u)$.*
(26.2.3) *If $\varphi \in CJ(K^n(r), \mathbb{R}^n)$ and $\{u \in K^n(r) \mid 0 \in \varphi(u)\} \subset \mathrm{Int}\, K^n(\widetilde{r})$ for some $0 < \widetilde{r} < r$, then the restriction $\widetilde{\varphi}$ of φ to $K^n(\widetilde{r})$ is in $CJ(K^n(r), \mathbb{R}^n)$ and $\mathrm{Deg}(\widetilde{\varphi}) = \mathrm{Deg}(\varphi)$.*
(26.2.4) *Let $\varphi_1, \varphi_2 \in CJ(K^n(r), \mathbb{R}^n)$ be two maps of the form*

$$\begin{aligned} \varphi_1 &= f_1 \circ F_1, \quad K^n(r) \xrightarrow{F_1} X \xrightarrow{f_1} \mathbb{R}^n, \\ \varphi_2 &= f_2 \circ F_2, \quad K^n(r) \xrightarrow{F_2} Y \xrightarrow{f_2} \mathbb{R}^n, \end{aligned}$$

where $X, Y \in$ ANR. *If there exists a continuous map* $h: X \to Y$ *such that the diagram*



is commutative, that is $F_2 = h \circ F_1$ *and* $f_1 = f_2 \circ h$, *then* $\mathrm{Deg}(\varphi_1) = \mathrm{Deg}(\varphi_2)$.

(26.2.5) *If* φ_1, φ_2 *are homotopic in* $CJ(K^n(r), \mathbb{R}^n)$ *then* $\mathrm{Deg}(\varphi_1) = \mathrm{Deg}(\varphi_2)$.

PROOF. Let $\varphi = f \circ F \in CJ(K^n(r), \mathbb{R}^n)$ be a map of the form

$$K^n(r) \xrightarrow{F} X \xrightarrow{f} \mathbb{R}^n, \quad \text{where } X \in \text{ANR}.$$

Analogously to the the proof of (22.3.3) we can find $\varepsilon_0 > 0$ such that no element of $a(\varphi; \varepsilon_0)$ has zero value on $S^{n-1}(r)$. By (22.3.6), there is a $\rho_0 > 0$ such that $f \circ g \in a(\varphi, \varepsilon_0)$ for $g \in a(F, \rho_0)$. At last, by (23.9) there is a $\delta_0 > 0$ such that for $g_1, g_2 \in a(F, \delta_0)$ the maps $f \circ g_1$, $f \circ g_2$ are homotopic and the joining homotopy has no zero value on $S^{n-1}(r)$.

By the homotopy property of the Brouwer degree (cf. [Br1-M], [Br2M-M], [Do-M], [Ro-M]) we have

$$\deg(f \circ g_1) = \deg(f \circ g_2).$$

So, we can put:

$$\mathrm{Deg}(\varphi) = \deg(f \circ g), \quad \text{where } g \in a(F, \delta_0). \tag{26.2.6}$$

The topological degree $\mathrm{Deg}(\varphi)$ of φ is well defined since it does not depend on the choice of g.

Then properties (26.2.1)–(26.2.5) follow directly from (26.2.6), (22.3), (24.2) and the properties of the Brouwer degree for singlevalued mappings. □

(26.3) REMARK. Note that our definition in formula (26.2.6) depends not only on $\varphi \in CJ(K^n(r), \mathbb{R}^n)$ but also on the decompositions F and f of φ. In fact, to be precise we should use the following notation

$$\mathrm{Deg}(\varphi; f, F), \quad \text{where } \varphi = f \circ F.$$

For better understanding of the above remark let us present the following example.

(26.4) EXAMPLE. We let $K^2 = K^2(1)$ and $S^1 = S^1(1)$. Moreover, we shall identify $\mathbb{R}^2$ with the field of complex numbers. For given $z \in \mathbb{R}^2$ by $|z|$ we will denote its modulus (also called "absolute value"). We shall write z in trigonometric from:

$$z = |z|(\cos s + i\ \sin s).$$

Let $\varphi\colon K^2 \multimap K^2$ be the mapping defined as follows:

$$\varphi(z) = \{|z| \cdot x \mid x \in S^1\} \quad \text{for } x \in K^2.$$

We shall also consider $F\colon K^2 \multimap K^2$ defined by

$$F(|z|(\cos s + i \sin s)) = \{|z|(\cos(s+t) + i \sin(s+t)),\ t \in [0, 3\pi/4]\}$$

and

$$f_1, f_2\colon K^2 \to K^2, \quad f_1(z) = z^2, \quad f_2(z) = z^3.$$

Of course $F \in J(K^2, K^2)$ and $\varphi = f_1 \circ F$ or $\varphi = f_2 \circ F$. Since $f_1 \subset \varphi$ and $f_2 \subset \varphi$ we deduce that $f_1, f_2 \in a(\varphi, \varepsilon)$ for every $\varepsilon > 0$. Therefore,

$$\mathrm{Deg}(\varphi; f_1 \circ F) = \deg(f_1) = 2 \quad \text{and} \quad \mathrm{Deg}(\varphi; f_2 \circ F) = \deg(f_2) = 3.$$

The topological degree $\mathrm{Deg}(\varphi)$ of φ depends on its decomposition. Therefore, in what follows, the notation $\mathrm{Deg}(\varphi)$ we will denote the degree of φ with respect to the fixed decomposition $\varphi = f \circ F$.

The following proposition is useful in applications.

(26.5) PROPOSITION. *Let $\varphi_1, \varphi_2 \in CJ(K^n(r), \mathbb{R}^n)$ be two maps of the form*

$$\begin{aligned} \varphi_1 &= f_1 \circ F_1, \quad K^n(r) \xrightarrow{F_1} X \xrightarrow{f_1} \mathbb{R}^n, \\ \varphi_2 &= f_2 \circ F_2, \quad K^n(r) \xrightarrow{F_2} Y \xrightarrow{f_2} \mathbb{R}^n, \end{aligned}$$

where $X, Y \in \mathrm{ANR}$, such that

(26.5.1) $\quad 0 \notin \lambda\varphi_1(u) + (1-\lambda)\varphi_2(u)$ *for every* $(u, \lambda) \in S^{n-1}(r) \times [0,1]$.

Then $\mathrm{Deg}(\varphi_1) = \mathrm{Deg}(\varphi_2)$.

PROOF. Consider the following two diagrams:

$$K^n(r) \xrightarrow{G} X \times Y \xrightarrow{g_i} \mathbb{R}^n, \quad i = 1, 2,$$

where for every $u \in K^n(r)$ and $(x, y) \in X \times Y$,

$$G(u) = F_1(u) \times F_2(u), \quad g_1(x, y) = f_1(x), \quad g_2(x, y) = f_2(y).$$

Clearly, the maps $\psi_1 = g_1 \circ G$ and $\psi_2 = g_2 \circ G$ are in $CJ(K^n(r), \mathbb{R}^n)$. In order to verify that ψ_1 and ψ_2 are homotopic in $CJ(K^n(r), \mathbb{R}^n)$, consider the maps $\chi\colon K^n(r) \times [0, 1] \multimap X \times Y$ and $h\colon X \times Y \to \mathbb{R}^n$ given, respectively, by

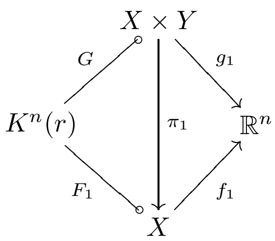
$$\chi(u, \lambda) = G(u), \quad h(x, y, \lambda) = (1 - \lambda)g_1(x, y) + \lambda g_2(x, y).$$

Observe that χ is an u.s.c. R_δ-valued homotopy, and h is a continuous homotopy satisfying conditions (26.1.1) and (26.1.2) of Definition 26.1 ((26.1.1) with $F_1 = F_2 = G$). To check (26.1.3), let $(u, \lambda) \in S^{n-1}(r) \times [0, 1]$ and $(x, y) \in \chi(u, \lambda)$ be arbitrary. Then we have:

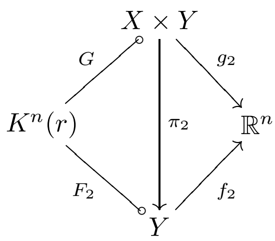
$$h(x, y, \lambda) = (1 - \lambda)f_1(x) + \lambda f_2(y) \in (1 - \lambda)\varphi_1(u) + \lambda\varphi_2(u).$$

Hence, in view of (26.5.1), it follows that $h(x, y, \lambda) \neq 0$, proving (26.1.3). Thus ψ_1 and ψ_2 are homotopic in $CJ(K^n(r), \mathbb{R}^n)$ and, by (26.2.5), $\mathrm{Deg}(\psi_1) = \mathrm{Deg}(\psi_2)$.

On the other hand, the following two diagrams



and



where $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$, are commutative. By (26.2.4), it follows that $\mathrm{Deg}(\psi_1) = \mathrm{Deg}(\varphi_1)$ and $\mathrm{Deg}(\psi_2) = \mathrm{Deg}(\varphi_2)$, thus $\mathrm{Deg}(\varphi_1) = \mathrm{Deg}(\varphi_2)$, completing the proof. □

From Theorem (26.2) we shall deduce some fixed point results. More details will be presented in the next chapter by using homological methods as a tool.

First we shall recall the following well known property of the Brouwer topological degree.

(26.6) Proposition (see [Br2-M], [Do-M], [Ro-M]). *Let $i\colon K^n(r) \to \mathbb{R}^n$ be the inclusion map:*

$$i(x) = x \quad \textit{for every } x \in K^n(r).$$

Then the Brouwer degree $\deg(i)$ *of* i *is equal to* 1.

Moreover, for given subsets $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ $CJ_0(A, B)$ will denote the following class of mappings:

$$CJ_0(A,B) = \{\varphi\colon A \multimap B \mid \varphi = f \circ F,\ F\colon A \multimap X,\ F \text{ is u.s.c. with } R_\delta\text{-values and } f : X \to B \text{ is continuous}\},$$

where $X \in \mathrm{ANR}$. We prove the following:

(26.7) THEOREM (Brouwer Fixed Point Theorem). *If* $\varphi \in CJ_0(K^n(r), K^n(r))$, *then* $\mathrm{Fix}(\varphi) \neq \emptyset$.

PROOF. Let $\varphi = f \circ F$ where $F\colon K^n(r) \multimap X$ and $f\colon X \to K^n(r)$. We can assume, without loss of generality, that $\mathrm{Fix}(\varphi) \cap S^{n-1}(r) = \emptyset$. Then we will consider the mappings:

$$\begin{aligned} &G{:}K^n(r) \multimap X \times K^n(r), && G(x) = \{(y,x) \mid y \in G(x)\} && \text{for every } x \in K^n(r),\\ &g{:}X \times K^n(r) \to \mathbb{R}^n, && g(y,x) = x - f(y) && \text{for every } y \in X \text{ and } x \in K^n(r). \end{aligned}$$

Note, that in view of (1.2.4) and (1.9) we have $(X \times K^n(r)) \in \mathrm{ANR}$.

Consequently, we get $\psi = g \circ G \in CJ(K^n(r), \mathbb{R}^n)$. We define the homotopy

$$h\colon X \times K^n(r) \times [0,1] \to \mathbb{R}^n, \quad h(y,x) = x - t\,f(y).$$

Then for $x \in S^1$ we have $x \neq t\,f(y)$ for every $t \in [0,1]$ and $y \in F(x)$ because $\mathrm{Fix}(\varphi) \cap S^{n-1} = \emptyset$ and $\varphi(K^n) \subset K^n$. Therefore, we see that ψ is homotopic to the inclusion $i\colon K^n(r) \to \mathbb{R}^n$. Consequently, $\mathrm{Deg}(\psi) = \deg(i) = 1$. So, from (26.2.2) we infer that $0 \in \psi(x_0)$ for some $x_0 \in K^n(r)$ and what means $x_0 \in \mathrm{Fix}(\varphi)$. The proof is completed. □

Let us make a simple observation that the Brouwer Fixed Point Theorem remains true when we replace $K^n(r)$ by a space A such that:

$$A \text{ is homeomorphic to } K^n(r) \quad \text{or} \quad A \text{ is a retract of } K^n(r).$$

(26.8) THEOREM (Nonlinear Alternative). *Assume that* $\varphi \in CJ_0(K^n(r), \mathbb{R}^n)$. *Then* φ *has at least one of the following properties:*

(26.8.1) $\mathrm{Fix}(\varphi) \neq \emptyset$,

(26.8.2) *there is an* $x \in S^{n-1}(r)$ *with* $x \in \lambda\varphi(x)$ *for some* $0 < \lambda < 1$.

Proof. Proceeding as in the proof of Theorem (26.7) we obtain the homotopy

$$h: X \times K^n(r) \times [0,1] \to \mathbb{R}^n, \quad h(y,x) = x - t\, f(y).$$

If the homotopy h has no zero on $S^{n-1}(r)$ then, as we known, $\mathrm{Fix}(\varphi) \neq \emptyset$. If for some $x \in S^{n-1}(r)$, $x - t\, f(y) = 0$, then $x \in t\,\varphi(x)$ for some $t \in (0,1)$ (because for $t = 1$ we have assumed that $\mathrm{Fix}(\varphi) \cap S^{n-1}(r) = \emptyset$) and the proof is completed. □

In fact we are able to prove the following:

(26.9) Corollary. *Theorems* (26.7) *and* (26.8) *are equivalent.*

For the proof it is sufficient to see that if $\varphi \in CJ_0(K^n(r))$ then the possibility (26.8.2) cannot occur.

Finally, we shall show, using approximation arguments, the famous Borsuk–Ulam Theorem.

(26.10) Theorem. *Let $\varphi \in A_0(S^n(r), \mathbb{R}^n)$. Then there is a point $x_0 \in S^n(r)$ such that*

$$\varphi(x_0) \cap \varphi(-x_0) \neq \emptyset.$$

Proof. For every $k = 1, 2, \ldots$ we take $f_n \in a(\varphi, 1/k)$. Then in view of the classical Borsuk–Ulam Theorem (see [DG-M]) we have:

$$f_k(x_k) = f_k(-x_k) \quad \text{for some } x_k \in S^n,\ k = 1, 2, \ldots$$

Since $f_k \in a(\varphi, 1/k)$ there are $z_k \in O_{1/k}(x_k)$, $z_k' \in O_{1/k}(-x_k)$, $y_k \in \varphi(z_k)$ and $y_k' \in \varphi(z_k')$ such that:

$$\|y_k - f_k(x_k)\| < \frac{1}{k} \quad \text{and} \quad \|y_k' - f_k(-x_k)\| < \frac{1}{k}.$$

Since S^n is compact we can assume that:

$$\lim_k x_k = x = \lim_k z_k.$$

Let m be a natural number and let $k_m \geq 4m$ be such that

$$\varphi(z_{k_m}) \subset O_{1/4m}(\varphi(x)) \quad \text{and} \quad \varphi(z_{k_m}') \subset O_{1/4m}(\varphi(-x)).$$

Then there are $u_{k_m} \in \varphi(x)$ and $u_{k_m}' \in \varphi(-x)$ such that:

$$\|y_{k_m} - u_{k_m}\| < \frac{1}{4m} \quad \text{and} \quad \|y_{k_m}' - u_{k_m}'\| < \frac{1}{4m}.$$

It implies that

$$\|f_{k_m}(x_{k_m}) - u_{k_m}\| < \frac{1}{2m} \quad \text{and} \quad \|f_{k_m}(-x_{k_m}) - u'_{k_m}\| < \frac{1}{2m}.$$

Finally, we obtain

$$\begin{aligned}\|u_{k_m} - u'_{k_m}\| < \|f_{k_m}(x_{k_m}) - u_{k_m}\| + \|f_{k_m}(-x_{k_m}) - u'_{k_m}\| \\ + \|f_{k_m}(x_{k_m}) - f_{k_m}(-x_{k_m})\| < \frac{1}{2m} + \frac{1}{2m} + 0 < \frac{1}{m}\end{aligned}$$

for every $m = 1, 2, \ldots$ Hence $\lim_m u_{k_m} = \lim_m u'_{k_m}$ ($\varphi(x)$ is compact) and the proof is completed. □

(26.11) REMARK. Observe that if $\varphi(x)$ is a closed subset of $\mathbb{R}^n$ we are getting (from the above proof) only that $\operatorname{dist}(\varphi(x), \varphi(-x)) = 0$.

27. Topological degree for mappings with non-compact values in $\mathbb{R}^n$

Since we would like to consider mappings with closed nonempty, but not necessarily compact, values we need the appropriate notion of approximation. As we observed in Section 14 the notion of PC_X^∞ subsets of X differs from ∞-proximally connected sets (cf. (2.17)–(2.20)). Roughly speaking in the case of arbitrary closed sets we have to replace ε-approximation by α-approximation or W-approximation, where α is an open covering and W is an open neighbourhood of the graph Γ_φ for given multivalued map φ.

Let X, Y be spaces, $A \subset X$, $\varphi\colon X \multimap Y$ a set-valued map and let $W \subset X \times Y$ be an open neighbourhood of Γ_φ. A map $f\colon A \to Y$ is a W-approximation (on the graph) of φ, if $\Gamma_f \subset W$. By *admissible open covering of* Γ_φ we mean the family $\alpha = \{U_x^\alpha \times W_x^\alpha \mid x \in U_x^\alpha \text{ and } \varphi(x) \subset W_x^\alpha,\ x \in X\}$, where U_X^α and W_x^α are open in X and Y, respectively. We write $\alpha \in \mathcal{U}(\varphi)$. One can see that the set $|\alpha| = \bigcup_{x\in X}(U_x^\alpha \times W_x^\alpha)$ is an open neighbourhood of Γ_φ in $X \times Y$. Let $\alpha, \beta \in \mathcal{U}(\varphi)$. We will say that β is *a refinement* of α ($\beta \subseteq \alpha$) if $U_x^\beta \subset U_x^\alpha$ and $W_x^\beta \subset W_x^\alpha$ for every $x \in X$. We say that $f\colon A \to Y$ is an *α-approximation of* φ if $\Gamma_f \subset |\alpha|$ or, equivalently

$$\text{for all } x \in A \text{ exists } y \in X \text{ such that } (x, f(x)) \in U_y^\alpha \times W_y^\alpha$$

(we denote it by the symbol $f \in a(A, \varphi, \alpha)$ or $f \in a(\varphi, \alpha)$ if $A = X$).

Define a *diameter* of $\alpha \in \mathcal{U}(\varphi)$ as

$$\delta(\alpha) = \max\Big\{\sup_{x\in X} \delta(U_x^\alpha),\ \sup_{x\in X} \inf\{r > 0 \mid W_x^\alpha \subset N_r(\varphi(x))\}\Big\}.$$

The following example shows that there are maps which have α-approximations for every $\alpha \in \mathcal{U}(\varphi)$ but do not have W-approximations for some open $W \supset \Gamma_\varphi$.

(27.1) Example. Let $\varphi\colon [-1,1] \multimap R$ be defined by the formula

$$\varphi(x) = \begin{cases} \left\{\dfrac{1}{x}\right\} & \text{if } x \neq 0, \\ R & \text{if } x = 0. \end{cases}$$

Let

$$\begin{aligned} W_1 &= \left\{(x,y) \;\middle|\; x \in [-1,0) \text{ and } y \in \left(\frac{2}{x}, \frac{1}{2x}\right)\right\}, \\ W_2 &= \left\{(x,y) \;\middle|\; x \in (0,1] \text{ and } y \in \left(\frac{1}{2x}, \frac{2}{x}\right)\right\}, \\ W_3 &= \left\{(x,y) \;\middle|\; x \in \left(-\frac{1}{2}, 0\right) \text{ and } y > \frac{1}{3x}\right\} \\ &\quad \cup \left\{(x,y) \;\middle|\; x \in \left(0, \frac{1}{2}\right) \text{ and } y < \frac{1}{3x}\right\} \cup (\{0\} \times R) \end{aligned}$$

and $W = W_1 \cup W_2 \cup W_3$. Then $\Gamma_\varphi \subset W$ and there is no W-approximation of φ because W_1, W_2 and W_3 are pairwise disjoint. However, for each $\alpha \in \mathcal{U}(\varphi)$ there exists $f \in a(\varphi, \alpha)$.

Note that the compactness of values of φ implies an equivalence of W-approximation and α-approximation in the following sense: φ has a W-approximation for every $W \supset \Gamma_\varphi$ if and only if φ has an α-approximation for every $\alpha \in \mathcal{U}(\varphi)$. In fact, $|\alpha|$ is obviously an open neighbourhood of the graph of φ, where $|\alpha| = \bigcup\{U \mid U \in \alpha\}$. Thus it is sufficient to show that φ has a W-approximation for every $W \supset \Gamma_\varphi$ whenever φ has an α-approximation for every $\alpha \in \mathcal{U}(\varphi)$.

Let $W \supset \Gamma_\varphi$ and $x \in X$. The set $\{x\} \times \varphi(x) \subset W$ is compact, thus there is $\varepsilon > 0$ such that $N_\varepsilon(x) \times N_\varepsilon(\varphi(x)) = N_\varepsilon(\{x\} \times \varphi(x)) \subset W$. Let $U_x^\alpha = N_\varepsilon(x)$, $W_x^\alpha = N_\varepsilon(\varphi(x))$ and $\alpha = \{U_x^\alpha \times W_x^\alpha \mid x \in X\}$. Now, every α-approximation of φ is also W-approximation of φ.

Some important facts are summarized in the following proposition (cf. (22.3)).

(27.2) Proposition.

(27.2.1) *Let X, Y be spaces, $\varphi\colon X \multimap Y$ an u.s.c. map, P be a compact space and let $r\colon P \to X$ be a map. Then for each $\alpha \in \mathcal{U}(\varphi \circ r)$ there is $\beta \in \mathcal{U}(\varphi)$ such that $f \circ r \in a(\varphi \circ r, \alpha)$ whenever $f \in a(\varphi, \beta)$.*

(27.2.2) *Let X, Y be spaces, C a compact subset of X, $y \in Y$ and $\varphi\colon X \multimap Y$, be an u.s.c. multivalued map. If $\varphi_+^{-1}(y) \cap C = \emptyset$ then there exists $\varepsilon > 0$ such that for every $\alpha \in \mathcal{U}(\varphi)$, $\delta(\alpha) \leq \varepsilon$ and for every $f \in a\ (\varphi, \alpha)$ we have $f^{-1}(y) \cap C = \emptyset$.*

(27.2.3) *Let X, Y be spaces, C a compact subset of X and $\varphi\colon X \multimap Y$ be an u.s.c. multivalued map. Then for every $\alpha \in \mathcal{U}(\varphi|_C)$ there exists $\beta \in \mathcal{U}(\varphi)$ such that $g = f|_C \in a(\varphi|_C)$ whenever $f \in a(\varphi, \beta)$.*

(27.2.4) *Let X, Y be spaces, X compact, and $\chi: X \times [0,1] \multimap Y$ an u.s.c. map. Then for every $t \in [0,1]$ and for every $\alpha \in \mathcal{U}(\chi_t)$ there exists $\beta \in \mathcal{U}(\chi)$ such that $h_t \in a(\chi_t, \alpha)$ whenever $h \in a\ (\chi, \beta)$, where $\chi_t, h_t: X \to Y$ are defined in the following way: $\chi_t(x) = \chi(x,t)$ and $h_t(x) = h(x,t)$ for every $x \in X$.*

PROOF. To prove (27.2.1). Let $\alpha = \{U_p^\alpha \times W_p^\alpha \mid p \in P\}$. For every $p \in P$ there is $\varepsilon(p) > 0$ such that $N_{\varepsilon(p)}(p) \subset U_p^\alpha$.

Let $x \in r(P)$. For $p \in P$ such that $x = r(p)$ we have $\varphi(x) \subset W_p^\alpha$ thus there is $\eta(x,p) > 0$ such that $\varphi(N_{\eta(x,p)}(x)) \subset W_p^\alpha$. By the continuity of r we have:

$$\text{for all } p \in P \text{ there exists } \theta(p),\ 0 < \theta(p) < \varepsilon(p),\qquad r(N_{\theta(p)}(p)) \subset N_{(1/2)\eta(r(p),p)}(r(p)).$$

By the compactness of P, $P = \bigcup_{i=1}^n N_{\theta(p_i)}(p_i)$. Let $\delta := \min\{(1/2)\eta(r(p_i), p_i)\}$.

For $x \in r(P)$ we define $W_x^\beta := \bigcap\{W_{p_i}^\alpha \mid \varphi(x) \subset W_{p_i}^\alpha\}$ and $U_x^\beta := N_\delta(x)$. For $x \notin r(P)$ we have: If there is i, $1 \le i \le n$ such that $\varphi(x) \subset W_{p_i}^\alpha$, then $W_x^\beta := \bigcap\{W_{p_i}^\alpha \mid \varphi(x) \subset W_{p_i}^\alpha\}$. If $\varphi(x) \not\subset W_{p_i}^\alpha$, $1 \le i \le n$, then W_x^β is an arbitrary open neighbourhood of $\varphi(x)$. For $x \notin r(P)$ we define $U_x^\beta := N_\delta(x)$.

Now, let $\beta = \{U_x^\beta \times W_x^\beta \mid x \in X\}$ and let $f \in a(\varphi, \beta)$. Take $p \in P$. There exists $x \in X$ such that $(r(p), f(r(p))) \in U_x^\beta \times W_x^\beta$. Moreover, there is i, $1 \le i \le n$, such that $p \in N_{\theta(p_i)}(p_i)$. Therefore, $r(p) \in N_{(1/2)\eta(r(p_i),p_i)}(r(p_i))$ and $p \in U_{p_i}^\alpha$ since $\theta(p_i) < \varepsilon(p_i)$. But by the definition of β we have $d(r(p), x) < \delta$, thus $x \in N_{\eta(r(p_i),p_i)}(r(p_i))$. This implies $\varphi(x) \subset W_{p_i}^\alpha$. Hence, $W_x^\beta \subset W_{p_i}^\alpha$. Finally, $(p, f \circ r(p)) \in U_{p_i}^\alpha \times W_{p_i}^\alpha$ and the proof is completed.

To prove (27.2.2). Suppose that

$$\text{for all } \varepsilon > 0 \text{ there exists } \alpha \in \mathcal{U}(\varphi), \delta(\alpha) \le \varepsilon \quad \text{and there exists } f \in a(\varphi, \alpha) \text{ such that } f^{-1}(y) \cap C \ne \emptyset.$$

Consider the sequence $\{\varepsilon_n\}$, $0 < \varepsilon_n < 1/n$. Now,

$$\text{for any } n \ge 1 \text{ there exists } \alpha_n \in \mathcal{U}(\varphi), \delta(\alpha_n) \le 1/n,\quad \text{there exists } f_n \in a(\varphi, \alpha_n), \text{ and there exists } x_n \in C \text{ such that } f_n(x_n) = y.$$

Let $\alpha_n = \{U_x^n \times W_z^n \mid z \in X\}$. Then, for any $n \ge 1$ there is $z_n \in X$ such that $x_n \in U_{z_n}^n$ and $y = f_n(x_n) \in W_{z_n}^n$. But $d(x_n, z_n) < 1/n$ and $y \in N_{1/n}(\varphi(z_n))$, therefore there exists $t_n \in \varphi(z_n)$ such that $d(y, t_n) < 1/n$.

We may assume without loss of generality (since C is compact), that $x_n \to x \in C$. Then $z_n \to x$ and $\varphi(z_n) \ni t_n \to y$. This implies that $y \in \varphi(x)$ since Γ_φ is closed. We have a contradiction which completes the proof of (27.2.2).

To prove (27.2.3) and (27.2.4). These are easy consequences of (27.2.1). It is sufficient to take $r := i: C \to X$, $i(x) = x$, in (27.2.3) and $r := i_t: X \to X \times [0,1]$, $i_t(x) = (x,t)$ in (27.2.4). The proof of (27.2) is completed. □

(27.3) DEFINITION. Let X, Y be spaces, $C \subset X$ be a compact subset and $y \in Y$.

(27.3.1) $\widetilde{A}_0(X,Y)$ (resp. $\widetilde{A}_0(X)$) is a class of all u.s.c. maps $\varphi: X \multimap Y$ (resp. $\varphi: X \multimap X$) such that for every $\alpha \in \mathcal{U}(\varphi)$ there is $f \in a(\varphi,\alpha)$.

(27.3.2) $\widetilde{A}(X,Y)$ (resp. $\widetilde{A}(X)$) is a class of all u.s.c. maps $\varphi: X \multimap Y$ (resp. $\varphi: X \multimap X$) such that $\varphi \in \widetilde{A}_0(X,Y)$ (resp. $\varphi \in \widetilde{A}_0(X)$) and for each $\alpha \in \mathcal{U}(\varphi)$ there is $\beta \in \mathcal{U}(\varphi)$ such that, if $f, g \in a(\varphi,\beta)$ then there exists a continuous mapping $h: X \times [0,1] \to Y$ (resp. $h: X \times [0,1] \to X$) such that $h(x,0) = f(x)$, $h(x,1) = g(x)$ for each $x \in X$ and $h_t = h(\,\cdot\,,t) \in a(\varphi,\alpha)$ for every $t \in [0,1]$.

(27.3.3) $\widetilde{A}_C(X,Y;y)$ is a class of all maps $\varphi \in \widetilde{A}(X,Y)$ such that $\varphi_+^{-1}(y) \cap C = \emptyset$.

Now we shall define a notion of homotopy in $\widetilde{A}_C(X,Y;y)$.

(27.4) DEFINITION. Two maps ψ and φ in $\widetilde{A}_C(X,Y;y)$ *are homotopic* (in $\widetilde{A}_C(X,Y;y)$) if there exists $\chi \in A_0(X \times [0,1], Y)$ such that $\chi(x,0) = \varphi(x)$, $\chi(x,1) = \psi(x)$ for each $x \in X$ and $y \notin \chi(x,t)$ for each $x \in C$, $t \in [0,1]$. If φ and ψ are homotopic then we write $\varphi \sim_C \psi$.

(27.5) PROPOSITION. *If C is a closed subset of a compact space X then the relation $\sim_C$ is an equivalence.*

PROOF. Obviously, the relation $\sim_C$ is reflexive and symmetric. Let $\varphi_1 \sim_C \varphi_2$ and $\varphi_2 \sim_C \varphi_3$, that is, there are two homotopies χ_1 and χ_3 joining respectively φ_1 with φ_2 and φ_2 with φ_3. We define $\chi: X \times [0,1] \to Y$,

$$\chi(x,t) = \begin{cases} \chi_1(x,3t) & \text{for } t \in [0,1/3], \\ \varphi_2(x) & \text{for } t \in (1/3,2/3], \\ \chi_3(x,3t-2) & \text{for } t \in (2/3,1]. \end{cases}$$

We can see that $\chi(x,t) \not\ni y$ for every $(x,t) \in C \times [0,1]$. We show that $\chi \in \widetilde{A}_0(X \times [0,1], Y)$.

Let $\alpha \in \mathcal{U}(\chi)$, $\alpha = \{U^\alpha_{z,t} \times W^\alpha_{z,t} \mid (z,t) \in X \times [0,1]\}$. Denote $X_1 = X \times [0,1/3]$, $X_2 = X \times [1/3,2/3]$ and $X_3 = X \times [2/3,1]$. Then $X \times [0,1] = X_1 \cup X_2 \cup X_3$.

Let $\overline{\chi_i} = \chi|_{X_i}$. We define $\alpha_i \in \mathcal{U}(\overline{\chi_i})$, $\alpha_i = \{(X_i \cap U^\alpha_{(z,t)} \times W^\alpha_{(z,t)} \times W^\alpha_{(z,t)} \mid (z,t) \in Z_i\}$, $i = 1,2,3$. We find for every $(z,t) \in X_2$ a number $\eta(z,t) > 0$ such that $N_{\eta(z,t)}((z,t)) \subset U^\alpha_{(z,t)}$ and $\chi(N_{\eta(z,t)}((z,t))) \subset W^\alpha_{(z,t)}$.

We have $N_{\eta(z,t)}((z,t)) = N_{\eta(z,t)}(t)$ and $X_2 \cap N_{\eta(z,t)}((z,t)) = N_{\eta(z,t)}(z) \times (N_{\eta(z,t)}(t) \cap [1/3,2/3])$. By the compactness of X_2, $X_2 = \bigcup_{j=1}^n U'_{(z_j,t_j)}$, where

$U'_{(z_j,t_j)} = N_{(1/2)\eta(z_j,t_j)}(z_j) \times (N_{(1/2)\eta(z_j,t_j)}(t_j) \cap [1/3, 2/3])$. Let $\lambda > 0$ be the Lebesgue number of this covering.

Define $\beta = \{U_z^\beta \times W_z^\beta \mid z \in X\} \in \mathcal{U}(\varphi_2)$ in the following way:

$$U_z^\beta := N_{\lambda/2}(z), \quad W_z^\beta := \bigcap \{W^\alpha_{(z_j,t_j)} \mid \varphi_2(z) \subset W^\alpha_{(z_j,t_j)}\}.$$

Take $h_0 \colon X_2 \to Y$ such that $h_0(\,\cdot\,, 3t-1) \in a(\varphi_2, \beta)$ for each $t \in [1/3, 2/3]$. Then for every $(x,t) \in X_2$ there is $z \in X$ such that $x \in U_z^\beta$ and $h_0(x, 3t-1) \in W_z^\beta$. For $(x,t) \in X_2$ we find (z_j, t_j) such that $(x,t) \in U'_{(z_j,t_j)}$. Then $d\,(x,z) < \lambda/2$ and $d(x, z_j) < (1/2)\eta(z_j, t_j)$, thus $z \in N_{\eta(z_j,t_j)}(z_j)$. Hence, $(z, t_j) \in U'_{(z_j,t_j)} \subset U^\alpha_{(z_j,t_j)} \cap X_2$ and $\varphi_2(z) = \chi(z, t_j) \subset W^\alpha_{(z_j,t_j)}$. This implies $W_z^\beta \subset W^\alpha_{(z_j t_j)}$ and $h_0 \in a(\overline{\chi_2}, \alpha_2)$.

Let $\beta_2 \in \mathcal{U}(\varphi_2)$ be such that $\beta_2 \subseteq \beta$ and for any $f, g \in a\ (\varphi_2, \beta_2)$ there is $h_2 \colon X \times [0,1] \to Y$ such that $h_2(\,\cdot\,, 1) = g$ and $h_2(\,\cdot\,, t) \in a\ (\varphi_2, \beta)$ for every $t \in [0,1]$.

By Proposition (27.2.4), there are $\beta_i \in \mathcal{U}(\chi_i)$, $i = 1, 3$ such that $\beta_i \subseteq \alpha_i$ and $h_1(\,\cdot\,, 1) \in a(\varphi_2, \beta_2)$, $h_3(\,\cdot\,, 0) \in a(\varphi_2, \beta_2)$ whenever $h_i \in a(\chi_i, \beta_i)$.

Let $h_2 \colon X \times [0,1] \to Y$ joins $h_1(\,\cdot\,, 1)$ and $h_3(\,\cdot\,, 0)$ as above. Define

$$h(x,t) = \begin{cases} h_1(x, 3t) & \text{for } t \in [0, 1/3), \\ h_2(x, 3t-1) & \text{for } t \in (1/3, 2/3], \\ h_3(x, 3t-2) & \text{for } t \in (2/3, 1]. \end{cases}$$

Now, $h \in a(\chi, \alpha)$ and this ends the proof. □

Now, let $[\widetilde{A}_C(X, Y; y)]$ denote the set of all homotopy classes $[\varphi]_C$ of the relation $\sim_C$, $S_C(X, Y; y)$ a class of all continuous mappings $f \colon X \to Y$ such that $f(x) \neq y$ for each $x \in C$ and $[S_C(X, Y; y)]$ the set of all homotopy classes $[f]^1_C$ of ordinary singlevalued homotopy without y as a value on C.

The following theorem is crucial in the sequel:

(27.6) THEOREM. *If C is a closed subset of a compact space X and Y is a space with the property:*

(27.6.1) *for every space Z and for every $\varepsilon > 0$ there exists $\delta > 0$ such that each two continuous δ-near mappings $f, g \colon Z \to Y$ are ε-homotopic,*

then one can construct a bijection $F \colon [\widetilde{A}_C(X, Y; y)] \to [S_C(X, Y; y)]$.

PROOF. Let $\varphi \in \widetilde{A}_C(X, Y; y)$. By Proposition (27.2.2), there exists $\varepsilon > 0$ such that for every $\alpha_0 \in \mathcal{U}(\varphi)$, $\delta(\alpha_0) \leq \varepsilon$ and for every $f \in a(\varphi, \alpha_0)$, $f^{-1}(y) \cap C = \emptyset$. Let $\alpha_0 \in \mathcal{U}(\varphi)$ be such that $\delta(\alpha_0) \leq \varepsilon$.

By the definition of a class $\widetilde{A}(X, Y)$ there is $\alpha \in \mathcal{U}(\varphi)$, $\alpha \subseteq \alpha_0$ such that each $f, g \in a(\varphi, \alpha)$ can be joined by a homotopy $h \colon X \times [0,1] \to Y$ such that $h(\,\cdot\,, t) \in a(\varphi, \alpha_0)$ for every $t \in [0,1]$.

Let $f \in a(\varphi, \alpha)$. Define $F\colon [\widetilde{A}_C(X, Y; y)] \to [S_C(X, Y; y)]$,

$$F([\varphi]_C) = [f]^1_C.$$

We shall check the correctness of the above definition. It is easy to see that this definition does not depend on the choice of α_0. Let $\varphi \sim_C \psi$. We show that $F([\varphi]_C) = F([\psi]_C)$. Let $\chi \in A_0(X \times [0, 1], Y)$ be the homotopy joining φ and ψ in $A_C(X, Y; y)$ (see Definition (27.4)). By Proposition (27.2.2), there is $\tau \in \mathcal{U}(\chi)$ such that for any $h \in a(\chi, \tau)$, $h(x, t) \neq y$ for every $x \in C$, $t \in [0, 1]$. By Proposition (27.2.4), there exists τ', $\tau' \subseteq \tau$, such that, for any $h \in a(\chi, \tau')$, $h_0 \in a(\varphi, \alpha(\varphi))$ and $h_1 \in a(\psi, \alpha(\psi))$ where $\alpha(\varphi), \alpha(\psi)$ are such that $F([\varphi]_C) = [h_0]^1_C$ and $F([\psi]_C) = [h_1]^1_C$. Let $h \in a(\chi, \tau')$. Thus we have $F([\varphi]_C) = [h_0]^1_C = [h_1]^1_C = F([\psi]_C)$ and hence the definition is correct.

For the proof of surjectivity of F it is sufficient to show that $S_C(X, Y; y) \subset \widetilde{A}_C(X, Y; y)$. Let $f \in S_C(X, Y; y)$ and let $\alpha \in \mathcal{U}(f)$, $\alpha = \{U^\alpha_z \times W^\alpha_z \mid z \in X\}$. For every $z \in X$ there is $\eta(z) > 0$ such that $N_{\eta(z)}(z) \times N_{\eta(z)}(f(z)) \subset U^\alpha_z \times W^\alpha_z$. By the continuity of f, for every $z \in X$ there exists $\gamma(z)$, $0 < \gamma(z) < \eta(z)$ such that $f(N_{\gamma(z)}(z)) \subset N_{(1/2)\eta(z)}(f(z))$. By the compactness of X, $X = \bigcup_{i=1}^n N_{\gamma(z_i)}(z_i)$. For every $x \in C$ we have $f(x) \neq y$ hence $r = \min_{x \in C}\{d(f(x), y)\} > 0$.

Let $\varepsilon = \min\{r, (1/2)\eta(z_1), \ldots, (1/2)\eta(z_n)\}$. By the property of the space Y there exists δ, $0 < \delta < \varepsilon$ such that any two δ-near mappings are ε-homotopic. By the compactness of X there is η, $0 < \eta < \min\{\eta(z_1), \ldots, \eta(z_n)\}$ such that $d(f(x), f(z)) < \delta/2$ whenever $d(x, z) < \eta$.

Consider $\beta \in \mathcal{U}(f)$, $\beta = \{(N_\eta(z) \cap U^\alpha_z) \times (N_{\delta/2}(f(z)) \cap W^\alpha_z) \mid z \in X\}$. One can see that $\beta \subseteq \alpha$ and if $g \in a(f, \beta)$ then for every $x \in X$ there exists $z \in X$ such that $(x, g(x)) \in N_\eta(z) \times N_{\delta/2}(f(z))$. Hence $d(x, z) < \eta$ what implies $d(f(x), f(z)) < \delta/2$ and $d(g(x), f(z)) < \delta/2$. Thus $d(f(x), g(x)) < \delta$ what means that f and g are δ-near.

Let $g, g' \in a(f, \beta)$. From the above discussion one can see that g and g' are homotopic in $S_C(X, Y; y)$. Let $h\colon X \times [0, 1] \to Y$ be that homotopy joining g and g'. For every $t \in [0, 1]$ and for every $x \in X$ we have $d(h(x, t), f(x)) < \varepsilon$.

Let $t \in [0, 1]$ and $x \in X$. There is i, $1 \le i \le n$ such that $x \in N_{\gamma(z_i)}(z_i) \subset U^\alpha_{z_i}$. Then $f(x) \in N_{(1/2)\eta(z_i)}(f(z_i))$. This implies $H(x, t) \in N_{\eta(z_i)}(f(z_i)) \subset W^\alpha_{z_i}$. Thus $h(\cdot, t) \in a(f, \alpha)$ for every $t \in [0, 1]$ what proves that $f \in A_C(X, Y; y)$.

Now, we shall prove that F is injective. It is sufficient to show that for any $\varphi \in \widetilde{A}_C(X, Y; y)$ there is $\beta \in \mathcal{U}(\varphi)$ such that each $f \in a(\varphi, \beta)$ is in $[\varphi]_C$. In fact, suppose that it is true. Let $\varphi, \psi \in \widetilde{A}_C(X, Y; y)$ and suppose that $F([\varphi]_C) = F([\psi]_C) = [f]^1_C$, where f is such that $f \in a(\varphi, \alpha')$, $f \in a(\psi, \alpha'')$, $\alpha' \subseteq \beta(\varphi)$, $\alpha' \subseteq \alpha(\varphi)$, $\alpha'' \subseteq \beta(\psi)$, $\alpha'' \subseteq \alpha(\psi)$. $\beta(\varphi), \beta(\psi)$ are chosen by the assumption mentioned above and $\alpha(\varphi), \alpha(\psi)$ are chosen by the definition of F. Then $f \in [\varphi]_C \cap [\psi]_C$ and thus $[\varphi]_C = [\psi]_C$.

Let $\varphi \in \widetilde{A}_C(X,Y;y)$. By the compactness of C and since Γ_φ is closed, there is $\alpha \in \mathcal{U}(\varphi),\ \alpha = \{U_z^\alpha \times W_z^\alpha \mid z \in X\}$ such that $y \notin \overline{W_z^\alpha}$ for every $z \in C$.

Take, for every $z \in X$, the set $U_z \subset U_z^\alpha$ such that $\varphi(U_z) \subset W_z^\alpha$. By the compactness of C and $\overline{X \setminus C}$

$$X = \bigcup_{i=1}^{m}\{U_{x_i} \mid x_i \in C\} \cup \bigcup_{i=m+1}^{k}\{U_{x_i} \mid x_i \in X \setminus C\}$$

and $C \subset U := \bigcup_{i=1}^{m}\{U_{x_i} \mid x_i \in C\}$.

Let $0 < \gamma := \min\{\mathrm{dist}(y, \overline{W_{x_i}^\alpha}) \mid i = 1, \dots, m\}$. There is $\theta > 0$ such that $N_\theta(C) \subset U$.

Define $\alpha' \in \mathcal{U}(\varphi),\ \alpha' = \{U_z^{\alpha'} \times W_z^{\alpha'} \mid z \in X\}$ in the following way:

$$U_z^{\alpha'} := U_z \cap N_\theta(z), \quad W_z^{\alpha'} = \bigcap\{W_{x_i}^\alpha \mid \varphi(z) \subset W_{x_i}^\alpha\}.$$

There exists $\beta \in \mathcal{U}(\varphi)$, $\beta \subseteq \alpha'$ such that for every $f \in a(\varphi,\beta)$, $f^{-1}(y) \cap C = \emptyset$ and β is the same as in Definition (27.3.2) for φ and α'. Let $f \in a(\varphi,\beta)$. Define $\chi\colon X \times [0,1] \to Y$ by the formula

$$\chi(x,t) = \begin{cases} \varphi(x) & \text{for } t \in [0,1/3), \\ \mathrm{cl}\,(\bigcup\{W_z^{\alpha'} \mid d(z,x) \le \theta/2\}) & \text{for } t \in [1/3,2/3], \\ f(x) & \text{for } t \in (2/3,1]. \end{cases}$$

We see that χ has closed values. It is easy to check that χ is a u.s.c. map.

Now, we show that $\chi(x,t) \not\ni y$ for every $(x,t) \in C \times [0,1]$. Let $(x,t) \in C \times [0,1]$. If $t \in [0,1/3)$, then $\chi(x,t) = \varphi(x) \not\ni y$. If $t \in (2/3,1]$, then $\chi(x,t) = f(x) \neq y$.

Let $t \in [1/3,2/3]$. Then $\chi(x,t) = \mathrm{cl}\,(\bigcup\{W_z^{\alpha'} \mid d(z,x) \le \theta/2\})$. Let $z \in X$ be such that $d(z,x) \le \theta/2$. Then $z \in N_\theta(x)$ and, hence, $z \in U$, thus there is $x_i \in C$ such that $z \in U_{x_i}$. This implies $\varphi(z) \subset W_{x_i}^\alpha$ and hence $W_z^{\alpha'} \subset W_{x_i}^\alpha$. Therefore, $y \notin \overline{W_z^{\alpha'}}$ and $\mathrm{dist}(y, \overline{W_z^{\alpha'}}) \ge \gamma$. We conclude that $\mathrm{dist}(y, \chi(x,t)) \ge \gamma$ and, finally, $y \notin \chi(x,t)$.

Now, we shall show that $\chi \in \widetilde{A}_0(X \times [0,1], Y)$. Let $\sigma \in \mathcal{U}(\chi)$, $\sigma = \{U_{(z,t)}^\sigma \times W_{(z,t)}^\sigma \mid (z,t) \in X \times [0,1]\}$. We show that there is $h \in a(\chi,\sigma)$.

Take, for every $(z,t) \in X \times [0,1]$, the set $U_{(z,t)} := N_{\eta(z,t)}((z,t)) = N_{\eta(z,t)}(z) \times N_{\eta(z,t)}(t) \subset U_{(z,t)}^\sigma$ such that $\chi(U_{(z,t)}) \subset W_{(z,t)}^\sigma$. By compactness, $X \times [0,1/3] \subset \bigcup_{i=1}^{p} U'_{(z_i,y_i)}$, where $U'_{(z_i,t_i)} = N_{(1/2)\eta(z_i,t_i)}(z_i) \times N_{(1/2)\eta(z_i,t_i)}(t_i)$ and $(z_i,t_i) \in X \times [0,1/3]$ for every $i = 1, \dots, p$. Let $\lambda = \min\{\eta(z_i,t_i) \mid i = 1, \dots, p\}$.

Define $\sigma' \in \mathcal{U}(\varphi)$, $\sigma' = \{U_z^{\sigma'} \times W_z^{\sigma'} \mid z \in X\}$ as follows:

$$U_z^{\sigma'} := U_z^\beta \cap N_{\lambda/2}(z), \quad W_z^{\sigma'} := \bigcap\{W_{(z_i,t_i)}^\sigma \mid z \in N_{\eta(z_i,t_i)}(Z_i)\} \cap W_z^\beta.$$

Let $g \in a(\varphi, \sigma')$. Then $g \in a(\varphi, \beta)$ and there is $k\colon X \times [1/3, 2/3] \to Y$ such that $k(\,\cdot\,, 1/3) = g$, $k(\,\cdot\,, 2/3) = f$ and $k(\,\cdot\,, t) \in a(\varphi, \alpha')$ for every $t \in [1/3, 2/3]$.

Define $h\colon X \times [0,1] \to Y$ by the formula

$$h(x,t) = \begin{cases} g(x) & \text{for } t \in [0, 1/3), \\ k(x,t) & \text{for } t \in [1/3, 2/3], \\ f(x) & \text{for } t \in (2/3, 1]. \end{cases}$$

It is sufficient to show that $h \in a(\chi, \sigma)$. Let $(x,t) \in X \times [0,1]$.

If $t \in [0, 1/3)$, then there is (z_i, t_i) such that $(x,t) \in U'_{(z_i,t_i)} \subset U^{\sigma}_{(z_i,t_i)}$. There exists $z \in X$ such that $z \in U^{\sigma'}_z$ and $h(x,t) = g(x) \in W^{\sigma'}_z$. Thus $d(x,z) < \lambda/2$ and $d(x, z_i) < (1/2)\eta(z_i, t_i)$. Hence $z \in N_{\eta(z_i,t_i)}(z_i)$. This implies $W^{\sigma'}_z \subset W^{\sigma}_{(z_i,t_i)}$.

If $t \in [1/3, 2/3]$, then $h(x,t) = k(x,t)$ and there is $z = z(x,t)$ such that $x \in U^{\alpha'}_z$ and $k(x,t) \in W^{\alpha'}_z \subset \chi(x,t)$. If $t \in (2/3, 1]$, then $h(x,t) = f(x) = \chi(x,t)$.

Finally, $h \in a(\chi, \sigma)$ and a proof of Theorem (27.6) is completed. □

(27.7) Definition (cf. (2.16)). For spaces X, Y, let $C \subset X$ be a compact subset and $y \in Y$ we define the following classes of J-maps:

(27.7.1) $$\widetilde{J}(X,Y) = \{\varphi : X \multimap Y \text{ u.s.c.} \mid \varphi(x) \in PC^{\infty}_Y \text{ for all } x \in X\},$$

(27.7.2) $$\widetilde{J}_C(X,Y;y) = \{\varphi \in J(X,Y) \mid \varphi_+^{-1}(y) \cap C = \emptyset\}.$$

(27.8) Remark. It is easy to check that $\varphi \in \widetilde{J}(X,Y)$ if and only if φ is u.s.c. and for each $x \in X$, and for each open set $W_x \subset Y$ such that $\varphi(x) \subset W_x$ and for $n \geq 1$ there exists an open set $V_x \subset W_x$, $\varphi(x) \subset V_x$ such that, for any k, $0 \leq k \leq n$ and a continuous map $g\colon \partial\Delta^k \to V_x$, there exists a continuous extension $\overline{g}\colon \Delta^k \to W_x$ of g.

Let Δ^k denote the standard k-dimensional simplex in $\mathbb{R}^k$. The following fact is obvious.

(27.9) Lemma. *Let X, Y, Z be spaces, $r\colon Z \to X$ be continuous and $\varphi \in \widetilde{J}(X,Y)$. Then $\varphi \circ r \in \widetilde{J}(X,Y)$.*

The following Lemma will be needed in the proof of the main result of this section.

(27.10) Lemma. *Let X, Y be spaces, X be compact and $\varphi \in \widetilde{J}(X,Y)$. Then for each $n \geq 1$ and $\alpha \in \mathcal{U}(\varphi)$ there exists $\beta \in \mathcal{U}(\varphi)$, $\beta \subseteq \alpha$ such that, for each $z \in X$ and k, $0 \leq k \leq n$, if $g\colon \partial\Delta^k \to W^{\beta}_x$ is continuous, then there is a continuous extension $\overline{g}\colon \Delta^k \to W^{\alpha}_x$ of g.*

Proof. Let $\alpha \in \mathcal{U}(\varphi)$, $\alpha = \{U^{\alpha}_z \times W^{\alpha}_z \mid z \in X\}$. Then for any $z \in X$ there exists $W^{\beta}_z \subset W^{\alpha}_z$ such that (27.8) holds for z, W^{α}_z and W^{β}_z. Let $U^{\beta}_z \ni z$ be such that $U_z^\beta \subset U_z^\alpha$ and $\varphi(U_z^\beta) \subset W_z^\beta$, by u.s.c. Then $\beta = \{U_z^\beta \times W_z^\beta \mid z \in X\}$ is a good covering of Γ_φ. □

Now, we prove the following crucial theorem:

(27.11) THEOREM (cf. (23.5)). *Let P be a finite polyhedron, P_0 its subpolyhedron, Y be a space and $\varphi \in J(P,Y)$. For any $\alpha \in \mathcal{U}(\varphi)$ there is $\beta \in \mathcal{U}(\varphi)$ such that, for any $f_0 \in a(P_0, \varphi, \beta)$ there exists an extension $f\colon P \to Y$ of f_0 being an α-approximation of φ.*

PROOF. The idea of the proof is similar to that in (23.5). Let $\alpha \in \mathcal{U}(\varphi)$ and let $N = \dim P \geq N_0 = \dim P_0$. For every $z \in P$ there is $\eta_N(z) > 0$ such that $N_{\eta_N(z)}(z) \subset U_z^\alpha$ and $\varphi(N_{\eta_N(z)}(z)) \subset W_z^\alpha$.

Consider the $P = \bigcup_{z\in P} N_{\eta_N(z)/4}(z) = \bigcup_{i=1}^{r(N)} N_{\eta_N(z_i^N)/4}(z_i^N)$, by the compactness of P. Let $\theta_N > 0$ be the Lebesgue number of this covering and $\eta_N = \min\{\eta_N(z_i^N)\}$. Define for every $z \in P$.

$$W_z^{\alpha_N} = \begin{cases} W_{z_i}^{\alpha_N} & \text{if } z = z_i^N, \\ \bigcap\{W_{z_i}^{\alpha_N} \mid z \in N_{\eta_N}(z_i^N)\} \cap W_z^\alpha & \text{if } z \neq z_i^N \text{ for every } 1 \leq i \leq r(N), \end{cases}$$

$$U_z^{\alpha_N} = N_{\eta_N/4}(z) \cap N_{\eta_N(z)}(z) \quad \text{and} \quad \alpha_N := \{N_{\eta_N/4}(z) \times W_z^{\alpha_N} \mid z \in P\}.$$

Suppose that α_{k+1} is constructed for same $k = 0, \ldots, N-1$. There is $\overline{\alpha_k} \in \mathcal{U}(\varphi)$ such that, for any $z \in P$, $U_z^{\overline{\alpha_k}} \subset U_z^{\alpha_{k+1}}$, $W_z^{\overline{\alpha_k}} \subset W_z^{\alpha_{k+1}}$ and every $g\colon \partial\Delta^n \to W_z^{\overline{\alpha_k}}$ may be extended to $\overline{g}\colon \Delta^n \to W_z^{\alpha_{k+1}}$ for any $1 \leq n \leq N$.

Take for any $z \in P$ a positive number $\eta_k(z) < \eta_{k+1}$ such that $N_{\eta_k(z)}(z) \subset U_z^{\overline{\alpha_k}}$ and $\varphi(N_{\eta_k(z)}(z)) \subset W_z^{\overline{\alpha_k}}$. By the compactness of P

$$P = \bigcup_{z\in P} N_{(1/4)\eta_k(z)}(z) = \bigcup_{i=1}^{r(k)} N_{(1/4)\eta_k(z_i^k)}(z_i^k).$$

Let $\theta_k > 0$ be the Lebesgue number of this covering and $\eta_k = \min\{\eta_k(z_i^k) \mid i = 1, \ldots, r(k)\}$. Define for every $z \in P$.

$$W_z^{\alpha_k} = \begin{cases} W_{z_i^k}^{\overline{\alpha_k}} & \text{if } z = z_i^k, \\ \bigcap\{W_{z_i^k}^{\overline{\alpha_k}} \mid z \in N_{\eta_k(z_i^k)}(z_i^k)\} \cap W_z^\alpha & \text{if } z \neq z_i^k \text{ for every } 1 \leq i \leq r(k), \end{cases}$$

$$U_z^{\alpha_k} = N_{\eta_k/4}(z) \cap N_{\eta_k(z)}(z) \quad \text{and} \quad \alpha_k := \{N_{\eta_k/4}(z) \times W_z^{\alpha_k} \mid z \in P\}.$$

Now, we have defined $\alpha_0 := \{U_z^{\alpha_0} \times W_z^{\alpha_0} \mid z \in P\}$. Let $\beta := \alpha_0$ and $f_0\colon P_0 \to Y$ be a β-approximation of φ.

Let (T, T_0) be a triangulation of (P, P_0) such that $\delta(T) < \min\{\theta_N, \dots, \theta_0\}$. It implies that for every simplex $S \in T$ and for every k, $0 \le k \le N$ there exists z_i^k such that $S \subset N_{\eta_k(z_i^k)/4}(z_i^k)$.

By T^k, $0 \le k \le N$ (T_0^k, $0 \le k \le N_0$), we denote a k-dimensional skeleton of T and T_0, respectively. Denote $P^k = |T^k|$ and $P_0^k = |T_0^k|$. Note that T_0^k is a subcomplex of T^k for every $k \le N_0$ and $P^N = P$, $P_0^{N_0} = P_0$.

Now, we shall construct a sequence of maps $\{f^k \colon P^k \to Y\}$ such that

(27.11.1) $\quad f^k|_{P_0^k} = f_0|_{P_0^k} \quad$ for any k, $0 \le k \le N_0$,

(27.11.2) $\quad f^{k+1}|_{P^k} = f^k \quad$ for any k, $0 \le k \le N$,

(27.11.3) $\quad f^k$ is a β_k-approximation of φ $\quad$ for any k, $0 \le k \le N$,

where $\beta_k = \{U_z^{\beta_k} \times W_z^{\beta_k} \mid z \in P\}$ is such that $U_z^{\beta_k} = N_{\eta_k/2}(z) \cap N_{\eta_k(z)}(z)$ and $W_z^{\beta_k} = W_z^{\alpha_k}$.

Let $P^0 = \{x_1, \dots, x_q, x_{q+1}, \dots x_r\}$, where $x_i \in P_0^0$ for $0 \le i \le q$ and $x_i \notin P_0^0$ for $q+1 \le i \le r$. If $0 \le i \le q$ then we put $f^0(x_i) := f_0(x_i)$. For $q+1 \le i \le r$ we put $f^0(x_i) \in \varphi(x_i)$. One can see that f^0 satisfies (27.11.1)–(27.11.3).

Let $k \le N-1$. Suppose that we have defined f^j satisfying (27.11.1)–(27.11.3) for all $0 \le j \le k$. We shall define f^{k+1}.

Let S be an arbitrary $(k+1)$-dimensional simplex in T. There is $z_i^k \in P$ such that $S \subset N_{\eta_k(z_i^k)/4}(z_i^k)$. Let $x \in \partial S$. By assumption, there exists $z \in P$ such that $(x, f^k(x)) \in U_z^{\beta_k} \times W_z^{\beta_k}$. Then $d(x, z) < \eta_k/2$, what implies $d(z, z_i^k) < (1/4)\eta_k(z_i^k) + (1/2)\eta_k(z_i^k) < \eta_k(z_i^k)$. Hence $W_z^{\beta_k} = W_z^{\alpha_k} \subset W_{z_i^k}^{\alpha_k}$ which gives $f^k(x) \in W_{z_i^k}^{\alpha_k}$. Thus $(x, f^k(x)) \in N_{\eta_k(z_i^k)/4}(z_i^k) \times W_{z_i^k}^{\alpha_k}$.

(27.11.4) If $k+1 \le N_0$ and $S \in T_0^{k+1}$ then we put $f^{k+1}|_S = f_0|_S$.

(27.11.5) If $k+1 > N_0$ or $S \notin T_0^{k+1}$ then there is $f^{k+1} \colon S \to W_{z_i^k}^{\alpha_{k+1}}$ such that $f^{k+1}|_{\partial S}$.

We shall show that f^{k+1} is a β_{k+1}-approximation of φ. Let $x \in S$.

If (27.11.4) holds, then $(x, f^{k+1}(x)) = (x, f_0(x)) \in U_z^{\alpha_0} \times W_z^{\alpha_0}$ for some $z \in P$. There exists $z_j^k \in P$ such that $z \in N_{\eta_k(z_j^k)/4}(z_j^k)$. Thus $d(x, z) < \eta_0/4 \le \eta_k(z_j^k)/4$ and $d(z, z_j^k) < \eta_k(z_j^k)/4$. It follows that $x \in N_{\eta_k(z_j^k)/2}(z_j^k) \subset N_{\eta_{k+1}/2}(z_j^k)$ and $x \in N_{\eta_{k+1}(z_j^k)}(z_j^k)$. Hence, $x \in U_{z_j^k}^{\beta_{k+1}}$.

Moreover, there exists $z_{i(0)}^0 \in P$ such that $d(z, z_{i(0)}^0) < \eta_0(z_{i(0)}^0)/4 < (\eta_1)/4)$, what implies $W_z^{\alpha_0} \subset W_{z_{i(0)}^0}^{\alpha_0}$. There exists $z_{i(1)}^1 \in P$ such that $d(z, z_{i(1)}^1) < \eta_1(z_{i(1)}^1)/4$. Therefore, $d(z_{i(0)}^0, z_{i(1)}^1) < \eta_1(z_{i(1)}^1)/2$ and $W_{z_{i(0)}^0}^{\alpha_1} \subset W_{z_{i(1)}^1}^{\alpha_1}$. Since $W_{z_{i(0)}^0}^{\alpha_0} \subset W_{z_{i(0)}^0}^{\alpha_1}$, $W_z^{\alpha_0} \subset W_{z_{i(1)}^1}^{\alpha_1}$.

It is easy to check, by induction, that $W_z^{\alpha_0} \subset W^{\alpha_k}_{z_j^{k+1}} \subset W^{\alpha_{k+1}}_{z_j^k} = W^{\beta_{k+1}}_{z_j^k}$. Hence $(x, f^{k+1}(x)) \in U^{\beta_{k+1}}_{z_j^k} \times W^{\beta_{k+1}}_{z_j^k}$.

If (27.11.5) holds, then $(x, f^{k+1}(x)) \in N_{\eta_k(z_i^k)/4}(z_i^k) \times W^{\alpha_{k+1}}_{z_i^k}$. Hence $x \in N_{\eta_{k+1}/2}(z_i^k) \cap N_{\eta_{k+1}(z_i^k)}(z_i^k)$ and $(x, f^{k+1}(x)) \in U^{\beta_{k+1}}_{z_i^k} \times W^{\beta_{k+1}}_{z_i^k}$.

Now, we conclude that f^{k+1} is a β_{k+1}-approximation of φ. This completes the proof. □

As an immediate consequence of the above theorem we obtain:

(27.12) Corollary. *If P is a finite polyhedron, Y is an arbitrary metric space and $\varphi \in \widetilde{J}(P, Y)$, then for every $\alpha \in \mathcal{U}(\varphi)$ there exists $f \in a(\varphi, \alpha)$.*

(27.13) Theorem. *If P is a finite polyhedron, Y is an arbitrary metric space and $\varphi \in J(P, Y)$ then for every $\alpha \in \mathcal{U}(\varphi)$ there exists $\beta \in \mathcal{U}(\varphi)$ such that for any two maps $f, g \in a(\varphi, \beta)$ one can find a homotopy $h\colon P \times [0,1] \to Y$ such that $h_0 = f$, $h_1 = g$ and $h_t \in a\ (\varphi, \alpha)$ for every $t \in [0,1]$.*

Proof. Consider the polyhedron $P' = P \times [0,1]$ (with the canonical triangulation). Let $P'_0 = (P \times \{0\}) \cup (P \times \{1\})$. Define $\varphi'\colon P \times [0,1] \to Y$, $\varphi'(x,t) := \varphi(x)$ for each $(x,t) \in P \times [0,1]$.

Let $\alpha \in \mathcal{U}(\varphi)$, $\alpha = \{U_z^\alpha \times W_z^\alpha \mid z \in P\}$. For any $t \in [0,1]$ and $z \in P$ define:

$$U^{\alpha'}_{(z,t)} := U_z^\alpha \times [0,1], \quad W^{\alpha'}_{(z,t)} := W_z^\alpha.$$

Let $\alpha' = \{W^{\alpha'}_{(z,t)} \times W^{\alpha'}_{(z,t)} \mid (z,t) \in P'\}$. By Theorem (27.11), there is $\beta' \in \mathcal{U}(\varphi')$ such that for any $h_0 \in a\ (P'_0, \varphi', \beta')$ there exists an extension $h\colon P' \to Y$ of h_0 which is an α'-approximation of φ'.

Notice that for any $z \in P$ there exists $\gamma(z) > 0$ such that $N_{\gamma(z)}((z,0)) \subset U^{\beta'}_{(z,0)}$ and $N_{\gamma(z)}((z,1)) \subset U^{\beta'}_{(z,1)}$. Define for every $z \in P$:

$$U_z^\beta := N_{\gamma(z)}(z), \quad W_z^{\beta'} := W^{\beta'}_{(z,0)} \cap W^{\beta'}_{(z,1)}$$

and $\beta := \{U_z^\beta \times W_z^\beta \mid z \in P\}$. Let $f, g \in a\ (\varphi, \beta)$. Define $h_0\colon P'_0 \to Y$,

$$h_0(x,t) = \begin{cases} f(x) & \text{for } t = 0, \\ g(x) & \text{for } t = 1. \end{cases}$$

For any $(x,t) \in P \times [0,1]$

$$((x,t), h_0(x,t)) = \begin{cases} ((x,0), f(x)) & \text{for } t = 0, \\ ((x,1), g(x)) & \text{for } t = 1. \end{cases}$$

There is $z \in P$ such that $(x, f(x)) \in U_z^\beta \times W_z^\beta$, thus $(x, 0) \in N_{\gamma(z)}((z,0)) \subset U_{(z,0)}^{\beta'}$ and $f(x) \in W_{(z,0)}^{\beta'}$. Analogously, one can find $p \in P$ such that $((x,1), g(x)) \in U_{(p,1)}^{\beta'} \times W_{(p,1)}^{\beta'}$. This implies that h_0 is a β'-approximation of φ'.

Let $h\colon P' \to Y$ be an extension of h_0 and let $(x,t) \in P'$. There exists a $(z,s) \in P'$ such that $((x,t), h(x,t)) \in U_{(z,s)}^{\alpha'} \times W_{(z,s)}^{\alpha'}$. By the definition of α' we have $x \in U_z^\alpha$ and $h_t(x) = h(x,t) \in W_z^\alpha$. We conclude that $h_t \in a(\varphi, \alpha)$ for every $t \in [0,1]$. The proof is completed. □

The facts (27.12) and (27.13) imply $\widetilde{J}(P,Y) \subset \widetilde{A}(P,Y)$.

Using the above results we will construct the topological degree for maps with proximally ∞-connected values. Note that the values do not have to be compact (cf. Section 26).

Let U be an open, bounded subset of a space $\mathbb{R}^n$ and $\varphi \in \widetilde{J}_{\partial U}(\overline{U}, \mathbb{R}; y)$. Since Γ_φ is closed, the set $\mathcal{F} = \varphi_+^{-1}(y)$ is closed in U and there is $V \subset \overline{V} \subset U$ such that $\mathcal{F} \subset V$ and $\overline{V}$ is a finite polyhedron. The fact that $V \subset U$ is an open neighbourhood of the set $\mathcal{F}$ with $\overline{V}$ being a polyhedron will be denoted by the symbol $V \in N^p(\mathcal{F}, U)$. We see that $\varphi_V := \varphi|_{\overline{V}}\colon \overline{V} \to \mathbb{R}^n$ is an element of $\widetilde{J}_{\partial V}(\overline{V}, \mathbb{R}^n; y)$. By (27.14) and Theorem (27.6), there is $f\colon \overline{V} \to \mathbb{R}^n$ an α_V-approximation of φ_V such that $F([\varphi]_{\partial V}) = [f]_{\partial V}^1$. Define

$$\mathrm{Deg}(\overline{U}, \varphi, y) := \mathrm{Deg}(\overline{V}, \varphi_V, y) := \deg(\overline{V}, f, y),$$

where $\deg(\overline{V}, f, y)$ stands for the Brouwer degree (cf. [Ro-M]).

By Theorem (27.6) and the properties of a topological degree for singlevalued maps, we can see that this definition is correct. We shall show that it does not depend on the choice of V. In fact, suppose that we have two sets $V, W \in N^p(\mathcal{F}, U)$. Let $O \in N^p(\mathcal{F}, U)$ be such that $\overline{O} \subset V \cap W$. Let $\varphi_O = \varphi|_{\overline{O}}$ and α_O be the same as in definition of $\mathrm{Deg}(\overline{O}, \varphi_O, y)$. There exist $\alpha_V \in \mathcal{U}(\varphi_V)$ and $\alpha_W \in \mathcal{U}(\varphi_W)$ such that, if $f \in a(\varphi_V, \alpha_V)$, $g \in a(\varphi_W, \alpha_W)$ then $f|_{\overline{O}}, g|_{\overline{O}} \in a(\varphi_O, \alpha_O)$ and $\mathrm{Deg}(\overline{V}, \varphi_V, y) = \deg(\overline{V}, f, y)$, $\mathrm{deg}(\overline{W}, \varphi_W, y) = \deg(\overline{W}, g, y)$. Let $f \in a(\varphi_V, \alpha_V)$ and $g \in a(\varphi_W, \alpha_W)$. Then, applying the excision property of the Brouwer degree,

$$\begin{aligned}\mathrm{Deg}(\overline{V}, \varphi_V, y) &= \deg(\overline{V}, f, y) = \deg(\overline{O}, f|_{\overline{O}}, y) = \mathrm{Deg}(\overline{O}, \varphi_O, y)\\ &= \deg(\overline{O}, g|_{\overline{O}}, y) = \deg(\overline{W}, g, y) = \mathrm{Deg}(\overline{W}, \varphi_W, y).\end{aligned}$$

In the following proposition we collect some properties of Deg:

(27.14) PROPOSITION. *Let U be an open and bounded subset of a space $\mathbb{R}^n$ and $\varphi, \psi \in \widetilde{J}_{\partial U}(\overline{U}, \mathbb{R}^n; y)$.*

(27.14.1) (Additivity) *Let $U_1, U_2 \subset \mathbb{R}^n$ be open, $U_1 \cup U_2 \subset U$, $U_1 \cap U_2 = \emptyset$ and $\varphi_+^{-1}(y) \cap (\overline{U} \setminus (U_1 \cup U_2)) = \emptyset$. Then* $\mathrm{Deg}(\overline{U}, \varphi, y) = \mathrm{Deg}(\overline{U_1}, \varphi_{U_1}, y) + \mathrm{Deg}(\overline{U_2}, \varphi_{U_2}, y)$.

(27.14.2) (Existence) *If* $\mathrm{Deg}(\overline{U},\varphi,y)\neq 0$, *then* $\varphi_+^{-1}(y)\cap U\neq\emptyset$.

(27.14.3) (Excision) *Let* $V\subset U$ *be open and such that* $(\overline{U}\setminus V)\cap\varphi_+^{-1}(y)=\emptyset$. *Then* $\mathrm{Deg}(\overline{U},\varphi,y)=\mathrm{Deg}(\overline{V},\varphi_V,y)$.

(27.14.4) (Homotopy) *If* $\chi\in A_0(\overline{U}\times[0,1],\mathbb{R}^n)$ *joins* φ *and* ψ *in* $A_{\partial U}(\overline{U},\mathbb{R}^n;y)$, *then* $\mathrm{Deg}(\overline{U},\varphi,y)=\mathrm{Deg}(\overline{U},\psi,y)$.

PROOF. (27.14.1) Let $V_1\subset U_1$, $\alpha_{V_1}\in\mathcal{U}(\varphi_{V_1})$, $V_2\subset U_2$ and $\alpha_{V_2}\in\mathcal{U}(\varphi_{V_2})$ be the same as in definition of $\mathrm{Deg}(\overline{U_1},\varphi_{U_1},y)$ and $\mathrm{Deg}(\overline{U_2},\varphi_{U_2},y)$. Define $V=V_1\cup V_2$. Then V is an appropriate set for defining $\mathrm{Deg}(\overline{U},\varphi,y)$. Let $\alpha(V)$ be such that for any $f\in a(\varphi_V,\alpha(V))$ we have $\deg(\overline{V},\varphi_V,y)=\deg(\overline{V},f,y)$ and $f|_{V_i}\in a(\varphi_{V_i},\alpha_{V_i})$ for $i=1,2$. Now, take $f\in a(\varphi_V,\alpha(V))$. Then

$$\begin{aligned}\mathrm{Deg}(\overline{U},\varphi,y)&=\mathrm{Deg}(\overline{V},\varphi_V,y)=\deg(\overline{V},f,y)\\&=\deg(\overline{V_1},f|_{V_1},y)+\deg(\overline{V_2},f|_{V_2},y)\\&=\mathrm{Deg}(\overline{U_1},\varphi_{U_1},y)+\mathrm{Deg}(\overline{U_2},\varphi_{U_2},y).\end{aligned}$$

(27.14.2) The proof is a consequence of compactness of $\overline{U}$, u.s.c. of φ and the existence property for singlevalued maps. We omit details.

(27.14.3) Use (27.14.1) for $U_1=V$, $U_2=U\setminus\overline{V}$ and (27.14.2) for $\overline{U_2}$.

(27.14.4) Suppose that $\chi\in A_0(\overline{U}\times[0,1],\mathbb{R}^n)$ joins φ and ψ in $A_{\partial U}(\overline{U},\mathbb{R}^n;y)$. Let $V\in N^p(\mathcal{F},U)$ be such that $\chi(x,t)\neq y$ for any $x\in\overline{U}\setminus V$, $t\in[0,1]$. Then both $\deg(\overline{V},\varphi_V,y)$, $\deg(\overline{V},\psi_V,y)$ are well defined and $\chi_V:=\chi|_{\overline{V}\times[0,1]}\in A_0(\overline{V}\times[0,1],\mathbb{R}^n)$ joins φ_V and ψ_V in $A_{\partial V}(\overline{V},\mathbb{R}^n;y)$. By Proposition (27.2.4), we can find $h\colon\overline{V}\times[0,1]\to\mathbb{R}^n$ such that $h(x,t)\neq y$ for every $(x,t)\in\partial V\times[0,1]$ and $\deg(\overline{V},h_0,y)=\mathrm{Deg}(\overline{V},\varphi_V,y)$ and $\deg(\overline{V},h_1,y)=\mathrm{Deg}(\overline{V},\psi_V,y)$. So by the homotopy property of the Brouwer degree we have

$$\mathrm{Deg}(\overline{V},\varphi_V,y)=\mathrm{Deg}(\overline{V},\psi_V,y)$$

and the proof is completed. □

We shall end this section by giving some applications of the above results. The first observation is negative.

(27.15) EXAMPLE. (The nonlinear alternative is not true for mappings in space $\widetilde{J}(K^2,\mathbb{R}^2)$). Consider $\varphi\colon K^2\multimap\mathbb{R}^2$ defined as follows:

$$\varphi(x,y)=\begin{cases}\Gamma_f & \text{if } x=0,\\ (\|x\|,f(\|x\|)) & \text{if } x\neq 0,\end{cases}$$

where $f\colon(0,1]\to R$, $f(t)=1+1/t$. Obviously, $x\notin\varphi(x)$ for each $x\in K^2$. It is easy to see that $x\notin\lambda\varphi(x)$ for every $x\in S^1$ and $\lambda\in(0,1)$ but $\varphi\in\widetilde{J}(K^2,\mathbb{R}^2)$.

Observe that by the same arguments as in the proof of (26.7) or (26.8) we can obtain the following weaker version of the nonlinear alternative.

(27.16) THEOREM. *If $\varphi \in \widetilde{J}(K^n(r), \mathbb{R}^n)$ then φ has at least one of the following properties:*

(27.16.1) $\mathrm{Fix}(\varphi) \neq \emptyset$,
(27.16.2) *there is $x_0 \in S^{n-1}$ such that $\mathrm{dist}(\{x_0\}, A) = 0$, where $A = \{t \cdot \varphi(x_0) \mid t \in (0,1)\}$.*

We can also get the following version of the Borsuk–Ulam Theorem (cf. (26.10) and (26.11):

(27.17) THEOREM. *If $\varphi \in \widetilde{A}_0(S^n(r), \mathbb{R}^n)$ then there exists $x_0 \in S^n(r)$ such that:*

$$\mathrm{dist}(\varphi(x_0), \varphi(-x_0)) = 0.$$

(27.18) REMARK. We would like to point out that the problem of extending the topological degree onto the class $\widetilde{CJ}(K^n(r), \mathbb{R}^n)$ is open.

28. Topological degree in normed spaces

In this section by E we shall denote a real normed space. We let

$$K(r) = \{x \in E \mid \|x\| \leq r\}, \quad S(r) = \{x \in E \mid \|x\| = r\}, \quad P = E \setminus \{0\}.$$

Moreover, we shall use the notations of earlier sections. First we define:

$$CJ_C(K(r), E) = \{\Phi = f \circ F \mid \text{where } \Phi \in J(K(r), X),\ f\colon X \to E,$$
$$X \in \mathrm{ANR},\ \mathrm{Fix}\phi \cap S(r) = \emptyset \text{ and } \mathrm{cl}\,\Phi(K(r)) \text{ is compact}\}.$$

In what follows, with given $\Phi \in CJ(K(r), E)$ we shall associate $\varphi\colon K(r) \multimap E$, $\varphi = j - \Phi$ given as follows:

$$\varphi(x) = j(x) - \Phi(x) = x - \phi(x)$$

for every $x \in K(r)$, where $j\colon K(r) \to E$, $j(x) = x$ is the inclusion map. Note that if $\Phi \in CJ_C(K(r), E)$ then $\varphi(S(r)) \subset P$. We claim more:

(28.1) PROPOSITION. *If $\Phi \in CJ_C(K(r), E)$ then $\varphi \in CJ(K(r), E)$.*

PROOF. In fact assume that $\Phi = f \circ F\colon K^n(r) \xrightarrow{F} X \xrightarrow{f} E$. Then we let:

$$K^n(r) \xrightarrow{\widetilde{F}} K^n(r) \times X \xrightarrow{\widetilde{f}} E,$$

where $\widetilde{F}(x) = \{(x, y) \mid y \in F(x)\}$ and $\widetilde{f}(x, y) = x - f(y)$. Since $(K^n(r) \times X) \in \mathrm{ANR}$ (cf. (1.2.4)) and $\varphi = \widetilde{f} \circ \widetilde{F}$ we infer that $\varphi \in CJ(K(r), E)$ and the proof is completed. $\square$

In what follows φ is called a *compact vector field* associated with Φ. Proposition (28.1) allows us to define:

$$CJ_{CV}(K(r),E) = \{\varphi \in CJ(K(r),E) \mid \varphi \text{ is a compact vector field associated with some } \Phi \in CJ_C(B(r),E)\}.$$

We shall use also the following general property.

(28.2) THEOREM. *Assume that $\Psi\colon X \multimap E$ is an u.s.c. and compact map, i.e. $\operatorname{cl}\Psi(x)$ is compact. Let $\psi\colon X \multimap E$ be a compact vector field associated with Ψ, where $X \subset E$. Then ψ is a closed map, i.e. for every closed $A \subset X$ the set $\psi(A)$ is closed.*

PROOF. Let A be a closed subset of X. Since Ψ is compact then there exists a compact subset $K \subset E$ such that $\Psi(X) \subset K$. Let $\{y_n\} \subset \psi(A)$ and $\lim_n y_n = y$. For the proof it is sufficient to show that $y \in \psi(A)$. We can assume that

$$y_n = x_n - z_n, \quad n = 1, 2, \dots,$$

where $x_n \in A$ and $z_n \in \Psi(x_n)$ for every n. Since $\{z_n\} \subset K$ and K is compact we can assume that:

$$\lim_n z_n = z.$$

Consequently, we deduce that $\lim_n x_n = x$ and $x \in A$ because A is closed. Since Ψ is u.s.c. we deduce that $z \in \Psi(x)$ but $\lim_n y_n = y = \lim_n(x_n - z_n) = x - z$ and hence $y \in \psi(A)$. The proof is completed. □

Now, we are going to define a topological degree on $CJ_{CV}(K(r),E)$. Let $\varphi \in CJ_{CV}(K(r),E)$. Then from (28.2) we deduce that:

$$\varphi(S(r)) \subset P$$

and $\varphi(S(r))$ is a closed subset of E. Hence $\delta = \operatorname{dist}(\varphi(S(r)),0) > 0$. Let $\varphi = j - \Phi$ and let $K = d\Phi(K(r))$. Then K is a compact subset of E. We have a compact inclusion $i\colon K \to E$, $i(x) = x$.

Let $\varepsilon > 0$ be such that $\varepsilon < \delta/2$. By using Schauder Approximation Theorem (see (12.9)) to the map i we get a finite dimensional subspace $E^{n(\varepsilon)}$ of E and a compact map $i_\varepsilon\colon K \to E$ such that:

(28.2.1) $$\|x - i_\varepsilon(x)\| < \varepsilon \quad \text{for every } x \in K,$$

(28.2.2) $$i_\varepsilon(K) \subset E^{n(\varepsilon)},$$

(28.2.3) the maps i and i_ε are homotopic.

Assume that $\Phi = f \circ F \colon K^n(r) \xrightarrow{F} X \xrightarrow{\widetilde{f}} K \xrightarrow{i} E$ (we know that $f(X) \subset K$). Now we can consider the map

$$\Phi_\varepsilon = f_\varepsilon \circ F \colon K^n(r) \xrightarrow{F} X \xrightarrow{f_\varepsilon} E,$$

where $f_\varepsilon = i_\varepsilon \circ \widetilde{f}$. It follows from (28.2.1) that $\Phi_\varepsilon \in JC_C(K^n(r), E)$. In view of (28.2.2) $f_\varepsilon(X) \subset E^{n(\varepsilon)}$. So we can define the map $\Phi_{n(\varepsilon)} \in CJ_C(K^{n(\varepsilon)}(r), E^{n(\varepsilon)})$ by putting.

$$\Phi_{n(\varepsilon)} = \widetilde{f}_\varepsilon \circ F, \quad \text{where } \widetilde{f}_\varepsilon \colon X \to E^{n(\varepsilon)}$$

is defined by the formula: $\widetilde{f}_\varepsilon(x) = f_\varepsilon(x)$.

(28.3) Remark. Let us observe that $\varphi_{n(\varepsilon)} \in CJ(K^{n(\varepsilon)}(r), E^{n(\varepsilon)})$, where $\varphi_{n(\varepsilon)}$ is a compact vector field associated with $\Phi_{n(\varepsilon)}$.

Therefore we can define:

(28.4) Definition. Let $\varphi \in CJ_{CV}(K(r), E)$. We define the topological degree $\mathrm{Deg}(\varphi)$ of φ as follows:

$$\mathrm{Deg}(\varphi) = \mathrm{Deg}(\varphi_{n(\varepsilon)}),$$

where $\varphi_{n(\varepsilon)}$ is obtained by the above procedure and $\mathrm{Deg}(\varphi_{n(\varepsilon)})$ is defined in (26.2).

In view of (28.2.3) any two approximations i_ε and $i_{\varepsilon'}$ of i are homotopic so by the homotopy property (26.2.5) it follows that Definition (28.4) is correct.

By standard arguments we deduce from (26.2):

(28.5) Theorem. *The topological degree* $\mathrm{Deg}\colon CJ_{CV}(K(r), E) \to Z$ *defined in* (28.4) *satisfies the following properties.*

(28.5.1) *If* $\varphi \in CJ_{CV}(K(r), E)$ *is of the form* $\varphi = f \circ F$ *and* F *is singlevalued then* $\mathrm{Deg}(\varphi) = \deg(\varphi)$, *where* $\deg(\varphi)$ *stands for Leray–Schauder topological degree* (*cf.* [Gr1-M], [Gr2-M], [Gr3-M] *or* [Ro-M]).

(28.5.2) *If* $\mathrm{Deg}(\varphi) \neq 0$ *then there exists* $x \in K(r)$ *such that* $0 \in \varphi(x)$.

(28.5.3) *If* $\varphi \in CJ_{CV}(K(r), E)$ *and* $\{u \in K(r) \mid 0 \in \varphi(u)\} \subset \mathrm{Int}\, K(\widetilde{r})$, $0 < \widetilde{r} < r$, *then the restriction* $\widetilde{\varphi}$ *of* φ *to* $K(\widetilde{r})$ *is in* $CJ_{CV}(K(\widetilde{r}), E)$ *and* $\mathrm{Deg}(\varphi) = \mathrm{Deg}(\widetilde{\varphi})$.

(28.5.4) *If* φ_1, φ_2 *are homotopic in* $CJ_{CV}(K(r), E)$ *then* $\mathrm{Deg}(\varphi_1) = \mathrm{Deg}(\varphi_2)$, *where homotopy in* $CJ_{CV}(K(r), E)$ *means that joining homotopy is a compact vector field* (*cf.* (26.1)).

(28.5.5) *Let* $\Phi_1, \Phi_2 \in JC_C(K(r), E)$ *and assume that:*

$$x \notin \{\lambda\Phi_1(u) + (1-\lambda)\Phi_2(u) \mid \text{for every } (u, \lambda) \in S(r) \times [0,1]\}$$

then $\mathrm{Deg}(\varphi_1) = \mathrm{Deg}(\varphi_2)$.

The proof of (28.5) is left to the reader. Now we will sketch possible consequences of (28.5).

Note, that there are standard topological facts which follow from a topological degree theory (cf. [Br2-M], [Do-M], [Gr1-M], [L1-M], [LR-M], [Ro-M] [Sp-M] and [Wa-M]).

The inclusion $j\colon K(r) \to E$ is a compact vector field because $\Phi(x) = 0$ for every $x \in K(r)$ a compact map and $j = j - \Phi$. Now by the construction of topological degree we have:

$$j_{n(\varepsilon)}\colon K^{n(\varepsilon)}(r) \to E^{n(\varepsilon)}, \quad j_{n(\varepsilon)}(x) = x$$

for every $x \in K^{n(\varepsilon)}(r)$. So, in view of (26.6), we obtain

$$\operatorname{Deg}(j) = \deg(j) = \deg(j_{n(\varepsilon)}) = 1. \tag{28.6}$$

Below we give examples of mappings with degree different from zero.

(28.7) Example. Let $\Phi\colon E \multimap E$ be of the from $\Phi = f \circ F\colon E \overset{F}{\multimap} X \overset{f}{\longrightarrow} E$, $x \in \mathrm{ANR}$, where F has R_δ-values and Φ is u.s.c. and compact. Let $r > 0$ be such that $\operatorname{cl}\Phi(E) \subset \operatorname{Int} K(r)$. If $\varphi\colon K(r) \multimap E$ is a compact vector field associated with φ then $\operatorname{Deg}(\varphi) = 1$. In fact, consider the homotopy $\chi\colon K(r) \times [0,1] \multimap E$ given as follows:

$$\chi(x,t) = x - t \cdot \Phi(x).$$

Then $\chi(S(r) \times [0,1]) \subset P$ and for every $t \in [0,1]$ the map $\chi(\,\cdot\,,t)$ is a compact vector field. Consequently from (28.5.5) we deduce that

$$\operatorname{Deg}(\varphi) = \operatorname{Deg}(j) = 1.$$

(28.8) Remark. From (28.7) and (28.5.2) we deduce that $\operatorname{Fix}(\Phi) \neq \emptyset$ provided Φ is the same as in (28.7).

Now we are able to prove:

(28.9) Proposition. *Let $X \in \mathrm{AR}$ and $\Phi\colon X \multimap X$ be a compact u.s.c. map of the form:*

$$X \overset{F}{\multimap} Y \overset{f}{\longrightarrow} X,$$

where F is u.s.c. with R_δ-values, $Y \in \mathrm{ANR}$ and f is continuous. Then $\operatorname{Fix}(\Phi) \neq \emptyset$.

Proof. In view of the Arens–Eells embedding Theorem (1.6) we can assume that X is a closed subset of a normed space E. Consequently, it follows from (1.8.1) that there exists a retraction $r\colon E \to X$. We have diagram:

$$E \overset{r}{\longrightarrow} X \overset{F}{\multimap} Y \overset{f}{\longrightarrow} X \overset{j}{\longrightarrow} E,$$

where j is the inclusion map.

We let $F_1 = F \circ r$ and $f_1 = j \circ f$. Then we get a map $\Psi = f_1 \circ F_1$ which satisfies assumptions of (28.8) and hence $\mathrm{Fix}(\psi) \neq \emptyset$. Now by using standard arguments we deduce that $\mathrm{Fix}(\Phi) \neq \emptyset$ and the proof is completed. □

(28.10) EXAMPLE. Let $\varphi \in CJ_{CV}(K(r), E)$ be such that $(-\lambda x) \notin \varphi(x)$ for all $x \in S(r)$ and $\lambda > 0$. Then $\mathrm{Deg}(\varphi) = 1$.

In fact assume that φ is associated with $\Phi \in CJ_C(K(r), E)$. We consider the homotopy $\chi\colon K(r) \times [0,1] \multimap E$ defined by:

$$\chi(x, 1) = x - t\phi(x).$$

Assume that for some $x \in S(r)$ and $t > 0$ we have $0 \in \chi(x, t)$. It implies that

$$0 = x - ty \quad \text{for every } y \in \phi(x).$$

Consequently $x \in f \cdot \Phi(x)$ and hence $(1 - 1/t) \cdot x \in \varphi(x)$ but it contradicts our assumption. Therefore $\chi(S(r) \times [0,1]) \subset P$ and $\mathrm{Deg}(\varphi) = \deg(1) = 1$.

29. Topological degree of vector fields with non-compact values in Banach spaces

In this section all metric spaces are assumed to be complete. We would like to point out that the problem of defining topological degree on $\widetilde{J}(K(r), E)$ for arbitrary normed space E is still open (cf. [Ga-1], [Da1-M], [BM-7]). In this section we will restrict our considerations to the case of closed convex and bounded subsets of a Banach space E.

First, following Section 4 for a complete metric space X by $B(X)$ we shall denote the complete metric space of closed bounded and nonempty subsets of X with the Hausdorff metric d_H. If $X = E$ is a Banach space then we will consider $CB(E)$ to be a subspace of $B(E)$ defined as follows:

$$CB(E) = \{A \in B(E) \mid A \text{ is convex}\}.$$

We start with the following theorem:

(29.1) THEOREM. *Let $\varphi\colon Y \to B(X)$ be a continuous and compact map where Y is a metric space. Then there exists a compact subset $K \subset X$ such that:*

(29.1.2) $$\varphi(y) \cap K \neq \emptyset \quad \textit{for every } y \in Y.$$

For the proof of (29.1) we need the following lemma.

(29.2) Lemma. *Let $F: Y \to B(X)$ be a continuous compact map and A be a compact subset of X such that:*

(29.2.1) *$d_H(A, F(y)) < \varepsilon$ for some $\varepsilon > 0$ and every $y \in Y$.*

Then for every $\alpha > 0$ there exists a compact set $B \subset X$ such that:

(29.2.2) *$A \subset B \subset O_\varepsilon(A)$,*

(29.2.3) *the set $B \setminus A$ is finite,*

(29.2.4) *$d_H(B, F(y)) < \alpha$ for every $y \in Y$.*

Proof. Let $\varepsilon > 0$ be a given positive real number. Since F is compact the set $F(Y) \in B(X)$ is a relatively compact subset of a complete space $B(X)$. Therefore for given $\varepsilon > 0$ we can find a finite ε-net $F(y_1), \dots, F(y_k)$ of $F(Y)$, i.e. for every $y \in Y$ there is $i = 1, \dots, k$ such that $d_H(F(y_i), F(y)) < \varepsilon$, so in particular $F(y_i) \subset O_\varepsilon(F(y))$.

Now, for every $i = 1, \dots, k$ we choose a point $x_i \in F(y_i)$ such that $\operatorname{dist}(\{x_i\}, A) < \varepsilon$. It is possible, owing to (29.2.1) and the compactness of A. Let $B = A \subset \{x_1, \dots, x_k\}$. Now, it is evident that B satisfies (29.2.2)–(29.2.4). □

Proof of Theorem (29.1). First, we let $\varepsilon = 1$ and choose a finite 1-set of $\varphi(Y)$ in $B(X)$. Let $\varphi(y_1), \dots, \varphi(y_l)$ be the above set. We choose $x_i \in \varphi(y_i)$, $i = 1, \dots, k$.

Let $A_0 = \{x_1, \dots, x_l\}$. Then A_0 is compact and $d_H(A_0, \varphi(y)) < 1$ for every $y \in Y$. So A_0 satisfies all assumptions of Lemma (29.2). By applying Lemma (29.2) to A_0 and $\varepsilon = 1/2$ we get a compact (finite) set A_1 such that: $A_0 \subset A_1 \subset O_\varepsilon(\varphi(A_0))$ and (29.2.3) and (29.2.4) are satisfied.

Consequently, by induction we can construct a sequence $\{A_n\}$ of finite subsets of X such that:

(29.2.5) $$A_n \subset A_{n+1} \subset O_{1/2^{n+1}}(A_n),$$

(29.2.6) $$d_H(A_n, \varphi(y)) < \frac{1}{2^n} \quad \text{for every } y \in Y \text{ and } n = 1, 2, \dots$$

We let
$$B = \operatorname{cl}\Big(\bigcup_{n=0}^{\infty}(A_n)\Big).$$

Now, from (29.2.5), we deduce that for every $\varepsilon > 0$ B possesses a finite ε-net. In fact, we choose $k \in N$ such that
$$\sum_{n=k}^{\infty} \frac{1}{2^k} < \varepsilon,$$
then A_k is the needed ε-set of B and since X is complete we obtain that B is compact. Moreover, from (29.2.6) we infer $d_H(B, \varphi(y)) = 0$ for every $y \in Y$.

Consequently compactness of B implies that $B \cap \varphi(y) \neq \emptyset$ for every $y \in Y$ and the proof is completed. □

(29.3) COROLLARY. *Let $F: Y \to B(E)$ be a continuous compact map such that for every $y \in Y$ we have $F(y) \in CB(E)$, where E is a Banach space. Then there exists an u.s.c. compact map $\varphi: X \multimap E$ with convex values such that:*

(29.3.1) $$\varphi(y) \subset F(y) \quad \text{for every } y \in Y.$$

PROOF. By applying theorem (29.1) to the map F we get a compact set $K \subset E$ such that:

$$F(y) \cap K \neq \emptyset \quad \text{for every } y \in Y.$$

We let:

$$\widetilde{\varphi}: Y \multimap E, \quad \widetilde{\varphi}(y) = F(y) \cap K.$$

Then from (14.7) we deduce that $\widetilde{\varphi}$ is u.s.c. As we already remarked (see (3.6)) by the Mazur Theorem the closed convex ball of a compact subset in a Banach space is again a compact set. So we are allowed to define:

$$\varphi: Y \multimap E, \quad \varphi(y) = \overline{\mathrm{conv}}(\widetilde{\varphi}(y)) \quad \text{for every } y \in Y.$$

Then from (14.12) we deduce that φ is u.s.c. and by applying again the Mazur Theorem we deduce that φ is compact. The proof of (29.3) is completed. □

Now we are able to define the topological degree for vector fields with convex closed bounded values in Banach spaces.

First, let us introduce some notations. We let:

$$\widetilde{J}_C(K(r), E) = \{F: K(r) \to B(E) \mid F \text{ continuous, compact,} \\ F(x) \in CB(E) \text{ and } x \notin F(x) \text{ for every } x \in S(r)\}.$$

We have of course:

$$\widetilde{F}_C(K(r), E) \subset J(K(r), E).$$

As before by $\widetilde{J}_{CV}(K(r), E)$ we shall define the set of all associated vector fields, i.e. a map $f: K(r) \to B(E)$ belongs to $J_{CV}(K(r), E)$ if and only if there is $F \in \widetilde{J}_C(K(r), E)$ such that:

$$f(x) = \{x - y \mid y \in F(x)\}.$$

Note that for $f \in J_{CV}(K(r), E)$ we have $0 \notin f(x)$ for every $x \in S(r)$.

(29.4) Definition. Two compact vector fields $f, g \in \widetilde{J}_{CV}(K(r), E)$ are *homotopic* provided there exists a compact vector field $h \in \widetilde{J}_{CV}(K(r), \times[0,1], E)$ such that:

(29.4.1) $$h(x, 0) = f(x),$$

(29.4.2) $$h(x, 1) = g(x),$$

where $h(x,t) = \{x - y \mid y \in H(x,t)\}$ and $H\colon K(r) \times [0,1] \to B(E)$ is continuous compact with closed convex values and $x \notin H(x,t)$ for every $x \in S(r)$ and $t \in [0,1]$.

We prove:

(29.5) Theorem. *There is a function* $\mathrm{Deg}\colon \widetilde{J}_{CV}(K(r), E) \to Z$ *which satisfies the following conditions:*

(29.5.1) *If* $\mathrm{Deg}(f) \neq 0$*, then there is* $x \in K(r)$ *such that* $0 \in f(x)$*.*

(29.5.2) *If* $\{x \in K(r) \mid 0 \in f(x)\} \subset K(\widetilde{r})$ *for some* $0 < \widetilde{r} < r$ *and* $f \in \widetilde{F}_{CV}(K(r), E)$*, then* $\widetilde{f} = f|_{B(\widetilde{r})} \in \widetilde{J}_{CV}(K(\widetilde{r}), E)$ *and* $\mathrm{Deg}(f) = \mathrm{Deg}(\widetilde{f})$*.*

(29.5.3) *If* $f, g \in \widetilde{J}_{CV}(K(\widetilde{r}), E)$ *are homotopic, then* $\mathrm{Deg}(f) = \mathrm{Deg}(g)$*.*

(29.5.4) *If* $f\colon K(r) \to E$ *is a singlevalued compact vector field then*

$$\mathrm{Deg}(f) = \deg(f),$$

where $\deg(f)$ *stands for the Leray–Schauder degree* (cf. [Br2-M], [Gr4-M], [L1-M], [Ni-M] or [Ro-M]).

Proof. Let $f \in \widetilde{J}_{CV}(K(r), E)$ be of the form:

$$f(x) = \{x - y \mid y \in F(x)\},$$

where $F \in \widetilde{J}_C(K(r), E)$. By applying Corollary (29.3) to F, we get a compact map $\Phi\colon K(r) \multimap E$ with convex values such that $\Phi(x) \subset F(x)$ for every $x \in K(r)$. Note that, if $\Psi\colon K(r) \multimap E$ is a second map satisfying Lemma (29.3) then for any fixed $t \in [0,1]$ the map $\chi_t\colon K(r) \multimap E$ defined by

$$\chi_t(x) = t \cdot \Phi(x) + (1 - t)\Psi(x)$$

is again an u.s.c. compact map with convex values such that:

$$\chi_t(x) \subset F(x) \quad \text{for every } x \in K(r),$$

and moreover, $x \notin \chi_t(x)$ for every $x \in S(r)$. It means that the topological degree $\mathrm{Deg}(j - \Phi)$ and $\mathrm{Deg}(j - \Psi)$ is well defined (cf. the preceding section) and

$$\mathrm{Deg}(j - \Phi) = \mathrm{Deg}(j - \Psi).$$

Therefore, the following definition:

$$\mathrm{Deg}(f) = \mathrm{Deg}(j - \Phi) \tag{29.5.5}$$

is correct. □

Now, properties (29.5.1)–(29.5.4) are easy consequences of the previous section and (29.3).

Applying Theorem (29.5), by standard arguments (cf. Sections 27, 28) we obtain:

(29.6) THEOREM. *Let $F\colon K(r) \to B(E)$ be a continuous compact map such that $F(x) \subset K(r)$ and $F(x) \in CB(E)$ for every $x \in K(r)$. Then* $\mathrm{Fix}(F) \neq \emptyset$.

Theorem (29.6) is an interesting generalization of the Kakutani–Brouwer fixed point theorem (cf. [Ma-M]). Note that F. S. De Blasi and J. Myjak in [BM-7] proved Theorem (29.5) but without property (29.5.1). Finally, we would like to add that the topological degree for multivalued mappings with non-compact values were studied in [Da1-M] and [Ga-1].

(29.7) REMARK. Let us remark that in all considerations of this section it is sufficient to assume that $F\colon K(r) \to B(E)$ be u.s.c. considered as the map $F\colon K(r) \multimap E$ instead of continuity of F (of course we keep that F be compact).

30. Topological essentiality

In this section we will be looking at a more general construction than topological degree — the essentiality, also called topological transversality. Topological essentiality can be defined on a larger class of mappings than topological degree but yields less information. So, one can consider topological essentiality as a weaker from of the topological degree theory.

First the concept of topological essentiality was systematicaly studied in [Gr4-M] (cf. also [Gr2-M], [DG1-M]).

Let E, E_1 be two Banach spaces and let $K(r) \subset E$. We will consider:

$$\begin{aligned} \widetilde{J}_{CU}(K(r), E_1) &= \{F\colon K(r) \to B(E_1) \mid \varphi \text{ is compact with convex values and } F \\ &\qquad \text{considered as a multivalued map from } K(r) \text{ to } E_1 \text{ is u.s.c.}\}, \\ \widetilde{J}^0_{CU}(K(r), E_1) &= \{F \in \widetilde{J}_{C0}(K(r), E_1) \mid F(x) = \{0\} \text{ for all } x \in S(r)\}, \\ \widetilde{J}_0(K(r), E_1) &= \{F\colon K(r) \to CB(E_1) \mid 0 \notin F(x) \text{ for every } x \in S(r) \\ &\qquad \text{and } F \text{is u.s.c. as a map from } K(r) \text{ to } E_1\}. \end{aligned}$$

We can now define:

(30.1) DEFINITION. A map $F \in \widetilde{J}_0(K(r), E_1)$ is called *essential* (with respect to $\widetilde{J}^0_{CU}(K(r), E_1)$) if for every $G \in \widetilde{J}^0_{CU}(K(r), E_1)$ there exists a point $x \in K(r)$ such that:

$$F(x) \cap G(x) \neq \emptyset.$$

Let us enumerate several properties of the above defined essentiality.

(30.2) PROPERTY (Existence). *If $F \in \widetilde{J}_0(K(r), E_1)$ is essential, then there exists $x \in \operatorname{Int} K(r)$ such that $0 \in F(x)$. In fact, we take $G(x) = \{0\}$ for every $x \in K(r)$. Then $G \in \widetilde{J}^0_{CU}(K(r), E_1)$ and our claim follows from* (30.1).

(30.3) PROPERTY (Compact perturbation). *If $F \in \widetilde{J}_0(K(r), E_1)$ is essential and $G \in \widetilde{J}^0_{CU}(K(r), E_1)$, then $(F + G) \in \widetilde{J}_0(K(r), E_1)$ and $(F + G)$ is essential.*

Property (30.3) is self-evident.

(30.4) PROPERTY (Coincidence). *Assume that $F \in \widetilde{J}_0(K(r), E_1)$ is essential and $H \in \widetilde{J}_{CU}(K(r), E_1)$. Let $A = \{x \in K(r) \mid F(x) \cap (tH)(x) \neq \emptyset,$ for some $t \in [0, 1]\}$. If $A \subset \operatorname{Int}(K(r)$, then F and H have a coincidence.*

PROOF. First observe that the essentiality of F implies that A is nonempty. Moreover, A is closed and such that $A \cap S(r) = \emptyset$.

Let $s\colon K(r) \to [0, 1]$ be an Urysohn function such that $s(x) = 1$ for $x \in A$ and $s(x) = 0$ for $x \in S(r)$. We define the map $G\colon K(r) \multimap E_1$ as follows:

$$G(x) = s(x) \cdot H(x) \quad \text{for every } x \in K(r).$$

Then $G \in \widetilde{J}^0_{CU}(B(r), E_1)$ and since F is essential, we get

$$F(x_0) \cap G(x_0) \neq \emptyset \quad \text{for some } x_0 \in K(r).$$

This implies that $x_0 \in A$ and hence $S(x_0) = 1$. Finally, we get $F(x_0) \cap H(x_0) \neq \emptyset$ and the proof is completed. □

(30.5) PROPERTY (Normalization). *The inclusion map $i\colon B(r) \to E$ is essential.*

PROOF. Let $G \in \widetilde{J}^0_{CU}(K(r), E)$. We let:

$$A = \{x \in K(r) \mid x \in (t \cdot G)(x), \text{ for some } t \in [0, 1]\}.$$

Then A is a closed nonempty subset of $K(r)$ such that $0 \in A$ and $A \subset \operatorname{Int} K(r)$. We consider an Urysohn function $s\colon E \to [0, 1]$ such that $s(x) = 1$ for $x \in A$ and $s(x) =$ for $x \notin \operatorname{Int} K(r)$. We consider $H\colon K(r) \multimap K(r)$ defined as:

$$H(x) = s(x) \cdot G(\rho(x)),$$

where $\rho\colon E \to K(r)$ is the retraction map defined as follows:

$$\rho(x) = \begin{cases} r(x/\|x\|) & \text{for } x \notin K(r), \\ x & \text{for } x \in K(r). \end{cases}$$

It follows from (29.6) that $\operatorname{Fix}(H) \neq \emptyset$. If $x \notin \operatorname{Int} K(r)$, then $s(x) = 0$ and $x = 0$ but $0 \in B(r)$ so we get a contradiction. Therefore, we deduce that $x \in \operatorname{Int} K(r)$. So, $x = i(x) \in G(x)$ and the proof is completed. □

(30.6) PROPERTY (Localization). *Assume that $F \in \widetilde{J}_0(K(r), E_1)$ is an essential map such that the set*

$$A = \{x \in K(r) \mid 0 \in F(x)\} \subset \operatorname{Int} K(\widetilde{r}) \quad \textit{for some } 0 < \widetilde{r} < r.$$

Then the restriction $\widetilde{F}$ of F to $K(\widetilde{r})$ is an essential map in $\widetilde{J}_0(K(\widetilde{r}), E_1)$.

PROOF. We know that $A \neq \emptyset$. We let:

$$B = \{x \in K(\widetilde{r}) \mid x \in (F(x) \cap (tG)(x)) \neq \emptyset, \text{ for some } t \in [0,1]\},$$

where $G \in \widetilde{J}^0_{CU}(K(\widetilde{r}), E_1)$. Then $A \subset B$. Again let $s\colon K(r) \to [0,1]$ be an Urysohn function such that $s(x) = 1$ for some $x \in A$ and $s(x) = 0$ for $x \notin \operatorname{Int} K(\widetilde{r})$. Moreover, we consider $\rho\colon K(r) \to K(\widetilde{r})$ defined as follows:

$$\rho(x) = \begin{cases} x & \text{for } x \in K(\widetilde{r}), \\ \widetilde{r} \cdot (x/\|x\|) & \text{for } x \notin K(\widetilde{r}). \end{cases}$$

We define the map $H\colon K(\widetilde{r}) \multimap E_1$ by the formula:

$$H(x) = s(x) \cdot G(\rho(x))$$

for every $x \in K(r)$. Obviously $H \in J^0_{CU}(K(r), E_1)$. Since F is essential there is a point $x \in K(r)$ such that $F(x) \cap H(x) \neq \emptyset$. It is easy to see that $x \in K(\widetilde{r})$ and this ends the proof. □

(30.7) PROPERTY (Homotopy). *Let $F \in \widetilde{J}_0(K(r), E_1)$ be an essential map. Let $H\colon B(r) \to B \subset (E_1)$ be compact u.s.c. map such that:*

(30.7.1) $H(x,0) = \{0\}$ *for every* $x \in S(r)$,
(30.7.2) $\{x \in K(r) \mid F(x) \cap H(x,t) \neq \emptyset$ *for some* $t \in [0,1]\} \subset \operatorname{Int} K(r)$.

Then the map $F_1(x) = \{u - v \mid u \in F(x)$ and $v \in H(x,1)\}$ belongs to $\widetilde{J}_0(K(r), E_1)$ and is essential.

PROOF. Let $G \in \widetilde{J}^0_{CU}(K(r), E_1)$. We let:

$$A = \{x \in K(r) \mid F(x)\} \cap \{u + v \mid u \in G(x) \text{ and } v \in H(x,t), \text{ for some } t \in [0,1]\}.$$

Since $(G + H(\cdot\,,0)) \in \widetilde{J}^0_{CU}$ and F is essential we obtain that $A \neq \emptyset$. Evidently A is closed. Let $s\colon K(r) \to [0,1]$ be an Urysohn function such that $s(x) = 1$, for $x \in A$ and $s(x) = 0$ for $x \in S(r)$. We define the map G_1 by putting:

$$G_1(x) = \psi(x) + \chi(x, s(x)) = \{u + v \mid u \in \psi(x) \text{ and } v \in \chi(x, s(x))\}.$$

Then $G_1 \in \widetilde{J}^0_{CU}(K(r), E_1)$ and our assertion follows from essentiality of F. □

(30.8) PROPERTY (Continuation). *Let $F \in \widetilde{J}_0(K(r), E_1)$ be an essential mapping. Assume that F is proper, i.e. $\{x \in K(r) \mid F(x) \cap K \neq \emptyset\}$ is compact for every compact $K \subset E_1$. Assume further that $H\colon K(r) \times [-1,1] \multimap CB(E_1)$ is a compact u.s.c. map such that $H(x,0) = \{0\}$ for every $x \in S(r)$. Then there exists $\varepsilon > 0$ such that the mapping $(F - H)(\cdot\,,t))$, $(F - H)(\cdot\,,t))(x) = \{u - v \mid u \in F(x)$ and $v \in H(x,t)\}$ belongs to $\widetilde{J}_0(K(r), E_1)$ and is essential for every $t \in (-\varepsilon, \varepsilon)$.*

PROOF. According to the homotopy property it is sufficient to show that there exists $\varepsilon > 0$ such that $F(x) \cap H(x,t) = \emptyset$ for every $x \in S(r)$ and $t \in (-\varepsilon, \varepsilon)$. □

But this condition is easy to verify by contradiction.

(30.9) REMARK. Observe that in the proofs of all the properties here we were using essentially the following facts:

(30.9.1) for any two maps F and G from $J_{CU}(K(r), E_1)$ and $\widetilde{J}^0(K(r), E_1)$ the mappings $F + G$, $F - G$, sF belong to the respective classes; and

(30.9.2) the Brouwer Fixed Point Theorem holds for these maps (with respect to essentiality).

Therefore, it is possible to repeat all results of this section for arbitrary classes of multivalued mappings satisfying the above properties.

(30.10) REMARK. Observe that the technique of essential mappings allows us to obtain the same type topological results as by using the topological degree theory. Below we will prove only the nonlinear alternative but, for example, the Leray–Schauder alternative, the Birkhoff–Kellogg theorem and Borsuk's theorem on antipodes are also possible. A systematic approach to these applications of the topological degree theory we will present in Section 33.

(30.11) THEOREM. *Let $F \in \widetilde{J}_{CU}(K(r), E_1)$ be essential and $G \in \widetilde{J}_{CU}(K(r), E_1)$. If $F(x) \cap G(x) = \emptyset$ for every $x \in S(r)$ then at least one of the following conditions holds:*

(30.11.1) *there exists $x \in K(r)$ such that $F(x) \cap G(x) \neq \emptyset$,*

(30.11.2) *there exists $\lambda \in (0,1)$ and $x \in S(r)$ such that $F(x) \cap (\lambda G(x)) \neq \emptyset$.*

To prove (30.11) it is sufficient to apply the homotopy property for F and H, where $H(x,t) = t \cdot G(x)$ for $x \in K(r)$, $t \in [0,1]$.

Now, if $E = E_1$ then from (30.11) and the normalization property immediately follows:

(30.12) PROPERTY (Nonlinear alternative). *Let* $G \in \widetilde{J}_{CU}(K(r), E)$. *Then at least one of the following conditions is satisfied:*

(30.12.1) $\mathrm{Fix}(G) \neq \emptyset$,
(30.12.2) *there exists* $x \in S(r)$ *and* $\lambda \in (0,1)$ *such that* $x \in (\lambda G(x))$.

31. Random fixed points

A systematic study of Random Operators was initiated in the 1950s in the Praque school of probabilists. Today the research in random operators includes such areas as operator valued random variables and functions and their properties, random equations, random dynamical systems, measure-theoretic problems and obviously random fixed point theorems.

In this section we would like to give a short presentation of the random topological degree theory. We assume X to be a separable metric space and Ω be a complete measure space (cf. Section 19).

(31.1) DEFINITION. Let A be a closed subset of X and $\varphi\colon \Omega \times A \multimap X$ be a multivalued map with compact values. We will say that φ is a *random operator* provided the following two conditions are satisfied.

(31.1.1) φ is product-measurable,
(31.1.2) $\varphi(\omega, \cdot\,)$ is u.s.c. for every $\omega \in \Omega$.

(31.2) DEFINITION. Assume that $\varphi\colon \Omega \times A \multimap X$ is a random operator. A measurable map: $\xi\colon \Omega \multimap A$ is called a *random fixed point* for φ provided we have:

$$\xi(\omega) \in \varphi(\omega, \xi(\omega)) \quad \text{for every } \omega \in \Omega.$$

We prove the following:

(31.3) PROPOSITION. *Let* $\varphi\colon \Omega \times A \multimap X$ *be a random operator such that for every* $\omega \in \Omega$ *the set of fixed points of* $\varphi(\omega, \cdot\,)$ *is nonempty* $(\mathrm{Fix}(\varphi(\omega, \cdot\,)) \neq \emptyset)$. *Then* φ *has a random fixed point.*

PROOF. For the proof we define the multivalued map $F\colon \Omega \multimap A$ defined by:

$$F(\omega) = \mathrm{Fix}(\varphi(\omega, \cdot\,)) \quad \text{for every } \omega \in \Omega.$$

To deduce that the graph Γ_F of F is measurable we consider $f\colon \Omega \times A \to [0, +\infty)$ defined as follows:

$$f(\omega, x) = \mathrm{dist}(x, \varphi(\omega, x)) \quad \text{for every } (\omega, x) \in \Omega \times A.$$

In view of (19.16), f is measurable. But $\Gamma_F = f^{-1}(\{0\})$. So, Γ_F is measurable and hence, in view of (19.17), F possesses a measurable selector $\xi\colon \Omega \to A$ i.e. $\xi(\omega) \in F(\omega)$ for every $\omega \in \Omega$. Observe that ξ is the needed random fixed point of φ. The proof is completed. □

Note, that having (31.3) one can formulate in a natural way the random version of the Banach fixed point theorem, the Brouwer fixed point theorem, and the Schauder fixed point theorem for both single and multivalued random operators.

From (31.3) it follows that the deterministic fixed theorem implies the respective random fixed point theorem. Roughly speaking the same is true in the case of the topological degree theory. We will sketch it below.

We denote by $M(\Omega \times K^n(r), \mathbb{R}^n)$ a family of random operators such that the following two conditions are satisfied:

(31.4) for every $\varphi \in M(\Omega \times K^n(r), \mathbb{R}^n)$ we have $0 \notin \varphi(\Omega \times S^{n-1}(r))$;

(31.5) for every $\omega \in \Omega$ for the map $\varphi(\omega, \cdot\,)\colon K^n(r) \multimap \mathbb{R}^n$ the topological degree $\operatorname{Deg}(\varphi(\omega, \cdot\,))$ is defined (cf. earlier sections of this chapter).

By homotopy in $M(\Omega \times K^n(r), \mathbb{R}^n)$ we will understand a random homotopy, i.e.

$$\chi\colon \Omega \times K^n(r) \times [0,1] \multimap \mathbb{R}^n,$$

which is product-measurable u.s.c. with respect to the last variable, such that:

$$0 \notin \chi(\Omega \times S^{n-1}(r) \times [0,1]),$$
$$\chi(\,\cdot\,,\,\cdot\,,t) \in M(\Omega \times K^n(r), \mathbb{R}^n) \quad \text{for every } t \in [0,1].$$

Let $\varphi \in M(\Omega \times B^n(r), \mathbb{R}^n)$. We define the random degree $\operatorname{Deg}_{\mathrm{ra}}(\varphi)$ of φ as follows:

(31.6) $$\operatorname{Deg}_{\mathrm{ra}}(\varphi) = \{\operatorname{Deg}(\varphi(\omega, \cdot\,) \mid \omega \in \Omega\}.$$

Then we have:

(31.7) Property (Existence). *If $0 \notin \operatorname{Deg}_{\mathrm{ra}}(\varphi)$ then there exists a measurable map $\xi\colon \Omega \to B^n(r)$ such that:*

$$0 \in \varphi(\omega, \xi(\omega)) \quad \textit{for } \omega \in \Omega.$$

The proof is strictly analogous to the proof of (31.3).

(31.8) Property (Localization). *If $\varphi \in M(\Omega \times K^n(r), \mathbb{R}^n)$ and $\{x \in K^n(r) \mid$ exists $\omega \in \Omega$ $0 \in \varphi(\omega, x)\} \subset \operatorname{Int} K(\widetilde{r})$ for some $0 < \widetilde{r} < r$. Then the restriction $\widetilde{\varphi}$ of φ to $\Omega \times K^n(\widetilde{r})$ belongs to $M(\Omega \times K^n(\widetilde{r}), \mathbb{R}^n)$ and*

$$\operatorname{Deg}_{\mathrm{ra}}(\varphi) = \operatorname{Deg}_{\mathrm{ra}}(\widetilde{\varphi}).$$

(31.9) PROPERTY (Homotopy). *If φ and ψ are homotopic in $M(\Omega\times K^n(r),\mathbb{R}^n)$ then*

$$\mathrm{Deg}_{\mathrm{ra}}(\varphi)=\mathrm{Deg}_{\mathrm{ra}}(\psi).$$

Proofs of (31.8) and (31.9) are strictly analogous to the proofs of respective properties in deterministic case.

Finally, note that we can do exactly the same as above if we replace $\mathbb{R}^n$ by an arbitrary separable Banach space E.

CHAPTER IV

HOMOLOGICAL METHODS IN FIXED POINT THEORY OF MULTIVALUED MAPPINGS

In this chapter we would like to present a systematic study of the fixed point theory for multivalued maps by using homological methods. Homological methods were initiated in 1946 by S. Eilenberg and D. Montgomery in their celebrated paper [EM]. Using methods of homology we can obtain stronger results than those obtained by means of the approximation methods as used in Chapter III. Hence in this chapter the results will be formulated in a more general form than previously.

32. Acyclic mappings

In this section we would like to study general properties of acyclic mappings including the Lefschetz fixed point theorem. Note that the class of acyclic mappings was introduced by S. Eilenberg and D. Montgomery in the paper [EM].

(32.1) Definition. An u.s.c. map $\varphi\colon X \multimap Y$ with compact values is called *acyclic* provided for every $x \in X$ the set $\varphi(x)$ is acyclic (cf. Section 8).

Denote by $\mathrm{AC}(X,Y)$ the class of all acyclic mappings. Let us remark that in particular we have:

$$J(X,Y) \subset \mathrm{AC}(X,Y),$$

i.e. u.s.c. mappings with R_δ-values are acyclic mappings. Unfortunately the class of acyclic maps is not closed with respect to the composition law.

(32.2) Example. Let $\varphi\colon S^1 \to S^1$ be the map given by the formula:

$$\varphi(x) = \{y \in S^1 \mid \|x-y\| \leq 3/2\}.$$

Then φ is acyclic but the composition $\psi = \varphi \circ \varphi\colon S^1 \to S^1$ is no longer acyclic because $\psi(x) = S^1$ for every $x \in S^1$.

We have the following:

(32.3) PROPOSITION. *If* $\varphi\colon X \multimap Y$ *is an acyclic map then the natural projection* $p_\varphi\colon \Gamma_\varphi \to X$, $p_\varphi(x,y) = x$, *is a Vietoris map.*

PROOF. Since $p_\varphi^{-1}(x)$ is homeomorphic to $\varphi(x)$ for every $x \in X$ we get that $p_\varphi^{-1}(x)$ is acyclic. Now, let $A \subset X$ be a compact set. Then

$$p^{-1}(A) \subset A \times \varphi(A).$$

Since $p^{-1}(A)$ is closed and in view of (14.10), $\varphi(A)$ is compact, so $p^{-1}(A)$ is compact as a closed subset of the compact set $A \times \varphi(A)$. Therefore p is proper and the proof is completed. □

Let $\varphi\colon X \multimap Y$ be an acyclic map. In view of (32.3) we can associate with φ the diagram:

$$X \overset{p_\varphi}{\Longleftarrow} \Gamma_\varphi \overset{q_\varphi}{\longrightarrow} Y$$

in which p_φ is a Vietoris map. By applying to the above diagram the Čech homology functor with compact carriers we obtain the following diagram:

$$H_n(X) \underset{\sim}{\overset{(p_\varphi)_{*n}}{\Longleftarrow}} H_n(\Gamma_\varphi) \overset{(q_\varphi)_*}{\longrightarrow} H_n(Y)$$

for every $n = 1, 2, 3\ldots$, in which $(p_\varphi)_{*n}$ is an isomorphism. Therefore, we can define the induced by φ linear $\varphi_* : H_*(X) \to H_*(Y)$ by putting:

$$\varphi_* = \{\varphi_{*n}\},$$

where $\varphi_{*n} = (q_\varphi)_* \circ (p_\varphi)_{*n}^{-1}$, $n = 1, 2, \ldots$ It is easy to see that if $\varphi = f$ is a singlevalued map then φ_* is equal to the induced linear map f_* defined in Section 7.

(32.4) DEFINITION. Assume that $\varphi\colon X \multimap X$ is an acyclic map.

(32.4.1) If the graded linear space $H_*(X)$ is of finite type then we define the (ordinary) Lefschetz number $\lambda(\varphi)$ of φ by letting:

$$\lambda(\varphi) = \lambda(\varphi_*).$$

(32.4.2) If $\varphi_*\colon H_*(X) \to H_*(X)$ is a Leray endomorphism then we define the (generalized) Lefschetz number $\Lambda(\varphi)$ of φ by:

$$\Lambda(\varphi) = \Lambda(\varphi_*).$$

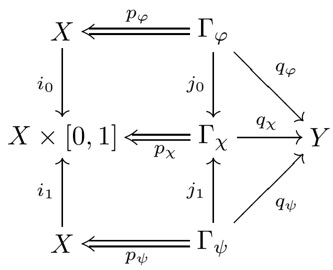
(32.5) Definition. Two acyclic maps $\varphi, \psi\colon X \multimap Y$ are called *homotopic* (written $\varphi \sim \psi$) provided there exists an acyclic map $\chi\colon X \times [0,1] \multimap Y$ such that:

(32.5.1) $\chi(x, 0) = \varphi(x)$ for every $x \in X$,
(32.5.2) $\chi(x, 1) = \psi(1)$ for every $x \in X$.

We prove:

(32.6) Proposition. *If two acyclic mappings $\varphi, \psi\colon X \multimap Y$ are homotopic then $\varphi_* = \psi_*$.*

Proof. Let $\chi\colon X \times [0,1] \multimap Y$ be joining homotopy such that $\chi(x, 0) = \varphi(x)$ and $\chi(x, 1) = \psi(x)$ for every $x \in X$. Consider the commutative diagram:



in which $i_0(x) = (x, 0)$, $i_1(x) = (x, 1)$, $j_0(x, y) = (x, 0, y)$, $j_1(x, y) = (x, 1, y)$. Since $(i_0)_* = (i_1)_*$ ($i_0 \sim i_1$!), by applying to the above diagram the functor H we obtain our assertion and the proof is completed. □

From (32.6) immediately follows:

(32.7) Corollary. *If two acyclic mappings φ, ψ are homotopic then*

(32.7.1) $\lambda(\varphi) = \lambda(\psi)$ *provided $H_*(X)$ is of a finite type,*
(32.7.2) $\Lambda(\varphi) = \Lambda(\psi)$ *provided φ_* or ψ_* is a Leray endomorphism.*

We shall prove the following theorem:

(32.9) Theorem (The Lefschetz Fixed Point Theorem for acyclic mappings). *If $X \in \mathrm{ANR}$ and $\varphi\colon X \multimap X$ is an acyclic compact map, i.e. $\operatorname{cl}\varphi(X)$ is a compact subset of X, then φ_* is a Leray endomorphism and $\Lambda(\varphi) \neq 0$ implies that*

$$\mathrm{Fix}(\varphi) \neq \emptyset.$$

Proof. We have the diagram:

$$X \overset{p_\varphi}{\Longleftarrow} \Gamma_\varphi \overset{q}{\longrightarrow} X$$

in which p_φ is a Vietoris map and q is compact because φ is compact. By definition, $\varphi_* = (q_\varphi)_* \circ (p_\varphi)_*^{-1}$, so from (12.11) it follows that φ_* is a Leray endomorphism.

Assume that $\Lambda(\varphi) = \Lambda(\varphi_*) = \Lambda((q_\varphi)_* \circ (p_\varphi)_*^{-1}) \neq 0$. Then, in view of (12.11), there is a point $(x, y) \in \Gamma_\varphi$ such that $p(x, y) = q(x, y)$. It implies that $x = p(x, y) = q(x, y) = y$. So, $(x, x) \in \Gamma_\varphi$ and therefore $x \in \varphi(x)$. The proof is completed. □

Since any $X \in \mathrm{AR}$ is acyclic we obtain:

(32.10) COROLLARY. *If $X \in \mathrm{AR}$ and $\varphi\colon X \multimap X$ is an acyclic compact map, then $\mathrm{Fix}(\varphi) \neq \emptyset$.*

In fact, from (2.12) we know that $\Lambda(\varphi) = 1$. Note that Theorem (32.9) for compact ANRs was proved in 1946 by S. Eilenberg and D. Montgomery ([EM]).

Theorem (32.9) can be proved also for compact AANR spaces.

(32.11) THEOREM. *Let $X \in \mathrm{AANR}$ and $\varphi\colon X \multimap X$ be an acyclic map. Then X is a space of a finite type and $\lambda(\varphi) \neq 0$ implies that $\mathrm{Fix}(\varphi) \neq \emptyset$.*

PROOF. Let $X \in \mathrm{AANR}$. By using Theorem (3.4) we obtain a compact ANR-space Y such that X is an approximative retract of Y. Therefore, for every $\varepsilon > 0$ we have a ε-retraction $r_\varepsilon\colon Y \to X$.

Let $i\colon X \to Y$, $i(x) = x$, be the inclusion map. Of course, in view of (2.10) we infer that Y is of a finite type.

We have two maps $i \circ r_\varepsilon$, $\mathrm{id}_Y\colon Y \to Y$ which are ε-close. So, from (6.1) we obtain that

$$(\mathrm{id}_Y)_* = i_* \circ (r_\varepsilon)_*$$

and hence $i_*\colon H_*(X) \to H_*(Y)$ is an epimorphism. But Y is of a finite type so, X must also be of a finite type and the proof of the first part of our theorem is completed.

Now assume that $\lambda(\varphi) \neq 0$. Then by taking $Y \in \mathrm{ANR}$ as above we have the diagram:

$$\begin{array}{ccc}
H_*(X) & \xrightarrow{\quad i_* \quad} & H_*(Y) \\
{\scriptstyle (q_\varphi)_*\circ(p_\varphi)_*^{-1}} \uparrow & \nwarrow {\scriptstyle (q_\varphi)_*\circ(p_\varphi)_*^{-1}\circ(r_\varepsilon)_*} & \uparrow {\scriptstyle i_*\circ(q_\varphi)_*\circ(p_\varphi)_*^{-1}\circ(r_\varepsilon)_*} \\
H_*(X) & \xrightarrow[\quad i_* \quad]{} & H_*(Y)
\end{array}$$

By applying (6.1) again, we infer that for sufficiently small ε the above diagram is commutative. Then:

$$\Lambda(\varphi) = \Lambda(\varphi_\varepsilon) \neq 0,$$

where $\varphi_\varepsilon = i \circ \varphi \circ r_\varepsilon\colon Y \multimap Y$ is an acyclic map (cf. (10.1)). Therefore, from (32.9) for every $\varepsilon > 0$ (sufficiently small) $\mathrm{Fix}(\varphi_\varepsilon) \neq \emptyset$.

So we can construct a sequence $\{y_n\}$ of points in Y such that $y_n \in \varphi_{\varepsilon_n}(y_n)$ and $\{\varepsilon_n\} \downarrow 0$. Let $x_n = r_{\varepsilon_n}(y_n)$. We can assume, without loss of generality, that

$\lim_n y_n = y$ and $\lim_n x_n = x$. Then we have $y_n \in \varphi(r_{\varepsilon_n}(y_n)) = \varphi(x_n)$. Since the graph Γ_φ of φ is closed we obtain $y \in \varphi(x)$.

But $\varphi_{\varepsilon_n}(Y) \subset X$ so $y_n \in X$ for every n and therefore $d(y_n, r_{\varepsilon_n}(x_n)) < \varepsilon_n$. Hence $y = \lim_n y_n = \lim_n r_\varepsilon(x_n) = x$. Finally, we obtain $x \in \varphi(x)$ and the proof is completed. □

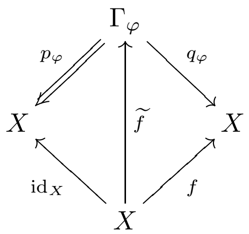
(32.12) Corollary. *If X is an acyclic* AANR*, then any acyclic map $\varphi\colon X \multimap X$ has a fixed point.*

It is interesting to establish connections between acyclic maps and its continuous selections. Namely, we shall prove:

(32.13) Proposition. *Assume that $\varphi\colon X \multimap X$ is an acyclic map and $f\colon X \to X$ is its selection, i.e. $f(x) \in \varphi(x)$ for every $x \in X$. Then:*

(32.13.1) $f_* = \varphi_*$,
(32.13.2) *f_* is a Leray endomorphism if and only if φ_* is a Leray endomorphism,*
(32.13.3) *$\Lambda(f_*) = \Lambda(\varphi_*)$ provided φ_* or f_* is a Leray endomorphism.*

Proof. It is evident that (32.13.2) follows from (32.13.1) and that (32.13.3) follows from (32.13.2). So, for the proof it is sufficient to prove (32.13.1). Consider the following commutative diagram:



in which $\widetilde{f}(x) = (x, f(x))$. By applying to the above diagram the homology functor H, we obtain: $\varphi_* = q_{\varphi_*} \circ p_{\varphi_*}^{-1} = f_* \circ (\mathrm{id}_X)_*^{-1} = f_* \circ \mathrm{id}_{X^*} = f_*$ and the proof is completed. □

(32.14) Remark. Observe that (32.13.1) remains true for an arbitrary acyclic maps $\varphi, \psi\colon X \multimap Y$ such that $\psi \subset \varphi$ i.e. $\psi(x) \subset \varphi(x)$ for every $x \in X$.

33. Strongly acyclic maps

Recall that the notion of strongly acyclic sets was introduced in the last part of Section 9.

(33.1) Definition. An u.s.c. multivalued map $\varphi\colon X \multimap S^n$ is called *strongly acyclic* (shortly $\varphi \in \mathrm{SAC}(X)$) if $\varphi(x)$ is a strongly acyclic subset of S^n.

From (9.10) we obtain:

$$\text{(33.2)} \qquad \mathrm{SAC}(X) \subset \mathrm{AC}(X, S^n).$$

Observe, that if the map $\varphi: X \multimap S^n$ is strongly acyclic, then the map $\widetilde{\varphi}: X \multimap S^n$, $\widetilde{\varphi}(x) = S^n \setminus \varphi(x)$ has open graph and values to be infinitely connected.

In what follows, a continuous selection $f: X \to S^n$ of $\widetilde{\varphi}$ is called co-selection for φ. We have:

(33.3) PROPOSITION. *Let X be a finite dimensional space and let $\varphi: X \multimap S^n$ be a strongly acyclic map. If $f: X \to S^n$ is a co-selection of φ then there exists a strongly acyclic homotopy $\chi: X \times [0,1] \multimap S^n$ joining φ with $\alpha \circ f$ such that $f(x) \notin \chi(x,t)$ for every $(x,t) \in X \times [0,1]$ and $\chi(x,t)$ is homeomorphic to $\varphi(x)$ for every $x \in X$ and $t \neq 1$, where $\alpha: S^n \to S^n$, $\alpha(x) = -x$, is the antipodal map.*

PROOF. Fix a point $p \in S^n$. Let $e_p: S^n \setminus \{p\} \to \mathbb{R}^n$ denote the stereographic projection of $S^n \setminus \{p\}$ onto $\mathbb{R}^n$. We consider $h_p: (S^n \setminus \{p\}) \times [0,1] \to S^n$ defined as follows:

$$h_p(u,t) = e_p^{-1}(t \cdot e_p(-p) + (1-t)e_p(u))$$

for every $u \in S^n \setminus \{p\}$ and $t \in [0,1]$.

Then the homotopy $\chi: X \times [0,1] \multimap S^n$ given by:

$$\chi(x,t) = \{h_{f(x)}(u,t) \mid u \in \varphi(x)\}$$

satisfies (33.3) and the proof is completed. □

Now we define a fixed point index for strongly acyclic maps. By $\mathcal{K}_{\mathrm{SA}}$ we denote the class of all triples (S^n, U, φ), where U is an open subset of S^n and $\varphi \in \mathrm{SAC}(U)$ is such that $\mathrm{Fix}(\varphi)$ is compact; $n = 1, 2, \ldots$

If (S^n, U, φ_1) and (S^n, U, φ_2) are two triples in $\mathcal{K}_{\mathrm{SA}}$, then by homotopy joining φ_1 and φ_2 we shall understand a strongly acyclic map $\chi: U \times [0,1] \multimap S^n$ such that $\chi(\cdot\,,0) = \varphi_1$, $\chi(\cdot\,,1) = \varphi_2$ and the set $\bigcup_{t\in[0,1]} \mathrm{Fix}(\chi(\cdot\,,t))$ is compact. Such a homotopy χ is called an SAC-homotopy joining φ_1 and φ_2.

The following fact allows us to define a fixed point index for strongly acyclic mappings.

(33.4) PROPOSITION. *Let $(S^n, U, \varphi) \in \mathcal{K}_{\mathrm{SA}}$. Then:*

(33.4.1) *there is a* SAC-*homotopy $\chi: U \times [0,1] \multimap S^n$ such that $\chi(\cdot\,,0) = \varphi$ and $\chi(\cdot\,,1)$ is singlevalued,*

(33.4.2) *if $f, g: U \to S^n$ are singlevalued and* SAC-*homotopic to φ, then there is a singlevalued homotopy $h: U \times [0,1] \to S^n$ joining f and g such that h is an* SAC-*homotopy, i.e. the set $\bigcup_{t\in[0,1]} \mathrm{Fix}(h(\cdot\,,t))$ is compact.*

PROOF. Since $\mathrm{Fix}(\varphi)$ is compact, there is an open set V such that $\mathrm{Fix}(\varphi) \subset V \subset \mathrm{cl}\, V \subset U$. Therefore, the inclusion map $i: U \setminus V \to S^n$, $i(x) = x$, is a co-selection for φ on $U \setminus V$. By Theorem (16.3), we can extend the map i to a co-selection $g: U \to S^n$ for φ. By Proposition (33.3), there is an SAC-homotopy χ joining φ and $\alpha \circ g$ such that $g(x) \notin \chi(x,t)$ for every $t \in [0,1]$ and $x \in U$, therefore, $\bigcup_{t\in[0,1]} \mathrm{Fix}(\chi(\,\cdot\,,t)) \subset V \subset \overline{V}$ and hence it is compact. Thus we have proved (33.4.1).

Now, let χ_1, χ_2 be two homotopies joining φ with f and φ with g respectively. Define an SAC-homotopy χ joining f and g by putting:

$$\chi(x,t) = \begin{cases} \chi_1(x, 1-2t) & \text{for } t \le 1/2, \\ \chi_2(x, 2t-1) & \text{for } t \ge 1/2. \end{cases}$$

By the assumption, there is an open set $V \subset \mathrm{cl}\, V \subset U$ such that

$$\bigcup_{t\in[0,1]} \mathrm{Fix}(\chi(\,\cdot\,,t)) \subset V.$$

Therefore, the projection $p_0: (U \setminus V) \times [0,1] \to S^n$, $p(x,t) = x$ is a co-selection for the restriction $\chi\,|_{(U\setminus V)\times[0,1]}$ of χ to $(U \setminus V) \times [0,1]$. Again, by (16.3) we can extend p_0 to a co-selection $p: U \times [0,1] \to S^n$ for χ. Let $\psi: U \times [0,1] \times [0,1] \multimap S^n$ denote an SAC-homotopy joining χ and $\alpha \circ p$ with properties such as in (33.3). Let $q = \psi(\,\cdot\,, 0, \,\cdot\,)$ and $r = \psi(\,\cdot\,, 1, \,\cdot\,)$ Since $\psi(x,t,s)$ is homeomorphic to $\chi(x,t)$ for $s \neq 1$ and $\chi(\,\cdot\,,0) = f$, $\chi(\,\cdot\,,1) = g$ we conclude that q and r are singlevalued. Define $h: U \times [0,1] \to S^n$ by

$$h(x,t) = \begin{cases} q(x,3t) & \text{for } t \le 1/3, \\ (\alpha \circ p)(x, 3t-1) & \text{for } 1/3 \le t \le 2/3, \\ r(x, 3-3t) & \text{for } t \ge 2/3. \end{cases}$$

Then h is continuous and

$$\bigcup_{t\in[0,1]} \mathrm{Fix}(h(\,\cdot\,,t)) \subset V \subset \mathrm{cl}\, V \subset U$$

hence $\bigcup_{t\in[0,1]} \mathrm{Fix}(h(\,\cdot\,,t))$ is compact. The proof of (33.4.2) is completed. □

(33.5) DEFINITION. The *fixed point index* $J_{\mathrm{SA}}(S^n, U, \varphi)$ of a triple $(S^n, U, \varphi) \in \mathcal{K}_{\mathrm{SA}}$ is equal to the usual fixed point index $i(f)$ of a singlevalued map $f: U \to S^n$, which is SAC-homotopic to φ (for the definition of $i(f)$ see, for example, [Do-M]).

Observe that from (16.3) and (33.4) it follows that such a map f exists and that $J_{\mathrm{SA}}(S^n, U, \varphi)$ does not depend on the choice of f. So Definition (33.5) is correct.

(33.6) REMARK. Since we consider homology with rational coefficients Q, the index $J_{\mathrm{SA}}(S^n, U, \varphi)$ of the triple (S^n, U, φ) is formally a rational number.

In the following theorem, we list some properties of the index J_{SA}.

(33.7) THEOREM. *The fixed point index $J_{\mathrm{SA}}:\mathcal{K}_{\mathrm{SA}} \to Q$ has the following properties:*

(33.7.1) (Normalization) *If* $(S^n, S^n, \varphi) \in \mathcal{K}_{\mathrm{SA}}$ *then* $J_{\mathrm{SA}}(S^n, S^n, \varphi) = \lambda(\varphi)$,

(33.7.2) (Fixed Points) *If* $J_{\mathrm{SA}}(S^n, U, \varphi) \neq 0$ *then* $\mathrm{Fix}(\varphi) \neq \emptyset$,

(33.7.3) (Homotopy) *If* $\chi: U \times [0,1] \multimap S^n$ *is an* SA*-homotopy, then*

$$J_{\mathrm{SA}}(S^n, U, \chi(\,\cdot\,,0)) = J_{\mathrm{SA}}(S^n, U, \chi(\,\cdot\,,1)),$$

(33.7.4) (Additivity) *If* $(S^n, U, \varphi) \in \mathcal{K}_{\mathrm{SA}}$ *and* $\mathrm{Fix}(\varphi) \subset \bigcup_{i=1}^k U_i$, *where* U_i *are open disjoint subsets of* U *then* $(S^n, U_i, \varphi_i) \in \mathcal{K}_{\mathrm{SA}}$ *and*

$$J_{\mathrm{SA}}(S^n, U, \varphi) = \sum_{i=1}^{k} J_{\mathrm{SA}}(S^n, U_i, \varphi_i),$$

where φ_i *is the restriction of* φ *to* U_i.

(33.7.5) (Contraction and Topological Invariance) *If* $(S^n, U, \varphi) \in \mathcal{K}_{\mathrm{SA}}$ *and* $i: S^m \to S^n$ *is a homeomorphic embedding such that* $\varphi(u) \subset i(S^m)$ *and* $i^{-1} \circ \varphi: U \multimap S^m$ *is a strongly acyclic map then*

$$J_{\mathrm{SA}}(S^n, U, \varphi) = J_{\mathrm{SA}}(S^m, i^{-1}(U), i^{-1} \circ \varphi \circ i).$$

The proof of (33.7) is quite easy and follows from the respective properties of the usual fixed point index for singlevalued mappings (cf. [Br1-M], [Do-M], [Sp-M] for example) and from Definition (33.5). Concerning (33.7.2) observe that if $\mathrm{Fix}(\varphi) = \emptyset$ then the inclusion $i: U \to S^n$, $i(x) = x$, is a co-selection of φ. We recommend to the reader to deduce Theorem (33.7) from (33.5) and the respective theorem for fixed point index of singlevalued mappings.

Finally, let us add another simple theorem on the uniqueness of the fixed point index for strongly acyclic mappings.

(33.8) PROPOSITION. *Let* $\mathrm{ind}: \mathcal{K}_{\mathrm{SA}} \to Q$ *be a function satisfying the axiom of homotopy* (33.7.3) *and let the restriction of* ind *to singlevalued maps be equal to the usual fixed point index. Then* $\mathrm{ind} = J_{\mathrm{SA}}$.

PROOF. Let $(S^n, U, \varphi) \in \mathcal{K}_{\mathrm{SA}}$. By (33.4.1), φ is admissibly homotopic to a singlevalued map $f: U \to S^n$. Therefore, $\mathrm{ind}\,(S^n, U, \varphi) = i(S^n, U, f) = J_{\mathrm{SA}}(S^n, U, \varphi)$ and the proof is completed. □

34. The fixed point index for acyclic maps of Euclidean neighbourhood retracts

In this section the fixed point index defined in Section 33 for strongly acyclic mappings will be taken up on the class of all acyclic mappings. First, we shall explain precisely connections between the classes $\mathrm{SAC}(X)$ and $\mathrm{AC}(X, S^n)$.

(34.1) PROPOSITION. *$\varphi \in \mathrm{SAC}(X)$ if and only if $\varphi \in \mathrm{AC}(X, S^n)$ and the set $S^n \setminus \varphi(x)$ is simply connected, i.e. the first homotopy group $\pi_1(S^n \setminus \varphi(x))$ of $S^n \setminus \varphi(x)$ is trivial for every $x \in X$.*

PROOF. The assertion that $\varphi \in \mathrm{SAC}(X)$ implies that $\varphi \in \mathrm{AC}(X, S^n)$ and $\pi_1(S^n \setminus \varphi(x)) = 0$ for every $x \in X$ follows automatically from the definition.

Now let $x \in X$ and $\varphi(x)$ be acyclic such that $S^n \setminus \varphi(x)$ is simply connected. Then using the exactness of the cohomology sequence of the pair $(S^n, \varphi(x))$ and the Lefschetz Duality Theorem (see Section 9), we gather that

$$H_q(S^n \setminus \varphi(x)) = H^{n-q}(S^n, \varphi(x)) = \begin{cases} Q & \text{for } q = 0, \\ 0 & \text{for } q > 0. \end{cases}$$

This shows that $S^n \setminus \varphi(x)$ is path connected and, in particular $\pi_1(S^n \setminus \varphi(x), u) = 0$, $u \in S^n \setminus (x)$, for every $u \in S^n \setminus \varphi(x)$. The Hurewicz theorem asserts that $\pi_m(S^n \setminus \varphi(x), u) = 0$ for $m \geq 2$ and the proof is completed. $\square$

(34.2) REMARK. Observe that if $n = 1, 2$ then the notions of acyclicity and strong acyclicity coincide, i.e. $\mathrm{SAC}(X) = \mathrm{AC}(X, S^n)$, $n = 1, 2$. For $n = 1$ it is evident since arcs of a circumference are the only acyclic subsets of S^1. Now suppose that A is compact acyclic subset of S^2. We shall show that any loop $f\colon S^1 \to S^2 \setminus A$ is homotopic to a constant map. There is $\varepsilon > 0$ such that $f(S^1) \cap O_\varepsilon(A) = \emptyset$. As above, we see that $S^2 \setminus A$ is path connected and therefore connected. Then there is a topological disc $h\colon D^2 \to D$ such that $A \subset D \subset O_\varepsilon(A)$ (see [Bo-M, p. 132]).

Now by applying (9.12) there is a homeomorphism $\widetilde{h}\colon S^2 \to S^2$ such that $\widetilde{h}(x) = h(x)$ for every $x \in D^2$. Evidently, $\widetilde{h}^{-1}(f(S^1)) \subset S^2 \setminus D^2$, hence $f(S^1)$ is contractible to a point in $S^2 \setminus D$ and therefore, $\pi_1(S^2 \setminus A) = 0$.

As we mentioned in Section 9, the situation is getting complicated for $n = 3$ onwards. It is not difficult to construct a set $A \subset S^3$ (so called Antoine arc) homeomorphic to the unit interval, thus acyclic, such that the set $S^3 \setminus A$ is not simply connected.

(34.3) PROPOSITION. *Let $\varphi\colon X \multimap \mathbb{R}^n$ be a multivalued map and let $i_n\colon \mathbb{R}^n \to S^n$, $i_n(x) = x$. Then the map $i_n \circ \varphi\colon X \to S^n$ is strongly acyclic if and only if $\varphi \in \mathrm{AC}(X, \mathbb{R}^n)$ and the set $\mathbb{R}^n \setminus \varphi(x)$ is simply connected.*

PROOF. Observe that in view of (34.2) the situation is clear if $n = 1, 2$. Moreover, the sufficiency of the second condition is self evident. Necessity: Suppose that $i_n(\varphi(x))$ is strongly acyclic; hence $\varphi(x)$ is acyclic. Let $U_1(x) = S^n \setminus (i_n(\varphi(x) \cup \{\infty\})$. The set $U_1(x)$ is homeomorphic with $\mathbb{R}^n \setminus \varphi(x)$. The Alexander Duality Theorem shows that $U_1(x)$ is path connected. Let U_2 be any sufficiently small, simply connected neighbourhood of ∞ in S^n. The sets $U_1(x)$, U_2 and $U_1(x) \cap U_2 = U_2 \setminus \{\infty\}$ are path connected and $U_1(x) \cup U_2 = S^n \setminus i_n(\varphi(x))$. By the Van Kampen Theorem, the homeomorphism $\tau\colon \pi_1(U_1(x)) \to \pi_1(U_1(x) \cup U_2)$ induced by the embedding $U_1(x) \to U_1(x) \cup U_2$ is an epimorphism with the kernel being the minimal normal subgroup of $\pi_1(U_1(x))$ containing $\eta(\pi_1(U_1(x) \cup U_2))$, where η is induced by the embedding $U_1(x) \cap U_2 \to U_1$. Since $n \geq 3$ one can take U_2 such that $U_1(x) \cap U_2$ is simply connected. Hence $\pi_1(U_1(x)) = 0$ and the proof is completed. □

In the same spirit we have the following proposition.

(34.4) PROPOSITION. *Let $\varphi\colon X \multimap S^n$ be an acyclic map and let $j_n\colon S^n \to S^{n+1}$ be the embedding $j_n(u_1, \ldots, u_{n+1}) = (u_1, \ldots, u_{n+1}, 0)$. Then the map $j_n \circ \varphi\colon X \multimap S^{n+1}$ is strongly acyclic.*

PROOF. Let $x \in X$ an $B(x) = j_n(\varphi(x))$. Denote by e_+ (resp. e_-) the north (resp. the south) pole of S^{n+1}. For $u \in S^{n+1}$, $u \neq e_\pm$ denote by $r(u)$ the common point of the great circle joining e_+ with e_- passing true u and $j_n(S^n)$. By $[u, r(u)]$ we denote the arc of the great circle joining u with $r(u)$. Let $U_1(x) = \{u \in S^{n+1} \mid u \neq e_\pm,\ r(u) \notin B(x)\}$. The set $U_1(x)$ is open and path-connected. Indeed, if $a_1, a_2 \in U_1(x)$ and $b_i = j_n^{-1}(a_i)$, $i = 1, 2$, then in view of the Lefschetz Duality Theorem, $S^n \setminus \varphi(x)$ is path-connected, there is a path l in $S^n \setminus \varphi(x)$ joining b_1 with b_2. Hence, the arc $[a_1, r(a_1)] \cup l \cup [a_2, r(a_2)]$ joins a_1 with a_2 in $U_1(x)$. The acyclicity of $\varphi(x)$ implies that $A \neq S^n$ i.e. there is a point $y \in S^n \setminus \varphi(x)$ and $\varepsilon > 0$ such that $O_\varepsilon(y) \cap \varphi(x) = \emptyset$.

Let $U_2 = \{u \in S^{n+1} \mid u \in S^{n+1} \setminus j_n(S^n) \text{ or } \|u - j_n(y)\| < \varepsilon\}$. The set U_2 is open contractible, hence simply connected. Moreover, $U_1(x) \cap U_2$ is path connected and nonempty. If $\tau_1\colon \pi_1(U_1(x)) \to \pi_1(U_1(x) \cup U_2)$ and $\tau_2\colon \pi_1(U_2) \to \pi_1(U_1(x) \cup U_2)$ are homomorphisms induced by the embeddings $U_1(x) \to U_1(x) \cup U_2$, $U_2(x) \to U_1(x) \cup U_2$, respectively, then by the Van Kampen theorem $\pi_1(S^{n+1} \setminus j_n(\varphi(x))) = \pi_1(U_1(x) \cup U_2)$ is generated by $\tau_1(\pi_1(U_1(x)))$ and $\tau_2(\pi_1(U_2)) = 0$. To see that $\pi_1(U_1(x)) = 0$ it is enough to show that any loop $f\colon S^1 \to U_1(x)$ may be homotopically deformed in $S^{n+1} \setminus j_n(\varphi(x))$ to a constant one. Let $e\colon S^{n+1} \setminus \{e_+\} \to \mathbb{R}^{n+1}$ be a stereographic projection such that $e \circ j_n(S^n) = S^n$ and $e(e_-) = 0$. It is easy to see that $e(\varphi_1(x)) = \{\lambda z \mid \lambda > 0,\ z \in e(j_n(\varphi(x)))\}$. Let $K = e(f(S^1)) \subset e(U_1(x))$. Since K is compact, then there is $\alpha > 0$ such that $\alpha \cdot K \subset B^{n+1}(r)$, where $r < 1$. Therefore, K may be deformed to a point in $B^{n+1}(r) \cup e(U_1(x)) \subset \mathbb{R}^{n+1} \setminus e(j_n(\varphi(x)))$. At last, $j_n(\varphi(x))$ being acyclic is strongly acyclic in view of Proposition (34.1) and the proof is completed. □

Now, we are ready to construct the fixed point index for acyclic mappings. The following stand the base of our definition of the fixed point index for acyclic mappings. We let:

$$\mathcal{K} = \{(X, U, \varphi) \mid X \in \text{ANR and } X \text{ is locally compact finite dimensional,}\\ \varphi\colon U \multimap X \text{ is acyclic with } \mathrm{Fix}(\varphi) \text{ compact and } U \text{ is open in } X\}.$$

By homotopy in $\mathcal{K}$ we shall understand an acyclic map $\chi\colon U \times [0,1] \multimap X$ such that the set $\bigcup_{t\in[0,1]} \mathrm{Fix}(\chi(\,\cdot\,,t))$ is compact.

(34.5) THEOREM. *Let $(X, U, \varphi) \in \mathcal{K}$ and assume that X is compact. Then there exist a homeomorphic embedding $i\colon X \to S^n$, an open subset V of S^n and a continuous map $r\colon V \to U$ such that $(r \circ i)|_U = \mathrm{id}_U$, $i^{-1}(V) = U$ and the multivalued map $i \circ \varphi \circ r\colon V \multimap S^n$ is strongly acyclic with $(S^n, V, i \circ \varphi \circ r) \in \mathcal{K}_{\mathrm{SA}}$. Moreover, the fixed point index $J_{\mathrm{SA}}(i \circ \varphi \circ r)$ does not depend on the choice of i, V, or r.*

PROOF. Since X is a compact metric and finite dimensional space by using the Menger–Nöbeling embedding theorem (see [DG-M]) there exists $m > 0$ and a homeomorphic embedding $j\colon X \to \mathbb{R}^m$.

We consider natural embedding $k\colon \mathbb{R}^m \to \mathbb{R}^m \times \mathbb{R}^2$, $k(x) = (x, 0)$ and $k'\colon \mathbb{R}^{m+2} \to S^{m+2} = \mathbb{R}^{m+2} \cup \{\infty\}$, $k'(u) = u$. We let $n = m + 2$ and $i = k' \circ k \circ j$. Then $i\colon X \to S^n$ is the needed embedding. In view of (34.4) the map $i \circ \varphi\colon U \multimap S^{n+2}$ is strongly acyclic.

Now let $i(U) = i(X) \cap W$ for some open subset W of S^n. Then $i(U)$ is closed in W and since $U \in$ ANR there is an open subset V of W and a retraction $p\colon V \to i(U)$. Put $r = i^{-1} \circ p$. Then $(r \circ i)|_U = \mathrm{id}_U$ and, of course, $i \circ \varphi \circ r$ is strongly acyclic.

Now assume that we have the second embedding $i'\colon X \to S^{n'}$, open set V' in $S^{n'}$ and the map $r'\colon V' \to U$ such that $(r' \circ i')|_U = \mathrm{id}_U$, $(i')^{-1}(V') = U$ and $i' \circ \varphi \circ r'\colon V' \multimap S^{n'}$ is strongly acyclic.

Observe that the sets $\mathrm{Fix}(i \circ \varphi \circ r)$ and $\mathrm{Fix}(i' \circ \varphi \circ r')$ are homeomorphic to $\mathrm{Fix}(\varphi)$ so $(S^n, V, i \circ \varphi \circ r) \in \mathcal{K}_{\mathrm{SA}}$ and $(S^{n'}, V', i' \circ \varphi \circ r') \in \mathcal{K}_{\mathrm{SA}}$. We can assume, without loss of generality, that $n \le n'$ and consequently there is a homeomorphic embedding $l\colon S^n \to S^{n'}$. Let $k = \max\{6, n'+3, 2\dim X+2\}$ and consider a natural embedding $j_0\colon S^{n'} \to \mathbb{R}^k$. Let $j = j_0 \circ l \circ i$ and $j' = j_0 \circ i'$. From (9.10) and (9.12) it follows that there is a homeomorphism $f\colon \mathbb{R}^n \to \mathbb{R}^k$ such that $f|_{j'(X)} = j \circ (j')^{-1}$. Let to $e\colon \mathbb{R}^k \to S^k = \mathbb{R}^k \cup \{\infty\}$ be the natural embedding and let $e = e \circ j$, $e' = e \circ j'$. Putting $f(\infty) = \infty$ we get a homeomorphism $F\colon S^k \to S^k$ such that $F|_{e'(X)} = e \circ (e')^{-1}$. There are two open subsets W, W' of S^k and two continuous maps $p\colon W \to V$ and $p'\colon W' \to V'$ such that $e^{-1}(W) = {e'}^{-1}(W')$, $p \circ e_0 \circ j_0 \circ l|_V = \mathrm{id}_V$, $p' \circ e_0 \circ j_0|_{V'} = \mathrm{id}_{V'}$. From (33.7.5) (the proof of the previous part of this theorem shows that $e \circ \varphi: U \multimap S^k$ and $e' \circ \varphi: U \multimap S^k$ are strongly acyclic maps) we obtain:

(34.5.1)
$$J_{\mathrm{SA}}(S^n, V, i \circ \varphi \circ r) = J_{\mathrm{SA}}(S^k, W, e \circ \varphi \circ r \circ p),$$
$$J_{\mathrm{SA}}(S^m, V', i' \circ \varphi \circ r') = J_{\mathrm{SA}}(S^k, W', e' \circ \varphi \circ r' \circ p').$$

Moreover, from (33.7.5) we can also obtain:

(34.5.2)
$$\begin{aligned} J_{\mathrm{SA}}(S^k, W', e' \circ \varphi \circ r' \circ p') &= J_{\mathrm{SA}}(S^k, F(W'), F \circ e' \circ \varphi \circ r' \circ p' \circ F^{-1}) \\ &= J_{\mathrm{SA}}(S^k, F(W'), e \circ \varphi \circ r' \circ p' \circ F^{-1}). \end{aligned}$$

Let $M = W \cap F(W')$, $P = r \circ p|_M$, $P' = r' \circ p' \circ F^{-1}|_M$. Then $P \circ e|_U = \mathrm{id}_U$ and $p' \circ e|_U = \mathrm{id}_U$. Consider S^k as a subset of $\mathbb{R}^{k+1}$. For every $x \in U$ there is a $\delta > 0$ such that $2/\|z\| \in M$ for every $z \in \kappa(e(x), \delta) = \{z \in \mathbb{R}^{k+1} \mid \|z - e(x)\| < \delta\}$. Let $M' = \bigcup\{Z_x \mid x \in U\}$. Then M' is an open subset of M containing $e(U)$.

Therefore, we obtain from (33.7.4):

(34.5.3)
$$J_{\mathrm{SA}}(S^k, W, e \circ \varphi \circ r \circ p) = J_{\mathrm{SA}}(S^k, M', e \circ \varphi \circ P|_{M'}),$$
$$J_{\mathrm{SA}}(S^k, F(W'), e \circ \varphi \circ r' \circ p' \circ F^{-1}) = J_{\mathrm{SA}}(S^k, M', e \circ \varphi \circ P'|_{M'}).$$

Now observe that the multivalued map $\chi: M' \times [0,1] \multimap S^k$ given by:

$$\chi(z,t) = e \circ \varphi \circ P(te \circ P(z) + (1-t)e \circ P'(z))\|te \circ P(z) + (1-t)e' \circ P'(z)\|^{-1}$$

is a strongly acyclic homotopy joining $e \circ \varphi \circ P$ and $e \circ \varphi \circ P'$ such that $\bigcup\{\mathrm{Fix}\chi(\,\cdot\,, t) \mid t \in [0,1]\}$ is homeomorphic to $\mathrm{Fix}(\varphi)$. Therefore, we obtain from (33.5.3):

(34.5.4)
$$J_{\mathrm{SA}}(S^k, M', e \circ \varphi \circ P|_{M'}) = J_{\mathrm{SA}}(S^k, M', e \circ \varphi \circ P'|_{M'}).$$

Combining equalities (33.5.1)–(33.5.4) we obtain the desired result and the proof is completed. □

Theorem (34.4) allows us to define the fixed point index in the compact case, but we would like to do it for Euclidean neighbourhood retracts.

We need the following fact, whose easy proof follows directly from (33.5) and, therefore, is left to the reader.

(34.6) PROPOSITION. *Let $(X, U, \varphi) \in \mathcal{K}$. Then there exists a compact subset X' of X and an open subset U' of U such that $\mathrm{Fix}(\varphi) \subset U' \subset X'$ and $\varphi(U') \subset X'$. Moreover, if $i: X' \to S^n$ is an embedding, V is an open subset of S^n, $r: V \to U'$ is such that $r \circ i|_{U'} = \mathrm{id}_{U'}$, $i^{-1}(V) = U'$ and $i \circ \varphi \circ r$ is strongly acyclic then $(S^n, V, i \circ \varphi \circ r) \in \mathcal{K}_{\mathrm{SA}}$ and the index $J_{\mathrm{SA}}(S^n, V, i \circ \varphi \circ r)$ does not depend on the choice of X', U', i, V, r.*

Now, we are in a position to define the fixed point index on $\mathcal{K}$.

(34.7) DEFINITION. Let $(X, U, \varphi) \in \mathcal{K}$, and U' and X' be chosen as in (34.6). Let $i: X' \to S^n$, $V \subset S^n$ and $r: V \to U'$ be defined as in (34.5), i.e. i is a homeomorphic embedding, V is open is S^n, r is continuous, $r \circ i|_{U'} = \mathrm{id}_{U'}$, $i^{-1}(V) = U'$ and $i \circ \varphi \circ r$ is strongly acyclic. The fixed point index $I(X, U, \varphi)$ of (X, U, φ) is defined as follows:

$$I(X, U, \varphi) = J_{\mathrm{SA}}(S^n, V, i \circ \varphi \circ r).$$

Let us observe that by Proposition (34.6) the above definition is correct.

Now we prove that the fixed point index on locally compact finite-dimensional ANRs is uniquely determined by usual properties (cf. [Br1-M], [Do-M]).

(34.8) THEOREM. *Let $\mathcal{K}'$ denote the subclass of $\mathcal{K}$ consisted of all $(X, U, \varphi) \in \mathcal{K}$, for which X is a locally compact* ANR. *Let $J: \mathcal{K}' \to Q$ be a function satisfying the following properties:*

(34.8.1) *If $(X, U, \varphi) \in \mathcal{K}'$ and $\varphi(x) = x_0$ is a constant map, then*

$$J(X, U, \varphi) = \begin{cases} 1 & \text{provided } x_0 \in U, \\ 0 & \text{provided } x_0 \notin U. \end{cases}$$

(34.8.2) (Homotopy) *If $\chi: U \times [0,1]$ is a homotopy in $\mathcal{K}'$, then*

$$J(X, U; \chi(\,\cdot\,, 0)) = J(X, U; \chi(\,\cdot\,, 1)).$$

(34.8.3) (Additivity) *If $(X, U, \varphi) \in \mathcal{K}'$ and $\mathrm{Fix}(\varphi) \subset \bigcup\{U_i \mid i = 1, \ldots, k\}$, where U_i are open and disjoint subsets of U then:*

$$J(X, U, \varphi) = \sum_{i=1}^{k} J(X, U_i, \varphi_i),$$

where $\varphi_i = \varphi|_{U_i}$.

(34.8.4) (Topological Invariance) *If $(X, U, \varphi) \in \mathcal{K}'$ and $h: X \to Y$ is a homeomorphism, then*

$$J(X, U, \varphi) = J(Y, h(U), h \circ \varphi \circ h^{-1}).$$

(34.8.5) (Contraction) *If $(X, U, \varphi) \in \mathcal{K}'$, $\varphi(U) \subset Y$ and Y is a locally compact* ANR *then $J(X, U, \varphi) = J(Y, U \cap Y, \varphi|_{U \cap Y})$.*

Then $J = I$.

PROOF. Let $(X, U, \varphi) \in \mathcal{K}$. By Proposition (34.6) there are an open subset U' of U and a compact subset X' of X such that $\mathrm{Fix}(\varphi) \subset U' \subset X'$, $\varphi(U') \subset X'$. Since X is locally compact, then there is a compact subset Z of X such that

$X' \subset \operatorname{Int} Z$. By Theorem (34.5) there are an embedding $i: Z \to S^n$, an open subset V of S^n and a continuous mapping $r: V \to U'$ such that $i^{-1}(V) = U'$, $r \circ i|_{U'} = \mathrm{id}_{U'}$ and $i \circ \varphi \circ r$ is strongly acyclic. From (34.8.3)–(34.8.5) it follows that

$$J(X, U, \varphi) = J(\operatorname{Int} Z, U', \varphi|_{U'}) = J(S^n, V, i \circ \varphi \circ r).$$

Observe that properties (34.8.3)–(34.8.5) uniquely determine a fixed point index for singlevalued maps on polyhedra (see [Br1-M], when one can easily see that it is sufficient to use a contraction and a topological invariance instead of a commutativity). Therefore, by Proposition (33.8), $J|_{\mathcal{K}_{\mathrm{SA}}} = J_{\mathrm{SA}}$. Hence $J(S^n, V, i \circ \varphi \circ r) = J_{\mathrm{SA}}(S^n, V, i \circ \varphi \circ r)$. From (34.7) it follows that $I(X, U, \varphi) = J_{\mathrm{SA}}(S^n, V, i \circ \varphi \circ r)$, which completes the proof. □

(34.9) REMARK. In the last three sections we considered acyclic sets with respect to the Čech homology with coefficients in the field of rational numbers Q. But as we have already observed in the category C_1 of open subsets in Euclidean spaces (see (9.1)) the Čech and singular homology functor are equivalent.

Consequently in the case of strong acyclic maps we can consider Z-acyclicity with respect to the singular homology functor with integer coefficients Z. Hence, the fixed point index J_{SA} can be defined as a function

$$J_{\mathrm{SA}}: \mathcal{K}_{\mathrm{SA}} \to Z$$

(cf. (33.8)). Therefore, in view of (34.7) and (34.9) we can consider the fixed point index I on $\mathcal{K}$ as a function:

$$I: \mathcal{K} \to Z.$$

Now, we can formulate the following theorem:

(34.10) THEOREM. *The fixed point index $I: \mathcal{K} \to Z$ has the following properties:*

(34.10.1) (Normalization) *If $(X, X, \varphi) \in \mathcal{K}$ and φ is compact, i.e.* $\operatorname{cl}\varphi(X)$ *is a compact set, then: $I(X, X, \varphi) = \Lambda(\varphi)$.*

(34.10.2) (Fixed Points) *If $I(X, U, \varphi) \neq 0$ then* $\operatorname{Fix}(\varphi) \neq \emptyset$.

(34.10.3) (Homotopy) *If $\chi: U \times [0, 1] \multimap X$ is a homotopy in $\mathcal{K}$ then:*

$$I(X, U, \chi(\,\cdot\,, 0)) = I(X, U, \chi(\,\cdot\,, 1)).$$

(34.10.4) (Additivity) *If $(X, U, \varphi) \in \mathcal{K}$ and* $\operatorname{Fix}(\varphi) \subset \bigcup\{U_i \mid i = 1, \dots, k\}$, *where U_i are open disjoint subsets of U then:*

$$I(X, U, \varphi) = \sum_{i=1}^{k} I(X, U_i, \varphi_i), \quad \textit{where } \varphi_i = \varphi|_{U_i}.$$

(34.10.5) (Commutativity) *Let* $\varphi_i\colon U_i \multimap X$ *be two* Z*-acyclic maps, where* U_i *is an open subset of a finite-dimensional space* X_i, $i,j = 1,2$ *and* $i \neq j$. *Assume that* I_i *is a locally compact* ANR *and* $\varphi_j \circ \varphi_i|_{\varphi_i^{-1}(U_j)}$ *is* Z*-acyclic,* $i,j = 1,2$, $i \neq j$. *Assume also that:*

$$y \in \varphi_1(x) \text{ and } x \in \varphi_2(y) \text{ and } (x \in \varphi_1^{-1}(U_2) \text{ or } y \in \varphi_2^{-1}(U_1)) \\ \Rightarrow x \in \varphi_1^{-1}(U_2) \text{ and } y \in \varphi_2^{-1}(U_1).$$

Then $\mathrm{Fix}(\varphi_1 \circ \varphi_2|_{\varphi_2^{-1}(U_2)})$ *is compact if and only if* $\mathrm{Fix}(\varphi_2 \circ \varphi_1|_{\varphi_2^{-1}(U_2)})$ *is compact, and if so, then:*

$$I(X_1, \varphi_1^{-1}(U_2), \varphi_2 \circ \varphi_1) = I(X_2, \varphi_2^{-1}(U_1), \varphi_1 \circ \varphi_2).$$

(34.10.6) (Mod-p) *Let* $\varphi\colon U \multimap X$ *be* Z*-acyclic, where* U *is an open subset of a finite-dimensional space* X. *Let* V *be an open subset of* $\varphi^{-p+1}(U)$, *where* p *is a prime number. Assume that* U *is a locally compact* ANR, $(\varphi|_V)^p$ *is acyclic and: if* $y \in \varphi^i(x)$ *and* $x \in \varphi^{p-i}(y)$, $0 < i < p$ *and* $x \in V$ *then* $y \in V$. *When* $\mathrm{Fix}((\varphi|_V)^p)$ *is compact, then so is* $\mathrm{Fix}\,\varphi|_V$ *and*

$$I(X, V, \varphi|_V) \equiv I(X, V, (\varphi|_V)^p) \mod p.$$

The proof of (34.10) is straightforward, in view of (33.7) and (34.8), (34.7). We also recommend [Bi-2], [FV], [SeS].

Above we have defined the fixed point index for acyclic (resp. Z-acyclic) mappings on Euclidean Neighbourhood Retracts. The definition presented here is elementary if we compare with [SeS] or [Cal1]. The question to define the fixed point index for arbitrary ANRs remains open. We will come back to this question later.

35. The Nielsen number

The aim of this section is to present the basic notions of the Nielsen fixed point theory (cf. [Br1-M]) for a class of acyclic maps of Euclidean neighbourhood retracts (cf. previous section).

First we shall define the Reidemeister relation.

(35.1) DEFINITION. Let $A \subset X$. We will say that A has a property $(*)$ in X if and only if it is nonempty connected and there exists an open neighbourhood U of A in X such that each loop in U is homotopic (with fixed ends) in X to the constant loop.

Observe that we can demand U to be path-connected provided X is locally path-connected.

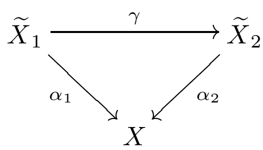
Recall that the space $\widetilde{X}$ is called a *covering* of a locally path connected space X provided there exists a continuous map $\alpha\colon \widetilde{X} \to X$ such that for every $x \in X$ there exists an open neighbourhood U_x of x in X such that

$$\alpha^{-1}(U_x) = \bigcup_{i=1} \{V_t \mid t \in T\},$$

where V_t is open in $\widetilde{X}$, $V_t \cap V_s = \emptyset$, $t \neq s$, and $\alpha|_{V_t}\colon V_t \to U_x$ is a homeomorphism for every $i = 1, \ldots, k$.

In what follows we will consider a covering of X as a pair $(\widetilde{X}, \alpha)$, where $\widetilde{X}$ and α are defined above.

Let $(\widetilde{X}_1, \alpha_1)$ and $(\widetilde{X}_2, \alpha_2)$ be two covering spaces for X. A continuous map $\gamma\colon \widetilde{X}_1 \to \widetilde{X}_2$ is called a *homomorphism of covering spaces* provided the following diagram is commutative:



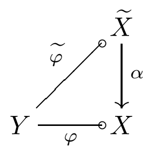
Now, a covering $(\widetilde{X}, \alpha)$ of X is called *universal* if for any covering (Y, β) of X there exists homeomorphism $\gamma\colon \widetilde{X} \to Y$. In fact it is not difficult to show that $(\widetilde{X}, \gamma)$ is a covering space for Y.

It is well known that any ANR-space X admits a universal covering space $(\widetilde{X}, \alpha)$ (see [Br2-M], [Ji-M], [Sp-M]).

(35.2) Definition. A multivalued map $\varphi\colon Y \multimap X$ will be called *m-map* provided it is u.s.c. with compact values and $\varphi(x)$ has property $(*)$ in X.

(35.3) Remark. In what follows we assume that X admits a universal covering space. Let us fix a universal covering $\alpha\colon \widetilde{X} \to X$ of X.

(35.4) Definition. A m-map $\widetilde{\varphi}\colon Y \multimap \widetilde{X}$ such that the diagram:



commutes will be called a *lift of the m-map* φ.

Note that for every $y \in Y$ the sets $\varphi(y)$ and $\widetilde{\varphi}(y)$ are homeomorphic, i.e. $\alpha|_{\widetilde{\varphi}(y)}\colon \widetilde{\varphi}(y) \to \varphi(y)$ is a homeomorphism.

First, we prove the following:

(35.5) THEOREM. *If Y is path-connected and simply connected then for any m-map $\varphi\colon Y \multimap X$ and points $y_0 \in Y$, $\widetilde{x}_0 \in \widetilde{X}$ such that $p(\widetilde{x}_0) \in \varphi(y_0)$ there exists a unique lift $\widetilde{\varphi}\colon Y \multimap \widetilde{X}$ satisfying $\widetilde{x}_0 \in \widetilde{\varphi}(y_0)$.*

PROOF. *Case 1.* First, assume that $Y = [0,1]$, $y_0 = 0$. Let $t \in [0,1]$ and let U_t denote the set from (35.1) (for the set $\varphi(t)$) which we assume to be connected. Since φ is upper semicontinuous so there exists an open subset $A_t \subset [0,1]$ containing t for which $\varphi(A_t) \subset U_t$. The family $\{A_t\}_{t\in[0,1]}$ forms an open covering of the interval $[0,1]$. Let $\lambda > 0$ be its Lebesgue number and suppose that $\lambda < 1/n$. Consider the interval $[0,1/n]$. There is a set A_t containing this interval such that $\varphi(A_{t_1}) \subset U_{t_1}$. Each loop from U_{t_1} is trivial in X, hence $\alpha^{-1}(U_{t_1})$ splits into the sum of disjoint connected components each of them mapped homeomorphically by α onto U_{t_1}. Let $s_1\colon U_{t_1} \to \alpha^{-1}(U_{t_1})$ denote the inverse map onto the component containing $\widetilde{x}_0$. We define for $t \in [0,1/n]$, $\widetilde{\varphi}(t) = s_1\varphi(t)$. Then we choose an arbitrary point $\widetilde{x}_1 \in \widetilde{\varphi}(1/n)$ and extend $\widetilde{\varphi}$ the same way onto the interval $[0,2/n]$. Following this procedure we obtain the desired lift.

Case 2. $Y = [0,1]^2$, $y_0 = (0,0)$. The proof is similar.

Case 3. The general case. Choose an arbitrary point $y \in Y$. Let ω be a path in Y joining y_0 with y. We apply (a) to the map $\varphi_\omega\colon [0,1] \to X$, $\varphi_\omega = \varphi \circ \omega$, and get a lift $(\widetilde{\varphi}_\omega)\colon [0,1] \to X$ such that $(\widetilde{\varphi}_\omega) \ni \widetilde{x}_0$. We define $\widetilde{\varphi}(y) = (\widetilde{\varphi}_\omega)(1)$.

This definition is correct: if ω' is another path joining y_0 with y, then they are fixed end homotopic and by Case 2 $(\widetilde{\varphi}_\omega)(1) = (\widetilde{\varphi}_{\omega'})(1)$. □

(35.6) COROLLARY. *Let $\varphi\colon X \multimap X$ be an m-map and let $\widetilde{x}_1, \widetilde{x}_2 \in \widetilde{X}$ be such points that $\alpha(\widetilde{x}_2) \in \varphi\alpha(\widetilde{x}_1)$. Then there exists a unique m-map $\widetilde{\varphi}\colon \widetilde{X} \multimap \widetilde{X}$ for which the diagram*

$$\begin{array}{ccc} \widetilde{X} & \overset{\widetilde{\varphi}}{-\,-\multimap} & \widetilde{X} \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha} \\ X & \underset{\varphi}{-\,-\multimap} & X \end{array}$$

commutes and $\widetilde{x}_2 \in \widetilde{\varphi}(\widetilde{x}_1)$.

PROOF. We apply (35.1) to the diagram

$$\begin{array}{ccc} & & \widetilde{X} \\ & & \downarrow{\scriptstyle\alpha} \\ \widetilde{X} & \underset{\varphi}{-\,-\multimap} & X \end{array}$$

Let us denote by lift(φ) the set of all m-maps $\varphi: X \multimap X$ for which the diagram

$$\begin{array}{ccc} \widetilde{X} & \overset{\widetilde{\varphi}}{- - \multimap} & \widetilde{X} \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha} \\ X & \underset{\varphi}{- - \multimap} & X \end{array}$$

commutes.

We will call the elements of lift(φ) the lifts of m-map φ. Let us recall that the set of all (singlevalued) maps $\beta: \widetilde{X} \to \widetilde{X}$ such that the diagram commutes

$$\begin{array}{ccc} \widetilde{X} & \overset{\beta}{- - - \to} & \widetilde{X} \\ & {\scriptstyle\alpha}\searrow \quad \swarrow{\scriptstyle\alpha} & \\ & X & \end{array}$$

forms a group (isomorphic to the fundamental group of the space X). We will denote this group by ϑ. □

(35.7) COROLLARY. *Let us fix one element $\widetilde{\varphi} \in$ lift(φ). Then each lift of φ is of the form $\beta \circ \widetilde{\varphi}$, where $\beta \in \vartheta$ and $\beta\widetilde{\varphi} = \gamma\widetilde{\varphi}$ if and only if $\beta = \gamma$.*

Now we define an equivalence relation R on the set lift(φ):

$$(35.8) \qquad \widetilde{\varphi} R \widetilde{\varphi'} \quad \text{if and only if} \quad \widetilde{\varphi'} = \gamma\widetilde{\varphi}\gamma^{-1} \quad \text{for some } \gamma = \vartheta.$$

Following the singlevalued case (see [Ji-M]) we will call it the Reidemeister relation and denote the quotient set by:

$$\bigtriangledown(\varphi) = \text{lift}(\varphi)/R.$$

The elements of $\bigtriangledown(\varphi)$ are called Reidemeister classes of the m-map φ.

(35.9) REMARK. The above definition of $\bigtriangledown(\varphi)$ depends on the choice of the universal covering. Nevertheless, one can prove that the sets of Reidemeister classes got from different universal coverings are in natural one to one correspondence.

The number of elements of the set $\bigtriangledown(\varphi)$ will be called the *Reidemeister number* of the m-map φ. Now we are going to check that it is a homotopy invariant.

(35.10) DEFINITION. Two m-maps $\varphi, \psi: Y \multimap X$ are called *m-homotopic* if and only if there exists a m-map $\chi: Y \times I \multimap X$ such that $\chi(y, 0) = \varphi(y)$ and $\chi(y, 1) = \psi(y)$.

(35.11) COROLLARY. *Let $\varphi, \psi\colon X \multimap X$ be two m-maps and let $\chi\colon X \times I \multimap X$ be a m-homotopy joining them. Then for any $\widetilde{\varphi} \in \operatorname{lift}(\varphi)$ there exists a unique m-map $\widetilde{\chi}\colon \widetilde{X} \times I \multimap \widetilde{X}$ such that the diagram*

$$\begin{array}{ccc} \widetilde{X} \times I & \xrightarrow{\ \widetilde{\chi}\ } & \widetilde{X} \\ {\scriptstyle \alpha\times \mathrm{id}}\big\downarrow & & \big\downarrow{\scriptstyle \alpha} \\ X \times I & \xrightarrow[\ \chi\ ]{} & X \end{array}$$

commutes and $\widetilde{\chi}(\widetilde{x}, 0) = \widetilde{\varphi}(\widetilde{x})$.

Thus, $\widetilde{\chi}(\,\cdot\,, 1) \in \operatorname{lift}(\psi)$ and this way the homotopy χ determines a bijection between the sets $\operatorname{lift}(\varphi)$ and $\operatorname{lift}(\psi)$. This bijection preserves the Reidemeister relation and induces a one to one correspondence between the sets $\triangledown(\varphi)$ and $\triangledown(\psi)$.

(35.12) THEOREM. *The Reidemeister numbers of the m-homotopic m-maps are equal.*

Now we are going to define the so called Nielsen relation. In order to do this we shall study the fixed point set $\operatorname{Fix}(\varphi)$ of the m-map $\varphi\colon X \multimap X$.

(35.13) DEFINITION. Let $x, x' \in \operatorname{Fix}(\varphi)$. We will say that x and x' are *Nielsen equivalent* $x \approx |_{\widetilde{N}} x'$ if and only if there exists a lift $\widetilde{\varphi} \in \operatorname{lift}(\varphi)$ such that $x, x' \in \alpha(\operatorname{Fix}(\widetilde{\varphi}))$ and the quotient set $\operatorname{Fix}(\varphi)/N$ will be denoted by $\Delta(\varphi)$.

Observe that for singlevalued $\varphi = f$ the above definition coincide with the classical Nielsen relation (cf. [Ji-M]).

(35.14) LEMMA. *If $x \in \operatorname{Fix}(\varphi)$ then there exists an open subset V_x containing x such that $y \in V_x \cap \operatorname{Fix}(\varphi)$ implies $x \underset{N}{\sim} y$.*

PROOF. Let $x \in \operatorname{Fix}(\varphi)$ and let U_x be the corresponding neighbourhood of $\varphi(x) \subset X$ from (35.1). Since φ is upper semi-continuous, there exists an open subset V_x containing x such that $\varphi(V_x) \subset U_x$. We may assume that V_x is path-connected and that $V_x \subset U_x$. Let $y \in V_x \cap \operatorname{Fix}(\varphi)$. We will show that $x \underset{N}{\sim} y$. Let $\alpha\colon \widetilde{X} \to X$ denote again a universal covering and let us fix a point $\widetilde{x} \in \alpha^{-1}(x)$. From (35.6) we obtain a lift $\varphi\colon \widetilde{X} \to \widetilde{X}$ such that $\widetilde{x} \in \widetilde{\varphi}(x)$. Consider the restriction of $\widetilde{\varphi}$

$$\widetilde{\varphi}\colon \alpha^{-1}(V_x) \to \alpha^{-1}(U_x).$$

The two above sets are disjoint sums of connected components, each of them mapped homeomorphically by α onto V_x and U_x, respectively. Denote by $\widetilde{V}_{\tilde{x}}, \widetilde{U}_{\tilde{x}}$

the components containing $\widetilde{x}$. We obtain a commutative diagram

$$\begin{array}{ccc} \widetilde{V}_{\widetilde{x}} & \xrightarrow{\ \widetilde{\varphi}\ } & \widetilde{U}_{\widetilde{x}} \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha} \\ V_x & \xrightarrow[\ \varphi\ ]{} & U_x \end{array}$$

where the vertical lines are homeomorphisms and $\widetilde{V}_{\widetilde{x}} \subset \widetilde{U}_{\widetilde{x}}$. Now it is obvious that if $y \in \mathrm{Fix}(\varphi) \cap V_x$ and $\widetilde{y} \in \widetilde{V}_x$ satisfy $\alpha(\widetilde{y}) = y$ then $\widetilde{y} \in \widetilde{\varphi}(\widetilde{y})$. Thus $x, y \in \alpha(\mathrm{Fix}(\widetilde{\varphi}))$, so $x \underset{N}{\sim} y$. □

(35.15) Lemma. *Let $\varphi: X \multimap X$ be a m-map. Then*

(35.15.1) $$\mathrm{Fix}(\varphi) = \bigcup \{\alpha(\mathrm{Fix}(\widetilde{\varphi})) \mid \widetilde{\varphi} \in \mathrm{lift}(\varphi)\},$$

(35.15.2) *for any two lifts $\widetilde{\varphi}, \widetilde{\varphi}'$ of φ the sets $\alpha(\mathrm{Fix}(\widetilde{\varphi})), \alpha(\mathrm{Fix}(\widetilde{\varphi}'))$ are either equal or disjoint,*

(35.12.3) $\alpha(\mathrm{Fix}(\widetilde{\varphi})) = \alpha(\mathrm{Fix}(\widetilde{\varphi}')) \neq \emptyset$ *implies* $\widetilde{\varphi} \underset{R}{\sim} \widetilde{\varphi}'$.

Proof. Similar to the singlevalued case [BoJ]. Let $x \in \mathrm{Fix}(\varphi)$. Let us consider $L_x = \{\widetilde{\varphi} \in \mathrm{lift}(\varphi) \mid x \in \alpha(\mathrm{Fix}(\widetilde{\varphi}))\}$. Then (35.15.1) implies that L_x is nonempty and it follows from (35.15.2) and (35.15.3) that L_x is exactly one Reidemeister class. On the other hand (b) and (35.15.3) imply that $L_x = L_y$ if and only if $x \underset{N}{\sim} y$. Thus we obtain the injective map $v: \Delta(\varphi) \to \nabla(\varphi)$ given by the formula $v[x] = L_x$. □

There are two equivalent definitions of the Nielsen relation for singlevalued maps (see [Ji-M] or [Br2-M]). The first using universal coverings we have already generalized onto the case of m-maps. Let us recall the second one (more popular for singlevalued maps).

(35.16) Definition. Let $f: X \to X$ denote a singlevalued self-map of a topological space X. Then two points $x, x' \in \mathrm{Fix}(f)$ are called *equivalent* if and only if there is a path $\omega: I \to X$ joining them such that ω and $f \circ \omega$ are fixed end homotopic.

The last approach can not be simply applied to multivalued case since the composition $\varphi \circ \omega$ is generally no longer a path. Nevertheless we will show how to generalize this definition onto the case of m-maps. This approach seems to be more convenient in calculations.

Let us recall:

(35.17) Definition. Let X be a metric space. The category which objects are points of X and morphisms from x to x' are the fixed end homotopy classes of paths joining these points is called the *fundamental groupoid* of the space X.

We denote the set of morphisms between the points x and x' by $\Pi(X; x, x')$ and the whole fundamental groupoid by $\Pi(X)$.

(35.18) Remark. Any continuous singlevalued map $f\colon X \to Y$ induces a functor $\Pi(f)\colon \Pi(X) \to \Pi(Y)$ given by $\Pi(f)(x) = f(x)$, $\Pi(f)[\omega] = [f\omega]$.

Using these notations we may formulate the obvious:

(35.19) Lemma. *Let $f\colon X \to X$ be a continuous singlevalued self-map. Two points $x, x' \in \operatorname{Fix}(f)$ are equivalent in the sense of* (35.16) *if and only if the map*

$$\Pi(f)\colon \Pi(X;\, x, x') \to \Pi(X;\, x, x')$$

has a fixed point.

Our aim is to generalize the notion of fundamental groupoid to extend the definition (35.16) onto the case of m-maps. Then we will check that this extension coincides with (35.13).

Let X denote again a connected, locally path-connected, semi-locally simply-connected space and let A_0, A_1 be two subsets of X satisfying the property $(*)$ of (35.1). Then the sets $\Pi(X; a_0, a_1)$ for $(a_0, a_1) \in A_0 \times A_1$ may be identified as follows: let U_i be a path-connected neighbourhood of A_i from (35.1), let $a_i, a_i' \in A_i$ and let ω_i be a path in U_i joining the points a_i and a_i' $(i = 0, 1)$. Then we identify $\Pi(X; a_0, a_1) \ni [\alpha]$ with $[\omega_0^{-1} * \alpha * \omega_1] \in \Pi(X; a_0', a_1')$ and define the quotient set

$$\widehat{\Pi}(X; A_0, A_1) = \bigcup_{\substack{a_0 \in A_0 \\ a_1 \in A_1}} \Pi(X; a_0, a_1)/\sim . \tag{35.20}$$

Let $(a_0, a_1) \in A_0 \times A_1$ and denote by

$$i_{a_0,a_1}\colon \Pi(X; a_0, a_1) \to \widetilde{\Pi}(X; A_0, A_1)$$

the natural bijection.

(35.21) Remark. When A_0 and A_1 are single points then (35.20) agrees with (35.17).

(35.22) Definition. The generalized fundamental groupoid of the space X is the category which objects are subsets of X satisfying property $(*)$ of (35.1) and $\widehat{\Pi}(X; A_0, A_1)$ is the set of morphisms between the objects A_0 and A_1. We will denote this category by $\widehat{\Pi}(X)$.

(35.23) Remark. $\Pi(X)$ may be regarded as a subcategory of $\widehat{\Pi}(X)$.

(35.24) Lemma. *Let X be a connected space admitting a universal covering and let Y be an arbitrary topological space. Then each m-map $\varphi: Y \multimap X$ induces a functor*

$$\widehat{\Pi}(\varphi): \Pi(Y) \to \widehat{\Pi}(X)$$

which coincides with $\Pi(\varphi): \Pi(Y) \to \Pi(X)$ when φ is singlevalued map.

Proof. We define $\widehat{\Pi}(\varphi)(y) = \varphi(y)$ for each $y \in Y$. Let $[\omega] \in \Pi(Y; y_0, y_1)$. Let us fix a universal covering $\alpha: \widetilde{X} \to X$ and points $x_0 \in \varphi(y_0)$, $x_1 \in \varphi(y_1)$, $\widetilde{x}_0 \in \alpha^{-1}(x_0)$. Then by (35.5) the diagram

$$\begin{array}{ccccc} & & & & \widetilde{X} \\ & & & & \big\downarrow{\scriptstyle \alpha} \\ [0,1] & \overset{\omega}{\dashrightarrow} & Y & \xrightarrow{\ \varphi\ } & X \end{array}$$

admits a unique lift $\widetilde{\varphi}_\omega$ such that $\widetilde{x}_0 \in \widetilde{\varphi}_\omega(0)$. Let $\{\widetilde{x}_1\} = \widetilde{\varphi}_\omega(1) \cap \alpha^{-1}(x_1)$ and let τ be a path in $\widetilde{X}$ joining $\widetilde{x}_0$ with $\widetilde{x}_1$. We define

$$\widehat{\Pi}(\varphi)[\omega] = i_{(x_0,x_1)}[\alpha\tau] \in \widehat{\Pi}(X; \varphi(y_0), \varphi(y_1)). \tag{35.25}$$

One can check that the above definition does not depend on the choice of the covering $\widetilde{X}$, the points $x_0, x_1, \widetilde{x}_0$ and the path τ. Thus we get the desired functor $\widehat{\Pi}(\varphi)$ and proof is completed. □

Now, we are able to modify (35.16) (cf. (35.11)).

(35.26) Definition. Two fixed points x, x' of the m-map $\varphi: X \multimap X$ are in $\underset{N'}{\sim}$ relation if and only if the maps

$$\widehat{\Pi}(\varphi), i_{(x,x')}: \Pi(X; x, x') \to \widehat{\Pi}(X; \varphi(x), \varphi(x'))$$

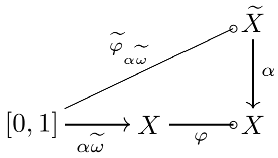
have a coincidence point.

(35.27) Theorem. *The relations $\underset{N}{\sim}, \underset{N'}{\sim}$ are equal.*

Proof. Let $x \underset{N}{\sim} x'$. Then there exists a lift $\widetilde{\varphi}$

$$\begin{array}{ccc} \widetilde{X} & \overset{\widetilde{\varphi}}{\multimap} & \widetilde{X} \\ {\scriptstyle \alpha}\big\downarrow & & \big\downarrow{\scriptstyle \alpha} \\ X & \underset{\varphi}{\multimap} & X \end{array}$$

such that $x, x' \in \alpha(\operatorname{Fix}(\widetilde{\varphi}))$. Let us choose two points $\widetilde{x}, \widetilde{x}' \in X$ such that $\alpha(\widetilde{x}) = x$, $\alpha(\widetilde{x}') = x'$, and $\widetilde{x}, \widetilde{x}' \in \operatorname{Fix}\widetilde{\varphi}$. Let $\widetilde{\omega}$ denote the path in $\widetilde{X}$ joining the points $\widetilde{x}$ and $\widetilde{x}'$. Then the commutative diagram

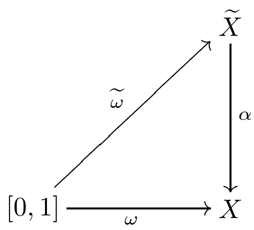


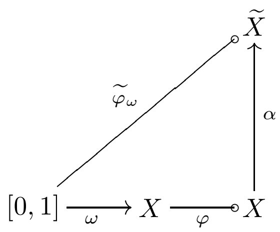
gives us $\widehat{\Pi}(\varphi)[\alpha\widetilde{\omega}]$ hence $x \underset{N'}{\sim} x'$.

Assume now that $x \underset{N'}{\sim} x'$. Then there exists a path ω joining x and x' in X such that

$$\widehat{\Pi}(\varphi)[\omega] = i_{(x,x')}[\omega] \in \widehat{\Pi}(X, \varphi(x), \varphi(x')). \tag{35.28}$$

Let us fix a point $\widetilde{x} \in \alpha^{-1}(x)$ and suppose that the lifts





satisfy $\omega(0) = \widetilde{\omega}(\widetilde{\varphi})(0)$. Then by (35.28) and (35.26) $\widetilde{\omega}(1) \in \widetilde{\varphi}_{\omega}(1)$. Take the lift $\widetilde{\varphi}$ such that $\widetilde{x} \in \operatorname{Fix}\widetilde{\varphi}$. Then $\widetilde{x} \in \widetilde{\varphi}(\widetilde{x}) = \widetilde{\varphi} \circ \widetilde{\omega}(0)$ and we get two lifts $\widetilde{\varphi}_{\omega}, \widetilde{\varphi} \circ \widetilde{\omega}$ of the map $\varphi\omega$ such that $\widetilde{\varphi}_{\omega}(0) \cap \widetilde{\varphi} \circ \widetilde{\omega}(0) \neq \emptyset$ hence $\widetilde{\varphi}_{\omega} = \widetilde{\varphi} \circ \widetilde{\omega}$. In particular $\widetilde{\omega}(1) \in \widetilde{\varphi}_{\omega}(1) = \widetilde{\varphi} \circ \widetilde{\omega}(1)$ so $\widetilde{\omega}(1) \in \operatorname{Fix}\widetilde{\varphi}$. The equality $\alpha\widetilde{\omega}(1) = \omega(1) = x'$ implies $x, x' \in \alpha(\operatorname{Fix}\widetilde{\varphi})$ so $x \underset{N}{\sim} x'$. □

(35.29) Corollary. *Let $\varphi: X \multimap X$ denote a m-map and let $x, x' \in \operatorname{Fix}\varphi$. Suppose that there exists a path joining them such that the composition $\varphi \circ \omega: I \multimap X$ admits a continuous singlevalued selector τ satisfying: $\tau(0) = x$, $\tau(1) = x'$ and the paths ω, τ are fixed end homotopic. Then $x \underset{N}{\sim} x'$.*

Proof. It follows from the definition of the induced map (35.24) that $\widehat{\Pi}(\varphi)[\omega] = i_{(x,x')}[\tau] \in \widehat{\Pi}(X, \varphi(x), \varphi(x'))$ hence $x \underset{N'}{\sim} x'$ and the theorem (35.27) gives us $x \underset{N}{\sim} x'$. The proof is completed. □

In what follows we shall assume that X is a finite dimensional Euclidean neighbourhood retract and $\varphi: X \multimap X$ is an acyclic compact m-map. Then for any open

$U \subset X$ such that $\mathrm{Fix}(\varphi) \cap \partial U = \emptyset$ we have $(X, U, \varphi) \in \mathcal{K}$ and therefore the fixed point index $I(X, U, \varphi)$ is well defined.

Now let $A \subset \mathrm{Fix}(\varphi)$ be one of the Nielsen classes of $\mathrm{Fix}(\varphi)$. Let us choose an open $U \subset X$ such that $U \cap \mathrm{Fix}(\varphi) = A$.

(35.30) Definition. The class A is called *essential* if and only if $I(X, U, \varphi) \neq 0$, where U is chosen above.

(35.31) Definition. The number of essential classes of φ is called *Nielsen number* $N(\varphi)$ of φ.

We prove:

(35.32) Theorem. *Let $\chi: X \times [0,1] \multimap X$ be an acyclic compact homotopy which is a m-map. Then $N(\chi(\,\cdot\,, 0)) = N(\chi(\,\cdot\,, 1))$.*

Proof. For a subset $Z \subset X \times [0,1]$ and $t \in [0,1]$ we denote $Z_t = \{x \in X \mid (x,t) \in Z\}$. Let $\overline{\chi}: X \times [0,1] \multimap X \times [0,1]$ be the "fat" homotopy defined by $\overline{\chi}(x,t) = (\chi(x,t), t)$. Then $\overline{\chi}$ is also a m-map and for $A \in \Delta(\overline{\chi})$ either $A_t \in \Delta(\chi_t)$ or $A_t = \emptyset$. Moreover, for $B \in \Delta(\chi_t)$ there exists exactly one class $A \in \Delta(\chi)$ such that $A_t = B$.

Now, we are going to prove that if $A \in \Delta(\overline{\chi})$ and U is an open subset of $X \times [0,1]$ such that $U \cap \mathrm{Fix}(\overline{\chi}) = A$, then

$$I(X, U_0, \chi_0) = I(X, U_1, \chi_1).$$

It is enough to show that the number $I(X, U_t, \chi_t)$ is a locally constant function of $t \in [0,1]$.

Let us fix $t_0 \in [0,1]$. The compactness of A gives us open neighbourhoods V, W such that $t_0 \in U \subset [0,1]$, $A_1 \subset W \subset U_{t_0}$ and $A \cap (X \times V) \subset W \times V \subset U$. We may assume V to be connected. Then for arbitrary $t \in V$ from the fixed point index properties we infer:

$$I(X, U_t, \chi_t) = I(X, W, \chi_t) = I(X, U_{t_0}, \chi_{t_0}).$$

Let $B \in \Delta(\chi_0)$ be an essential class and let $A \in \Delta(\overline{\chi})$ be the only class satisfying $B = A_0$. Then $A_1 \in \nabla(\overline{\chi})$ is also essential and it proves $N(\chi_0) \leq N(\chi_1)$.

By the same arguments we can prove the opposite inequality and the proof is completed. □

Note that from the definition of the Nielsen number we obtain:

(35.33) Corollary. *The map φ has at least $N(\varphi)$ fixed points.*

Now let $f: X \to X$ be a (singlevalued) selection of φ. Then, of course, $\mathrm{Fix}(f) \subset \mathrm{Fix}(\varphi)$. Let $x, y \in \mathrm{Fix}(f)$ be such that $x \underset{N}{\sim} y$ with respect to f. Observe, that $x \underset{N}{\sim} y$ with respect to f if and only if $x \underset{N}{\sim} y$ with respect to φ. Since the fixed point index $I(X, U, \varphi)$ is equal to $I(X, U, f)$ we obtain:

(35.34) PROPOSITION. *If $f \subset \varphi$ and φ is a m-map for which the Nielsen number is defined, then $N(f) = N(\varphi)$.*

(35.35) PROPOSITION. *Let $X \in$ ANR and $\varphi: X \multimap X$ be a compact u.s.c. map such that $\varphi(x) \in$ AR for every $x \in X$. Then φ is a m-map.*

PROOF. Let $A = \varphi(x)$. For the proof it is sufficient to show that A has the property $(*)$ in X.

Since $X \in$ ANR, we can assume without loss of generality (see (1.8)) that there is an open U in a normed space such that $X \subset U$ and $r: U \to X$ is a retraction. On the other hand, since $A \in$ AR, there is a retraction $r_1: X \to A$. Let us denote

$$U_1 = \{x \in U \mid [x, r_1(x)] \subset U\},$$

where $[a, b]$ denotes the closed interval in E joining a and b, $a, b \in E$. Then U_1 is open and $A \subset U_1$. Let us put $V_1 = X \cap U_1$. Then the formula:

$$h(x,t) = r((1-t)x + tr_1(x))$$

gives a homotopy between the inclusion $i: V_1 \to X$ and the retraction $r_1: V_1 \to A$. Hence each loop in V_1 may be deformed in X into A, and since A is contractible ($A \in$ AR) we proved our claim. The proof of (35.35) is completed. □

(35.36) REMARK. We recommend [AGJ] for some generalizations of the Nielsen theory presented in this section.

36. n-Acyclic mappings

The notion of acyclic maps can be generalized to the so called n-acyclic (acyclic in dimension n) maps.

For acyclic mappings we have the induced linear map on Čech homology with compact carriers in all dimensions $k \geq 0$, meanwhile, for n-acyclic mappings we have the induced linear map only in dimension $k = n$.

In what follows by acyclic set we shall understand a compact nonempty space A with trivial Čech cohomology with integer coefficients Z, i.e. $H^0(A) = Z$ and $H^i(A) = 0$ for $i > 0$. By a Vietoris map $p: Y \Rightarrow X$ we mean (as in Section 8) a proper mapping such that $p^{-1}(x)$ is acyclic in the above sense. Observe that if p is a proper map, then p is closed.

(36.1) DEFINITION. A map $p: Y \to X$ is called a *n-Vietoris map*, $n \geq 1$ if the following two conditions are satisfied:

(36.1.1) p is proper and surjective,
(36.1.2) $\operatorname{rd}_X M^i(p) \leq n-2-i$ for $i = 0, 1, \dots$, where $M^i(p)$ is defined in Section 8.

(36.2) Remark. If $\operatorname{rd}_X M^i(p) < 0$, then we let $M^i(p) = \emptyset$.

Assume that $p: Y \to X$ is a n-Vietoris map. Then we have:

$$m^n(p) = 1 + \max_{0 \le i \le n-1} (\operatorname{rd}_X M^i(p) + i) \le 1 + \max_{0 \le i \le n-1} [(n-2-i)+i] = n-1.$$

Therefore, from (8.14) we obtain:

(36.3) Theorem. *If $p: Y \to X$ is a n-Vietoris map, then*

$$p^{*k}: H^k(X) \xrightarrow{\sim} H^k(Y)$$

is an isomorphism for every $k \ge n$.

Now we are in position to define the class of n-acyclic maps.

(36.4) Definition. An u.s.c. map $\varphi: X \multimap Y$ with compact values is called a *n-acyclic*, $n \ge 1$ map (written $\varphi \in \mathrm{AC}_n(X,Y)$) provided:

(36.4.1) $\operatorname{rd}_X M^i(\varphi) \le n-2-i$, $i = 1, 2, \ldots$, where $M^i(\varphi) = \{x \in X \mid H^i(\varphi(x)) \ne 0\}$ for $i > 0$ and $M^0(\varphi) = \{x \in X \mid H^0(\varphi(x)) \ne Z\}$.

(36.5) Remark. If $\varphi \in \mathrm{AC}_n(X,Y)$, then the natural projection $p_\varphi: \Gamma_\varphi \to X$ is a n-Vietoris map.

(36.6) Definition. Let $\varphi \in \mathrm{AC}_n(X,Y)$. Then we define the induced homomorphisms $\varphi^{*k}: H^k(Y) \to H^k(X)$, $k \ge n$, by putting:

$$\varphi^{*k} = (p_\varphi^{*k})^{-1} \circ q_\varphi^{*k}.$$

Observe, that $\mathrm{AC}_1(X,Y) = \mathrm{AC}(X,Y)$ if we consider acyclic maps with respect to H^* with coefficients in Z. Moreover, we have the following diagram:

(36.7) $$\mathrm{AC}_1(X,Y) \subset \mathrm{AC}_2(X,Y) \subset \mathrm{AC}_3(X,Y) \subset \ldots \subset \mathrm{AC}_n(X,Y) \subset \ldots$$

(36.8) Remark. It follows from Theorem (5.1) that acyclic mappings considered in last three sections coincide with acyclic mappings considered with respect to Čech cohomology functor with rational coefficients Q.

(36.9) Definition. Two mappings $\varphi, \psi \in \mathrm{AC}_n(X,Y)$ are called *homotopic* (written $\varphi \sim \psi$) provided there exists a map $\chi \in \mathrm{AC}_n(X \times [0,1], Y)$ such that:

$$\chi(x,0) = \varphi(x) \quad \text{and} \quad \chi(x,1) = \psi(x) \quad \text{for every } x \in X;$$

then χ is called a *homotopy joining* φ and ψ.

The proof of the following proposition is strictly analogous to the proof of (32.6).

(36.10) Proposition. *If $\varphi \sim \psi$, then $\varphi^{*k} = \psi^{*k}$ for every $k \geq n$.*

As an easy consequence of the universal coefficients formula (see (5.7)) we obtain:

(36.11) Proposition. *If $\varphi\colon X \multimap Y$ is acyclic with respect to $H^*(\,\cdot\,, Q)$ then φ is acyclic with respect to $H^*(\,\cdot\,, Z)$.*

Proposition (36.11) explains the reason why we consider acyclicity in the above sense.

Now we shall show that the topological degree theory can be developed for n-acyclic mappings.

(36.12) Definition. A space A is called a *n-cohomological sphere*, $n \geq 1$, provided $H^*(A)$ is isomorphic to the Čech cohomology (with integer coefficients) of n-dimensional sphere $S^n \subset \mathbb{R}^{n+1}$.

For example, $P^{n+1} = \mathbb{R}^{n+1} \setminus \{0\}$ is a n-cohomological sphere. Of course, any A homeomorphic to S^n or P^{n+1} is a n-cohomological sphere, too.

In what follows, for any n-cohomological sphere A we identify $H^n(A)$ with the ring of integers, i.e. $H^n(A) \approx Z$.

(36.13) Definition. Let A and B be two n-cohomological spheres and let $\varphi \in \mathrm{AC}_n(A, B)$. We define the topological degree $\mathrm{Deg}(\varphi)$ of φ as an integer defined as follows:

$$\mathrm{Deg}(\varphi) = \varphi^{*n}(1).$$

From (36.10) directly follows:

(36.14) Property (Homotopy Property). *If A, B are n-cohomological spheres and $\varphi, \psi \in \mathrm{AC}_n(A, B)$ are homotopic, then* $\mathrm{Deg}(\varphi) = \mathrm{Deg}(\psi)$.

According to Section 29, by $K^{n+1}(r)$ we denote the closed ball in $\mathbb{R}^{n+1}$ with radius r and the center at zero point of $\mathbb{R}^{n+1}$. Moreover, $S^n(r) = \partial K^{n+1}(r)$ is the n-dimensional sphere.

We let:

$$\begin{aligned}\mathrm{AC}_n[(K^{n+1}(r), S^n(r)), (\mathbb{R}^{n+1}, P^{n+1})]\\ = \{\varphi \in \mathrm{AC}_n(K^{n+1}(r), \mathbb{R}^{n+1}) \mid \varphi(S^n(r)) \subset P^{n+1}\}.\end{aligned}$$

If $\varphi, \psi \in \mathrm{AC}_n[(K^{n+1}(r), S^n(r)), (\mathbb{R}^{n+1}, P^{n+1})]$ then we will say that φ is homotopic to ψ (written $\varphi \sim \psi$) provided there exists $\chi \in \mathrm{AC}_n[(K^{n+1}(r) \times [0,1], S^n(r) \times [0,1]), (\mathbb{R}^{n+1}, P^{n+1})]$ such that:

$$\chi(x, 0) = \varphi(x) \quad \text{and} \quad \chi(x, 1) = \psi(x) \quad \text{for every } x \in K^{n+1}(r).$$

Let $\varphi \in \mathrm{AC}_n[(K^{n+1}(r), S^n(r)), (\mathbb{R}^{n+1}, P^{n+1})]$. We define a restriction of φ as the map $\widehat{\varphi}\colon S^n(r) \multimap P^{n+1}$ defined as follows:

$$\widehat{\varphi}(x) = \varphi(x) \quad \text{for every } x \in S^n(r).$$

Since $S^n(r)$ is closed in $K^{n+1}(r)$ we obtain:

(36.15) PROPOSITION. *If map $\varphi \in \mathrm{AC}_n[(K^{n+1}(r), S^n(r)), (\mathbb{R}^{n+1}, P^{n+1})]$ then $\widehat{\varphi} \in \mathrm{AC}_n(S^n(r), P^{n+1})$.*

Proposition (36.15) allows us to define:

(36.16) DEFINITION. Let $\varphi \in \mathrm{AC}_n[(K^{n+1}(r), S^n(r)), (\mathbb{R}^{n+1}, P^{n+1})]$. Then we define the topological degree $\mathrm{Deg}(\varphi)$ of φ by letting:

$$\mathrm{Deg}(\varphi) = \mathrm{Deg}(\widehat{\varphi}).$$

Then we get the following homotopy property:

(36.17) PROPOSITION. *If $\varphi, \psi \in \mathrm{AC}_n[(K^{n+1}(r), S^n(r)), (\mathbb{R}^{n+1}, P^{n+1})]$ are homotopic then:*

$$\mathrm{Deg}(\varphi) = \mathrm{Deg}(\psi).$$

Moreover, similarly as we have proved in (32.13), one can show the following

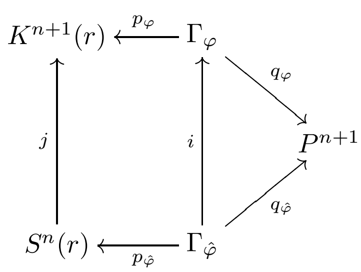
(36.18) PROPOSITION. *Assume that two maps $\psi, \varphi \in \mathrm{AC}_n(A, B)$ (resp. $\psi, \varphi \in \mathrm{AC}_n((K^{n+1}(r), S^n(r)), (\mathbb{R}^{n+1}, P^{n+1}))$). If $\psi \subset \varphi$, i.e. $\psi(x) \subset \varphi(x)$ for every x, then*

$$\mathrm{Deg}(\varphi) = \mathrm{Deg}(\psi).$$

Now, we shall show fixed point theory consequences following from the above topological degree.

(36.19) EXISTENCE PROPERTY. *Let $\varphi \in \mathrm{AC}_n((K^{n+1}(r), S^n), (\mathbb{R}^{n+1}, P^{n+1}))$. If $\mathrm{Deg}(\varphi) \neq 0$ then there exists $x \in K^{n+1}(r)$ such that $0 \in \varphi(x)$.*

PROOF. Assume to the contrary that $\varphi(K^{n+1}(r)) \subset P^{n+1}$. Then we have the following commutative diagram:



in which i, j are the respective inclusions. Applying to the above diagram the functor H^n we obtain:

$$(p_{\hat\varphi}^{*n})^{-1} \circ q_{\hat\varphi}^{*n} = j^{*n}(p_\varphi^{*n})^{-1} q_\varphi^{*n},$$

but $j^{*n} = 0$ because $H^n(K^{n+1}(r)) = 0$ and hence $\mathrm{Deg}(\varphi) = 0$, so we obtain a contradiction; the proof is completed. □

(36.20) THEOREM. *Let $\varphi \in \mathrm{AC}_n(K^{n+1}(r), \mathbb{R}^{n+1})$ be a multivalued map such that $\varphi(S^n(r)) \subset K^{n+1}(r)$. Then* $\mathrm{Fix}(\varphi) \neq \emptyset$.

PROOF. Assume to the contrary that $\mathrm{Fix}(\varphi) = \emptyset$. Consider the diagram:

$$K^{n+1}(r) \xleftarrow{p_\varphi} \Gamma_\varphi \xrightarrow{q_\varphi} \mathbb{R}^{n+1}$$

and let $f\colon \Gamma_\varphi \to P^{n+1}$ be defined as follows:

$$f(x, y) = x - y \quad \text{for every } (x, y) \in \Gamma_\varphi.$$

We define the multivalued map $\psi\colon K^{n+1}(r) \multimap \mathbb{R}^{n+1}$ by putting:

$$\psi(x) = f(p_\varphi^{-1}(x)).$$

Since for every $x \in K^{n+1}(r)$, the set $\psi(x) = \{x - y \mid y \in \varphi(x)\}$ is homeomorphic to $\varphi(x)$, we infer $\psi \in \mathrm{AC}_n((K^{n+1}(r), S^n(r)), (\mathbb{R}^{n+1}, P^{n+1}))$. To obtain a contradiction it is enough to show that: $\mathrm{Deg}(\psi) \neq 0$. In order to do this we define a homotopy

$$\chi\colon K^{n+1}(r) \times [0, 1] \multimap \mathbb{R}^{n+1}$$

by the formula:

$$\chi(x, t) = \{x - ty \mid y \in \varphi(x)\}.$$

Observe again that for every (x, t), the set $\chi(x, t)$ is homeomorphic to $\varphi(x)$ and hence

$$\chi \in \mathrm{AC}_n(K^{n+1}(r), \mathbb{R}^{n+1}).$$

Moreover, let us assume that $x - ty = 0$ for some $x \in S^n(r)$, $y \in \varphi(x)$ and $0 < t < 1$. Then $1 = \|x\| = t\|y\| \leq t < 1$. Consequently, $\chi(S^n(r)) \subset P^{n+1}$. Therefore, $\mathrm{Deg}(\chi(\,\cdot\,, 0)) = \mathrm{Deg}(\chi(\,\cdot\,, 1)) = \mathrm{Deg}(\psi)$. On the other hand $\chi(x, 0) = x$ for every $x \in K^{n+1}$ and consequently

$$\chi(\,\cdot\,, 0)^{*n} = i^{*n}\colon H^n(P^{n+1}) \xrightarrow{\sim} H^n(S^n)$$

is an isomorphism. Finally, we obtain $\mathrm{Deg}(\psi) \neq 0$ which is a contradiction; the proof is completed. □

As an immediate consequence of (36.20) we obtain the following generalization of the classical Brouwer fixed point theorem.

(36.21) COROLLARY. *If $\varphi \in \mathrm{AC}_n(K^{n+1}(r), K^{n+1}(r))$ then* $\mathrm{Fix}(\varphi) \neq \emptyset$.

37. Theorem on antipodes for n-acyclic mappings

In this section we discuss the classical Borsuk theorem on antipodes for n-acyclic and acyclic mappings.

The following theorem is an extension of the Borsuk's theorem on antipodes for singlevalued mappings.

(37.1) THEOREM. *Let $\varphi\colon S^n \multimap P^{n+1}$ be an n-admissible map. If for every $x \in S^n$ there exists an n-hyperplane through 0 strictly separating $\varphi(x)$ and $\varphi(-x)$, then $\mathrm{Deg}(\varphi) \neq 0$.*

PROOF. Let $y \in S^n$. Then we define the set U_y by putting

$$U_y = \{x \in S^n \mid \langle y, z\rangle > 0 \text{ for each } z \in \varphi(x) \text{ and } \langle y, z\rangle < 0 \text{ for each } z \in \varphi(-x)\},$$

where $\langle y, z\rangle$ denotes the inner product in $\mathbb{R}^{n+1}$.

Since φ is compact, U is open. From the assumption we have that: for every $x \in S^n$ there is $y \in S^n$ such that $\langle y, z\rangle > 0$ for $z \in \varphi(x)$ and $\langle y, z\rangle < 0$ for $z \in \varphi(-x)$. This implies that $S^n = \bigcup_{y\in S^n} U_y$. Since S^n is compact, there exists a finite subcover $\{U_{y_i}\}_{i=1,\dots,m}$. Let $\{g_t\}_{i=1,\dots,m}$ be a subordinated partition of unity.

Consider $g\colon S^n \to P^{n+1}$ defined by $g(x) = \sum_i (g_i(x) - g_i(-x))y_i$. The map g is odd, thus $\deg(g) \neq 0$. For the proof it suffices to show that g and φ are homotopic (in $\mathrm{AC}_n(S^n, P^{n+1})$). For this we define a map $\chi\colon S^n \times I \multimap P^{n+1}$ by putting

$$\chi(x,t) = \{t \circ g(x) + (1-t)z \mid z \in \varphi(x)\}.$$

If $\chi(x,t)$ contains 0 for some $t \in [0,1]$ and $x \in S^n$, then there is $z \in \varphi(x)$, with

$$-(1-t)z = \sum_i (g_i(x) - g_i(-x))y_i. \tag{37.1.1}$$

Observe that:

$$\begin{aligned} &\text{(i) } \langle y_i, z\rangle > 0 \quad \text{for some } i = 1,\dots,m,\\ &\text{(ii) } \langle y_i, z\rangle < 0 \quad \text{implies } g_i(x) = 0,\\ &\text{(iii) } \langle y_i, z\rangle > 0 \quad \text{implies } g_i(-x) = 0. \end{aligned} \tag{37.1.2}$$

Taking the inner product of both sides of (37.1.1) with z and applying (37.1.2) we obtain $-(1-t)\langle z, z\rangle > 0$, a contradiction. For every $k \geq 0$ we have $M^k(\chi) \subset M^k(\varphi) \times I$ and hence we deduce

$$\mathrm{rd}_{\,S^n\times I} M^k(\chi) \leq (n-1-2-k) + 1 = n-2-k.$$

This implies that χ is an n-acyclic homotopy joining g with φ; the proof is completed. □

Using (4.4) and (6.1) we deduce the following

(37.2) COROLLARY. *Suppose $\varphi\colon K^{n+1} \multimap \mathbb{R}^{n+1}$ is an n-admissible and for $x \in S^n$ there is an n-hyperplane through* 0 *strictly separating $\varphi(x)$ and $\varphi(-x)$. If $\widetilde{\varphi} = \varphi|_{S^n}$ is $(n-1)$-acyclic then φ vanishes in some interior point of K^{n+1}.*

For acyclic mappings Theorem (37.2) can be formulated in a more general form.

Let M be a compact n-cohomological sphere. An acyclic map $\Phi\colon M \multimap M$ is called an *involution*, if the condition $(x,y) \in \Gamma_\Phi$ implies that $(y,x) \in \Gamma_\Phi$, i.e. the graph Γ_Φ of Φ is symmetric.

We recall the well known fact for singlevalued maps (see [Go1-M] and references therein).

(37.3) PROPOSITION. *Let $g\colon M \to M$ be a singlevalued involution and let $f\colon M \to S^n$ be such that $f(x) \neq f(g(x))$ for every $x \in M$. Then the induced homomorphism $f^{*n}\colon H^n(S^n) \to H^n(M)$ is non trivial, i.e. $f^{*n} \neq 0$.*

Now we are able to prove:

(37.4) THEOREM. *Let $\Phi\colon M \multimap M$ be an acyclic involution and let $\varphi\colon M \multimap P^{n+1}$ be an acyclic map such that the following condition is satisfied:*

(37.4.1) *every radius with origin at the zero point of $\mathbb{R}^{n+1}$ has an empty intersection with the set $\varphi(x)$ or $\varphi(\Phi(x))$ for every $x \in M$.*

Then $\mathrm{Deg}(\varphi) \neq 0$.

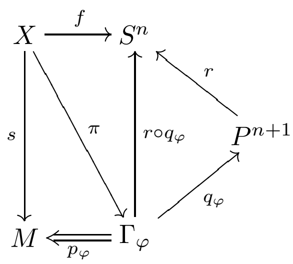
PROOF. Consider the diagram:

$$M \xLeftarrow{p_\varphi} \Gamma_\varphi \xrightarrow{q_\varphi} P^{n+1}.$$

Define the set X by putting:

$$X = \{(x,x',y,y') \mid x \in M,\ x' \in \Phi(x),\ y \in \varphi(x),\ y' \in \varphi(x')\}.$$

Of course, X is a compact set. Consider the diagram:



in which

$$s(x,x',y,y') = x, \qquad f(x,x',y,y') = \frac{q_\varphi(y)}{\|q_\varphi(y)\|},$$
$$\pi(x,x',y,y') = (x,y), \qquad r(z) = \frac{z}{\|z\|}.$$

It is easy to see that the above diagram commutes. Observe that the map s has the following decomposition:

$$(x,x',y,y') \to (x,x',y) \to (x,x') \to x.$$

Since the maps given in the above decomposition of s are determined by Vietoris mappings p_φ and p_Φ, in view of (8.10), we conclude that s is a Vietoris map. Therefore, X is a n-cohomological sphere.

Now we define the singlevalued involution $g\colon X \to X$ as follows:

$$g(x,x'y,y') = (x',x,y',y).$$

We prove that $f(x,x',y,y') \neq f(g(x,x',y,y'))$. Indeed, we have:

$$f(x,x',y,y') = \frac{q_\varphi(y)}{\|q_\varphi(y)\|} \quad \text{and} \quad f(g(x,x',y,y')) = \frac{q_\varphi(y')}{\|q_\varphi(y')\|}.$$

So, our claim follows from (37.4.1). Therefore, from (37.3) we obtain that $f^{*n} \neq 0$. Consequently, from the commutativity of the above diagram we obtain:

$$(p_\varphi^{*n})^{-1} \circ (q^{*n}) = (s^{*n})^{-1} f^{*n} (r^{*n})^{-1} \neq 0$$

and hence $\operatorname{Deg}(\varphi) \neq 0$; the proof is completed. □

Now we shall formulate some consequences of (37.4).

(37.5) Corollary. *Let $\Phi\colon M \multimap M$ be an involution and let $\varphi\colon M \multimap S^n$ be an acyclic map such that the following condition is satisfied:*

(37.5.1) $\varphi(x) \cap \varphi(y) = \emptyset$ *for every $x \in M$ and $y \in \Phi(x)$.*

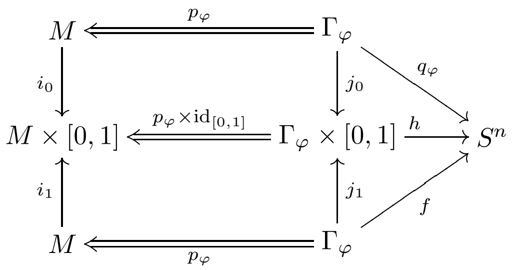
Then $\operatorname{Deg}(\varphi) \neq 0$.

For the proof of (37.5) observe that (37.5.1) implies (37.4.1).

Now we prove the following:

(37.6) Corollary. *Let $\Phi\colon M \multimap M$ and $\varphi\colon M \multimap S^n$ be as in* (37.5). *Then* $\varphi(M) = S^n$.

Proof. Assume to the contrary that there exists a point $u_0 \in S^n \setminus \varphi(M)$. Consider the following diagram:



in which i_0, i_1, j_0, j_1 are the respective inclusions, $f(x,y) = u_0$ for every $(x,y) \in \Gamma_\varphi$. Let

$$h(x,y,t) = \frac{ty + (t-1)u_0}{\|ty + (t-1)u_0\|} \quad \text{for every } x,\ y,\ t.$$

Since $q_\varphi(x,y) = y \neq u_0$ for every $y \in Y$ the map h is well defined. Since f is a constant map we get $f^{*n} = 0$. So from the commutativity of the above diagram we obtain:

$$(p_\varphi^{*n})^{-1} q_\varphi^{*n} = (p_\varphi^{*n})^{-1} f^{*n} = 0,$$

and $\mathrm{Deg}(\varphi) = 0$; a contradiction. □

From (37.6) immediately follows:

(37.7) COROLLARY. *Let $\Phi: M \multimap M$ be an involution and let $\varphi: M \multimap \mathbb{R}^n$ be acyclic. Then there exists a point $(x,y) \in \Gamma_\Phi$ such that $\varphi(x) \cap \varphi(y) \neq \emptyset$.*

Assuming $M = S^n$ and $\Phi = -\mathrm{id}_{S^n}$ from (37.7) we obtain:

(37.8) COROLLARY. *If $\varphi: S^n \multimap \mathbb{R}^n$ is an acyclic map then there exists a point $x \in S^n$ such that $\varphi(x) \cap \varphi(-x) \neq \emptyset$.*

Note that (37.8) is a multivalued generalization of the well known Borsuk–Ulam theorem.

Now we shall discuss the multivalued version of the Bourgin–Yang theorem. To do it we need the notion of genus.

Let X be a space and $\alpha: X \to X$ be a fixed point free (singlevalued) involution, i.e. $\alpha(x) \neq x$ and $\alpha^2(x) = \alpha(\alpha(x)) = x$. In what follows by (X,α) we will denote a space X with a fixed point free involution α.

Let (X,α) and (Y,β) be two spaces with fixed point free involutions. A map $f: X \to Y$ is called *equivariant* if the following diagram is commutative:

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ X & \xrightarrow[\ f\ ]{} & Y \end{array}$$

(37.9) DEFINITION. The genus $\gamma(X,\alpha)$ of a pair (X,α) is the minimal nonnegative integer k such that there exists an equivariant map $f: X \to S^k$, where in S^k the involution β is the antipodal map $\beta: S^k \to S^k$, $\beta(x) = -x$, for every $x \in S^k$; we let $\gamma(X,\alpha) = \infty$ if no such k exists.

It is well known that:

(37.10) PROPOSITION. *If X is a compact space with a fixed point free involution α then*

$$\gamma(X, \alpha) \leq \dim X.$$

The proof of (37.10) can be found in [CF]. We omit it here because it requires some developed methods of equivariant topology. Recall that if V is a linear subspace of $\mathbb{R}^n$ then the codimension $\operatorname{codim} V$ of V in $\mathbb{R}^n$ is defined by:

$$\operatorname{codim} V = n - \dim V.$$

In what follows if $X \subset \mathbb{R}^{n+1} \setminus \{0\}$ is a nonempty symmetric set then the genus $\gamma(X, \beta)$ of X with respect to the antipodal map β will be denoted shortly by $\gamma(X)$.

(37.11) PROPOSITION. *Let $X \subset \mathbb{R}^{n+1} \setminus \{0\}$ be a nonempty and symmetric set and let $k = \gamma(X)$. If V is a linear subspace of $\mathbb{R}^n$ then $\gamma(X \cap V) \geq k - \operatorname{codim} V$.*

PROOF. Suppose, on the contrary, that $\gamma(X \cap V) = q < k - p$, where $p = \operatorname{codim} V$. That means, according to the definition of γ, that there exists an add map $f_0: X \cap V \to S^q$. Extend f_0 to a continuous map $\widetilde{f}: X \to \mathbb{R}^{q+1}$ and let $f(x) = (1/2)(\widetilde{f}(x) - \widetilde{f}(-x))$; $f: X \to \mathbb{R}^{q+1}$ be an odd continuous extension of f_0.

Let $V^\perp$ denote the orthogonal complement of V in $\mathbb{R}^n$ and $P: \mathbb{R}^n \to V^\perp$ be the orthogonal projection. Let $A: V^\perp \to \mathbb{R}^p$ be a linear isomorphism. Define $g_0: X \to \mathbb{R}^{p+q+1} \setminus \{0\}$ by $q_0(x) = (A \circ P(x), f(x))$ (note that we identify $\mathbb{R}^p \times \mathbb{R}^{q+1}$ with $\mathbb{R}^{p+q+1}$) and let $q(x) = g_0(x)/\|g_0(x)\|$. Clearly, $g: X \to S^{p+q}$ is odd and continuous; therefore, $\gamma(X) \leq p + q < p + k - p = k$. We have obtained a contradiction which completes the proof. □

From (37.11) by induction we deduce:

(37.12) PROPOSITION. *Let $X \subset \mathbb{R}^{n+k} \setminus \{0\}$, $k \geq 1$, be a nonempty and symmetric set. If $\gamma(X) \geq k$ then there are at least $(k+1)$-mutually orthogonal points in X.*

Now we are able to prove the Bourgin–Yang theorem for multivalued mappings.

(37.13) THEOREM (Bourgin–Yang). *If $\varphi: S^{n+k} \multimap \mathbb{R}^n$ is an acyclic map, then $\gamma(A(\varphi)) \geq k$, where $A(\varphi) = \{x \in S^{n+k} \mid \varphi(x) \cap \varphi(-x) \neq \emptyset\}$.*

PROOF. First observe that $A(\varphi)$, in view of (37.8), is nonempty. Moreover, it is evidently symmetric and compact subset of S^{n+k}.

We shall proceed by contradiction. So, let us assume to the contrary that $\gamma(A(\varphi)) < k$. Of course we can assume that $k \geq 1$. We let $\gamma(A(\varphi)) = p < k$. Then from (37.9) we obtain an add continuous map $f: A(\varphi) \to S^p$. Let $\widetilde{f}: S^{n+k} \to \mathbb{R}^{p+1}$ be a continuous (not necessarily odd) extension of f. Consider:

$$\widetilde{\varphi}: S^{n+k} \multimap \mathbb{R}^n \times \mathbb{R}^{p+1} = \mathbb{R}^{n+p+1} \subset \mathbb{R}^{n+k}$$

defined by:

$$\widetilde{\varphi}(x) = \{(y, \widetilde{f}(x)) \mid y \in \varphi(x)\}.$$

Since the set $\widetilde{\varphi}(x)$ is homeomorphic to $\varphi(x)$ we deduce that $\widetilde{\varphi}$ is acyclic. By applying the Borsuk-Ulam theorem to $\widetilde{\varphi}$ (see (37.8)) we obtain a point $\widetilde{x} \in S^{n+k}$ such that

$$\widetilde{\varphi}(\widetilde{x}) \cap \widetilde{\varphi}(-\widetilde{x}) \neq \emptyset.$$

It means that $\varphi(\widetilde{x}) \cap \varphi(-\widetilde{x}) \neq \emptyset$ i.e. $\widetilde{x} \in A(\varphi)$ and $f(\widetilde{x}) = f(-\widetilde{x})$. Hence, $\widetilde{f}(\widetilde{x}) = f(\widetilde{x}) = -f(-\widetilde{x}) = -\widetilde{f}(-\widetilde{x}) \in S^p$. Consequently, $\widetilde{f}(\widetilde{x}) \neq \widetilde{f}(-\widetilde{x})$ and we have obtained a contradiction with $\widetilde{f}(\widetilde{x}) = \widetilde{f}(-\widetilde{x})$; the proof is completed. □

(37.14) REMARK. By letting $k = 0$ from (37.13) we deduce (37.8). In fact the above proof shows that (37.8) implies (37.13). So the Borsuk–Ulam theorem and the Bourgin–Yang theorem are equivalent.

Now, from (37.13) and (37.10) we obtain:

(37.15) COROLLARY. *If $\varphi\colon S^{n+k} \multimap \mathbb{R}^n$ is an acyclic map then $\dim A(\varphi) \geq k$.*

From (37.13) and (37.12) we deduce:

(37.16) COROLLARY. *If $\varphi\colon S^{n+k} \multimap \mathbb{R}^n$, $k \geq 1$ is an acyclic map, then there exist $(k+1)$-mutually orthogonal points in $A(\varphi)$.*

Finally, note that (37.13) can be expressed in terms of a cohomological index (see [GG-1]).

38. Theorem on invariance of domain

In this section we shall show that the Brouwer Invariance of Domain Theorem may be generalized to acyclic maps. As usual, for a subset $A \subset \mathbb{R}^{n+1}$ by $\operatorname{Int} A$ we denote its interior in $\mathbb{R}^{n+1}$ moreover, as before, for $a_0 \in \mathbb{R}^{n+1}$ and $r > 0$ by $B(a_0, r)$ we denote the open ball in $\mathbb{R}^{n+1}$ with the center a_0 and radius r.

We start with the following cohomological characterization of an interior point (see [ES-M, p. 394]).

(38.1) PROPOSITION. *Let A be a compact subset, $a_0 \in A$ and $j\colon A \setminus B_0(a_0, r) \to A$ be the inclusion map. The point $a_0 \in \operatorname{Int} A$ if and only if there exists a positive number r_0 such that for every $0 < r < r_0$ the homomorphism*

$$j^{*n}\colon H^n(H) \to H^n(A \setminus B(a_0, r))$$

is not an epimorphism.

Now we will prove the following two lemmas.

(38.2) LEMMA. *Let A be a compact subset of $\mathbb{R}^{n+1}$, $a_0 \in A$. The point $a_0 \in$ $\operatorname{Int} A$ if and only if there exists an acyclic map $\varphi\colon K^{n+1} \multimap A$ such that:*

$$\varphi(S^n) \subset A \setminus \{a_0\}, \tag{38.2.1}$$

$$\operatorname{Deg}(\varphi) \neq 0. \tag{38.2.2}$$

PROOF. First, let us observe that if $a_0 \in \operatorname{Int} A$ then we can find even a single-valued map with the properties (38.2.1) and (38.2.2).

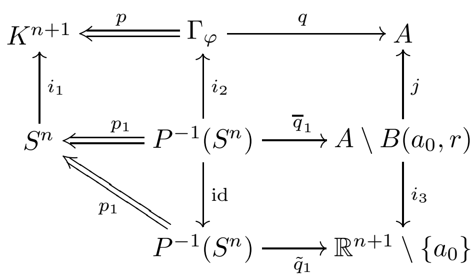
Conversely, assume that there exists an acyclic map $\varphi\colon K^{n+1} \to A$ satisfying (38.2.1) and (38.2.2). We prove that $a_0 \in \operatorname{Int} A$. Consider the diagram:

$$K^{n+1} \overset{p_\varphi}{\Longleftarrow} \Gamma_\varphi \overset{q_\varphi}{\longrightarrow} A,$$

in what follows for simplicity we will write $p = p_\varphi$ and $q = q_\varphi$.

Define a map $p_1\colon p^{-1}(S^n) \to S^n$ and $q_1\colon p^{-1}(S^n) \to A \setminus \{a_0\}$ by putting $p_1(y) = p(y)$, $q_1(y) = q(y)$ for every $y \in p^{-1}(S^n)$. The set $q_1(p^{-1}(S^n))$ is compact. Let $r_0 = \operatorname{dist}(a_0,\ q_1(p^{-1}(S^n)))$. Then $r_0 > 0$. Consider the open ball $B(a_0, r)$, $0 < r < r_0$ and the inclusion map $j\colon A \setminus B(a_0, r) \to A$. We claim that j^{*n} is not an epimorphism.

Indeed, we have the commutative diagram



in which i_1, i_2, i_3 are inclusions, $\widetilde{q}_1, \overline{q}_1$ are given by $\overline{q}_1(y) = \widetilde{q}_1(y) = q_1(y)$ for each $y \in p^{-1}(S^n)$. From the assumption we have $(p_1^{*n})^{-1}\widetilde{q}_1^{*n} \neq 0$.

This implies that $(p_1^{*n})^{-1}\overline{q}_1^{*n} i_3^{*n} \neq 0$ and hence $\overline{q}_1^{*n} \neq 0$. Assume that j^{*n} is an epimorphism. Then we obtain

$$i_1^{*n}(p^{*n})^{-1}q^{*n} = (p_1^{*n})^{-1}\overline{q}_1^{*n} j^{*n} \neq 0,$$

which is a contradiction. Since j^{*n} is not an epimorphism, from (38.1) we obtain $a_0 \in \operatorname{Int} A$, and the proof of (38.2) is completed. □

An acyclic map $\varphi\colon X \multimap Z$ is called an ε-map if the condition $\varphi(x) \cap \varphi(x') \neq \emptyset$ implies $d(x, x') < \varepsilon$ for each $x, x' \in X$.

(38.3) LEMMA. *Let* $\varphi\colon K^{n+1} \multimap \mathbb{R}^{n+1}$ *be a* 1*-map. Then*

(38.3.1) $$\varphi(S^n) \subset \mathbb{R}^{n+1} \setminus \{z_0\} \quad \textit{for each } z_0 \in \varphi(0),$$

(38.3.2) $$\mathrm{Deg}(\varphi) \neq 0.$$

PROOF. Let $z_0 \in \varphi(0)$. We prove that $z_0 \notin \varphi(S^n)$. Assume that $z_0 \in \varphi(x)$ for some $x \in S^n$. Then we have $\varphi(0) \cap \varphi(x) \neq \emptyset$ and from the assumption we deduce that $\|x\| < 1$, which is a contradiction.

Now we prove (38.3.2). We have the diagram:

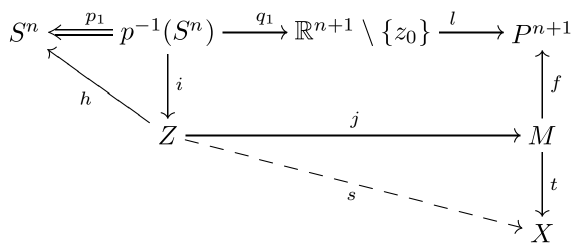
$$K^{n+1} \overset{p}{\Longleftarrow} \Gamma_\varphi \overset{q}{\longrightarrow} \mathbb{R}^{n+1}, \quad \text{in which } p = p_\varphi,\ q = q_\varphi.$$

Let $y_0 \in p^{-1}(0)$ be a point such that $q(y_0) = z$. Define the maps $p_1\colon p^{-1}(S^n) \to S^n$, $q_1\colon p^{-1}(S^n) \to \mathbb{R}^{n+1} \setminus \{z_0\}$ by putting $p_1(y) = p(y)$, $q_1(y) = q(y)$ for each $y \in p^{-1}(S^n)$. For the proof it is sufficient to show that $\deg(p_1, q_1) \neq 0$, where $deg(p_1, q_1) = [(p_1^{*n})^{-1} \circ q_1^{*n}](1)$ and $1 \in H^n(\mathbb{R}^n \setminus \{z_0\}) = Z$ is a fixed generator.

Define the following sets:

$$\begin{aligned} X &= \{(x, x') \in K^{n+1} \times K^{n+1} \mid \|x - x'\| = 1\}, \\ M &= \{(x, x', y, y') \mid (x, x') \in X,\ y \in p^{-1}(x),\ y' \in p^{-1}(x')\}, \\ Z &= \{(x, x', y, y') \mid (x, x', y, y') \in M,\ x' = 0\}. \end{aligned}$$

It is easy to see that X, M, Z are compact sets. Consider the diagram:



in which

$$\begin{aligned} i(y) &= (p_1(y), 0, y, y_0), & h(x, 0, y, y') &= x, \\ j(x, 0, y, y') &= (x, 0, y, y'), & t(x, x', y, y') &= (x, x'), \\ s(x, 0, y, y') &= (x, 0), & t(z) &= z - z_0, \\ f(x, x', y, y') &= q(y) - q(y'). \end{aligned}$$

Since φ is an 1-map, we have $f(x, x', y, y') \neq 0$. It is evident that the above diagram commutes.

As in the proof of theorem on antipodes (Section 37), we deduce that h^*, s^*, t^* are isomorphisms. Hence the commutativity of the above diagram implies that j^* and i^* are isomorphisms. This implies that M has the cohomology of S^n.

Define the involution $g\colon M \to M$ by putting $g(x,x',y,y') = (x',x,y',y)$. Then $f(g(x,x',y,y')) \neq f(x,x',y,y')$. Applying the Theorem On Antipodes to the maps f, g, we obtain $f^{*n} \neq 0$. From the commutativity of the above diagram we have $q_1^{*n} l^{*n} \neq 0$. Finally, we obtain $q_1^{*n} \neq 0$, and this implies that $\mathrm{Deg}(\varphi) \neq 0$. The proof of (38.3) is completed. □

(38.4) Remark. It is evident that Lemma (38.2) remains true for any closed ball in $\mathbb{R}^{n+1}$ with radius ε and for any ε-map, where ε is a positive real number.

We will prove now two theorems of the Brouwer Invariance of Domain Theorem type for acyclic maps.

(38.5) Theorem. *Let $\varepsilon > 0$ be a positive real number. If $\varphi\colon \mathbb{R}^{n+1} \multimap \mathbb{R}^{n+1}$ is an ε-map then $\varphi(\mathbb{R}^{n+1})$ is an open subset of $\mathbb{R}^{n+1}$.*

Proof. Let $y \in \varphi(\mathbb{R}^{n+1})$. We prove that $y \in \operatorname{Int} \varphi(\mathbb{R}^{n+1})$. Assume that $y \in \varphi(x)$ for some $x \in \mathbb{R}^{n+1}$. Let K_ε^{n+1} be a closed ball in $\mathbb{R}^{n+1}$ with the center at x and radius ε.

Since φ is an acyclic map, we deduce that $\varphi(K_\varepsilon^{n+1})$ is a compact set. We have $y \in \varphi(K_\varepsilon^{n+1})$. Let ψ be the restriction of φ to the ball K^{n+1}. Then ψ is an ε-map and hence we have $\psi(S_\varepsilon^n) \subset \mathbb{R}^{n+1} \setminus \{y\}$, where S_ε^n denotes the boundary of K_ε^{n+1}. Therefore, Lemma (38.2) (cf. Remark (38.4)) implies that $0 \notin \mathrm{Deg}(\psi, y)$ and from (38.2) we obtain $y \in \operatorname{Int} \varphi(\mathbb{R}^{n+1})$. The proof of (38.5) is completed. □

(38.6) Theorem. *Let U be an open subset of $\mathbb{R}^{n+1}$ and $\varphi\colon U \multimap \mathbb{R}^{n+1}$ an acyclic map. Assume further that for any two points $x_1, x_2 \in U$ the condition $x_1 \neq x_2$ implies $\varphi(x_1) \cap \varphi(x_2) = \emptyset$. Then $\varphi(U)$ is an open subset of $\mathbb{R}^{n+1}$.*

Proof. From the assumption we infer that φ is an ε-map for each $\varepsilon > 0$. Let $y \in \varphi(U)$. We prove that $y \in \operatorname{Int} \varphi(U)$. Assume that $y \in \varphi(x)$ for some $x \in U$. Since U is open, there exists an $\varepsilon > 0$ such that the set K_ε^{n+1} is contained in U, where K_ε^{n+1} is a closed ball with center at x and radius ε. Let ψ be the restriction of φ to the set K_ε^{n+1}. Since ψ is an ε-map, we have $y \notin \psi(S_\varepsilon^n)$, where S_ε^n is the boundary of K_ε^{n+1}. Applying lemmas (38.3) and (38.2), as in the proof of (38.5), we obtain $y \in \operatorname{Int} \varphi(U)$. The proof of (38.6) is completed. □

Note that for n-acyclic mappings the above theorem is not true in general.

39. n-Acyclic compact vector fields in normed spaces

In this section E will denote a (real) normed space with $\dim E = +\infty$. Let $E^{k+1} \subset E^{k+2}$ be two subspaces of E such that $\dim E^{k+1} = k+1$ and $\dim E^{k+2} =$

$k+2$. Denote by E_+^{k+2}, E_-^{k+2} two closed half-spaces of E^{k+2} such that $E^{k+1} = E_+^{k+2} \cap E_-^{k+2}$ and the respective unit half-spheres $S_+^{k+1} = E_+^{k+2} \cap S$, $S_-^{k+1} = E_-^{k+2} \cap S$, where S is the unit sphere in E.

Clearly, $S^k = S \cap E^{k+1} = S_+^{k+1} \cap S_-^{k+1}$. Then $(S^{k+1}, S_+^{k+1}, S_-^{k+1})$ is a triad and consequently the Mayer–Vietoris homomorphism $\Delta\colon H^k(S^k) \xrightarrow{\sim} H^{k+1}(S^{k+1})$ is an isomorphism (see Chapter I).

In what follows we shall make use of the following:

(39.1) LEMMA. *Let $p, q\colon Y \to S^{k+1}$ be two mappings such that:*

(39.1.1) *p is a n-Vietoris map, $n \le k$,*
(39.1.2) *$q(p^{-1}(S^{k+1})) \subset S_-^{k+1}$,*

Then the following diagram commutes:

$$\begin{array}{ccc} H^k(S^k) & \xrightarrow{\Delta} & H^{k+1}(S^{k+1}) \\ {\scriptstyle \overline{q}^{*k}(\overline{p}^{*k})^{-1}}\big\downarrow & & \big\downarrow{\scriptstyle q^{*k+1}(p^{*k+1})^{-1}} \\ H^k(S^k) & \xrightarrow[\Delta]{} & H^{k+1}(S^{k+1}) \end{array}$$

where $\overline{q}, \overline{p}\colon p^{-1}(S^k) \to S^k$ are the respective restrictions of p and q.

PROOF. Let $Y_+ = p^{-1}(S_+^{k+1})$ and $Y_- = p^{-1}(S_-^{k+1})$. Since p is a n-Vietoris map, $n \le k$, we infer that (Y, Y_+, Y_-) is a k-triad. Observe that $Y_+ \cap Y_- = p^{-1}(S^k)$. By assumptions (39.1.1) and (39.1.2), p, q are maps between triads (Y, Y_+, Y_-) and $(S^{k+1}, S_+^{k+1}, S_-^{k+1})$. Consequently our lemma follows from the Mayer–Vietoris theorem (see Chapter I). □

Let X be a subset of E and let $\Phi\colon X \multimap E$ be a u.s.c. multivalued map. We define a multivalued vector field $\varphi\colon X \to E$ associated with Φ by putting $\varphi = i - \Phi$, where $(i-\Phi)(x) = \{x - y \mid y \in \Phi(x)\}$ for every $x \in X$. We say that φ is a compact vector field provided Φ is compact.

(39.2) PROPOSITION. *If $\varphi\colon X \to E$ is a compact vector field associated with Φ then $\varphi(X)$ is a closed subset of E.*

PROOF. Let $\{u_n\} \subset \varphi(X)$ and $\lim_n u_n = u$. We have to prove that $u \in \varphi(X)$. Since $\varphi = i - \Phi$ we have the following expression:

$$u_n = x_n - y_n, \quad \text{where } x_n \in X \text{ and } y_n \in \Phi(x_n).$$

Observe that $\{y_n\} \subset \Phi(X) \subset \overline{\Phi(X)}$ and $\overline{\Phi(X)}$ is compact. Therefore, we can assume that $\lim_n y_n = y$. Consequently, we deduce that $\lim_n x_n = x \in X$ and hence $u = (x - y) \in \varphi(x) \subset \varphi(X)$. The proof of (39.2) is completed. □

(39.3) Definition. A multivalued vector field $\varphi\colon X \multimap E$ associated with Φ is called *n-acyclic* (*acyclic*) provided Φ is a n-acyclic (acyclic).

We will say that $x_0 \in X$ is a *singular* point of the compact vector field $\varphi\colon X \multimap E$ provided $0 \in \varphi(x_0)$. We let

$$\mathrm{AV}_n(S,P) = \{\varphi\colon S \multimap P \mid \varphi \text{ is a compact } n\text{-acyclic vector field associated with } \Phi\colon S \multimap E\},$$

where $S = E \setminus \{0\}$. Let $\varphi_1 = i - \Phi_1$ and $\varphi_2 = i - \Phi_2$ be two compact n-acyclic vector fields in $\mathrm{AV}_n(S,P)$. We will say that φ_1 and φ_2 are homotopic in $\mathrm{AV}_n(S,P)$ provided there exists a compact n-acyclic vector field $\eta = (i - \chi)\colon S \times [0,1] \multimap P$ such that $\chi(x,0) = \Phi_1(x)$ and $\chi(x,1) = \Phi_2(x)$.

Now we are going to define the topological degree Deg on the class $\mathrm{AV}_n(S,P)$. Let $\varphi = (i - \Phi) \in \mathrm{AV}_n(S,P)$. We consider the following diagram:

$$S \xLeftarrow{p_\Phi} \Gamma_\Phi \xrightarrow{q_\Phi} E,$$

in which p_Φ is a n-Vietoris map. In what follows we let for simplicity $Y = \Gamma_\Phi$, $p = p_\Phi$ and $q = q_\Phi$. We let:

$$\widetilde{q}\colon Y \to P, \quad \widetilde{q}(y) = p(y) - q(y),$$

where

$$y = (x,u) \in \Gamma_\Phi, \quad p(y) = x \quad \text{and} \quad q(y) = u.$$

In view of Proposition (39.2) we obtain $\operatorname{dist}(0, \varphi(S)) = \delta > 0$.

Let $0 < \varepsilon < \delta$. By applying the Schauder Approximation Theorem (12.9) to q and $\varepsilon > 0$ we get a map $q_\varepsilon\colon Y \to E^{k+1}$ such that $\|q(y) - q_\varepsilon(y)\| < \varepsilon$ for every $y \in Y$. Consider the diagram:

$$S^k \xLeftarrow{p_k} Y_k \xrightarrow{q_k} E^{k+1} \quad \text{and} \quad S^k \xLeftarrow{p_k} Y_k \xrightarrow{\widetilde{q}_k} P^{k+1} = E^{k+1} \setminus \{0\},$$

where $Y_k = p^{-1}(S^k)$, p_k is the respective restriction of p, q_k is the respective restriction of q_ε and $\widetilde{q}_k(y) = p_k(y) - q_k(y)$. Observe that $\widetilde{q}_k$ is well defined. Indeed, since $\|x - q(y)\| \geq \delta$ and $y \in p^{-1}(x)$, we have:

$$\begin{aligned}\|\widetilde{q}_k(y)\| &= \|p_k(y) - q(y)\| = \|x - q_\varepsilon(y)\| \\ &\geq \|x - q(y)\| - \|q(y) - q_\varepsilon(y)\| \geq \delta - \varepsilon > 0.\end{aligned}$$

Therefore, we can consider the following diagram

$$S^k \xLeftarrow{p_k} Y_k \xrightarrow{\widetilde{q}_k} P^{k+1}.$$

Since $n < k$ we have:

$$Z \approx H^k(P^{k+1}) \xrightarrow{(\widetilde{q}_k)^{*k}} H^k(Y_k) \xrightarrow[\sim]{[(p_k)^{*k}]^{-1}} H^k(S^k) \approx Z$$

We let:

$$\operatorname{Deg}(\varphi) = [(p_k)^{*k}]^{-1}[(q_k)^{*k}(1)]. \tag{39.4}$$

To prove that definition (39.4) is correct, we shall prove the following two lemmas.

(39.5) LEMMA. *Let $\varphi = i - \Phi \colon S \multimap P$ be an n-acyclic compact vector field and $p = p_\varphi \colon \Gamma_\Phi \to S$ and $q = q_\varphi \colon \Gamma_\Phi \to E$ be two natural projections. Assume further that $q_\varepsilon, q'_\varepsilon \colon \Gamma_\Phi \to E^{k+1}$ are two ε-approximations of q, where $\varepsilon < \operatorname{dist}(0, \varphi(S)) = \delta > 0$. Then $(\widetilde{q}_k)^{*k} = (\widetilde{q}'_k)^{*k}$, where $\widetilde{q}_k, \widetilde{q}'_k \colon Y_k = p^{-1}(S^k) \to P^{k+1}$ are defined as above.*

PROOF. Consider a homotopy $h \colon Y_k \times [0,1] \to E^{k+1}$ defined as follows:

$$h(u,t) = t \cdot \widetilde{q}_k(u) + (1-t)\widetilde{q}'_k(u).$$

It is sufficient to show that $h(Y_k \times [0,1]) \subset P^{k+1}$. In fact, we have:

$$\begin{aligned}
\|t\widetilde{q}_k(u) + {} & (1-t)\widetilde{q}'_k(u)\| \\
&= \|t(p_k(u) - q_k(u)) + (1-t)(p_k(u) - q'_k(u))\| \\
&= \|p_k(u) - tq_\varepsilon(u) + (1-t)q'_\varepsilon(u)\| \\
&= \|p(u) - q(u) - [t(q_\varepsilon(u) - q(u)) + (1-t)(q'_\varepsilon(u) - q(u))]\| \geq \delta - \varepsilon > 0;
\end{aligned}$$

the proof is completed. □

(39.6) LEMMA. *Let $\varphi = i - \Phi$ and p, q be the same as in (5.1). Assume that E^{k+1}, E^{k+2} are two subspaces of E such that $E^{k+1} \subset E^{k+2}$. Let $q_\varepsilon \colon \Gamma_\Phi \to E^{k+1}$ be an ε-approximation of q with $\varepsilon < \operatorname{dist}(0, \varphi(S)) = \delta$ and let $q'_\varepsilon : \Gamma_\Phi \to E^{k+2}$ be defined as follows, $q'_\varepsilon(u) = q_\varepsilon(u)$ for every u. Then:*

$$[(p_k)^{*k}]^{-1}(\widetilde{q}_k)^{*k}(1) = [p_{k+1}^{*k+1}]^{-1}(\widetilde{q}_{k+1})^{*k+1}(1).$$

PROOF. We define $r \colon P^{k+2} \to S^{k+1}$, $r(z) = z/\|z\|$. Applying Lemma (39.1) to the pair (p_{k+1}, rq_{k+1}) we obtain our claim; the proof is completed. □

Note that from (39.5) and (39.6) it immediately follows that definition (39.4) is correct.

The following two properties are self-evident:

(39.7) PROPOSITION. *Let $\varphi, \psi: S \multimap P$ be two n-acyclic compact vector fields. Then:*

(39.7.1) $\varphi \sim \psi$ *implies that* $\operatorname{Deg}(\varphi) = \operatorname{Deg}(\psi)$*;*
(39.7.2) $\varphi \subset \psi$*, i.e.* $\varphi(x) \subset \psi(x)$ *for every* $x \in S$*, implies that* $\operatorname{Deg}(\varphi) = \operatorname{Deg}(\psi)$.

Let us remark that the above topological degree theory will be generalized to mappings called n-admissible compact vector fields in one of the following sections that. We will also present topological consequences of our degree theory then.

40. Admissible mappings

The class of acyclic and hence n-acyclic mappings is not closed with respect to the composition law (see Example (32.2)). Therefore, we are going to extend the class of acyclic mappings to the class of, so called, admissible mappings (see [Go1-M]), which is closed with respect to the composition law.

(40.1) DEFINITION ([Go1-M]). A multivalued map $\varphi: X \multimap Y$ is called *admissible* (*strongly admissible*) provided there exists a (metric) space Γ and two mappings $p: \Gamma \Longrightarrow X$, $q: \Gamma \to Y$ such that:

(40.1.1) p is a Vietoris map,
(40.1.2) $q(p^{-1}(x)) \subset \varphi(x)$ $(q(p^{-1}(x)) = \varphi(x))$ for every $x \in X$.

(40.2) REMARK. By the term Vietoris map we understand Vietoris with respect to the Čech homology functor with compact carriers and coefficients Q (cf. Chapter I). But, of course, we can consider Vietoris mappings with respect to the Čech cohomology functor if necessary.

First, note that any acyclic map $\varphi: X \multimap Y$ is strongly admissible. In fact, it is enough to take $\Gamma = \Gamma_\varphi$ and $p = p_\varphi$, $q = q_\varphi$.

In what follows, a pair of mappings (p, q) satisfying (40.1) is called a *selected pair* of φ (written $(p, q) \subset \varphi$ or $(p, q) = \varphi$, when φ is strongly admissible).

(40.3) REMARKS.

(40.3.1) Observe that the map $\psi: S^1 \multimap S^1$, $\psi(z) = S^1$, for every $z \in S^1$ is admissible. Let $f: S^1 \to S^1$ be an arbitrary (continuous) map. Then $(\mathrm{id}_{S^1}, f) \subset \varphi$. We will see later that ψ is even strongly admissible as a composition of strongly admissible maps.

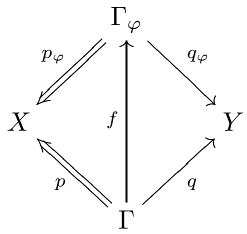
(40.3.2) Observe that if $\varphi: X \multimap Y$ is strongly admissible then $\varphi(x)$ is compact connected for every $x \in X$. So the map $\varphi: [0,1] \multimap [0,1]$ given by

$$\varphi(t) = \begin{cases} \{t\} & \text{for } t \neq 0, \\ \{0,1\} & \text{for } t = 0. \end{cases}$$

is admissible but not strongly admissible.

(40.4) PROPOSITION. *If $\varphi\colon X \multimap Y$ is acyclic and $(p,q) \subset \varphi$, then $\varphi_* = q_* p_*^{-1}$.*

PROOF. We have the following commutative diagram:

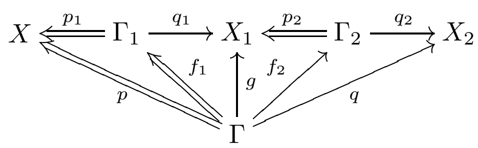


in which $f(z) = (p(z), q(z))$. Now, by applying the functor H to the above diagram we obtain our claim. □

(40.5) THEOREM. *Let $\varphi\colon X \multimap X_1$ and $\psi\colon X_1 \multimap X_2$ be two admissible maps. Then the composition $\psi \circ \varphi\colon X \multimap X_2$ is an admissible map and for every selected pairs $(p_1, q_1) \subset \varphi$ and $(p_2, q_2) \subset \psi$ there exists a selected pair (p, q) of $\psi \circ \varphi$ such that*

$$q_{2*} \circ (p_{2*})^{-1} \circ q_{1*} \circ (p_{1*})^{-1} = q_* \circ p_*^{-1}.$$

PROOF. Let $(p_1, q_1) \subset \varphi$ and $(p_2, q_2) \subset \psi$. Consider the commutative diagram



in which, for each $(z_1, z_2) \in \Gamma$,

$$\Gamma = \{(z_1, z_2) \in \Gamma_1 \times \Gamma_2 \mid q_1(z_1) = p_2(z_2)\}, \quad p(z_1, z_2) = p_1(z_1),$$
$$q(z_1, z_2) = q_2(z_2), \quad f_1(z_1, z_2) = z_1, \quad f_2(z_1, z_2) = z_2, \quad g(z_1, z_2) = q_1(z_1).$$

Since $f_1^{-1}(z_1)$ is homeomorphic to $p_2^{-1}(q_1(z_1))$ and p_2 is a Vietoris map we deduce that f_1 is a Vietoris map. Hence p, as the composition $p_1 \circ f_1$, is a Vietoris map. Moreover, we have $q(p^{-1}(x)) \subset \psi(\varphi(x))$ for each $x \in X$. Applying to the above diagram the functor H, we obtain

$$q_{2*}(p_{2*})^{-1} \circ q_{1*}(p_{1*})^{-1} = q_* p_*^{-1}$$

and the proof (40.5) is completed. □

(40.6) THEOREM. *If $\varphi\colon X \multimap X_1$ and $\psi\colon X_1 \multimap X_2$ are two s-admissible maps then the composition $\psi\circ\varphi\colon X \multimap X_2$ is an s-admissible map and for every $(p_1, q_1) = \varphi$ and $(p_2, q_2) = \psi$ there exists a $(p, q) = \psi \circ \varphi$ such that*

$$q_{2*}(p_{2*})^{-1} \circ q_{1*}(p_{1*})^{-1} = q_* p_*^{-1}.$$

The proof of (40.6) is strictly analogous to (40.5). Note that from (40.6) it follows that the composition of two acyclic maps is strongly admissible.

Let $\varphi\colon X \multimap Y$ be an admissible map. We will define the set $\{\varphi\}_*$ of induced by φ linear mappings on homology of X into homology of Y. We let:

$$\{\varphi\}_* = \{q_* p_*^{-1} \mid (p, q) \subset \varphi\} \quad \text{and} \quad \{\varphi\}_* = \{q_* p_*^{-1} \mid (p, q) = \varphi\}$$

in the case of a strongly admissible map.

(40.7) EXAMPLE. Let $\psi\colon S^n \multimap S^n$ be the map defined as follows:

$$\psi(z) = S^n \quad \text{for every } z \in S^n.$$

Since $(\mathrm{id}_{S^n}, f) \subset \psi$ for any $f\colon S^n \to S^n$ we see that $\{\psi\}_*$ is an infinite set. On the other hand, ψ is also strongly admissible. Observe that the set:

$$\{\psi\}_* = \{q_* p_*^{-1} \mid (p, q) = \psi\}$$

is not a singleton. In fact, let $\varphi_1\colon S^n \to S^n$, $\varphi_1(z) = \{y \in S^n \mid \|y - z\| \le 3/2\}$ (cf. Example (32.2)). Since $\mathrm{id}_{S^n} \subset \varphi_1$ it follows that $\varphi_{1*} = \mathrm{id}_{H(S^n)}$ (see (40.5)).

Let $\varphi_2(x) = \varphi_1(-x)$. Then $(-\mathrm{id}_{S^n}) \subset \varphi_2$, so again in view of (40.5) we have $(-\mathrm{id}_{S^n})_* = \varphi_{2*}$. By applying Theorem (40.7), there are two selected pairs $(p, q) = \varphi_1 \circ \varphi_1$, $(p', q') = \varphi_1 \circ \varphi_2$ such that $q_* p_*^{-1} = \varphi_{1*} \circ \varphi_{1*}$ and $q'_*(p'_*)^{-1} = \varphi_{2*}\varphi_{1*}$. Finally, if $n = 2k$ we obtain that $q_* p_*^{-1} \ne q'_*(p'_*)^{-1}$. Since $\psi = \varphi_1 \circ \varphi_1 = \varphi_2 \circ \varphi_1$ we conclude that the set $\{\psi\}_*$ consists of at least two elements when ψ is regarded as a strongly admissible map.

We will prove:

(40.8) PROPOSITION. *Let $\varphi, \psi\colon X \multimap Y$ be two admissible maps. If $\varphi \subset \psi$, then $\{\varphi\}_* \subset \{\psi\}_*$.*

For the proof of (40.8) it is sufficient to see that if $(p, q) \subset \varphi$ then $(p, q) \subset \psi$.

Now from (40.8) and (40.4) we obtain:

(40.9) COROLLARY. *Let $\psi\colon X \multimap Y$ be an acyclic map and $\varphi\colon X \multimap Y$ an admissible map. If $\varphi \subset \psi$ then the set $\{\varphi\}_*$ is a singleton and $\{\varphi\}_* = \{\psi_*\}$.*

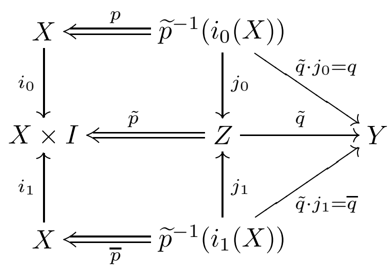
(40.10) Definition. Two admissible maps $\varphi, \psi\colon X \multimap Y$ are called *homotopic* (written $\varphi \sim \psi$) provided there exists an admissible map $\chi\colon X \times I \multimap Y$, $I = [0,1]$, such that:

$$\chi(x,0) \subset \varphi(x) \quad \text{and} \quad \chi(x,1) \subset \psi(x) \quad \text{for every } x \in X.$$

(40.11) Theorem. *Let $\varphi, \psi\colon X \multimap Y$ be two admissible maps. Then $\varphi \sim \psi$ implies that there exist selected pairs $(p,q) \subset \varphi$ and $(\overline{p}, \overline{q}) \subset \psi$ such that:*

$$q_* \circ p_*^{-1} = \overline{q}_* \circ \overline{p}_{1*}^{-1}.$$

Proof. Let $(\widetilde{p}, \widetilde{q}) \subset \chi$. Consider the commutative diagram



where $i_0(x) = (x,0)$, $i_1(x) = (x,1)$ for each $x \in X$, j_0, j_1 are inclusions and $p, \overline{p}$ are given as the first coordinates of $p(z)$ for every $z \in \widetilde{p}^{-1}(i_0(X))$ and $z \in \widetilde{p}^{-1}(i_1(X))$ respectively. Then $p, \overline{p}$ are Vietoris maps and we have $(p,q) \subset \varphi$, $(\overline{p}, \overline{q}) \subset \psi$. Observe that $i_{0*} = i_{1*}$ is a linear isomorphism. This and the commutativity of the above diagram imply $q_* \circ p_*^{-1} = \overline{q}_* \circ \overline{p}_*^{-1}$. This proves Theorem (40.11). $\square$

(40.12) Corollary. *Let $\varphi, \psi\colon X \multimap Y$ be two admissible maps. Then $\varphi \sim \psi$ implies $\{\varphi\}_* \cap \{\psi\}_* \neq \emptyset$.*

(40.13) Corollary. *Let $\varphi, \psi\colon X \to Y$ be two acyclic maps. Then $\varphi \sim \psi$ implies $\varphi_* = \psi_*$.*

(40.14) Example. Let $\varphi_1, \psi\colon S^n \multimap S^n$ be the same as in Example (40.7). Consider the homotopy $\chi\colon S^n \times I \multimap S^n$ defined by:

$$\chi(x,t) = \varphi_1(x) \quad \text{for every } x \in S^n.$$

Then χ is a homotopy joining φ_1 with ψ but $\{\varphi_1\}_*$ is an infinite set.

(40.15) DEFINITION. An admissible map $\varphi\colon X \multimap X$ is called a *Lefschetz map* provided the linear map $q_* \circ p_*^{-1}\colon H(X) \to H(X)$ is a Leray endomorphism for every selected pair $(p, q) \subset \varphi$ (see Section 11).

For a Lefschetz map $\varphi\colon X \multimap X$ we define the Lefschetz set $\mathbf{\Lambda}(\varphi)$ of φ by putting:

$$\mathbf{\Lambda}(\varphi) = \{\Lambda(q_* p_*^{-1}) \mid (p, q) \subset \varphi\},$$

where $\Lambda(q_* p_*^{-1})$ denotes the generalized Lefschetz number of $q_* \circ p_*^{-1}$ (see again Section 11).

Note that if φ is an acyclic Lefschetz map then $\mathbf{\Lambda}(\varphi) = \{\Lambda(\varphi)\} = \{\Lambda(q_* p_*^{-1})\}$ is a singleton.

We have the following simple properties:

(40.16) PROPOSITION. *Let $\varphi, \psi\colon X \multimap X$ be two Lefschetz maps.*

(40.16.1) *If $\varphi \subset \psi$, then $\mathbf{\Lambda}(\varphi) \subset \mathbf{\Lambda}(\psi)$;*
(40.16.2) *If φ and ψ are acyclic and $\varphi \subset \psi$ or $\varphi \sim \psi$, then φ is a Lefschetz map if and only if ψ is a Lefschetz map and in this case $\Lambda(\varphi) = \Lambda(\psi)$.*

(40.17) EXAMPLE. Let X be a space which is not of a finite type. Let $f\colon X \to X$, $f(x) = x_0$ for every $x \in X$ and $\varphi(x) = X$ for every $x \in X$. We have $f \subset \varphi$ and $\mathrm{id}_X \subset \varphi$ but f_* is a Leray endomorphism and $\mathrm{id}_{H(X)}$ is not a Leray endomorphism.

41. The Lefschetz fixed point theorem for admissible mappings

The aim of this section is to formulate the Lefschetz fixed point theorem for admissible mappings of AANR-spaces.

The notion of an AANR-space was introduced in (3.1). Now we need some special properties of AANR-spaces related directly to the Lefschetz fixed point theorem.

(41.1) DEFINITION. An AANR-space X is said to be admissible provided there exists a homeomorphism $h\colon X \to E$ mapping X onto a closed subset $h(X)$ of a normed space E and an open neighbourhood U of $h(X)$ in E such that the following two conditions are satisfied:

(41.1.1) $h(X)$ is an approximative retract of U,
(41.1.2) the inclusion $i\colon h(X) \to U$ induces a monomorphism $i_*\colon H(h(X)) \to H(U)$.

(41.2) PROPOSITION. *Every* ANR *is an admissible* AANR.

PROOF. Let $X \in$ ANR. Using the Arens–Eells embedding theorem, we obtain a homeomorphism h mapping X into a normed space E such that

(41.2.1) $h(X)$ is closed in E,

(4.1.2.2) there exists a retraction $r: U \to h(X)$, where U is an open neighbourhood of $h(X)$ in E.

Then the inclusion $i: h(X) \to U$ is the right inverse of r and we have $ri = \mathrm{id}_{h(X)}$. We infer that $r_* i_* = \mathrm{id}_{H(h(X))}$ and this implies that i_* is a monomorphism. □

(41.3) PROPOSITION. *Every compact* AANR *is an admissible* AANR.

PROOF. Using the Arens–Eells embedding theorem we may assume without loss of generality that X is an approximative retract of some open neighbourhood U of X in a normed space E. Since X is of finite type (see (6.1)), we deduce that there exists an $\varepsilon_0 > 0$ such that for every two maps $f, g: X \to X$, the condition $\|f(x) - g(x)\| < \varepsilon_0$ implies $f_* = g_*$.

Choose an $\varepsilon > 0$ such that $\varepsilon < \varepsilon_0$ and consider the two maps id, $r_\varepsilon \circ i: X \to X$, where $r: U \to X$ is an ε-retraction and $i: X \to U$ an inclusion map. By Theorem (6.1), we infer that $\mathrm{id}_{H(X)} = (r_\varepsilon)_* \cdot i_*$ and this implies that $i_*: H(X) \to H(U)$ is a monomorphism. □

(41.4) PROPOSITION. *Every acyclic* AANR *is an admissible* AANR.

For the proof (41.4) observe that if X is an acyclic space and $X \subset Y$ then the inclusion $i: X \to Y$ induces a monomorphism $i_*: H(X) \to H(Y)$.

The following lemma is of importance:

(41.5) LEMMA. *Let X be an* AANR. *Assume that X is an approximative retract of an open subset U in a space E and $i: X \to U$ induces a monomorphism $i_*: H(X) \to H(U)$. Then for every compact subset $K \subset X$ there exists a positive real number $\varepsilon(K)$ such that for every $\varepsilon < \varepsilon(K)$ and for every ε-retraction $r_\varepsilon: U \to X$ we have:*

$$(r_\varepsilon)_* i_* j_* = j_*, \quad \text{where } j: K \to X \text{ is the inclusion map.}$$

PROOF. Let $\varepsilon(K) > 0$ be a number smaller than the distance $\mathrm{dist}(K, \partial U)$ from the compact set K to the boundary ∂U of U in E. From the definition of $\varepsilon(K)$ we infer that for each $x \in X$ and $\varepsilon < \varepsilon(K)$ the interval $t \cdot ir_\varepsilon ij(x) + (1-t) \cdot ij(x)$, where $0 \le t \le 1$, is entirely contained in U. This implies that $ir_\varepsilon ij$ and ij are homotopic for every $\varepsilon < \varepsilon(K)$. Since i_* is a monomorphism, we get $(r_\varepsilon)_* i_* j_* = j_*$ for each $\varepsilon < \varepsilon(K)$ and the proof is completed. □

Before starting the main result of this section in full generality, we shall first consider the following special case:

(41.6) LEMMA. *Let U be an open subset of a normed space and let $\varphi: U \multimap U$ be an admissible compact map. Then:*

(41.6.1) *φ is a Lefschetz map, and*
(41.6.2) *$\mathbf{\Lambda}(\varphi) \neq \{0\}$ implies that* $\mathrm{Fix}(\varphi) \neq \emptyset$.

PROOF. Let $(p, q) \subset \varphi$ be a selected pair. Since φ is compact, so q is compact, too. By (12.10) $q_* p_*^{-1}$ is a Leray endomorphism and hence φ is a Lefschetz map. Assume that $\mathbf{\Lambda}(\varphi) \neq \{0\}$. Therefore, there is $(p, q) \subset \varphi$ such that $\Lambda(q_* p_*^{-1}) \neq 0$. Consequently p and q have a coincidence point u, i.e. $p(u) = q(u)$ (see again (12.10)). Thus $x = p(u)$ is a fixed point for φ and the proof of (41.6) is completed. □

We are able now to prove the principal result of this section.

(41.7) THEOREM. *Let X be an admissible* AANR*-space and $\varphi: X \multimap X$ be a compact admissible map. Then:*

(41.7.1) *φ is a Lefschetz map, and*
(41.7.2) *$\mathbf{\Lambda}(\varphi) \neq \{0\}$, implies that* $\mathrm{Fix}(\varphi) \neq \emptyset$.

PROOF. Since X is an admissible AANR, we may assume that there exists an open subset of a normed space E such that the following two conditions are satisfied:

(41.7.3) X is an approximative retract of U,
(41.7.4) the inclusion $i: X \to U$ induces a monomorphism $i_*: H(X) \to H(U)$.

Let $r_n: U \to X$ be $(1/n)$-retraction. We have

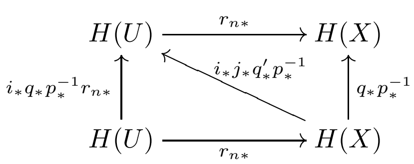
(41.7.5) $\|r_n(x) - x\| < 1/n$ for each $x \in X$ and for every n.

Let $p, q: Y \to X$ be a pair of maps such that $(p, q) \subset \varphi$. Consider for each n an admissible compact map $\psi_n: U \multimap U$ given by $\psi_n = iq\varphi r_n$. Using (40.6) we choose a selected pair $(p_n, q_n) \subset \psi_n$ such that

(41.7.6) $q_{n*} p_{n*}^{-1} = i_* q_* p_*^{-1} r_{n*}$, for each n.

Since q is a compact map, we infer that the set $A = \overline{q(Y)}$ is compact.

Consider for each n the diagram



where $q': Y \to A$ is given by $q'(y) = q(y)$ for each $y \in Y$ and $j: A \to X$ is an inclusion. From Lemma (41.5) we obtain $r_{n*} i_* j_* = j_*$ for all $n > n_0$. Since $j_* q'_* = (j \circ q')_* = q_*$, we deduce that the above diagram commutes for each

$n > n_0$. Consequently, from (11.4) and (41.7.6) we conclude that $q_* p_*^{-1}$ is a Leray endomorphism. Thus the assertion (41.7.1) is proved.

To prove (41.7.2) assume that $\mathbf{\Lambda}(\varphi) \neq \{0\}$. Then there exists a selected pair $(p, q) \subset \varphi$ such that $\Lambda(q_* p_*^{-1}) \neq 0$. Let $(p_n, q_n) \subset \psi_n$, where p_n, q_n and ψ_n are obtained as in the first part of the proof. Then from (11.4) and (41.7.6) we have

$$\Lambda(q_{n*} p_{n*}^{-1}) = \Lambda(i_* q_* p_*^{-1} r_{n*}) = \Lambda(q_* p_*^{-1}) \neq 0, \quad \text{for each } n > n_0.$$

This, in view of (41.6), implies that ψ_n has a fixed point for each $n > n_0$. We find a sequence $\{x_n\}$ in the compact set A such that:

(41.7.7) $x_n \in \psi_n(x_n)$ for each $n > n_0$.

Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that

(41.7.8) $\lim_k x_{n_k} = x$.

Then from (41.7.5) we obtain

(41.7.9) $\lim_k r_{n_k}(x_{n_k}) = x$.

Conditions (41.7.7)–(41.7.9) give

(41.7.10) $\{r_{n_k}(x_{n_k})\} \to x$, $x_{n_k} \in q\varphi_p r_{n_k}(x_{n_k})$ and $\{x_{n_k}\} \to x$, where $\varphi_p = p^{-1}$.

Finally, then by u.s.c. of $\psi = q \circ \varphi_p$ in view of (41.7.10) $x \in \psi(x) = q \circ \varphi_p(x) = qp^{-1}(x) \subset \varphi(x)$ and the proof of Theorem (41.7) is completed. □

We now draw a few immediate consequences of Theorem (41.7)

(41.8) COROLLARY. *Let X be an* ANR *or a compact* AANR *and let $\varphi\colon X \to X$ be an admissible compact map. Then*

(41.8.1) *φ is a Lefschetz map, and*
(41.8.2) $\mathbf{\Lambda}(\varphi) \neq \{0\}$

implies that φ has a fixed point.

For acyclic maps we obtain the following:

(41.9) COROLLARY. *Let X be an admissible* AANR *or, in particular, either of the following:*

(41.9.1) *an* ANR*,*
(41.9.2) *a compact* AANR*.*

If $\varphi\colon X \multimap X$ is a compact acyclic map then

(41.9.3) *φ is Lefschetz map, and*
(41.9.4) $\Lambda(\varphi) \neq 0$

implies that φ has a fixed point.

From (41.9) and (40.17) we deduce:

(41.10) COROLLARY. *Let X be an admissible* AANR *and let $\varphi, \psi: X \multimap X$ be two compact acyclic maps which satisfy one of the following conditions:*

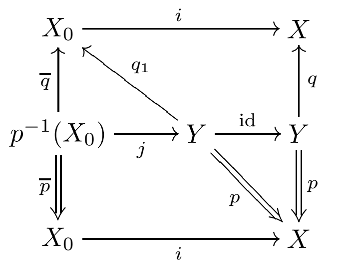
(41.10.1) *φ is a selector of ψ,*

(41.10.2) *φ is homotopic to ψ.*

Then both φ and ψ are Lefschetz maps, $\Lambda(\varphi) = \Lambda(\psi)$, and $\Lambda(\psi) \neq 0$ implies that φ has a fixed point.

(41.11) COROLLARY. *Let X be an admissible* AANR *and $\varphi: X \multimap X$ an admissible compact map. Assume further that $\varphi(X)$ is contained in an acyclic subset X_0 of X. Then $\mathbf{\Lambda}(\varphi) = \{1\}$ and φ has a fixed point.*

PROOF. Let $p, q: Y \to X$ be a pair of maps such that $(p, q) \subset \varphi$. Write the diagram



in which p, q, q_1 are restrictions of p and q respectively, and i, j are inclusions. Then its image under H also commutes. Since $\Lambda(q_* p_*^{-1}) = 1$ from (11.4), we have $\Lambda(q_* p_*^{-1}) = 1$ for every $(p, q) \subset \varphi$, and from Theorem (41.7) we obtain (41.11). □

A space X has the fixed point property within the class of admissible compact maps provided any admissible compact map $\varphi: X \multimap X$ has a fixed point.

(41.12) COROLLARY. *Let X be an acyclic* AANR *or, in particular, any of the following:*

(41.12.1) *an acyclic* ANR*,*

(41.12.2) *a contractible open set in a normed space.*

Then X has the fixed point property within the class of admissible compact maps.

This simply follows from (41.12) and (6.1). Similarly, from (41.12) and (6.1), we have

(41.13) COROLLARY (The Schauder Fixed Point Theorem). *Let X be a convex subset of a normed space. Then X has the fixed point property within the class of admissible compact maps.*

42. The Lefschetz fixed point theorem for non-compact admissible mappings

The aim of this section is to extend the Lefschetz fixed point theorem onto a class of non-compact mappings: the class of compact absorbing contractions. We define:

(42.1) Definition. A multivalued map $\varphi: X \to X$ is called a *compact absorbing contraction*, if there exists an open set $U \in X$ such that $\operatorname{cl}\varphi(U)$ is a compact subset of U and $X \subset \bigcup_{i=0}^{\infty} \varphi^{-i}(U)$.

Evidently, any compact map $\varphi: X \multimap X$ is a compact absorbing contraction; then we can take $U = X$.

In what follows we will use the following notion: $\varphi \in \mathrm{CAC}(X)$ if and only if $\varphi: X \multimap X$ is admissible and a compact absorbing contraction.

(42.2) Proposition. *If $\varphi \in \mathrm{CAC}(X)$ then for every selected pair $(p, q) \subset \varphi$ the homomorphism:*

$$\widetilde{q}_* \circ \widetilde{p}_*^{-1}: H(X, U) \to H(X, U)$$

is weakly nilpotent, where for $p, q: \Gamma \to X$ we define $\widetilde{p}, \widetilde{q}: (\Gamma, p^{-1}(U)) \to (X, U)$, $\widetilde{p}(u) = p(u)$ and $\widetilde{q}(u) = q(u)$ for every $u \in \Gamma$.

Proof. For any compact $K \subset X$ one can find n such that $(qp^{-1})^n(K) \subset U$. Since we consider the Čech homology functor with compact carriers then our claim holds true. □

Now, we shall prove the following:

(42.3) Theorem. *Let $X \in \mathrm{ANR}$ and $\varphi \in \mathrm{CAC}(X)$. Then φ is a Lefschetz map and $\mathbf{\Lambda}(\varphi) \neq \{0\}$ implies that $\mathrm{Fix}(\varphi) \neq \emptyset$.*

Proof. Let $\varphi: X \multimap X$ be an admissible compact absorbing contraction map. Since $\varphi(U) \subset \operatorname{cl}\varphi(U) \subset U$, consider $\varphi': U \multimap U$, $\varphi'(x) = \varphi(x)$. Let $(p, q) \subset \varphi$ be a selected pair of φ. Then $q(p^{-1}(U)) \subset \varphi(U)$. Let $p, q: Y \to X$. Then we define $q', p': p^{-1}(U) \to U$, $p'(u) = p(u)$, $q'(u) = q(u)$. Observe that $(p', q') \subset \varphi'$. Since φ' is compact, in view of (41.8), $q'_*(p'_*)^{-1}$ is a Leray endomorphism. Consider the maps $p'', q'': (Y, p^{-1}(U)) \to (X, U)$; p'' is a Vietoris map and, in view of (42.2) $q''_* \circ (p''_*)^{-1}$ is weakly nilpotent. Consequently, from (11.5), (11.8) and (41.8) we deduce that $\Lambda(q_* p_*^{-1}) = \Lambda(q'_*(p'_*)^{-1})$. So, φ is a Lefschetz map.

Now, if we assume that $\Lambda(q_* p_*^{-1}) \neq 0$ for some $(p, q) \subset \varphi$, then $\Lambda(q'_*(p'_*)^{-1}) \neq 0$ and by using once again (41.8) we get $\mathrm{Fix}(\varphi') \neq \emptyset$ but it implies that $\mathrm{Fix}(\varphi) \neq \emptyset$ and the proof is completed. □

Now, we would like to show how large the class $\mathrm{CAC}(X)$ is.

(42.4) DEFINITION. An u.s.c. multivalued map $\varphi: X \multimap Y$ is called *locally compact* provided that, for each $x \in X$, there exists a subset V of X such that $x \in V$, and the restriction $\varphi|_V$ is compact.

(42.5) DEFINITION. A multivalued locally compact map $\varphi: X \multimap X$ is called *eventually compact* if there exists an iterate $\varphi^n: X \multimap X$ of φ such that φ^n is compact.

(42.6) DEFINITION. A multivalued locally compact map $\varphi: X \multimap X$ is called a *compact attraction* if there exists a compact K of X such that for each open neighbourhood V of K we have $X \subset \bigcup_{i=0}^{\infty} \varphi^{-i}(V)$ and $\varphi^n(x) \subset V$ implies that $\varphi^m(x) \subset V$ for every $m \geq n$ and every $x \in X$, the compact K is then called an *attractor* for φ.

(42.7) DEFINITION. A multivalued locally compact map $\varphi: X \multimap X$ is called *asymptotically compact* if the set $C_\varphi = \bigcap_{n=0}^{\infty} \varphi^n(X)$ is a nonempty, relatively compact subset of X. The set C_φ is called the *center* of φ.

Note that any multivalued eventually compact map is a compact attraction and asymptotically compact map.

(42.8) LEMMA. *Any eventually compact map is a compact absorbing contraction map.*

PROOF. Let $\varphi: X \multimap X$ be an eventually compact map such that $K' = \overline{\varphi^n(X)}$ is compact. Define $K = \bigcup_{i=0}^{n-1} \varphi^i(K')$, we have

$$\varphi(K) \subset \bigcup_{i=1}^{n} \varphi^i(K') \subset K \cup \varphi^n(X) \subset K \cup K' \subset K.$$

Since φ is locally compact, there exists an open neighbourhood V_0 of K such that $L = \overline{\varphi(V_0)}$ is compact, where $\overline{\varphi(V_0)} = \operatorname{cl} \varphi(V_0)$.

There exists a sequence $\{V_1, \ldots, V_n\}$ of open subsets of X such that $L \cap \overline{\varphi(V_i)} \subset V_{i-1}$ and $K \cup \varphi^{n-i}(L) \subset V_i$ for all $i = 1, \ldots, n$. In fact, if $K \cup \varphi^{n-i}(L) \subset V$, and $0 \leq i < n$, since $K \cup \varphi^{n-i}(L)$ and $CV_i \cap L$ are disjoint compact sets of X, there exists an open subset W of X such that

$$K \cup \varphi^{n-i}(L) \subset W \subset \overline{W} \subset V_i \cup CL.$$

Define $V_{i+1} = \varphi^{-1}(W)$; since $\varphi(K) \cup \varphi(\varphi^{n-(i+1)}(L)) \subset K \cup \varphi^{n-i}(L) \subset W$, we have $K \cup \varphi^{n-(i+1)}(L) \subset V_{i+1}$, and $\varphi(V_{i+1}) \subset \overline{W} \subset V_i \cup CL$ implies $L \cap \overline{\varphi(V_{i+1})} \subset V_i$. Beginning with $K \cup \varphi^n(L) \subset K \subset V_0$, we define, by induction $V_1, \ldots, V_n$ with the desired properties.

Putting $U = V_0 \cap V_1 \cap \ldots \cap V_n$, we have $K' \subset K \subset U$ and

$$\varphi(U) \subset \varphi(V_0) \cap \varphi(V_1) \cap \ldots \cap \varphi(V_n) \subset L \cap \overline{\varphi(V_1)} \cap \ldots \cap \overline{\varphi(V_n)},$$

hence

$$\overline{\varphi(U)} \subset (L \cap \overline{\varphi(V_1)}) \cap \ldots \cap (L \cap \overline{\varphi(V_n)}) \cap L \subset V_0 \cap \ldots \cap V_{n-1} \cap V_n = U,$$

but $\overline{\varphi(U)}$ is compact since $\overline{\varphi(U)} \subset L$. Moreover,

$$X = \bigcup_{i=1}^{n} \varphi^{-i}(K') \subset \bigcup_{i=0}^{\infty} \varphi^{-i}(U). \qquad \square$$

(42.9) PROPOSITION. *Any compact attraction map is a compact absorbing contraction map.*

PROOF. Let $\varphi\colon X \multimap X$ be a compact attraction map, K, a compact attractor for φ and W, an open set of X such that $K \subset W$ and $L = \overline{\varphi(W)}$ is compact. We have $L \subset X \subset \bigcup_{i=0}^{\infty} \varphi^{-i}(W)$ hence, since L is compact, there exists $n \in N$ such that $L \subset \bigcup_{i=0}^{n} \varphi^{-i}(W)$. Define $V = \bigcup_{i=0}^{n} \varphi^{-i}(W)$. Then

$$X \subset \bigcup_{i=0}^{\infty} \varphi^{-i}(W) \subset \bigcup_{i=0}^{\infty} \varphi^{-i}(V),$$

$$\varphi(V) \subset \bigcup_{i=0}^{n} \varphi^{-i+1}(W) \subset \varphi(W) \cup V \subset L \cup V \subset V$$

and

$$\varphi^{n+1}(V) \subset \bigcup_{i=0}^{n} \varphi^{n-i+1}(W) = \bigcup_{j=0}^{n} \varphi^{j+1}(W) \subset \bigcup_{j=0}^{n} \varphi^{j}(L),$$

which is compact and included in V, since $L \subset V$ and $\varphi(V) \subset V$ implies that $\varphi^j(L) \subset V$ for all $j \in N$. Consider the restriction $\varphi'\colon V \to V$ of φ. $\varphi'\colon V \to V$ is an eventually compact map, since V is an open set. By Lemma (42.8), there exists an open subset U of V, hence of X, such that $\operatorname{cl}\varphi'(U) = \operatorname{cl}\varphi(U)$ is a compact subset of U and $V \subset \bigcup_{n=0}^{\infty} {\varphi'}^{-n}(U) \subset \bigcup_{n=0}^{\infty} \varphi^{-n}(U)$. Hence

$$X \subset \bigcup_{n=0}^{\infty} \varphi^{-n}(W) \subset \bigcup_{n=0}^{\infty} \varphi^{-n}(V) \subset \bigcup_{n=0}^{\infty} \varphi^{-n}(U). \qquad \square$$

From Theorem (42.3) and Proposition (42.9) we deduce:

(42.10) COROLLARY. *Let $X \in \mathrm{ANR}$ and $\varphi\colon X \multimap X$ be an admissible compact attraction map. Then φ is a Lefschetz map and $\mathbf{\Lambda}(\varphi) \neq \{0\}$ implies that* $\mathrm{Fix}(\varphi) \neq \emptyset$.

(42.11) LEMMA. *Let $\varphi\colon X \multimap X$ be an u.s.c. multivalued map, $C_\varphi = \bigcap_{i=0}^{\infty} \varphi^i(X)$ and let V be an open subset of X such that $C_\varphi \subset V$. Then, for each compact K of X, there exists $n \in N$ such that $\bigcap_{i=0}^{\infty} \varphi^i(K) \subset V$.*

PROOF. The family $\big\{\bigcap_{i=0}^{n} \varphi^i(K) \cap CV\big\}_{n\in N}$ of closed subsets of the compact K, has an empty intersection, hence there exists a finite empty intersection. □

(42.12) LEMMA. *Let $\varphi\colon X \multimap X$ be an u.s.c. multivalued map,*

$$C_\varphi = \bigcap_{i=0}^{\infty} \varphi^i(X), \quad U_\varphi = \bigg\{x \in X \;\bigg|\; \overline{\bigcup_{i=0}^{\infty} \varphi^i(X)} \textit{ is compact}\bigg\}$$

and V, an open subset of X such that $C_\varphi \subset V$. Then $U_\varphi \subset \bigcup_{i=0}^{\infty} \varphi^{-i}(V)$.

PROOF. Let $x \in U_\varphi$, $K = \overline{\bigcup_{n=0}^{\infty} \varphi^n(X)}$ is compact. By Lemma (42.11), there exists $n \in N$ such that $\varphi^n(x) \subset \bigcap_{i=0}^{\infty} \varphi^i(K) \subset V$. □

(42.13) DEFINITION. A multivalued map $\varphi\colon X \multimap X$ is called a map *with compact orbits* if $\bigcup_{n=0}^{\infty} \varphi^n(x)$ is relatively compact for every $x \in X$.

(42.14) PROPOSITION. *Any asymptotically compact map with compact orbits, is a compact attraction map.*

PROOF. Let $\varphi\colon X \multimap X$ be an asymptotically compact map with compact orbits, then $U_\varphi = X$ so $\overline{C}_\varphi$ is a compact attractor for φ and φ is a compact attraction map. □

(42.15) LEMMA. *Let X be a space and $\varphi\colon X \multimap X$ an asymptotically compact map with the center C_φ. Then there exists an open subset V of X such that $C_\varphi \subset V$, $\varphi(V) \subset V$ and $\overline{\varphi(V)}$ is compact.*

PROOF. Let U be an open subset of X such that $\overline{C}_\varphi \subset U$ and $K = \overline{\varphi(U)}$ is compact. By Lemma (42.11), there exists $n \in N$ such that $\bigcap_{i=0}^{n} \varphi^i(K) \subset U$. Define $V = \bigcap_{i=0}^{n} \varphi^{-i}(U)$. Since $\varphi(C_\varphi) \subset C_\varphi$, we have that $C_\varphi \subset V$. Moreover,

$$\begin{aligned}\varphi(V) \subset \bigcap_{i=0}^{n} \varphi^{-i}(\varphi(U)) \subset \bigcap_{i=0}^{n} \varphi^{-i}(K) \\ &= \bigcap_{i=0}^{n} \varphi^{i-n}(K) \subset \varphi^{i-n}\bigg(\bigcap_{i=0}^{n} \varphi(K)\bigg) \subset \varphi^{i-n}(U),\end{aligned}$$

hence $\varphi(V) \subset \bigcap_{i=0}^{n-1} \varphi^{-i}(U) \cap \varphi^{-n}(U) = V$. Since $\varphi(V) \subset \varphi^i(U) \subset K$, $\overline{\varphi(V)}$ is compact. □

43. n-Admissible mappings

As we have observed, admissible mappings stand as a natural generalization of acyclic mappings. In the same spirit we can generalize n-acyclic mappings to n-admissible mappings.

(43.1) DEFINITION. A multivalued map $\varphi: X \multimap Z$ is called n-admissible, provided there exists a pair $p: Y \to X$ and $q: Y \to Z$ of maps such that: p is a n-Vietoris map and $q(p^{-1}(x)) \subset \varphi(x)$ for every $x \in X$; in that case we write $(p,q) \subset \varphi$.

Since any Vietoris map is n-Vietoris for every $n \geq 1$ so any admissible map is n-admissible. For a n-admissible map $\varphi: X \multimap Z$ and for every $k \geq n$ we define the set of induced homomorphisms $\{\varphi\}^{*k}$ on cohomology by putting:

$$\{\varphi\}^{*k} = \{(p^{*k})^{-1} q^{*k} \mid (p,q) \subset \varphi\}$$

(compare (8.13)).

(43.2) DEFINITION. Two compact n-admissible mappings $\varphi, \psi: X \multimap Z$ are called *homotopic* provided there exists a compact n-admissible map $\chi: X \times [0,1] \multimap Z$ such that $\chi(x,0) \subset \varphi(x)$ and $\chi(x,1) \subset \psi(x)$ for every $x \in X$; we write $\varphi \sim \psi$.

Strictly analogously to the case of admissible mappings, one can prove:

(43.3) PROPOSITION. *If $\varphi \sim \psi$ then, for every $k \geq n$, we have $\{\varphi\}^{*k} \cap \{\psi\}^{*k} \neq \emptyset$.*

(43.4) REMARK. In fact, (43.3) can be formulated in a more precise form, namely, one can prove that if $\varphi \sim \psi$, then there exist $(p,q) \subset \varphi$ and $(\overline{p},\overline{q}) \subset \psi$ such that for every $k \geq n$ we have:

$$(p^{*k})^{-1} q^{*k} = (\overline{p}^{*k})^{-1} \overline{q}^{*k}.$$

Now, we shall show that for n-admissible mappings of subsets of Euclidean spaces, the topological degree will be defined as for n-acyclic or acyclic maps (see Section 40). We will sketch it here, but later we would like to present in details the topological degree theory for n-admissible mappings in normed spaces.

Let $\varphi: S_1^n \multimap S_2^n$ be an n-admissible map, where S_i^n $(i = 1,2)$ are two n-cohomological spheres. We fix generators $\beta_i \in H^n(S_i^n)$, $i = 1,2$. For a pair $(p,q) \subset \varphi$ we let:

$$(p^{*n})^{-1}(q^{*n}(\beta_2)) = \deg(p,q) \cdot \beta_1.$$

Consequently, let us define $\{\mathrm{Deg}(\varphi)\} = \{\deg(p,q) \mid (p,q) \subset \varphi\}$. We have

(43.5) Proposition.

(43.5.1) *If* $\varphi \sim \psi$ *then* $\{\mathrm{Deg}(\varphi)\} \cap \{\mathrm{Deg}(\psi)\} \neq \emptyset$,

(43.5.2) *If* $\varphi \subset \psi$ *then* $\{\mathrm{Deg}(\varphi)\} \subset \{\mathrm{Deg}(\psi)\}$,

(43.5.3) *If* φ *is acyclic then* $\{\mathrm{Deg}(\varphi)\}$ *is a singleton and for two acyclic maps such that* $\varphi \sim \psi$ *we have* $\{\mathrm{Deg}(\varphi)\} = \{\mathrm{Deg}(\psi)\}$.

Now, all results contained in Section 40 can be generalized to the case of n-admissible (resp. admissible) maps. Since all proofs are strictly analogous to those presented in Section 40, we will list all mentioned results only.

(43.6) Theorem. *Let* $\varphi\colon K^{n+1} \multimap \mathbb{R}^{n+1}$ *be a* n*-admissible map such that* $\varphi(S^n) \subset K^{n+1}$. *Then* $\mathrm{Fix}(\varphi) \neq \emptyset$.

(43.7) Theorem. *Let* $\varphi\colon S^n \multimap S^n$ *be an* n*-admissible map such that* $\{\mathrm{Deg}(\varphi)\} \neq \{0\}$ *then* $\varphi(S^n) = S^n$.

(43.8) Theorem (On antipodes for n-admissible maps). *Let* $\varphi\colon S^n \multimap P^{n+1}$ *be an* n*-admissible map. If for every* $x \in S^n$ *there exists a* n*-dimensional subspace* $E^n \subset \mathbb{R}^{n+1}$ *strictly separating* $\varphi(x)$ *and* $\varphi(-x)$ *then* $0 \notin \{\mathrm{Deg}(\varphi)\}$.

(43.9) Theorem (On antipodes for admissible maps). *Let* $\varphi\colon S^n \multimap P^{n+1}$ *be an admissible map such that the following condition is satisfied:*

(43.9.1) *every radius with origin of the zero point of* $\mathbb{R}^{n+1}$ *has an empty intersection with the set* $\varphi(x)$ *or* $\varphi(-x)$ *for every* $x \in S^n$.

Then $0 \notin \{\mathrm{Deg}(\varphi)\}$.

(43.10) Theorem (Borsuk–Ulam). *If* $\varphi\colon S^n \multimap \mathbb{R}^n$ *is an admissible map then there exists* $x \in S^n$ *such that* $\varphi(x) \cap \varphi(-x) \neq \emptyset$.

(43.11) Theorem (Bourgin–Yang). *If* $\varphi\colon S^{n+k} \multimap \mathbb{R}^n$ *is an admissible then the genus* $\gamma(A(\varphi)) \geq k$, *where* $A(\varphi) = \{x \in S^{n+k} \mid \varphi(x) \cap \varphi(-x) \neq \emptyset\}$.

(43.12) Theorem (On invariance of domain). *Let* U *be an open subset of* $\mathbb{R}^{n+1}$ *and* $\varphi\colon U \multimap \mathbb{R}^{n+1}$ *be a strongly admissible map such that* $x_1 \neq x_2$ *implies* $\varphi(x_1) \cap \varphi(x_2) = \emptyset$ *for* $x_1, x_2 \in U$. *Then* $\varphi(U)$ *is an open subset of* $\mathbb{R}^{n+1}$.

Now, let E be a normed space and X a subset of E. Assume further that $\Phi\colon X \multimap E$ is a multivalued map. We define a multivalued vector field $\varphi\colon X \multimap E$ associated with Φ by putting:

$$\varphi = I - \Phi, \quad \text{i.e.} \quad \varphi(x) = \{x - y \mid y \in \Phi(x)\} \quad \text{for every } x \in X.$$

(43.13) DEFINITION. A map $\varphi\colon X \multimap Y \subset E$ is called an *n-admissible* (*s-admissible*) *compact vector field* if and only if there exists an n-admissible (s-admissible) compact map $\Phi\colon X \multimap E$ such that $\varphi = I - \Phi$.

If Φ is an admissible compact map then $\varphi = I - \Phi$ is called an *admissible compact vector field*.

A point $x_0 \in X$ is called a *singular point of the vector field* $\varphi\colon X \multimap Y$ *if image* $\varphi(x_0)$ contains the origin 0 of E. If there are no singular points, we say that φ is *singularity free* (written $\varphi\colon X \multimap P$).

(43.14) DEFINITION. Two n-admissible (s-admissible) compact vector fields $\varphi_1 = I - \Phi_1$, $\varphi_2 = I - \Phi_2$ ($\varphi_1, \varphi_2\colon X \multimap Y \subset E$) are said to be *homotopic*, written $\varphi_1 \sim \varphi_2$, provided there exists a map $\chi\colon X \times J \to Y$, where J is a unit interval, which can be represented in the form $\chi(x,t) = x - \mathbb{X}(x,t)$, where $\mathbb{X}\colon X \times J \multimap E$ is an n-admissible (s-admissible) compact homotopy between φ_1 and φ_2.

The following evident remark is of importance (cf. (39.2)).

(43.15) REMARK. Let A be a closed subset of E and let $\varphi\colon A \multimap E$ be an n-admissible compact vector field. Then the image $\varphi(A)$ is a closed subset of E.

(43.16) REMARK. Consider two maps of the form $X \xleftarrow{p} Y \xrightarrow{q} E$ such that X is a subset of E and p is a Vietoris n-map. Define a map $\widetilde{q}\colon Y \to E$ by putting $\widetilde{q}(y) = p(y) - q(y)$. Then $I - \varphi_{p,q} = \varphi_{p,\widetilde{q}}$, and hence

(43.16.1) if q is a compact map, then $I - \varphi_{p,q}$ is an n-admissible compact vector field,

(43.16.2) every n-admissible compact field is an n-admissible map,

where $\varphi_{p,q}(x) = q(p^{-1}(x))$.

Let E be a Banach space and let $\varphi\colon S \multimap P$ be an n-admissible compact vector field from the unit sphere S to $P = E \setminus \{0\}$.

Consider an arbitrary but fixed selected pair $(p,q) \subset \Phi$ of the from $S \xleftarrow{p} Y \xrightarrow{q} E$. First, for such a pair $(p,q) \subset \Phi$ we define an integer $\deg(p,q)$ which is called the *degree* of (p,q). Then we obtain a positive number δ such that $\operatorname{dist}(0,\varphi(S)) = \delta$. We observe that $\operatorname{dist}(0,(I - \varphi_{p,q})(S)) \geq \delta$. Let ε be a positive number such that $\varepsilon < \delta$. Since Φ is a compact map we infer that q is also compact. Applying the approximation theorem to the map q and the number ε, we obtain a map $q_\varepsilon\colon Y \to E^{k+1}$ such that $\|q(y) - q_\varepsilon(y)\| < \varepsilon$ for every $y \in Y$. We may assume without loss of generality that $k+1 \geq 2$ and $k+1 \geq n$.

Let $Y_k = p^{-1}(S^k)$, where $S^k = S \cap E^{k+1}$. Consider the diagram

$$S^k \xleftarrow{p_k} Y_k \xrightarrow{q_k} E^{k+1},$$

in which p_k and q_k are restrictions of p and q, respectively. So, we obtain a pair (p_k, q_k) and a diagram:

$$S^k \xleftarrow{p_k} Y_k \xrightarrow{\widetilde{q}_k} P^{k+1},$$

in which $\widetilde{q}_k(y) = p_k(y) - q_k(y)$. We observe that $\widetilde{q}_k(y) \neq 0$ for every $y \in Y_k$. Indeed, since $\|x - q(y)\| \geq \delta$ for every $x \in S$ and $y \in p^{-1}(x)$, we have

$$\begin{aligned} \|q_k(y)\| &= \|p_k(y) - q_k(y)\| = \|x - q(y)\| = \|(x - q_k(y)) - (q_k(y) - q(y))\| \\ &\geq \|x - q(y)\| - \|q(y) - q_k(y)\| \geq \delta - \varepsilon > 0. \end{aligned}$$

We define $\deg(p, q)$ of the pair $(p, q) \subset \Phi$ by putting $\deg(p, q) = \deg(p_k, \widetilde{q}_k)$ where $\deg(p_k, \widetilde{q}_k)$ is given in (43.5).

(43.17) LEMMA. *Let $\varphi = I - \Phi\colon S \multimap P$ be an n-admissible compact vector field and let (p, q) be a selected pair of Φ of the from $S \xleftarrow{p} Y \xrightarrow{q} E$. Assume further that $q_\varepsilon, q'_\varepsilon\colon Y \to E^{k+1}$ are two ε-approximations of q. Then* $\deg(p_k, \widetilde{q}_k) = \deg(p_k, \widetilde{q}'_k)$.

PROOF. Define the map $h\colon Y_k \times [0, 1] \to E^{k+1}$ by putting

$$h(y, t) = t\widetilde{q}_k(y) + (1 - t)\widetilde{q}'_k(y).$$

Then h is a homotopy between $\widetilde{q}_k$ and $\widetilde{q}'_k$. We will prove that $h(y, t) \neq 0$ for each $y \in Y_k$ and $t \in [0, 1]$:

$$\begin{aligned} &\|t\widetilde{q}_k(y) + (1 - t)\widetilde{q}'_k(y)\| \\ &\quad = \|t(p_k(y) - q_k(y)) + (1 - t)(p_k(y) - q'_k(y))\| \\ &\quad = \|p_k(y) - tq_\varepsilon(y) + (1 - t)q'_\varepsilon(y)\| \\ &\quad = \|p(y) - q(y) - [t(q_\varepsilon(y) - q(y)) + (1 - t)(q'_\varepsilon(y) - q(y))]\| \geq \delta - \varepsilon > 0. \end{aligned}$$

Therefore we have $\widetilde{q}'^{*}_k = \widetilde{q}^{*}_k$ and the proof is completed. □

(43.18) LEMMA. *Let $\varphi = I - \Phi$ and $(p, q) \subset \Phi$ be as in* (43.17)*. Assume further that E^{k+1}, E^{k+2} are two subspaces of E such that $E^{k+1} \subset E^{k+2}$. If $q_\varepsilon\colon Y \to E^{k+1}$ is an ε-approximation of q and $q'_\varepsilon\colon Y \to E^{k+2}$ is the map given by $q_\varepsilon(y) = q'_\varepsilon(y)$ for every $y \in Y$ then* $\deg(p_k, \widetilde{q}_k) = \deg(p_{k+1}, \widetilde{q}'_{k+1})$.

PROOF. Define a map $r\colon P^{k+2} \to S^{k+1}$ by putting $r(z) = z/\|z\|$. We orient S^{k+1} and P^{k+2} so that $\deg(p_{k+1}, \widetilde{q}'_{k+1}) = \deg(p_{k+1}, r\widetilde{q}'_{k+1})$. Applying (43.5) to the pair (p_{k+1}, rq_{k+1}), we obtain (43.18). □

Finally, from (43.17) and (43.18) we deduce that $\deg(p, q)$ of the pair (p, q) is well defined.

Now, we define $\mathrm{Deg}(I - \Phi)$ of an n-admissible compact vector field $\varphi = I - \Phi\colon S \multimap P$ by putting

$$\{\mathrm{Deg}(I - \Phi)\} = \{\deg(p, q) \mid (p, q) \subset \Phi\}. \tag{43.19}$$

(43.20) PROPOSITION. *Let $\varphi, \psi: S \multimap P$ be two n-admissible compact vector fields. Then*

(43.20.1) $\varphi \sim \psi$ *implies* $\{\mathrm{Deg}(\varphi)\} \cap \{\mathrm{Deg}(\psi)\} \neq \emptyset$,
(43.20.2) $\varphi \subset \psi$ *implies* $\{\mathrm{Deg}(\varphi)\} \subset \{\mathrm{Deg}(\psi)\}$.

PROOF. Let $\chi = I - \mathbb{X}$ be a homotopy between φ and ψ and let (p, q) be a selected pair of χ. The set $\chi(S \times [0, 1])$ is closed and does not contain the origin. Then from the above construction of the degree for the selected pairs we obtain (43.20.1). The proof of (43.20.2) is evident. $\square$

(43.21) PROPOSITION. *Let $\varphi = I - \Phi: S \multimap P$ be an admissible compact vector field. If Φ is an acyclic map then the set* $\mathrm{Deg}(\varphi)$ *is a singleton.*

PROOF. Let $(p, q) \subset \Phi$ be a selected pair of Φ of the form $S \xleftarrow{p} Y \xrightarrow{q} E$. Consider the commutative diagram

$$\begin{array}{ccccc} S & \xleftarrow{p_\Phi} & \Gamma_\Phi & \xrightarrow{q_\Phi} & E \\ & {}_{p}\nwarrow & \uparrow{\scriptstyle f} & \nearrow_{q} & \\ & & Y & & \end{array}$$

in which $f(y) = (p(y), q(y))$ for each $y \in Y$. Let $(q_\Phi)_\varepsilon: \Gamma_\Phi \to E^{k+1}$ be an ε-approximation of q_Φ. For the proof we take an ε-approximation of q such that $q_\varepsilon(y) = (q_\Phi)_\varepsilon(f(y))$ for each $y \in Y$. Denote by Γ_k the graph of $\Phi|_{S^k}$ ($S^k = S \cap E^{k+1}$). Let

$$(p_\Phi)_k: \Gamma_k \to S^k \quad \text{and} \quad (q_\Phi)_k: \Gamma_k \to E^{k+1}$$

be restrictions of p_Φ and $(q_\Phi)_\varepsilon$, respectively. Finally, we obtain (cf. the definition of $\deg(p, q)$ in this section) the commutative diagram

$$\begin{array}{ccccc} S^k & \xleftarrow{(p_\Phi)_k} & \Gamma_k & \xrightarrow{(\tilde{q}_\Phi)_k} & P^{k+1} \\ & {}_{p_k}\nwarrow & \uparrow{\scriptstyle \overline{f}} & \nearrow_{\tilde{q}_k} & \\ & & p^{-1}(S^k) & & \end{array}$$

in which p_k and $\overline{f}$ are restrictions of p and f, respectively, and the map $(\widetilde{q}_\Phi)_k: \Gamma_k \to P^{k+1}$ is given by $(\widetilde{q}_\Phi)_k(x, y) = (p_\Phi)_k(x, y) - (q_\Phi)_k(x, y)$ for each $(x, y) \in \Gamma_k$. The map $\widetilde{q}_k: p^{-1}(S^k) \to P^{k+1}$ is given by $\widetilde{q}_k(y) = p_k(y) - q_\varepsilon(y)$ for each $y \in p^{-1}(S^k)$. Now, the proof of (43.21) is evident. $\square$

(43.22) EXAMPLE. Let E be a Banach space and $y_0 \in E$ be a point such that $\|y_0\| > 1$. Consider the map $\Phi: S \multimap E$ given by $\Phi(x) = \{0, y_0\}$ for each $x \in S$.

Clearly, Φ is an admissible and compact map. We have the following selected pairs of Φ:

(43.22.1) $(\mathrm{id}_S, f) \subset \Phi$, where $f\colon S \to E$ is given by $f(x) = 0$ for $y \in S$,
(43.22.2) (id_S, g), where $g\colon S \to E$ is given by $g(x) = y_0$ for each x.

Moreover, we infer that $\deg(\mathrm{id}_S, f) \neq 0$ and $\deg(\mathrm{id}_S, g) = 0$ and hence $\mathrm{Deg}(I - \Phi)$ is not a singleton.

Now, we prove the following:

(43.23) THEOREM. *Let $\varphi\colon S \multimap P$ be an n-admissible compact vector field such that $\{\mathrm{Deg}(\varphi)\} \neq \{0\}$. Then for every $x \in S$ there is a positive real number $\lambda > 0$ such that $\lambda x \in \varphi(S)$.*

PROOF. Suppose that there exists an $x_0 \in S$ such that

$$L_{x_0} = \{\lambda x_0 \mid \lambda \geq 0\} \cap \varphi(S) = \emptyset.$$

Let $\varepsilon = (1/2)\min(\mathrm{dist}(\varphi(S), L_{x_0}), d(0, \varphi(S)))$. We observe that $\mathrm{dist}(\varphi(S), L_{x_0}) > 0$, and by assumption we have $\varepsilon > 0$. Let $(p, q) \subset \Phi$ be a selected pair of the form $S \xleftarrow{p} Y \xrightarrow{q} E$ such that $\deg(p, q) \neq 0$. We take an ε-approximation $q_\varepsilon\colon Y \to E^{k+1}$ such that $x_0 \in E^{k+1}$.

Consider the diagram

$$S^k \xleftarrow{p_k} Y_k \xrightarrow{\widetilde{q}_k} P^{k+1}$$

(cf. the definition of $\deg(p, q)$ in this section). Since q_ε is an ε-approximation of q, we obtain

$$\widetilde{q}_k(p_k^{-1}(S^k)) \subset O_\varepsilon(\varphi(S)),$$

where $O_\varepsilon(\varphi(S))$ as usual is an ε-neighbourhood of $\varphi(S)$ in E, hence

$$L_{x_0} \cap \widetilde{q}_k(p_k^{-1}(S^k)) = \emptyset. \tag{43.23.1}$$

Consider the map $\psi_k\colon S^k \to P^{k+1}$ given by $\psi_k(x) = \widetilde{q}_k(p_k^{-1}(x))$ for each $x \in S^k$. Then $(p_k, \widetilde{q}_k) \subset \psi_k$ is a selected pair of ψ_k and hence ψ_k is an n-admissible map. Moreover, $\deg(p_k, \widetilde{q}_k) = \deg(p, q) \neq 0$.

Let $r\colon P^{k+1} \to S$ be a retraction ($r(x) = x/\|x\|$). Then $\mathrm{Deg}(r\psi_k) \neq \{0\}$ and from (43.23.1) we have $x_0 \notin (r\psi_k)(S^k)$, but this contradicts (43.7). The proof of (43.23) is completed. □

Let $\varphi = I - \Phi\colon K \multimap E$ be an n-admissible compact vector field such that $\varphi(S) \subset P$, where S, as usually, is the boundary of the closed ball K. By $\varphi|_S\colon S \multimap P$ we denote the restriction of φ to the pair (S, P). We infer that $\varphi|_S$ is a n-admissible compact vector field on S. In this case with every selected pair $(p, q) \subset \Phi$ we associate a pair $(p_1, q_1) \subset \Phi|_S$ as follows: let $p: Y \to K$, $q: Y \to E$ be two maps such that $(p, q) \subset \Phi$; then $p_1: p^{-1}(S) \to S$, $q_1: p^{-1}(S) \to E$ are given as restrictions of p and q, respectively. Evidently $(p_1, q_1) \subset \Phi|_S$. We define degree $\operatorname{Deg}(\varphi; 0)$ of φ by putting

$$\{\operatorname{Deg}(\varphi; 0)\} = \{\deg(p_1, q_1) \mid (p, q) \subset \Phi\}. \tag{43.24}$$

Clearly, $\{\operatorname{Deg}(\varphi; 0)\} \subset \{\operatorname{Deg}(\varphi|_S)\}$. Let $\varphi: K \multimap E$ be an n-admissible compact vector field such that $\varphi(S) \subset E \setminus \{z_0\}$. By $(\varphi - z_0): K \multimap E$ we denote the n-admissible compact vector field given by

$$(\varphi - z_0)(x) = \{y - z_0 \mid y \in \varphi(x)\}$$

for each $x \in K$. Observe that $(\varphi - z_0)(S) \subset P$. We define $\operatorname{Deg}(\varphi; z_0)$ by putting

$$\operatorname{Deg}(\varphi; z_0) = \operatorname{Deg}(\varphi - z_0; 0). \tag{43.25}$$

The following lemma is of importance:

(43.26) Lemma. *Let $\varphi: K \multimap E$ be an n-admissible compact vector field such that $\varphi(S) \subset P$. If $\{\operatorname{Deg}(\varphi; 0)\} \neq \{0\}$, then there exists a point $x_0 \in K$ such that $0 \in \varphi(x_0)$.*

Proof. Let $\varphi = I - \Phi$, where $\Phi: K \multimap E$ is an n-admissible compact map. Assume that $0 \notin \varphi(x)$ for all $x \in K$. First, we obtain a positive number δ such that $\operatorname{dist}(0, \varphi(K)) = \delta$. Let ε be a positive number such that $\varepsilon < \delta$. Let (p, q) be a selected pair of Φ of the form $K \xleftarrow{p} Y \xrightarrow{q} E$. Let q_ε be a ε-approximation of q. Then, as in the definition of the degree $\deg(p, q)$, we obtain the following diagram:

$$K^{k+1} \xleftarrow{p_k} Y_k \xrightarrow{\widetilde{q_k}} P^{k+1}.$$

Consider the map $\psi: K^{k+1} \multimap P^{k+1}$ given by $\psi(x) = q_k(p_k^{-1}(x))$ for each $x \in K^{k+1}$. Then we have $\operatorname{Deg}(\psi; 0) = \{0\}$. Consequently, $\deg(p_1, q_1) = 0$, where (p_1, q_1) is a pair associated with (p, q). Since (p, q) is an arbitrary selected pair of Φ, we obtain $\operatorname{Deg}(\varphi; 0) = \{0\}$ and the proof is completed. □

The following theorem is an extension of the well known Rothe theorem to the case of n-admissible maps (see [Gr1-M]).

(43.27) Theorem. *If $\Phi: K \multimap E$ is an n-admissible compact map such that $\Phi(S) \subset K$ then Φ has a fixed point.*

Proof. Let $\varphi: K \multimap E$ be an n-admissible compact vector field given by $\varphi = I - \Phi$. We may assume without loss of generality that $\varphi(S) \subset P$ and by Lemma (43.26) it suffices to prove that $\operatorname{Deg}(\varphi; 0) \neq \{0\}$. For this purpose let

$$\psi(x, t) = x - t\Phi(x) \quad \text{for an arbitrary } x \in S,\ 0 \le t \le 1.$$

It follows from our assumption that for an arbitrary $z \in \psi(x,t)$ we have

$$\|z\| = \|x - ty\| \geq \|x\| - t\|y\| > 0 \quad \text{for } 0 \leq t < 1$$

and thus $\psi\colon S\times[0,1] \multimap P$. It is evident that $\psi(S\times[0,1])$ is a closed subset of E and hence $\operatorname{dist}(0,\psi(S\times[0,1])) = \delta > 0$. Let $(p,q) \subset \Phi$ be a selected pair of the form $K \xleftarrow{p} Y \xrightarrow{q} E$ and let $S \xleftarrow{p_1} p^{-1}(S) \xrightarrow{q_1} E$ be the pair associated with (p,q) (cf. the definition of $\operatorname{Deg}(\varphi;0)$). Let $q_\varepsilon\colon p^{-1}(S) \to E^{k+1}$ be an ε-approximation of q_1, where $0 < \varepsilon < \delta$. We put $S^k = S \cap E^{k+1}$ and $Y_k = p^{-1}(S^k)$.

We have the diagram

$$S^k \xleftarrow{p_k} Y_k \xrightarrow{q_k} E^{k+1},$$

in which p_k, q_k are restrictions of p_1 and q_1, respectively. Define the map $\widetilde{q}_k\colon Y_k \to P^{k+1}$ by putting $\widetilde{q}_k(y) = p_k(y) - q_k(y)$ for each $y \in Y_k$. We claim that $\deg(p_1,q_1) = \deg(p_k,\widetilde{q}_k) \neq 0$. In this order, consider the map $f\colon Y_k \to P^{k+1}$ given by $f(y) = p_k(y)$ and a homotopy $h\colon Y_k\times[0,1] \to P^{k+1}$ given by $h(y,t) = p_k(y) - tq_k(y)$. Since $\varphi(S) \subset P$ and q_ε is an ε-approximation of $q_1, 0 < \varepsilon < \delta$, we deduce that $h(Y_k\times[0,1]) \subset P^{k+1}$. Then the maps f and $\widetilde{q}_k$ are homotopic and hence $f^* = \widetilde{g}_k^*$. Finally, we obtain

$$\deg(p_1,q_1) = \deg(p_k,\widetilde{q}_k) = \deg(p_k,f) \neq 0,$$

and the proof is completed. □

In fact, from the above proof we infer that $\{\operatorname{Deg}(\varphi;0)\}$ is a singleton and $0 \neq \operatorname{Deg}(\varphi;0)$.

(43.28) Theorem. *Let $\varphi = I - \Phi\colon S \multimap P$ be an n-admissible compact vector field. Suppose that there exists a real positive number η such that for each $x \in S$ there exists a subspace E_x of E, of codimension equal to* 1, *strictly separating $O_\eta(\varphi(x))$ and $O_\eta(\varphi(-x))$. Then $0 \notin \operatorname{Deg}(\varphi)$.*

Proof. Consider a selected pair $(p,q) \subset \Phi$ of the form $S \xleftarrow{p} Y \xrightarrow{q} E$. Let $\varepsilon_0 = \min(\eta,\operatorname{dist}(0,\varphi(S)))$. By the assumption, $\varepsilon_0 > 0$. We take an ε-approximation $q_\varepsilon\colon Y \to E^{k+1}$ of q, with $0 < \varepsilon < \varepsilon_0$ $(k \geq n)$. Consider the diagram

$$S^k \xleftarrow{p_k} p^{-1}(S^k) \xrightarrow{\widetilde{q}_k} P^{k+1},$$

where p_k is the restriction of p to the pair $(S^k, p^{-1}(S^k))$ and $\widetilde{q}_k(y) = p_k(y) - q_k(y)$ for each $y \in p^{-1}(S^k)$. Then we have an n-admissible map $\psi\colon S^k \multimap P^{k+1}$ given as the composition $\psi = \widetilde{q}_k p_k^{-1}$.

Let $E_x^k = E_x^k \cap E^{k+1}$. Observe that $\dim E_x^k = k$. Since q_ε is an ε-approximation of q, $\varepsilon < \eta$, by the assumption we have $\psi(x) \subset O_\eta(\varphi(x))$ for each $x \in S^k$. This implies that E^k_x strictly separates $\psi(x)$ and $\psi(-x)$ for each $x \in S^k$. Applying Theorem (43.9) to ψ, we obtain $0 \notin \operatorname{Deg}(\psi)$ and hence $0 \neq \deg(p,q)$. Since (p,q) is an arbitrary selected pair of Φ, we have $0 \notin \operatorname{Deg}(\varphi)$ and the proof of theorem is completed. □

From (43.26) and (43.28) we obtain

(43.29) COROLLARY. *Let $\varphi\colon K \multimap E$ be an n-admissible compact vector field such that $\varphi|_S$ satisfies all the assumptions of* (43.21). *Then there is a point $x_0 \in K \setminus S$ such that $0 \in \varphi(x_0)$.*

Now, for admissible compact vector fields we prove a stronger version of Theorem (43.28)

(43.30) THEOREM. *Let $\varphi\colon S \multimap P$ be an admissible compact vector field. Suppose that there exists $\eta > 0$ such that the following condition is satisfied:*

(43.30.1) *every half-ray $L_y = \{z \in E \mid z = ty \text{ for some } t \geq 0\}$ has an empty intersection with the set $O_\eta(\varphi(x))$ or $O_\eta(\varphi(-x))$ for each $x \in S$.*

Then $0 \notin \operatorname{Deg}(\varphi)$.

OUTLINE OF THE PROOF. Consider the admissible map ψ given in the same way as in the proof of (43.28). Applying Theorem (43.8) to the map ψ, we deduce (43.30). □

From (43.26) and (43.30) we infer

(43.31) COROLLARY. *Let $\varphi\colon K \multimap E$ be an admissible compact vector field such that $\varphi|_S$ satisfies all the assumptions of* (43.30). *Then there is a point $x_0 \in K \setminus S$ such that $0 \in \varphi(x_0)$.*

Now we would like to present the infinite-dimensional version of the Bourgin–Yang theorem. Let L, N be two linear closed subspaces of E. If for any $x \in E$ there exists a unique decomposition $x = y + z$, where $y \in L$ and $z \in N$ then we say that E is a *direct sum* of L and N; in this case we write $E = L \oplus N$. Note that if L is a finite-dimensional (respectively finite codimensional and closed) linear subspace of E then there exists a closed linear subspace $N \subset E$ such that $E = L \oplus N$ and $\operatorname{codim} N = \dim L$ (respectively $\dim N = \operatorname{codim} L$).

Throughout the rest of the section, let E_k, $k = 1, 2, \ldots$, denote an arbitrary but fixed closed, k-codimensional subspace of E. In what follows we assume that for each k we are given a direct sum decomposition $E = E_k \oplus L^k$. We also let

$$S = S_E = \{x \in E \mid \|x\| = 1\}.$$

The following two lemmas play a crucial role in our considerations:

(43.32) LEMMA. *Suppose $\varphi\colon S \multimap E_k$ is an admissible compact vector field and $\varepsilon > 0$. Then there exists a finite dimensional subspace $V \subset E_k$ and an admissible map $\widetilde{\varphi}\colon S \cap (V \oplus L^k) \multimap V$ such that $\widetilde{\varphi}(x) \subset O_\varepsilon(\varphi(x))$, for each $x \in S \cap (V \oplus L^k)$.*

PROOF. Let Φ denote the compact part of φ. Then we have a diagram

$$S \xleftarrow{p} \Gamma \xrightarrow{q} E$$

in which p is a Vietoris map, q is compact and $q(p^{-1}(x)) \subset \Phi(x)$, for each $x \in S$. Denote by $\pi_1\colon E \to E_k$, $\pi_2\colon E \to L^k$ the linear projections determined by the direct sum decomposition $E = E_k \oplus L^k$. Note, that π_1, π_2 are continuous and for each $x \in E$ there is a unique decomposition $x = \pi_1(x) + \pi_2(x)$, $\pi_1(x) \in E_k$, $\pi_2(x) \in L^k$. Define $q_1\colon \Gamma \to E_k$, $q_2\colon \Gamma \to L^k$ by $q_i = \pi_i \circ q$, $i = 1, 2$. Clearly q_1, q_2 are continuous compact maps. For a given $\varepsilon > 0$ the Schauder Approximation Theorem implies the existence of a finite dimensional subspace $V \subset E_k$ and a continuous map $q_3\colon \Gamma \to V$ such that:

$$\|q_3(y) - q_1(y)\| < \varepsilon, \quad \text{for every } y \in \Gamma. \tag{43.32.1}$$

Since $\varphi(x) \subset E_k$, for all $x \in S$, $y \in \Gamma$ implies $\pi_2(p(y) - q(y)) = 0$. Thus for $y \in \Gamma$ we have

$$\pi_1(p(y) - q_3(y) - q_1(y)) = \pi_2(p(y) - q(y)) + \pi_2(q_2(y) - q_3(y)) = 0.$$

Therefore,

$$(p(y) - q_3(y) - q_2(y)) \in E_k, \quad \text{for } y \in \Gamma. \tag{43.32.2}$$

Let $\Sigma = S \cap (V \oplus L^k), \widetilde{\Gamma} = p^{-1}(\Sigma)$ and define the multivalued map $\widetilde{\varphi}\colon \Sigma \multimap V$ by $\widetilde{\varphi}(x) = \{p(y) - q_3(y) - q_2(y) \mid x = p(y)\}$. From (43.32.2) it follows that the above definition is correct, i.e. $\widetilde{\varphi}(x) \subset V$, for $x \in \Sigma$. If we let $\widetilde{p}\colon \widetilde{\Gamma} \to \Sigma$ to be the restriction of p and $\widetilde{q}(y) = p(y) - q_3(y) - q_2(y)$, then we obtain a diagram

$$\Sigma \xleftarrow{\widetilde{p}} \widetilde{\Gamma} \xrightarrow{\widetilde{q}} V,$$

in which $\widetilde{p}$ is a Vietoris map and $\widetilde{q}$ is continuous. Since $\widetilde{\varphi}(x) = \widetilde{q}(\widetilde{p}^{-1}(x))$ for each $x \in \Sigma$, $\widetilde{\varphi}$ is admissible. Finally, (43.32.1) implies that $\widetilde{\varphi}(x) \subset O_\varepsilon(\varphi(x))$ and the proof is completed. □

(43.33) LEMMA. *Assume that X is a closed, bounded and symmetric (i.e. $x \in X$ implies $(-x) \in X$) subset of E. Assume further that $\varphi: X \multimap E_k$ is an s-admissible compact vector field such that $\varphi(x) \cap \varphi(-x) = \emptyset$, for each $x \in X$. Then there exists $\varepsilon > 0$ such that $O_\varepsilon(\varphi(x)) \cap O_\varepsilon(\varphi(-x)) = \emptyset$, for each $x \in X$.*

PROOF. Assume, to the contrary, that the conclusion is false. Then, taking $\varepsilon = 1/n$, $n = 1, 2, \ldots$, we obtain sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, $x_n \in X$, $y_n \in \varphi(x_n)$, $z_n \in \varphi(-x_n)$ such that:

(43.33.1) $$\|y_n - z_n\| < \frac{1}{n}, \quad \text{for each } n.$$

Let Φ denote the compact part of φ. There exist $u_n \in \Phi(x_n)$, $v_n \in \Phi(-x_n)$ such that:

(43.33.2) $$y_n = x_n - u_n, \quad z_n = -x_n - v_n.$$

Using (43.33.1) we obtain

(43.33.3) $$\|x_n - u_n + x_n + v_n\| \leq \frac{2}{n}.$$

Since Φ is compact, we may assume without loss of generality, that

(43.33.4) $$\lim_n u_n = u_0, \ \lim_n v_n = v_0.$$

Thus, in view of (43.33.3), there exists $x_0 = \lim_n x_n = (1/2)(u_0 - v_0)$. Moreover, since X is closed, $x_0 \in X$. By (43.33.2) the sequences $\{y_n\}, \{z_n\}$ are convergent and

$$y_0 = \lim_n y_n = x_0 - u_0 = -\frac{1}{2}(u_0 + v_0) = -x_0 - v_0 = \lim_n z_n = z_0. \qquad \square$$

(43.34) THEOREM (Borsuk–Ulam). *If $\varphi: S \multimap E_1$ is an s-admissible compact vector field then there exists a point $x \in S$ such that: $\varphi(x) \cap \varphi(-x) \neq \emptyset$.*

PROOF. Assume to the contrary, that $\varphi(x) \cap \varphi(-x) = \emptyset$, for each $x \in S$. Applying Lemma (43.33) we obtain $\varepsilon > 0$ such that:

(43.34.1) $$O_\varepsilon(\varphi(x)) \cap O_\varepsilon(\varphi(-x)) = \emptyset, \quad \text{for each } x \in S.$$

From Lemma (43.32) it follows that there exist a finite dimensional subspace $V \subset E$, and a s-admissible map $\widetilde{\varphi}: S \cap (V \oplus L^1) \multimap V$ such that $\widetilde{\varphi}(x) \subset O_\varepsilon(\varphi(x))$, for $x \in S \cap (V \oplus L^1)$. Therefore, in view of (43.34.1), $\widetilde{\varphi}(x) \cap \widetilde{\varphi}(-x) = \emptyset$, for $x \in S \cap (V \oplus L^1)$, which contradicts Theorem (43.10) and the proof of (43.34) is completed. $\square$

We consider an admissible compact vector field $\varphi: S \multimap E_k$, $k \geq 1$. We let

$$A(\varphi) = \{x \in S \mid \varphi(x) \cap \varphi(-x) \neq \emptyset\}.$$

(43.35) LEMMA. *If $\varphi: S \multimap E_k$ is an admissible compact vector field then $A(\varphi)$ is nonempty, symmetric and compact.*

It is evident that $A(\varphi)$ is symmetric and, by (43.34) nonempty. The proof of a compactness of $A(\varphi)$ is strictly analogous to the proof of Lemma (43.33) and therefore it is omitted here.

We will prove the genus version of the Bourgin–Yang theorem.

(43.36) THEOREM. *If $\varphi: S \multimap E_k$ is an admissible compact vector field then $\gamma(A(\varphi)) \geq k-1$.*

PROOF. Since for $k = 1$ our assertion follows from Theorem (43.34), we may assume $k \geq 2$. Assume, to the contrary, that $\gamma(A(\varphi)) = p < k-1$. Choose a $(p+1)$-dimensional subspace $V \subset L^k$. There exists an odd continuous map $f: S(\varphi) \to S_V = \{x \in V \mid \|x\| = 1\}$. Let $\widetilde{f}: S \to V$ be a continuous compact (not necessarily odd) extension of f. Define an admissible compact vector field $\widetilde{\varphi}: S \multimap E_k \oplus V$ by

$$\widetilde{\varphi}(x) = \{x - y \mid y = z + \widetilde{f}(x),\ z \in \Phi(x)\},$$

where Φ denotes the compact part of φ. Applying Theorem (43.34) we get a point $x_0 \in S$ such that $\widetilde{\varphi}(x_0) \cap \widetilde{\varphi}(-x_0) \neq \emptyset$. This implies $x_0 \in A(\varphi)$ and $\widetilde{f}(x_0) = \widetilde{f}(-x_0)$. Since $x_0 \in A(\varphi)$, we get a contradiction and the proof is completed. □

Now, we denote by K a closed ball in a Banach space E with the center 0 and radius ε, and by S, the boundary of K in E. Let A be a subset of E.

A compact admissible field $\varphi: A \multimap E$ is called an *ε-field* provided the condition:

$$\text{if } \varphi(x_1) \cap \varphi(x_2) \neq \emptyset \text{ then } \|x_1 - x_2\| < \varepsilon$$

is satisfied for any $x_1, x_2 \in A$. A compact admissible field $\varphi: A \multimap E$ is called an *ε-field in the narrow sense* if for some constant $\eta > 0$ the condition:

$$\text{if } O_\eta(\varphi(x_1)) \cap O_\eta(\varphi(x_2)) \neq \emptyset \text{ then } \|x_1 - x_2\| < \varepsilon$$

is satisfied for every $x_1, x_2 \in A$.

The proof of the theorem on the invariance of domain for ε-fields in the narrow sense is based on the following lemmas.

(43.37) LEMMA. *Let $\varphi: K \multimap E$ be an ε-field in the narrow sense. Then:*

(43.37.1) *$\varphi(S) \subset E \setminus \{y_0\}$ for each $y_0 \in \varphi(0)$, and*
(43.37.2) *$0 \notin \operatorname{Deg}(\varphi, y_0)$.*

PROOF. For the proof of (43.37.1) we observe that if $\varphi(0) \cap \varphi(x) \neq \emptyset$ for some $x \in K$ then $O_\eta(\varphi(0)) \cap O_\eta(\varphi(x)) \neq \emptyset$ and this implies that $\|x\| < \varepsilon$; hence

$\varphi(S) \subset E \setminus \{y_0\}$ for each $y_0 \in \varphi(0)$. Let $\Phi\colon K \to E$ be a compact part of φ, i.e. $\varphi = I - \Phi$. Consider a selected pair $(p,q) \subset \Phi$ of the form $K \xleftarrow{p} Y \xrightarrow{q} E$ and a point $y_0 \in E$ such that $y_0 \in \varphi(0)$. Let $\delta = \min(\eta, \operatorname{dist}(y_0, \varphi(S)))$, where φ is an ε-field in the narrow sense with the constant η. It is evident that δ is a positive real number. We take a δ-approximation $q_\delta\colon Y \to E^{k+1}$ of the compact map q such that $y_0 \in E^{k+1}$. Let $K^{k+1} = K \cap E^{k+1}$ and $Y_k = p^{-1}(K^{k+1})$. We have the diagram

$$K^{k+1} \xleftarrow{p_k} Y_k \xrightarrow{\widetilde{q}_k} E^{k+1},$$

in which p_k is the restriction of p to the pair (Y_k, K^{k+1}) and $\widetilde{q}_k$ is given by $\widetilde{q}_k(y) = p_k(y) - q_\delta(y)$ for each $y \in Y_k$.

Let $\varphi_k\colon K^{k+1} \multimap E^{k+1}$ be a multivalued map given by $\varphi_k(x) = \widetilde{q}_k(p_k^{-1}(x))$ for each $x \in K^{k+1}$. Then φ_k is an admissible map. We assert that φ_k is an ε-map. Indeed, because $0 < \delta \le \eta$, we have $\varphi_k(x) \subset O_\delta(\varphi(x)) \subset O_\eta(\varphi(x))$ for each $x \in K^{k+1}$ and hence the condition $\varphi_k(x_1) \cap \varphi_k(x_2) \neq \emptyset$ implies $O_\eta(\varphi(x_1)) \cap O_\eta(\varphi(x_2)) \neq \emptyset$. Then, by assumption, we obtain $\|x_1 - x_2\| < \varepsilon$ and φ_k is an ε-map.

Applying Lemma (43.9) to the map φ_k, we obtain $0 \notin \operatorname{Deg}(\varphi_k; y_0)$ and hence $\deg(p,q) \neq 0$. Since (p,q) is an arbitrary selected pair of Φ, we obtain (43.37.2) and the proof is completed. □

(43.38) Lemma. *If $\varphi\colon K \multimap E$ is an admissible compact vector field and $y_0 \notin \varphi(S)$ then for every $y_1 \in E$ such that $\|y_0 - y_1\| < \operatorname{dist}(y_0, \varphi(S))$ we have*

$$\operatorname{Deg}(\varphi; y_0)\} \cap \{\operatorname{Deg}(\varphi; y_1)\} \neq \emptyset.$$

Proof. Consider the map $\chi\colon K \times [0,1] \multimap E$ given by $\chi(x,t) = x - \mathbf{X}(x,t)$, where $\chi(x,t) = \Phi(x) + (ty_1 + (1-t)y_0)$. It is evident that $\chi(S \times [0,1]) \subset E \setminus \{0\}$ and $\chi(S \times [0,1])$ is a closed subset of E. Therefore, χ is a homotopy between $\varphi - y_0$ and $\varphi - y_1$ and our assertion follows from the homotopy property of the topological degree. □

(43.39) Remark. It is possible to prove that $\operatorname{Deg}(\varphi; y_0) = \operatorname{Deg}(\varphi; y_1)$ for φ as in (43.37) but we only need (43.37).

We now prove the main result of this section.

(43.40) Theorem. *If $\varphi\colon E \multimap E$ is an ε-field in the narrow sense then $\varphi(E)$ is an open subset of E.*

Proof. Let $y_0 \in \varphi(x_0)$ be a point in $\varphi(E)$. Consider the closed ball $K_\varepsilon = K(y_0, \varepsilon)$ and let $\psi = \varphi|_{K_\varepsilon}$ be the restriction of φ to K_ε. Then $\psi\colon K_\varepsilon \multimap E$ is an ε-field in the narrow sense. Applying Lemma (43.37) to ψ we obtain $0 \notin \operatorname{Deg}(\psi, y_0)$.

Let $y_1 \in E$ be a point such that $\|y_0 - y_1\| < \operatorname{dist}(y_0, \psi(S_\varepsilon))$. Then from (43.38) and (43.37) we infer $\operatorname{Deg}(\psi; y_1) \neq \{0\}$ and, in view of (5.10), we have $y_1 \in \psi(K_\varepsilon)$. This implies that $B(y_0, \delta) \subset \psi(K_\varepsilon) \subset \varphi(E)$, where $B(y_0, \delta)$ is the open ball in E with center y_0 and radius $\delta = \operatorname{dist}(y_0, \psi(S_\varepsilon))$. The proof of (43.40) is completed.□

Because E is a connected space, from Theorem (43.40) we deduce:

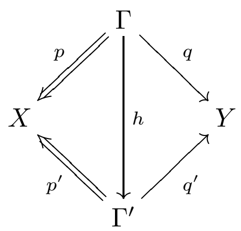
(43.41) COROLLARY. *If $\varphi: E \to E$ is an ε-field in the narrow sense, then $\varphi(E) = E$.*

44. Category of morphisms

Given two spaces X and Y let $D(X, Y)$ be the set of all diagrams of the form:

$$X \overset{p}{\Longleftarrow} \Gamma \overset{q}{\longrightarrow} Y,$$

in which p is a Vietoris map and q is a continuous map. Every such a diagram will be denoted briefly by (p, q). Given two diagrams $(p, q), (p', q') \in D(X, Y)$ we will write $(p, q) \simeq (p', q')$ if there exists a homeomorphism $h: \Gamma \to \Gamma'$ such that the following diagram is commutative:



i.e. $p' \circ h = p$ and $q' \circ h = q$. Clearly, "$\simeq$" is an equivalence relation in $D(X, Y)$.

(44.1) DEFINITION. The equivalence class of a diagram $(p, q) \in D(X, Y)$ with respect to $\simeq$ is denoted by

$$\varphi = \{X \overset{p}{\Longleftarrow} \Gamma \overset{q}{\longrightarrow} Y\}: X \to Y$$

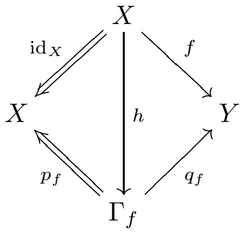
and is called a *morphism* from X to Y.

In what follows by $\mathcal{M}(X, Y)$ we will denote the set of all such morphisms. We will denote morphisms by Greek letters $\varphi, \psi, \chi, \ldots$, and the singlevalued maps by Latin letters $f, g, h, p, q, \ldots$; we will identify a map $f: X \to Y$ with the morphism:

$$f = \{X \overset{\mathrm{id}_X}{\Longleftarrow} X \overset{f}{\longrightarrow} Y\}: X \to Y.$$

Note that for a map $f\colon X \to Y$ the following two diagrams are equivalent:

(44.1.1)

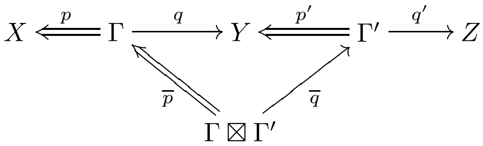


where the homeomorphism $h\colon X \to \Gamma_f$ is defined as follows: $h(x) = (x, f(x))$.

To compose two morphisms we shall use the fibre product construction (see (8.11)). Let

$$\varphi = \{X \xLeftarrow{p} \Gamma \xrightarrow{q} Y\}\colon X \to Y \quad \text{and} \quad \psi = \{Y \xLeftarrow{p'} \Gamma' \xrightarrow{q'} Z\}\colon Y \to Z$$

be two morphisms. Then we have the following commutative diagram:



in which $\Gamma \boxtimes \Gamma'$ is the fibre product of q and p' and $\overline{p'}$ and $\overline{q}$ are the respective pull-backs.

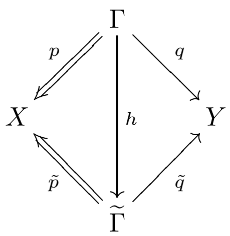
Since the composition of two Vietoris maps is a Vietoris map, too (see (8.10)) we are allowed to define:

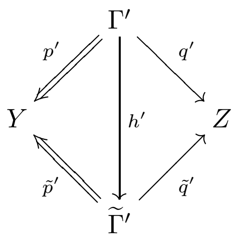
(44.2) Definition. For two morphisms $\varphi\colon X \to Y$ and $\psi\colon Y \to Z$ given as above, we define the composition $\psi \circ \varphi$ of ψ and φ by letting:

$$\psi \circ \varphi = \{\, X \xLeftarrow{p \circ \overline{p'}} \Gamma \boxtimes \Gamma' \xrightarrow{q \circ \overline{q}'} Z \,\}\colon X \to Z$$

(44.3) Proposition. *Definition* (44.2) *does not depend on the choice of diagrams* (p, q) *and* (p', q') *for* φ *and* ψ*, respectively.*

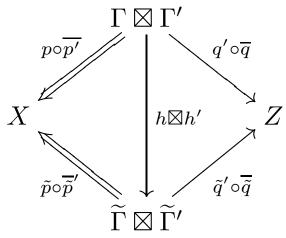
Proof. Assume that $(p, q) \simeq (\widetilde{p}, \widetilde{q})$ and $(p', q') \simeq (\widetilde{p}', \widetilde{q}')$. We have to prove that $(p \circ \overline{p'}, q' \circ \overline{q}) \simeq (\widetilde{p} \circ \overline{\widetilde{p}'}, \widetilde{q}' \circ \overline{\widetilde{q}})$. In this order we consider the following two commutative diagrams:





where h and h' are the respective homeomorphisms.

Now let us consider the following diagram:



in which $(h\boxtimes h')(u,v) = (h(u), h'(v))$. Then $h\boxtimes h'$ is a well defined homeomorphism and it is easy to see that the above diagram is commutative. Hence the proof is completed. $\square$

Observe that metric spaces as objects and morphisms form a category. The homology functor H_* extends over this category. Namely, for a morphism $\varphi = \{X \overset{p}{\Longleftarrow} \Gamma \overset{q}{\longrightarrow} Y\}\colon X \to Y$ we define the induced linear map

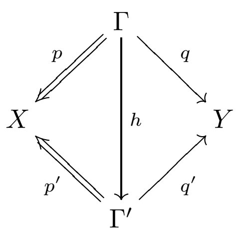
$$H_*(\varphi) = \varphi_*\colon H_*(X) \to H_*(Y)$$

by putting:

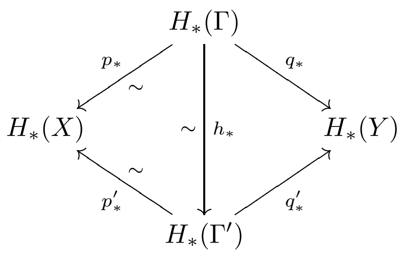
$$\varphi_* = q_* \circ p_*^{-1}. \tag{44.4}$$

(44.5) Proposition. *The induced map φ_* does not depend on the choice of the diagram (p,q).*

Proof. Assume that the following diagram is commutative:



where h is a homeomorphism. By applying functor H_* to the above diagram we obtain:



Since p_*, p'_* and h_* are isomorphisms, from the commutativity of the above diagram we infer:

$$q_* \circ p_*^{-1} = q'_* \circ (p'_*)^{-1}$$

and the proof is completed. □

(44.6) PROPOSITION. *If $\varphi\colon X \to Y$ and $\psi\colon Y \to Z$ are two morphisms then we obtain:*

$$(\psi \circ \varphi)_* = \psi_* \circ \varphi_*.$$

PROOF. In fact, we have the commutative diagram:

$$\begin{array}{ccccccccc} X & \stackrel{p}{\Longleftarrow} & \Gamma & \stackrel{q}{\longrightarrow} & Y & \stackrel{p'}{\Longleftarrow} & \Gamma' & \stackrel{q'}{\longrightarrow} & Z \\ & & & \overline{p'}\nwarrow\!\!\nwarrow & & \nearrow \overline{q} & & & \\ & & & & \Gamma \boxtimes \Gamma' & & & & \end{array}$$

By applying functor H_* to the above diagram we obtain:

$$(\psi \circ \varphi)_* = q'_* \circ \overline{q}_* \circ (\overline{p'}_*)^{-1} \circ p_*^{-1} = (q'_* \circ (p'_*)^{-1}) \circ (q_* \circ p_*^{-1}) = \psi_* \circ \varphi_*$$

and the proof is completed. □

Note that the above definition does not depend on the choice of the representative (p,q). Now, having a morphism φ, we have defined a multivalued map $\psi\colon X \multimap Y$ determined by morphism ψ which assigns to every $x \in X$ the compact set $\varphi(x) = q(p^{-1}(x))$, where (p,q) is a representative of φ. Sometimes, for simplicity, we will use the same notation for a morphism and a multivalued map determined by this morphism.

Since any Vietoris map is proper, we conclude that a multivalued map determined by a morphism is u.s.c. with compact values.

We will say also that a morphism φ is *compact* if the multivalued map ψ, determined by φ, is a compact map.

(44.7) PROPOSITION. *Any acyclic map is determined by a morphism. Moreover, any strongly admissible map is determined by a morphism, too.*

PROOF. Let $\psi\colon X \multimap Y$ be an acyclic map. Then we have a diagram:

$$X \stackrel{p_\psi}{\Longleftarrow} \Gamma_\psi \stackrel{q_\psi}{\longrightarrow} Y.$$

It is evident that ψ is determined by the morphism

$$\varphi = \{X \stackrel{p_\psi}{\Longleftarrow} \Gamma_\psi \stackrel{q_\psi}{\longrightarrow} Y\}\colon X \to Y.$$

If $\psi\colon X \multimap Y$ is strongly admissible, then according to the definition there is a diagram:

$$X \xLeftarrow{p} \Gamma \xrightarrow{q} Y$$

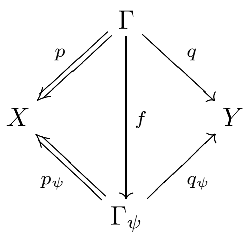
such that $\psi(x) = q(p^{-1}(x))$. Hence ψ is determined by the morphism:

$$\varphi = \{X \xLeftarrow{p} \Gamma \xrightarrow{q} Y\}\colon X \to Y$$

and the proof is completed. □

(44.8) Remarks.

(44.8.1) The pair (p_ψ, q_ψ) representing the morphism determined by an acyclic map ψ will be called the *generic pair*. In spite of the fact an acyclic map may be determined by different morphisms but the following relation holds. Assume that a morphism represented by a pair (p, q) determines an acyclic map ψ. Then the diagram:



is commutative, where $f(u) = (p(u), q(u))$ for every $u \in \Gamma$.

(44.8.2) We have seen already that any (singlevalued) map is determined by exactly one morphism (cf. (44.8.1)).

(44.9) Example. Let $\psi\colon \{p\} \multimap [0,1]$ be an acyclic map defined as follows:

$$\psi(p) = [0,1].$$

Let $\Gamma = [0,1] \times [0,1]$ and $p\colon \Gamma \to \{p\}$, $p(t,s) = p$ and $q\colon \Gamma \to [0,1]$, $q(t,s) = s$ for every $(t,s) \in \Gamma$. Then $\psi(p) = q(p^{-1}(p))$ but the pair (p,q) is not equivalent to the pair (p_ψ, q_ψ). So ψ is determined by two different morphisms. Observe that Γ_ψ is homeomorphic to $[0,1]$ but Γ is not homeomorphic to $[0,1]$.

Let $\varphi = \{X \xLeftarrow{p} \Gamma \xrightarrow{q} Y\}\colon X \to Y$ be a morphism and B be a subset of X. In such a case one can define the *restriction* of φ to the set B as the morphism:

$$\varphi|_B = \{B \xLeftarrow{\widetilde{p}} p^{-1}(B) \xrightarrow{\widetilde{q}} Y\}\colon B \to Y,$$

where $\widetilde{p}(u) = p(u)$ and $\widetilde{q}(u) = q(u)$ for every $u \in p^{-1}(B)$.

(44.10) PROPOSITION. *Let* $\varphi_j \in \mathcal{M}(X, Y_j)$, $j = 1, 2$. *Then there exists a morphism* $\varphi \in \mathcal{M}(X, Y_1 \times Y_2)$ *(denoted by* (φ_1, φ_2)*) such that*

$$\varphi(x) = \varphi_1(x) \times \varphi_2(x) \quad \textit{for every } x \in X.$$

PROOF. Indeed let (p_j, q_j), $j = 1, 2$ be the representative pairs for φ_1 and φ_2, respectively. We have the diagram:

$$\begin{array}{ccccc} & & \Gamma_1 & \xrightarrow{q_1} & Y_1 \\ & \overset{p_1}{\swarrow\!\!\swarrow} & & & \\ X & \overset{p}{\Longleftarrow} & \Gamma_1 \boxtimes \Gamma_2 & \xrightarrow{q} & Y_1 \times Y_2 \\ & \underset{p_2}{\nwarrow\!\!\nwarrow} & & & \\ & & \Gamma_2 & \xrightarrow{q_2} & Y_2 \end{array}$$

in which $\Gamma_1 \boxtimes \Gamma_2$ is the fibre product of p_1 and p_2 and $q(u, v) = (q_1(u), q_2(v))$, $p(u, v) = p_1(u) = p_2(v)$. Then we let:

$$(\varphi_1, \varphi_2) = \{X \overset{p}{\Longleftarrow} \Gamma_1 \boxtimes \Gamma_2 \xrightarrow{q} Y_1 \times Y_2\}\colon X \to Y_1 \times Y_2.$$

It left to the reader to verify that the above definition is correct. The proof of (44.10) is completed. □

(44.11) PROPOSITION. *Let* $\varphi_j \in \mathcal{M}(X_j, Y_j)$, $j = 1, 2$. *Then there exists a morphism* φ *(denoted by* $(\varphi_1 \times \varphi_2)$*) such that* $(\varphi_1 \times \varphi_2) \in \mathcal{M}(X_1 \times X_2, Y_1 \times Y_2)$, *which determines the map* $(x_1, x_2) \to \varphi_1(x_1) \times \varphi_2(x_2)$ *for every* $(x_1, x_2) \in X_1 \times X_2$.

PROOF. Having representative pairs (p_j, q_j), $j = 1, 2$ for φ_j, $j = 1, 2$ respectively, we let $\varphi = \varphi_1 \times \varphi_2$ to be the equivalence class of the pair $(p_1, \times p_2, q_1 \times q_2)$, and the proof is completed. □

Let E be a real normed space. Consider two morphisms:

$$\varphi = \{X \overset{p_0}{\Longleftarrow} \Gamma \xrightarrow{q_0} E\}\colon X \to E \quad \text{and} \quad \psi = \{X \overset{p_1}{\Longleftarrow} \Gamma \xrightarrow{q_1} E\}\colon X \to E$$

and a continuous map $f\colon X \to R$.

We would like to define two new morphisms:

$$(\varphi + \psi) \in \mathcal{M}(X, E) \quad \text{and} \quad (f \cdot \varphi) \in \mathcal{M}(X, E).$$

Let

$$\oplus\colon E \times E \to E \quad \text{and} \quad \odot\colon E \to R$$

be the algebraic operations in linear space E. In view of (44.11) we have:

$$(\varphi, \psi)\colon X \to E \times E \quad \text{and} \quad (f, \varphi)\colon X \to R \times E.$$

Consequently, we are allowed to define:

(44.12) Definition. We let

$$\varphi + \psi = \oplus(\varphi, \psi), \quad f \cdot \varphi = \odot(\alpha, \varphi).$$

It follows from the composition law for morphisms and (44.11) that the Definition (44.12) is correct.

(44.13) Remark. The notion of morphism was introduced in [GGr-2], where a weaker equivalence relation between diagrams was considered. Note that that in this section, for simplicity's sake, we have considered only Vietoris mappings with respect to the Čech homology functor with compact carriers, but all formulations and results would be exactly the same if we considered the notion of Vietoris maps with respect to the Čech cohomology functor.

We will end this section by introducing the notion of a homotopy of morphisms.

(44.14) Definition. Two morphisms $\varphi, \psi \in \mathcal{M}(X, Y)$ are called homotopic (written $\varphi \sim \psi$), provided there exists a morphism $\chi \in \mathcal{M}(X \times [0,1], Y)$ such that:

$$\chi(x, 0) = \varphi(x) \quad \text{and} \quad \chi(x, 1) = \psi(x)$$

for every $x \in X$, i.e.

$$\varphi = \chi \circ i_0 \quad \text{and} \quad \psi = \chi \circ i_1,$$

where $i_0, i_1 \colon X \to X \times [0,1]$ are defined $i_0(x) = (x, 0)$, $i_1(x) = (x, 1)$.

Since homology (cohomology) functor satisfies the composition law in the case of morphisms we conclude:

(44.15) Proposition. *If $\varphi \sim \psi$ then $\varphi_* = \psi_* (\varphi^* = \psi^*)$.*

We will go back to the homotopy properties of morphisms in Section 46.

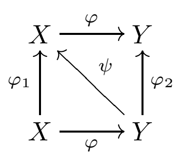
45. The Lefschetz fixed point theorem for morphisms

A morphism $\varphi \colon X \to X$ is called a *Lefschetz morphism* provided $\varphi_* \colon H(X) \to H(X)$ is a Leray endomorphism; for such φ we define the generalized Lefschetz number by putting

$$\Lambda(\varphi) = \Lambda(\varphi_*).$$

Immediately from (11.4) one can deduce:

(45.1) Proposition. *If the following diagram of metric spaces and morphisms is commutative:*



then if one of the vertical arrows is a Lefschetz morphism, then so is the other and in that case $\Lambda(\varphi_1) = \Lambda(\varphi_2)$.

Let $\varphi = \{X \overset{p}{\Longleftarrow} \Gamma \overset{q}{\longrightarrow} X\}\colon X \to X$ be a morphism. Recall that the coincidence set $\varkappa_{p,q}$ of (p,q) (see Section 12) is defined as follows:

$$\varkappa_{p,q} = \{y \in \Gamma \mid p(y) = q(y)\}.$$

(45.2) Definition. We say that the morphism $\varphi = \{X \overset{p}{\Longleftarrow} \Gamma \overset{q}{\longrightarrow} X\}\colon X \to X$ has a coincidence provided the set $\kappa(\varphi) = p(\kappa_{p,q})$ is non-empty.

Note that the above definition does not depend on the choice of a representative (p,q). Moreover, $\kappa(\varphi) \neq \emptyset$ if and only if $\kappa_{p,q} \neq \emptyset$ for some representative (p,q) of φ.

Let $\varphi = \{X \overset{p}{\Longleftarrow} \Gamma \overset{q}{\longrightarrow} X\}\colon X \to X$ by a morphism and $\psi\colon X \multimap X$, $\psi(x) = q(p^{-1}(x))$ be a multivalued map determined by φ. Then we have:

(45.3) Proposition. $\kappa(\varphi) \neq \emptyset$ *if and only if* $\mathrm{Fix}(\psi) \neq \emptyset$.

Proof of (45.3) is self-evident.

Let us observe that from our definitions and provided properties it follows that all results of the Section 43, about n-admissible maps, can be taken up on the class of the newly defined morphisms.

46. Homotopical classification theorems for morphisms

In this section we will study the homotopy properties of morphisms. We prove some generalizations of the classical Hopf classification theorems. The notion of homotopy for morphisms was introduced already in Section 44 (see (44.14)).

(46.1) Remark. Throughout this book we have assumed that all considered topological spaces are metrizable. In this section we will not keep this assumption however. We shall assume instead, that all considered spaces are paracompact. First note (see [Sp-M]) that theorem (8.14) is true for paracompact spaces. Since not every proper map is closed in paracompact spaces, in such a case, by Vietoris map $p\colon W \Longrightarrow X$ we shall understand a closed map such that for every $x \in X$ the set $p^{-1}(x)$ is nonempty, compact and acyclic (so p is a perfect surjection with acyclic fibres). This assumption implies that the multivalued map $\psi\colon X \multimap Y$, determined by a morphism

$$\varphi = \{X \overset{p}{\Longleftarrow} W \overset{q}{\longrightarrow} Y\}\colon X \to Y,$$

is u.s.c. where p is a perfect surjection with acyclic fibres.

Concluding, in this section (only!) we will consider morphisms of the following type

$$\varphi = \{X \stackrel{p}{\Longleftarrow} W \stackrel{q}{\longrightarrow} Y\}: X \to Y,$$

where X, Y are metric spaces, W is a paracompact space and p is a perfect surjection with acyclic fibres with respect to the Čech cohomology functor (cf. Remark (44.13)).

The reason for the above changes is very simple. Namely, for a continuous map $f: X \to Y$ we will consider three new spaces:

$$X \oplus Y, \quad X \cup_f Y \quad \text{and} \quad Z_f$$

which are not metric in general (even if X and Y are metric).

Recall the definitions of $X \oplus Y$, $X \cup_f Y$ and Z_f (see [Sp-M] for details): By $X \oplus Y$ we shall understand the disjoint union of X and Y. The space $X \cup_f Y$ is the space obtained by *attaching* X to Y by means of $f: A \to Y, A \subset X$, i.e. the quotient space obtained from $X \oplus Y$ by identifying each point $x \in A$ with $f(x) \in Y$. By the cylinder Z_f of $f: X \to Y$ we mean the space $X \times [0,1] \cup_q Y$, where $q: X \times \{1\} \to Y$, $q(x,1) = f(x)$.

(46.2) PROPOSITION. *The homotopy relation '$\sim$' is an equivalence relation in $\mathcal{M}(X,Y)$.*

For the proof of (46.2) we need the following lemma:

(46.3) LEMMA. *Assume that $X_i = X \times [(1/2)i, (1/2)(i+1)]$, $i = 0,1$ and let $j_i: X_0 \cap X_1 \to X_i$, $r_i: X_i \to X \times [0,1]$ be inclusions. If $\varphi_i \in \mathcal{M}(X_i, Y)$ and $\varphi_0 j_0 = \varphi_1 j_1$, then there is $\varphi \in \mathcal{M}(X \times I, Y)$ such that $\varphi \circ r_i = \varphi_i$, $i = 0,1$ and $I = [0,1]$.*

PROOF. Let $X_i \stackrel{p_i}{\Longleftarrow} W_i \stackrel{q_i}{\longrightarrow} Y$ represent φ_i, $i = 0,1$. We have a diagram:

$$X_0 \cap X_1 \stackrel{p}{\Longleftarrow} W \stackrel{q}{\longrightarrow} Y$$

and mappings $f_i: W \to p_i^{-1}(X_0 \cap X_1) \subset W_i$ such that $p = p_i \circ f_i$, $q = q_i \circ f_i$, $i = 0,1$. Let $W_I = W \times I$, define $g: W \times \{0,1\} \to W_0 \oplus W_1$ by $g(w,i) = f_i(w)$, $w \in W$, $i = 0,1$ and put $\overline{W} = W_I \cup_g W_0 \oplus W_1$. Moreover, let $h_i: W_i \to \overline{W}$, $i = 0,1$, and $h: W_i \to W$ be the respective quotient maps. We shall define a cotriad $X \times I \stackrel{\overline{p}}{\longleftarrow} \overline{W} \stackrel{\overline{q}}{\longrightarrow} Y \in D_m$ and Vietoris maps $\overline{f}_i: \overline{W}_i \to W_i$, where $\overline{W}_i = \overline{p}^{-1}(X_i)$, such that $p_i \overline{f}_i = \overline{p}|_{\overline{W}_i}$ and $q_i \overline{f}_i = \overline{q}|_{\overline{W}_i}$ for $i = 0,1$.

To this end consider a map $\overline{p}: \overline{W} \to X \times I$ given by the formulae:

$$\begin{aligned} \overline{p}(h_i(w_i)) &= p_i(w_i), \quad \text{for } w_i \in W_i,\ i = 0,1, \\ \overline{p}(h(w,t)) &= p(w), \qquad \text{for } w \in W,\ t \in I. \end{aligned}$$

One can easily see that $\overline{p}$ is well-defined and $\overline{p}$ is a continuous surjection. For any closed subset $C \subset \overline{W}$,

$$\overline{p}(C) = p_0(h_0^{-1}(C)) \cup p\pi(h^{-1}(C)) \cup p_1^{-1}(C)),$$

where $\pi\colon W_I \to W$ is the projection. For any $x \in X$ we have:

$$\overline{p}^{-1}(x) = \begin{cases} h_i(p_i^{-1}(x)) & \text{for } x \in X_i \setminus (X_0 \cap X_1),\ i = 0,1, \\ h(p^{-1}(x) \times I) & \text{for } x \in X_0 \cap X_1. \end{cases}$$

Observe that

$$h^{-1}(\overline{w}) = \begin{cases} (w,t) & \text{for } t \in (0,1), \\ f_i^{-1}(f_i(w)) \times \{i\} & \text{for } t = i = 0,1; \end{cases}$$

hence $h\colon p^{-1}(x) \times I \to \overline{p}^{-1}(x)$ is a Vietoris map. Consequently, we infer that $\overline{p}$ is a Vietoris map.

Define $\overline{q} : \overline{W} \to Y$ similarly as $\overline{p}$ above.

Since $\overline{W}_i = \overline{p}^{-1}(X_i) = h_i(W_i) \cup h(W_I)$, we define $\overline{f}_i\colon \overline{W}_i \to W_i$ by the formula

$$\begin{aligned} \overline{f}_i(h_i(w_i)) &= w_i, && w_i \in W_i, \\ \overline{f}_i(h(w,t)) &= f_i(w), && w \in W,\ t \in I,\ i = 0,1. \end{aligned}$$

Analogously, we can check that $\overline{f}_i$ are Vietoris maps and $p_i \circ \overline{f}_i = \overline{p}_i|_{W_i}$, $q_i \circ \overline{f}_i = \overline{q}|_{W_i}$, $i = 0,1$. At last, it is enough to represent $\varphi \in \mathcal{M}(X \times I, Y)$ by $(\overline{p}, \overline{q})$ and the proof is completed. □

We are in a position to prove (46.2).

PROOF OF (46.2). It is easy to see that the relation $\sim$ is symmetric and, by Lemma (46.3) above, transitive. However, for a morphism $\varphi \in \mathcal{M}(X,Y)$ we can build its self-homotopy. Namely, if $X \overset{p}{\Longleftarrow} W \overset{q}{\longrightarrow} Y$ represents φ, then we have a diagram:

$$X \times I \overset{p\times \mathrm{id}}{\Longleftarrow} W \times I \overset{q\times\mathrm{id}}{\longrightarrow} Y, \quad \overline{q}(x,t) = q(x)$$

which gives a needed homotopy. The proof of (46.2) is completed. □

We let $\mathcal{M}[X,Y] = \mathcal{M}(X,Y)/_{\sim}$.

Now we would like to study the problem what are the conditions under which a given morphism $\varphi \in \mathcal{M}(X,Y)$ is homotopic to a map $f\colon X \to Y$.

(46.4) PROPOSITION. *Let* $\varphi = \{X \overset{p}{\Longleftarrow} W \overset{q}{\longrightarrow} Y\}: X \to Y$ *and assume there exists* $f: X \to Y$ *such that the following diagram:*

$$\begin{array}{ccc} W & \xrightarrow{\;q\;} & Y \\ {\scriptstyle p}\Big\Downarrow & \nearrow_{f} & \\ X & & \end{array}$$

is homotopically commutative, i.e. $f \circ p \sim q$ *(as singlevalued mappings). Then* $\varphi \sim f$.

PROOF. Let $Z = Z_p$ be the cylinder of p. Define $\overline{p}: Z \to X \times I$ and $\overline{q}: Z \to Y$ by:

$$\begin{aligned} \overline{p}(w,t) &= (p(w),t), & \overline{q}(w,t) &= f(p(w)) & &\text{for } w \in W,\ t \in J, \\ \overline{p}(x) &= (x,1), & \widetilde{q}(x) &= f(x) & &\text{for } x \in X. \end{aligned}$$

Since the inclusion $i: W \to Z$ is a cofibration, there is $\overline{h}: Z \times J \to Y$ such that $h(z,0) = \overline{q}(z)$ and $\overline{h}|_{W \times I} = h$, where h is a homotopy joining $f \circ p$ and q.

Let $\overline{q}: Z \to Y$ be defined as $\overline{q} = \overline{h}(\,\cdot\,,1)$. Then $(\overline{p},\overline{q})$ represents a morphism $\chi: X \times I \to Y$ which is a homotopy joining φ with f and the proof is completed.□

For given two spaces X, Y, by $[X,Y]$ we shall denote the set of all homotopy classes of continuous mappings from X to Y, i.e. $[X,Y] = C(X,Y)/_{\sim}$, where $C(X,Y)$ is the set of all continuous mappings from X to Y and '$\sim$' stands for the homotopy relation.

The most general classification theorem is the following (see [Kr2-M], or [Kr1]).

(46.5) THEOREM. *Let* X *and* Y *be two spaces such that*

(46.5.1) $\dim X < +\infty$,
(46.5.2) $Y \in \mathrm{ANR}$ *and* $\pi_n(Y)$ *is finitely generated for every* $n \geq 1$,
(46.5.3) X *or* Y *is compact.*

Then there exists a one to one correspondence $T: \mathcal{M}[X,Y] \to [X,Y]$.

The proof of (46.5) is quite complicated and needs several topological constructions including the obstruction theory, Eilenberg–MacLane complexes and some other facts and, therefore, is beyond the scope of this monograph (we recommend [Kr2-M] or [Kr1] for details).

(46.6) REMARK. Note that if (46.5) is considered in the relative version, i.e. for $\varphi: (X,X') \to (Y,Y')$ we have to assume also that $Y' \in \mathrm{ANR}$ (see again [Kr2-M] or [Kr1]).

We will see in the next section that (46.5) is useful in the fixed point index theory for morphisms.

47. The fixed point index for morphisms

We shall define the fixed point index for morphisms in two steps. At first, for morphisms in normed spaces, and then for morphisms of arbitrary ANRs. Let E be a normed space and assume that A, X are closed subsets of E such that $A \subset X$. By $\mathcal{M}_A(X, E)$ we shall denote the set of all compact morphisms from X to E such that

$$\mathrm{Fix}(\psi_\varphi) \cap A = \emptyset,$$

where ψ_φ is a multivalued mapping determined by φ. Note, that in this section by a Vietoris map we mean 'Vietoris with respect to the Čech cohomology functor H^*'. Similarly, let $C_A(X, E)$ denote the set of all compact maps from X to E. Within the class of $\mathcal{M}_A(X, E)$ we consider the following notion of a homotopy. If $\varphi_j \in \mathcal{M}_A(X, E)$, $j = 0, 1$ then we say that φ_0 and φ_1 are homotopic (written $\varphi_0 \simeq \varphi_1$) whenever, there exists a compact morphism $\chi \in \mathcal{M}(X \times [0, 1], E)$ such that

$$\psi_{\chi(\,\cdot\,, t)} \cap A = \emptyset \quad \text{for every } t \in [0, 1].$$

(47.1) ASSUMPTION. In this section we shall assume finite-dimensionality of the fibres $p^{-1}(x)$ for every considered Vietoris mapping p.

First we shall prove the following lemma:

(47.2) LEMMA. *Let* $\varphi = \{X \stackrel{p}{\Longleftarrow} W \stackrel{q}{\longrightarrow} E\}\colon X \to E$ *be a morphism such that* $\mathrm{Fix}(\psi_\varphi) \cap A = \emptyset$. *Then there is a unique (up to homotopy in* $C_A(X, E)$*) map* $F \in C_A(X, E)$ *such that there is a compact homotopy* $H\colon W \times [0, 1] \to E$ *joining* $F \circ p$ *with* q *and* $p(w) \neq H(w, t)$ *for* $w \in W$, $t \in [0, 1]$.

PROOF. For any $\varepsilon > 0$ there is a finite-dimensional linear subspace E_ε of E such that $A \cap E_\varepsilon \neq \emptyset$ and a Schauder projection $\pi_\varepsilon\colon K \to E_\varepsilon$ such that $\|\pi_\varepsilon(x) - x\| < \varepsilon$ for $x \in K$ (if $\dim E < \infty$ then, e.g. $E_\varepsilon = E$ and π_ε is the inclusion $K \to E$).

(1) By standard arguments one shows the existence of $\varepsilon > 0$ such that $\{w \in W \mid \|p(w) - \pi q(w)\| < 2\varepsilon\} \cap \overline{W} = \emptyset$, where we put $\pi := \pi_\varepsilon$. Let $L := E_\varepsilon$ and assume, without loss of generality, that $\dim L > m + 1$. In order to facilitate the notion, we let

$$\begin{aligned} X_L &:= X \cap L, & A_L &:= A \cap L,\\ W_L &:= p^{-1}(X_L), & \overline{W}_L &:= p^{-1}(A_L),\\ p_L &:= p|W_L\colon W_L \to X_L, & q_L &:= \pi q|W_L\colon W_L \to L. \end{aligned}$$

Clearly the spaces W_L and $\overline{W}_L$ are compact.

(2) Evidently $p_L - q_L\colon (W_L, \overline{W}_L) \to (L, L \setminus 0)$. By (8.15) there is a unique (up to homotopy) map $f_L\colon (X_L, A_L) \to (L, L \setminus 0)$ such that $h_L\colon f_L p_L \simeq p_L -$

$q_L\colon (W_L, \overline{W}_L) \to (L, L \setminus 0)$. Let $F\colon X \to L \subset E$ be a compact extension of the map $X_L \ni x \to x - f_L(x) \in L$. Clearly, $\operatorname{Fix}(F) \cap A = \emptyset$, hence $F \in C_A(X, E)$. Let $H_1\colon W \times I \to L \subset E$ be a compact extension of the map

$$W \times \{0,1\} \cup W_L \times I \ni (w,t) \to \begin{cases} fp(w) & \text{for } w \in W,\ t = 0, \\ \pi q(w) & \text{for } w \in W,\ t = 1, \\ p_L(w) - h_L(w,t) & \text{for } w \in W_L,\ t \in I. \end{cases}$$

It is easy to see that, for $w \in \overline{W}$, $t \in I$, $p(w) \neq H_1(w,t)$. Now, $H_1(\,\cdot\,, 0) = Fp$, $H_1(\,\cdot\,, 1) = \pi q \simeq q$ through the linear homotopy $H_2(w,t) = (1-t)\pi q(w) + tq(w)$ for $w \in W$, $t \in I$. Altogether, taking into account the choice of ε, we have established the existence part of the assertion.

(3) To show the uniqueness of $[F]$, suppose that $G \in C_A(X, E)$ and there is a compact homotopy $H\colon G_p \simeq q\colon W \to E$ and $H(w,t) \neq p(w)$ for $w \in \overline{W}$, $t \in I$. In view of the choice of ε, we can assume that $H\colon G_p \simeq \pi q$.

First, let us suppose that H is a finite-dimensional map, i.e. $H(W \times I) \subset N$ (in particular $G(X) \subset N$), where N is a linear subspace of $E, \dim N < \infty$, and $L \subset N$.

(i) If $N = L$ then defining $g\colon X_L \to L$ by $g(x) = x - G(x)$, $x \in X_L$, we see that $gp_L \simeq p_L - q_L$ and, by the uniqueness of $[f_L]$, $g \simeq f_L$. Hence $F \simeq G$ in $C_A(X, E)$.

(ii) Suppose that $L \subset N$. Replacing in (47.2.1) L by N, we define X_N, A_N, W_N and $\overline{W}_N$ and maps $p_N\colon W_N \to X_N$, $q_N := \pi q|_{W_N}\colon W_N \to L \subset N$. Moreover, let $g_N(x) = x - G(x)$ for $x \in X_N$. Therefore, $g_N p_N \simeq p_N - q_N\colon (W_N, \overline{W}_N) \to (N, N \setminus 0)$. Take any subspace $N', L \subset N' \subset N$ such that $N = N' \oplus Y$ (direct sum), where $\dim Y = 1$.

We claim that there are $F'\colon X \to N'$, $F' \simeq G$ in $C_A(X, E)$ and a compact homotopy $H'\colon F'p \simeq \pi q\colon W \to E$ such that $H'(W \times I) \subset N'$ and $p(w) \neq H'(w,t)$ for $w \in \overline{W}, t \in I$.

Let $X' := X \cap N'$, $A' := A \cap N'$, $W' := p^{-1}(X')$, $\overline{W}' := p^{-1}(A')$ and $p' := p|_{W'}\colon W' \to X'$, $q' := \pi q|_{W'}\colon W' \to L \subset N'$. Evidently, $(p' - q')(\overline{W}') \subset N' \setminus 0$; hence, again by (8.15), there is a map $f'\colon (X', A') \to (N', N' \setminus 0)$ such that $h'\colon f'p' \simeq p' - q'\colon (W', \overline{W}') \to (N', N' \setminus 0)$. Assume that f' is defined on N' (if not, take an arbitrary compact extension of f' onto N' with values in N') and define a map $f_N\colon X_N \to N$ by the formula $f_N(x) = f'(x') + y$, $x \in X$ and $x = x' + y$ is the unique representation: $x' \in N'$, $y \in Y$. Obviously, $f_N p_N(\overline{W}_N) \subset N \setminus 0$. Consider the map $d\colon W_N \times \{0,1\} \cup W' \times I \to N$ given by

$$d(w,t) = \begin{cases} f_N p_N(w) & \text{for } w \in W_N,\ t = 0, \\ p_N(w) - q_N(w) & \text{for } w \in W_N,\ t = 1, \\ h'(w,t) & \text{for } w \in W',\ t \in I. \end{cases}$$

Since $f_N p_N|W' = f'p'$ and $(p_N - q_N)|_{W'} = p' - q'$, we see that d is well-defined and compact. Now let $N^{\pm}$ be the open half spaces of N determined by N' and Y and let $W_N^{\pm} = p^{-1}(X \cap N^{\pm})$. Hence $W_N = W_N^{+} \cup W' \cup W_N^{-}$ and, for $w \in W_N^{\pm}$, $f_N p_N(w)$, $p_N(w) - q_N(w) \in N^{\pm}$. Thus there is a compact extension h of d over $W_N \times I$ such that $h(W_N^{\pm} \times I) \subset N^{\pm}$ showing that $h\colon f_N p_N \simeq p_N - q_N\colon (W_N, \overline{W}_N) \to (N, N \setminus 0)$. By the uniqueness of $[g_N]$, $f_N \simeq g_N\colon (X_N, A_N) \to (N, N \setminus 0)$. Define $F'\colon X \to N'$ as a compact extension of the map $X_N \ni x \to x - f_N(x)$ (observing that, if $x = x' + y \in X_N$, $x' \in N'$, $y \in Y$ then $x - f_N(x) = x' - f'(x') \in N'$). Clearly, $G \simeq F'$ in $C_A(X, E)$. Let $H'\colon W \times I \to N'$ be a compact extension of the map

$$W \times \{0,1\} \cup W' \times I \ni (w,t) \to \begin{cases} F'p(w) & \text{for } w \in W,\ t = 0, \\ \pi q(w) & \text{for } w \in W,\ t = 1, \\ p'(w) - h'(w,t) & \text{for } w \in W',\ t \in I. \end{cases}$$

One easily sees that, for $w \in \overline{W}$, $t \in I$, $p(w) \neq H'(w,t)$.

(iii) After at most $\operatorname{codim}_N L$ steps we get a map $F' \simeq G$ (in $C_A(X, E)$) and $H'\colon F' \simeq \pi q\colon W \to L$ with $H'(w,t) \neq p(w)$ for $w \in \overline{W}$, $t \in I$. In view of (i), $F' \simeq F$ hence $F \simeq G$ in $C_A(X, E)$.

If the original H is not finite-dimensional, then take $\varepsilon' > 0$, $\varepsilon' < \varepsilon$, such that $\{w \in W \mid \|\pi' H(w,t) - p(w)\| < \varepsilon' \text{ for some } t \in I\} \cap \overline{W} = \emptyset$, where $\pi\colon \operatorname{cl} H(W \times I) \to N \supset L$, $\dim N < \infty$, is a Schauder projection with $\|\pi'(x) - x\| < \varepsilon'$ for $x \in \operatorname{cl} H(W \times I)$.

Now $\pi' H\colon \pi' G p \simeq \pi' \pi q \simeq \pi q$ and, for $w \in \overline{W}$, $t \in I$, $p(w) \neq \pi' H(w,t)$. By (i)–(iii) above, $G \simeq \pi' G \simeq F$ in $C_A(X, E)$. This completes the proof of Lemma (47.2). □

Now, we prove the following:

(47.3) THEOREM. *Under the above assumptions there is a bijection:*

$$\mathcal{D}\colon \mathcal{M}_A[X, E] \xrightarrow{\sim} C_A[X, E],$$

where $\mathcal{M}_A[X, E]$ and $C_A[X, E]$ denote the sets of the respective homotopy classes of $\mathcal{M}_A(X, E)$ and $C_A(X, E)$.

Observe that in comparison to (46.5), where the finite dimensionality of the domain and the compactness of either the domain or the range were crucial, in the case above ($\dim E = \infty$) we have replaced these conditions by the compactness of the respective maps.

PROOF OF (47.3). If $\dim E < +\infty$ then the assertion follows almost immediately from (46.5). So we shall restrict our considerations to the case $\dim E = \infty$.

Let $\alpha \in \mathcal{M}_A[X,E]$, take an arbitrary $\varphi \in \alpha$ and put $K = \operatorname{cl} \psi_\varphi(X)$. Of course, the sets K and $\operatorname{Fix}(\psi_\varphi)$ are compact.

Suppose $X \overset{p}{\Longleftarrow} W \overset{q}{\longrightarrow} E$ is a representative of φ. Moreover, put $\overline{W} = p^{-1}(A)$. Now, we proceed by using (47.2). Take $F \in C_A(X,E)$ in (47.2) and put

$$\mathcal{D}(\alpha) = [F].$$

The verification that $\mathcal{D}$ is well defined (i.e. does not depend on the choice of $\varphi \in \alpha$ and $(p,q) \subset \varphi$) is strictly technical and, therefore, we left it to the reader. The proof of (47.3) is completed. □

Now, we would like to give a definition of the fixed point index for morphisms as presented in ([Kr2]; cf. also [Kr1-M], [Kr2-M]). Note that the mentioned definition is obtained in spirit of [Go1-M], [Bry], [BG-1], [Ku1] and [Ku2].

Let $\mathcal{A}$ denote the class of all triples (E,U,φ) where: E is a normed space, U is open and bounded in E, $\varphi\colon U \to E$ is a compact morphism such that $\operatorname{Fix}(\psi_\varphi)$ is a compact subset of U.

We say that triples (E_j,U_j,φ_j), $j = 0,1$, are homotopic, if $E_j = E$, $U_j = U \subset E$, $j = 0,1$ and there is a compact morphism $\chi\colon U \times [0,1] \to E$ which is a homotopy between φ_0 and φ_1 and $\operatorname{cl}\big(\bigcup_{t\in[0,1]} \operatorname{Fix}(\psi_{\varphi(t,\,\cdot\,)})\big)$ is a compact subset of U.

Consider a triple (E,U,φ) with $\dim E = n$. Let $\varphi = \{U \overset{p}{\Longleftarrow} W \overset{q}{\longrightarrow} E\}\colon U \to E$. There exists an open set $V \subset E$ such that:

$$\operatorname{Fix}(\psi_\varphi) \subset V \subset \operatorname{cl} V \subset U.$$

We let $X = \operatorname{cl} V$, $A = \partial V$. Then we can consider the following diagram (cf. Section 12).

$$(E,P) \overset{p-q}{\longleftarrow} (\widetilde{W},\overline{W}) \overset{p}{\Longrightarrow} (X,A)$$

where $\widetilde{W} = p^{-1}(X)$, $\overline{W} = p^{-1}(A)$, $p = E \setminus \{0\}$ and we keep the notation of p,q for the respective restrictions.

Consequently, we obtain the diagram:

$$(47.4)\quad (L,P) \overset{p-q}{\longleftarrow} (\widetilde{W},\overline{W}) \overset{p}{\Longrightarrow} (X,A) \overset{i_1}{\longrightarrow} (L, L\setminus V) \overset{i_2}{\longleftarrow} (L, L\setminus B) \overset{i_3}{\longrightarrow} (L,P),$$

in which B is an open ball in L such that $V \subset B$, $L = E$. Since $H^n(E,P) \approx H^n(S^n) \approx Z$, we can choose a generator $\kappa \in H^n(E,P)$. Similarly, as in Section 12 we define the fixed point index $\operatorname{ind}(E,U,\varphi)$ of (E,U,φ) as the number $d \in Z$ given by the equality

$$(47.5)\qquad (i_3^{*n})^{-1} \circ i_2^{*n} \circ (i_1^{*n})^{-1} \circ (p^{*n})^{-1} \circ (p-q)^{*n}(\kappa) = d \cdot \kappa.$$

From the functionality of H^* it follows immediately that this definition is correct. If (E, U, φ) is an arbitrary triple in $\mathcal{A}$, then again we choose V such that

$$\mathrm{Fix}(\psi_\varphi) \subset V \subset \mathrm{cl}\, V \subset U$$

and we put $X = \mathrm{cl}\, V$, $A = \partial V$. Then, clearly $\varphi|_X \in \mathcal{M}_A(X, E)$. Using the notation from the proof of (47.3) (where we replace φ by $\varphi|_X$) and from the proof of (47.2) (see (47.2.1)) we can get again the diagram (47.4) in which L is a finite dimensional subspace an q is a composition of the given q with π_ε.

So, our index $\mathrm{ind}\,(E, U, \varphi)$ defined in (47.5) is correct for arbitrary $(E, U, \varphi) \in \mathcal{A}$, i.e. it does not depend on choice of V, ε, π, L and B. Let us collect some properties of the index ind.

(47.6) PROPOSITION. *Let* $(E, U, \varphi) \in \mathcal{A}$.

(47.6.1) *If* φ *determines a singlevalued map* F, *then* $\mathrm{ind}\,(E, U, \varphi) = i(F, U)$, *where* $i(F, U)$ *is the ordinary fixed point index* (*for singlevalued maps*).

(47.6.2) (Units) *If, for each* $x \in U$, $\varphi(x) = \{x_0\}$ *then:*

$$\mathrm{ind}\,(E, U, \varphi) = \begin{cases} 1 & \text{if } x_0 \in U, \\ 0 & \text{if } x_0 \notin U. \end{cases}$$

(47.6.3) (Existence) *If* $\mathrm{ind}\,(E, U, \varphi) \neq 0$ *then* $\kappa(\varphi) \neq \emptyset$.

(47.6.4) (Additivity) *If* $\mathrm{Fix}(\psi_\varphi) \subset \bigcup_{i=1}^r U_i$, *where* U_i *are open disjoint subsets of* U *then* $(E, U_i, \varphi|_{U_i}) \in \mathcal{A}$ *for* $i = 1, \ldots, r$, *and*

$$\mathrm{ind}\,(E, U, \varphi) = \sum_{i=1}^r \mathrm{ind}\,(E, U_i, \varphi|_{U_i}).$$

(47.6.5) (Homotopy) *If* (E', U', φ') *and* (E, U, φ) *are homotopic then*

$$\mathrm{ind}\,(E, U, \varphi) = \mathrm{ind}\,(E', U', \varphi').$$

(47.6.6) (Contraction) *If* $\varphi' \in \mathcal{M}(U, E')$ *where* E' *is a linear subspace of* E, $j\colon E' \to E$ *is the inclusion and* $j \circ \varphi' = \varphi$ *then* $(E, U \cap E', \varphi'|_{U \cap E'}) \in \mathcal{A}$ *and*

$$\mathrm{ind}\,(E', U \cap E', \varphi'|_{U \cap E'}) = \mathrm{ind}\,(E, U, \varphi).$$

(47.6.7) (Strong Units) *If, for each* $x \in U$, $\varphi(x) = K \subset E$ *then*

$$\mathrm{ind}\,(E, U, \varphi) = \begin{cases} 1 & \text{if } K \cap U \neq \emptyset, \\ 0 & \text{if } K \cap U = \emptyset. \end{cases}$$

(47.6.8) (Multiplicativity) *Let E' be a normed space and U' a bounded open subset of E'. If $(E', U', \varphi') \in \mathcal{A}$ then $(E \times E', U \times U', \varphi \times \varphi') \in \mathcal{A}$ and*

$$\operatorname{ind}(E \times E', U \times U', \varphi \times \varphi') = \operatorname{ind}(E, U, \varphi) \cdot \operatorname{ind}(E', U', \varphi').$$

The proof of (47.6) depends on the application of the definition and the Čech cohomology functor and it is straightforward (strictly analogous as for singlevalued maps see [Do-M] or also Section 12).

(47.7) Remark. Note, similarly as for acyclic mappings, that if an index function satisfies property (47.6.1), Units, Additivity and Homology properties then it is equal to ind defined in (47.5).

Now, we are going to define the fixed point index for morphisms of arbitrary ANRs. Let $\mathcal{B}$ denote the class of all triples (X, U, φ), where $X \in \mathrm{ANR}$, U is an open subset of X and $\varphi: U \to X$ is a compact morphism such that $\mathrm{Fix}(\psi_\varphi)$ is a compact subset of U. Similarly, two triples $(X_j, U_j, \varphi_j) \in \mathcal{B}$, $j = 0, 1$ are homotopic if $X_j = X$, $U_j = U$, $j = 0, 1$ and there is a compact morphism $\chi: U \times [0, 1] \to X$ such that $\chi \circ i_j = \varphi_j$, $j = 0, 1$, $i_j: U \to U \times [0, 1]$, $i_j(x) = (x, j)$, $j = 0, 1$ and $\operatorname{cl} \bigcup_{t \in [0,1]} \mathrm{Fix}(\psi_{\varphi(t,\,\cdot\,)})$ is a compact subset of U.

Let $(X, U, \varphi) \in \mathcal{B}$. In view of the Arens–Eells embedding theorem, there is an embedding $i: X \to i(X) \subset E$ of X onto a closed subset $i(X)$ of a normed space E. There is an open set $V \subset E$ and a map $r: V \to U$ such that $r \circ i|_U = \mathrm{id}_U$ and $i^{-1}(V) = U$ (indeed, since $i(X)$ is a neighbourhood retract of E). Observe that $(E, U, i \circ \varphi \circ r) \in \mathcal{A}$. Therefore, we are allowed to let:

(47.7.1) $$\operatorname{ind}(X, U, \varphi) = \operatorname{ind}(E, V, i \circ \varphi \circ r).$$

(47.8) Theorem. *Let $(X, U, \varphi) \in \mathcal{B}$.*

(47.8.1) *The above definition (47.7.1) is correct i.e. it does not depend on the choice of i, E, V, r.*

(47.8.2) (Contraction) *If $\varphi(U) \in Y$, where $Y \subset X$ is an ANR then $(Y, Y \cap U, \varphi|_{Y \cap U}) \in \mathcal{B}$ and $\operatorname{ind}(X, U, \varphi) = \operatorname{ind}(Y, Y \cap U, \varphi|_{Y \cap U})$.*

(47.8.3) (Topological Invariance) *If $Y \in \mathrm{ANR}$ and $h: X \to Y$ is a homeomorphism then $(Y, h(U), h \circ \varphi \circ h^{-1}) \in \mathcal{B}$, and*

(47.8.4) *The defined index $\operatorname{ind}: \mathcal{B} \to Z$ satisfies the properties of Existence, Additivity, Homotopy, Multiplicativity, Strong Unity and the restriction of* ind *to singlevalued maps is equal to the ordinary index (as formulated in (47.6)).*

Proof. Let $i|_C: X \to E_k$ be an embedding of X onto a closed set $A_k = i_k(X)$ in a normed space E_k, $V_k \subset E_k$ be an open set such that $i_k^{-1}(V_k) = U$ and $r_k: V_k \to U$ be a map such that $r_k i_k = \mathrm{id}_{E_k}$, $k = 1, 2$.

We are to show that $\operatorname{ind}(E_1, V_1, \eta_1) = \operatorname{ind}(E_2, V_2, \eta_2)$, where $\eta_k = i_k \varphi r_k$, $k = 1, 2$. Observe that $h' = i_2 i_1^{-1}\colon A_1 \to A_2$ is a homeomorphism. Let $E = E_1 \times E_2$ and $j_k\colon E_k \to E$, $\pi_k\colon E \to E_k$ be given as follows:

$$j_1(x) = (x, 0), \quad \pi_1(x, y) = x, \quad j_2(x) = (0, x), \quad \pi_2(x, y) = y,$$

for $y, x \in E_k$, $k = 1, 2$. Clearly, $\pi_k j_k = \mathrm{id}_{E_k}$. We claim that there is a homeomorphism $h\colon E \to E$ such that $hj_1|_{A_1} = j_2 h'$. Indeed, by the Dugundji extension theorem, there are maps $h_1'\colon E_1 \to E_2$ and $h_2'\colon E_2 \to E_1$ such that $h_1|_{H_1} = h'$ and $h_2'|_{A_2} = h'^{-1}$. Define $h_k\colon E \to E_k$, $k = 1, 2$ by the formula

$$h_1(x, y) = (x, h_1'(x) + y), \quad h_2(x, y) = (x - h_2'(y), y)$$

and put $h = h_2 \circ h_1$, then h is needed homeomorphism.

(1) Now, let us introduce some notation. For $k = 1, 2$, we put $W_k = \pi_k^{-1}(V_k)$, $s_k = j_k i_k\colon X \to E$. Then $s_k^{-1}(W_k) = U$ and $s_2 = hs_1$. Let $W = W_2 \cap h(W_1)$ (observe that W is open and nonempty since $s_2(U) \subset W$) and let $f_1 = r_1 \pi_1 h^{-1}|_W\colon W \to U$, $f_2 = r_2 \pi_2|_W\colon W \to U$. Then $f_2 s_2|_U$, $f_1 s_2|_U = 1_U$.

(2) In view of (47.6.6) and (47.6.4)

$$\begin{aligned}\operatorname{ind}(E_1, V_1, \eta_1) &= \operatorname{ind}(E, W_1, j_1\eta_1\pi_1) = \operatorname{ind}(E, W_1, s_1\varphi r_1\pi_1)\\ &= \operatorname{ind}(E, h(W_1), s_2\varphi r_1 \pi_1 h^{-1}) = \operatorname{ind}(E, W, s_2\varphi f_1)\end{aligned}$$

and

$$\operatorname{ind}(E_2, V_2, \eta_2) = \operatorname{ind}(E_2, W_2, j_2\varphi_2\pi_2) = \operatorname{ind}(E, W, s_2\varphi f_2)$$

since, as one easily checks,

$$\operatorname{Fix}(s_2\varphi r_1\pi_1(h^{-1}|_{h(W_1)})) \subset W \quad \text{and} \quad \operatorname{Fix}(s_2\varphi r_2(\pi_2\,|_{W_2})) \subset W.$$

(3) For any number $\eta > 0$, let $N_\eta = \{x \in X \mid \inf\{d_X(x, z) \mid z \in \operatorname{Fix}(\varphi)\} < \eta\}$. Choose an $\alpha > 0$ such that $\operatorname{cl} N_{4\alpha} \subset U$ and $\varphi \mid \operatorname{cl} N_{4\alpha}$ determines a compact map and put $U' = N_{3\alpha}$. There is an $\varepsilon \in (0, \alpha)$ such that, for $x \in \operatorname{cl} N_{4\alpha}$, if $y \in \varphi(x)$ and $d_X(y, x) < \varepsilon$ then $x \in N_\alpha$.

(4) For any $x \in U'$, by the continuity of f_2 and since $s_2(x) \in W$ there is a $\delta_x > 0$ such that $B_x = \{z \in E \mid \|z - s_2(x)\|_E < \delta_x\} \subset W$ and $d_X(f_2(z), x) < \varepsilon/2$ for $z \in B_x$ (recall that $f_2 s_2(x) = x$). For $x \in U'$, let $W_x = f_1^{-1}s_2^{-1}(B_x) \cap f_2^{-1}s_2^{-1}(B_x)$ and put $W' = \bigcup\{W_x \mid x \in U'\}$. Clearly $s_2(U') \subset W'$. One easily checks, using (1), that $\operatorname{Fix}(s_2\varphi f_1), \operatorname{Fix}(s_2\varphi f_2) \subset W'$. Therefore, by (2) and (47.6.4)

$$\begin{cases} \operatorname{ind}(E_1, V_1, \eta_1) = \operatorname{ind}(E, W', s_2\varphi f_1),\\ \operatorname{ind}(E_2, V_2, \eta_2) = \operatorname{ind}(E, W', s_2\varphi f_2).\end{cases}$$

Consider the map $l\colon W' \times I \to E$ given by $l(z,t) = ts_2f_2(z)$ $t \in I$, then there is $x \in U'$ such that $z \in W_x$; hence $s_2f_2(z)$, $s_2f_1(z) \in B_x$ and $l(z,t) \in B_x$, as well. Now, define a morphism χ by the formula $\chi = s_2\varphi f_2 l$. Evidently χ determines a locally compact map. Moreover, it is a matter of a more or less straightforward calculation, based on the choice of ε (see (3) and the choice of δ_x), to show that the set $\{z \in W' \mid z \in \Psi(z,t) \text{ for some } t \in I\}$ is contained in $s_2(\operatorname{cl}\varphi(U') \cap \operatorname{cl} N_{2\alpha})$ and thus, is compact. Now since χ, as a homotopy, joins $s_2\varphi f_1|_{W'}$ with $s_2\varphi f_2$ by (47.6.5) we infer

$$\operatorname{ind}(E_1, V_1, \eta_1) = \operatorname{ind}(E_2, V_2, \eta_2).$$

The other properties are self-evident and follow from the respective properties in (47.6). Hence the proof is completed. □

Now, by using the same arguments as in Section 12 we obtain:

(47.9) Property (Normalization Property). *If* $(X, X, \varphi) \in \mathcal{B}$ *then* φ *is a Lefschetz morphism and* $\operatorname{ind}(X, X, \varphi) = \mathbf{\Lambda}(\varphi)$.

Note that (47.9) connects this section with Section 45.

48. Noncompact morphisms

A morphism $\varphi\colon X \to Y$ is called *locally compact* provided the determined multivalued map $\psi_\varphi\colon X \multimap Y$ is locally compact, i.e. for every $x \in X$ there exists an open neighbourhood U_x of x in X such that the restriction $\psi_\varphi|_{U_x}$ of ψ_φ to U_x is a compact map.

(48.1) Remark. In what follows we shall use the same notation φ for a morphism $\varphi\colon X \to Y$ and the associated map $\psi_\varphi\colon X \multimap Y$. So $\varphi\colon X \to Y$ is a morphism and $\varphi = \psi_\varphi\colon X \multimap Y$ is the multivalued map determined by φ.

(48.2) Lemma. *Assume that* $\varphi\colon U \to X$ *is a locally compact morphism with* $\operatorname{Fix}(\varphi)$ *compact, where* U *is a subset of* X. *Then there exists an open subset* V *of* U *such that* $\operatorname{Fix}(\varphi) \subset V$ *and* $\varphi|_V$ *is compact, where* $\operatorname{Fix}(\varphi) = \operatorname{Fix}(\psi_\varphi) = \kappa(\varphi)$.

Proof. Consider $\varphi\colon U \multimap X$. We choose an open neighbourhood $U_x \subset U$ of $x \in U$ such that $\varphi|_{U_x}\colon U_x \multimap X$ is compact. Then $\{U_x\}_{x \in \operatorname{Fix}(\varphi)}$ is a covering of $\operatorname{Fix}(\varphi)$. Let $\{U_{x_1}, \ldots, U_{x_k}\}$ be a finite sub-covering. We let $V = U_{x_1} \cup \ldots \cup U_{x_k}$ and the proof is completed. □

Let $\mathcal{B}_l$ be the family of all triples (X, U, φ) such that $X \in \mathrm{ANR}$, U is open in X, $\varphi\colon U \to X$ is a locally compact morphism such that $\operatorname{Fix}(\varphi)$ is compact. Of course, $\mathcal{B} \subset \mathcal{B}_l$.

Below we would like to show that the fixed point index $\operatorname{ind}: \mathcal{B} \to Z$ can be extended onto $\mathcal{B}_l$. Assume that $(X, U, \varphi) \in \mathcal{B}_l$. By using (47.2.1)**?**, we can choose $V \subset U$ such that $\operatorname{Fix}(\varphi) \subset U$ and $(X, V, \varphi|_V) \in \mathcal{B}$. We let:

$$\operatorname{ind}(X, U, \varphi) = \operatorname{ind}(X, V, \varphi|_V). \tag{48.3}$$

From Theorem (47.8) it follows that the above definition is correct. Moreover, we obtain that this index has all properties as formulated in (47.8). The most interesting problem is to show for which triples in $\mathcal{B}_l$ the normalization property (47.9) holds true.

The rest of this section is devoted to this problem.

(48.4) Remark. In what follows all considered morphisms are locally compact. Note, that if $X \subset E$ and $\dim E < +\infty$ then every morphism $\varphi: A \to X$, $A \subset X$ is locally compact. Moreover, for a subset $A \subset X$ by $\overline{A}$ we shall denote $\operatorname{cl}(A)$ in X.

(48.5) Definition. A morphism $\varphi: X \to X$ is called *eventually compact* provided there exists $n \in N$ such that:

$$\varphi^n = \underbrace{\varphi \circ \ldots \circ \varphi}_{n}$$

is compact.

(48.6) Definition. A morphism $\varphi: X \to X$ is called a *compact attraction* if there exists a compact $K \subset X$ such that, for each open neighbourhood V of K in X we have $X \subset \bigcup_{i=0}^{\infty} \varphi^{-1}(V)$ and if $\varphi^n(x) \subset V$, then for every $m \geq n$, $\varphi^m(x) \subset V$ for every $x \in X$.

(48.7) Definition. A morphism $\varphi: X \to X$ is called *asymptotically compact* if the set $C_\varphi = \bigcap_{n\geq 0} \varphi^n(X)$ is nonempty and relatively compact, i.e. $\overline{\bigcap_{n\geq 0} \varphi^n(X)}$ is compact (then V_φ is called the *center* or *core* of φ) and for every $x \in X$ the orbit $\bigcup_{n\geq 0} \varphi^n(x)$ is relatively compact.

(48.8) Definition. A morphism $\varphi: X \to X$ is called a *compact absorbing contraction* provided there exists an open $U \subset X$ such that:

(48.8.1) $\varphi(U) \subset U$,
(48.8.2) the restriction $\varphi|_U: U \to U$ of φ to U is compact,
(48.8.3) for every $x \in X$ there is $n = n_x$ such that $\varphi^{n_x}(x) \subset U$.

We would like to point out that every compact morphism is eventually compact and, also, a compact absorbing contraction.

Now, we shall explain connections between the above classes of morphisms. Our considerations will be strictly analogous to those presented in Section 42.

(48.9) PROPOSITION. *Any eventually compact morphism $\varphi\colon X \to X$ is a compact absorbing contraction morphism.*

PROOF. Let $\varphi\colon X \to X$ be an eventually compact morphism such that $K' = \overline{\varphi^n(X)}$ is compact. Define

$$K = \bigcup_{i=0}^{n-1} \varphi^i(K'),$$

we have

$$\varphi(K) \subset \bigcup_{i=1}^{n} \varphi^i(K') \subset K \cup \varphi^n(X) \subset K \cup K' \subset K.$$

Since φ is locally compact, there exists an open neighbourhood V_0 of K such that $L = \overline{\varphi(V_0)}$ is compact. There exists a sequence $\{V_1, \ldots, V_n\}$ of open subsets of X such that $L \cap \overline{\varphi(V_i)} \subset V_{i-1}$ and $K \cup \varphi^{n-i}(L) \subset V_i$ for all $i = 1, \ldots, n$. In fact, if $K \cup \varphi^{n-i}(L) \subset V$, and $0 \leq i < n$, since $K \cup \varphi^{n-i}(L)$ and $CV_i \cap L$ are disjoint compact sets of X, there exists an open subset W of X such that

$$K \cup \varphi^{n-i}(L) \subset W \subset \overline{W} \subset V_i \cup CL.$$

Define $V_{i+1} = \varphi^{-1}(W)$; since $\varphi(K) \cup \varphi(\varphi^{n-(i+1)}(L)) \subset K \cup \varphi^{n-i}(L) \subset W$, we have $K \cup \varphi^{n-(i+1)}(L) \subset V_{i+1}$ and $\varphi(V_{i+1}) \subset \overline{W} \subset V_i \cup CL$ implies $L \cap \overline{\varphi(V_{i+1})} \subset V_i$. Beginning with $K \cup \varphi^n(L) \subset K \subset V_0$, we define, by induction $V_1, \ldots, V_n$ with the desired properties.

Putting $U = V_0 \cap V_1 \cap \ldots \cap V_n$, we have $K' \subset K \subset U$ and

$$\varphi(U) \subset \varphi(V_0) \cap \varphi(V_1) \cap \ldots \cap \varphi(V_n) \subset L \cap \overline{\varphi(V_1)} \cap \ldots \cap \overline{\varphi(V_n)},$$

hence

$$\overline{\varphi(U)} \subset (L \cap \overline{\varphi(V_1)}) \cap \ldots \cap (L \cap \overline{\varphi(V_n)}) \cap L \subset V_0 \cap \ldots \cap V_{n-1} \cap V_n = U,$$

but $\overline{\varphi(U)}$ is compact since $\overline{\varphi(U)} \subset L$. Moreover,

$$X \subset \bigcup_{i=1}^{n} \varphi^{-i}(K') \subset \bigcup_{i=0}^{\infty} \varphi^{-i}(U)$$

and the proof is completed. □

(48.10) PROPOSITION. *Any compact attraction morphism is a compact absorbing contraction.*

PROOF. Let $\varphi\colon X \to X$ be a compact attraction morphism, K a compact attractor for φ and W, an open set of X such that $K \subset W$ and $L = \overline{\varphi(W)}$ is compact. We have $L \subset X \subset \bigcup_{i=0}^{\infty} \varphi^{-i}(W)$ hence, since L is compact, there exists $n \in N$ such that $L \subset \bigcup_{i=0}^{n} \varphi^{-i}(W)$. Define $V = \bigcup_{i=0}^{n} \varphi^{-i}(W)$; then

$$X \subset \bigcup_{i=0}^{\infty} \varphi^{-i}(W) \subset \bigcup_{i=0}^{\infty} \varphi^{-i}(V),$$
$$\varphi(V) \subset \bigcup_{i=0}^{n} \varphi^{-i+1}(W) \subset \varphi(W) \cup V \subset L \cup V \subset V$$

and

$$\varphi^{n+1}(V) \subset \bigcup_{i=0}^{n} \varphi^{n-i+1}(W) = \bigcup_{j=0}^{n} \varphi^{j+1}(W) \subset \bigcup_{j=0}^{n} \varphi^{j}(L),$$

which is compact and included in V, since $L \subset V$ and $\varphi(V) \subset V$ implies that $\varphi^i(L) \subset V$ for all $j \in N$. Consider the restriction $\varphi' \colon V \to V$ of φ; $\varphi' \colon V \to V$ is an eventually compact map, since V is an open set. By (47.9), there exists an open set U of V, hence of X such that $\overline{\varphi'(U)} = \overline{\varphi(U)}$ is a compact subset of U and $V \subset \bigcup_{n=0}^{\infty} \varphi'^{-n}(U) \subset \bigcup_{n=0}^{\infty} \varphi^{-n}(U)$; hence

$$X \subset \bigcup_{i=0}^{\infty} \varphi^{-i}(W) \subset \bigcup_{i=0}^{\infty} \varphi^{-i}(V) \subset \bigcup_{n=0}^{\infty} \varphi^{-n}(U),$$

and the proof is completed. □

(48.11) PROPOSITION. *Any asymptotically compact morphism is a compact absorbing contraction.*

PROOF. Observe that $C_\varphi = \overline{\bigcap_{n \geq 0} \varphi^n(X)}$ is a compact attractor. So, φ is a compact attraction morphism and in view of (48.10) our Proposition (48.11) is proved. □

It follows from the above that the class of compact absorbing contraction morphisms is a big one. Consequently, the following theorem is interesting:

(48.12) THEOREM. *If $X \in \mathrm{ANR}$ and $\varphi \colon X \to X$ is a compact absorbing contraction morphism then φ is a Lefschetz morphism and $\Lambda(\varphi) \neq 0$ implies that $\mathrm{Fix}(\varphi) \neq \emptyset$.*

PROOF. We choose U according to the definition of compact absorbing contraction morphisms. Let

$$\begin{aligned} \widetilde{\varphi} &\colon U \to U, & \widetilde{\varphi}(x) &= \varphi(x) \quad \text{for every } x \in U, \\ \overline{\varphi} &\colon (X, U) \to (X, U), & \overline{\varphi}(x) &= \varphi(x) \quad \text{for every } x \in X. \end{aligned}$$

Consider $\overline{\varphi}_* \colon H(X, U) \to H(X, U)$. Then $\overline{\varphi}_*$ is weakly nilpotent, because for every $x \in X$ there is $n = n_x$ such that $\varphi^{n_x}(x) \subset U$. Hence $\Lambda(\overline{\varphi}_*) = 0$.

Consequently, from (48.9) we infer that $\widetilde{\varphi}$ is a Lefschetz map. So, by applying (11.5) we deduce that φ is a Lefschetz morphism and $\Lambda(\overline{\varphi}) = \Lambda(\widetilde{\varphi}) + \Lambda(\varphi)$.

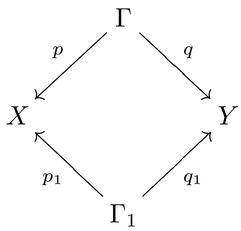
Assume that $\Lambda(\varphi) \neq 0$. Then $\Lambda(\widetilde{\varphi}) \neq 0$ and by using the existence property $\widetilde{\varphi}$ has a fixed point, so $\mathrm{Fix}(\varphi) \neq \emptyset$ and the proof is completed. $\square$

Many consequences of (48.12) can be obtained. We just mention the following:

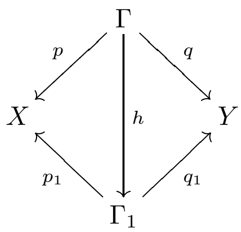
(48.13) COROLLARY. *If* $X \in \mathrm{AR}$ *and* $\varphi\colon X \to X$ *is a compact absorbing contraction morphism, then* $\mathrm{Fix}(\varphi) \neq \emptyset$.

49. n-Morphisms

The notion of a morphism is based on Vietoris maps. But instead of Vietoris maps we can dispose Vietoris n-maps. So similarly to n-acyclic maps we can consider also n-morphisms. Consider two diagrams



in which p and p_1 are n-Vietoris maps. We will say that (p, q) is equivalent to (p_1, q_1) provided there is a homeomorphism such that the diagram is commutative:



Clearly, the relation $(p, q) \sim (p_1, q_1)$ defined above is an equivalence relation. The equivalence class for a diagram:

$$X \xleftarrow{p} \Gamma \xrightarrow{q} Y$$

in which p is a n-Vietoris map is denoted by

$$\varphi = \{X \xleftarrow{p} \Gamma \xrightarrow{q} Y\}\colon X \to Y$$

and is called a *n-morphism* from X to Y. In what follows by $\mathcal{M}_n(X, Y)$ we shall denote the set of all n-morphisms from X to Y. Unfortunately a composition of two n-morphisms is not a n-morphism but all the results obtained before for n-acyclic maps can be generalized for n-morphisms. We left for the reader the respective formulations and proofs.

50. Multivalued maps with nonconnected values

Following B. O'Neill we would like to consider multivalued mappings with values consisting of 1 or n acyclic components (see [Ne1], [Ne2], [Dz1-M]). First we shall recall one example owed to J. Jezierski (see [Je]). A multivalued map $\varphi\colon X \multimap Y$ is called finitely-valued provided for every $x \in X$ the set $\varphi(x)$ is finite.

Let $C = \mathbb{R}^2$ denote the plane of complex numbers. Then $\varphi\colon C \to C$, $\varphi(z) = \sqrt[n]{z}$, is a finitely-valued map. Moreover, $\varphi(z)$ consists of one or n acyclic components. The following proposition shows us that multivalued mappings having one, two or three values are not useful in the fixed point theory.

(50.1) PROPOSITION. *One dimensional sphere $S^1 \subset C$ is* (1–2–3)-*contractible, i.e. there exists a homotopy $\chi\colon S^1 \times [0,1] \multimap S^1$ such that:*

(50.1.1) $\chi(x,0) = x$ *and* $\chi(x,1) = x_0$ *for every* $x \in S^1$,

(50.1.2) $\chi(x,t)$ *has one or two or three values.*

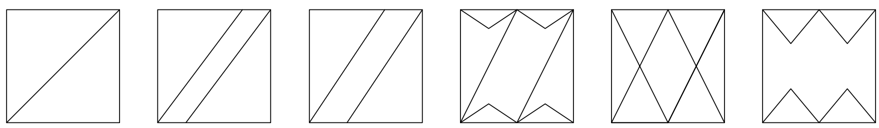
PROOF. We shall treat S^1 as a quotient space of R with respect to Z, i.e. $S^1 = R/Z = [0,1]/_{0\sim 1}$. We define a sequence of singlevalued homotopies:

$$A_1, A_2, B_1, B_2, B_3, C_1, C_2\colon S^1 \times [0,1] \to S^1$$

as follows:

$$A_1(x,t) = \max(0, (2x-t)\cdot(2-t)^{-1}),$$
$$A_2(x,t) = \max(1, 2x(2-t)^{-1}),$$
$$B_1(x,t) = \begin{cases} t(2^{-1} - |2x - 2^{-1}|) & \text{if } x \le 2^{-1}, \\ t(2^{-1} - |2x - 3\cdot 2^{-1}|) & \text{if } x \ge 2^{-1}, \end{cases}$$
$$B_2(x,t) = 1 - B_1(x,t),$$
$$B_3(x,t) = 2x,$$
$$C_1(x,t) = B_1(x, 1-t),$$
$$C_2(x,t) = B_2(x, 1-t).$$

We let $A = A_1 \cup A_2$, $B = B_1 \cup B_2 \cup B_3$, $C = C_1 \cup C_2$. Then $A \circ B \circ C$ is a desired (1–2–3) contraction of S^1 which we can illustrate as follows:



□

As an easy consequence of (50.1) we obtain:

(50.2) COROLLARY. *Two dimensional closed ball $K^2 \subset C$ fails* (1–2–3) *fixed point property, i.e. there is $\varphi\colon K^2 \multimap K^2$ such that φ is u.s.c. and $\varphi(x)$ consists of* 1, 2 *or* 3 *points.*

PROOF. Let $\chi\colon S^1 \times [0,1] \multimap S^1$ be (1-2-3) contraction of S^1 to a point $x_0 \in S^1$ (as defined in (50.1)). We define $\Phi\colon K^2 \multimap S^1$ by putting

$$\Phi(x) = \begin{cases} \chi(\|x\|^{-1} \cdot x, \|x\|) & \text{if } x \neq 0, \\ x_0 & \text{if } x = 0. \end{cases}$$

Then φ is (1–2–3) retraction of the ball K^2 onto S^1. Consequently, the map $\psi\colon K^2 \multimap K^2$ define by $\Psi(x) = -\Phi(x)$ is a desired fixed point free map and the proof is completed. □

(50.3) REMARK. Observe that all multivalued mappings considered in (50.1) and (50.2) are not only u.s.c. but also l.s.c. and consequently continuous.

According to (50.1) and (50.2) in what follows we shall consider multivalued mappings for which there is a fixed natural number n such that the values of these mappings consist of one or n acyclic components (like: $\Phi(z) = \sqrt[n]{z}$).

As initiated in 1957 by B. O'Neill (see [Ne1]) we shall see that multivalued mappings of this type are convenient in the fixed point theory. Recall that an u.s.c. map $\Phi\colon X \multimap Y$ is said to be acyclic provided the set $\Phi(x)$ is acyclic for every $x \in X$. We will denote the class of acyclic maps from X to Y by $\mathcal{A}_0(X,Y)$ instead of $\mathrm{AC}(X,Y)$ used earlier, i.e. $\mathrm{AC}(X,Y) = \mathcal{A}_0(X,Y)$.

Let m be a positive integer.

(50.4) DEFINITION. A multivalued map $\Phi\colon X \multimap Y$ is in the class $\mathcal{A}_m(X,Y)$ provided

(50.4.1) Φ is continuous,

(50.4.2) For each $x \in X$ the set $\Phi(x)$ consists of one or m acyclic components.

Note that $\mathcal{A}_1(X,Y) \subset \mathcal{A}_0(X,Y)$.

(50.5) EXAMPLES.

(50.5.1) Let C be the complex plane. Define a map $\Phi\colon C \multimap C$ by $\Phi(x) := \{z \mid z^m = x\}$. It is clear that $\Phi \in \mathcal{A}_m(C,C)$.

(50.5.2) Let $p\colon X \to B$ be a finite covering with B connected. Define the inverse map $\Psi\colon B \multimap X$, $\Psi(b) := p^{-1}(b)$. Since p is a local homeomorphism, Ψ is a continuous map and thus $\Psi \in \mathcal{A}_n(B,X)$, where n is the number of elements in $p^{-1}(b)$.

(50.5.3) Consider the map $\Phi\colon [-1,1] \multimap [-1,1]$ defined by

$$\Phi(x) := \begin{cases} \{1, x+1\} & \text{for } x \in [-1,0), \\ \{1,-1\} & \text{for } x = 0, \\ \{-1, x-1\} & \text{for } x \in (0,1]. \end{cases}$$

Then Φ is u.s.c. and 2-point-valued but not l.s.c. and hence $\Phi \notin \mathcal{A}_2([-1,1],[-1,1])$. Observe that Φ has no fixed points.

It is evident that the classes $\mathcal{A}_m$ are not closed under the composition law.

(50.6) DEFINITION. A *decomposition* $(\Phi_1,\dots,\Phi_n)$ of a multivalued map $\Phi\colon X \multimap Y$ is a sequence of maps

$$X = X_0 \xrightarrow{\Phi_1}\!\!\circ\, X_1 \xrightarrow{\Phi_2}\!\!\circ\, X_2 \xrightarrow{\Phi_3}\!\!\circ \cdots \xrightarrow{\Phi_{n-1}}\!\!\circ\, X_{n-1} \xrightarrow{\Phi_n}\!\!\circ\, X_n = Y,$$

such that $\Phi_i \in \mathcal{A}_{m_i}(X_{i-1}, X_i)$, $\Phi = \Phi_n \circ \ldots \circ \Phi_1$. We say the map Φ is *determined* by the decomposition $(\Phi_1,\dots,\Phi_n)$. The number n is said to be the *length* of the decomposition $(\Phi_1,\dots,\Phi_n)$. We will denote the class of decompositions by $\mathcal{DA}(X,Y)$.

(50.7) REMARK. The two decompositions

$$X_0 \xrightarrow{\Phi_1}\!\!\circ\, X_1 \xrightarrow{\Phi_2}\!\!\circ \cdots \xrightarrow{\Phi_n}\!\!\circ\, X_n, \quad X_0 \xrightarrow{\Phi_1}\!\!\circ\, \Phi_1(X_0) \xrightarrow{\Phi_2}\!\!\circ\, \Phi_2(\Phi_1(X_0)) \longrightarrow\!\!\circ \cdots \xrightarrow{\Phi_n}\!\!\circ\, X_n,$$

will be considered identical because they determine the same map Φ. But we will not identify a decomposition with the multivalued map it determines, as one map Φ may be determined by different decompositions.

(50.8) EXAMPLE. Let $f\colon X \to Y$ be a singlevalued continuous map. It admits the following decomposition:

$$\begin{aligned} &\Phi_1\colon X \multimap X \times \{0,1\}, && \Phi_1(x) := \{(x,0),(x,1)\}, \\ &\Phi_2\colon X \times \{0,1\} \to X, && \Phi_2(x,t) := f(x). \end{aligned}$$

Here $\Phi_1 \in \mathcal{A}_2(X, X \times \{0,1\})$ and $\Phi_2 \in \mathcal{A}_1(X \times \{0,1\}, X)$.

(50.9) REMARK. Elements of the class $\mathcal{A}_i(X,Y)$ can be identified with decompositions of length one. Therefore, $\mathcal{A}_i(X,Y) \subset \mathcal{DA}(X,Y)$.

The *restriction* of $(\Phi_1,\dots,\Phi_n) \in \mathcal{DA}(X,Y)$ to a subset $A \subset X$ and the *composition* of $(\Phi_1,\dots,\Phi_n) \in \mathcal{DA}(X,Y)$ and $(\Psi_1,\dots,\Psi_n) \in \mathcal{DA}(Y,Z)$ are defined in the obvious way. Now we introduce the notion of a homotopy in $\mathcal{DA}(X,Y)$.

(50.10) DEFINITION. Decompositions $(\Phi_1, \dots, \Phi_n)$, $(\Psi_1, \dots, \Psi_n) \in \mathcal{DA}(X, Y)$ are *homotopic* if there exists $(\Theta_1, \dots, \Theta_n) \in \mathcal{DA}(X \times [0,1], Y)$ such that

$$(\Theta_1, \dots, \Theta_n)|_{X \times \{0\}} = (\Phi_1, \dots, \Phi_n) \quad \text{and} \quad (\Theta_1, \dots, \Theta_n)|_{X \times \{1\}} = (\Psi_1, \dots, \Psi_n).$$

(50.11) PROPOSITION. *If Φ_i is homotopic to Ψ_i in $\mathcal{A}_{m_i}(X_{i-1}, X_i)$ for each $i = 1, \dots, n$ then the decompositions $(\Phi_1, \dots, \Phi_n)$ and $(\Psi_1, \dots, \Psi_n)$ are homotopic.*

PROOF. Let $H_i\colon X_{i-1} \times [0,1] \multimap X_i \in \mathcal{A}_{m_i}(X_{i-1} \times [0,1], X_i)$ be a homotopy joining Φ_i and Ψ_i. Define a map $\mathbb{X}_i\colon X_{i-1} \times [0,1] \multimap X_i \times [0,1]$ by the formula $\mathbb{X}_i(x,t) := (H_i(x,t), t)$. Then $(\mathbb{X}_1, \dots, \mathbb{X}_{n-1}, H_n)$ is a homotopy joining $(\Phi_1, \dots, \Phi_n)$ and $(\Psi_1, \dots, \Psi_n)$ (cf. (50.7)). $\square$

Let X, Y be two spaces.

(50.12) DEFINITION. An u.s.c. map $\Phi\colon X \multimap Y$ is *permissible* provided it admits a selector $\Psi\colon X \multimap Y$ which is determined by a decomposition $(\Psi_1, \dots, \Psi_n) \in \mathcal{DA}(X, Y)$. If Φ itself is determined by a decomposition $(\Phi_1, \dots, \Phi_n)$ then it is *strongly permissible* (s-permissible).

We will denote the class of permissible maps from X into Y by $\mathcal{P}(X, Y)$ and the class of s-permissible maps by s-$\mathcal{P}(X, Y)$.

(50.13) EXAMPLES.

(50.13.1) The map $\Phi\colon [-1,1] \multimap [-1,1]$ defined in (50.5.3) is an u.s.c. map which is not permissible.

(50.13.2) Consider the map $\Psi\colon [-1,1] \multimap [-1,1]$ defined by $\Psi(x) := \Phi(x) \cup \{x\}$, where Φ is the map from (50.5.3). Then Ψ is permissible because it admits the identity map as a selector. But it is not strongly permissible.

(50.13.3) Let $S^1 := \{z \in C \mid |z| = 1\}$. The map $\Phi\colon S^1 \multimap S^1$ defined by $\Phi(z) := \{w \mid w^2 = z\}$, is s-permissible (it is in the class $\mathcal{A}_2(S^1, S^1)$), but not admissible.

(50.14) PROPOSITION.

(50.14.1) *The composition of two permissible maps $\Phi\colon X \multimap Y$ and $\Psi\colon Y \multimap Z$ is permissible.*

(50.14.2) *The product of permissible maps is permissible.*

PROOF. The first part is evident. We prove (50.14.2). Observe that if $\Phi \in \mathcal{A}_m(X, Y)$ then $\Phi \times \mathrm{id} \in \mathcal{A}_m(X \times I, Y \times I)$. Let the decomposition

$$X_0 \xrightarrow{\Phi_1} X_1 \xrightarrow{\Phi_2} \cdots \xrightarrow{\Phi_n} X_n$$

determine a selector $\widetilde{\Phi}$ of $\Phi: X_0 \multimap X_n$ and the decomposition

$$Y_0 \xrightarrow{\Psi_1} Y_1 \xrightarrow{\Psi_2} \cdots \xrightarrow{\Psi_m} Y_m$$

determine a selector $\widetilde{\Psi}$ of $\Psi: Y_0 \multimap Y_m$. Then the decomposition

$$X_0 \times Y_0 \xrightarrow{\overline{\Phi}_1} X_1 \times Y_0 \xrightarrow{\overline{\Phi}_2} \cdots \xrightarrow{\overline{\Phi}_n} X_n \times Y_0 \xrightarrow{\overline{\Psi}_1} X_n \times Y_1 \xrightarrow{\overline{\Psi}_2} \cdots \xrightarrow{\overline{\Psi}_m} X_n \times Y_m,$$

where $\overline{\Phi}_i(x,y) := \Phi_i(x) \times \{y\}$, $\overline{\Psi}_j(x,y) := \{x\} \times \Psi_j(y)$, determines a selector of $\Phi \times \Psi$. □

(50.15) Definition. Two permissible maps $\Phi, \Psi: X \multimap Y$ are *homotopic* if there exists a permissible map $\Theta: X \times [0,1] \multimap Y$ such that $\Theta(x,0) \subset \Phi(x)$ and $\Theta(x,1) \subset \Psi(x)$ for $x \in X$.

(50.16) Proposition. *If two permissible maps $\Phi, \Psi: X \multimap Y$ are homotopic then they have selectors $\widetilde{\Psi}, \widetilde{\Phi}$ which are determined by homotopic decompositions.*

Proof. Let $\Theta: X \times [0,1] \multimap Y$ be a homotopy. Assume that there is a selector $\widetilde{\Theta} \subset \Theta$ determined by a decomposition $(\Theta_1, \ldots, \Theta_n)$. Then the restrictions $(\Theta_1, \ldots, \Theta_n)|_{X \times \{0\}}$ and $(\Theta_1, \ldots, \Theta_n)|_{X \times \{1\}}$ are homotopic and determine the desired selectors $\widetilde{\Psi}$ and $\widetilde{\Phi}$ and the proof is completed. □

Now we shall define an approximation system for multivalued maps and then the *index* for such systems. We will keep notations and notions presented at the end of Section 5. Let (K,τ) and (L,μ) be two finite simplicial complexes with triangulations τ and μ, respectively. Let $\Phi: (K,\tau) \multimap (L,\mu)$ be an u.s.c. map.

(50.17) Definition. Let k and l be two natural numbers. A chain mapping $\varphi: C_*(K,\tau^l) \to C_*(L,\mu^k)$ is called an (n,k)-*approximation* of Φ provided the following condition holds: for each simplex $\sigma \in \tau^l$ there exists a point $y(\sigma) \in K$ such that

$$\sigma \subset \mathrm{St}^n(y(\sigma),\tau^l), \quad \mathrm{carr}\,\varphi\sigma \subset \mathrm{St}^n(\Phi(y(\sigma)),\mu^k).$$

(50.18) Definition. A graded set $A(\Phi) = \{A(\Phi)_j\}_{j\in N}$, where $A(\Phi)_j \subset \hom(C_*(K,\tau'),C_*(L,\mu'))$, is called an *approximation system* (A-system) for Φ provided there is an integer $n = n(A)$ such that:

(50.18.1) If $\varphi \in A(\Phi)_j$ then $\varphi = \varphi_1 \circ b$, where φ_1 is an (n,j)-approximation of Φ,

(50.18.2) For every $j \in N$ there exists $j_1 \in N$ such that for $m \geq j_1$ and for all $\varphi = \varphi_1 \circ b \in A(\Phi)_l$ and $\psi = \psi_1 \circ b \in A(\Phi)_m$ the diagram

$$\begin{array}{ccc} C_*(K,\tau^{l_1}) & \xrightarrow{\varphi_1} & C_*(L,\mu^l) \\ \chi\uparrow & & \uparrow\chi \\ C_*(K,\tau^{m_1}) & \xrightarrow[\psi_1]{} & C_*(L,\mu^m) \end{array}$$

with $m_1 \geq l_1$ is homotopy commutative with a chain homotopy D satisfying the following *smallness* condition:

(50.18.3) For any simplex $\sigma \in \tau^{m1}$ there exists a point $z(\sigma) \in K$ such that

$$\sigma \subset \mathrm{St}^n(z(\sigma), \tau^j), \quad \operatorname{carr} D\sigma \subset \mathrm{St}^n(\Phi(z(\sigma)), \mu^j).$$

(50.19) DEFINITION. Let $\Phi_1, \Phi_2 \colon K \multimap L$ be u.s.c. maps and let $H \colon K \times [0,1] \multimap L$ be an u.s.c. homotopy joining Φ_1 and Φ_2. Let $A(\Phi_1)$ and $A(\Phi_1)$ be A-systems for Φ_1 and Φ_2, respectively. They are *H-homotopic* provided there is an integer $m \in N$ such that the following condition holds:

(50.19.1) For every $j \in N$ there is $j_1 \in N$ such that for any $l \geq j_1$ there are $\varphi = \varphi_1 \circ b \in A(\Phi_1)_l$ and $\psi = \psi_2 \circ b \in A(\Phi_2)_l$ such that $\psi_1, \varphi_1 \colon C_*(K, \tau^{l_1}) \to C_*(L, \mu^l)$ are chain homotopic with an *H-small homotopy* D, i.e.

(50.19.2) For $\sigma \in \tau^{l_1}$ there is a point $d(\sigma) \in K$ such that

$$\sigma \subset \mathrm{St}^m(d(\sigma), \tau^j), \quad \operatorname{carr} D\sigma \subset \mathrm{St}^m(H(d(\sigma) \times I), \mu^j).$$

(50.20) PROPOSITION. *Let $\Phi_1 \colon K \multimap L$, $\Phi_2 \colon L \multimap M$ be u.s.c. maps and let $A(\Phi_1)$, $A(\Phi_2)$ be A-systems for Φ_1, Φ_2, respectively. Then the graded set $A = \{A_j\}$, where*

$$A_j = A(\Phi_2)_j \circ A(\Phi_1)_j := \{\varphi = \varphi_2 \circ \varphi_1 \mid \varphi_2 \in A(\Phi_2)_j,\ \varphi_1 \in A(\Phi_1)_j\}$$

is an A-system for the composition $\Phi_2 \circ \Phi_1$.

A simple example of an A-system is the family of all chain mappings induced by simplical approximations of a given singlevalued continuous map (see [SeS]). Let $U \subset K$ be open and polyhedral and let $\Phi \colon \overline{U} \multimap K$ be an u.s.c. map such that $x \notin \Phi(x)$ for $x \in \partial U$. Let $A(\Phi)$ be an A-system for Φ. Then the *index* $I_A(K, \Phi, U) \in F$ is defined as follows:

Let denote by $p_U \colon C_*(K, \tau^k) \to C_*(\overline{U}, \tau^k)$ the natural linear projection. Let $\varphi \in A(\Phi)_k$. Then the "*local Lefschetz number*"

$$\lambda(p_U \circ \varphi) := \sum_{i=0}^{\dim K} (-1)^i \operatorname{tr}(p_U \circ \varphi)_i$$

is defined, here we consider homology with coefficients in an arbitrary ring F, for example $F = Q$ or $F = Z$. It has been proved in [SeS] that for sufficiently large k_0 the above element of F is independent of the choice of $\varphi \in A(\Phi)_k$ ($k \geq k_0$), because all the approximations are small homotopic (see (50.5.2)).

(50.21) DEFINITION. $I_A(K,\Phi,U) := \lambda(p_U \circ \varphi)$ for $\varphi \in A(\Phi)_k$, $k \geq k_0$.

(50.22) PROPOSITION (Additivity). *Let U_1, U_2 be open, disjoint and polyhedral subsets of U and $\Phi\colon \overline{U} \multimap K$ an u.s.c. mapping such that* $\operatorname{Fix}\Phi \subset U_1 \cup U_2$. *If $A(\Phi)$ is an A-system for Φ then*

$$I_A(K,\Phi,U) = I_A(K,\Phi,U_1) + I_A(K,\Phi,U_2).$$

(50.23) COROLLARY (Excision). *Let $U_1 \subset U$ be an open and polyhedral subset of K and* $\operatorname{Fix}\Phi \subset U_1$. *Then*

$$I_A(K,\Phi,U) = I_A(K,\Phi,U_1).$$

(50.24) PROPOSITION (Homotopy Invariance). *Let $H\colon \overline{U} \times [0,1] \multimap K$ be an u.s.c. homotopy such that $x \notin H(x,t)$ for $x \in \partial U$ and $t \in [0,1]$. Let A_0, A_1 be H-homotopic A-systems for $H_0 = H(\,\cdot\,,0)$, $H_1 = H(\,\cdot\,,1)$, respectively. Then*

$$I_{A_0}(K,H_0,U) = I_{A_1}(K,H_1,U).$$

(50.25) REMARK. Because of (50.22) one can define $I_A(K,\Phi,V)$, where V is open and not polyhedral, if $\Phi\colon L \multimap K$, $V \subset L \subset K$, and L is a subpolyhedron of K. Then one puts $I_A(K,\Phi,V) := I_A(K,\Phi,U)$ where $U \subset V$ is polyhedral and $\operatorname{Fix}\Phi|_{\overline{V}} \subset U$.

(50.26) PROPOSITION (Commutativity). *Let $W \subset K$ be open and let $\Phi_1\colon K \multimap L$, $\Phi_2\colon L \multimap K$ be u.s.c. maps. Assume that $x \notin \Phi_2 \circ \Phi_1(x)$ for $x \in \partial W$ and $y \notin \Phi_1 \circ \Phi_2(y)$ for $y \in \partial(\Phi_2^{-1}(W))$. Assume further that if $y \in \operatorname{Fix}\Phi_1 \circ \Phi_2 - \overline{\Phi_2^{-1}(W)}$ then $\Phi_2(y) \cap \operatorname{Fix}\Phi_2 \circ \Phi_1|_{\overline{W}} = \emptyset$. Then for any A-systems $A_1 = A(\Phi_1)$, $A_2 = A(\Phi_1)$*

$$I_{A_1 \circ A_2}(L,\Phi_1 \circ \Phi_2,\Phi_2^{-1}(W)) = I_{A_2 \circ A_1}(K,\Phi_2 \circ \Phi_1,W).$$

(50.27) PROPOSITION (Mod-p Property). *Let $F = Z_p$, p prime. Let $W \subset K$ be open and $\Phi\colon K \multimap K$ an u.s.c. map such that $x \notin \Phi^p(x)$ for $x \in \partial W$. Assume that if $y \in \operatorname{Fix}\Phi^p - W$ then $\Phi^k(y) \cap \operatorname{Fix}\Phi^p|_{\overline{W}} = \emptyset$ for $k < p$. Then*

$$I_A(K,\Phi,W) = I_{A^p}(K,\Phi^p,W).$$

The detailed proofs of the above properties are given in [SeS]. Now we shall use nerves of coverings for constructing chain approximations of decompositions of multivalued maps. We will use here the notation introduced in Chapter I, Section 5. The symbol i^α_β will stand for the simplicial map $N(\alpha) \to N(\beta)$ defined there. It was also defined for the induced map of chain complexes.

First we will prove the following technical lemma:

(50.28) LEMMA. *Let* $\Phi \in \mathcal{A}_m(X,Y)$, $\alpha_0 \in \operatorname{Cov} X$, $\beta_0 \in \operatorname{Cov} Y$, $n \in N$. *There exist sequences of coverings* $\alpha_i \in \operatorname{Cov} X$, $\beta_i \in \operatorname{Cov} Y$,

$$\alpha_{n+1} \geq \alpha_n \geq \ldots \geq \alpha_0, \quad \beta_{n+1} \geq \beta_n \geq \ldots \geq \beta_0,$$

such that for each simplex $s \in N(\alpha_i)$ *there are a point* $a\ (s) \in X$ *and a covering* $\beta_{i-1}(s) \in \operatorname{Cov} Y\, (\beta_i \geq \beta_{i-1}(s) \geq \beta_{i-1})$ *with the following properties:*

(50.28.1) $\operatorname{supp} s \subset \operatorname{St}(a(s), \alpha_{i-1})$;

(50.28.2) $\Phi(\operatorname{St}(\operatorname{supp} s, \alpha_i)) \subset \operatorname{St}(\Phi(s(s)), \beta_{i-1}(s))$;

(50.28.3) *If* $C_j(a(s))$ *are the components of* $\Phi(a(s))$ *then sets* $\operatorname{St}^2(C_j(a(s)), \beta_{i-1}(s))$ *are pairwise disjoint;*

(50.28.4) *For* $y \in \operatorname{St}(\operatorname{supp} s, \alpha_i)$ *and* $j = 1, \ldots, m$

$$\Phi(y) \cap \operatorname{St}(C_j(a(s)), \beta_{i-1}(s)) \neq \emptyset,$$

(50.28.5) $$(i_{\beta_{i-1}}^{\beta_{i-1}(s)})_*\colon \widetilde{H}_*(N(\beta_{i-1}(s))|_{\operatorname{St}^2(C_j(a(s)),\beta_{i-1}(s))}) \to \widetilde{H}_*(N(\beta_{i-1})|_{\operatorname{St}(\Phi(a(s)),\beta_{i-1})}) \quad \text{is a zero homomorphism.}$$

PROOF. Let $n = 0$ and $x \in X$. Since every component C_j of $\Phi(x)$ is acyclic, by (5.12) there is $\beta = \beta_0(x) \in \operatorname{Cov} Y$ such that the sets $\operatorname{St}^2(C_j, \beta)$ are pairwise disjoint and the homomorphisms

$$(i_{\beta_0}^{\beta})_*\colon \widetilde{H}_*(N(\beta)|_{\operatorname{St}^2(C_j,\beta)}) \to \widetilde{H}_*(N(\beta_0)|_{\operatorname{St}(\Phi(x),\beta_0)})$$

are trivial. By the continuity of Φ there is a neighbourhood $\mathcal{O}_x$ of x such that:

(i) $\Phi(\mathcal{O}_x) \subset \operatorname{St}(\Phi(x), \beta)$,

(ii) for each $y \in \mathcal{O}_x$, $\Phi(y) \cap \operatorname{St}(C_j, \beta) \neq \emptyset$,

(iii) the covering $\{\mathcal{O}_x\}_{x\in X}$ is a refinement of α_0.

We choose a finite subcovering $\{\mathcal{O}_{x_i}\}_{i=1}^k$. Let α_1 be a star-refinement of $\{\mathcal{O}_{x_i}\}$. For a simplex $s \in N(\alpha_1)$ we define $\alpha(s) := x_i$ if $\operatorname{supp} s \subset \mathcal{O}_{x_i}$ and $\beta_0(s) := \beta_0(x_i)$. Let β_1 be a common refinement of all $\beta_0(x_i)$. The above procedure can be continued inductively. □

Such sequences (α_i, β_i) of coverings as in Lemma (50.28) will be called *squeezing sequences* for $(\Phi, \alpha_0, \beta_0)$.

Recall that the *Kronecker index* of a 0-chain $c = \sum c_i\sigma_i$ is the sum $\sum c_i$ (see (SE-M]). Let us denote by $N^{(n)}(\alpha)$ the n-skeleton of the simplicial complex $N(\alpha)$.

(50.29) DEFINITION. Let $\alpha, \overline{\alpha} \in \mathrm{Cov}_f(X)$, $\beta, \overline{\beta} \in \mathrm{Cov}_f(Y)$ and $\Phi \in \mathcal{A}_m(X,Y)$. A chain map $\varphi\colon C_*(N^{(n)}(\overline{\alpha})) \to C_*(N^{(n)}(\overline{\beta}))$ is called an (α,β)-*approximation* of Φ provided

(50.29.1) φ multiplies the Kronecker index by m.
(50.29.2) For each simplex $s \in N^{(n)}(\overline{\alpha})$ there is a point $p(s) \in X$ such that

$$\operatorname{supp} s \subset \mathrm{St}(p(s),\alpha), \quad \operatorname{supp} \varphi s \subset \mathrm{St}(\Phi(p(s)),\beta).$$

(50.29.3) If $\dim s = 0$ then for every component $C_j = C_j(p(s))$ of $\Phi(p(s))$,

$$\operatorname{supp} \varphi s \subset \mathrm{St}(C_j,\beta) \neq \emptyset.$$

(50.30) THEOREM. *Let $\Phi \in \mathcal{A}_m(X,Y)$, $\alpha \in \mathrm{Cov}\, X$, $\beta \in \mathrm{Cov}\, Y$. For every $n \in N$ there exist a refinement $\overline{\alpha}$ of α and an (α,β)-approximation $\varphi\colon C_*(N^{(n)}(\overline{\alpha})) \to C_*(N^{(n)}(\beta))$ of Φ.*

PROOF. From (50.28) we obtain a squeezing sequence (α_i,β_i) with $\alpha_0 = \alpha$ and $\beta_0 = \beta$. Define $\overline{\alpha} := \alpha_{n+1}$. We will construct the desired chain map φ inductively.

$k = 0$. Let s_0 be a vertex of $N(\overline{\alpha})$. By (50.28), we obtain a point $a(s_0) \in X$. If the set $\Phi(a(s_0))$ is connected then we define $\varphi_0 s_0 := m\overline{a}$, where $\overline{a}$ is an arbitrary vertex of $N(\beta_{n+1})$ with $\operatorname{supp}\overline{a} \subset \mathrm{St}(\Phi(a(s_0)),\beta_n(s_0))$. If $\Phi(a(s_0))$ consists of m components then $\varphi_0 s_0 := a_1 + \ldots + a_m$, where a_i are vertices of $N(\beta_{n+1})$ such that $\operatorname{supp} a_i \subset \mathrm{St}(C_i(a(s_0)),\beta_n(s_0))$. So, we have defined $\varphi_0\colon C_0(N(\alpha_{n+1})) \to C_0(N(\beta_{n+1}))$. We would like to extend it to 1-chains. Let s be a 1-simplex in $N(\overline{\alpha})$. Then $\partial s = s_1 - s_0$. Since $a\ (s_0)$ and $a\ (s_1)$ belong to $\mathrm{St}(\operatorname{supp} s,\alpha_n)$,

$$\Phi(a(s_0)) \cup \Phi(a(s_1)) \subset \mathrm{St}(\Phi(a(s)),\beta_{n-1}(s)).$$

Let $\varphi_0\partial s = \sum a_i - \sum b_i$ with $a_i, b_i \in C_0(N(\beta_{n+1}))$. If $\Phi(a(s))$ is connected then by (50.28.5)

$$i^{\beta_{n+1}}_{\beta_{n-1}}\Big(\sum(a_i - b_i)\Big) = \sum \partial c_i, \quad \text{where } c_i \in C_1(N(\beta_{n-1})).$$

If $\Phi(a(s)) = \bigcup_{i=1}^n C_i(a(s))$ then for each pair a_i, b_i

$$\operatorname{supp}(a_i - b_i) \subset \mathrm{St}(C_i(a(s)),(\beta_{n-1}(s)).$$

Thus by (50.28.5), we obtain

$$i^{\beta_{n+1}}_{\beta_{n-1}}(a_i - b_i) = \partial c_i$$

for some $c_i \in C_1(N(\beta_{n-1}))$ such that $\operatorname{supp} c_i \subset \operatorname{St}(C_i(a(s)), \beta_{n-1})$. Let us define $\varphi_1 s := \sum c_i$. Hence we have obtained the following commutative diagram:

$$\begin{array}{ccccc}
C_0(N(\alpha_{n+1})) & \xrightarrow{\varphi_0} & C_0(N(\beta_{n+1})) & \xrightarrow{i^{\beta_{n+1}}_{\beta_{n-1}}} & C_0(N(\beta_{n-1})) \\
\uparrow{\scriptstyle\partial} & & & & \uparrow{\scriptstyle\partial} \\
C_1(N(\alpha_{n+1})) & & \xrightarrow[\varphi_1]{\qquad\qquad} & & C_1(N(\beta_{n-1})
\end{array}$$

Therefore, a chain map $\varphi_1 \colon C_*(N^{(1)}(\alpha_{n+1})) \to C_*(N^{(1)}(\beta_{n-1}))$ is defined (on 0-chains $\varphi_1 c := i\varphi_0 c$). Now, assume that

$$\varphi_{k-1} \colon C_*(N^{(k-1)}(\alpha_{n+1})) \to C_*(N^{(k-1)}(\beta_{n-k+1}))$$

is defined and satisfies conditions (50.29) with $\alpha = \alpha_{n+1}$ and $\beta = \beta_{n-k})$). As in the step $0 \Rightarrow 1$, we will define $\varphi_k \colon C_k(N(\alpha_{n+1})) \to C_k(N(\beta_{n-k}))$. Let $s \in N(\alpha_{n+1})$ be a k-simplex and $\overline{s} := i^{\alpha_{n+k}}_{\alpha_{n-k+1}}(s)$. From (50.28) we obtain a point $a\ (\overline{s})$ and a covering $\beta_{n-k}(\overline{s})$. Let $\partial s = \sum s_i$. $\dim s_i = k-1$. By our inductive assumption we have $p(s_i)$ such that

$$\operatorname{supp} s_i \subset \operatorname{St}(p(s_i), \alpha_{n-k+1}), \quad \operatorname{supp} \varphi_{k-1} s_i \subset \operatorname{St}(\Phi(p(s_i)), \beta_{n-k+1}).$$

But we also have the following inclusion:

$$\Phi(p(s_i)) \subset \operatorname{St}(\Phi(a(s)), \beta_{n-k}(\overline{s})).$$

Therefore,

$$\operatorname{supp} \varphi_{k-1}\partial s \subset \operatorname{St}^2(\Phi(a(s)), \beta_{n-k}(\overline{s})).$$

Because of our assumption, $\varphi_{k-1}\partial s \in C_{k-1}(N(\beta_{n-k+1}))$ is a cycle. Hence by (50.27.5) there is a chain $c \in C_k(N(\beta_{n-k})|_{\operatorname{St}(\Phi(a(\overline{s})),\beta_{n-k})})$ such that

$$i^{\beta_{n-k+1}}_{\beta_{n-k}} \varphi_{k-1}\partial s = \partial c.$$

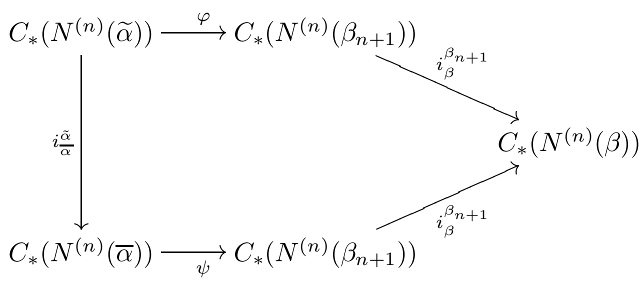
Let us define $\varphi_k s := c$ and $p(s) := a\ (\overline{s})$. For chains of lower dimension we put $\varphi_k := i \circ \varphi_{k-1}$. Then φ_n is the desired approximation φ. □

(50.31) Definition. A chain homotopy $D \colon C_*(N(\overline{\alpha})) \to C_*(N(\overline{\beta}))$ is (Φ, α, β)-*small* provided for each simplex $s \in N(\overline{\alpha})$ there exists a point $c(s) \in X$ such that

$$\operatorname{supp} s \subset \operatorname{St}(c(s), \alpha), \quad \operatorname{supp} Ds \subset \operatorname{St}(\Phi(c(d)), \beta).$$

Let $\Phi \in \mathcal{A}_m(X, Y)$. We will prove the following:

(50.32) Proposition. *For every two n-close (α,β)-approximations $i^{\beta_{n+1}}_{\beta}\circ\varphi$, and $i^{\beta_{n+1}}_{\beta}\circ\psi$ of Φ with the same squeezing sequences $(\alpha_i,\beta_i)_{i=0}^{n+1}$ the diagram*



is homotopy commutative with a (Φ,α,β)-small homotopy D.

Proof. Let s_0 be a vertex in $N(\widetilde{\alpha})$ and let $\overline{s}_0 := i^{\widetilde{\alpha}}_{\overline{\alpha}}(s_0)$, $\widetilde{s}_0 = \pi^{\widetilde{\alpha}}_{\alpha_{n+1}}(s_0)$. According to (50.23) we choose points $p(s_0)$, $p(\overline{s}_0)$, respectively for s_0 and $\overline{s}_0$. From (50.28) we obtain a point $a(\widetilde{s}_0)$. Since $\operatorname{supp} s_0 \subset \operatorname{supp}\overline{s}_0 \subset \operatorname{supp}\widetilde{s}_0$, it follows that $p(s_0)$, $p(\overline{s}_0)\in \operatorname{St}(\operatorname{supp}\widetilde{s}_0,\alpha_{n+1})$. Therefore,

$$\Phi(p(s_0)\cup p(\overline{s}_0))\subset \operatorname{St}(\Phi(a(\widetilde{s}_0)),\beta_n(\widetilde{s}_0)).$$

Since φ and ψ are approximations of Φ,

$$\operatorname{supp}\varphi s_0\subset \operatorname{St}(\Phi(p(s_0)),\beta_{n+1}),\quad \operatorname{supp}\psi s_0\subset \operatorname{St}(\Phi(p(\overline{s}_0)),\beta_{n+1}).$$

Therefore,

$$\operatorname{supp}\varphi s_0\cup\operatorname{supp}\psi s_0\subset \operatorname{St}^2(\Phi(a(\widetilde{s}_0)),\beta_n(\widetilde{s}_0)).$$

Moreover, conditions (50.28.3) and (50.28.4) ensure that if $\varphi s_0=\sum_{i=1}^m a_i$ and $\psi\overline{s}_0=\sum_{i=1}^m b_i$ then there are m cycles of the form a_i-b_i with

$$\operatorname{supp}(a_i-b_i)\subset \operatorname{St}^2(C_i(a(\widetilde{s}_0)),\beta_n(\widetilde{s}_0)).$$

Because of (50.28.5) there is a chain $c\in C_1(N(\beta_n))$ such that

$$\operatorname{supp} c\subset \operatorname{St}(\Phi(a(\widetilde{s}_0)),\beta_n),\quad \partial c=\sum_{i=1}^m(a_i-b_i)=\varphi s_0-\psi\overline{s}_0.$$

We define $D_0 s_0 := c$ and $c(s_0) := a\ (\widetilde{s}_0)$. The construction of the chain homotopy follows inductively: Assume that $D_i\colon C_i(N^{(n)}(\widetilde{\alpha}))\to C_{i+1}(N^{(n)}(\beta_{n-k+1}))$ are defined for $i<k$ and satisfy the conditions:

(50.32.1) $i^{\beta_{n+1}}_{\beta_{n-k+1}}\varphi c - i^{\beta_{n+1}}_{\beta_{n-k+1}}\psi i^{\widetilde{\alpha}}_{\overline{\alpha}}\, c = \partial D_i c + D_{i-1}\partial c,$

(50.32.1) For each simplex $s \in N(\widetilde{\alpha})$, $\dim s = i$, there is a point $c(s) \in X$ such that

$$\operatorname{supp} s \subset \operatorname{St}(c(s), \alpha_{n-i}), \quad \operatorname{supp} D_i s \subset \operatorname{St}(\Phi(c(s)), \beta_{n-i}).$$

Let $s \in N(\widetilde{\alpha})$ be a k-simplex. We consider the simplex $\overline{s} := i_{\widetilde{\alpha}}^{\overline{\alpha}} s$ and $\widetilde{s} := i_{\alpha_{n-k+1}}^{\widetilde{\alpha}} s$. From (50.28) we obtain points $p(s)$ and $p(\overline{s})$. From (50.27) we have $a\ (\widetilde{s})$. They satisfy

$$p(s), p(\overline{s}) \in \operatorname{St}(\operatorname{supp} \widetilde{s}, \alpha_{n-k+1}), \quad \Phi(p(s) \cup p(\overline{s})) \subset \operatorname{St}(\Phi(a(\widetilde{s})), \beta_{n-k}(\widetilde{s})).$$

Therefore,

$$\operatorname{supp} i_{\beta_{n-k+1}}^{\beta_{n+1}} \varphi s \subset \operatorname{St}^2(\Phi(a(\widetilde{s})), \beta_{n-k}(\widetilde{s})), \quad \operatorname{supp} i_{\beta_{n-k+1}}^{\beta_{n+1}} \psi \overline{s} \subset \operatorname{St}^2(\Phi(a(\widetilde{s})), \beta_{n-k}(\widetilde{s})).$$

Let $\partial s = \sum s_j$ and $\dim s_j = k-1$. By the inductive assumption we obtain that $\operatorname{supp} s_j \subset \operatorname{St}(c(s_j), \alpha_{n-k+1})$. Hence $c(s_j) \in \operatorname{St}(\operatorname{supp} \widetilde{s}, \alpha_{n-k+1})$ and thus $\Phi(c(s_j)) \subset \operatorname{St}(\Phi(a(\widetilde{s})), \beta_{n-k}(\widetilde{s}))$. Therefore,

$$\operatorname{supp} D_{k-1} s_j \subset \operatorname{St}(\Phi(c(s_j)), \beta_{n-k+1}) \subset \operatorname{St}^2(\Phi(a(\widetilde{s})), \beta_{n-k}(\widetilde{s})).$$

Let us consider the chain $c := i_{\beta_{n-k+1}}^{\beta_{n+1}} \varphi s - i_{\beta_{n-k+1}}^{\beta_{n+1}} \psi \overline{s} - D_{k-1} \partial s$. We deduce from the above that $\operatorname{supp} c \subset \operatorname{St}^2(\Phi(a(\widetilde{s})), \beta_{n-k}(\widetilde{s}))$.

On the other hand we have

$$\begin{aligned} \partial c &= \partial i \varphi s - \partial i \psi s - \partial D_{k-1} \partial s = i \partial \varphi s - i \partial \psi i s - \partial D_{k-1} \partial s \\ &= i \varphi \partial s - i \psi i \partial s - \partial D_{k-1} \partial s = \partial D_{k-1} \partial s + D_{k-2} \partial \partial s - \partial D_{k-1} \partial s = 0. \end{aligned}$$

Therefore, by (50.30.5), there exists a chain $\overline{c} \in C_{k+1}(N(\beta_{n-k}))$ such that $\partial \overline{c} = c$ and $\operatorname{supp} \overline{c} \subset \operatorname{St}(\Phi(a(\widetilde{s})), \beta_{n-k})$. We define $D_k s := \overline{c}$ and $c(s) := a(\widetilde{s})$. For $i < k$ we compose D_i with $i_{\beta_{n-k}}^{\beta_{n-k+1}}$. Since i can only enlarge supports, the D_i satisfy (50.32.1), (50.32.2). In the n-th step one obtains a desired (Φ, α, β)-small homotopy D. □

(50.33) PROPOSITION. *For arbitrary two n-close (α, β)-approximations*

$$\varphi \colon C_*(N^{(n)}(\overline{\alpha}) \to C_*(N^{(n)}(\beta)) \quad \text{and} \quad \psi \colon C_*(N^{(n)}(\widetilde{\alpha})) \to C_*(N^{(n)}(\beta))$$

of Φ there exists $\alpha' \in \operatorname{Cov}(X)$ such that $\varphi \circ i_{\overline{\alpha}}^{\alpha'}$ and $\psi \circ i_{\widetilde{\alpha}}^{\alpha'}$ are (Φ, α, β)-small homotopic.

PROOF. Let $\varphi = i_{\beta}^{\beta_{n+1}} \circ \varphi_1$ and $\psi = i_{\beta}^{\overline{\beta}_{n+1}} \circ \psi_1$. There exist common refinements: $\overline{\overline{\alpha}}$ of $\overline{\alpha}$, $\widetilde{\alpha}$, α_{n+1}, $\overline{\alpha}_{n+1}$, and $\overline{\overline{\beta}}$ of β_{n+1}, $\overline{\beta}_{n+1}$. We construct an $(\overline{\overline{\alpha}}, \overline{\overline{\beta}})$-approximation

$\varrho\colon C_*(N^{(n)}(\alpha')) \to C_*(N^{(n)}(\overline{\overline{\beta}}))$ of Φ using (50.29). From (50.32) we deduce that $\varphi\circ i^{\alpha'}_{\overline{\alpha}}$ is small homotopic to $i^{\overline{\overline{\beta}}}_{\beta}\circ\varrho$ with a homotopy D_1, $\psi\circ i^{\alpha'}_{\underset{\sim}{\alpha}}$ is small homotopic to $i^{\overline{\overline{\beta}}}_{\beta}\circ\varrho$ with a homotopy D_2. Therefore, D_1+D_2 is a small homotopy joining $\varphi\circ i^{\alpha'}_{\overline{\alpha}}$ and $\psi\circ i^{\alpha'}_{\underset{\sim}{\alpha}}$. □

(50.34) REMARK. Note that the only place where the continuity of the map Φ is important is the proof (50.27.3) and (50.27.4). If Φ is an acyclic map then we can assume it to be u.s.c. only.

(50.35) LEMMA. *Let $\Phi_1\colon X_1 \multimap X_2$, $\Phi_2\colon X_2\multimap X_3$ be u.s.c. maps, X_1,X_2,X_3 compact spaces and $\alpha\in\operatorname{Cov}(X_1)$, $\gamma\in\operatorname{Cov}(X_3)$. There exists $\beta\in\operatorname{Cov}(X_2)$ such that for each $y\in X_1$ there is a point $u(y)\in X_1$ such that*

$$y\in\operatorname{St}(u(y),\alpha),\quad \Phi_2(\operatorname{St}^2(\Phi_1(y),\beta))\subset\operatorname{St}(\Phi_2\circ\Phi_1(u(y)),\gamma).$$

PROOF. The set $\operatorname{St}(\Phi_2\circ\Phi_1(y),\gamma)$ is a neighbourhood of the compact set $\Phi_2\circ\Phi_1(y)$ in X_3. Thus for every $x\in\Phi_1(y)$ there exists a covering $\overline{\beta}(x)\in\operatorname{Cov}(X_2)$ such that $\Phi_2(\operatorname{St}(x,\overline{\beta}(x)))\subset\operatorname{St}(\Phi_2\circ\Phi_1(y),\gamma)$. So, the set $U:=\bigcup\{\operatorname{St}(x,\overline{\beta}(x))\colon x\in\Phi_1(y)\}$ is a neighbourhood of $\Phi_1(y)$ such that $\Phi_2(U)\subset\operatorname{St}(\Phi_2\circ\Phi_1(y),\gamma)$. Since $\Phi_1(y)$ is a compact set, there is $\beta(y)\in\operatorname{Cov}(X_2)$ such that $\operatorname{St}^3(\Phi_1(y),\beta(y))\subset U$. Hence,

$$\Phi_2(\operatorname{St}^3(\Phi_1(y),\beta(y)))\subset\operatorname{St}(\Phi_2\circ\Phi_1(y),\gamma).$$

Now, it is sufficient to show that there is $\beta\in\operatorname{Cov}(X_2)$ such that for every $x\in X_1$ there is $u(y)\in X_1$ such that

$$y\in\operatorname{St}(u(y),\alpha),\quad \operatorname{St}^2(\Phi_1(y),\beta)\subset\operatorname{St}^3(\Phi_1(u(y)),\beta(u(y))).$$

Suppose on the contrary that for every $\beta\in\operatorname{Cov}(X_2)$ there is $y_\beta\in X_1$ such that for each $z\in\operatorname{St}(y_\beta,\alpha)$

$$\operatorname{St}^2(\Phi_1(y_\beta),\beta)\not\subset\operatorname{St}^3(\Phi_1(z),\beta(z)). \tag{50.35.1}$$

We can assume that the net $\{y_\beta\}$ converges to y_0, because X_1 is compact. The set $\operatorname{St}(\Phi_1(y_0),\beta(y_0))$ is a neighbourhood of $\Phi_1(y_0)$ in X_2. Since Φ_1 is u.s.c. there is a covering $\alpha_0\ge\alpha$ such that $\Phi_1(\operatorname{St}(y_0,\alpha_0))\subset\operatorname{St}(\Phi_1(y_0),\beta(y_0))$. There exists $\beta_1\in\operatorname{Cov}(X_2)$ such that, for every $\widetilde{\beta}\ge\beta_1$, $y_{\widetilde{\beta}}\in\operatorname{St}(y_0,\alpha_0)$, and consequently

$$\Phi_1(y_{\widetilde{\beta}})\subset\operatorname{St}(\Phi_1(y_0),\beta(y_0)).$$

Hence, $\operatorname{St}^2(\Phi_1(y_{\widetilde{\beta}}),\widetilde{\beta})\subset\operatorname{St}^3(\Phi_1(y_0),\beta(y_0))$ and $y_0\in\operatorname{St}(y_{\widetilde{\beta}},\alpha_0)\subset\operatorname{St}(y_{\widetilde{\beta}},\alpha)$. A contradiction with (50.35.1). □

(50.36) DEFINITION. Let $(\Phi_1, \dots, \Phi_k) \in \mathcal{DA}(X_0, X_k)$, $\Phi_i \in \mathcal{A}_{m_i}(X_{i-1}, X_i)$. Let $\alpha, \overline{\alpha} \in \operatorname{Cov}(X_0)$, $\beta, \overline{\beta} \in \operatorname{Cov}(X_k)$. A chain map $\varphi: C_*(N^{(n)}(\overline{\alpha})) \to C_*(N^{(n)}(\overline{\beta}))$ is an (α,β)-*approximation* of $(\Phi_1, \dots, \Phi_k)$ provided that:

(50.36.1) φ multiplies the Kronecker index by $m = m_1 \dots m_k$;

(50.36.2) For each simplex $s \in N(\overline{\alpha})$ there is a point $p(s) \in X_0$ and $r \in N$ such that

$$\operatorname{supp} s \subset \operatorname{St}^r(p(s), \alpha), \quad \operatorname{supp} \varphi s \subset \operatorname{St}^r(\Phi(p(s)), \beta),$$

where Φ is a map determined by $(\Phi_1, \dots, \Phi_k)$.

(50.37) PROPOSITION. *Let $\Phi_1 \in \mathcal{DA}(X_1, X_2)$, $\Phi_2 \in \mathcal{DA}(X_2, X_3)$. Let the coverings $\alpha \in \operatorname{Cov}(X_1)$ and $\gamma \in \operatorname{Cov}(X_3)$ be given. There is a covering $\beta \in \operatorname{Cov}(X_2)$ such that if $\varphi: C_*(N^{(n)}(\widetilde{\alpha})) \to C_*(N^{(n)}(\widetilde{\beta}))$ is an (α,β)-approximation of Φ_1 and $\psi: C_*(N^{(n)}(\widetilde{\beta})) \to C_*(N^{(n)}(\gamma))$ is a (β,γ)-approximation of Φ_2 then $\psi \circ \varphi$ is an (α,γ)-approximation of $\Phi_2 \circ \Phi_1 \in \mathcal{DA}(X_1, X_3)$.*

PROOF. (50.36.1) is evident. We prove (50.36.2). For simplicity we assume that $r_1 = r_2 = 1$. Let $\beta \in \operatorname{Cov}(X_2)$ be obtained from (50.35). If φ is an (α,β)-approximation of Φ_1, then for each simplex $s \in N(\widetilde{\alpha})$ there is a point $p(s) \in X_1$ such that

$$\operatorname{supp} s \subset \operatorname{St}(p(s), \alpha), \quad \operatorname{supp} \varphi s \subset \operatorname{St}(\Phi_1(p(s)), \beta).$$

From (50.35) we obtain a point $u = u(p(s)) \in X_1$ such that

$$p(s) \in \operatorname{St}(u, \alpha), \quad \Phi_2(\operatorname{St}^2(\Phi_1(p(s)), \beta)) \subset \operatorname{St}(\Phi_2 \circ \Phi_1(u), \gamma).$$

Therefore, $\operatorname{supp} s \subset \operatorname{St}^2(u, \alpha)$.

Let $\varphi s = \sum a_i s_i$. For every s_i there is $p(s_i) \in X_2$ such that

$$\operatorname{supp} s_i \subset \operatorname{St}(p(s_i), \beta), \quad \operatorname{supp} \psi s_i \subset \operatorname{St}(\Phi_2(p(s_i)), \gamma).$$

Hence, $p(s_i) \in \operatorname{St}^2(\Phi_1(p(s)), \beta)$ and

$$\operatorname{supp} \psi s_i \subset \operatorname{St}(\Phi_2(\operatorname{St}^2(p(s), \beta)), \gamma) \subset \operatorname{St}^2(\Phi_2 \circ \Phi_1(u), \gamma).$$

Since $\psi\varphi s = \sum a_i \psi s_i$, we obtain

$$\operatorname{supp} s \subset \operatorname{St}^2(u, \alpha), \quad \operatorname{supp} \psi\varphi s \subset \operatorname{St}^2(\Phi_2 \circ \Phi_1(u), \gamma).$$

The proof is finished. $\square$

(50.38) Corollary. *Let* $(\Phi_1, \dots, \Phi_k) \in \mathcal{DA}(X_0, X_k)$. *Let* $\alpha \in \operatorname{Cov}(X_0)$, $\beta \in \operatorname{Cov}(X_k)$, $n \in N$. *Then there exists an* (α, β)*-approximation* $\varphi\colon C_*(N^{(n)}(\overline{\alpha})) \to C_*(N^{(n)}(\beta))$ *of* $(\Phi_1, \dots, \Phi_k)$ *which is a composition of chain approximations of the* Φ_i.

Proof. The proof proceeds by induction on the length k. For $k = 1$ the assertion follows from (50.29) and (50.37) is used in the inductive step. □

(50.39) Proposition. *Compositions of small homotopic and sufficiently fine chain approximations are also small homotopic.*

Proof. Let $\Phi_1 \in \mathcal{DA}(X_1, X_2)$, $\Phi_2 \in \mathcal{DA}(X_2, X_3)$ and let $\alpha, \overline{\alpha} \in \operatorname{Cov}(X_1)$, $\beta, \overline{\beta} \in \operatorname{Cov}(X_2)$, $\gamma \in \operatorname{Cov}(X_3)$. Assume that the following diagram is given:

$$\begin{array}{ccccc}
C_*(N^{(n)}(\overline{\alpha})) & \xrightarrow{\ \varphi_1\ } & C_*(N^{(n)}(\overline{\beta})) & \xrightarrow{\ \varphi_2\ } & C_*(N^{(n)}(\gamma)) \\
\Big\downarrow{\scriptstyle \mathrm{id}} & \text{(I)} & \Big\downarrow{\scriptstyle \mathrm{id}} & \text{(II)} & \Big\downarrow{\scriptstyle \mathrm{id}} \\
C_*(N^{(n)}(\overline{\alpha})) & \xrightarrow[\ \psi_1\ ]{} & C_*(N^{(n)}(\overline{\beta})) & \xrightarrow[\ \psi_2\ ]{} & C_*(N^{(n)}(\gamma))
\end{array}$$

where (I) is a homotopy commutative with a (Φ_1, α, β)-small homotopy D_1 and (II) is a homotopy commutative with a (Φ_2, β, γ)-small homotopy D_2 (cf. 50.35). Let the covering β be fine enough to satisfy (50.35). We prove that $D := \psi_2 D_1 + D_2 \varphi_1$ is a chain homotopy joining $\varphi_2\varphi_1$ and $\psi_2 \circ \psi_1$ that it is (Φ, α, γ)-small, where $\Phi = \Phi_2 \circ \Phi_1$. Let $s \in N^{(n)}(\overline{\alpha})$ be a simplex. Then

$$\begin{aligned}
\psi_2\psi_1 s - \varphi_2\varphi_1 s &= \psi_2\psi_1 s - \psi_2\varphi_1 s + \psi_2\varphi_1 s - \varphi_2\varphi_1 s \\
&= \psi_2(\partial D_1 s + D_1 \partial s) + \partial D_2 \varphi_1 s + D_2 \partial \varphi_1 s \\
&= \partial(\psi_2 D_1 s + D_2 \varphi_1 s) + (\psi_2 D_1 + D_2 \varphi_1)\partial s.
\end{aligned}$$

Therefore, D is a chain homotopy. Now, we check the *smallness condition*. From (50.31) we obtain a point $c(s) \in X_1$ such that

$$\operatorname{supp} s \subset \operatorname{St}(c(s), \alpha), \quad \operatorname{supp} D_1 s \subset \operatorname{St}(\Psi_1(c(s)), \beta).$$

By (50.35), we find a point $u = u(c(s)) \in X_1$ such that

$$c(s) \in \operatorname{St}(u, \alpha), \quad \Phi_2(\operatorname{St}^2(\Phi_1(c(s)), \beta)) \subset \operatorname{St}(\Phi_2 \circ \Phi_1(u), \gamma).$$

Thus $\operatorname{supp} s \subset \operatorname{St}^2(u, \alpha)$. We show that $\operatorname{supp} Ds \subset \operatorname{St}^2(\Phi_2 \circ \Phi_1(u), \gamma)$. But $\operatorname{supp} Ds = \operatorname{supp} \psi_2 D_1 s \cup \operatorname{supp} D_2 \varphi_1 s$. Let $D_1 s = \sum a_i s_i$. For each s_i there is $p(s_i) \in X_2$ such that

$$\operatorname{supp} s_i \subset \operatorname{St}(p(s_i), \beta), \quad \operatorname{supp} \psi_2 s_i \subset \operatorname{St}(\Phi_2(p(s_i), \gamma).$$

Hence $p(s_i) \in \operatorname{St}^2(\Phi_1(c(s)), \beta)$ and

$$\operatorname{supp} \psi_2 s_i \subset \operatorname{St}(\Phi_2 \operatorname{St}^2(\Phi_1(c(s)), \beta), \gamma) \subset \operatorname{St}^2(\Phi_2 \circ \Phi_1(u), \gamma).$$

Therefore,

$$\operatorname{supp} \psi_2 D_1 s = \bigcup_i \operatorname{supp} \psi_2 s_i \subset \operatorname{St}^2(\Phi_2 \circ \Phi_1(u), \gamma).$$

Note that $\operatorname{supp} \varphi s \subset D_1 s$. Let $\varphi_1 s = \sum b_j s_j$. By (50.31), for each $s_j \in N^{(n)}(\overline{\beta})$ there is a point $c(s_j) \in X_2$ such that

$$\operatorname{supp} s_j \subset \operatorname{St}(c(s_j), \beta), \quad \sup D_2 s_j \subset \operatorname{St}(\Phi_2(c(s_j)), \gamma).$$

But we note that $c(s_j) \in \operatorname{St}^2(\Psi_1(c(s)), \beta)$ and consequently

$$\operatorname{supp} D_2 s_j \subset \operatorname{St}^2(\Phi_2 \circ \Phi_1(u), \gamma).$$

Therefore,

$$\operatorname{supp} D_2 \varphi_1 s = \bigcup_j \operatorname{supp} D_2 s_j \subset \operatorname{St}^2(\Phi_2 \circ \Phi_1(u), \gamma)$$

and our proof is finished. □

51. A fixed point index of decompositions for finite polyhedra

In this section we shall use the previous results to construct a fixed point index theory on compact polyhedra.

Let X and Y be two compact spaces. Let $(\Phi_1, \ldots, \Phi_k) \in \mathcal{DA}(X, Y)$. Then each chain approximation $\varphi\colon C_*(N^{(n)}(\alpha)) \to C_*(N^{(n)}(\beta))$ of $(\Phi_1, \ldots, \Phi_k)$ induces a homomorphism

$$\varphi_*\colon H_*(N^{(n)}(\alpha)) \to H_*(N^{(n)}(\beta)).$$

Assume that the chain approximations considered are compositions of n-close approximations of the Φ_i. From (50.33) and (50.39) we deduce that for sufficiently fine coverings α, β the diagram

$$\begin{array}{ccc} H_*(N^{(n)}(\overline{\alpha})) & \xrightarrow{\varphi_*} & H_*(N^{(n)}(\overline{\beta})) \\ \Big\uparrow {\scriptstyle i^{\tilde{\alpha}}_{\bar{\alpha}*}} & & \Big\uparrow {\scriptstyle i^{\tilde{\beta}}_{\bar{\beta}*}} \\ H_*(N^{(n)}(\widetilde{\alpha})) & \xrightarrow[\psi_*]{} & H_*(N^{(n)}(\widetilde{\beta})) \end{array}$$

commutes, where $\widetilde{\alpha} \geq \overline{\alpha} \geq \alpha$, $\widetilde{\beta} \geq \overline{\beta} \geq \beta$ and φ, ψ are (α, β)-approximations of $(\Phi_1, \ldots, \Phi_k)$. Therefore, one can define the *induced homomorphism*

$$(\Phi_1, \ldots, \Phi_k)_*\colon \check{H}_*(X) \to \check{H}_*(Y)$$

by the formula

$$(51.1)\qquad (\Phi_1,\ldots,\Phi_k)_q := \varprojlim\{\varphi_{*q}\colon H_q(N^{(n)}(\alpha)) \to H_q(N^{(n)}(\beta))\}$$

for $q < n$. This homomorphism is nontrivial in the sense of O'Neill (see [Ne1]), i.e. it is a nonzero homomorphism of the 0-th homology vector spaces.

Let (X,A) and (Y,B) be compact pairs and let Φ be a map determined by $(\Phi_1,\ldots,\Phi_k) \in \mathcal{DA}(X,Y)$ such that $\Phi(A) \subset B$. Let $\alpha \in \operatorname{Cov} X$ and $\beta \in \operatorname{Cov} Y$ be such that $\Phi(\mathrm{St}^2(A,\alpha)) \subset \mathrm{St}(B,\beta)$. Let $\widetilde{\beta} \in \operatorname{Cov} Y$ be a star-refinement of β. Consider a chain map $\varphi\colon C_*(N^{(n)}(\overline{\alpha})) \to C_*(N^{(n)}(\overline{\beta}))$ which is an $(\alpha,\widetilde{\beta})$-approximation of $(\Phi_1,\ldots,\Phi_k)$.

If a chain $c \in C_*(N^{(n)}(\overline{\alpha}))$ is such that $\operatorname{supp} c \subset \mathrm{St}(A,\overline{\alpha})$ then $\operatorname{supp}\varphi c \subset \mathrm{St}^2(B,\widetilde{\beta}) \subset \mathrm{St}(B,\beta)$. This follows from (50.36.2) (without loss of generality we can assume that φ satisfies (50.36.2) with $r=1$). Therefore, $\overline{\varphi} = i^{\overline{\beta}}_{\beta} \circ \varphi$ is a chain map of the relative chain complexes:

$$\overline{\varphi}\colon C_*(N^{(n)}(\overline{\alpha}), N^{(n)}(\overline{\alpha})|_{\mathrm{St}(A,\widetilde{\alpha})}) \to C_*(N^{(n)}(\beta), N^{(n)}(\beta)|_{\mathrm{St}(B,\beta)}).$$

Now, a formula similar to (51.1) gives a definition of the relative induced homomorphism ([ES-M]):

$$(51.2)\qquad (\Phi_1,\ldots,\Phi_k)_*\colon \check{H}_*(X,A) \to \check{H}_*(Y,B).$$

Let (X,A) and (Y,B) be two arbitrary pairs. If Φ is determined by $(\Phi_1,\ldots,\Phi_k) \in \mathcal{DA}(X,Y)$ then by (14.9), the image $\Phi(C)$ of any compact set $C \subset X$ is a compact subset of Y. Assume that $\Phi(A) \subset B$. Then the procedure given in Section 5 of Chapter I can be applied. Therefore, we obtain the induced homology homomorphism $(\Phi_1,\ldots,\Phi_k)_*\colon H_*(X,A) \to H_*(Y,B)$, where H is the Čech homology functor with compact carriers and coefficients in the field Q.

Let (K,τ) be a polyhedron with a fixed triangulation τ. The *covering associated with the triangulation* τ is $\alpha(\tau) := \{\mathrm{st}(v_i,\tau) := \operatorname{Int}\mathrm{St}(v_i,\tau)\}$, where v_i are vertices of τ. There are simplicial maps $\theta\colon (K,\tau) \to N(\alpha(\tau))$ and $\lambda\colon N(\alpha(\tau)) \to (K,\tau)$ defined on vertices by $\theta(v) := \mathrm{st}(v,\tau)$ and $\lambda(\mathrm{st}(v,\tau)) := v$. These maps define the canonical simplicial isomorphism between the complexes (K,τ) and $N(\alpha(\tau))$. Moreover, $\operatorname{carr} s \subset \operatorname{supp}\theta s$ and $\operatorname{supp}\sigma \subset \mathrm{St}(\operatorname{carr}\lambda\sigma, \alpha(\tau))$.

Let $(\Phi_1,\ldots,\Phi_k) \in \mathcal{DA}(K,L)$. Let τ be a triangulation of K and μ a triangulation of L. We define $A_j(\Phi_1,\ldots,\Phi_k)$ to be the set of chain maps $\varphi\colon C_*(K,\tau^j) \to C_*(L,\mu^j)$, which are of the form $\varphi = \lambda\circ\varphi_k\circ\ldots\circ\varphi_1\circ\theta\circ b$, where b is the standard subdivision map, λ and θ are induced by the above-named isomorphism for μ^j, τ^j, respectively, and φ_i are n-close chain approximations of Φ_i; $n = \dim K$.

(51.3) PROPOSITION. *The graded set* $\{A_j(\Phi_1, \dots, \Phi_k)\}_j$ *is an* A*-system for the map* Φ *determined by* $(\Phi_1, \dots, \Phi_k)$.

PROOF. From (50.30) and (50.38) we deduce that the sets $A_j(\Phi_1, \dots, \Phi_k)$ are nonempty for arbitrarily large j. Condition (50.18.2) follows from Propositions (50.33) and (50.39). □

(51.4) DEFINITION. The above A-system is said to be an *induced* A*-system* of the decomposition $(\Phi_1, \dots, \Phi_k)$. We denote it by $A_*(\Phi_1, \dots, \Phi_k)$.

Since every element φ of $A_j(\Phi_1, \dots, \Phi_k)$ induces a homology homomorphism, using (50.18.2) we obtain the induced homomorphism

$$(A_*(\Phi_1, \dots, \Phi_k))_* \colon H_*(K) \to H_*(L).$$

A straightforward consequence of the definitions is given by

(51.5) PROPOSITION. *If* X *and* Y *are compact polyhedra and* $(\Phi_1, \dots, \Phi_k) \in \mathcal{DA}(X, Y)$ *then*

$$(\Phi_1, \dots, \Phi_k)_* = (A_*(\Phi_1, \dots, \Phi_k))_*.$$

Let X be a compact polyhedron and let Φ be a map determined by a decomposition $(\Phi_1, \dots, \Phi_k) \in \mathcal{DA}(X, X)$. Let U be an open subset of X such that $x \notin \Phi(x)$ for $x \in \partial U$.

(51.6) DEFINITION. We define a *fixed point index* $i(X, (\Phi_1, \dots, \Phi_k), U)$ of a decomposition $(\Phi_1, \dots, \Phi_k)$ with respect to U by

$$i(X, (\Phi_1, \dots, \Phi_k), U) := I_{A_*(\Phi_1, \dots, \Phi_k)}(X, \Phi, U).$$

(51.7) PROPOSITION. *Definition* (51.6) *does not depend on the triangulation* τ *of* X.

PROOF. Let $A_*(\Phi_1, \dots, \Phi_k)$ and $\overline{A}_*(\Phi_1, \dots, \Phi_k)$ be constructed for triangulations τ and τ_0, respectively. Given $j \in N$ there is an integer j_1 such that $\alpha(\tau_0^{j_1}) \geq \alpha(\tau^j)$. Let $\psi \colon C_*(N^{(n)}(\alpha(\tau_0^{j_1}))) \to C_*(N^{(n)}(\alpha(\tau_0^{j_1})))$ be an approximation of Φ such that

$$\lambda \circ \psi \circ \theta \circ b \in \overline{A}_{j_1}(\Phi_1, \dots, \Phi_k).$$

Let l be an integer such that $\alpha(\tau^l) \geq \alpha(\tau_0^{l_1})$.

We define a chain map $\varphi \colon C_*(N^{(n)}(\alpha(\tau^l))) \to C_*(N^{(n)}(\alpha(\tau^j)))$ by the formula

$$\varphi := i^{\alpha(\tau_0^{j_1})}_{\alpha(\tau^j)} \circ \psi \circ i^{\alpha(\tau^l)}_{\alpha(\tau_0^{l_1})}.$$

It is also an approximation of Φ and $\lambda \circ \varphi \circ \theta \circ b \in A_j(\Phi_1, \dots, \Phi_k)$. Note that the only difference between φ and ψ is in the natural maps i. Therefore, one easily verifies that A_* and $\overline{A}_*$ have the same index. □

(51.8) LEMMA. *Let $H: X \times I \to Y \in \mathcal{A}_m(X \times I, Y)$ be a homotopy ($I = [0,1]$). Then for every pair $\alpha \in \operatorname{Cov} X$ and $\beta \in \operatorname{Cov} Y$ there are (α, β)-approximations φ_0 of H_0 and φ_1 of H_1 which are chain homotopic with an (H, α, β)-small homotopy D (see* (50.18.2)).

PROOF. For every $x \in X$ there is a neighbourhood $\mathcal{O}_x \subset X$ of x such that $H(\mathcal{O}_x \times I) \subset \operatorname{St}(H(\{x\}) \times I), \beta)$. We can assume that the covering $\{\mathcal{O}\}_{x \in X}$ is a refinement of α. Let $\{\mathcal{O}_{x_i}\}_{i=1}^k$ be a finite subcovering of $\{\mathcal{O}_x\}$. Let $\overline{\alpha}$ be a star-refinement of $\{\mathcal{O}_{x_i}\}$. Now, we consider a covering γ of $X \times I$, $\gamma := \{U \times I \mid U \in \overline{\alpha}\}$. There exists a (γ, β)-approximation of H, $\psi: C_*(N^{(n+1)}(\widetilde{\gamma})) \to C_*(N^{(n+1)}(\beta))$. We can assume that $\widetilde{\gamma}$ consists of set of the form $U \times V$, where $U \subset X$ and $V \subset I$. Then $N(\widetilde{\gamma}) = N(\widetilde{\alpha}) \times N(\eta)$, where $\widetilde{\alpha} = \{U\}$, $\eta = \{V\}$ and $\widetilde{\gamma} = \{U \times V\}$. We can also assume that the complex $N(\eta)$ is 1-dimensional. Let us define, for $i = 0, 1$, $\varphi_i := \psi|_{N(\widetilde{\alpha}) \times \{V_i\}}$, where V_i are such that $i \in V_i$. Obviously $N(\widetilde{\alpha}) \times \{V_i\} \approx N(\widetilde{\alpha})$ and one easily verifies that the φ_i are (α, β)-approximations of H_i, $i = 0, 1$.

Let $s \in N(\widetilde{\alpha})$ be a simplex and consider the chain $s \times I = \sum s_j$, where s_j are simplex of $N(\widetilde{\gamma})$. We define $Ds := \psi(s \times I)$. Since ψ is an approximation of H, for each s_j there is a point $p(s_j) \in X \times I$ such that

$$\operatorname{supp} s_j \subset \operatorname{St}(p(s_j), \gamma), \quad \operatorname{supp} \psi s_j \subset \operatorname{St}(H(p(s_j)), \beta).$$

Since $\widetilde{\alpha} \geq \overline{\alpha}$, there is a set $\mathcal{O}_{x_i}$ such that $\operatorname{St}(\operatorname{supp} s, \overline{\alpha}) \subset \mathcal{O}_{x_i}$. Therefore,

$$H(\operatorname{St}(\operatorname{supp} s, \overline{\alpha}) \times I) \subset \operatorname{St}(H(\{x_i\} \times I), \beta).$$

Let $p(s_j) = (p_j, t_j)$. Then $p_j \in \operatorname{St}(\operatorname{supp} s\overline{\alpha})$ and hence

$$\operatorname{supp} \psi s_j \subset \operatorname{St}(H(\{x_i\} \times I), \beta).$$

We put $d(s) := x_i$ (see (50.18.2)) and the proof is complete. □

(51.9) PROPOSITION. *Let X and Y be compact polyhedra and let $(\Phi_1, \ldots, \Phi_k)$, $(\Psi_1, \ldots, \Psi_k) \in \mathcal{DA}(X, Y)$ be homotopic. If $H: X \times I \to Y$ is the map determined by the joining homotopy then the induced A-systems $A_*(\Phi_1, \ldots, \Phi_k)$ and $A_*(\Psi_1, \ldots, \Psi_k)$ are H-homotopic* (see (50.18)).

PROOF. Without loss of generality we can assume that $\Phi_2 = \Psi_2, \ldots, \Phi_k = \Psi_k$ and Φ_1 is homotopic to Ψ_1 in $\mathcal{A}_{m_1}(X, X_1)$. By (51.8), there are arbitrarily fine approximations of Φ_1 and Ψ_1 which are h_1-small homotopic (where H is determined by $(h_1, \ldots, h_k)$). We obtain the desired H-small homotopic approximations of $(\Phi_1, \ldots, \Phi_k)$ and $(\Psi_1, \ldots, \Psi_k)$ by composing those from (51.8) with the approximations of the maps $h_2, \ldots, h_k$. □

(51.10) THEOREM. *The fixed point index satisfies the following properties:*

(51.10.1) (Additivity) *Let Φ be determined by the decomposition $(\Phi_1, \dots, \Phi_k) \in \mathcal{DA}(X, X)$, where X is a compact polyhedron. Assume that U_1 and U_2 are open disjoint subsets of an open set $U \subset X$ and $\operatorname{Fix}\Phi|_{\overline{U}} \subset U_1 \cup U_2$. Then*

$$i(X, (\Phi_1, \dots, \Phi_k), U) = i(X, (\Phi_1, \dots, \Phi_k), U_1) + i(X, (\Phi_1, \dots, \Phi_k), U_2).$$

(51.10.2) (Homotopy Invariance) *Let $(\Phi_1, \dots, \Phi_k)$ and $(\Psi_1, \dots, \Psi_k)$ be homotopic and let the map $H\colon X \times [0, 1] \to Y$ determined by the homotopy satisfy $x \notin H(x, t)$ for $x \in \partial U$ and $t \in [0, 1]$. Then*

$$i(X, (\Phi_1, \dots, \Phi_k), U) = i(X, (\Psi_1, \dots, \Psi_k), U).$$

(51.10.3) (Normalization) *If $(\Phi_1, \dots, \Phi_k) \in \mathcal{DA}(X, X)$ and*

$$\lambda(\Phi_1, \dots, \Phi_k)_* = \sum (-1)^j \operatorname{tr}(\Phi_1, \dots, \Phi_k)_*)_j$$

then $i(X, (\Phi_1, \dots, \Phi_k), X) = \lambda(\Phi_1, \dots, \Phi_k)_*$.

(51.10.4) (Commutativity) *Let X, Y be compact polyhedra and $\Phi \in \mathcal{DA}(X, Y)$, $\Psi \in \mathcal{DA}(Y, X)$. Denote by the same letters the maps determined by Φ and Ψ. Let W be an open subset of X such that $x \notin \Psi \circ \Phi(x)$ for $x \in \partial W$, $y \notin \Phi \circ \Psi(y)$ for $y \in \partial(\Psi^{-1}(W))$ and*

$$\Psi(\operatorname{Fix}\Phi \circ \Psi - \Psi^{-1}(W)) \cap \operatorname{Fix}\Psi \circ \Phi|_{\overline{W}} = \emptyset.$$

Then $i(X, \Psi \circ \Phi, W) = i(Y, \Phi \circ \Psi, \Psi^{-1}(W))$.

(51.10.5) (Mod-p Property) *Let $F = Z_p$, p prime. Let $\Phi \in \mathcal{DA}(X, X)$ and denote by same letter the map determined by Φ. Let $W \subset X$ be open and assume that $x \notin \Phi^p(x)$ for $x \in \partial W$ and*

$$\Phi^k(\operatorname{Fix}\Phi^p - W) \cap \operatorname{Fix}\Phi^p|_{\overline{W}} = \emptyset \quad \text{for } k < p.$$

Then $i(X, \Phi, W) = i(X, \Phi^p, W)$.

PROOF. (51.10.1) is an immediate consequence of (50.21) (see (50.24)). The homotopy invariance follows from (50.23) and (51.9). The normalization property is a consequence of (51.5) and the Hopf trace theorem (see [Sp-M]). The last two properties follow from (50.25) and (50.26), respectively. □

52. Fixed point index of decompositions for compact ANRs

In this section we extend the fixed point index theory to the case of compact ANRs. We need five lemmas which are strictly analogous to (22.2) but for the clarity of this section we shall present the respective formulations.

(52.1) LEMMA. *Let A be a closed subset of a compact space X. Let $\Phi: X \multimap X$ be an u.s.c. map such that* $\mathrm{Fix}(\Phi)\cap A = \emptyset$. *Then there exists $\delta = \delta(\Phi, A) > 0$ such that: if* $\mathrm{dist}(x, A) < \delta$ *then* $\mathrm{dist}(x, \Phi(x)) > \delta$.

(52.2) LEMMA. *Let U be an open subset of a compact* ANR-*space X and let $\Phi: X \multimap X$ be an u.s.c. map such that* $\mathrm{Fix}(\Phi)\cap\partial U = \emptyset$. *If $\varepsilon < \delta\ (\Phi, \partial U)$ and P_ε ε-dominates X with $r_\varepsilon: P \to X$ and $s_\varepsilon: X \to P$ then the map $\Psi = s_\varepsilon \circ \Phi \circ r_\varepsilon: P \multimap P$ is fixed point free on $\partial(r_\varepsilon^{-1}(U))$.*

(52.3) LEMMA. *Let X_1, X_2, X_3 be compact metric spaces, $\Phi_1: X_1 \multimap X_2$ and $\Phi_2: X_2 \multimap X_3$ u.s.c. maps, and $\varepsilon_0 > 0$. There exists $\varepsilon > 0$ such that for every point $y \in X_1$ there is a point $u(y) \in X_1$ such that*

$$y \in O_{\varepsilon_0}(u(y)), \quad \Phi_2(O_{2\varepsilon}(\Phi_1(y))) \subset O_{\varepsilon_0}(\Phi_2 \circ \Phi_1(u(y))).$$

(52.4) LEMMA. *Let X_1, X_2 be two compact metric spaces and let $\Phi_1: X_1 \multimap X_2$ and $\Phi_2: X_2 \multimap X_1$ be u.s.c. maps. Then for each $\varepsilon > 0$ there is $\varepsilon_0 > 0$ such that for $\eta \in (0, \varepsilon_0)$ and u.s.c. maps $\Phi_{1\eta}: X_1 \multimap X_2$, $\Phi_{2\eta}: X_2 \multimap X_1$ with $\Phi_{i\eta}(x) \subset O_\eta(\Phi_i(x))$ we have*

$$\mathrm{Fix}\, \Phi_{2\eta} \circ \Phi_{1\eta} \subset O_\varepsilon(\mathrm{Fix}\, \Phi_2 \circ \Phi_1).$$

(52.5) LEMMA. *Let U be an open subset of a metric space Z and let $C \subset Z$ be such that $C\cap\overline{U} = \emptyset$. For $\delta < d(C, U)$ and a continuous singlevalued map $f: Z \to Z$ such that $d(x, f(x)) < \delta$ we have $f^{-1}(U) \cap C = \emptyset$.*

In this section we will denote by the same letter a decomposition $\Phi \in \mathcal{DA}(X, X)$ and the map $\Phi: X \multimap X$, which determines the decomposition. Usually, the assumptions that are made for the map. The index is defined for decompositions.

Let X be a compact metric ANR and $\Phi \in \mathcal{DA}(X, X)$. Let U be an open subset of X such that $\mathrm{Fix}\, \Phi \cap \partial U = \emptyset$.

(52.6) DEFINITION. Let $\varepsilon < (1/4)\delta(\Phi, \partial U)$. Let P be a finite polyhedron ε-dominating X with maps $r: P \to X$ and $s: X \to P$. We define the *fixed point index* of the decomposition Φ with respect to U by

$$i(X, \Phi, U) := i(P, \Psi, r^{-1}(U)), \quad \text{where } \Psi = s \circ \Phi \circ r.$$

(52.7) PROPOSITION. *Definition* (52.6) *does not depend on the choice of an* ε*-domination.*

PROOF. Let $\varepsilon_1, \varepsilon_2 < (1/4)\delta(\Phi, \partial U)$ and let $r_i\colon P_i \to X$, $s_i\colon X \to P_i$ be such that $r_i \circ s_i$ is ε_i-homotopic to id_X, $i = 1, 2$. We shall prove that

$$i(P_1, s_1 \circ \Phi \circ r_1, r_1^{-1}(U)) = i(P_2, s_2 \circ \Phi \circ r_2, r_2^{-1}(U))$$

in four steps.

(52.7.1) $$i(P_2, s_2 \circ \Phi \circ r_2, r_2^{-1}(U)) = i(P_2, T, r_2^{-1}(U)),$$

where $T := s_2 \circ r_1 \circ s_1 \circ \Phi \circ r_1 \circ s_1 \circ r_2$. Let $H\colon X \times I \to X$ be an ε_1-homotopy with $H(x, 0) = x$ and $H(x, 1) = r_1 \circ s_1(x)$. Consider the composition

$$P_2 \times I \xrightarrow{r_2'} X \times I \xrightarrow{H'} X \times I \to \overset{\Phi'}{\cdots} \to X \times I \xrightarrow{H} X \xrightarrow{s_2} P_2,$$

where $r_2' = (r_2, \mathrm{id})$, $H' = (H, \mathrm{id})$, $\Phi' = (\Phi, \mathrm{id})$. In order to apply the homotopy property (51.10.2) we have to show that for each $t \in [0, 1]$ the map

$$\Psi := s_2 \circ H_t \circ \Phi \circ H_t \circ r_2\colon P_2 \to P_2$$

has no fixed points in the set $\partial(r_2^{-1}(U))$. Note that $\partial(r_2^{-1}(U)) \subset r_2^{-1}(\partial U)$. Actually, we will prove that $y \notin \Psi(y)$ for $y \in X$ such that $d(r_2(y), \partial U) < (1/4)\delta(\Phi, \partial U)$. First, note that if $x \in \partial U$ is such that $d(r_2(y), \partial U) = d(r_2(y), x)$ then

$$\begin{aligned} d(H_t \circ r_2(y), \partial U) &\le d(H_t \circ r_2(y), x) \le d(H_t \circ r_2(y), r_2(y)) + d(r_2(y), x) \\ &< \varepsilon_1 + \frac{1}{4}\delta(\Phi, \partial U) < \frac{1}{2}\delta(\Phi, \partial U). \end{aligned}$$

From (52.1) we obtain $d(\Phi \circ H_t \circ r_2(y), r_2(y)) > (3/4)\delta(\Phi, \partial U)$. For every $z \in \Phi \circ H_t \circ r_2(y)$ we have

$$d(H_t(z), z) < \varepsilon_1 < \frac{1}{4}\delta(\Phi, \partial U)$$

and therefore,

(52.7.2) $$d(H_t(z), r_2(y)) > \frac{1}{2}\delta(\Phi, \partial U).$$

Now, suppose that for some y such that $d(r_2(y), \partial U) < (1/4)\delta(\Phi, \partial U)$ we have $y \in \Psi(y)$. Then $r_2(y) \in r_2 \circ \Psi(y)$. But for each $z \in \Phi \circ H_t \circ r_2(y)$ we have

(72.7.3) $$d(r_2 \circ s_2 \circ H_t(z), H_t(z)) < \varepsilon_2 < \frac{1}{4}\delta(\Phi, \partial U).$$

Now, (52.7.3) together with (52.7.2) imply that

$$d(r_2 \circ s_2 \circ H_t(z), r_2(y)) > \frac{1}{4}\delta(\Phi, \partial U) > 0.$$

Therefore, $d(r_2 \circ \Psi(y), r_2(y)) > 0$ which is impossible because $r_2(y) \in r_2 \circ \Psi(y)$. So, the homotopy property (51.10.2) gives (52.7.1).

(52.7.4)
$$i(P_2, T, r_2^{-1}(U)) = i(P_2, T, S^{-1}(U)),$$

where $T := s_2 \circ r_1 \circ s_1 \circ \Phi \circ r_1 \circ s_1 \circ r_2$, $S := r_1 \circ s_1 \circ r_2$.

For the proof of (52.7.4) we apply the additivity (51.10.1). First, we show that the map $T\colon P_2 \to P_2$ has no fixed points in the set $A := r_2^{-1}(U) \setminus S^{-1}(U)$. Let $y \in A$. Then $r_2(y) \in U$ but $S(y) \notin U$. Consider the map $c\colon [0,1] \to X$, $c(t) := H(r_2(y), t)$. The path $c([0,1])$ meets U at $c(0)$ and $X \setminus U$ at $c(1)$. So, there is $t_0 \in [0,1]$ such that $c(t_0) \in \partial U$. But

$$d(r_2(y), H(r_2(y), t_0)) < \varepsilon_1 < \frac{1}{4}\delta(\Phi, \partial U),$$

hence $d(r_2(y), \partial U) < (1/4)\delta(\Phi, \partial U)$. The argument from the first step implies that $y \notin T(y)$. By the additivity property (51.10.1) one obtains

$$i(P_2, T, r_2^{-1}(U)) = i(P_2, T, r_2^{-1}(U) \cap S^{-1}(U)).$$

The same argument used for set $B := S^{-1}(U) \setminus r_2^{-1}(U)$ implies that

$$i(P_2, T, S^{-1}(U)) = i(P_2, T, r_2^{-1}(U) \cap S^{-1}(U)).$$

The last two equalities imply (52.7.4)

(52.7.5)
$$i(P_2, T, S^{-1}(U)) = i(P_1, R, r_1^{-1}(U)),$$

where $R := s_1 \circ r_2 \circ s_2 \circ r_1 \circ s_1 \circ \Phi \circ r_1$.

For the proof of (52.7.5) we apply the commutativity (51.10.4). Let $K\colon X \times [0,1] \multimap X$ be an ε_2-homotopy such that $K(x,0) = x$ and $K(x,1) = r_2 \circ s_2(x)$. We wish to show that the map

$$R_t := s_1 \circ K_t \circ H_t \circ \Phi \circ r_1\colon P_1 \to P_1$$

has no fixed points in the set $\partial(r_1^{-1}(U))$ for $t \in [0,1]$. Let $y \in P_1$ be such that $r_1(y) \in \partial U$. Then $d(r_1(y), \Phi \circ r_1(y)) > \delta(\Phi, \partial U)$. For each $z \in \Phi \circ r_1(y)$ we have

$$d(K_t \circ H_t(z), z) \le d(K_t \circ H_t(z), H_t(z)) + d(H_t(z), z) < \varepsilon_2 + \varepsilon_1 < \frac{1}{2}\delta(\Phi, \partial U).$$

Since $\delta(\Phi,\partial U) < d(r_1(y),z) \le d(z,K_t\circ H_t(z)) + d(r_1(y),K_t\circ H_t(z))$, we have

$$d(r_1(y),K_t\circ H_t\circ\Phi\circ r_1(y)) > \frac{1}{2}\delta(\Phi,\partial U). \tag{52.7.6}$$

Suppose, that for some $y\in r_1^{-1}(\partial U)$ we have $y\in R_t(y)$. Then $r_1(y)\in r_1\circ R_t(y)$. But for each $z\in\Phi\circ r_1(y)$ we have

$$d(r_1\circ s_1\circ K_t\circ H_t(z),K_t\circ H_t(z)) < \varepsilon_1 < \frac{1}{4}\delta(\Phi,\partial U).$$

Combining this with (52.7.6) we obtain

$$d(r_1\circ R_t(y),r_1(y)) > \frac{1}{4}\delta(\Phi,\partial U) > 0,$$

which establishes the contradiction with our assumption. In particular, taking $t=1$, we have shown that the map $R=R_1$ has no fixed points in the set $\partial(r_1^{-1}(U))$.

Let us consider the compositions $h := s_2\circ r_1\circ s_1\circ\Phi\circ r_1$, $k := s_1\circ r_2$. We have just shown that $k\circ h$ has no fixed points in the set $\partial(r_1^{-1}(U))$. Now, we check the last two assumptions for the commutativity property (51.10.4). Denote $W := r_1^{-1}(U)$. Let $y\in\partial(k^{-1}(W)) = k^{-1}(\partial W)$. If $y\in h\circ k(y) = T(y)$ then $k(y)\in s_1\circ r_2\circ T(y) = k\circ h\circ s_1\circ r_2(y)$ and $s_1\circ r_2(y)\in\partial(r_1^{-1}(U))$, which is impossible.

Let $y\in \operatorname{Fix} h\circ k - \overline{k^{-1}(W)}$. We have to prove that $k(y)\in\operatorname{Fix} k\circ h|_{\overline{W}}$. But this follows from $\operatorname{Fix} k\circ h|_{\overline{W}}\subset k^{-1}(W)$. Therefore, (51.10.4) implies (52.7.5).

$$i(P_1,R,r_1^{-1}(U)) = i(P_1,s_1\circ\Phi\circ r_1,r_1^{-1}(U)). \tag{52.7.7}$$

We take the homotopy

$$L\colon P_1\times I \xrightarrow{r_1'} X\times I \xrightarrow{\Phi}\!\!\circ \cdots \xrightarrow{\Phi}\!\!\circ X\times I \xrightarrow{H'}\!\!\circ X\times I \xrightarrow{K}\!\!\circ X \xrightarrow{s_1} P_1,$$

where $r_1' = (r_1,\mathrm{id})$, $\Phi' = (\Phi,\mathrm{id})$, $H' = (H,\mathrm{id})$. We have already proved in the third step that $y\notin L(y,t)$ for $y\in\partial(r_1^{-1}(U))$ and $t\in I$. By the homotopy property (51.10.2) we obtain (52.7.7) and this completes the proof of (52.7). □

Now we verify the properties of the index.

(52.8) PROPOSITION. *Let U be an open subset of a compact* ANR *X. Let $\Phi\in\mathcal{DA}(X,X)$ and let $U_1,U_2\subset U$ be open, disjoint and such that* $\operatorname{Fix}\Phi|_{\overline{U}}\subset U_1\cup U_2$. *Then*

$$i(X,\Phi,U) = i(X,\Phi,U_1) + i(X,\Phi,U_2).$$

PROOF. Applying (52.1) to the map Φ and $A := (U \setminus U_1) \cup U_2$ we find δ which is smaller than $\delta(\Phi, \partial U)$ and $\delta(\Phi, \partial U_i)$. Let $\varepsilon < (1/4)\delta$ and let P be a compact polyhedron which ε-dominates X with maps $r: P \to X$ and $s: X \to P$. Then

$$i(X, \Phi, U) = i(P, \Psi, r^{-1}(U)), \quad i(X, \Phi, U_j) = i(P, \Psi, r^{-1}(U_j)),$$

where $\Psi = s \circ \Phi \circ r$, $j = 1, 2$. We have to show that Ψ has no fixed points in the set $r^{-1}(U) \setminus r^{-1}(U_1 \cup U_2) = r^{-1}(U - U_1 \cup U_2)$. Suppose $y \in \Psi(y)$ and let $z \in \Phi(r(y))$ be such that $r(y) = (r \circ s)(z)$. Therefore,

$$d(r(y), \Phi(r(y))) \leq d(r(y), z) = d((r \circ s)(z), z) < \varepsilon < \frac{\delta}{4},$$

and we have arrived at a contradiction with (52.1). Hence the additivity (51.10.1) implies our assertion. □

(52.9) PROPOSITION. *Let $H \in \mathcal{DA}(X \times I, X)$. If U is an open subset of the compact* ANR*, X such that the maps $H_t: X \multimap X$ have no fixed points on ∂U for $t \in I$ then*

$$i(X, H_0, U) = i(X, H_1, U).$$

PROOF. Let $\mathbb{X} = (H, \mathrm{id}): X \times I \multimap X \times I$. If $x \in \partial U$ then $\mathbb{X}(x, t) = (x, t)$ means that $H(x, t) = H_t(x) = x$ contrary to the assumption, and so $\mathbb{X}$ has no fixed points in the set $\partial U \times I$. Apply (52.1) to the map $\mathbb{X}$ and $A = \partial U \times I$, with the natural Cartesian metric d' in $X \times I$. We find δ such that

$$d'((x, t), \partial U \times I) < \delta \quad \text{implies} \quad d'((x, t), \mathcal{H}(x, t)) > \delta.$$

But

$$d'((x, t), \partial U \times I) = d(x, \partial U), \quad d'((x, t), \mathbb{X}(x, t)) = d(x, H(x, t)) = d(x, H_t(x)).$$

Therefore, we may assume that $\delta(H_t, \partial U)$ is the same for all $t \in I$. Denote it by δ. For $\varepsilon < \delta/4$ we have a polyhedron P ε-dominating X with maps $s: X \to P$ and $r: P \to X$. By definition, $i(X, H_t, U) = i(P, s \circ H_t \circ r, r^{-1}(U))$. By (52.2), the family $s \circ H_t \circ r$ satisfies the assumption of the homotopy property (51.10.2). □

(52.10) PROPOSITION. *Let X be a compact* ANR *and $\Phi \in \mathcal{DA}(X, X)$. Then*

$$i(X, \Phi, X) = \lambda(\Phi_*).$$

PROOF. The assertion follows from (51.10.3) and from the fact that $r_* \circ s_* = \mathrm{id}_*$ (see (6.1)). □

(52.11) PROPOSITION. *Let X, X' be compact* ANR*s and $\Phi \in \mathcal{DA}(X, X')$, $\Psi \in \mathcal{DA}(X', X)$. Let U be an open subset of X such that $x \notin \Psi \circ \Phi(x)$ for $x \in \partial U$, $y \notin \Phi \circ \Psi(y)$ for $y \in \partial(\Psi^{-1}(U))$ and*

$$\Psi(\operatorname{Fix} \Phi \circ \Psi \setminus \Psi^{-1}(U)) \cap \operatorname{Fix} \Psi \circ \Phi|_{\overline{U}} = \emptyset.$$

Then $i(X', \Phi \circ \Psi, \Psi^{-1}(U)) = i(X, \Psi \circ \Phi, U)$.

PROOF. From (52.1) we obtain $\delta(\Psi \circ \Phi, \partial U)$ and $\delta(\Phi \circ \Psi, \partial(\Psi^{-1}(U)))$. Let δ be the smaller of these two. Next, we find ε for $\varepsilon_0 = \delta/4$ from (52.3) ($\varepsilon < \varepsilon_0$) with $\Phi_1 = \Phi$ and $\Phi_2 = \Psi$. Again from (52.3) we find ε' for $\varepsilon_0 = \varepsilon$ and $\Phi_1 = \Psi$ and $\Phi_2 = \Phi$. Let P be a polyhedron ε-dominating X with maps $r_1: P \to X$, $s_1: X \to P$. Let P' be a polyhedron ε'-dominating X with $r_2: P' \to X$, $s_2: X \to P'$. Since $\varepsilon < \delta/4 < \delta(\Psi \circ \Phi, \partial U)$,

$$i(X, \Psi \circ \Phi, U) = i(P, s_1 \circ \Psi \circ \Phi \circ r_1, r_1^{-1}(U)).$$

Since $\varepsilon' < (1/4)\delta(\Phi \circ \Psi, \partial(\Psi^{-1}(U)))$,

$$i(X', \Phi \circ \Psi, \Psi^{-1}(U)) = i(P', s_2 \circ \Phi \circ \Psi \circ r_2, (\Psi \circ r_2)^{-1}(U)).$$

We have to prove that the right sides of the above equalities are equal. We will do it in four steps:

(52.11.1) $$i(P, s_1 \circ \Psi \circ \Phi \circ r_1, r_1^{-1}(U)) = i(P, \Psi_1 \circ \Phi_1, r_1^{-1}(U)),$$

where $\Psi_1 = s_1 \circ \Psi \circ r_2$, $\Phi_1 = s_2 \circ \Phi \circ r_1$,

(52.11.2) $$i(P, \Psi_1 \circ \Phi_1, r_1^{-1}(U)) = i(P', \Phi_1 \circ \Psi_1, (r_1 \circ \Psi_1)^{-1}(U)),$$

(52.11.3) $$i(P', \Phi_1 \circ \Psi_1, (r_1 \circ \Psi_1)^{-1}(U)) = i(P', \Phi_1 \circ \Psi_1, (\Psi \circ r_2)^{-1}(U)),$$

(52.11.4) $$i(P', \Phi_1 \circ \Psi_1, (\Psi \circ r_2)^{-1}(U)) = i(P', s_2 \circ \Phi \circ \Psi \circ r_2, (\Psi \circ r_2)^{-1}(U)).$$

Proof of (52.11.1). We apply the homotopy property (51.10.2). Let $H': X \times I \to X$ be an ε'-homotopy such that $H'(x, 0) = x$, $H'(x, t) = r_2 \circ s_2(x)$ and $H'_t(x) := H'(x, t)$. We have to prove that $\Theta_t := s_1 \circ \Psi \circ H_t \circ \Phi \circ r_1$ has no fixed points in the set $r_1^{-1}(\partial U)$. Let $y \in r_1^{-1}(\partial U)$ be such that $y \in \Theta_t(y)$. Then $r_1(y) \in r_1 \circ \Theta_t(y)$ and therefore,

$$r_1(y) \in r_1 \circ s_1 \circ \Psi(O_{\varepsilon'}(\Phi \circ r_1(y))).$$

By the definition of ε', there is a point $u = u(r_1(y))$ such that

$$r_1(y) \in (r_1 \circ s_1)(O_{\delta/4}(\Psi \circ \Phi(u))) \subset O_{\delta/2}(\Psi \circ \Phi(u)).$$

Therefore, $u \in O_{3\delta/4}((\Psi \circ \Phi)(u))$ and $d(u, \partial U) < \delta$ which contradicts the choice of δ.

Proof of (52.11.2). We verify the assumptions of the commutativity property of Theorem (51.10). First, note that the map $\Psi_1 \circ \Phi_1$ has no fixed points in $\partial(r_1^{-1}(U))$, as has been proved above ($t = 1$). We let $B_1 = \mathrm{Fix}(\Psi \circ \Phi|_U)$, $B_2 = \mathrm{Fix}(\Phi \circ \Psi|_{\Psi^{-1}(U)})$, $B_3 = \mathrm{Fix}(\Phi \circ \Psi) \setminus \Psi^{-1}(U)$, Since B_i, $i = 1, 2, 3$ are compact, there exists ε_1 such that

$$d(O_{\varepsilon_1}(B_1), X \setminus U) = \delta_1 > 0, \quad d(O_{\varepsilon_1}(B_2), X' \setminus \Psi^{-1}(U)) = \delta_2 > 0,$$
$$d(O_{\varepsilon_1}(B_3), \Psi^{-1}(U)) = \delta_3 > 0, \qquad d(\Psi(O_{\varepsilon_1}(B_3)), U) = \delta_4 > 0,$$
$$d(\Psi(O_{\varepsilon_1}(B_2)), X \setminus U) = \delta_5 > 0.$$

Let $\varepsilon_2 \leq (1/2)\min\{\varepsilon_1, \delta_1, \delta_2, \delta_3, \delta_4, \delta_5\}$. Let $\varepsilon_3 < \varepsilon_2$ satisfy (52.3) for $\varepsilon_0 = \varepsilon_2$. Without loss of generality we can assume that ε' is smaller than ε_3. Suppose $y \in \Phi_1 \circ \Psi_1(y)$. Then $r_2(y) \in \mathrm{Fix}\, r_2 \circ \Phi_1 \circ s_1$ and therefore,

$$r_2(y) \in O_{\varepsilon_2}(\mathrm{Fix}\, \Phi \circ \Psi) = O_{\varepsilon_2}(B_3).$$

Let $C := \overline{\Psi(O_{\varepsilon_2}(B_3))}$. We have $\varepsilon < \varepsilon_2 < (1/2)d(C, \overline{U})$ and from (52.8) we obtain $(r_1 \circ s_1)^{-1}(U) \cap C = \emptyset$. Hence, $d(O_{\varepsilon_1}(B_3), V) > 0$.

Let $D := \overline{\Psi(O_{\varepsilon_1}(B_2))}$. Since $\varepsilon < (1/2)d(D, X \setminus U)$, by (52.8) we obtain $(r_1 \circ s_1)^{-1}(X \setminus U) \cap D = \emptyset$ and thus $D \subset (r_1 \circ s_1)^{-1}(U)$. Therefore, $\overline{O_{\varepsilon_1}(B_2)} \subset V$. Hence, we deduce that $r_2(y) \notin \partial V$ and therefore $v \notin \partial(r_2^{-1}(V))$. It remains to prove the following implication:

(52.11.5) $$y \in \mathrm{Fix}\, \Phi_1 \circ \Psi_1 \setminus r_2^{-1}(V) \Rightarrow \Psi_1(y) \cap \mathrm{Fix}\, \Psi_1 \circ \Phi_1|_{r_1^{-1}(U)} = \emptyset.$$

Actually we prove a stronger one:

$$(52.11.5) \Rightarrow \Psi_1(y) \cap r_1^{-1}(U) = \emptyset.$$

Suppose y satisfies (52.11.5). Then $r_2(y) \in \mathrm{Fix}\, r_2 \circ \Phi_1 \circ s_1 \setminus V$. Therefore, $r_2(y) \in O_{\varepsilon_2}(B_3) \supset O_{\varepsilon_1}(B_3)$. But $\Psi(O_{\varepsilon_1}(B_3)) \cap \overline{U} = \emptyset$ and by the choice of $\varepsilon' < \delta_4$ we obtain $(r_1 \circ s_1 \circ \Psi)(O_{\varepsilon_1}(B_3)) \cap \overline{U} = \emptyset$. Therefore, $\Phi_1(y) \cap r_1^{-1}(U) = \emptyset$ and applying (5.10.4) we finish the proof of (52.11.2).

Proof of (52.11.3). Note that we have just proved above that the map $\Phi_1 \circ \Psi_1$ has no fixed points in the sets $A_1 := r_2^{-1}(V) \setminus (\Psi \circ r_2)^{-1}(U)$ and $A_2 := (\Psi \circ r_2)^{-1}(U) \setminus r_2^{-1}(V)$.

So, applying twice the additivity property (51.10.1) one obtains (52.11.3).

Proof of (52.11.4). We apply again the homotopy property. We only have to show that the map $\Theta_t' := s_2 \circ \Phi \circ H_t \circ \Psi \circ r_2$ has no fixed points in the set $r_2^{-1}(\partial(\Psi^{-1}(U)))$. Let $y \in \Theta_t'(y)$. Then $r_2(y) \in (r_2 \circ \Theta_t')(y)$ and therefore, by (52.4), $r_2(y) \in O_{\varepsilon_2}(\mathrm{Fix}\, \Phi \circ \Psi)$. The considerations in the proof (52.11.3) imply that $r_2(y) \notin \partial(\Psi^{-1}(U))$. This finishes the proof of (52.11). $\square$

(52.12) PROPOSITION. *Let $F = Z_p$, p a prime number. Let $\Phi = \mathcal{DA}(X,X)$ and let U be an open subset of X such that* $\operatorname{Fix}\Phi^p \cap \partial U = \emptyset$ *and*

$$\Phi^k(\operatorname{Fix}\Phi^p \setminus U) \cap \operatorname{Fix}\Phi^p|_{\overline{U}} = \emptyset \quad \textit{for } k < p.$$

Then $i(X,\Phi^p,U) \equiv i(X,\Phi,U) \pmod p$.

PROOF. As in the proof of (52.11) we choose ε such small that for the polyhedron P ε-dominating X with maps $r\colon P \to X$ and $s\colon X \to P$, the map $s \circ \Phi \circ r$ satisfies the assumptions of (51.10.1). We omit the details. □

53. Fixed point index of decompositions for arbitrary ANRs

In this section we extend results from Section 52 to arbitrary ANRs. Let $X \in$ ANR and let $\Phi \in \mathcal{DA}(X,X)$ determine a compact map $\Phi\colon X \multimap X$. Assume further that U is an open subset of X such that $\operatorname{Fix}(\Phi) \cap \partial U = \emptyset$. We can assume, without loss of generality, that X is a closed subset of a normed space E (see (1.6)). Let V be an open neighbourhood of X in E and $r\colon V \to X$ a retraction map. We let

$$\Psi\colon V \multimap V, \quad \Psi = i \circ \Phi \circ r,$$

where $i\colon X \to V$ is the inclusion map. Then $\Psi \in \mathcal{DA}(V,V)$ determine a compact map and $\operatorname{Fix}(\Psi) \cap r^{-1}(U) = \emptyset$ (cf. (52.2)). By using (1.13), we get a compact ANR-space K such that

$$\operatorname{cl}\Psi(V) \subset K \subset V.$$

We let also $W = K \cap r^{-1}(U)$. Then $i(K,\Psi|_K,W)$ is well defined. We let:

$$i(X,\Phi,U) = i(K,\Psi|_K,W). \tag{53.1}$$

It is an easy exercise to see that the above definition does not depend on the possible choices. Moreover, we obtain:

(53.2) PROPOSITION. *The index defined in* (53.1) *satisfies the following properties:*

(53.2.1) *If $U_1, U_2 \subset U$ are open disjoint and if* $\operatorname{Fix}(\Phi|_{\overline{U}}) \subset U_1 \cup U_2$ *then:*

$$i(X,\Phi,U) = (X,\Phi,U_1) + i(X,\Phi,U_2).$$

(53.2.2) *Let $\chi \in \mathcal{DA}(X \times [0,1], X)$ determine a compact homotopy $\chi\colon X \times [0,1] \multimap X$ such that $x \notin \chi(x,t)$ for all $x \in \partial U$ and $t \in [0,1]$. Then $i(X,\chi_0,U) = i(X,\chi_1,U)$.*

(53.2.3) *Let* $W \subset X$ *be an open set and let* $\Phi \in \mathcal{DA}(X,Y)$, $\Psi \in \mathcal{DA}(Y,X)$ *determine compact maps. Assume that* $x \notin \Psi(\Phi(x))$ *for* $x \in \partial W$, $y \notin \Phi(\Psi(y))$ *for* $y \in \partial(\Psi^{-1}(W))$ *and*

$$\psi(\mathrm{Fix}(\Phi \circ \Psi \setminus \Psi^{-1}(W)) \cap \mathrm{Fix}(\Psi \circ \Phi|_{\overline{W}}) = \emptyset.$$

Then $i(X, \Psi \circ \Phi, W) = i(Y, \Phi \circ \Psi, \Psi^{-1}(W))$.

(53.2.4) *Let* $W \subset X$ *be an open subset such that*

$$\Phi^k(\mathrm{Fix}(\Phi^p) \setminus W) \cap \mathrm{Fix}(\Phi^p|_{\overline{W}})) = \emptyset,$$

for $k < p$, p *is a prime number and* $\mathrm{Fix}(\Phi^p) \cap \partial W = \emptyset$. *Then*

$$i(X, \Phi, W) \equiv i(X, \Phi^p, W) \pmod p.$$

(53.2.5) *If* $\Phi \in \mathcal{DA}(X,X)$ *determines a compact map* $\Phi\colon X \multimap X$ *then* Φ_* *is a Leray endomorphism and* $i(X, \Phi, X) = \Lambda(\Phi_*)$.

Proofs of properties (53.2.1)–(53.2.4) are straightforward consequences of (53.1) and the respective properties for compact ANRs. To deduce (53.2.5) it is enough to use, additionally, the commutativity property of the Leray endomorphisms (see Property (11.4)).

(53.3) Remark. We would like to point out that definition (53.1) can be extended to compositions in $\mathcal{DA}(X,X)$ which determine compact absorbing contractions, where $X \in \mathrm{ANR}$. In fact assume $\Phi \in \mathcal{DA}(X,X)$ determines compact absorbing contraction $\Phi\colon X \multimap X$ and V is an open set in X such that $\mathrm{Fix}(\Phi) \cap \partial V = \emptyset$.

According to the definition of compact absorbing contraction maps we can find an open set $U \subset X$ such that:

(i) $\overline{\Phi(U)}$ is a compact subset of U, and
(ii) for every $x \in X$ there is n_x such that $\Phi^{n_x}(x) \subset U$.

So $\Phi_1\colon U \multimap U$, $\Phi_1(x) = \Phi(x)$ is a compact map and, in view of (ii), $\mathrm{Fix}(\Phi) = \mathrm{Fix}(\Phi_1) \cap W = \emptyset$, where $W = U \cap V$. Therefore, we can let:

(53.3.1) $$i(X, \Phi, V) = i(U, \Phi_1, W).$$

Then, it is easy to verify that definition (53.3.1) is correct and (53.2) holds fine in that case.

We shall end this section by applying the above method to the, defined earlier, permissible maps. Let $X \in \mathrm{ANR}$ and let $\Phi\colon X \multimap X$ be a permissible map of compact absorbing contraction. Thus there exists a decomposition $(\Phi_1, \ldots, \Phi_k) \in$

$\mathcal{DA}(X,X)$ which determines a selector of Φ. So, similarly to admissible maps, we can define the fixed point index set $ii(X,\Phi,U)$ of Φ and the Lefschetz set $\mathbf{\Lambda}(\Phi)$ of Φ by putting:

$$ii(X,\Phi,U)=\{i(X,(\Phi_1,\dots,\Phi_k),U)\mid \Phi_k\circ\ldots\circ\Phi_1\subset\Phi\},$$
$$\mathbf{\Lambda}(\Phi)=\{\Lambda((\Phi_1,\dots,\Phi_k)_*)\mid \Phi_k\circ\ldots\circ\Phi_1\subset\Phi\}.$$

It is an easy exercise for the reader to formulate properties of the above sets (see [Dz1-M]). We restrict our considerations to the following two facts which are simple consequences of obtained earlier results.

(53.4) THEOREM (The Lefschetz Fixed Point Theorem). *If $X\in$ ANR and $\Phi\in\mathcal{P}(X,X)$ is a compact absorbing contraction map then the Lefschetz set $\mathbf{\Lambda}(\Phi)$ of Φ is not empty and $\mathbf{\Lambda}(\Phi)\neq\{0\}$ implies that* $\operatorname{Fix}(\Phi)\neq\emptyset$.

(53.5) COROLLARY (The Schauder Fixed Point Theorem). *Let $X\in$ AR and $\Phi\in\mathcal{P}(X,X)$ be a compact absorbing contraction map then* $\operatorname{Fix}(\Phi)\neq\emptyset$.

54. Spheric mappings

Let A be a compact subset of the Euclidean space $\mathbb{R}^{n+1}$, then the set $\mathbb{R}^{n+1}\setminus A$ consists of two different parts, namely:

(54.1) the unbounded component $\mathcal{D}A$ of A, and

(54.2) the bounded part BA of A to be the union of all bounded components of $\mathbb{R}^{n+1}\setminus A$.

Of course $\mathcal{D}A\cap BA=\emptyset$, $\mathcal{D}A\neq\emptyset$, $\mathcal{D}A$ is connected and we have:

$$\mathbb{R}^{n+1}=BA\cup A\cup\mathcal{D}A.$$

Moreover, we let $\widetilde{A}=BA\cup A=\mathbb{R}^{n+1}\setminus\mathcal{D}A$.

The following property is an immediate consequence of the above notations:

(54.3) Let A be a compact subset of X, and X be a compact subset of $\mathbb{R}^{n+1}$. If $\mathbb{R}^{n+1}\setminus X$ is connected, then $\widetilde{A}\subset X$.

(54.4) DEFINITION. A compact subset $A\subset\mathbb{R}^{n+1}$ is called *spheric* provided the set $\widetilde{A}$ is acyclic (with respect to the Čech cohomology functor).

By using the Alexander duality theorem (see Chapter I) we can obtain the following examples of spheric sets.

(54.5) EXAMPLES.

(54.5.1) Any acyclic set A is spheric.

(54.5.2) Any compact connected subset of $\mathbb{R}^2$ is spheric.
(54.5.3) If A has the same Čech cohomology as S^n then A is spheric.

It follows from (54.5.2) that the multivalued map $\varphi: K^2 \multimap K^2$ considered in Chapter II (see (13.5.9)) is continuous and has spheric values but we see that $\mathrm{Fix}(\varphi) = \emptyset$.

So, the Brouwer fixed point theorem is not true for u.s.c. mappings with spherical values. In order to understand the reasons for this we let:

$$\delta(\varphi) = \{x \in K^2 \mid x \in B\varphi(x)\}.$$

Note that for φ as in (13.5.9) we have $\delta(\varphi) = \{0\}$.

In what follows we have assumed that all multivalued mappings are u.s.c. with compact values. Let X be a compact of $\mathbb{R}^n$. Therefore, $X \in \mathrm{AR}$.

(54.6) Definition. Let $\varphi: X \multimap X$ be a multivalued map. We will say that $\varphi \in \delta(X)$ if and only if the set $\delta(\varphi) = \{x \in X \mid x \in B\varphi(x)\}$ is an open subset of X.

Observe that φ considered in (13.5.9) is not an element of $\delta(K^2)$. Note, that if $\varphi: X \multimap X$ is acyclic or there is m such that $\varphi(x)$ consists of one or m-acyclic components then $\varphi \in \delta(X)$.

Now, with a map $\varphi: X \multimap \mathbb{R}^{n+1}$ we shall associate the map $\widetilde{\varphi}: X \multimap \mathbb{R}^{n+1}$ by putting

$$\widetilde{\varphi}(x) = B\varphi(x) \cup \varphi(x) = \mathbb{R}^n \setminus \mathcal{D}\varphi(x) \quad \text{for every } x \in X.$$

(54.7) Remark. Since $X \in \mathrm{AR}$ we obtain: if $\varphi: X \multimap X$ is a multivalued map then $\widetilde{\varphi}(x) \subset X$, for every $x \in X$, i.e. $\widetilde{\varphi}$ can be considered as a map from X to X.

We will prove the following:

(54.8) Proposition. *If $\varphi: X \multimap \mathbb{R}^{n+1}$ then $\widetilde{\varphi}: X \multimap \mathbb{R}^{n+1}$ is u.s.c., in particular if φ is a map from X to X then $\widetilde{\varphi}$ is an u.s.c. map from X to X.*

Proof. Let $x_0 \in X$ and $\varepsilon > 0$ be given. Let $O_\varepsilon(\widetilde{\varphi}(x_0)) = \{y \in \mathbb{R}^n \mid \text{there exists } z \in \widetilde{\varphi}(x_0) \ni \|y - z\| < \varepsilon\}$ be the open ε-neighbourhood of $\widetilde{\varphi}(x_0)$ in $\mathbb{R}^{n+1}$. Since $\widetilde{\varphi}(x_0)$ is compact we can choose $r > 0$ such that $O_\varepsilon(\widetilde{\varphi}(x_0)) \subset B(0, r)$, where $B(0, r)$ denotes the open ball in $\mathbb{R}^{n+1}$ with center 0 and the radius r. Let us consider the compact set

$$K = (\mathbb{R}^{n+1} \setminus O_\varepsilon(\widetilde{\varphi}(x_0))) \setminus (\mathbb{R}^{n+1} \setminus \overline{B(0, r)}).$$

We can cover K by a finite number of open balls $B(a_i, \varepsilon/2)$ in $\mathbb{R}^{n+1}$, $i = 1, \ldots, k$. Consequently $B(a_1, \varepsilon/2), \ldots, B(a_k, \varepsilon/2)$, $\mathbb{R}^{n+1} \setminus \overline{B(0, r)}$ is a covering of the set $\mathbb{R}^{n+1} \setminus O_\varepsilon(\widetilde{\varphi}(x_0))$.

We choose an arbitrary point $a \in \mathbb{R}^{n+1} \setminus \overline{B(0,r)}$. By joining it with all points $a_1, \ldots, a_k$ we get a continuum C such that.

$$C \cup (\mathbb{R}^{n+1} \setminus O_\varepsilon(\widetilde{\varphi}(x_0)) \subset \mathbb{R}^{n+1} \setminus \widetilde{\varphi}(x_0).$$

We let $\delta = \min\{\varepsilon/2, \operatorname{dist}(C, \widetilde{\varphi}(x_0))\} = \inf\{\|u - v\|,\ u \in C \wedge v \in \widetilde{\varphi}(x_0)\}$. Since φ is u.s.c. there exists an open neighbourhood U_{x_0} of x_0 in X such that $\varphi(x) \subset O_\delta(\varphi(x_0))$ for every $x \in U_{x_0}$.

Then we have $\mathbb{R}^{n+1} \setminus O_\varepsilon(\widetilde{\varphi}(x_0)) \subset \mathbb{R}^{n+1} \setminus O_\delta(\widetilde{\varphi}(x_0))$. But $\mathbb{R}^{n+1} \setminus O_\delta(\widetilde{\varphi}(x_0)) \subset \mathbb{R}^{n+1} \setminus \varphi(x)$ for every $x \in U_{x_0}$. Consequently we obtain:

$$\widetilde{\varphi}(x) = \mathbb{R}^{n+1} \setminus D\varphi(x) \subset O_\delta(\widetilde{\varphi}(x_0))$$

and the proof is completed. □

We let $\mathcal{F}(X) = \{\varphi\colon X \multimap X \mid \varphi \text{ is u.s.c. with compact values and } \operatorname{Fix}(\varphi) \neq \emptyset\}$. We have seen in the last two chapters, that the class $\mathcal{F}(X)$ is quite rich.

(54.9) Proposition. *A multivalued map $\varphi\colon X \multimap X$ is called spherical (written $\varphi \in S(X)$) provided $\widetilde{\varphi} \in \mathcal{F}(X)$.*

(54.10) Theorem. *$S(X) \subset \mathcal{F}(X)$.*

Proof. There are two possible cases:

(i) the boundary ∂X of X in $\mathbb{R}^{n+1}$ is equal to X,
(ii) $\partial X \neq X$.

In the case (ii) we have $\widetilde{\varphi} = \varphi$ so our claim holds true because $\widetilde{\varphi} \in \mathcal{F}(X)$.

Now, let assume on the contrary that $\operatorname{Fix}\varphi = \emptyset$. It means that the set $\delta(\varphi) = \{x \in X \mid x \in B\varphi(x)\} \neq \emptyset$ and hence $\delta(\varphi)$ is open because $\varphi \in \delta(X)$. On the other hand $\delta(\varphi) = \operatorname{Fix}\widetilde{\varphi}$ so it is also a closed subset of X. To obtain contradiction it is sufficient to observe that $X \setminus \delta(\varphi) \neq \emptyset$. Indeed, we have $\partial X \neq \emptyset$ (X is compact!) but if $x \in \partial X$, then $x \notin B\varphi(x)$ and consequently $x \notin \delta(\varphi)$. The proof is completed. □

Now we will look carefully at the class $\delta(X)$ of multivalued mappings.

We shall show that the set $\delta(\varphi) = \{x \in X \mid x \in B\varphi(x)\}$ is open in terms of the Borsuk continuity of the map φ (see Section 20).

(54.11) Theorem. *If $\varphi\colon X \multimap X$ is Borsuk continuous then $\varphi \in \delta(X)$.*

Proof. For the proof we consider the multivalued map $B\varphi\colon X \multimap X$ defined as $(B\varphi)(x) = B(\varphi(x))$. It is sufficient to show that, if φ is Borsuk continuous then the graph $\Gamma_{B\varphi} = \{(x,a) \in X \times X \mid a \in B\varphi(x)\}$ is an open subset of $X \times X$.

Let $(x, a) \in \Gamma_{B\varphi}$. We take two real numbers $\alpha, \beta > 0$ such that $\alpha + \beta = \operatorname{dist}(a, \varphi(x))$. Evidently,

(54.11.1) $$B(a,\alpha) \cap O_\beta(\varphi(x)) = \emptyset.$$

Since φ is Borsuk continuous there is an open neighbourhood V of x in X such that $d_C(\varphi(x), \varphi(y)) < \beta$ for every $y \in V$. Therefore,

(54.11.2) $$\varphi(y) \subset O_\delta(\varphi(x)) \quad \text{for every } y \in V.$$

We will prove that $V \times B(a,\alpha)$ is an open neighbourhood of (x, a) in $\Gamma_{B\varphi}$. Assume to the contrary that there exist $z \in V$ and $b \in B(a,\alpha)$ such that $b \in B\varphi(z)$.

From (54.11.1) and (54.11.2) we infer that $b \notin \varphi(z)$ and hence $b \in \mathcal{D}\varphi(z)$. Obviously, $b \in B\varphi(x)$. It follows that

$$d_C(\varphi(x), \varphi(y)) \geq \inf\{|f| \mid f \in C(\varphi(x), \varphi(y))\} \geq B,$$

which is a contradiction. □

From (54.11), (54.10) and (54.5.3) we infer:

(54.12) THEOREM. *Let $X \subset \mathbb{R}^2$ and $\varphi\colon X \multimap X$ be an u.s.c. map with compact connected values. If φ is Borsuk continuous then* $\operatorname{Fix}(\varphi) \neq \emptyset$.

To finish this section we would like to add that some topological invariants, e.g. the topological degree, the Lefschetz number, the fixed point index etc, can be defined for spherical type of mappings (see [Da-M]).

Note, that in Definition (54.6) instead of assuming that $\delta(\varphi)$ is open we can assume that the graph $\Gamma_{B\varphi}$ of the map $B\varphi\colon X \multimap X$ is open in $X \times X$ (cf. [Da-M]). Throughout this paper we have assumed that $X \in \mathrm{AR}$. Let us remark that most of the results hold true for acyclic approximative neighbourhood retracts (see Chapter I).

Finally, let us formulate two open problems:

(54.13) OPEN PROBLEM. Does the Brouwer fixed point theorem hold true for compositions of spheric mappings?

(54.14) OPEN PROBLEM. Does Theorem (54.12) hold true for $X \subset \mathbb{R}^{n+1}$, $n > 2$?

To re-formulate all the results of this section for an arbitrary Banach space E, would constitute another problem.

It may be done but some essential changes are necessary, namely:

(54.15) compact sets have to be replaced by closed and bounded subsets of E;

(54.16) mostly acyclic sets have to be replaced by convex sets;

(54.17) instead of the Čech cohomology functor we have to consider the so called Gęba–Granas cohomology functor $H^{\infty-n}$ (see [Da-M]).

(54.18) in the definition of the Borsuk metric of continuity one has to consider compact vector fields instead of continuous functions.

For more information on the infinite dimensional case see [Da-M]. More about spheric mappings we shall present in Section 81.

CHAPTER V

CONSEQUENCES AND APPLICATIONS

Results obtained in Chapter III and Chapter IV have many applications in fixed point theory, nonlinear analysis, games theory, mathematical economics, and other fields.

Applications to nonlinear analysis (or more precisely to differential inclusions) will be presented in Chapter VI. This chapter is devoted to applications in the others of the aforementioned fields. In Chapters III and IV we considered several different classes of multivalued mappings; it is possible to generalize the results for all of those classes of mappings.

In this section we shall deal with the so called class of admissible mappings or, simply, mappings determined by morphisms. Formulations of the respective results for other classes of mappings will be left to the reader.

55. Birkhoff–Kellogg, Rothe and Poincaré theorems

We shall keep notations used in previous chapters. First we will formulate the finite dimensional version of the Birkhoff–Kellogg theorem.

(55.1) THEOREM. *If $\varphi\colon S^{2n} \multimap P^{2n+1}$ is a compact, $(2n)$-admissible map, then there exists $x_0 \in S^{2n}$ and a real number $\lambda_0 \neq 0$, such that $\lambda_0 x_0 \in \varphi(x_0)$.*

PROOF. Assume to the contrary: for each $x \in S^{2n}$ and for each $\lambda \neq 0$, $\lambda x \notin \varphi(x)$. Let $(p,q) \subset \varphi$. We have the diagram

$$S^{2n} \xleftarrow{\ p\ } \Gamma \xrightarrow{\ q\ } P^{2n+1}$$

and by assumption $\lambda x \notin (q\varphi_p)(x)$ for each $x \in S^{2n}$ and $\lambda \neq 0$. Let $\overline{p} = i \cdot p$, where $i\colon S^{2n} \to P^{2n+1}$ is the inclusion mapping. Define a homotopy $h\colon \Gamma \times [0,1] \to P^{2n+1}$ by putting:

$$h(y,t) = t \cdot q(y) + (1-t)\overline{p}(y).$$

From assumptions immediately follows that $h(\Gamma \times [0,1]) \subset P^{2n+1}$, so $q^{*2n} = \overline{p}^{*2n}$, and consequently

$$(p^{*2n})^{-1} q^{*2n} = (p^{*2n})^{-1} p^{*2n} i^{*2n} = i^{*2n}. \qquad (55.1.1)$$

Let $\widetilde{p} = j \circ p$, where $j: S^{2n} \to P^{2n+1}$ is a map given as follows $j(x) = -x$, for each $x \in S^{2n}$. Let $f: \Gamma \times [0,1] \to P^{2n+1}$ be a homotopy defined by

$$f(y,t) = tq(y) + (1-t)\widetilde{p}(y).$$

It is easy to see that $f(\Gamma \times [0,1]) \subset P^{2n+1}$, so $q^{*2n} = \widetilde{p}^{*2n}$ and consequently

$$(p^{*2n})^{-1} q^{*2n} = (p^{*2n})^{-1} p^{*2n} j^{*2n} = j^{*2n}. \tag{55.1.2}$$

From (55.1.1) and (55.1.2) we have $i^{*2n} = j^{*2n}$. On the other hand $j^{*2n} = (-1)^{2n+1} i^{*2n}$ and i^{*2n} is an isomorphism. Therefore, $j^{*2n} \neq i^{*2n}$ and we obtain a contradiction and the proof is completed. □

Now we are able to formulate the general version of the Birkhoff–Kellogg theorem.

(55.2) THEOREM (Birkhoff–Kellogg). *Let E be an infinite dimensional normed space. If $\varphi: S \multimap P$ is a compact, n-admissible map (for some n), then there exists $x_0 \notin S$ and a real number $\lambda_0 \neq 0$, such that $\lambda_0 x_0 \in \varphi(x_0)$.*

PROOF. By assumption $\overline{\varphi(S)}$ is a compact subset of P, so the distance

$$\operatorname{dist}(0, \overline{\varphi(S)}) = \inf_{x \in \overline{\varphi(S)}} \|x\| = \delta > 0.$$

We choose a natural number m_0 such that $1/m < \delta/2$, for each $m \geq m_0$. Let $(p,q) \subset \varphi$, where $p: \Gamma \to S$ and $q: \Gamma \to P$. Applying the Schauder Approximation Theorem to the compact map q and for $\varepsilon = 1/m$, $m \geq m_0$, we obtain a map $q_m: \Gamma \to P$ such that:

(55.2.1) $\|q_m(y) - q(y)\| < 1/m$, for all $y \in \Gamma$, and
(55.2.2) $q_m(\Gamma) \subset P^{k(m)} \subset E$, $P^{k(m)} = E^{k(m)} \setminus \{0\}$.

The condition (55.2.2) follows automatically from the inequality: $1/m < \delta/2$. We can assume that $k(m) \geq n$ and $k(m)$ is odd for each $m \geq m_0$. Now, for each $m \geq m_0$ we define a map $\varphi_m: S^{k(m)-1} \multimap P^{k(m)}$ by putting

$$\varphi_m(x) = q_m(\varphi_p(x)), \quad \text{for each } x \in S^{k(m)-1}.$$

Because q_m is compact, so φ_m is compact, too. Moreover, the pair

$$S^{k(m)-1} \xleftarrow{p} p^{-1}(S^{k(m)-1}) \xrightarrow{q_m} P^{k(m)},$$

is a selected pair for φ_m, so φ_m is an n-admissible map. Applying Theorem (55.1) to φ_m we obtain a point $x_m \in S$ and a real number $\lambda_m \neq 0$ such that

$$\lambda_m x_m \in \varphi_m(x_m), \quad \text{for every } m \geq m_0.$$

Let y_m be a point in Γ such that

$$q_m(y_m) = \lambda_m x_m \quad \text{and} \quad p(y_m) = x_m, \quad \text{for every } m \geq m_0.$$

Let $z_m = q(y_m)$. Then the sequence $\{z_m\}$ is contained in the compact set $\overline{\varphi(S)}$. Therefore, we can assume, without loss of generality, that $\lim z_m = z_0$. Consequently from the inequality

$$\|q_m(y_m) - q(y_m)\| < \frac{1}{m}$$

it follows that $\lim q_m(y_m) = z_0$. Because $\|x_m\| = 1$, the sequence $\{\lambda_m\}$ is bounded. Therefore we can assume without loss of generality, that $\lim \lambda_m = \lambda_0$ and from $\delta/2 \geq \|\lambda_m x_m\| = |\lambda_m|$ we deduce that $\lambda_0 \neq 0$. It implies that $\lim x_m = x_0$ and $x_0 \in S$. Now we have:

(55.2.3) $z_m \in (q\varphi_p)(x_m)$, and
(55.2.4) $\lim z_m = z_0$, $\lim x_m = x_0$,

so, from (2.1)–(2.3) it follows that $z_0 \in (q \circ \varphi_p)(x_0) \subset \varphi(x_0)$. Because $z_0 = \lambda_0 x_0$ we found a point $x_0 \in S$ and a real number $\lambda_0 \neq 0$ such that $\lambda_0 x_0 \in \varphi(x_0)$. The proof of Theorem (55.2) is completed. □

(55.3) Remark. In fact, we have shown that for any two maps $p\colon Y \to S$ and $q\colon Y \to P$ such that:

(55.3.1) p is a Vietoris n-map,
(55.3.2) q is a compact map,

there exists $y_0 \in Y$ and $\lambda_0 \neq 0$ such that $\lambda_0 p(y_0) = q(y_0)$. The same remark is true in the finite dimensional case (cf. (55.1)).

(55.4) Remark. Moreover, let us observe that the formulation of the Birkhoff–Kellogg theorem in terms of n-admissible maps is equivalent to the formulation in terms of such pairs of maps p, q for which the above conditions (55.3.1) and (55.3.2) are satisfied.

Let us formulate a generalization of the Rothe fixed point theorem. Let K be the unit ball in a normed space E and $S = \partial K$ be the unit sphere in E.

(55.5) Theorem (Rothe). *Let $\Phi\colon K \multimap E$ be an n-admissible compact map such that $\Phi(S) \subset K$. Then* $\mathrm{Fix}(\Phi) \neq \emptyset$.

Proof. Let $\varphi\colon K \multimap E$ be an n-admissible compact vector field given by $\varphi = I - \Phi$. We may assume without loss of generality, that $\varphi(S) \subset P$. It suffices to prove that $\mathrm{Deg}(\varphi; 0) \neq \{0\}$. For this purpose let

$$\psi(x,t) = x - t\Phi(x) \quad \text{for an arbitrary } x \in S,\ 0 \leq t \leq 1.$$

It follows from our assumption that for an arbitrary $z \in \psi(x,t)$ we have

$$\|z\| = \|x - ty\| \geq \|x\| - t\|y\| > 0 \quad \text{for } 0 \leq t \leq 1$$

and thus $\psi: S \times [0,1] \multimap P$. It is evident that $\psi(S \times [0,1])$ is a closed subset of E and hence $d(0, \psi(S \times [0,1])) = \delta > 0$.

Let $(p,q) \subset \Phi$ be a selected pair of the form $K \xleftarrow{p} Y \xrightarrow{q} E$ and let the pair (p_1, q_1) of the form $S \xleftarrow{p_1} p^{-1}(S) \xrightarrow{q_1} E$ be associated with (p,q) (cf. the definition of $\operatorname{Deg}(\varphi;0)$). Let $q_\varepsilon: p^{-1}(S) \to E^{k+1}$ be an ε-approximation of q_1, where $0 < \varepsilon < \delta$. We put $S^k = S \cap E^{k+1}$ and $Y_k = p^{-1}(S^k)$. We have the diagram

$$S^k \xleftarrow{p_k} Y_k \xrightarrow{q_k} E^{k+1},$$

in which p_k, q_k are restrictions of p_1 and q_ε, respectively.

Define the map $\widetilde{q}_k: Y_k \to P^{k+1}$ by putting $\widetilde{q}_k(y) = p_k(y) - q_k(y)$ for each $y \in Y_k$. We claim that $\deg(p_1, q_1) = \deg(p_k, \widetilde{q}_k) \neq 0$. In this order, consider the map $f: Y_k \to P^{k+1}$ given by $f(y) = p_k(y)$ and a homotopy $h: Y_k \times [0,1] \to P^{k+1}$ given by $h(y,t) = p_k(y) - tq_k(y)$. Since $\varphi(S) \subset P$ and q_ε is an ε-approximation of q_1, $0 < \varepsilon < \delta$, we deduce that $h(Y_k \times [0,1]) \subset P^{k+1}$. Then the maps f and $\widetilde{q}_k$ are homotopic and hence $f^* = \widetilde{q}_k^*$. Finally, we obtain

$$\deg(p_1, q_1) = \deg(p_k, \widetilde{q}_k) = \deg(p_k, f) \neq 0,$$

and the proof is completed. □

Now we shall apply the Rothe result to obtain the Leray–Schauder alternative. Let $\varphi: E \multimap E$ be an n-admissible map which is completely continuous, i.e. the restriction $\varphi|_B$ of φ to any bounded set $B \subset E$ is a compact map. We let:

$$L(\varphi) = \{x \in E \mid x \in \lambda \cdot \varphi(x) \text{ for some } 0 < \lambda < 1\}.$$

(55.6) THEOREM (Leray–Schauder Alternative). *Under the above assumptions we have:*

either $L(\varphi)$ *is unbounded or* $\operatorname{Fix}(\varphi) \neq \emptyset$.

PROOF. Assume that $L(\varphi)$ is bounded and let $K(0,r)$ be a closed ball in E containing $L(\varphi)$ in its interior. The $\widetilde{\varphi} = \varphi|_{K(0,r)}: K(0,r) \multimap E$ is a compact admissible map such that $\varphi(\partial K(0,r)) \subset K(0,r)$. So (54.6) follows from (54.5). □

(55.7) THEOREM (Poincaré type of Coincidence Theorem). *Let $\varphi: K \multimap E$ be an admissible compact vector field and $\Psi: K \multimap E$ be a compact admissible map. Assume further that:*

(55.7.1) $0 \notin \operatorname{Deg}(\varphi)$, (55.7.2) $\lambda \cdot \varphi(x) \cap \psi(x) = \emptyset$ *for every* $\lambda > 1$ *and* $x \in S$, *where* ψ *is the field associated with* Ψ.

Then there exists $x \in K$ *such that* $\varphi(x) \cap \psi(x) \neq \emptyset$.

PROOF. We consider the map $(\varphi - \psi): K \multimap E$ given by $(\varphi - \psi)(x) = \varphi(x) - \psi(x) = \{y - u \mid y \in \varphi(x),\ u \in \psi(x)\}$. We may assume, without loss of generality, that $0 \notin (\varphi - \psi)(x)$ for every $x \in S$. Of course $(\varphi - \psi)$ is an admissible map. We have to prove that $0 \in (\varphi - \psi)(K)$. In order to do that we consider:

$$\chi: K \times [0,1] \multimap E, \quad \chi(x,t) = x - (\Phi(x) + t\Psi(x)),$$

where Φ is a compact part of φ. We will prove that $\chi(S \times [0,1]) \subset P$. For $t = 0$ and $t = 1$ it is evident. Assume that for some $0 < t_0 < 1$ there exists $x_0 \in S$ such that $0 \in \chi(x_0, t_0)$. Then we have $(1/t_0)\varphi(x_0) \cap \psi(x_0) \neq \emptyset$ and, since $1/t_0 > 1$, from (55.7.2) we get a contradiction. Consequently, in view of (55.7.1) and a homotopy property of the topological degree we infer that $\mathrm{Deg}(\chi(\,\cdot\,, 1)) \neq 0$ so, our claim holds true and the proof is completed. □

56. Fixed point property and families of multivalued mappings

For a space X we denote by $\mathrm{Cov}_f(X)$ the direct set of all finite open coverings of X. Let $\varphi: X \multimap X$ be a multivalued map and $\alpha \in \mathrm{Cov}_f(X)$. A point $x \in X$ is said to be an α-fixed point for φ provided that there exists a member $U \in \alpha$ such that $x \in U$ and $\varphi(x) \cap U \neq \emptyset$. Moreover, if $\alpha, \beta \in \mathrm{Cov}(X)$ and α refines β, then every α-fixed point for φ is also β-fixed point for φ.

Let X be a compact space. We will say that X has *fixed point property* with respect to admissible maps provided that any admissible map $\varphi: X \multimap X$ has a fixed point.

(56.1) LEMMA. *Let* $\varphi: X \multimap X$ *be an u.s.c. map. Assume that there exists a cofinal family of coverings* $\mathcal{D} \subset \mathrm{Cov}(X)$ *such that* φ *has an* α*-fixed point for every* $\alpha \in \mathcal{D}$. *Then* φ *has a fixed point.*

PROOF. Suppose that φ has no fixed points. Then for each $x \in X$ there are open neighbourhoods V_x and $U_{\varphi(x)}$ of x and $\varphi(x)$, respectively, such that $V_x \cap U_{\varphi(x)} = \emptyset$. From the u.s.c. of φ we deduce that the set $V = \varphi^{-1}(U_{\varphi(x)})$ is an open neighbourhood of x in X. Let $W_x = V_x \cap V$; then we have

(56.1.1) $\varphi(W_x) \subset U_{\varphi(x)}$, and
(56.1.2) $W_x \cap U_{\varphi(x)} = \emptyset$.

Since X is a compact space, we infer that there exists a finite number of sets $W_{x_1}, \ldots, W_{x_n}$ such that $X = \bigcup_{i=1}^{n} W_{x_i}$. Putting $\beta = \{W_{x_1}, \ldots, W_{x_n}\}$, we get a covering of X such that φ has no β-fixed point. If α is a member of $\mathcal{D}$ that refines β then φ has no α-fixed point, and thus we obtain a contradiction. □

Let $\{X_i\}_{i\in I}$ be a family of compact spaces indexed by an infinite set I and let $X = \bigtimes_{i\in I} X_i$ be their topological product. Denote by $\mathcal{J} = \{J\}$ the family of all finite subsets of I; given $J \in \mathcal{J}$, we put $X_J = \bigtimes_{i\in J} X_i$.

(56.2) THEOREM. *An infinite product $X = \bigtimes_{i\in I} X_i$ of compact spaces has the fixed point property within the class of admissible maps if and only if every finite product $X_J = \bigtimes_{i\in J} X_i$ ($J \in \mathcal{J}$) has the fixed point property within the class of admissible maps.*

PROOF. Choose in each X_i a point x_i^0 and define $\widetilde{X}_J \subset X$ as follows:

$$\{x_i\} \in \widetilde{X}_J \Leftrightarrow \begin{cases} x_i \in X_i & \text{for } i \in J, \\ x_i = x_i^0 & \text{for } i \notin J. \end{cases}$$

Clearly, we may identify $\widetilde{X}_J$ with X_J. Next we define a subset $\mathcal{D} = \{\alpha\} \subset \operatorname{Cov}(X)$ as follows: $\alpha \in \mathcal{D}$ provided α is a finite covering consisting of open sets of the form $U_J = \bigtimes_{i\in J} U_i$ with U_i open in X_i and $U_i = X_i$ for all $i \notin J$. By the theorem of Tychonoff and taking into account the definition of the product topology, we conclude that $\mathcal{D}$ is cofinal in $\operatorname{Cov}(X)$. Let $\alpha \in \mathcal{D}$; it follows from the definition of the set $\mathcal{D}$ that α determines a finite set of essential indices $J(\alpha)$. Take $r_\alpha: X \to \widetilde{X}_{J(\alpha)}$ to be the projection and $s_\alpha: \widetilde{X}_{J(\alpha)} \to X$ the inclusion.

Assume that every finite product $X_J = \bigtimes_{i\in J} X_i$ has the fixed point property within the class of admissible maps. Let $\varphi: X \multimap X$ be an admissible map. We prove that φ has a fixed point. Let $p, q: Y \to X$ be a selected pair of φ. Consider the map $\psi: X \multimap X$ given by $\psi = q \circ \varphi_p$. Then ψ is a u.s.c. admissible maps. For each $\alpha \in \mathcal{D}$, consider the map $\psi_\alpha: \widetilde{X}_{J(\alpha)} \multimap \widetilde{X}_{J(\alpha)}$ given by $\psi_\alpha = r_\alpha \psi s_\alpha$. Then ψ_α is a u.s.c., admissible map for each $\alpha \in \mathcal{D}$. By the assumption, there exists a point $x^\alpha \in \widetilde{X}_{J(\alpha)}$ such that

$$x^\alpha \in \psi_\alpha(x^\alpha) = r_\alpha \psi s_\alpha(x^\alpha) = r_\alpha \psi(x^\alpha), \quad \text{for each } \alpha \in \mathcal{D}. \tag{56.2.1}$$

Let U be a member of α such that $x^\alpha \in U$. Then from (56.2.1) we deduce that $\psi(x^\alpha) \cap U \neq \emptyset$. This implies that x^α is an α-fixed point of ψ, and hence from (56.1) we infer that ψ has a fixed point. Finally, since $\psi(x) \subset \varphi(x)$ for each $x \in X$, we conclude that φ has a fixed point.

Conversely, assume that X has the fixed point property within the class of admissible maps and that there exists a finite set $J \in \mathcal{J}$ such that X_J has not the fixed point property within the class of admissible maps. We may assume, without loss of generality, that there is an admissible map $\psi: \widetilde{X}_J \multimap \widetilde{X}_J$ such that

$x \notin \psi(x)$, for each $x \in \widetilde{X}_J$. Let $r_J: X \to \widetilde{X}_J$ be the projection and $s_J: \widetilde{X}_J \to X$ the inclusion. Then we have the admissible map $\varphi: X \to X$ given by $\varphi = s_J\psi r_J$. By assumption there exists a point $x \in X$ such that

$$x \in \varphi(x) = s_J\psi r_J(x).$$

This implies that $r_J(x) \in r_J s_J \psi(r_J(x))$ and thus we obtain a contradiction. The proof of (56.2) is completed. □

As an immediate consequence we obtain the following two corollaries:

(56.3) COROLLARY. *An arbitrary Tychonoff cube has the fixed point property within the class of admissible maps.*

(56.4) COROLLARY. *Every retract of a Tychonoff cube has the fixed point property within the class of admissible maps.*

In what follows by K we will denote the compact approximative retract and by

$$\chi(K) = \lambda(\mathrm{id}_K)$$

its Euler characteristic. A multivalued semi-flow on K is a strongly admissible map $\varphi: K \times [0, +\infty) \multimap K$ such that:

(i) $\varphi(\,\cdot\,, 0): K \multimap K$ is an acyclic map,
(ii) $x \in \varphi(x, 0)$ for every $x \in K$,
(iii) $\varphi(\varphi(x,t), \tau) = \bigcup_{y \in \varphi(x,t)} \varphi(y, \tau) \subset \varphi(x, t+\tau)$ for every $x \in K$, $t, \tau \in \mathbb{R}_+ = [0, +\infty)$.

A fixed point for the multivalued semi-flow is a point $x \in K$ such that $x \in \varphi(x,t)$ for all $t \in \mathbb{R}_+$.

(56.5) THEOREM. *If $\chi(K) \neq 0$, then any multivalued semi-flow on K must have a fixed point.*

PROOF. Let $\varphi_t = \varphi(\,\cdot\,, t)$. Consider the homotopy $(x, \rho) \to \varphi(x, (1-\rho)t_0)$. Then we see that φ_0 is homotopic to φ_{t_0}. Since, in view of (i), φ_0 is acyclic we obtain

$$\lambda(\varphi_0) = \chi(K) \neq 0.$$

Moreover, since $\varphi_0 \sim \varphi_{t_0}$ we infer $\lambda(\varphi_0) \in \boldsymbol{\lambda}(\varphi_{t_0})$ for every $t_0 \in \mathbb{R}_+$.

Consequently $\boldsymbol{\lambda}(\varphi_{t_0}) \neq \{0\}$. Now, in view of the Lefschetz fixed point theorem, we conclude: $\mathrm{Fix}(\varphi_t) \neq \emptyset$ for every $t \in \mathbb{R}_+$.

Let $A_n = \{x \in K \mid x \in \varphi(x, 2^{-n})\}$; each A_n is nonempty and compact. Moreover, from (iii) we obtain:

$$\varphi\left(x, \frac{1}{2^n}\right) = \varphi\left(x, \frac{1}{2^{n+1}} + \frac{1}{2^{n+1}}\right) \supset \varphi\left(\varphi\left(x, \frac{1}{2^{n+1}}\right), \frac{1}{2^{n+1}}\right)$$

so, it implies that $A_{n+1} \subset A_n$. Therefore, $\bigcap_n A_n \neq \emptyset$. Let $x_0 \in \bigcap_n A_n$. Since $x_0 \in \varphi(x_0, 2^{-n})$ for every $n = 1, 2, 3, \ldots$, it implies (cf. (iii)) that $x_0 \in \varphi(x_0, m \cdot 2^{-n})$ for every $n = 1, 2, \ldots$ and $m = 1, 2, \ldots$.

Because the set $\{n \cdot 2^{-n}\}$ of dyadic rationals is dense in $\mathbb{R}_+$, upper semicontinuity of φ assures that $x_0 \in \varphi(x_0, t)$ for every $t \in \mathbb{R}_+$ and the proof is completed. □

(56.6) REMARK. Observe that if in the definition of multivalued semi-flow we replace the condition (i) by the following:

$$\text{(i}') \qquad 0 \notin \boldsymbol{\lambda}(\varphi_0),$$

then (56.5) holds true.

By J^n we will denote the n-th Cartesian product of $[0, 1]$.

(56.7) THEOREM. *Let $\varphi: K \times J^n \multimap K$ be a strongly admissible map and $K \in$ ANR. Assume that $0 \notin \boldsymbol{\lambda}(\varphi_0)$, where $\varphi_0 = \varphi(\,\cdot\,, (0, \ldots, 0))$. Then there exists $x \in K$ such that:*

$$\dim\{t \in J^n \mid x \in \varphi(x, t)\} \geq n - k.$$

For the proof of (56.7) we need some additional notions and facts. Let Y, X be two compact spaces and $p: Y \to X$ be a map; p is called universal, if for any map $g: Y \to X$ there exists a coincidence point, i.e. there is $y \in Y$ such that $p(y) = q(y)$.

We have proved in (12.11) that if X is a compact AR-space then any Vietoris map $p: Y \to X$, for arbitrary compact Y, is universal.

Note the following well known fact (see [Go-4]):

$$\text{(56.8)} \qquad \text{if } p: Y \to J^n \text{ is universal then } \dim Y \geq n.$$

PROOF OF (56.7). Let $A = \{(x, t) \in K \times J^n \mid x \in \varphi(x, t)\}$. Since $0 \notin \boldsymbol{\lambda}(\varphi_0)$, in view of the Lefschetz fixed point theorem, the set A is nonempty. Moreover, A is compact (φ is u.s.c.). Let $g: A \to J^n$ be defined as $g(x, t) = t$. We claim that g is universal. Indeed, let $h: A \to J^n$ be a map. By using the Dugundji extension theorem let $\widetilde{h}: K \times J^n \to J^n$ be an extension of h. The map $\psi: K \times J^n \multimap K \times J^n$ defined by putting:

$$\psi(x, t) = \{(y, s) \in K \times J^n \mid y \in \varphi(x, t),\ s = \widetilde{h}(x, t)\}$$

is clearly strongly admissible.

Now, by using the homotopy argument we deduce that $\boldsymbol{\lambda}(\psi) \neq \{0\}$. Since $(K \times J^n) \in$ ANR we obtain $\mathrm{Fix}(\psi) \neq \emptyset$.

If $(x, t) \in \psi(x, t)$ then $x \in \varphi(x, t)$ and $t = \widetilde{h}(x, t) = h(x, t) = g(x)$. So g is universal and hence from (56.8) we obtain $\dim A \geq n$.

Let $f\colon K \times J^n \to K$ be defined as $f(x,t) = x$. Thus by the generalized Hurewicz theorem (see [HW-M]) relating maps and dimension we obtain:

$$\dim f^{-1}(x) \geq n - \dim K \quad \text{for some } x \in K.$$

Such a point x satisfies our hypothesis and the proof is completed. □

57. The Lefschetz fixed point theorem for pairs of spaces

In this section we will consider morphisms

$$\varphi = \{(X,A) \overset{p}{\Longleftarrow} (\Gamma,\Gamma_0) \overset{q}{\longrightarrow} (X,A)\}\colon (X,A) \to (X,A)$$

and determined by them multivalued mappings. We shall use the same notation φ for a morphism and its determined mapping.

In what follows we assume also that $X, A \in \mathrm{ANR}$ and φ determines a compact map. Let

$$\varphi = \{(X,A) \overset{p}{\Longleftarrow} (\Gamma,\Gamma_0) \overset{q}{\longrightarrow} (X,A)\}\colon (X,A) \to (X,A)$$

be such a morphism. We will denote by

$$\varphi_X = \{X \overset{\widetilde{p}}{\Longleftarrow} \Gamma \overset{\widetilde{q}}{\longrightarrow} X\}\colon X \to X \quad \text{and} \quad \varphi_A = \{A \overset{\overline{p}}{\Longleftarrow} \Gamma_0 \overset{\overline{q}}{\longrightarrow} A\}\colon A \to A$$

the morphisms induced by φ.

In view of (11.5) and the normalization property of the fixed point index we obtain:

(57.1) PROPOSITION. *The generalized Lefschetz number $\Lambda(\varphi) = \Lambda(\varphi^*)$ of φ is well defined and*

$$\Lambda(\varphi) = \Lambda(\varphi_X) - \Lambda(\varphi_A).$$

Assume $\varphi\colon (X,A) \multimap (X,A)$ is a compact multivalued map determined by a morphism φ such that $\Lambda(\varphi) \neq 0$ ($X, A \in \mathrm{ANR}$). We would like to get some information about fixed points of φ in $X \setminus A$.

We start with an example:

(57.2) EXAMPLE. Let $f\colon ([0,1],\{0,1\}) \to ([0,1],\{0,1\})$ be a map defined as follows:

$$f(x) = x^2, \quad \text{for every } x \in [0,1].$$

Let $f_1 = f|_{[0,1]}$ and $f_2 = f|_{\{0,1\}}$. We have $\lambda(f) = \lambda(f_1) - \lambda(f_2) = 1 - 2 = -1 \neq 0$ but f has no fixed points in $[0,1] \setminus \{0,1\}$.

We prove the following:

(57.3) THEOREM. *Let $X, A \in$ ANR, $A \subset X$ and let $\varphi\colon (X,A) \multimap (X,A)$ be a compact multivalued map (i.e. φ_X and φ_A are compact) determined by a morphism φ such that $\Lambda(\varphi) \neq 0$. Then φ has a fixed point in* $\mathrm{cl}\,(X \setminus A)$.

PROOF. Assume to the contrary that $\mathrm{Fix}(\varphi) \cap \mathrm{cl}\,(X \setminus A) = \emptyset$. It implies that $\mathrm{Fix}(\varphi) \subset \mathrm{Int}_X A$. Let $U = \mathrm{Int}_X A$. Then U is an open set in X such that $U \subset A$ and $\mathrm{Fix}(\varphi_X) \subset U$. Therefore, from the additivity and contraction properties of the index we obtain:

$$\mathrm{Ind}(X, \varphi_X, U) = \mathrm{Ind}(A, \varphi_A, U). \tag{57.3.1}$$

But from the normalization property of the index we obtain:

$$\Lambda(\varphi_X) = \mathrm{Ind}(X, \varphi_X, U), \tag{57.3.2}$$

$$\Lambda(\varphi_A) = \mathrm{Ind}(A, \varphi_A, U). \tag{57.3.3}$$

So, in view of (57.1) we obtain $\Lambda(\varphi) = \Lambda(\varphi_X) - \Lambda(\varphi_A) = 0$, a contradiction and the proof is completed. □

As an application of (57.3) we obtain Krasnosiel'skiĭ's theorem. Let X be a closed cone in a normed space. Given two real numbers r_0, r such that $0 < r_0 < r$ we let:

$$\begin{aligned} S_{r_0} &= \{x \in X \mid \|x\| = r_0\}, & K^-_{r_0} &= \{x \in X \mid \|x\| \leq r_0\}, \\ A_{r_0,r} &= \{x \in X \mid r_0 \leq \|x\| \leq r\}, & K^+_r &= \{x \in X \mid \|x\| \geq r\}. \end{aligned}$$

We prove:

(57.4) THEOREM (Krasnosiel'skiĭ). *Let $\varphi\colon X \multimap X$ be a compact mapping determined by a morphism such that $\varphi(S_{r_0}) \subset K^-_{r_0}$ and $\varphi(S_r) \subset K^+_r$. Then* $\mathrm{Fix}(\varphi) \neq \emptyset$.

PROOF. Let $A = K^-_{r_0} \cup K^+_r$. It follows from our assumptions that φ maps A into itself. Denote by $\widetilde{\varphi}\colon (X,A) \multimap (X,A)$ the respective map of pairs and let $\varphi_A\colon A \multimap A$ be the restriction of φ to (A,A). Since $X, A \in$ ANR we obtain:

$$\Lambda(\widetilde{\varphi}) = \Lambda(\varphi) - \Lambda(\varphi_A).$$

But $\Lambda(\varphi) = 1$ (X is a convex subset of E) and $\Lambda(\varphi_A) = 2$ (A consists of two contractible components) hence $\Lambda(\widetilde{\varphi}) = 1 - 2 = -1$ and (57.4) follows from (57.3). □

58. Repulsive and ejective fixed points

There are several classifications of fixed points for singlevalued mappings: essential and inessential, repulsive and ejective or Sharkhowskiĭ's classification (see [Bro4], [Bro5], [For], [Go1] and [Sh]). In the case of multivalued maps the situation is much more complicated. We are able to consider only repulsive and ejective fixed points (see [BGK], [FP], [GP]).

Let X be a space and $x \in X$. By $\mathcal{U}(x)$ we will denote the family of all open neighbourhoods of x in X. Let X be a compact space and $\varphi\colon X \multimap X$ be an u.s.c. map.

(58.1) Definition. A fixed point $\alpha \in \operatorname{Fix}(\varphi)$ is called *repulsive relative to* $U \in \mathcal{U}(\alpha)$ if, for any $V \in \mathcal{U}(\alpha)$, there exists an integer $n(V) \geq 1$ such that $\varphi^n(X \setminus V) \subset X \setminus \overline{U}$ for all $n \geq n(V)$. If there is $U \in \mathcal{U}(\alpha)$ such that α is repulsive relative to U then α is called *repulsive*. The set of all repulsive fixed points is denoted by $\operatorname{Fix}_r(\varphi)$.

(58.2) Definition. A fixed point $\alpha \in \operatorname{Fix}(\varphi)$ is called *ejective relative to* $U \in \mathcal{U}(\alpha)$ if, for any $x \in \overline{U} \setminus \{\alpha\}$, there exists an integer $n \geq 1$ such that $\varphi^n(x) \subset X \setminus \overline{U}$. If there is $U \in \mathcal{U}(\alpha)$ such that α is ejective relative to U then α is called *ejective*. The set of all ejective fixed points is denoted by $\operatorname{Fix}_e(\varphi)$.

As an immediate consequence of the above definitions we obtain:

$$\operatorname{Fix}_r(\varphi) \subset \operatorname{Fix}_e(\varphi).$$

A simple example shows that the converse is not true even for singlevalued mappings.

(58.3) Example. Consider a function $f\colon [0,1] \to [0,1]$ given by

$$f(x) = 2(-x^2 + x)$$

and the open neighbourhood $U = [0, 1/4)$ of the origin. It is easy to see that 0 is ejective relative to U. However, this fixed point is not repulsive because $f(1) = 0$.

First we prove the following:

(58.4) Proposition. *Let X be a compact space, $\varphi\colon X \multimap X$ be continuous and $\alpha \in \operatorname{Fix}(\varphi)$. The following statements are equivalent:*

(58.4.1) *α is repulsive,*

(58.4.2) *there exists $U \in \mathcal{U}(\alpha)$ such that, for any $x \in X \setminus \{\alpha\}$, there exists an integer $n(x) \geq 1$ such that $\varphi^n(x) \subset X \setminus \overline{U}$ for all $n \geq n(x)$.*

PROOF. It is obvious that (58.4.1) implies (58.4.2). Suppose that (58.4.2) holds We show that there exists $M \geq 1$ such that $X \setminus U \subset \bigcup_{i=1}^{M} \varphi^{-i}(X \setminus \overline{U})$. But $\bigcup_{i=1}^{\infty} \varphi^{-i}(X \setminus \overline{U})$ is an open covering of a compact set, hence we can find the wanted number M. Thus, we have

$$U \supset X \setminus \bigcup_{i=1}^{M} \varphi^{-i}(X \setminus \overline{U}) = \bigcap_{i=1}^{M}(X \setminus \varphi^{-i}(X \setminus \overline{U}) = \bigcap_{i=1}^{M}(\varphi_{+}^{-i}(\overline{U})) \supset \bigcap_{i=1}^{M}(\varphi_{+}^{-i}(U)).$$

Let $N = U \cap \bigcap_{i=1}^{M-1}(\varphi_{+}^{-i}(U))$. By continuity of φ the set N is open, $\alpha \in N$ and $\overline{N} \subset \overline{U}$.

We show that $\varphi_{+}^{-1}(N) \subset N$. In fact $x \in \varphi_{+}^{-1}(N)$ if and only if $\varphi(x) \cap U \neq \emptyset$ and for every i, $1 \leq i \leq M-1$, $\varphi(x) \cap \varphi_{+}^{-i}(U) \neq \emptyset$. It means that $x \in \bigcap_{i=1}^{M}(\varphi_{+}^{-i}(U))$. Thus $x \in U$ and $x \in \bigcap_{i=1}^{M-1}(\varphi_{+}^{-i}(U))$. This implies that $\varphi_{+}^{-n-1}(N) \subset N$ and $\overline{\varphi_{+}^{-n-1}(N)} \subset \overline{\varphi_{+}^{-n}(N)}$ for every $n \geq 0$. Now,

$$\alpha \in \bigcap_{n=1}^{\infty} \varphi_{+}^{-n}(N) \subset \bigcap_{n=1}^{\infty} \overline{\varphi_{+}^{-n}(N)} \subset \bigcap_{n=1}^{\infty} \varphi_{+}^{-n}(\overline{N}) \subset \bigcap_{n=1}^{\infty} \varphi_{+}^{-n}(\overline{U}) = \{\alpha\}.$$

Thus $\bigcap_{n=1}^{\infty} \overline{\varphi_{+}^{-n}(N)} = \{\alpha\}$. Let $W \in \mathcal{U}(\alpha)$ be such that $\overline{W} \subset N$. We assert that α is a repulsive fixed point relative to W.

Let $V \in \mathcal{U}(\alpha)$. We show that there exists $n(V) \geq 1$ such that $\overline{\varphi_{+}^{-n(V)}(N)} \subset V$. Suppose that for every $n \geq 1$ we have $\overline{\varphi_{+}^{-n}(N)} \cap (X \setminus V) \neq \emptyset$, i.e. for every $n \geq 1$ there exists $x_n \in \overline{\varphi_{+}^{-n}(N)}$ such that $x_n \in X \setminus V$. By a compactness of $X \setminus V$ there is a subsequence $x_{n_k} \to x \in X \setminus V$. But $x \in \bigcap_{k=1}^{\infty} \overline{\varphi_{+}^{-n_k}(N)} = \{\alpha\}$. This is a contradiction.

Now, for every $n \geq n(V)$ we have $\varphi_{+}^{-n}(N) \subset V$ and thus $\bigcup_{n=n(V)}^{\infty} \varphi_{+}^{-n}(N) \subset V$. Therefore,

$$X \setminus V \subset \bigcap_{n=n(V)}^{\infty} (X \setminus \varphi_{+}^{-n}(N)) = \bigcap_{n=n(V)}^{\infty} \varphi_{-}^{-n}(X \setminus N).$$

This implies that for every $n \geq n(V)$ we have $\varphi^{n}(X \setminus V) \subset X \setminus N \subset X \subset \overline{W}$; the proof is completed. □

(58.5) PROPOSITION. *Let X be a compact space, $\varphi\colon X \multimap X$ be continuous and $\alpha \in \mathrm{Fix}(\varphi)$. The following statements are equivalent:*

(58.5.1) *α is repulsive,*

(58.5.2) *α is ejective and $\alpha \notin \varphi(X \setminus \{\alpha\})$ (α is strongly ejective).*

PROOF. If α is repulsive then, obviously, (58.5.2) is also true. Let α be ejective relative to $U \in \mathcal{U}(\alpha)$ and $\alpha \notin \varphi(X \setminus \{\alpha\})$. For every point $x \in X \setminus \{\alpha\}$ we have

$\operatorname{dist}(\varphi(x), \alpha) \geq 2\gamma(x) > 0$ for some $\gamma(x)$. For every $x \in X \setminus U$ there exists $\varrho(x) > 0$ such that $\varphi(y) \subset N_{\gamma(x)}(\varphi(x))$ whenever $y \in N_{\varrho(x)}(x)$. Then $\operatorname{dist}(\varphi(y), \alpha) \geq \gamma(x) > 0$. By a compactness of $X \setminus U$ there are $z_1, \ldots, z_k$ and $N_1 = N_{\varrho(z_1)}(z_1), \ldots, N_k = N_{\varrho(z_k)}(z_k)$ such that $X \setminus U \subset \bigcup_{i=1}^{k} N_k$.

Define $\gamma := \min\{\gamma(z_1), \ldots, \gamma(z_k)\}$. Let $V \in \mathcal{U}(\alpha)$ be such that $\overline{V} \subset U \cap N_\gamma(\alpha)$. Then for every $x \in X \setminus U$ there is i, $1 \leq i \leq k$, such that $x \in N_i$, and then $\operatorname{dist}(\varphi(x), \alpha) \geq \gamma(z_i) \geq \gamma$. This implies $\varphi(x) \subset X \setminus \overline{V}$ and for every $x \in X \setminus \{\alpha\}$ there is $n(x) \geq 1$ such that $\varphi^{n(x)}(x) \subset X \setminus \overline{V}$. The set $X \setminus \overline{V}$ is open and φ is u.s.c. so, for every $x \in X \setminus \{\alpha\}$ there exists $N(x) \in \mathcal{U}(x)$ such that $\varphi^{n(x)}(N(x)) \subset X \setminus \overline{V}$. Let $C = X \setminus V$. This set is compact and $\alpha \notin C$. Choose for every $x \in C$ the set $V(x) \in \mathcal{U}(x)$ such that $\overline{V(x)} \subset N(x)$. Then $C \subset \bigcup_{i=1}^{r} V(x_i)$ for some $r \geq 1$ and for every i, $1 \leq i \leq r$, $\varphi^{n(x_i)}(\overline{V(x_i)}) \subset X \setminus \overline{V} \subset C$.

Define $K = \bigcup_{i=1}^{r} \bigcup_{s=0}^{n(x_i)-1} \varphi^s(\overline{V(x_i)})$. This set is compact, $\alpha \notin K$ and $C \subset K$. We show that $\varphi^n(K) \subset K$ for every $n \geq 1$. First, note that $\varphi^n(K) \subset \bigcup_{i=1}^{r} \bigcup_{s=0}^{n(x_i)-1} \varphi^{n+s}(\overline{V(x_i)})$. Now, if $n + s < n(x_i)$ then $\varphi^{n+s}(\overline{V(x_i)}) \subset K$. If $n+s \geq n(x_i)$ then $\varphi^{n+s}(\overline{V(x_i)}) = \varphi^{n+s-n(x_i)}(\varphi^{n(x_i)}(\overline{V(x_i)}))$. But $\varphi^{n(x_i)}(\overline{V(x_i)}) \subset X \setminus \overline{V} \subset C \subset K$. Therefore, by induction, $\varphi^{n+s}(\overline{V(x_i)}) \subset K$.

Let $x \in X \setminus \{\alpha\}$. There exists $n(x) \geq 1$ such that $\varphi^{n(x)}(x) \subset X \setminus \overline{V} \subset K$. Hence, $\varphi^n(x) \subset K$ for every $n \geq n(x)$. Now, if $W \in \mathcal{U}(\alpha) < \overline{W} \subset X \setminus K$ then α is repulsive relative to W; the proof is completed. □

(58.6) THEOREM. *Let X be a compact* ANR *and $\varphi\colon X \multimap X$ a multivalued map determined by a morphism. Suppose that there exists an open subset $V \subset X$ such that*

(58.6.1) *the inclusion $i\colon V \to X$ induces an isomorphism on homology,*

(58.6.2) *there exists a natural number n_0 such that $\varphi^n(X \setminus V) \subset X \setminus \overline{V}$, for every $n \geq n_0$.*

Then $i(X, \varphi, V) = 0$.

PROOF. Observe that $\operatorname{Fix}(\varphi^n) \cap \partial V = \emptyset$ for every $n \in N$. Indeed, if $\operatorname{Fix}(\varphi^n) \cap \partial V \neq \emptyset$ for some n, then we choose $k \in N$ such that $k \cdot n > n_0$. Then $\operatorname{Fix}(\varphi^n) \cap \partial V \neq \emptyset$ implies $\operatorname{Fix}(\varphi^{kn}) \cap \partial V \neq 0$ in contradiction to $\varphi^n(X \setminus V) \subset X \setminus \overline{V}$ (see (58.6.2)). Therefore, the index $\operatorname{ind}(X, \varphi^n, V)$ is well defined for every n. It follows from the mod-p property of the fixed point index that, if

$$i(X, \varphi^n, V) = 0 \quad \text{for every } n \geq n_0,$$

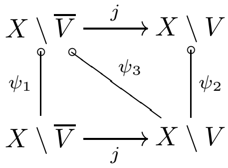
then $i(X, \varphi, V) = 0$. So, we shall prove that $i(X, \varphi^n, V) = 0$ for every $n \geq n_0$. First, from the additivity and normalization properties of the fixed point index we obtain:

$$i(X, \varphi^n, V) + i(X, \varphi^n, X \setminus \overline{V}) = \Lambda(\varphi^n). \tag{58.6.3}$$

We consider $\psi_1\colon X \setminus \overline{V} \multimap X \setminus \overline{V}$, $\psi_2\colon X \setminus V \multimap X \setminus V$, $\psi_3\colon X \setminus V \multimap X \setminus \overline{V}$ the respective mappings defined by φ^n (cf. (58.6.2)). From (58.6.1) and (11.4.1) we deduce:

(58.6.4) $$\Lambda(\varphi^n) = \Lambda(\psi_2).$$

On the other hand we have the following commutative diagram:



in which $j\colon X \setminus \overline{V} \to X \setminus V$ is the inclusion map. So, from (11.4) we deduce that:

(58.6.5) $$\Lambda(\psi_2) = \Lambda(\psi_1).$$

Since the open set $X \setminus \overline{V}$ of an ANR-space X is again ANR from the contraction and normalization properties of the fixed point index we obtain:

(58.6.6) $$i(X, \varphi^n, X \setminus \overline{V}) = i(X \setminus \overline{V}, \psi_1, X \setminus \overline{V}) = \Lambda(\psi_1).$$

Consequently, from (58.6.3)–(58.6.6) we infer: $i(X, \varphi^n, V) = 0$, and the proof is completed. □

As an immediate consequence of (58.6) we obtain:

(58.7) Corollary. *Let X and φ be the same as in* (58.6). *Let $\alpha \in \mathrm{Fix}_r(\varphi)$ be a repulsive point relative to $U \in \mathcal{U}(\alpha)$. Assume that:*

(58.7.1) *there exists $V \in U(\alpha)$ such that $V \subset \overline{V} \subset U$ and $i\colon X \setminus V \to X$ induces an isomorphism on homology.*

Then $i(X, \varphi, U) = 0$.

Proof. Observe that α is the only fixed point in U (by definition). So,

$$i(X, \varphi, U) = i(X, \varphi, V)$$

and our claim holds by (58.6). □

Now from (58.7) and the additivity property of the fixed point index we obtain:

(58.8) Corollary. *Let X and φ be as in Theorem* (58.6). *Assume that $\mathrm{Fix}_r(\varphi) = \{\alpha_1, \dots, \alpha_k\}$ and α_i is repulsive relative to $U_i \in \mathcal{U}(\alpha_i)$, $i = 1, \dots, k$ and the assumption* (58.7.1) *holds true for every U_i, $i = 1, \dots, k$. If $\Lambda(\varphi) \neq 0$ then there exists $\alpha \in \mathrm{Fix}(\varphi)$ such that $\alpha \notin \mathrm{Fix}_r(\varphi)$.*

In particular, if in (58.8) we assume that $X \in \mathrm{AR}$ then we have:

(58.9) COROLLARY. *If $X \in \mathrm{AR}$ and $\varphi\colon X \multimap X$ is a multivalued map determined by a morphism then $Fix(\varphi) \setminus \mathrm{Fix}_r(\varphi) \neq \emptyset$.*

PROOF. Since X is acyclic, so $\Lambda(\varphi) = 1$. If φ has no repulsive fixed point then, there exists $x \in \mathrm{Fix}(\varphi) \setminus \mathrm{Fix}_r(\varphi)$. So, we can assume that $\mathrm{Fix}_r(\varphi) \neq \emptyset$. If the set $\mathrm{Fix}_r(\varphi)$ is finite, then our claim follows from (58.8).

If it is not, then we choose a sequence $\{x_n\} \subset \mathrm{Fix}_r(\varphi)$ such that $\lim_n x_n = x \in \mathrm{Fix}(\varphi)$ (observe that by definition any repulsive fixed point is isolated). □

Since $\mathrm{Fix}_r(\varphi) \subset \mathrm{Fix}_e(\varphi)$ it would be interesting to find sufficient conditions for fixed points which are not belonging to $\mathrm{Fix}_e(\varphi)$.

Recall that we have assumed X to be a compact ANR-space and $\varphi\colon X \multimap X$ to be a multivalued map determined by a morphism. In what follows we shall keep the above assumptions. We start with the following:

(58.10) PROPOSITION. *Let $F \subset \mathrm{Fix}(\varphi)$ be an open and closed in $\mathrm{Fix}(\varphi)$. Assume that $\varphi(X \setminus F) \subset X \setminus F$ and the restriction $\varphi'\colon X \setminus F \multimap X \setminus F$ is a compact map. Denote by $\overline{\varphi}\colon (X, X \setminus F) \multimap (X, X \setminus F)$ the map determined by φ. Then $i(X, \varphi, W) = \Lambda(\overline{\varphi}) = \Lambda(\varphi) - \Lambda(\varphi')$, for any open neighbourhood W of F in X such that $W \cap (\mathrm{Fix}(\varphi) \setminus F) = \emptyset$.*

Proposition (58.10) immediately follows from the Lefschetz fixed point theorem, normalization and additivity properties of the fixed point index and (58.5).

(58.11) PROPOSITION. *Assume that $\varphi(X \setminus \mathrm{Fix}_e(\varphi)) \subset X \setminus \mathrm{Fix}_e(\varphi)$, $\#\mathrm{Fix}_e(\varphi) < +\infty$. Denote by $\varphi'\colon X \setminus \mathrm{Fix}_e(\varphi) \multimap X \setminus \mathrm{Fix}_e(\varphi)$ and $\overline{\varphi}\colon (X, X \setminus \mathrm{Fix}_e(f)) \multimap (X, X \setminus \mathrm{Fix}_e(\varphi))$ the respective maps determined by φ. Then $\Lambda(\overline{\varphi}) = 0$ and $\Lambda(\varphi) \neq 0$ implies that $\mathrm{Fix}(\varphi) \setminus \mathrm{Fix}_e(\varphi) \neq \emptyset$.*

PROOF. From (58.10) we deduce that $\Lambda(\varphi') \neq 0$. So by the Lefschetz fixed point theorem applied to φ' we obtain $\mathrm{Fix}(\varphi') = \mathrm{Fix}(\varphi) \setminus \mathrm{Fix}_e(\varphi) \neq \emptyset$ and the proof is completed. □

For the remaining part of this section we will keep, in addition to the previously made assumptions, the assumptions of proposition (58.11).

(58.12) COROLLARIES.

(58.12.1) *If φ is homotopic to a constant map, then $\mathrm{Fix}(\varphi) \setminus \mathrm{Fix}_e(\varphi) \neq \emptyset$.*

(58.12.2) *If X or $X \setminus \mathrm{Fix}_e(\varphi)$ is acyclic, then $\Lambda(\overline{\varphi}) = 0$ implies that $\mathrm{Fix}(\varphi) \setminus \mathrm{Fix}_e(\varphi) \neq \emptyset$.*

(58.12.3) *If X and $X \setminus \mathrm{Fix}_e(\varphi)$ are acyclic then $\mathrm{Fix}(\varphi) \setminus \mathrm{Fix}_e(\varphi) \neq \emptyset$.*

(58.12.4) *Let $B \subset \mathrm{Fix}_e(\varphi)$ and $i\colon X \setminus \mathrm{Fix}_e(\varphi) \to X \setminus B$ be the inclusion map. If $i_*\colon H_*(X \setminus \mathrm{Fix}_e(\varphi)) \xrightarrow{\sim} H_*(X \setminus B)$ is an isomorphism, then the set $\mathrm{Fix}_e(\varphi)$ in (3.2) can be replaced by $\overline{B}$.*

(58.12.5) *If the inclusion* $i: X \setminus \mathrm{Fix}_e(\varphi) \to X$ *induces an isomorphism on homologies and* $\Lambda(\varphi) \neq 0$*, then* $\mathrm{Fix}(\varphi) \setminus \mathrm{Fix}_e(\varphi) \neq \emptyset$.

(58.12.6) *Assume that* $Y \subset X$ *is a closed subset such that* $\varphi(X) \subset Y$. *If for every* $x \in \mathrm{Fix}_e(\varphi)$ *we have* $H_*(Y, Y \setminus \{x\}) = 0$ *and* $\Lambda(\varphi) \neq 0$*, then* $\mathrm{Fix}(\varphi) \setminus \mathrm{Fix}_e(\varphi) \neq \emptyset$.

For some further results we recommend [FP], [GP], [Pei].

59. Condensing and k-set contraction mappings

To learn about condensing maps it is useful to start with the notion of k-set contraction and condensing pairs of maps. As in Chapter IV, by a pair (p, q) we mean the following diagram:

$$X \overset{p}{\Longleftarrow} \Gamma \overset{q}{\longrightarrow} Y$$

in which p is Vietoris and q continuous. Such a pair (p, q) is called compact provided q is compact.

Let E be a Banach space. By $\gamma: \mathcal{B}(E) \to \mathbb{R}_+$ we will denote the measure of non-compactness function, i.e. γ is a function satisfying all properties of (4.10). In particular, we can let $\gamma = \alpha$ to be the Kuratowski measure of compactness or $\gamma = \beta$ to be the Hausdorff measure of non-compactness (see Section 4).

(59.1) Definition. Let A and C be two subsets of E. A pair $A \overset{p}{\Longleftarrow} \Gamma \overset{q}{\longrightarrow} C$ is called a *k-set contraction pair*, if there exists a real number k, $0 \leq k < 1$, such that for every bounded $B \subset A$ the following condition is satisfied:

(59.1.1) $$\gamma(q(p^{-1}(B))) \leq k \cdot \gamma(B);$$

(p, q) is called a *condensing pair*, if for every bounded and no relatively compact $B \subset A$ we have

(59.1.2) $$\gamma(q(p^{-1}(B))) < \gamma(B).$$

It is evident that any compact pair is k-set contraction with $k = 0$ and any k-set contraction pair is condensing. Moreover, let us observe that if (p, q) is a condensing pair then for any bounded $B \subset A$ the set $q(p^{-1}(B))$ is bounded.

(59.2) Proposition. *Let* $A \overset{p}{\Longleftarrow} \Gamma \overset{q}{\longrightarrow} C$ *be a condensing pair, where* A *is a bounded and closed subset of* E. *Then* $\mathrm{Fix}(p, q)$ *is a compact set, where as before* $\mathrm{Fix}(p, q) = \{x \in A \mid x \in q(p^{-1}(x)\}$.

Proof. Indeed, we have $\mathrm{Fix}(p, q) \subset q(p^{-1}(\mathrm{Fix}(p, q)))$, hence

$$\gamma(\mathrm{Fix}(p, q)) \leq \gamma(q(p^{-1}(\mathrm{Fix}(p, q)))) < \gamma(\mathrm{Fix}(p, q)).$$

So, by (4.10) we deduce that $\overline{\mathrm{Fix}(p,q)}$ is compact. Because $\mathrm{Fix}(p,q) = \overline{\mathrm{Fix}(p,q)}$ the proof is completed. □

We will say that the pair (p,q) satisfies the Palais–Smale condition provided for every sequence $\{u_n\} \subset \Gamma$, the property

$$\lim_n (p(u_n) - q(u_n)) = 0$$

implies that there exists a convergent subsequence of $\{u_n\}$.

(59.3) PROPOSITION. *Let (p,q) be the same as in* (59.2). *Then the pair (p,q) satisfies the Palais–Smale condition.*

PROOF. Let $\lim_n (p-q)(y_n) = 0$. We put $x_n = p(y_n) - q(y_n)$, $u_n = p(y_n)$. Then $\{x_n\} \subset E$ and $\{u_n\} \subset A$. By assumption $\gamma(\{x_n\}) = 0$. We will show that $\gamma(\{u_n\}) = 0$. Because $q(y_n) \in q(p^{-1}(u_n))$ we have

$$\gamma(q(\{y_n\})) \leq \gamma(q(p^{-1}(\{u_n\}))) \leq k \cdot \gamma(\{u_n\}).$$

On the other hand, $u_n = x_n + q(y_n)$ so, in view of (4.10.2), we obtain

$$\gamma(\{u_n\}) \leq \gamma(\{x_n\}) + \gamma(\{q(y_n)\}) = \gamma(q(\{y_n\})).$$

The above two inequalities imply that $\gamma(\{u_n\}) = 0$. Therefore the set $p^{-1}\overline{(\{u_n\})}$ is compact (p is proper!), so from the sequence $\{y_n\}$ in $p^{-1}(\{u_n\})$ we can choose a convergent subsequence and the proof is completed. □

Let A be a bounded closed subset of E and let C be a convex closed subset of E. Consider a k-set contraction pair (p,q) from A to C. We will associate with such a pair (p,q) a compact pair $(\widetilde{p},\widetilde{q})$ such that $\mathrm{Fix}(p,q) = \mathrm{Fix}(\widetilde{p},\widetilde{q})$. In order to do it we define a decreasing sequence $\{K_n\}$ of closed bounded and convex subsets of C by putting

$$K_1 = \overline{\mathrm{conv}}(q(p^{-1}(A))), \dots, K_n = \overline{\mathrm{conv}}(q(p^{-1}(A \cap K_{n-1}))), \dots$$

It is evident that $q(p^{-1}(K_n \cap A)) \subset K_{n+1}$ and $\mathrm{Fix}(p,q) \subset K_n$ for every n. There are two possibilities, namely,

(59.4) $\qquad K_n \neq \emptyset, \quad$ for each n,

(59.5) $\qquad K_i \neq \emptyset, \quad$ for $i = 1, \dots, m$ and $K_{m+j} = \emptyset$, for each j.

If (59.5) holds then we choose a point $x_0 \in K_m$ and we define

(59.6) $\qquad \widetilde{q} : \Gamma \to C \quad$ by putting $q(y) = x_0$ and $\widetilde{p} = p$.

Then $(\widetilde{p},\widetilde{q})$ is a compact pair such that $\mathrm{Fix}(p,q) = \mathrm{Fix}(\widetilde{p},\widetilde{q}) = \emptyset$.

(59.7) Lemma. *Assume that* (59.5) *holds and let* $x_1 \in K_m$. *Then there exists a compact homotopy* $h\colon \Gamma \times [0,1] \to C$ *joining* $\widetilde{q}$ *with* $\widetilde{q}_1$ *such that*

$$\mathrm{Fix}(p,h) = \{x \in A \mid x \in h(p^{-1}((x) \times \{t\})),\ \textit{for every } t\} = \emptyset,$$

where $\widetilde{q}_1\colon \Gamma \to C$ *is given by the formula* $\widetilde{q}_1(y) = x_1$.

For the proof of Lemma (59.7) it is sufficient to consider a homotopy $h\colon \Gamma \times [0,1] \to C$ given as follows:

$$h(y,t) = (1-t)x_0 + tx_1.$$

(59.8) Remark. By comparing (59.6) and (59.7) we can say that, if (59.5) holds, then the pair $(\widetilde{p}, \widetilde{q})$ is defined uniquely up to homotopy.

(59.9) Lemma. *If* (59.4) *holds, then* $K_\infty = \bigcap_{n=1}^{\infty} K_n$ *is a compact convex and nonempty set which contains* $\mathrm{Fix}(p,q)$.

Proof. First, we claim that

(59.9.1) $\gamma(K_n) \leq k^n \cdot \gamma(A)$, for each n, where k is given for considered k-set contraction pair (p,q).

We prove (59.9.1) by induction. Since

$$\gamma(K_1) = \gamma(\overline{\mathrm{conv}}(q(p^{-1}(A)))) = \gamma(q(p^{-1}(A))) \leq k \cdot \gamma(A),$$

our assertion holds for $n = 1$. Now assume that (59.9.1) is true for every $m < n$. Then we obtain:

$$\begin{aligned}\gamma(K_n) &= \gamma(\overline{\mathrm{conv}}(q(p^{-1}(A \cap K_{n-1}))))\\ &= \gamma(q(p^{-1}(A \cap K_{n-1}))) \leq k \cdot \gamma(A \cap K_{n-1})\\ &\leq k \cdot \gamma(K_{n-1}) \leq k \cdot k^{n-1}\gamma(A) = k^n\gamma(A)\end{aligned}$$

and thus finish the proof of (59.9.1). Now, from (59.9.1) it follows that $\lim_n \gamma(K_n) = 0$. Therefore, our claim follows from (4.14). □

We associate with given k-set contraction pair (p,q):

$$A \overset{p}{\Longleftarrow} \Gamma \overset{q}{\longrightarrow} C$$

the pair $(\widetilde{p}, \widetilde{q})$:

$$A \cap K_\infty \overset{\widetilde{p}}{\Longleftarrow} p^{-1}(A \cap K_\infty) \overset{\widetilde{q}}{\longrightarrow} K_\infty \tag{59.10}$$

by putting $\widetilde{p}(u) = p(u)$ and $\widetilde{q}(u) = q(u)$. Since $\widetilde{q}\,\widetilde{p}^{-1}(A \cap K_\infty) \subset K_\infty$, in view of (59.9) we get $\mathrm{Fix}(p,q) = \mathrm{Fix}(\widetilde{p}, \widetilde{q})$. Observe, that if $A = C$, then the condition (59.5) cannot occur.

Since $(\widetilde{p}, \widetilde{q})$ is a compact pair, then from the Lefschetz fixed point theorem for admissible (or determined by morphisms) maps we obtain:

(59.11) PROPOSITION. *If C is a bounded closed and convex subset of E and (p,q) is a k-set contraction pair from C to C, then* $\mathrm{Fix}(p,q) \neq \emptyset$.

We prove:

(59.12) THEOREM. *If C is a bounded closed and convex subset of E and (p,q) is a condensing pair from C to C, then* $\mathrm{Fix}(p,q) \neq \emptyset$.

For the proof of (59.12) we need some additional facts. Let $\varepsilon > 0$. A point $u \in \Gamma$ is called an *ε-coincidence* for (p,q), if $\|p(u) - q(u)\| < \varepsilon$.

(59.13) LEMMA. *If (p,q) has an ε-coincidence for every $\varepsilon > 0$ and satisfies the Palais–Smale condition, then* $\mathrm{Fix}(p,q) \neq \emptyset$.

PROOF. Let $\varepsilon_n = 1/n$ and $\{u_n\} \subset \Gamma$ be a sequence of ε_n-coincidence points of (p,q), $n = 1,2,\ldots$. Then $\lim_n (p(u_n) - q(u_n)) = 0$. So, from the Palais–Smale condition we obtain that there exists a convergent subsequence $\{u_{n_k}\}$ of $\{u_n\}$.

Let $u = \lim_k u_{n_k}$. Then $p(u) = q(u)$, so the set $\varkappa(p,q)$ of coincidence points is nonempty and consequently $\mathrm{Fix}(p,q) \neq \emptyset$. □

PROOF OF (59.12). We can assume without loss of generality, that $0 \in C$. For each $n = 1,2,\ldots$ we define a map $q_n\colon \Gamma \to C$ by putting:

$$q_n(u) = \left(1 - \frac{1}{n}\right) \cdot q(u).$$

Then (p,q) is an $(1 - (1/n))$-set contraction, $n \geq 2$. So, from (59.11) for every $n \geq 2$ we obtain a point $u_n \in \Gamma$ such that $p(u_n) = q(u_n)$. On the other hand we have:

$$\begin{aligned}\|p(u_n) - q(u_n)\| &\leq \|p(u_n) - q_n(u_n)\| + \|q(u_n) - q_n(u_n)\| \\ &= \frac{1}{n}\|q(u_n)\| \leq \frac{1}{n} \cdot \mathrm{diam}(C),\end{aligned}$$

where $\mathrm{diam}(C)$ denotes the diameter of C. It implies that (p,q) has ε-coincidence for every $\varepsilon > 0$ and hence our theorem follows from (59.13) and (59.4); the proof is completed. □

Let C be a convex closed subset of E and $(p,q)\colon \overline{U} \overset{p}{\Longleftarrow} \Gamma \overset{q}{\longrightarrow} C$ be a k-set contraction pair such that $\mathrm{Fix}(p,q) \cap \partial U = \emptyset$, i.e. (p,q) has no fixed points on the boundary ∂U of U in C, where U is an open subset of C.

Following (59.10) we obtain a compact pair

$$(\widetilde{p},\widetilde{q})\colon \overline{U} \cap K_\infty \overset{\widetilde{p}}{\Longleftarrow} p^{-1}(\overline{U} \cap K'_\infty) \overset{\widetilde{q}}{\longrightarrow} C \cap K_\infty.$$

For simplicity let us denote $U_1 = U \cap K_\infty$ and $C_1 = C \cap K_\infty$, $\Gamma_1 = p^{-1}(\overline{U} \cap K_\infty)$. Then we have a compact pair

$$\overline{U}_1 \overset{\widetilde{p}}{\Longleftarrow} \Gamma_1 \overset{\widetilde{q}}{\longrightarrow} C_1,$$

where U_1 is open in C_1 and C_1 is a convex nonempty compact subset of E.

Now, by using the Schauder Approximation Theorem, for given $\varepsilon > 0$ we can find a $n(\varepsilon)$-dimensional subspace $E^{n(\varepsilon)}$ of E and an ε-approximation $q_\varepsilon\colon \Gamma_1 \to E^{n(\varepsilon)}$ of $\widetilde{q}$. We let $V = U_1 \cap E^{n(\varepsilon)}$ and $C_\varepsilon = C_1 \cap E^{n(\varepsilon)}$. Then we obtain a diagram:

$$\overline{V} \overset{p_1}{\Longleftarrow} \widetilde{p}^{-1}(\overline{V}) \overset{q_\varepsilon}{\longrightarrow} C_\varepsilon.$$

As we have seen in Chapter 3 for sufficiently small $\varepsilon > 0$ such that $\mathrm{Fix}(p_1, q_\varepsilon) \cap \partial V = \emptyset$ and q_ε is homotopic to $q_{\varepsilon'}$ for $\varepsilon, \varepsilon' \leq \varepsilon_0$, for some $\varepsilon > 0$.

Let $r\colon \mathbb{R}^n \to C_\varepsilon$ be a retraction (C_ε is convex and closed, so $C_\varepsilon \in \mathrm{AR}$). Then $r^{-1}(V)$ is an open subset of $\mathbb{R}^n$ and we have the following commutative diagram:

$$\begin{array}{ccccccc}
\overline{r^{-1}(V)} & \overset{r}{\longrightarrow} & \overline{V} & \overset{p_1}{\Longleftarrow} & \widetilde{p}^{-1}(V) & \overset{q_\varepsilon}{\longrightarrow} & C_\varepsilon \\
 & {\scriptstyle p_\varepsilon}\nwarrow\!\!\nwarrow & \uparrow{\scriptstyle f} & \nearrow{\scriptstyle g} & & & \downarrow{\scriptstyle i} \\
 & & \Gamma_\varepsilon & & \xrightarrow[\overline{q}_\varepsilon = i \circ q_\varepsilon \circ g]{} & & E^{n(\varepsilon)}
\end{array}$$

in which $\Gamma_\varepsilon = \{(x,y) \in \overline{r^{-1}(V)} \times \widetilde{p}^{-1}(V) \mid r(x) = p_1(y)\}$, $p_\varepsilon(x,y) = x$, $f(x,y) = r(x)$, $g(x,y) = y$. Moreover, we obtain:

$$\mathrm{Fix}(p_\varepsilon, \overline{q}_\varepsilon) = \mathrm{Fix}(p_1, q_\varepsilon) \subset V.$$

But for the pair $(p_\varepsilon, \overline{q}_\varepsilon)$ the coincidence index $I(p_\varepsilon, \overline{q}_\varepsilon)$ is well defined (see (12.4)).

We let:

$$I(p,q) = I(p_\varepsilon, \overline{q}_\varepsilon). \tag{59.14}$$

Then $I(p,q)$ is called the coincidence index for the k-set contraction pair (p,q). Note that by a standard argument, used already several times, we can see that definition (59.14) is correct for a given retraction r.

The following problem remains open (see [GK]).

(59.15) Does Definition (59.14) depend on the choice of a retraction map r?

Note that (59.15) is a slight reformulation of the definition of a topological degree for n-admissible mappings.

We shall make use of the following two properties of the coincidence index defined in (59.14).

(59.16) PROPERTY (Existence). *If* $I(p,q) \neq 0$, *then* $\mathrm{Fix}(p,q) \neq \emptyset$.

(59.17) PROPERTY (Homotopy). *Let U be an open subset of C, where C is a convex closed subset of a normed space E. Let $p\colon \Gamma \Rightarrow \overline{U}$ be a Vietoris map and let $h\colon \Gamma \times [0,1] \to C$ be a continuous map. Assume further that the following two conditions are satisfied:*

(59.17.1) $\mathrm{Fix}(p,h) \cap \partial U = \emptyset$, *where* $\mathrm{Fix}(p,h) = \{x \in U \mid x \in h(p^{-1}(x,t))$ *for some* $t \in [0,1]\}$,

(59.17.2) $\gamma(h(p^{-1}(B \times [0,1]))) \leq k \cdot \gamma(B)$ *for every* $B \subset \overline{U}$ *and some* $0 \leq k < 1$.

Then $I(p,h_0) = I(p,h_1)$, *where* $h_i(x) = h(x,i)$, $i = 0,1$.

The standard proofs of (59.16) and (59.17) are left to the reader.

Now we will generalize the non-linear alternative and the Leray–Schauder alternative from the case of k-set contraction singlevalued maps to the case of k-set contraction pairs. Till the end of this section we will assume that C is a convex and closed subset of E which contains the zero point 0 of E.

(59.18) THEOREM (The Non-Linear Alternative). *Let U be an open bounded subset of C such that $0 \in U$ and let (p,q) be a k-set contraction pair from $\overline{U}$ to C. Then at least one of the following properties holds:*

(59.18.1) $\varkappa(p,q) \neq \emptyset$,

(59.18.2) *there is an* $x \in \partial U$ *such that* $x \in (\lambda \cdot q(p^{-1}(x)))$ *for some* $\lambda > 1$.

PROOF. We can assume without loss of generality, that $\mathrm{Fix}(p,q) \cap \partial U = \emptyset$. For the proof consider a homotopy $h\colon \Gamma \times [0,1] \to C$ defined by the formula $h(y,t) = t \cdot q(y)$. Then h satisfies (59.17.2) and it is a homotopy joining q with the constant map $q_1, q_1(y) = 0$. If $\mathrm{Fix}(p,h) \cap \partial U = \emptyset$, then from (59.17) and (59.16) we deduce that $\mathrm{Fix}(p,q) \neq \emptyset$, so (59.18.1) holds. If $\mathrm{Fix}(p,h) \cap \partial U \neq \emptyset$, then we can take a point $x_0 \in \partial U$ such that $x_0 \in (t_0 \cdot q(p^{-1}(x_0)))$ for some $0 < t_0 < 1$. Consequently, for $\lambda = 1/t_0 > 1$ we have $x_0 \in \lambda \cdot q(p^{-1}(x_0)))$ and the proof is completed. □

(59.19) COROLLARY. *Assume* (p,q) *is as in* (59.18). *Assume further that for every $x \in \partial U$ and for every $u \in q(p^{-1}(x))$ one of the following conditions holds:*

(59.19.1) $$\|u\| \leq \|x\|,$$

(59.19.2) $$\|u\| \leq \|x-u\|,$$

(59.19.3) $$\|u\|^2 \leq \|x\|^2 + \|x-u\|^2.$$

Then $\varkappa(p,q) \neq \emptyset$.

For the proof of (59.19) it is sufficient to note that each of conditions (59.19.1)–(59.19.3) implies that the second property of the non-linear alternative cannot occur.

For a pair (p,q) from C to C and for a subset $A \subset C$, by (p_A, q_A) we will denote a pair defined as follows:

$$p_A\colon p^{-1}(A) \Rightarrow A, \quad p_A(y) = p(y),$$
$$q_A\colon p^{-1}(A) \to C, \quad q_A(y) = q(y).$$

(59.20) THEOREM (The Leray–Schauder Alternative). *Let (p,q) be a pair from C to C such that for any open and bounded $U \subset C$ the pair (p_U, q_U) is a k-set contraction. Let $G(p,q) = \{x \in C \mid x \in (\lambda \cdot q(p^{-1}(x)))$, for some $0 < \lambda < 1\}$. Then either $G(p,q)$ is unbounded or $\varkappa(p,q) \neq \emptyset$.*

PROOF. Assume $G(p,q)$ is bounded. We choose an open ball $B(0,r)$ in E containing $G(p,q)$ in its interior. Let $U = B(0,r) \cap C$. Then $(p_U, q_U) \in \mathcal{C}(\overline{U}, C)$ and no $x \in \partial U$ can satisfy the second property of the non-linear alternative. By using once again (59.18) to the pair (p_U, q_U) we have $\emptyset \neq \varkappa(p_U, q_U) \subset \varkappa(p,q)$ and the proof is completed. □

(59.21) REMARK. Finally, let us remark that all results of this section can be formulated for k-set contraction and condensing admissible maps or morphisms; φ is a k-set contraction (condensing) admissible map if there exists a k-set contraction (condensing) pair (p,q) such that $(p,q) \subset \varphi$. In the definition of the k-set contraction (condensing) morphism we consider the equivalence relation in family of all k-set contraction (condensing) pairs (p,q).

(59.22) REMARK. Note that in Section 60 we will continue the study of k-set contraction and condensing maps in the framework of so called compacting mappings.

60. Compacting mappings

The aim of this section is to show a way of generalizing some of the results presented in Section 59. In this section we assume that all multivalued mappings are determined by morphisms. We start with some notations and notions.

(60.1) DEFINITION. A closed subset of X of a Banach space E is called a *special* ANR provided there exists a family $\{C_j\}_{j\in J}$ of closed convex subsets of E such that $X = \bigcup_{j\in J} C_j$ and this union is locally finite, i.e. for every $x \in X$ there exists a finite set $J_x \subset J$, such that $x \notin C_j$ for every $j \in J \setminus J_x$ (written $X \in$ s-ANR).

Note that a special ANR is an ANR-space. In fact it follows from the so called Second Hanner Theorem, which states that any space locally ANR is an ANR-space, and from (1.10.1).

If $X \in$ s-ANR and X is a finite union $X = \bigcup_{j=1}^n C_j$ of closed convex subsets in E, then we will write $X \in$ sf-ANR.

We shall use the following lemma:

(60.2) Lemma. *Let* $C \in$ sf-ANR *be the union* $C = \bigcup_{j=1}^{n} C_j$ *of closed convex subsets of* E. *Then there exists a polyhedron* P *such that* $P = \bigcup_{i=1}^{m} P_i$, *where* $P_i = \operatorname{conv}\{x_1, \dots, x_{m_i}\}$ *for some* $x_1, \dots, x_{m_i} \in C$ *and a continuous map* $\pi\colon C \to C$ *such that* $\pi(C_i) \subset P_i \subset C_i$ *for all* $i \leq n$.

Lemma (60.2) is strictly technical so the proof is omitted here. For details see [Nu].

(60.3) Definition. Let $X \in$ s-ANR and U be an open subset of X. A multi-valued map $\varphi\colon U \multimap X$ (determined by a morphism) is called *compacting* provided there exists an open set $W \subset U$ and a sequence $\{K_n\}$ such that $K_n \in$ sf-ANR for every n and the following conditions are satisfied:

(60.3.1) $\operatorname{Fix}(\varphi) \subset W \subset \overline{W} \subset U$,
(60.3.2) $W \subset K_1 \subset X$,
(60.3.3) $\varphi(W \cap K_n) \subset K_{n+1} \subset K_n$ for any $n \geq 1$,
(60.3.4) $\lim_{n\to\infty} \gamma(K_n) = 0$, where γ, as in Section 59, denotes the measure of non-compactness.

We shall prove the following:

(60.4) Proposition. *Suppose that* $X \in$ s-ANR, U *is an open subset of* X *and* $f\colon U \to X$ *is a continuous map such that* $S = \{x \in U \mid f(x) = x\}$ *is compact. Assume that there is an open neighbourhood* W *of* S *such that* $f|_W$ *is a* k*-set-contraction with* $k < 1$. *Then* f *is compacting.*

Proof. X has a locally finite covering $\{C_\alpha \mid \alpha \in A\}$ by closed, convex hull of B in the overlying Banach space. By the local finiteness of the covering and the *compactness of* $\overline{\operatorname{conv}} f(W)$, there exists a neighbourhood W_1 of S, $\overline{W}_1 \subset W$, such that $(W_1 \cup \overline{\operatorname{conv}} f(W_1)) \cap C_\alpha$ is empty except for α in a finite index set A_1. Define $K_1 \in \mathcal{F}_0$ by $K_1 = \bigcup_{\alpha\in A_1} C_\alpha$ and for $n \geq 1$ define $\{K_n\}$ inductively by $K_{n+1} = (\overline{\operatorname{conv}} f(W_1 \cap K_n)) \cap X$. Since f is a k-set-contraction, $k < 1$, $\gamma(K_{n+1}) \leq k^n \gamma(W_1) \to 0$. It is also not hard to see that $K_n \supset K_{n+1}$, $f(W_1 \cap K_n) \subset K_{n+1}$ and $W_1 \subset K_1$. Thus f is compacting; the proof is completed. □

Now assume that $\varphi\colon W \multimap X$ is compacting with W and $\{K_n\}$ satisfying the conditions of (2.8). Since $K_i \in$ sf-ANR there exists $m(i)$ and $C_{i1}, \dots, C_{im(i)}$ closed convex such that $K_i = \bigcup_{j=1}^{m(i)} C_{ij}$ and $\partial(C_{ij}) < \gamma(K_i) + i^{-1}$.

We may now choose n and by Lemma (60.2), $\pi_n\colon K_1 \to K_1$ such that $\pi_n(K_1) \subset P_n \subset X$, where P_n is a polyhedron such that

$$P_n = \bigcup_{i=1}^{m} \bigcup_{j=1}^{m(i)} P_{ij} \quad \text{and} \quad \pi_n(C_{ij}) \subset P_{ij} \subset C_{ij}.$$

Thus $\pi_n(K_i) \subset K_i$ for any $i \leq n$ and if $x \in K_i$ we have $\|\pi_n(x) - x\| \leq \gamma(K_i) + i^{-1}$. Since the fixed point index for maps determined by morphisms on polyhedra is defined, as we have already observed, we can let:

$$i(X, \varphi, U) = \lim_{n\to\infty} i(P_n, \pi_n \circ \varphi, W \cap P_n) = i(P_n, \pi_n \circ \varphi, W \cap P_n) \tag{60.5}$$

if n is big enough. The proof of a correctness of the above definition is quite long and technically complicated. We recommend [FV1].

We shall restrict our considerations to the case of the fixed point index defined in (60.5) having the following properties:

- existence,
- excision,
- additivity,
- homotopy,
- commutativity,
- mod p.

To obtain the normalization property of the above fixed point index one more assumption about φ is needed. We have to assume that φ is a compact absorbing contraction and compacting mapping. We have proved in Chapter IV that for compact absorbing contractions the Lefschetz fixed point theorem is true, so it is sufficient to see that the respective Lefschetz number and the fixed point index are equal.

There are still some open problems concerning compacting and compact absorbing contractions:

One of those is to find relations between:

- compacting, condensing, k-set contraction mappings on one hand;
- eventually compact mappings with compact attractors, compact absorbing contractions, asymptotically compact — on the other hand.

61. Fixed points of differentiable multivalued maps

In this section we prove several fundamental fixed point principles for u.s.c. maps with convex values which are k-set contractions, $k < 1$ and differentiable at the origin or infinity. We will base on the works of G. Fournier and D. Violette (cf. [FV1], [FV2]).

Let E, E' be two Banach spaces. A multivalued map $F: E \multimap E'$ is called *homogeneous*, if $F(tx) = t \cdot F(x)$ for any $x \in E$ and $t \in R$. We say that F is semi-linear positive if $F(\sum_{i=1}^n t_i x_i) \leq \sum_{i=1}^n t_i \cdot F(x_i)$ for every $x_i \in E$, $t_i \geq 0$ and $\sum_{i=1}^n t_i \leq 1$. In this case $F(0) = 0$ and $F(\operatorname{conv} A) \subset \operatorname{conv} F(A)$. A real number λ is an eigenvalue of $F: E \multimap E$ if there exists $x \in E$ such that $\lambda x \in F(x)$; then x is called an eigenvector corresponding to λ.

Let U be an open subset of E.

(61.1) DEFINITION. A multivalued map $T: U \multimap E'$ is *differentiable* at the point $x \in U$ if there exists an u.s.c. multivalued map $S_x: T(x) \times E \to E'$ such that the map $S_{x,z}: E \to E'$ defined by $S_{x,z}(h) = S_x(z,h)$ is u.s.c., homogeneous and that if $\|h\| < \delta$, then

$$d_H\bigg(T(x+h), \bigcup_{z \in T(x)} (z + S_x(z,h))\bigg) \leq \varepsilon \|h\|,$$

where d_H stand for the Hausdorff distance. The map S_x is called a *differential* of T at x. If T is differentiable at every point of U, T is said to be *differentiable on* U. We do not have the uniqueness of the differential at a point. Moreover, our differential is not necessarily a map with convex values.

(61.2) EXAMPLE. Let $T: \mathbb{R}^n \multimap \mathbb{R}^n$ be defined by

$$T(x) = \operatorname{conv}(T_1(x), \dots, T_n(x)), \quad \text{where } T_i: \mathbb{R}^n \to \mathbb{R}^n$$

is a singlevalued differentiable maps on an open subset U of $\mathbb{R}^n$. Then the map T is differentiable on U and the map $S_x: T(x) \times \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$S_x(z,h) = \bigg\{ \sum_{i=1}^n a_i DT_i(x)(h) \;\bigg|\; \sum_{i=1}^n a_i T_i = z,\ a_i \geq 0 \text{ and } \sum_{i=1}^n a_i = 1 \bigg\}$$

is a differential of T at $x \in U$.

The differentiable multivalued maps have the following properties:

(61.3) PROPOSITION. *Let $T: U \multimap E'$ be a multivalued map differentiable at $x \in U$. Then there exists $\delta > 0$ and there exists $k > 0$ such that $\|S_x(z,h)\| \leq k\|h\|$ for $\|h\| < \delta$ and for every $z \in T(x)$.*

(61.4) PROPOSITION. *If $T: U \multimap E'$ is a multivalued map differentiable at the point $x \in U$, then T is continuous at that point.*

(61.5) DEFINITION. Let $T: U \multimap E'$ be a multivalued map such that $T(x)$ is compact for every $x \in U$, where $U = \{x \in E \mid \|x\| > M > 0\}$. T is *differentiable at infinity* if for any $\varepsilon > 0$, there exists $\delta > 0$ such that $\|h\| > \delta$ implies $d_H(T(h), S(h)) \leq \varepsilon \|h\|$, where $S: E \multimap E'$ is an u.s.c., homogeneous and semilinear positive map. The map S is called a *differential* of T at infinity.

(61.6) DEFINITION. Let $T: U \multimap E'$ be a differentiable map on U. We shall say that T is *continuously differentiable* on U if there exists a map $(x,z) \to S_{x,z}$ such that

(61.6.1) it is continuous on $\bigcup_{x \in U} \{x \times T(x)\}$,
(61.6.2) S_x is a differential of T at x.

(61.7) Proposition. *Let $U \subset E$ be an open set containing the point $0 \in E$ and let $F\colon U \multimap E$ be a multivalued differentiable at 0 map such that $F(0) = 0$. If S_0 is a differential of F at 0 then the m-th iteration F^m of F is differentiable at 0 and S_0^m is a differential of F^m at 0.*

Proof. Since $S_{0,0}$ is homogeneous, $S_{0,0}(0) = 0$ and hence $S_{0,0}^m(0) = 0$.

Since F is differentiable at 0, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$d_H(F(h), S_{0,0}(h)) \leq \varepsilon \|h\| \quad \text{for every } h \in N_\delta(0). \tag{61.7.1}$$

Since F^i is u.s.c. at 0 and $F(0) = 0$, we can assume that $F^i(h) \subset N_\delta(0)$ for all $i \leq m$ if $h \in N_{\delta_1}(0)$ where $\delta_1 > 0$. By (61.7.1) we have (3)

$$d_H(F^i(h), S_{0,0}(F^{i-1}(h))) \leq \varepsilon \|F^{i-1}(h)\| \quad \text{for all } i \leq m.$$

In particular,

$$d_H(F^m(h), S_{0,0}(F^{m-1}(h))) \leq \varepsilon \|F^{m-1}(h)\|.$$

But $d_H(F^{m-1}(h), S_{0,0}(F^{m-2}(h))) \leq \varepsilon \|F^{m-2}(h)\|$ and $S_{0,0}$ is semi-linear positive so,

$$d_H(F^m(h), S_{0,0}^2(F^{m-2}(h))) \leq \varepsilon \|F^{m-1}(h)\| + \varepsilon \|S_{0,0}\| \, \|F^{m-2}(h)\|.$$

We also have

$$\begin{aligned} d_H(F^{m-2}(h), S_{0,0}(F^{m-3}(h))) &\leq \varepsilon \|F^{m-3}(h)\| \\ d_H(F^m(h), S_{0,0}^3(F^{m-3}(h))) &\leq \varepsilon \|F^{m-1}(h)\| + \varepsilon \|S_{0,0}\| \, \|F^{m-2}(h)\| \\ &\quad + \varepsilon \|S_{0,0}\|^2 \|F^{m-3}(h)\| \end{aligned}$$

and by the finite induction, we have

$$d_H(F^m(h), S_{0,0}^m(h)) \leq \varepsilon \sum_{i=0}^{m-1} \|F^{m-i-1}(h)\| \, \|S_{0,0}\|^i.$$

By the finite induction, we will show that

$$d_H(F^m(h), S_{0,0}^m(h)) \leq \varepsilon x_m \|h\|,$$

where $x_m = \sum_{i=0}^{m-1} (\varepsilon + \|S_{0,0}\|)^i \|S_{0,0}\|^{m-1-i}$ and $x_1 = 1$.

The case $m = 1$ is (61.7.1). Suppose that it is true for $k < m$. Let us show this for $k + 1$. By (61.7.1) with $m = k + 1$, we have

$$d_H(F^{k+1}(h), S_{0,0}^{k+1}(h)) \leq \varepsilon \sum_{i=0}^{k} \|F^{k-i}(h)\| \, \|S_{0,0}\|^i.$$

(3) where $\|F^j(h)\| = \operatorname{dist}(0, F^j(h))$.

Let $\varepsilon x_{k+1}\|h\| = \varepsilon\sum_{i=0}^{k}\|F^{k-i}(h)\|\,\|S_{0,0}\|^i$, then

$$\begin{aligned}\varepsilon x_{k+1}\|h\| &= \varepsilon\|F^k(h)\|_\varepsilon + \sum_{i=1}^{k}\|F^{k-i}(h)\|\,\|S_{0,0}\|^i\\ &= \varepsilon\|F^k(h)\| + \varepsilon\bigg(\sum_{j=0}^{k-1}\|F^{k-1-j}(h)\|\,\|S_{0,0}\|^j\bigg)\|S_{0,0}\|\end{aligned}$$

writing down $j = i-1$

$$\begin{aligned}&= \varepsilon x_k\|h\|\,\|S_{0,0}\| + \varepsilon\|F^k(h)\|\\ &\leq \varepsilon x_k\|h\|\,\|S_{0,0}\| + \varepsilon(\varepsilon x_k\|h\| + \|S_{0,0}\|^k\|h\|)\end{aligned}$$

by the induction hypothesis

$$= \varepsilon((\varepsilon + \|S_{0,0}\|)x_k + \|S_{0,0}\|^k)\|h\| = \varepsilon x_{k+1}\|h\|$$

since

$$\begin{aligned}x_{k+1} &= \sum_{i=0}^{k}(\varepsilon + \|S_{0,0}\|^i)\|S_{0,0}\|^{k-i} = \|S_{0,0}\| + \sum_{j=1}^{k}(\varepsilon + \|S_{0,0}\|)^i\|S_{0,0}\|^{k-i}\\ &= \|S_{0,0}\|^k + (\varepsilon + \|S_{0,0}\|)\sum_{i=0}^{k-1}(\varepsilon + \|S_{0,0}\|)^j\|S_{0,0}\|^{k-j-1}\end{aligned}$$

writing down $j = i-1$

$$= \|S_{0,0}\|^k + (\varepsilon + \|S_{0,0}\|)x_k.$$

As a consequence $d_H(F^{k+1}(h), S_{0,0}^{k+1}(h)) \leq \varepsilon x_{k+1}\|h\|$. □

(61.8) PROPOSITION. *Let $U \subset E$ be an open set containing 0. If $F\colon U \multimap E$ is a multivalued map differentiable at 0 such that $F(0) = 0$ and F is a k-set contraction, then $S_{0,0}$ is a k-set contraction.*

PROOF. By Proposition (61.7), the m-th iteration F^m of F is differentiable at 0 and S_0^m is a differential of F^m at 0. Also, $F^m(0) = 0$ since $F(0) = 0$. Then the fact that F^m is a k-set contraction implies that its differential S_0^m at 0 is also one. □

(61.9) PROPOSITION. *Let $U = \{x \in E \mid \|x\| > M > 0\}$ and $F\colon U \multimap E$ be a multivalued map differentiable at infinity. If S is a differential of F at infinity such that $0 \notin \overline{S(h)}$ for $\|h\| = 1$, then the m-th interaction F^m of F is differentiable at infinity and S^m is a differential of F^m at infinity.*

PROOF. Since S is homogeneous, $S(0) = 0$ and $S^m(0) = 0$. Since F is differentiable at infinity, then for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$d_H(F(h), S(h)) \leq \varepsilon\|h\| \quad \text{if } \|h\| > \delta.$$

Since $0 \notin \overline{S(h)}$ for $\|h\| = 1$, there exists $\delta^1 > 0$ such that $F^i(h) \subset E \setminus N_\delta(0)$ for every $i \le m$ if $\|h\| > \delta^1$. We then have $d_H(F^i(h), S(F^{i-1}(h))) \le \varepsilon \|F^{i-1}(h)\|$ for all $i \le m$.

By proceeding similarly as in the proof of (61.7), we obtain the conclusion. □

(61.10) PROPOSITION. *Let $U = \{x \in E \mid \|x\| > M > 0\}$ and let $F\colon U \multimap E$ be a multivalued map differentiable at infinity such that F is a k-set contraction, $k < 1$. If S is a differential of F at infinity such that $0 \notin \overline{S(h)}$ for $\|h\| = 1$, then S is a k-set contraction.*

PROOF. We use the proposition (61.8), and obtain the conclusion by proceeding similarly as in (61.8) □

Now we will formulate and prove our main results of this paper.

(61.11) THEOREM (Expansion at the origin). *Let $C \subset E$ be a cone and $F\colon C \multimap C$ be a u.s.c. multivalued map with convex values such that $F(0) = 0$. Assume F to be differentiable at $0 \in C$ and let S_0 be a differential of F at 0. Assume, furthermore, that there exists a positive integer m such that:*

(61.11.1) *F^m is a k-set contraction,*
(61.11.2) *$S^m_{0,0}(h)$ is convex for every h,*
(61.11.3) $\operatorname{dist}(0, \overline{S^m_{0,0}(\partial B(0,1))}) > \gamma(S^m_{0,0})$,
(61.11.4) *the eigenvalues of $S^m_{0,0}$ are strictly greater than 1.*

Then $i(C, F^m, B(0,r)) = 0$ if r is small enough.

For the proof of (61.11) we need some lemmas.

(61.12) LEMMA. *Let C be a cone in E. Then there exists $y \in C$ such that $\|x + \lambda y\| \ge \|x\|$ for every $x \in C$ and for all $\lambda \ge 0$.*

Lemma (61.12) is a well known fact from functional analysis (see [Be4-M]).

(61.13) LEMMA. *Let C be a cone in E and $S\colon C \multimap C$ be a homogeneous map such that S is a k-set contraction and all eigenvalues of S are different from 1. Then there exists $\varepsilon > 0$ such that $\operatorname{dist}(X, S(x)) > \varepsilon \cdot \|x\|$ for $\|x\| \neq 0$.*

PROOF. Since S is homogeneous, it sufficient to show that there exists $\varepsilon > 0$ such that $\operatorname{dist}(X, S(x)) > \varepsilon$. We proceed by contradiction and assume that there exists a sequence $\{x_n\} \subset C$, $\|x_n\| = 1$ such that $\operatorname{dist}(x_n, S(x_n))$ goes to 0 when $n \to \infty$. Let $z_n \in S(x_n)$ be such that $\lim_n \|x_n - z_n\| = 0$. Then $\gamma(\{x_n\}) = \gamma(\{z_n\}) \le \gamma(S(x_n)) \le k \cdot \gamma(\{x_n\})$, where $k < 1$. But this is true only if $\gamma(\{x_n\}) = 0$. Let $y \in \overline{\{x_n\}}$. Then $\|y\| = 1$ and $y \in S(y)$, a contradiction since 1 is not an eigenvalue of S. □

Now, by using (61.13) contradiction argument we obtain:

(61.14) LEMMA. *Let C be a cone in E and let $F: C \multimap C$ be a multivalued map which is a k-set contraction and differentiable at 0 such that $F(0) = 0$. If S_0 is a differential of F at 0 and all eigenvalues of $S_{0,0}$ are different from 1, then 0 is an isolated fixed point of F.*

Finally, note that from (61.13) it is easy to obtain the following:

(61.15) LEMMA. *Let C be a cone in E and $F: C \multimap C$ a k-set contraction map differentiable at infinity. If S is a differential of F at infinity and all eigenvalues of S are different from 1, then F has no fixed point if $\|x\|$ is big enough.*

PROOF OF (61.11). There are three important steps to show this result.

Step 1. We will show that F is homotopic to $S_{0,0}$ on $B(0,r)$ if r is small enough.

Let G be the homotopy given by:

$$G(h,t) = (H_h(F(h),t),t),$$

where $H_h(y,t) = H(y,h,t) = (1-t)y + t\rho_h(y)$, $y \in F(h)$, $h \in C$, $t \in [0,1]$ and ρ_h is the projection on the convex compact $F(0) + S_{0,0}(h) = S_{0,0}(h)$, i.e. ρ_h is the set of all elements of $S_{0,0}(h)$ which are the nearest to y. Then ρ_h is a multivalued u.s.c. map with convex values. So, G is a homotopy in the class of mappings determined by morphisms.

By Lemma (61.14), G has no fixed point on $\partial B(0,r)$ if r is small enough and G is compacting. By the homotopy property of the fixed point index we obtain:

$$i(C,F,B(0,r)) = i(C,\rho \circ g, B(0,r)),$$

where $g(h) = F(h) \times \{h\}$ and $\rho(y,h') = \rho'_h(y)$ for all $(y,h') \in F(h) \times C$.

Now, we consider the homotopy G' defined as follows:

$$G'(h,t) = (1-t)\rho_h(F(h)) + tS_{0,0}(h) \subset S_{0,0}(h)$$

and hence we deduce:

$$i(C,\rho \circ g, B_r(0)) = i(C,g,B(0,r)) = i(C,S_{0,0},B(0,1)).$$

Step 2. We show now that $S_{0,0}$ is homotopic to $\lambda S_{0,0}$, if $r/\|S_{0,0}(\partial B(0,r))\| < \lambda < 1/\gamma(S_{0,0})$. By the choice of λ, the map $\lambda \cdot S_{0,0}$ is a k-set contraction. Now, we consider the homotopy H' defined by:

$$H'(h,s) = s \cdot S_{0,0}(h) \quad \text{for } 1 \leq s \leq \lambda.$$

Then we obtain

$$i(C,S_{0,0},B(0,r)) = i(C,\lambda \cdot S_{0,0},B(0,r)).$$

Step 3. By (61.11), let $y_0 \in C$ be such that

$$\operatorname{dist}(0, p \cdot y_0 + \lambda S_{0,0}(h)) \geq \lambda \operatorname{dist}(0, S_{0,0}(h))$$

for all $p \geq 0$. Choose $p > (r+\lambda \cdot \|S_{0,0}(\partial B(0,r))\|)/\|y_0\|$ and consider the homotopy H'' defined by:

$$H''(h,t) = tpy_0 + \lambda(S_{0,0}(h)) \quad \text{for all } t \in [0,1].$$

Then we obtain $i(C, \lambda S_{0,0}, B(0,r)) = i(C, py_0 + \lambda S_{0,0}, B(0,r)) = 0$ and the proof is completed. □

(61.16) THEOREM (Compression at the origin). *Let $C \subset E$ be a cone. Let $F\colon C \multimap C$ be a k-set contraction map with convex values and differentiable at $0 \in C$. Let S_0 be a differential of F at 0 such that:*

(61.16.1) *$S_{0,0}(h)$ is convex for every h,*
(61.16.2) *the eigenvalues of $S_{0,0}$ belong to the interval $[0,1)$.*

Then $i(C, F, B(0,r)) = 1$ if r is small enough.

PROOF. As in the Step 1 of the proof of (61.11) we can show that $i(C, F, B(0,r)) = i(C, S_{0,0}, B(0,r))$ if r is small enough. The homotopy H' defined by

$$H'(x,t) = t \cdot S_{0,0}(h), \quad t \in [0,1]$$

has no fixed points on $\partial B(0,r)$. So, by the homotopy property of the index we obtain:

$$i(C, S_{0,0}, B(r,0)) = i(C, f_0, B(0,r)) = 1,$$

where $f_0\colon B(0,r) \to C$ is defined by $f_0(h) = 0$ for every $h \in B(0,r)$ and the proof is completed. □

(61.17) LEMMA. *Under the same hypotheses of Theorem* (61.16)*, the map G defined in the proof of Theorem* (61.11) (*see, Step 1*) *is a k-set contraction.*

PROOF. We have $G(h,t) \subset \operatorname{conv}(F(h) \cup S_{0,0}(h))$ and so

$$G(A \times [0,1]) \subset \operatorname{conv}(F(A) \cup S_{0,0}(A))$$

for every bounded subset A of E. It follows that:

$$\gamma(G(A \times [0,1])) \leq \max(\gamma(F(A)), \gamma(S_{0,0}(A)))$$

and thus G is a k-set contraction since F and $S_{0,0}$ are. □

We will end this section by formulating next four results.

(61.18) THEOREM (Expansion at infinity). *Let $C \subset E$ be a cone. Let $F: C \multimap C$ be an u.s.c. multivalued map with convex values such that $F(0) = 0$, F is a k-set contraction and differentiable at the infinity. Let S be a differential of F at infinity such that:*

(61.18.1) *$S(h)$ is convex for all h,*

(61.18.2) $\operatorname{dist}(0, S(\partial B(0,r))) > \gamma(S)$, *$r > 0$,*

(61.18.3) *the eigenvalues of S are strictly greater than* 1.

Then $i(C, F, B(0,r)) = 0$ if r is big enough.

PROOF. The proof is similar to the proof of (61.11) but we must substitute $S_{0,0}$ by S and use Lemma (61.15) and Proposition (61.8). □

(61.19) THEOREM (Compression at infinity). *Let $C \subset E$ be a cone. Let $F: C \multimap C$ be an u.s.c. multivalued map with convex values such that $F(0) = 0$, F is a k-set contraction and differentiable at infinity. Let S be a differential of F at infinity such that:*

(61.19.1) *$S(h)$ is convex for all h,*

(61.19.2) *the eigenvalues of S belong to the interval* $[0, 1)$.

Then $i(C, F, B(0,r)) = 1$ if r is big enough.

PROOF. Similar to the proof (61.16). □

If we combine the previous theorems, we obtain the following two theorems.

(61.20) THEOREM (Expansion at the origin and compression at infinity). *Let $C \subset E$ be a cone and let $F: C \multimap C$ be an u.s.c. multivalued map with convex values such that $F(0) = 0$, F is a k-set contraction and differentiable at the origin* 0 *and infinity. Assume furthermore that conditions* (61.11.1)–(61.11.4) *are satisfied for $R_1 > 0$ and conditions* (61.19.1)–(61.19.2) *are satisfied for $R_2 > 0$. Let $U = \{x \in C \mid R_1 < \|x\| < R_2\}$. Then $i(C, F, U) = 1$ and thus F has a non-trivial fixed point in U.*

(61.21) THEOREM (Compression at the origin and expansion at infinity). *Let $C \subset E$ be a cone and let $F: C \multimap C$ be an u.s.c. multivalued map with convex values such that $F(0) = 0$, F is a k-set contraction and differentiable at the origin* 0 *and infinity. Assume furthermore that conditions* (61.16.1)–(61.16.2) *are satisfied for $R_1 > 0$ and conditions* (61.18.1)–(61.18.3) *are satisfied for $R_2 > 0$. Let $U = \{x \in C \mid R_1 < \|x\| < R_2\}$. Then $i(U, F, C) = -1$; and thus F has a non-trivial fixed point in U.*

PROOF. By combining the previous theorems. □

62. The generalized topological degree for acyclic mappings

In the bifurcation theory a generalized topological degree for mappings acting between spheres of different dimensions is needed. First, we shall do it for strongly acyclic mappings (see Section 33). To provide necessary constructions we shall appeal to the homotopy groups of sphere. For the unit sphere $S^n \subset \mathbb{R}^{n+1}$ we let $\pi_m(S^n)$, $m = 1, 2, \ldots$ be the m-th homotopy group of S^n. According to the Section 33 by $\mathrm{SA}(m,n)$ (resp. $C(m,n)$) $m \geq n$, we denote the class of all triples (φ, U, y) (resp. (f, U, y)), where U is an open subset of S^n, $y \in S^n$ and $\varphi\colon (\operatorname{cl} U, \partial U) \multimap (S^n, S^n \setminus \{y\})$ is an SA-map (resp. $f\colon (\operatorname{cl} U, \partial U) \to (S^n, S^n \setminus \{y\})$ is a continuous map), where a SA-map means a strongly acyclic map.

The next lemma plays a role in what follows.

(62.1) LEMMA. *Let U be an open subset of a finite-dimensional metric space X and let $y \in S^n$. If $\varphi\colon (\operatorname{cl} U, \partial U) \multimap (S^n, S^n \setminus \{y\})$ is an* SA*-map, then there is an* SA*-map $\varphi'\colon (X, X \setminus U) \multimap (S^n, S^n \setminus \{y\})$ such that $\varphi'|_U = \varphi|_U$. If $\psi'\colon (X, X \setminus U) \multimap (S^n, S^n \setminus \{y\})$ is an* SA*-map such that $\psi'|_U = \varphi|_U$ and $f, g\colon X \to S^n$ are co-selections of φ', ψ', respectively, then the maps f, g are homotopic.*

PROOF. There is $\delta > 0$ such that $\varphi(x) \cap B(y, \delta) = \emptyset$ for $x \in \partial U$. Define $Z = S^n \setminus B(y, (\delta/2))$. The set Z is strongly acyclic and a map $\varphi'\colon X \multimap S^n$ given by the formula

$$\varphi'(x) = \begin{cases} Z & \text{for } x \in X \setminus U, \\ \varphi(x) & \text{for } x \in U, \end{cases}$$

satisfies our requirements. Indeed, it is upper semi-continuous since its graph is closed and has strongly acyclic values. Now, consider an SA-map ψ' as above and a map $f'\colon X \setminus U \to S^n$ given by $f'(x) = y$ for $x \in X \setminus U$. By Proposition (33.4) there is a co-selection f'' of φ' such that $f''|X \setminus U = f'$. Evidently f, f'' are co-selections of φ' and g, f'' are co-selections of ψ'. Therefore, once again by Proposition (33.4) f and g are homotopic. □

We shall define a function

$$\deg\colon \mathrm{SA}(m,n) \to \pi_m(S^n).$$

Let $(\varphi, U, y) \in \mathrm{SA}(m,n)$. By (62.1), there is an SA-map $\varphi'\colon (S^m, S^m \setminus U) \to (S^n, S^n \setminus \{y\})$ such that $\varphi'|_U = \varphi|_U$. Let $f' \in C(m,n)$ be an arbitrary co-selection of φ'. We define the *generalized topological degree* $\deg(\varphi, U, y)$ of φ on U over y by the formula:

$$\deg(\varphi, U, y) = [\alpha \circ f'] \in \pi_m(S^n),$$

where $\alpha\colon S^n \to S^n$, $\alpha(x) = -x$, is the antipodal map and $[\alpha \circ f']$ is the homotopy class of $\alpha \circ f'\colon S^m \to S^n$. It follows from (62.1) that the above definition does not depend on the choice of φ' and f'. So, our definition is correct. If $\varphi = f$ is a singlevalued map, then the above definition coincides with the generalized degree defined for continuous mappings (see [Kr2-M]).

(62.2) PROPOSITION. *Under the above assumptions:*

(62.2.1) (Existence) *If* $\deg(\varphi, U, y) \neq 0 \in \pi_m(S^n)$*, then* $L \subset \varphi(U)$*, where* L *is a component of the set* $S^n \setminus \varphi(\partial U)$ *that contains* y*.*

(62.2.2) (Localization) *If* $V \subset U$ *is open in* S^m *and* $y \in S^n \setminus \varphi(x)$ *for* $x \in U \setminus V$*, then* $(\varphi, V, y) \in \mathrm{SA}(m, n)$ *and* $\deg(\varphi, U, y) = \deg(\varphi, V, y)$*.*

PROOF. If $y \notin \varphi(U)$, then as a choice of f' we can take a constant map $f'\colon S^m \to S^n$, $f'(x) = y$, in that case $[\alpha \circ f'] = 0$. It is clear that $\deg(\varphi, U, y)$ does not depend on a small perturbation of y. It follows that the assertion (62.2.1) holds. The proof of (62.2.2) is self-evident. □

The localization property (62.2.2) can be generalized, however, under stronger assumptions concerning the dimensions m, n.

(62.3) PROPOSITION. *Let* $n \leq m < 2n - 1$*. If* U_1*,* U_2 *are open disjoint subsets of* U *and* $\varphi\colon (\mathrm{cl}\, U, \mathrm{cl}\, U \setminus (U_1 \cup U_2)) \multimap (S^n, S^n \setminus \{y\})$ *is strongly acyclic, then:*

$$\deg(\varphi, U, y) = \deg(\varphi, U_1, y) + \deg(\varphi, U_2, y).$$

The proof of (62.3) relies on the following lemma which is well known in algebraic topology (see [Sp-M]).

(62.4) LEMMA. *Let* X *and* $F\colon (X, \{p\}) \to (S^n \times S^n, \{(y, y)\})$ *be a continuous map. Then there is a map* $G\colon (X, \{p\}) \to (S^n \vee S^n, \{(y, y)\})$ *homotopic to* F *relative to the set* $F^{-1}(S^n \cup S^n)$*, where* $S^n \vee S^n = S^n \times \{y\} \cup \{y\} \times S^n$*.*

PROOF. By Proposition (62.3) $\deg(\varphi, U, y) = \deg(\varphi, U_1 \cup U_2, y)$. Let $A_i = S^m \setminus U_i$, $i = 1, 2$. Obviously, $A_1 \cup A_2 = S^m$. Let fix $p \in A_1 \cap A_2$. Without any loss of generality we may assume that $p = (1, 0, \dots, 0)$. For $\varphi_i = \varphi|_{\mathrm{cl}\, U_i}$, we construct an SA-map $\varphi_i'\colon (S^m, A_i) \multimap (S^n, S^n \setminus \{y\})$ such that $\varphi_i'|_{U_i} = \varphi|_{U_i}$ and a co-selection $f_i'\colon S^m \to S^n$ of φ_i' $(i = 1, 2)$. It is clear that we may assume that $f_i'(x) = y$ for $x \in A_i$, $i = 1, 2$.

Consider a map $f'\colon (S^m, \{p\}) \to (S^n \times S^n, \{(y, y)\})$ given by $f'(x) = (f_1'(x), f_2'(x))$ for $x \in S^m$. Observe that, actually, $f'\colon S^m \to S^n \vee S^n$. Hence, if $\Omega\colon S^n \vee S^n \to S^n$ is defined via $\Omega(z, y) = z$ or $\Omega(y, z) = z$, then the map $\Omega \circ f'\colon S^m \to S^n$ is a well defined co-selection of an arbitrary SA-map $\varphi'\colon (S^m, A_1 \cap A_2) \multimap (S^n, S^n \setminus \{y\})$ such that $\varphi'|_{U_1 \cup U_2} = \varphi|_{U_1 \cup U_2}$. Hence, $\deg(\varphi, U_1 \cup U_2, y) = [\alpha \circ \Omega \circ f']$ and it is sufficient to show that $[f_1'] + [f_2'] = [\Omega \circ f']$. To this end let us recall the definition of the sum $[f_1'] + [f_2']$.

Let $\theta^{\pm}\colon (S^m,\{p\})\times I \to (S^m,\{p\})$ be a deformation such that $\theta^{\pm}(x,1)=p$ for any $x\in S^m_{\pm}$, and let $g_i^{\pm}=f_1'\circ\theta^{\pm}(\,\cdot\,,1)$, $i=1,2$. Obviously, $g_i^{\pm}(S^m_{\pm})=\{y\}$. If $g=(g_1^+,g_2^-)\colon S^m\to S^n\vee S^n$ then, by the definition,

$$[f_1']+[f_2']=[\Omega\circ g].$$

Therefore, we have to show that maps $\Omega\circ g$ and $\Omega\circ f'$ are homotopic. Since f_1' and g_1^+ are homotopic $\{\mathrm{rel}\,p\}$, f_2' and g_2^- are homotopic $\{\mathrm{rel}\,p\}$, then f' and g are homotopic $\{\mathrm{rel}\,p\}$, as well. Let $F\colon (S^m,\{p\})\times I\to (S^n\times S^n,\{(y,y)\})$ be a homotopy joining f' with g, i.e. $F(\,\cdot\,,0)=f'$, $F(\,\cdot\,,1)=g$. We see that $S^m\times\{0\}\cup S^m\times\{1\}\subset F^{-1}(S^n\vee S^n)$. In view of Lemma (62.4), since $m+1<2n$, there is a map $G\colon S^m\times I\to S^n\vee S^n$ such that $G|_{S^m\times\{0\}\cup S^m\times\{1\}}=F|_{S^m\times\{0\}\cup S^m\times\{1\}}$, i.e. $G(\,\cdot\,,0)=f'$, $G(\,\cdot\,,1)=g$. Hence $\Omega\circ f'$ and $\Omega\circ g$ are joined by the homotopy $\Omega\circ G$; the proof is completed. □

In order to prove the next property we shall need the following lemma.

(62.5) LEMMA. *Let X be a metric space with* $\mathrm{diam}(X)<M$*, A be a closed subset of X and $f\colon A\to Y$ be a continuous map. If Y is a topological n-disc, $h\colon D^n\to Y$ is a homeomorphism and $Z=h(S^{n-1})$, then there is a continuous map $F^*\colon X\to Y$ such that $f^*|_A=f$ and $f^*(X\setminus A)\subset Y\setminus Z$.*

PROOF. Let $r\colon X\to I$ be given by $r(x)=1-M^{-1}\,\mathrm{dist}(x,A)$, $x\in X$. By the Tietze theorem, there is $f'\colon X\to D^n$ such that $h\circ f'|_A=f$. Consider a map $f''\colon X\to D^n$ given by $f''(x)=r(x)f'(x)$, $x\in X$. It is easy to verify that a map $f^*=h\circ f''$ satisfies the assertion. □

Assume now that $(\varphi,U,y)\in \mathrm{SA}(m+1,n+1)$, where y belongs to the equatorial sphere of S^{n+1} identified with S^n, and let $U_0=U\cap S^{m+1}_+\cap S^{m+1}_-=U\cap S^m$.

(62.6) PROPOSITION. *Suppose that $\varphi(\mathrm{cl}\,U\cap S^{m+1}_{\pm})\subset S^{n+1}_{\pm}$ and let $B=\mathrm{cl}\,U_0$, where the closure is taken with respect to S^m. Then $B\subset\mathrm{cl}\,U$ and, for $x\in B\setminus U_0$, $y\notin\varphi(x)$. If $\varphi_0\colon B\multimap S^n$ is given by $\varphi_0(x)=\varphi(x)$, $x\in B$, then*

$$\deg(\varphi,U,y)=\sum(\deg(\varphi_0,U_0,y)),$$

where $\sum\colon \pi_m(S^n)\to\pi_{m+1}(S^{n+1})$ is the suspension homomorphism.

PROOF. Let $Z=S^{n+1}\setminus B(y,(\delta/2)$, $Z_0=S^n\cap Z$, where $\delta>0$ is such that $\varphi(x)\cap B(y,\delta)=\emptyset$ for $x\in\mathrm{bd}\,U$. Define SA-maps $\varphi\colon S^{m+1}\multimap S^{n+1}$, $\varphi_0'\colon S^m\multimap S^n$ by the formula

$$\varphi'(x)=\begin{cases} Z & \text{for } x\notin U,\\ \varphi(x) & \text{for } x\in U,\end{cases}\qquad \varphi_0'(x)=\begin{cases} Z_0 & \text{for } x\notin U_0,\\ \varphi_0(x) & \text{for } x\in U_0.\end{cases}$$

Let $f_0'\colon S^m \to S^n$ be a co-selection of φ_0' such that $f_0'(x) = y$ for all $x \in S^m \setminus U_0$. By the definition, $\deg(\varphi_0, U_0, y) = [a \circ f_0'] \in \pi_m(S^n)$. Define $C_\pm = S_\pm^{m+1} \setminus U$ and continuous maps $f_\pm\colon C_\pm \cup S^m \to S^n$ by the formula

$$f_\pm(x) = \begin{cases} y & \text{for } x \in C_\pm, \\ f_0'(x) & \text{for } x \in S^m. \end{cases}$$

In view of Lemma (62.5), we construct maps $f_\pm'\colon S_\pm^{m+1} \to S_\pm^{n+1}$ such that $f_\pm'|_{C_\pm \cup S^m} = f_\pm$ and $f_\pm'(S_\pm^{m+1} \cup C_\pm) \subset S_\pm^{n+1}$. Next, we define $f'\colon S^{m+1} \to S^{n+1}$ by the formula

$$f'(x) = \begin{cases} f_+'(x) & \text{for } x \in S_+^{m+1}, \\ f_-'(x) & \text{for } x \in S_-^{m+1}. \end{cases}$$

Evidently, f' is a co-selection of φ', hence $\deg(\varphi, U, y) = [\alpha \circ f'] \in \pi_{m+1}(S^{n+1})$. On the other hand, observe that $\alpha \circ f'$ is homotopic to the suspension $S(\alpha \circ f_0')$. This completes the proof since $[\alpha \circ f'] = [S(\alpha \circ f_0')] = \sum([\alpha \circ f_0'])$. □

In particular, let $(\varphi, U, y) \in \mathrm{SA}(m, n)$ and put $W = SU$. We may treat W as a subset of S^{m+1}. In view of Proposition (62.6) and (62.2.2), we have the following corollary.

(62.7) COROLLARY (Suspension). *For any open subset $V \subset S^{m+1}$ such that $V \subset W$, if $y \notin \varphi(x)$ for $x \in \operatorname{cl} U \setminus (V \cap S^m)$, then*

$$\deg(S\varphi, V, y) = \sum(\deg(\varphi, U, y)).$$

The next proposition is self-evident.

(62.8) PROPOSITION (Homotopy). *If* SA*-maps $\varphi_0, \varphi_1\colon (\operatorname{cl} U, \operatorname{bd} U) \multimap (S^n, S^n \setminus \{y\})$ are* SA*-homotopic, the* $\deg(\varphi_0, U, y) = \deg(\varphi_1, U, y)$.

Now we shall prove a result concerning the uniqueness of the degree deg introduced above.

(62.9) THEOREM. *If $D\colon \mathrm{SA}(m, n) \to \pi_m(S^n)$ is a function satisfying the homotopy property and the restriction of D to triples $(\varphi, U, y) \in \mathrm{SA}(m, n)$, with φ being singlevalued, is equal to the degree d, then $D = \deg$.*

PROOF. Let $(\varphi, U, y) \in \mathrm{SA}(m, n)$ and let $\varphi'\colon (S^m, S^m \setminus U) \multimap (S^n, S^n \setminus \{y\})$ be an SA-map such that $\varphi'|_U = \varphi|_U$. If f' is a co-selection of φ' such that $f'(x) = y$ for $x \in S^m \setminus U$, then we see that $f'' = f'|_{\operatorname{cl} U}$ is a co-selection of φ and $(\alpha \circ f'', U, y) \in \mathrm{SA}(m, n)$. By Lemma (62.1), φ and $\alpha \circ f''$ are SA-homotopic, hence

$$D(\varphi, U, y) = D(\alpha \circ f'', U, y) = d(\alpha \circ f'', U, y) = [\alpha \circ f'] = \deg(\varphi, U, y).$$

On the other hand, if $(f, U, y) \in C(m, n)$, then it is well known that by the uniqueness of the topological degree one has $\deg(f, U, y) = \deg_B(f, U, y)$, where $\deg_B$ stands for the ordinary Brouwer degree. So the generalized degree deg introduced above is the unique extension of the Brouwer degree for maps in $\mathrm{SA}(m, n)$ with values in $\pi_m(S^n)$. □

We would like to extend now the generalized degree theory to the class of acyclic maps. As above, let $A(m, n)$, $m \geq n$ be the class of all triples (φ, U, y), where U is open in S^m, $y \in S^m$ and $\varphi\colon (\mathrm{cl}\, U, \partial U) \multimap (S^n, S^n \setminus \{y\})$ is an acyclic map.

A deformation retract of $S^n \setminus \{z\}$, has the form

$$H^n(S^n) = Z \in x \to c \cdot x \in Z = H^n(S^n),$$

where $c \in Z$. This number c is called the degree of φ on U over y and denoted by $\deg_y(\varphi, U)$. It is not difficult to show that $\deg_y$ has the suspension property and is consistent with the Brouwer degree for singlevalued maps. Precisely, if $(f, U, y) \in C(n, n) \subset A(n, n)$, then $\deg_y(f, U) = \deg_B(f, U, y)$.

Now, let $(\varphi, U, y) \in A(m, n)$ and $K = \varphi_+^{-1}(y)$. We identify S^m with the equatorial sphere of S^{m+1} and let V be an open subset of S^{m+1} such that $K \subset C \subset \mathrm{cl}\, V \subset SU$. The map $\psi = S\varphi|_{\mathrm{cl}\, V}\colon \mathrm{cl}\, V \multimap S^{n+1}$ is, in view of Proposition (34.4), strongly acyclic and $y \notin \psi(x)$ for $x \in \mathrm{bd}\, V$. Therefore $(\psi, V, y) \in \mathrm{SA}(m+1, n+1)$ and we are in a position to define

$$\mathrm{Deg}(\varphi, U, y) = \deg(\psi, U, y) \in \pi_{m+1}(S^{n+1}).$$

One can easily see this definition is correct since, by (62.22.2), it does not depend on the choice of V.

(62.10) Remark. Assume that $(\varphi, U, y) \in \mathrm{SA}(m, n) \subset A(m, n)$. By Corollary (62.7), $\mathrm{Deg}(\varphi, U, y) = \sum(\deg(\varphi, U, y))$.

The degree Degdefined above has similar properties to those of degree deg.

(62.11) Theorem. *Suppose that U, V, φ, K, y are as above.*

(62.11.1) *If $\mathrm{Deg}(\varphi, U, y) \neq 0 \in \pi_{m+1}(S^{n+1})$ and L is a component of $S^n \setminus \varphi(\partial U)$ that contains y, then $L \subset \varphi(U)$.*

(62.11.2) *If $m < 2n$, $U = U_1 \cup \ldots \cup U_k$ and the sets $K_i = K \cap U_i$, $i = 1, 2, \ldots, k$ are pairwise disjoint, then*

$$\mathrm{Deg}(\varphi, U, y) = \sum_{i=1}^{k} \mathrm{Deg}(\varphi, U_i, y).$$

(62.11.3) $\mathrm{Deg}(S\varphi, V, y) = \sum(\mathrm{Deg}(\varphi, U, y))$.

(62.11.4) *If* $\varphi\colon (\operatorname{cl} U, \partial U) \times I \multimap (S^n, S^n \setminus \{y\})$ *is acyclic, then* $\operatorname{Deg}(\varphi_0, U, y) = \operatorname{Deg}(\varphi_1, U, y)$*, where* $\varphi_i = \varphi(\,\cdot\,, i)$*,* $i = 0, 1$.

We end this section with the following observation concerning the uniqueness of the degree Deg.

(62.12) PROPOSITION.

(62.12.1) *If* $D\colon A(m,n) \to \pi_{m+1}(S^{n+1})$ *is a function satisfying the suspension property* (62.11.3) *and* $D|\mathrm{SA}(m,n) = \sum \circ \deg$*, then* $\sum \circ D = \sum \circ \operatorname{Deg}$.

(62.12.2) *For any* $(\varphi, U, y) \in A(n,n)$*,* $\operatorname{Deg}(\varphi, U, y) = \deg_y(\varphi, U)$.

PROOF. Let $(\varphi, U, y) \in A(m,n)$ and let V have the same meaning as above. Then $D(S\varphi, V, y) = \sum(D(\varphi, U, y))$. On the other hand,

$$D(S\varphi, V, y) = \sum(\deg(S\varphi, V, y)) = \sum(\operatorname{Deg}(\varphi, U, y)).$$

Since $\deg_y$ satisfies the suspension property, in view of Theorem (62.9), we have $\deg_y(\varphi, U) = \deg_y(S\varphi, V, y) = \deg(S\varphi, V, y) = \operatorname{Deg}(\varphi, U, y)$. □

63. The bifurcation index

When dealing with the phenomenon of a bifurcation on solutions of inclusions (e.g. multivalued equations) with parameters some different homotopy invariants are needed. In this section we will introduce an invariant available for acyclic maps and inclusions involving this type of maps.

Let $\varphi\colon U \multimap \mathbb{R}^n$, where U is an open subset of $\mathbb{R}^m = \mathbb{R}^k \times \mathbb{R}^n$, i.e. $m = n + k$, $n, k \geq 1$, be an acyclic map. Define:

$$\begin{aligned} T &= \{\lambda \in \mathbb{R}^n \mid (\lambda, 0) \in U\}, \\ U_0 &= U \cap (\mathbb{R}^k \times \{0\}), \quad \text{i.e. } U_0 = T \times \{0\}, \\ S &= \{(\lambda, x) \in U \setminus U_0 \mid 0 \in \varphi(\lambda, x)\}. \end{aligned}$$

We assume that $0 \in \varphi(\lambda, 0)$ for all $\lambda \in T$. Then the set $\mathcal{B}(\varphi) = \{\lambda \in T \mid (\lambda, 0) \in \operatorname{cl} S\}$ is called the set of *bifurcation points* for φ. We will assume that $\mathcal{B}(\varphi)$ is compact.

In order to define the bifurcation index $\mathrm{BI}(\varphi)$ of φ we shall need some auxiliary objects. Let us consider a continuous function $\alpha\colon T \to (0, +\infty)$ such that:

$$0 < \alpha(\lambda) < \operatorname{dist}((\lambda, 0), \partial U \cup \operatorname{cl} S).$$

Next let

$$\begin{aligned} X &= \{(\lambda, x) \in \mathbb{R}^m \mid \lambda \in T,\ \|x\| = \alpha(\lambda)\}, \\ X^+ &= \{(\lambda, x) \in \mathbb{R}^m \mid \lambda \in T,\ \|x\| < \alpha(\lambda)\}. \end{aligned}$$

Observe that $\mathrm{cl}X^+ = X^+ \cup X \subset U$ and put $X^- = U \setminus \mathrm{cl}\, X^+$. It is easy to see that $S \subset X^-$ and $\mathcal{B}(\varphi) \times \{0\} \subset X$.

Let $f: U \to R$ be a continuous function such that:

$$\begin{cases} f(\lambda, x) < 0 & \text{for } (\lambda, x) \in X^-, \\ f(\lambda, x) = 0 & \text{for } (\lambda, x) \in X, \\ f(\lambda, x) > 0 & \text{for } (\lambda, x) \in X^+. \end{cases}$$

We consider a multivalued map $\Phi = (\varphi, f): U \multimap \mathbb{R}^{n+1}$, i.e. $\Phi(\lambda, x) \times \{f(\lambda, x)\}$, $(\lambda, x) \in U$. Evidently, Φ is an SA-map and it is easy to see that $\Phi_+^{-1}(0) = \mathcal{B}(\varphi) \times \{0\}$. Hence, $\Phi_+^{-1}(0)$ is compact in U and we may define

$$\mathrm{BI}(\varphi) = \deg(\Phi, U, 0) \in \pi_m(S^{n+1}).$$

This definition is correct, since by Proposition (62.8) it does not depend on the choice of α and the complementing function f. The element $\mathrm{BI}(\varphi)$ of the group $\pi_m(S^{n+1})$ is called the *bifurcation index* of φ.

We shall now collect some properties of the bifurcation index.

(63.1) Theorem (Existence and structure of solutions). *If* $\mathrm{BI}(\varphi) \neq 0$, *then* $\mathcal{B}(\varphi) \neq \emptyset$ *and, moreover, there exists a connected subset* C *of* S *such that* $\mathrm{cl}\, C \cap U_0 = \mathrm{cl}\, C \cap (\mathcal{B}(\varphi) \times \{0\}) \neq \emptyset$ *and* C *is not contained in any compact subset of* U.

Proof. Let $A = S^m \setminus i_m(U)$ (recall that $i_m: \mathbb{R}^m \to S^m = \mathbb{R}^m \cup \{\infty\}$ is the embedding). We shall prove that $B = i_m(\mathcal{B}(\varphi) \times \{0\})$ and A cannot be separated in the compact space $Z = B \cup i_m(S) \cup A$. If so, then in virtue of [Al], Proposition 5, there is a connected subset $C \subset i_m(S)$ such that $\mathrm{cl}\, C \cap B \neq \emptyset$ and $\mathrm{cl}\, C \cap A \neq \emptyset$, and this is what we actually require.

Suppose, to the contrary, that A and B can be separated in Z. Hence, there is an open set $W \subset U$ such that

$$\mathcal{B}(\varphi) \times \{0\} \subset W, \quad \mathrm{cl}(i_m(W)) \cap A = \emptyset, \quad i_m(S) \cap \mathrm{bd}(i_m(W)) = \emptyset.$$

Therefore, $\mathrm{cl}\, W \subset U$, $\mathrm{cl}\, W$ is compact and $S \cap \mathrm{bd}\, W = \emptyset$.

Let $W' = X^+ \cup W$ and let $X' = \mathrm{bd}\, W'$. It is easy to see that $X' = X^- \cap \partial W \cup X \cap (U \setminus \mathrm{cl}\, W) \cup X \cap \partial W$. We define a continuous function $f': U \to R$ such that

$$f'(\lambda, x) \begin{cases} > 0 & \text{for } (\lambda, x) \in W', \\ = 0 & \text{for } (\lambda, x) \in X', \\ < 0 & \text{for } (\lambda, x) \in U \setminus \mathrm{cl}\, W', \end{cases}$$

and an SA-map $\Phi' = (\varphi, f'): U \multimap \mathbb{R}^{n+1}$.

First, observe that $\Phi'^{-}(0) = \{y \in U \mid 0 \in \Phi'(y)\} = \emptyset$. Indeed, if $0 \in \Phi'(\lambda, x)$, then $(\lambda, x) \in X'$ and $0 \in \varphi(\lambda, x)$; if $(\lambda, x) \in X^{-} \cap \partial W$, then $x \neq 0$, hence $(\lambda, x) \in S$; if $(\lambda, x) \in X \cap (U \setminus \operatorname{cl} W) \cup X \cap \partial W$, then $x = 0$ and $\alpha(\lambda) = 0$, i.e. $\lambda \in \mathcal{B}(\varphi)$. In both cases we obtain a contradiction.

Let an SA-map $\chi: U \times I \multimap \mathbb{R}^{n+1}$ be given by the formula

$$\chi(\lambda, x, t) = \varphi(\lambda, x) \times \{(1-t)f(\lambda, x) + tf'(\lambda, x)\}$$

for $(\lambda, x) \in U$ and $t \in I$. We see that $\chi_0 = \Phi$ and $\chi_1 = \Phi'$. One can easily show that the set

$$\{(\lambda, x) \in U \mid 0 \in \chi(\lambda, x, t) \text{ for some } t \in T\}$$

is contained in $\operatorname{cl} X^{-} \cap \operatorname{cl} W$, hence it is compact.

Evidently $\deg(\Phi', U, 0) = \deg(\Phi, U, 0) = \mathrm{BI}(\varphi) \neq 0$, i.e. $\Phi_{+}^{-1}(0) \neq \emptyset$ - a contradiction. In other words, we have proved that if $\mathrm{BI}(\varphi) \neq 0$, then there exists a connected branch C of nontrivial solutions such that $\operatorname{cl} C \cap \mathcal{B}(\varphi) \times \{0\} \neq \emptyset$, and either C is unbounded or $\operatorname{cl} C \cap \partial U \neq \emptyset$. □

(63.2) Corollary (Compactness). *If the set* $\operatorname{cl} S$ *is compact in* U*, then*

$$\mathrm{BI}(\varphi) = 0.$$

(63.3) Proposition (Localization). *If* $V \subset U$ *is open and* $\mathcal{B}(\varphi) \times \{0\} \subset V$*, then* $\mathrm{BI}(\varphi) = \mathrm{BI}(\varphi|_V)$*. Thus* $\mathrm{BI}(\varphi)$ *depends only on the behaviour of* φ *on a neighbourhood of* $\mathcal{B}(\varphi) \times \{0\}$*. In particular, if* φ *is defined on a larger open set* $U' \supset U$ *such that* $(U' \setminus U) \cap \mathbb{R}^k \times \{0\} = \emptyset$ *then* $\mathrm{BI}(\varphi) = \mathrm{BI}(\varphi|_U)$*.*

Proof. Follows from the localization and homotopy properties of deg. □

We are now in a position to state one of the main results of this section.

Assume that there is an acyclic map $\psi: W \multimap \mathbb{R}^n$, where W is open in $\mathbb{R}^m$, $U \subset W$ such that $\psi|_U = \varphi$ and $0 \in \psi(\lambda, 0)$ for any $(\lambda, 0) \in W_0 = W \cap \mathbb{R}^k \times \{0\}$, ($\varphi$ satisfies all the assumptions from the beginning of this section). Let

$$P = \{(\lambda, x) \in W \setminus W_0 \mid 0 \in \psi(\lambda, x)\}.$$

(63.4) Corollary (Global Bifurcation). *If* $\mathrm{BI}(\varphi) = \mathrm{BI}(\psi|_U) \neq 0$*, then there exists a connected branch* $C \subset P$ *such that* $\operatorname{cl} C \cap \mathcal{B}(\varphi) \times \{0\} \neq \emptyset$ *and at last one of the following occurs:*

(63.4.1) C *is unbounded,*
(63.4.2) $\operatorname{cl} C \cap \partial W \neq \emptyset$,
(63.4.3) *there is a point* $\lambda_0 \in \mathbb{R}^k \setminus T$ *(i.e.* $(\lambda_0, 0) \in W_0 \setminus U_0$*) such that* $(\lambda_0, 0) \in \operatorname{cl} C$*.*

Thus ψ has bifurcation points outside U connected to $\mathcal{B}(\varphi) \times \{0\}$ in $\operatorname{cl} P$.

Proof. Let $A = S^m \setminus i_m(W)$. If the sets A and $B = i_m(\mathcal{B}(\varphi) \times \{0\})$ cannot be separated in a compact space $Z = B \cup i_m(P) \cup A$, then in view of [Al], (63.4.1) or (63.4.2) holds.

If A and B can be separated in Z, then there is an open set V such that $\mathcal{B}(\varphi) \times \{0\} \subset V \subset \operatorname{cl} V \subset W$, $P \cap \partial V = \emptyset$ and $\operatorname{cl} V$ is compact. Therefore

$$P' = P \cap \operatorname{cl} V \subset V$$

and P' is compact.

If $V_0 = V \cap \mathbb{R}^k \times \{0\} \subset U_0$, then in view of Proposition (63.3) $\mathrm{BI}(\psi|_V) = \mathrm{BI}(\varphi)$ and, by Corollary (63.2) $\mathrm{BI}(\psi|_V) = 0$, which is a contradiction.

Hence $D = \operatorname{cl} V_0 \setminus U_0$ is compact nonempty. Consider a compact space $Z' = D \cup P' \cup \mathcal{B}(\varphi) \times \{0\}$. If D and $\mathcal{B}(\varphi) \times \{0\}$ can not be separated, then conclusion (63.4.3) holds. Suppose to the contrary that D and $\mathcal{B}(\varphi) \times \{0\}$ can be separated. Then there are open disjoint sets $U'', U' \subset W$ such that $U'' \cup U' \supset Z'$, $D \subset U''$ and $\mathcal{B}(\varphi) \times \{0\} \subset U'$. Hence $\{(\lambda, x) \in U' \mid x \neq 0,\ 0 \in \psi(\lambda, x)\}$ is compact (in U'). Thus, by Proposition (63.3) $\mathrm{BI}(\varphi) = \mathrm{BI}(\psi|_{U'})$. But, in view of Corollary (63.2), $\mathrm{BI}(\psi|_{U'}) = 0$, which is a contradiction. □

Now we shall proceed with collecting some properties of the bifurcation index.

As before, by restricting dimensions we may generalize the property of localization.

(63.5) Proposition (Additivity). *Assume that $k < n + 1$.*

(63.5.1) *If U_1, U_2 are open disjoint, $U = U_1 \cup U_2$, then*

$$BI(\varphi) = \mathrm{BI}(\varphi|_{U_1}) + \mathrm{BI}(\varphi|_{U_2}).$$

(63.5.2) *If U_1, U_2 are open subsets of U, $\mathcal{B}(\varphi) \times \{0\} \subset U_1 \cup U_2$ and $\mathcal{B}(\varphi) \times \{0\} \cap \partial(U_1 \cap U_2) = \emptyset$, then $\mathrm{BI}(\varphi|_{U_1 \cap U_2})$, $BI(\varphi|_{U_1})$ and $\mathrm{BI}(\varphi|_{U_2})$ are defined, and*

$$\mathrm{BI}(\varphi) = \mathrm{BI}(\varphi|_{U_1}) + \mathrm{BI}(\varphi|_{U_2}) - \mathrm{BI}(\varphi|_{U_1 \cap U_2}).$$

Proof. Part (63.5.1) follows easily from the additivity property of deg (observe for this reason we needed $n + k < 2(n + 1) - 1$, i.e. $k < n + 1$). (63.5.2) is just a restatement of (63.5.1). □

An important property of the bifurcation index is the homotopy invariance. We will give two versions of this fact.

(63.6) PROPOSITION (Homotopy Invariance I). *Let* $\varphi: U \times I \multimap \mathbb{R}^n$ *be an acyclic map such that* $0 \in \varphi(\lambda, 0, t)$ *for all* $\lambda \in T$, $t \in I$. *If the set* $\bigcup_{t\in I} B(\varphi_t)$ *is compact in* T, *then* $\mathrm{BI}(\varphi_0) = \mathrm{BI}(\varphi_1)$.

PROOF. Is obvious. □

However, the above homotopy invariance is not sufficient since in general, one cannot avoid the bifurcation points of φ_t escaping through the boundary of T during the homotopy (i.e. when t runs through I). For instance, this is the case when maps φ_0, φ_1 are singlevalued close together and one considers a linear homotopy joining them.

Hence we have to formulate another homotopy invariance property.

(63.7) THEOREM (Homotopy Invariance II). *Let* $\varphi: U \times I \multimap \mathbb{R}^n$ *be an acyclic map such that* $\mathrm{BI}(\varphi_i)$ *is defined for* $i = 0, 1$. *Suppose that there are open sets* $V, W \subset T$ *such that* $B(\varphi_0) \cup B(\varphi_1) \subset V \subset \mathrm{cl}\, V \subset W$ *and* $\mathrm{cl}\, V$ *is compact. Let* $G = W \setminus V$. *If there is* $\varepsilon > 0$ *such that* $G \times D^n_\varepsilon \subset U$ *and, for any* $t \in I$, $(\lambda, x) \in G \times S^{n-1}_\varepsilon$, $0 \notin \varphi(\lambda, x)$ *and, for any* $\lambda \in G$, $0 < |x| \le \varepsilon$, $i = 0, 1$, $0 \notin \varphi(\lambda, x, i)$, *then* $\mathrm{BI}(\varphi_0) = \mathrm{BI}(\varphi_1)$.

Observe that Proposition (63.6) follows as a consequence from Theorem (63.5). In fact, under the hypotheses of Proposition (63.6), $B = \bigcup_{t\in I} B(\varphi_t)$ is compact in T; hence taking any open sets V, W such that $B \subset V \subset \mathrm{cl}\, V \subset W \subset \mathrm{cl}\, W \subset T$ and as $\mathrm{cl}\, W$ is compact, we can put $\varepsilon = (1/2)\,\mathrm{dist}(\mathrm{cl}\, W \setminus V, \mathrm{cl}\, S \cup \partial U)$, where $S = \{(\lambda, x) \in U \setminus U_0 \mid 0 \in \varphi(\lambda, x, t) \text{ for some } t \in I\}$.

PROOF OF (63.7). Put $U' = U \cap W \times \mathbb{R}^n$. For $i = 0, 1$ let

$$\begin{gathered}
S_i = \{(\lambda, x) \in U \setminus U_0 \mid 0 \in \varphi(\lambda, x, i)\}, \\
\alpha_i(\lambda) = \min\left\{1, \frac{1}{2}\mathrm{dist}((\lambda, 0), \mathrm{cl}\, S_i \cup \partial U)\right\}, \quad \text{for } \lambda \in T, \\
X_i = \{(\lambda, x) \in \mathbb{R}^m \mid \lambda \in \mathbb{R},\ |x| = \alpha_i(\lambda)\}
\end{gathered}$$

and $f_i: U \to \mathbb{R}$ continuous such that

$$f_i(\lambda, x) \begin{cases} > 0 & \text{for } |x| < \alpha_i(\lambda), \\ = 0 & \text{for } (\lambda, x) \in X_i, \\ < 0 & \text{for } \lambda \notin T \text{ or } |x| > \alpha_i(\lambda). \end{cases}$$

According to the definition and Proposition (63.3) $\mathrm{BI}(\varphi_i) = \deg((\varphi_i, f_i), U', 0)$, $i = 0, 1$. Because of our assumption we can modify α_i in such a manner that $\alpha_i(\lambda) = \varepsilon$ for $\lambda \in G$.

Let $\alpha = \min\,\{\alpha_0, \alpha_1\}$, $X = \{(\lambda, x) \in \mathbb{R}^m \mid \lambda \in W,\ |x| = \alpha(\lambda)\}$. We see that $X \subset U'$. Define a continuous function $f\colon U' \to R$ such that

$$f(\lambda, x)\begin{cases} > 0 & \text{for } |x| < \alpha(\lambda), \\ = 0 & \text{for } (\lambda, x) \in X, \\ < 0 & \text{for } |x| > \alpha(\lambda). \end{cases}$$

Using the homotopy property it is easy to see that $\mathrm{BI}(\varphi_i) = \deg((\varphi_i, f), U', 0)$, $i = 0, 1$.

Consider an SA-homotopy $\chi\colon U' \times I \multimap \mathbb{R}^{n+1}$ given by the formula

$$\chi(\lambda, x, t) = \varphi(\lambda, x, t) \times \{f(\lambda, x)\}, \quad (\lambda, x) \in U',\ t \in I.$$

We easily see that, for $(\lambda, x) \in U'$, $t \in I$, if $\lambda \in G$, then $0 \notin \varphi(\lambda, x, t)$. Therefore, the set $\{(\lambda, x) \in U' \mid 0 \in \varphi(\lambda, x, t)$ for some $t \in I\}$ is contained in $X \cap \operatorname{cl} V \times \mathbb{R}^n$ and is compact. Hence $\mathrm{BI}(\varphi_0) = \deg((\varphi_0, f), U', 0) = \deg((\varphi_1, f), U', 0) = \mathrm{BI}(\varphi_1)$ and the proof is completed. □

One can easily formulate a version of the suspension property for BI. We confine ourselves to the statement of the following so called stability property.

(63.8) PROPOSITION (Stability). *Let* $\mathrm{id}\colon \mathbb{R} \to \mathbb{R}$ *be the identity map. If* $\varphi \times \mathrm{id}\colon U \times \mathbb{R} \multimap \mathbb{R}^{n+1}$ *is given by the formula* $(\varphi \times \mathrm{id})(\lambda, x, y) = \varphi(\lambda, x) \times \{y\}$, *then*

$$\mathrm{BI}(\varphi \times \mathrm{id}) = \sum(\mathrm{BI}(\varphi)),$$

where (as before) $\sum\colon \pi_m(S^{n+1}) \to \pi_{m+1}(S^{n+2})$.

We leave the easy proof to the reader.

64. Multivalued dynamical systems

In this section we assume that (X, d) is a metric locally compact space. We will consider only discrete multivalued flows. We will use the following notations taken from [KaM] and [Mr1]. For $A \subset X$ we let $\operatorname{bd} A = \partial A$ namely, the boundary of A in X, $B_\varepsilon(A) = O_\varepsilon(A) = \{x \in X \mid \operatorname{dist}(x, A) < \varepsilon\}$, $\varepsilon > 0$, $\operatorname{diam} A = \delta(A)$ to be the diameter of A. Moreover, we denote the sets of all integers, nonnegative integers, and nonpositive integers by $\mathbb{Z}, \mathbb{Z}^+, \mathbb{Z}^-$, respectively. By an interval I in $\mathbb{Z}$ we understand the intersection $[a, b] \cap \mathbb{Z}$, for some $[a, b] \subset R$.

(64.1) DEFINITION. An u.s.c. mapping $F\colon X \times \mathbb{Z} \multimap X$ with compact values is called a *discrete multivalued dynamical system* (dmds) if the following conditions are satisfied:

(64.1.1) For all $x \in X$, $F(x, 0) = \{x\}$,

(64.1.2) For all $n, m \in \mathbb{Z}$ with $n, m \geq 0$ and all $x \in X$,

$$F(F(x, n), m) = F(x, n + m),$$

(64.1.3) For all $x, y \in X$, $y \in F(x, -1)$ if and only if $x \in F(y, 1)$.

We use the notation $F^n(x) := F(x, n)$ and $F(x) := F^1(x)$. Thus F^n coincides with a superposition of $F: X \to P(X)$ or its inverse F^{-1}. In what follows, F is a given dmds.

(64.2) DEFINITION. Let I be an interval in $\mathbb{Z}$ with $0 \in I$. A singlevalued mapping $\sigma: I \to X$ is a *solution for F through* $x \in X$ if $\sigma(n + 1) \in F(\sigma(n))$ for all $n, n + 1 \in I$, and $\sigma(0) = x$.

Note that if $\sigma: I \to X$ is a solution for F then $\sigma(n) \in F^n(\sigma(0))$ for all $n \in I$. (The proof is straightforward by induction on m and k, where $I = [-k, m]$, $k, m \in \mathbb{Z}^+$). The existence of a solution through x forces $F^n(x)$ to be nonempty for $n \in I$.

Given a subset $N \subset X$, we introduce the following notation:

$$\begin{aligned} \operatorname{inv}^+ N &:= \{x \in N \mid \text{ there exists a solution } \sigma : \mathbb{Z}^+ \to N \text{ for } F \text{ through } x\}, \\ \operatorname{inv}^- N &:= \{x \in N \mid \text{ there exists a solution } \sigma : \mathbb{Z}^- \to N \text{ for } F \text{ through } x\}, \\ \operatorname{inv} N &:= \{x \in N \mid \text{ there exists a solution } \sigma : \mathbb{Z} \to N \text{ for } F \text{ through } x\}. \end{aligned}$$

By (64.1) we have: $\operatorname{inv} N = \operatorname{inv}^+ N \cap \operatorname{inv}^- N$.

Let $\operatorname{diam}_N F := \sup\{\operatorname{diam} F(x) \mid x \in N\}$ and $\operatorname{dist}(A, B) := \min \{d(x, y) \mid x \in A,\ y \in B\}$, $A, B \subset X$. We also put $F_+^{-1}(A) := \{x \in X \mid F(x) \cap A \neq \emptyset\}$ (called the *weak inverse image of* A), $A \subset X$.

(64.3) DEFINITION. A compact subset $N \subset X$ is called

(a) an *isolating neighbourhood for F* if

$$B_{\operatorname{diam}_N F}(\operatorname{inv} N) \subset \operatorname{Int} N, \tag{64.3.1}$$

(b) an *isolating block for F* if

$$B_{\operatorname{diam}_N F}(F^{-1}(N) \cap N \cap F(N)) \subset \operatorname{Int} N. \tag{64.3.2}$$

A straightforward verification shows that (64.3.2) implies (64.3.1).

(64.4) DEFINITION. Let N be an isolating neighbourhood for F. A pair $P = (P_1, P_2)$ of compact subsets $P_2 \subset P_1 \subset N$ is called an *index pair* it the following conditions are satisfied:

(64.4.1) $F(P_i) \cap N \subset P_i$, $i = 1, 2$,
(64.4.2) $F(P_1 \setminus P_2) \subset N$,
(64.4.3) $\operatorname{inv} N \subset \operatorname{Int}(P_1 \setminus P_2)$.

Our first aim is to prove the following result:

(64.5) THEOREM. *Let N be an isolating neighbourhood for F and W be a neighbourhood of* $\operatorname{inv} N$. *Then there exists an index pair P for N with $P_1 \setminus P_2 \subset W$.*

The proof is based on several lemmas. First, given $N \subset X$, $x \in N$, and $n \in \mathbb{Z}^+$, the following notation will be used:

$$F_{N,n}(x) := \{y \in N \mid \text{ there exists a solution } \sigma\colon [0,n] \to N \text{ for } F \text{ such that } \sigma(0) = x \text{ and } \sigma(n) = y\};$$

$$F_{N,-n}(x) := \{y \in N \mid \text{ there exists a solution } \sigma\colon [-n,0] \to N \text{ for } F \text{ such that } \sigma(-n) = y \text{ and } \sigma(0) = x\};$$

$$F^+_N(x) := \bigcup_{n\in\mathbb{Z}} F_{N,n}(x), \quad F^-_N(x) := \bigcup_{n\in\mathbb{Z}} F_{N,-n}(x).$$

(64.6) PROPOSITION. *If $N \subset X$ is compact, then $F_{N,n}\colon N \to 2^N$ is u.s.c. for any $n \in \mathbb{Z}$.*

PROOF. It is enough to prove the assertion for $n \in \mathbb{Z}^+$ since the case for a negative n is analogous. Suppose that $F_{N,n}$ is not u.s.c. Then there exists an open subset U of N, a point $x \in N$ with $F_{N,n}(x) \subset U$ and a convergent sequence $x_k \to x$, $\{x_k\} \subset N$ with $F_{N,n}(x_k) \cap (N \setminus U) \neq \emptyset$. Consequently for each k, there exists a solution $\sigma_k\colon [0,n] \to N$ for F through x_k such that $\sigma_k(m_k) \in N \setminus U$ for some $m_k \in [0,n]$. By passing to a subsequence we may assume that $m_k \equiv m \in [0,n]$ for all k. Since $N \setminus U$ is compact, we may assume that $\sigma_k(i) \to y_i \in N$ for $i \in [0,n]$, moreover, $y_m \in N \setminus U$. We define $\sigma(i) = y_i$, $I \in [0,n]$. By the closed graph property of F, $\sigma(i+1) \in F(\sigma(i))$, moreover, $\sigma(0) = x$ and $\sigma(m) \in N \setminus U$. That contradicts the hypothesis $F_{N,n}(x) \subset U$. The proof is completed. □

(64.7) LEMMA. *Let $N \subset X$ be compact and suppose that, for all $n \in \mathbb{Z}^+$, $D(F_{(N,n)}) \neq \emptyset$. Then* $\operatorname{inv} N \neq \emptyset$. *Moreover,* $\operatorname{inv}^{(\pm)} N = \bigcap\{D(F_{N,n}) \mid n \in \mathbb{Z}$, *(respectively $n \in \mathbb{Z}^+$, $n \in \mathbb{Z}^-$)*$\}$.

PROOF. It is easy to see that $\{D(F_{N,n})\}_{n=0,1,\dots}$ is a decreasing sequence of compact sets, therefore its intersection K is nonempty.

We prove that $\operatorname{inv}^+ N = K$. The proof for $\operatorname{inv}^- N$ is analogous and the conclusion for $\operatorname{inv} N$ follows from (64.3.1).

The inclusion $\operatorname{inv}^+ N \subset K$ is obvious. Suppose that $x \in K$. Then, for each $n \in \mathbb{Z}^+$ there exists a solution $\sigma_n: [0,n] \to N$ for F through x. We construct a solution $\sigma: \mathbb{Z}^+ \to N$ for F through x by recurrence. Evidently, $\sigma(0) := x$. Suppose that $\sigma|_{[0,n]}$ is constructed and there is a sequence $\{k_i\}$ with $k_i > n$ such that $\sigma_{k_i} \to \sigma(n)$ as $i \to \infty$. By the compactness of N and passing again to a subsequence, we may assume that $\sigma_{k_i}(n+1) \to y_{n+1} \in N$ as $i \to \infty$. By the closed graph property, $y_{n+1} \in F(\sigma(n))$, so, it remains to put $\sigma(n+1) := y_{n+1}$. □

(64.8) LEMMA. *Let $N \subset X$ be compact. Then*

(64.8.1) *The sets $\operatorname{inv}^+ N$, $\operatorname{inv}^- N$ and $\operatorname{inv} N$ are compact,*
(64.8.2) *If A is compact with $\operatorname{inv}^- N \subset A \subset N$ then $F_N^+(A)$ is compact.*

PROOF. (64.8.1) Since N is compact and $F_{N,n}$ is u.s.c. the set $D(F_{N,n})$ is compact for any given n. The intersection of a family of compact sets is compact, hence the conclusion.

(64.8.2) It is sufficient to show that $F_N^+(A)$ is closed. Let $\{y_k\}$ be a sequence of points in N, $\sigma_k: [0,n_k] \to N$ a solution for F with $\sigma(0) \in A$ and $\sigma_k(n_k) = y + k$, for each k, and let $y_k \to y \in N$. We need to show that $y \in F_N^+(A)$.

Case 1. $\{n_k\}$ is bounded: Then, by passing to a subsequence if necessary, we may assume that $n_k = n$ for all k. Since N and A are compact, we may assume that $\sigma_k(i) \to y_i \in N$, $i = 0, \ldots, n$, $y_0 \in A$, and $y_n = y$. By the closed graph property, $y_{i+1} \in F(y_i)$, $i, i+1 \in [0,n]$, one may therefore, define $\sigma(i) := y_i$, $i = 0, 1, \ldots, n$. It shows that $y \in F_{N,n}(A)$.

Case 2. $\{n_k\}$ is unbounded: Then, by passing to a subsequence, we may assume that $\{m_k\}$ is increasing and, by restricting the interval $[0,n_k]$, that $n_k = k$. Let $\sigma_k'(i) = \sigma_k(i+k)$, $i \in [-k,0] \to N$ is a solution for F with $\sigma_k'(-k) \in N$ and $\sigma_k'(0) = y_k$. Then $y_k \in D(F_{N,-k})$ and, by the same argument as in the proof of Lemma (64.7), $y \in \operatorname{inv}^- N \subset A$, the proof is completed. □

(64.9) LEMMA. *Let $K \subset N$ be compact subsets of X such that $K \cap \operatorname{inv}^+ N = \emptyset$ (respectively, $K \cap \operatorname{inv}^- N = \emptyset$). Then*

(64.9.1) *$F_{N,n}(K) = \emptyset$ for all but finitely many $n > 0$ (respectively, $n < 0$),*
(64.9.2) *The mapping F_N^+ (respectively, F_N^-) is u.s.c. on K,*
(64.9.3) *$F_N^+(K) \cap \operatorname{inv}^+ N = \emptyset$ (respectively, $F_N^-(K) \cap \operatorname{inv}^- N = \emptyset$).*

PROOF. (64.9.1) Assume that $K \cap \operatorname{inv}^+ N = \emptyset$. By Lemma (64.7), for each $x \in K$ there exists n_x such that $F_{N,n_x}(x) = \emptyset$. Since F_{N,n_x} is u.s.c. there exists V_x such that $F_{N,n_x}(V_x) = \emptyset$. Let $\{V_{x_1}, \ldots, V_{x_k}\}$ be a finite covering of K. Now, if $m \geq \max\{n_{x_i} \mid i = 1, \ldots, k\}$, then $F_{N,m}(K) = \emptyset$. The proof for F_N^- is analogous.

(64.9.2) Follows from (64.9.1) and Proposition (64.6) since the union of finitely many u.s.c. maps is u.s.c.

(64.9.3) is straightforward. The proof is completed. □

(64.10) Lemma. *Let $N \subset X$ be compact. Then for any neighbourhood V of* $\operatorname{inv}^- N$ *there exists a compact neighbourhood A of* $\operatorname{inv}^- N$ *such that $F_N^+(A) \subset V$.*

Proof. By Lemma (64.5), there exists $m \in \mathbb{Z}^+$ such that $F_{N,-m}(N \setminus V) = \emptyset$. Since $F_{N,m}$ is u.s.c. one can find for every $x \in \operatorname{inv}^- N$ a compact neighbourhood V_x of x such that $F_{N,m}(V_x) \subset V$. By the compactness of $\operatorname{inv}^- N$, one can select a finite covering $\{V_{x_1}, \ldots, V_{x_k}\}$ of $\operatorname{inv}^- N$. Let $A = \bigcup_{i=1}^k V_{x_i}$. Then A is a compact neighbourhood of $\operatorname{inv}^- N$ such that $F_{N,m}(A) \subset V$. It remains to show that $F_N^+(A) \subset V$. Indeed, let $y \in F_N^+(A)$. Then there exists $n > 0$ and $y \in A$ such that $y \in F_{N,n}(x)$. If $n \leq m$, we are already done. If $n > m$, we note that $x \in F_{N,-n}(y) \subset F_{N,-m}(y)$ and (64.3) implies that $y \in V$; the proof is completed.□

Proof of (64.5). Since $\operatorname{inv} N \subset \operatorname{Int} N$, we may assume that $W \subset \operatorname{Int} N$. We may also assume that $F(W) \subset \operatorname{Int} N$. Indeed, let

$$0 < \varepsilon < \operatorname{dist}(\operatorname{inv} N, \operatorname{bd} N) - \operatorname{diam} F$$

and let $\gamma = \varepsilon + \operatorname{diam} F$. Then $B_\gamma(\operatorname{inv} N) \subset \operatorname{Int} N$ and we may intersect W with the open neighbourhood $F^{-1}(B_\gamma(\operatorname{inv}^- N))$, respectively, such that $U \cap V \subset W$, and let A be given for V by Lemma (64.10). We define

$$P_1 = F_N^+(A), \quad P_2 = F_N^+(P_1 \setminus U). \tag{64.10.1}$$

Then $P_1 \subset V$ and $P_1 \setminus U \subset P_2$ which implies that $P_1 \setminus P_2 \subset U$. Therefore, $P_1 \setminus P_2 \subset U \cap V \subset W$. We verify that (P_1, P_2) is the index pair. P_1 is compact by Lemma (64.8) and P_2 is compact by Lemma (64.9.2), since $P_1 \setminus U$ is compact. Next, $P_2 \subset F_N^+(P_1) \subset P_1$. To verify (64.9.1), let $x \in P_i$ and $y \in F(x) \cap N$. Then there exists a solution $\sigma\colon [0, n] \to N$ with $\sigma(n) = x$ and $\sigma(0) \in A$ in the case $i = 1$, $\sigma(0) \in P_1 \setminus U$ in the case $i = 2$, so one may extend σ to $[0, n+1]$ by putting $\sigma(n+1) = y$. Hence $y \in P_i$. Since $P_1 \setminus P_2 \subset W$ and $W \subset \operatorname{Int} N$, (64.9.2) is verified. In order to verify (64.9.3), observe that P_1 is a neighbourhood of $\operatorname{inv}^- N$ and, by (64.9.3) $N \setminus P_2$ is a neighbourhood of $\operatorname{inv}^+ N$. Therefore, $P_1 \setminus P_2 = P_1 \cap (N \setminus P_2)$ is a neighbourhood of $\operatorname{inv}^- N \cap \operatorname{inv}^+ N = \operatorname{inv} N$; the proof is completed. □

We shall now discuss several properties of index pairs which will be used in the next section.

(64.11) Proposition.

(64.11.1) *If P is an index pair for N, then $(P_1 \cup F(P_2)) \setminus (P_2 \cup F(P_2)) = P_1 \setminus P_2$.*

(64.11.2) *If P and Q are index pairs for N, then so is $P \cap Q$.*

(64.11.3) *If $P \subseteq Q$ are index pairs for N, then so are $(P_1, P_1 \cap Q_2)$ and $(P_1 \cup Q_2, Q_2)$.*

Proof. (64.11.1) It is verified that $(P_1 \cup F(P_2)) \setminus (P_2 \cup F(P_2)) = (P_1 \setminus P_2) \setminus F(P_2) \subset P_1 \setminus P_2$. For proving the inverse inclusion, let $x \in P_1 \setminus P_2$. Then $x \in N$ and if $x \in F(P_2)$, the property (64.9.1) of index pairs implies that $x \in P_2$, a contradiction.

(64.11.2) Verification of (64.9.1) and (64.9.2) is obvious. For (64.9.3), let us note that

$$\operatorname{Int}(P_1 \setminus P_2) \cap \operatorname{Int}(Q_1 \setminus Q_2) \subset \operatorname{Int}((P_1 \cap Q_1) \setminus (P_2 \cup Q_2)) \subset \operatorname{Int}((P_1 \cap Q_1) \setminus (P_2 \cap Q_2)).$$

(64.11.3) is a routine verification; the proof is completed. □

(64.12) Lemma. *Let $P \subset Q$ be index pairs for N which differ by at most one coordinate, i.e. $P_1 = Q_1$ or $P_2 = Q_2$. Define a pair of sets $G(P, Q)$ by*

$$G_i(P, Q) = P_i \cup (F(Q_i) \cap N), \quad i = 1, 2.$$

Then

(64.12.1) *If $P_i = Q_i$ then $G_i(P, Q) = P_i = Q_i$, $i = 1, 2$,*

(64.12.2) *$P \subset G(P, Q) \subset Q$,*

(64.12.3) *$G(P, Q)$ is an index pair,*

(64.12.4) *$F(Q_i) \cap N \subset G_i(P, Q)$, $i = 1, 2$.*

Proof. (64.12.4) is obvious and (64.12.1) is immediate from the property (64.9.1) of index pairs.

(64.12.2) The first inclusion is obvious and the second is an immediate consequence of the first one and the property (64.9.1) satisfied by Q.

(64.12.3) For (64.9.1), let $x \in G_i(P, Q)$ and $y \in F(x) \cap N$. If $x \in P_i$ then obviously $y \in G_i(P, Q)$. If $x \in F(Q_i) \cap N$ then $x \in Q_i$ hence, $y \in F(Q_i) \cap N \subset G_i(P, Q)$. For (64.9.2) let us note that

$$G_1(P, Q) \setminus G_2(P, Q) \subset Q_1 \setminus G_2(P, Q) \subset Q_1 \setminus P_2$$

and either $Q_1 \setminus P_2 = Q_1 \setminus Q_2$, or $(F(Q_1) \setminus P_2) \subset N$. For (64.9.3), let us note that $\operatorname{inv} N \subset \operatorname{Int}(P_1 \setminus P_2) \cap \operatorname{Int}(Q_1 \setminus Q_2)$ so, (64.9.3) will follow if we verify that $(P_1 \setminus P_2) \cap (Q_1 \setminus Q_2) \subset G_1(P, Q) \setminus G_2(P, Q)$. Indeed, let $y \in (P_1 \setminus P_2) \cap (Q_1 \setminus Q_2)$. Then $y \in G_1(P, Q)$ and it remains to show that $y \notin F(Q_2) \cap N$. For, if $y \in F(Q_2) \cap N$, then by (a) $y \in Q_2$, a contradiction and the proof is completed. □

(64.13) Lemma. *Let $P \subseteq Q$ be index pairs for N which differ by at most one coordinate. Then there exists a sequence of pairs*

$$P = Q^n \subset Q^{n-1} \subset \ldots \subset Q^1 \subset Q^0 = Q$$

with the following properties:

(64.13.1) *If $P_i = Q_i$ then $Q_i^k = P_i = Q_i$ for all $k = 1, 2, \ldots, n-1$, $i = 1, 2$.*
(64.13.2) *Q^k is an index pair for all $k = 1, \ldots, n-1$.*
(64.13.3) *$F(Q_i^k) \cap N \subset Q_i^{k+1}$, $i = 1, 2$, $k = 0, \ldots, n-1$.*

Proof. Let Q^k be given by the recurrence formula $Q^0 = Q$, $Q^{k+1} = G(P, Q^k)$, $k = 0, 1, \ldots$. By Lemma (64.12) and an induction on k, $\{Q^k\}$ is a decreasing sequence of pairs containing P and satisfying (64.13.1)–(64.13.3) for all $k \in \mathbb{Z}^+$.

It remains to show that $A^n = P$ for some n. Indeed, suppose that the inclusion $P \subset Q^k$ is strict for all k, i.e. if $i \in \{1, 2\}$ is such $Q_i \setminus P_i \neq \emptyset$ then $Q_i^k \setminus P_i \neq \emptyset$. Let us choose $\sigma(k) \in Q_i^k \setminus P_i$. Then $\sigma(k) \in F(Q_i^{k-1}) \cap N$, so there exists $\sigma(k-1) \in Q_i^{k-1}$ with $\sigma(k) \in F(\sigma(k-1))$. If $\sigma(k-1) \in P_i$ then, by property **(a)?** of index pair, $\sigma(k) \in P_i$, a contradiction. Therefore, $\sigma(k-1) \in Q_i^{k-1} \setminus P_i$.

By the reverse recurrence, one may construct a solution $\sigma_k \colon [0, k] \to Q_i \setminus \operatorname{Int} P_i$ such that $\sigma(j) \in Q_i^i \setminus P_i$, $j = 0, \ldots, k$, $k \in \mathbb{Z}^+$. By Lemma (64.7), $\operatorname{inv} N \subset \operatorname{Int}(P_1 \setminus P_2) \subset \operatorname{Int} P_1$. If $i = 2$, we get $\emptyset \neq \operatorname{inv}(Q_2 \setminus \operatorname{Int} P_2) \subset \operatorname{inv} Q_2$. On the other hand $\operatorname{inv} Q_2 \subset Q$ and $\operatorname{inv} Q_2 \subset \operatorname{Int}(Q_1 \setminus Q_2) \subset Q_1 \setminus Q_2$ implies that $\operatorname{inv} Q_2 = \emptyset$, a contradiction. □

We shall consider now, a dmds $F \colon X \times \mathbb{Z} \multimap X$ such that the map $F = F^1 \colon X \multimap X$ is determined by a morphism. If P is an index pair for an isolating neighbourhood $N \subset X$ we let

$$S(P) := (P_1 \cup F(P_2), P_2 \cup F(P_2)),$$
$$T(P) := T_N(P) := (P_1 \cup (X \setminus \operatorname{Int} N), P_2 \cup (X \setminus \operatorname{Int} N)).$$

(64.14) Proposition. *If P is an index pair for N then*

(64.14.1) *$F(P) \subset S(P) \subset T(P)$,*
(64.14.2) *The inclusions $i_{P,S(P)}, i_{S(P),T(P)}$ and, consequently, $i_{P,T(P)}$ induce isomorphisms in the Čech cohomology.*

Proof. (64.14.1) If $y \in F(P_i)$ then either $y \in P_i$ or $y \in N$. In the second case $y \in F(P_2)$ which proves the first inclusion. Since $F(P_2) \subset F(P_1)$ and $X \setminus N \subset X \setminus \operatorname{Int} N$, the second inclusion follows by the same argument.

(64.14.2) By Proposition (64.12.1), $S_1(P) \setminus S_2(P) = P_1 \setminus P_2$. We also have $T_1(P) \setminus T_2(P) = (P_1 \setminus P_2) \cap \operatorname{Int} N = P_1 \setminus P_2$, hence (64.14.2) follows from the strong excision property for cohomology (see [Sp-M]). □

Now, let $F_{P,T(P)}\colon P \multimap T(P)$ be the restriction of F to $(P, T(P))$ and let $i_P = i_{P,T(P)}$. The endomorphism $I_P = (F_{P,T(P)})^{x_0}(i_P)^{x-1}$ of $H^*(P)$ is called the *index map* associated with the index pair P.

So, we have a pair $(H^*(P), I_P)$. Now, following Section 11 we can apply the Leray construction to the above pair. Then we get $\widetilde{H^*(P)} = H^*(P)|_{N(I_P)}$ and the isomorphism $\widetilde{I}_P\colon \widetilde{H^*(P)} \xrightarrow{\sim} \widetilde{H^*(P)}$. Let $L(H^*(P), I_P) = (\widetilde{H^*(P)}, \widetilde{I}_P)$. Then $(LH^*(P), I_P)$ is called the Leray reduction of the Čech cohomology for P.

(64.15) Remark. In Section 11 we have presented the Leray construction for graded vector spaces only. Here we deal with graded Abelian groups, but the respective construction is strictly similar (cf. [Mr2]).

A compact subset K of X is called an *isolated invariant set* if $K = \operatorname{inv} N$ for an isolating neighbourhood N containing K. In such a case we say that N is an *isolating neighbourhood of K for F*. The main result of this section is the following:

(64.16) Theorem. *Let K be an isolated invariant set. Then setting $C(K, F)$ $:= (H^*(P), I_P)$ is independent of the choice of an isolating neighbourhood N of K for F and of an index pair P for N.*

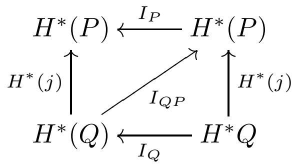
The module $C(K, F)$ given by the above theorem (denoted shortly by $C(K)$ if F is clear from the context) is called the cohomological Conley index of K.

Proof of Theorem (64.16). We need to show that if M and N are two isolating neighbourhoods of K, P an index pair for N and Q an index pair for M then $L(H^*(P), I_P) = L(H^*(Q), I_Q)$. The proof will be given in several steps.

Step 1. We consider the following special case

(64.16.1) $M = N$,
(64.16.2) $P \subset Q$,
(64.16.3) $F(Q) \subset T_N(P)$.

By (64.12.4) we may consider the map $F_{Q,T(P)}\colon Q \to P(T_N(P))$, $F_{Q,T(P)}(x) = F(x)$, and the induced homomorphism $I_{Q,P} := H^*(i_P)^{-1}$. $I_{Q,P} := H^*(F_{Q,T(P)}) \circ H^*(i_P)^{-1}$. We obtain the commutative diagram



where $j\colon P \to Q$ is the inclusion. Then $LH^*(j)\colon L(H^*(Q), I_Q) \to L(H^*(P), I_P)$ is an isomorphism (cf. [Mr2]).

Step 2. We lift the assumption (64.12.4). Let $\{Q^k\}_{k=0,\ldots,n-1}$ be such that the pair of index pairs $Q^{k+1} \subseteq Q^k$ satisfies the assumptions (64.12.2)–(64.12.4), so their corresponding Leray reductions are isomorphic. Since $Q^0 = Q$ and $Q^n = P$, the conclusion follows.

Step 3. We lift the assumption (64.12.3). Put $R_1 := P_1 \cup Q_2$, $R_2 := P_1 \cap Q_2$. By Proposition (64.11.3), (P_1, R_2) and (R_1, Q_2) are index pairs. We have the commutative diagram of inclusions

$$\begin{array}{ccc} (P_1, R_2) & \xrightarrow{\ j_2\ } & (R_1, Q_2) \\ {\scriptstyle i_1}\big\uparrow & & \big\downarrow{\scriptstyle i_3} \\ (P_1, P_2) & \xrightarrow[\ j\ ]{} & (Q_1, Q_2) \end{array}$$

It is clear that the pair of index pairs $(P_1, P_2) \subset (Q_1, Q_2)$ satisfies the assumption (64.12.3) and so, does $(R_1, Q_2) \subset (Q_1, Q_2)$, so by Step 2, the inclusions i_1 and i_3 induce isomorphisms of the Leray reductions of the corresponding cohomologies. Since $P_1 \setminus R_2 = P_1 \setminus Q_2 = R_1 \setminus Q_2$, the inclusion j_2 induces an isomorphism in cohomologies by the strong excision property for cohomology.

Step 4. Now, only (64.16.1) is assumed. By Proposition (64.12.1) $P \cap Q$ is an index pair, hence the conclusion of Step 3 can be applied for pairs $P \cap Q \subset P$ and $P \cap Q \subset Q$.

Step 5. If $M \neq N$, one may always assume that $M \subset N$ since otherwise $M \cap N$ can be considered which is also an isolating neighbourhood of K. By Step 4, it is sufficient to show the existence of one index pair P for N and one index pair Q for M such that $L(H^*(P), I_P)$ and $L(H^*(Q), I_Q)$ are isomorphic.

By Theorem (64.5), there exists an index pair P for N such that $P_1 \setminus P_2 \subset \operatorname{Int} M \cap F^{-1}(\operatorname{Int} M)$. It is easily verified that $Q := (M \cap P_1, M \cap P_2)$ is an index pair for M. Since $Q_1 \setminus Q_2 = M \cap (P_1 \setminus P_2) = P_1 \setminus P_2$, the inclusion $Q \subseteq P$ induces an isomorphism in cohomology, by the strong excision property; the proof is completed. □

Let $\Lambda \in R$ be a closed interval and $F\colon \Lambda \times X \times \mathbb{Z} \multimap X$ an u.s.c. map with compact nonempty values such that, for each $\lambda \in \Lambda$, $F_\lambda\colon X \times \mathbb{Z} \multimap X$ given by

$$F_\lambda(x, n) = F(\lambda, x, n)$$

is a discrete multivalued dynamical system. Given a compact subset $N \subset X$ and $\lambda \in \Lambda$, the sets $\operatorname{inv}^{(\pm)} N$ with respect to F_λ are denoted by $\operatorname{inv}^{(\pm)}(N, \lambda)$. We will discuss the following *homotopy property* (called also *continuation property*) of the Conley index:

(64.17) THEOREM. *Let $\lambda_0 \in A$ and let N be an isolating neighbourhood for F_{λ_0}. Then:*

(64.17.1) *N is an isolating neighbourhood for F_λ for all λ sufficiently close to F_{λ_0}.*
(64.17.2) *$C(\mathrm{inv}(N,\lambda))$ does not depend on λ (provided λ is as in (64.17.1)).*

We prove here the assertion (64.17.1) only as the proof of (64.17.2) is exactly the same as in the singlevalued case.

(64.18) LEMMA. *Let $N \subset X$ be compact. Then the mappings $\lambda \multimap \mathrm{inv}^+(N,\lambda)$, $\lambda \multimap \mathrm{inv}^-(N,\lambda)$, and $\lambda \multimap \mathrm{inv}(N,\lambda)$, $\lambda \in \Lambda$, are u.s.c.*

PROOF. We prove the assertion for the first mapping as the other two proofs can be done by extending the same argument to negative integers.

Suppose that $\lambda \multimap \mathrm{inv}^+(N,\lambda)$ is not u.s.c. at $\lambda_0 \in \Lambda$. Then there exists an open U and a sequence $\lambda_n \to \lambda_0$ such that $\mathrm{inv}^+(N,\lambda_0) \subseteq U$ but $\mathrm{inv}^+(N,\lambda_n) \subseteq N \setminus U$. Let $x_n \in \mathrm{inv}^+(N,\lambda_n)$ since $N \setminus U$ is compact, we may assume that $x_n \to x \in N \setminus U$. In order to achieve a contradiction, we have to show that $x \in \mathrm{inv}^+(N,\lambda_N)$. Indeed, let $\sigma_n\colon \mathbb{Z}^+ \to N$ be a solution for F_{λ_n} with $\sigma_N(0) = x_n$. Then $\sigma_n(k) \subset \mathrm{inv}^+(N,\lambda_n) \subseteq N \setminus U$ for all $k = 1, 2, \ldots$. We construct a solution $\sigma\colon \mathbb{Z}^+ \to N \setminus U$ for F_λ by induction on k. Let $\sigma(0) = \lim_n \sigma_n(0) = x$. Let $\sigma(k)$ be constructed for a given k, such that $\sigma(k) = \lim_i \sigma_{n_i}(k)$, where $\{\sigma_{n_i}(k)\}_i$ is a subsequence of $\{\sigma_n(k)\}_n$ convergent in $N \setminus U$. By again passing to a subsequence, we may assume that $\{\sigma_{n_i}(k+1)\}_i$ is convergent and define $\sigma(k+1)$ to be its limit. Since $\sigma_n(k+1) \in F(\lambda_n(k))$ for all n, the closed graph property of f implies that $\sigma(k+1) \in F(\lambda, \sigma(k))$, the proof is completed. □

PROOF OF THEOREM (64.17). By the compactness of N, the condition (64.3.1) implies that

$$B_{\mathrm{diam}_N F_{\lambda_0}+3\varepsilon}(\mathrm{inv}(N,\lambda_0)) \subset \mathrm{Int}\, N$$

for some $\varepsilon > 0$. Since F is u.s.c. $F_\lambda(x) \subset B_\varepsilon(F_{\lambda_0}(x))$ for all λ close to λ_0 and all $x \in N$. Again by the compactness of N,

$$\mathrm{diam}_N F_\lambda < \mathrm{diam}_N F_{\lambda_0} + 2\varepsilon$$

for all λ close to λ_0. By Lemma (64.18), $\mathrm{inv}(N,\lambda) \subset B_\varepsilon(\mathrm{inv}(N,\lambda_0))$ for all λ close to λ_0, hence we obtain

$$\begin{aligned} B_{\mathrm{diam}_N F_\lambda(\mathrm{inv}(N,\lambda))} &\subset B_{\mathrm{diam}_N F_{\lambda_0}+2\varepsilon}(B_\varepsilon(\mathrm{inv}(N,\lambda_0)) \\ &= B_{\mathrm{diam}_N F_{\lambda_0}+3\varepsilon}(\mathrm{inv}(N,\lambda_0) \subset \mathrm{Int}\, N. \end{aligned}$$ □

By the same arguments as for singlevalued mappings one proves the following *additivity property* of the Conley index:

(64.19) THEOREM. *Let $K \subset X$ be an isolated invariant set for a* dmds $F: X \times \mathbb{Z} \to P(x)$*, which is a disjoint sum of two other isolated invariant sets K_1, K_2. Then $C(K) = C(K_1) \times C(K_2)$.*

For details concerning singlevalued Conley index see [Co-M].

65. Minimax theorems for ANRs

There are several minimax theorems in the literature. In this section we shall present a version related to ANRs and acyclic mappings. Note that minimax theorems are useful in such areas as the game theory or the mathematical economy.

We first prove an intersection theorem. If $S \subset X \times Y$ then let $S(x) = \{y \in Y \mid (x,y) \in S\}$ and $S^{-1}(y) = \{x \in X \mid (x,y) \in S\}$.

(65.1) THEOREM. *Suppose that $X \in$* AR *and Y is a compact finite-dimensional* ANR*. Suppose $S, T \subset X \times Y$ satisfy:*

(65.1.1) *S is open in $X \times Y$ and $S^{-1}(y)$ is contractible and nonempty for all $y \in Y$,*

(65.1.2) *T is closed in $X \times Y$ and $T(x)$ is acyclic for all $x \in X$.*

Then $S \cap T \neq \emptyset$.

PROOF. Define $\varphi: Y \multimap X$ by $\varphi(y) = S^{-1}(y)$. Then φ is a multivalued map with open graph Γ_φ and contractible values. Theorem (16.4) gives a continuous selection $g: Y \to X$, $g \subset \varphi$. Define $\psi: X \multimap X$ by $\psi(x) = g(T(x))$ for every $x \in X$. Then ψ as a composition of two acyclic maps is an admissible and compact map (Y is compact!). Since $X \in$ AR we infer from the Lefschetz Fixed Point Theorem for admissible maps that $\mathrm{Fix}(\psi) \neq \emptyset$. Let $x \in \psi(x) = g(T(x))$. Then $x = g(y)$ with $y \in T(x)$. So $x \in S^{-1}(y)$, and thus $(x,y) \in S \cap T$, proving the theorem. □

(65.2) THEOREM. *Let*

$$\alpha = \sup_{x \in X} \min_{y \in Y} f(x,y) \quad \text{and} \quad \beta = \min_{y \in Y} \sup_{x \in X} f(x,y).$$

Here X is an acyclic ANR*, Y a compact finite dimensional* ANR*, and $f: X \times Y \to \mathbb{R}$ a function satisfying the following conditions:*

(65.2.1) *$f(x, \cdot\,)$ is lower semicontinuous, for all x in X,*

(65.2.2) *$\{(x,y) \mid f(x,y) > \alpha\}$ is open,*

(65.2.3) *$\{x \in X \mid f(x,y) > \alpha\}$ is contractible or empty, for all y in Y,*

(65.2.4) *$\{y \in Y \mid f(x,y) \leq \alpha\}$ is acyclic, for all x in X.*

Then $\alpha = \beta$.

PROOF. Observe that α and β both exist (possibly infinite) and $\alpha \leq \beta$, so we need only to prove $\alpha \geq \beta$.

Define $S(x) = \{(x, y) \mid f(x, y) > \alpha\}$. Then S is open (by (65.2.2)) and $S^{-1}(y)$ is contractible or empty for all y (by (65.2.3)). Suppose, to the contrary, that $S^{-1}(y)$ is not empty, for all y.

Define $T(x) = \{(x, y) \mid f(x, y) \leq \alpha\}$. Then T is closed by (65.2.2) and $T(x)$ is acyclic for all x in X by (65.2.4). It follows easily from definition of α that $T(x)$ is nonempty for all y in Y.

The hypotheses of the intersection theorem are satisfied so, $S \cap T$ must be nonempty. But this is not true so, $S^{-1}(y)$ must be empty for some y, say $\overline{y}$. Then $f(x, \overline{y}) \leq \alpha$, for all x, so $\sup_{x \in X} f(x, \overline{y}) \leq \alpha$, and $\beta \leq \alpha$, proving the theorem. □

Small modifications of the proof yield the following version.

(65.3) Theorem. *Let*

$$\alpha = \max_{x \in X} \inf_{y \in Y} f(x, y) \quad \text{and} \quad \beta = \inf_{y \in Y} \max_{x \in X} f(x, y).$$

Here X is a compact finite dimensional ANR*, Y an acyclic* ANR*, and $f \colon X \times Y \to R$ a function satisfying the following conditions:*

(65.3.1) *$f(\,\cdot\,, y)$ is upper semicontinuous, for all y in Y,*
(65.3.2) *$\{(x, y) \mid f(x, y) < \beta\}$ is open,*
(65.3.3) *$\{(y \in Y \mid f(x, y) < \beta\}$ is contractible or empty, for all x in X,*
(65.3.4) *$\{(x \in X \mid f(x, y) \geq \beta\}$ is acyclic, for all y in Y.*

Then $\alpha = \beta$.

Now a result for a continuous f will be proved. First, recall that a function $h \colon X \to R$ is *quasiconcave* if $\{x \mid h(x) > t\}$ is convex for each t, and is *quasiconvex* if $\{x \mid h(x) < t\}$ is convex for each t. The next definition describes larger classes of functions.

(65.4) Definition. $h \colon X \to R$ is

(65.4.1) *t-upper acyclic* if $\{x \in X \mid h(x) > t\}$ is acyclic or empty,
(65.4.2) *t-lower acyclic* if $\{x \in X \mid h(x) < t\}$ is acyclic or empty.

(65.5) Theorem. *Let*

$$\alpha = \max_{x \in X} \min_{y \in Y} f(x, y) \quad \text{and} \quad \beta = \min_{y \in Y} \max_{x \in X} f(x, y).$$

Suppose X is a compact Hausdorff space, Y a compact acyclic ANR*, and $f \colon X \times Y \to R$ a continuous function satisfying:*

(65.5.1) *$f(\,\cdot\,, y)$ is t-upper acyclic, for all y in Y, all t near β,*
(65.5.2) *$f(x, \,\cdot\,)$ is t-lower acyclic, for all x in X, all t near α.*

Then $\alpha = \beta$.

PROOF. Since $\alpha \le \beta$, we need to prove only $\alpha \ge \beta$. Let $\varepsilon > 0$, $S^{-1}(y) = \{x \in X \mid f(x,y) \ge \beta - \varepsilon\}$ and $T(x) = \{y \in Y \mid f(x,y) \le \alpha + \varepsilon\}$. From the definition of α and β, S^{-1} and T have nonempty values. From (65.5.1) and (65.5.2) follows, that T and S^{-1} have acyclic values. Therefore, the map $\varphi_\varepsilon = T \circ S^{-1}\colon Y \multimap Y$ is admissible and compact. Since Y is a compact acyclic ANR, we obtain a fixed point $x_0 \in \varphi_\varepsilon(x_0)$. So, there is $y_0 \in S^{-1}(x_0)$ such that $x_0 \in T(y_0)$. Hence, $f(x_0,y_0) \ge \beta - \varepsilon$, $f(x_0,y_0) \ge \alpha - \varepsilon$ and $\beta \le \alpha + 2\varepsilon$ for all small ε so, $\beta \le \alpha$ and the proof is completed. □

(65.6) REMARK. Note that in a natural way the intersection theorem (65.1) can be generalized for n subsets $S_1, \dots, S_n \subset X_1 \times \dots \times X_n$.

Below we will consider some similar problem which arises from the game theory. Let $X_1, \dots, X_n \in \mathrm{AR}$. We will use the following notations:

$$X = \times\{X_i \mid i = 1, \dots, n\}, \qquad x = (x_1, \dots, x_n) \in X,$$
$$X_{\widehat{i}} = \times\{X_j \mid i \ne j\}, \qquad x_{\widehat{i}} = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n\} \in X_{\widehat{i}}.$$

Given $K \subset X$, K_i, $K_{\widehat{i}}$ denote the images of K by the projections of X onto X_i, $X_{\widehat{i}}$, respectively. For any $y \in K_{\widehat{i}}$ we define the cross-section of K at y as

$$S_i(y) = \{z \in K_i \mid \text{there exists } x \in K,\ x_i = z \text{ and } x_{\widehat{i}} = y\}.$$

We are concerned with the existence of a solution $\overline{x} \in K$ of the following system of maximization problems:

$$(\mathrm{P}_i) \qquad \max\{f_i(\overline{x}_1, \dots, \overline{x}_{i-1}, z, \overline{x}_{i+1}, \dots, \overline{x}_n) \mid z \in S_{\widehat{i}}(x_i)\} = V_i(x_{\widehat{i}}),$$

where $f_1, \dots, f_n$ are real-valued functions on K; then $\overline{x}$ is called the social equilibrium.

It is clear that, when the functions f_i are continuous and K is compact the maximum in (P_i) is attained for each i, but not necessarily at $\overline{x}_i$ as in the example presented below. In this example we will illustrate also that the quasi-concavity of f_i is necessary to get social equilibrium.

(65.7) EXAMPLE. Consider

$$K = [0,1] \times [0,1] \subset \mathbb{R}^2, \quad f_1(x,y) = (y-x)^2, \quad \text{and} \quad f_2(x,y) = (x+y-1)^2.$$

Then

$$M_1(y) = \begin{cases} \{1\} & \text{if } 0 \le y < 1/2, \\ \{0,1\} & \text{if } y = 1/2, \\ \{0\} & \text{if } 1/2 < y \le 1, \end{cases} \quad \text{and} \quad M_2(x) = \begin{cases} \{0\} & \text{if } 0 \le x < 1/2, \\ \{0,1\} & \text{if } x = 1/2, \\ \{1\} & \text{if } 1/2 < x \le 1. \end{cases}$$

It is clear that neither $f_1(\,\cdot\,,y)$ nor $f_2(x,\,\cdot\,)$ is quasi-concave, and that there exists no $(\overline{x},\overline{y})\in K$ satisfying $\overline{x}\in M_1(\overline{y})$ and $\overline{y}\in M_2(\overline{x})$.

We will need the following additional assumptions on K and f_i. A subset $K\subset X$ is said to have the *continuous cross-section property*, if for $i=1,\dots,n$ the multivalued map:

$$S_i\colon K_{\widehat{i}}\multimap K_i,\quad y\multimap S_i(y)$$

is continuous. Our result is presented in the following:

(65.8) THEOREM. *Let K be a compact retract of X with the continuous cross-section property and $f_1,\dots,f_n\colon K\to\mathbb{R}$ be continuous functions such that, for any $i=1,\dots,n$ and x in K, the map*

$$z\to f_i(x_1,\dots,x_{i-1},z,x_{i+1},\dots,x_n) \tag{65.8.1}$$

defined on $S_i(x_{\widehat{i}})$ is quasiconcave. Assume further that for every $i=1,\dots,n$ and $x_{\widehat{i}}\in K_{\widehat{i}}$ the set $M_i(x_{\widehat{i}})=\{z\in S_i(x_{\widehat{i}})\mid V_i(x_{\widehat{i}})=f_i(x_1,\dots,x_{i-1},z,x_{i+1},\dots,x_n)\}$ is acyclic. Then there exists a solution $\overline{x}\in K$ to the system of problems $(\mathrm{P_i})$, *$i=1,\dots,n$.*

PROOF. Consider the so-called marginal functions $V_i\colon K_{\widehat{i}}\to\mathbb{R}$ and marginal set-valued maps $M_i\colon K_{\widehat{i}}\multimap K_i$ defined by

$$\begin{aligned} V_i(x_{\widehat{i}})&=\max\{f_i(x_1,\dots,x_{i-1},z,x_{i+1},\dots,x_n)\mid z\in S_i(x_{\widehat{i}})\},\\ M_i(x_{\widehat{i}})&=\{z\in S_i(x_{\widehat{i}})\mid V_i(x_{\widehat{i}})=f_i(x_1,\dots,x_{i-1},z,x_{i+1},\dots,x_n)\},\end{aligned}$$

where $i=1,\dots,n$. The maps M_i are upper semicontinuous. The compactness of K yields that $M_i(x_{\widehat{i}})$ is nonempty and compact for all x in K. Since the function defined in (65.8.1) is quasiconcave, $M_i(x_{\widehat{i}})$ is also convex. The set $C=\bigtimes\{K_i\mid i=1,\dots,n\}$ is a compact AR-space. Moreover, the map $F\colon C\multimap C$ defined by

$$F(x)=\{y\in C\mid y_i\in M_i(x_{\widehat{i}})\ i=1,\dots,n\}$$

is upper semicontinuous. Indeed, F is the composition of the continuous map $x\to(x_{\widehat{1}},x_{\widehat{2}},\dots,x_{\widehat{n}})$ with the upper semicontinuous product map

$$M_1\times\dots\times M_n\colon\bigtimes\{K_{\widehat{i}}\mid i=1,\dots,n\}\multimap C;$$

hence, F is an acyclic map and consequently $\mathrm{Fix}(F)\neq\emptyset$. Let $\overline{x}\in F(\overline{x})$. Then by an easy observation we deduce that $\overline{x}$ is a social equilibrium and the proof is completed. □

It seems that any compact convex set in $\mathbb{R}^2$ has the continuous cross-section property, but this is not true in $\mathbb{R}^3$, as it is shown in the following example.

(65.9) Example. Let K be the cone in $\mathbb{R}^3$ with vertex $(1,0,1)$ and base $\{(x,y,0) \mid x^2+y^2 \leq 1\}$. For any pair $(x,y) \neq (1,0)$ with $x^2+y^2 = 1$, $S_3(x,y)$ is the singleton $\{(x,y,0)\}$, but $S_3(1,0)$ is the line segment $[(1,0,0),(1,0,1)]$. Therefore, S_3 is discontinuous at $(1,0)$.

For more applications of multivalued maps to mathematical economy we recommend [Bor-M]. Finally, note that in Section 66 there are also some applications to mathematical economics.

Note that the notion of equilibrium is used also in a more general situation. Namely, equilibrium theorems provide sufficient conditions for the existence of an equilibrium (or a zero) for a given multifunction Φ under certain constraints, that is, a solution of the inclusion $0 \in \Phi(x)$ is required to belong to a certain constraints set X. Many important problems in nonlinear analysis can be reduced to equilibrium problems, for example the problem of existence of critical points for smooth and non-smooth functions, the problem of existence of stationary solutions to differential inclusions, etc.

The well known equilibrium result is the following theorem:

(65.10) Theorem. *Let K be a compact convex subset of a normed space E. Let $\Phi\colon K \multimap E$ be an u.s.c. map with closed convex values. If Φ satisfies the tangency condition*

$$\Phi(x) \cap T_K(x) \neq \emptyset \quad \text{for all } x \in K, \tag{65.10.1}$$

then Φ has an equilibrium.

The proof of (65.10) is omitted here as below we will prove a generalization of this theorem. We will present new results on the existence of equilibrium (or zero) of multivalued maps on compact ANR-s which are locally convex. First result of this type was proved in [Pl1].

Recall that a multivalued map $\Phi\colon X \multimap E$, where E is a Banach space, is called *upper hemi-continuous* if for every linear continuous functional $\mu\colon E \to R$, the extended real function, $x \to \sup\{\mu(y) \mid y \in \Phi(x)\}$ is upper semicontinuous. Any u.s.c. map $\Phi\colon X \multimap E$, where E is supplied in weak topology is upper hemi-continuous. Consequently, if $\dim E < +\infty$ then the notion of upper hemi-continuity and upper semicontinuity coincide provided values of the considered map are compact and convex.

In what follows we will always assume that K is a compact neighbourhood retract of a Banach space E. So, we can assume without loss of generality that:

(A1) K is a compact locally convex subset of a Banach space with a given retraction $r\colon O_\delta(K) \to K$ for a fixed $\delta > 0$.

Note that the class of sets satisfying (A1) is quite large. For instance, if K is compact and convex, or if K is a locally convex proximate neighbourhood retract (see Chapter 1), then K satisfies (65.10.1).

Let $T_K^{\perp}(x)$ be the polar cone to $T_K(x)$. For a given $\Phi\colon K \multimap E$ we impose the following condition:

(65.10.2) $$\inf_{y\in\Phi(x)} \langle p, y\rangle \le 0 \quad \text{for all } x \in K \text{ and all } p \in T_K^{\perp}(x).$$

Observe that the condition (65.10.1) is sufficient for (65.10.2) and is equivalent too, whenever Φ has compact values. Assume K satisfies (A1), D is the unit closed ball in E and D^* its dual in E^*. For any $\varepsilon > 0$ we let:

$$G_\varepsilon\colon K \times D^* \multimap K,$$
$$G_\varepsilon(x,p) = \Big\{ y \in K \mid \|y - x\| \le \varepsilon \text{ and } \sup_{z\in K\cap \overline{B(x,\varepsilon)}} \langle p, y - z\rangle \le 0 \Big\}.$$

Note that G_ε is an u.s.c. map with compact values. We prove:

(65.11) THEOREM. *Assume K satisfies* (A1) *and the Euler characteristic* $\chi(K) = \lambda(\mathrm{id}_K)$ *of the set K is different from zero. If $\Phi\colon K \multimap E$ is upper hemi-continuous with closed convex values and satisfies* (65.10.2)*, then Φ has an equilibrium.*

PROOF. Suppose that $0 \notin \Phi(x)$ for each $x \in K$. By the separation theorem, for each $x \in K$ there is $p_x \in E^*$ such that $\inf\{\langle p_x, y\rangle \mid y \in \Phi(x)\} > 0$, i.e. $\sup_{y\in\Phi(x)}\langle -p_x, y\rangle < 0$. Since Φ is upper hemi-continuous, the set

$$U(x) := \Big\{ z \in K \mid \sup_{y\in\Phi(z)} \langle -p_x, y\rangle < 0 \Big\}$$

is an open neighbourhood of x, and the collection $\mathcal{U} := \{U(x)\}_{x\in K}$ constitutes an open covering of K. Let $\{\lambda_x\}_{x\in K}$ be a locally finite partition of unity subordinated to $\mathcal{U}$. Let us define a continuous map $f\colon K \to E^*$ by the formula

$$f(z) := \sum_{x\in K} \lambda_x(z) p_x, \quad \text{for } z \in K.$$

Then, for any $z \in K$, we have $\sup_{y\in\Phi(z)}\langle -f(z), y\rangle < 0$; hence, $f(z) \ne 0$. Indeed, let $\{x_1, \dots, x_k\} = \{x \in K \mid z \in U(x)\}$. Then $f(z) = \sum_{i=1}^k \lambda_i(z) p_i$, where $\lambda_i = \lambda_{x_i}$, $p_i = p_{x_i}$, $i = 1, \dots, k$. Since $z \in U(x_i)$, it follows that $\sup_{y\in\Phi(z)}\langle -p_i, y\rangle < 0$, and we are done.

We are going to prove the existence of an element $\overline{x} \in K$ such that $f(\overline{x}) \in T_K^{\perp}(x)$. This together with the condition (65.10.2) will lead to a contradiction.

Since $K \subset E$ satisfies (A1) we obtain

(A2) there exists $\eta > 0$, $\eta < \delta/2$ such that $\|x - r(x)\| < \eta$ for all $x \in O_\eta(K)$.

Let $n \in N$ be such that $0 < 1/n < \eta$, where η is given by (A2). Consider the map $\Psi_n: K \multimap 2^K$ defined by

$$\Psi_n(x) := r\left[\overline{\text{conv}}\left\{G_{1/n}\left(x, \frac{f(x)}{\|f(x)\|}\right)\right\}\right], \quad \text{for } x \in K.$$

Evidently, Ψ_n is admissible for every n. Moreover, for any $x \in K$, $\Psi_n(x) \in B(x, 2\eta)$. Indeed, if $y \in \Psi_n(x)$ then $y \in r(z)$, $z \in \text{conv}\{G_{1/n}(x, f(x))\} \subset B(x, 1/n) \subset B(x, \eta)$. Therefore, in view of (A2), $\|y - r\| \le \|r(z) - z\| + \|z - x\| \le 2\eta < \delta$. Observe that $\text{Fix}(\Psi_n) \neq \emptyset$ for every n. Let $x_n \in \Psi_n(x_n)$. The compactness of K allows us to assume $\lim_n x_n = \overline{x} \in K$. Hence, $\lim_n f(x_n) = f(\overline{x})$ and $f(\overline{x}) \in T_K^{\perp}(\overline{x})$, a contradiction. The proof of (65.11) is completed. □

Since (65.10.1) implies (65.10.2) we have:

(65.12) Corollary. *Suppose that K and Φ are as in* (65.11) *and the condition* (65.10.1) *is satisfied, then Φ has an equilibrium.*

(65.13) Remark. Note that if K is convex then Corollary (65.12) is equivalent to (65.1).

The following example shows that the assumption in (A1) on K to be locally convex is essential.

(65.14) Example. Let $K = S_1 \cup S_2$, where

$$S_1 = \{(x, y) \in \mathbb{R}^2 \mid (x - 1)^2 + y^2 = 1\},$$
$$S_2 = \{(x, y) \in \mathbb{R}^2 \mid (x + 1)^2 + y^2 = 1\}.$$

Let $f: K \to \mathbb{R}^2$ be given by the formula

$$f(x, y) = \begin{cases} (y, 1 - x) & \text{if } (x, y) \in S_1, \\ (-y, 1 + x) & \text{if } (x, y) \in S_2. \end{cases}$$

Then for every $(x, y) \in K$ we have $f(x, y) \in T_K(x, y)$, but f has no zeros.

For admissible mappings one can prove the following:

(65.15) Theorem. *Let $K \subset \mathbb{R}^n$ satisfies* (A1) *and let $\Phi: K \multimap \mathbb{R}^n$ be an admissible map such that the following condition is satisfied:*

(65.15.1) *for all $x \in K$, all $y \in T_K^{\perp}(x)$ and all $z \in \Phi(x)$*
if $(y, z) \neq 0$ then $\langle y, z\rangle < \|y\|\,\|z\|$.

Then Φ has an equilibrium provided $\chi(K) \neq 0$.

Proof. Consider the duality map $J: \mathbb{R}^n \multimap \mathbb{R}^n$ defined by:

$$J(x) = \{y \in K_0 \mid \langle y, x\rangle = \|x\|\},$$

where K_0 denotes the unit closed ball in $\mathbb{R}^n$. Of course J is an u.s.c. convex valued mapping.

Consequently the map $\Psi = J \circ \Phi$ is admissible and hence the map:

$$\Psi_\varepsilon(x) = r(\overline{\text{conv}}\{G_\varepsilon(x) \times \Psi(x)\})$$

is admissible.

By homotopy argument, we obtain that $\text{Fix}(\Psi_\varepsilon) \neq \emptyset$ for every $\varepsilon > 0$. Let $x_\varepsilon \in \Psi_\varepsilon(x_\varepsilon)$. Then $x_\varepsilon \in r(\overline{\text{conv}}\{G_\varepsilon(x_\varepsilon, y_\varepsilon)\}$, i.e. $y_\varepsilon \in T_K(x_\varepsilon)$. We can assume that for $\varepsilon_n = 1/n$ we have

$$\lim_n (x_{\varepsilon_n}, y_{\varepsilon_n}) = (\overline{x}, \overline{y}).$$

Thus $\overline{y} = \Psi(\overline{x})$. Consequently $\|\overline{y}\| = 1$ and there exists $\overline{z} \in \Phi(\overline{x})$ with $\langle \overline{y}, \overline{z} \rangle = \|\overline{z}\|$. On the other hand since $y_\varepsilon \in T_K(x_\varepsilon)$, it follows that $\overline{y} \in T_K^{\perp}(\overline{x})$. This contradicts (65.15.1) and the proof is completed. □

66. KKM-mappings

We begin with some notations and terminology as presented in [GrL1] and [GrL2]. Given multivalued map $\varphi: X \multimap Y$, its inverse $\varphi^{-1}: Y \multimap X$ is given by $\varphi^{-1}(y) = \{x \in X, | \ y \in \varphi(x)\}$ and its dual $\varphi^*: Y \multimap X$ is given by $\varphi^*(y) = X \setminus \varphi^{-1}(y)$.

(66.1) Definition. Let E be a normed space and $X \subset E$ be an arbitrary subset. A map $\varphi: X \multimap E$ is called a *Knaster–Kuratowski–Mazurkiewicz map* (or simply *KKM-map*) provided for each finite set $\{x_1, \ldots, x_n\} \subset X$ we have

$$\text{conv}\{x_1, \ldots, x_n\} \subset \bigcup_{i=1}^n \varphi(x_i).$$

(66.2) Proposition. *If $X \subset E$ is convex and $\varphi: X \multimap E$ satisfies the following two conditions:*

(66.2.1) *$x \in \varphi(x)$ for every $x \in X$,*
(66.2.2) *$\varphi^*(y)$ is convex for every $y \in E$,*

then φ is a KKM-*map.*

Proof. Let $\{x_1, \ldots, x_n\} \subset X$ and $\overline{y} \in \text{conv}\{x_1, \ldots, x_n\}$. We have to show

$$\overline{y} \in \bigcup_{i=1}^n \varphi(x_i).$$

Since $\overline{y} \in \varphi(\overline{y})$, we see that $\overline{y} \notin \varphi^*(\overline{y})$ and therefore, $\text{conv}\{x_1, \ldots, x_n\} \not\subset \varphi^*(\overline{y})$. Since $\varphi^*(\overline{y})$ is convex, at least one point x_i does not belong to $\varphi^*(\overline{y})$, which implies that $\overline{y} \in \varphi(x_i)$ and hence the proof is completed. □

There are natural examples of KKM-maps.

(66.3) Example. Let C be a convex subset of E and let $f\colon C \to R$ be a convex function, i.e. $f(\sum t_i x_i) \le \sum t_i f(x_i)$ for any convex combination $\sum \lambda_i x_i$ in C. For each $x \in C$ we let:

$$\varphi(x) = \{y \in C \mid f(y) \le f(x)\}.$$

We show that $\varphi\colon C \multimap C$ is a KKM-map. For a contradiction let $y = \sum_{i=1}^n t_i x_i$, $x_i \in C$ and assume that $y \in \bigcup_{i=1}^n \varphi(x_i)$. Then $f(x_i) < f(y)$ for $i = 1, \dots, n$ and this means that each x_i lies in $\{x \mid f(x) < f(y)\}$. Since this set is convex we have a contradiction

$$f(y) = f\Big(\sum t_i x_i\Big) < f(y).$$

(66.4) Example. Let $E = (E, \|\ \ \|)$ be a normed linear space, $C \subset E$ be convex and $f\colon C \to E$ be a map. For each $x \in C$ let

$$\varphi(x) = \{y \in C \mid \|f(y) - y\| \le \|f(y) - x\|\}.$$

We show that $\varphi\colon C \multimap C$ is a KKM-map. Indeed, let $y_0 = \sum \lambda_i x_i$ be a convex combination in C. If $y_0 \notin \bigcup_{i=1}^n \varphi(x_i)$ we would have $\|f(y_0) - y_0\| > \|f(y_0) - x_i\|$ for each $i = 1, \dots, n$, i.e. that each x_i lies in the open ball $\{x \in E \mid \|f(y_0) - x\| < \|f(y_0) - y_0\|\}$. Since this ball is convex it contains $y_0 \in \operatorname{conv}\{x_1, \dots, x_n\}$ and we have a contradiction $\|f(y_0) - y_0\| < \|f(y_0) - y_0\|$.

(66.5) Example. Let $E = (H, \langle\cdot\rangle)$ be a Hilbert space and C a convex subset of H. Assume that $f\colon C \to H$ is a singlevalued map. We define $\varphi\colon C \multimap C$ by putting:

$$\varphi(x) = \{y \in C \mid \langle f(y), y - x\rangle \le 0\}.$$

It is easy to verify that φ is a KKM-map.

Before proving the general principle for KKM-maps we need two more notions.

A subset $X \subset E$ is called *finitely closed* provided for any finite dimensional subspace E_0 of E the intersection $X \cap E_0$ is closed in E_0.

Let $\{A_i\}_{i \in J}$ be a family of subsets of E. We will say that $\{A_i\}_{i \in J}$ has the finite intersection property provided for every finite subset $J_0 \in J$ we have

$$\bigcap\{A_i \mid i \in J_0\} \ne \emptyset.$$

The following result represents a version of the well known Knaster–Kuratowski–Mazurkiewicz theorem proved in 1929, which was used in their elementary proof of the Brouwer's fixed point theorem.

(66.6) THEOREM. *Let X be an arbitrary subset of E and $\varphi\colon X \multimap E$ be a* KKM-*map such that each $\varphi(x)$ is finitely closed (and nonempty). Then the family $\{\varphi(x) \mid x \in X\}$ has the finite intersection property.*

PROOF. We argue by contradiction, so assume that $\bigcap_{i=1}^{n} \varphi(x_i) = \emptyset$. Working in the finite-dimensional subspace L spanned by $\{x_1, \ldots, x_n\}$, let d be the Euclidean metric in L and $C = \operatorname{conv}\{x_1, \ldots, x_n\} \subset L$. Note that because each $L \cap \varphi(x_i)$ is closed in L, and since $\bigcap_{i=1}^{n} L \cap \varphi(x_i) = \emptyset$ by hypothesis, the function $g\colon C \to R$ given by $x \to \sum_{i=1}^{n} \operatorname{dist}(x, L \cap \varphi(x_i))$ does not vanish. We now define a continuous map $f\colon C \to C$ by setting

$$f(x) = \frac{1}{g(x)} \sum_{i=1}^{n} \operatorname{dist}(x, L \cap \varphi(x_i)) \cdot x_i.$$

By Brouwer's Fixed Point Theorem f would have a fixed point $x_0 \in C$. Letting $I = \{i \mid \operatorname{dist}(x_0, L \cap \varphi(x_i) \neq 0\}$, the fixed point x_0 cannot belong to $\bigcup\{\varphi(x_i) \mid i \in I\}$; however,

$$x_0 = f(x_0) \in \operatorname{conv}\{x_i \mid i \in I\} \subset \bigcup\{\varphi(x_i) \mid i \in I\}$$

and, with this contradiction, the proof is completed. □

There are many consequences and applications of (66.6). We restrict our considerations only to the most important.

As an immediate corollary we obtain:

(66.7) THEOREM. *Let $X \subset E$ be an arbitrary subset of E and $\varphi\colon X \multimap E$ a* KKM-*map. If all sets $\varphi(x)$ are closed in E and if one is compact, then $\bigcap\{\varphi(x) \mid x \in X\} \neq \emptyset$.*

Theorem (66.7) was proved in 1961 by Ky Fan.

Now, by using (66.6) we prove the Mazur–Schauder theorem proved in 1936.

(66.8) THEOREM. *Let E be a reflexive Banach space, i.e. E^* is isomorphic to E, and C a closed convex subset of E. Let $f\colon C \to \mathbb{R}$ be a convex and lower semicontinuous map,* (*i.e.* $\{x \in C \mid f(x) > \lambda\}$ *is open for each* $\lambda \in \mathbb{R}$) *and coercive* (*i.e.* $\|f(x)\| \to \infty$ *as* $\|x\| \to \infty$). *Then the function f attains its minimum.*

PROOF. Let $d = \inf f(x)$; because f is coercive, we can find a number $\rho > 0$ such that $K = \overline{B}(0, \rho) \cap C \neq \emptyset$ and $f(x) > d + 1$ for all $x \in C \setminus K$. Since $d = \inf f(x)$ it is enough now to show that there is a point $x_0 \in K$ such that $f(x_0) \leq f(x)$ for all $x \in K$. For each $x \in K$, let $\varphi(x) = \{y \in K \mid f(y) \leq f(x)\}$; since $d = \inf f(x)$, the theorem will be proved by showing $\bigcap \varphi(x) \neq \emptyset$. Since $\varphi\colon K \multimap E$ is a KKM-map, the conclusion is obtained by observing that in the weak topology of E each $\varphi(x)$ (being closed and convex) is compact. The proof is completed. □

Now we prove the following:

(66.9) THEOREM (Ky Fan, 1968). *Let C be a compact convex subset of a normed space E and let $f\colon C \to E$ be a continuous map such that for each x with $x \neq f(x)$, the line segment $[x, f(x)]$ contains at least two points of C. Then* $\mathrm{Fix}(f) \neq \emptyset$.

PROOF. Assume $f(x) \neq x$ for all $x \in C$. Then we would have $\inf_{y\in C} \|f(y) - y\| > 0$. Define $\varphi\colon C \multimap C$ by $\varphi(x) = \{y \in C \mid \|f(y) - y\| \leq \|f(y) - x\|\}$. Since φ is a compact valued KKM-map, we obtain a point $y_0 \in C$ such that

$$0 < \|f(y_0) - y_0\| \leq \|f(y_0) - x\|.$$

Now, the same simple argument as in (66.6) gives a contradiction $\|f(y_0) - y_0\| < \|f(y_0) - y_0\|$. The proof is completed. □

As an immediate application of (66.9) we derive a fixed point theorem for inward and outward maps in the sense of B. Halpern. Let C be a convex subset of a normed space E; for each $x \in C$, let

$$I_C = \{y \in C \mid \text{ there exists } y_0 \in C \text{ and } \lambda > 0 \text{ such that } y = x + \lambda(y_0 - x)\},$$
$$O_C = \{y \in C \mid \text{ there exists } y_0 \in C \text{ and } \lambda > 0 \text{ such that } y = x - \lambda(y_0 - x)\}.$$

A map $f\colon C \to E$ is said to be *inward* (resp. *outward*) if $f(x) \in I_C(x)$ (resp. $f(x) \in O_C(x)$).

(66.10) THEOREM. *Let C be a convex compact subset of a normed E. Then every continuous inward (resp. every continuous outward) map $f\colon C \to E$ has a fixed point.*

PROOF. The case of an inward map follows directly from (66.9); if f is outward then $g\colon C \to E$ given by $x \to 2x - f(x)$ is inward with the same set of fixed points as f and the conclusion follows. □

We would like to point out that the minimax theorem in the case of $X = Y = C$ being convex, can be obtained from (66.6). In Section 65 we have proved a little more general theorem (see (65.6)).

Now we will show applications of (66.6) to variational inequalities. Let $(H, \langle\,\cdot\,\rangle)$ be a Hilbert space and $C \subset H$. We recall that a map $f\colon C \to H$ is called *monotone* on C if $\langle f(x) - f(y), x - y\rangle \geq 0$ for all $x, y \in C$. We will say that $f\colon C \to H$ is a *one finitely continuous* if $f|_{H_0\cap C}$ is continuous for each one-dimensional subspace $H_0 \subset H$.

(66.11) THEOREM. *Let H be a Hilbert space, C a closed bounded convex subset of H, and $f: C \to H$ monotone and hemi-continuous. Then there exists a $y_0 \in C$ such that $\langle f(y_0), y_0 - x\rangle \leq 0$ for all $x \in C$.*

PROOF. For each $x \in C$, let $\varphi(x) = \{y \in C \mid \langle f(y), y - x\rangle \leq 0\}$; the theorem will be proved showing $\bigcap\{\varphi(x) \mid x \in C\} \neq \emptyset$.

We know that $\varphi: C \multimap H$ is a KKM-map. Consider now the map $\Psi: C \multimap H$ given by

$$\Psi(x) = \{x \in C \mid \langle f(x), y - x\rangle \leq 0\};$$

we show that

Step 1. $\varphi(x) \subset \Psi(x)$ for each $x \in C$.

For, let $y \in \varphi(x)$, so that $0 \geq \langle f(y), y - x\rangle$. By the monotonicity of $f: C \to H$, we have $\langle f(y) - f(x), y - x\rangle \geq 0$, so $0 \geq \langle f(y), y - x\rangle \geq \langle f(x), y - x\rangle$ and $y \in \Psi(x)$.

Step 2. Because of Step 1, it is enough to show $\bigcap\{\varphi(x) \mid x \in C\} \subset \bigcap\{\Psi(x) \mid x \in C\}$.

Assume $y_0 \in \bigcap \Psi(x)$. Choose any $x \in C$ and let $z_t = tx + (1-t)y_0 = y_0 - t(y_0 - x)$. Because C is convex, we have $z_t \in C$ for each $0 \leq t \leq 1$. Since $y_0 \in \Psi(z_t)$ for each $t \in [0,1]$, we find that $\langle f(z_t), y_0 - z_t\rangle \leq 0$ for all $t \in [0,1]$. This means that $t\langle f(z_t), y_0 - x\rangle \leq 0$ for all $t \in [0,1]$ and, in particular, that $\langle f(z_t), y_0 - x\rangle \leq 0$ for $0 < t \leq 1$. Now let $t \to 0$; the continuity of f on the ray joining y_0 and x gives $f(z_t) \to f(y_0)$ and therefore, that $\langle f(y_0), y_0 - x\rangle \leq 0$. Thus, $y_0 \in \varphi(x)$ for each $x \in C$ and $\bigcap \Psi(x) = \bigcap \varphi(x)$.

Step 3. Now, we can equip H with the weak topology. Then each $\Psi(x)$, being the intersection of the closed half-space $\{y \in H \mid \langle f(x), y\rangle \leq \langle f(x), x\rangle\}$ with C is closed convex and bounded and therefore, weakly compact.

Thus, all the requirements in (66.8) are satisfied; therefore, $\bigcap\{\varphi(x) \mid x \in C\} \neq \emptyset$ and, as we have observed, the proof is completed. □

(66.12) COROLLARY. *Let C be a closed bounded convex subset of H and F: $C \to C$ a non-expansive map i.e. $\|Fx - Fy\| \leq \|x - y\|$ for all $x, y \in C$. Then F has a fixed point.*

PROOF. Putting $f(x) = x - F(x)$ for $x \in C$, we verify by simple calculation that $f: C \to H$ is a continuous monotone map; so, by Theorem (66.11), there exists $y_0 \in C$ such that $\langle y_0 - Fy_0, y_0 - x\rangle = \langle fy_0, y_0 - x\rangle \leq 0$ for all $x \in C$. By taking in the above inequality $x = F(y_0)$ we obtain $y_0 = Fy_0$, and the proof is completed. □

(66.13) COROLLARY. *Let $C \subset H$ be a closed convex set. Then for each $x_0 \in H$ there is a unique $y_0 \in C$ with $\|x_0 - y_0\| = \inf\{\|x_0 - x\| \mid x \in C\}$.*

PROOF. Uniqueness being evident, let $f\colon C \to H$ be given by $y \to y - x_0$; clearly f is continuous and monotone. By (66.11), there is $y_0 \in C$ with $\langle y_0 - x_0, y_0 - x\rangle \leq 0$ for all $x \in C$; this being equivalent to $\|x_0 - y_0\| = \inf_C \|x_0 - x\|$, the assertion of the theorem follows. □

67. Topological dimension of the set of fixed points

In Section 21 we have proved the Banach contraction principle and a topological characterization of the set of fixed points. In this section, we would like to prove some additional properties of the set of fixed points for multivalued contractions.

Again let E be a Banach space and X its closed convex nonempty subset. Let $F\colon X \to B(X)$ be a contraction map. We are interested in the following problem: when does $\dim F(x) \geq n$ for every $x \in X$ imply $\dim \operatorname{Fix}(F) \geq n$?

Before giving some particular answers we shall collected some important properties.

(67.1) PROPOSITION. *If $F\colon X \to C(X)$ is a contraction map, then the set* $\operatorname{Fix}(F)$ *is compact.*

PROOF. Assume $\operatorname{Fix}(F)$ is not compact. Since it is complete, it cannot be precompact. Thus, there exists some $\delta > 0$ and some sequence $\{x_i\}$ in $\operatorname{Fix}(F)$ such that $d(x_i, x_j) \geq \delta$ for any two different i and j. Put

$$\rho = \inf\{r \mid \text{there exists } a \in X \text{ such that } B(a, r) \text{ contains infinitely many } x_i\text{'s}\}.$$

Since, for every $a \in X$, $B(a, \delta/2)$ contains at most one x_i, one has $\rho \geq \delta/2 > 0$. Let fix ε such that

$$0 < \varepsilon < \rho\,\frac{1-q}{1+q},$$

where $q < 1$ is a contraction constant and choose $a \in X$ such that $J = \{i \mid x_i \in B(a, \rho + \varepsilon)\}$ is infinite. For each $i \in J$

$$d(x_i, F(a)) \leq d_H(F(x_i), F(a)) \leq q d(x_i, a) < q(\rho + \varepsilon),$$

and we can choose some $y_i \in F(a)$ such that $d(x_i, y_i) < q(\rho + \varepsilon)$. By the compactness of $F(a)$, there is a $b \in F(a)$ such that $J' = \{i \in J \mid d(y_j, b) < \varepsilon\}$ is finite. Then for each $i \in J'$:

$$d(x_i, b) < q(\rho + \varepsilon) + \varepsilon = q\rho + \varepsilon(1 + q) = r < q\rho + \rho(1 - q) = \rho,$$

and this contradicts the definition of ρ since $B(b, r)$ contains infinitely many x_i's. □

(67.2) PROPOSITION. *If $F: X \to B(X)$ is a contraction, then there exists a bounded convex and closed subset $B \subset X$ such that* $\mathrm{Fix}(F) \subset B$ *and B is F-invariant, i.e. $F(B) \subset B$.*

PROOF. It follows from (21.1) that $\mathrm{Fix}(F) \neq \emptyset$. Let $x_0 \in \mathrm{Fix}(F)$. Consider a closed ball $B(x_0, r) = \{y \in X \mid \|y - x_0\| \leq r\}$, where r is closed in such a way that $B(x_0, r)$ is F-invariant. For every $y \in B(x_0, r)$ we have:

$$d_H(F(x_0), F(y)) \leq k \cdot \|x_0 - y\| \leq k \cdot r,$$

where F is a contraction with the constant k, $0 \leq k < 1$. Thus, if $r \geq \delta(F(x_0))$ and $z \in F(y)$ we have:

$$\|z - x_0\| \leq d_H(F(y), F(x_0)) + \delta(F(x_0)) \leq kr + r.$$

So, $B(x_0, r)$ is F-invariant provided $r \geq (1 - k)^{-1} \cdot \delta(F(x_0))$, where $\delta(F(x_0))$ denotes the diameter of F. Therefore, we have to check only that $\mathrm{Fix}(F) \subset B(x_0, r)$. Let $x_1 \in \mathrm{Fix}(F)$. Then

$$\begin{aligned}\|x_1 - x_0\| &\leq \mathrm{dist}(x_1, F(x_0)) + \delta(F(x_0)) \\ &\leq d_H(F(x_1), F(x_0)) + \delta(F(x_0)) \leq k\|x_1 - x_0\| + \delta(F(x_0))\end{aligned}$$

and hence

$$\|x_1 - x_0\| \leq (1 - k)^{-1}\delta(F(x_0)),$$

so the proof is completed. □

The proof of the following proposition is a simple exercise:

(67.3) PROPOSITION. *Any contraction $F: X \to C(X)$ is a condensing map with respect to the Hausdorff measure of noncompactness γ (see Section* 4*).*

(67.4) PROPOSITION. *Let $F: X \to C(X)$ be a contraction. If $f: X \to X$, $f \subset F$, is a continuous selection of F, then* $\mathrm{Fix}(f) \neq \emptyset$.

PROOF. Since, in view of (67.3), F is condensing so, $f \subset F$ is also condensing. On the other hand, from (67.2) it follows that we can assume, without loss of generality, that X is also bounded. Hence our claim follows from (59.12). □

We shall make use of the following:

(67.5) LEMMA. *Let T be a compact space and $\varphi: T \multimap E$ be a l.s.c. map with closed convex values. Assume further that* $\dim T < n$, $0 \in \varphi(x)$ *and* $\dim \varphi(x) \geq n$ *for every $x \in T$. Then there exists a continuous selection $f \subset F$ such that $f(x) \neq 0$ for each $x \in T$.*

Our main result of this section is the following:

(67.6) THEOREM. *Let $F: X \to C(X)$ be a contraction. If* $\dim F(x) \geq n$ *for each $x \in X$, then* $\dim \mathrm{Fix}(F) \geq n$.

PROOF. It follows from (67.1) that $\mathrm{Fix}(F)$ is compact and from (21.1) that $\mathrm{Fix}(F) \neq \emptyset$. Consider a map $(i-F)\colon \mathrm{Fix}(F) \multimap E$, $(i-F)(x) = \{x-y \mid y \in F(x)\}$. Then $(i-F)$ satisfies the assumptions of (67.5) instead of $\dim \mathrm{Fix}(F) < n$. So, let us assume that $\dim \mathrm{Fix}(F) < n$. By (67.5), there exists a continuous selection f_0 of $F|_{\mathrm{Fix}(F)}$ without fixed points. Using Michael's selection theorem we can extend f_0 to a map $f\colon X \to X$ without fixed points such that $f \subset F$ and we have a contradiction with (67.4). □

By a similar consideration we can obtain the following:

(67.7) THEOREM. *Let $F: X \to B(X)$ be a contraction with the constant k such that $0 \leq k < 1/2$. Assume that* $\mathrm{Fix}(F)$ *is compact and* $\dim F(x) \geq n$ *for every $x \in X$. Then* $\dim \mathrm{Fix}(F) \geq n$.

68. On the basis problem in normed spaces

Let E be a normed space. A sequence $\{x_n\}$ of vectors of E is said to be a *basis* (or Schauder basis) for E, provided that every $x \in E$ has a unique representation as the sum of the series

$$x = \sum_{n=1}^{\infty} t_n x_n, \quad t_n \in \mathbb{R},\ n = 1, 2, \ldots$$

It is well known that not every separable Banach space has a basis. Therefore, following Day ([Day]) we introduce the notion of biorthogonal system in E. Let B be the unit closed ball in E, E^* be the conjugate space and B^* the conjugate unit ball.

A pair $\{\{b_n\}, \{\beta_n\}\}$ of sequences $\{b_n\} \subset B$ and $\{\beta_n\} \subset B^*$ is called a *biorthogonal system* for E provided:

(i) the sequence $\{b_i\}$ is a Schauder basis for the subspace
$$L = \mathrm{span}\{b_1, b_2, \ldots\} \subset E,$$
(ii) $\|b_n\| = \|\beta_n\| = 1$ for all n,
(iii) setting $P_m = \sum_{i \leq m} \beta_i(x) b_i(x)$, for each m, the linear operator $P_m\colon L \to L$ is a projection, i.e. $P_m^2 = \mathrm{id}_L$ and $\|P_m\| \leq 1 + (1/m)$.

By using the Borsuk theorem on antipodes for multivalued mappings one can prove the following:

(68.1) THEOREM. *Any normed space possesses a biorthogonal system.*

PROOF. Take b_1 of norm 1 in B and choose β_1, by the Hahn–Banach theorem, such that $\|\beta_1\| = \|b_1\| = \beta_1(b_1) = 1$. If $b_1, \dots, b_m$ and $\beta_1, \dots, \beta_m$, and, if necessary, certain auxiliary $\gamma_1, \dots, \gamma_k$ in B^* have bee chosen, the choice of b_{m+1} is made to depend on the Borsuk–Ulam theorem in the following way.

Let L_m be the linear ball spanned on b_i, $i \le m$, and let $S_m = S \cap L_m$ and $\gamma_j^{-1}(K) \cap L_m$ contains S_m and the intersection of these sets is a polyhedron W_m in L_m. It may happen that W_m contains only points of norm $\le 1 + (1/m)$; if this is not case take enough elements of norm 1 in L_m^*, say $\alpha_1, \dots, \alpha_n$, that W_m intersected with all the sets $\alpha_q^{-1}(K)$ is a polyhedron in L_m which all lies within the sphere of radius $1 + (1/m)$. Let γ_{k+q} be an element of B^* of the norm 1 which is an extension of α_q, $q = 1, \dots, n$. Let A_m be the (infinite-dimensional) intersection of the hype planes $\beta_i^{-1}(0)$, $i \le m$ and $\gamma_j^{-1}(0)$, $j \le k + n$.

In A_m any $(m+1)$-dimensional subspace Λ'_m and in Λ'_m consider the unit sphere $S'_m = S \cap \Lambda'_m$.

Define a mapping $\varphi\colon S'_m \multimap L_m$ by putting:

$$\varphi(x) = \{y \in L_m \mid \|x + y\| = \|x + L_m\|\},$$

where $\|x + L_m\| = \inf\{\|x + z\|, z \in L_m\}$. Then φ is an u.s.c. map with compact convex values and $\varphi(x) = -\varphi(-x)$. Thus $\mathrm{Deg}(\varphi) \ne 0$ and hence there is $x \in \Lambda'_m$ such that $0 \in \varphi(x)$. Let b_{m+1} be any point of S'_m such that $0 \in \varphi(b_{m+1})$. Let β_{m+1} in B^* be chosen so that it is of norm 1 vanishes on the b_i, $i \le m$ and is 1 at b_{m+1}. This induction process defines sequences $\{b_i\}$ and $\{\beta_i\}$. If L' is the union of all the L_m, then for each m the function P_m is defined in B and has in L'_m the norm $1 + (1/m)$. The set of those x in L, for which $\lim_m P_m(x) = x$ includes all of the b_i and is closed in L, so it is all of L; the proof is completed. $\square$

CHAPTER VI

FIXED POINT THEORY APPROACH TO DIFFERENTIAL INCLUSIONS

The aim of this chapter is to give a systematic and unified account of topics in fixed point theory methods of differential inclusions which lie on the border line between topology and ordinary differential equations.

It is well known that the topological degree is a fundamental tool for proving the existence of various kinds of solutions of nonlinear equations and for investigating the structure of sets of such solutions. Since the original classical work of Leray and Schauder many authors have made contributions to the problem of extending the Leray–Schauder degree and applying it to problems in analysis. These generalizations include extensions of the Lefschetz fixed point theory and the fixed point index theory on ANRs for singlevalued mappings to multivalued case. In this chapter we shall concentrate our considerations on the topological degree theory for multivalued mappings which are compositions of R_δ-valued mappings with singlevalued mappings. This degree theory gives us a tool for investigating the following types of questions about differential inclusions:

- existence problems;
- topological characterization of the set of solutions for Cauchy problems;
- periodic problems.

We shall study the above problems in the case when our multivalued right hand side of the differential inclusion considered is defined on the whole space $\mathbb{R}^n$ or on a certain compact subset A of $\mathbb{R}^n$, the so called proximate retracts.

69. Aronszajn type of results

In this section we shall present results about the topological structure of the set of solutions of the Cauchy problem for some nonlinear ordinary differential equations as owed to N. Aronszajn in 1942.

First, we prove a result concerning the topological structure of the set of solutions for some nonlinear equations.

(69.1) THEOREM. *Let X be a space, $(E, \|\cdot\|)$ a Banach space and $f\colon X \to E$ a proper map, i.e. f is continuous and for every compact $K \subset E$ the set $f^{-1}(K)$*

is compact. Assume further that for each $\varepsilon > 0$ a proper map $f_\varepsilon\colon X \to E$ is given and the following two conditions are satisfied:

(69.1.1) $\|f_\varepsilon(x) - f(x)\| < \varepsilon$, *for every* $x \in X$;

(69.1.2) *for any* $\varepsilon > 0$ *and* $u \in E$ *such that* $\|u\| \le \varepsilon$, *the equation* $f_\varepsilon(x) = u$ *has exactly one solution.*

Then the set $S = f^{-1}(0)$ *is* R_δ.

PROOF. Let $\{\varepsilon_n\}$ be a sequence of positive real numbers such that $\lim_n \varepsilon_n = 0$. Since $S = f^{-1}(0)$ and f is proper we infer that S is compact. To see that $S \neq \emptyset$, in view of (69.1.2) we choose $x_n \in X$ such that $f_n(x_n) = 0$ for every n, where f_n is short for f_{ε_n}. Then we have:

$$\|f(x_n)\| = \|f(x_n) - f_n(x_n)\| < \varepsilon_n.$$

Thus $\lim_n f(x_n) = 0$ and hence the set $\{f(x_n)\} \cup \{0\}$ is compact. Therefore, $\{x_n\} \subset f^{-1}(\{(f(x_n))\} \cup \{0\})$, and we may assume without loss of generality that $\lim_n x_n = x$. Then from the continuity of f we conclude that $x \in S$, and consequently we have that S is a compact nonempty set.

Now, it follows from (69.1.1) that $f_n(S) \subset B(0, \varepsilon_n)$. Let Q_n denote the convex closure of the compact set $f_n(S) \subset E$. Then Q_n, as a compact convex set, is an AR-space and hence contractible. Moreover, we have

$$Q_n \subset B(0, \varepsilon_n) \subset B(0, \rho).$$

Let $A_n = f_n^{-1}(Q_n)$. The mapping f_n restricted to the pair (A_n, Q_n) is a homeomorphism (this follows easily from our assumptions), so A_n is a compact AR-space and in particular contractible. To conclude that S is a compact R_δ we shall show that the sequence $\{A_n\}$ satisfies the conditions (2.15.1)–(2.15.3).

Clearly, S is contained in A_n for every n, so S is contained in the inferior set-theoretic limit of the sequence $\{A_n\}$. Now, let x be a point in the superior set-theoretic limit of the sequence $\{A_n\}$, so that $x \in A_{n_i}$, for some subsequence $\{A_{n_i}\}$ of $\{A_n\}$. This implies that $\|f_{n_i}(x)\| < \varepsilon_{n_i}$, for every n_i. Hence, $f(x) = 0$, which implies that $x \in S$. Thus the superior set-theoretic limit of $\{A_n\}$ is contained in S. Hence, S is the set-theoretic limit of $\{A_n\}$.

Now, to verify the condition (2.15.3) it suffices to show that each open neighbourhood V of S in X contains at least one member of the sequence $\{A_n\}$, as the set-theoretic limit remains unchanged if finitely many members of $\{A_n\}$ are omitted. Suppose now that V is an open neighbourhood of S in X such that A_n is not contained in V for any n. So, there exists a sequence $\{x_n\}$ in X such that $x_n \in A_n$ for every n, and $x_n \notin V$ for any n. Then we have

$$\|f_n(x_n)\| < \varepsilon_n, \quad \|f_n(x_n) - f(x_n)\| < \varepsilon_n,$$

and hence,

$$\|f(x_n)\| < 2\varepsilon_n \quad \text{for every } n.$$

This implies that $\lim f(x_n) = 0$. Since f is proper we deduce that $\{x_n\}$ contains a convergent subsequence $\{x_{n_i}\}$. Let $x = \lim x_{n_i}$. Then $f(x) = 0$. Thus $x \in S$ and $x_{n_i} \notin V$ for every i, which is a contradiction. This proves that each neighbourhood V of S contains at least one member of the sequence $\{A_n\}$. Hence the conditions of Proposition (2.15) are verified. Thus S is an R_δ-set; the proof is completed. □

The following result is a slight reformulation of Lemma 1 in [Sz].

(69.2) THEOREM. *Let $E = C([0,a],\mathbb{R}^m)$ be the Banach space of continuous maps with the usual max-norm. If $F: E \to E$ is a compact map and $f: E \to E$ is a compact vector field associated with F, i.e. $f(u) = u - F(u)$, such that the following conditions are satisfied:*

(69.2.1) *there exists an $x_0 \in \mathbb{R}^m$ such that $F(u)(0) = x_0$, for every $u \in K(0,r)$;*

(69.2.2) *for every $\varepsilon \in [0,a]$ and for every $u, v \in E$ if $u(t) = v(t)$ for each $t \in [0,\varepsilon]$, then $F(u)(t) = F(v)(t)$ for each $t \in [0,\varepsilon]$;*

then there exists a sequence $f_n: E \to E$ of continuous proper mappings satisfying the conditions (69.1.1) *and* (69.1.2) *with respect to f.*

SKETCH OF PROOF. For the proof it is sufficient to define a sequence $F_n: E \to E$ of compact maps such that:

(69.2.3) $$F(x) = \lim_{n\to\infty} F_n(x), \quad \text{uniformly in } x \in E,$$

and

(69.2.4) $$f_n: E \to E, \quad f_n(x) = x - F_n(x), \quad \text{is a one-to-one map.}$$

To do this we additionally define the mappings $r_n: [0,a] \to [0,a]$ by putting:

$$r_n(t) = \begin{cases} 0 & \text{for } t \in [0, a/n], \\ t - a/n & \text{for } t \in (a/n, a]. \end{cases}$$

Now, we are able to define the sequence $\{F_n\}$ as follows:

(69.2.5) $$F_n(x)(t) = F(x)(r_n(t)), \quad \text{for } x \in E,\ n = 1, 2, \ldots.$$

It is easily seen that F_n is a continuous and compact mapping, $n = 1, 2, \ldots$. Since $[r_n(t) - t] \le a/n$ we deduce from the compactness of F and (69.2.5) that

$$\lim_{n\to\infty} F_n(x) = F(x), \quad \text{uniformly in } x \in E.$$

Now, we shall prove that f_n is a one-to-one map. Assume that for some $x, y \in E$ we have $f_n(x) = f_n(y)$. This implies that $x - y = F_n(x) - F_n(y)$. If $t \in [0, a/n]$, then we have

$$x(t) - y(t) = F(x)(r_n(t)) - F(y)(r_n(t)) = F(x)(0) - F(y)(0).$$

Thus, in view of (69.2.1), we obtain $x(t) = y(t)$, for every $t \in [0, a/n]$.

Finally, by successively repeating the above procedure n times we infer that

$$x(t) = y(t), \quad \text{for every} \quad t \in [0, a].$$

Therefore, f_n is a one-to-one map and the proof is completed. □

Now, from Theorems (69.1) and (69.2) we obtain:

(69.3) Corollary. *Assume that f and F are as in Theorem (69.2). Then $f^{-1}(0) = \mathrm{Fix}(F)$ is an R_δ-set.*

For a given map $g: [0, a] \times \mathbb{R}^n \to \mathbb{R}^n$ and $x_0 \in \mathbb{R}^n$ we shall consider the following Cauchy problem:

$$\begin{cases} x'(t) = g(t, x(t)), \\ x(0) = x_0. \end{cases} \tag{69.4}$$

In our considerations g is a Carathéodory mapping. By a solution of (69.4) we shall understand an absolutely continuous map $x: [0, a] \to \mathbb{R}^n$ such that $x'(t) = g(t, x(t))$ for almost all $t \in [0, a]$ and $x(0) = x_0$. If the right hand side g is continuous, then every solution $x(\,\cdot\,)$ is C^1 regular and satisfies $x'(t) = g(t, x(t))$ for every $t \in [0, a]$.

We denote by $S(g, 0, x_0)$ the set of all solutions of the Cauchy problem (69.4).

(69.5) Theorem. *Let $g: [0, a] \times \mathbb{R}^n \to \mathbb{R}^n$ be an integrably bounded Carathéodory mapping. Then $S(g, 0, x_0)$ is R_δ.*

Proof. We define the integral operator $F: C([0, a], \mathbb{R}^n) \to C([0, a], \mathbb{R}^n)$ by putting

$$F(u)(t) = x_0 + \int_0^t g(r, u(r))\, dr \quad \text{for every } u \text{ and } t. \tag{69.5.1}$$

Then $\mathrm{Fix}(F) = S(g, 0, x_0)$. It is easy to see that F satisfies all the assumptions of Theorem (69.2). Consequently we deduce Theorem (69.5) from Corollary (69.3) and the proof is completed. □

Now, let g be a Carathéodory map with linear growth. Assume further that $u \in S(g, 0, x_0)$. Then we have (cf. (69.5.1))

$$u(t) = F(u)(t) = x_0 + \int_0^t g(r, u(r))\, dr,$$

and consequently

$$\|u(t)\| \leq \|x_0\| + \int_0^t \mu(r)\,dr + \int_0^t \mu(r)\|u(r)\|\,dr.$$

Therefore, from the well known Gronwall inequality we obtain

$$\|u(t)\| \leq (\|x_0\| + \gamma)\exp(\gamma) \quad \text{for every } t,$$

where $\gamma = \int_0^a \mu(r)\,dr$.

We define $g_0\colon [0,a] \times \mathbb{R}^n \to \mathbb{R}^n$ by putting

$$g_0(t,x) = \begin{cases} g(t,x) & \text{if } \|x\| \leq M \text{ and } t \in [0,a], \\ g(t, Mx/\|x\|) & \text{if } \|x\| \geq M \text{ and } t \in [0,a], \end{cases}$$

where $M = (\|x_0\| + \gamma)\exp(\gamma)$.

(69.6) Proposition. *If g is a Carathéodory map with linear growth, then:*

(69.6.1) *g_0 is Carathéodory and integrably bounded; and*
(69.6.2) *$S(g_0, 0, x_0) = S(g, 0, x_0)$.*

The proof of Proposition (69.6) is straightforward.

Now, from Theorem (69.5) and Proposition (69.6) we obtain immediately:

(69.7) Corollary. *If $g\colon [0,a] \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory map and has linear growth then $S(g,0,x_0)$ is a R_δ-set.*

Finally, let us recall the following classical result from the theory of ordinary differential equations.

(69.8) Theorem. *If $f\colon [0,a] \times \mathbb{R}^n \to \mathbb{R}^n$ is an integrably bounded, locally-measurable Lipschitz map, then the set $S(f,0,x_0)$ is a singleton for every $x_0 \in \mathbb{R}^n$.*

Later we shall make use of the following:

(69.9) Theorem. *Let E be a normed space, X a metric space and $F\colon E \times X \to E$ a continuous (singlevalued) map such that for any compact subset $A \subset X$ the closure $\overline{F(E \times A)}$ of $F(E \times A)$ is a compact subset of E. Then the (multivalued) map $\varphi\colon X \multimap E$ defined as follows:*

$$\varphi(x) = \operatorname{Fix}(F(\,\cdot\,, x))$$

is an u.s.c. mapping.

Proof. It follows from the Schauder Fixed Point Theorem that the set $\varphi(x)$ is compact and nonempty for every $x \in X$.

Let $x_0 \in X$ and let U be an open neighbourhood of $\varphi(x_0)$ in E. It is enough to prove that there exists $r > 0$ such that for every $x \in B(x_0, r)$ we have $\varphi(x) \subset U$. Assume to the contrary that for every $n = 1, 2, \ldots$ there exists $x_n \in B(x_0, 1/n)$ and $y_n \in \mathrm{Fix}(F(\,\cdot\,, x_n))$ such that $y_n \notin U$.

We let $A = \overline{\{x_n\}}$. So, $A = \{x_n\} \cup \{x_0\}$. Consequently, in view of our assertion, we can assume that $\lim_{n\to\infty} y_n = y_0$. Then $y_n \notin U$ and $y_0 \in \mathrm{Fix}(F(\,\cdot\,, x_0)) = \varphi(x_0) \subset U$, so we obtain a contradiction. □

We shall conclude this section by introducing the following notion:

(69.10) Definition. A space X is called *R_δ-contractible* provided there exists a multivalued homotopy $\chi\colon [0,1] \times X \multimap X$ such that:

(69.10.1) $x \in \chi(1, x)$, for every $x \in X$;
(69.10.2) $\chi(0, x) = A$, for every $x \in X$;
(69.10.3) $\chi(t, x)$ is an R_δ-set, for every $t \in [0,1]$ and $x \in X$;
(69.10.4) χ is a u.s.c. map,

where A is an R_δ-subset of X.

Let us remark that any R_δ-contractible space has the same homology as one-point space $\{p\}$ so, that it is acyclic and in particular connected.

70. Solution sets for differential inclusions

Let $\varphi\colon [0,a] \times \mathbb{R}^n \multimap \mathbb{R}^n$ be a multivalued map. In the present section we consider the differential inclusion

$$\begin{cases} x'(t) \in \varphi(t, x(t)), \\ x(0) = x_0. \end{cases} \tag{70.1}$$

An absolutely continuous map $x\colon [0,a] \to \mathbb{R}^n$ is called a *solution* of (70.1), if $x'(t) \in \varphi(t, x(t))$ almost everywhere in $[0,a]$ (a.e. $t \in [0,a]$) and $x(0) = x_0$.

Consider the differential inclusion

$$x'(t) \in \varphi(t, x(t)).$$

The connection between differential inclusions and control systems is well known. If $f\colon [0,a] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a continuous function and A a compact subset of $\mathbb{R}^m$, then the set of trajectories for the system:

$$x'(t) = f(t, x(t), u(t)), \quad u(t) \in A,$$

coincides with the set of solutions to the above differential inclusion, where

$$\varphi(t, x) = \{f(t, x, u) \mid u \in A\}.$$

Differential inclusions can thus provide a convenient abstract framework for the study of certain control problems. Solutions of the above inclusion, or more precisely of (70.1), have been studied mainly under two separate kind of assumptions:

(B1) φ is u.s.c. with convex values;

(B2) φ is l.s.c. possibly with nonconvex values.

In this section we shall look at the case (B1). Then in Section 71 we shall deal with the case (B2).

Let $S(\varphi, 0, x_0)$ denote the set of all solutions of (70.1). We are now going to characterize the topological structure of $S(\varphi, 0, x_0)$. First, we prove the following:

(70.2) THEOREM. *If $\varphi\colon [0,a] \times \mathbb{R}^n \multimap \mathbb{R}^n$ is integrably bounded and mLL-selectionable, then $S(\varphi, 0, x_0)$ is contractible for every $x_0 \in \mathbb{R}^n$.*

PROOF. Let $f \subset \varphi$ be measurable, locally Lipschitz. By Theorem (69.8), the following Cauchy problem:

$$\begin{cases} x'(t) = f(t, x(t)), \\ x(t_0) = u_0, \end{cases} \tag{70.2.1}$$

has exactly one solution for every $t_0 \in [0,a]$ and $u_0 \in \mathbb{R}^n$. For the proof it is sufficient to define a homotopy $h\colon S(\varphi, 0, x_0) \times [0,1] \to S(\varphi, 0, x_0)$ such that

$$h(x,s) = \begin{cases} x & \text{for } s = 1 \text{ and } x \in S(\varphi, 0, x_0), \\ \widetilde{x} & \text{for } s = 0, \end{cases}$$

where $\widetilde{x} = S(f, 0, x_0)$ is exactly one solution given for the Cauchy problem (70.2.1). We put

$$h(x,s)(t) = \begin{cases} x(t) & \text{if } 0 \le t \le sa, \\ S(f, sa, x(sa))(t) & \text{if } sa \le t \le a. \end{cases}$$

Then h is a continuous homotopy contracting $S(\varphi, 0, x_0)$ to the point $S(f, 0, x_0)$.□

(70.3) THEOREM. *If $\varphi\colon [0,a] \times \mathbb{R}^n \to \mathbb{R}^n$ is an integrably bounded, Ca-selectionable or in particular C-selectionable map, then the set $S(\varphi, 0, x_0)$ is R_δ-contractible.*

PROOF. The proof is strictly analogous to that of Theorem (70.2). We replace the singlevalued homotopy $h\colon S(\varphi, 0, x_0) \times [0,1] \to S(\varphi, 0, x_0)$ by a multivalued homotopy $\chi\colon [0,1] \times S(\varphi, 0, x_0) \multimap S(\varphi, 0, x_0)$ as follows:

$$\chi(s,x)(t) = \begin{cases} x(t) & \text{if } 0 \le t \le sa, \\ S(f, sa, x(sa))(t) & \text{if } sa < t \le a, \end{cases}$$

where $f \subset \varphi$ and $S(f, sa, x(sa))$ is an R_δ-set according to Corollary (69.7). □

Observe that if $\varphi\colon [0,a]\times\mathbb{R}^n \multimap \mathbb{R}^n$ is an intersection of the decreasing sequence $\varphi_k\colon [0,a]\times\mathbb{R}^n \multimap \mathbb{R}^n$, i.e. $\varphi(t,x) = \bigcap_{k=1}^{\infty}\varphi_k(t,x)$ and $\varphi_{k+1}(t,x)\subset\varphi_k(t,x)$ for almost all $t\in[0,a]$ and for all $x\in\mathbb{R}^n$, then

$$S(\varphi,0,x_0) = \bigcap_{k=1}^{\infty} S(\varphi_k,0,x_0). \tag{70.4}$$

From theorems (70.2) and (70.3) we obtain:

(70.5) THEOREM. *Let $\varphi\colon [0,a]\times\mathbb{R}^n \multimap \mathbb{R}^n$ be a multivalued map.*

(70.5.1) *If φ is σ-mLL-selectionable, then the set $S(\varphi,0,x_0)$ is an intersection of a decreasing sequence of contractible sets;*

(70.5.2) *If φ is σ-Ca-selectionable, then the set $S(\varphi,0,x_0)$ is an intersection of a decreasing sequence of R_δ-contractible spaces.*

We can now formulate the main result of this section.

(70.6) THEOREM. *If $\varphi\colon [0,a]\times\mathbb{R}^n \multimap \mathbb{R}^n$ is an u-Carathéodory map with convex compact values and having linear growth, then the set $S(\varphi,0,x_0)$ is an R_δ-set for every $x_0\in\mathbb{R}^n$.*

PROOF. Using similar arguments as for (69.7) we can find $r>0$ such that if $x(\,\cdot\,)$ is a solution of (70.1), then $\|x(t)\|<r$ for every $t\in[0,a]$. We put

$$\varphi_r(t,x) = \begin{cases} \varphi(t,x) & \text{if } \|x\|\le r \text{ and } t\in[0,a],\\ \varphi(t,rx/\|x\|) & \text{if } \|x\|\ge r \text{ and } t\in[0,a]. \end{cases}$$

It is obvious that φ_r is an integrably bounded u-Carathéodory map and also $S(\varphi_r,0,x_0) = S(\varphi,0,x_0)$. Now, the proof is analogous to the proof of (70.3) (comp. [Go2-M]). □

Finally, as a simple corollary from Theorem (70.6) we obtain:

(70.7) COROLLARY. *If $\varphi\colon [0,a]\times\mathbb{R}^n \multimap \mathbb{R}^n$ is a u.s.c. bounded mapping with convex compact values, then $S(\varphi,0,x_0)$ is an R_δ-set for every $x_0\in\mathbb{R}^n$.*

In the following we describe the dependence of the set of solutions on the initial values and parameters. Let Λ be a compact space and $\varphi\colon [0,a]\times\Lambda\to\mathbb{R}^n$ be a nonempty convex compact valued map such that:

(70.8.1) $t\to\varphi(t,x,\lambda)$ is u.s.c. measurable for every $(x,\lambda)\in\mathbb{R}^n\times\Lambda$;

(70.8.2) $(x,\lambda)\to\varphi(t,x,\lambda)$ is u.s.c. for almost all $t\in[0,a]$;

(70.8.3) φ is integrably bounded.

For a given $\lambda\in\Lambda$ we let:

$$\varphi_\lambda\colon [0,a]\times\mathbb{R}^n\to\mathbb{R}^n,\quad \varphi_\lambda(t,x)=\varphi(t,x,\lambda)\quad\text{for every } t,x.$$

(70.9) PROPOSITION. *Suppose that* $\varphi\colon [0,a]\times\mathbb{R}^n\times\Lambda \multimap \mathbb{R}^n$ *satisfies the conditions* (70.8.1)–(70.8.3). *Let* $P\colon \mathbb{R}^n\times\Lambda \multimap C([0,a],\mathbb{R}^n)$ *be a map defined as follows*

$$P(x,\lambda) = S(\varphi_\lambda, 0, x).$$

Then P *is u.s.c. map with* R_δ*-values.*

PROOF. It follows from Theorem (70.6) that P has R_δ-values. To prove the upper semicontinuity of P we shall consider the integral operator:

$$\Phi\colon C([0,a],\mathbb{R}^n)\times\mathbb{R}^n\times\Lambda \to C([0,a],\mathbb{R}^n)$$

defined by the formula:

$$\Phi(u,x,\lambda) = \bigg\{ v \;\bigg|\; v(t) = x + \int_0^t w(\tau)\,d\tau,\\ \text{where } w(\tau)\in\varphi(\tau,u(\tau),\lambda) \text{ for every } \tau\in[0,a] \bigg\}.$$

Note that w is a measurable selection of $\varphi(t,u(t),\lambda))$ which exists by the Kuratowski–Ryll–Nardzewski Selection Theorem. Now, it follows from (70.8.3) that w is integrable.

It is easy to see that $S(\varphi_\lambda,0,x) = \operatorname{Fix}\Phi(\,\cdot\,,x,\lambda)$.

On the other hand, it is easy to verify that Φ satisfies all the assumptions of Theorem (69.9), and consequently our result follows from that theorem. □

Some other characterization results are possible to obtain by applying Theorem (21.15). Namely, we shall give a topological characterization of the set of solutions of some boundary value problems for differential inclusions of order k.

Let E be a separable Banach space and let $\varphi\colon [0,a]\times E^k \multimap E$ be a multivalued mapping, where $E^k = E\times\ldots\times E$ (k-times). We shall consider the following problem:

$$\text{(70.10)}\qquad \begin{cases} x^{(k)}(t)\in\varphi(t,x(t),x'(t),\ldots,x^{(k-1)}(t)),\\ \quad x(0) = x_0,\\ \quad x'(0) = x_1,\\ \ldots\ldots\ldots\ldots\ldots\ldots\\ x^{(k-1)}(0) = x_{k-1}, \end{cases}$$

where the solution $x\colon [0,a]\to E$ is understood in the sense of t almost everywhere (a.e. $t\in[0,a]$) and $x_0,\ldots,x_{k-1}\in E$.

Observe that for $k=1$ problem (70.10) reduces to the well-known Cauchy problem for differential inclusions. In what follows we shall denote by $S(\phi,x_0,\ldots,x_{k-1})$ the set of all solutions of (70.10).

Our first application of Theorem (21.15) is the following:

(70.11) THEOREM. *Assume that φ is a mapping with compact values. Assume further that the following conditions hold:*

(70.11.1) *φ is bounded, i.e. there is an $M > 0$ such that $\|y\| \le M$ for every $t \in [0,a]$, $x \in E^k$ and $y \in \varphi(t,x)$;*

(70.11.2) *the map $\varphi(\,\cdot\,,x)$ is measurable for each $x \in E^k$;*

(70.11.3) *φ is a Lipschitz map with respect to the second variable, i.e. there exists an $L > 0$ such that for every $t \in [0,a]$ and for every $z = (z_1,\dots,z_k)$, $y = (y_1,\dots,y_k) \in E^k$ we have:*

$$d_H(\varphi(t,z),\varphi(t,y)) \le L \sum_{i=1}^{k} \|z_i - y_i\|.$$

Then the set $S(\varphi,x_0,\dots,x_{k-1})$ of all solutions of the problem (70.10) *is an* AR-*space.*

PROOF. For the proof we define (singlevalued) mappings $h_j\colon M([0,a],E) \to AC^j$, $j = 0,\dots,k-1$, by putting

$$(h_j(z))(t) = x_0 + tx_1 + \ldots + \frac{t^j}{j!}x_j + \int_0^t \int_0^{s_1} \ldots \int_0^{s_j} z(s)\,ds\,ds_j \ldots ds_1,$$

where $AC^j = \{u \in C^j([0,a],E) \mid u^{(j)}$ is absolutely continuous$\}$ and for $u \in AC^j$ we put:

$$\|u\| = \|u\|_{C^j} + \sup_{t\in[0,a]} \operatorname{ess}\{\|u^{(j+1)}(t)\|\}.$$

Now, consider a multivalued mapping $\psi\colon M([0,a],E) \multimap M([0,a],E)$ defined as follows:

$$\psi(x) = \{z \in M([0,a],E) \mid z(t) \in \varphi(t,h_{k-1}(x)(t),\dots,h_0(x)(t)),\ \text{for } t \in [0,a]\}.$$

It follows from the Kuratowski–Ryll–Nardzewski Selection Theorem and (70.11.1) that ψ is well defined (with closed decomposable values in $M([0,a],E)$). Moreover, it is easy to see that $h_{k-1}(\mathrm{Fix}(\psi)) = S(\varphi,x_0,\dots,x_{k-1})$. Consequently, since h_{k-1} is a homeomorphism onto its image in view of Theorem (21.15), it is sufficient to show that ψ is a contractive mapping. We shall do this by using the $M([0,a],E)$-version of Bielecki's method and the Kuratowski–Ryll–Nardzewski Theorem. In fact it is enough to see that for every $u,z \in M([0,a],E)$ and for every $y \in \psi(u)$ there is a $v \in \psi(z)$ such that

(70.11.4) $$\|y - v\|_1 \le \alpha\|u - z\|_1,$$

where $\alpha \in [0,1)$ and $\|w\|_1 = \sup\operatorname{ess}_{t\in[0,a]}\{e^{-Lakt}\|w(t)\|\}$ is the Bielecki norm in $M([0,a],E)$. Observe that using Theorem (19.7) for ψ and z, we get a mapping

$v \in \psi(z)$ and now (70.2.1) follows directly from (70.11.3). The proof of Theorem (70.11) is complete. □

If we impose more conditions on φ, then we obtain even better information about the set $S(\varphi, x_0, \ldots, x_{k-1})$; namely we prove the following:

(70.12) THEOREM. *Let E be a separable Banach space and let $\varphi\colon [0,a] \times E^k \multimap E$ be a map with convex compact values satisfying* (70.11.2) *and the following additional conditions:*

(70.12.1) *the mapping $\varphi(t, \cdot\,)$ is completely continuous for every $t \in [0,a]$, i.e. u.s.c. and maps bounded sets into compact sets;*
(70.12.2) *the set $\varphi(A)$ is compact, for every compact $A \subset [0,a] \times E^k$;*
(70.12.3) *for every $t \in [0,a]$, $x \in E^k$ and $y \in \varphi(t,x)$, we have*

$$\|y\| \le u(t) + v(t)\|x\|,$$

where $u, v\colon [0,a] \to \mathbb{R}$ are integrable functions.

Then $S(\varphi, x_0, \ldots, x_{k-1})$ is an R_δ-set.

For the proof of Theorem (70.12) we need the following lemma.

(70.13) LEMMA. *Let φ satisfies all the assumptions of Theorem* (70.12). *Then there exists a multivalued mapping $\psi\colon [0,a] \times E^k \multimap E$ such that the following conditions are satisfied:*

(70.13.1) *ψ is compact, i.e. $\psi([0,a] \times E^k)$ is contained in a compact subset of E;*
(70.13.2) *$\psi(\,\cdot\,, x)$ is measurable for every $x \in E^k$, and $\psi(t, \cdot\,)$ is u.s.c. for every $t \in [0,a]$;*
(70.13.3) *$S(\psi, x_0, \ldots, x_{k-1}) = S(\varphi, x_0, \ldots, x_{k-1})$.*

Since the proof of Lemma (70.13) contains many technical details we shall only sketch it.

SKETCH OF PROOF OF LEMMA (70.13). First observe that, in view of (70.12.3), and by using the Gronwall inequality, there exists a constant $D > 0$ such that $\|x\| \le D$, for every $x \in S(\varphi, x_0, \ldots, x_{k-1})$. Then we define: $\gamma\colon [0,a] \times E^k \multimap E$ as follows:

$$\gamma(t,z) = \begin{cases} \varphi(t,z) & \text{for every } \ \|z\| \le D, \\ \varphi(t, Dz/\|z\|) & \text{for every } \ \|z\| > D. \end{cases}$$

From (70.12.3) we deduce that γ is bounded from above by an integrable function $w\colon [0,a] \to \mathbb{R}$, i.e. $\|y\| \le w(t)$ provided $y \in \varphi(t,x)$.

Next, for every $j = 0, \ldots, k-1$ we define a map

$$\gamma_j\colon C([0,a],E) \times \ldots \times C([0,a],E) \to C([0,a],E)$$

by putting:

$$\gamma_j(u_1,\dots,u_k) = \{y \in C([0,a],E) \mid y(t) = h_{k-1-j}(z)(t) \text{ and } z(t) \in \varphi(t,u_1(t),\dots,u_k(t)),\ \text{a.e. } t \in [0,a]\},$$

where each h_j is the singlevalued mappings defined in the proof of (70.11) and $z \in M([0,a],E)$. In the next step we deduce that all mappings γ_j are bounded by a common constant M. Let us denote by B the open ball in $C([0,a],E)$ with center at zero and radius M. Using separability of $C([0,a],E)$ we infer that for every $j = 0,\dots,k-1$, the set

$$A_j = \mathrm{cl}\,(\mathrm{conv}(\gamma_j(B\times\dots\times B)([0,a])))$$

is a compact and convex subset of E. This allows us to consider the following retractions:

$$r_j\colon E \to A_j, \quad j = 0,\dots,k-1.$$

Finally, we are ready to define the needed map ψ as follows:

$$\psi(t,z) = \gamma(t, r_0(z_1),\dots,r_{k-1}(z_k)),$$

for every $t \in [0,a]$ and $z = (z_1,\dots,z_k) \in E^k$. We leave it to the reader to verify that ψ satisfies (70.13.1)–(70.13.3). □

Proof of Theorem (70.12). For a given φ, we consider the map ψ given by Lemma (70.13). In view of (70.13.3) it is sufficient to prove that $S(\psi,x_0,\dots,x_{k-1})$ is an R_δ-set. We obtain it using the (modified) version of Theorem (70.11) and an approximation method as presented in Chapter III. It is well known that ψ can be approximated by a decreasing sequence $\psi_n\colon [0,a]\times E^k \multimap E$ of compact locally Lipschitz mappings satisfying the assumption of Theorem (70.11) and such that $S(\psi_n,x_0,\dots,x_{k-1})$ is a compact AR-space (from Theorem (70.11) and Corollary (2.14)), and

$$S(\psi,x_0,\dots,x_{k-1}) = \bigcap_{n\geq 1} S(\psi_n,x_0,\dots,x_{k-1}) \tag{70.14}$$

Consequently (70.14) implies that $S(\psi,x_0,\dots,x_{k-1})$ is an R_δ-set and the proof is completed. □

71. The lower semicontinuous case

In this section we shall consider the Cauchy problem:

$$\begin{cases} x'(t) \in \psi(t,x(t)), \\ x(0) = x_0, \end{cases} \tag{71.1}$$

for ψ to be l.s.c. or l-Carathéodory map.

Our first observation is evident:

(71.2) PROPOSITION. *If $\psi\colon [0,a]\times\mathbb{R}^n \multimap \mathbb{R}^n$ is l.s.c. with closed convex values and with the linear growth, then $S(\psi, x_0)\neq\emptyset$.*

PROOF. In view of the Michael selection theorem, there is a continuous map $f\colon [0,a]\times\mathbb{R}^n\to\mathbb{R}^n$ such that $f\subset\psi$. Since f has linear growth we obtain that $S(f,x_0)\subset S(\psi,x_0)$. But from the Peano theorem $S(f,x_0)\neq\emptyset$ and hence we get $S(\psi,x_0)\neq\emptyset$; the proof is completed. □

(71.3) REMARK. In fact we have proved (cf. (70.3)) that $S(\psi,x_0)$ is an R_δ-contractible set (not necessarily compact!).

Now, we prove the following:

(71.4) THEOREM. *Let $\psi\colon [0,a]\times\mathbb{R}^n\multimap\mathbb{R}^n$ be a bounded l.s.c. map with compact values. Then there exists an u.s.c. map $\varphi\colon [0,a]\times\mathbb{R}^n\multimap\mathbb{R}^n$ with compact convex values such that for every $x_0\in\mathbb{R}^n$ we have $S(\varphi,x_0)\subset S(\psi,x_0)$.*

For the proof of (71.4) we need some propositions. First, let us consider the cone Γ_M in $\mathbb{R}^{n+1}$ defined as follows:

$$\Gamma_M = \{(t,x)\in R\times\mathbb{R}^n \mid \|x\|\le t\cdot M\},$$

where M is an upper bound of ψ.

By applying Theorem (18.5) to ψ and Γ we get a Γ-continuous map $f\colon [0,a]\times\mathbb{R}^n\to\mathbb{R}^n$ such that $f\subset\Psi$. Evidently, we have:

$$S(f,x_0)\subset S(\Psi,x_0). \tag{71.5}$$

Now, we shall consider the multivalued regularization of f called also the Krassovskiĭ regularization of f. Namely, we define the multivalued map $\varphi(f)\colon [0,\varepsilon]\times\mathbb{R}^n\multimap\mathbb{R}^n$ by putting:

$$\varphi(f)(t,x) = \bigcap_{\varepsilon>0}\overline{\operatorname{conv}}\{f(s,y)\mid (s,y)\in[0,a]\times\mathbb{R}^n,\ |s-t|<\varepsilon,\ \|y-x\|<\varepsilon\}; \tag{71.6}$$

then $\varphi(f)$ is called the Krassovskiĭ regularization.

The following result is crucial in what follows:

(71.7) THEOREM. *Assume f is as above. Then we have:*

(71.7.1) *$\varphi(f)$ is u.s.c. bounded with compact convex values;*
(71.7.2) *$S(\varphi(f),x_0)=S(f,x_0)$.*

For the proof of (71.7) we need:

(71.8) Lemma. *Let $x(\,\cdot\,)$ be a solution of* (70.1). *Let J be the set of all times $t \in [0,a]$ such that:*

(71.8.1) $x'(t) \in \Psi(t, x(t))$,

(71.8.2) *there exists a decreasing to t sequence $\{t_k\} \subset [0,a]$, with $\{x'(t_k)\} \to x'(t)$, $x'(t_k) \in \Psi(t_k, x(t_k))$ for all k.*

Then the Lebesgue measure $\mu(J)$ of J is equal to a.

Proof. Let J_1 be the set of all times $t \in [0,a]$ where (i) holds. From the definition of solution of (70.1) we obtain that $\mu(J_1) = a$. Fix any $\varepsilon > 0$. Since x is an absolutely continuous map there exists a measurable u such that $u(t) = x'(t)$ for every $t \in J_2$, where $J_2 \subset K_1$ and $\mu(J_2) > a - \varepsilon$. Clearly, (71.8.2) holds for all points in J_2 which are density points of J_2 hence $\mu(J) \geq \mu(J_2) > a \ - \varepsilon$. Since ε was arbitrary, the Lemma (71.8) is proved. □

Proof of (71.7). The proof of (71.7.1) is self-evident because $\varphi(t)$ is an intersection of u.s.c. bounded and compact convex valued maps. So we are going to prove (71.7.2). Of course $f \subset \varphi(f)$ so $S(f, x_0) \subset S(\varphi(f), x_0)$. Therefore it is sufficient to show that $S(\varphi(f), x_0) \subset S(f; x_0)$. In what follows we will denote by φ the map $\varphi(f)$ for simplicity. Let us assume that for every $y \in \varphi(t,x)$ and $(t,x) \in [0,a] \times \mathbb{R}^n$ we have

$$\|y\| \leq L < M,$$

where M is chosen according to Ψ. Let $x(\,\cdot\,)$ be a solution of (70.1) for $\varphi = \varphi(f)$. Define $J \subset [a,b]$ to be the set of times t such that

(i) $x'(t) \in F(t, x(t))$, and

(ii) there exists a sequence of times t_k, strictly decreasing to t, such that $x'(t_k) \in \varphi(t_k, x(t_k))$ and $x'(t)$.

By Lemma (71.8), J has full measure in $[a,b]$. We claim that $x'(t) = f(t, x(t))$ for every $t \in J$. Assume, to the contrary, that $t \in J$ but

$$\|x'(t) - f(t, x(t))\| = \varepsilon > 0. \tag{71.8.1}$$

Using the Γ_M-continuity of f at the point $(t, x(t))$, choose $\delta > 0$ such that

$$\|f(s,y) - f(t, x(t))\| < \varepsilon/2, \tag{71.8.2}$$

whenever $t \leq s < t + \delta$, $\|y - x(t)\| \leq M(s-t)$. Let $t_k \to t$ be a sequence with the properties stated in (ii). Then there exists k large enough so that

$$0 < t_k - t < \delta, \tag{71.8.3}$$

$$\|x'(t_k) - x'(t)\| < \varepsilon/2, \tag{71.8.4}$$

for all $k > k$. The boundedness assumption $\|f(t,x)\| < L$ implies that $F(t,x) \subseteq \overline{B}(0,L)$ for all t,x. Our solution $x(\,\cdot\,)$ is therefore Lipschitz continuous with constant L. In particular,

$$\|x(t_k) - x(t)\| \le L(t_k - t) < M(t_k - t).$$

Using (71.8.2) and (71.8.3) we conclude that

$$F(t_k, x(t_k)) \subseteq \overline{B}\bigg(f(t,x(t)), \frac{\varepsilon}{2}\bigg), \tag{71.8.5}$$

for all $k > k$, hence

$$\|x'(t_k) - f(t,x(t))\| \le \frac{\varepsilon}{2}. \tag{71.8.6}$$

Comparing (71.8.1) with (71.8.4) and (71.8.6) we obtain a contradiction, proving (71.7.2) □

Proof of (71.4). Let $M > 0$ be an upper bound of Ψ. We choose Γ_M-continuous selection f of Ψ and consider its Krassovskiĭ regularization $\varphi(f)$. In view of (71.5) and (71.7) we have

$$S(\Psi; x_0) \supset S(f, x_0) = S(\varphi(f), x_0) \neq \emptyset$$

so our claim holds; the proof is completed. □

The method of proof of (71.4) suggests the following notion.

(71.9) Definition. A bounded multivalued map $\Psi\colon [0,a] \times \mathbb{R}^n \multimap \mathbb{R}^n$ with nonempty values is said to be *regular* if there is a map $\varphi\colon [0,a] \times \mathbb{R}^n \multimap \mathbb{R}^n$, called regular quasi selection of Ψ, satisfying the properties:

(71.9.1) φ is u-Carathéodory bounded with compact convex values,
(71.9.2) $\varphi(t,x) \cap \Psi(t,x) \neq \emptyset$, for every $(t,x) \in [0,a] \times \mathbb{R}^n$,
(71.9.3) each solution $x\colon [0,a] \to \mathbb{R}^n$ of the differential inclusion $x'(t) \in \varphi(t,x(t))$ is also a solution of $x'(t) \in \Psi(t,x(t))$.

The following result shows the significance of the class of regular maps.

(71.10) Proposition. *Let $\Psi\colon [0,a] \times \mathbb{R}^n \multimap \mathbb{R}^n$ be a bounded multivalued map with compact values. Then Ψ is regular if:*

(71.10.1) *Ψ is u-Carathéodory,*
(71.10.2) *Ψ is l-Scorza Dragoni,*
(71.10.3) *Ψ is l-Carathéodory.*

For the proof of (71.10) a generalized version of the directionally continuous selection is needed (cf. Section 18). Note that the mentioned proof of this generalized version is strictly analogous to the one presented in Section 18 and therefore it is omitted here.

For any regular map $\Psi\colon [0,a]\times\mathbb{R}^n \multimap \mathbb{R}^n$, we set:

(71.11) $$U(\Psi) = \{\varphi\colon [0,a]\times\mathbb{R}^n \multimap \mathbb{R}^n \mid \varphi \text{ is a regular quasi-selection of } \Psi\}.$$

We state below some immediate properties of regular maps, we shall use later.

(71.12) Proposition. *Let $\Psi\colon [0,a]\times\mathbb{R}^n \multimap \mathbb{R}^n$ be a regular map. Then, for each $\varphi\in U(\Psi)$, the solution set map $P_\varphi\colon \mathbb{R}^n \multimap C([0,a],\mathbb{R}^n)$, $P_\varphi(x) = S(\varphi,x)$ is u.s.c. with R_δ-values and*

$$P_\varphi(x)\subset P_\Psi(x),$$

where $P_\Psi\colon\mathbb{R}^n\multimap C([0,a],\mathbb{R}^n)$, $P_\Psi(x) = S(\Psi,x)$.

Proposition (71.12) easily follows from (70.7) and (70.9).

(71.13) Proposition. *Let $\Psi\colon [0,a]\times\mathbb{R}^n \multimap \mathbb{R}^n$ be regular, and let $\varphi_1,\varphi_2\in U(\Psi)$. Then, for every $t_0\in[0,a]$, the map $\chi_{t_0}; [0,a]\times\mathbb{R}^n\multimap\mathbb{R}^n$ given by:*

$$\chi_{t_0}(t,x) = \begin{cases} \varphi_1(t,x) & \text{if } 0\le t< t_0,\\ \overline{\operatorname{conv}}\{\varphi_1(t_0,x)\cup\varphi_2(t_0,x)\} & \text{if } t=t_0,\\ \varphi_2(t,x) & \text{if } t_0<t\le a \end{cases}$$

is also in $U(\Psi)$.

(71.14) Proposition. *Let $\Psi\colon [0,a]\times\mathbb{R}^n \multimap \mathbb{R}^n$ be regular, and let $\varphi_1,\varphi_2\in U(\Psi)$. Then the map $\chi\colon [0,a]\times\mathbb{R}^n\times[0,1]\multimap\mathbb{R}^n$ given by:*

$$\chi(t,x,\lambda) = \begin{cases} \varphi_1(t,x) & \text{if } 0\le t<\lambda a,\\ \overline{\operatorname{conv}}\{\varphi_1(\lambda\, a,x)\cup\varphi_2(\lambda a,x)\} & \text{if } t=\lambda a,\\ \varphi_2(t,x) & \text{if } \lambda a<t\le a \end{cases}$$

satisfies the properties:

(71.14.1) $\chi(\,\cdot\,,\,\cdot\,,\lambda)\in U(\Psi)$ *for every* $\lambda\in[0,1]$,

(71.14.2) $\chi(t,x,0) = \varphi_2(t,x)$ *for every* $(t,x)\in(0,a]\times\mathbb{R}^n$,

(71.14.3) $\chi(t,x,1) = \varphi_1(t,x)$ *for every* $(t,x)\in[0,a)\times\mathbb{R}^n$.

Finally note the following:

(71.15) Remark. All the existence results presented in this section can be also obtained by using the Fryszkowski selection theorem and the Schauder fixed point theorem for singlevalued mappings.

72. Periodic solutions for differential inclusions in $\mathbb{R}^n$

In this section we consider the problem of existence of a solution $x(\,\cdot\,)$ to the following periodic problem:

$$\begin{cases} x'(t) \in \varphi(t, x(t)), \\ x(0) = x(a), \end{cases} \tag{72.1}$$

where $\varphi\colon [0,a] \times \mathbb{R}^n \multimap \mathbb{R}^n$ is a multivalued map. This problem plays a central role in a qualitive theory of differential equations. Among the topological methods an important role is played by the topological degree method applied to the Poincaré (also called Poincaré–Andronov) translation operator. This method was developed by M. A. Krasnosiel'skiĭ (cf. [KZ-M]) in the singlevalued case, i.e. when we have a unique solution for the Cauchy problem considered. In the case of nonuniqueness (or, in particular, for differential inclusions) we need the multivalued Poincaré operator (see [DyG], [Go2-M], [GP1]). Then we are able to find periodic solutions using the topological degree theory for multivalued maps. Similarly as in [KZ-M] we use the guiding function method adopted to differential inclusions to obtain a sufficient condition.

In this section we shall assume that $\varphi\colon [0,a] \times \mathbb{R}^n \multimap \mathbb{R}^n$ is u-Carathéodory map and has nonempty compact convex values and linear growth. By Proposition (70.9), the map $P\colon \mathbb{R}^n \multimap C([0,a],\mathbb{R}^n)$ defined by $P(x) = S(\varphi, 0, x)$ is u.s.c. with R_δ-values. Consider also the evaluation maps:

$$\begin{aligned} e_t\colon C([0,a],\mathbb{R}^n) &\to \mathbb{R}^n, & e_t(x) &= x(t) - x(0), \\ e\colon C([0,a],\mathbb{R}^n) \times [0,a] &\to \mathbb{R}^n, & e(x,t) &= x(t) - x(0). \end{aligned}$$

So, we have the diagram:

$$\mathbb{R}^n \xrightarrow{P}\!\!\circ\; C([0,a],\mathbb{R}^n) \xrightarrow{e_a} \mathbb{R}^n;$$

the composition $P_a = e_a \circ P$ is called the *Poincaré translation operator.*

Now, it is evident that problem (72.1) is equivalent to the problem of existence of a point $x \in \mathbb{R}^n$ such that $0 \in P_a(x)$. Therefore, in terms of the topological degree theory we can state the following theorem:

(72.2) THEOREM. *Assume that the topological degree* $\deg(P_a, K_r^n)$ *with respect to* K_r^n *is defined; i.e.* $x \notin P_a(x)$ *for every* x *such that* $\|x\| = r$*. If* $\deg(P_a, K_r^n) \neq 0$, *then problem* (72.1) *has a solution.*

In fact, our theorem follows immediately from the existence property of the topological degree.

In view of Theorem (72.2) a sufficient condition for $\deg(P_a, K_r^n)$ to be different from zero is needed. We shall obtain such a condition by using the guiding function method (also called potential function method). We will start by defining the notion of a potential (guiding) mapping.

(72.3) DEFINITION. A C^1-function $V\colon \mathbb{R}^n \to \mathbb{R}$ is said to be a *direct potential* if it satisfies the following condition:

(72.3.1) there exists an $r_0 > 0$ such that $\operatorname{grad} V(x) \neq 0$ for any $x \in \mathbb{R}^n$ with $\|x\| \geq r_0$, where $\operatorname{grad} V = (\partial V/\partial x_1, \ldots, \partial V/\partial x_n)$ denotes the gradient of the function V.

It follows from the additivity property of the Brouwer degree and (72.3.1) that for any $r \geq r_0$ we have

$$\deg(\operatorname{grad} V, K_r^n) = \deg(\operatorname{grad} V, K_{r_0}^n).$$

The above formula enables us to define the index, $\operatorname{Ind} V$, of the direct potential V by letting

$$\operatorname{Ind} V = \deg(\operatorname{grad} V, K_r^n), \quad \text{where } r \geq r_0. \tag{72.3.2}$$

It is important to have an example of a direct potential whose index is different from zero. First of all it is well known that if a direct potential satisfies the coercitivity condition:

$$\lim_{\|x\| \to \infty} V(x) = \infty, \tag{72.3.3}$$

then $\operatorname{Ind} V \neq 0$.

By an easy homotopy argument we obtain:

(72.4) PROPOSITION. *If $U, V\colon \mathbb{R}^n \to \mathbb{R}$ are direct potentials for which there exists an $r_0 > 0$ such that*

$$\langle \operatorname{grad} V(x), \operatorname{grad} U(x) \rangle > -\|\operatorname{grad} V(x)\|\, \|\operatorname{grad} U(x)\|,$$

for any $x \in \mathbb{R}^n$ with $\|x\| \geq r_0$, then $\operatorname{Ind} V = \operatorname{Ind} U$.

(72.5) EXAMPLE. Let $V, U\colon \mathbb{R}^2 \to \mathbb{R}$ be two direct potentials defined as follows:

$$U(x, y) = 1 - \exp(-x^2) + y^2, \quad V(x, y) = x^2 + y^2.$$

Then U does not satisfy the coercitivity condition but from Proposition (72.4) we obtain that $\operatorname{Ind} U = \operatorname{Ind} V \neq 0$. In fact, $\operatorname{Ind} U = \operatorname{Ind} V = 1$.

A relationship between the notion of potential and differential inclusions is stated in the following definition:

(72.6) Definition. Let $\varphi\colon [0,a] \times \mathbb{R}^n \multimap \mathbb{R}^n$ be a set-valued map and let $V\colon \mathbb{R}^n \to \mathbb{R}$ be a direct potential; V is called a *guiding function* for φ if the following condition is satisfied:

(72.6.1) exists $r_0 > 0$ for all $\|x\| \geq r_0$ for all $t \in [0,a]$

there exists $y \in \varphi(t,x)$ such that $\langle y, \operatorname{grad} V(x)\rangle \geq 0$;

where $\langle \cdot\,,\,\cdot \rangle$ denotes the inner product in $\mathbb{R}^n$.

Now we are able to formulate the main result of this section.

(72.7) Theorem. *If $\varphi\colon [0,a] \times \mathbb{R}^n \multimap \mathbb{R}^n$ is a u-Carathéodory map with convex compact values and linear growth, and if there exists a guiding function $V\colon \mathbb{R}^n \to \mathbb{R}$ for the map φ such that $\operatorname{Ind} V \neq 0$, then the periodic problem*

$$\begin{cases} x'(t) \in \varphi(t, x(t)), \\ x(0) = x(a) \end{cases}$$

has a solution.

For the proof of Theorem (72.7) we need some additional notation and two lemmas.

For a given direct potential V we define the induced vector field $W_V\colon \mathbb{R}^n \to \mathbb{R}^n$ by the following formula:

$$W_V(x) = \begin{cases} \operatorname{grad} V(x) & \text{if } \|\operatorname{grad} V(x)\| \leq 1, \\ \dfrac{\operatorname{grad} V(x)}{\|\operatorname{grad} V(x)\|} & \text{if } \|\operatorname{grad} V(x)\| > 1. \end{cases}$$

(72.8) Lemma. *Let $r_0 > 0$ be a constant chosen for the direct potential V (cf. (72.3.2)). Then for every $r > r_0 + a$ there exists a $t_r \in [0,a]$ such that for any solution $x\colon [0,a] \to \mathbb{R}^n$ of the differential equation*

(72.8.1) $$x'(t) = W_V(x(t))$$

which satisfies $\|x(0)\| = r$ the following conditions hold:

(72.8.2) $$\langle x(t) - x(0), \operatorname{grad} V(x(0))\rangle > 0 \quad \textit{for } t \in (0, t_r];$$

(72.8.3) $$x(t) - x(0) \neq 0 \quad \textit{for } t \in (0, a].$$

Proof. Since the field W_V is continuous, there exists an $\varepsilon_r > 0$ such that product $\langle W_V(z_0), W_V(z)\rangle > 0$ for every $z, z_0 \in \mathbb{R}^n$, $\|z_0\| = r$, $\|z - z_0\| < \varepsilon_r$. Moreover, since W_V is bounded there exists $t_r \in (0,a)$ such that $\|x(s) - x(0)\| < \varepsilon_r$

for every solution x of (72.8.1) and every $s \in (0, t_r]$. Now let x be a solution of (72.8.1) such that $\|x(0)\| = r$. Then we have

$$\langle x(s) - x(0), \operatorname{grad} V(x(0))\rangle = \int_0^s \langle W_V(x(\tau)), \operatorname{grad} V(x(0))\rangle \, d\tau > 0$$

for every $s \in (0, t_r]$, which completes the proof of (1). Further, we obtain

$$\begin{aligned} V(x(t)) - V(x(0)) &= \int_0^t \langle \operatorname{grad} V(x(\tau)), x'(\tau)\rangle \, d\tau \\ &= \int_0^t \langle \operatorname{grad} V(x(\tau)), W_V(x(\tau))\rangle \, d\tau > 0 \end{aligned}$$

and this completes the proof of the lemma. □

(72.9) Lemma. *Suppose $r_0 > 0$. If a solution $x(\cdot)$ of the differential inclusion $x'(t) \in \varphi(t, x(t))$ satisfies the condition*

$$\|x(0)\| > \left(x_0 + \int_0^a \mu(\tau)\, d\tau\right) \exp\left(\int_0^q \mu(\tau)\, d\tau\right) \ (= r_\mu),$$

then $\|x(t)\| > r_0$ for every $t \in [0, a]$.

Proof. Suppose that there exists a solution x and $t_0 \in [0, a]$ such that $\|x(0)\| > r_\mu$ and $\|x_0(t_0)\| \le r_0$. For every $t \in [0, t_0]$ we let: $y(t) = x(t_0 - t)$, $\xi(t) = \mu(t_0 - t)$, $\psi(t, x) = -\varphi(t_0 - t, x)$. Obviously $y'(t) \in \psi(t, y(t))$. As for every $z \in \psi(t, x)$

$$\|z\| \le \xi(t)(1 + \|x\|),$$

then using the Gronwall inequality we obtain

$$\|y(t)\| \le \left(\|y(0)\| + \int_0^t \xi(\tau)\, d\tau\right) \exp\left(\int_0^t \xi(\tau)\, d\tau\right) \le r_\mu,$$

for every $t \in [0, t_0]$. Thus $\|x(0)\| = \|y(t_0)\| \le r_\mu$, and we obtain a contradiction. □

Now we are able to prove Theorem (72.7).

Proof of Theorem (72.7). Choose $r_0 > 0$ according to (72.3.1) and (72.6.1), and define a map $B\colon \mathbb{R}^n \to \mathbb{R}^n$ by putting

$$B(x) = \{y \in \mathbb{R}^n \colon \langle y, \alpha(x) \operatorname{grad} V(x)\rangle \ge 0\},$$

where $\alpha(x) = 0$ for $\|x\| \le r_0$ and $\alpha(x) = 1$ for $\|x\| > r_0$.

It is easy to check that the graph Γ_B of the map B is closed. Next we define a map $\varphi_V\colon [0, a] \times \mathbb{R}^n \multimap \mathbb{R}^n$ by

$$\varphi_V(t, x) = \varphi(t, x) \cap B(x).$$

It is easy to see that φ_V is a u-Carathéodory map. Now, by using Lemma (72.9), we choose $r \geq r_0 + a$ such that for every $\kappa \in [0,1]$, $\|x_0\| \geq r$, and for every solution $x\colon [0,a] \to \mathbb{R}^n$ of the Cauchy problem

$$\begin{cases} x'(t) \in \kappa W_V(x(t)) + (1-\kappa)\varphi_V(t,x(t)), \\ x(0) = x_0, \end{cases}$$

the following estimate holds:

$$\|x(t)\| \geq r_0 \quad \text{for every } t \in [0,a].$$

We choose t_r according to Lemma (72.8) and define a decomposable homotopy $H_1\colon K_r^n \times [0,1] \multimap \mathbb{R}^n$ by putting

$$H_1(x_0,\lambda) = (1-\lambda)\operatorname{grad} V(x_0) + \lambda e_{t_r}(S_{W_V}(x_0))$$

Now, in view of the second version of the homotopy property (see (26.5)) and Lemma (72.8), we obtain

$$\operatorname{Deg}(e_{t_r} \circ S_{W_V}, K_r^n) = \{\operatorname{Ind} V\}. \tag{72.10}$$

We define:

$$k(\lambda) = \begin{cases} 1 & \text{for } \lambda \in [0,1/2), \\ 2-2\lambda & \text{for } \lambda \in [1/2,1], \end{cases}$$

$$h(\lambda) = \begin{cases} 2(a-t_r)\lambda + t_r & \text{for } \lambda \in [0,1/2), \\ a & \text{for } \lambda \in [1/2,1]. \end{cases}$$

The map $G\colon [0,a] \times \mathbb{R}^n \times [0,1] \multimap \mathbb{R}^n$ given by:

$$G(t,x,\lambda) = k(\lambda)W_V(x) + (1-k(\lambda))\varphi_V(t,x),$$

satisfies all the assumptions of Proposition (8.9). So, the map $\chi\colon K_r^n \times [0,1] \multimap C([0,a],\mathbb{R}^n) \times [0,a]$ given by:

$$\chi(x_0,\lambda) = S_{G(\,\cdot\,,\,\cdot\,,\lambda)}(x_0) \times \{h(\lambda)\},$$

is upper semicontinuous, and $\chi(x_0,\lambda)$ is an R_δ-set for every $(x_0,\lambda) \in \mathbb{R}^n \times [0,1]$. Hence, the homotopy $H\colon K_r^n \times [0,1] \to \mathbb{R}^n$ given by $H = e \circ \chi$ is decomposable.

Now, we show that $0 \notin H(x_0,\lambda)$, for $\|x_0\| = r$ and $\lambda \in [0,1]$.

If $\lambda \in [0,1/2]$, then there exists a solution $x\colon [0,a] \to \mathbb{R}^n$ of the Cauchy problem

$$\begin{cases} x'(t) = W_V(x(t)), \\ x(0) = x_0, \end{cases}$$

such that $z = x(h(\lambda)) - x_0$, and by Lemma (72.8.2) we have $z \neq 0$.

If $\lambda \in [1/2, 1)$ and $z \in H(x_0, \lambda)$, then there exists a solution $x\colon [0, a] \to \mathbb{R}^n$ of the Cauchy problem

$$\begin{cases} x'(t) \in G(t, x(t), \lambda), \\ x(0) = x_0, \end{cases}$$

such that $z = x(a) - x_0$. Consequently we have

$$\langle k(\lambda)W_V(x(t)) + (1 - k(\lambda))y, \operatorname{grad} V(x(t))\rangle > 0$$

for every $y \in \varphi_V(t, x(t))$. This implies that $\langle x'(t), \operatorname{grad} V(x(t))\rangle > 0$ for almost all $t \in [0, a]$. Therefore, we obtain

$$V(x(a)) - V(x(0)) = \int_0^a \langle x'(t), \operatorname{grad} V(x(t))\rangle \, dt > 0,$$

and hence $0 \notin H(x, \lambda)$, for every $\|x\| = r$ and $\lambda \in [0, 1)$. If there is an $x \in S_r^{n-1}$ such that $0 \in H(x, 1)$, then the conclusion of the theorem holds true. If not, then H is a homotopy in $D_{S_r^{n-1}}(K_r^n, \mathbb{R}^n)$.

Finally, from (26.2.5) and (72.10) we deduce $\operatorname{Ind} V \in \deg(H(\,\cdot\,, 1), K_r^n)$. Since V is a guiding function for φ, we infer in view of (26.2.2) that $0 \in H(x, 1)$ for some $x \in K_r^n$ and this completes the proof. □

(72.11) Remark. If we change condition (72.6.1) in the definition of a guiding function to the following one:

$$\begin{aligned} &\text{exists } r_0 > 0, \text{ for all } \|x\| \geq r_0, \text{ for all } t \in [0, a], \\ &\qquad\qquad \text{there exists } y \in \varphi(t, x) \text{ such that } \langle y, \operatorname{grad} V(x)\rangle \leq 0, \end{aligned}$$

then Theorem (72.7) holds true.

Indeed, by the substitution $t \to (a - t)$, we reduce this case to the previous one.

Now, let $\psi\colon [0, a] \times \mathbb{R}^n \multimap \mathbb{R}^n$ be a regular map. We shall consider the periodic problem (72.1) for ψ. It follows from (71.12) that the Poincaré operator P_a^ψ for ψ has a selection P_a^φ for every $\varphi \in U(\psi)$ and, in addition, $P_a^\varphi \in CJ(B^n(r), \mathbb{R}^n)$.

(72.12) Definition. A C^1-function $V\colon \mathbb{R}^n \to R$ is said to be a *strong guiding function* for ψ if it is a direct potential and the following condition is satisfied:

$$\begin{aligned} &\text{there exists } r_0 > 0 \text{ for all } \|x\| \geq r_0, \text{ for all } t \in [0, a], \\ &\qquad\qquad \text{for all } y \in \psi(t, x) \text{ such that } \langle y, \operatorname{grad} V(x)\rangle \geq 0. \end{aligned}$$

We prove:

(72.13) THEOREM. *Let* $\psi\colon [0,a]\times\mathbb{R}^n \multimap \mathbb{R}^n$ *be a regular map having a strong guiding function* V *with* $\operatorname{Ind}(V)\neq 0$. *Then the periodic problem* (72.1) *for* ψ *has a solution.*

PROOF. Let $\varphi\in U(\psi)$. From the definition of $U(\psi)$, we have:

(72.13.1) φ is u-Carathéodory, with nonempty compact convex values,
(73.13.2) $\varphi(t,x)\cap\psi(t,x)\neq\emptyset$ for every $(t,x)\in[0,a]\times\mathbb{R}^n$,
(73.13.3) $P_a^{\varphi}(x_0)\subset P_a^{\psi}(x_0)$, for every $x_0\in\mathbb{R}^n$.

In view of (73.13.3), it suffices to show that (70.1) for φ has a solution. Observe that, in general, V is not a guiding function for φ, because φ is not necessarily a selection of ψ. To overcome this difficulty we introduce an auxiliary map.

In fact, Fix$r_0>0$ as in (72.8.1). Define $\widetilde{\varphi}\colon[0,a]\times\mathbb{R}^n\multimap\mathbb{R}^n$ by

$$\widetilde{\varphi}(t,x)=\begin{cases}\varphi(t,x) & \text{if } (t,x)\in[0,a]\times B^n(r_0),\\ \varphi(t,x)\cap H(x) & \text{if } (t,x)\in[0,a]\times(\mathbb{R}^n\setminus B^n(r_0)),\end{cases}$$

where $H(x)=\{y\in\mathbb{R}^n\mid\langle y,\operatorname{grad}V(x)\rangle\geq 0\}$. It is simple to check that $\widetilde{\varphi}$ is u-Carathéodory, with nonempty compact convex values. Moreover, let $(t,x)\in[0,a]\times\mathbb{R}^n$, $\|x\|>r_0$ be arbitrary. Take $y\in\varphi(t,x)\cap\psi(t,x)$ (a nonempty set by (73.13.2)!). Since V is a C^1 strong guiding function for ψ, we have that V is a guiding function for $\widetilde{\varphi}$ and hence from (72.7) the periodic problem for $\widetilde{\varphi}$ has a solution. Clearly, it gives a periodic solution for φ and consequently for ψ; the proof is completed. □

73. Differential inclusions on proximate retracts

In the present section we survey the current results concerning the existence problem, topological characterization of the set of solutions, and periodic solutions of differential inclusions on subsets of Euclidean spaces. Specifically, we shall deal with these problems on the compact subsets of Euclidean spaces called proximate retracts (see Chapter II).

Let us remark that, in particular, convex sets and smooth manifolds with boundary or without boundary are proximate retracts.

In what follows we shall assume that $A\subset\mathbb{R}^n$ is a compact proximate retract and $\varphi\colon[0,a]\times A\multimap\mathbb{R}^n$ is an integrably bounded Carathéodory map with compact convex values and the following Nagumo-type condition:

(73.1) $$\varphi(t,x)\cap T_A(x)\neq\emptyset\quad\text{for all } t\in[0,a] \text{ and all } x\in A$$

where $T_A(x)$ is the Bouligand cone to A at x as defined in Chapter I.

First we shall study the Cauchy problem for φ, i.e.

$$\begin{cases} x'(t) \in \varphi(t, x(t)) & \text{a.e. for } t \in [0, a], \\ x(0) = x_0 & \text{for } x_0 \in A. \end{cases} \tag{73.2}$$

Let $B \subset A$. By a *B-viable solution* of (73.2) we understand an absolutely continuous map $x: [0, a] \to B$ such that $x'(t) \in \varphi(t, x(t))$, a.e. $t \in [0, a]$ and $x(0) = x_0$. By $S_B(\varphi, 0, x_0)$ we denote the set of all B-viable solutions of (73.2).

We are going to prove that under the above assumptions $S_A(\varphi, 0, x_0)$ in an R_δ-set.

The following example shows us that if we remove that assumption the A is a proximate retract, then the set $S_A(\varphi, 0, x_0)$ may even be disconnected.

(73.3) Example. Let

$$\begin{aligned} S_1 &= \{(x, y) \in \mathbb{R}^2 \mid (x-1)^2 + y^2 = 1\}, \\ S_2 &= \{(x, y) \in \mathbb{R}^2 \mid (x+1)^2 + y^2 = 1\}, \\ A &= S_1 \cup S_2, \end{aligned}$$

and define $f: [0, 1] \times A \to \mathbb{R}^2$ by

$$f(t, (x, y)) = \begin{cases} (y, 1-x) & \text{for } (x, y) \in S_1, \\ (-y, 1+x) & \text{for } (x, y) \in S_2. \end{cases}$$

It is easy to see that the set $S(f, 0, (0, 0))$ is disconnected and hence is not R_δ.

Note that if $A \subset \mathbb{R}^n$ is an arbitrary compact set, then $S_A(\varphi, 0, x_0) \neq \emptyset$. We shall prove the following:

(73.4) Theorem. *If $A \in$ PANR (see Definition (3.8)) and $\varphi: [0, a] \times A \multimap \mathbb{R}^n$ is an integrably bounded Carathéodory map with compact convex values and satisfies (73.1), then for every $x_0 \in A$ the set $S_A(\varphi, 0, x_0)$ is R_δ.*

To prove Theorem (73.4) we need some additional information. Since $A \in$ PANR there exists an open neighbourhood U of A in $\mathbb{R}^n$ and a metric retraction $r: U \to A$ (cf. Proposition (3.10)). We define $\widetilde{\varphi}: [0, a] \times \mathbb{R}^n \multimap \mathbb{R}^n$ to be an extension of φ as follows:

$$\widetilde{\varphi}(t, x) = \begin{cases} \alpha(x) \cdot \varphi(t, r(x)) & \text{for } x \in U \text{ and } t \in [0, a], \\ \{0\} & \text{for } x \notin U \text{ and } t \in [0, a], \end{cases}$$

where $\alpha: \mathbb{R}^n \to [0, 1]$ is a Urysohn function such that $\alpha(x) = 1$ for $x \in A$, and $\alpha(x) = 0$ for $x \notin U$.

(73.5) LEMMA. *Assume now that φ satisfies*

$$\varphi(t,x) \subset T_A(x) \quad \textit{for all } t \in [0,a] \textit{ and for all } x \in A. \tag{73.1$'$}$$

If $x\colon [0,a] \to \mathbb{R}^n$ is an absolutely continuous function such that $x(0) \in A$ and $x'(t) \in \widetilde{\varphi}(t,x(t))$ a.e. $t \in [0,a]$, then $x(t) \in A$ for every $t \in [0,a]$.

PROOF. For the proof we define a map $d\colon [0,a] \to \mathbb{R}$ by letting

$$d(t) = \operatorname{dist}(x(t),A), \qquad \text{for } \ t \in [0,a].$$

We would like to prove that $d(t) = 0$ for every $t \in [0,a]$. We have:

$$|d(t) - d(s)| = |\operatorname{dist}(x(t),A) - \operatorname{dist}(x(s),A)| \leq \|x(t) - x(s)\|,$$

so d is an absolutely continuous function. Let $t_0 \in [0,a]$ and $x'(t_0) \in \widetilde{\varphi}(t_0, x_0(t_0))$. If $x_0(t_0) \in U$, then $x'(t_0) \in T_A(r(x(t_0)))$ and hence

$$\liminf_{h\to 0^+} \frac{\operatorname{dist}(r(x(t_0)) + hx'(t_0), A)}{h} = 0. \tag{73.5.1}$$

We have:

$$\begin{aligned}\operatorname{dist}(x(t_0+h),A) - \operatorname{dist}(x(t_0),A) \leq{}& \operatorname{dist}(r(x(t_0)) + hx'(t_0), A) \\ &+ \|x(t_0+h) - x(t_0) - hx'(t_0)\|\end{aligned}$$

and from (73.5.1) we obtain

$$\liminf_{h\to 0^+} \frac{d(t_0+h) - d(t_0)}{h} \leq 0. \tag{73.5.2}$$

If $x(t_0) \notin U$, then $x'(t_0) = 0$ and

$$d(t_0+h) - d(t_0) \leq \|x(t_0+h) - x(t_0) - hx'(t_0)\|.$$

Therefore, in this case (73.5.2) holds, too. Since, d is differentiable almost everywhere and its derivative $d'(t) \leq 0$, a.e. $t \in [0,a]$ so it is non-increasing, but $d(0) = 0$ and hence $d(t) = 0$ for every $t \in [0,a]$. □

From the above lemma it follows that:

$$S_A(\varphi, 0, x_0) = S_{\mathbb{R}^n}(\widetilde{\varphi}, 0, x_0) \quad \text{provided } x_0 \in A$$

and φ satisfies (73.1′).

Thus, from this and Theorem (70.4) we obtain:

(73.6) Corollary. *If φ satisfies (73.1′) and $x_0 \in A$, then $S_A(\varphi, 0, x_0)$ is an R_δ-set.*

Now we are able to prove Theorem (73.4).

Proof of Theorem (73.4). Let $r\colon U \to A$ be the metric retraction. According to (3.10) we choose $\varepsilon > 0$ such that $O_{2\varepsilon}(A) \subset U$ and $\mathrm{cl}\,(O_\varepsilon(A)) \in \mathrm{PANR}$.

Define a map $T\colon \mathrm{cl}\,(O_\varepsilon(A)) \to \mathbb{R}^n$ by

$$T(x) = \{y \in \mathbb{R}^n \colon \langle y, x - r(x)\rangle \le 0\}.$$

It is easy to see that the graph Γ_T of T is a closed subset in $\mathrm{cl}\,(O_\varepsilon(A)) \times \mathbb{R}^n$ and moreover, in view of Proposition (3.10), the multivalued mapping

$$\psi_\varepsilon\colon [0,a] \times \mathrm{cl}\,(O_\varepsilon(A)) \multimap \mathbb{R}^n$$

defined by

$$\psi_\varepsilon(t,x) = \varphi(t, r(t)) \cap T(x)$$

is Carathéodory integrably bounded and satisfies (73.1) (cf. (3.1) and (3.11)). Therefore from Corollary (73.6) we obtain that $S_{\mathrm{cl}\,(O_\varepsilon(A))}(\psi_\varepsilon, 0, x_0)$ is R_δ provided $x_0 \in A$.

Finally, let us observe that for $x_0 \in A$ we have

$$S_A(\varphi, 0, x_0) = \bigcap_{n=1}^{\infty} S_{\mathrm{cl}\,(O_{1/n}(A))}(\psi_{1/n}, 0, x_0).$$

So, our theorem follows from Corollary (73.6) and (2.14). □

Now, keeping the above assumptions on A and φ we shall consider the following periodic problem:

$$\text{(73.7)} \qquad \begin{cases} x'(t) \in \varphi(t, x(t)) & \text{a.e. } t \in [0,a], \\ x(0) = x(a) \in A. \end{cases}$$

To solve problem (73.7) we consider the following diagram:

$$A \overset{P}{\multimap} C([0,a], \mathbb{R}^n) \xrightarrow{w_s} \mathbb{R}^n,$$

where $P(x) = S_A(\varphi, 0, x)$ and $w_s(x) = x(s)$ for $x \in C([0,a],\mathbb{R}^n)$ and $s \in [0,a]$.

Observe that for every $u \in S_A(\varphi, 0, x)$ and for every $t \in [0,a]$ we have $u(t) \in A$, so the composition $P_a = w_a \circ P$ is a map from A to A.

Consider the multivalued homotopy $\chi\colon A \times [0,1] \multimap A$ defined by

$$\chi(x,\lambda) = w_{\lambda a}(S_A(\varphi, 0, x)).$$

Then $\chi(x,0) = x$ and $\chi(x,1) = P_a(x)$ and our theorem follows from (69.10).

Therefore we have proved the following theorem

(73.8) THEOREM. *Assume that* $A \in \mathrm{PANR}$ *is compact, and that* $\varphi\colon [0,a] \times A \multimap \mathbb{R}^n$ *is an integrably bounded Carathéodory map with compact convex values and satisfying* (70.1). *If* $\chi(A) \neq 0$, *then problem* (73.7) *has a solution.*

Indeed, under our assumptions the Lefschetz number $\lambda(P_a) \neq 0$, so P_a has a fixed point which is a solution of (73.7).

Finally, for regular mappings Theorem (73.8) can be formulated as follows:

(73.9) COROLLARY. *Assume that* $A \in \mathrm{PANR}$ *is compact, and that* $\psi\colon [0,a] \times A \multimap \mathbb{R}^n$ *is a regular map satisfying* (73.1'). *If* $\chi(A) \neq 0$, *then* (73.7) *has a solution.*

PROOF. We take $\varphi \in U(\psi)$. Then condition (73.1) for φ is satisfied and (73.9) follows from (73.8). □

74. Implicit differential inclusions

The aim of this section is to show that, using the topological degree method as a tool, many types of differential equations (inclusions) whose right hand sides depend on the derivative can be reduced very easily to differential inclusions with right hand sides not depending on the derivative. We apply this method only to the following types of differential equations, but some other applications are also feasible:

- ordinary differential equations of first or higher order (e.g. the satelite equations);
- hyperbolic differential equations;
- elliptic differential equations.

We shall formulate all results in the simplest possible form. For more general formulations see [BiG] or [BiGP].

In this section we will assume X to be the closed ball $K_r \subset \mathbb{R}^n$ or $\mathbb{R}^n$. By the dimension, $\dim A$ of a compact subset of X we shall mean the topological covering dimension.

Following [BiG] we recall:

(74.1) PROPOSITION. *Let* A *be a compact subset of* X *such that* $\dim A = 0$. *Then for every* $x \in A$ *and for every open neighbourhood* U *of* x *in* X *there exists an open neighbourhood* $V \subset U$ *of* x *in* X *such that the boundary* ∂V *of* V *in* X *has empty intersection with* A, *i.e.* $\partial V \cap A = \emptyset$.

The proof of (74.1) is quite easy by a contradiction argument.

In the Euclidean space $\mathbb{R}^n$ we can identify a notion of the Brouwer degree with the fixed point index (cf. [Do-M]).

Namely, let U be an open bounded subset of $\mathbb{R}^n$ and let $g\colon \overline{U} \to \mathbb{R}^n$ be a continuous singlevalued map such that $\mathrm{Fix}(g) \cap \partial U = \emptyset$. We let $\widetilde{g}\colon \overline{U} \to \mathbb{R}^n$;

$$\widetilde{g}(x) = x - g(x), \quad x \in \overline{U},$$

$$i(g, U) = \deg(\widetilde{g}, U), \tag{74.2}$$

where $\deg(\widetilde{g}, U)$ denotes the Brouwer degree of $\widetilde{g}$ with respect to U; then $i(g, U)$ is called a *fixed point index* of g with respect to U.

Now all the properties of the Brouwer degree can be reformulated in terms of the fixed point index (cf. [Do-M]). So, in the case we are considering, it is exactly the same to use the topological degree or the fixed point index.

We shall start with the following:

(74.3) PROPOSITION. *Let $g\colon X \to X$ be a compact map. Assume further that the following two conditions are satisfied:*

(74.3.1) $\dim \mathrm{Fix}(g) = 0$*;*

(74.3.2) *there exists an open subset $U \subset X$ such that*

$$\partial U \cap \mathrm{Fix}(g) = \emptyset \quad \textit{and} \quad i(g, U) \neq 0.$$

Then there exists a point $z \in \mathrm{Fix}(g)$ for which we have:

(74.3.3) *for every open neighbourhood U_z of z in X there exists an open neighbourhood V_z of z in X such that*

$$V_z \subset U_z, \quad \partial V_z \cap \mathrm{Fix}(g) = \emptyset \quad \textit{and} \quad i(g, V_z) \neq 0.$$

PROOF. Let Γ be the family of all subsets A of $\mathrm{Fix}(g) \cap U$ which are compact, nonempty, and such that for every open neighbourhood W of A in X there is an open neighbourhood V of A in X which satisfies the following three conditions:

$$V \subset W, \quad \partial V \cap \mathrm{Fix}(g) = \emptyset \quad \text{and} \quad i(g, V) \neq 0.$$

It follows from (74.3.2) that Γ is a nonempty family. Let Γ be partially ordered by an inclusion. We are going to apply the famous Kuratowski–Zorn Lemma. Let $\{A_i\}$ be a chain in Γ and put $A_0 = \bigcap_{i \in I} \{A_i\}$. To prove that $A_0 \in \Gamma$ assume that W is an open neighbourhood of A_0 in X. We claim that there is an $i \in I$ such that $A_i \subset W$. Indeed, if we assume the contrary then we get a family $B_i = (X \setminus W) \cap A_i$, $i \in I$, of compact nonempty sets which has nonempty compact intersection B_0. Then $B_0 \subset X \setminus W$ and $B_0 \subset A_0$, so we obtain a contradiction and hence $A_0 \in \Gamma$. Consequently, in view of the Kuratowski–Zorn Lemma, Γ has a minimal element A_*.

We claim that A_* is a singleton. Let $z \in A_*$. It is sufficient to prove that $\{z\} \in \Gamma$. Since $A_* \in \Gamma$ we obtain an open neighbourhood U_* of A_* in X with the following properties: $U_* \subset U$, $\partial U_* \cap \operatorname{Fix}(g) = \emptyset$ and $i(g, U_*) \neq 0$. Let W be an arbitrary open neighbourhood of z in X. Using Proposition (74.1) we can choose an open neighbourhood U_z of z in $U_* \cap W$ such that $\operatorname{Fix}(g) \cap \partial U_z = \emptyset$. Since A_* is a minimal element of Γ the compact set $A_* \setminus U_z$ is not in Γ, and hence there exists an open set $V \subset U_*$ such that $(A_* \setminus U_z) \subset V \subset U_*$, $\operatorname{Fix}(g) \cap \partial V = \emptyset$, $V \cap U_z = \emptyset$, $i(g, V) = 0$ and $i(g, U_*) = i(g, V \cup U_z)$. Now, from the additivity property of the fixed point index we have

$$i(g, U_*) = i(g, U_z) + i(g, V) \neq 0$$

and consequently $i(g, U_z) \neq 0$. This implies that $\{z\} \in \Gamma$ and the proof of (74.3) is completed. □

Now, we are going to consider a more general situation. Let Y be a locally arcwise connected space and let $f: Y \times X \to X$ be a compact map. In what follows we shall assume that f satisfies the following condition:

(74.4) $\quad$ for all $y \in Y$ exists U_y such that U_y is open in X and $i(f_y, U_y) \neq 0$,

where $f_y: X \to X$ is given by the formula $f_y(x) = f(y, x)$ for every $x \in X$. Observe that in particular, if X is an absolute retract, then (74.4) holds automatically. We associate with a map $f: Y \times X \to X$ satisfying the above conditions the following multivalued map:

$$\varphi_f: Y \to X, \quad \varphi_f(y) = \operatorname{Fix}(f_y).$$

Then from (74.4) it follows that φ_f is well defined. Moreover, we obtain:

(74.5) Proposition. *Under all of the above assumptions the map* $\varphi_f: Y \multimap X$ *is u.s.c.*

Let us remark that, in general, φ_f is not an l.s.c. map. Below we would like to formulate a sufficient condition which guarantees that φ_f has an l.s.c. selection. To this end we shall add one more assumption. Namely, we assume that f satisfies the following condition:

(74.6) $$\dim \operatorname{Fix}(f_y) = 0 \quad \text{for all } y \in Y.$$

Now, in view of (74.4) and (74.6), we are able to define the map $\psi_f: Y \multimap X$ by putting $\psi_f(y) = \operatorname{cl}\{z \in \operatorname{Fix}(f_y) \mid z \text{ satisfies condition (74.3.3)}\}$, for every $y \in Y$.

We prove the following:

(74.7) THEOREM. *Under all of the above assumptions we have:*

(74.7.1) ψ_f *is a selection of* φ_f*;*
(74.7.2) ψ_f *is an l.s.c. map.*

PROOF. Since (74.7.1) follows immediately from the definition we shall prove (74.7.2). To do this we let $\eta_f\colon Y \multimap X$:

$$\eta_f(y) = \{z \in \mathrm{Fix}(f_y) \mid z \text{ satisfies (74.3.3)}\}.$$

For the proof it is sufficient to show that η_f is l.s.c. Let U be an open subset of X and let $y_0 \in Y$ be a point such that $\eta_f(y_0) \cap U \neq \emptyset$. Assume further that $x_0 \in \eta_f(y_0) \cap U$. Then there exists an open neighbourhood V of x_0 in X such that $V \subset U$ and $i(f_{y_0}, V) \neq 0$. Since φ_f is a u.s.c. map and Y is locally arcwise connected we can find an open arcwise connected W in Y such that $y_0 \in W$ and for every $y \in W$ we have:

$$\mathrm{Fix}(f_y) \cap \partial V = \emptyset. \tag{74.7.3}$$

Let $y \in W$ and let $\delta\colon [0,1] \to W$ be an arc joining y_0 with y, i.e. $\delta(0) = y_0$ and $\delta(1) = y$. We define a homotopy $h\colon [0,1] \times V \to X$ by putting $h(t,x) = (\delta(t), x)$. Then it follows from (74.7.3) that h is a well defined homotopy joining f_{y_0} with f_y and hence we obtain $i(f_{y_0}, V) = i(f_y, V) \neq 0$; so $\mathrm{Fix}(f_y) \cap V \neq \emptyset$ and our assertion follows from (74.2). □

Observe that the condition (74.6) is quite restrictive. Therefore, it is interesting to characterize the topological structure of all mappings satisfying (74.6). We shall do this in the case when $Y = A$ is a closed subset of $\mathbb{R}^n$ and $X = \mathbb{R}^n$.

By $C(A\times\mathbb{R}^n, \mathbb{R}^n)$ we shall denote the Banach space of all compact (singlevalued) maps from $A \times \mathbb{R}^n$ into $\mathbb{R}^n$ with the usual supremum norm. Let

$$Q = \{f \in C(A \times \mathbb{R}^n, \mathbb{R}^n) \mid f \text{ satisfies (74.6)}\}.$$

Let us formulate the following well known result from functional analysis (see [BiG]):

(74.8) THEOREM. *The set Q is dense in $C(A \times \mathbb{R}^n, \mathbb{R}^n)$.*

Let us remark that all of the above results remain true for X being an arbitrary ANR-space (see [BiG]).

Now, we shall apply the above results.

(74.9) Ordinary differential equations of first order. According to the above considerations we let $Y = [0,1] \times \mathbb{R}^n$, $X = \mathbb{R}^n$, and let $f: Y \times X \to X$ be a compact map. Then f satisfies condition (74.4) automatically so we shall assume only (74.6). Let us consider the following equation:

(74.9.1) $$x'(t) = f(t, x(t), x'(t)),$$

where the solution is understood in the sense of almost everywhere, $t \in [0,1]$ (a.e. $t \in [0,1]$).

We shall associate with (74.9.1) the following two differential inclusions:

(74.9.2) $$x'(t) \in \varphi_f(t, x(t)),$$

(74.9.3) $$x'(t) \in \psi_f(t, x(t)),$$

where φ_f and ψ_f are defined as before, and by a solution of (74.9.2) or (74.9.3) we mean an absolutely continuous function which satisfies (74.9.2) (resp. (74.9.3)) in the sense of a.e. $t \in [0,1]$.

Denote by $S(f)$, $S(\varphi_f)$ and $S(\psi_f)$ the set of all solutions of (74.9.1)–(74.9.3), respectively. Then we obtain $S(\psi_f) \subset S(f) = S(\varphi_f)$. But the map ψ_f is l.s.c. so we obtain $S(\psi_f) \neq \emptyset$. Thus we have proved: $\emptyset \neq S(\psi_f) \subset S(\varphi_f) = S(f)$. Observe that in (74.9.2) and (74.9.3) the right hand side does not depend on the derivative.

(74.10) Ordinary differential equations of higher order. Let $Y = [0,1] \times \mathbb{R}^{kn}$, $X = \mathbb{R}^n$, and let $f: Y \times X \to X$ be a compact map. To study the existence problem for the following equation:

$$x^{(k)}(t) = f(t, x(t), \ldots, x^{(k)}(t))$$

we consider the following two differential inclusions

$$x^{(k)}(t) \in \varphi_f(t, x(t), x'(t), \ldots, x^{(k-1)}(t)),$$
$$x^{(k)}(t) \in \psi_f(t, x(t), x'(t), \ldots, x^{(k-1)}(t)).$$

(74.11) Hyperbolic equations. Now let $Y = [0,1] \times [0,1] \times \mathbb{R}^{3n}$, $X = \mathbb{R}^n$, and let $f: Y \times X \to X$ be a compact map. Consider the following hyperbolic equation:

(74.11.1) $$u_{ts}(t,s) = f(t, s, u(t,s), u_t(t,s), u_s(t,s), u_{ts}(t,s)),$$

where the solution $u: [0,1] \times [0,1] \to \mathbb{R}^n$ is understood in the sense of a.e. $(t,s) \in [0,1] \times [0,1]$.

As above, we associate with (74.11.1) the following two differential inclusions:

(74.11.2) $$u_{ts}(t,s) \in \varphi_f(t, s, u(t,s), u_t(t,s), u_s(t,s)),$$

(74.11.3) $$u_{ts}(t,s) \in \psi_f(t, s, u(t,s), u_t(t,s), u_s(t,s)).$$

Then it is evident that the set of all solutions of (74.11.1) is equal to the set of all solutions of (74.11.2) and every solution of (74.11.3) is a solution of (74.11.2). So, the inclusions (74.11.2) and (74.11.3) give us a full information about (74.11.1).

(74.12) Elliptic differential equations. Let K_r^n denote the closed ball in $\mathbb{R}^n$ with center at 0 and radius r. Now we put $Y = K_r^n \times \mathbb{R}^{2n}$, $X = \mathbb{R}^n$, and let $f\colon Y \times X \to X$ be a compact map. Since (74.4) is satisfied we assume only (74.6). We consider the following elliptic equation:

$$\Delta(u)(z) = f(z, u(z), D(u)(z), \Delta(u)(z)), \qquad \text{a.e. } z \in K_r^n,$$

where Δ denotes the Laplace operator and $D(u)(z) = u_{z_1}(z) + \ldots + u_{z_n}(z)$, $z = (z_1, \ldots, z_n)$. Then we consider the following two differential inclusions:

$$\Delta(u)(z) \in \varphi_f(z, u(z), D(u)(z)), \tag{74.12.1}$$

$$\Delta(u)(z) \in \psi_f(z, u(z), D(u)(z)), \tag{74.12.2}$$

and we have exactly the same situation as in (74.11) or (74.10).

We shall end our applications with the following remarks.

(74.13) Remark. Observe that all results of this section, except (74.12), remain true if we replace the Euclidean space $\mathbb{R}^n$ by an arbitrary Banach space.

(74.14) Remark. Let us observe that if we replace (74.9)–(74.12) by the respective differential inclusions, then we obtain all results of this section without any change.

(74.15) Remark. Finally, let us remark that some another method for implicit differential equations was considered by B. Ricceri in [Ri3].

75. Concluding remarks and comments

Chapter I. For further studies connected with the material of this chapter see: [BPe-M], [Bo-M], [Br1-M], [De4-M], [Do-M], [ES-M], [Go1-M], [Gr1-M], [Gr2-M], [Gr3-M], [HW-M], [Sp-M].

Chapter II. This chapter contains some new results presented in Section 21. The main result is Theorem (21.15) where a topological characterization of the set of fixed points for some contraction mappings is presented. The mentioned result was proved in [GMS] (see also [GM]) as a generalization of an earlier results proved by B. Ricceri ([Ri1]) for mappings with convex values and, later, by A. Bressan, A. Cellina and A. Fryszkowski ([BCF]) for mappings with decomposable values.

For more details concerning the material of this chapter we recommend: [APNZ-M], [Au-C], [AuE-M], [Be-M], [BrGMO1-M], [CV-M], [Ki-M], [LR-M].

Chapter III. The approximation (on the graph) method for multivalued maps were initiated by J. Von Neumann in 1933. Then this method was developed by many authors (see: [ACZ1], [ACZ2], [ACZ3], [ACZ4], [Bee1], [Bee4], [CL1], [GGK1], [GGK2], [GGK3], [GL], [HC]).

Note, that recently W. Kryszewski obtained some important new results in this direction. For details we recommend [Kr1-M] and [Kr2-M].

Chapter IV. Homological methods in the fixed point theory of multivalued maps was initiated in 1946 by S. Eilenberg and D. Montgomery (cf. [EM]) where the Lefschetz Fixed Point Theorem for acyclic mappings was proved. Later, the Lefschetz Fixed Point Theorem, the Fixed Point Index, and the Topological Degree Theory for several classes of multivalued mappings were studied by many authors. Below we recommend more essential works in this area: [BgGMO-M], [Cu-M], [Da-M], [Dz1-M], [Go1-M], [Go4-M], [LR-M], [Ma-M], [We-M], [Bi2], [Bou1], [Bou2], [Bry], [BG-2], [Bg-3], [Cal1], [FG1], [Go2]–[Go12], [GGr1], [GGr2], [GR], [Ja1]–[Ja4], [JP], [Kr1], [Kr2], [Ne1], [Pa], [Po1]–[Po3], [SS1].

There are some other classes of multivalued mappings for which the fixed point theory was studied. Let us mention the so called multivalued mappings with multiplicity, defined by S. Darbo in 1958 ([Da]).

Let $\varphi\colon X \multimap Y$ be a multivalued map with compact (nonempty) values. Two points $(x_1, y_1), (x_2, y_2) \in \Gamma_\varphi$ are equivalent $((x_1, y_1) \sim (x_2, y_2))$ if and only if $x_1 = x_2$ and y_1, y_2 are in the same connected component of $\varphi(x_1) = \varphi(x_2)$. This defines a new set $\widetilde{\Gamma}_\varphi = \Gamma_\varphi|_\sim$ with elements denoted by $(x, C(x))$; $C(x)$ denotes also a connected component of $\varphi(x)$ as a subset of Y.

In what follows, a map $m\colon \widetilde{\Gamma}_\varphi \to Q$ is called the multiplicity function for φ. Note that in the above definition Q can be replaced by an arbitrary ring without zero divisors.

Let $\varphi\colon X \multimap Y$ be a multivalued map with multiplicity-function $m\colon \widetilde{\Gamma}_\varphi \to Q$; φ is an *m-map* (*map with multiplicity*) if the following two conditions are satisfied:

(i) $\varphi(x)$ consists of finitely many connected components for each $x \in X$;

(ii) for all $x_0 \in X$ with $\varphi(x_0) = C_1(x_0) \cup \ldots \cup C_s(x_0)$, $s = s(x_0)$, and disjoint open neighbourhoods U_i of $C_i(x_0)$ in Y there exists a neighbourhood U of x_0 such that:

$$\varphi(U) \subset \bigcup_{i=1}^{s} U_i$$

and, for all $x \in U$, $i = 1, \ldots, s$,

$$m(x_0, C_i(x_0)) = \sum_{C(x) \subset U_i} m(x, C(x)).$$

In [HaS] it is showed that using the chain approximation technique the fixed point index theory for mappings with multiplicity is possible to develop. Note that mappings with multiplicity are defined using algebraic or combinatorial approach instead of geometrical approach presented by us in Chapter IV. This is the reason why we did not consider the class of mappings with multiplicity in Chapter IV. We shall present details concerning the above notion in Chapter VII.

Finally, note that the class of small multivalued mappings was considered by H. Schirmer (cf. [Sch1]–[Sch3]).

The last two chapters of our monograph are devoted to applications of the material in the main two Chapters (III and IV).

Chapter V. In Chapter V we concentrate mainly on consequences of the Lefschetz Fixed Point Theorem and the fixed point index (Sections 55–60).

Some results concerning minimax theorem and consequently applications to mathematical economy are presented in Sections 65–66 (for further results see: [AuE-M], [BK1], [BK2], [Bor-M], [CV-M], [DG-M], [Wie-M], [GrL1], [GrL2], [GF], [Mcc1], [Las]).

In Chapter V we discuss also the bifurcation problem for multivalued mappings (see Section 63). A nontypical application of the Borsuk theorem on antipodes for convex valued mappings is presented in Section 68, where the so called Day's result is presented.

Chapter VI. The last chapter is devoted to the topological approach to differential inclusions. We discuss the Cauchy problem both for differential inclusions with u.s.c. and l.s.c. right hand sides. Then the Aronszajn type of results are presented. We study also the periodic problem and implicit differential inclusions. Chapter VI is based on [Go2-M] but we recommend also a very rich literature on this subject, namely: [Au-M], [AuC-M], [De1-M], [Fi1-M], [Fi2-M], [Fr-M], [Go3-M], [Go4-M], [GGL-M], [Ki-M], [LR-M], [Pr-M], [To-M], [And1]–[And5], [AGG], [AGJ], [AGL], [AZ1], [AZ2], [Au2], [BaP], [BaF], [BiGP], [Bl1], [Bl2], [BM1]–[BM7], [BP1]–[BP7], [Bog1]–[Bog3], [Bre1]–[Bre7], [BCF], [BC1], [BC2], [Ce2], [Ce3], [CC1], [CC2], [CCF], [COZ], [Dar], [De1], [DyG], [Fi1]–[Fi4], [Fry4], [GaP2], [GN], [GNZ1], [Had1]–[Had3], [HP], [Hu], [KNOZ], [Ko], [KaPa], [La], [LO2], [MNZ1]–[MNZ3], [Ma1]–[Ma3], [Ni], [NoZ1], [NoZ2], [Ob1]–[Ob3], [Pap1]–[Pap7], [Pl1], [Pl2], [To1], [Za].

CHAPTER VII

RECENT RESULTS

During last five years several new important results in the topological fixed point theory of multivalued mappings were obtained. The aim of this chapter is to survey the above mentioned results together with their historical development and motivation.

76. Periodic invariants; the Euler–Poincaré characteristic

Let $L\colon E \to E$ be a Leray endomorphism. Using notations of the Section 11, we define the Euler–Poincaré characteristic $\chi(L)$ of L by putting:

$$\chi(L) = \sum_n (-1)^n \dim(\widetilde{E}_n). \tag{76.1}$$

We start with the following lemma:

(76.2) LEMMA. *Let $L\colon E \to E$ be an endomorphism and let*

$$L^m = \underbrace{L \circ \ldots \circ L}_{m\text{-}times}$$

be its mth iterate, i.e. $L^m = \{L_n^m\}_{n\in\mathbb{N}}$, for every $n \in \mathbb{N}$. Then L is a Leray endomorphism if and only if L^m is a Leray endomorphism, for $m = 1, 2, \ldots$ and

$$\chi(L) = \chi(L^m), \quad m \geq 1.$$

PROOF. It is sufficient to observe that for every $m = 1, 2, \ldots$ and for every $n \in \mathbb{N}$ we have $N(L_n) = N(L_n^m)$. □

Starting from now until the end of this section we shall assume that K is the field of complex numbers.

(76.3) LEMMA. *Let $L\colon E \to E$ be an endomorphism of a finite dimensional space and let λ_j, $1 \leq j \leq \dim E$ be roots of the characteristic polynomial of L. Then*

$$\operatorname{tr}(L^m) = \sum_j (-\lambda_j)^m, \quad m = 1, 2, \ldots$$

PROOF. We can represent L as a triangular matrix with roots $\lambda_1, \dots, \lambda_s$, $s = \dim E$, on the main diagonal. Then for the mth iteration L^m of L the matrix is also triangular (keeping the same basis) and on the main diagonal, we get mth powers $(\lambda_1)^m, \dots, (\lambda_s)^m$ of $\lambda_1, \dots, \lambda_s$, respectively. It proves our lemma. □

Now, let $L = \{L_n\}\colon E \to E$ be a Leray endomorphism in GV. For such a L, we are able to assign the power series $S(L)$ in K by letting:

$$(76.4)\quad S(L)(z) = \Lambda(L) + \sum_{m=1}^{\infty} \Lambda(L^m) \cdot z^m = \sum_{m=0}^{\infty} \Lambda(L^m) \cdot z^m, \quad \text{where } L^0 = \mathrm{id}_E\,.$$

(76.5) THEOREM. *Let $L = \{L_n\}\colon E \to E$ be a Leray endomorphism. Let W_n be the characteristic polynomial of $\widetilde{L}_n\colon \widetilde{E}_n \to \widetilde{E}_n$ with roots $\lambda_{n,j} \neq 0$, $1 \leq j \leq \dim \widetilde{E}_n$, let $W(\lambda) = \prod_n (-1)^n W_n$ and $T(\lambda) = (\log W(\lambda))' = W'(\lambda)/W(\lambda)$. Then, for $0 < |z| < \min |\lambda_{n,j}|^{-1} = r$, we get*

$$S(L)(z) = \sum_{n,j} \frac{(-1)^n}{1 - \lambda_{n,j}} = \frac{1}{z} T\Big(\frac{1}{z}\Big).$$

PROOF. By using (76.3), for $|z| < r$, we get

$$\begin{aligned} S(L)(z) &= \sum_{m=0}^{\infty} \lambda(\widetilde{L}^m) \cdot z^m = \sum_{m=0}^{\infty} \sum_i (-1)^i tr(\widetilde{L}_i^m) \\ &= \sum_{m=0}^{\infty} \sum_{i,j} (-1)^i \lambda_{i,j}^m z^m = \sum_{i,j} \frac{(-1)^i}{1 - \lambda_{i,j} z}. \end{aligned}$$

So taking the logarithmic derivative for $\lambda > r - 1$, we get

$$T(\lambda) = \sum_{i,j} (-1)^i \frac{(-1)^i}{z - \lambda_{i,j}}$$

and consequently, we have

$$\sum_{i,j} \frac{(-1)^i}{1 - \lambda_{i,j} z} = \sum_{i,j} \frac{1}{z} \frac{(-1)^i}{z^{-1} - \lambda_{i,j}} = \frac{1}{z} \cdot T\Big(\frac{1}{z}\Big),$$

for $0 < |z| < r$. The proof is completed. □

It implies that $S(L)$ can be represented in a unique form as a factor of two polynomials W_1, W_2, i.e. $S(L) = W_1/W_2$ such that:

$$(W_1, W_2) = 1, \quad \deg W_1 < \deg W_2.$$

This allows us to define the natural number $P(L)$ of L by putting

$$(76.6)\qquad P(L) = \begin{cases} \deg W_2 & \text{if } S(f) \neq 0, \\ 0 & \text{if } S(f) = 0. \end{cases}$$

Below we shall summarize the properties of the above considered invariants Λ, χ, P, S.

(76.7) PROPERTIES.

(76.7.1) *There exists a natural number m such that $\Lambda(L^m) \neq 0$ if and only if $P(L) \neq 0$.*

(76.7.2) *if $\chi(L) \neq 0$ then $P(L) \neq 0$.*

(76.7.3) *If $P(L) \neq 0$, then for every natural number l there is a natural number m such that $l \leq m \leq l + P(L)$ and $\Lambda(L^m) \neq 0$.*

(76.7.4) *Let $L: E \to E$ and $L': E' \to E'$ be two Leray endomorphisms. If there exists a natural number l such that $\Lambda(L^m) = \Lambda((L')^m)$, for every natural number $l \leq m < l + P(L) + P(L')$, then $\Lambda(L^k) = \Lambda((L')^k)$, for all natural k and $\chi(L) = \chi(L')$.*

The proof of (76.7) is straightforward. For details concerning Sections 1–3 we recommend [Bow1]–[Bow3] and [Gr1], [Gr2].

Now, we shall apply the above notions to multivalued mappings. It is possible to do it in terms of admissible mappings or morphisms. It is more convenient to consider the case of morphisms (see Sections 44–48).

Let $\varphi: X \to X$ be a CAC morphism. In spite of the Lefschetz number $\Lambda(\varphi)$, we can define:

$$\Lambda(\varphi) = \Lambda(\varphi_*), \quad S(\varphi) = S(\varphi_*), \quad P(\varphi) = P(\varphi_*).$$

Now, Lemma (76.2) can be formulated in the following form:

(76.8) PROPOSITION. *A morphism $\varphi: X \to X$ is a Lefschetz morphism if and only if any iterate φ^n of φ is a Lefschetz morphism and in such a case we have $\Lambda(\varphi) = \Lambda(\varphi^n)$.*

Recall that a morphism $\varphi: X \to X$ is called a Lefschetz morphism provided $\varphi_*: H(X) \to H(X)$ is a Leray endomorphism.

From (76.7) we infer:

(76.9) PROPOSITION. *Let $\varphi: X \to X$ be a morphism. We have:*

(76.9.1) *$\Lambda(\varphi) \neq 0$ implies $P(\varphi) \neq 0$,*

(76.9.2) *$P(\varphi) \neq 0$ if and only if $\Lambda(\varphi^n) \neq 0$ for some $n \geq 1$,*

(76.9.3) *$P(\varphi) = k \neq 0$, then for any natural $m \geq 0$ at least one of the coefficients $\Lambda(\varphi^{m+1}), \ldots, \Lambda(\varphi^{m+k})$ of the series $S(\varphi)$ must be different from zero, $m \geq 0$.*

Let $\varphi: X \to X$ be a morphism. A point $x \in X$ is called periodic for φ with period n provided $x \in \varphi^n(x)$, $n \geq 1$. Observe that any fixed point of φ is periodic with period n, for arbitrary $n \geq 1$.

(76.10) THEOREM (Periodic Point Theorem). *Let $X \in$ ANR and $\varphi\colon X \to X$ be a* CAC*-morphism. If $\lambda(\varphi) \neq 0$ or $P(\varphi) \neq 0$, then φ has a periodic point with period n, where $m+1 \leq n \leq m+P(\varphi)$ and $m \geq 0$ is an arbitrary natural number.*

PROOF. It follows from Theorem (48.12) that φ is a Lefschetz morphism. In view of Proposition (76.9.1) it is sufficient to assume that $P(\varphi) \neq 0$. Applying (76.9.3), for any $m \geq 0$ we get n such that $\Lambda(\varphi^n) \neq 0$, where $m+1 \leq n \leq m+P(\varphi)$. Since composition of CAC morphisms is CAC again, we deduce from Theorem (48.12) that $\mathrm{Fix}(\varphi^n) \neq \emptyset$.

Of course, if $x \in \mathrm{Fix}(\varphi^n)$, then x is periodic point of φ with period n. Hence the proof is complete. □

(76.11) REMARK. It can be easily checked that $\Lambda(\varphi)$ and $P(\varphi)$ are homotopic invariants.

77. The coincidence Nielsen number

In Section 35 we discussed the Nielsen number theory for multivalued mappings so called m-mappings. In 2000–2005 years the problem of developing Nielsen theory for multivalued mappings was taken up by several authors see: [AnGo-M], [AGJ-1]–[AGJ-4]. Note that in [AnGo-M] nonmetric case is considered. In this section we shall study the Nielsen theory for morphisms of ANR-s. As we observed in Section 44 a morphism $\varphi\colon X \to X$ is an abstract class of a pair (p, q), where

$$X \overset{p}{\Longleftarrow} \Gamma \overset{q}{\longrightarrow} X.$$

In what follows by a multivalued map we shall understand a morphism represented by a fixed pair (p, q) of the above form.

As first, we give an example which demonstrates that the multivalued setting is not a direct extension of the singlevalued one.

As in the last two sections, by a multivalued map we shall understand an admissible map represented by a fixed pair (p, q) of the form

$$X \overset{p}{\Longleftarrow} \Gamma \overset{q}{\longrightarrow} Y.$$

For the clarity of our explanation, we shall present below some necessary well-known notions.

At first, we give an example which demonstrates that the multivalued setting is not a direct extension of the single-valued one.

(77.1) EXAMPLE. Let us consider the unit circle $S^1 = \mathbb{R}/\mathbb{Z}$ with the correspondence

$$\mathbb{R}/\mathbb{Z} \ni [t] \leftrightarrow e^{2\pi t i} \in S^1,$$

define a family of maps $p_\varepsilon\colon S^1 \to S^1$, $0 < \varepsilon \le 1/2$,

$$p_\varepsilon[t] = \begin{cases} [t/2\varepsilon] & \text{for } 0 \le t \le \varepsilon, \\ [1/2] & \text{for } \varepsilon \le t \le 1-\varepsilon, \\ [(1-t)/2\varepsilon + 1] & \text{for } 1-\varepsilon \le t \le 1, \end{cases}$$

and put $q[t] = [kt]$, for a fixed $k \in \mathbb{Z}$.

Let us note that $p_\varepsilon^{-1}[y]$ is one point, for $[y] \neq [1/2]$, while $p_\varepsilon^{-1}[1/2] = \{[t] \mid \varepsilon \le t \le 1-\varepsilon\}$ is an arc. Thus, any counter image is contractible. Let us fix a number ε_0 satisfying $0 < \varepsilon_0 \le 1/2k$. Then $\{(p_\varepsilon, q)\}$, where ε runs through the interval $[\varepsilon_0, 1/2]$, is a homotopy between the multivalued maps $(p_{1/2}, q)$ and (p_ε, q). But since we can observe that $p_{1/2} = \mathrm{id}_{S^1}$, $(p_{1/2}, q)$ corresponds to the single-valued map $q[t] = [kt]$. It is known (see e.g. [KTs-M]) that the *Nielsen number* $N(q) = |k-1|$, by which we know that any single-valued map homotopic to q has at least $|k-1|$ fixed points. On the other hand, we will show that $[x] \in qp_{\varepsilon_0}^{-1}[x]$ only for $[x] = [0]$ or $[x] = [1/2]$. (In fact, for $0 < x < 1/2$, $p_\varepsilon^{-1}[x] = [2\varepsilon_0 x]$, so $qp_{\varepsilon_0}^{-1}[x] = [2k\varepsilon_0 x]$, but the assumption $0 < 2k\varepsilon_0 < 1$ implies $0 < 2k\varepsilon_0 x < x < 1/2$ which gives $[2k\varepsilon_0 x] \neq [x]$). If $1/2 < x < 1$, then

$$p_{\varepsilon_0}^{-1}[x] = [1 - 2\varepsilon_0(1-x)],$$

so

$$qp_{\varepsilon_0}^{-1}[x] = [k - 2k\varepsilon_0(1-x)] = [1 - 2k\varepsilon_0(1-x)].$$

It follows from the assumption $0 < 2k\varepsilon_0 < 1$ that $0 < 1 - 2k\varepsilon_0(1-x) < 1$, and subsequently $[x] = [1 - 2k\varepsilon_0(1-x)]$ if and only if $x = 1 - 2k\varepsilon_0(1-x)$. The last equality, however, yields $2k\varepsilon_0(1-x) = 1-x$, i.e. $2k\varepsilon_0 = 1$, after dividing by $x - 1 \neq 0$, which contradicts the assumption $0 < 2k\varepsilon_0 < 1$.

Thus, a multivalued homotopy of a single-valued map with the Nielsen number $N(q) = |k-1|$ gives the fixed point set

$$\mathrm{Fix}(p,q) = \{x \in X \mid x \in qp^{-1}(x)\}$$

consisting of only two elements!

The above example seems to suggest that the Nielsen fixed point theory fails to be extended to the multivalued case. On the other hand, we can observe that, in this example, the restriction $q\colon p_{\varepsilon_0}^{-1}[1/2] \to S^1$ covers the point $[1/2]k$-times. This observation encourages us to estimate the *number of coincidences* $C(p,q) = \{z \in \Gamma \mid p(z) = q(z)\}$ of the pair (p,q) rather than the *number of fixed points* $\mathrm{Fix}(qp^{-1})$ of the map qp^{-1}.

Let $X \overset{p_0}{\Leftarrow} \Gamma \overset{q_0}{\longrightarrow} Y$ and $X \overset{p_1}{\Leftarrow} \Gamma \overset{q_1}{\longrightarrow} Y$ be two maps. We say that (p_0, q_0) is *homotopic* to (p_1, q_1) (written $(p_0, q_0) \sim (p_1, q_1)$) if there exists a multivalued map $X \times I \overset{p}{\longleftarrow} \overline{\Gamma} \overset{q}{\longrightarrow} Y$ such that the following diagram is commutative:

$$\begin{array}{ccccc} X & \overset{p_i}{\Longleftarrow} & \Gamma & \overset{q_i}{\longrightarrow} & Y \\ {\scriptstyle k_i}\downarrow & & {\scriptstyle f_i}\downarrow & \nearrow {\scriptstyle q} & \\ X \times I & \underset{p}{\Longleftarrow} & \overline{\Gamma} & & \end{array}$$

for $k_i(x) = (x, i)$, $i = 0, 1$, and for some $f_i\colon \Gamma \to \overline{\Gamma}$, $i = 0, 1$, i.e. $k_0 p_0 = p f_0$, $q_0 = q f_0$, $k_1 p_1 = p f_1$ and $q_1 = q f_1$.

If $(p_0, q_0) \sim (p_1, q_1)$ and $h\colon Y \to Z$ is a continuous map, then we write $(p_0, hq_0) \sim (p, hq)$. We say that a multivalued map $X \overset{p}{\Leftarrow} \Gamma \overset{q}{\longrightarrow} Y$ *represents a single-valued map* $\rho\colon X \to Y$ if $q = p\rho$. Now, we assume that $X = Y$ and we are going to estimate the cardinality of the coincidence set $C(p, q)$. We begin by defining a Nielsen-type relation on $C(p, q)$. This definition requires the following conditions on $X \overset{p}{\Leftarrow} \Gamma \overset{q}{\longrightarrow} Y$:

(77.2) X and Y are metric connected and locally contractible spaces (observe that then they admit universal coverings; moreover, connected ANRs satisfied this assumption),

(77.3) $p\colon \Gamma \Rightarrow X$ is a Vietoris map,

(77.4) for any $x \in X$, the restriction $q_1 = q|_{p^{-1}(x)}\colon p^{-1}(x) \to Y$ admits a lift $\widetilde{q}_1$ to the universal covering space $(p_Y\colon \widetilde{Y} \to Y)$:

$$\begin{array}{ccc} & & \widetilde{Y} \\ & \overset{\widetilde{q}_1}{\nearrow} & \Downarrow {\scriptstyle p_Y} \\ p^{-1}(x) & \underset{q_1}{\longrightarrow} & Y \end{array}$$

Consider a single-valued map $\rho\colon X \to Y$ between two spaces admitting universal coverings $p_X\colon \widetilde{X} \Rightarrow X$ and $p_Y\colon \widetilde{X} \Rightarrow Y$. Let $\theta_X = \{\alpha\colon \widetilde{X} \to \widetilde{X} \mid p_X \alpha = p_X\}$ be the group of natural transformations of the covering p_X. Then the map ρ admits a lift $\widetilde{\rho}\colon \widetilde{X} \to \widetilde{Y}$. We can define a homomorphism $\widetilde{\rho}_!\colon \theta_X \to \theta_Y$ by the equality

$$\widetilde{q}(\alpha \cdot \widetilde{x}) = \widetilde{q}_!(\alpha)\widetilde{q}(\widetilde{x}) \quad (\alpha \in \theta_X,\ \widetilde{x} \in \widetilde{X}).$$

It is well-known (see for example [Sp-M]) that there is an isomorphism between the fundamental group $\pi_1(X)$ and θ_X which may be described as follows. We fix points $x_0 \in X$, $\widetilde{x} \in \widetilde{X}$ and a loop $\omega\colon I \to X$ based at x_0. Let $\widetilde{\omega}$ denote the unique lift of ω starting from $\widetilde{x}_0$. We subordinate to $[\omega] \in \pi_1(X, x_0)$ the unique transformation from θ_X sending $\widetilde{\omega}(0)$ to $\widetilde{\omega}(1)$. Then the homomorphism $\widetilde{\rho}_!\colon \theta_X \to \theta_Y$ corresponds to the induced homomorphism between the fundamental groups $\rho_{\#}\colon \pi_1(X, x_0) \to \pi_1(Y, \rho(x_0))$.

We will show that, under the assumptions (77.2)–(77.4), a multivalued map (p, q) admits a lift to a multivalued map between the universal coverings. These lifts will split the coincidence set $C(p, q)$ into Nielsen classes. Besides that, we will also show that the pair (p, q) induces a homomorphism $\theta_X \to \theta_Y$ giving the Reidemeister set in this situation.

We start with the following lemma.

(77.5) LEMMA. *Suppose we are given Y, a paracompact locally contractible space, Γ a topological space, $\Gamma_0 \subset \Gamma$ a compact subspace, $q\colon \Gamma \to Y$, $\widetilde{q}_0\colon \Gamma_0 \to Y$ continuous maps for which the diagram*

$$\begin{array}{ccc} \Gamma_0 & \xrightarrow{\ \widetilde{q}_0\ } & \widetilde{Y} \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p_Y} \\ \Gamma & \xrightarrow[\ q\ ]{} & Y \end{array}$$

commutes (here, $p_Y\colon \widetilde{Y} \to Y$ denotes the universal covering). In other words, $\widetilde{q}_0$ is a partial lift of q. Then $\widetilde{q}_0$ admits an extension to a lift onto an open neighbourhood of Γ_0 in Γ.

PROOF. Let us fix a covering $\{W_i\}$ of the space Y consisting of open connected sets satisfying: if $\operatorname{cl} W_i \cap \operatorname{cl} E_j \neq \emptyset$, then $\operatorname{cl} W_i \cup \operatorname{cl} W_j$ is contained in a contractible subset of Y.

Let $\{W_j\}$ denote the covering consisting of connected components of the covering $\{p_Y^{-1}\widetilde{W}_i\}$. We notice that the restriction of p_Y to any of sets $\widetilde{W}_j$ is a homeomorphism.

Let $\{U_i\}$ be a finite covering of Γ_0 such that $\{\operatorname{cl} U_i\}$ is subcovering of $\{\widetilde{q}_0^{-1}\widetilde{W}_i\}$. For any U_i, we fix an open subset $V_i \subset \Gamma$ satisfying $V_i \cap \Gamma_0 = U_i$.

We can assume that $\operatorname{cl} V_i$ is disjoint with $\bigcup\{\operatorname{cl} U_j \mid \operatorname{cl} U_j \cap \operatorname{cl} U_i = \emptyset\}$ (notice that the sets $\operatorname{cl} U_i$ and $F_i = \bigcup\{\operatorname{cl} U_j \mid \operatorname{cl} U_j \cap \operatorname{cl} U_i = \emptyset\}$ are disjoint and closed. Hence, there exists an open subset $S \subset \Gamma$ satisfying $\operatorname{cl} U_i \subset S \subset \operatorname{cl} S \subset \Gamma \setminus F_i$, and so we can put $V_i := V_i \cap S$).

Let $V_i' = V_i - \bigcup\{\operatorname{cl} U_j \mid \operatorname{cl} U_j \cap \operatorname{cl} U_i = \emptyset\}$. Then

(77.5.1) $V_i' = V_i \cap \Gamma_0 = U_i$,

(77.5.2) if $V_i' \cap V_i \neq \emptyset$, then $\operatorname{cl} U_i \cap \operatorname{cl} U_j \neq \emptyset$.

For any V_i', we fix $\widetilde{W}_{\alpha(i)}$ satisfying $\widetilde{q}_0(\operatorname{cl} U_i) \subset \widetilde{W}_{\alpha(i)}$ and we put $V_i'' = V_i' \cap q^{-1}(\widetilde{W}_{\alpha(i)})$. The covering $\{V_i'\}$ also satisfies the above conditions (77.5.1) and

(77.5.2). For any i, we denote by $\varphi_i\colon \widetilde{W}_{\alpha(i)} \to \widetilde{W}_\alpha$ the homomorphism inverse to the projection P_Y.

Now, we can define an extension of the lift $\widetilde{q}_0$ onto the neighbourhood $\bigcup V_i'$. We define the map $\widetilde{q}_i\colon V_i'' \to \widetilde{Y}$ by the formula $\widetilde{q}_i = \varphi q(x) \in \widetilde{W}_{\alpha(i)} \subset \widetilde{Y}$.

It remains to show that the maps $\widetilde{q}_i$ and $\widetilde{q}_j$ are consistent. Let $V_i'' \cap V_j'' \neq \emptyset$. Then there is a point $x \in \operatorname{cl} U_i \cap \operatorname{cl} U_j$ which implies $\widetilde{q}_0(x) \in \widetilde{W}_{\alpha(i)} \cap \widetilde{W}_{\alpha(j)}$. Let $S \subset Y$ be a contractible set containing $\pi_Y \widetilde{W}_{\alpha(i)} \cup \widetilde{W}_{\alpha(j)}$ and let $\widetilde{S}$ be the component of $p_Y^{-1}(S)$ containing $\widetilde{Q}_0(x)$. Then $\widetilde{W}_{\alpha(i)} \cup \widetilde{W}_{\alpha(j)} \subset \widetilde{S}$, and so the values of the sections $\widetilde{q}_i$, $\widetilde{q}_j$ are contained in $\widetilde{S}$ which implies that they must be consistent. $\square$

(77.6) LEMMA. *Suppose we are given a multivalued map $X \overset{p}{\Longleftarrow} \Gamma \overset{q}{\longrightarrow} Y$ satisfying* (77.2)*, where X is simply-connected. Then there exists a map $\widetilde{q}\colon \Gamma \to \widetilde{Y}$ making the diagram*

$$\begin{array}{ccc} & & \widetilde{Y} \\ & {\scriptstyle \widetilde{q}} \nearrow & \downarrow {\scriptstyle p_Y} \\ \Gamma & \xrightarrow[q]{} & Y \end{array}$$

commutative.

PROOF. *Case 1.* Let $X = [0,1]$. Then Γ is compact. Let us fix $t_0 \in [0,1]$. By Lemma (77.5), there exists an open set U, $\Gamma_{t_0} = p^{-1}(t_0) \subset U \subset \Gamma$ and a lift $\widetilde{q}\colon U \to \widetilde{V}$. Since Γ is compact, there exists $\varepsilon \geq 0$ satisfying $p^{-1}[t_0 - \varepsilon, t_0 + \varepsilon] \subset U$. Thus, for any to $t_0 \in [0,1]$, any lift $\widetilde{q}\colon \Gamma_{t_0} \to \widetilde{Y}$ extends onto $p^{-1}[t_0 - \varepsilon, t_0 + \varepsilon]$, for an $\varepsilon > 0$. Now, if we fix a sufficiently fine division $0 = t_0 < t_1 < \ldots < t_n = 1$, then we can extend any lift from Γ_0 onto $p^{-1}[t_0, t_1]$, and subsequently onto the whole $\Gamma = p^{-1}[0,1]$.

Case 2. $X = [0,1]$. The proof is similar.

Case 3. X is an arbitrary simply-connected space satisfying (77.2). Fix a point $x_0 \in X$ and a lift $\widetilde{q}_{x_0}\colon \Gamma_{x_0} \to \widetilde{Y}$. Let $x_1 \in X$ be another point. We choose a path $\omega\colon I \to X$ satisfying $\omega(i) = \chi_i$, for $i = 0, 1$. Now, we can apply Case 1 to the induced fibering $\omega^* = \{(z,t) \in \Gamma \times I \mid p(z) = \omega(t)\}$ and the obtained lift $\omega\colon \Gamma \to \widetilde{Y}$ defines, for $t = 1$, a lift on $\Gamma_{x_1} = \omega^*_{x_1} = \{(z,t) \in \omega^* \mid t = 1\}$. If we take another path ω' from x_0 to x_1, then there is a homotopy $H\colon I \times I \to Y$ joining these two paths, because Y is simply-connected. Now, (b) shows that both obtained lifts coincide. $\square$

Consider again a multivalued map $X \overset{p}{\longleftarrow} \Gamma \overset{q}{\longrightarrow} Y$ satisfying (77.2). Define

$$\widetilde{\Gamma} = \{(\widetilde{x}, z) \in \widetilde{X} \times \Gamma \mid p_X(\widetilde{x}) = p(z)\}$$

(a pullback). This gives the diagram

$$\begin{array}{ccccc} \widetilde X & \xleftarrow{\ \widetilde p\ } & \widetilde\Gamma & \xrightarrow{\ \widetilde q\ } & \widetilde Y \\ {\scriptstyle p_X}\big\downarrow & & {\scriptstyle p_\Gamma}\big\downarrow & & \big\downarrow{\scriptstyle p_Y} \\ X & \xleftarrow[\ p\ ]{} & \Gamma & \xrightarrow[\ q\ ]{} & Y \end{array}$$

where $\widetilde p(\widetilde x, z) = \widetilde x$ and $p_\Gamma(\widetilde x, z) = z$. Notice that the restrictions of $\widetilde p$ are homeomorphic on fibres.

Now, we can apply Lemma (77.5) to the multivalued map $\widetilde X \overset{\widetilde p}{\Longleftarrow} \widetilde\Gamma \overset{qp_\Gamma}{\longrightarrow} Y$, and so we get a lift $\widetilde q\colon \widetilde\Gamma \to \widetilde Y$ such that the diagram

$$\begin{array}{ccccc} \widetilde X & \overset{\widetilde p}{\Longleftarrow} & \widetilde\Gamma & \xrightarrow{\ \widetilde q\ } & \widetilde Y \\ {\scriptstyle p_X}\big\downarrow & & {\scriptstyle p_\Gamma}\big\downarrow & & \big\downarrow{\scriptstyle p_Y} \\ X & \underset{p}{\Longleftarrow} & \Gamma & \xrightarrow[\ q\ ]{} & Y \end{array}$$

is commutative. Let us note that the lift $\widetilde p$ is given by the above formula, but $\widetilde q$ is not precise. We fix such a $\widetilde q$.

Observe that $p\colon \Gamma \Rightarrow X$ and the lift $\widetilde p$ induce a homomorphism $\widetilde p^{\,!}\colon \theta_X \to \theta_\Gamma$ by the formula $\widetilde p^{\,!}(\alpha)(\widetilde x, z) = (\alpha\widetilde x, z)$. It is easy to check that the homomorphism $\widetilde p^{\,!}$ is an isomorphism (any natural transformation of $\widetilde\Gamma$ is of the form $\alpha\cdot(\widetilde x, z) = (\alpha\widetilde x, z)$) and that $\widetilde p^{\,!}$ is inverse to $\widetilde p_!$. Recall that the lift $\widetilde q$ defines a homomorphism $\widetilde q_!\colon \theta_\Gamma \to \theta_Y$ by the equality $\widetilde q(\lambda) = \widetilde q_!(\lambda)\widetilde q$.

In the sequel, we will consider the composition $\widetilde q_!\widetilde p^{\,!}\colon \theta_X \to \theta_Y$.

(77.7) LEMMA. *Let a multivalued map (p, q) satisfying (77.2) represent a single-valued map ρ, i.e. $q = \rho p$. Let $\widetilde\rho$ be the lift of ρ which satisfies $\widetilde q = \widetilde\rho\widetilde p$. Then $\widetilde\rho_!\widetilde p^{\,!} = \rho_!$.*

PROOF. $\widetilde\rho_!\widetilde p^{\,!} = (\widetilde\rho\widetilde p)_!\widetilde p^{\,!} = \widetilde\rho_!\widetilde p_!\widetilde p^{\,!} = \rho_!$. □

Now, we are in a position to define the Nielsen classes. Consider a multivalued self-map $X \overset{p}{\Longleftarrow} \Gamma \overset{q}{\longrightarrow} X$ satisfying (77.2). By the above consideration, we have a commutative diagram

$$\begin{array}{ccc} \widetilde\Gamma & \xrightarrow{\ \widetilde p, \widetilde q\ } & \widetilde X \\ {\scriptstyle p_\Gamma}\big\downarrow & & \big\downarrow{\scriptstyle p_X} \\ \Gamma & \xrightarrow[\ p,q\ ]{} & X \end{array}$$

Following the single-valued case (see [BJ-M]), we can prove

(77.8) Lemma.

(77.8.1) $C(p,q) = \bigcup_{\alpha\in\theta_X} p_\Gamma C(\widetilde{p},\alpha\widetilde{q})$,

(77.8.2) *if* $p_\Gamma C(\widetilde{p},\alpha\widetilde{q}) \cap p_\Gamma C(\widetilde{p},\beta\widetilde{q})$ *is not empty, then there exists a* $\gamma\in\theta_X$ *such that* $\beta = \gamma\circ\alpha\circ(\widetilde{q}_!\widetilde{p}^!\gamma)^{-1}$,

(77.8.3) *the sets* $p_\Gamma C(\widetilde{p},\alpha\widetilde{q})$ *are either disjoint or equal.*

Proof. (77.8.1) Let $p(z) = q(z)$ and $\widetilde{z}\in p_\Gamma^{-1}(z)$. Then $\widetilde{p}(\widetilde{z}), \widetilde{q(\widetilde{(z)})} \in p_X^{-1}(p(z))$. Thus, there exists $\alpha\in\theta_X$ such that $\widetilde{p}(\widetilde{z}) = \alpha\widetilde{q}(\widetilde{z})$, which implies $\widetilde{z}\in C(\widetilde{p},\alpha\widetilde{q})$.

(77.8.2) Let $z\in p_\Gamma C(\widetilde{p},\alpha\widetilde{q}) \cap p_\Gamma C(\widetilde{p},\beta\widetilde{q})$. Then there exist, $\widetilde{x}, \widetilde{x}'\in\widetilde{X}$ such that $(\widetilde{x},z)\in C(\widetilde{x},\alpha\widetilde{q})$, $(\widetilde{x}',z)\in C(\widetilde{x},\beta\widetilde{q})$ and $\widetilde{x} = \alpha\widetilde{q}(\widetilde{x},z)$, $\widetilde{x}' = \beta\widetilde{q}(\widetilde{x}',z)$. On the other hand, $\widetilde{p}_X\widetilde{x}' = pz$ implies $\widetilde{x}' = \gamma\widetilde{x}$, for a $\gamma\in\theta_X$. Thus,

$$\gamma\widetilde{x} = \widetilde{x}' = \beta\widetilde{q}(\widetilde{x}',z) = \beta\widetilde{q}(\gamma\widetilde{x},z) = \beta(\widetilde{q}_!\widetilde{p}^!\gamma)\widetilde{q}(\widetilde{x},z) = \beta(\widetilde{q}_!p^!\gamma)\alpha^{-1}(\widetilde{x}),$$

which implies $\gamma = \beta(\widetilde{q}_!p^!\gamma)\alpha^{-1}$ and $\beta = \gamma\alpha(\widetilde{q}_!p!\gamma)^{-1}$.

(77.8.3) It remains to prove that $p_\Gamma C(\widetilde{p},\alpha\widetilde{q}) = p_\Gamma C(\widetilde{p},\gamma\alpha(\widetilde{q}_!\widetilde{p}^!\gamma)^{-1}\widetilde{q})$. Let $(\widetilde{x},z)\in C(\widetilde{p},\gamma\alpha(\widetilde{q}_!p^!\gamma)^{-1}\widetilde{q})$. Then $\widetilde{x} = \gamma\alpha\widetilde{q}_!(\widetilde{p}^!\gamma)^{-1})\widetilde{q}(\widetilde{x},z)$, $\widetilde{x} = \gamma\alpha\widetilde{q}(\gamma^{-1}\widetilde{x},z)$. Hence, $\gamma^{-1}\widetilde{x} = \alpha\widetilde{q}(\gamma^{-1}\widetilde{x},z)$ and $\widetilde{p}(\gamma^{-1}\widetilde{x},z) = \alpha\widetilde{q}(\gamma^{-1}\widetilde{x},z)$. Thus, $p_!(\gamma^{-1})\cdot(\widetilde{x},z) = (\gamma^{-1}\widetilde{x},z)\in C(\widetilde{p},\alpha\widetilde{q})$, which implies $\widetilde{p}_\Gamma(\widetilde{x},z) = \widetilde{p}_\Gamma(p^!(\gamma)(\widetilde{x},z))\in p_\Gamma C(\widetilde{x},\alpha\widetilde{q})$. □

Define an action of θ_X on itself by the formula $\gamma\circ\alpha = \gamma\alpha(\widetilde{q}_!p^!\gamma)$. The quotient set will be called the *set of Reidemeister classes* and will be denoted by $R(p,q)$. The above lemma defines an injection

$$\text{Set of Nielsen classes} \to R(p,q),$$

given by $A\to[\alpha]\in R(p,q)$, where $\alpha\in\theta_X$ satisfies $A = p_\Gamma(C(\widetilde{p},\alpha\widetilde{q}))$.

Now, we are going to prove that our definition does not depend on $\widetilde{q}$.

Let us recall that the homomorphism $\widetilde{q}_!\colon\theta_\Gamma\to\theta_Y$ is defined by the relation $\widetilde{q}\alpha = \widetilde{q}_!(\alpha)\widetilde{q}$, for $\alpha\in\theta_\Gamma$. If $\widetilde{q}' = \gamma\widetilde{q}$ is another lift of q ($\widetilde{\gamma}\in\theta_\Gamma$), then the induced homomorphism $\widetilde{q}'_!\colon\theta_\Gamma\to\theta_Y$ is defined by the relation $\widetilde{q}'\alpha = \widetilde{q}'_!(\alpha)\widetilde{q}'$.

(77.9) Lemma. *If* $\widetilde{q}' = \gamma\cdot\widetilde{q}$ *is another lift of* q*, then* $\gamma\cdot\widetilde{q}'_!(\alpha)\cdot\gamma^{-1} = \widetilde{q}'_!(\alpha)$*, for all* $\alpha\in\theta_\Gamma$*.*

Proof. The equalities $\widetilde{q}' = \gamma\cdot\widetilde{q}$ and $\widetilde{q}'(\alpha\widetilde{u}) = \widetilde{q}'_!(\alpha)\widetilde{q}'(\widetilde{u})$ imply $\gamma\cdot\widetilde{q}(\alpha\widetilde{u}) = \widetilde{q}'_!(\alpha)\cdot\gamma\cdot\widetilde{q}(\widetilde{u})$, by which $\gamma\cdot\widetilde{q}_!(\alpha)\cdot\widetilde{q}(\widetilde{u}) = \widetilde{q}'_!(\alpha)\cdot\gamma\cdot\widetilde{q}'(\widetilde{u})$. Thus, $\gamma\cdot\widetilde{q}_!(\alpha) = \widetilde{q}'_!(\alpha)\cdot\gamma$ and finally $\widetilde{q}'_!(\alpha)\cdot\widetilde{q}(\widetilde{u}) = \gamma\cdot\widetilde{q}_!(\alpha)\cdot\gamma^{-1}$. □

(77.10) Theorem. *Let us fix two lifts* $\widetilde{q}$ *and* $\widetilde{q}'$*. Let* $\gamma\in\theta_X$ *denote the unique transformation satisfying* $\widetilde{q} = \gamma\cdot\widetilde{q}'$*. Then* $\alpha,\beta\in\theta_X$ *are in the Reidemeister relation with respect to* $\widetilde{q}$ *if and only if so are* $\alpha\cdot\gamma^{-1}, \beta\cdot\gamma^{-1}$ *with respect to* $\widetilde{q}'$*.*

PROOF. Suppose that $\beta = \delta\cdot\alpha\cdot\widetilde{q}_!\widetilde{p}^!(\delta^{-1})$. Then $\beta\cdot\gamma^{-1} = \delta\cdot\alpha\cdot\widetilde{q}_!\widetilde{p}^!(\delta^{-1})\gamma^{-1} = \delta\cdot\alpha\cdot\gamma^{-1}(\gamma\cdot\widetilde{q}_!\widetilde{p}^!(\delta^{-1})\gamma^{-1}) = \delta(\alpha\cdot\gamma^{-1})\widetilde{q}_!\widetilde{p}^!(\delta^{-1})\gamma^{-1})\delta\cdot\alpha\cdot\gamma^{-1}\cdot\gamma\cdot\widetilde{q}_!\widetilde{p}^!(\delta^{-1})$. Then

$$\delta\cdot\alpha\cdot\gamma^{-1}\cdot\gamma\cdot\widetilde{q}_!\widetilde{p}^!(\delta^{-1}) = \delta(\alpha\cdot\gamma^{-1})\widetilde{q}'_!\widetilde{p}^!(\delta^{-1}). \qquad \square$$

The above consideration shows that the Reidemeister sets obtained by different lifts of q are canonically isomorphic. That is why we write $R(p,q)$ omitting tildes.

(77.11) THEOREM. *If $X\times\mathcal{I} \xleftarrow{p} \Gamma \xrightarrow{q} Y$ is a homotopy satisfying (77.2)–(77.4), then the homomorphism $\widetilde{q}_{t!}\widetilde{p}_t^{\,!}\colon\theta_X \to \theta_Y$ does not depend on $t\in[0,1]$, where the lifts used in the definitions of these homomorphisms are restrictions of some fixed lifts p, q of the given homotopy.*

PROOF. The commutative diagram of maps

$$\begin{array}{ccccc}
X & \overset{p_t}{\Longleftarrow} & \Gamma_t & \overset{q_t}{\longrightarrow} & Y \\
{\scriptstyle i_{X,t}}\downarrow & & {\scriptstyle i_{\Gamma,t}}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\
X\times\mathcal{I} & \underset{p}{\Longleftarrow} & \Gamma & \underset{q}{\longrightarrow} & Y
\end{array}$$

where $i_{X,t}(x) = (x,t)$, $i_{\Gamma,t}(\widetilde{x},z) = (\widetilde{x},t,z)$, induces the commutative diagram of homomorphisms

$$\begin{array}{ccccc}
\theta_X & \overset{p_t^!}{\longrightarrow} & \theta_{\Gamma_t} & \overset{q_{t!}}{\longrightarrow} & \theta_Y \\
{\scriptstyle (i_{X,t})^!}\uparrow & & {\scriptstyle (i_{\Gamma,t})^!}\uparrow & & \uparrow{\scriptstyle \mathrm{id}} \\
\theta_{X\times\mathcal{I}} & \underset{p^!}{\longrightarrow} & \theta_\Gamma & \underset{q_!}{\longrightarrow} & \theta_Y
\end{array}$$

and it remains to notice that $(i_{x,t})^!\colon\theta_{X\times\mathcal{I}}\to\theta_X$ is an isomorphism. $\square$

(77.12) REMARK. If (p,q) represents a single-valued map $\rho\colon X\to Y(q=\rho p)$, then $\widetilde{q}_!\widetilde{p}^!$ equals $\widetilde{\rho}_!$ (here the chosen lifts satisfy $\widetilde{q} = \widetilde{\rho}\widetilde{p}$).

Indeed. Let us fix a point $(\widetilde{x},z)\in\widetilde{\Gamma}$, and $\alpha\in\theta_X$. Then

$$\widetilde{q}(\alpha\widetilde{x},z) = \widetilde{\rho}\widetilde{p}(\alpha\widetilde{x},z) = \widetilde{\rho}(\alpha\widetilde{x}) = \widetilde{\rho}_!(\alpha)\widetilde{\rho}(\widetilde{x}) = \widetilde{\rho}_!(\alpha)\widetilde{\rho}\widetilde{p}(\widetilde{x},z) = \widetilde{\rho}_!(\alpha)\widetilde{q}(\widetilde{x},z).$$

On the other hand, $\widetilde{q}(\alpha\widetilde{x},z) = \widetilde{q}(\widetilde{p}_!(\alpha)(\widetilde{x},z)) = \widetilde{q}_!\widetilde{p}_!(\alpha)\cdot\widetilde{q}(\widetilde{x},z)$. Since the natural transformations $\widetilde{\rho}_!(\alpha)$, $\widetilde{q}_!\widetilde{p}_!(\alpha)\in\theta_Y$ coincide at the point $\widetilde{q}(\widetilde{x},z)\in\widetilde{Y}$, they are equal.

Below we shall define the Nielsen relation modulo a subgroup.

Let us point out that the above theory can be modified onto the relative case. Consider again a multivalued pair (p,q) satisfying (77.2)–(77.4). Let $H\subset\theta_X$, $H'\subset\theta_Y$ be normal subgroups. Then the action of H on $\widetilde{X}$ gives the quotient space $\widetilde{X}_H$ and the map $p_{XH}\colon \widetilde{X}_H \to X$ is also a covering. Similarly, we get $p_{YH'}\colon \widetilde{Y}'_H \to Y$. On the other hand, the action of H on $\widetilde{\Gamma}$ given by $h\circ(\widetilde{x}, z) = (h\widetilde{x}, z)$ determines the quotient space $\widetilde{\Gamma}_H$ with the natural map $\widetilde{p}_H\colon \widetilde{\Gamma}_H \to \widetilde{X}_H$ induced by $\widetilde{p}$. Assume that $\widetilde{q}_!\widetilde{p}^!(H) \subset H'$. Observe that this condition does not depend on the choice of the lifts $\widetilde{p}$, $\widetilde{q}$, because the subgroups H, H' are the normal divisors. Thus, $\widetilde{q}\colon \widetilde{\Gamma} \to \widetilde{Y}$ induces a map $\widetilde{q}_H\colon \widetilde{\Gamma}_H \to \widetilde{Y}_{H'}$ and the diagram

$$\begin{array}{ccccc}
\widetilde{X}_H & \overset{\widetilde{p}_H}{\Longleftarrow} & \widetilde{\Gamma}_H & \overset{\widetilde{q}_H}{\longrightarrow} & \widetilde{Y}_H \\
{\scriptstyle p_{XH}}\big\downarrow & & {\scriptstyle p_{\Gamma H}}\big\downarrow & & \big\downarrow{\scriptstyle p_{YH'}} \\
X & \underset{p}{\Longleftarrow} & \Gamma & \underset{q}{\longrightarrow} & Y
\end{array}$$

commutes. Now, we can get the homomorphisms $\widetilde{q}_{H!}\widetilde{p}_H{}^!\colon \theta_{XH} \to \theta_{YH'}$, where θ_{XH}, $\theta_{YH'}$ denote the groups of natural transformations of $\widetilde{X}_H$ and $\widetilde{Y}_{H'}$, respectively.

Assuming $X = Y$ and $H = H'$. We can give

(77.13) Lemma.

(77.13.1) $C(p,q) = \bigcup_{\alpha\in\theta_{XH}} p_{\Gamma H} C(\widetilde{p}_H, \alpha\widetilde{q}_H)$,

(77.13.2) *if* $p_{\Gamma H}C(\widetilde{p}_H, \alpha\widetilde{q}_H) \cap p_{\Gamma H}C(\widetilde{p}_H, \beta\widetilde{q}_H)$ *is not empty, then there exists a* $\gamma \in \theta_{XH}$ *such that* $\beta = \gamma\circ\alpha\circ(\widetilde{q}_{H!}\widetilde{p}_H{}^!\gamma)^{-1}$,

(77.13.3) *the sets* $p_{\Gamma H}C(\widetilde{p}_H, \alpha\widetilde{q}_H)$ *are either disjoint or equal.*

Hence, we get the splitting of $C(p,q)$ into the H-Nielsen classes and the natural injection from the set of H-Nielsen classes into the set of Reidemeister classes modulo H, namely, $R_H(p,q)$.

Now, we would like to exhibit the classes which do not disappear under any (admissible) homotopy. For this, we need however (besides (77.2)–(77.4)) the following two assumptions on the pair $X \overset{p}{\Longleftarrow} \Gamma \overset{q}{\longrightarrow} Y$.

(77.14) Let X be a connected retract of an open set in a paracompact locally convex space, p is a Vietoris map and $\mathrm{cl}(q(\Gamma)) \subset X$ is compact, i.e. q is a compact map.

(77.15) There exists a normal subgroup $H \subset \theta_X$ of a finite index satisfying

$$\widetilde{q}_!\widetilde{p}^!(H) \subset H.$$

(77.16) Definition. We call a pair (p,q) *N-admissible* if it satisfies (77.2)–(77.4), (77.14) and (77.15).

(77.17) Remark. The pairs satisfying (77.14) are called admissible.

Let us recall that, under the assumption (77.14), the *Lefschetz number* $\Lambda(p,q) \in \mathbb{Q}$ is defined. This is a homotopy invariant (with respect to the homotopies satisfying (77.14)) and $\Lambda(p,q) \neq 0$ implies $C(p,q) \neq \emptyset$ (comp. Section 6).

The assumption (77.15) gives rise to the commutative diagram

$$\begin{array}{ccccc}
\widetilde{X}_H & \xLeftarrow{\;\widetilde{p}_H\;} & \widetilde{\Gamma}_H & \xrightarrow{\;\widetilde{q}_H\;} & \widetilde{Y}_H \\
{\scriptstyle p_{XH}}\downarrow & & {\scriptstyle p_{\Gamma H}}\downarrow & & \downarrow{\scriptstyle p_{YH'}} \\
X & \xLeftarrow[\;p\;]{} & \Gamma & \xrightarrow[\;q\;]{} & Y
\end{array}$$

where the coverings p_{XH}, $p_{\Gamma H}$, p_{YH} are finite, because the subgroup $H \in \theta_{XH}$ has a finite index. Now, we can observe that the pair $(\widetilde{p}_H, \alpha\widetilde{q}_H)$, for any $\alpha \in \theta_{XH}$, also satisfies (77.14) ($\widetilde{p}^{-1}(\widetilde{x}) = p^{-1}(x)$, $\mathrm{cl}(\alpha\widetilde{q}_H(\widetilde{\Gamma}_H)) \subset p_{XH}^{-1}(\mathrm{cl}\, q(\Gamma))$ and the last set is compact, because the covering p_{XH} is finite).

Let $A = p_{\Gamma H} C(\widetilde{p}, \alpha\widetilde{q})$ be a Nielsen class of an N-admissible pair (p,q). We say that (the N-Nielsen class) A is *essential* if $\Lambda(\widetilde{p}, \alpha\widetilde{q}) \neq 0$. The following lemma explains that this definition it correct, i.e. does not depend on the choice of α.

(77.18) LEMMA. *If $p_{\Gamma H} C(\widetilde{p}, \alpha\widetilde{q}) = p_{\Gamma H} C(\widetilde{p}, \alpha'\widetilde{q}) \neq \emptyset$, for some $\alpha, \alpha' \in \theta_{XH}$, then $\Lambda(\widetilde{p}, \alpha\widetilde{q}) = \Lambda(\widetilde{p}, \alpha'\widetilde{q})$.*

PROOF. Since α, α' represent the same element in $R_H(p,q)$, there exists $\gamma \in \theta_{XH}$ such that $\alpha' = \gamma \circ \alpha \circ \widetilde{q}_! \widetilde{p}^!(\gamma^{-1})$. Thus,

$$\begin{aligned}
\Lambda(\widetilde{p}, \alpha'\widetilde{q}) &= \mathrm{Tr}((\widetilde{p}^*)^{-1}(\alpha' \cdot \widetilde{q})^*) = \mathrm{Tr}((\widetilde{p}^*)^{-1}(\gamma \circ \alpha \circ (\widetilde{q}_! \widetilde{p}^!(\gamma^{-1})) \circ \widetilde{q})^*) \\
&= \mathrm{Tr}((\widetilde{p}^*)^{-1}(\gamma \circ \alpha \circ \widetilde{q}_!(\widetilde{p}^!(\gamma^{-1})))^*) \\
&= \mathrm{Tr}((\widetilde{p}^*)^{-1}((\widetilde{p}^!(\gamma^{-1})))^* \circ (\alpha\widetilde{q})^* \circ \gamma^*) \\
&= \mathrm{Tr}((\widetilde{p}^* \widetilde{p}^!(\gamma))^*)^{-1} \circ (\alpha\widetilde{q})^* \circ \gamma^*) = \mathrm{Tr}((\widetilde{\gamma}^*)^{-1}(\widetilde{p}^*)^{-1} \circ (\alpha\widetilde{q})^* \circ \gamma^*) \\
&= \mathrm{Tr}((\widetilde{p}^*)^{-1} \circ (\alpha\widetilde{q})^*) = \Lambda(\widetilde{p}, \alpha\widetilde{q}). \qquad \square
\end{aligned}$$

(77.19) DEFINITION. Let (p,q) be an N-admissible multivalued map (for a subgroup $H \subset \theta_X$). We define the *Nielsen number* modulo H as the number of essential classes in θ_{X_H}. We denote this number by $N_H(p,q)$.

(77.20) REMARK. Observe that the above method allows us to define only essential classes (and the Nielsen number) modulo a subgroup of a finite index in $\theta_X = \pi_1 X$. The problem how to get similar notions in an arbitrary case we leave open.

The following theorem is an easy consequence of the homotopy invariance of the Lefschetz number.

(77.21) THEOREM. *$N_H(p,q)$ is a homotopy invariant (with respect to N-admissible homotopies) $X \times [0,1] \overset{p}{\Longleftarrow} \Gamma \overset{q}{\longrightarrow} X$. Moreover, (p,q) has at least $N_H(p,q)$ coincidences.*

The following theorem shows that the above definition is consistent with the classical Nielsen number for single-valued maps.

(77.22) THEOREM. *If an N-admissible map (p,q) is N-admissibly homotopic to a pair (p',q'), representing a single-valued map p (i.e. $q' = \rho p'$), then (p,q) has at least $N_H(\rho)$ coincidences (here H denotes also the subgroup of $\pi_1 X$ corresponding to the given $H \subset \theta_X$ in (77.4)).*

PROOF. Consider a covering space $p\colon \widetilde{X}_H \to X$, corresponding to H. So, ρ admits a lift $\widetilde{\rho}\colon \widetilde{X}_H \to \widetilde{X}_H$ and in the diagram

$$\begin{array}{ccccc} \widetilde{X}_H & \xleftarrow{\ \widetilde{p}'\ } & \widetilde{\Gamma}_H & \xrightarrow{\ \widetilde{q}'\ } & \widetilde{Y}_H \\ {\scriptstyle p_{XH}}\big\downarrow & & {\scriptstyle p_{\Gamma H}}\big\downarrow & & \big\downarrow{\scriptstyle p_{YH'}} \\ X & \xleftarrow[\ p'\ ]{} & \Gamma & \xrightarrow[\ q'\ ]{} & Y \end{array}$$

we can put $\widetilde{q}' = \widetilde{\rho}\widetilde{p}'$. Thus, a homotopy between (p',q') and (p,q) lifts onto the coverings and we get lifts $(\widetilde{p},\widetilde{q})$. Since

$$\widetilde{q}_! \widetilde{p}^{\,!} = (\widetilde{q}')_! (\widetilde{p}')^{\,!} = (\widetilde{\rho}\widetilde{p}')_! \widetilde{p}'^{\,!} = \widetilde{\rho},$$

there is a natural bijection between the Reidemeister sets $R(\rho)$ and $R(p,q)$. It remains to show that the essential classes correspond to the essential classes in the both Reidemeister sets. Consider a class $[\alpha] \in R_H(\rho)$. This class is essential if and only if the index of $p_H(\mathrm{Fix}(\alpha\widetilde{\rho}))$ is non-zero. But $\mathrm{ind}\,(\alpha\widetilde{\rho}) = \Lambda(\alpha\widetilde{\rho})$ is a non-zero multiplicity of $\mathrm{ind}\,(p_{XH}(\mathrm{Fix}(\alpha\widetilde{\rho})))$, i.e. it is also non-zero. Thus,

$$0 \neq \Lambda(\alpha\widetilde{\rho}) = \Lambda(\widetilde{p}', \alpha\widetilde{\rho}'\widetilde{p}') = \Lambda(\widetilde{p}', \alpha\widetilde{q}') = \Lambda(\widetilde{p}, \alpha\widetilde{q}),$$

and $[\alpha] \in R_H(p,q)$ is also essential. □

Although in the general case the theory, presented in the previous sections, requires special assumptions on the considered pair (p,q), we shall see that in the case of multivalued self-maps on a torus it is enough to assume that this pair satisfies only (77.9), i.e. it is admissible. We will do this by showing that in the case of any pair satisfying (77.9), it is homotopic to a pair representing a single-valued map.

(77.23) LEMMA. *For any compact space X, if $\widetilde{H}^1(X;\mathbb{Q}) = 0$, then*

$$\widetilde{H}^1(X;\mathbb{Z}) = 0.$$

PROOF. Recall that $\widetilde{H}^k(X;\mathbb{Q}) = \varinjlim H^k(N(\alpha);\mathbb{Q})$, where $N(\alpha)$ denotes the nerve of a covering α. Since X is compact, we can consider only finite coverings. So, by the Universal Coefficient Formula (see [Sp-M, Theorem 5.5.10]), the natural homomorphism

$$H^k(X_\alpha;\mathbb{Z}) \otimes \mathbb{Q} \to H^k(X_\alpha;\mathbb{Q})$$

is an injection. Since $\widetilde{H}^k(X;\mathbb{Q}) = \varinjlim H^k(N(\alpha);\mathbb{Q})$ and the direct limit functor is exact, the homomorphism $\widetilde{H}^q(X;\mathbb{Z}) \otimes \mathbb{Q} \to \widetilde{H}^k(X;\mathbb{Q})$ is also mono. Thus, $\widetilde{H}^q(X;\mathbb{Z}) \otimes \mathbb{Q} = 0$, which implies that any element in $\widetilde{H}^q(X;\mathbb{Z})$ is a torsion. By another Universal Coefficient Formula (see [Sp-M, Theorem 5.5.3]),

$$H^1(X_\alpha;\mathbb{Z}) = \mathrm{Hom}(H_1(X_\alpha;\mathbb{Z},\mathbb{Z}) \otimes \mathrm{Ext}(H_0(X_\alpha;\mathbb{Z}),\mathbb{Z}).$$

Since $H_0(X_\alpha;\mathbb{Z})$ is free, $\mathrm{Ext} = 0$. Now, $H^1(X_\alpha;\mathbb{Z}) = \mathrm{Hom}(H_1(H_\alpha;\mathbb{Z}),\mathbb{Z})$ is torsion free and $\widetilde{H}^1(X;\mathbb{Z})$, as the direct limit of torsion free groups, is also torsion free. Therefore, $\widetilde{H}^1(X;\mathbb{Z})$ must be zero. □

(77.24) THEOREM. *Any multivalued self-map (p,q) on the torus satisfying* (77.9) *is admissible homotopic to a pair representing a single-valued map.*

PROOF. At first, we prove that (p,q) satisfies (77.4), i.e. that the restriction $q\colon p^{-1}(x) \to \mathbb{T}^n$ admits a lift to the universal cover $\mathbb{R}^n \to \mathbb{T}^n$ (for any $x \in \mathbb{T}^n$). It is enough to show that any such restriction is contractible. On the other hand, since any map into the n-torus $\mathbb{T}^n = S^1 \times \ldots \times S^1$ splits into n-maps into the circle, it is enough to show that any map from $p^{-1}(x)$ to S^1 is contractible. By the well-known Hopf theorem (see e.g. [Sp-M]),

$$[p^{-1}(x), S^1] = \widetilde{H}^1(p^{-1}(x);\mathbb{Z}).$$

On the other hand, Lemma (77.23) implies that $\widetilde{H}^1(p^{-1}(x);\mathbb{Z}) = 0$. Thus, any restriction $q\colon p^{-1}(x) \to S^1$ is contractible. □

Now, we prove that $X \overset{p}{\Longleftarrow} \Gamma \overset{q}{\longrightarrow} \mathbb{T}^n$ is homotopic to a pair representing a single-valued map. By the above consideration, there is a commutative diagram

$$\begin{array}{ccccc}
\widetilde{X} & \overset{\widetilde{p}}{\Longleftarrow} & \widetilde{\Gamma} & \overset{\widetilde{q}}{\longrightarrow} & \mathbb{R}^n \\
{\scriptstyle p_X}\downarrow & & {\scriptstyle p_\Gamma}\downarrow & & \downarrow{\scriptstyle p_T} \\
X & \underset{p}{\Longleftarrow} & \Gamma & \underset{q}{\longrightarrow} & \mathbb{T}^n
\end{array}$$

which gives rise to the induced homomorphism $\widetilde{q}_!\widetilde{p}^!\colon \theta_X \to \theta_{\mathbb{T}^n}$. Since the torus is a $K(\pi,1)$ space and there is an isomorphism between the groups θ_X and $\pi_1 X$, there exists a single-valued map $\rho\colon X \to \mathbb{T}^n$ such that the induced homomorphism $\widetilde{\rho}\colon \theta_X \to \theta_{\mathbb{T}^n}$ coincides with $\widetilde{q}_!\widetilde{p}^!$. We will show that (p,q) is homotopic to $(p,\rho p)$. Define a homotopy $\widetilde{q}_t\colon \widetilde{\Gamma} \to \mathbb{R}^n$ by putting

$$\widetilde{q}_t(\widetilde{x},z) = (1-t)\widetilde{q}(\widetilde{x},z) + t\widetilde{\rho}(\widetilde{x}).$$

The equalities

$$\begin{aligned}
\widetilde{q}_t\widetilde{p}^!(\gamma)(\widetilde{x},z) &= (1-t)\widetilde{q}(\widetilde{x},z) + t\widetilde{\rho}(\widetilde{x}) = (1-t)\widetilde{q}(p^!(\gamma)(\widetilde{x},z)) + t\widetilde{\rho}\widetilde{p}(\widetilde{p}^!(\gamma)(\widetilde{x},z))\\
&= (1-t)\widetilde{q}_!p^!(\gamma)\widetilde{q}(\widetilde{x},z) + t\widetilde{\rho}(\widetilde{p}_!(\widetilde{p}^!(\gamma))\widetilde{p}(\widetilde{x},z)\\
&= (1-t)\widetilde{q}_!p^!(\gamma)\widetilde{q}(\widetilde{x},z) + t\widetilde{\rho}(\gamma)\widetilde{p}(\widetilde{x},z)\\
&= \widetilde{\rho}_!(\gamma)[(1-t)\widetilde{q}(\widetilde{x},z) + t\widetilde{\rho}(\widetilde{x})] = \widetilde{\rho}_!(\gamma)\widetilde{q}_t(\widetilde{x},z)
\end{aligned}$$

verify that $\widetilde{q}_t\widetilde{p}^!(\gamma)(\widetilde{x},z) = \widetilde{\rho}_!(\gamma)\widetilde{q}_t(\widetilde{x},z)$.

Since any natural transformation on $\widetilde{\Gamma}$ is of the form, $\widetilde{p}^!(\gamma)$ for some $\gamma \in \theta_X$, $\widetilde{q}_t(\gamma)$ induces a homotopy $q_t\colon \Gamma \to \mathbb{T}^n$ for which the diagram

$$\begin{array}{ccccc}
\widetilde{X} & \overset{\widetilde{p}}{\Longleftarrow} & \widetilde{\Gamma} & \overset{\widetilde{q}_t}{\longrightarrow} & \mathcal{R}E^n\\
{\scriptstyle p_X}\downarrow & & {\scriptstyle p_\Gamma}\downarrow & & \downarrow{\scriptstyle p_T}\\
X & \underset{p}{\Longleftarrow} & \Gamma & \underset{q_t}{\longrightarrow} & \mathbb{T}^n
\end{array}$$

commutes and the obtained homotopy satisfies (77.2) ($\widetilde{p}$ does not vary). For $t=1$, we get $\widetilde{q}_1(\widetilde{x},z) = \rho(\widetilde{x}) = \rho\widetilde{p}(\widetilde{x},z)$ which implies $q_1(z) = \rho p(z)$.

(77.25) Theorem. *Let* $\mathbb{T}^n \overset{p}{\Longleftarrow} \Gamma \overset{q}{\longrightarrow} \mathbb{T}^n$ *be such that* p *is a Vietoris map. Let* $\rho\colon \mathbb{T}^n \to \mathbb{T}^n$ *be a single-valued map representing a multivalued map homotopic to* (p,q) (*according to Theorem* (77.24), *such a map always exists*). *Then* (p,q) *has at least* $N(\rho)$ *coincidences.*

Proof. Let us recall (cf. [BBPT]) that $N(\rho) = |\Lambda(\rho)| = |\det(I-A)|$, where A is an integer $(n\times n)$-matrix representing the induced homotopy homomorphism $\rho_\#\colon \pi_1\mathbb{T}^n \to \pi_1\mathbb{T}^n$. Moreover, if $\det(I-A) \neq 0$, then $\operatorname{card}(\pi_1(\mathbb{T}^n)/\operatorname{Im}(\rho_\#)) = |\det(I-A)|$.

The case $N(\rho)=0$ is obvious. Assume that $N(\rho)\neq 0$. By Theorem (77.24), it is enough to find a subgroup (of a finite index) $H \subset \pi_1\mathbb{T}^n = \mathbb{Z}^n$ satisfying

(77.24.1) $\rho_\#(H) \subset H$, and
(77.24.2) $N_H(\rho) = N(\rho)$.

We define $H = \{z - \rho_{\#}(x) \mid x \in \pi_1 X\}$. Then (77.24.1) is clear.

Recall that, for any endomorphism of an abelian group $\rho\colon G \to G$, the Reidemeister set is the quotient group $R(\rho) = G/(\mathrm{Im}(\mathrm{id} - \rho)$. In our case, $H = \mathrm{Im}(\mathrm{id} - \rho)$ and the natural map $G \to G/H$ induces the bijection between $R(\rho) = G/H$ and $R_H(\rho) = (G/H)/(H/H)$. Thus, we get the bijection $R_H(\rho) = R(\rho)$. Finally, we notice that all the Nielsen classes of ρ have the same index ($= \mathrm{sign}(\det(I - A))$. Thus, all involving classes in $R_H(\rho)$ in $R(\rho)$ are essential, which proves $N_H(\rho) = N(\rho)$. □

Now, we shall generalize the above construction from the compact case into the case of CAC-morphism.

As in the single-valued case, the definition of a Nielsen number is done in two stages: at first, $C(p, q)$ is split into disjoint classes (Nielsen classes) and then we define essential classes.

Fix a universal covering ([4]) $p_X\colon \widetilde{X} \to X$. We define $\widetilde{\Gamma} = \{(\widetilde{x}, z) \in \widetilde{X} \times \Gamma \mid p_X(\widetilde{x}) \supset p(z)\}$ (pullback) and the map $\widetilde{p}\colon \widetilde{\Gamma} \to \widetilde{X}$ by $\widetilde{p}(\widetilde{x}, z) = \widetilde{x}$.

Since in our case X is a connected ANR the following property is satisfied automatically.

PROPERTY A. *For any $x \in X$, the restriction $q_1 = q|_{p^{-1}(x)}\colon p^{-1}(x) \to X$ admits a lift $\widetilde{q}_1$, making the diagram*

$$\begin{array}{ccc} & \overset{\widetilde{q}_1}{\nearrow} & \widetilde{X} \\ & & \big\downarrow p_X \\ p^{-1}(x) & \xrightarrow[q_1]{} & X \end{array}$$

commutative.

(77.26) REMARK. Note that a sufficient condition for guaranteeing Property A is, for example, that $p^{-1}(x)$ is an ∞-proximally connected set, for every $x \in X$ (see [KM]). It is well-known (see (2.21)) that, on ANR-spaces, any ∞-proximally connected compact (nonempty) subset is an R_δ-set and vice versa.

(77.27) LEMMA. *If (p, q) satisfies* (CAC + A)*, then there is a lift $\widetilde{q}\colon \Gamma \to \widetilde{X}$ making commutative the diagram*

$$\begin{array}{ccccc} \widetilde{X} & \xleftarrow{\ \widetilde{q}\ } & \widetilde{\Gamma} & \xrightarrow{\ \widetilde{p}\ } & \widetilde{X} \\ {\scriptstyle p_X}\big\downarrow & & \big\downarrow{\scriptstyle p_\Gamma} & & \big\downarrow{\scriptstyle p_X} \\ X & \xleftarrow[\ p\ ]{} & \Gamma & \xrightarrow[\ q\ ]{} & X \end{array}$$

([4]) In what follows, we shall assume that $\widetilde{X}$ is a retract of an open set in a locally convex space. For metric ANRs, this assumption is satisfied automatically (see the proof of Lemma 2.5 in [AGJ2]).

Proof. Notice that the assumptions (10.2)–(10.4) in Section 10 are satisfied. Let

$$\theta_X = \{\alpha\colon \widetilde{X} \to \widetilde{X} \mid p_X\alpha = p_X\}$$

denote the group of covering transformations of the covering $\widetilde{X}$. Similarly, we define θ_Γ.

The lifts $\widetilde{p}$, $\widetilde{q}$ define homomorphisms:

$$\begin{aligned} \widetilde{p}^{\,!}\colon \theta_X &\to \theta_\Gamma \quad \text{by the formula } p^{\,!}(\alpha)(\widetilde{x}, z) = (\alpha\widetilde{x}, z),\\ \widetilde{q}_!\colon \theta_\Gamma &\to \theta_X \quad \text{by the equality } \widetilde{q}\cdot\alpha = \widetilde{q}_!(\alpha)\cdot\widetilde{q}. \end{aligned}$$

Let us recall that θ_X is isomorphic with $\pi_1(X)$ and if (p,q) represents a single-valued map (i.e. $qp^{-1}(x) = \eta(x)$, for a single-valued map $\eta\colon X \to X$), then $\widetilde{q}_!\widetilde{p}^{\,!}\colon \theta_X \to \theta_X$ is equal to the homomorphism $\widetilde{\eta}_!\colon \theta_X \to \theta_X$ given by $\widetilde{\eta}\cdot\alpha = \widetilde{\eta}_!(\alpha)\cdot\widetilde{\eta}$, where $\widetilde{\eta}$ is given by the formula $\widetilde{\eta}(\widetilde{x}) = \widetilde{q}\widetilde{p}^{-1}(x)$. However, the homomorphism $\widetilde{\eta}_!$ corresponds to the induced map $\eta_\#\colon \pi_1(X) \to \pi_1(X)$.

Thus, the composition $\widetilde{q}_!\widetilde{p}^{\,!}\colon \theta_X \to \theta_X$ can be considered as a generalization of the induced homotopy homomorphism. □

Property B. *There is a normal subgroup $H \subset \theta_X$ of a finite index (θ_X/H-finite), invariant under the homomorphism $q_!p^{\,!}$ ($q_!p^{\,!}(H) \subset H$).*

(77.28) Remark. In particular, if X is a connected space such that the fundamental group $\pi_1(X)$ of X is abelian and finitely generated, then X satisfies Property B (see [Sp-M]). Observe also that if (p,q) is admissibly homotopic to a single-valued map f, then Property B holds true (see Section 10).

Let us note that $(\mathrm{CAC} + \mathrm{A} + \mathrm{B})$ makes the diagram

$$\begin{array}{ccccc} \widetilde{X}_H & \xleftarrow{\ \widetilde{q}_H\ } & \widetilde{\Gamma}_H & \xrightarrow{\ \widetilde{p}_H\ } & \widetilde{X}_H \\ \downarrow{\scriptstyle p_{XH}} & & \downarrow{\scriptstyle p_{\Gamma H}} & & \downarrow{\scriptstyle p_{XH}} \\ X & \xleftarrow[\ p\ ]{} & \Gamma & \xrightarrow[\ q\ ]{} & X \end{array}$$

commutative, where $p_{XH}\colon \widetilde{X}_H \to X$ is a covering corresponding to the normal subgroup $H\Delta\theta_X \sim \pi_1 X$ and $\widetilde{\Gamma}_H$ is a pullback. As above, we can define homomorphisms $\widetilde{p}_H^{\,!}\colon \theta_{XH} \to \theta_{\Gamma H}$, $\widetilde{q}_{H!}\colon \theta_{\Gamma H} \to \theta_{XH}$, where $\theta_{XH} = \{\alpha\colon \widetilde{X}_H \to \widetilde{X}_H \mid p_{XH}\alpha = p_{XH}\}$.

(77.29) Lemma. *We have:*

(77.29.1) $C(p,q) = \bigcup_{\alpha\in\theta_{XH}} p_{\Gamma H} C(\widetilde{p}_H, \alpha\widetilde{q}_H)$,

(77.29.2) *if $p_{\Gamma H}C(\widetilde{p}_H, \alpha\widetilde{q}_H) \cap p_{\Gamma H}C(\widetilde{p}_H, \beta\widetilde{q}_H)$ is not empty, then there exists a $\gamma \in \theta_{XH}$ such that $\beta = \gamma\circ\alpha\circ(\widetilde{q}_{H!}\widetilde{p}_H^{\,!}\gamma)^{-1}$,* (77.29.3) *the sets* $p_{\Gamma H}C(\widetilde{p}_H, \alpha\widetilde{q}_H)$ *are either disjoint or equal.*

Thus, $C(p,q)$ *splits into disjoint subsets* $p_{\Gamma H}C(\widetilde{p}_H, \alpha \circ \widetilde{q}_H)$ *called Nielsen classes modulo a subgroup* H*.*

Now, we shall define essential classes. We consider the diagram

$$\begin{array}{ccccc} \widetilde{X}_H & \xleftarrow{\alpha\widetilde{q}_H} & \widetilde{\Gamma}_H & \xrightarrow{\widetilde{p}_H} & \widetilde{X}_H \\ \downarrow{\scriptstyle p_{XH}} & & \downarrow{\scriptstyle p_{\Gamma H}} & & \downarrow{\scriptstyle p_{XH}} \\ X & \xleftarrow[p]{} & \Gamma & \xrightarrow[q]{} & X \end{array}$$

(77.30) Lemma. *The multivalued map* $(\widetilde{p}_H, \widetilde{q}_H)$ *is a* CAC.

Proof. Since $\widetilde{p}_{\Gamma H}$ is a homeomorphism between $\widetilde{p}_H^{-1}(x)$ and $\widetilde{p}^{-1}(px)$, $\widetilde{p}_H$ is Vietoris. If $U \subset X$ satisfies the definition of the CAC for (p,q), then $\widetilde{U} = p_{XH}^{-1}(U)$ satisfies the same for $(\widetilde{p}_H, \alpha\widetilde{q}_H)$. To see the last relation, we note that

$$\operatorname{cl}\widetilde{\varphi}(\widetilde{U}) \subset \operatorname{cl}(p_{XH}^{-1}(\varphi(U))) \subset \operatorname{cl}(p_{XH}^{-1}(\operatorname{cl}(\varphi(U)))).$$

Since $\operatorname{cl}(\varphi(U))$ is compact and covering p_{XH} is finite, $p_{XH}^{-1}(\operatorname{cl}(\varphi(U)))$ is also compact. Thus, so is $\operatorname{cl}\varphi(\widetilde{U})$. □

(77.31) Definition. A Nielsen class $\operatorname{mod} H$ of the form $p_{\Gamma H}C(\widetilde{p}_H, \alpha\widetilde{q}_H)$ is called *essential* if $\Lambda(\widetilde{p}_H, \alpha\widetilde{q}_H) \neq 0$.

By Lemma (77.18) this definition is correct, i.e.

$$\text{if} \quad p_{\Gamma H}C(\widetilde{p}_H, \alpha\widetilde{q}_H) = p_{\Gamma H}C(\widetilde{p}_H, \beta\widetilde{q}_H), \quad \text{then} \quad \Lambda(\widetilde{p}_H, \alpha\widetilde{q}_H) = \Lambda(\widetilde{p}_H, \beta\widetilde{q}_H).$$

(77.32) Definition. The number of essential classes of (p,q) mod a subgroup H is called the *H-Nielsen number* and is denoted by $N_H(p,q)$.

Now, we can give two main theorems of this section.

(77.33) Theorem. *A multivalued map* (p,q) *satisfying* $(\mathrm{CAC} + \mathrm{A} + \mathrm{B})$ *has at least* $N_H(p,q)$ *coincidence points.*

Proof. We show that each essential H-Nielsen class is nonempty. Consider an essential class $p_{\Gamma H}C(\widetilde{p}_H, \alpha\widetilde{q}_H)$. Then $\Lambda(\widetilde{p}_H, \alpha\widetilde{q}_H) \neq 0$ implies a point $\widetilde{z} \in C(\widetilde{p}_H, \alpha\widetilde{q}_H)$, by which $p_{\Gamma H}C(\widetilde{p}_H, \alpha\widetilde{q}_H)$ is nonempty as required. □

(77.34) Theorem. $N_H(p,q)$ *is a homotopy invariant* (*with respect to homotopies satisfying* $(\mathrm{CAC} + \mathrm{A} + \mathrm{B})$.

Proof. Let the map (p_t, q_t) be such a homotopy. It is enough to show that the class $p_{\Gamma H}C(\widetilde{p}_{0H}, \alpha\widetilde{q}_{0H})$ is essential if and only if the same is true for the class $p_{\Gamma H}C(\widetilde{p}_{1H}, \alpha\widetilde{q}_{1H})$. However, this is implied by the eguality of Lefschetz numbers

$$\Lambda(\widetilde{p}_{0H}, \alpha\widetilde{q}_{0H}) = \Lambda(\widetilde{p}_{1H}, \alpha\widetilde{q}_{1H}).$$ □

78. Fixed points of symmetric product mappings

In 1957 C. N. Maxwell [Max] proved the Lefschetz fixed point theorem for symmetric product mappings on compact polyhedra. Later this theorem was studied by many authors, namely by: H. Schirmer, S. Masih, N. Rallis, D. Miklaszewski and others. For the present state of this theory see for example: [Mik1-M], [Mik1], [Mik2], [Mas1], [Mas2] and [Ral].

Note that any mapping into symmetric product can be treat as a finite valued multivalued map.

In this section all considered spaces are compact metric. Let $M = \{1, \dots, m\}$ be a finite set and let G be the subgroup of the group of all permutations of M. For a compact metric space X by X^m we shall denote its m-th cartesian product. Then G acts on X^m by the formula:

$$(78.1) \quad x \cdot \sigma = (x_{\sigma(1)}, \dots, x_{\sigma(m)}), \quad \text{for any } x = (x_1, \dots, x_m) \in X^m \text{ and } \sigma \in G.$$

(78.2) Definition. The orbit space $SP^m_G X$ of the action (78.1) is called *m-th symmetric product of X with respect to G.*

The orbit Gx of a point $x \in X^m$ is defined by $Gx = \{x \cdot \sigma \mid \sigma \in G\}$. Of course Gx is an element of $SP^m_G X$. We let,

$$q: X^m \to SP^m_G X, \quad q(x) = Gx.$$

If (X, d) is a metric space, then we define a metric d_1 in $SP^m_G X$ by putting

$$d_1(q(x), q(y)) = \min\{d(x, y \cdot \sigma) \mid \sigma \in G\}.$$

In what follows we shall use the same symbol d for the metric in X and in $SP^m_G X$. Any continuous function $f: Y \to SP^m_G X$ will be called a mapping into the symmetric product of X.

A point $c \in X$ is called a coordinate of the point $a \in SP^m_G X$ if and only if there exists a point $x \in X^m$ and $j \in \{1, \dots, m\}$ such that $p_j(x) = c$ and $q(x) = a$, where $p_j: X^m \to X$ is defined by

$$p_j(x_1, \dots, x_j, \dots x_m) = x_j, \quad j = 1, \dots, m.$$

(78.3) Definition. Let $f: Y \to SP^m_G X$ be a map and $Y \subset X$. A point $x \in Y$ is called a fixed point of f if and only if y is a coordinate of $f(y)$.

We let $\mathrm{Fix}(f) = \{y \in Y \mid y \text{ is a fixed point of } f\}$.

We recall the following well-known result (see [Mik1-M]).

(78.4) THEOREM. *If $\mathcal{A}$ is one from the following classes of spaces:*

(78.4.1) *finite polyhedra,*
(78.4.2) ANR*s,*
(78.4.3) *contractible spaces,*

then so is $(SP_G^m X) \in \mathcal{A}$ provided $X \in \mathcal{A}$.

In what follows $\check{H}$ is the Čech homology functor with coefficients in $\mathbb{Q}$ or $\mathbb{Z}$ (comp. Chapter I).

(78.5) THEOREM. *Let A be a closed subset of X. We have:*

(78.5.1) *the homomorphism $q_*\colon \check{H}(X^m, A^m; \mathbb{Q}) \to \check{H}(SP_G^m X, SP_G^m A; \mathbb{Q})$ is an epimorphism,*
(78.5.2) *there exists exactly one homomorphism*

$$\mu\colon \check{H}(SP_G^m X, SP_G^m X; \mathbb{Q}) \to H(X, A; \mathbb{Q})$$

such that

$$\mu \circ q_* = \sum_{i=1}^{m} p_{i_*}.$$

PROOF. It is well known that there exists a homomorphism

$$\tau\colon \check{H}(SP_G^m X, SP_G^m X; \mathbb{Q}) \to \check{H}(X^m, A^m; \mathbb{Q})$$

so called transfer homomorphism, with the following two properties:

(78.5.3) $q_* \circ \tau = \#G \cdot \mathrm{id}$,
(78.5.4) $\tau \circ q_* = \sum_{\sigma \in G} \sigma_*$,

where $\#G$ is the number of elements in G.

Now, from (78.5.3) we deduce that q_* is an epimorphism. To prove (78.5.2) it is sufficient to show that $\ker q_* \subset \ker(\sum_{i=1}^m p_{i_*})$. It follows from (78.5.4). Indeed, we have:

$$\bigg(\sum_{i=1}^{m} p_{i_*}\bigg) \circ \tau \circ q_* = \sum_{i=1}^{m} \sum_{\sigma \in G} p_{i*} \circ \sigma_* = \sum_{\sigma \in G} \sum_{i=1}^{m} p_{\sigma(i)*} = \#G \cdot \sum_{j=1}^{m} p_{j_*}$$

and the proof is completed. □

(78.6) REMARK. In what follows homomorphism μ defined in (78.5.2) is called the *trace homomorphism.*

(78.7) PROPOSITION. *If G acts transitively on $\{0,\dots,m\}$, then the trace homomorphism $\mu\colon \check{H}_k(SP^m_G S^k) \to \check{H}_k(S^k)$ is an isomorphism.*

PROOF. Let us denote $M = \{1,\dots,m\}$, $s_0 \in S^k$ and $i\colon S^k \to SP^M_G S^k$ be the map defined as follows $i(x) = q(x, s_0, \dots, s_0)$. Let $\psi^c_j\colon (S^k)^M \to (S^k)^M$ be defined by the formula:

$$\psi^c_j(x) = y \text{ provided } p_i(y) = s_0 \text{ for } j \neq i \text{ and } p_j(y) = p_j(x), \text{ for } i,j \in M.$$

then we have $i \circ p_j = q \circ \psi^c_j$. Consequently by a natural calculations one can show that i_* is an inverse to μ. In fact

$$\mu \circ i_* = \mu \circ q_*(\mathrm{id}) \times s_0 \times \circ \times s_0)_* = \sum_{j\in M} p_{j_*} \circ (\mathrm{id}) \times s_0 \times \circ \times s_0) = \mathrm{id}$$

and

$$i_* \circ \mu \circ q_* = i_* \circ \sum_{j\in M} p_{j_*} = q_\circ \sum_{j\in M} \psi^c_{j_*} = q_*$$

and the proof is completed. □

Now, we prove:

(78.8) PROPOSITION. *Let $O(G)$ be the set of all orbits of the G-action on M. Then*

$$(\overline{p}_{J*})_{J\in O(G)}\colon H_k(SP^M_G \mathbb{S}^k) \to \bigoplus_{J\in O(G)} H_k(SP^J_{G(J)} \mathbb{S}^k)$$

is an isomorphism.

PROOF. Since the symmetric products of $\mathbb{S}^k$ are $(k-1)$-connected, one can use the Hurewicz isomorphisms and behavior of homotopy groups with respect to the cartesian products to write the assertion in the equivalent form:

$$((\overline{p}_{J*})_{J\in O(G)})_*\colon H_k(SP^M_G \mathbb{S}^k) \to H_k\Big(\prod_{J\in O(G)} SP^J_{G(J)} \mathbb{S}^k\Big)$$

is an isomorphism. Let $X = \mathbb{S}$, $s_0 \in X$, $c = (s_0,\dots,s_0) \in X^M = \prod_{I\in O(G)} X^I$. From (78.2) take $\psi^c_J\colon \prod_I X^I \to \prod_I X^I$ with $c_I = p_I(c)$ and

$$\psi^b_J\colon \prod_I SP^I_{G(I)} X \to \prod_I SP^I_{G(I)} X$$

with $b_I = q_I \circ p_I(c)$ for $I,J \in O(G)$. Let $r_J\colon \prod_I SP^I_{G(I)} X \to SP^J_{G(J)} X$ be the projection and $i_J\colon X^J \to X^M$ be the unique map such that

$$p_I \circ i_J = \begin{cases} p_I(c) & \text{if } I \neq J, \\ \mathrm{id} & \text{if } I = J, \end{cases}$$

for $I, J \in O(G)$. There is an induced map $\overline{i}_J\colon SP^J_{G(J)}X \to SP^M_G X$ such that $\overline{i}_J \circ q_J = q \circ i_J$. A direct calculation shows that $\psi^b_J = (\overline{p}_I \circ \overline{i}_J \circ r_J)_{I\in O(G)}$ and $\psi^c_J = i_J \circ p_J$. Thus

$$((\overline{p}_I)_{I\in O(G)})_* \circ \left(\sum_J \overline{i}_{J_*} \circ r_{J_*}\right) = \sum_J \psi^b_{J_*} = \mathrm{id}$$

and

$$\begin{aligned}\left(\sum_J \overline{i}_{J_*} \circ r_{J_*}\right) \circ ((\overline{p}_I)_{I\in O(G)})_* \circ q_* &= \sum_J \overline{i}_{J_*} \circ \overline{p}_{J_*} \circ q_* = \sum_J \overline{i}_{J_*} \circ q_{J_*} \circ p_{J_*} \\ &= \sum_J q_* \circ i_{J_*} \circ p_{J_*} = q_* \circ \sum_J \psi^c_{J_*} = q_*.\end{aligned}$$

The proof is completed. □

Let $O(G)$ be the set of all orbits of the G-action on M and ν be a generator of $H_k(\mathbb{S}^k)$.

Let $\mathbb{Z}$ be set of integers. For $d \in \mathbb{Z}^{O(G)}$ let $\overline{d}\colon H_k(\mathbb{S}^k) \to \bigoplus_{J\in O(G)} H_k(\mathbb{S}^k)$ be a homomorphism defined by the formula $\overline{d}(\nu) = (d(J)\nu)_{J\in O(G)}$.

(78.9) Definition. The topological degree $\deg(f)$ of the symmetric product map $f\colon \mathbb{S}^k \to SP^M_G \mathbb{S}^k$ is defined to be a function $d \in \mathbb{Z}^{O(G)}$ such that the homomorphism

$$H_k(\mathbb{S}^k) \xrightarrow{f_*} H_k(SP^M_G\mathbb{S}^k) \xrightarrow{p_{J_*}} \bigoplus_{J\in O(G)} H_k(SP^J_{G(J)}\mathbb{S}^k) \xrightarrow{\bigoplus \nu_J} \bigoplus_{J\in O(G)} H_k(\mathbb{S}^k)$$

is equal with $\overline{d}$.

We may now formulate the main result concerning topological degree.

(78.10) Theorem. *Symmetric product maps of the same degree are homotopic.*

Proof follows directly from (78.7), (78.8) and the Hurewicz isomorphism theorem.

(78.11) Proposition. *If G acts transitively on M, $f_i\colon S^k \to S^k$ for $i \in M$ and $f = q^M_G \circ (f_i)_{i\in M}$, then* $\deg(f) = \sum_{i\in M} \deg(f_i)$.

Proof of Proposition (78.11) follows immediately from (78.5).

C. N. Maxwell (see [Max]) defined a degree Deg for symmetric product maps with respect to the n-th symmetric group. By definition $\mathrm{Deg} = n^{-1}\deg$. Our approach comes from D. Miklaszewski (see [Mik1-M]).

In fact for symmetric product mappings the fixed point index can be defined by using homological apparatus. Below we shall formulate only most important parts of this theory. In what follows by $\mathcal{A}_G$ we shall denote a class of all triples (X, f, U), where X is a compact ANR, U is an open subset of X and $f\colon \overline{U} \to SP^m_G X$ is a continuous map such that $\mathrm{Fix}(f) \subset U$.

(78.12) Definition. A function $\mathrm{Ind}\colon \mathcal{A}_G \to \mathbb{Q}$ is called the *fixed point index*, if the following axioms are satisfied:

(78.12.1) normalization,
(78.12.2) additivity,
(78.12.3) homotopy,
(78.12.4) commutativity,

(compare Sections 34, 47, 51–53).

We shall formulate (78.12.4) because the respective formulation is not standard.

(78.12.4) (Commutativity). Let X and Y be compact ANR-s, U be an open subset of X, $f\colon Y \to X$ and $g\colon \overline{U} \to SP^m_G Y$ be two continuous functions. If $(X, \overline{f} \circ g, U) \in \mathcal{A}_G$, then $(Y, g \circ f, f^{-1}(U)) \in \mathcal{A}_G$ and $\mathrm{Ind}(X, \overline{f} \circ g, U) = \mathrm{Ind}(Y, g \circ f, f^{-1}(U))$, where $\overline{f}\colon SP^m_G Y \to SP^m_G X$ is induced by f.

(78.13) Definition. If $f\colon X \to SP^m_G X$ is a continuous map, then we define the Lefschetz number $\Lambda(f)$ of f by letting:

$$\Lambda(f) = \Lambda(\mu \circ f_*),$$

where μ is defined in (78.5).

The construction of the fixed point point index $\mathrm{Ind}\colon \mathcal{A}_G \to \mathbb{Q}$ is given in two steps.

At first, we define it for X to be a finite polyhedron and then we use the fact that any compact ANR-space is homotopically dominated by a finite polyhedron (see (2.23)).

Note that the fixed point index for symmetric product mappings of polyhedra was defined in 1979 by S. Masih ([Mas2]). For complete information we recommend [Mik1-M].

Using the fixed point index we can prove the following:

(78.14) Theorem (Relative Lefschetz Fixed Point Theorem). *Let X, A be two compact* ANR*s and $A \subset X$. Assume further that*

$$f\colon (X, A) \to (SP^m_G X, SP^m_G A)$$

is a continuous map. If $\Lambda(f) \neq 0$, then $\mathrm{Fix}(f) \cap \overline{X \setminus A} \neq \emptyset$.

The proof of (78.14) is strictly analogous to the proof of (57.3).

Finally, we shall consider the case when $G = S_m$ is the full permutation group of the set $\{1, \dots, m\}$. For short, we shall use the following notations $X_m = SP^m_{S_m} X$. Let $f\colon X \to X_m$ by a symmetric product map and $x, y \in \mathrm{Fix}(f)$. We shall say that x and y belongs to the same class F provided there exists a path $C\colon [0,1] \to X^n$ such that $p_1(C(0)) = x$, $p_1(C(1)) = y$ and $f \circ p_1 \circ C$ is homotopic to $q \circ C$ rel $\{0,1\}$.

A class F is called *essential* provided there exists an open neighbourhood U of F in X such that for every open V of F in X, $V \subset U$ we have $\mathrm{Ind}(X, f, V) \neq 0$.

We define the Nielsen number $N(f)$ of f to be the number of essential classes.

We prove the following:

(78.15) Proposition. *If X is pathwise connected compact* ANR*-space and $f\colon X \to X_n$, $n \geq 2$ is a continuous symmetric product map, then f has at most one fixed point class.*

Proof. Let x, y be fixed points of f. There are $\overline{x}, \overline{y} \in X^{n-1}$ such that $f(x) = q(x, \overline{x})$, $f(y) = q(y, \overline{y})$. Let C_1 be a path from x to y and $\overline{D}$ be a path from $\overline{x}$ to $\overline{y}$. The projection $q\colon X^n \to X_n$ induces an epimorphism of fundamental groups. Since X^n is a pathwise connected space, this result does not depend on the choice of the basepoint of the fundamental group of X^n, in particular there is a lop $(E, \overline{E})$ based in $(x, \overline{x})$ such that $(f \circ C_1) * (q(C_1, \overline{D}))^{-1} \equiv q(E, \overline{E})$. Let $(x', x")$ be coordinates of $\overline{x}$ in $X \times X^{n-2}$ and $(E', E") = \overline{E}$ (if $n = 2$ then the second coordinate should be omitted). Let F be a path from x to x', $G = F^{-1} * E * F$, $\overline{H} = (G * E', x" * E")$. Then $q(G, \overline{E}) \equiv q((G, \overline{x}) * (x', \overline{E})) = q(x', \overline{X})$ and $f \circ C_1 \equiv q((E, \overline{E}) * q(C_1, \overline{D})) \equiv q(F, \overline{x}) * q(G, \overline{E}) * q(F^{-1}, \overline{x}) * q(C_1, \overline{D}) \equiv q(C_1, \overline{H} * \overline{D})$, i.e. the points x, y are in the same class and the proof is completed. □

As a simple consequence of (78.15) we get:

(78.16) Corollary. *If X is a pathwide connected compact* ANR *and $f\colon X \to X_n$ is a symmetric product map, then:*

(78.15.1) $\Lambda(f) = 0$ *implies that* $N(f) = 0$,

(78.15.2) $\Lambda(f) \neq 0$ *implies that* $N(f) = 1$.

In another words (78.16) says that the Nielsen theory for symmetric product mappings is trivial because it is reduced to the Lefschetz number.

79. The category of weighted maps

The class of weighted mappings was introduced in 1958 by G. Darbo (see [Pej-M], [Pej1]–[Pej3], [Pej-5], [Shi-M], [Ski1], [Ski2], [SeS]).

Let Ω be a commutative ring with unit. Usually, we shall consider $\Omega = \mathbb{Z}$ to be the ring of integers. Let X, Y be two (metric) spaces.

(79.1) Definition. A pair $\psi = (\nu_\psi, \omega_\psi)$ is called a weighted map (ω-map for short) defined on X with values in Y provided:

$$\nu_\psi\colon X \multimap Y \quad \text{and} \quad \omega_\psi\colon X \times Y \to \Omega$$

are two mappings for which the following conditions are satisfied:

(79.1.1) ν_ψ is an u.s.c. map with finite values, i.e. $\#\nu_\psi(x) < \infty$, for every $x \in X$,
(79.1.2) if $y \notin \nu_\psi(x)$, then $\omega_\psi(x,y) = 0$,
(79.1.3) for any open $U \subset Y$ and any $x \in X$, if $\nu_\psi(x) \cap \delta U = \emptyset$, then there is an open neighbourhood V of X such that

$$\sum_{y \in U} \omega_\psi(x,y) = \sum_{y \in U} \omega_\psi(x',y),$$

for every $x' \in V$, where δU stands for the boundary of U in Y.

In what follows ν_ψ is called the support and ω_ψ the weight of $\psi = (\nu_\psi, \omega_\psi)$.

(79.2) Example. Let $f\colon X \to Y$ be a continuous mapping. We define $\omega_f\colon X \times X \to Z$ defined as follows:

$$\omega_f(x,y) = \begin{cases} 0 & \text{if } y \neq f(x), \\ 1 & \text{if } y = f(x). \end{cases}$$

Evidently $\psi = (f, \omega_f)$ is a weighted map.

(79.3) Example. Let m be a given natural number and $\psi\colon X \multimap Y$ be a continuous multivalued map such that

$$\#\varphi(x) = 1 \text{ or } m, \quad \text{for very } x \in X.$$

We define $\omega_\varphi\colon X \times Y \to Z$ by formula:

$$\omega_f(x,y) = \begin{cases} 0 & \text{if } y \neq \varphi(x), \\ m & \text{if } \{y\} = \varphi(x), \\ 1 & \text{if } \#\varphi(x) = m \text{ and } y \in \varphi(x). \end{cases}$$

It is not difficult to verify that $\psi = (\varphi, \omega_\varphi)$ is a ω-map.

(79.4) Example. Let $C([a,b])$ be the space of all continuous mappings for $[a,b]$ into $\mathbb{R}$ with the usual maximum norm. Assume further that $0 \in (a,b)$.

Let $\varphi\colon C([a,b]) \multimap \mathbb{R}$ be defined as follows:

$$\varphi(x) = \left\{ \int_a^0 x(s)\,ds, \int_0^b x(s)\,ds \right\}.$$

Then $\#\varphi(x) = 1$ or 2. So (79.4) is a special case of (79.3).

In what follows we shall deal with some another ω-mappings.

Assume that $\psi = (\nu_\psi, \omega_\psi): X \multimap Y$ is a ω-map. It is easy to see that that the map $\omega_\psi: X \times Y \to \Omega$ given by the following formula:

$$\omega_\psi(x, y) = 0 \quad \text{for every } x \in X \text{ and } y \in Y$$

satisfies all assumptions of (79.1), such a ω_ψ is called the trivial weight. Observe that any multivalued map $y: X \multimap Y$ can be equipped in the trivial weight ω_ψ. So the main problem is the possibility to equipped given multivalued map in nontrivial weight.

Let $SP^M Y$ be a full symmetric product of Y, i.e. $SP^m Y = SP^m_G Y$ and G is the group of all permutations of the set $\{1, \ldots, m\}$.

Note that any continuous map $f: X \to SP^m Y$ can be regarded as a weighted map. In fact, if we denote by $p: SP^m Y \multimap Y$ a map defined by

$$p([y_1^{\sigma(1)}, \ldots, y_m^{\sigma(m)}]) = \{y_1, \ldots, y_s\},$$

then $\nu_f: X \multimap Y$ and $\omega_f: X \times Y \to Z$ can be defined as follows:

$$\nu_f(x) = p(f(x)),$$

$$\omega_f(x, y) = \begin{cases} \sigma(i) & \text{if } y \in p(f(x)), \\ 0 & \text{if } y \notin p(f(x)), \end{cases}$$

for every $x \in X$ and $y \in Y$. Then (ν_f, ω_f) is a weighted map indexed by f.

First, we shall prove the following:

(79.5) PROPOSITION. *Assume $\psi = (\nu_\psi, \omega_\psi): X \multimap Y$ is a ω-map and X is a connected space. Then for every $x, x' \in X$ and $y \in Y$ we have:*

$$\sum_y \omega_\psi(x, y) = \sum_y \omega_\psi(x', y).$$

PROOF. Assume to the contrary that for some $x_0, x_0' \in X$, we have

$$\sum_y \omega_\psi(x_0, y) \neq \sum_y \omega_\psi(x_0', y).$$

We let:

$$U = \Big\{ x \in X \;\Big|\; \sum_{y \in Y} \omega_\psi(x, y) = \sum_{y \in Y} \omega_\psi(x_0', y) \Big\},$$

$$W = \Big\{ x \in X \;\Big|\; \sum_{y \in Y} \omega_\psi(x, y) \neq \sum_{y \in Y} \omega_\psi(x_0', y) \Big\}.$$

It follows from (79.1.4) that U and W are open subset of X. Evidently $U \cap W = \emptyset$ and $U \cup W = X$. Since $x_0 \in W$ and $x_0' \in U$ and X is connected we get a contradiction. □

Below, we shall show an example of the multivalued map φ with 1 or 2 values, for which every weight ω_φ is trivial.

(79.6) EXAMPLE. Let $\varphi\colon [0,1] \multimap [0,1]$ be defined as follows:

$$\omega_\varphi(x,y) = \begin{cases} \{1\} & \text{for } 0 \le x < 1/2, \\ \{0,1\} & \text{for } x = 1/2, \\ \{0\} & \text{for } 1/2 < x \le 1. \end{cases}$$

Evidently φ is an u.s.c. map without fixed points. Let $\omega_\varphi\colon [0,1]\times[0,1] \to \Omega$ be a given weight for φ. We shall show that $\omega_\varphi(x,y) = 0$, for every $x, y \in [0,1]$.

To show it we need the following lemma:

(79.7) LEMMA. *If $\varphi\colon X \multimap Y$ is a ω-map and X is a connected space, then*

$$\sum_{y\in Y} \omega_\varphi(x,y) = \sum_{y\in Y} \omega_\varphi(x',y),$$

for every $x, x' \in X$.

PROOF. Assume to the contrary that there are two points $x_0, x_0' \in X$ such that:

$$\sum_{y\in Y} \omega_\varphi(x_0,y) \neq \sum_{y\in Y} \omega_\varphi(x_0',y).$$

Then, we define:

$$X_1 = \Big\{ x \in X \ \Big|\ \sum_{y\in Y} \omega_\varphi(x,y) = \sum_{y\in Y} \omega_\varphi(x_0,y) \Big\},$$
$$X_2 = \Big\{ x \in X \ \Big|\ \sum_{y\in Y} \omega_\varphi(x,y) = \sum_{y\in Y} \omega_\varphi(x_0',y) \Big\}.$$

Then $Y = X_1 \cup X_2$ and X_1, X_2 are nonempty disjoint and closed subset of Y, so we get a contradiction. □

Now, we come back to the Example (79.6).

Let $x_0 = 1/2$ and $U = (2/3, 1]$. Then by using (79.1.4) we get an open neighbourhood $(x_0 - \varepsilon, x_0 + \varepsilon)$ of x_0 such that

$$\sum_{y\in(2/3,1]} \omega_\varphi(x_0,y) = \sum_{y\in(2/3,1]} \omega_\varphi(x',y),$$

for every $x' \in (x_0 - \varepsilon, x_0 + \varepsilon)$. Moreover, we have

$$\sum_{y\in(2/3,1]} \omega_\varphi(x_0,y) = \omega_\varphi(x_0,y)$$

and

$$\sum_{y\in(2/3,1]} \omega_\varphi(x_0, y) = \begin{cases} \omega_\varphi(x', y), & \text{for } x' \in (x_0 - \varepsilon, x_0], \\ 0, & \text{for } x' \in (x_0, x_0 + \varepsilon), \end{cases}$$

Hence, we get $\omega_\varphi(x_0, 1) = 0$. Similarly, we deduce that $\omega_\varphi(x_0, 0) = 0$. Since, we have

$$\sum_{y\in[0,1]} \omega_\varphi(x, y) = \sum_{y\in\varphi(x)} \omega_\varphi(x, y) = \omega_\varphi(x, \varphi(x)), \quad \text{for } x \neq \frac{1}{2}.$$

Then in, view of (79.7) we deduce that

$$\omega_\varphi(x, y) = 0, \quad \text{for every } x, y \in [0, 1].$$

Now, we shall list most important properties of ω-maps.

(79.8) PROPERTIES.

(79.8.1) *If* $\varphi = (\nu_\varphi, \omega_\varphi)$, $\psi = (\nu_\psi, \omega_\psi)\colon X \multimap Y$ *are two* ω*-maps, then*

$$\varphi \cup \psi = (\nu_{\varphi\cup\psi}, \omega_{\varphi\cup\psi})\colon X \multimap Y$$

is a ω*-map too, where*

$$\nu_{\varphi\cup\psi}(x) = \nu_\varphi(x) \cup \nu_\psi(x),$$
$$\omega_{\varphi\cup\psi}(x, y) = \omega_\varphi(x, y) + \omega_\psi(x, y),$$

for every $x \in X$, $y \in Y$.

(79.8.2) *If* $\varphi = (\nu_\varphi, \omega_\varphi)\colon X \multimap Y$ *is a* ω*-map and* $\lambda \in \Lambda$*, then*

$$\lambda\varphi = (\nu_{\lambda\varphi}, \omega_{\lambda\varphi})\colon X \to Y$$

is a ω*-map too, where* $\nu\lambda\varphi(x) = \nu_\varphi(x)$ *and* $\omega_{\lambda\varphi}(x, y) = \lambda \cdot \omega_\varphi(x, y)$*, for* $x \in X$ *and* $y \in Y$.

(79.8.3) *Let* $\varphi = (\nu_\varphi, \omega_\varphi)\colon X \multimap Y$ *and* $\psi = (\nu_\psi, \omega_\psi)\colon Y \multimap Z$ *be two* ω*-maps, then* $\varphi \circ \psi = (\nu_{\psi\circ\varphi}, \omega_{\psi\circ\varphi})\colon X \multimap Z$ *is a* ω*-map too, where*

$$\nu_{\psi\circ\varphi}(x) = (\nu_\psi \circ \nu_\varphi)(x)$$

and

$$\omega_{\psi\circ\varphi}(x, z) = \sum_{y\in Y} \omega_\varphi(x, y) \cdot \omega_\psi(y, z),$$

for every $x \in X$, $y \in Y$ *and* $z \in Z$.

PROOF. The proof of (79.8.1) and (79.8.2) is straightforward. So we shall prove only (79.8.3).

Let U be an open subset of Z and let $x_0 \in X$ be a point such that:

$$(\nu_\psi \circ \nu_\varphi)(x_0) \cap \delta U = \emptyset.$$

We have to show that there exists an open neighbourhood W_{x_0} of x_0 in X such that:

$$\sum_{z\in U} \omega_{\psi\circ\varphi}(x_0, z) = \sum_{z\in U} \omega_{\psi\circ\varphi}(x', z), \quad \text{for } x' \in W_{x_0}. \tag{79.8.4}$$

We have:

$$\begin{aligned}\sum_{z\in U} \omega_{\psi\circ\varphi}(x_0, z) &= \sum_{z\in U} \Big(\sum_{y\in Y} \omega_\varphi(x_0, y) \circ \omega_\psi(y, z) \Big) \\ &= \sum_{y\in Y} \Big(\sum_{z\in U} \omega_\varphi(x_0, y) \circ \omega_\psi(y, z) \Big) = \sum_{y\in Y} \omega_\varphi(x_0, y) \cdot \Big(\sum_{z\in U} \omega_\psi(y, z) \Big)\end{aligned}$$

We let:

$$\nu_\varphi(x_0) = \{y_0^1, \dots, y_0^k\} \quad \text{and} \quad a_y = \sum_{z\in U} \omega_\psi(y, z).$$

Then, we obtain

$$\begin{aligned}\sum_{y\in Y} \omega_\varphi(x_0, y) \cdot \sum_{z\in U} \omega_\psi(y, z) &= \sum_{y\in Y} \omega_\varphi(x_0, y) \cdot a_y \\ &= \sum_{y\in \nu_\varphi(x_0)} \omega_\varphi(x_0, y) \cdot a_y = \sum_{i=1}^{k} \omega_\varphi(x_0, y_0^i) \cdot a_{y_0^i}.\end{aligned}$$

Hence, we get

$$\sum_{z\in U} \omega_{\psi\circ\varphi}(x_0, z) = \sum_{i=1}^{k} \omega_\varphi(x_0, y_0^i) \cdot a_{y_0^i}. \tag{79.8.5}$$

So, for the proof it is sufficient to show that right hand sides of (79.8.4) and (79.8.5) are equal for some open neighbourhood of x_0 in X.

We consider open neighbourhoods $V_{y_0^i}$ of y_0^i in Y, $i = 1, \dots k$ such that

$$V_{y_0^i} \cup V_{y_0^j} = \emptyset, \quad \text{for } i \neq j \tag{79.8.6}$$

and

$$a_{y_0^i} = \sum_{z\in U} \omega_\psi(y_0^i, z) = \sum_{z\in U} \omega_\psi(y, z) = a_y. \tag{79.8.7}$$

Now, for every $i = 1, \dots, k$ we choose open set $W_i \subset X$ such that

$$\omega_\varphi(x_0, y_)^i) = \sum_{y \in V_{y_0^i}} \omega_\varphi(x', y) = \sum_{y \in \varphi(x') \cap V_{y_0^i}} \omega_\varphi(x', y),$$

for $x' \in W_i$. Since ν_φ is u.s.c. we can choose an open neighbourhood W_{x_0} of x_0 in Y such that

$$\nu_\varphi(\overline{W_{x_0}}) \subset \bigcup_{i=1}^{k} V_{y_0^i}. \tag{79.8.8}$$

Finally, we let $W_0 = W_{x_0} \cap W_1 \cap \dots W_k$. Then for any $x' \in W_0$ we obtain

$$\begin{aligned}
\sum_{z \in U} \omega_{\psi\circ\varphi}(x', z) &= \sum_{z \in U} \left(\sum_{y \in Y} \omega_\varphi(x', y) \circ \omega_\psi(y, z) \right) \\
&= \sum_{y \in Y} \sum_{z \in U} \omega_\varphi(x', y) \circ \omega_\psi(y, z) = \sum_{y \in Y} \omega_\varphi(x', y) \circ \sum_{z \in U} \omega_\psi(y, z) \\
&= \sum_{y \in Y} \omega_\varphi(x', y) \circ a_y = \sum_{y \in \nu_{\varphi(x')}} \omega_\varphi(x', y) \cdot a_y.
\end{aligned}$$

From the other hand form (79.8.9) we deduce

$$\begin{aligned}
\sum_{y \in \nu_{\varphi(x')}} \omega_\varphi(x', y) \circ a_y &= \sum_{y \in \nu_{\varphi(x')} \cap \left(\bigcup_{i=1}^{k} V_{y_0^i} \right)} \omega_\varphi(x', y) \\
&= \sum_{i=1}^{k} \sum_{y \in \nu_{\varphi(x')} \cap V_{y_0^i}} \omega_\varphi(x', y) \circ a_y.
\end{aligned}$$

In view of (79.8.7) $a_y = a_{y_0}$, for every $y \in V_{y_0^i}$. Consequently (by using also (79.8.8)) we get

$$\begin{aligned}
\sum_{i=1}^{k} \sum_{y \in \nu_{\varphi(x')} \cap V_{y_0^i}} \omega_\varphi(x', y) \circ a_y &= \sum_{i=1}^{k} \sum_{y \in \nu_{\varphi(x')} \cap V_{y_0^i}} \omega_\varphi(x', y) \circ a_{y_0^i} \\
&= \sum_{i=1}^{k} a_{y_0^i} \Bigg(\sum_{y \in \nu_{\varphi(x')} \cap V_{y_0^i}} \omega_\varphi(x', y) \Bigg) = \sum_{i=1}^{k} a_{y_0^i} \omega_\varphi(x', y)
\end{aligned}$$

and the proof of (79.8.4) is completed. □

(79.9) REMARK. If we shall consider metric spaces as objects, ω-maps as morphisms and the composition law defined in (79.7.3), then we get a category $\mathcal{E}_\Omega$ so called the category of metric spaces and ω-maps. Of course $\mathcal{E}$ is a subcategory of $\mathcal{E}_\omega$ (comp. Section 5).

For two metric spaces X and Y by $\mathbb{W}(X,Y)$ we shall denote the set of all ω-maps from X to Y (with the weight in a fixed ring Ω).

According to (79.8.1) and (79.8.2) we have two operators in $\mathbb{W}(X,Y)$. If we introduce in $\mathbb{W}(X,Y)$ the following equivalence relation

$$\varphi = (\nu_\varphi, \omega_\varphi) \sim \psi = (\nu_\psi, \omega_\psi) \Leftrightarrow \omega_\varphi = \omega_\psi.$$

Then the factor set $\mathbb{W}(X,Y)/_\sim = \langle X, Y\rangle$ possess an algebraic structure of Ω-module.

(79.10) PROPOSITION. *Let $\varphi_i) = (\nu_{\varphi_i}, \omega_{\varphi_i})\colon X_i \multimap Y_i$, $i = 1,2$ be two weighted mappings. Then $\varphi_1 \times \varphi_2 = (\nu_{\varphi_1\times\varphi_2}, \omega_{\varphi_1\times\varphi_2})$ is a weighted map, where*

$$\nu_{\varphi_1\times\varphi_2}\colon X_1 \times X_2 \multimap Y_1 \times Y_2$$

and

$$\omega_{\varphi_1\times\varphi_2}\colon (X_1 \times X_2) \times (Y_1 \times Y_2) \to \Omega$$

are defined as follows

$$\begin{aligned}&\nu_{\varphi_1\times\varphi_2}(x_1,x_2) = \nu_{\varphi_1}(x_1) \times \nu_{\varphi_2}(x_2),\\ &\omega_{\varphi_1\times\varphi_2}((x_1,x_2),(y_1,y_2)) = \omega_{\varphi_1}(x_1,y_1)\cdot\omega_{\varphi_2}(x_2,y_2),\end{aligned}$$

for every $x_1 \in X_1$, $x_2 \in X_2$, $y_1 \in Y_1$ and $y_2 \in Y_2$.

PROOF. First, observe that $\nu_{\varphi_1\times\varphi_2}$ is an u.s.c. map. Observe also that the condition (79.1.1) is automaticaly satisfied. So we have to prove (79.1.2) and (79.1.3).

Let $(y_1,y_2) \notin \nu_{\varphi_1\times\varphi_2}(x_1,x_2)$. Then $y_1 \notin \nu_{\varphi_1}(x_1)$ or $y_2 \notin \nu_2(x_2)$ but in any case we have $\omega_{\varphi_1\times\varphi_2}((x_1,x_2),(y_1,y_2)) = \omega_{\varphi_1}(x_1,y_1)\cdot\omega_{\varphi_2}(x_2,y_2) = 0$. Consequently (79.1.2) holds true.

Now, we are going to prove (79.1.3). Let $U \subset Y_1 \times Y_2$ be an open set and let $x^0 = (x_1^0, x_2^0) \in X_1 \times X_2$ be such that

$$\nu_{\varphi_1\times\varphi_2}(x_1^0,x_2^0) = \{y^1,\dots,y^n\},$$

where $y^i = (y_1^i, y_2^i) \in Y_1 \times Y_2$, for $i = 1,\dots,n$.

It follows from (79.8.6) that there exists $1 \le m \le n$ such that:

$$\{y^{i_1},\dots,y^{i_m}\} \subset U \quad\text{and}\quad \nu_{\varphi_1\times\varphi_2}(x^0)\setminus\{y^{i_1},\dots,y^{i_m}\} \subset Y_1 \times Y_2 \setminus \overline{U}.$$

We choose open sets $U_1, \dots, U_n$ in $Y_1 \times Y_2$ such that

$$\begin{aligned}
&y^i = (y_1^i, y_2^i) \in U_i, && i = 1, \dots, n,\\
&U_i = U_1^i \times U_2^i \subset U, && i = 1, \dots, m,\\
&U_i \subset Y_1 \times Y_2 \setminus \overline{U}, && i = m+1, \dots, n,\\
&U_i \cap U_j = \emptyset, && i \neq j.
\end{aligned}$$

Then we have

$$\begin{aligned}
\sum_{(y_1,y_2)\in U} \omega_{\varphi_1\times\varphi_2}((x_1^0, x_2^0), (y_1, y_2)) &= \sum_{(y_1,y_2)\in U} \omega_{\varphi_1}((x_1^0, y_1) \cdot \omega_{\varphi_2}(x_2^0, y_2)\\
&= \sum_{(y_1,y_2)\in U\cap(U_1\cup\ldots\cup U_m)} \omega_{\varphi_1}((x_1^0, y_1) \cdot \omega_{\varphi_2}(x_2^0, y_2)\\
&= \sum_{j=1}^{m} \sum_{(y_1,y_2)\in U_j} \omega_{\varphi_1}((x_1^0, y_1) \cdot \omega_{\varphi_2}(x_2^0, y_2)\\
&= \sum_{j=1}^{m} \sum_{(y_1,y_2)\in U_j^1\times U_j^2} \omega_{\varphi_1}((x_1^0, y_1) \cdot \omega_{\varphi_2}(x_2^0, y_2)\\
&= \sum_{j=1}^{m} \Bigg(\sum_{y_1\in U_1^j} \omega_{\varphi_1}((x_1^0, y_1) \Bigg) \cdot \Bigg(\sum_{y_2\in U_2^j} \omega_{\varphi_2}(x_2^0, y_2) \Bigg).
\end{aligned}$$

Since φ_1 and φ_2 are weighted mappings, we have:

$$\nu_{\varphi_1}(x_1^0) \cap \delta U_1^j = \emptyset \quad \text{and} \quad \nu_{\varphi_2}(x_2^0) \cap \delta U_2^j = \emptyset.$$

Therefore, we can choose open neighbourhoods $V_{x_1^0}^j$ and $V_{x_2^0}^j$ of x_1^0 and x_2^0 such that

$$\sum_{y_1\in U_1^j} \omega_{\varphi_1}(x_1^0, y_1) = \sum_{y_1\in U_1^j} \omega_{\varphi_1}(x_1, y_1),$$

where $x_1 \in V_{x_1^0}^j$, $j = 1, \dots, m$. Moreover, we have:

$$\sum_{y_2\in U_2^j} \omega_{\varphi_2}(x_2^0, y_2) = \sum_{y_2\in U_2^j} \omega_{\varphi_2}(x_2, y_2),$$

for $x_2 \in V_{x_2^0}^j$, $j = 1, \dots, m$.

Now, since $\nu_{\varphi_1\times\varphi_2}$ is u.s.c. we are able to find an open neighbourhood V_{x^0} of x^0 such that

$$\overline{V}_{x^0} = \overline{V}_{x_1^0} \times \overline{V}_{x_2^0} \quad \text{and} \quad \nu_{\varphi_1\times\varphi_2}(\overline{V}_{x^0}) \subset \bigcup_{i=1}^{n} U_i.$$

We let $V_{x_1^0}$ and $V_{x_2^0}$ as follows:

$$V_{x_1^0} = V^1_{x_1^0} \cap \ldots \cap V^m_{x_1^0} \cap V^n_{x_1^0} \quad \text{and} \quad V_{x_2^0} = V^1_{x_2^0} \cap \ldots \cap V^m_{x_2^0} \cap V^n_{x_2^0}.$$

Now, it is easy to see that for $V_{x^0} = V_{x_1^0} \times V_{x_2^0}$ (79.1.3) holds true and the proof is completed □

As a special case of (79.9) we get:

(79.11) Corollary. *Let $\varphi = (\nu_\varphi, \omega_\varphi)\colon X \multimap Y$ and $\psi = (\nu_\psi, \omega_\psi)\colon X \multimap Z$ be two weighted mappings. Then the map $\varphi \times_\Delta \psi = (\nu_\psi \times_\Delta \nu_\psi, \omega_\varphi \times_\Delta \omega_\psi)\colon X \times X \multimap Y \times Z$ is a weighted map, where*

$$\nu_{\varphi\times_\Delta\psi}(x) = \nu_\varphi(x) \times \nu_\psi(x) \quad \textit{and} \quad \omega_{\varphi\times_\Delta\psi}(x,(y,z)) = \omega_\varphi(x,y)\cdot\omega_\psi(x,z),$$

for every $x \in X$, $y \in Y$ and $z \in Z$.

(79.12) Definition. Let $\varphi = (\nu_\varphi, \omega_\varphi)$ and $\psi = (\nu_\psi, \omega_\psi)$ be two weighted mappings from X into Y. We shall say that φ is ω-homotopic to ψ (written $\varphi \sim_\omega \psi$) provided there exists a weighted map $h = (\nu_h, \omega_h)\colon X \times [0,1] \multimap Y$ such that the following two conditions are satisfied:

$$\omega_h((x,0),y) = \omega_\psi(x,y) \quad \text{and} \quad \omega_h((x,1),y) = \omega_\varphi(x,1),$$

for every $x \in X$ and $y \in Y$.

(79.13) Remark. Observe that we are not assume that

$$h(x,0) = \nu_\psi(x) \quad \text{and} \quad h(x,1) = \nu_\varphi(x).$$

It is easy to see that "$\sim_\omega$" is an equivalence relation in the class of all weighted mappings from X to Y.

We shall end this section by considering index $I_\omega(\varphi)$ of ω-map $\varphi(\nu_\varphi, \omega_\varphi)$.

(79.14) Definition. Let X be a connected space and let $\varphi = (\nu_\varphi, \omega_\varphi)\colon X \multimap Y$ be a weighted map. Then we let:

$$I_\omega(\varphi) = \sum_{y\in Y} \omega_\varphi(x,y) \quad \text{for some } x \in X.$$

(79.15) Remark. Observe that in (79.5) we have proved that $I_\omega(\varphi)$ does not depend on the choice of $x \in X$ (comp. (79.7)).

In the following proposition we shall list some important properties of the above defined index.

(79.16) PROPOSITION. *We have:*

(79.16.1) *if* $\varphi = (\nu_\varphi, \omega_\varphi), \psi = (\nu_\psi, \omega_\psi)\colon X \multimap Y$ *are* ω*-homotopic, then*

$$I_\omega(\varphi) = I_\omega(\psi),$$

(79.16.2) *if* $\varphi = (\nu_\varphi, \omega_\varphi)\colon X \multimap Y$ *and* $\psi = (\nu_\psi, \omega_\psi)\colon Y \multimap Z$ *are two weighted mappings, then*

$$I_\omega(\psi \circ \varphi) = I_\omega(\psi) \cdot I_\omega(\varphi).$$

(79.15.3) *if* $\varphi = (\nu_\varphi, \omega_\varphi)\colon X_1 \multimap Y_1$ *and* $\psi = (\nu_\psi, \omega_\psi)\colon X_2 \multimap Y_2$ *are two weighted mappings, then*

$$I_\omega(\psi \times \varphi) = I_\omega(\varphi) \cdot I_\omega(\psi),$$

(79.16.4) *if* $\varphi = f\colon X \to Y$ *is a continuous singlevalued mapping, then* $I_\omega(f) = 1$.

PROOF. (79.16.1) Let $h = (\nu_h, \omega_h)\colon X \times [0,1] \multimap Y$ be a ω-homotopy joining φ and ψ. Then we have

$$I_\omega(h) = \sum_{y\in Y} \omega_h((x_0,1),y) = \sum_{y\in Y} \omega_\varphi(x_0,y) = \sum_{y\in Y} \omega_h((x_0,0),y) = \sum_{y\in Y} \omega_\psi(x_0,y).$$

Observe that $X \times [0,1]$ is connected provided X is connected.

(79.16.2) Let $x_0 \in X$. Then we have

$$\begin{aligned}
I_\omega(\psi \circ \varphi) &= \sum_{z\in Z} \omega_{\psi\circ\varphi}(x_0,z) = \sum_{z\in Z} \Big(\sum_{y\in Y} \omega_\varphi(x_0,y)\cdot\omega_\psi(y,z)\Big) \\
&= \sum_{y\in Y}\sum_{z\in Z} \omega_\varphi(x_0,y)\cdot\omega_\psi(y,z) = \sum_{y\in Y} \omega_\varphi(x_0,y)\cdot\sum_{z\in Z}\omega_\psi(y,z) \\
&= \sum_{y\in Y} \omega_\varphi(x_0,y)\cdot I_\omega(\psi) = I_\omega(\psi)\cdot\sum_{y\in Y}\omega_\varphi(x_0,y) = I_\omega(\psi)\cdot I_\omega(\varphi).
\end{aligned}$$

(79.16.3) Let $x_1 \in X_1$ and $x_2 \in X_2$. We have

$$\begin{aligned}
I_\omega(\psi\times\varphi) &= \sum_{(y_1,y_2)\in Y_1\times Y_2} \omega_{\psi\times\varphi}((x_1,x_2),(y_1,y_2)) \\
&= \sum_{(y_1,y_2)\in Y_1\times Y_2} \omega_\varphi(x_1,y_1)\cdot\omega_\varphi(x_2,y_2) \\
&= \sum_{y_1\in Y_1}\sum_{y_2\in Y_2} \omega_\varphi(x_1,y_1)\cdot\omega_\varphi(x_2,y_2) \\
&= \sum_{y_1\in Y_1} \omega_\varphi(x_1,y_1)\cdot\sum_{y_2\in Y_2}\omega_\varphi(x_2,y_2) = I_\omega(\varphi)\cdot I_\omega(\psi).
\end{aligned}$$

(79.16.4) is self evident. □

Finally, let $\varphi = (\nu_\varphi, \omega_\varphi), \psi = (\nu_\psi, \omega_\psi): X \multimap Y$ be two weighted mappings and assume that Y is a normed space (over $\mathbb{R}$).

We define algebraic sum of φ and ψ by letting:

$$\varphi + \psi = (\nu_{\varphi+\psi}, \omega_{\varphi+\psi}),$$

where, for every $x \in X$, $y, z \in Y$,

$$\nu_{\varphi+\psi}(x) = \{u + v \mid u \in \varphi(x) \text{ and } v \in \psi(x)\},$$
$$\omega_{\varphi+\psi}(x, y) = \sum_{z \in Y} \omega_\varphi(x, y - z) \cdot \omega_\psi(x, z).$$

To show that $\varphi + \psi = (\nu_{\varphi+\psi}, \omega_{\varphi+\psi})$ is a weighted map we consider two (continuous) singlevalued mappings:

$$\Delta: X \to X \times X, \qquad \Delta(x) = (x, x),$$
$$f: Y \times Y \to Y, \qquad f(x, y) = x + y.$$

Then we have

$$f \circ (\varphi \times \psi) \circ \Delta = (\nu_{f \circ(\varphi\times\psi)\circ\Delta}, \omega_{f\circ(\varphi\times\psi)\circ\Delta}) = (\varphi + \psi) = (\nu_{\varphi+\psi}, \omega_{\varphi+\psi})$$

and our claim follows from (79.8.3).

Now, let $\varphi = (\nu_\varphi, \omega_\varphi): X \multimap Y$ be a weighted map and let $s: X \to \mathbb{R}$ be a continuous singlevalued map. We let $s \circ \varphi = (\nu_{s\circ\varphi}, \omega_{s\circ\varphi}): X \multimap Y$ where

$$\nu_{s\circ\varphi}(x) = \{s(x) \cdot u \mid u \in \nu_\varphi(x)\},$$
$$\omega_{s\circ\varphi}(x, y) = \begin{cases} \omega_\varphi\left(x, \dfrac{y}{s(x)}\right) & \text{if } s(x) \neq 0, \\ \displaystyle\sum_{z \in Y} \omega_\varphi(x, z) & \text{if } s(x) = 0. \end{cases}$$

Then, we have

$$s \circ \varphi = (\nu_{s\circ\varphi}, \omega_{s\circ\varphi}) = f \circ (s \times \varphi) \circ \Delta = (\nu_{f\circ(s\times\varphi)\circ\Delta}, \omega_{f\circ(s\times\varphi)\circ\Delta}),$$

where $f: \mathbb{R} \times Y \to Y$, $f(k, y) = k \cdot y$.

Consequently, in view of (79.8.3) $s \circ \varphi$ is a weighted map. Moreover, we have:

$$\begin{aligned} I_\omega(\psi + \varphi) &= I_\omega(f \circ (\varphi \times \psi) \circ \Delta) \\ &= I_\omega(f) \cdot I_\omega(\varphi \times \psi) \cdot I_\omega(\Delta) = I_\omega(\varphi) \cdot I_\omega(\psi), \\ I_\omega(s \cdot \varphi) &= I_\omega(f \circ (s \times \psi) \circ \Delta) \\ &= I_\omega(f) \cdot I_\omega(s) \cdot I_\omega(\varphi) \cdot I_\omega(\Delta) = I_\omega(\varphi). \end{aligned}$$

80. Darbo homology functor and its applications to fixed point problems

It is well known that homology theory has many direct applications to the fixed point theory. The homology functor, so called Darbo homology, is very useful in problems connected with the fixed point theory of weighted mappings. Note, that S. Darbo constructed his functor in 1958. Roughly speaking Darbo extended the singular homology functor from the category $\mathcal{E}$ of metric spaces and continuous (single valued) maps onto the category $\mathcal{E}_\Omega$ of metric spaces and ω-mappings.

Below, we shall restrict our consideration only to the necessary result from the point of view of our applications.

Let Δ^n be the standard n-dimensional simplex with vertices

$$e_0 = (1, 0, \dots, 0), \dots, e_n = (0, \dots, 0, 1) \quad \text{in } \mathbb{R}^{n+1}$$

and let X be a metric space. We let $C_n(X) = \langle \Delta_n, X\rangle$, where $\langle \Delta_n, X\rangle$ is the set of all ω-homotopy classes of ω-mappings from Δ_n to X.

As we remarked in the proceeding section $\langle \Delta_n, X\rangle$ is an Ω-module. Now, similarly as for singular homology, we shall define the boundary homomorphism

$$\delta_n \colon C_n(x) \to C_{n-1}(x).$$

To do it first we let:

$$d_n^i \colon \Delta_{n-1} \to \Delta_n, \qquad d_n^i(e_j) = \begin{cases} e_j & \text{if } j < i, \\ e_{j+1} & \text{if } j \geq i, \end{cases}$$

where $i = 0, \dots, n$.

Secondly, we define a ω-map $d_n \colon \Delta_{n-1} \to \Delta_n$ by putting

$$d_n = \bigcup_{i=0}^{n} (-1)^i d_n^i.$$

Finally, for every $[\varphi] \in C_n(X)$, we let:

$$\delta_n \colon C_n(X) \to C_{n-1}(X), \qquad \delta_n([\varphi]) = [\varphi \circ d_n].$$

Now, it is easy to see that $(C(X), \delta)$ is a chain complex. Consequently homology of $(C(x), \delta)$ are called *Darbo homology* of X, i.e.

$$H(X) = \{H_n(X)\}_{n\geq 0},$$

where $H_n(X) = H_n(C(X))$, for every $n \geq 0$. Moreover, for a pair (X, A) of spaces in ε_Ω we let:

$$H(X, A) = H(C(x)/C(A)),$$

i.e.

$$H_n(X,A) = H_n(C(X)/C(A)), \quad n = 0,1,\ldots$$

Now, if $\varphi = (\nu_\varphi, \omega_\varphi)\colon X \multimap Y$ is weighted map, then we define the induced homomorphism $\varphi_*\colon H(X) \to H(Y)$ by letting:

$$\varphi_* = (C(\varphi)_*),$$

where $C(\varphi)_* = \{C_n(\varphi)_*\}$ and $C_n(\varphi)(\sigma) = \varphi \circ \sigma$, for every $\sigma \in C_n(X)$.

Now, let us remark that the above defined homology functor satisfies all Eilenberg–Steenrod axioms and moreover it is a functor with compact carriers.

Now, let $\varphi = (\nu_\varphi, \omega_\varphi)\colon X \multimap X$ be a weighted map. We define the Lefschetz number $\Lambda(\varphi)$ of φ by putting

$$\Lambda(\varphi) = \Lambda(\varphi_*)$$

provided $\varphi_*\colon H(X) \to H(X)$ is a Leray endomorphism; now for simplicity we take Ω to be a field.

All properties of the Lefschetz number formulated in sections 10 and 11 are true in the case of ω-mappings.

Using the Darbo homology functor, in the place of the Čech homology functor with compact carriers, we can get for ω-mappings the same topological invariants as for admissible mappings or for morphism (see Sections 41, 42, 52, 53) i.e. the Lefschetz fixed point theorem, the fixed point index and the topological degree theory. For details we recommend [Ski-M], [Pej5], [ScS], [Ski1] and [Ski2].

Finally, we would like point out that approximation methods are also possible in the case of weighted mappings (comp. Chapter III). To see what types of approximation results are possible for weighted mappings we need some notions.

(80.1) Definition. Let $\psi\colon X \multimap Y$ be an u.s.c. mapping with compact values and let V be an open subset of Y such that $\psi(x) \cap \delta V = \emptyset$, for some $x \in X$. Then the triple (ψ, V, x) is called *admissible*.

Let AT be the set of all admissible triples and let Ω be a ring.

(80.2) Definition. A map $I_{\mathrm{loc}}\colon \mathrm{AT} \to \Omega$ is called a *local index* provided the following conditions are satisfied:

(80.2.1) if $I_{\mathrm{loc}}(\psi, V, x) \neq 0$, then $\psi(x) \cap V \neq \emptyset$,

(80.2.2) for every admissible triple (ψ, V, x), there exists an open neighbourhood U_x of x in X such that $I_{\mathrm{loc}}(\psi, V, x) = I_{\mathrm{loc}}(\psi, V, x')$, for every $x' \in U_x$,

(80.2.3) if $\psi(x) \cap V \subset \bigcup_{j=1}^k V_j \subset V$, where V_j, $j = 1, \ldots, k$ is an open subset of Y and $V_i \cap V_j = \emptyset$, for $i \neq j$, then

$$I_{\mathrm{loc}}(\psi, V, x) = \sum_{j=1}^{k} I_{\mathrm{loc}}(\psi, V_j, x).$$

(80.3) Definition. An u.s.c. map $\psi: X \multimap Y$ with compact values is called *m-map* provided for any admissible triple (ψ, V, x) the local index $I_{\mathrm{loc}}(\psi, V, x)$ of (ψ, V, x) is well defined.

Observe that if $\psi = (\nu_\psi, \omega_\psi)$ is a ω-map, then ψ is a m-map, if we let:

$$I_{\mathrm{loc}}(\psi, V, x) = \sum_{y \in V} \omega_\psi(x, y).$$

Several examples of m-maps can be given. Note that, in particular, any acyclic map $\varphi: X \multimap Y$ is a m-map (for more examples see [Ski-M]).

Let X be a metric space and $K \subset X$ be compact. We shall say that $K \in \omega$-PC_X^∞ (comp. (2.17)) provided for every open U in X such that $K \subset U$ there exists open neighbourhood $V \subset U$ of K such that for every $n = 0, 1, \dots$ and for every ω-map $\psi: \delta\Delta^{n+1} \to V$ there exists a ω-extension $\widetilde{\psi}: \Delta^n \to U$ of ψ.

(80.4) Theorem. *Let X be a compact* ANR*-space, $A \in$* ANR *be a closed subset of X and Y be an arbitrary space. If $\psi: X \multimap Y$ be a m-map with ω-PC_Y^∞ values, then for every $\varepsilon > 0$ there exists $\delta > 0$ such that, if $\varphi_0: A \multimap Y$ is a ω-map such that $\Gamma_{\nu_{\varphi_0}} \subset O_\delta(\Gamma_\psi)$, then there exists a ω-map $\psi: X \multimap Y$ such that $\nu_{\varphi_0}(x) = \nu_\varphi(x)$, for every $x \in A$ and $\Gamma_{\nu_\varphi} \subset O_\varepsilon(\Gamma_\psi)$.*

For possible applications of (80.4) see Chapter III.

81. More about spheric mappings

The notion of spheric mappings was considered in Section 54. During last four years important generalizations of all results presented in Section 54 were obtained. Most important new results belongs to D. Miklaszewski (see [Mik1-M]).

Note, that in this section we would like to survey these results. In particular, we shall present the full answer on the open problem (54.14). Unfortunately the problem (54.13) is still open.

The notion of a Borsuk continuous map and a Hausdorff continuous map was introduced in Section 20. Let us recall only that d_C stands for the continuity metric of Borsuk and d_H stands for the Hausdorff metric. We shall use also the homotopy metric d_h defined by K. Borsuk.

Let Y be a metric space and let $\mathrm{ANR}(Y)$ be the family of all compact ANRs contained in Y. The Borsuk metric of homotopy d_h is defined on $\mathrm{ANR}(Y)$, i.e.

$$d_h: \mathrm{ANR}(Y) \times \mathrm{ANR}(Y) \to [0, \infty).$$

To define d_h let us fix $t \geq 0$ and a locally contractible compact subset $A \subset Y$. We define $\varphi_A(t)$ to be the lower bound of the set, which is composed of 1 and $s \geq t$

such that every set $T \subset A$ with the diameter $\operatorname{diam} T \leq t$ is contractible in a set $S \subset A$ with $\operatorname{diam} S \leq s$.

We say, that sets from the class $\Theta \subset 2^Y$ are equally locally contractible (e.l.c.), if

$$\text{for all } \varepsilon > 0 \text{ there exists } \delta > 0 \text{ such that } \varphi_A(\delta) < \varepsilon \text{ for all } A \in \Theta.$$

Let $\Gamma_{\mathrm{sub}}(\varphi_A) = \{(t,u) \mid u \leq \varphi_A(t) \text{ and } t \geq 0\}$ and $\lambda_A(t) = \sup\{s \mid (t,s) \in \operatorname{conv}(\Gamma_{\mathrm{sub}}(\varphi_A))\}$. Then

$$\rho_h(A,B) = \rho_c(A,B) + \sup\{|\lambda_A(t) - \lambda_B(t)| \mid t \geq 0]. \tag{81.1}$$

Let $\varphi\colon X \multimap \mathbb{R}^n$ be a multivalued map with compact values. Denote by $B\varphi(x)$ the union of all bounded components of $\mathbb{R}^n \setminus \varphi(x)$. Then we let (see Section 54):

$$\widetilde{\varphi}(x) = \varphi(x) \cup B\varphi(x).$$

A map $\varphi\colon K^n \multimap K^n$ with compact values is called *spheric* provided the following conditions are satisfied:

(81.2.1) φ is u.s.c.

(81.2.2) the graph $\Gamma(B\varphi)$ of the mapping $B\varphi$ is an open subset of $K^n \times \mathbb{R}^n$,

(81.2.3) the mapping $\widetilde{\varphi}$ has a fixed point, where K^n denotes the unit closed ball in $\mathbb{R}^n$.

Note that the above definition is slightly different as (54.9).

Recall that in Section 54 we have proved the following two theorems:

(81.3) THEOREM.

(81.3.1) *Every spheric map has a fixed point.*

(81.3.2) *Every d_c-continuous mapping $\varphi\colon K^2 \multimap K^2$ with compact connected values is a spheric map.*

According to the open problem (54.14) we would like to present the following four results (all proved by D. Miklaszewski) which gives us an answer on (54.14).

(81.4) THEOREM ([5]. *There is a fixed point free mapping $\varphi\colon K^n \multimap K^n$ with compact connected values which is d_c-continuous, for every $n \geq 4$.*

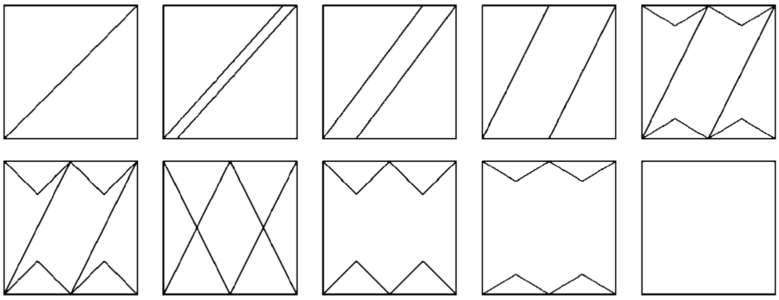
(81.5) THEOREM. *Every d_h-continuous mapping $\varphi\colon K^n \multimap K^n$ has a continuous selector and hence has a fixed point.*

([5]) Note, that for $n = 2$ any d_c-continuous map with compact connected values has a fixed point (see (54.12)). For $n = 3$ problem (54.14) is still open.

(81.6) Theorem. *Every d_H-continuous mapping $\varphi\colon K^n \multimap K^n$ with eLC^{n-2} values, such that the mapping $\widetilde{\varphi}$ has eLC^{n-1} values is spheric and has a fixed point.*

(81.7) Theorem. *Every d_c-continuous mapping $\varphi\colon K^n \multimap K^n$, $n \neq 6$, such that for every $x \in K^n$ the set $\varphi(x)$ is homeomorphic either to a one point space or to the $(n-2)$-sphere S^{n-2}, has a fixed point.*

In the proof of (81.4) we need an example of J. Jezierski ([Je]) of d_H-continuous homotopy $\chi^J\colon S^1 \times [0,1] \multimap S^1$ joining the (singlevalued) identity map id_{S^1} and (singlevalued) constant map with values being finite sets.



This picture shows graphs of χ^J_t for $t = 0, 1/9, \dots, 8/9$ ("as time goes by") under an obvious identification of $S^1 \times S^1$ and $J \times J$, where $J = [0,1]$.

Proof of (81.4). Let us recall that

$$\begin{aligned} d_c(X,Y) &= \max\{d(X,Y), d(Y,X)\},\\ d(X,Y) &= \inf\{\max\{\|\alpha(x) - x\| \mid x \in X\}\}, \end{aligned}$$

where the infimum is taken over all continuous functions $\alpha\colon X \to Y$; $(X, Y \subset K^4)$. The disc K^4 will be identified with $K^2 \times K^2$. Set $H = (\chi x^J)^{-1}$. This formula yields a ρ_s-continuous homotopy $H\colon S^1 \times I \multimap S^1$ joining $H(z,0) = \{z_0\}$ and $H(z,1) = \{z\}$ such that $H(z,t)$ is a finite subset of S^1, which has at most 3 elements for every $(z,t) \in S^1 \times I$. The multivalued retraction $r\colon K^2 \multimap S^1$ is the standard one:

$$r(0) = \begin{cases} \{z_0\}, & \text{for } \|x\| \leq \dfrac{1}{2},\\[2mm] H\left(\dfrac{x}{\|x\|}, 2\|x\| - 1\right), & \text{for } \|x\| \in \left[\dfrac{1}{2}, 1\right]. \end{cases}$$

We define $J\colon K^2 \multimap S^1$ by

$$J(0) = \begin{cases} -r(3x), & \text{for } \|x\| \leq \dfrac{1}{3},\\[2mm] \left\{\dfrac{x}{\|x\|}\right\}, & \text{for } \|x\| \in \left[\dfrac{1}{3}, 1\right]. \end{cases}$$

Of course, J is ρ_s-continuous and has finite values. Since $\rho_s = \rho_c$ on finite sets, J is d_c-continuous. The mapping J is fixed point free, moreover, $x \notin [1/2,1]J(x)$ for every $x \in K^2$. This is easy to check that the join of sets $A \subset S^1 \times \{0\}$ and $B \subset \{0\} \times S^1$ in $K^2 \times K^2$ is well defined by

$$A * B = \{(1-t)a + tb \mid t \in [0,1], a \in A, b \in B\}.$$

Let $\varphi_1, \varphi_2, \varphi\colon K^2 \multimap K^2 \times K^2$ be given by

$$\varphi_1(x,y) = J(x) \times \{0\}, \quad \varphi_2(x,y) = \{0\} \times J(y), \quad \varphi(p) = \varphi_1(p) * \varphi_2(p).$$

We check now that φ is d_c-continuous. Take an $\varepsilon > 0$. Since φ_i, is ρ_c-continuous, there is a positive δ such that

$$\text{if } \|p-q\| < \delta \text{ then } \rho_c(\varphi_i(p), \varphi_i(q)) < \varepsilon, \text{ for } i = 1,2.$$

Fix $p, q \in K^2 \times K^2$ with $\|p-q\| < \delta$. By the definition of ρ_c, there is a continuous map $\alpha_i\colon \varphi_i(p) \to \varphi_i(q)$ such that $\|\alpha_i(v) - v\| < \varphi$, for every $v \in \varphi_i(p)$. Let $\alpha_1 * \alpha_2\colon \varphi(p) \to \varphi(q)$ be the join of maps α_1 and α_2. Take $x = (1-t)u_1 + tu_2 \in f(p)$ with $u_i \in \varphi_i(p)$. Thus $\|\alpha_1 * \alpha_2(x) - r\| = \|(1-t)\alpha_1(u_1) + t\alpha_2(u_2) - (1-t)u_1 - tu_2\| \le (1-t)\|\alpha_1(u_1) - u_1\| + t\|\alpha_2(u_2) - u_2\| < \varepsilon$. Hence $d_c(\varphi(p), \varphi(q)) < \varepsilon$. Likewise $d_c(\varphi(p), \varphi(q)) < \varepsilon$.

The mapping φ is fixed point free. Otherwise, there is $(x,y) \in K^2 \times K^2$ such that

$$(x,y) \in \varphi(x,y) = \{((1-t)a, tb) \mid t \in [0,1], a \in J(x), b \in J(y)\}.$$

Thus $x = (1-t)a \in [1/2,1]J(x)$, for $t \in [0,1/2]$ and $y = tb \in [1/2,1]J(y)$, for $t \in [1/2,1]$, a contradiction.

The values of φ being joins of some finite sets are compact and connected (these are graphs of 4 homotopy types: $\cdot, \circ, \ominus, \oplus$). □

Before proving (81.5) we need some additional notions and results.

As we already seen the homology theory will provide us here with the basic tools of proving that some set-valued mappings have fixed points. In the other words, we show that a homology property of graphs forces that the corresponding mappings have fixed points. We call these mappings the Brouwer mappings.

Let $\check{H}_*$ denote the Čech homology functor, F be a field. For simplicity we shall denote $K^n = B$, $S^{n-1} = S$ and $\Gamma(\varphi|_A)$ – the graph of the restriction $\varphi|_A$ of $\varphi\colon B \multimap B$ to A. Moreover, as usually $p\colon \Gamma(\varphi|_A) \to A$ is projection $p = p_{\varphi|_A}$.

(81.8) Definition. The upper-semicontinuous compact-valued map $\varphi\colon B \multimap B$ is called an *F-Brouwer mapping* if and only if

$$\check{H}_n(\Gamma(\varphi|B), \Gamma(\varphi|S); F) \xrightarrow{i_*} \check{H}_n(B \times B, S \times B; F)$$

induced by inclusion is a non-zero homomorphism.

From now on we consider only upper-semicontinuous compact-valued mappings.

(81.9) LEMMA. *The following conditions are equivalent:*

(81.9.1) *φ is an F-Brouwer mapping,*
(81.9.2) *$i_*\colon \check{H}_n(\Gamma(\varphi|B),\Gamma(\varphi|S);F) \to \check{H}_n(B\times B, S\times B;F)$ is an epimorphism,*
(81.9.3) *$p_*\colon \check{H}_n(\Gamma(\varphi|B),\Gamma(\varphi|S);F) \to \check{H}_n(B,S;F)$ is non-zero,*
(81.9.4) *p_* is an epimorphism.*

PROOF. The equivalence (81.9.3) to (81.9.4) follows from the fact, that

$$\check{H}_n(B,S;F) = F$$

and F is a field. The homomorphism $j_*\colon \check{H}_n(B,S;F) \to \check{H}_n(B\times B, S\times B;F)$ which is induced by the homotopy equivalence $j(x)=(x,0)$, is an isomorphism. Moreover, $j\circ p \sim i$, which proves the lemma. □

First, we prove

(81.10) THEOREM. *Every F-Brouwer mapping has a fixed point.*

PROOF. On the contrary, suppose that an F-Brouwer mapping φ has no fixed point. Let $\Delta = \{(x,x) \mid x\in B\}$. The following diagram

$$\begin{array}{ccc}
\check{H}_n(\Gamma(\varphi|B),\Gamma(\varphi|B)) & \xrightarrow{\ i_*\ } & H_n(B\times B, S\times B) \\
\Big\| & & \Big\uparrow \\
\check{H}_n(\Gamma(\varphi|B),\Gamma(\varphi|B)) & \longrightarrow & H_n(B\times B\setminus\Delta, S\times B\setminus\Delta)
\end{array}$$

with all arrows induced by inclusions, is commutative. (We omit $(\check{\ })$ for some Čech homology groups which are isomorphic to the singular ones). Since $S\times B\setminus\Delta$ is the deformation retract of $B\times B\setminus\Delta$, $H_n(B\times B\setminus\Delta, S\times B\setminus\Delta)=0$ and $i_*=0$, a contradiction.

It remains to define a suitable deformation retraction. For $x\neq y\in B$ we denote by $s(x,y)$ the unique point $s\in S$ such that $s=y+\lambda(x-y)$ for a positive number λ. Define $r\colon (B\times B\setminus\Delta)\times I \to B\times B\setminus\Delta$ by the formula $r((x,y),t) = ((1-t)x+ts(x,y),y)$. It follows that $r\colon \mathrm{id}\sim r_1$ and r_1 is a strong deformation retraction from $B\times B\setminus\Delta$ onto $S\times B\setminus\Delta$. □

The mapping $\psi\colon B \multimap B$ is called a selector of φ if $\psi(x)\subset\varphi(x)$ for every $x\in B$. The inclusion $(\Gamma(\psi|B),\Gamma(\psi|S))\subset(\Gamma(\varphi|B,\Gamma|S)$ implies the following

(81.11) LEMMA. *Every map having an F-Brouwer selector is F-Brouwer mapping too.*

Any compact neighbourhood U of $\Gamma(\varphi|B)$ in $B\times B$ determines a set-valued map $\varphi_U\colon B\multimap B$ such that $\varphi_U(x)=\{y\in B \mid (x,y)\in U\}$. We have

$$(\Gamma(\varphi_U|B),\Gamma(\varphi_U|S) = (U, U\cap(S\times B)).$$

Recall that on the category of compact pairs functors $\check{H}_*$ and $\mathrm{Hom}_F \circ \check{H}_*$ are naturally isomorphic. (see Chapter I and [Go1-M]). The above fact, the continuity of the Čech cohomology functor vH_* (see [EM-M]) and the formula $\mathrm{Hom}(\,\cdot\,,F)\circ$ dir lim $=$ inv lim $\circ\, \mathrm{Hom}(\,\cdot\,,F)$ give

$$\check{H}_n(\Gamma(\varphi|B)), \Gamma(\varphi|S)) = \mathrm{inv}\lim\{\check{H}_n(U, U\cap(S\times B))\}.$$

We say that the set-valued map $\varphi\colon B \multimap B$ is approximable by F-Brouwer mappings if for every compact neighborhood U of $\Gamma(\varphi|B)$ in $B\times B$ the map φ_U has an F-Brouwer selector. We have the following generalization of Lemma (81.11).

(81.12) Lemma. *Every map approximable by F-Brouwer mappings is an F-Brouwer mapping too.*

The proof of (81.12) is strightforward.

The map having F-acyclic values is called an F-acyclic map.

(81.13) Lemma. *The composition $B \xrightarrow{\varphi} B \xrightarrow{\psi} B$ of an F-acyclic map ψ and an F-Brouwer mapping φ is an F-Brouwer mapping.*

Proof. Let C be B or S. Consider the following commutative diagram

$$\begin{array}{ccccc} \Gamma(\varphi|C) & \xrightarrow{\quad q \quad} & B & \xleftarrow{\quad p \quad} & \Gamma(\varphi|B) \\ & \nwarrow{\scriptstyle \overline{p}} & & \nearrow{\scriptstyle \overline{q}} & \\ & & \Gamma(\varphi|C)*\Gamma(\varphi|B) & & \end{array}$$

where $q(x,y)=y$, $p(y_1,z)=y_1$,

$$\Gamma(\varphi|C)*\Gamma(\psi|B) = \{(x,y,y,z) \mid (x,y)\in\Gamma(\varphi|C), (y,z)\in\Gamma(\psi|B)\},$$

and $\overline{p}(x,y,y,z)=(x,y)$, $\overline{q}(x,y,y,z)=(y,z)$. Assumption that ψ is an F-acyclic map implies that p, $\overline{p}$ are Vietoris maps and $\overline{p}_*\colon \check{H}_*(\Gamma(\varphi|C)*\Gamma(\psi|B)) \to \check{H}_*(\Gamma(\varphi|C))$ is an isomorphism in the following commutative diagram

$$\begin{array}{ccccc} \check{H}_n(\Gamma^{\varphi}_B*\Gamma^{\psi}_B, \Gamma^{\varphi}_S*\Gamma^{\psi}_B) & \xrightarrow{\pi_*} & \check{H}_n(\Gamma^{\psi\circ\varphi}_B*\Gamma^{\psi\circ\varphi}_B) & \xrightarrow{i_*} & \check{H}_n(B\times B, S\times B) \\ \Big\downarrow{\scriptstyle \overline{p}_*} & & & & \Big\uparrow{\scriptstyle j_*} \\ \check{H}_n(\Gamma^{\varphi}_B*\Gamma^{\psi}_B) & & & & \\ \Big\downarrow{\scriptstyle p_*} & & & & \\ \check{H}_n(B,S) & = & = & = & \check{H}_n(B,S) \end{array}$$

where $\pi(x,y,y,z)=(x,z)$, $j(x)=(x,0)$ and $\Gamma^{\chi}_C=\Gamma(\chi|C)$. Since $j_*p_*\overline{p}_*$ is an epimorphism, i_* is an epimorphism too. $\square$

(81.14) Definition. We call $\varphi: B \multimap B$ an 1-S^k-*mapping* if and only if is ρ_c-continuous and for every x in B, $\varphi(x)$ is homeomorphic to either a singleton or the topological k-sphere.

The motivation for this definition comes from [Go9] and [Da-M]. Results of these papers show that 1-S^0-mappings and 1-S^{n-1}-mappings of B^n have fixed points, though for different reasons. The 1-S^0-mappings (called bimaps) are equipped with the fixed point index, (see Chapter IV). These mappings can be considered as the single-valued maps of B^n into its second symmetric product. The fixed point theory for this case was developed in Chapter IV. The 1-S^{n-1}-mappings are simplest spheric mappings which have been studied in [Da-M]. The main idea was there to "fill" each value $\varphi(x) \cong S^{n-1}$ with the bounded component $B\varphi(x)$ of $\mathbb{R}^n \setminus \varphi(x)$ and consider the mapping $\widetilde{\varphi}$ with acyclic values $\widetilde{\varphi}(x)(x) = \varphi(x) \cup B\varphi(x)$; (note, that $\widetilde{\varphi}(x)$ does not have to be a disc). Of course, both methods mentioned above do not apply to 1-S^k-mappings with $0 < k < n-1$.

Now, we describe the method of the approximation of 1-S^k-mappings by the mappings from the same class, but having the more regular set of all these points, where the corresponding values are spheres.

(81.15) Definition. Let U be an open subset of $B = B^n$ and $\varepsilon > 0$. We say that an $n-1$-dimensional piecewise linear manifold M ε-approximates $\operatorname{bd} U = \operatorname{bd}_{\mathbb{R}^n} U$ in U, if there exists a compact n-dimensional p.l. manifold K such that $\partial K = M$ and $U \supset K \supset U \setminus O_\varepsilon(\operatorname{bd}_B U)$.

Let us observe that for every U and ε there is a p.l. manifold K such that ∂K ε-approximates $\operatorname{bd} U$ in U: it suffices to take a simplicial decomposition of B with mesh $\leq \varepsilon/2$ and define K to be a small regular neighbourhood of the union of all simplices intersecting $U \setminus O_\varepsilon(\operatorname{bd}_B U)$.

Let $\varphi, \psi: B \multimap B$ be mappings. We say that ψ ε-approximates φ, if $\psi(x) \subset O_\varepsilon \varphi(x)$ for every $x \in B$.

(81.16) Lemma. *Let $\varphi: B \multimap B$ be an 1-S^k mapping and $U_\varphi = \{x \in B \mid \varphi(x) \cong S^k\}$. Then for every $\varepsilon > 0$ there is an $r > 0$ such that for any compact p.l. manifold K with ∂K r-approximating* $\operatorname{bd} U$ *in U there is an 1-S^k-mapping ψ with $U_\psi = \operatorname{Int}_B K$, which ε-approximates φ.*

Proof. Fix an $\varepsilon > 0$. Take $r > 0$ such that $\operatorname{diam} \varphi(x) < \varepsilon$ for all $x \in O_{2r}(\operatorname{bd}_B U)$. Let K be a p.l. manifold with ∂K r-approximating $\operatorname{bd} U$ in U. Take $\rho < r$. Then

$$U \supset K \supset K \setminus O_\rho(\operatorname{bd}_B K) \supset U \setminus O_{2r}(\operatorname{bd}_B U).$$

For every compact convex subset C of $\mathbb{R}^n$ we will denote by $s(C)$ the Steiner point of C (see [Mi-M]). We have $s(C) \in C$ and $\|s(C_1) - s(C_2)\| < n \cdot \rho_s(C_1, C_2)$. Let

$$\lambda(x) = \rho^{-1} \cdot \min(\rho, \operatorname{dist}(x, B \setminus K)), \quad b(x) = s(\operatorname{cl}(\operatorname{conv}(\varphi(x)))),$$
$$\varphi(x) = b(x) + \lambda(x) \cdot (\varphi(x) - b(x)), \quad \text{for } x \in B. \tag{81.16.1}$$

Since $\psi(x) \subset \operatorname{cl}(\operatorname{conv}(\varphi(x)))$ for every x and $\{x \mid \psi(x) \neq \varphi(x)\} \subset 0_{2r}(\operatorname{bd}_B U)$, ψ is an ε-approximation of φ. One can check that ψ is a ρ_c-continuous mapping which takes values homeomorphic to S^k on $\operatorname{Int}_B K$ and which is single-valued elsewhere.$\square$

(81.17) THEOREM. *Let $\varphi\colon B \multimap B$ be an 1-S^k-mapping with $0 < k \neq 4$ and $U = \{x \in B \mid \varphi(x) \cong S^k\}$. Assume that for every $\varepsilon > 0$ there exists a p.l. manifold M, which ε-approximates* $\operatorname{bd} U$ *in U and satisfies the inequality*

$$\dim H_k(\Gamma(\varphi|M_i); Z_2) > \dim H_k(M_i; Z_2) \tag{81.17.1}$$

for all components M_i of M. Then φ is a Z_2-Brouwer mapping.

PROOF. *Case 1.* $U \subset \operatorname{Int} B$.

Fix $\varepsilon > 0$. Take $r > 0$ from the Lemma (81.16). Choose K with $M = \partial K$, r-approximating $\operatorname{bd} U$ in U and satisfying (81.17.1). Define ψ be the ε-approximation of φ, which is the one we have described in the proof of Lemma (81.16).

By Lemma (81.12), it suffices to prove that φ is a Z_2-Brouwer mapping. Consider the following diagram

$$\begin{array}{ccccc}
H_n(\Gamma(\psi|B), \Gamma(\psi|S)) & \longrightarrow & H_{n-1}(\Gamma(\psi|S)) & \xrightarrow{i_*} & H_{n-1}(\Gamma(\psi|B)) \\
{\scriptstyle p_*}\downarrow & & \downarrow{\scriptstyle \cong} & & \\
H_n(B, S) & \xrightarrow[\cong]{} & H_{n-1}(S) & = & Z_2
\end{array}$$

with the first row exact; $n = \dim B$. The right vertical arrow represents an isomorphism because $\varphi|S$ is singlevalued. Note that the condition on φ to be a Z_2-Brouwer mapping ($p_* \neq 0$) is equivalent to $i_* = 0$. We shall define a Z_2-cycle which generates $H_{n-1}\Gamma(\psi|S)$ and which is zero in $H_{n-1}\Gamma(\psi|B)$.

There exists a simplicial decomposition $\mathcal{T}$ of B and a subcomplex $\mathcal{K}$ of $\mathcal{T}$ such that $K = |\mathcal{K}|$. Let us denote by K_i — components of K, by M_{ij} — components of ∂K_i, and by $\mathcal{K}_i$, $\mathcal{M}_{ij}$ — corresponding subcomplexes of the simplicial decomposition $\mathcal{T}$ of B. Let $\mathcal{S} \subset \mathcal{T}$ be such that $S = |\mathcal{S}|$. Fix a linear order in the set of all vertices of $\mathcal{T}$. Ordered and singular simplices determined by $\sigma \in \mathcal{T}$ will be denoted by the same letter σ. If $\varphi|\sigma$ is single-valued then a denotes the singular simplex $\widetilde{\sigma}(x) = (\sigma(x), \varphi(\sigma(x)))$. We use the same notation for chains. All considered chain complexes have Z_2-coefficients. For every $\mathcal{T}' \subset \mathcal{T}$ let $\sum \mathcal{T}'(p)$ denote the chain equal to the sum of all p-simplices of $\mathcal{T}'$. If $1_S = \sum \mathcal{S}(n-1)$, $1_{ij} = \sum \mathcal{M}_{ij}(n-1)$, $c = \sum(\mathcal{T} \setminus \mathcal{K}(n)$ then $1_S = \partial c + \sum_{i,j}$, 1_{ij}, $1_S = \partial\widetilde{c} + \sum_{i,j} 1_{ij}$ and $\widetilde{1_S}$ is a generator of $H_{n-1}\Gamma(\psi|S)$. It suffices to prove that $\sum_j \widetilde{1_{ij}} = 0$ in $H_{n-1}\Gamma(\psi|K_i)$.

Without loss of generality we can assume that K is connected and we omit the index i. There exists a neighbourhood N_1 of ∂K in K (the collar of ∂K in K) and a homeomorphism $h_1\colon N_1 \to \partial K \times [0,2]$ such that $h_1(x) = (x,0)$, for $x \in \partial K$. For simplicity of notation we write $N_1 = \partial K \times [0,2]$. Let $N = \partial K \times [0,1] \subset N_1$ and $L = \mathrm{cl}(K \setminus N)$. We define a homeomorphism $h\colon L \to K$ by formulae: $h(y) = y$ for $y \in L \setminus \partial K \times [1,2]$, $h(x,t) = (x, 2t-2)$ for $(x,t) \in \partial K \times [1,2]$. In particular, $h(x,1) = (x,0)$, i.e. $h(\partial L) = \partial K$.

Let $M'_j = h^{-1}(M_j)$ and $1'_j = h^{-1}l_j$. Of course M'_j is a component of $M' = \partial L$ and the cycle $1'_j$ is a generator of $H_{n-1}(M'_j)$. We assume that ρ, (chosen in the proof of Lemma (81.16) is small enough, i.e. that $0_\rho(\partial K) \subset N$ and consequently, $\psi = \varphi$ on M'.

Consider the following commutative diagram

$$\begin{array}{ccccc} & & H_{n-1}(\Gamma(\psi|\partial K)) & = & H_{n-1}(\Gamma(\psi|\partial K)) \\ & & \downarrow{\scriptstyle (0,u)} & & \downarrow{\scriptstyle v} \\ H_{n-1}(\Gamma(\psi|\partial L)) & \xrightarrow{(-\beta,\alpha)} & H_{n-1}(\Gamma(\psi|\partial L)) \oplus H_{n-1}(\Gamma(\psi|\partial N)) & \longrightarrow & H_{n-1}(\Gamma(\psi|\partial K)) \end{array}$$

where α, β, u, v are induced by inclusions and the second row is a segment of the Mayer–Vietoris exact sequence. We are reduced to proving that $v(\sum_j \widetilde{1_j}) = 0$, which is equivalent to $(0, u(\sum_j \widetilde{1_j})) \in \mathrm{im}(-\beta, \alpha)$. Rows of the next diagram are segments of Gysin exact sequences:

$$\begin{array}{ccccc} H_{n-k-1}\partial L & \xrightarrow{\gamma} & H_{n-1}(\Gamma(\psi|\partial L)) & \xrightarrow{p_*} & H_{n-1}\partial L \\ \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \eta} \\ H_{n-k-1}L & \xrightarrow[\varepsilon]{} & H_{n-1}(\Gamma(\psi|L)) & \xrightarrow[\pi]{} & H_{n-1}L \end{array}$$

We first prove that $(0, \sum_j, 1'_j) \in \mathrm{im}(\beta, p_*)$. The first row of the above diagram is the direct sum of following exact sequences:

$$H_{n-k-1}M'_j \xrightarrow{\gamma_j} H_{n-1}\Gamma(\psi|M'_j) \xrightarrow{p_{j*}} H_{n-1}M'_j.$$

By the Poincaré duality,

$$\begin{aligned} \dim H_{n-1}M'_j &= 1, \\ \dim H_{n-1}\Gamma(\psi|M'_j) &= \dim H_k\Gamma(\psi|M'_j), \\ \dim H_{n-k-1}M'_j &= \dim H_k M'_j. \end{aligned}$$

By (81.17.1) we can deduce, $\dim H_k\Gamma(\psi|M_j') > \dim H_k M_j'$. Hence γ_j is not an epimorphism, $p_{j*} \neq 0$, p_* is onto, p_* is an epimorphism. Another epimorphism is δ. This follows from the Mayer–Vietoris exact sequence:

$$H_{n-k-1}\partial L \to H_{n-k-1}L \oplus H_{n-k-1}\mathrm{cl}(\mathbb{R}^3 \setminus L) \to H_{n-k-1}\mathbb{R}^3.$$

Since p_* is onto, there exists $z \in H_{n-1}\Gamma(\varphi|\partial L)$ such that $p_*z = \sum_j 1_j'$. Of course, $\eta\sum_j 1_j' = 0$. Hence $0 = \eta p_* z = \pi\beta z$, $\beta z \in \mathrm{im}\varepsilon$. Let $y \in H_{n-k-1}L$ and $a \in H_{n-k-1}\partial L$ be such that $\varepsilon y = \beta z$ and $\delta a = y$. Thus $\beta z = \varepsilon\delta a = \beta\gamma a$, $\beta(z-\gamma a) = 0$, $(\beta, p_*)(z-\gamma a) = (0, p_* z) = (0, \sum_j 1_j')$. It remains to prove that $p_* x = \sum_j 1_j'$ implies that $\alpha x = u(\sum_j \widetilde{1_j'})$ for $x \in H_{n-1}\Gamma(\varphi|\partial L)$, (here $x = z - \gamma a$). This is a corollary from the following diagram:

$$\begin{array}{ccccc}
H_{n-1}\Gamma(\psi|\partial L) & \xrightarrow{p_*} & H_{n-1}\partial L & \xrightarrow{p_*} & H_{n-1}\partial K \\
\downarrow{\scriptstyle\alpha} & & & & \downarrow{\scriptstyle(\mathrm{id},\varphi)_*} \\
H_{n-1}\Gamma(\psi|L) & & \xleftarrow[u]{\hspace{4em}} & & H_{n-1}\Gamma(\varphi|\partial K)
\end{array}$$

If $p_*x = \sum_j 1_j'$, then $h_*\sum_j 1_j' = \sum_j 1_j$, $(\mathrm{id},\varphi)_*\sum_j 1_j = \sum_j \widetilde{1_j}$ and finally $\alpha x = u(\sum_j \widetilde{1_j})$. The only point remaining concerns the commutativity of the above diagram. Let $r\colon N \to \partial K$ be the retraction $r(x,s) = x$ and $\overline{r}\colon \Gamma(\varphi|N) \to \Gamma(\varphi|\partial K)$ be given by $\overline{r}(x,y) = (r(x), \varphi(r(x)))$. If $\overline{r}$ is a strong deformation retraction then $u = (\overline{r}_*)^{-1}$ and reversing the lower arrow makes the corresponding diagram of mappings commutative. We now prove that this is the case.

Define $\rho\colon N \times I \to N$ by the formula $\rho((x,s),t) = (x,(1-t)s)$ for $(x,s) \in \partial K \times [0,1] = N$. Of course $\rho\colon \mathrm{id}_N \sim r$. By the Homotopy Lifting Property, there exists $\overline{\rho}\colon \Gamma(\varphi|N) \times I \to \Gamma(\varphi|N)$ which makes the following diagram

$$\begin{array}{ccccc}
\Gamma(\varphi|N) \times \{0\} & & \longrightarrow & & \Gamma(\varphi|N) \\
\downarrow & & & \nearrow{\scriptstyle\widehat{\rho}} & \downarrow{\scriptstyle p} \\
\Gamma(\varphi|N) \times I & \xrightarrow[p\times\mathrm{id}]{} & N \times I & \xrightarrow[\rho]{} & N
\end{array}$$

commutative. Recall that $\varphi(x) = b(x) + \lambda(x)(\varphi(x) - b(x))$, (see (81.16.1)). Let $h\colon \Gamma(\varphi|N\setminus\partial K) \to \Gamma(\varphi|N\setminus\partial K)$ be a homeomorphism defined by the formula

$$h(x,y) = (x, b(x) + \lambda(x)(y - b(x))).$$

One can check that $\overline{\rho}\colon \Gamma(\psi|N) \times I \to \Gamma(\psi|N)$ defined by

$$\overline{\rho}((x,y),t) = \begin{cases} h(\widetilde{\rho}(h^{-1}(x,y),t)) & \text{for } x \in N\setminus\partial K \text{ and } t \neq 1, \\ (\rho(x,1), \varphi(\rho(x,1))) & \text{for } x \in \partial K \text{ or } t = 1, \end{cases}$$

is continuous and $\widetilde{\rho}\colon \mathrm{id} \sim \overline{r}$, which proves theorem.

Case 2. $U \cap \partial B \neq \emptyset$.

We replace φ by $\eta\colon 2B \multimap 2B$,

$$\psi = \begin{cases} \varphi(x) & \text{if } \|x\| \in [0,1], \\ (2 - \|x\|)\varphi\left(\dfrac{x}{\|x\|}\right) & \text{if } \|x\| \in [1,2], \end{cases}$$

which is singlevalued on $2S$ and satisfies all assumptions which were made on φ. Let us check the condition (81.17.1) for ψ. Let $V = U \cap \partial B$. The new U is the set

$$U_1 = U \cup [1,2) \cdot V \cong U \times \{0\} \cup V \times [0,1).$$

Take an $\varepsilon > 0$. We find the $n-1$-manifold $L \subset V$ such that ∂L ε-approximates $\mathrm{bd}_{\partial B} V$ in V. Then we find the n-manifold $K \subset U$ such that ∂K δ-approximates $\mathrm{bd}\, U$ in U and satisfies (81.17.1). For simplicity of notation we may assume that ∂K is connected. If $\delta > 0$ is sufficiently small, then L is contained in ∂K (even with a collar). Then the set

$$K_1 = K \sup[1, 2-\delta] \cdot L \cong K \times \{0\} \cup L \times I$$

is a p.l. manifold and ∂K_1 well approximates $\mathrm{bd}\, U_1$ in U_1. We have

$$\partial K_1 = (\partial K \setminus L) \times \{0\} \sup (\partial L) \times I \cup L \times \{1\}.$$

Since $(L, \partial L)$ is a Borsuk pair, the set $(\partial L) \times I \cup L \times \{1\}$ is a strong deformation retract of $L \times I$ and consequently, ∂K_1 is a strong deformation retract of the set $C = (\partial K) \times \{0\} \cup L \times I$. Of course, also ∂K is a strong deformation retract of C. Observe, that the conditions (81.17.1) for ∂K and for ∂K_1 are equivalent. The Case 1 now implies that ψ is a Z_2-mapping.

It suffices to prove that if ψ is a Z_2-Brouwer mapping, so is φ. Let $P = 2B \setminus \mathrm{Int}(B)$. Consider the following diagram

$$\begin{array}{ccccc} \check{H}_n(\Gamma(\varphi|B), \Gamma(\varphi|S))I & \xrightarrow{\cong} & \check{H}_n(\Gamma(\psi|2B), \Gamma(\psi|P)) & \xleftarrow{\cong} & \check{H}_n(\Gamma(\psi|2B), \Gamma(\phi|2S)) \\ \Big\downarrow{\scriptstyle p_*} & & \Big\downarrow & & \Big\downarrow{\scriptstyle p_*^{\psi}} \\ \check{H}_n(B, S) & \xrightarrow[\cong]{} & \check{H}_n(2B, P) & \xleftarrow[\cong]{} & \check{H}_n(2B, 2S) \end{array}$$

All horizontal arrows represent isomorphisms: left arrows are excisions, on the right-hand side $2S$ and $\Gamma(\eta|2S)$ are strong deformation retracts of P and $\Gamma(\eta|P)$. Thus $p_*^{\psi} \neq 0$ implies that $p_* \neq 0$. $\square$

The next result is in author's opinion the "crown" of this part.

(81.18) THEOREM. *Every* 1-S^{n-2}*-mapping of* B^n *is a* Z_2*-Brouwer mapping and has a fixed point.*

We will need the following lemma, which states that the raising to the n-th power in the Z_2-cohomology algebra of any closed n-manifold in $\mathbb{R}^{n+1}$ is a trivial operation.

(81.19) LEMMA. *Let* $M \subset \mathbb{R}^{n+1}$ *be an* n*-dimensional compact connected topological manifold without boundary,* $n \geq 2$*. Then* $x^n = 0$ *for every* $x \in H^1(M; Z_2)$*.*

The situation described in hypotheses of this lemma is known very well in the literature. Let us gather some facts before the proof. First, $M \subset \mathbb{R}^{n+1} \cup \{\infty\} \cong S^{n+1}$; $S^{n+1} \setminus M = U \cup V$, ($U$, V-connected). The closures $A = \overline{U}$, $B = \overline{V}$ are ANRs. By the Alexander duality, $H^n(A; Z_2) = H^n(B; Z_2) = 0$. Let $i\colon M \to A$, $j\colon M \to B$ be inclusions. The Mayer–Vietoris exact sequence shows that $\varphi\colon H^s(A; Z_2) \oplus H^s(B; Z_2) \to H^s(M; Z_2)$, $\varphi(\alpha, \beta) = i^*\alpha + j^*\beta$ is an isomorphism for $1 \leq s < n$. Moreover, (see [Sp-M]), we have:

(81.19.1) $Sq^{n-1}y = 0$ for every $y \in H^r(M; Z_2)$, $1 \leq r < n$;
(81.19.2) $i^*Sq^1 a \cup j^*b = i^* \sup j^*Sq^1 b$ for all $a \in H^r(A; Z_2)$, $b \in H^{n-1-r}(B; Z_2)$;
(81.19.3) $Sq^i u^k = \binom{k}{i} u^{k+i}$ if $\dim(u) = 1$.

PROOF OF LEMMA (81.19).

Case 1. Let $n \neq 2^m - 1$ for every natural m. Since $0 = Sq^{n-r}x^r = \binom{r}{n-r}$, by (81.19.1), (81.19.3), it suffices to find r such that $\binom{r}{n-r}$ is odd and $1 \leq n - r \leq r < n$. If $n = 2t$ then $r = t$ satisfies the above conditions. If $n = 2t - 1$ then $t \neq 2^{m-1} - 1$ for every m. Thus $t = 2^{i-1} + j$ for some $i \geq 2$ and $1 \leq j \leq 2^{i-1} - 1$. It is easy to check that $\binom{2^i-1}{k}$ is odd for every $k = 0, 1, \ldots, 2^i - 1$, (by induction on i, $(x + y)^q \mod 2 = x^q + y^q$ for $q = 2^i$, so $\binom{2^i}{k}$ is even for $k = 1, \ldots, 2^i - 1$), and $r = 2^i - 1$ satisfies $1 < n - r < r < n$.

Case 2. Let $n = 2^m - 1$. Then, by (81.19.2),

$$\begin{aligned}
x^n &= (i^*\alpha + j^*\beta)^n \\
&= \sum_{k=0}^{n} \binom{n}{k} i^*\alpha^k \cup j^*\beta^{n-k} = \sum_{k=1}^{n-1} i^*\alpha^k \cup j^*\beta^{n-k} \\
&= \sum_{p=1}^{(n-1)/2} (i^*\alpha^{2p-1} \cup j^*\beta^{n-2p+1} + i^*\alpha^{2p} \cup j^*\beta^{n-2p}) \\
&= \sum_{p=1}^{(n-1)/2} \left(i^*\alpha^{2p-1} \cup j^*\binom{n-2p}{1}\beta^{n-2p+1} + i^*\binom{2p-1}{1}\alpha^{2p} \cup j^*\beta^{n-2p}\right) \\
&= \sum_{p=1}^{(n-1)/2} (i^*\alpha^{2p-1} \cup j^*Sq^1\beta^{n-2p} + i^*Sq^1\alpha^{2p-1} \cup j^*\beta^{n-2p}) = 0. \qquad \square
\end{aligned}$$

Proof of Theorem (81.18). By Theorem (81.17) it suffices to check the inequality (81.17.1). Let $\varphi: B^n \multimap B^n$ be a 1-S^{n-2}-mapping, M — a closed $n-1$-manifold in B^n, $p: \Gamma(\varphi|M) \to M$ — a projection. Observe that p is a locally trivial bundle with the fibre $S = S^{n-2}$. Let us denote by w_j the jth Stiefel–Whitney class of p. Set $\Gamma = \Gamma(\varphi|M)$. Consider the bundle $p^\Delta: \Gamma^\Delta \to M$ and the map $g: \Gamma^\Delta \to \mathbb{R}^n$; $g((x,y),(x,z)) = y$. Recall that c^Δ denotes the 1st Stiefel–Whitney class of the S^0-bundle $\Gamma^\Delta \to \Gamma^\Delta/Z_2$ with the Z_2 action given by the transposition $T((x,y),(x,z)) = ((x,z),(x,y))$. We must have $(c^\Delta)^n = 0$, for otherwise, $y = z$ for a $((x,y),(x,z)) \in \Gamma^\Delta$, a contradiction. Therefore, applying twice Lemma (81.19), we see that

$$\begin{aligned}
0 = (c^\Delta)^{n-1} \cup c^\Delta &= \sum_{j=1}^{n-1} (q^\Delta)^*(w_j^\Delta) \cup (c^\Delta)^{n-1-j} \cup c^\Delta \\
&= (q^\Delta)^*(w_1^\Delta) \cup (c^\Delta)^{n-1} + \sum_{j=2}^{n-1} (q^\Delta)^*(w_j^\Delta) \cup (c^\Delta)^{n-j} \\
&= \sum_{j=1}^{n-1} (q^\Delta)^*(w_1^\Delta \cup w_j^\Delta) \cup (c^\Delta)^{n-1-j} + \sum_{j=1}^{n-2} (q^\Delta)^*(w_{j+1}^\Delta) \cup (c^\Delta)^{n-1-j} \\
&= \sum_{j=1}^{n-2} (q^\Delta)^*(w_1^\Delta \cup w_j^\Delta + w_{j+1}^\Delta) \cup (c^\Delta)^{n-1-j}.
\end{aligned}$$

This gives $w_{j+1}^\Delta = w_1^\Delta \cup w_j^\Delta$ for $j = 1, \ldots, n-2$, and $w_{n-1} = w_{n-1}^\Delta = (w_1^\Delta)^{n-1} = 0$. This proves the theorem. □

For $n = 3$ Theorem (81.18) sounds especially visually:

(81.20) Theorem. *Every ρ_C-continuous mapping of the closed 3-dimensional disc, taking values which are points or knots, has a fixed point.*

We give an alternative proof of this special case of Theorem (81.18), which is based on the other lemma.

(81.21) Lemma. *If M_g is a closed orientable surface of genus g then*

$$\widetilde{K}(M_g) = (Z_2)^{2g+1}.$$

For the proof of (81.21) see [Mik1-M].

Proof of Theorem (81.20). The structural group of the locally trivial bundle with fibre S^1 reduces to $O(2)$, (see [Mik1-M]). For this reason we can rewrite the proof of Theorem (81.18) omitting the triangles in all symbols $(\,\cdot\,)^\Delta$. We now change our last argument in that proof.

Let $\overrightarrow{p}$ be the vector bundle with fibre $\mathbb{R}^2$, which corresponds to p. By Lemma (81.19), $\overrightarrow{p} \oplus \overrightarrow{p}$ represents zero in $\widetilde{K}(M)$. This gives $\overrightarrow{p} \oplus \overrightarrow{p} \oplus \overrightarrow{\theta} = \overrightarrow{\Theta}$ for some trivial vector bundles $\overrightarrow{\theta}$, $\overrightarrow{\Theta}$. It follows that

$$1 = w(p \oplus p) = w(p) \cup w(p) = (1 + w_1 + w_2)^2 = 1 + [w_1]^2 + [w_2]^2 = 1 + [w_1]^2.$$

Therefore $[w_1]^2 = 0$, which finishes the proof. □

Finally, let us inform that just appeared a monograph by D. Miklaszewski (see [Mik2-M]) which is devoted to spheric mappings.

82. A coincidence index involving Fredholm operators

In this section we shall present recent results obtained by D. Gabor and W. Kryszewski (see: [Kr2-M], [GaDKr1]–[GaDKr3]). To do it cohomotopy methods are more suitable than homological ones. Therefore, we recall some results from the cohomotopy theory.

For two pairs (X, A) and (Y, B) of (metric) spaces by $[X, A; Y, B]$ we shall denote the set of homotopy classes $[f]$ of (continuous) maps $f\colon (X, A) \to (Y, B)$.

Any map $p\colon (\Gamma, \Gamma') \to (X, A)$ induces a transformation

$$p^{\#}\colon [X, A; Y, B] \to [\Gamma, \Gamma'; Y, B]$$

by the formula: $p^{\#}([f]) = [f \circ p]$, for every $[f] \in [X, A; Y, B]$.

Simiraly, if $q\colon (Y, B) \to (Y', B')$, then we shall consider the transformation

$$q_{\#}\colon [X, A; Y, B] \to [X, A; Y, B]$$

defined by $q_{\#}([f]) = [q \circ f]$, for every $[f] \in [X, A; Y, B]$.

For convenience we recall some basic notions. Given a pair (X, A) of spaces and $n \geq 0$, in the set $\pi^n(X, A) := [X, A; S^n, s_0]$ ([6]), where S^n stands for the unit n-dimensional sphere and $s_0 = (1, 0, \dots, 0) \in \mathbb{R}^{n+1}$ is the base point, we distinguish the element 0, i.e. the homotopy class of the constant map $0\colon x \mapsto s_0$. In the set $\pi^n(X)$ we distinguish the element 1_X being the homotopy class of the map $c\colon (X, \emptyset) \to (S^0, s_0)$, defined by $c(x) = s_1 \neq s_0$ for $x \in X$. If $A \neq \emptyset$, then contracting A to the point $*$, we get the pair $(X/A, *)$ and the quotient projection $f\colon (X, A) \to \pi^n(X/A, *)$; it is clear that $f^{\#}\colon \pi^n(X/A, *) \to \pi^n(X, A)$ is a bijection for all $n > 0$. If $f\colon (X, A) \to (Y, B)$ is the homotopy equivalence, then, for all $n \neq 0$, $f^{\#}\colon \pi^n(Y, B) \to \pi^n(X, A)$ is a bijection.

The following *excision property* is satisfied:

- if A, B are closed subsets of X and $n \geq 0$, then the inclusion $e\colon (A, A \cap B) \to (A \cup B, B)$ induces a bijection $e^{\#}\colon \pi^n(A \cup B, B) \to \pi^n(A, A \cap B)$.

([6]) We also write $\pi^n(X) := \pi^n(X, \emptyset)$.

For any $n \geq 0$, there exists the so-called *coboundary operator* $\delta\colon \pi^n(A) \to \pi^{n+1}(X,A)$ which preserves the zero elements and is natural, i.e. if $f\colon (X,A) \to (Y,B)$ and $g := f|A\colon A \to B$, then $\delta \circ g^\# = f^\# \circ \delta$. In a similar manner one defines the coboundary operator for a triple (X,A,B) (i.e. $B \subset A \subset X$ are closed) $\delta^\#\colon \pi^n(A,B) \to \pi^{n+1}(X,A)$ putting $\delta^\# := \delta \circ k^\#$ where $k\colon A \to (A,B)$ is the inclusion. In particular, given a triad $(X;A,B)$ (i.e. $A,B \subset X$ are closed), the coboundary operators $\delta^\#\colon \pi^n(A\cup B,B) \to \pi^{n+1}(X,A\cup B)$ and $\delta^\#\colon \pi^n(A,A\cap B) \to \pi^{n+1}(X,A\cup B)$ are defined (the last one is defined via identification of $\pi^n(A\cup B,B)$ with $\pi^n(A,A\cap B)$ through the excision bijection $\pi^n(A\cup B,B) \to \pi^n(A,A\cap B)$).

It is well-known that $\pi^n(X,A)$ admits the structure of an abelian group provided the covering dimension $\dim X < \infty$ and the Čech cohomology (with integer coefficients) $H^q(X,A) = 0$ for $q \geq 2n-1$ (see [Kr2-M]). If, additionally, $B \subset A$ is closed, $H^q(A,B) = 0$ for $q \geq 2n-1$, then $\delta^\#\colon \pi^n(A,B) \to \pi^{n+1}(X,A)$ is a homomorphism; if $f\colon (X,A) \to (Y,B)$, $\dim Y < \infty$ and $H^q(Y,B) = 0$ for $q \geq 2n-1$, then $f^\#\colon \pi^n(Y,B) \to \pi^n(X,A)$ is a group homomorphism.

Given a triad $(X;A,B)$, the *cohomotopy* and the *modified cohomotopy* sequences:

$$\pi^n(X,A\cup B) \xrightarrow{j^\#} \pi^n(X,B) \xrightarrow{j^\#} \pi^n(A\cup B,B) \xrightarrow{\delta^\#} \pi^{n+1}(X,A\cup B) \xrightarrow{j^\#} \cdots,$$
$$\pi^n(X,A\cup B) \xrightarrow{j^\#} \pi^n(X,B) \xrightarrow{(i\circ e)^\#} \pi^n(A,A\cap B) \xrightarrow{\delta^\#} \pi^{n+1}(X,A\cup B) \xrightarrow{j^\#} \cdots,$$

where $j\colon (X,B) \to (X,A\cup B)$, $i\colon (A,B) \to (X,B)$ are the inclusions and $e\colon (A,A\cap B) \to (A\cup B,-B)$ is the excision, are defined. It is well-known that $\mathrm{Im} j^\# \subset \ker i^\#$, $\mathrm{Im}\, i^\# \subset \ker \delta^\#$ (hence $\mathrm{Im}\,(i\circ e)^\# \subset \ker \delta^\#$, as well), $\mathrm{Im}\,\delta^\# \subset \ker j^\#$ and $\mathrm{Im}\, j^\# \supset \ker i^\#$ (where, Im stands for the image and, ker for the preimage of 0, e.g. $\ker \delta^\# := (\delta^\#)^{-1}(0)$). If $\dim X < \infty$ and, for $q > 2n-1$, $H^q(X,A\cup B) = H^q(A\cup B,B) = 0$, then the above sequences consist of homomorphisms and are exact. These sequences reduce to the cohomotopy sequence of a triple (X,A,B) if $B \subset A$.

Let $\xi \in \pi^n(X,A)$ and $\eta \in \pi^m(Y,B)$, $n,m \geq 0$, be represented by $u\colon (X,A) \to (S^n,s_0)$ and $v\colon (Y,B) \to (S^m,s_0)$, respectively. Then $u \times v\colon (X,A) \times (Y,B) = (X\times Y, X\times B\cup A\times Y) \to (S^n\times S^m, S^n\times\{s_0\}\cup\{s_0\}\times S^m)$ is given by $(u\times v)(x,y) = (u(x),v(y))$. The map $f_\#\colon [X\times Y, X\times B\cup A\times Y; S^n\times S^m, S^n\times\{s_0\}\cup\{s_0\}\times S^m] \to [X\times Y, X\times B\cup A\times Y; S^{n+m},s_0] = \pi^{n+m}(X\times Y, X\times B\cup A\times Y)$, induced by the quotient projection $f\colon (S^n\times S^m, S^n\times\{s_0\}\cup\{s_0\}\times S^m) \to (S^{n+m},s_0)$, is a bijection. We define

$$\xi \otimes \nu := f_\#[u\times v] \in \pi^{n+m}((X,A)\times(Y,B)). \tag{82.1}$$

It is easy to see that the defined above *external product*

$\otimes\colon \pi^n(X,A) \times \pi^m(Y,B) \to \pi^{n+m}((X,A)\times(Y,B))\pi^{n+m}(X\times Y, X\times N\cup A\times Y)$

has the following properties:

(82.2) PROPERTIES.

(82.2.1) *For any* $\xi \in \pi^n(X,A)$, $\nu \in \pi^m(Y,B)$ *and* $\eta \in \pi^p(Z,C)$,

$$(\xi \otimes \nu) \otimes \eta = \xi \otimes (\nu \otimes \eta)$$

in $\pi^{n+m+p}(X \times Y \times Z, X \times Y \times C \cup X \times B \times X \sup A \times Y \times Z)$.

(82.2.2) *If* $p\colon (X_1, A_1) \to (X,A)$, $q\colon (Y_1,B_1) to (Y,B)$, *then*

$$(p \times q)^{\#}(\xi \otimes \nu) = p^{\#}(\xi) \otimes q^{\#}(\nu),$$

for any $\xi \in \pi^n(X,A)$ *and* $\nu \in \pi^m(Y,B)$,

(82.2.3) *For any* $\xi \in \pi^n(X,A)$, $\xi \otimes 1_Y = \mathrm{pr}^{\#}(\xi)$ *in* $\pi^n(X \times Y, A \times Y$, *where* $\mathrm{pr}\colon (X \times Y, A \times Y) \to (X,A)$ *is the projection.*

(82.2.4) *For any* $\xi \in \pi^n(A)$, $\nu \in \pi^m(Y,B)$,

$$\delta^{\#}(\xi \otimes \nu) = \delta(\xi) \otimes \nu,$$

where $\delta\colon \pi^n(A) \to \pi^{n+1}(X,A)$ *is the coboundary operator of the pair* (X,A) *and* $\delta^{\#}\colon \pi^{n+m}(A \times Y, A \times Y \cap X \times B) \to \pi^{n+m+1}(X \times Y, A \times Y \cup X \times B)$ *is the coboundary operator from the modified cohomotopy sequence of the triad* $(X \times Y; A \times Y, X \times B)$.

Let $m \geq 1$; for any $n \geq 0$, we make the following identifications

$$\pi^n(S^m, s_0) \equiv \pi^n(D^m, S^{m-1}) \equiv \pi^n(\mathbb{R}^m, \mathbb{R}^m \setminus B^m(0,\rho)),$$
$$\pi^n(S^m) \equiv \pi^n(\mathbb{R}^{m+1} \setminus B^m(0,\rho)),$$

where $\rho > 0$, justified by the existence of bijections between the respective sets (being group isomorphisms provided $2n-1 > m$). If $2n-2 > m$, then $\delta\colon \pi^{n-1}(S^{m-1}) \to \pi^n(D^m, S^{m-1})$ is an isomorphism.

The set $\pi^n(S^m, s_0)$ and the homotopy group $\pi_m(S^n, s_0)$ coincide as sets (if, additionally $2n-1 > m$, then they also do as groups). Hence, if $m < n$, then $\pi^n(S^m, s_0) = 0$. Recall that given $k \geq 0$, if $n \geq k+2$, then the homotopy groups $\pi_{n+k}(S^n, s_0)$ and $\pi_{n+1+k}(S^{n+1}, s_0)$ are canonically isomorphic (through the Freudenthal suspension homomorphism $\mathcal{E}$) with $\Pi_k\colon = \pi_{2k+2}(S^{k+2}, s_0)$. Moreover, by the results of Serre, the *stable homotopy groups* of spheres Π_k are abelian and finite for any $k > 0$. Later on, if $m < 2n-1$, then we identify $\pi^n(S^m, s_0)$ with Π_{m-n}.

Given $\xi \in \Pi_{i_1}, \nu \in \Pi_{i_2}$ $(i_1, i_2 \geq 0)$, regarding Π_{i_1} and Π_{i_2} as $\pi^n(S^m, s_0)$ and $\pi^l(S^m, s_0)$, where $m - n = i_1$, $2n-1 > m$ and $k - l = i_2$, $2l - 1 > k$, respectively, we may consider an element

$$\xi \otimes \nu \in \pi^{n+l}(S^{m+k}, s_0) = \Pi_{i_1+i_2}. \tag{82.3}$$

This definition is correct, i.e. does not depend on the choice of m, n, k and l, up to the above mentioned identifications.

(82.4) DEFINITION. A compact space Y is called a *cell-like set*, if there exists an ANR-space Z and an embedding $i\colon Y \to Z$ such that the set $i(Y)$ is contractible in any of its neighbourhoods $U \subset Z$.

It is not difficult to see that given a cell-like space Y, an ANR-space Z and an embedding $i\colon Y \to Z$ the set $i(A)$ is contractible in an arbitrary neighbourhood in Z; i.e. cell-likeness is an absolute property. The cartesian product of two cell-like sets is cell-like. It is evident that any compact convex or contractible, or an R_δ-set is cell-like. It is also easy to see that cell-like sets are acyclic (in the sense of the Čech homology theory); however there are examples of acyclic sets which are not cell-like.

(82.5) DEFINITION. A proper surjection $p\colon \Gamma \to X$ is a *cell-like map*, if for every $x \in X$ the fibre $p^{-1}(x)$ is a cell like set.

Evidently, any cell-like map is a Vietoris map and there are examples of Vietoris mappings which are not cell-like mappings.

We have proved in Chapter I that composition of Vietoris mappings is again a Vietoris mapping. Note that this is no longer true for cell-like mappings.

Recall that a multivalued mappings $\varphi\colon X \multimap Y$ is called admissible if there exists a diagram $X \overset{p}{\Longleftarrow} \Gamma \overset{q}{\longrightarrow} Y$ in which p is a Vietoris map such that $\varphi(x) = q(p^{-1}(x))$, for every $x \in X$.

(82.6) REMARK. In this section we shall consider only admissible mappings for which there exists a diagram $X \overset{p}{\Longleftarrow} \Gamma \overset{q}{\longrightarrow} Y$ in which p is a cell-like map such that $\varphi(x) = q(p^{-1}(x))$; such an admissible map $\varphi\colon X \multimap Y$ we shall call cell-like admissible (for short c-l admissible).

We shall use the following result proved by W. Kryszewski (see [Kr2-M]).

(82.7) THEOREM. *Let (K, L) be a pairs of* ANR*s. If* $\dim X < \infty$ *and* $p\colon (\Gamma, \Gamma') \Rightarrow (X, A)$, *where* $\Gamma' = p^{-1}(A)$, *is a cell-like map, then*

$$p^{\#}\colon [X, A; K, L] \to [\Gamma, \Gamma'; K, L]$$

is a bijection.

As a simple corollary we get

(82.8) COROLLARY. *If* $\dim X < \infty$ *and* $p\colon (\Gamma, \Gamma') \Rightarrow (X, A)$ *is a cell-like map, with* $\Gamma' = p^{-1}(A)$, *then, for each* $n \geq 0$,

$$p^{\#}\colon \pi^n(X, A) \to \pi^n(\Gamma, \Gamma')$$

is a bijection. In particular, if $\pi^q(X, A) = 0$ *for* $q \geq 2n - 1$, *then* $p^{\#}$ *is a group isomorphism.*

Now, let X and X' be two Banach spaces. A bounded linear map $L\colon X \to X'$ is a *Fredholm operator*, if the kernel $\mathrm{Ker}(L)$ and cokernel $\mathrm{Coker}(L) = X'|\mathrm{Im}(L)$ are finite dimensional Banach spaces.

The Fredholm index $i(L)$ of L is given by:

$$i(L) = \dim \mathrm{Ker}(l) - \dim \mathrm{Coker}(L).$$

Observe that if $L\colon \mathbb{R}^n \to \mathbb{R}^m$ is a linear map, then L is a Fredholm operator and $i(L) = n - m$.

Until end of this section we shall consider only Fredholm operators with non-negative indices.

Assume $L\colon X \to X'$ is a Fredholm operator, $\Omega \subset X$ is an open set and $\varphi\colon \Omega \multimap X'$ is a c-l admissible map. The aim of this section is to study the following coincidence problem:

$$L(X) \in \varphi(x). \tag{82.9}$$

In this order we shall present a brief description the so called *coincidence index*, i.e. a homotopy invariant responsible for the existence of solutions to (82.8).

The first assume that $X = \mathbb{R}^m$, $X' = \mathbb{R}^n$, $m, n \leq 1$. Let $U \subset \mathbb{R}^m$ be open and bounded and let (p, q) be an c-l-admissible pair of maps determining φ such that $\mathrm{cl} \xleftarrow{p} \Gamma \xrightarrow{p} \mathbb{R}^n$ and $q(p^{-1}(\mathrm{bd}\, U)) \subset \mathbb{R}^n \setminus \{0\}$. Then $q(p^{-1}(\mathrm{bd}\, U)) \subset \mathbb{R}^n \setminus B^n(0,\rho)$ for some $\rho > 0$. Consider the following sequence of maps

$$\begin{aligned}(\mathbb{R}^n, \mathbb{R}^n \setminus B^n(0,\rho)) \xleftarrow{q} (p^{-1}(\mathrm{cl}\, U, p^{-1}(\mathrm{bd}\, U)) \xrightarrow{p} (\mathrm{cl}\, U, \mathrm{bd}\, U) \xrightarrow{i_1} \\ \xrightarrow{i_1} (\mathbb{R}^m, \mathbb{R}^m \setminus U) \xleftarrow{i_2} (\mathbb{R}^m, \mathbb{R}^m \setminus B^m(0,d))\end{aligned} \tag{82.10}$$

where $d > 0$ is such that $\mathrm{cl}\, U \subset B(0,d)$, i_1, i_2 are inclusions, and the corresponding sequence on the level of cohomotopy sets

$$\begin{aligned}\pi^n(S^n, s_0) \equiv \pi^n(\mathbb{R}^n, \mathbb{R}^n \setminus B^n(0,\rho)) \xrightarrow{q^{\#}} \pi^n(p^{-1}(\mathrm{cl}\, U, p^{-1}(\mathrm{bd}\, U)) \\ \xleftarrow{p^{\#}} \pi^n(\mathrm{cl}\, U, \mathrm{bd}\, U) \xleftarrow{i_1^{\#}} \pi^n(\mathbb{R}^m, \mathbb{R}^m \setminus U) \xrightarrow{i_2^{\#}} \pi^n(\mathbb{R}^m, \mathbb{R}^m \setminus B^m(0,d)) \\ \equiv \pi^n(S^m, s_0).\end{aligned} \tag{82.11}$$

By (82.11), $p^{\#}$ is a bijection; so does $i_1^{\#}$ by the excision property. Hence the sequence (82.11) defines the transformation

$$\mathcal{K}\colon \pi^n(S^n, s_0) \equiv \pi^n(\mathbb{R}^n, \mathbb{R}^n \setminus B^n(0,\rho)) \to \pi^n(\mathbb{R}^m, \mathbb{R}^m \setminus B^m(0,d)) \equiv \pi^n(S^m, s_0),$$
$$\mathcal{K} := i_2^{\#} \circ (i_1^{\#})^{-1} \circ (p^{\#})^{-1} \circ q^{\#}.$$

The following definition is correct since it evidently does not depend on the choice of p and d.

(82.12) DEFINITION. By the generalized degree of the pair (p,q) on the set U in 0 we understand the element

$$\deg((p,q),U,0) := \mathcal{K}(\nu^n) \in \pi^n(\mathbb{R}^m, \mathbb{R}^m \setminus B^m(0,d)) \equiv \pi^n(S^m, s_0),$$

where ν^n corresponds to the homotopy class of the identity map $\mathrm{id}\colon S^n \to S^n$ in $\mathbb{Z} \equiv \pi(S^n, s_0) \equiv \pi^n(\mathbb{R}^n, \mathbb{R}^n \setminus B^n(0,\rho))$.

The degree defined above has all expected properties: existence, homotopy invariance (the notion of an admissible homotopy will be described below) and localization; the additivity property holds if $m < 2n-1$.

If (p,q) is as above, but U is not bounded, then the degree $\deg((p,q),U,0)$ is still defined (as usual by means of the localization property) provided the set $\{x \in \mathrm{cl}\, U \mid 0 \in q(p^{-1}(x))\}$ is compact.

(82.13) REMARKS.

(82.13.1) It is clear that if $m = n$, then $\deg((p,q),U,0)$ agrees with the usual topological degree defined in Chapter III. If $m < n$, then $\deg((p,q),U,0) = 0$. Therefore it is natural to assume that $m \geq n$.

(82.13.2) Suppose that $m \geq n$. For $k \geq 1$, by the kth suspension of (p,q) we mean the pair $(S^k p, S^k q)$ where $S^k p$, $S^k q$ are the (unreduced) kth suspensions of p and q, respectively. Additionally let $S^0 p = p$, $S^0 q = q$. It is clear that, for any $k \geq 0$, $S^k p$ is a cell-like map, $\deg((S^k p, S^k q), S^k U, 0) \in \pi^{n+k}(S^{m+k}, s_0)$ is defined and if $m < 2n-1$, then it is equal (remember the identification $\pi^n(S^m, s_0) \equiv \pi^{n+k}(S^{m+k}, s_0)$) to $\deg((p,q),U,0)$.

(82.13.3) The following procedure is performed in order to define the so-called 'stable' degree Deg. Suppose that $m \geq n$, take $k \geq \max\{0, m-2n+2\}$ (then $m+k < 2(n+k)-1$) and define

$$\mathrm{Deg}((p,g),U,0) := \deg((S^k p, S^k q), S^k U, 0) \in \pi^{n+k}(S^{m+k}, s_0) \equiv \Pi_{m-n}.$$

The stable degree thus defined has also all the usual properties mentioned above and $\mathrm{Deg}((p,q),U,0) = \deg((p,q),U,0)$ (up to the suspension isomorphism) provided $n \leq m < 2n-1$.

(82.13.4) Let $U = B^m$ and put $\widetilde{p} := p|_{p^{-1}(S^{m-1})}\colon p^{-1}(S^{m-1}) \to S^{m-1}$, $\widetilde{q} := q|_{p^{-1}(S^{m-1})}\colon p^{-1}(S^{m-1}) \to \mathbb{R}^n \setminus B^n(0,\rho)$. Consider a commutative diagram

$$\begin{array}{ccc}
\mathbb{Z} \equiv \pi^n(S^n, s_0) \equiv \pi^n(\mathbb{R}^n, \mathbb{R}^n \setminus B^n(0,\rho)) & \xrightarrow{(p^{\#})^{-1} \circ q^{\#}} & \pi^n(D^m, S^{m-1}) \equiv \pi^n(S^m, s_0) \\
\cong \Big\uparrow \delta_1 & & \Big\uparrow \delta \\
\mathbb{Z} \equiv \pi^{n-1}(S^{n-1}) \equiv \pi^{n-1}(\mathbb{R}^n \setminus B^n(0,\rho)) & \xrightarrow[(\widetilde{p^{\#}})^{-1} \circ \widetilde{q}^{\#}]{} & \pi^{n-1}(S^{m-1})
\end{array}$$

where δ_1, δ are the respective coboundary operators. It is clear that the upper row defines the degree, i.e. $\xi := \deg((p,g), B^m, 0) = (p^{\#})^{-1} \circ g^{\#}(\nu^n)$. If $\xi_0 := (\widetilde{p}^{\#})^{-1} \circ \widetilde{q}^{\#}(\nu^{n-1}$ (where ν^{n-1} is the element in $\pi^{n-1}(\mathbb{R}^n \setminus B^n(0,\rho))$ corresponding to the homotopy class of $\mathrm{id}\colon S^{n-1} \to S^{n-1}$), then $\delta(\xi_0) = \xi$. Observe that if $n \le m < 2n-2$, then δ is an isomorphism; hence one may identify ξ with ξ_0 up to this isomorphism.

(82.13.5) Suppose that Y, Y' are finite-dimensional Banach spaces, $m := \dim Y \ge n := \dim Y'$, and let $U \subset Y$ be open. Suppose that an admissible pair $\mathrm{cl}\, U \overset{p}{\longleftarrow} \Gamma \overset{q}{\longrightarrow} Y'$ such that $q(p^{-1}\mathrm{bd}\, U)) \subset Y' \setminus \{0\}$ is given (if U is not bounded, then one assumes additionally that the set $\{x \in \mathrm{cl}\, U \mid 0 \in q(p^{-1}(x))\}$ is compact). Suppose further that $\eta\colon Y \to \mathbb{R}^m$ and $\eta'\colon Y' \to \mathbb{R}^n$ are isomorphisms determining orientations in Y and Y', respectively. Then we put

$$\deg((p,q), U, 0) := \deg((\eta \circ p, \eta' \circ q), \eta(U), 0).$$

This definition is correct since $\eta \circ p$ is a cell-like map. In a similar manner one defines the 'stable' degree Deg.

These degrees heavily depend on the orientations η and η': a change of any of these orientations may effect a change of the 'sign' of the degree; in particular, the nontriviality of the degree is not effected by such a change.

(82.13.6) The degree Deg is stable with respect to the suspension operator. It is not the case with deg; it essentially depends on specific choice of numbers m and n. On the other hand deg seems to be more precise. For example: if $m = 9$ and $n = 5$, then $\pi_9(S^5) = \mathbb{Z}_2$ and the suspension map $\mathcal{E}\colon \pi_8(S^4) \to \pi_9(S^5)$ is an epimorphism (see [Sp-M]). Moreover, $\pi_8(S^4) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and $\Pi_4 \cong \pi_{10}(S^6) = 0$; therefore, for $f\colon (D^9, S^8) \to (D^5, S^4)$ such that $\mathcal{E}[f|_{S^8}]$ is a nontrivial element of $\pi_9(S^5)$, we get $\deg(f, B^9, 0) \ne 0$ in $\pi^5(S^9)$ whereas $\mathrm{Deg}(f, B^9, 0) = 0$ in $\pi^6(S^{10})$.

Now, let us get back to the general situation: let X, X' be infinite dimensional Banach spaces, $L\colon X \to X'$ be a Fredholm operator with nonnegative index $i(L) = k$. Since both $\mathrm{Ker}(L)$ and $\mathrm{Im}(L)$ are direct summands in X and X', respectively, there exist continuous linear projections $P\colon \times \to X$ and $Q\colon X' \to X'$ such that $\mathrm{Ker}(L) = \mathrm{Im}(P)$ and $\mathrm{Ker}(Q) = \mathrm{Im}(L)$. Clearly X, X' split into the (topological) direct sums

$$\mathrm{Ker}(P) \oplus \mathrm{Ker}(L) = X, \quad \mathrm{Im}(Q) \oplus \mathrm{Im}(L) = X'.$$

Moreover, since $\mathrm{Im}(L)$ is a closed subspace of X', the Banach theorem implies that $L|_{\mathrm{Ker}(P)}\colon \mathrm{Ker}(P) \to \mathrm{Im}(L)$ is a linear homeomorphism. It also implies that L is a *proper* map when restricted to any closed set $A \subset X$ such that $P(A)$ is bounded.

Let us fix orientations in $\mathrm{Ker}(L)$ and $\mathrm{Coker}(L)$. It is clear that the orientation in $\mathrm{Coker}(L)$ induces the orientation on $\mathrm{Im}(Q)$ in a unique way. If Y' is a finite-dimensional subspace of X' of the form $Y' = \mathrm{Im}(Q) \oplus \widetilde{Y}'$, where $\widetilde{Y}' \subset \mathrm{Im}(L)$, and an orientation in Y' is given, then so is in Y'. Let $K_P\colon \mathrm{Im}(L) \to \mathrm{Ker}(P)$ be the (topological) isomorphism inverse to $L|_{\mathrm{Ker}(P)}$. In the (finite-dimensional) space $L^{-1}(Y') = K_P(\widetilde{Y}') \oplus \mathrm{Ker}(L)$ we choose an orientation which agrees on $K_P(Y')$ with that induced by K_P from $\widetilde{Y}'$ and on $\mathrm{Ker}(L)$ with the original one.

Let $\Omega \subset X$ with $\mathrm{int}\,\Omega \neq \emptyset$ and consider c.l. admissible pair $\Omega \xleftarrow{p} \Gamma \xrightarrow{q} X'$ such that:

(i) q is a compact map;

(ii) the *coincidence* set $\kappa(p,q) = \{x \in \Omega \mid L(x) \in q(p^{-1}(x))\}$ is compact and contained in the interior $\mathrm{int}\,\Omega$.

Choose an open bounded set $U \subset E$, such that

$$\kappa(p,q) \subset U \subset \mathrm{cl}\,U \subset \mathrm{int}\,\Omega.$$

Since $L|_{\mathrm{cl}\,U}$ is proper, the map $\mathrm{cl} \ni \times \mapsto L(x) - q(p^{-1}))$ is closed. Moreover, for any $x \in \mathrm{bd}\,U$, $0 \notin L(x) - q(p^{-1}))$; therefore there is $\varepsilon_0 > 0$ such that

$$\inf\{\|y\| \mid y \in L(x) - q(p^{-1})),\ x \in \mathrm{bd}\,U\} > \varepsilon_0.$$

Take $0 < \varepsilon \leq \varepsilon_0$ and let $\pi_\varepsilon\colon \mathrm{cl}\,q(p^{-1}(U)) \to X'$ be a Schauder projection of the compact set $\mathrm{cl}\,q(p^{-1}))$ into a finite dimensional subspace $\kappa(p,q)$ of X', such that

$$\|\pi_\varepsilon(y) - y\| < \varepsilon \quad \text{for } y \in \mathrm{cl}\,q(p^{-1})).$$

Denote by $\widetilde{Y}'$ the finite dimensional subspace of $\mathrm{Im}(L)$ such that $Z \subset Y' := \mathrm{Im}(Q) \oplus \widetilde{Y}'$ and fix an arbitrary orientation on Y'. Put $Y := L^{-1}(Y')$ (according to the above remarks Y and Y' are 'canonically' oriented; these orientations depend on the orientations of $\mathrm{Ker}(L)$, $\mathrm{Coker}(L)$ and of $\widetilde{Y}'$) and let $U_Y = U \cap Y$. It is clear that the closure $\mathrm{cl}\,U_Y$ (in Y) is contained in $\mathrm{cl}\,U \cap Y$ and its boundary $\mathrm{bd}\,U_Y$ (relative to Y) is contained in $\mathrm{bd}\,U \cap Y$. Further let $p_Y = p|_{p^{-1}(\mathrm{cl}\,U_Y)}$, $q_Y = \pi_\varepsilon \circ q|_{p^{-1}(\mathrm{cl}\,U_Y)}$ and $L_Y = L|_Y\colon Y \to Y'$. Observe, that p_Y is a cell-like map and L_Y is a Fredholm operator of index

$$i(L_Y) = \dim Y - \dim Y' = k.$$

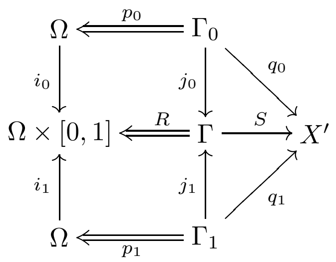
Enlarging $\widetilde{Y}'$ if necessary we may assume that $n := \dim Y' \geq k+2$ (this is possible since $\dim X' = \infty$). Putting $m := \dim Y = n + k$ we arrive in a finite dimensional situation described earlier in Remark (82.13.5).

(82.14) Definition. By the *generalized coincidence L-index of a pair* (p,q) we understand the element

$$\mathrm{Ind}_L((p,q),\Omega) = \deg((p_Y, L_Y \circ p_Y - q_Y), U_Y, 0) \in \Pi_{m-n} = \Pi_k.$$

The above definition is correct, i.e. it does not depend on the choice of all auxiliary objects: P, Q, U, ε, π_ε, $\widetilde{Y}'$ and the orientation on $\widetilde{Y}'$; this was carefully proved in [Kr2-M] and [GaDKr1]. As in the case discussed in Remark (82.13.5), a change of orientation in either $\operatorname{Ker}(L)$ or $\operatorname{Coker}(L)$ does not destroy the nontriviality of the index.

In order to enlist the most important properties of the introduced index Ind_L, let us first introduce the notion of *homotopy.* c.l. admissible pairs $\Omega \xleftarrow{p_k} \Gamma_k \xrightarrow{q_k} X'$ (or maps determined by them), $k = 0, 1$, are *L-homotopic* if there exist c.l. admissible pair $\omega \times [0,1] \xleftarrow{R} \Gamma \xrightarrow{S} X'$ with a compact S such that the set $\{x \in \Omega \mid L(x) \in S(R^{-1}(x,t))$ for some $t \in [0,1]\}$ is compact and contained in $\operatorname{int}\Omega$, and maps $j_k\colon \Gamma_k \to \Gamma$, $k = 0, 1$, such that the following diagram commutes



where $i_k(x) = (x, k)$ for $k = 0, 1$ and $x \in \Omega$.

(82.15) Theorem (comp. [Kr2-M], [GaDKr1], [GaDKr2]). *The index* Ind_L *has the following properties:*

(82.15.1) (Existence) *If* $\operatorname{Ind}_L((p,q),\Omega) \neq 0$*, then* $\kappa(p,q) \neq \emptyset$*, i.e. there is* $x \in \Omega$ *such that* $L(x) \in q(p^{-1}(x))$.

(82.15.2) (Localization) *If* $\Omega' \subset \Omega$*,* $\operatorname{int}\Omega' \neq \emptyset$*,* $\kappa(p,q) \subset \operatorname{int}\Omega'$*, then* $\operatorname{Ind}_L((p,q),\Omega')$ *is well defined and equal to* $\operatorname{Ind}_L((p,q),\Omega)$.

(82.15.3) (Homotopy Invariance) *If* (p_0,q_0)*,* (p_1,q_1) *are L-homotopic, then*

$$\operatorname{Ind}_L((p_0,q_0),\Omega) = \operatorname{Ind}_L((p_1,q_1),\Omega).$$

(82.15.4) (Additivity) *If sets* $\Omega_1, \Omega_2 \subset \Omega$ *are disjoint, have nonempty interiors and* $\kappa(p,q) \subset \operatorname{int}\Omega_1 \cup \operatorname{int}\Omega_2$*, then*

$$\operatorname{Ind}_L((p,q),\Omega) = \operatorname{Ind}_L((p,q),\Omega_1) + \operatorname{Ind}_L((p,q),\Omega_2).$$

(82.15.5) (Restriction) *If* $q(p^{-1}(\Omega)) \subset Y'$*, where* Y' *is a closed subspace of* X'*, then*

$$\operatorname{Ind}_L((p,q),\Omega) = \operatorname{Ind}_{L'}(p',q'),\Omega'),$$

where $\Omega' = \Omega \cap X'$*,* $X' := L^{-1}(Y' + \operatorname{Im}(Q))$*,* $p' = p|_{p^{-1}(\Omega')}$*,* $q' = q|_{p^{-1}(\Omega')}$ *and* $L' = L|_{X'}$.

The restriction property immediately implies that

(82.16) Corollary. *If* $q(p^{-1}(\Omega)) \subset Y' := \mathrm{Im}(Q) \oplus W$, *where* W *is a finite dimensional subspace of* $\mathrm{Im}\, L$, *then*

$$\mathrm{Ind}_L((p,q),\Omega) = \mathrm{Deg}((p_1, L' \circ p_1 - q_1), \Omega \cap T, 0),$$

where $X' := L^{-1}\mathrm{Im}(Q) \oplus W)$, $p_1 = p|_{p^{-1}(\mathrm{cl}\,\Omega \cap X')}\colon p^{-1}(\mathrm{cl}\,\Omega \cap X') \to X'$, $q_1 = q|_{p^{-1}(\mathrm{cl}\,\Omega \cap X')}\colon p^{-1}(\mathrm{cl}\,\Omega \cap X') \to Y'$ *and* $L' = L|_{X'}\colon X' \to Y'$.

Now, we shall apply the above topological invariants to the problem of solving systems of nonconvex inclusions involving Fredholm operators of nonnegative index.

Let X, Y, X' and Y' be Banach spaces, D be an open subset of X and let Ω be a subset of $D \times Y$. We shall consider a system of set-valued equations of the following form

$$\begin{cases} L_1(y) \in F(x,y), \\ L_2(x) \in G(x,y), \end{cases} \tag{82.17}$$

where $L_1\colon Y \to Y'$, $L_2\colon X \to X'$ are Fredholm operators and $F\colon \Omega \multimap Y'$, $G\colon \Omega \multimap X'$ are set-valued maps.

The operator $L\colon X \times Y \to X' \times Y'$, given by $L(x,y) := (L_1(x), L_2(y))$ for $x \in X$, $y \in Y$, is Fredholm with the index $\mathrm{ind}\,(L) = \mathrm{ind}\,(L_1) + \mathrm{ind}\,(L_2)$. If the map $\Omega \ni (x,y) \multimap \Psi(x,y) = G(x,y) \times F(x,y)$ is admissible and compact, the index $\mathrm{Ind}_L(\Psi,\Omega)$ is defined and nontrivial, then solutions to problem (82.17) exist. We shall look for (separated) conditions stated in terms of F and G implying the existence of solutions to (82.7); (82.10); hence, we shall employ the alternative method, which lets us to relax assumptions concerning the coincidence index.

Let $\mathrm{pr}\colon D \times Y \to D$ be the projection (i.e. $\mathrm{pr}(x,y) = x$ for $x \in D$, $y \in Y$),

$$\kappa(F,L) = \{(x,y) \in \Omega \mid L_1(y) \in F(x,y)\} \quad \text{and} \quad D_F := \mathrm{pr}(\kappa(F,L)) \subset D.$$

It is clear that $\kappa(F,L)$ is closed in Ω and coincides with the graph of the solution map S_F defined as follows

$$D_F \ni x \mapsto S_F(x) := \{y \in Y \mid (x,y) \in \kappa(F,L)\}.$$

Let $\mathrm{pr}_{\kappa(F,L)} := \mathrm{pr}|_{\kappa(F,L)}\colon \kappa(F,L) \to D_F$, let P_1 denote a (fixed) bounded linear projector of Y onto $\mathrm{Ker}(L_1)$ and let $\mathcal{P}\colon X \times Y \to Y$ be given by $\mathcal{P}(x,y) = P_1(y)$ for $x \in X$, $y \in Y$. One gets easily the following

(82.18) Lemma. S_F *is a multivalued map* (*i.e. is u.s.c. with compact values*) *if and only if* $\mathrm{pr}_{\kappa(F,L)}\colon \kappa(F,L) \to D_F$ *is a proper map. This holds e.g. if* Ω *is closed in* $D \times Y$ *and one of the following conditions is satisfied:*

(82.18.1) $Y = \mathbb{R}^m$, $Y' = \mathbb{R}^n$ *with* $n \le m$, Ω *is locally bounded over* X (7),

(82.18.2) Y, Y' *are arbitrary Banach spaces,* F *is compact and* $\mathcal{P}(\Omega)$ *is bounded.*

Further on we assume that:

(A1) $\mathrm{pr}_{\kappa(F,L)}\colon \kappa(F,L) \to D_F$ is proper;
(A2) F is a compact admissible set-valued map;
(A3) $\kappa(F,L) \subset \operatorname{int}\Omega$;
(A4) for any $a \in D$, $\operatorname{int}\Omega_a \ne \emptyset$, where $\Omega_a := \{y \in Y \mid (a,y) \in \Omega\}$.

Suppose that an admissible pair $\Omega \overset{p}{\longleftarrow} \Gamma \overset{q}{\longrightarrow} Y'$ determines F, let $a \in D$,

$$\Gamma_a = \{(y,\gamma) \in \Omega_a \times \Gamma \mid p(\gamma) = (a,y)\},$$

and define maps $p_a\colon \Gamma_a \to \Gamma_a$ and $q_a\colon \Gamma_a \to Y'$ by

$$p_a(y,\gamma) = y, \quad q_a(y,\gamma) = q(\gamma), \quad \text{for } (y,\gamma) \in \Gamma_a.$$

Note that (p_a, q_a) is a c-l-admissible pair and $q(p^{-1}(a,y)) = q_a(p_a^{-1}(y))$ for $y \in \Omega_a$. Moreover, (p_a, q_a) determines a compact set-valued map $\Omega_a \ni y \mapsto F(a,\,\cdot\,)$; in particular q_a is compact.

Assume that the spaces $\mathrm{Ker}(L_1)$ and $\mathrm{Coker}(L_1)$ are oriented in an arbitrary manner. In view of assumptions (A3), (A4), for each $a \in D$, the set $\{y \in \Omega_a \mid L_1(y) \in F(a,y)\} = \kappa(F,L) \cap \mathrm{pr}^{-1}(a)$ is compact and contained in $\operatorname{int}\Omega_a$. Hence the index

$$\mathrm{Ind}L_1((p,q)(a,\,\cdot\,),\Omega_a) := \mathrm{Ind}_{L_1}((p_a,q_a),\Omega_a)$$

is well-defined.

(82.19) Proposition. *If, for some* $a \in D$, $\mathrm{Ind}_{L_1}((p,g)(a,\,\cdot\,),\Omega_a) \ne \emptyset$ *and* D_0 *is a pathwise component of* D *containing* a*, then* $D_0 \subset D_F$.

To prove this fact observe first that $S_F(a) \ne \emptyset$, i.e. $a \in D_F$, in view of the existence property of Ind_{L_1} and let us present a result which will be useful here and in the sequel.

(82.20) Lemma. *There are numbers* $\varepsilon, \eta > 0$ *and closed neighbourhoods* W_1, W *of* $S_F(a)$ *such that* $W_1 \subset \operatorname{int} W$ *and*

$$\begin{aligned}(D^X(a,\varepsilon) \times V) \cap \kappa(F,L) \subset D^X(a,\varepsilon) \times W_1 \subset (D^X(a,\varepsilon) \times Y) \cap \mathcal{O}_\eta)(\kappa(F,L))\\ \subset (D^X(a,\varepsilon) \times Y) \cap \mathrm{cl}\,\mathcal{O}_\eta)(\kappa(F,L)) \subset D^X(a,\varepsilon) \times W \subset \operatorname{int}\Omega.\end{aligned}$$

(7) We say that Ω is locally bounded over X if, each $x \in X$ has a neighourhood N in X such that $(N \times Y) \cap \Omega$ is bounded.

PROOF. By (A3) above, there is $\rho > 0$ such that $D^X(a,\rho) \times \operatorname{cl}\mathcal{O}_\rho)(S_F(a)) \subset \operatorname{int}\Omega$. The upper semicontinuity of S_F implies that there is $0\eta \leq \rho/2$ such that

$$S_F(D^X(a,2\eta)) \subset \mathcal{O}_{\rho/2}(S_F(a)).$$

It is now easy to see that

$$(D^X(a,\eta) \times Y) \cap \operatorname{cl}\mathcal{O}_\eta)(\kappa(F,L)) \subset D^X(a,\rho) \times \operatorname{cl}\mathcal{O}_\rho)(S_F(a)).$$

Next, we choose $\rho_1 > 0$ such that

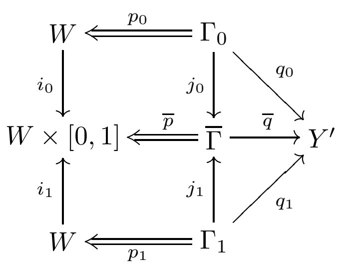
$$D^X(a,\rho_1) \times \operatorname{cl}\mathcal{O}_{\rho_1})(S_F(a)) \subset (B^X(a,\eta) \times Y) \cap \mathcal{O}_\eta)(\kappa(F,L))$$

and, again using the upper semicontinuity of S_F, a number $0 < \varepsilon \leq \rho_1$ such that $S_F(D^X(a,\varepsilon)) \subset \mathcal{O}_{\rho_1})(S_F(a))$. Then putting $W_1 := \operatorname{cl}\mathcal{O}_{\rho_1})(S_F(a))$ and $W := \operatorname{cl}\mathcal{O}_\rho)(S_F(a))$ we get the desired assertion. □

PROOF OF PROPOSITION (82.19). We shall prove that $B^X(a,\varepsilon) \subset D_F$. Let $b \in B^X(a,\varepsilon)$, then $W \subset \operatorname{int}\Omega_b$. Let

$$\begin{aligned} \Gamma_0 &= \Gamma_a \cap (W \times \Gamma), & \Gamma_1 &= \Gamma_b \cap (W \times \Gamma), \\ p_0 &= p_a|_{\Gamma_0}\colon \Gamma_0 \to W, & p_1 &= p_b|_{\Gamma_1}\colon \Gamma_1 \to W, \\ q_0 &= q_a|_{\Gamma_0}, & q_1 &= q_b|_{\Gamma_1}. \end{aligned}$$

Observe that pairs (p_0,q_0) and (p_1,q_1) are admissible and determine compact maps. Moreover, they are homotopic. Indeed, the following diagram



where

$$\begin{gathered} \overline{\Gamma} = \{(y,\gamma,t) \subset W \times \Gamma \times [0,1] \mid p(\gamma) = ((1-t)a + tb, y)\}, \\ \overline{p}(y,\gamma,t) = (y,t), \quad \overline{q}(y,\gamma,t) = q(\gamma), \quad \text{for } (y,\gamma,t) \in \overline{\Gamma}, \\ j_k(y,\gamma) = (y,\gamma,t), \quad \text{for } (y,\gamma) \in \Gamma_k \ (k = 0,1), \end{gathered}$$

is commutative. The pair $(\overline{p}, \overline{q})$ is admissible and determines a compact map. If, for some $y \in W$ and $t \in [0,1]$, $L_1(y) \in \overline{q}(\overline{p}^{-1}(y,t))$, then

$$((1-t)a + tb, y) \in (D^X(a,\varepsilon) \times Y) \cap \kappa(F,L) \subset D^X(a,\varepsilon) \times W_1;$$

i.e. $y \subset \operatorname{int} W$. Therefore the set $\{y \in Y \mid L_1(y) \in \overline{q}(\overline{p}^{-1}(y,t))$, for some $t \in [0,1]\}$ is a compact subset of W and does not intersect the boundary $\operatorname{bd} W$. Hence, by the localization and the homotopy invariance properties of Ind_{L_1}, we get that

$$\begin{aligned} o \neq \mathrm{Ind}_{L_1}((p,q)(a,\,\cdot\,), \Omega_a) &= \mathrm{Ind}_{L_1}((p_0,q_0), W) \\ &= \mathrm{Ind}_{L_1}((p_1,q_1), W) = \mathrm{Ind}_{L_1}((p,q)(b,\,\cdot\,), \Omega_b). \end{aligned}$$

This implies that $b \in D_F$. The connectedness argument ends the proof. □

(82.21) Remark. In particular we have proved that the map

$$a \mapsto \mathrm{Ind}_{L_1}(p_a, q_a), \Omega_a)$$

is constant on path components of the set D.

Now, we are in a position to present the main results of this section, i.e. theorems concerning the existence of solutions to the system of the form (82.17). For the rest of this section, if $A \subset D$, then

$$Z_A := \mathrm{pr}^{-1}(A) \cap \kappa(F,L) = (A \times Y) \cap \kappa(F,L) \quad \text{and} \quad \mathrm{pr}_A \colon = \mathrm{pr}|_{Z_A} \colon Z_A \to A.$$

Observe that $\mathrm{pr}_A(Z_A) = \mathrm{pr}(Z_A) = \mathrm{pr}_{\kappa(F,L)}(Z_A) = A \cap D_F$.

Assume that $Y = \mathbb{R}^m$, $Y' = \mathbb{R}^n$, $X = \mathbb{R}^k$ and $X' = \mathbb{R}^l$, where $1 \le n < m$ and $1 \le l \le k$. This implies that $\mathrm{ind}\,(L_1) = m - n$, $\mathrm{ind}\,(L_2) = k - l$. Let $m < 2n - 1$.

As before, let $D \subset \mathbb{R}^k$ be open, $\Omega \subset D \times \mathbb{R}^m$ and let $F \colon \Omega \multimap \mathbb{R}^n$, $G \colon \Omega \multimap \mathbb{R}^l$ be c-l-admissible maps. We assume that hypotheses (A1)–(A4) are satisfied (note that (A2) holds automatically since $\dim Y < \infty$). In order to simplify the notation we transform the system (82.17) and get

$$\begin{cases} 0 \in \varphi(x,y) := L_1(y) - F(x,y), \\ 0 \in \psi(x,y) := L_2(y) - G(x,y), \end{cases}$$

where $(x,y) \in \Omega$. It is evident that maps φ and ψ are admissible.

If $\Omega \overset{p}{\Longleftarrow} \Gamma \overset{q}{\longrightarrow} \mathbb{R}^n$ determines φ, then

$$\kappa(F,L) = \{(x,y) \in \Omega \mid 0 \in q(p^{-1}(x,y))\}.$$

For any $a \in D$, the degree

$$\deg((p,q)(a,\,\cdot\,), \Omega_a, 0) := \deg((p_a,q_a), \Omega_a, 0) \in \pi^n(S^m, s_0) \equiv \Pi_{m-n}$$

(where p_a and q_a are defined as above) is well-defined and the function $D \ni a \mapsto \deg((p,q)(a,\,\cdot\,),\Omega_a,0)$ is constant on path components of D.

We are now going to establish the first existence result. For the rest of this section we assume that:

(B1) there is $x_0 \in D$ such that $\eta_0 := \deg((p,q)(x_0,\,\cdot\,),\Omega_{x_0},0) \neq 0$ in $\pi^n(S^m,s_0)$.

(B2) There are a number $r > 0$ and an admissible map $\Phi\colon D^k(x_0,r) \multimap \mathbb{R}^l$, determined by an admissible pair (u',v'), such that $D^k(x_0,r) \subset D$ and, for $(x,y) \in Z_{S^{k-1}(x_0,r)}$, if $\mu \cdot \Phi(x) \cap \psi(x,y) \neq \emptyset$, then $\mu \geq 0$.

(B3) $0 \notin \Phi(S^{k-1}(x_0,r))$, $\xi := \deg((u',u'),D^k(x_0,r),0) \neq 0 \in \pi^l(S^k,s_0)$ and $\xi \otimes \eta_0 \neq 0$ in $\pi^{l+n}(S^k \times S^m, S^k \times \{s_0\} \cup \{s_0\} \times S^m) \equiv \pi^{l+n}(S^{k+m},s_0)$.

It is easy to see that assumptions (B2), (B3) constitute an *a priori* bounds condition of sorts. To see this better, consider the following example.

(82.22) EXAMPLE. Assume $x_0 = 0$, $\eta_0 = \deg((p,q)(0,\,\cdot\,),\Omega_0,0) \neq 0$, let $k = l$ and suppose that $\operatorname{ind}(L_2) = 0$. In this case, one may build an appropriate map Φ suitably complementing the operator L_2 itself.

(82.22.1) Let L_2 be an isomorphism. If we put $\Phi = L_2$ then (B3) is satisfied since Φ is an isomorphism: in this case $xi := \deg((u',v'),D^k(0,r),0) \neq O$ in $\pi^k(S^k,s_0)$ for any admissible pair determining Φ (in fact $\xi = \pm\nu^k$) and $\xi \otimes \eta \neq 0$ (comp. [Kr2-M]).

If, additionally, $D = \mathbb{R}^k$ and, e.g. the map G is bounded, i.e. there is $R > 0$ such that $\|G(x,y)\| \leq< R$ for all $(x,y) \in \Omega$, then we see that (B2) is satisfied provided that r is large enough. The same holds if the set of all solutions of the family of systems

$$(82.22.1)_\lambda \qquad \begin{cases} 0 \in \varphi(x,y), \\ L_2(x) \in \lambda G(x,y), \end{cases}$$

where $\lambda \in (0,1)$, is bounded.

Finally, if we suppose that, there is $r > 0$ such that, for any solution (x,y) with $\|x\| = r$ of the system $(82.22.1)_\lambda$ for some λ, one has $\lambda \geq 1$, then assumption (B2) is satisfied again. The last condition holds e.g. if there is $r > 0$ such that, for any $(x,y) \in Z_{S^{k-1}(0,R)}$, one of the following of conditions (of the Rothe or Altman type) is satisfied

$$\sup_{z \in G(x,y)} \|z\| \leq \|L_2(x)\|,$$

$$\sup_{z \in G(x,y)} \|z\|^k \leq \inf_{z \in G(x,y)} \|z - L_2(x)\|^k + \|L_2(x)\|^k, \quad \text{where } k > 1.$$

(82.22.2) If L_2 is not an isomorphism, then let $P_2, Q_2\colon \mathbb{R}^k \to \mathbb{R}^k$ be linear projections such that $\operatorname{Ker}(P_2) \oplus \operatorname{Ker}(L_2) = \operatorname{Im}(Q_2) \oplus \operatorname{Im}(L_2) = \mathbb{R}^k$. Additionally assume that Q_2 is an orthogonal projection. Since $\operatorname{ind}(L_2) = 0$ (i.e. $\dim \operatorname{Ker}(L_2) = \dim \operatorname{Im}(Q_2)$), there exists an isomorphism $J\colon \operatorname{Ker}(L_2) \to \operatorname{Im}(Q_2)$. Define $\Phi\colon \mathbb{R}^k \to \mathbb{R}^k$ by the formula $\Phi(x) = J \circ P_2(x) + L_2(x)$ and suppose that there exists $r > 0$ such that, for $(x, y) \in Z_{S^k(0,r)}$,

$$\sup_{z \in G(x,y)} \langle \Phi(x), z\rangle \leq \|L_2(x)\|^2.$$

where $\langle \cdot\,, \cdot \rangle$ stands for the inner product in $\mathbb{R}^k$. Then again conditions (B2) and (B3) are satisfied.

The general assumption $k > l$ of the possible dimension defect requires to introduce a nontrivial 'reference' mapping Φ in (B2); in the ordinary fixed point theory usually the identity plays such a role.

(82.22) THEOREM. *Let $1 \leq n \leq m < 2n - 1$, $1 \leq l \leq k < 2l - 2$ and $l > m - n + 1$. If assumptions* (A1)–(A4) *and* (B1)–(B3) *are satisfied, then the system* (82.22.1) *has a solution.*

PROOF. In order to simplify the notation (and without loss of generality), further on we assume that $x_0 = 0$ and $r = 1$. Hence $D^k(x_0, r) = D^k$ and $\operatorname{bd} D^k(x_0, r) = S^{k-1}$.

In view of the above mentioned identifications, we treat η_0 as the element of $\pi^n(\mathbb{R}^m, \mathbb{R}^m \setminus B^m) \equiv \pi^n(S^m, s_0)$. Assume that $D^k \xleftarrow{u'} \Lambda' \xrightarrow{v'} \mathbb{R}^l$ is an admissible pair determining Φ. According to Remark (82.12.4), $\xi = \deg(u', v'), D^k, 0) \neq 0$ in $\pi^l(D^k, S^{k-1})$. Hence

$$\xi \otimes \eta_0 \neq 0 \quad \text{in } \pi^{l+n}((D^k, S^{k-1}) \times (\mathbb{R}^m, \mathbb{R}^m \setminus B^m)).$$

Again by Remark (82.12.4), $\xi = \delta(\xi_0)$, where δ is the coboundary operator of the pair (D^k, S^{k-1}) and $\xi_0 = (u^\#)^{-1} \circ ov^\#(\nu^{l-1} \in \pi^{l-1}(S^{k-1})$ (where, as above, ν^{l-1} corresponds in $\pi^{l-1}(\mathbb{R}^l \setminus B^l(0, \rho))$ to the homotopy class in $\pi^{l-1}(S^{l-1})$ of the identity $S^{l-1} \to S^{l-1}$), $u := u'|_{u'^{-1}(S^{k-1})}$ and $v := v'|_{u'^{-1}(S^{k-1})}$. Since $k < 2l - 2$, δ is an isomorphism. Moreover, we deduce:

$$\xi \otimes \eta_0 = \delta(\xi_0) \otimes \eta_0 = \delta^\#(\xi_0 \otimes \eta_0)$$

where $\delta^\#$ is the coboundary operator in the modified cohomotopy sequence of the triad $(D^k \times \mathbb{R}^m, S^{k-1} \times \mathbb{R}^m, D^k \times (\mathbb{R}^m \setminus B^m))$, i.e.

$$\begin{aligned}
\delta^\# : \pi^{l-1+n}&(S^{k-1} \times (\mathbb{R}^m, \mathbb{R}^m \setminus B^m)) \\
&= \pi^{l-1+n}(S^{k-1} \times (\mathbb{R}^m, S^{k-1} \times \mathbb{R}^m \cap D^k \times (\mathbb{R}^m \setminus B^m)) \\
&\to \pi^{l-1+n}(D^k \times (\mathbb{R}^m, S^{k-1} \times \mathbb{R}^m \cup D^k \times (\mathbb{R}^m \setminus B^m)) \\
&\qquad = \pi^{l+n}((D^k, S^{k-1}) \times (\mathbb{R}^m, \mathbb{R}^m \setminus B^m)).
\end{aligned}$$

Since $2(l-1+n)-l > m+k$ and $l-1+n > m$, we gather that $\delta^{\#}$ is an isomorphism. Therefore

$$\xi_0 \otimes \eta_0 \neq 0 \quad \text{in } \pi^{l-1+n}(S^{k-1} \times (\mathbb{R}^m, \mathbb{R}^m \setminus B^m)).$$

We shall now make use of the following result (for details see [Kr2-M])

(82.23) LEMMA. *The following composition of maps*

$$\pi^{l-1}(S^{k-1}) \xrightarrow{(\mathrm{pr}_{S^{k-1}})^{\#}} \pi^{l-1}(Z_{S^{k-1}}) \xrightarrow{\delta} \pi^l(Z_{D^k}, Z_{S^{k-1}}),$$

where δ is the coboundary operator for the pair $(Z_{D^k}, Z_{S^{k-1}})$, maps ξ_0 onto a nontrivial element, i.e. $\delta \circ \mathrm{pr}^{\#}_{S^{k-1}}(\xi_0) \neq 0$ in $\pi^l(Z_{D^k}, Z_{S^{k-1}})$. Moreover, the composition

$$\pi^{l-1}(S^{k-1}) \xrightarrow{\delta} \pi^l(D^k, S^{k-1}) \xrightarrow{(\mathrm{pr}|_{D^k})^{\#}} \pi^l(Z_{D^k}, Z_{S^{k-1}})$$

maps ξ_0 onto a nontrivial element.

Now, we shall establish the existence of solutions of the map

$$\Theta\colon D_F \to \mathbb{R}^l, \quad \Theta(x) = \psi(\{x\} \times S_F(x)), \quad x \in D_F.$$

It is clear that if there is $x \in D_F$ such that $0 \in \Theta(x)$, then (82.22.1) has a solution.

The map ψ is admissible; let an admissible pair $\Omega \xleftarrow{p'} \Gamma' \xrightarrow{q'} \mathbb{R}^l$ determine ψ. It is easy to see that Θ is then determined by the (no more admissible) pair

$$D_F \xleftarrow{p_\Theta} \Gamma_\Theta \xrightarrow{q_\Theta} \mathbb{R}^l,$$

where $\Gamma_\Theta := \{(x, y, \gamma') \in \kappa(F, L) \times \Gamma' | (x, y) = p'(\gamma')\}$ and

$$p_\Theta(x, y, \gamma') = x, \quad q_\Theta(x, y, \gamma') = q'(\gamma')$$

for $(x, y, \gamma') \in \Gamma_\Theta$, i.e. for $x \in D_F$, $\Theta(x) = q_\Theta(p_\Theta^{-1}(x))$. Maps p_Θ, q_Θ admit factorizations

$$p_\Theta\colon \Gamma_\Theta \xleftarrow{\widetilde{p}} \kappa(F, L) \xleftarrow{\mathrm{pr}_{\kappa(F,L)}} D_F, \quad q_\Theta\colon \Gamma_\Theta \xleftarrow{\widetilde{q}} \Gamma' \xleftarrow{q'} \mathbb{R}^l,$$

where $\widetilde{p}(x, y, \gamma') = (x, y)$, $\widetilde{q}(x, y, \gamma') = \gamma'$ for $(x, y, \gamma') \in \Gamma_\Theta$. It is easy to see that $\widetilde{p}$ is a cell-like map.

To simplify the notation, let

$$\begin{aligned} A &:= D^k, & \Gamma_{\Theta,A} &: \widetilde{p}^{-1}(Z_A), & \widetilde{p}_A &:= \widetilde{p}|_{\Gamma_{\Theta,A}}, \\ B &:= S^{k-1}, & \Gamma_{\Theta,B} &: \widetilde{p}^{-1}(Z_B), & \widetilde{p}_B &:= \widetilde{p}|_{\Gamma_{\Theta,B}}. \end{aligned}$$

Clearly $\widetilde{p}_A$ and $\widetilde{p}_B$ are cell-like maps. Moreover, let

$$p_{\Theta,A} := \mathrm{pr}_A \circ \widetilde{p}_A \colon \Gamma_{\Theta,A} \to A, \quad p_{\Theta,B} := \mathrm{pr}_B \circ \widetilde{p}_B \colon \Gamma_{\Theta,B} \to B$$

and

$$q_{\Theta,B} = q_\Theta|_{\Gamma_{\Theta,B}}.$$

In view of Lemma (82.23), the composition

$$\pi^{l-1}(B) \xrightarrow{(\mathrm{pr}_B)^\#} \pi^{l-1}(Z_B) \xrightarrow{\delta} \pi^l(Z_A, Z_B),$$

transforms ξ_0 onto a nontrivial element. Since $\widetilde{p}_A^\# \circ \delta = \delta_\Theta \circ \widetilde{p}_B^\#$ (where δ_Θ denotes the coboundary operator of the pair $(\Gamma_{\Theta,A}, \Gamma_{\Theta,B})$) and $\widetilde{p}_A^\#$, $\widetilde{p}_B^\#$ are bijections, we see that the composition

$$\pi^{l-1}(B) \xrightarrow{(\mathrm{pr}_B)^\#} \pi^{l-1}(Z_B) \xrightarrow{(\mathrm{pr}_B)^\#} \pi^{l-1}(\Gamma_{\Theta,B}) \xrightarrow{\delta_\Theta} \pi^l(\Gamma_{\Theta,A}, \Gamma_{\Theta,B})$$

transforms ξ_0 onto a nontrivial element in $\pi^l(\Gamma_{\Theta,A}, \Gamma_{\Theta,B})$. This implies also that

$$\delta_\Theta \circ p_{\Theta,B}^\#(\xi_0) \neq 0. \tag{82.24}$$

Observe that if $0 \in q_\Theta(\Gamma_{\Theta,B})$, then there is $(x, y) \in Z_B$ such that $0 \in \psi(x, y)$, i.e. (x, y) is a solution to (82.22.1) and we are done. Hence assume that, there is $\rho > 0$ such that $q_\Theta(\Gamma_{\Theta,B}) \subset \mathbb{R}^l \setminus B^l(0, \rho)$. Without loss of generality we may assume that ρ is such that $\Phi(B) \subset \mathbb{R}^l \setminus B^l(0, \rho)$, i.e.

$$v(\Lambda) \subset \mathbb{R}^l \setminus B^l(0, \rho), \quad \text{where } \Lambda := {u'}^{-1}(B).$$

Assume for a while that the following diagram

$$\begin{array}{ccccc} & & \pi^{l-1}(\mathbb{R}^l \setminus B^l(0,\rho)) & & \\ & \swarrow^{(u^\#)^{-1}\circ v^\#} & \downarrow q^\#_{\Theta,B} & & \\ \pi^{l-1}(B) & \xrightarrow[p^\#_{\Theta,B}]{} & \pi^{l-1}(\Gamma_{\Theta,B}) & \xrightarrow[\delta_\Theta]{} & \pi^l(\Gamma_{\Theta,A}, \Gamma_{\Theta,B}) \end{array} \tag{82.25}$$

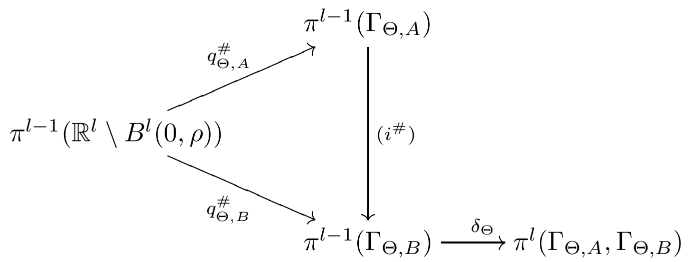
is commutative. Therefore, in view of (82.24),

$$0 \neq \delta_\Theta \circ p_{\Theta,B}^\#(\xi_0) = \delta_\Theta \circ p_{\Theta,B}^\#(u^\#)^{-1} \circ v^\#(\nu^{l-1}) = \delta_\Theta \circ q_{\Theta,B}(\nu^{l-1}). \tag{82.26}$$

This means that $0 \in \Theta(A)$. Indeed, assume to the contrary that $0 \notin \Theta(A)$; therefore $0 \notin q_\Theta(\Gamma_{\Theta,A})$. Without loss of generality we may assume then that

$q_\Theta(\Gamma_{\Theta,A}) \subset \mathbb{R}^l \setminus B^l(0,\rho)$. Hence the diagram

(82.27)



where $i\colon \Gamma_{\Theta,B} \to \Gamma_{\Theta,A}$ is the inclusion, is commutative. This, however, implies that

$$\delta_\Theta \circ p_{\Theta,B} = \delta_\Theta \circ i^{\#} \circ q_{\Theta,A} = 0$$

in view of the properties of the cohomotopy sequence of the pair $\Gamma_{\Theta,A}, \Gamma_{\Theta,B}$. The obtained contradiction with (82.26) shows that $0 \in \Theta(A)$ and that the system (82.22.1) has a solution.

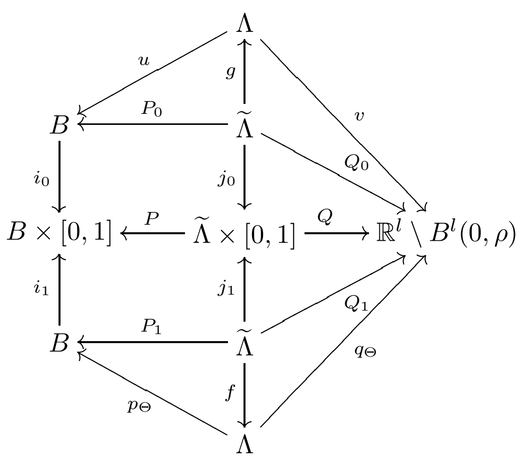
It now remains to show the commutativity of the diagram (82.25). To this end let

$$\widetilde{\Lambda} := \{(\gamma,\lambda) \in \Gamma_{\Theta,B} \times \Lambda \mid u(\lambda) = p_\Theta(\gamma)\}$$

and consider maps $P\colon \widetilde{\Lambda} \times [0,1] \to B \times [0,1]$ and $Q\colon \widetilde{\lambda} \times [0,1] \to \mathbb{R}^l$ given by

$$P(\gamma,\lambda,t) = (p_\Theta(\gamma,t), \quad Q(\gamma,\lambda,t) = (1-t)v(\lambda) + tq_\Theta(\gamma)$$

for $(\gamma,\lambda) \in \widetilde{\Lambda}$ and $t \in [0,1]$. It is clear that $Q(\,\cdot\,,\cdot\,,0)$ and $Q(\,\cdot\,,\cdot\,,1)$ have no zeros on $\widetilde{\Lambda}$. If $Q(\gamma,\lambda,t) = 0$, for some $(\gamma,\lambda) \in \widetilde{\Lambda}$ and $t \in (0,1)$, then $((t-1)/t)\Phi(x) \cap \Theta(x) \neq \emptyset$, where $x := p_\Theta(\gamma) \in B$: contradiction with (B2). Hence we may assume, without loss of generality, that $Q(\widetilde{\Lambda} \times [0,1]) \subset \mathbb{R}^l \setminus B^l(0,\rho)$. Consider a diagram



where

$$Q_0(\gamma,\lambda)=v(\lambda),\quad Q_1(\gamma,\lambda)=q_\Theta(\lambda),\quad g(\gamma,\lambda)=\lambda,\quad f(\gamma,\lambda)=\gamma,$$
$$j_k(\gamma,\lambda)=(\gamma,\lambda,k)\quad\text{and}\quad P_k(\gamma,\lambda)=p_\Theta(\lambda)\quad(\text{where } k=0,1)$$

for $(\gamma,\lambda)\in\widetilde{\Lambda}$ and $i_k(x)=(x,k)$ for $x\in B$ and $k=0,1$. It is easy to see that this diagram is commutative; moreover, observe that $f\colon\widetilde{\Lambda}\to\Gamma_{\Theta,B}$ is and admissible map; hence on the level of cohomotopy, $u^\#$, $f_0^\#$, $i_0^\#$, $j_0^\#$ and $j_1^\#$ are isomorphisms. Therefore, after easy diagram chasing we infer that

$$q_\Theta^\# = p_\Theta^\#\circ(u^\#)^{-1}\circ v^\#.$$

This shows the commutativity of (82.25) and concludes the proof. □

Now, we come back to the infinite dimensional case.

Having Theorem (82.22) we are ready to provide the main result of this chapter. As before we suppose that X, X' and Y, Y' are arbitrary Banach spaces, $L_1\colon Y\to Y'$ and $L_2\colon X\to X'$ are Fredholm operators with nonnegative indices $\operatorname{ind}(L_1)=i_1$, $\operatorname{ind}(L_2)=i_2$ such that:

(C1) $\dim X'>\max\{i_1+1,i_2+2\}$ and $\dim Y'>i_1+1$.

This condition is automatically satisfied if $\dim X'=\dim Y'=\infty$. Assume that (the finite-dimensional spaces) $\operatorname{Ker}(L_1)$, $\operatorname{Ker}(L_2)$, $\operatorname{Coker}(L_1)$ and $\operatorname{Coker}(L_2)$ are oriented in an arbitrary manner. Let P_i, and Q_i be the projections related to L_i, $i=1,2$. We also suppose $\mathcal{P}(\Omega)$ is bounded, conditions (A2)–(A4) hold (in view of Lemma (82.18) ((A1) is also satisfied) and

(C2) there is $x_0\in D$ and an admissible $\Omega\xleftarrow{p}\Gamma\xrightarrow{q}Y'$ determining F such that
$$\eta_0:=\operatorname{Ind}_{L_1}((p,q)(x_0;\,\cdot\,),\Omega_{x_0})\neq 0\quad\text{in } \Pi_{i_1},$$

(C3) there are a number $r>0$ and a compact admissible map $\Psi\colon D^X(x_0,r)\multimap X'$ determined by an admissible pair (u,v) such that $D^X(x_0,r)\subset D$ and, for $(x,y)\in Z_{S^X(x_0,r)}$, if $\mu(L_2(x)-\Psi(x))\cap(L_2(x)-G(x,y))\neq\emptyset$, then $\mu\geq 0$,

(C4) for $x\in S^X(x_0,r)$ $L_2(x)\notin\Psi(x)$, $\xi:=\operatorname{Ind}_{L_2}((u,v),D^X(x_0,r))\neq 0\in\Pi_{i_2}$ and $\xi\oplus\eta_0\neq O$ in $\Pi_{i_1+i_2}$.

(82.28) Remark. Similarly as above, note that condition (C3) plays a role similar to that of an a priori bounds assumption frequently present in the ordinary fixed-point theory. If L_2 is an isomorphism, $x_0=0$ and $\Psi\equiv 0$, then condition (C4) is satisfied (see Example (82.22)). Condition (C3) may be checked directly (see e.g. applications below) or, if additionally $D = X$, then (C3) is satisfied provided the set of all solutions to the parametrized, by $\lambda \in (0,1)$, system

$$\begin{cases} L_1(y) \in F(x,y), \\ L_2(x) \in \lambda G(x,y) \end{cases}$$

is bounded: in this case (C3) holds immediately. Some other sufficient conditions for (C3) may be stated in the same spirit as in Example (82.22).

(82.29) THEOREM. *Under the above assumptions, if G is compact, then the system* (82.22.1) *has a solution.*

PROOF. In order to simplify the setting, without loss of generality we suppose that $x_0 = 0$ and $r = 1$. Hence $D^X(x_0, r) = D^X$ and $S^X(x_0, r) = S^X$.

Sets $\operatorname{cl} F(\Omega)$, $\operatorname{cl} G(\Omega)$ and $\operatorname{cl} \Psi(D^X)$ are compact; therefore, for each integer $n \leq 1$, there are finite dimensional linear subspaces $\widetilde{Y}'_n \subset \operatorname{Im}(L_1)$, $\widetilde{X}'_n, \widetilde{X}''_n \subset \operatorname{Im}(L_2)$and Schauder projections $f_n \colon \operatorname{cl} F(\Omega) \to Y'_n := \widetilde{Y}'_n \oplus \operatorname{Im}(Q_1)$, $g_n \colon \operatorname{cl} G(\Omega) \to X'_n := \widetilde{X}'_n \oplus \operatorname{Im}(Q_2)$ and $h_n \colon \operatorname{cl} \Psi(D^X) \to \widetilde{X}'_n \oplus \operatorname{Im}(Q_2)$, such that $\|f_n(y) - y\| < n^{-1}$, $\|g_n(x) - x\| < n^{-1}$ and $\|h_n(z) - z\| < n^{-1}$ for $y \in \operatorname{cl} F(\Omega)$, $x \in \operatorname{cl} G(\Omega)$ and $z \in \operatorname{cl} \Psi(D^X)$, respectively. It is clear that, without loss of generality, we may assume that $\widetilde{X}'_n = \widetilde{X}''_n$. For any $n \geq 1$, let us put

$$\begin{aligned} Y_n &:= L_1^{-1}(Y'_n); & X_n &:= L_2^{-1}(X'_n); & & \\ \Omega_{n,m} &:= \Omega \cap (X_n \times Y_m); & D_n &:= D^X \cap X_n; & S_n &:= S^X \cap X_n; \\ F_m &:= f_m \circ F; & G_n &:= g_n \circ G; & \Psi_n &:= h_n \circ \Psi, \end{aligned}$$

As in the definition of Ind, there is $\varepsilon > 0$ such that

$$\inf\{\|z\| \mid z \in L_2(x) - v(u^{-1}(x)), x \in S^X\} > \varepsilon. \tag{82.30}$$

Hence, there is $N \geq 1$ such that $L_2(x) \notin \Psi_n(x)$ for $n \geq N$ and $x \in \operatorname{bd} D_n$.

Fix $n_0 \geq N$.

Claim. There is $n \geq n_0$ such that, for any $m \geq n$, if $x \in S_n$, $(x,y) \in \Omega_{n,m}$, $L_1(y) \in F_m(x,y)$ and $\mu(L_2(x) - \Psi_n(x)) \cap (L_2(x) - g_n \circ G(x,y)) \neq \emptyset$, then $\mu \geq 0$.

First observe that if, for some $(x,y) \in Z_{S^X}$, $0 \in L_2(x) - G(x,y)$, then (x,y) is a solution to (82.17). Hence, instead of (C3), we may suppose that:

(C3)' if $\mu(L_2(x) - \Psi(x,y)) \cap (L_2(x) - G(x,y)) \neq \emptyset$ for some $(x,y) \in Z_{S^X}$, then $\mu > 0$.

Suppose now to the contrary that the Claim is not true. Hence, for any $n \geq n_0$, there is $m_n \geq n$, $x_n \in S_n$, $y_n \in Y_{m_n}$, $z_n \in X'$ and $\mu_n < 0$ such that $L_1(y_n) \in F_{m_n}(x_n, y_n)$ and $z_n \in \mu_n(L_2(x_n) - \Psi(x_n)) \cap (L_2(x_n) - G_n(x_n, y_n))$.

Since, for all $n \geq n_0$, $\|x_n\| = 1$ and $z_n \in L_2(x_n) - G_n(x_n, y_n)$ we see that the sequence (z_n) is bounded. On the other hand, $z_n \in \mu_n(L_2(x_n) - \Psi_n(x_n))$; in view of (82.30), we see that the sequence (μ_n) is also bounded. Hence, passing to a subsequence, $\mu_n \to \mu_0 \leq 0$ as $n \to \infty$. Now, for all $n \geq n_0$, there are $z'_n \in G(x_n, y_n)$ and $z''_n \in \Psi(x_n)$ such that $Z_n = L_2(x_n) - g_n(z'_n)$ and $z_n = \mu_n(L_2(x_n) - h_n(z''_n))$. Without loss of generality we may suppose that $z'_n \to z'_0$ and $z''_n \to z''_0$ as $n \to \infty$. Hence $L_2(x_n) - z_n = g_n(z'_n) \to z'_0$ and $\mu_n L_2(x_n) - z_n = \mu_n h_n(z''_n) \to \mu_0 z''_o$. Therefore $(\mu_n - 1)z_n \to (z''_0 - z'_0)$. This implies that $z_n \to z_0 := \mu_0(\mu_0 - 1)^{-1}(z''_0 - z'_0)$. Hence $L_2(x_n) \to z_0 + z'_0$ and, again passing to a subsequence, $x_n \to x_0 \in B$ and $L_2(x_0) = z_0 + z'_0$.

Moreover, for any n, there is $y'_n \in F(x_n, y_n)$ such that $\|L_1(y_n) - y'_n\| < m_n^{-1}$. Passing to a subsequence if necessary, we may suppose that $y'_n \to y'_0 \in \operatorname{cl} F(\Omega \cap \operatorname{pr}^{-1}(S^X))$, as $n \to \infty$. Obviously $L_1(y_n) \to y'_0$, too. According to our assumptions (for almost all $n \geq 1$), the sequence $(\mathcal{P}(x_n, y_n))$ is bounded; hence, passing to a subsequence, $y_n \to y_0$.

Finally, by the upper semicontinuity of F, G and Ψ, $L_1(y_0) \in F(x_0, y_0)$, $(x_0, y_0) \in Z_{S^X}$ and $z_0 \in \mu_0(L_2(x_0) - \Psi(x_0)) \cap (L_2(x_0) - G(x_0, y_0))$: a contradiction with (C3)' ends the proof of the claim.

Fix $n \geq n_0$ given in Claim. The map S_F is upper semicontinuous, D_n is compact; hence $Z_{D_n} = \{(x, y) \in \Omega \cap (D_n \times Y) \mid L_1(y) \in F(x, y)\}$ is compact and contained in $\operatorname{int} \Omega$. Hence, there is $\delta > 0$ such that the δ-neighbourhood $\mathcal{O}_\delta(Z_{D_n})$ is contained in $\operatorname{int} \Omega$. Take $m \geq n$ such that $m^{-1} < \delta$; then $\{(x, y) \in \Omega \cap (D_n \times Y) \mid L_1(y) \in F_m(x, y)\}$ is contained in $\mathcal{O}_\delta(Z_{D_n})$.

In this way we get finite-dimensional spaces $X_n \subset X$, $X'_n \subset X'$, $Y_m \subset Y$ and $Y'_m \subset Y'$. Let $L_1^m := L_1|_{Y_m}$ and $L_2^n := L_2|_{X_n}$. It is clear that $\dim X_n - \dim X'_n = i_2$ and $\dim Y_m - \dim Y'_m = i_1$. According to (C1), and enlarging $\widetilde{Y}'_m$ and $\widetilde{X}'_n$ if necessary, we get that

$$
\begin{aligned}
&1 \leq \dim X'_n \leq \dim X_n < 2 \dim X'_n - 2; \\
&1 \leq \dim Y'_m \leq \dim Y_n < 2 \dim Y'_m - 1; \\
&\dim X'_n > \dim Y_m - \dim Y'_m + 1.
\end{aligned} \tag{82.31}
$$

Further, let $\varphi\colon \Omega_{n,m} \to Y'_m$, $\psi\colon \Omega_{m,n} \multimap X'_n$ and $\Phi\colon D_n \to X'_n$ be given by

$$
\varphi(x, y) = L_1^m(x) - F_m(x, y), \quad \psi(x, y) = L_2^n(x) - G_n(x, y) \quad for (x, y) \in \Omega_{m,n}
$$
$$
\Phi(x) = L_2(x) - \Psi_n(x) \quad textfor x \in D_n.
$$

It is easy to see that maps φ, ψ and Φ are admissible: they are determined by pairs (p_m, q_m), (p'_n, q'_n) and (u_n, v_n), respectively, where $p_m := p|_{p^{-1}(\Omega_{m,n})}$, $q_m := L_1^m \circ p_m - (f_m \circ q|_{p^{-1}(\Omega_{m,n})})$, $p'_m := p'|_{p'^{-1}(\Omega_{m,n})}$, $q'_n := L_2^n \circ p'_n - (g_n \circ q'|_{p'^{-1}(\Omega_{m,n})})$,

$u_n := u|_{u^{-1}}(D_n)$ and $v_n := L_2^n \circ u_n - (h_n \circ v|_{u^{-1}}(D_n))$. According to the definition of the generalized index

$$\deg((p_m, q_m)(0, \cdot\,), (\Omega_{m,n})_0, 0) = \mathrm{Ind}_{L_1}((p, q)(0, \cdot\,), \Omega_0),$$
$$\deg((u_n, v_n), D_n, 0) = \mathrm{Ind}_{L_2}((u, v), D^X).$$

By the definition of $\otimes$ in stable homotopy groups and taking into account the dimension restrictions,

$$\deg((p_m, q_m)(0, \cdot\,), (\Omega_{m,n})_0, 0) \otimes \deg((u_n, v_n), D_n, 0) = \xi \otimes \eta_0 \neq 0.$$

Thus we see that all assumptions of Theorem (82.22) are satisfied.

Summing up, for any $n_0 \geq N$, there are $m \geq n \geq n_0$ such that the system

$$\begin{cases} L_1(y) \in F_m(x, y), \\ L_2(x) \in G_n(x, y) \end{cases}$$

has a solution $(x_m, y_m) \in \Omega \cap (D_n \times Y_m)$. Using the upper semicontinuity of F and G, after passing to a subsequences, we get the solution $(x_0, y_0) \in \Omega \cap (D^X \times Y)$ of (82.17). □

Finally, following D. Gabor and W. Kryszewski ([GaDKr-1]), we shall establish the existence of solutions to the following problem

$$\begin{cases} x'(t) \in g(t, x(t), y(t)), & \text{for a.a. } t \in I := [0, T], \\ x(0) = x(T), & \\ y'(t) \in f(t, x(t), y(t)), & \text{for a.a. } t \in I, \\ O \in l(x, y), & \end{cases} \tag{82.32}$$

where $T > 0$, $f: I \times \mathbb{R}^n \times \mathbb{R}^m \multimap \mathbb{R}^m$, $g: I \times \mathbb{R}^n \times \mathbb{R}^m \multimap \mathbb{R}^n$ are Carathéodory multifunctions with convex compact values and $l: C(I, \mathbb{R}^n) \times C(I, \mathbb{R}^m) \multimap \mathbb{R}^k$, $1 \leq k \leq m$, is a set-valued map ($C(I, P\mathbb{R}^d)$ stands, as usual, for the space of continuous maps endowed with the standard sup-norm $\|\cdot\|$). The map l may be viewed as the system of nonlocal boundary value data. By a solution we mean a pair of absolutely continuous functions $x: I \to \mathbb{R}^n$, $y: I \to \mathbb{R}^m$ satisfying (82.32).

Recall that a multifunction $h: I \times \mathbb{R}^n \times \mathbb{R}^m \multimap \mathbb{R}^d$ is Carathéodory if:

(i) for almost all (a.a) $t \in I$, $h(t, \cdot\,, \cdot\,): \mathbb{R}^n \times \mathbb{R}^m \multimap \mathbb{R}^m$ is u.s.c.,
(ii) for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$, $h(\,\cdot\,, x, y): I \multimap \mathbb{R}^m$ is measurable.

By the *Nemytski operator generated by* h we mean a multifunction assigning, to each pair of measurable functions $x: I \to \mathbb{R}^n$, $y: I \to \mathbb{R}^m$, the set

$$N_h(x, y) := \{z: I \to \mathbb{R}^d \mid z(t) \in h(t, x(t), y(t)) \text{ a.e. on } I; z \text{ is measurable}\}.$$

It is well-know that N_h is well-defined, i.e. $N_h(x,y) \neq \emptyset$.

Assume that f, g are Carathéodory multifunctions such that

(f_1) there is an integrable function $\alpha \in L^1(I,\mathbb{R})$ such that, for any $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$ and a.a. teI, if $z \in f(t,x,y)$, then $\|z\| \leq \alpha(t)(1 + \|x\| + \|y\|)$,

(g_1) there is a function $\beta \in L^1(I,\mathbb{R})$ such that, for any $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$ and a.a. $t \in I$, if $z \in g(t,x,y)$, then $\|z\| \leq \beta(t)(1 + \|x\| + \|y\|)$.

Additionally we assume that there is a smooth C^1-function $V\colon \mathbb{R}^n \to \mathbb{R}$ such that:

(V_1) there is $r_1 > 0$ such that, for any $x \in \mathbb{R}^n$, if $\|x\| > r_1$, then $\langle \nabla V(x), x\rangle > 0$,

(V_2) there is $r_2 > 0$ such that, for any $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$ and $t \in I$, then there is $z \in g(t,x,y)$ such that $\langle \nabla V(x), z\rangle \geq 0$.

Further on assume also that $l\colon C(I,\mathbb{R}^n) \times C(I,\mathbb{R}^m) \multimap \mathbb{R}^k$, where $1 \leq k \leq m$, is a set-valued map such that:

(l_1) l is an admissible map and maps bounded sets onto bounded ones,

(l_2) there is $\rho > 0$ such that, for any $x \in C(I,\mathbb{R}^n)$ and $y \in C(I,\mathbb{R}^m)$, if $0 \in l(x,y)$, then $\|y(t_x)\| \leq \rho$ for some $t_x \in I$.

(l_3) there is $R > 0$ such that, for all $y \in C(I,\mathbb{R}^m)$, if $\|y(0)\| \geq R$, then $0 \notin l(0,y)$ and there is a an admissible pair (u,v) determining l such that

$$\operatorname{Deg}((\overline{u},\overline{v})(0,\,\cdot\,), B_c(0,R), 0) \neq 0 \in \Pi_{m-k},$$

where $B_c(0,R) = \{y \in C_c(I,\mathbb{R}^m) \mid \|y\| \leq R\}$ and $(\overline{u},\overline{v})$ is the restriction of the pair (u,v) to $C(I,\mathbb{R}^n) \times C_c(I,\mathbb{R}^m)$.

(82.33) Remark.

(82.33.1) Observe that if e.g. $V(x) = \|x\|^2$ for $x \in \mathbb{R}^n$, then condition (V_1) is satisfied automatically and (V_2) means that, for all $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$ and $t \in I$, if $\|x\| > r_2$, then $g(t,x,y)$ has a nonempty intersection with the half space $\{z \in \mathbb{R}^n \mid \langle x, z\rangle \geq 0\}$.

(82.33.2) Let, for $t \in I$, $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$,

$$\overline{g}(t,x,y) = \begin{cases} g(t,x,y) & \text{if } \|x\| \leq r_2, \\ \{z \in g(t,x,y) \mid \langle \nabla V(x), z\rangle \geq 0\} & \text{if } \|x\| > r_2. \end{cases}$$

It is easy to see that $\overline{g}\colon I \times \mathbb{R}^n \times \mathbb{R}^m \multimap \mathbb{R}^n$ is a Carathéodory multifunction (with nonempty compact convex values) having the sublinear growth and, for all $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$ and $t \in I$, if $\|x\| \geq r_2$ and $z \in \overline{g}(t,x,y)$, then $\langle \nabla V(x), z\rangle \geq 0$. Moreover, if $x'(t) \in \overline{g}(t,x(t),y(t))$, then $x'(t) \in g(t,x(t),y(t))$. Hence, in what follows, replacing g by $\overline{g}$ if necessary, we assume that g satisfies the same properties as $\overline{g}$ does.

(82.33.3) Taking, if necessary, $\max\{\alpha(t),\beta(t),1\}$ we may assume that, in (f_1) and (g_1), $\alpha(t)=\beta(t)\ge 1$ for all $t\in I$. Moreover, we may assume without loss of generality that, in (V_1) and (V_2), $r_1=r_2$.

(82.33.4) By (f_1) and (g_1), if functions x, y are continuous and, for a measurable function z, $z(t)\in f(t,x(t),y(t))$ (resp. $z(t)\in g(t,x(t),y(t))$) for a.a. $t\in I$, then $z\in L^1\mathbb{R}^m)$ (resp. $z\in L^1(I,\mathbb{R}^n)$). Therefore $N_f\colon C(I,\mathbb{R}^n)\times C(I,\mathbb{R}^m)\multimap L^1(I,\mathbb{R}^m)$ (resp. $N_g\colon C(I,\mathbb{R}^n)\times C(I,\mathbb{R}^m)\multimap L^1(I,\mathbb{R}^m)$).

(82.34) THEOREM. *Under assumptions* (l_1)–(l_3)*,* (f_1)*,* (g_1) *and* (V_1)–(V_2)*, the system* (82.32) *has a solution.*

PROOF. Let $a:=\int_0^T\alpha(s)\,ds$ and

$$r:=(r_1+\rho+1)e^{2a}. \tag{82.34.1}$$

Step 1. Let $X:=C(I,\mathbb{R}^n)$, $X':=X$; $Y:=C(I,\mathbb{R}^m)$, $Y':=Y_0\oplus\mathbb{R}^k$, where $Y_0:=\{y\in Y\mid y(0)=0\}$, and let $L_1\colon Y\to Y'$ be given, for $y\in Y$, by $L_1(y)=(y-y(0),0)$. It is evident that L is a linear continuous operator, $\operatorname{Ker}(L_1)=C_c(I,\mathbb{R}^m)$, $\operatorname{Im}(L_1)=Y_0\oplus\{0\}$, i.e. L_1 is a Fredholm operator of index $m-k$.

Consider the multivalued transformation F which assigns to $(x,y)\in X\times Y$ a subset of Y' given by

$$F(x,y)=\{J(z)\mid z\in N_f(x,y)\}\times l(x,y),$$

where $N_f\colon X\times Y\multimap L^1(I,\mathbb{R}^m)$ is the Nemytski operator generated by f and $J\colon L^1(I,\mathbb{R}^m)\to Y_0$ is given, for $z\in L^1(I,\mathbb{R}^m)$, by

$$J(z)(t)=\int_0^t z(s)\,ds,\quad t\in I.$$

It is clear that the problem

$$\begin{cases} y'\in f(t,x,y),\\ 0\in l(x,y)\end{cases} \tag{82.34.2}$$

is equivalent to

$$L_1(y)\in F(x,y) \tag{82.34.3}$$

with $x\in X$, $y\in Y$. The values of N_f is are weakly compact, closed and convex (it is a consequence of the Dunford–Pettis Theorem). In view of the Ascoli–Arzela theorem the values of $J\circ N_f$ are compact convex, $J\circ N_f\colon X\times Y\multimap Y_0$ is upper semicontinuous and compact (when restricted to a bounded subset in $X\times Y$).

Hence in view of (l_1), $F: X \times Y \multimap Y'$ is an admissible set-valued map being compact when restricted to a bounded subset of $X \times Y$.

Next define the multivalued transformation G which assigns to $(x, y) \in X \times Y$ a subset of X given, for $(x, y) \in X \times Y$, by

$$G(x, y) = \{K(z - x) \mid z \in N_g(x, y)\},$$

where $N_g: X \times Y \multimap L^1(I, \mathbb{R}^n)$ is the Nemytski operator of g and $K: L^1(I, \mathbb{R}^n) \to X$ is given, for $z \in L^1(I, \mathbb{R}^n)$, by

$$K(z)(t) = e^t \frac{e^T}{1 - e^T} \int_0^T e^{-s} z(s)\, ds + e^t \int_0^T e^{-s} z(s)\, ds, \quad t \in I.$$

Moreover, let $L_2 := I_X: X \to X$ be the identity operator. As above, it is easy to see that $G: X \times Y \multimap X$ is an admissible map (upper semicontinuous with, in fact, compact convex values); moreover, G is compact when restricted to a bounded set. Since, for each $z \in L^1(I, \mathbb{R}^n)$, $K(z)$ is a periodic function, we see that the coincidence problem

$$L_2(x) \in G(x, y), \tag{82.34.4}$$

with $X \in X$, $y \in Y$, is equivalent to the inclusion

$$\begin{cases} x' \in g(t, x, y), \\ x(0) = x(T). \end{cases} \tag{82.34.5}$$

Within the introduced setting problem (82.32) is equivalent to the following one

$$\begin{cases} L_1(y) \in F(x, y), \\ L_2(x) \in G(x, y). \end{cases} \tag{82.34.6}$$

Step 2. We shall prove that the system (82.34.6) satisfies assumptions of Theorem (82.29) (with $D = B^X(0, r + 1)$ where r was given in (82.34.1). To this end let

$$\Omega := \{(x, y) \in D \times Y \mid \|y\|_Y \leq M\},$$

where

$$M := \max\{[\rho + (2 + r)a]e^a, (R + a)e^a\} + 1.$$

Clearly Ω, satisfies assumption (A4). Let

$$Z := \{(x, y) \in \Omega \mid L_1(y) \in F(x, y)\}.$$

If $(x, y) \in Z$, then $l(x, y) = 0$. $\|x\|_X \le r + 1$ and $y' \in f(t, x, y)$. Hence, by (f_1) and (l_2),

$$\|y(t)\| \le \|y(t_x)\| + \int_{t_x}^{t} \alpha(s)(2 + r + \|y(s)\|)\, ds.$$

Therefore, by the Gronwall inequality, $\|y\|_Y \le [\rho + (2 + r)a]e^a < M$. This shows that $Z \subset \operatorname{int}\Omega$ and assumption (A3) is satisfied. Since $F|_\Omega$ is compact and admissible, Ω is bounded, we see that assumptions (A2) and (A1) are also satisfied (see Lemma (82.18)).

Step 3. According to Remark (82.33) we put $\Psi \equiv 0$; since (C1) is obviously satisfied, it now remains to check assumptions (C2) and (C3).

As concerns (C2), we have to show that there is an admissible pair (p, q) determining F such that $\operatorname{Ind}_{L_1}((p, q)(0, \cdot\,), \Omega_0) \ne 0 \in \Pi_{m-k}$.

Suppose that an admissible pair $\Omega \xleftarrow{r} \Delta \xrightarrow{s} Y_0$, determines $J \circ N_f|_\Omega$ and let $\Omega \xleftarrow{u} \Sigma \xrightarrow{v} \mathbb{R}^k$ be an admissible pair determining l and such that, according to (l_3), $\operatorname{Deg}((u, v)(0, \cdot\,), B^Y(0, R) \cap C_c(I, \mathbb{R}^m), 0) \ne 0$ in Π_{m-k}.

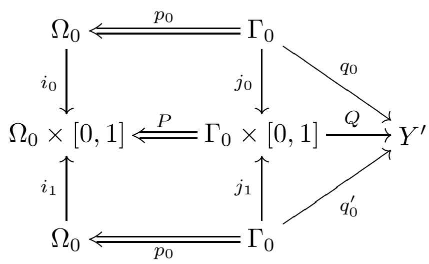
Observe that the map F is determined by the c-l-admissible pair $\Omega \overset{p}{\Longleftarrow} \Gamma \xrightarrow{q} Y'$ where $\Gamma := \{(\tau, \sigma) \in \Delta \times \Sigma \mid r(\tau) = u(\sigma)\}$, $p(\tau, \sigma) := r(\tau)$ and $q(\tau, \sigma) := (s(\tau), v(\sigma))$ for all $(\tau, \sigma) \in \Gamma$.

The index $\operatorname{Ind}_{L_1}((p, q)(0, \cdot\,), \Omega_0)$ is well-defined and

(82.34.7) $$\operatorname{Ind}_{L_1}((p, g)(0, \cdot\,), \Omega_0) := \operatorname{Ind}_{L_1}((p_0, q_0), \Omega_0),$$

where $\Omega \overset{p_0}{\Longleftarrow} \Gamma_0 \xrightarrow{q_0} Y'$ $\Gamma_0 := \{y, \tau, \sigma) \in \Omega_0 \times \Gamma \mid p(\tau, \sigma) = (0, y)\}$, $p_0(y, \tau, \sigma) = y$ and $q_0(y, \tau, \sigma) = q(\tau, \sigma) = (s(\tau), v(\sigma))$ for $(y, \tau, \sigma) \in \Gamma_0$.

Consider the following diagram



where $j_k(y, \tau, \sigma) = (y, \tau, \sigma, k)$ $(k = 0, 1)$, $P(y, \tau, \sigma, \lambda) = (y, \lambda)$, $Q(y, \tau, \sigma, \lambda) = ((1 - \lambda)s(\tau), v(\sigma))$, $q_0' = (0, v(\sigma))$ for all $\lambda \in [0, 1]$ and $(y, \tau, \sigma) \in \Gamma_0$. It is easy to see that this diagram is commutative.

Suppose that, for some $\lambda \in [0, 1]$ and $y \in \Omega_0$, $L_1(y) \in Q(P^{-1}(y, \lambda))$. This means that there is $(\tau, \sigma) \in \Gamma$ such that $r(\tau) = u(\sigma) = p(\tau, \sigma) = (0, y)$ and $L_1(y) = ((1 - \lambda)s(\tau), v(\sigma))$ i.e. $0 \in l(0, y)$ and $y - y(0) \in (1 - \lambda)J \circ N_f(0, y)$, i.e.

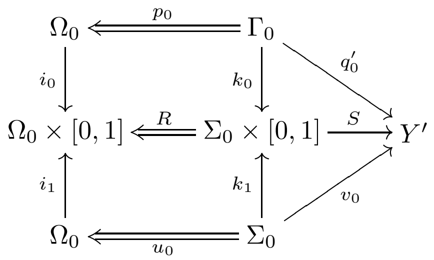
$y'(t) \in (1-\lambda)f(t, 0, y(t))$ for a.a. $t \in I$. By (l_3), the first inclusion implies that $\|y(0)\| < R$. Hence, for all $t \in I$,

$$\|y(t)\| \leq R + \int_0^t \alpha(s)(1 + \|y(s)\|)\, ds.$$

By the Gronwall inequality, $\|y\|_Y \leq (R+a)e^a < M$. The homotopy invariance of the index Ind_{L_1}, we see that

$$\mathrm{Ind}_{L_1}((p_0, q_0), \Omega_0) = \mathrm{Ind}_{L_1}((p_0, q_0'), \Omega_0). \tag{82.34.8}$$

Now, let $\Sigma_0 := \{(y, \sigma) \in \Omega_0 \times \Sigma \mid u(\sigma) = (0, y)\}$. Consider a diagram:



where $k_1(y, \sigma) = (y, \sigma, 1)$, $R((y, \sigma, \lambda) = (y, \lambda)$, $S(y, \sigma, \lambda) = (0, v(\sigma))$ for all $\lambda \in [0,1]$, $u_0(y, \sigma) = y$, $v_0(y, \sigma = (0, v(\sigma))$, $k_0(y, \tau, \sigma) = (y, \sigma, 0)$. As above this diagram is commutative and, if for some $\lambda \in [0,1]$ and $y \in \Omega_0$, $L_1(y) \in S(R^{-1}(y, \lambda))$, then there is $\sigma \in \Sigma$ such that $u(\sigma) = (0, y)$ and $L_1(y) = (0, v(\sigma))$ i.e. $0 \in l(0, y)$ and $y \in C_c(I, \mathbb{R}^m)$. Hence, by ($l_3$), $\|y\|_Y \leq R < M$. Once more, by the homotopy invariance of the index Ind_{L_1} we get

$$\mathrm{Ind}_{L_1}((p_0, q_0'), \Omega_0) = \mathrm{Ind}_{L_1}((u_0, v_0), \Omega_0). \tag{82.34.9}$$

But observe, that in fact $v_0(u^{-1}(\Omega_0)) \subset \{0\} \oplus \mathbb{R}^k = \mathrm{Coker}(L_1)$. Hence, by Corollary (82.16),

$$\begin{aligned} \mathrm{Ind}_{L_1}((u_0, v_0), \Omega_0) &= \mathrm{Deg}((u_0', -v_0'), \Omega_0 \cap \mathrm{Ker}(L_1), 0) \\ &= \mathrm{Deg}((u_0', -d \circ v_0'), \Omega_0 \cap \mathrm{Ker}(L_1), 0), \end{aligned} \tag{82.34.10}$$

where

$$\begin{aligned} u_0' &= u_0|_{u_0^{-1}(\Omega_0 \cap \mathrm{Ker}(L_1))} \colon u_0^{-1}(\Omega_0 \cap \mathrm{Ker}(L_1)) \to \Omega_0 \cap \mathrm{Ker}(L_1), \\ v_0' &= v_0|_{u_0^{-1}(\Omega_0 \cap \mathrm{Ker}(L_1))} \colon u_0^{-1}(\Omega_0 \cap \mathrm{Ker}(L_1)) \to \{0\} \oplus \mathbb{R}^k \end{aligned}$$

and $d\colon \{0\} \oplus \mathbb{R}^k \to \mathbb{R}^k$ is a natural projection.

Now one can easily check that by assumption (l_3)

$$\begin{aligned}(82.34.11)\qquad \operatorname{Deg}((u_0', -d\circ v_0'), \Omega_0\cap \operatorname{Ker}(L_1), 0) &= \operatorname{Deg}((\overline{u}, -\overline{v})(0,\,\cdot\,), \Omega_0\cap\operatorname{Ker}(L_1), 0)\\ &= \operatorname{Deg}((\overline{u}, -\overline{v})(0,\,\cdot\,), B(0,R)\cap C_c(I,\mathbb{R}^m)) \neq 0.\end{aligned}$$

In view of (82.34.7)–(82.34.11), we see that $\operatorname{Ind}_{L_1}((p,q)(0,\,\cdot\,),\Omega_0)\neq 0$ in Π_{m-k}. This establishes condition (C2).

In order to show (C3) suppose that there is $\mu<0$ and $(x,y)\in Z$ with $\|x\|_X = r$ such that $\mu L_2(x)\in L_2(x)-G(x,y)$. This means that $x(0)=x(T)$, $l(x,y)=0$ and, for a.a. $t\in I$,

$$\begin{cases} y'\in f(t,x(t),y(t)),\\ x'(t)\in g(t,x(t),y(t))+(1-\lambda)x(t),\end{cases}$$

where $\lambda := (1-\mu)^{-1}\in(0,1)$. We shall see that this can not occur.

Claim. $\min_{t\in I}\|x(t)\|>r_1$.

Suppose to the contrary that there is $t_1\in I$ such that

$$(82.34.12)\qquad \|x(t_1)\|\le r_1.$$

Since $y'(t)\in f(t,x(t),y(t))$, there is an integrable function $z\in N_f(x,y)$ such that $y'(f)=z_1(t)$. Recalling (l_2), $y(t)=y(t_x)+\int_{t_x}^t z_1(s)\,ds$. By ($f_1$), for any $t\in I$,

$$\|y(t)\|\le y(t_x)\| + \int_{t_x}^t \alpha(s)(1+\|x(s)\|+\|y(s)\|)\,ds.$$

Similarly, since $x'(t)\in\lambda g(t,x(t),y(t))+(1-\lambda)x(t)$, there is an integrable $z_2\in N_g(x,y)$ such that $x(t)=x(t_1)+\int_{t_1}^t \lambda z_2(s)+(1-\lambda)x(s))ds$. By ($g_1$) and since $\alpha(s)\ge 1$, for all $t\in I$,

$$\|x(t)\|\le x(t_1)\| + \int_{t_1}^t \alpha(s)(1+\|x(s)\|+\|y(s)\|)ds.$$

Hence, by (82.34.12) and (l_2), for all $t\in I$,

$$\|x(s)\|+\|y(s)\|\le (r_1+\rho)+\int_{t_1}^t \alpha(s)(1+\|x(s)\|+\|y(s)\|)\,ds + \int_{t_x}^t \alpha(s)(1+\|x(s)\|+\|y(s)\|)\,ds.$$

By Lemma (82.35) given below,

$$r\le \max_{t\in I}(\|x(t)\|+\|y(t)\|)\le (r_1+\rho+1)e^{2a}-1 = r-1.$$

The contradiction establishes the claim.

Since $x(0) = x(T)$, $\|x(t)\| > r_1$ on I and $0 < \lambda < 1$, by (V_1) and (V_2), we have

$$\begin{aligned} 0 = V(x(T)) - V(x(0)) &= \int_0^T \langle \nabla V(x(t)), x'(t)\rangle \, dt \\ &= \lambda \int_0^T \langle \nabla V(x(t)), z_2(t)\rangle \, dt + (1-\lambda)\int_0^T \langle \nabla V(x(t)), x(t)\rangle \, dt > 0. \end{aligned}$$

The obtained contradiction concludes the proof. □

(82.35) LEMMA. *Let $p \in C(I, \mathbb{R})$, $q \in L^1(I, \mathbb{R})$, $q \geq 0$, $A \in \mathbb{R}$ and $a, b \in I$. If, for all $t \in I$,*

$$p(t) \leq A + \int_a^t (1 + p(s))q(s)\, ds + \int_b^t (1 + p(s))q(s)\, ds,$$

then, for all $t \in I$,

$$p(t) \leq (A + 1)\exp\bigg(2\int_0^t q(s)\, ds\bigg) - 1.$$

PROOF. Let

$$h(t) = A + 1 + \int_a^t (1 + p(s))q(s)\, ds + \int_b^t (1 + p(s))q(s)\, ds, \quad t \in I,$$

i.e. for $t \in I$, $1 + p(t) \leq h(t)$. Then $h'(t) = 2(1 + p(t))q(t) \leq 2h(t)q(t)$. Hence, for any $t \in I$,

$$\frac{d}{dt}\bigg[h(t)\exp\bigg(-\int_0^t 2q(s)\, ds\bigg)\bigg] = \exp\bigg(-\int_0^t 2q(s)\, ds\bigg)[h'(t) - 2q(t)h(t)] \leq 0$$

and

$$h(t)\exp\bigg(-\int_0^t 2q(s)\, ds\bigg) \leq h(0) = A + 1.$$ □

We recommend [Kr2-M], [GaDKr1], [GaDKr2] for further applications.

83. Fixed points of monotone-type multivalued operators

In the first part of this section we shall define a topological degree for monotone-type multivalued operators in reflexive Banach spaces. Then in the second part some abstract fixed point results for monotone operators in ordered spaces will be formulated.

For a given reflexive Banach space E by E^* we shall denote its dual space. In what follows $\langle\,\cdot\,,\,\cdot\,\rangle$ stand for the pairing between E and E^*.

We recall the Browder–Ton Theorem (see [BT]):

(83.1) THEOREM. *For any reflexive Banach space E there exists a separable Hilbert space H and a linear completely continuous* ([8]) *injection $h: H \to E$ such that $h(H)$ is dense in E.*

Assume that $h: H \to E$ is the same as in (83.1). We define the map $h^*: E \to H^*$ by putting:

$$h^*(f)(v) = f(h(v)), \quad \text{for every } f \in E^* \text{ and } v \in H.$$

Now, we define a map $\widehat{h}: E^* \to H$ by letting:

$$\widehat{h}(f) = v_f \quad \text{for every } f \in E^*,$$

where v_f is an unique element of H for which we have:

$$\langle v, v_f \rangle = h^*(f)(v),$$

for every $v \in H$, where $\langle v, v_f \rangle$ is the inner product of v and v_f in H.

It is easy to see that (comp. [BM]):

(83.2) COROLLARY. *The map $\widehat{h}: E^* \to H$ is a linear completely continuous injection.*

Let X be a metric space and $g: X \to E^*$ be a map. We shall say that g is *demicontinuous* provided for every $\{x_n\} \subset X$ if $\{x_n\} \to x$, then $\{g(x_n)\} \rightharpoonup g(x)$ i.e. the sequence $\{g(x_n)\}$ is weakly convergent to $g(x)$; g is called *bounded* if it maps bounded sets of X into bounded sets in E^*.

Let us consider an open bounded subset U of E, where E is a reflexive Banach space and let $\overline{U}$ be the closure of U in E. We shall consider a pair of the following type:

$$\overline{U} \overset{p}{\Longleftarrow} \Gamma \overset{q}{\longrightarrow} E^*.$$

The pair $(p, q): \overline{U} \to E^*$ is called *nonotone* if for every $x_1, x_2 \in \overline{U}$ and for every $y_1 \in q(p^{-1}(x_1))$, $y_2 \in q(p^{-1}(x_2))$ we have:

$$\langle x_1 - x_2, y_1 - y_2 \rangle \geq 0.$$

We shall associate to every pair $\overline{U} \overset{p}{\Longleftarrow} \Gamma \overset{q}{\longrightarrow} E^*$ the pair

$$\overline{U} \overset{p}{\Longleftarrow} \Gamma \xrightarrow{h \circ \widehat{h} \circ q} E$$

where h and $\widehat{h}$ are defined in (83.1) and (83.2).

From (83.1) we deduce:

([8]) i.e. h is continuous and for any bounded $A \subset H$ the set $h(A)$ is relatively compact in E.

(83.3) PROPOSITION. *If* $(p,q)\colon \overline{U} \to \overline{E}^*$ *is d-admissible* (*i.e. q is semicontinuous and bounded*) *and bounded, then* $(p, h\circ \widehat{h}\circ q)\colon \overline{U}\to E^*$ *is compact and admissible.*

The proof of (83.3) is straightforward.

Instead of the pair $(p, h\circ\widehat{h}\circ q)$ for every $\varepsilon > 0$, we can define a pair $(p, \varepsilon\cdot h\circ\widehat{h}\circ q)$, where $(p,\varepsilon\cdot h\circ\widehat{h}\circ q)(y) = h(\widehat{(q(y)})/\varepsilon$. Of course under assumptions of (83.3), we have that $(p,\varepsilon\cdot h\circ\widehat{h}\circ q)$ is compact and admissible.

We let

(83.4) $\varphi_\varepsilon\colon \overline{U}\to E$ to be defined as follows:

$$\varphi_\varepsilon(x) = \left\{ x+y \;\middle|\; y \in \left(\frac{1}{\varepsilon}\cdot h\circ\widehat{h}\circ q\right)(p^{-1}(x))\right\}.$$

Below we shall introduce the class of (monotone-type) mappings for which we shall able to define the topological degree.

(83.5) DEFINITION. A multivalued map $(p,q)\colon \overline{U} \multimap E^*$ is *acceptable* provided the following conditions are satisfied:

(83.5.1) (p,q) is d-admissible and bounded,

(83.5.2) for every sequence $\{x_n\}\subset \overline{U}$ such that $\{x_n\}\varepsilon x$ if there exists a sequence $\{y_n\}\subset E^*$ with $y_n\in q(p^{-1}(x_n))$ and such that $\langle x_n, y_n\rangle \le 0$, for every $n=1,2,\ldots$, then $\{x_n\}\to x$.

We let:

$$\mathcal{A}_c(\overline{U},E) = \{(p,q)\colon \overline{U}\multimap E^* \mid (p,q) \text{ is acceptable}\},$$
$$\mathcal{A}_{c\partial U}(\overline{U},E) = \{(p,q)\in \mathcal{A}(\overline{U},E) \mid 0\notin q(p^{-1}(\partial U))\}.$$

(83.6) PROPOSITION. *Assume that* $(p,q)\in\mathcal{A}_c(\overline{U},E^*)$. *Then:*

(83.6.1) *for every* $\varepsilon>0$ *the map*

$$\varphi_\varepsilon\colon \overline{U}\to E,\quad \varphi_\varepsilon(x) = \left\{x+y \;\middle|\; y\in\frac{1}{\varepsilon}\cdot h(\widehat{h}(q(p^{-1}(x))))\right\}$$

is a compact admissible vector field of the type $\mathrm{id}_E - (\widetilde{p},\widetilde{q})$, (*comp.* [GL]),

(83.6.2) *if for* $\{\varepsilon_n\}\to 0$ *there exists* $x_n\in\overline{U}$ *such that* $0\in\varphi_\varepsilon(x_n)$, *then there exists* $x\in\overline{U}$, *for which* $0\in q(p^{-1}(x))$.

PROOF. Observe that (83.6.1) is a simple consequence of (83.3). Therefore, we shall prove (83.6.2). Let us assume for the simplicity that $\varepsilon = 1/n$ and $\varphi_n = \varphi_{\varepsilon_n}$. Let $0\in\varphi_n(x_n)$, for every $n=1,2,\ldots$ and $y_n\in q(p^{-1}(x_n))$ be such that $0 =$

$x_n + nh(\widehat{h}(y_n))$. Since U is bounded and (p,q) is bounded, we can assume without loss of generality that $\{x_n\} \rightharpoonup x$ and $\{y_n\} \rightharpoonup y$. From:

$$0 = x_n + n \cdot (h(\widehat{h}(y_n))),$$

we get:

$$h(\widehat{h}(y_n)) = -\frac{1}{n} x_n.$$

Therefore, we infer $\{h(\widehat{h}(y_n))\} \to 0 = h(\widehat{h}(y))$ and $h \circ \widehat{h}$ is a linear injection, we obtain $y = 0$.

According to definition of $\widehat{h}$, we get:

$$\langle y_n, x_n \rangle = -n\langle y_k, h(\widehat{h}(y_k))\rangle = -n\|\widehat{h}(y_n)\|_H^2 \leq 0.$$

Now, by using (83.6.2), we get that $\{x_n\} \to x$ and from demicontinuity of (p,q), we get that $0 \in q(p^{-1}(x))$. □

Now, we shall define the topological degree for mappings in $\mathcal{A}_{c\partial U}(\overline{U}, E^*)$.

(83.7) Lemma. *Assume that $(p,q) \in \mathcal{A}_{\partial U}(\overline{U}, E^*)$. Then there exists $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$, we have $0 \notin \varphi_\varepsilon(\partial U)$, where φ_ε is defined in* (83.6.1).

Proof. Assuming to the contrary, we get a contradiction with (83.6.2). □

We let:

$$\mathcal{A}_{c\partial U}(\overline{U}, E) = \{\varphi\colon \overline{U} \to E \mid \varphi \text{ is a compact admissible vector field,} \\ 0 \notin \varphi(\partial U \text{ and } \varphi \text{ is determined by a pair } (\overline{p}, \overline{q})\}.$$

In view of (83.7) and (83.6), we see that:

(83.8) Proposition. *Assume $(p,q) \in \mathcal{A}_{c\partial U}(\overline{U}, E)$ and ε_0 is chosen according to* (83.7)*. Then $\varphi_\varepsilon \in \mathcal{A}_{c\partial U}(\overline{U}, E)$, for every $0 < \varepsilon < \varepsilon_0$. Moreover, if for every $0 < \varepsilon_1, \varepsilon_2 < \varepsilon$, we have:*

$$\deg_{\mathrm{LS}}(\varphi_{\varepsilon_1}) = \deg_{\mathrm{LS}}(\varphi_{\varepsilon_2}),$$

where $\deg_{\mathrm{LS}}$ *stands for the Leray–Schauder topological degree on* $\mathcal{A}_{\partial U}(\overline{U}, E)$ (*see: Chapter* IV).

Proof. From Lemma (83.7), we get that $\varphi_\varepsilon \in \mathcal{A}_{c\partial U}(\overline{U}, E)$, for every $0 < \varepsilon < \varepsilon_0$. Observe that if $0 < \varepsilon_1, \varepsilon_2 < \varepsilon_0$, then the formula:

$$t \cdot \varphi_{\varepsilon_1} + (1-t)\varphi_{\varepsilon_2} = \mathrm{id} + \left(\frac{t}{\varepsilon_1} + \frac{1-t}{\varepsilon_2}\right) h\widehat{h}qp^{-1} = \varphi_{\varepsilon_t},$$

where
$$\frac{1}{\varepsilon_t} = t \cdot \frac{1}{\varepsilon_1} + (1-t)\frac{1}{\varepsilon_2},$$
gives us the homotopy joining φ_{ε_2} with φ_{ε_2} and the hypothesis follows from the homotopy property of the Leray–Schauder degree. □

We define the function:

(83.9) $$\deg\colon \mathcal{A}_{c\partial U}(\overline{U}, E^*) \to Z$$

by putting:
$$\deg((p,q)) = \deg_{\mathrm{LS}}(\varphi_\varepsilon),$$
where $0 < \varepsilon < \varepsilon_0$ and ε_0 is chosen fro (p,q) according to (83.7).

It follows from (83.8) that Definition (83.9) is correct.

Now, it is easy to see that the topological degree defined in (83.9) satisfies the following standard properties:

(83.10.1) Existence property,
(83.10.2) Additivity property,
(83.10.3) Homotopy property.

We left to the reader to formulate (83.10.1)–(83.10.3) and also natural topological consequences of the above notion (see [BM], Chapters III, IV).

Now, let us recall some notions and results connected with ordered spaces (for details we recommend [DG-M], [He-Hu] and [JaJa].

A *partially ordered set* is a pair $(P, \leq)$, where P is a nonempty set and $\leq$ is a relation in P which is *reflexive* ($p \leq p$ for all $p \in P$), *weakly antisymmetric* (for $p, q \in P$, $p \leq q$ and $q \leq p$ imply $p = q$) and *transitive* (for $p, q, r \in P$, $p \leq q$ and $q \leq r$ imply $p \leq r$). A nonempty subset C of P is said to be a *chain* if given $p, q \in C$, either $p \leq q$ or $q \leq p$. If every chain in $(P, \leq)$ has a supremum, then $(P, \leq)$ is called *chain-complete*. A mapping $F\colon P \to P$ is said to be *isotone* or *increasing* if it preserves ordering, i.e. given $p, q \in P$, $p \leq q$ implies that $F(p) \leq F(q)$.

We have:

(83.11) THEOREM (Knaster–Tarski). *Let $(P, \leq)$ be a partially ordered set in which every chain has a supremum. Assume that $F\colon P \to P$ is isotone and there is an element $p_0 \in P$ such that $p_0 \leq F(p_0)$. Then F has a fixed point.*

By substituting an inverse ordering $\geq$ for $\leq$ in Theorem (83.11), we obtain the following dual version.

(83.12) THEOREM. *Let $(P, \leq)$ be a partially ordered set in which every chain has an infimum. Assume that $F\colon P \to P$ is isotone and there is an element $p_0 \in P$ such that $F(p_0) \leq p_0$. Then F has a fixed point.*

(83.13) REMARK. In what follows a partially ordered space $(P, \leq)$ is called also a *poset*.

By an ordered topological space we mean a poset P equipped with such a topology of closed subset $\{C_a\}_{a\in P}$ defined as follows:

$$C_a = \{x \in P \mid a \leq x\}$$

and closed subsets $\{C^a\}_{a\in P}$ given by:

$$C^a = \{x \in P \mid x \leq a\}.$$

In particular each ordered interval $[a, b]$ defined by

$$[a, b] = \{x \in P \mid a \leq x \leq b\}$$

is closed in ordered topological space P. If the topology of P is determined by a metric, we say that P is an ordered metric space. In particular any reflexive Banach space is a metric ordered space with cone ordering (see [JaJa] or [HeHu]).

Let $P = (P, \leq)$ be a poset and let $F\colon P \multimap P$ be a multivalued mapping with nonempty values. We consider:

$$M(F) = M = \{x \in P \mid x \leq y \text{ for some } y \in F(x)\}.$$

In what follows we shall consider only such multivalued mappings $F\colon P \multimap P$ for which $M(F) \neq \emptyset$. If the set $M_+ = \{x \in P \mid x < y \text{ for some } y \in F(x)\}$ is nonempty, then we denote by $f\colon M_+ \to P$ a choice function which satisfies $x < f(x) \in F(x)$ for each $x \in M_+$.

Now, we shall present some results contained in [HeHu].

(83.14) LEMMA. *Given a multifunction $F\colon P \multimap P$ assume there is $a \in M$ such that the transfinite sequence $(x_\mu)_{\mu\in\Lambda}$ defined by*

(83.14.1) *$x_0 = a$, and $0 < \mu \in \Lambda$ if and only if $x_\mu = f(\sup\{x_\nu \mid \nu < \mu\})$ exists has a supremum x in M.*

Then x is a fixed point of F.

PROOF. To see that (83.14.1) defines a transfinite sequence; denote by $\mathcal{K}$ the class of all the transfinite sequences which satisfy

$$x_0 = a, \quad \text{and if } 0 < \mu \in \Lambda \text{ then } x_\mu = f(\sup\{x_\nu \mid \nu < \mu\}) \text{ exists.}$$

It is easy to see by transfinite induction that the sequences of $\mathcal{K}$, are strictly increasing (whence $\mathcal{K}$ is a set) and compatible, and that their union also belongs to $\mathcal{K}$. This union, being the longest sequence in $\mathcal{K}$, is the one which satisfies (83.14.1). If this sequence $(x_\mu)_{\mu\in\Lambda}$ has the supremum x in M, then it must be a fixed point of F, for otherwise we could extend $(x_\mu)_{\mu\in\Lambda}$ by the term $f(x)$. □

By a slight reinterpretation of (83.14.1) we obtain.

(83.15) Proposition. *A multifunction $F\colon P \multimap P$ has a fixed point, if each increasing transfinite sequence in $F(P)$ has an upper bound in M.*

Proof. Obviously, $M \neq \emptyset$. If $a \in M$ is not a fixed point of F, consider 'sup' in the condition (83.14.1) as the least of the upper bounds in M_+ with respect to a well-ordering of M_+. Any upper bound $x \in M$ of the so obtained sequence $(x_\mu)_{\mu\in\Lambda}$ is a fixed point of F. Moreover, $x \not< y$ for each $y \in F(x)$. □

In the case when $M = P$ we have.

(83.16) Corollary. *Assume that a multifunction $F\colon P \multimap P$ is such that for each $x \in P$ there is $y \in F(x)$ for which $x \leq y$. If each increasing transfinite sequence in $F(P)$ has an upper bound, then F has a fixed point.*

If P is an ordered topological space we obtain the following consequence of Lemma (83.14).

(83.17) Theorem. *A multifunction $F\colon P \multimap P$ has a fixed point, if its values are compact subsets of P, and if it satisfies condition $M(F) \neq \emptyset$ and conditions*

(83.17.1) *$x_1 \leq y_1 \in F(x_1)$ and $x_1 \leq x_2$ imply $y_1 \leq y_2$ for some $y_2 \in F(x_2)$,*
(83.17.2) *each increasing transfinite sequence in $\bigcup F(P)$ has a cluster point in P.*

Proof. Given $a \in M$, let $(x_\mu)_{\mu\in\Lambda}$ be defined by (83.14.1). By Lemma (83.14) it suffices to show that $(x_\mu)_{\mu\in\Lambda}$ has the supremum in M. Condition (83.17.2) implies by Lemma (83.14) that $(x_\mu)_{\mu\in\Lambda}$ converges to its least upper bound x. If a is the only term of $(x_\mu)_{\mu\in\Lambda}$, then $x = a \in M$. Otherwise, for $0 < \mu \in \Lambda$ there is by (83.14.1) $y_m u \in P$ such that $y_\mu \leq x_\mu \in F(y_\mu)$, whence $y_\mu \leq x$. This implies by condition (83.17.1) that, there corresponds $z_\mu \in F(x)$ for which $x_\mu \leq z_\mu$. Moreover, $[x_\nu) \cap F(x) \subset [x_\mu) \cap Fx$ whenever $\mu < v \in \Lambda$, whence $\{[x_\mu) \cap Fx \mid \mu \in \Lambda\}$ is a family of closed and nonempty subsets of Fx having the finite intersection property. Since $F(x)$ is compact, then the intersection of this family is nonempty. Thus, there is $y \in Fx$ such that $x_\mu \leq y$ for each $\mu \in \Lambda$. Since x is the least upper bound of $(x_\mu)_{\mu\in\Lambda}$ then $x \leq y$. This proves that $x \in M$, whence x is by Lemma (83.14) a fixed point of F. □

As another special case of Lemma (83.14) we have:

(83.18) THEOREM. *Let $F: P \multimap P$ be a multifunction whose values are closed and directed sets, and which satisfies conditions $M(F) \neq \emptyset$, (83.17.1) and (83.17.2). If P satisfies the first axiom of countability, then F has a, fixed point.*

An ordered metric space P is called *regularly* (resp. *fully regularly*) ordered, if each monotone and order (resp. metrically) bounded ordinary sequence of P converges. For instance, each closed subset of an ordered normed space with (fully) regular order cone is (fully) regularly ordered metric space.

Below, we shall formulate next consequences of Lemma (83.14). For details we recommend [HeHu].

Let P be an ordered metric space. We say that a multifunction $F: P \multimap P$ is *bounded above* (resp. *bounded*) if $\bigcup F(P)$ is bounded above (resp. metrically bounded) in P.

(83.19) PROPOSITION. *Let P be a regularly (resp. fully regularly) ordered metric space. A multifunction $F: P \multimap$ has a fixed point, if it is bounded above (resp. bounded) and satisfies the following conditions: $M(F) \neq \emptyset$, (83.17.1) and if Fx is closed and directed or compact for each $x \in P$.*

From Proposition (83.19) it follows:

(83.20) PROPOSITION. *Let P be a closed subset of an ordered reflexive Banach space E with normal order cone. A multifunction $F: P \multimap P$ has a fixed point, if it is bounded and satisfies: $M(F) \neq \emptyset$, (83.17.1) and if its values are closed and directed.*

(83.21) PROPOSITION. *Let P be a weakly closed subset of an ordered reflexive Banach space E, which is separable or its order cone is normal. A multifunction $F: P \multimap P$ has a fixed point, if it is bounded and satisfies: $M(F) \neq \emptyset$, (83.17.1) and if its values are weakly closed.*

84. Multivalued Poincaré operators

In Section 72 we defined so called *Poincaré multivalued operator* associated with the Cauchy problem for first order differential inclusions. Note that for the first time, the above operator was considered in 1983 (see [DyG]).

In this section we would like to survey current results connected with the above mentioned operators. Let us explain also that Poincaré idea of the translation operator along the trajectories of differential systems comes back to the end of the nineteenth century. Since it was effectively applied for investigating periodic orbits by A.-A. Andronov (comp. [KZ-M]) in the late 20's and by N. Levinson in 1944.

By Poincaré operators we mean the translation operator along the trajectories of the associated differential system and the first return (or section) map defined on the cross section of the torus by means of the flow generated by the vector field. The translation operator is sometimes also called as Poincaré–Andronov or Levinson or, simply, T-operator.

In the classical theory (see [KZ-M] and the references therein), both these operators are defined to be single-valued, when assuming, among other things, the uniqueness of the initial value problems. At the absence of uniqueness one usually approximates the right-hand sides of given systems by the locally lipschitzean ones (implying already uniqueness), and then applies the standard limiting argument. This might be, however, rather complicated and is impossible for discontinuous right-hand sides.

On the other hand, set-valued analysis allows us to handle effectively also with such classically troublesome situations. In particular, the class of admissible maps in the sense of [Go1-M] has been shown to be very useful with this respect, because generalized topological invariants like the Brouwer degree, the fixed point index or the Lefschetz index with properties similar to those of their classical analogues can be defined, and subsequently applied for them.

Hence, in our contribution we develop at first the Rothe-type generalization of the Brouwer fixed-point theorem for admissible maps. Then we introduce some conditions under which the Marchaud right-hand sides of differential inclusions determine admissible Poincaré operators. Finally, we present simple applications of the obtained results to the existence of forced nonlinear oscillations and, in the single-valued case, to the multiplicity criterion for the target problem.

In this section we need the following formulation of the Rothe (or so called Bohl) theorem (comp. (55.5)).

(84.1) THEOREM. *Let K be a compact convex subset of $\mathbb{R}^n$, with nonempty interior and let $\varphi\colon K \multimap \mathbb{R}^n$ be an admissible map such that $\varphi(\partial K) \subset K$, where ∂K denotes the boundary of K. Then* $\operatorname{Fix}(\varphi) \neq \emptyset$.

The proof of (84.1) is strictly analogous to the proof of (55.5). Note also that from the Lefschetz fixed point theorem for admissible mappings it follows (comp. (41.12) and (41.13)).

(84.2) THEOREM. *If K is a compact convex nonempty subset of $\mathbb{R}^n$ and $\varphi\colon K \multimap K$ is a admissible, then* $\operatorname{Fix}(\varphi) \neq \emptyset$.

Now, consider the differential inclusion

$$\theta' \in f(t,\theta), \tag{84.3}$$

where $\theta = (\theta_1, \dots, \theta_n)$, $\theta' = (\theta_1', \dots, \theta_n')$ and $f(t,\theta) = (f_1(t,\theta), \dots, f_n(t,\theta))$. Assume that $f\colon \mathbb{R} \times \mathbb{R}^n \multimap \mathbb{R}^n$ is bounded in t, and linearly bounded in θ, upper semi-continuous with nonempty, compact, convex values. Then all solutions of (84.3) entirely exist in the sense of Carathéodory (i.e. are locally absolutely continuous and satisfy (84.3)). Moreover, for any compact interval, the solution set is R_δ (see Section 69).

If $\theta(t, X) := \theta(t; 0, X)$ is a solution of (84.3) with $\theta(0, X) = X$, then we can define the *Poincaré–Andronov map* (*translation operator at the time* T) along the trajectories of (84.3) as follows:

(84.4)
$$\Phi_T : \mathbb{R}^n \multimap \mathbb{R}^n,$$
$$\Phi_T(X) := \{\theta \mid \theta(\,\cdot\,, X) \text{ is a solution of (84.3) satisfying } \theta(0, X) = X\}.$$

We recall the following important property (comp. Section 72).

(84.5) LEMMA. *Φ_T given by* (84.4) *is an admissible map.*

PROOF. $\Phi_T : \mathbb{R}^n \multimap \mathbb{R}^n$ can be considered as the composition of two maps, namely $\Phi_T = \varphi \circ \psi$, or more precisely

$$\mathbb{R}^n \overset{\varphi}{\multimap} \mathrm{AC}([0, T], \mathbb{R}^n) \overset{\psi}{\multimap} \mathbb{R}^n,$$

where $\varphi: X \multimap \{\theta \mid \theta$ is a solution of (84.3) with $\theta(0, X) = X\}$ is known to be acyclic and $\psi: y \to y(T)$, which is obviously continuous. Since every composition of an acyclic and a continuous map is admissible as required (see Section 40), we are done. □

Observe that no uniqueness restriction has been imposed on. Hence, assuming furthermore that

$$f(t + T, \theta) \equiv f(t, \theta), \tag{84.6}$$

where T is a positive constant, system (84.3) admits a T-periodic solution as far as Φ_T in (84.4) has a fixed point.

If, for example, $\Phi_T(S^{n-1}) \subset B^n$, where $S^{n-1} = \partial B^n$, and $B^n \subset \mathbb{R}^n$ is a closed ball centered at the origin, or any other set with the fixed-point property as indicated in Theorem (84.1) or Corollary (84.2), then (84.3) admits, under the above assumptions, including (84.6), a harmonic, i.e. a T-periodic solution. Similarly, if for some $k \in \mathbb{N}$, $\Phi_{kT}(S^{n-1}) \subset B^n$, then by the same reasoning (84.3) admits a subharmonic, i.e. a kT-periodic solution. This is certainly also true because of $\mathrm{Deg}(i - \Phi_T, B^n) \neq \{0\}$, where Deg denotes the generalized Brouwer degree of an admissible map (see Section 51), where $i: B^n \to \mathbb{R}^n$, $i(x) = x$ is the inclusion map.

This can be expressed in terms of bounding functions or guiding functions as follows.

(84.7) THEOREM. *Let a continuous T-periodic in t and locally lipschitzean in θ bounding function V exist such that*

(84.7.1) $V(t,0) = 0$ *for* $\|\theta\| = r$, *uniformly w.r.t.* $t \in [0,T]$,

(84.7.2) $V(t,0) < 0$ *for* $\|\theta\| < r$, *uniformly w.r.t.* $t \in [0,T]$,

(84.7.3) $\limsup_{h\to 0^+}[V(t+h,\theta+hY) - V(t,\theta)]/h < 0$ *for each* $Y \in f(t,\theta)$ *and* $\|\theta\| = r$, *uniformly w.r.t.* $t \in [0,T]$,

where r is a suitable positive constant which may be large. Then system (84.3) *admits, under* (84.6)*, a harmonic.*

PROOF. In the single-valued case this result is well-known, when using C^1-bounding functions. For the differential inclusions, a similar type of results has also been developed in [DyG] (see also [BGP]), but using again only autonomous bounding functions. Thus, our statement represents only a slight generalization and can be proved quite analogously, when following the same geometrical ideas.□

(84.8) REMARK. Replacing conditions (84.7.1)–(84.7.3) by

$$\lim_{\|\theta\|\to\infty} V(t,\theta) = \infty, \quad \text{uniformly w.r.t. } t \in [0,T],$$

and

$$\limsup_{h\to 0^+} \frac{1}{h}[V(t+h,\theta+hY) - V(t,\theta)] < 0 \quad \text{for each } Y \in f(t,\theta)$$

and $\|\theta\| \geq r$, uniformly w.r.t. $t \in [0,T]$, where r is a positive constant which may be large, we obtain, under (84.6), a subharmonic of (84.3) (for the single-valued case see e.g. [And3] and the references therein).

(84.9) REMARK. In the single-valued case, the dissipativity in the sense of N. Levinson, i.e. the uniform ultimate boundedness of all solutions of (84.3) (which can be expressed quite equivalently in terms of guiding functions with the same properties as in Remark (84.8), is sufficient for the existence of harmonics (see e.g. [And3]). So, we can conjecture that the same is true in the set-valued case.

The situation is, however, much more interesting when, for example, (84.3) is only partially dissipative, i.e. if only

$$\limsup_{t\to\infty} \|(\theta_1(t), \dots, \theta_j(t))\| \leq D, \quad 1 \leq j < n$$

holds w.r.t. some part of components of every solution $\theta(t)$ of (84.3), where D is a positive constant common for all solutions of (84.3). It is clear that Theorem (84.1) is this time insufficient for applications.

In [AGL], we have developed the appropriate abstract apparatus for considering such a situation, mainly using the generalized fixed point index technique, which can be expressed in terms of two bounding functions as follows (for more details, in the single-valued case, see e.g. [KZ-M]).

(84.10) THEOREM. *Let continuous T-periodic in t and locally lipschitzean in θ bounding functions V and W exist such that*

(84.10.1) $V(t,\theta) = 0$ *for* $\|\theta_j\| = r$, *uniformly w.r.t. all* $\widehat{\theta}_j \in \mathbb{R}^{n-j}$ *and* $t \in [0,T]$,

(84.10.2) $V(t,\theta) < 0$ *for* $\|\theta_j\| < r$, *uniformly w.r.t. all* $\widehat{\theta}_j \in \mathbb{R}^{n-j}$ *and* $t \in [0,T]$,

(84.10.3) $\limsup_{h\to 0^+}[V(t+h,\theta+hY) - V(t,\theta)]/h < 0$ *for each* $Y \in f(t,\theta)$, $\|\theta_j\| = r$, $\widehat{\theta}_j \in \mathbb{R}^{n-j}$ *and* $t \in [0,T]$,

(84.10.4) $W(t,\theta) = 0$ *for* $\|\widehat{\theta}_j\| = s$, *uniformly w.r.t.* $\|\theta_j\| \le r$ *and* $t \in [0,T]$,

(84.10.5) $W(t,\theta) > 0$ *for* $\|\widetilde{\theta}_j\| > s$, *uniformly w.r.t.* $\|\theta_j\| \le r$ *and* $t \in [0,T]$,

(84.10.6) $\liminf_{h\to 0^+}[W(t+h,\theta+hY) - W(t,\theta)]/h > 0$ *for each* $Y \in f(t,\theta)$, $\|\widehat{\theta}_j\| = s$, $\|\theta_j\| \le r$ *and* $t \in [0,T]$,

where $\theta = (\theta_j \oplus \widehat{\theta}_j)$, $1 \le j < n$, *i.e.* $\theta_j := (\theta_1,\dots,\theta_j)$, $\widehat{\theta}_j := (\theta_{j+1},\dots,\theta_n)$ *and* r, s *are suitable positive constants which may be large. Then system* (84.3) *admits, under* (84.6), *a harmonic.*

(84.11) REMARK. The application of the Dini derivatives above is more appropriate on the boundary of nonconvex bound sets (in the sense [GM-M]) G, H. The approach developed in [AGL] allows us, certainly under a modification in the spirit of e.g. [GM-M], to take for this goal the domains which are star-shaped.

(84.12) REMARK. For the existence of subharmonics, the bounding functions V, W satisfying (84.10.1)–(84.10.6) can be replaced by guiding functions like in Remark (84.8) for $j = n$ (see e.g. [AGG]). In the single-valued case, the existence of suitable guiding functions can be shown to imply again the existence of harmonics, when using the abstract results in [AGZ], provided $f(t,\theta)$ represents a periodic perturbation of an autonomous function. So, we can again conjecture that the same is true in the set-valued case.

Now, system (84.3) will be considered on the cylinder $\mathcal{C}^{n+1} = \mathbb{R}_0^+ \times \mathbb{T}^n$ or, in the autonomous case, on the torus $\mathbb{T}^n = \mathbb{R}^n/\omega\mathbb{Z}^n$, where $\omega\mathbb{Z}$ denotes all the integer multiples of a positive constant ω. Thus, the natural restriction imposed on the right-hand side of (84.3), besides the boundedness in $t \in \mathbb{R}_0^+$, reads

$$f(t,\dots,\theta_j+\omega,\dots) \equiv f(t,\dots,\theta_j,\dots) \quad \text{for } j = 1,\dots,n. \tag{84.13}$$

Consider still the $(n-1)$-dimensional subtorus $\Sigma \subset \mathbb{T}^n$ given by

$$\sum_{j=1}^{n} \theta_j = 0 \pmod{\omega}$$

and assume, additionally, that

$$\inf_{(t,\theta)\in\mathcal{C}^{n+1}} \sum_{i=1}^{n} f_i(t,\theta) > 0 \quad \text{or} \quad \sup_{(t,\theta)\in\mathcal{C}^{n+1}} \sum_{i=1}^{n} f_i(t,\theta) < 0. \tag{84.14}$$

Then we can define the *Poincaré* (*first-return*) *map* Φ on the cross section Σ as follows:

$$\Phi(p)_{\{\tau(p)\}}\colon \Sigma \multimap \Sigma, \quad \Phi(p)_{\{\tau(p)\}} := \{\theta(\tau(p), p)\}, \tag{84.15}$$

where $\Phi_0(p) = p \in \Sigma$ and $\{\tau(p)\}$ denotes the least time for p to return back to Σ, when taking into account each branch of $\theta(t,p)$. Indeed, (84.15) implies that $\sum_{i=1}^n \theta_i'(t,p) \neq 0$ for every solution $\theta(t,p)$ of (84.3) and almost all $t \geq 0$ by which the map $\tau(p)\colon \sigma \multimap [\omega/E, \omega/\varepsilon]$ is well defined, where ε, E are positive constants such that

$$0 < \varepsilon \leq \inf_{(t,\theta)\in\mathcal{C}^{n+1}} \left|\sum_{i=1}^n f_i(t,\theta)\right| \leq \sup_{(t,\theta)\in\mathcal{C}^{n+1}} \left|\sum_{i=1}^n f_i(t,\theta)\right| \leq E.$$

Moreover, (84.14) geometrically means that the trajectories of (84.3), associated to (84.15)), intersect σ in a transversal way, which will be essential into the future.

Let us note that $\{\tau(p)\}$ is, even without (84.14), lower semi-continuous.

Observe that $\Phi_{\{\tau(p)\}}$ may be, as in the foregoing section, the fixed T time map. This appears if, for example,

$$\sum_{i=1}^n f_i(t,\theta) \equiv \{F(t)\},$$

where $F(t)$ is a T-periodic function such that $|\int_0^T F(t)\,dt| = \omega > 0$, because then $\tau(p) = T$ for all $p \in \Sigma$.

(84.17) Lemma. *$\Phi_{\{\tau(p)\}}$ given by* (84.16) *is, under* (84.14)*, admissible.*

Proof. $\Phi_{\{\tau(p)\}}$ can be considered as the composition of two maps, namely $\Phi_{\{\tau(p)\}} = \varphi \circ \psi$, or more precisely

$$\Sigma \overset{\varphi}{\multimap} \mathrm{AC}*\left(\left[0, \frac{\omega}{\varepsilon}\right], \mathbb{R}^n\right) \overset{\psi}{\multimap} \Sigma,$$

where AC$*$ means the space of all absolutely continuous functions with the properties (cf. (84.14))

$$E \geq \left|\sum_{i=1}^n y_i'(t,p)\right| \geq \varepsilon > 0, \quad \text{for almost all } t \in \left[0, \frac{\omega}{\varepsilon}\right], \tag{84.18}$$

$$\varepsilon t \leq \left|\sum_{i=1}^n y_i(t,p)\right| \leq Et, \qquad \text{for } t \in \left[0, \frac{\omega}{\varepsilon}\right], \tag{84.19}$$

Here, $\varphi: p \multimap \{\theta(t,p) \mid \theta(t,p)$ is a solution of (84.3) with $\theta(0,p) = p\}$ is known to be acyclic (see e.g. [BGP, Theorem 5.7]) and $\psi: y(t,p) \to y(\tau(y),p) \in \Sigma$, which will obviously be continuous as far as $\tau(y)$ is so.

Observe that, because of the "sterisque" properties (84.18), (84.19), $\tau(y)$ is again well denned and, moreover,

$$\sum_{i=1}^{n} y_i(\tau(y)) - y_i(0) = \pm\omega. \tag{84.20}$$

Hence, applying to (84.20) a suitable implicit function theorem for maps without continuous differentiability (see e.g. [AuE-M, Theorem 7.5.8]), the map $y \to \tau(y)$ can easily be verified, under (84.18), to be Lipschitz-continuous, as required.

Since the composition of acyclic and continuous maps is admissible, the proof is complete. □

In the single-valued case, Lemma (84.17) has the following direct consequence.

(84.21) LEMMA. *If f is, additionally, continuous and locally lipschitzean in θ, then $\Phi_{\{\tau(p)\}}$ associated to the equation $\theta' = f(t,\theta)$, is continuous.*

Since no torus has a fixed-point property (neither in the classical nor in the generalized sense), the analogue of Theorem (84.1) or (84.2) for Σ or $\mathbb{T}^n$ are impossible. Nevertheless, we can define there the generalized Lefschetz number Λ for admissible maps and, moreover, $\Phi_{\{\tau(p)\}}(p)$ in (84.16) can be shown to be, under (84.14), homotopic in the sense of admissible maps to the identity I. Thus, if for example $\Lambda(C^{-1}(\Phi_{\{\tau(p)\}}(p))) \neq \{0\}$, where $C: \Sigma \to \Sigma$ is a diffeomorphism, and subsequently $C^{-1}(\Phi_{\{\tau(p)\}}(p)): \Sigma \multimap \Sigma$ is admissible, then $C^{-1} \circ \Phi$ has a fixed point, i.e. the "target problem" $C(p) \in \Phi_{\{\tau(p)\}}(p)$ is solvable. Because of the invariance under admissible homotopy, the problem turns out to be equivalent to the computation of $\Lambda(C^{-1}) \neq 0$. This can be however performed, under natural restrictions, by means of a sum of the local indices, (see e.g. [Br1-M]).

Therefore, we can give the following statement concerning the target problem for (84.3) on the torus Σ, expressed by means of the condition

$$C(\theta(0,p)) = \theta(t*,p), \quad \text{for some } t* > 0, \tag{84.22}$$

for more details see [And3] and references therein.

(84.23) THEOREM. *Let all the above regularity assumptions be satisfied, jointly with* (84.14). *Assume that $C: \Sigma \to \Sigma$ is a diffeomorphism having finitely many, but at least one, simple fixed points, $\gamma_1, \ldots, \gamma_r$ on Σ and*

$$\sum_{k=1}^{r} \operatorname{sgn} \det(I - dC_{\gamma_k}^{-1}) \neq 0, \tag{84.24}$$

where $dC_{\gamma_k}^{-1}$ *denotes the derivative of* C^{-1} *at* $\gamma_k \in \Sigma$. *Then problem* (84.3)–(84.22) *admits a solution.*

(84.25) Corollary. *In the single-valued case, problem* (84.3)–(84.22) *admits, under the assumptions of Theorem* (84.23) *at least* r *geometrically distinct solutions.*

In the single-valued case, it is namely well-known (see [Br2-M]) that $N(C^{-1} \circ \Phi) = \|\Lambda(C^{-1} \circ \Phi)\|$ on Σ. Therefore, since the Nielsen index $N(C^{-1} \circ \Phi)$ determines the lower estimate of fixed points of $C^{-1} \circ \Phi$ on Σ, the absolute value of the nonzero number in (84.24) designates at the same time the lower estimate of desired solutions. Although the Nielsen index has also been generalized for admissible maps (see e.g. [KM] and the references therein), no definition seems directly available here.

Taking, in particular, $C := \sigma_{+,-}\colon \mathbb{T}^n \to \mathbb{T}^n$, where the shifts σ_+ or σ_- are denned by the rules $\sigma_{+,-}(\theta_1, \dots, \theta_n) = \pm(\theta_2, \dots, \theta_n, \theta_1)$, respectively, we can prove the following statement concerning the corresponding sort of nonlinear rotations (periodic oscillations of the second kind); for more details see [And2].

(84.26) Theorem. *Let an autonomous system* (84.3) *determine a* $\sigma_{+,-}$ *equivariant flow on* $\mathbb{T}^n$, *i.e.*

$$\begin{aligned}(84.26.1)\quad f_i(\theta_1, \dots, \theta_n) = (\pm 1)^{i+1} f_i((\pm 1)^{i+1}\theta_i, \dots, (\pm 1)^{i+1}\theta_n,\\ (\pm 1)^{i+1}\theta_1, \dots, (\pm 1)^{i+1}\theta_{i-1}), \quad \textit{for } i = 1, \dots, n.\end{aligned}$$

Let, furthermore, f_i *be* ω*-periodic in each variable* θ_j, $i, j = 1, \dots, n$, (*see* (84.12)) *and*

$$(84.26.2)\qquad \sum_{i=1}^{n} (\pm 1)^i f_i(\theta) > 0 \quad \textit{or} \quad \sum_{i=1}^{n} (\pm 1)^i f_i(\theta) < 0,$$

respectively. Then system (84.3) *admits nontrivial splay-phase or anti-splay-phase orbits* $\theta(t)$, *respectively, provided* n *is even in the latter case, i.e.* $\theta'(t+T) = \theta'(t)$ *for almost all* $t \in (-\infty, \infty)$, *where* T *is a suitable positive constant and*

$$\theta(t) = \bigg(\varphi(t), \pm\varphi\bigg(t + \frac{1}{n}T\bigg), \dots, \varphi\bigg(t + \frac{n-2}{n}T\bigg), \pm\varphi\bigg(t + \frac{n-1}{n}T\bigg)\bigg),$$

respectively.

(84.27) Corollary. *The assertion of Theorem* (84.26) *remains valid for* (*not necessarily autonomous*) (T/n)*-periodic in* t *system* (84.3), *provided the nonautonomous analogue to* (84.26.1) *holds and, instead of* (84.26.2)

$$\sum_{i=1}^{n} (\pm 1)^i f_i(t, \theta) \equiv \{F(t)\},$$

where F is a (T/n)-periodic function such that

$$\int_0^{T/n} F(t)\,dt = \pm\omega,$$

respectively. For nonautonomous, (T/n)-periodic in t, systems (84.3)*, we obtain in fact the whole one-parameter family* (*i.e. generically, infinitely many*) *of such subharmonics of the second kind.*

It is so, because the associated first return map becomes the translation (fixed time) operator, $\Phi_{\{\tau(p)\}}(p) = \Phi_{T/n}(p)$ as already pointed out. In the nonautonomous case, it has meaning to apply the admissible translation operator

$$\Phi_{t_0+T/n}(p) := \{\theta(t_0 + T/n, t_0, p)\}, \quad \text{for each } t_0 \in [0, T/n],$$

where $\theta(t, t_0, p)$ is this time a solution of (84.3) with $\theta(t_0, t_0, p) = p \in \Sigma$. Thus, $\Phi_{t_0+T/n}(\Sigma) \subset \Sigma$, and using the same approach, we obtain for each value of $t_0 \in [0, T/n]$ the desired solutions.

(84.28) Example. As the simplest application of Theorem (84.10), consider the planar system under nonlinear, periodic in t, perturbation:

$$(\theta' + A\theta^T) \in g(t, \Theta),$$

where $\theta = (\theta_1, \theta_2)$, $A = \operatorname{diag}(a_1, a_2)$ is a constant matrix and $g = (g_1, g_2)^T$ satisfies all the above regularity assumptions. Suppose, furthermore, that $a_1 > 0$, $a_2 < 0$,

$$\lim_{|\theta_1|\to\infty} \frac{g_1(t,\theta)}{|\theta_1|} = 0,$$

uniformly w.r.t. $t \in [0, T]$ and $\theta_2 \in (-\infty, \infty)$;

$$\lim_{|\theta_2|\to\infty} \frac{g_2(t,\theta)}{|\theta_2|} = 0,$$

uniformly w.r.t. $t \in [0, T]$ and $|\theta)1| \leq r$.

Defining the bounding functions V, W as

$$V(\theta_1) := \|\theta_1\| - r, \quad W(\theta_2) = \|\theta_2\| - s,$$

where r, s are sufficiently big positive constants, one can readily check that Theorem (84.10) applies. Consequently, we have a harmonic.

(84.29) EXAMPLE. As an application of Corollary (84.25), consider the system

$$\theta' = f(t,\theta) \quad \text{for } n = 2, \text{ satisfying (84.14)},$$

where f is 2π-periodic in each variable θ_j, $j = 1,2$ ($n = 2$), and define the diffeomorphism $C\colon \Sigma \to \Sigma$ by

$$C(\theta) := (\theta_1 + c\sin\theta_1, \theta_2 + c\sin\theta_2), \quad \text{where } c \in (0,1).$$

One can easily check that $C^{-1}(\theta)$ as well as $C(\theta)$ have two fixed points on Σ, namely $\gamma_0 = (0,0)$ and $\gamma_\pi = (\pi,\pi)$.

Since there are still (using the theorem about the derivative of the inverse function)

$$dC_0^{-1} = \operatorname{diag}\left(\frac{1}{1+c}, \frac{1}{1+c}\right), \quad dC_\pi^{-1} = \operatorname{diag}\left(\frac{1}{1-c}, \frac{1}{1-c}\right),$$

i.e.

$$I - dC_0^{-1} = \operatorname{diag}\left(\frac{1}{1+c}, \frac{1}{1+c}\right), \quad I - dC_\pi^{-1} = \operatorname{diag}\left(\frac{1}{1-c}, \frac{1}{1-c}\right),$$

we obtain

$$\det(I - dC_0^{-1}) = \left(\frac{c}{1+c}\right)^2, \quad \text{and} \quad \det(I - dC_\pi^{-1}) = \left(\frac{c}{1-c}\right)^2.$$

So, we can conclude that the Nielsen index $N(C^{-1}) = 2$, by which our system admits, according to Corollary (84.25), at least two geometrically distinct solutions $\theta(t)$ satisfying (84.22), i.e.

$$[\theta_1(0) + c\sin\theta_1(0), \theta_2(0) + c\sin\theta_2(0)] = [\theta_1(t*), \theta_2(t*)]$$

for some $t* > 0$.

Finally, we shall consider the following example.

(84.30) EXAMPLE. As an application of Corollary (84.27), consider the system

$$\theta_i' = (\pm 1)^{i+1} p(t) \mp nC\sin\theta_i + C\sum j = 1^n (\pm 1)^{j+i+1}\sin\theta_j, \quad i = 1,\dots,n,$$

where C is an arbitrary constant and $p(t)$ is a continuous function such that $p(t + T/n) \equiv p(t)$, $\int_0^{T/n} p(t)\,dt = \pm 2\pi$, respectively. Since all the assumptions of Corollary (84.27) are evidently satisfied, we have a one-parameter family of subharmonics of the second kind with the components equally staggered in phase.

85. Multivalued fractals

Multivalued fractals are usually considered as fixed points of certain induced union operators, called the Hutchinson–Barnsley operators, in hyperspaces of compact subsets of the original spaces endowed with the Hausdorff metric. In this section we are going to present current results connected with multivalued fractals. For details we recommend [AnGo-M], [AFGL], [AG].

It is well known that Hutchinson [Hut] and Barnsley [Bar-M] initiated the way to define and construct fractals as compact invariant subsets of complete metric space with respect to union of contractions. Consequently, they proved the following.

(85.1) THEOREM (Hutchinson–Barnsley). *Assume that (X,d) is a complete metric space and*

(85.1.1) $$\{f_i\colon X\to X,\ i=1,\dots,n;\ n\in\mathbb{N}\}$$

is a system of contractions, i.e.

(85.1.2) $$d(f_i(x),f_i(y))\le L_i\,d(x,y),\quad \text{for all } x,y\in X,\ i=1,\dots,n,$$

where $L_i\in[0,1)$, $i=1,\dots,n$. Then there exists exactly one compact invariant subset $A^*\subset X$ of the Hutchinson–Barnsley map

$$F(x):=\bigcup_{i=1}^{n} f_i(x),\quad x\in X,$$

called the attractor (fractal) of (85.1.1) or, equivalently, exactly one fixed-point $A^*\in\mathcal{K}(X):=\{A\subset X\mid A \text{ is nonempty and compact}\}$ of the induced Hutchinson–Barnsley operator

$$F^*(A):=\overline{\bigcup_{x\in A}F(x)}\quad\Big(=\bigcup_{x\in A}F(x)\Big),\quad A\in\mathcal{K}(X),$$

in the hyperspace $(\mathcal{K}(X),d_H)$, where d_H as before stands for the Hausdorff metric, defined by

$$d_H(A,B):=\inf\{\varepsilon>0\mid A\subset O_\varepsilon(B) \text{ and } B\subset O_\varepsilon(A)\},$$

where $O_\varepsilon(C)=\{x\in X\mid d(x,C)<\varepsilon\}$, for any nonempty, bounded, closed set $C\subset X$. Moreover,

(85.1.3) $$\lim_{m\to\infty} d_H(F^{*^m}(A),A^*)=0,\quad \text{for every } A\in\mathcal{K}(X),$$

and

$$d_H(A, A^*) \le \frac{1}{1-L}\, d_H(A, F^*(A)),$$

where $(1 >)\, L = \max_{i=1,\dots,n} L_i$.

The proof is apparently based on a simple application of the Banach contraction principle and one usually speaks about fractals associated with an *iterated function system* (IFS) (85.1.1). The same is true for a system of multivalued contractions with compact values, provided the Hausdorff metric appears in (85.1.2). Hence, one can speak analogously about an *iterated multifunction system* (IMS).

The further development consists mainly in replacing the Banach contraction principle by

(i) other metric fixed-point theorems, e.g. for weakly contractive or nonexpansive maps, then we speak about *metric fractals*,

(ii) topological (Schauder-type) fixed-point theorems, then we speak about *topological fractals*,

(iii) order-theoretic (Knaster–Tarski or Tarski–Kantorovitch-type) fixed-point theorems, where invariant sets are deduced, under extremely weak assumptions, from subinvariant sets and the related fixed-points in hyperspaces are only deduced a posteriori, then we speak about *Tarski's fractals*,

(iv) putting the accent on the constructive part (cf. (85.1.3)), where the fractals can be expressed in terms of the Kuratowski limits of stable or semistable maps, then one can distinguish between fractals and *semifractals*, respectively.

Let $\{A_n\}$ be a sequence of subsets of X. The (*topological*) *lower Kuratowski limit* $\operatorname{Li} A_n$, and the (*topological*) *upper Kuratowski limit* $\operatorname{Ls} A_n$, are defined as follows (see [Ku-M]):

- $x \in \operatorname{Li} A_n$ if, for every $\varepsilon > 0$, there is a positive integer n_0 such that $A_n \cap B(x, \varepsilon) \neq \emptyset$, for $n \ge n_0$.
- $x \in \operatorname{Ls} A_n$ if, for every $\varepsilon > 0$, the condition $A_n \cap B(x, \varepsilon) \neq \emptyset$ is satisfied, for infinitely many n.
- If $\operatorname{Li} A_n = \operatorname{Ls} A_n$, then this set is called the (*topological*) *Kuratowski limit* $\operatorname{Lim} A_n$.

$\operatorname{Li} A_n$, $\operatorname{Ls} A_n$ and $\operatorname{Lim} A_n$ are always closed sets, and $\operatorname{Li} A_n = \operatorname{Li} \overline{A_n}$, $\operatorname{Ls} A_n = \operatorname{Ls} \overline{A_n}$.

For a metric space (X, d), the *hyperspace* of compact sets of X endowed with the induced Hausdorff metric d_H will be denoted $(\mathcal{K}(X), d_H)$.

(85.2) Definition. We say that a metric space (X, d) is a *hyper absolute neighbourhood retract* (a *hyper absolute retract*) if $(\mathcal{K}, d_H)$ is an absolute neighbourhood retract (an absolute retract), written $X \in \mathrm{HANR}$ ($X \in \mathrm{HAR}$).

For our convenience we shall use the following notions.

(85.3) Definition. A map $\varphi\colon X \multimap X$ is called the (*Nadler*) *multivalued contraction* if there exists a nonnegative constant $L < 1$ such that

$$d_H(\varphi(x), \varphi(y)) \le L d(x, y) \quad \text{for all } x, y \in X.$$

When referring to (the least) L, one often speaks about a *Lipschitz constant* or a *contraction rate*.

(85.4) Definition. A multivalued map $\varphi\colon X \multimap X$ is said to be (*Edelstein*) *contractive* if

$$d_H(\varphi(x), \varphi(y)) < d(x, y) \quad \text{for all } x \neq y \in X.$$

Obviously, every Nadler contraction is Edelstein contractive, but not reversely. Although the Edelstein contractive map is Hausdorff-continuous, it is well-known that it need not admit a fixed-point. As an example, the mapping

$$\mathbb{R} \ni x \overset{f}{\longmapsto} \ln(1 + e^x) \in \mathbb{R}$$

is Edelstein contractive, but fixed-point free (for more details, see e.g. [GoeKi-M], [KS-M]).

(85.5) Definition. A multivalued map $\varphi\colon X \multimap X$ is called to be a *weak contraction* if a continuous and nondecreasing function $\psi\colon [0, \infty) \to [0, \infty)$ exists such that

(85.5.1) $0 < \psi(t) < t$, for $t \in (0, \infty)$, and $\psi(0) = 0$,
(85.5.2) $\lim_{t\to\infty}(t - \psi(t)) = \infty$,
(85.5.3) $d_H(\varphi(x), \varphi(y)) \le \psi(d(x, y))$, for all $x, y \in X$.

The notion of an abstract (Hölderian) η-contraction reads as follows.

(85.6) Definition. Let $\eta\colon (0, \infty) \to [0, \infty)$ be an abstract *comparison function*. A multivalued function $\varphi\colon X \multimap X$ with (closed) bounded values is called an *η-contraction* if

$$d_H \varphi(x), \varphi(y)) \le \eta(d(x, y)) \quad \text{for all } x \neq y.$$

To guarantee suitable properties of η-contractions, we need to impose on η some additional restrictions. Below we quote typical requirements (holding for positive t's):

(a) $\eta(t) < t$,
(b) if $t_1 \le t_2$ then $\eta(t_1) \le \eta(t_2)$,
(c) $\limsup_{r\to t^+} \eta(r) < t$,
(d) $\limsup_{r\to t} \eta(r) < t$,
(e) $\lim_{r\to\infty}(r - \eta(r)) = \infty$.

Recall that

$$\limsup_{r\to t}\eta(r)=\inf_{\delta>0}\sup_{0<|r-t|<\delta}\eta(r),\quad \limsup_{r\to t^+}\eta(r)=\inf_{\delta>0}\sup_{t<r<t+\delta}\eta(r).$$

The behaviour of η in 0 is not significant for our discussion (cf. [Xu1]). Obviously, (d) is stronger than (c). Let us also note that an upper semicontinuous function with (a) satisfies (d) and a right continuous function with (a) satisfies (c).

(85.7) LEMMA. *If η satisfies* (b) *and* (c), *then it satisfies* (a) *and* (d), *too.*

PROOF. (b) $\wedge$ (c) $\Rightarrow$ (a). Suppose (a) is not satisfied. Then $\eta(t)\ge t$, for some $t>0$. By (c), there exists $r>t$ such that $\eta(r)<t$. Thanks to (b),

$$\eta(t)\ge t>\eta(r)\ge\eta(t),$$

a contradiction.

(b) $\wedge$ (c) $\Rightarrow$ (d). In view of the above property, we can use (a) and (b) to get

$$\inf_{\delta>0}\sup_{r\in(t-\delta,t)}\eta(r)\le\eta(t)<t.$$

This together with (c) yields (d). □

Let η satisfy (d). Denote $\overline{\eta}(t):=\limsup_{r\to t}\eta(r)$, $\widetilde{\eta}(t):=\sup\{\overline{\eta}(\tau)\mid\tau\le t\}$. Obviously, $\overline{\eta}(t)<t$, and $\widetilde{\eta}$ is nondecreasing (i.e. fulfills (b)). The following observation (cf. [Mt]) allows us to reduce the case of a nonmonotone comparison function to a monotone one.

(85.8) LEMMA. *If η satisfies* (a) *and* (d), *then its modification $\widetilde{\eta}$ satisfies* (b) *and* (d).

PROOF. We only need to check (d). By the hypothesis,

$$\text{there exists }\delta>0\text{ such that }\sup_{r\in(t-\delta,t+\delta)}\eta(r)<t.\tag{85.8.1}$$

Now, let us make the following estimates

$$\begin{aligned}\sup_{r\in(t-\delta,t+\delta)}\widetilde{\eta}(r)&=\max\{\sup_{r\in(0,t-\delta]}\overline{\eta}(r),\sup_{r\in(t-\delta,t+\delta)}\overline{\eta}(r)\}\\&\le\max\{t-\delta,\sup_{r\in(t-\delta,t+\delta)}\inf_{\varepsilon>0}\sup_{s\in(r-\varepsilon,r+\varepsilon)}\eta(s)\}\\&\le\max\{t-\delta,\sup_{r\in(t-\delta,t+\delta)}\eta(r)\},\end{aligned}\tag{85.8.2}$$

where the last inequality uses the fact that, for any $\delta>0$ and $r\in(t-\delta,t+\delta)$, there exists $\varepsilon>0$ such that $(r-\varepsilon,r+\varepsilon)\subset(t-\delta,t+\delta)$. The assertion follows from (85.8.1) and (85.8.2). □

(85.9) LEMMA. *Let* $\varphi\colon X \multimap X$ *be an* η*-contraction satisfying* (b). *Then*

$$\varphi(B(x,r)) \subset O_{\eta(r)+\varepsilon}(\varphi(x)) \quad \text{for every } x \in X,\ \varepsilon > 0,\ r > 0.$$

PROOF. For $x' \in B(x,r)$, one has

$$d_H(\varphi(x'),\varphi(x)) \le \eta(d(x',x)) \le \eta(r) < \eta(r)+\varepsilon.$$

Thus, $\varphi(x') \subset O_{\eta(r)+\varepsilon}(\varphi(x))$. □

Although there exist Edelstein contractive maps without any reasonable comparison function, the situation changes if we assume compactness of the metric space X. Then any Edelstein contractive map is just a weak contraction as follows from the following lemma (for single-valued maps, cf. Proposition A.4.5, p. 141 in [Wi-M]).

(85.10) LEMMA. *Let* $\varphi\colon X \multimap X$ *be Edelstein contractive on a compact* X. *Then* φ *is an* η*-contraction for* η *defined as*

$$\eta(t) := \sup\{d_H(\varphi(x),\varphi(y)) \mid d(x,y) \le t\}.$$

This comparison function is continuous and satisfies all conditions (a)–(e).

PROOF. At first, observe that, by the compactness assumption, (e) is trivial. Condition (b) follows immediately from definition, too. Moreover, (b) and (c) imply, according to Lemma (85.7), (a) and (d). Therefore, we only need to check that φ is indeed an η-contraction with (c). We have

$$d_H(\varphi(x_0),\varphi(y_0)) \le \sup\{d_H(\varphi(x),\varphi(y)) \mid d(x,y) \le d(x_0,y_0)\} = \eta(d(x_0,y_0))$$

which implies contractivity. To check (c), suppose the contrary. Then there exists $t \le r_n \to t$ such that $\eta(r_n) \ge t$. Since the supremum in the formula for η is attained ($d_H(\varphi(\cdot),\varphi(\cdot))$ is continuous), we can find sequences $\{x_n\}_{n=1}^\infty$, $\{y_n\}_{n=1}^\infty$ with $\eta(r_n) = d_H(\varphi(x_n),\varphi(y_n))$ and $d(x_n,y_n) \le r_n$. By compactness, we can assume that $x_n \to x$, $y_n \to y$. Thus, $t \le \eta(r_n) \to d_H(\varphi(x),\varphi(y))$ and $d(x,y) = \lim_{n\to\infty} d(x_n,y_n) \le t$. Altogether, this leads to a contradiction, namely

$$t \le d_H(\varphi(x),\varphi(y)) < d(x,y) \le t. \qquad \square$$

Now, it will be convenient to define condensing maps w.r.t. a measure of noncompactness (MNC).

Let (X, d) be a metric space and $\mathcal{B}$ be the set of all bounded subsets of X. The function $\alpha\colon \mathcal{B} \to [0, \infty)$, where

$$\alpha(B) := \inf\{\delta > 0 \mid B \text{ admits a finite covering by the sets of diameter} \leq \delta\}$$

is called the *Kuratowski measure of noncompactness* and the function $\gamma\colon \mathcal{B} \to [0, \infty)$, where

$$\gamma(B) := \inf\{\varepsilon > 0 \mid B \text{ has a finite } \varepsilon\text{-net}\} \quad \text{i.e. } B \subset \bigcup_{i=1}^{m} B(x_i, \varepsilon),$$

is called the *Hausdorff measure of noncompactness*. These MNC are related as follows:

$$\gamma(B) \leq \alpha(B) \leq 2\gamma(B)$$

Moreover, they satisfy the following properties (comp. Chapter I, Section 4):

(μ_1) $\mu(B) = 0$ if and only if $\overline{B}$ is compact,
(μ_2) if $B_1 \subset B_2$ then $\mu(B_1) \leq \mu(B_2)$,
(μ_3) $\mu(\overline{B}) = \mu(B)$,
(μ_4) if $\{B_i\}_{i=1}^{\infty}$ is a decreasing sequence of nonempty closed sets $B_i \in \mathcal{B}$ with $\lim_{i\to\infty} \mu(B_i) = 0$, then $\bigcap_{i=1}^{\infty} B_i \neq \emptyset$,
(μ_5) $\mu(B_1 \cup B_2) = \max\{\mu(B_1), \mu(B_2)\}$,

where μ denotes either α or γ.

(85.11) Definition. Letting $\mu := \alpha$ or $\mu := \gamma$, a bounded mapping $F\colon X \supset U \multimap X$, i.e. $F(B) \in \mathcal{B}$, for any $\mathcal{B} \ni B \subset U$, is then said to be *$\mu$-condensing* (shortly, *condensing*) if $\mu(F(B)) < \mu(B)$, whenever $\mathcal{B} \ni B \subset U$ and $\mu(B) > 0$ or, equivalently, if $\mu(F(B)) \geq \mu(B)$ implies $\mu(B) = 0$, whenever $\mathcal{B} \ni B \subset U$.

Analogously, a bounded mapping $F\colon X \supset U \multimap X$ is said to be a *k-set contraction* w.r.t. μ (shortly, a *k-contraction* or a *set-contraction*) if $\mu(F(B)) \leq k\mu(B)$, for some $k \in [0, 1)$, whenever $\mathcal{B} \ni B \subset U$.

Obviously, any set-contraction is condensing. Furthermore, compact maps or (Nadler) contractions with compact values (in vector spaces, also their sum) are well-known to be set-contractions, and so condensing. For more details, see e.g. [AKPRS-M].

The following statement allows us to deduce condensity from η-contraction for maps with compact values satisfying the above conditions (b) and (c), i.e. in particular, from weak contractions in the sense of Definition (85.6).

(85.12) Proposition. *Let $\varphi\colon X \multimap X$ be an η-contraction with compact values, and let the comparison function η satisfy* (b) *and* (c). *Then φ is μ-condensing, where $\mu := \alpha$ or $\mu := \gamma$ (see Definition* (85.11)).

Proof. Fix $r > \mu(A) > 0$ and find a finite r-net $\{x_i\}_i$ of A, i.e. $\bigcup_i B(x_i, r) \supset A$. Thus, by Lemma (85.9),

$$\varphi(A) \subset \bigcup_i \varphi(B(x_i, r)) \subset \bigcup_i O_{\eta(r)+\varepsilon}(\varphi(x_i)).$$

Putting compact $K = \bigcup_i \varphi(x_i)$, we have

$$\bigcup_i O_{\eta(r)+\varepsilon}(\varphi(x_i)) \subset O_{\eta(r)+\varepsilon}(K) \subset \bigcup_j B(z_j, \eta(r) + 2\varepsilon),$$

where $\{z_j\}_j$ is a finite ε-net for K. So, $\mu(\varphi(A)) \leq \eta(r) + 2\varepsilon$. Since ε and r were arbitrary, we get

$$\mu(\varphi(A)) \leq \eta(r) < \mu(A).$$

If $\mu(A) = 0$, then $\mu(\varphi(A)) = 0$, because φ maps compacta onto compacta. □

In order to derive the existence of fractals for weak contractions from analogous theorems for bounded condensing maps (see Theorem (85.21) below), one needs to localize a bounded fractal. More precisely, we must ensure the existence of a closed bounded subinvariant set (which contains all closed bounded invariant sets. In fact, by contractivity, there is only one such invariant set). The following Proposition (85.13) demonstrates that this holds almost automatically.

(85.13) Proposition. *Let $\varphi: X \multimap X$ be an η-contraction with* (*closed*) *bounded values, and let the comparison function η satisfy* (b), (c) *and* (e). *Then, for any $x_0 \in X$, there exists r_0 such that*

$$\varphi(B(x_0, r)) \subset B(x_0, r), \quad \textit{for all } r \geq r_0.$$

Proof. Take σ such that $\varphi(x_0) \subset B(x_0, \sigma)$. Fix $\varepsilon > 0$ and, in view of property (e), find r_0 such that $r - \eta(r) > \sigma + 2\varepsilon$, for all $r \geq r_0$. To r, we can assign $\rho > r$ such that $\eta(\rho) < \eta(r) + \varepsilon$, which gives

$$r - \eta(\rho) > r - (\eta(r) + \varepsilon) > \sigma + 2\varepsilon - \varepsilon = \sigma + \varepsilon.$$

Now, we have

$$\begin{aligned}\varphi(B(x_0, r)) &\subset \varphi(B(x_0, \rho)) \subset O_{\eta(\rho)+\varepsilon}(\varphi(x_0)) \\ &\subset O_{\eta(\rho)+\varepsilon}(B(x_0, \sigma)) \subset B(x_0, \eta(\rho) + \sigma + \varepsilon) \subset B(x_0, r),\end{aligned}$$

when applying Lemma (85.9) to the second inclusion. □

Our primer interest are metric multivalued fractals. More precisely, we are interested in the existence and the topological structure of sets of fractals for systems of maps approximated by multivalued contractions.

Let X be a complete AR-space. For $k \geq 1$, let $\varphi_{i,k}: X \multimap X$, $i = 1, \dots, n$, be contractions with Lipschitz constants $L_{i,k} < 1$. Consider the Hutchinson–Barnsley operators

$$F_k^*: \mathcal{K}(X) \to \mathcal{K}(X), \quad F_k^*(A) = \bigcup_{i=1}^{n} \varphi_{i,k}(A).$$

Then each F_k^* is a contraction with a Lipschitz constant $L_k = \max_{i=1,\dots,n}\{L_{i,k}\}$. Indeed, let $d_H(A, B) = D$. Take any $y \in \bigcup_{i=1}^k \varphi_{i,k}(A)$. Thus, for some $i \in \{1, \dots, n\}$ and any $\delta > 0$, there exist $x \in A$ and $z \in B$ such that $y \in \varphi_{i,k}(x)$ and $d(x, z) < D + \delta$. This implies that

$$\operatorname{dist}(y, \varphi_{i,k}(z)) \leq L_{i,k}(D + \delta) \leq L_k(D + \delta),$$

and so $y \in O_{L_k(D+\delta)}(\bigcup_{i=1}^k \varphi_{i,k}(B))$. Similarly, we can prove that

$$\bigcup_{i=1}^{k} \varphi_{i,k}(B) \subset O_{L_k(D+\delta)}\Big(\bigcup_{i=1}^{k} \varphi_{i,k}(A)\Big).$$

Since δ was arbitrary, we obtain that

$$d_H(F_k^*(A), F_k^*(B)) \leq L_k d_H(A, B).$$

Below we study the topological structure of the set of fractals (invariant sets of the Hutchinson–Barnsley map F or fixed-points of the Hutchinson–Barnsley operator F^*) for maps being approximated by contractions. In particular, nonexpansive maps can be considered with this respect.

We need the following

(85.14) Definition. We say that a family of maps $\{\varphi_i: X \multimap X\}_{i=1}^n$ satisfies the *Palais–Smale condition* (shortly PS-*condition*) if each φ_i maps compact sets onto compact sets and, for the Hutchinson–Barnsley operator F^*, one has:

(85.14.1) if $\{A_k\} \subset \mathcal{K}(X)$ is a sequence with $d_H(A_k, F^*(A_k)) \to 0$, then $\{A_k\}$ has a subsequence convergent w.r.t. d_H.

The following result is important because of possible applications. Moreover, it relates to one of classes of maps satisfying the PS-condition.

(85.15) Proposition. *Let X be a metric space and let $\{\varphi_i: X \multimap X\}_{i=1}^n$ be a family of compact u.s.c. maps with compact values. Then the family $\{\varphi_i\}_{i=1}^n$ satisfies the* PS-*condition.*

Proof. Take any sequence $\{A_k\} \subset \mathcal{K}(X)$. Let $K \subset X$ be a compact set such that $\varphi_i(X) \subset K$, for every $i \in \{1, \dots, n\}$. Since $\mathcal{K}(K) \subset \mathcal{K}(X)$ is a compact subspace and $F^*(A_k) \in \mathcal{K}(K)$, for every $k \geq 1$ (see e.g. [AnGo-M]), one can choose a convergent subsequence $F^*(A_{k_l}) \to A \in \mathcal{K}(K)$. Using our assumption, one obtains $A_{k_l} \to A$. □

Now, we are in a position to prove the main result of this subsection.

(85.16) THEOREM. *Let X be a complete absolute retract and let maps $\varphi_i: X \multimap X$, $i = 1, \ldots, n$, be approximated by multivalued contractions $\varphi_{i,k}: X \multimap X$, $i = 1, \ldots, n$, $k \geq 1$, in the following way:*

$$d_H(\varphi_{i,k}(x), \varphi_i(x)) \leq \min\left\{\frac{1}{k}, c(1 - L_k)\right\}$$

for every $i=1, \ldots, n$ and $k \geq 1$, and some constant c (here $L_k = \max_{i=1,\ldots,n}\{L_{i,k}\}$, where $L_{i,k}$ is a contraction rate for $\varphi_{i,k}$). Let F^* be the Hutchinson–Barnsley operator for $\{\varphi_1, \ldots, \varphi_n\}$ and assume that $\{\varphi_1, \ldots, \varphi_n\}$ satisfies the PS*-condition. Then* $\operatorname{Fix} F^*$ *is an R_δ-set, i.e. it is an intersection of compact contractible sets.*

PROOF. Since X is an AR-space, it is contractible. Therefore, there is a multivalued homotopy $H_k: X \times [0,1] \multimap X$ such that

$$H_k(\,\cdot\,, 0) = \mathrm{id}_X \quad \text{and} \quad H_k(\,\cdot\,, 1) = \bigcup_{i=1}^{n} \varphi_{i,k}(\,\cdot\,).$$

This induces a homotopy in the hyperspace $(\mathcal{K}(X), d_H)$, $H_k^*: \mathcal{K}(X) \times [0,1] \to \mathcal{K}(X)$, which joins $\mathrm{id}_{\mathcal{K}(X)}$ with F_k^*.

Since (X, d) is complete, so is $(\mathcal{K}(X), d_H)$. Thus, there is a unique fixed-point A_k^* of F_k^*, according to the Banach principle.

Consider the set

$$\mathbf{A}_k = \{A \in \mathcal{K}(X) \mid d_H(A, F_k^*(A)) \leq \min\left\{\frac{1}{k}, c(1 - L_k)\right\}.$$

Since F_k^* and the distance function d_H are continuous, one has

$$\mathbf{A}_k = (d_H(\,\cdot\,, F_k^*(\,\cdot\,)))^{-1}\left(\left[0, \min\left\{\frac{1}{k}, c(1 - L_k)\right\}\right]\right)$$

and so, the set $\mathbf{A}_k$ is closed. We show that it is also contractible. Observe that, for every $A \in \mathbf{A}_k$,

$$\begin{aligned} d_H(A, A_k^*) &\leq d_H(A, F_k^*(A)) + d_H(F_k^*(A), F_k^*(A_k^*)) + d_H(F_k^*(A_k^*), A_k^*) \\ &\leq \min\left\{\frac{1}{k}, c(1 - L_k)\right\} + L_k d_H(A, A_k^*), \end{aligned}$$

and subsequently

$$d_H(A, A_k^*) \le \frac{1}{1-L_k} \min\left\{\frac{1}{k}, c(1-L_k)\right\} \le c.$$

Hence, $\mathbf{A}_k$ is bounded. Moreover, $A_k^* \in \mathbf{A}_k$.

Let $A \in \mathbf{A}_k$. Then

$$d_H((F_k^*)^2(A), F_k^*(A)) \le L_k\, d_H(F_k^*(A), A) \le \min\left\{\frac{1}{k}, c(1-L_k)\right\},$$

where F_k^{*2} denotes the second iterate of F_k^*. Thus, $F_k^{*m}(A) \in \mathbf{A}_k$, for any $m \ge 0$. We know that, by the Banach algorithm, $A_k^* = \lim_{m\to\infty} F_k^{*m}(A)$, for every $A \in \mathcal{K}(X)$. This allows us to construct the homotopy $\chi\colon \mathbf{A}_k \times [0,1] \to \mathbf{A}_k$,

$$\chi(A,t) = \begin{cases} H_k^*(F_k^{*m}(A), (m+1)(m+2)t - m(m+2)) \\ \qquad \text{whenever } \dfrac{m}{m+1} \le t \le \dfrac{m+1}{m+2}, \\ A_k^* \quad \text{for } t = 1. \end{cases}$$

This homotopy deforms $\mathbf{A}_k$ onto A_k^*.

Notice that, if $A = F^*(A)$, then

$$d_H(A, F_k^*(A)) = d_H(F^*(A), F_k^*(A)) \le \min\left\{\frac{1}{k}, c(1-L_k)\right\},$$

for every $k \ge 1$. So, $A \in \mathbf{A}_k$ and, consequently, $\operatorname{Fix} F^* \subset \mathbf{A}_k$.

We prove the nonemptiness of $\operatorname{Fix} F^*$. One has

$$d_H(A_k^*, F^*(A_k^*)) = d_H(F_k^*(A_k^*), F^*(A_k^*)) \le \frac{1}{k}. \tag{85.16.1}$$

In view of the PS-condition, there is a subsequence $A_{k_l}^*$ convergent to A^*. Because of continuity of F^*, one obtains $F^*(A_{k_l}^*) \to F^*(A^*)$. On the other hand, (85.16.1) gives $F^*(A_{k_l}^*) \to A^*$, so $A^* = F^*(A^*)$.

It is also easy to prove that $\operatorname{Fix} F^*$ is compact. To do this, take a sequence $\{A_k\} \subset \operatorname{Fix} F^*$, i.e. $d_H(A_k, F^*(A_k))(= 0) \to 0$, so by the PS-condition, there is a convergent subsequence.

Now, we show that

- for every open neighbourhood U of $\operatorname{Fix} F^*$ in $\mathcal{K}(X)$, one can find $k_0 \ge 1$ such that $\mathbf{A}_k \subset U$, for every $k \ge k_0$.

Indeed, otherwise, there is $A_{k_l} \in \mathbf{A}_{k_l}$ with $A_{k_l} \notin U$, $k_l \to \infty$. By the definition of $\mathbf{A}_{k_l}$,

$$d_H(A_{k_l}, F^*(A_{k_l}))\left(\le \frac{2}{k_l}\right) \to 0.$$

Applying the PS-condition, one obtains (up to a subsequence)

$$A_{k_l} \to A = F^*(A) \in \operatorname{Fix} F^*.$$

But this is impossible, because $\mathcal{K}(X) \setminus U \ni A_{k_l} \to A \in \mathcal{K}(X) \setminus U$.

Since $\operatorname{Fix} F^* \subset \mathbf{A}_k$, $k \geq 1$, we have that $\operatorname{Fix} F^* = \operatorname{Lim} \mathbf{A}_k$, where $\operatorname{Lim} \mathbf{A}_k$ denotes the topological limit of $\{\mathbf{A}_k\}$. By the variant of the Browder–Gupta lemma (cf. Chapter VI, Section 69 or [AnGo-M, Chapter II.1]), we conclude that $\operatorname{Fix} F^*$ is an R_δ-set. □

(85.17) REMARK. One can readily check that the mapping $\varphi(x) = [0, x]$, $x \in [0, 1]$, is nonexpansive, and satisfying the assumptions of Theorem (85.16), but not a weak contraction in the sense of Definition (85.5). The domain $[0, 1]$ is obviously the maximal compact invariant set of φ. The whole family $\{[0, x] \mid x \in [0, 1]\}$ of compact invariant sets of φ, guaranteed by Theorem (85.16), is much richer, namely an R_δ-set.

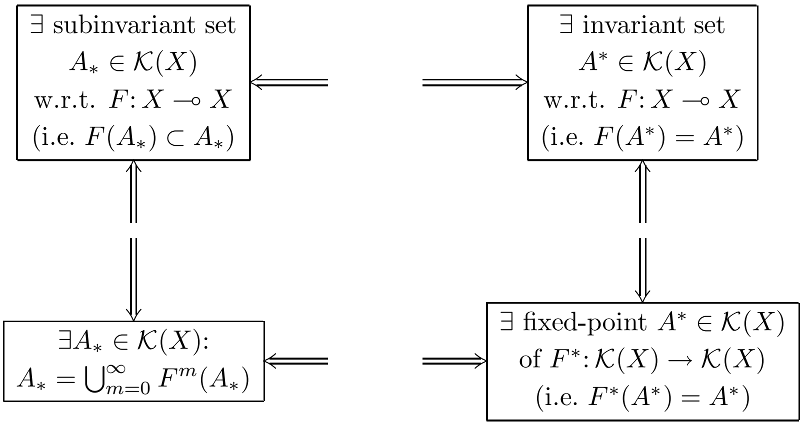
Let us start this subsection with a simple observation that although the generalized Lefschetz number $\Lambda(\varphi)$ of a (single-valued, continuous) endomorphism $\varphi: A \to A$, homotopic to $\operatorname{id}|_A$ on an annulus A ($\in$ ANR) equals zero, because $\Lambda(\varphi) = \Lambda(\operatorname{id} A) = \chi(A) = 0$, for the one of the induced endomorphism $\varphi^*: \mathcal{K}(A) \to \mathcal{K}(A)$, we have that $\Lambda(\varphi^*) = \Lambda(\operatorname{id}|_{\mathcal{K}(A)}) = \chi(\mathcal{K}(A)) = 1$, because $\mathcal{K}(A) \in$ AR), where $\chi(\,\cdot\,)$ stands for the Euler–Poincaré characteristic. Subsequently, although φ need not admit a fixed-point (if, for instance, $\varphi \neq \operatorname{id} A \mod 2\pi$ is a rotation, then φ is apparently fixed-point free), there must always be a fixed-point of φ^* in $\mathcal{K}(A)$.

Furthermore, if $\varphi: X \multimap X$ is a (multivalued) map and $A \subset X$ is a nonempty subset such that $A \subset \varphi(A)$, then it follows on the basis of the Knaster–Tarski fixed-point theorem (see Section 84) that there exists an invariant set $A^* \subset X$ w.r.t. φ, i.e. $A^* = \varphi(A^*)$. On the other hand, in spite of the existence of a fixed-point of an u.s.c. map with compact, convex values $\varphi: \mathbb{R} \multimap \mathbb{R}$ such that e.g. $[0, 1] \subset \varphi([0, 1])$ (see Lemma 9.9 from Chapter III.9 in [AnGo-M]), for $n > 1$, there need not exist a fixed-point for a continuous (single-valued) function $f: \mathbb{R}^n \to \mathbb{R}^n$ such that $[0, 1]^n \subset f([0, 1]^n)$.

These simple examples demonstrate that fractals can also exist on sets without fixed-points. On the other hand, we must prove

(i) either the existence of fixed-points in the given hyperspace $(\mathcal{K}(X), d_H)$,
(ii) or the existence of compact invariant sets in the original space (X, d).

Here, we will concentrate on the second approach. A bit surprisingly, from the point of view of the sole existence of fractals, this approach appears to be the most efficient of all. This is due to the application of the special case of the *Knaster–Tarski fixed-point theorem*, which can be schematically expressed for quasi-compact maps (i.e. $F^*: \mathcal{K}(X) \to \mathcal{K}(X)$) as follows:



(85.18) Remark. Observe that, besides satisfying the assumption $F(A_*) \subset A_* \in \mathcal{K}(X)$, the quasi-compact mapping $F: \multimap X$ can be quite arbitrary, because the induced map $F^*: \mathcal{K}(X) \to \mathcal{K}(X)$, as any other map, always fulfils the monotonicity (isotonicity) condition: if $A_1 \le A_2$ then $F^*(A_1) \le F^*(A_2)$, for all $A_1, A_2 \in \mathcal{K}(X)$, provided (as in our case) $A_1 \le A_2$ in $\mathcal{K}(X)$ if and only if $A_1 \subset A_2$ in X. The general version of the Knaster–Tarski theorem (cf. [DG-M, p. 25]) namely holds for monotone (isotone) maps on partially ordered sets whose chains have an infimum (again trivially satisfied here). Moreover, the implied invariant set $A^* \in \mathcal{K}(X)$ is *minimal*, i.e. if $B^* \subset A^*$ then $B^* = A^*$, for every invariant set $B^* \in \mathcal{K}(X)$ w.r.t. $F: X \multimap X$. Actually, one can easily check that there are no other minimal subinvariant sets, but minimal invariant sets.

(85.19) Remark. As a nontrivial example of quasi-compact maps, we can mention the class of u.s.c. maps with compact values (see e.g. [AnGo-M]). If, in particular, $F: X \multimap X$ is a compact u.s.c. map, then its center (core) $\bigcap_{m=1}^{\infty} F^m(X)$ can be verified to be a nonempty, compact (see [Go1-M, p. 25]), subinvariant subset of X w.r.t. F, because

$$F\Big(\bigcap_{m=1}^{\infty} F^m(X)\Big) \subset \bigcap_{m=1}^{\infty} F(F^m(X)) = \bigcap_{m=1}^{\infty} F^{m+1}(X) = \bigcap_{m=1}^{\infty} F^m(X).$$

Thus, taking

$$\mathcal{K}(X) \ni A_* := \bigcap_{m=1}^{\infty} F^m(X) = \bigcap_{m=1}^{\infty} \overline{\bigcup_{x \in X} F^m(x)},$$

where $\{\varphi_i: X \multimap X,\ i = 1, \dots, n\}$ is a system of compact u.s.c. maps, and subsequently so is $F(x): \bigcup_{i=1}^n \varphi_i(x)$, $x \in X$, the Knaster–Tarski theorem implies the existence of a minimal fractal for the system $\{\varphi_i: X \multimap X,\ i = 1, \dots, n\}$.

Hence, for the application of the above scheme, it is enough to prove the existence of a compact subinvariant set $A_* \subset X$ of the quasi-compact Hutchinson–Barnsley map $F: X \multimap X$, $F(x) := \bigcup_{i=1}^n \varphi_i(x)$, $x \in X$. We will do it for a rather general system of condensing multivalued functions. For the definition, based on the notion of measure of noncompactness (MNC) with the important properties (μ_1)–(μ_5).

One can readily check that if $F: X \multimap X$ is condensing, then F is quasi-compact. Hence, considering the Hutchinson–Barnsley map

$$F(x) := \bigcup_{i=1}^n \varphi_i(x), \quad x \in X,$$

where $\{\varphi_i: X \multimap X,\ i = 1, \dots, n\}$ is a system of condensing maps, and the induced Hutchinson–Barnsley operator

$$F^*(A) := \overline{\bigcup_{x \in A} F(x)}, \quad A \in \mathcal{K}(X),$$

F becomes quasi-compact, as required, because (cf. (μ_5))

$$0 = \mu(A) \geq \max_{i=1,\dots,n} \{\mu(\varphi_i(A))\} = \mu\bigg(\bigcup_{i=1}^n \varphi_i(A)\bigg) = \mu(F(A)), \quad \text{for } A \in \mathcal{K}(X),$$

and subsequently (cf. (μ_1)) $F^*(A) = \overline{F(A)} \in \mathcal{K}(X)$.

The following lemma is a slight (multivalued) modification of Lemma 1.6.11 in [AKPRS-M].

(85.20) Lemma. *Let (X, d) be a complete metric space, $F: X \multimap X$ be a condensing map such that $F(X)$ is bounded, and $F^*: \mathcal{B} \to \mathcal{B}$ be the corresponding Hutchinson–Barnsley operator on the family of bounded sets $\mathcal{B}$, i.e. $F^*(B) = \overline{F(B)}$, for $B \in \mathcal{B}$. Then*

$$\lim_{m \to \infty} \mu(F^{*\,m}(X)) = 0.$$

Proof. We will prove this lemma only for $\mu := \gamma$, because for $\mu := \alpha$ the proof will follow from the relation $\gamma(B) \leq \alpha(B) \leq 2\gamma(B)$.

Hence, denoting the bounded set (by the hypothesis) $Q := F(X) = F^*(X)$ and $R_m := F^{*\,m}(Q)$, we have $R_m \subset Q$. Defining

$$\Sigma := \bigg\{ A = \bigcup_{m=1}^\infty A_m \;\bigg|\; A_m \text{ is a finite subset of } R_m \bigg\}$$

and applying Lemma 1.6.10 in [AKPRS-M], there exists $A_s \in \Sigma$ such that $\gamma(A_s) = \sup_{A\in\Sigma} \gamma(A)$, i.e. $A_s = \bigcup_{m=1}^{\infty} A_{m,s}$.

Furthermore, denoting $R_0 := Q$, we can associate, with each $x \in A_{m,s} \subset R_m = F^*(R_{m-1})$, $y_x \in F^{*\,m-1}(Q) = R_{m-1}$ such that $x \in F^*(\{y_x\})$. Taking

$$B_{m-1} := \{y_x \mid x \in A_{m,s}\}, \quad B := \bigcup_{m=1}^{\infty} B_m \in \Sigma,$$

we obtain $\gamma(B) \le \gamma(A_s)$. Moreover,

$$F^*(B) = F^*\bigg(\bigcup_{m=1}^{\infty} B_m\bigg) \supset \bigcup_{m=1}^{\infty} F^*(B_m) \supset \bigcup_{m=1}^{\infty} A_{m+1,s} = \bigcup_{m=2}^{\infty} A_{m,s},$$

thanks to the inclusion

$$F^*(B_m) \supset \bigcup_{x\in A_{m+s,s}} F^*(\{y_x\}) \supset A_{m+1,s},$$

and subsequently

$$\gamma(F^*(B)) \ge \gamma\bigg(\bigcup_{m=2}^{\infty} A_{m,s}\bigg) = \gamma(A_s) \ge \gamma(B).$$

In view of γ-condensity of F, we arrive at $\gamma(A_s) = \gamma(B) = 0$, because B is bounded, and since X is complete, $\overline{B}$ is compact. After all, $\mu(A) = 0$, for all $A \in \Sigma$.

The limit

$$\lim_{m\to\infty} \gamma(F^{*\,m}(Q)) = \lim_{m\to\infty} \gamma(F^{*\,m}(X))$$

exists, because it is related to a nonincreasing sequence bounded from below.

Assume, on the contrary, that it is positive. Then there exists $\varepsilon_0 > 0$ such that $\gamma(F^{*\,m}(Q)) > \varepsilon_0$. Choose $x_1 \in F^*(Q)$ and select $x_2 \in F^{*\,2}(Q)$ with $d(x_1, x_2) \ge \varepsilon_0$. Proceeding inductively, we get an infinite sequence $\{x_m\}_{m=1}^{\infty}$, $x_m \in F^{*\,m}(Q)$, such that $d(x_k, x_l) \ge \varepsilon_0$, for $k \ne l$.

Having namely $\{x_k \mid k = 1, \dots, m\}$, we can find $x_{m+1} \in F^{*\,m+1}(Q)$ with $\operatorname{dist}(x_{n+1}, \{x_1, x_2, \dots, x_m\}) \ge \varepsilon_0$; otherwise, $\{x_k \mid k = 1, \dots, m\}$ would determine a finite ε_0-net for $F^{*\,m+1}(Q)$.

On the other hand, the set $A := \{x_m \mid m = 1, 2, \dots\}$ belongs to Σ, and so $\gamma(\{x_m \mid m = 1, 2, \dots\}) = 0$. Thus, $\{x_m\}_{m=1}^{\infty}$ must contain a convergent subsequence. By the separation property of this sequence, due to its construction, it is impossible which completes the proof. □

We are ready to give the main existence result of this part.

(85.21) THEOREM. *Let (X, d) be a complete metric space and let $\{\varphi_i: X \multimap X, i = 1, \ldots, n\}$ be a system of condensing maps such that $\varphi_i(X)$ is bounded, for every $i = 1, \ldots, n$. Then there exists a minimal, nonempty, compact, invariant set $A^* \subset X$ w.r.t. the Hutchinson–Barnsley map*

$$F(x) := \bigcup_{i=1}^{n} \varphi_i(x), \quad x \in X,$$

i.e. a minimal fractal $A^* \in \mathcal{K}(X)$ with $F^*(A^*) = A^*$, where

$$F^*(A) := \overline{\bigcup_{x \in A} F(x)}, \quad A \in \mathcal{K}(X),$$

is the Hutchinson–Barnsley operator.

PROOF. Because of (μ_5), $F: X \multimap X$ is condensing and since $F^*(A) = \overline{F(A)}$, for every $A \in \mathcal{K}(X)$, it is also quasi-compact, i.e. $F^*: \mathcal{K}(X) \to \mathcal{K}(X)$, as pointed out above. Thus, the particular form (see the above scheme) of the Knaster–Tarski theorem can be applied here. For this, we need a subinvariant $A_* \in \mathcal{K}(X)$ w.r.t. F.

Since, for

$$A := \overline{F(X)} \in \mathcal{B}, \quad \{\underbrace{\overline{F} \circ \ldots \circ \overline{F(A)}}_{m\text{-times}}\}_{m=1}^{\infty}$$

becomes a decreasing sequence of nonempty, closed sets with (see Lemma (85.20))

$$\lim_{m \to \infty} \mu(\underbrace{\overline{F} \circ \ldots \circ \overline{F(A)}}_{m\text{-times}}) = 0,$$

we get (cf. (μ_4)) that

$$A_* := \bigcap_{m=1}^{\infty} \underbrace{\overline{F} \circ \ldots \circ \overline{F(A)}}_{m\text{-times}} \neq \emptyset.$$

The nonempty center (core) A_* can be verified to be compact by means of the well-known Kuratowski theorem. It also satisfies the inclusion

$$\text{(85.21.1)} \qquad \begin{aligned} \overline{F(A_*)} &= \overline{F\bigg(\bigcap_{m=1}^{\infty} \underbrace{\overline{F} \circ \ldots \circ \overline{F(A)}}_{m\text{-times}}\bigg)} \subset \bigcap_{m=1}^{\infty} \overline{F\bigg(\underbrace{\overline{F} \circ \ldots \circ \overline{F(A)}}_{m\text{-times}}\bigg)} \\ &= \bigcap_{m=1}^{\infty} \underbrace{\overline{F} \circ \ldots \circ \overline{F(A)}}_{(m+1)\text{-times}} = \bigcap_{m=1}^{\infty} \underbrace{\overline{F} \circ \ldots \circ \overline{F(A)}}_{m\text{-times}} = A_*. \end{aligned}$$

Thus, applying the Knaster–Tarski theorem (see the above scheme), there exists a minimal fixed-point $A^* \in \mathcal{K}(X)$ of the Hutchinson–Barnsley operator $F^*: \mathcal{K}(X) \to \mathcal{K}(X)$. □

A more transparent proof of Lemma (85.20) can be done for a system of set-contractions in a complete metric space X.

Hence, consider a system $\{\varphi_i: X \multimap X,\ i = 1,\dots,n\}$ of k_i-set contractions φ_i, i.e.

$$\mu(\varphi_i(B)) \le k_i\mu(B), \quad \text{for all } B \in \mathcal{B},$$

where $k_i \in [0,1)$, $i = 1,\dots,n$, and assume that $\varphi_i(X)$ is bounded, for every $i = 1,\dots,n$. For the associated Hutchinson–Barnsley map $F: X \multimap X$, we get immediately that $\overline{F(X)} \subset X$, $F(X) \in \mathcal{B}$, and (cf. (μ_1), (μ_5)),

$$\mu(\overline{F(B)}) = \mu(F(B)) \le k\mu(\overline{B}) = k\mu(B), \quad \text{for all } B \in \mathcal{B},$$

where $1 > k = \max\{k_1,\dots,k_n\}$, and subsequently

$$\mu(\underbrace{\overline{F} \circ \dots \circ \overline{F(\overline{B})}}_{m\text{-times}}) \le k^m\mu(\overline{B}) = k^m\mu(B), \quad \text{for all } B \in \mathcal{B}.$$

Therefore,

$$0 \le \lim_{m\to\infty} \mu(\underbrace{\overline{F} \circ \dots \circ \overline{F(\overline{B})}}_{m\text{-times}}) \le \mu(B) \lim_{m\to\infty} k^m = 0.$$

(85.22) Corollary. *Let X be a complete metric space and $\{\varphi_i: X \multimap X,\ i = 1,\dots,n\}$ be a system of set-contractions such that $\varphi_i(X)$ is bounded, for every $i = 1,\dots,n$. Then there exists a minimal fractal $A^* \in \mathcal{K}(X)$ with $F^*(A^*) = A^*$ or, equivalently, with $\overline{F(A^*)} = A^*$.*

(85.23) Remark. One can readily check that since $\emptyset \ne A^* \in \mathcal{K}(X)$ is evidently a minimal subinvariant set, it must be an invariant set w.r.t. $\overline{F}$ (see Remark (85.8)), i.e. $A^* := A_* = \overline{F(A_*)}$. In other words,

$$\overline{F\Big(\bigcap_{m=1}^{\infty} \underbrace{\overline{F} \circ \dots \circ \overline{F(X)}}_{(m+1)\text{-times}}\Big)} = \bigcap_{m=1}^{\infty} \underbrace{\overline{F} \circ \dots \circ \overline{F(A)}}_{(m+2)\text{-times}}$$

holds in (85.21.1), and subsequently the minimal fractal in Theorem (85.21) or Corollary (85.22) takes the form

$$A^* := \bigcap_{m=1}^{\infty} \underbrace{\overline{F} \circ \dots \circ \overline{F(A)}}_{m\text{-times}}. \tag{85.23.1}$$

The assumption in Theorem (85.21) that $\varphi_i(X)$ is, for every $i = 1,\dots,n$, bounded might be sometimes rather restrictive (cf. Theorem (85.1) or Theorem (85.16)). Therefore, one can alternatively assume e.g. that φ_i are l.s.c. and such that the related Hutchinson–Barnsley map F has a compact attractor for related orbits starting at some point $x_0 \in X$.

(85.24) Corollary. *Let $\{\varphi_i : X \multimap X,\ i = 1, \dots, n\}$ be a system of l.s.c. maps and suppose that there exists a compact set $K \subset X$ of a complete space X attracting, for some $x_0 \in X$, the orbit $\{F^m(x_0)\}_{m=1}^{\infty}$ of the related Hutchinson–Barnsley map $F(x) = \bigcup_{i=1}^{n} \varphi_i(x)$, i.e.*

for all $\varepsilon > 0$ there exists l such that $F^m(x_0) \subset B(K, \varepsilon)$ for all $m \geq l$

or equivalently, $e(F^m(x_0), K) \to 0$, when $m \to \infty$. Then the omega-limit set $\omega(x_0) = \bigcap_m \overline{\bigcup_{k \geq m} F^k(x_0)}$ is a nonempty, compact subinvariant set. Moreover, there exists a minimal compact invariant set of the Hutchinson–Barnsley map F.

Proof. For the subinvariance of $\omega(x_0)$ observe that

$$\begin{aligned} F\bigg(\bigcap_m \overline{\bigcup_{k \geq m} F^k(x_0)}\bigg) &\subset \bigcap_m F\bigg(\overline{\bigcup_{k \geq m} F^k(x_0)}\bigg) \\ &\subset \bigcap_m \overline{F\bigg(\bigcup_{k \geq m} F^k(x_0)\bigg)} \subset \bigg(\bigcap_m \overline{\bigcup_{k \geq m+1} F^k(x_0)}\bigg), \end{aligned}$$

and realize that F becomes l.s.c. too. The middle inclusion above is due to the fact that a multifunction F is l.s.c. if and only if $F(\overline{A}) \subset \overline{F(A)}$, for all $A \subset X$.

Since $\omega(x_0) = \operatorname{Ls} F^k(x_0)$ (the upper Kuratowski topological limit), we have that $\omega(x_0) \subset K$ is nonempty and compact. Thus, $\overline{F(\omega(x_0))}$ must be compact, too. It follows immediately from the Knaster–Tarski theorem that F admits a minimal compact invariant set. □

Another possibility is to assume e.g. that φ_i are u.s.c. locally compact maps such that the related Hutchinson–Barnsley map has a compact attractor.

(85.25) Corollary. *Let $\{\varphi_i : X \multimap X,\ i = 1, \dots, n\}$ be a system of locally compact u.s.c. maps such that the related Hutchinson–Barnsley map $F(x) = \bigcup_{i=1}^{n} \varphi_i(x)$ has a compact attractor $K \subset X$, i.e.*

for all $\varepsilon > 0$ and all $x \in X$ there exists $l = l_x$ such that

$F^m(x) \subset B(K, \varepsilon)$ for all $m \geq l$.

Then there exists a minimal compact invariant set of the Hutchinson–Barnsley map F.

Proof. It can be proved exactly as in Lemma 5.11 in [AnGo-M, Chapter 1.5, pp. 75–76] that a subset $U \subset X$ w.r.t. F exists such that $F(U) \subset U$, where $\overline{F(U)}$ is compact. Therefore, applying the Knaster–Tarski theorem, there exists a minimal compact invariant set of F. □

(85.26) Corollary. *Let X be complete and $\{\varphi_i : X \multimap X, i = 1, \dots, n\}$ be a system of weak contractions with compact values. Then there exists a unique compact invariant set of the Hutchinson–Barnsley map F. If X is still compact, then, for the same conclusion, the system can consist of Edelstein contractive maps.*

We finish this subsection by two examples illustrating the disadvantage of using only condensing maps.

(85.27) Example (Noncondensing Nadler contraction). Let X be an infinite dimensional Banach space. Define multivalued constant map $\varphi : X \multimap X$ as $\varphi(x) = D(0,1)$, i.e. a closed unit ball. Then φ is a multivalued contraction with closed bounded values (and an arbitrarily small Lipschitz constant), but it is not condensing (it sends a singleton onto the noncompact set). The unique invariant set is $A^* = D(0,1)$ (there are no unbounded invariant sets).

(85.28) Example (Noncondensing Edelstein contractivity). Let $e_1, e_2, e_3, \dots$ be an orthonormal base in ℓ^2 and $X = D(0,1)$ be the unit closed ball at 0. The (single-valued) function $f : X \to X$ is defined for $x = \sum_{i=1}^{\infty} \lambda_i \, e_i$ by

$$f(x) = \sum_{i=1}^{\infty} \left(1 - \frac{1}{i+1}\right) \cdot \lambda_i e_i .$$

This is bounded and shrinks distances:

$$\begin{aligned}
(85.28.1)\qquad & \left\| \sum_i \left(1 - \frac{1}{i+1}\right) \lambda_i e_i - \sum_i \left(1 - \frac{1}{i+1}\right) {\lambda_i}' e_i \right\| \\
&= \left\| \sum_i \left(1 - \frac{1}{i+1}\right)^2 (\lambda_i - {\lambda_i}') \cdot e_i \right\| \\
&= \sqrt{\sum_{i \neq i_0} \left(1 - \frac{1}{i+1}\right)^2 (\lambda_i - {\lambda_i}')^2 + \left(1 - \frac{1}{i_0+1}\right)^2 (\lambda_{i_0} - {\lambda_{i_0}}')^2} \\
&< \sqrt{\sum_{i \neq i_0} (\lambda_i - {\lambda_i}')^2 + (\lambda_{i_0} - {\lambda_{i_0}}')^2} = \left\| \sum_i \lambda_i e_i - \sum_i {\lambda_i}' e_i \right\|,
\end{aligned}$$

for $x \neq x'$, $x = \sum_i \lambda_i e_i$, $x' = \sum_i {\lambda_i}' e_i$. The inequality (85.28.1) holds because if $x \neq x'$, then $\lambda_{i_0} - {\lambda_{i_0}}' \neq 0$, for some i_0. The unique invariant set is $A^* = \{0\}$ (there is no uniform attractor in the sense of definition given in [Les]). The function f is neither condensing nor an η-contraction (for any "reasonable" η), because the sphere is attracted arbitrarily "slowly": $\|f(e_n) - f(0)\| = 1 - 1/(n+1) \to 1$, when $n \to \infty$.

As we shall see, it is not difficult to formulate a continuation principle for fractals. It might allow us to employ some properties (like localization or additivity) of the related indices. On the other hand, there are some obstructions in its nontrivial application.

Let $X \in \mathrm{HANR}$ and let

$$F_\lambda(x) := \bigcup_{i=1}^{n} \varphi_i(x,\lambda), \quad x \in X,\ \lambda \in [0,1],$$

be a one-parameter family of *Hutchinson–Barnsley maps*, where $\varphi_i\colon X \times [0,1] \multimap X$, $i = 1,\dots,n$, are Hausdorff-continuous compact maps. Then we have the induced *Hutchinson–Barnsley operators* $F^*_\lambda\colon \mathcal{K}(X) \to \mathcal{K}(X)$,

$$F^*_\lambda(A) := \bigcup_{x \in A} F_\lambda(x), \quad A \in \mathcal{K}(X),\ \lambda \in [0,1],$$

and F^*_λ becomes a compact (continuous) homotopy.

Thus, we can associate with F^*_λ the generalized Lefschetz number $\Lambda(F^*_\lambda) \in \mathbb{Z}$ as well as the fixed-point index $\operatorname{ind}(F^*_\lambda, U) \in \mathbb{Z}$, for every open set $U \subset \mathcal{K}(X)$ such that $\operatorname{Fix}(F^*_\lambda) \cap \partial U = \emptyset$ (for more details, see e.g. [Br1-M] or [DG-M]). If $\Lambda(F^*_\lambda) \neq 0$, for some $\lambda \in [0,1]$, we get a fixed-point A^* of F^*_λ (i.e. $F^*_\lambda(A^*) = A^*$), for such a $\lambda \in [0,1]$. Below, we analyze the properties of fixed-points.

At first, using the fixed-point index and the Nielsen equivalence relation in $\operatorname{Fix}(F^*_\lambda)$, we can distinguish between essential and inessential classes of fixed-points of F^*_λ; i.e. the class $C \subset \operatorname{Fix}(F^*_\lambda)$ (which can be checked to be isolated and compact) is *essential* if $\operatorname{ind}(F^*_\lambda, U) \neq 0$, for an open set $U \subset \mathcal{K}(X)$ with $U \cap \operatorname{Fix}(F^*_\lambda) = C$. Note that if $\Lambda(F^*_\lambda) \neq 0$, then at least one of the Nielsen classes is essential.

(85.29) DEFINITION. We call a multivalued fractal A^* of the multifunction system $\{\varphi_i(x,\lambda), i = 1,\dots,n\}$, for a given $\lambda \in [0,1]$, *homotopically essential* if A^* belongs to some essential Nielsen class for F^*_λ.

Because of the invariance under homotopy of the generalized Lefschetz number $\Lambda(F^*_\lambda)$ (see [Br1-M]), we are in a position to formulate the following first continuation principle for (multivalued) fractals.

(85.30) THEOREM. *Let $X \in \mathrm{HANR}$ and $\{\varphi_i\colon X \times [0,1] \multimap X,\ i = 1,\dots,n\}$ be a system of Hausdorff-continuous compact maps. Then an essential fractal exists for the system $\{\varphi_i(\,\cdot\,,0)\colon X \multimap X,\ i = 1,\dots,n\}$ if and only if the same is true for $\{\varphi_i(\,\cdot\,,1)\colon X \multimap X,\ i = 1,\dots,n\}$.*

(85.31) REMARK. Because of

(85.31.1) $\mathcal{K}(X) \in \mathrm{ANR}$,
(85.31.2) the well-defined $\Lambda(F^*_\lambda)$,

and the fixed-point set $\mathrm{Fix}(F_\lambda^*)$ being compact, we can even associate another invariant under homotopy, namely the Nielsen number $N(F_\lambda^*)$, allowing us to make a lower estimate of the number of fractals for the system $\{\varphi_i\colon X \times [0,1] \multimap X,\ i = 1, \dots, n\}$.

An interesting information can be obtained if we use the following notion of essentiality introduced by M. K. Fort in [For].

(85.32) Definition. We say that a multivalued fractal $A^* = F_\lambda^*(A^*)$, for a given λ, is *topologically essential* if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $G^*\colon \mathcal{K}(X) \to \mathcal{K}(X)$ has a fixed-point B^* with $d_H(A^*, B^*) < \varepsilon$, for any G^* satisfying $\rho(F_\lambda^*, G^*) < \delta$.

Here, $\rho(F_\lambda^*, G^*) := \sup\{d_H(F_\lambda^*(A), G^*(A) \mid A \in \mathcal{K}(X)\}$.

Following the Granas characterization of essentiality (see [DG-M]) by the fixed-point index, we obtain

(85.33) Proposition. *Let* $X \in \mathrm{HANR}$ *and* $\{\varphi_i\colon X \times [0,1] \multimap X,\ i = 1, \dots, n\}$ *be a system of Hausdorff-continuous locally compact maps. Assume that* A^* *is an isolated multivalued fractal, for some* $F_{\lambda_0}^*$, $\lambda_0 \in [0,1]$, *and* $\mathrm{ind}\,(F_{\lambda_0}^*, A^*) \neq 0$, *where we define*

$$\mathrm{ind}\,(F_{\lambda_0}^*, A^*) := \lim_{\delta \to 0} \mathrm{ind}\,(F_{\lambda_0}^*, B(A^*, \delta)),$$

using the localization property of the fixed-point index. Then A^* *is topologically essential. In particular, for every* $\varepsilon > 0$, *there is an open neighbourhood* $J \subset [0,1]$ *of* λ_0 *such that each* F_λ^*, $\lambda \in J$, *has a fixed-point in* $B(A^*, \varepsilon) \subset \mathcal{K}(X)$.

Proof. Since the maps φ_i are locally compact and A^* is compact, the index $\mathrm{ind}\,(F_{\lambda_0}^*, A^*)$ is well-defined. By the Granas characterization, it follows that A^* is topologically essential. Let $\varepsilon > 0$ be arbitrary. The Hausdorff-continuity of φ_i implies that $\rho(F_{\lambda_0}^*, F_\lambda^*)$ is sufficiently small (as in the definition of topological essentiality with the given ε), for each λ close to λ_0. This completes the proof. □

Obviously, if $X \in \mathrm{HAR}$, then $X \in \mathrm{HANR}$. The nontrivial example of X to be an HANR is that X is a locally connected metric space and to be an HAR is that X is a connected and locally connected metric space.

Thus, in order to have $X \in \mathrm{HANR}$, but $X\mathbb{N} \notin \mathrm{HAR}$, rather strong restrictions seem to be imposed on X like local connectedness, but disconnectedness of X.

Since, for any continuum X, $\mathcal{K}(X)$ is contractible (see [AFGL] and references therein) w.r.t. any ANR-space, if $\mathcal{K}(X)$ is not contractible (in itself), then $\mathcal{K}(X)$ cannot be an ANR-space.

(85.34) Example. A simple example of a continuum X such that $\mathcal{K}(X)$ is not locally contractible looks as in Figure below:

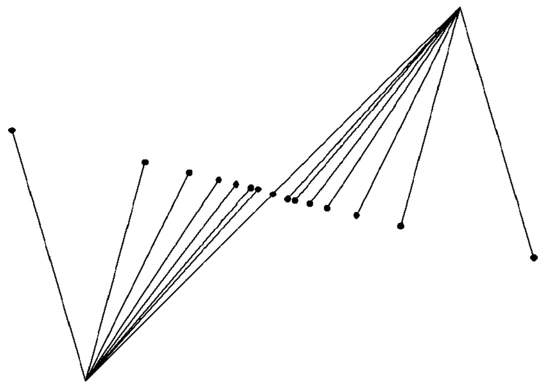


FIGURE 10. Continuum X such that $\mathcal{K}(X)$ is noncontractible

Since $\mathcal{K}(X)$ is not locally contractible, it cannot be an ANR-space.

Hence, for a continuum X, in order to have $X \in$ HANR, we must ask $\mathcal{K}(X)$ to be contractible. However, according to the Borsuk Theorem 9.1 in [Bo-M, p. 96], contractible ANR-spaces are exactly the same as AR-spaces. Therefore, for continua X, it has no meaning to ask $X \in$ HANR, but $X \notin$ HAR.

Let us emphasize that this is also related to nonlocally connected continua; for locally connected continua (i.e. for Peano's continua) X, we have always, according to the Arens–Eells embedding theorem that $X \in$ HAR.

As a consequence, we have only one possibility for a compact X such that $X \in$ HANR, but $X \notin$ HAR, namely to be disconnected. It is a question whether or not the same is true for a noncompact X.

From the point of view of applications, it is important to have at least simple examples of such a situation.

For this, we need two following lemmas

(85.35) LEMMA. *Let X be a (metric)* ANR*-space. Then the hyperspaces $\mathcal{K}_k(X) := \{A \in \mathcal{K}(X) \mid \operatorname{card} A \leq k\}$ of all nonempty, compact subsets of X consisting of at most k points, where $k \in \mathbb{N} \cup \{\infty\}$ (for $k = \infty$, we define $\mathcal{K}_\infty(X) = \bigcup_{k\in\mathbb{N}} \mathcal{K}_k(X)$), are also* ANR*-spaces. In particular, if $X \in$* AR*, then $\mathcal{K}_k(X) \in$* AR*, for every $k \in \mathbb{N} \cup \{\infty\}$.*

(85.36) EXAMPLE. As a trivial example of such a space, we have $X = \{x_i \in \mathbb{R} \mid i = 1, 2\} \in$ HANR, because $\mathcal{K}(X) = \mathcal{K}_2(X) = \{\{x_1\}, \{x_2\}, X\}$. Thus, if $F(x) := \{f_1(x)\} \cup \{f_2(x)\}$, $x \in X$, where $f_1(x_1) = x_2$, $f_1(x_2) = x_1$, and $f_2(x_1) = x_1$, $f_2(x_2) = x_2$, then $F^*(\{x_1\}) = \{f_1(x_1)\} \cup \{f_2(x_1)\} = X$, $F^*(x_2) = \{f_1(x_2)\} \cup \{f_2(x_2)\} = X$, $F^*(X) = f_1(\{x_1, x_2\}) \cup f_2(\{x_1, x_2\}) = X$. Therefore, X is a fractal of the system $\{f_1(x), f_2(x)\}$.

(85.37) Lemma. *If X is a locally continuum-connected, i.e. every point has a basis of neighbourhoods such that every two points of each can be joined by a continuum inside,* (*a connected and locally continuum-connected*) *metric space, then* $\mathcal{K}(X) \in \mathrm{ANR}$ ($\mathcal{K}(X) \in \mathrm{AR}$).

(85.38) Example. Let this time X be a union of two disjoint compact intervals, i.e. $X = I_1 \cup I_2$, $I_1 \subset \mathbb{R} \supset I_2$, $I_1 \cup I_2 = \emptyset$. Obviously, we have that $X \in \mathrm{ANR}$ and $\mathcal{K}_k(X) \subsetneq \mathcal{K}(X)$, for every $k \in \mathbb{N} \cup \{\infty\}$. Thus, Lemma (85.35) implies that $\mathcal{K}_k(X) \in \mathrm{ANR}$ and Lemma (85.37) implies that $X \in \mathrm{HANR}$. Taking $F_\lambda(x) := \varphi_1(x,\lambda) \cup \varphi_2(x,\lambda)$, $x \in X$, where φ_1 and φ_2 are Hausdorff-continuous ($\Rightarrow$ compact) maps such that $\varphi_1(I_1 \times [0,1]) \subset I_2$, $\varphi_1(I_2 \times [0,1]) \subset I_1$, and $\varphi_2(I_1 \times [0,1]) \subset I_1$, $\varphi_2(I_2 \times [0,1]) \subset I_2$, we get (see [AF1]) that F_λ^* defined by $F_\lambda^*(A) = \bigcup_{x\in A} F_\lambda(x)$, $A \in \mathcal{K}(X)$, $\lambda \in [0,1]$, is a compact (continuous) self-map, i.e. $F_\lambda^*\colon \mathcal{K}(X) \times [0,1] \to \mathcal{K}(X)$. Therefore, the generalized Lefschetz number $\Lambda(F_\lambda^*)$ is well-defined (see e.g. [Br1-M]), for every $\lambda \in [0,1]$, and if $\Lambda(F_0^*) \neq 0$ (for $\lambda = 0$), then according to Theorem (85.30) there exists a homotopically essential (multivalued) fractal for the system $\{\varphi_i\colon (\,\cdot\,,1)\colon X \multimap X,\ i = 1,2\}$. If, in particular, $\varphi_1(\,\cdot\,,0)(I_1) = \{x_2\} \subset I_2$, $\varphi_1(\,\cdot\,,0)(I_2) = \{x_1\} \subset I_1$, $\varphi_2(\cdot,0)(I_1) = \{x_1\} \subset I_1$, $\varphi_2(\,\cdot\,,0)(I_2) = \{x_2\} \subset I_2$, then $F_0(X) = \{x_1,x_2\}$ and, as in Example (85.36), we can see that $F_0^*(\{x_1,x_2\}) = \{x_1,x_2\}$, i.e. $\{x_1,x_2\}$ is a fractal of the system $\{\varphi_1(x,0),\varphi_2(x,0)\}$. Moreover, since $F_0^*(A) = \{x_1,x_2\}$, for every $A \in \mathcal{K}(X)$, i.e. F_0^* is constant in $\mathcal{K}(X)$, we have that $\Lambda(F_0^*) = 1$ (see e.g. [Br1-M]), as required. Taking $\varphi_i(x,\lambda)$, $\lambda \in [0,1]$, $i = 1,2$, as above, where $\varphi_1(I_1 \times \{0\}) = \{x_2\}$, $\varphi_1(I_2 \times \{0\}) = \{x_1\}$, $\varphi_2(I_1 \times \{0\}) = \{x_1\}$, $\varphi_2(I_2 \times \{0\}) = \{x_2\}$, there exists a (multivalued) fractal of the system $\{\varphi_1(x,1),\varphi_2(x,1)\}$.

The computation of the generalized Lefschetz number or (when we restrict ourselves only to open subsets of $\mathcal{K}(X) \in \mathrm{ANR}$) of the fixed-point index in hyperspaces is a very delicate problem. Some possibilities are indicated in the paper [RPS]. Before we show them, several notions which are typical in the frame of the Conley index theory, must be recalled.

Hence, defining the (semi)*invariant parts* of $N \subset U$, where U is locally compact, w.r.t. $F\colon X \supset U \multimap X$ as

$$
\begin{aligned}
\mathrm{Inv}^+(N,F) := \{x \in N \mid \sigma(i+1) \in F(\sigma(i)),\ &\text{for all } i \in \mathbb{N} \cup \{0\},\\
&\text{where } \sigma\colon \mathbb{N} \cup \{0\} \to N \text{ is a single-valued map with } \sigma(0) = x\},\\
\mathrm{Inv}^-(N,F) := \{x \in N \mid \sigma(i+1) \in F(\sigma(i)),\ &\text{for all } i \in \mathbb{Z} \setminus \mathbb{N},\\
&\text{where } \sigma\colon \mathbb{Z} \setminus \mathbb{N} \to N \text{ is a single-valued map with } \sigma(0) = x\},\\
\mathrm{Inv}(N,F) := \mathrm{Inv}^+(N,F) \cap \mathrm{Inv}^-(N,F),&
\end{aligned}
$$

we say that a compact invariant set $K \subset U$ (i.e. $F(K) = K$) is *isolated* w.r.t. F if there exists a compact neighbourhood N of K such that

$$O_{\mathrm{diam}_N F}(\mathrm{Inv}(N, F)) \subset \mathrm{int}\, K,$$

where

$$\mathrm{diam}_N F := \sup_{x \in N}\{\mathrm{diam}\, F(x)\},$$

or, equivalently,

$$\mathrm{dist}(\mathrm{Inv}(N, F), \partial N) > \mathrm{diam}_N F,$$

where ∂N stands for the boundary of N. The neighbourhood N is then called an *isolating neighbourhood* of K.

Let $F: X \supset U \multimap X$ be a (locally defined) compact Hausdorff-continuous map with compact values. A compact isolated invariant set $K \subset U$ is said to be an *attractor* if there exists an open neighbourhood $U_0 \subset U$ of K such that

(i) $F^m(U_0) = \underbrace{F \circ \ldots \circ F}_{m\text{-times}}(U_0) \subset U$, for every $m \in \mathbb{N}$,

(ii) for every open neighbourhood V of K, there is $m(V) \in \mathbb{Z}$ such that $F^n(U_0) \subset V$, for all $n \geq m(V)$.

We have proved in [AGFL] that $F: X \supset U \multimap X$ induces in a natural way the compact (continuous) single-valued map F^* in the hyperspace $(\mathcal{K}(X), d_H)$, i.e. $F^*|_{\mathcal{K}(U)}: \mathcal{K}(U) \to \mathcal{K}(X)$. Hence, let $K \subset U$ be a compact isolated invariant set and N be an isolating neighbourhood of K. Considering an open set W such that $K \subset W \subset N$, we have defined a locally compact (continuous) single-valued map $F^*|_{\mathcal{K}(W)}: \mathcal{K}(W) \to \mathcal{K}(X)$. Since $\mathrm{Fix}(F^*|_{\mathcal{K}(W)}) \subset \mathcal{K}(K)$, the set of fixed-points of $F^*|_{\mathcal{K}(W)}$ is a compact subset of $\mathcal{K}(K)$. Moreover, if $X \in$ HANR (e.g. a locally continuum-connected metric space, see Lemma (85.37)), then $\mathcal{K}(W)$ is obviously an open subset of the ANR-space $\mathcal{K}(X)$), and so the *fixed-point index*

$$\mathrm{ind}\,(F^*|_{\mathcal{K}(W)}, \mathcal{K}(W)) \in \mathbb{Z}$$

of $F^*|_{\mathcal{K}(W)}: \mathcal{K}(W) \to \mathcal{K}(X)$ in $\mathcal{K}(W)$ is well-defined (see e.g. [Br1-M]).

Following [RPS], we can also define the *Conley-type* (integer-valued) *index* $I_X(K, F)$ of the pair (K, F) just by identifying

$$I_X(K, F) := \mathrm{ind}\,(F^*|_{\mathcal{K}(W)}, \mathcal{K}(W)).$$

(85.39) REMARK. Because of the definition, the Conley-type index has all usual properties like a standard fixed-point index, namely the existence (Ważewski's property), homotopy, additivity, localization (excision), contraction (restriction), multiplicity and normalization properties. In particular, the additivity property reads as follows (see [RPS]):

$$I_X(K, F) = I_X(K_1, F) + I_X(K_2, F) + I_X(K_1, F) \cdot I_X(K_2, F),$$

where K is a compact isolated invariant set which is a disjoint union of two compact isolated invariant sets K_1 and K_2, i.e. $K = K_1 \cup K_2$, $K_1 \cap K_2 = \emptyset$. Moreover, it follows from the excision property of the related fixed-point index that $I_X(K, F)$ depends neither on the choice of the isolating neighbourhood N of K, nor on the open set W.

Theorem (85.30) can be therefore improved in terms of the Conley-type indices as follows.

(85.40) THEOREM. *Let $X \in \mathrm{HANR}$ (e.g. X be a locally continuum-connected metric space (see Lemma (85.37))) and $\{\varphi_i\colon X \times [0,1] \supset U \times [0,1] \multimap X,\ i = 1, \ldots, n\}$ be a system of Hausdorff-continuous compact maps. Assume that $N \subset U$, where $U \subset X$ is locally compact, is an isolating neighbourhood which is common for all the Hutchinson–Barnsley maps*

$$F_\lambda(x) := \bigcup_{i=1}^{n} \varphi_i(x, \lambda), \quad \text{for } x \in U,\ \lambda \in [0,1]. \tag{85.40.1}$$

Then an essential fractal $K_0 \subset N$ (i.e. $I_X(K_0, F_0) \neq 0$) exists for the system $\{\varphi_i(\,\cdot\,, 0)\colon X \supset U \multimap X,\ i = 1, \ldots, n\}$ if and only if an essential fractal $K_1 \subset N$ (i.e. $I_X(K_1, F_1) \neq 0$) exists for the system $\{\varphi_i(\,\cdot\,, 1)\colon X \supset U \multimap X,\ i = 1, \ldots, n\}$. Moreover, $I_X(K_0, F_0) = I_X(\mathrm{Inv}(N, F_0), F_0) = I_X(K_1, F_1) = I_X(\mathrm{Inv}(N, F_1), F_1)$.

Let us formulate the following lemma (see [AFGL]):

(85.41) LEMMA. *Let X be a locally continuum-connected metric space (see Lemma (85.37)) and let $F\colon U \multimap X$, where $U \subset X$ is locally compact, be a Hausdorff-continuous compact map. Let K be a compact isolated invariant set w.r.t. F which is a disjoint union of $p \in \mathbb{N}$ connected attractors. Then $I_X(K, F) = 2^{p-1}$.*

Applying Lemma (85.41) to Theorem (85.40), we immediately obtain

(85.42) COROLLARY. *Let X be a locally continuum-connected metric space and $\{\varphi_i\colon X \times [0,1] \supset U \times [0,1] \to X,\ i = 1, \ldots, n\}$ be a system of Hausdorff-continuous locally compact maps. Assume that $N \subset U$, where $U \subset X$ is locally compact, is an isolating neighbourhood which is common for all the Hutchinson–Barnsley maps (85.40.1), i.e. for every $\lambda \in [0,1]$. Let K_0 be a compact isolated invariant set w.r.t. $F_0(x) := \bigcup_{i=1}^{n} \varphi_i(x, 0), x \in U$, which is a disjoint union of $p \in \mathbb{N}$ connected attractors. Then, for the system $\{\varphi_i : (\,\cdot\,, 1)\colon X \supset U \multimap X,\ i = 1, \ldots, n\}$, there exists an (essential) fractal $K_1 \subset N$ such that $I_X(K_1, F_1) = 2^{p-1}$, where $F_1(x) := \bigcup_{i=1}^{n} \varphi_i(x, 1)$, $x \in U$.*

It was shown that we can sometimes obtain more information about the set of (multivalued) fractals, namely that we can recognize its topological structure.

The employed method there has been based on the Browder–Gupta type result (comp. Chapter VI). Now, we propose the second technique which uses inverse systems and multivalued inverse systems.

Here, we start with suitable definitions and preliminary results (comp. Chapter I).

Let Σ be a set ordered by the relation $\leq$, and, for every $\alpha \in \Sigma$, let X_α be a metric space (or, more generally, a Hausdorff space).

Assume that, for each $\alpha, \beta \in \Sigma$, $\alpha \leq \beta$, there is a multivalued u.s.c. map $\pi_\alpha^\beta \colon X_\beta \multimap X_\alpha$ with nonempty, compact values such that

(85.42.1) $\pi_\alpha^\alpha = \mathrm{id}_{X_\alpha}$, for every $\alpha \in \Sigma$;
(85.42.2) $\pi_\alpha^\beta \pi_\beta^\gamma = \pi_\alpha^\gamma$, for every $\alpha, \beta, \gamma \in \Sigma$, $\alpha \leq \beta \leq \gamma$.

Then the family

(85.42.3) $$S = \{X_\alpha, \pi_\alpha^\beta, \Sigma\}$$

is called a *multivalued inverse system* (cf. [Eng-M], for the corresponding notion of a (single-valued) inverse system).

(85.43) EXAMPLE. Every inverse system $\{X_\alpha, \pi_\alpha^\beta, \Sigma\}$ is obviously a multivalued inverse system, because each π_α^β is continuous, and so u.s.c. with compact values.

We define, in the product $\prod_{\alpha\in\Sigma} X_\alpha$, the following subset:

$$\varprojlim S := \Big\{ (x_\alpha) \in \prod_{\alpha\in\Sigma} X_\alpha \;\Big|\; x_\alpha \in \pi_\alpha^\beta(x_\beta) \text{ for all } \alpha \leq \beta \Big\},$$

and call it the *limit of a multivalued inverse system S*.

Let us observe that the projections

$$\pi_\alpha = p_\alpha|_{\varprojlim S} \colon \varprojlim S \to X_\alpha, \quad \pi_\alpha((x_\alpha)) = x_\alpha$$

satisfy the following condition:

$$\pi_\alpha((x_\alpha)) \in \pi_\alpha^\beta(\pi_\beta((x_\alpha))), \quad \text{for any } \alpha \leq \beta.$$

We can prove the following preliminary result.

(85.44) PROPOSITION. *The set $\varprojlim S$ is closed in $\prod X_\alpha$.*

In the proof, we shall use

(85.45) Lemma. *Let φ be an u.s.c. multivalued map from X to a (Hausdorff) space Y with compact values and $f\colon X \to Y$ be continuous. Then $\{x \in X \mid f(x) \in \varphi(x)\}$ is closed.*

Proof. We show that $\{x \in X \mid f(x) \notin \varphi(x)\}$ is open. Let x be such that $f(x) \notin \varphi(x)$. Then there are open sets $U \ni f(x)$ and $V \supset \varphi(x)$ such that $U \cap V = \emptyset$. Take $f^{-1}(U) \cap \varphi^{-1}(V) =: W$, where $\varphi^{-1}(V) := \{x \in X \mid \varphi(x) \subset V\}$. By the continuity of f and upper semicontinuity of φ, we know that W is an open neighbourhood of x with $f(z) \notin \varphi(z)$, for any $z \in W$. □

Proof of Proposition (85.44). Letting

$$M_{\gamma\delta} := \left\{(x_\alpha) \in \prod X_\alpha \;\middle|\; x_\gamma \in \pi_\gamma^\delta(x_\delta)\right\}, \quad \text{for } \gamma \le \delta,$$

then

$$M_{\gamma\delta} = \left\{x = (x_\alpha) \in \prod X_\alpha \;\middle|\; \pi_\gamma(x) \in \pi_\gamma^\delta \pi_\delta(x)\right\}.$$

Since projections π_α are continuous and maps π_α^β are u.s.c. with compact values, we can apply Lemma (85.45) to obtain that each set $M_{\gamma\delta}$ is closed.

Obviously, $\varprojlim S = \bigcap_{\gamma \le \sigma} M_{\gamma\sigma}$, so the limit is closed, too. □

One immediately obtains

(85.46) Proposition.

(85.46.1) *If X_α is compact, for every $\alpha \in \Sigma$, then $\varprojlim S$ is compact.*

(85.46.2) *If X_α is compact and nonempty, for every $\alpha \in \Sigma$, then so is $\varprojlim S$.*

(85.46.3) *If $S = \{X_n, \pi_n^p, \mathbb{N}\}$, the bonding maps π_n^p are single-valued and all X_n are compact R_δ-spaces, then $\varprojlim S$ is R_δ.*

Proof. Property (85.46.1) follows from the Tikhonov theorem which implies that $\prod_{\alpha\in\Sigma} X_\alpha$ is compact.

To prove (85.46.2), consider $M_\sigma = \bigcap_{\gamma\le\sigma} M_{\gamma\sigma}$ (see the previous proof) which is nonempty. Indeed, for every x_σ, we can take a fiber

$$\begin{cases} x_\gamma \in \pi_\gamma^\sigma(x_\sigma) & \text{for } \gamma \le \sigma, \\ x_\gamma \text{ — an arbitrary point in } X_\gamma & \text{for } \sigma \le \gamma,\ \sigma \ne \gamma. \end{cases}$$

Since the family $\{M_\sigma\}$ is centered and $\prod X_\alpha$ is compact, it follows by means of the Tikhonov theorem that $\bigcap_{\sigma\in\Sigma} M_\sigma \ne \emptyset$.

Since $\varprojlim S = \bigcap_{\sigma\in\Sigma} M_\sigma$, the proof of (85.46.2) is complete.

For (85.46.3), it is sufficient to notice that $M_p = \bigcap_{n\le p} M_{np}$ is homeomorphic to the R_δ-set $\prod_{i=p}^\infty X_i$, consequently, it follows that $\varprojlim S$ is an R_δ-set as well (comp. [AnGo-M]). □

Note that acyclicity is also inherited by the limit of an inverse system (see [Eng-M]). For more properties and details concerning multivalued inverse systems, see [AnGo-M].

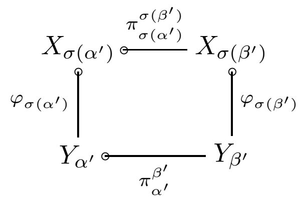
Consider two multivalued inverse systems

$$S = \{X_\alpha, \pi_\alpha^\beta, \Sigma\} \quad \text{and} \quad S' = \{Y_{\alpha'}, \pi_{\alpha'}^{\beta'}, \Sigma'\}.$$

By a multivalued map of S to S', we mean a family $\{\sigma, \varphi_{\sigma(\alpha')}\}$ consisting of a monotone function $\sigma\colon \Sigma' \to \Sigma$ (i.e. $\sigma(\alpha') \le \sigma(\beta')$, for any $\alpha' \le \beta'$) and multivalued maps $\varphi_{\sigma(\alpha')}\colon X_{\sigma(\alpha')} \multimap Y_{\alpha'}$, with the property

$$\pi_{\alpha'}^{\beta'} \varphi_{\sigma(\beta')} = \varphi_{\sigma(\alpha')} \pi_{\sigma(\alpha')}^{\sigma(\beta')} \quad \text{for any } \alpha' \le \beta'.$$

It means that the following diagram commutes



Now, let us concentrate on countable inverse systems, i.e. $\Sigma = \mathbb{N}$. For a given map $\{\mathrm{id}, \varphi_n\}\colon S \multimap S'$, we define a *limit map* $\varphi\colon \varprojlim S \multimap \varprojlim S'$ as follows:

$$\varphi(x) = \prod_{n=1}^{\infty} \varphi_n(x_n) \cap \varprojlim S' \quad \text{for every } x = (x_n) \in \varprojlim S.$$

This multivalued map has nonempty values. Indeed, let $y_1 \in \varphi_1(x_1)$. Then $y_1 \in \varphi_1 \pi_1^2(x_2) = \pi_1^2 \varphi_2(x_2)$, so there is $y_2 \in \varphi_2(x_2)$ such that $y_1 \in \pi_1^2(y_2)$. We proceed inductively and obtain a fiber in $\varphi(x)$.

Moreover, values of φ are compact, because each φ_n is compact valued.

Assume that $S = \{X_n, \pi_n^p, \mathbb{N}\}$ and $S' = \{Y_n, \rho_n^p, \mathbb{N}\}$ are two multivalued inverse systems. We define

$$\overline{\pi}_n^p\colon \mathcal{K}(X_p) \to \mathcal{K}(X_n), \quad \overline{\pi}_n^p(A) := \pi_n^p(A), \quad \text{for every } A \subset X_p,$$
$$\overline{\rho}_n^p\colon \mathcal{K}(Y_p) \to \mathcal{K}(Y_n), \quad \overline{\rho}_n^p(A) := \rho_n^p(A), \quad \text{for every } A \subset Y_p.$$

It is easy to see that $\overline{\pi}_n^p \circ \overline{\pi}_p^m = \overline{\pi}_n^m$, $\overline{\rho}_n^p \circ \overline{\rho}_p^m = \overline{\rho}_n^m$, and $\overline{\pi}_n^n = \mathrm{id}_{\mathcal{K}(X_n)}$, $\overline{\rho}_n^n = id_{\mathcal{K}(Y_n)}$. It means that S and S' induce (single-valued) inverse systems $\mathcal{K}(S) := \{\mathcal{K}(X_n), \overline{\pi}_n^p, \mathbb{N}\}$ and $\mathcal{K}(S') := \{\mathcal{K}(Y_n), \overline{\rho}_n^p, \mathbb{N}\}$.

Analogously, for a map $\{\mathrm{id}, \varphi_n\}: S \multimap S'$, we can define a map $\{\mathrm{id}, \overline{\varphi}_n\}: \mathcal{K}(S) \to \mathcal{K}(S')$, where $\overline{\varphi}_n: \mathcal{K}(X_n) \to \mathcal{K}(Y_n)$ is defined as $\overline{\varphi}(A) := \varphi(A)$, for every $A \in \mathcal{K}(X_n)$. We have $\overline{\rho}_n^p \circ \overline{\varphi}_p = \overline{\varphi}_n \circ \overline{\pi}_n^p$ which implies that $\{\mathrm{id}, \overline{\varphi}_n\}$ is indeed a map of inverse systems.

Now, $\{\mathrm{id}, \overline{\varphi}_n\}$ induces the limit map

$$\overline{\varphi}: \varprojlim \mathcal{K}(S) \to \varprojlim \mathcal{K}(S'), \quad \overline{\varphi}((A_n)) := (\overline{\varphi}_n(A_n)).$$

In order the above ideas to be connected with multivalued fractals, let us observe that, for any finite family of maps $\{\mathrm{id}, \varphi_n^i\}: S \multimap S'$, $i = 1, \ldots, k$, we have a sequence of multivalued maps $F_n: X_n \multimap Y_n$,

$$F_n(x) := \bigcup_{i=1}^{k} \varphi_n^i(x).$$

It is easy to check that $\rho_n^p \circ F_p = F_n \circ \pi_n^p$ which implies that $\{\mathrm{id}, F_n\}: S \multimap S'$ is a (multivalued) map of inverse systems. Therefore, it induces the map $\{\mathrm{id}, F_n^*\}: \mathcal{K}(S) \to \mathcal{K}(S')$, consisting of the Hutchinson–Barnsley operators and a limit map $F^*: \varprojlim \mathcal{K}(S) \to \varprojlim \mathcal{K}(S')$.

For studying the topological structure of the fixed-point set of the limit map F^*, we apply the following important observation.

(85.46) PROPOSITION (cf. [AnGo-M]). *Let $S = \{X_n, \pi_n^p, \mathbb{N}\}$ be a (multivalued) inverse system, and $\varphi: \varprojlim S \multimap \varprojlim S$ be a limit map induced by a map $\{\mathrm{id}, \varphi_n\}$. Then $\{\mathrm{Fix}(\varphi_n), \pi_n^p|_{\mathrm{Fix}(\varphi_n)}, \mathbb{N}\}$ forms a (multivalued) inverse system.*

Now, we are in a position to formulate a general principle for the topological structure of the sets of multivalued fractals. We keep the notation above.

(85.47) THEOREM. *Let $S = \{X_n, \pi_n^p, \mathbb{N}\}$ be a multivalued inverse system, and let $\{\mathrm{id}, \varphi_n^i\}: S \multimap S$, $i = 1, \ldots, k$, be a finite family of maps. If the sets of multivalued fractals, considered as compact invariant sets of $F_n: X_n \multimap X_n$ or, equivalently, as fixed-points of $F_n^*: \mathcal{K}(S) \to \mathcal{K}(S)$, are compact acyclic ($R_\delta$, nonempty), then the set of multivalued fractals of the limit map $F: \varprojlim S \multimap \varprojlim S$ induced by $\{\mathrm{id}, F_n\}$, considered again as compact invariant sets of $F: \varprojlim S \multimap \varprojlim S$ or, equivalently, as fixed-points of $F^*: \varprojlim \mathcal{K}(S) \to \varprojlim \mathcal{K}(S)$, is also compact acyclic (resp. R_δ, nonempty).*

The proof is a consequence of Propositions (85.46) and (85.47).

Finally, let us note that, by Proposition (85.47), if the sets $\mathrm{Fix}(F_n^*)$ are singletons, then the set of multivalued fractals of the limit map is also a singleton. It can bring an important information in differential problems on the half-line.

It is known that the Fréchet space of continuous maps $C := C([0,\infty),\mathbb{R}^n)$ can be treated as a limit of an inverse system of Banach spaces $C_m := C([0,m],\mathbb{R}^n)$, $m \in \mathbb{N}$, with the bonding maps $\pi_m^p(x) := x|_{[0,m]}$, for $x \in C_p$. It is easy to find an example of a family of contractions (even single-valued) $\varphi_m\colon C_m \to C_m$ with the same Lipschitz constant $L \in (0,1)$ such that the induced limit map φ on C is not a contraction. Then the map F^* induced by $\{\mathrm{id},\varphi_m\}$ is also not a contraction, but, by Theorem (85.48), the fixed-point set of F^* is a singleton.

In this section we presented most important, in our opinion, results obtained in [AFGL]. For further results we recommend again [AFGL]. Finally, let us describe that the theory of fractals is a modern topic which is actually very strong developing in many different directions.

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INDEX

Topological Fixed Point Theory and Its Applications

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