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To my parents Fathi and Mounira
To my wife Faten and our children Hadil, Hiba, and Youssef
and
To my brothers Imed, Aref, and my sister Alyssa

Afif Ben Amar

To my wife Alice and our children Aoife, Lorna, Daniel, and Niamh

Donal O'Regan

Preface

Fixed point theory is a powerful and fruitful tool in modern mathematics and may be considered as a core subject in nonlinear analysis. In the last 50 years, fixed point theory has been a flourishing area of research. In this book, we introduce topological fixed point theory for several classes of single- and multivalued maps. The selected topics reflect our particular interests.

The text is divided into seven chapters. In Chap. 1, we present basic notions in locally convex topological vector spaces. Special attention is devoted to weak compactness, in particular to the theorems of Eberlein–Šmulian, Grothendieck, and Dunford–Pettis. Leray–Schauder alternatives and eigenvalue problems for decomposable single-valued nonlinear weakly compact operators in Dunford–Pettis spaces are considered in Chap. 2. In Chap. 3, we present some variants of Schauder, Krasnoselskii, Sadovskii, and Leray–Schauder-type fixed point theorems for different classes of weakly sequentially continuous (resp. sequentially continuous) operators on general Banach spaces (resp. locally convex spaces). Sadovskii, Furi–Pera, and Krasnoselskii fixed point theorems and nonlinear Leray–Schauder alternatives in the framework of weak topologies and involving multivalued mappings with weakly sequentially closed graph are considered in Chap. 4. The results are formulated in terms of axiomatic measures of weak noncompactness. In Chap. 5, we present some fixed point theorems in a nonempty closed convex of any Banach algebras or Banach algebras satisfying a sequential condition ($\mathcal{P}$) for the sum and the product of nonlinear weakly sequentially continuous operators. We illustrate the theory by considering functional integral and partial differential equations. The existence of fixed points and nonlinear Leray–Schauder alternatives for different classes of nonlinear (*ws*)-compact operators (weakly condensing, 1-set weakly contractive, strictly quasi-bounded) defined on an unbounded closed convex subset of a Banach space is discussed in Chap. 6. We also discuss the existence of nonlinear eigenvalues and eigenvectors and surjectivity of quasi-bounded operators. In Chap. 7, we present some approximate fixed point theorems for multivalued mappings defined on Banach spaces. Weak and strong topologies play a role here and both bounded and unbounded regions are considered. A method is developed indicating how to use approximate fixed point theorems to prove the existence of approximate Nash equilibria for noncooperative games.

We hope the book will be of use to graduate students and theoretical and applied mathematicians who work in fixed point theory, integral equations, ordinary and partial differential equations, game theory, and other related areas.

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Contents

Chapter 1
Basic Concepts

In this chapter we discuss some concepts needed for the results presented in this book.

1.1 Topological Spaces: Some Fundamental Notions

Let X, Y be arbitrary sets. We use the standard notations $x \in X$ for "x is an element of X," $X \subset Y$ for "X is a subset of Y." The set of all subsets of X is denoted by $P(X)$. Let $\{X_i\}_{i \in I}$ be a family of sets. For the union of this family we use the notation $\bigcup_{i \in I} X_i$ and for intersection the notation $\bigcap_{i \in I} X_i$. If $I = \mathbb{N}$ we have a sequence of sets and we use respectively the notations $\bigcup_{n=1}^{\infty} X_n$ and $\bigcap_{n=1}^{\infty} X_n$. A mapping f of X into Y is denoted by $f : X \longrightarrow Y$. The domain of f is X and the image of X under f is called the range of f. For any $A \subset X$, we write $f(A)$ to denote the set $\{f(x) \ : \ x \in A\} \subset Y$. For any $B \subset Y, f^{-1}(B) = \{x \in X \ : f(x) \in B\}$. If $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are mappings, the composition mapping $x \longmapsto g(f(x))$ is denoted by $g \circ f$. We denote the empty set by $\emptyset$.

Definition 1.1. Let X be any nonempty. A subset τ of $\mathcal{P}(X)$ is said to be a topology on X if the following axioms are satisfied:

1. X and $\emptyset$ are members of τ,
2. the intersection of any two members of τ is a member of τ,
3. the union of any family of members of τ is again in τ

We say that the couple (X, τ) is a topological space. If τ is a topology on X the members of τ are then said to be τ-open subsets of X.

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Definition 1.2. Let (X, τ) be a topological space.

1. The closure of a subset A of X, denoted by $\overline{A}$ is the smallest closed subset containing A.
2. The interior of a subset A of X, denoted by A°, is the largest open subset of A.
3. The boundary of a subset A of X, denoted by ∂A, is the set $\overline{A} \setminus A^\circ$.
4. A subset D is dense in a subset A if $D \subseteq A \subseteq \overline{D}$.
5. A limit point or a cluster pointer or an accumulation point of a subset A is a point $x \in X$ such that each neighborhood of x contains at least one point of A distinct from x.
6. A subset A of X is compact if, for each open covering of A, there exists a finite subcovering. The set A is relatively compact if $\overline{A}$ is compact.
7. The space is locally compact if, for each $x \in X$, there is a neighborhood V_x of x such that $\overline{V_x}$ is compact.
8. A subset A of X is countably compact if, for each countable open covering of A, there is a finite subcovering.

Definition 1.3. A direct set is a nonempty set I with a relation $\leq$ such that

1. $\alpha \leq \alpha$ for all $\alpha \in I$,
2. if $\alpha \leq \beta$ and $\beta \leq \gamma$, then $\alpha \leq \gamma$,
3. for each pair α, β of elements of I, there is $\gamma_{\alpha,\beta}$ such that $\alpha \leq \gamma_{\alpha,\beta}$ and $\beta \leq \gamma_{\alpha,\beta}$.

Definition 1.4. Let (X, τ) be a topological space and I be a directed set. A function x from I into X is said to be a net in X. The expression $x(i)$ is usually denoted by x_i, and the net itself is denoted by $\{x_i\}_{i \in I}$. The set I is the index set for the net.

Definition 1.5. Let (X, τ) be a topological space. A net $\{x_i\}_{i\in I}$ is said to be convergent to a point $x_* \in X$ if for any neighborhood V of x_*, there exists an index $i_V \in I$ such that for any $i \in I$ satisfying $i_V \leq i$, we have that $x_i \in V$. If a net $\{x_i\}_{i\in I}$ is convergent to x_*, we write $\lim_{i\in I} x_i = x_*$

Remark 1.1. It is known that a subset A of X is closed, if and only if for any net $\{x_i\}_{i\in I}$ in A the condition $\lim_{i\in I} x_i = x_0$ implies $x_0 \in A$.

Definition 1.6. Let (X, τ_1), (Y, τ_2) be topological spaces and let $f : X \longrightarrow Y$ be a mapping. We say that f is continuous at a point $x \in X$, if for each τ_2-neighborhood V of $y = f(x)$, $f^{-1}(V)$ is a τ_1-neighborhood of x. If f is continuous at any $x \in X$, then in this case we say that f is continuous on X.

Definition 1.7. Let (X, τ) a topological space:

1. The space X is T_0 if for each pair of distinct points in X, at least one has a neighborhood not containing the other.
2. The space X is T_1 if, for each pair of distinct points in X, each has a neighborhood not containing the other.
3. The space is T_2 or Hausdorff or separated if, for each pair of distinct points x and y, there are disjoint neighborhoods V_x and V_y of x, and y, respectively.

4. The space is T_3 is regular if it is T_1 and, for each x and each closed subset F not containing x, there are disjoint open sets U and V such that $x \in U$ and $F \subset V$.
5. The space X is $T_{3\frac{1}{2}}$ or completely regular or Tychonoff if it is Hausdorff and, for each x and each closed F of X not containing x, there is a continuous function $\alpha : X \longrightarrow [0,1]$ such that $\alpha(x) = 0$ and $\alpha(y) = 1$ for each $y \in F$. In other words, X is completely regular if $C(X,[0,1])$ separates points from closed sets in X. Since singletons are closed in X, we deduce that $C(X,[0,1])$, also separates points in X.
6. The space is T_4 or normal if it is Hausdorff and, for each disjoint closed subsets $F_1, F_2 \subset X$, there are disjoint open subsets V_1 and V_2 such that $F_1 \subset V_1$ and $F_2 \subset V_2$.

Lemma 1.1 (Urysohn). *If F_1 and F_2 are disjoint closed sets in a normal space X, then there is a continuous function $\alpha \in C(X,[0,1])$ such that $\alpha = 0$ on F_1 while $\alpha = 1$ on F_2.*

Theorem 1.1 (Tietze's Extension). *If F is a closed subset of a normal space X, then each continuous function $\alpha \in C(F,[0,1])$ extends to a continuous function $\tilde{\alpha} \in C(X,[0,1])$ on all of X.*

Remark 1.2. From Urysohn's lemma, every normal space is completely regular. Thus, metric spaces and compact Hausdorff are completely regular.

Proposition 1.1. *Let X be a completely regular space. Let F_1, F_2 be disjoint subsets of X, with F_1 closed and F_2 compact. Then there exists a continuous function $\alpha : X \longrightarrow [0,1]$ such that $\alpha \equiv 0$ throughout F_1 and $\alpha \equiv 1$ throughout F_2.*

1.2 Normed Spaces and Banach Spaces

All linear spaces considered in this section are supposed to be over a field $\mathbb{K}$, which can be $\mathbb{R}$ or $\mathbb{C}$.

Definition 1.8. Given a linear space X and a topology τ on X. X is called a topological vector space if the following axioms are satisfied:

(1) $(x,y) \longrightarrow x + y$ is continuous on $X \times X$ into X.
(2) $(\lambda, x) \longrightarrow \lambda x$ is continuous on $\lambda \times X$ into X.

Remark 1.3. Note that we can extend the notion of Cauchy sequence, and therefore of completeness, to a topological vector space : a sequence x_n in a topological vector space is Cauchy if for neighborhood U of θ there exists N such that $x_m - x_n \in U$ for all $m,n \geq N$.

An important class of topological vector spaces is the class of normed vector spaces.

Definition 1.9. Let X be a linear space. A norm on X is a map $\|.\| : X \longrightarrow [0, \infty)$ such that

1. $\|x\| = 0 \iff x = 0 \qquad (x \in X)$,
2. $\|\lambda x\| = |\lambda| \, \|x\| \qquad (\lambda \in \mathbb{K}, x \in X)$,
3. $\|x + y\| \leq \|x\| + \|y\| \qquad (x, y \in X)$.

A linear space equipped with a norm is called a normed space.

Proposition 1.2. *Let $(X, \|.\|)$ be a normed space. Then the mapping*

$$d : X \times X \longrightarrow [0, \infty), (x, y) \longmapsto \|x - y\|$$

is a metric. We may thus speak of convergence, etc., in normed spaces.

Remark 1.4. Let $(X, \|.\|)$ be a normed space. The sets $B(\theta, 1) = \{x \in X : \|x\| < 1\}$ and $\overline{B}_1(\theta) = \{x \in X : \|x\| \leq 1\}$ are the open unit ball and the closed unit ball of X, respectively.

Definition 1.10. A normed space X is called a *Banach space* if the corresponding metric space is complete, i.e., every Cauchy sequence in X converges in X.

Now, we discuss some important properties of the first and second duals of a normed space.

Definition 1.11. The topological dual X^* of a normed space $(X, \|.\|)$ is a Banach space. The operator norm on X^* is also called the dual norm, also denoted by $\|.\|$. That is

$$\|\phi\| = \sup_{\|x\| \leq 1} |\phi(x)| = \sup_{\|x\| = 1} |\phi(x)| \,.$$

The topological dual of X' is called the second dual (or the double dual) of X and is denoted by X^{**}. The normed space X can be embedded isometrically in X^{**} in a natural way. Each $x \in X$ gives rise to a norm-continuous linear functional

$$\hat{x}(\phi) = \phi(x) \quad \text{for each } \phi \in X^*.$$

Lemma 1.2. *For each $x \in X$, we have $\|\hat{x}\| = \|x\| = \max_{\|\phi\| \leq 1} |\phi(x)|$, where $\|\hat{x}\|$ is the operator norm of $\hat{x}$ as a linear functional on the normed space X^*.*

Corollary 1.1. *The mapping $x \longmapsto \hat{x}$ from X into X^{**} is a linear isometry (a linear operator and an isometry), so X can be identified with a subspace $\hat{X}$ of X^{**}.*

When the linear isometry $x \longmapsto \hat{x}$ from a Banach space X into its double dual X^{**} is surjective, the Banach space is called reflexive. That is, we have the following definition.

Definition 1.12. A space X is called reflexive if $X = \hat{X} = X^{**}$.

1.3 Convex Sets

We start with some basic definitions and a few observations.

Definition 1.13. Let X be a linear space. A subset S of X is said to be **convex** if and only if $\lambda x + (1-\lambda)y \in S$ for every $x, y \in S$ and $\lambda \in [0, 1]$. That is, a convex set is one that contains all points on any "line segment" joining two of its members.

Lemma 1.3. *In any linear space*

1. *The sum of two convex sets is convex*
2. *Scalar multiples of convex sets are convex*
3. *A set S is convex if and only if $\alpha S + \beta S = (\alpha + \beta)S$ for all nonnegative scalars α and β.*
4. *The intersection of an arbitrary family of convex sets is convex.*
5. *In a topological vector space, both the interior and the closure of a convex set are convex.*

Definition 1.14. Let S be any set in a linear space X, and let $\mathcal{S}$ be the class of all convex subsets of X that contains S. We have $\mathcal{S} \neq \emptyset$ since $X \in \mathcal{S}$. Then, $\bigcap \mathcal{S}$ is a convex set in X which, obviously, contains S. Clearly, this set is the smallest (that is, $\supseteq -$ minimum) subset of X that contains S-it is called the **convex hull** of S and denoted by $co(S)$.

Remark 1.5. $S = co(S)$ iff S is convex.

Note

$$co(S) = \{\sum_{i=1}^{n} \lambda_i x_i : \lambda_i \geq 0 \text{ and } x_i \in S \text{ for all } i \leq n \text{ and } \sum_{i=1}^{n} \lambda_i = 1\}.$$

1.3.1 Cones

Definition 1.15. A nonempty subset C of linear space X is called a **convex cone** if it satisfies the following properties:

1. C is a convex set.
2. From $x \in C$ and $\lambda \geq 0$, it follows that $\lambda x \in C$.
3. From $x \in C$ and $-x \in C$, it follows that $x = \theta$

A cone can be characterized by 3) together with

$$x, y \in C \quad \text{and} \quad \lambda, \mu \geq 0 \text{ imply } \lambda x + \mu y \in C.$$

Examples 1.1. 1. The set $\mathbb{R}^n_+$ of all vectors $x = (\xi_1, \ldots, \xi_n)$ with nonnegative components is a cone in $\mathbb{R}^n$.

2. The set C_+ of all real continuous functions on $[a, b]$ with only nonnegative values is a cone in the space $C[a, b]$.

Remark 1.6. The set $C \subset l^p (1 \leq p < \infty)$, consisting of all sequences $(\xi_n)_{n\geq 1}$, such that for some $a > 0$

$$\sum_{n=1}^{\infty} |\xi_n|^p \leq a$$

is a convex set in l^p, but obviously, not a cone.

1.3.2 Ordered Vector Spaces

Definition 1.16. If a cone C is fixed in a linear space X, then an **order** can be introduced for certain pairs of vectors in X. Namely, if $x - y \in C$ for some $x, y \in X$ then we write $x \geq y$ or $y \leq x$ and say x is greater than or equal to y or y is smaller than or equal to x. The pair (X, C) is called an **ordered vector space** or a vector space **partially ordered** by the cone C. An element x is called **positive**, if $x \geq 0$ or, which means the same, if $x \in C$ holds. Moreover

$$C = \{x \in X : x \geq 0\}.$$

Remark 1.7. We consider the linear space $\mathbb{R}^2$ ordered by its first quadrant as the cone $C = \mathbb{R}^2_+$. Considering the vectors $x = (1, -1)$ and $y = (0, 2)$, neither the vector $x - y = (1, -3)$ nor $y - x = (-1, 3)$ is in C, so neither $x \geq y$ nor $x \leq y$ holds. An ordering in a linear space, generated by a cone, is always only a partial ordering.

It can be shown that the binary relation $\geq$ has the following properties:

1. $x \geq x \ \forall x \in X$ (reflexivity).
2. $x \geq y$ and $y \geq z$ imply $x \geq z$ (transitivity)
3. $x \geq y$ and $\alpha \geq 0, \alpha \geq 0, \alpha \in \mathbb{R}$, imply $\alpha x \geq \alpha y$.
4. $x_1 \geq y_1$ and $x_2 \geq y_2$ imply $x_1 + x_2 \geq y_1 + y_2$.

Example 1.1. In the real space $C[a, b]$ we define the natural order $x \geq y$ for two functions x and y by $x(t) \geq y(t), \forall t \in [a, b]$. Then $x \geq 0$ if and only if x is a nonnegative function in $[a, b]$. The corresponding cone is denoted by C_+.

1.3.3 Vector Lattices

Definition 1.17. An ordered vector space X is called a **vector lattice** or **linear lattice** or **Riesz space**, if for two arbitrary elements $x, y \in X$ there exist an element $z \in X$ with the following properties:

1. $x \leq z$ and $y \leq z$,
2. if $t \in X$ with $x \leq t$ and $y \leq t$, then $z \leq t$.

Such an element z is uniquely determined, is denoted by $x \vee y$, and is called the **supremum** of x and y (more precisely: supremum of the set consisting of the elements x and y)

In a vector lattice, there also exists the infimum for any x and y, which is denoted by $x \wedge y$.

Definition 1.18. A vector lattice in which every nonempty subset X that is order bounded from above has a supremum (equivalently, if every nonempty subset that is bounded from below has an infimum) is called a Dedekind or a K-space (Kantorovich space).

Example 1.2. The space $C[a, b]$ is a vector lattice.

Remark 1.8. For an arbitrary element x of a vector lattice X, the elements $x_+ = x \vee \theta$, $x_- = (-x) \vee \theta$ and $|x| = x_+ + x_-$ are called the positive part, negative part, and modulus of the element x, respectively. For every element $x \in X$ the three element $x_+, x_-, |x|$ are positive.

1.3.4 Ordered Normed Spaces

Definition 1.19. Let X be normed space with the norm $\|.\|$. A cone $X_+ \subset X$ is called a **solid**, if X_+ contains a ball (with positive radius), or equivalently, X_+ contains at least one interior point.

A cone X_+ is called **normal** if the norm in X is **semimonotonic**, i.e., there exists a constant $M > 0$ such that

$$0 \leq x \leq y \Longrightarrow \|x\| \leq M\|y.\|$$

A cone is called **regular** if every monotonically increasing sequence which is bounded above

$$x_1 \leq x_2 \leq \ldots \leq x_n \leq \ldots \leq z$$

is a Cauchy sequence in X. In a Banach space every closed regular cone is normal.

Examples 1.2. 1. The usual cones are solid in the space $\mathbb{R}$, $C[a,b]$, but in the spaces $L^p([a,b])$ and $l^p(1 \leq p < \infty)$ they are not solid.
2. The cones of the vectors with nonnegative components and the nonnegative functions in the spaces $\mathbb{R}^n$, c_0, l^p and L^p, respectively, are normal.
3. The cones in $\mathbb{R}^n$, l^p and L^p are regular.

1.3.5 Normed Vector Lattices and Banach Lattices

Definition 1.20. Let X be a vector lattice, which is a normed space at the same time. X is called a **normed lattice** or **normed vector lattice**, if the norm satisfies the condition

$$|x| \leq |y| \quad \text{implies} \quad \|x\| \leq \|y\| \quad \forall\, x, y \in X \quad \text{(monotonicity of the norm).}$$

A complete (with respect to the norm) normed lattice is called a **Banach lattice**.

Example 1.3. The spaces $C[a,b], L^p$ and l^p are Banach lattices.

Definition 1.21. Let S be a subset of a normed space X. The **closed convex hull** of S denoted by $\overline{co}(S)$, is defined as the smallest (that is, $\supseteq$ −minimum) closed and convex subset of X that contains S.

Let X be a normed space. Note

$$\overline{co}(S) := \bigcap \{A \in \mathcal{P}(X) : A \text{ is closed in } X, \text{ it is convex, and } S \subseteq A\}.$$

(Note, $\overline{co}(\emptyset) = \emptyset$.)

Clearly, we can view $\overline{co}(.)$ as a self-map on 2^X. Every closed and convex subset of X is a fixed point of this map, and $\overline{co}(S)$ is a closed and convex set for any $S \subseteq X$.

We have this following useful formula

Proposition 1.3. *Let X be a normed space. Then*

$$\overline{co}(S) = \overline{co(S)} \qquad \textit{for any } S \subseteq X.$$

Proof. Since $\overline{co(S)}$ is convex, it is a closed and convex subset of X that contains S, so $\overline{co}(S) \subseteq \overline{co(S)}$. The $\supseteq$ part follows from the fact that $\overline{co}(S)$ is a closed set in X that includes $co(S)$. ■

1.4 Locally Convex Vector Spaces

Definition 1.22. A seminorm on a linear space X is a map $p : X \longrightarrow [0, \infty)$ with the following properties:

1. $p(\lambda x) = |\lambda| p(x) \qquad (\lambda \in \mathbb{K}), x \in X.$
2. $p(x + y) \leq p(x) + p(y) \qquad (x, y \in X).$

Remark 1.9. If p is a seminorm on a linear space X then $F = \{x \in X : p(x) = 0\}$ is a linear subspace of X.

Definition 1.23. A linear space X is called **locally convex** if it is equipped with a family $\mathcal{P}$ of seminorms on X such that

$$\bigcap_{p \in \mathcal{P}} \{x \in X : p(x) = 0\} = \{\theta\}.$$

Example 1.4. Let X be a topological space, and let $\mathcal{C}(X)$ denote the vector space of all continuous functions on X. Let $\mathcal{K}$ be the collection of all compact subsets of X. For $K \in \mathcal{K}$, define

$$p_K(f) := \sup\{|f(x)| : x \in K\} \qquad (f \in \mathcal{C}(X)).$$

Then $\mathcal{C}(X)$ equipped with $(p_K)_{K \in \mathcal{K}}$ is a locally convex vector space.

Definition 1.24. Let X be a locally convex vector space. A subset U of X is defined as open if, for each $x_0 \in U$, there are $\epsilon > 0$ and $p_1, \ldots, p_n \in \mathcal{P}$ such that

$$\left\{x \in X : \max_{j=1,\ldots,n} p_j(x - x_0) < \epsilon\right\} \subset U.$$

Proposition 1.4. *Let X be a locally convex vector space. Then the collection of open subsets of X in Definition 1.24 is a topology on X.*

Proposition 1.5. *Let X be a locally convex vector space. Then a net $\{x_\alpha\}_\alpha$ in X converges to $x_0 \in X$ in the topology if and only if $p(x_\alpha - x_0) \longrightarrow 0$ for each $p \in \mathcal{P}$.*

Proof. Suppose that $x_\alpha \longrightarrow x_0$ in the topology. Fix $\epsilon > 0$ and $p \in \mathcal{P}$. Then $U := \{x \in X : p(x - x_0) < \epsilon\}$ is an open neighborhood of x_0. Hence, there is an index α_0 such that $x_\alpha \in U$, i.e., $p(x_\alpha - x_0) < \epsilon$ for all $\alpha \succ \alpha_0$. Hence $p(x_\alpha - x_0) \longrightarrow 0$.

Conversely, suppose that $p(x_\alpha - x_0) \longrightarrow 0$ for all $p \in \mathcal{P}$. Let U be a neighborhood of x_0, i.e., there is an open set $V \subset U$ with $x_0 \in V$. By Definition 1.24, there are $\epsilon > 0$ and $p_1, \ldots, p_n \in \mathcal{P}$ such that

$$\left\{x \in X : \max_{j=1,\ldots,n} p_j(x - x_0) < \epsilon\right\} \subset V.$$

Since $p_j(x_\alpha - x_0) \longrightarrow 0$ for $j = 1, \ldots, n$ there is an index α_0 such that

$$p_j(x_\alpha - x_0) < \epsilon \qquad (j = 1, \ldots, n, \alpha \succ \alpha_0).$$

This means, however, that $x_\alpha \in V \subset U$ for all $\alpha \succ \alpha_0$. ■

Remark 1.10. Let X be a locally convex vector space, and let Y be a finite dimensional subspace. Then, the relative topology on Y is induced by a norm.

Remark 1.11. A locally convex vector space is a topological vector space. In particular a locally convex vector space is completely regular.

Remark 1.12. Let X be locally convex vector space. If the number of seminorms is finite, we may add them to get a norm generating the same topology. If the number is countable, we may define a metric

$$d(x, y) = \sum_n 2^{-n} \frac{p_n(x-y)}{1 + p_n(x-y)},$$

so the topology is metrizable.

Definition 1.25. Let Y be a subset of a linear space X. A point $x_0 \in Y$ is said to be an **internal point** of provided for each $x \in X$, there is some $\lambda_0 > 0$ for which $x_0 + \lambda x$ belongs to Y if $|\lambda| \leq \lambda_0$.

Proposition 1.6. *Let X be a locally convex topological vector space.*

1. *A subset Y of X is open if and only if for each $x_0 \in X$ and $\lambda \neq 0$, $x_0 + Y$ and λY are open.*
2. *The closure of a convex subset of X is convex.*
3. *Every point is an open subset Y of X is an internal point of Y.*

Proposition 1.7. *Let X be a locally convex topological vector space and $\phi : X \longrightarrow \mathbb{R}$ be linear. Then ϕ is continuous if and only if there is neighborhood of the origin on which $|\phi|$ is bounded, that is, there is a neighborhood of the origin, Y, and an $M > 0$ for which*

$$|\phi| \leq M \ \text{ on } \ Y. \tag{1.1}$$

Proof. First suppose ϕ is continuous. Then it is continuous at $x = \theta$ and so, since $\phi(\theta) = \theta$, there is a neighborhood Y of θ such that $|\phi(x)| = |\phi(x) - \phi(\theta)| < 1$ for $x \in Y$. Thus $|\phi|$ is bounded on Y. To prove the converse, let Y be a neighborhood of θ and $M > 0$ be such (1.1) holds. For each $\lambda > 0, \lambda Y$ is also a neighborhood of θ and $|\phi| \leq \lambda M$ on λY. To verify the continuity of $\phi : X \longrightarrow \mathbb{R}$, let x_0 belong to X and $\epsilon > 0$. Choose λ so that $\lambda M < \epsilon$. Then $x_0 + \lambda Y$ is a neighborhood of x_0 and if x belongs to $x_0 + \lambda Y$, then $x - x_0$ belongs to λY so that

$$|\phi(x) - \phi(x_0)| = |\phi(x - x_0)| \leq \lambda M < \epsilon.$$

■

Theorem 1.2 (Analytic Form the Hahn–Banach Extension Theorem). *Let X be a linear space over $\mathbb{K}$ and p a semi-norm on X. Let X_0 be a linear subspace of X, and let f_0 be a linear functional on X_0 satisfying the relation*

$$|f_0(x)| \leq p(x) \quad \text{for all } x \in X_0. \tag{1.2}$$

Then there exists a linear functional f on X with the following properties:

$$f(x) = f_0(x) \quad \text{for all } x \in X_0, |f(x)| \leq p(x) \quad \text{for all } x \in X.$$

So, f is an extension of the functional f_0 onto the whole space X preserving the relation (1.2).

Remark 1.13. If X_0 is a linear subspace of a normed space X and f_0 is a continuous linear functional on X_0, then $p(x) = \|f_0\|\|x\|$ is a seminorm on X satisfying (1.2). Important consequences are

1. For every element $x \neq \theta$ there is a functional $f \in X^*$ with $f(x) = \|x\|$ and $\|f\| = 1$.
2. For every linear subspace $X_0 \subseteq X$ and $x_0 \notin X_0$ with the positive distance $d = \inf\limits_{x \in X_0} \|x - x_0\| > 0$ there is an $f \in X^*$ such that

$$f(x) = 0 \quad \text{for all } x \in X, f(x_0) = 1 \text{ and } \|f\| = \frac{1}{d}.$$

1.5 Weak and Weak* Topologies

To present some fixed point theory in a Banach spaces setting in this book we need to understand other topologies (different from the norm topology and weaker than it). The "weak topologies" arise naturally in this setting which is the subject of the present section.

Definition 1.26. Let X be a locally convex linear topological space and X^* is its topological dual. Then $\{p_\phi : \phi \in X^*\}$ with

$$p_\phi(x) = |\phi(x)| \qquad (x \in X, \phi \in X^*)$$

is a family of seminorms on X such that $\bigcap_{\phi \in X^*} \{x \in X : p_\phi(x) = 0\} = \{\theta\}$. The corresponding topology on X is called the weak topology on X and it is denoted by $\sigma(X, X^*)$.

Consequently, if X is a locally convex linear topological space and X^* is its topological dual, then $\sigma(X, X^*)$ the weak topology of X is a locally convex topology as well. Moreover, we have

Proposition 1.8. *Let X be a locally convex linear topological space with topological dual X^*. Then the dual of $(X, \sigma(X, X^*))$ is also X^*. That is, the dual space of X with respect to the $\sigma(X, X^*)$-topology is exactly X^*.*

If U is a weak neighborhood of θ of a locally convex linear topological space X then, by definition, there exists $\epsilon > 0$ and finitely many functionals $\varphi_n \in X^*$ such that $\{x : |\varphi_n(x)| < \epsilon \ \forall n\}$ is contained in U. Thus U contains the closed subspace $\ker(\varphi_1) \cap \ldots \cap \ker(\varphi_n)$.

Note that if C is any convex set in X then the closure of C is a closed convex set in X, and from the previous proposition, we obtain

Theorem 1.3 (Mazur). *Let X be a locally convex linear topological space, then:*

1. *The weak-closure (that is, the $\sigma(X, X^*)$-closure) of any convex set C in X coincides with the closure of C in the original topology of X.*
2. *The closed convex subsets of X and the weakly closed convex subsets of X are the same collections, that is, a convex set in X is weakly closed if and only if it is closed.*

Remark 1.14. In a normed linear space X, a convex set is norm closed if and only if it is weakly closed, while for a given linear subspace of X, its norm closure coincides with its weak closure.

On the dual space X^* we have two new topologies. We may endow it with the weak topology, the weakest one such that all functionals in X^{**} are continuous, or

Definition 1.27. Let X be a locally convex linear topological space and X^* is its topological dual. Then $\{p_x : x \in X\}$ with

$$p_x(\phi) = |\phi(x)| \qquad (x \in X, \phi \in X^*)$$

is a family of seminorms on X^* such that $\bigcap_{\phi \in X^*} \{x \in X : p_x(\phi) = 0\} = \{\theta\}$. The corresponding topology on X^* is called the weak* topology on X^* and it is denoted by $\sigma(X^*, X^{**})$.

Remark 1.15. 1. The weak* topology is weaker than the weak topology.
2. If X is reflexive, the weak and weak* topologies coincide.

1.6 Convergence and Compactness in Weak Topologies

We present the Eberlein–Šmulian criteria for weak compactness of subsets of a Banach space in this section.

Theorem 1.4. *If a sequence of elements of a Banach space converges weakly, then the sequence is norm bounded.*

Theorem 1.5. *If $x_n \rightharpoonup x$ in some Banach space, then* $\|x\| \leq \liminf_{n\longrightarrow\infty} \|x_n\|$.

Examples 1.3 (Weak and Weak Convergence).*

1. Consider weak convergence in $L^p(\Omega)$ where Ω is a bounded subset of $\mathbb{R}^n$. From the characterization of the dual of L^p we see that

$$f_n \xrightarrow{w^*} f \text{ in } L^\infty \Longrightarrow f_n \rightharpoonup f \text{ in } L^p \Longrightarrow f_n \rightharpoonup f \text{ in } L^q$$

whenever $1 \leq q \leq p < \infty$. In particular we claim that the complex exponentials $e^{2\pi inx} \xrightarrow{w^*} 0$ in $L^\infty([0,1])$ as $n \longrightarrow \infty$. This is simply the statement that

$$\lim_{n\longrightarrow\infty} \int_0^\infty g(x)e^{2\pi inx}dx = 0,$$

for all $g \in L^1([0,1])$, i.e., that the Fourier coefficients of an L^1 function tends to 0, which is known as the Riemann–Lebesgue Lemma. (Proof: Certainly true if g is a trigonometric polynomial. The trigonometric polynomials are dense in $C([0,1])$ by the Weirstrass Approximation Theorem, and $C([0,1])$ is dense in $L^1([0,1])$.) This is one common example of weak convergence which is not norm convergence, namely weak vanishing by oscillation.

2. Another common situation is weak vanishing to infinity. For a simple example, it is easy to see that the unit vectors in l_p converge weakly to zero for $1 < p < \infty$ (and weak* in l_∞, but not weakly in l_1). For another example let $f_n \in L^p(\mathbb{R})$ be a sequence of function which are uniformly bounded in L^p, and for which $f_{n|_{[-n,n]}} \equiv 0$. Then we claim that $f_n \longrightarrow 0$ weakly in L^p if $1 < p < \infty$. Thus we have to show that

$$\lim_{n\longrightarrow\infty} \int_{\mathbb{R}} f_n g dx = 0,$$

for all $g \in L^q$. Let $S_n = \{x \in \mathbb{R} \mid |x| \geq n\}$. Then $\lim_n \int_{S_n} |g|^q \, dx = 0$ (by the dominated convergence theorem). However

$$\left|\int_{\mathbb{R}} f_n g dx\right| = \left|\int_{S_n} f_n g dx\right| \leq \|f_n\|_{L^p}\|g\|_{L^q(S_n)} \leq C\|g\|_{L^q(S_n)} \longrightarrow 0.$$

The same proof shows that if the f_n are uniformly bounded they tend to 0 in L^∞ weak*. Note that the characteristic functions $\chi_{[n,n+1]}$ do not tend to zero weakly in L^1 however.

3. Consider the measure $\phi_n = 2n\chi_{[\frac{-1}{n},\frac{1}{n}]}dx$. Formally ϕ_n tends to the delta function δ_0 as $n \longrightarrow \infty$. Using the weak* topology on $C([-1,1])$ this convergence becomes: $\phi_n \xrightarrow{w^*} \delta_0$.

Theorem 1.6 (Alaoglu–Bourbaki Theorem). *Let X be a normed space. Then the closed unit ball of X^* is compact in the weak * topology on X^*.*

Proof. For each $x \in X$, let

$$K_x := \{\lambda \in \mathbb{K} : |\lambda| \leq \|x\|\}.$$

Since each K_x is closed and bounded, it is compact. By Tychonoff's theorem, $\prod_{x \in X} K_x$ is compact in the product topology. Embed the closed unit ball of X^* into $\prod_{x \in X} K_x$ via

$$B_1(\theta) \longrightarrow \prod_{x \in X} K_x, \qquad \phi \longmapsto (\phi(x))_{x \in X}.$$

Let $\{\phi_\alpha\}_\alpha$ be a net in the closed unit ball of X^*, we will show that it has a convergent subnet. The net $\{(\phi_\alpha(x))_{x \in X}\}_\alpha$ has a subnet $\{(\phi_\beta(x))_{x \in X}\}_\beta$ that converges in the product topology, i.e., for each $x \in X$, there is $\lambda_x \in K_x$ such that

$$\lambda_x = \lim_\beta \phi_\beta(x).$$

Define $\phi : X \longrightarrow \mathbb{K}$ by letting $\phi(x) := \lambda_x$ for $x \in X$. For $x, y \in X$ and $\mu \in \mathbb{K}$, we have

$$\phi(x+y) = \lambda_{x+y} = \lim_\beta \phi_\beta(x+y) = \lim_\beta \phi_\beta(x) + \lim_\beta \phi_\beta(y) = \lambda_x + \lambda_y = \phi(x) + \phi(y)$$

and

$$\phi(\mu x) = \lambda_{\mu x} = \lim_\beta \phi_\beta(\mu x) = \mu \lim_\beta \phi_\beta(x) = \mu \lambda_x = \mu \phi(x).$$

Hence, ϕ is linear. Moreover, note that, for $x \in X$ with $\|x\| \leq 1$

$$|\phi(x)| = |\lambda_x| \leq \|x\| \leq 1$$

because $\lambda_x \in K_x$. It follows that $\phi \in X^*$ lies in the closed unit ball. From the definition of the weak * topology, it is clear that the net $(\phi_\alpha)_\alpha$ converges to ϕ in the weak * topology of X^*. So, we obtain the weak * compactness of the closed unit ball of X^*. ■

Are all norm closed and bounded subsets of X weakly compact? The answer is no (in general).

Example 1.5. Consider the closed unit ball $B_1(\theta)$ of the Banach space c_0, a norm closed, bounded set. If it were weakly compact, every sequence in $B_1(\theta)$ would have to have a weak cluster point in $B_1(\theta)$. Now for each $n \in \mathbb{N}$, let e_n denote the nth-unit vector in $B_1(\theta)$ and consider the sequence $\{s_n\}_n \subseteq B_1(\theta)$ given by

$s_n = e_1 + e_2 + \ldots + e_n$ for each $n \in \mathbb{N}$. If $s \in B_1(\theta)$ were a weak cluster point of $\{s_n\}_n$, then for each $\phi \in (c_0)^*$, $\phi(s)$ would be a weaker cluster point of $\{\phi(s_n)\}_n$, that is, the values of $\phi(s_n)$ would be arbitrarily close to $\phi(s)$ infinitely often. But note, the value of $\phi(s_n)$, for any n, is a continuous linear functional on c_0, so let us denote it by ϕ_n. Of course, we have that $\phi_n(s_m) = 1$ for any $m \geq n$, so that $\phi_n(s)$ must have the value 1 for any n. That is, s must be the constant sequence of 1s, and hence not in c_0. Consequently, $B_1(\theta)$ is not weakly compact.

Proposition 1.9. *Let X be a normed linear space. Then the natural embedding $J : X \longrightarrow X^{**}$ is a topological homeomorphism between the locally convex topological vector spaces X and $J(X)$, where X has the weak topology and $J(X)$ has the weak * topology.*

Theorem 1.7 (Kakutani). *A Banach space is reflexive if and only if its closed unit ball is weakly compact.*

Corollary 1.2. *Every closed, bounded, convex subset of a reflexive Banach space is weakly compact.*

Proof. Let X be a Banach space. According to Kakutani's theorem, the closed unit ball of X is weakly compact. Hence so is any closed ball. According to Mazur's theorem, every closed, convex subset of X is weakly closed. Therefore any closed, convex, bounded subset of X is a weakly closed subset of a weakly compact set and hence must be weakly compact. ■

Definition 1.28. Let X be a topological space. A subset A of X is called

1. relatively compact if and only if A's closure is compact,
2. relatively sequentially compact if and only if every sequence of members of A contains a subsequence converging in X,
3. relatively countably compact if and only if every sequence of members of A has a cluster point in X.

Remark 1.16. In general, the concepts of relative compactness and relative sequential compactness are unrelated, but both, of course, imply relative countable compactness. As is well known, all three of these notions agree in metric spaces.

Theorem 1.8 (Eberlein–Šmulian Theorem). *Let X be a Banach space and $A \subseteq X$. Then the following assertions are equivalent:*

1. *A is relatively weakly compact,*
2. *A is relatively countably weakly compact,*
3. *A is relatively weakly sequentially compact.*

Remark 1.17. Let X be a Banach space. If $A \subseteq X$ is weakly compact, then A is bounded in X.

Corollary 1.3. *Let $B_1(\theta)$ be the closed unit ball of a Banach space X. Then $B_1(\theta)$ is weakly compact if and only if it is weakly sequentially compact.*

We now combine Kakutani's theorem and the Eberlein–Šmulian theorem.

Theorem 1.9 (Characterization of Weak Compactness). *Let $B_1(\theta)$ the closed unit ball of a Banach space X. Then the following three assertions are equivalent:*

1. *X is reflexive,*
2. *$B_1(\theta)$ is weakly compact,*
3. *$B_1(\theta)$ is weakly sequentially compact.*

Theorem 1.10 (Krein–Šmulian Theorem). *In a Banach space, the convex hull of a relatively weakly compact set is a relatively weakly compact set.*

1.7 Metrizability of Weak Topologies

We now establish some metrizable properties of weak topologies.

Theorem 1.11. *Let X be an infinite dimensional normed linear space. Then neither the weak topology on X nor the weak * topology on X^* is metrizable.*

Proof. To show that the weak topology on X is not metrizable, we argue by contradiction. Otherwise, there is a metric $\rho : X \times X \longrightarrow [0,\infty)$ that induces the weak topology on X. Fix a natural number n. Consider the weak neighborhood $\{x \in X : \rho(x,\theta) < \frac{1}{n}\}$ of θ. We may choose a finite subset F_n of X^* and $\epsilon_n > 0$ for which

$$\{x \in X : |\phi(x)| < \epsilon_n \text{ for all } \phi \in F_n\} \subseteq \{x \in X : \rho(x,\theta) < \frac{1}{n}\}.$$

Define W_n to be the linear space of F_n. Then

$$\cap_{\phi \in W_n} \ker\phi \subseteq \{x \in X : \rho(x,\theta) < \frac{1}{n}\}. \tag{1.3}$$

Since X is infinite dimensional, it follows from the Hahn–Banach theorem that X^* also is infinite dimensional. Choose $\phi_n \in X^* \sim W_n$. We infer that there is an $x_n \in X$ for which $\phi_n(x_n) \neq 0$ while $\phi(x_n) = 0$ for all $\phi \in F_n$. Define $u_n = \frac{n.u_n}{\|u_n\|}$. Observe that $\|u_n\| = n$ and by (1.3), that $\rho(x,\theta) < \frac{1}{n}$. Therefore $\{u_n\}$ is an unbounded sequence in X that converges weakly to θ. This contradicts the fact that every weakly compact subset of X is bounded. Therefore the weak topology is not metrizable.

To prove that the weak * topology on X^* is not metrizable, we once more argue by contradiction. Otherwise, there is a metric $\rho^* : X^* \times X^* \longrightarrow [0,\infty)$ that induces the weak * topology on X^*. Fix a natural number n. Consider the weak * neighborhood $\{\phi^* \in X^* : \rho^*(x,\theta) < \frac{1}{n}\}$ of θ. We may choose a finite subset A_n of X and $\epsilon_n > 0$ for which

$$\{\phi \in X^* : |\phi(x)| < \epsilon_n \text{ for all } x \in A_n\} \subseteq \{\phi \in X^* : \rho^*(\phi,\theta) < \frac{1}{n}\}.$$

Define X_n to be the linear span of A_n. Then

$$\{\phi \in X^* : \phi(x) = 0 \text{ for all } x \in X_n\} \subseteq \{\phi \in X^* : \rho^*(\phi, \theta) < \frac{1}{n}\}. \tag{1.4}$$

Since X_n is finite dimensional, it is closed and is a proper subspace of X since X is infinite dimensional. We know that there is a nonzero functional $\phi_n \in X^*$ which vanishes on X_n. Define $\varphi_n = \frac{n.\phi_n}{\|\phi_n\|}$. Observe that $\|\varphi_n\| = n$ and, by (1.4), that $\rho^*(\varphi_n, \theta) < \frac{1}{n}$. Therefore $\{\varphi_n\}$ is an unbounded sequence in X^* that converges pointwise to θ. This contradicts the Uniform Boundedness Theorem. Thus the weak * topology on X^* is not metrizable. ■

Finite dimensionality can be characterized in terms of weak topologies.

Theorem 1.12 (Finite Dimensional Spaces). *For a normed space X the following are equivalent.*

1. *The vector space X is finite dimensional.*
2. *The weak and norm topologies on X coincide.*
3. *The weak topology on X is metrizable.*
4. *The weak topology is countable.*

Corollary 1.4. *The weak interior of every closed or open ball in an infinite dimensional normed space is empty.*

Proof. Let X be an infinite dimensional normed space, and assume by way of contradiction that there exists a weak neighborhood W of zero and some $x \in B_1(\theta)$ such that $x + W \subseteq B_1(\theta)$. If $y \in W$, then $\|\frac{1}{2}y\| = \frac{1}{2}\|(x+y) - x\| \leq 1$, so $\frac{1}{2}W \subseteq B_1(\theta)$. This means that $B_1(\theta)$ is a weak neighborhood of zero, so (by Theorem 1.12) X is finite dimensional, a contradiction. Hence the closed unit ball $B_1(\theta)$ of X has an empty weak interior. ■

Corollary 1.5. *In any infinite dimensional normed space, the closed unit sphere is weakly dense in the closed unit ball.*

Theorem 1.13. *Let X be a normed linear space and W a separable subspace of X^* that separates points in X. Then the W-weak topology on the closed unit ball $B_1(\theta)$ of X is metrizable.*

Proof. Since W is separable, $B_1^*(\theta) \cap W$ also is separable, where $B_1^*(\theta)$ is the closed unit ball of X. Choose a countable dense subset $\{\phi_k\}_{k=1}^{\infty}$ of $B_1^*(\theta) \cap W$. Define $\rho : B_1(\theta) \times B_1(\theta) \longrightarrow \mathbb{R}$ by

$$\rho(x, y) = \sum_{k=1}^{\infty} \frac{1}{2^k} |\phi_k(x - y)| \quad \text{for all } x, y \in B_1(\theta).$$

This is properly defined since each ϕ_k belongs to $B_1^*(\theta)$. We first claim that ρ is a metric on $B_1(\theta)$. The symmetry and triangle inequality are inherited by ρ from the linearity of the $\phi_k's$. On the other hand, since W separates points in X, any dense subset of $B_1^*(\theta) \cap W$ also separates points in X. Therefore, for $x, y \in B_1(\theta)$ with $x \neq y$, there is a natural number k for which $\phi_k(x-y) \neq 0$ and therefore $\rho(x, y) > 0$. Thus ρ is a metric on $B_1(\theta)$. Observe that for each natural number n, since each ϕ_k belongs to $B_1^*(\theta)$,

$$\frac{1}{2^n}\left[\sum_{k=1}^{n} |\phi_k(z)|\right] \leq \rho(z, \theta) \leq \sum_{k=1}^{n} |\phi_k(z)| + \frac{1}{2^n} \quad \text{for all } z \in B_1(\theta).$$

We deduce from the previous inequalities and the denseness of $\{\phi_k\}_{k=1}^{\infty}$ in $B_1^*(\theta) \cap W$ that $\{z \in B_1(\theta) \mid \rho(z, \theta) < \frac{1}{n}\}_{n=1}^{\infty}$ is a base at the origin for the W-weak topology on B_1. Therefore the topology induced by the metric ρ is the W-weak topology on $B_1(\theta)$. ■

Corollary 1.6. *Let X be a normed space.*

1. *The weak topology on the closed unit ball of X is metrizable if X^* is separable.*
2. *The weak * topology on the closed unit ball $B_1^*(\theta)$ of X^* is metrizable if X is separable.*

Theorem 1.14. *Let X be a reflexive Banach space. Then the weak topology on the closed unit ball $B_1(\theta)$ is metrizable if and only if X is separable.*

Proof. Since X is reflexive, if X is separable, then X^* is separable. Therefore, by the previous corollary, if X is separable, then the weak topology on $B_1(\theta)$ is metrizable. Conversely, suppose the weak topology on $B_1(\theta)$ is metrizable. Let $\rho : B_1(\theta) \times B_1(\theta) \longrightarrow [0, \infty)$ be a metric that induces the weak topology on $B_1(\theta)$. Let n be a natural number. We may choose a finite subset F_n of X^* and $\epsilon_n > 0$ for which

$$\{x \in B_1(\theta) \ : \ |\phi(x)| < \epsilon_n \text{ for all } \phi \in F_n\} \subseteq \{x \in B_1(\theta) \ : \ \rho(x, \theta) < \frac{1}{n}\}.$$

Therefore

$$\left[\bigcap_{\phi \in F_n} \ker \phi\right] \cap B_1(\theta) \subseteq \{x \in B_1(\theta) \ : \ \rho(x, \theta) < \frac{1}{n}\}. \tag{1.5}$$

Define Z to be the closed linear span of $\bigcup_{n=1}^{\infty} F_n$. Then Z is separable since finite linear combinations, with rational coefficients, of the functional $\bigcup_{n=1}^{\infty} F_n$ is a countable dense subset of Z. We claim that $Z = X^*$. Otherwise, there is a nonzero $S \in (X^*)^*$, which vanishes of Z. Since X is reflexive, there is some $x_0 \in X$ for which $S = J(x_0)$. Thus $x_0 \neq \theta$ and $\phi_k(x_0) = 0$ for all k. According to (1.5), $\rho(x_0, \theta) < \frac{1}{n}$ for all n. Hence $x_0 \neq \theta$ but $\rho(x_0, \theta) = 0$. This is a contradiction. Therefore X^* is separable and so X also is separable. ■

1.8 Weak Compactness in $L^1(X,\mu)$: The Dunford–Pettis Theorem

For a measure space $(X,\mathcal{M},\mu)$, in general, the Banach space $L^1(X,\mu)$ is not reflexive, in which case, according to the Eberlein–Šmulian theorem, there are bounded sequences in $L^1(X,\mu)$ that fail to have weakly convergent subsequences.

As we see in the following example, for $[a,b]$ a nondegenerate closed, bounded interval, a bounded sequence in $L^1[a,b]$ may fail to have a weakly convergent subsequence.

Example 1.6. For $I = [0,1]$ and a natural n, define $I_n = [0,\frac{1}{n}]$ and $f_n = n.\chi_{I_n}$. Then $\{f_n\}$ is a bounded sequence in $L^1[0,1]$ since $\|f_n\| = 1$ for all n. We claim that $\{f_n\}$ fails to have a subsequence that converges weakly in $L^1[0,1]$. Indeed, suppose otherwise. Then there is a subsequence $\{f_{n_k}\}$ that converges weakly in $L^1[0,1]$ to $f \in L^1[0,1]$. For each $[c,d] \subseteq [0,1]$, integration against $\chi_{[c,d]}$ is a bounded linear functional on $L^1[0,1]$. Thus

$$\int_c^d f = \lim_{k\longrightarrow\infty} \int_c^d f_{n_k}.$$

Therefore

$$\int_c^d f = 0 \quad \text{for all } 0 < c < d \leq 1.$$

It follows that $f = 0$ almost everywhere on $[0,1]$. Therefore

$$0 = \int_0^1 f = \lim_{k\longrightarrow\infty} \int_0^1 f_{n_k} = 1.$$

This contradiction shows that $\{f_n\}$ has no weakly convergent subsequence.

Definition 1.29. Let $(X,\mathcal{M},\mu)$ a measure space. A subset $\mathcal{G}$ of $L^1(X,\mu)$ is said to be **uniformly integrable** (equi-integrable) provided for each $\epsilon > 0$, there is a $\delta > 0$ such that for any measurable set E

$$\text{if } \mu(E) < \delta, \text{ then } \sup_{f\in\mathcal{G}} \int_E |f|\, d\mu < \epsilon.$$

For finite measure spaces, we have the following characterization of uniform integrability.

Proposition 1.10. *For a finite measure space $(X,\mathcal{M},\mu)$ and a subset $\mathcal{G}$ of $L^1(X,\mu)$, the following two properties are equivalent:*

1. *The subset $\mathcal{G}$ is uniformly integrable.*

2. *For each* $\epsilon > 0$*, there is an* $M > 0$ *such that*

$$\sup_{f \in \mathcal{G}} \int_{\{x \in X \mid |f(x)| \geq M\}} |f| < \epsilon.$$

Remark 1.18. For a finite measure space $(X, \mathcal{M}, \mu)$, a subset $\mathcal{G}$ of $L^1(X, \mu)$ is uniformly integrable if

$$\sup_{f \in \mathcal{G}} \int_{\{x \in X \mid |f(x)| \geq M\}} |f| \, d\mu \longrightarrow 0 \text{ as } M \longrightarrow \infty.$$

This means that all of the elements of $\mathcal{G}$ can be truncated at height M with uniform error (in the L^1 norm).

Examples 1.4. 1. Any finite subset $\mathcal{G} = \{f_1, \ldots, f_n\}, n \geq 1$ is uniformly integrable. In fact, for each $i = 1 \ldots n$, from Chebyshev's inequality we have $\mu(\{x \in X \mid |f(x)| \geq M\}) \searrow 0$ as $M \nearrow \infty$. Thus, since for each $i = 1 \ldots n$, $E \longmapsto \int_E |f_i| \, d\mu$ is absolutely continuous with respect to μ, we get

$$\sup_{1 \leq i \leq n} \int_{\{x \in X \mid |f_i(x)| \geq M\}} |f| \, d\mu \longrightarrow 0 \text{ as } M \longrightarrow \infty.$$

2. If there exists an element $g \in L^1(X, \mu)$ such that $|f| \leq g$ for all $f \in \mathcal{G}$, then $\mathcal{G}$ is uniformly integrable.

Remark 1.19. For a finite measure space $(X, \mathcal{M}, \mu)$, if a subset $\mathcal{G}$ of $L^1(X, \mu)$ is uniformly integrable, then it is norm bounded. Indeed, if we choose $M \in \mathbb{R}$ such that $\int_{\{x \in X \mid |f(x)| \geq M\}} |f| \, d\mu \leq 1$ for all $f \in \mathcal{G}$, then $\|f\|_1 = \int_X |f| \, d\mu \leq M\mu(X) + 1$ for all $f \in \mathcal{G}$.

Proposition 1.11. *Let* $(X, \mathcal{M}, \mu)$ *be a finite measure space and* $\{f_n\}$ *be a sequence in* $L^1(X, \mu)$ *such that* $\lim\limits_{n \longrightarrow \infty} \int_E f_n d\mu$ *exists in* $\mathbb{R}$ *for every* $E \in \mathcal{M}$*. Then*

1. $\{f_n\}$ *is uniformly integrable and*
2. $\{f_n\}$ *converges weakly to some* $f \in L^1(X, \mu)$*, in particular* $\int_E f_n d\mu \longrightarrow \int_E f d\mu$ *for every* $E \in \mathcal{M}$*.*

Theorem 1.15 (Vitali–Hahn–Saks). *Let* $\{\mu_n\}$ *be a sequence of signed measures on a* σ*-algebra* $\mathcal{M}$ *such that* $\mu(E) = \lim\limits_{n \longrightarrow \infty} \mu_n(E)$ *exists in* $\mathbb{R}$ *for each* $E \in \mathcal{M}$*. Then* μ *is a signed measure on* $\mathcal{M}$*.*

Lemma 1.4. *For a finite measure space* $(X, \mathcal{M}, \mu)$ *and bounded uniformly integrable sequence* $\{f_n\} \in L^1(X, \mu)$*, there is a subsequence* $\{f_{n_k}\}$ *such that for each measurable subset E of X,*

$$\left\{ \int_E f_{n_k} d\mu \right\} \text{ is Cauchy.}$$

Theorem 1.16 (The Dunford–Pettis Theorem). *For a finite measure space $(X, \mathcal{M}, \mu)$ and bounded sequence $\{f_n\}$ in $L^1(X, \mu)$, the following two properties are equivalent:*

1. *$\{f_n\}$ is uniformly integrable over X.*
2. *Every subsequence of $\{f_n\}$ has a further subsequence that converges weakly in $L^1(X, \mu)$.*

Proof. First assume 1. It suffices to show that $\{f_n\}$ has a subsequence that converges weakly in $L^1(X, \mu)$. Without loss of generality, by considering positive and negative parts, we assume that each f_n is nonnegative. According to the preceding lemma, there is a subsequence of $\{f_n\}$ which we denote by $\{h_n\}$, such that for each measurable subset E of X,

$$\left\{ \int_E h_n d\mu \right\} \text{ is Cauchy.}$$

For each n, define the set function ν_n on $\mathcal{M}$ by

$$\nu_n(E) = \int_E h_n d\mu \quad \text{for all} \quad E \in \mathcal{M}.$$

Then, by the countable additivity over domains of integration, ν_n is a measure and it is absolutely continuous with respect to μ. Moreover, for each $E \in \mathcal{M}$, $\{\nu_n(E)\}$ is Cauchy. The real numbers are complete and hence we may define a real-valued set function ν on $\mathcal{M}$ by

$$\lim_{n \longrightarrow \infty} \nu_n(E) = \nu(E) \text{ for all } E \in \mathcal{M}.$$

Since $\{h_n\}$ is bounded in $L^1(X, \mu)$, the sequence $\{\nu_n(X)\}$ is bounded. Therefore, the Vitali–Hahn–Saks theorem tells us that ν is a measure on $(X, \mathcal{M})$ that is absolutely continuous with respect to μ. According to the Radon–Nikodym theorem, there is a function $f \in L^1(X, \mu)$ for which

$$\nu_n(E) = \int_E f d\mu \quad \text{for all} \quad E \in \mathcal{M}.$$

Since

$$\lim_{n \longrightarrow \infty} \int_E f_n d\mu = \int_E f d\mu \quad \text{for all } E \in \mathcal{M},$$

$$\lim_{n \longrightarrow \infty} \int_X f_n . \varphi d\mu = \int_X f . \varphi d\mu \quad \text{for every simple function} \quad \varphi.$$

By assumption, $\{f_n\}$ is bounded in $L^1(X,\mu)$. Furthermore, by a simple approximation lemma, the simple functions are dense in $L^\infty(X,\mu)$. Hence

$$\lim_{n\longrightarrow\infty}\int_X f_n.gd\mu = \int_X f.gd\mu \quad \text{for all } g \in L^\infty(X,\mu),$$

that is, $\{f_n\}$ converges weakly in $L^1(X,\mu)$ to f.

It remains to show that 2 implies 1. We argue by contradiction. Suppose $\{f_n\}$ satisfies 2 but fails to be uniformly integrable. Then there is an $\epsilon > 0$, a subsequence $\{h_n\}$ of $\{f_n\}$, and a sequence $\{E_n\}$ of measurable sets for which

$$\lim_{n\longrightarrow\infty}\nu_n(E_n) = 0 \;\text{ but }\; \int_{E_n} h_n d\mu \geq 0 \;\text{ for all } n. \tag{1.6}$$

By assumption 2 we may assume that $\{h_n\}$ converges weakly in $L^1(X,\mu)$ to h. For each n, define the measure ν_n on $\mathcal{M}$ by

$$\nu_n(E) = \int_E f d\mu \;\text{ for all }\; E \in \mathcal{M}.$$

Then each ν_n is absolutely continuous with respect to μ and the weak convergence in $L^1(X,\mu)$ of $\{h_n\}$ to h implies that

$$\{\nu_n(E)\} \text{ is Cauchy for all } E \in \mathcal{M}.$$

But the Vitali–Hahn–Saks theorem tells us that $\{\nu_n(E)\}$ is uniformly absolutely continuous with respect to μ and this contradicts (1.6). Therefore 2 implies 1 and the proof is complete. ■

Theorem 1.17. *For a finite measure space $(X,\mathcal{M},\mu)$, a subset $\mathcal{G}$ of $L^1(X,\mu)$ is relatively weakly compact if and only if it is uniformly integrable.*

Corollary 1.7. *Let $(X,\mathcal{M},\mu)$ be a finite measure space, $\mathcal{G}$ be a subset of $L^1(X,\mu)$ and a function $g \in L^1(X,\mu)$ such that*

$$|f| \leq g \text{ a.e. on } X \text{ for all } f \in \mathcal{G}.$$

Then $\mathcal{G}$ is relatively weakly compact.

Corollary 1.8. *Let $(X,\mathcal{M},\mu)$ be a finite measure space and $\{f_n\}$ a sequence in $L^1(X,\mu)$ that is dominated by the function $g \in L^1(X,\mu)$ in the sense that*

$$|f_n| \leq g \text{ a.e. on } X \text{ for all } n.$$

Then $\{f_n\}$ has a subsequence that converges weakly in $L^1(X,\mu)$.

Proof. The sequence $\{f_n\}$ is bounded in $L^1(X, \mu)$ and uniformly integrable. Apply the Dunford–Pettis theorem. ■

Corollary 1.9. *Let $(X, \mathcal{M}, \mu)$ be a finite measure space, $1 < p < \infty$, and $\{f_n\}$ a bounded sequence in $L^p(X, \mu)$. Then $\{f_n\}$ has a subsequence that converges weakly in $L^1(X, \mu)$.*

Proof. Since $\mu(X) < \infty$, we infer from Hölder's Inequality that $\{f_n\}$ is a bounded sequence in $L^1(X, \mu)$ and is uniformly integrable. Apply the Dunford–Pettis theorem. ■

1.9 The Dunford–Pettis Property

1.9.1 Weakly Compact Operators

Definition 1.30. Suppose that X and Y are Banach spaces. A linear operator T from X into Y is weakly compact if $T(D)$ is relatively weakly compact subset of Y whenever D is a bounded subset of X.

The collection of all weakly compact linear operators from X into Y is denoted by $\mathcal{W}(X, Y)$ or just $\mathcal{W}(X)$ if $X = Y$.

Proposition 1.12. *Every compact linear operator from a Banach space into a Banach space is weakly compact.*

Proposition 1.13. *Every weakly compact linear operator from a Banach space into a Banach space is bounded.*

The equivalence of the following characterization of weak compactness for linear operators is easily proved using elementary arguments and the Eberlein–Šmulian's theorem (see Theorem 1.8).

Proposition 1.14. *Suppose that T is a linear operator from a Banach space X into a Banach space Y. Then the following are equivalent*

- (i) *The operator T is weakly compact.*
- (ii) *The subset $T(B_1(\theta))$ is relatively weakly compact subset of Y.*
- (iii) *Every bounded sequence $\{x_n\}$ in X has a subsequence $\{x_{n_j}\}$ such that the sequence $\{Tx_{n_j}\}$ converges weakly.*

Remark 1.20. We have $\mathcal{W}(X, Y) \neq \emptyset$.

Proposition 1.15. *If X and Y are Banach spaces, then $\mathcal{W}(X, Y)$ is a closed subspace of $\mathcal{L}(X, Y)$.*

Proposition 1.16. *Suppose that X, Y and Z are Banach spaces that $T \in \mathcal{L}(X, Y)$ and that $S \in \mathcal{L}(Y, Z)$. If either T or S is weakly compact then ST is weakly compact.*

Proposition 1.17. *If X is a Banach space, then $\mathcal{W}(X)$ is a closed ideal in X.*

1.9.2 The Dunford–Pettis Property

Definition 1.31. Let X be a Banach space. We say that X has the **Dunford–Pettis property** if $\phi_n(x_n) \longrightarrow 0$ whenever $x_n \in X$ and $\phi_n \in X^*, n \in \mathbb{N}$, satisfy $x_n \rightharpoonup \theta$ in X and $\phi_n \rightharpoonup \theta$ in X^*.

A Dunford–Pettis space is a Banach space with the Dunford–Pettis property.

Examples 1.5. 1. Alexandre Grothendieck showed that every $C(K)$-space (for K compact and Hausdorff) has the Dunford–Pettis property, and further that given any Banach space X, anytime X^* has the Dunford–Pettis property, so does X.
2. l_1 has the Dunford–Pettis property. Indeed, if $x_n \rightharpoonup \theta$ in l_1, then $x_n \longrightarrow \theta$ by the Schur property of l_1. If, moreover, $\phi_n \in l_1^*$ are such that $\phi_n \rightharpoonup \theta$ in l_1^*, then $\sup \|\phi_n\| \leq C < \infty$ for some $C > 0$ and thus $|\phi_n(x_n)| \leq \|\phi_n\| \|x_n\| \leq C \|x_n\| \longrightarrow 0$. Since $c_0^* = l_1, c_0$ also has the Dunford–Pettis property.
3. Of course, by the classical Dunford–Pettis theorem, we know that every $L_1(\mu)$-space has the Dunford–Pettis property. You might also notice as $L_\infty(\mu)$ is isometrically a $C(K)$-space with $L_1(\mu)^* = L_\infty(\mu)$. Thus $L_1(\mu)$ has the Dunford–Pettis property.
4. Given any Banach space X, we let $C_0(X)$ denote the collection of all sequences i n X which converge to θ in norm, endowed with the supremum norm, and it can be shown that $C_0(X)$ has the Dunford–Pettis property.

Proposition 1.18. *Let X be a Banach space. Then the following are equivalent.*

(i) *X has the Dunford–Pettis property.*
(ii) *Every weakly compact operator from X into any Banach space maps weakly compact sets to norm compact sets.*

Proof. $(i) \Longrightarrow (ii)$: Assume that for some $\delta > 0$ and $x_n \rightharpoonup \theta$ we have $\|T(x_n)\| \geq \delta$ for all n. Let $\phi_n \in S_{Y^*}$ be such that $\phi(T(x_n))\|T(x_n)\|$ for all n. Since T^* is weakly compact, by eventual passing to a subsequence we may assume that for some $\phi \in X^*, T^*(\phi_n) \rightharpoonup \phi \in X^*$ in X^*. Since X has the Dunford–Pettis property, we have

$$0 = \lim(T^*(\phi_n) - \phi)(x_n) = \lim(\phi_n(T(x_n)) - \phi(x_n)) = \lim \|T(x_n)\|$$

as $\lim \phi(x_n) = 0$. This contradicts $\|T(x_n)\| \geq \delta > 0$ for all n.

$(ii) \Longrightarrow (i)$: Assume $x_n \rightharpoonup \theta$ in X and $\phi_n \rightharpoonup \theta$ in X^*. Define an operator $T : X \longrightarrow c_0$ by $T(x) = (\phi_1(x), \phi_2(x), \ldots)$. If e_n denotes the unit vector in l_1, then $T^*(e_n)(x) = e_n(T(x)) = \phi_n(x)$ for every n and every $x \in X$. Thus $T^*(e_n)$ is contained in the closed convex hull S of $\{\phi_n\}$ and so is $T^*(B_{l_1}(\theta))$. Since $\phi_n \rightharpoonup \theta, S$ is weakly compact by Krein's theorem. Thus T^* is a weakly compact operator and so is T by Gantmacher's theorem. Since $x_n \rightharpoonup \theta$, by (ii), $\|T(x_n)\| \longrightarrow 0$ and since $\{\phi_n\}$ is a bounded set in X^*, $|\phi_n(x_n)| \leq \sup_k |\phi_k(x_n)| \longrightarrow 0$ as $n \longrightarrow \infty$. Therefore (i) holds. ■

Remark 1.21. An infinite-dimensional reflexive Banach space cannot have the Dunford–Pettis property. Indeed, the identity mapping I_X on a reflexive Banach space X is weakly compact, so if X had the Dunford–Pettis property then $B_1(\theta)$ will be $\|.\|$-compact, and so $\dim X < \infty$, a contradiction.

1.10 Angelic Spaces

The term "angelic space" was introduced by Fremlin.

Definition 1.32. A topological space A is called a Fréchet–Urysohn space if for every $B \subset A$ and $x \in \overline{B}$ there is a sequence $\{x_n\} \subset B$ such that $x_n \longrightarrow x$.

Definition 1.33. A completely regular Hausdorff topological space A is called a g-space, if its relatively countably compact subsets are relatively compact.

Definition 1.34. A Hausdorff topological space X is said to be angelic space if for every relatively countably compact subset A of X the following two claims hold.

1. A is relatively compact.
2. If $b \in \overline{A}$, then there is a sequence in A that converges to b.

Obviously, if K is a compact topological space, K is a Fréchet–Urysohn space if and only if it is angelic. It can be said that a Hausdorff topological space X is angelic if and only if X is a g-space for which any compact subspace is a Fréchet–Urysohn space.

Example 1.7. Let $x_n = \sqrt{n}e_n \in l_2$, where e_n is the standard nth unit vector in l_2. Then $0 \in \overline{\{\sqrt{n}e_n\}}^w$. Let U the neighborhood of 0 given by vectors $x^1, x^2, \ldots, x^n \in l_2$ and $\varepsilon > 0$. Consider the element $y \in l_2$ defined by $y_i = \sum_{k=1}^{k=n} |x_i^k|$. Note that for an infinite number of indexes i we have $|y_i|^2 < \frac{\varepsilon}{i}$ since otherwise $y \notin l_2$. Therefore for an infinite number of indexes i we have $\left|\sqrt{i}e_i\right|(x^k) < \varepsilon$ for $k = 1, \ldots, n$, in particular $U \cap \{\sqrt{i}e_i\} \neq \emptyset$. Consequently there is no subsequence of $\{x_n\}$ that weakly converges to 0. Thus l_2 with its weak topology is not a Fréchet–Urysohn space.

Theorem 1.18. *If X is an angelic space, and $A \subset X$, then the following assertions are equivalent.*

1. *A is countably compact.*
2. *A is sequentially compact.*
3. *A is compact.*

Remark 1.22. In angelic spaces the classes of compact, countably compact, and sequentially compact sets coincide.

Remark 1.23. An important class of nonmetrizable spaces for which the equivalence also holds is provided by infinite dimensional Banach spaces endowed with their weak topology (the Eberlein–Šmulian theorem (Theorem 1.8)).

1.11 Normed Algebras

Definition 1.35. A vector space X over $\mathbb{K}$ is called an algebra, if a product $x.y \in X$ is also defined for every two elements $x, y \in X$, or with a simplified notation the product xy is defined so that for arbitrary $x, y, z \in X$ and $\alpha \in \mathbb{K}$ the following conditions are satisfied:

1. $x(yz) = (xy)z$,
2. $x(y + z) = xy + xz$,
3. $(x + y)z = xz + yz$,
4. $\alpha(xy) = (\alpha x)y = x(\alpha y)$.

An algebra X is said to be commutative if $xy = yx$ holds for two arbitrary elements x, y, and X is said to be unital if it possesses a (multiplicative) unit (this is also called an identity). Note that if X has an identity, then it is unique: since if e and e' are units, then $e = ee' = e'$.

Definition 1.36. An algebra X is called a **normed algebra** or **Banach algebra** if it is a normed linear space or a Banach space and the norm has the additional property

$$\|x.y\| \leq \|x\|\|y\|. \tag{1.7}$$

In a normed algebra all the above operations are continuous, i.e., additionally, if $x_n \longrightarrow x$ and $y_n \longrightarrow y$, then also $x_n y_n \longrightarrow xy$.

Remark 1.24. Every normed algebra can be completed to a Banach algebra, where the product is extended to the norm completion with respect to (1.7). Also, for any $x, x', y, y' \in X$, we have

$$\|xy - x'y'\| = \|x(y - y') + (x - x')y\| \leq \|x\|\|y - y'\| + \|x - x'\|\|y'\|$$

and so we see that

Remark 1.25. If e denotes the unit in the unital Banach algebra X, then $e = e^2$ and so we have $\|e\| \leq \|e\|\|e\|$, which implies that $\|e\| \geq 1$.

Lemma 1.5. *Let X be a Banach algebra with identity e. Then there is a norm $||| \,.\, |||$ on X, equivalent to the original norm, such that $(X, ||| \,.\, |||)$ is a unital Banach algebra with $||| \, e \, ||| = 1$.*

Examples 1.6. 1. Consider $C[0, 1]$, the Banach space of continuous complex-valued functions defined on the interval $[0, 1]$ equipped with the sup-norm namely, $\|f\| = \sup_{s\in[0,1]} |f(s)|$, and with multiplication defined point-wise

$$(fg)(s) = f(s)g(s), \quad \text{for } s \in [0, 1].$$

Then $C[0, 1]$ is a commutative unital Banach algebra, the constant function 1 is the unit element.

2. As above, but replace $[0, 1]$ by any compact topological space.
3. The linear space $W([0, 2\pi])$ of all complex-valued functions x continuous on $[0, 2\pi]$ and having an absolutely convergent Fourier series expansions, i.e.,

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{int},$$

with the norm $\|x\| = \sum_{n=-\infty}^{\infty} |c_n|$ and the usual multiplication.

4. The space $L(X)$ of all bounded linear operators on the normed space X with the operator norm and the usual algebraic operations, where the product TS of two operators is defined as the sequential application, i.e., $TS(x) = T(S(x)),\ x \in X$.
5. The space $L^1(-\infty, \infty)$ of all measurable and absolutely integrable functions on the real axis with the norm

$$\|x\| = \int_{-\infty}^{\infty} |x(t)|\, dt$$

is a Banach algebra if the multiplication is defined as the convolution $(x * y)(t) = \int_{-\infty}^{\infty} x(t-s)y(s)ds$.

6. Let D denote the closed unit disc in $\mathbb{C}$, and let X denote the set of continuous complex-valued functions on D which are analytic in the interior of D. Equip X with pointwise addition and multiplication and the norm

$$\|f\| = \sup\{|f(z)| : z \in \partial D\}$$

where ∂D is the boundary of D, that is, the unit circle. (That this is, indeed, a norm follows from the maximum modulus principle.) Then X is complete, and so is a (commutative) unital Banach algebra. X is called the disc algebra.

1.12 Measures of Weak Noncompactness

The theory of measures of weak noncompactness was introduced by De Blasi in [65]. This measure was used to establish existence results for weak solutions in a variety of settings (see [8, 20, 21, 26, 48, 49, 62, 86, 87, 127, 129, 159, 161]).

Definition 1.37. Let X be a Banach space and $\mathcal{B}$ the collection of all bounded sets of X. The measure of weak noncompactness $\beta : \mathcal{B} \longrightarrow \mathbf{R}_+$ is defined by

$$\beta(C) = \inf\{\varepsilon > 0 \ : \ \text{there exists a weakly compact set } D \text{ such that } C \subset D + \varepsilon B_1(\theta)\}.$$

Proposition 1.19. *If X is a Banach space and $\beta : \mathcal{B} \longrightarrow \mathbf{R}_+$ is the weak measure of noncompactness, then*

1. $\beta(C) = 0$ *if and only if* $\overline{C}^w$ *is weakly compact (regularity),*
2. $\beta(\lambda C) = |\lambda|\, \beta(C)$ *for all* $\lambda \in \mathbf{R}$ *and* $\beta(C_1 + C_2) \leq \beta(C_1) + \beta(C_2)$ *(seminorm),*
3. *if* $C_1 \subseteq C_2$*, then* $\beta(C_1) \leq \beta(C_2)$ *(monotonicity),*
4. $\beta(C_1) \cup C_2 = \max\{\beta(C_1), \beta(C_2)\}$ *(semi-additivity),*
5. $\beta(C) = \beta(\overline{C}^w)$,
6. $\beta(C) \leq diamC,$
7. $\beta(C) = \beta(coC)$.

Proposition 1.20. *If X is a Banach space and $B_1(\theta)$ denotes the unit closed ball of X, then*

1. *if X is reflexive, we have* $\beta(B_1(\theta)) = 0$,
2. *if X is nonreflexive, we have* $\beta(B_1(\theta)) = 1$.

Proposition 1.21. *If X is a Banach space and $C \subseteq X$ is bounded, then*

$$\beta(C + \lambda(B_1(\theta)) = \beta(C) + \lambda\beta(B_1(\theta)) \quad \textit{for all} \quad \lambda \geq 0.$$

Proposition 1.22. *If X is a Banach space, $\{C_n\}_{n\geq 1} \subset \mathcal{B}$ is a decreasing sequence of weakly closed sets in X and $\beta(C_n) \downarrow 0$ as $n \longrightarrow \infty$, then $\bigcap_{n\geq 1} C_n$ is nonempty and weakly compact.*

Proof. Let $x_n \in C_n, n \geq 1$. We have $\beta(\{x_n\}_{n\geq 1}) = \beta(\{x_n\}_{n\geq k}) \leq \beta(C_k)$ for all $k \geq 1$. Since $\beta(C_k) \downarrow 0$ as $k \to \infty$, we obtain $\beta(\{x_n\}_{n\geq 1}) = 0$ and so $\overline{\{x_n\}_{n\geq 1}}^w$ is weakly compact. From the Eberlein–Šmulian theorem (Theorem 1.8), we can find a subsequence $\{x_m\}_{m\geq 1}$ of $\{x_n\}_{n\geq 1}$ such that $x_n \rightharpoonup x$ in X as $m \to \infty$. Clearly $x \in C_n$ for all $n \geq 1$ (since all these sets are weakly closed) and so $x \in \bigcap_{n\geq 1} C_n \neq \emptyset$. Moreover, $\beta(\bigcap_{n\geq 1} C_n) \leq \beta(C_n)$ for all $n \geq 1$, hence $\beta(\bigcap_{n\geq 1} C_n) = 0$, which means that $\bigcap_{n\geq 1} C_n$ is weakly compact. ■

On the other hand it is rather difficult to express the De Blasi measure of weak noncompactness by a convenient formula. The first formula of this type was obtained by Appell and De Pascale in the Lebesgue space $L_1(a, b)$ [8]. This formula is very convenient and handy and based on the Dunford–Pettis theorem.

The De Blasi Measure of weak noncompactness β in $L_1(a,b)$ can be expressed by the formula

$$\beta(X) = \lim_{\varepsilon \to 0} \left\{ \sup_{x \in X} \left\{ \sup \left[\int_D |x(t)| dt : \ D \subset I, \ m(D) \leq \varepsilon \right] \right\} \right\}. \tag{1.8}$$

Also, using the concept of uniform integrability (equi-integrability), Banaŝ and Sadarangani [28] introduced a measure of weak noncompactness in the Lebesgue space $L_1(0,1)$ and showed that this measure is equal to the De Blasi measure of weak noncompactness.

Further assume that $a > 0$ is a fixed number. For an arbitrary function $x \in L_1(0,1)$ denote by $I(x,a)$ the set defined by

$$I(x,a) = \{t \in I \ : \ |x(t)| > a\}.$$

We introduce the function H defined on the family all bounded subsets of $L_1(0,1)$ by the formula

$$H(X) = \lim_{a \longrightarrow \infty} H_a(X),$$

where

$$H_a(X) = \left\{ \int_{I(x,a)} |x(t)| \, dt \ : \ x \in X \right\}.$$

Theorem 1.19. *$H(X) = \beta(X)$ for all bounded subsets X of $L_1(0,1)$.*

Now we describe some measures of weak noncompactness in the space $L^1(\mathbb{R}_+)$. First we recall the criterion for weak noncompactness due to Dieudonné [79].

Theorem 1.20. *A bounded set $X \subset L^1$ is relatively weakly compact if and only if*

1. *for any $\varepsilon > 0$ there is $\delta > 0$ such that if $|D| \leq \delta$ then $\int_D |x(t)| \, dt \leq \varepsilon, x \in X$*
2. *for any $\varepsilon > 0$ there is $T > 0$ such that $\int_T^\infty |x(t)| \, dt \leq \varepsilon$ for any $x \in X$.*

Further, take a nonempty subset X of L^1 and fix $\varepsilon > 0, x \in X$. Let

$$w(x,\varepsilon) = \sup \left[\int_D |x(t)| \, dt : D \subset \mathbb{R}_+, |D| \leq \varepsilon \right],$$

$$w(X,\varepsilon) = \sup[w(x,\varepsilon) : x \in X],$$

$$w_0(X) = \lim_{\varepsilon \longrightarrow 0} w(X,\varepsilon),$$

$$a(X) = \lim_{T \longrightarrow \infty} \left\{ \sup \left[\int_T^\infty |x(t)| \, dt : x \in X \right] \right\}$$

Finally, put

$$\gamma(X) = w_0(X) + a(X).$$

Lemma 1.6. *The function $\gamma(X)$ has the following properties:*

1. $\gamma(X) = 0 \iff X$ *is relatively weakly compact,*
2. $X \subset Y \Longrightarrow \gamma(X) \subset \gamma(Y)$,
3. $\gamma(X + Y) \leq \gamma(X) + \gamma(Y)$,
4. $\gamma(\lambda X) = |\lambda|\,\gamma(X)$ *for* $\lambda \in \mathbb{R}$,
5. $\gamma(coX) = \gamma(X)$, *where coX denotes the convex closure of the set X,*
6. $\gamma(X \cup Y) = \max[\gamma(X), \gamma(Y)]$,
7. $\gamma(B_r) = 2r$.

Proof. Note that the property 1) is a simple consequence of Theorem 1.20. The proof of properties 2)–6) is standard and follows from the definition of the function γ. In order to prove 7) it is enough to show that $\gamma(B_1) = 2$. Obviously $\gamma(B_1) \leq 2$. To show the converse inequality let us take the set $X \subset B_1, X = \{nP_{[n,n+1]} : n = 1, 2, \ldots\}$ where P_D will denote the characteristic function of a set D. It is easily seen that $w_0(X) = 1$ and $a(X) = 1$ so that $\gamma(B_1) \geq \gamma(X) = 2$. This completes the proof. ∎

Theorem 1.21. $\beta(X) \leq \gamma(X) \leq 2\beta(X)$.

Proof. Assume first that $\beta(X) = r$. Then for an arbitrary $\varepsilon > 0$ there is a weakly compact set Y such that $X \subset Y + (r + \varepsilon)B_1$. Hence, in view of the properties of the function γ described in Lemma 1.6 we get

$$\gamma(X) \leq \gamma(Y) + (r + \varepsilon)\gamma(B_1) = 2(r + \varepsilon)$$

which proves the right-hand side inequality.

Before proving the second inequality we introduce some auxiliary notations. Namely, for a fixed $b > 0$ let

$$\Omega(x, b) = [t \in \mathbb{R}_+ : |x(t)| > b],$$

$$\Omega_T(x, b) = [t \in [0, T] : |x(t)| > b].$$

Actually, $\Omega_T(x, b) \subset \Omega(x, b)$ for any $T > 0$. Further, for an arbitrary $x \in X$ we may write

$$x = xP_{[0,T]\setminus\Omega_T(x,b)} + xP_{\Omega_T(x,b)} + xP_{[T,\infty)}.$$

In what follows assume that $\varepsilon > 0$ and $T > 0$ are fixed and let $b \geq \frac{\sup[\|x\|\, x\in X]}{\varepsilon}$. Next, let

$$w^T(X,\varepsilon) = \sup[w(xP_{[0,T]},\varepsilon) : x \in X],$$
$$a^T(X) = \sup\left[\int_T^\infty |x(t)|\,dt : x \in X\right],$$
$$X_b^T = [xP_{[0,T]\setminus\Omega_T(x,b)} : x \in X],$$
$$B_T = [xP_{[0,T]} : x \in B_1],$$
$$B^T = [xP_{[0,\infty)} : x \in B_1],$$

and we see that the following inclusions hold

$$X \subset X_b^T + w^T(X,\varepsilon)B_T + a^T(X)B^T.$$

Hence, taking into account that the set X_b^T is relatively weakly compact, we arrive at the following inequality

$$\beta(X) \le w^T(X,\varepsilon) + a^T(X) \le w(X,\varepsilon) + a^T(X).$$

Passing with T to infinity we derive $\beta(X) \le \gamma(X)$. Thus the proof is complete. ■

1.13 The Superposition Operator

In this section we will denote by I an interval $[0,1]$, and by $L^1(I)$ the space of Lebesgue integrable functions (equivalence classes of functions) on I, with the standard norm $\|x\| = \int_0^1 |x(t)|\,dt$. Recall that by L^p we will denote the space of (equivalences classes of) functions x satisfying $\int_0^1 |x(t)|^p\,dt < \infty$.

Definition 1.38. Assume that $f(t,x) = f : I \times \mathbb{R} \to \mathbb{R}$ is a given function. For an arbitrary function $x : I \to \mathbb{R}$ denote by $\mathcal{N}_f x$ the function defined on I by the formula $(\mathcal{N}_f x)(t) = f(t,x(t))$. The operator $\mathcal{N}_f$ defined in such a way is said to be the *superposition operator* generated by the function f.

The first contribution to the theory of the superposition operator dates back to Carathéodory [56].

Definition 1.39. We say that the function $f = f(t,x)$ satisfies *Carathéodory conditions* if it is measurable in t for each $x \in \mathbb{R}$ and is continuous in x for almost all $t \in I$.

This theory received a new impetus after the fundamental paper of Krasnosel'skii [117] who showed a necessary and sufficient condition for the superposition operator to be continuous from the space L^p into L^q.

Theorem 1.22. *Let f satisfy the Carathéodory conditions. The superposition operator $\mathcal{N}_f$ generated by the function f maps continuously the space $L^p(I)$ into $L^q(I)$ $(p, q \geq 1)$ if and only if*

$$|f(t,x)| \leq a(t) + b\,|x|^{\frac{p}{q}} ,$$

for all $t \in I$ and $x \in \mathbb{R}$, where $a \in L^q(I)$ and $b \geq 0$.

Inspired by Krasnosel'skii's result some necessary and sufficient conditions were formulated which guarantee that the superposition operator is a continuous self-mapping of the space of continuous functions, the space of Hölder functions [45], the Orlicz space [177], the generalized Orlicz [173] space, and the Roumieu space [156].

The fundamental property of the superposition operator defined on the space L^1 is contained in the following theorem.

Theorem 1.23. *Assume that $f : I \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory conditions. Then the superposition operator $\mathcal{N}_f$ generated by f transforms the space L^1 into itself if and only if $|f(t,x)| \leq a(t) + b|x|$ for $t \in I$ and $x \in \mathbb{R}$, where $a(t)$ is a function from the space L^1 and b is a nonnegative constant. Moreover, the operator F is continuous on the space L^1.*

Remark 1.26. It should be noted that the superposition $\mathcal{N}_f$ takes its values in $L^\infty(I)$ if and only if the generating function f is independent on x (see [9]).

It is worthwhile mentioning that under the assumptions of the above theorem the superposition operator $\mathcal{N}_f$ need not be weakly sequentially continuous on the space L^1 or on a ball of L^1. Indeed this fact is a consequence of the following old result due to Shragin [186].

Theorem 1.24. *Assume that $f : I \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory conditions. Then the superposition operator $\mathcal{N}_f$ generated by f is weakly sequentially continuous on L^1 if and only if the generating function f has the form*

$$f(t,x) = a(t) + b(t)x ,$$

where $a \in L^1(I)$ and $b \in L^\infty(I)$.

1.14 Some Aspects of Continuity in L^1-Spaces

Let E be a Banach space with the norm $\|.\|$. For a sequence $\{x_n\} \subset E$ and $x \in E$ we write $x_n \longrightarrow x$ whenever the sequence $\{x_n\}$ converges to x (in the norm $\|.\|$). If $\{x_n\}$ converges weakly to x we will write that $x_n \rightharpoonup x$.

Definition 1.40. Let us assume that E_1, E_2 are Banach spaces and $X \subset E_1$ is a nonempty set. An operator $T : X \longrightarrow E_2$ is said to be weakly sequentially continuous on the set X if for every sequence $\{x_n\} \subset X$ and $x \in X$ such that $x_n \rightharpoonup x$ we have that $Tx_n \rightharpoonup Tx$. The operator T will be called demicontinuous provided $Tx_n \rightharpoonup Tx$ for $\{x_n\} \subset X$ and $x \in X$ such that $x_n \longrightarrow x$.

Obviously T is continuous if $x_n \longrightarrow x$ implies that $Tx_n \longrightarrow Tx$ for $\{x_n\} \subset X$ and $x \in X$.

Remark 1.27. Observe that every continuous operator is also demicontinuous. It is also easy to show that every weakly sequentially continuous operator is demicontinuous. In order to show that the converse implication is not true we give an example connected with the theory of the superposition operator.

Example 1.8. The superposition operator $\mathcal{N}_f$ transforms the space L^1 into itself and is continuous. Obviously $\mathcal{N}_f$ is also demicontinuous in this setting. On the other hand the operator $\mathcal{N}_f$ need not be weakly sequentially continuous which is a consequence of an old result due to Shragin [186] (see Theorem 1.24).

Let (a, b) be a given interval. For simplicity we assume that $(a, b) = (0, 1)$ and let $I = (0, 1)$. Let $S = S(I)$ denote the set of measurable (in Lebesgue sense) functions on I and let m stand for the Lebesgue measure in $\mathbb{R}$. The set S furnished with the metric

$$\varrho(x, y) = \inf[a + m\{s : |x(s) - y(s)| \geq a\} : a > 0]$$

becomes a complete metric space. Moreover, it is well known that the convergence generated by this metric coincides with convergence in measure. The compactness in such a space is called "compactness in measure" and such sets have very nice properties when considered as subsets of L^p-spaces of integrable functions ($p \geq 1$).

Lemma 1.7. *If a sequence $\{x_n\} \subset L^1$ and is compact in measure then this sequence converges in measure.*

Lemma 1.8. *A sequence $\{x_n\} \subset L^1$ converges in the norm of L^1 to a function $x \in L^1$ if and only if $\{x_n\}$ converges in measure to x and is weakly compact.*

In what follows let us suppose that X is a bounded subset of L^1 being compact in measure. Then we have the following results.

Theorem 1.25. *If $T : X \longrightarrow L^1$ is continuous then it is also weakly sequentially continuous.*

Proof. Fix arbitrarily a sequence $\{x_n\} \subset X$ being weakly convergent to $x \in X$. By assumption we have that the sequence $\{x_n\}$ is compact in measure. Hence and by Lemma 1.7 we deduce that $\{x_n\}$ converges in measure to x. On the other hand observe that the sequence $\{x_n\}$ is weakly compact. This fact in conjunction with Lemma 1.8 allows us to infer that $x_n \longrightarrow x$ in the norm of the space L^1. Thus in view of the assumption we obtain that $Tx_n \longrightarrow Tx$ which implies that $Tx_n \rightharpoonup Tx$ and completes the proof. ■

Theorem 1.26. *If $T : X \longrightarrow L^1$ is demicontinuous operator then T is weakly sequentially continuous on X.*

The proof may be obtained in the same way as in the proof of Theorem 1.25.

In what follows we give a result which summarizes and generalizes Theorems 1.25 and 1.26.

Theorem 1.27. *Let X be a subset of L^1 compact in measure and let $T : X \longrightarrow X$. Then the following three conditions are equivalent:*

a) T is continuous.
b) T is demicontinuous.
c) T is weakly sequentially continuous.

Proof. Taking into account the results established in Theorems 1.25 and 1.26 we see that it is enough to show that $c) \Longrightarrow a)$. Thus let us assume that T is weakly sequentially continuous on X and take a sequence $\{x_n\}$ contained in X and such that $x_n \longrightarrow x, x \in X$. Then $x_n \rightharpoonup x$ which in view of the assumption implies that $Tx_n \rightharpoonup Tx$. However $\{Tx_n\}$ is compact in measure and therefore from Lemma 1.7 we obtain that $\{Tx_n\}$ converges in measure to Tx. Since $\{Tx_n\}$ is also weakly compact (as weakly convergent) we conclude that $Tx_n \longrightarrow Tx$ in the norm. This ends the proof. ∎

The complete description of compactness in measure was given by Fréchet [84] but the following sufficient condition will be useful [122].

Theorem 1.28. *Let X be a bounded subset of the space L^1. Suppose there is a family of measurable subsets $\{\Omega_c\}_{0 \le c \le 1}$ of the interval I such that $m(\Omega_c) = c$. If for any $c \in I$ and for any $x \in X$ we have*

$$x(t_1) \le x(t_2)$$

for $t_1 \in \Omega_c$ and for $t_2 \notin \Omega_c$, then the set X is compact in measure.

It is clear that by putting $\Omega_c = [0, c) \cup D$ or $\Omega_c = [0, c) \setminus D$, where D is a set with measure zero, this family contains nonincreasing functions (possibly except for a set D). We will call the functions from this family "a.e. nonincreasing" functions. This is the case, when we choose an integrable and nonincreasing function y and all the functions equal a.e. to y satisfies the above condition. Thus we can write that elements from $L^1(I)$ belong to this class of functions. Due to the compactness criterion in the space of measurable functions (with the topology convergence in measure) we have a result concerning the compactness in measure of a subset X of $L^1(I)$ [19].

Lemma 1.9. *Let Ω be a bounded subset of $L^p(I)$ consisting of functions which are a.e. nondecreasing (or a.e. nonincreasing) on the interval I. Then Ω is compact in measure in $L^p(I)$.*

Proof. Let $R > 0$ be such that $\Omega \subset B_r(\theta)$. It is known that Ω is compact in measure as a subset of S. Since the compactness in measure is equivalent to sequential compactness, we are interested in studying the properties of the latter. By taking an arbitrary sequence $\{x_n\}_n$ in Ω we obtain that there exists a subsequence $\{x_{n_k}\}_k$ convergent in measure to some x in the space S. Since the balls in $L^p(I)$ spaces ($p \geq 1$) are closed in the topology of convergence in measure, we obtain $x \in B_r(\theta) \subset L^p(I)$ and finally $x \in \Omega$. ∎

1.15 Fixed Point Theory

We first state the Schauder fixed point theorem.

Theorem 1.29. *If Ω is a nonempty closed convex subset of a Banach space X and T is a continuous map from Ω to Ω whose image is countably compact, then T has a fixed point.*

Next we consider the Tikonov (Tychonoff) fixed point theorem.

Theorem 1.30. *Let Ω be a convex compact subset of a locally convex topological space X. If T is a continuous map on Ω into Ω, then T has a fixed point.*

Remark 1.28. Tychonoff's theorem contains as a special case the earlier result of Schauder asserting the existence of a fixed point for each weakly continuous self mapping of a weakly compact convex subset Ω of a separable Banach space.

It is not always possible to show that a given mapping between functional Banach spaces is weakly continuous, but quite often its weak sequential continuity offers no problem. This follows from the fact that Lebesgue's dominated convergence theorem is valid for sequences but not for nets.

Theorem 1.31 ([11, Theorem 1]). *Let X be a metrizable, locally convex topological vector space and let Ω be a weakly compact convex subset of X. Then any weakly sequentially continuous map $T : \Omega \longrightarrow \Omega$ has a fixed point.*

The proof of the last theorem is based on showing that any weakly sequentially continuous selfmap of the weakly compact Ω is in fact weakly continuous (due to the angelicity of the weak topology of metrizable locally convex spaces) and reducing the result to the Tychonoff fixed point theorem.

1.15.1 The Krasnosel'skii's Fixed Point Theorem

Many problems arising from diverse areas of natural science involve the study of solutions of nonlinear equations of the form

$$Ax + Bx = x, \qquad x \in \Omega,$$

where Ω is a closed and convex subset of a Banach space X, see for example [52, 55, 69, 70, 76]. Krasnosel'skii's fixed point theorem appeared as a prototype for solving equations of the previous type. The Krasnosel'skii's fixed point theorem in its original form is as follows.

Theorem 1.32. *Let Ω be a closed convex nonempty subset of a Banach space $(X, \|.\|$. Suppose that A and B map Ω into X and that*

(i) *$Ax + By \in \Omega$, for all $x, y \in \Omega$*
(ii) *A is continuous on Ω and $A(\Omega)$ is contained in a compact subset of X*
(iii) *B is a α contraction in X, with $\alpha \in [0, 1[$.*

Then there exist y in Ω such that

$$Ay + By = y.$$

Condition (i) can be quite restrictive, but this can be relaxed as follows.

Theorem 1.33. *Let Ω be a closed, convex and nonempty subset of a Banach space $(X, \|.\|)$. Suppose that $A : \Omega \longrightarrow X$ and $B : X \longrightarrow X$ such that*

(i) *B is a contraction with constant $\alpha < 1$,*
(ii) *A is continuous and $A(\Omega)$ is contained in a compact subset of X,*
(iii) $[x = Bx + Ay, y \in \Omega] \Longrightarrow x \in \Omega$.

Then there is a $y \in \Omega$ with $Ay + By = y$.

1.15.2 Leray–Schauder Theory

We state a Leray–Schauder result which is useful in applications.

Theorem 1.34 (Leray–Schauder). *Let Ω be an open bounded set in a real Banach space X and let $T : \overline{\Omega} \longrightarrow X$ be a completely continuous operator. Let $y \in \Omega$ be a point such that $x + \lambda T(x) \neq y$ for each $x \in \partial\Omega$ and $\lambda \in [0, 1]$, where $\partial\Omega$ denotes the boundary of the set Ω. Then the equation $(Id + T)(x) = y$ has at least one solution.*

The following version of this theorem is useful in applications:

Theorem 1.35. *Let T be a completely continuous operator in the Banach space X. If all solutions of the family of equations*

$$x = \lambda T(x) \quad (\lambda \in [0, 1]) \tag{1.9}$$

are uniformly bounded, i.e., $\exists\, c > 0$ such that $\forall \lambda$ and $\forall x$ satisfying (1.9) the a priori estimate $\|x\| \leq c$ holds, then the equation $x = T(x)$ has a solution.

1.15.3 Multivalued Maps

For a set Y, denote by $\mathcal{P}(Y)$ the power set of Y. By a multivalued map, we mean a map

$$T : X \longrightarrow \mathcal{P}(Y)$$

which thus assigns to each point $x \in X$ a subset $T(x) \subseteq Y$. Note that a map $S : X \longrightarrow Y$ can be identified with a multivalued map $S' : X \longrightarrow Y$ by setting $S'(x) = \{S(x)\}$.

For $T : X \longrightarrow \mathcal{P}(Y)$ and $M \subseteq X$ we define

$$T(M) = \bigcup_{x \in M} T(x),$$

and the graph $G(T)$ of T will be the set

$$G(T) = \{(x, y) \; : x \in X, y \in T(x)\}.$$

Definition 1.41. Let X and Y be topological spaces and $T : T : X \longrightarrow \mathcal{P}(Y)$ a multivalued map. T is called upper semicontinuous, if for every $x \in X$ and every open set V in Y with $T(x) \subseteq V$, there exists a neighborhood $U(x)$ such that $T(U(x)) \subseteq V$.

Using simple topological arguments, these definitions can be equivalently stated in a simpler formulation. The preimage $T^{-1}(A)$ of a set $A \subseteq Y$ under a multivalued map T is defined as

$$T^{-1}(A) = \{x \in X : T(x) \cap A \neq \emptyset\}.$$

Note that, unlike single-valued maps, the inclusion $T(T^{-1}(A)) \subseteq A$ need not hold. However, unless $T^{-1}(A) = \emptyset$, we have $T(T^{-1}(A)) \cap A \neq \emptyset$.

Proposition 1.23. *Let X and Y be topological spaces and $T : X \longrightarrow \mathcal{P}(Y)$ a multivalued map. Then, T is upper semicontinuous if and only if $T^{-1}(A)$ is closed for all closed sets $A \subseteq Y$.*

In some special cases, the graph of a map can be used to characterize upper semicontinuity.

Theorem 1.36. *Let X and Y be topological spaces and $T : X \longrightarrow \mathcal{P}(Y)$ a multivalued mapping. Assume that $T(x)$ is closed for all $x \in X$. Then T is upper semicontinuous if and only if $G(T)$ is closed in $X \times Y$.*

The Kakutani-fixed point theorem was the first fixed point result concerning multivalued mappings. It is a generalization of the fixed point theorem by Brouwer.

Theorem 1.37 (Kakutani Fixed Point Theorem). *Let Ω be a nonempty compact convex subset of $\mathbb{R}^n$. Let $T : \Omega \longrightarrow \mathcal{P}(\Omega)$ satisfy*

1. *for each $x \in \Omega$, $T(x)$ is nonempty closed and convex,*
2. *T is upper semicontinuous.*

Then T has a fixed point.

Theorem 1.38 (Fan–Glicksberg Fixed Point Theorem [95]). *Let X be a locally convex topological vector space and let $\Omega \subseteq X$ be nonempty compact and convex. Let $T : \Omega \longrightarrow \mathcal{P}(\Omega)$ satisfy*

1. *for each $x \in \Omega$, $T(x)$ is nonempty closed and convex,*
2. *T is upper semicontinuous.*

Then T has a fixed point.

A generalization of the previous theorem is the Himmelberg fixed point theorem [108].

Theorem 1.39 (Himmelberg Fixed Point Theorem). *Let Ω be a nonempty convex subset of a topological vector space X. Let $T : \Omega \longrightarrow \mathcal{P}(\Omega)$ satisfy*

1. *for each $x \in \Omega$, $T(x)$ is nonempty closed and convex,*
2. *T is upper semicontinuous.*
3. *$T(\Omega)$ is relatively compact.*

Then T has a fixed point.

Chapter 2
Nonlinear Eigenvalue Problems in Dunford–Pettis Spaces

In this chapter, we present some variants of Leray–Schauder type fixed point theorems and eigenvalue results for decomposable single-valued nonlinear weakly compact operators in Dunford–Pettis spaces.

2.1 Introduction

In this section we discuss operator equations of the form

$$GTx = \lambda x, \tag{2.1}$$

in appropriate spaces of functions, where by GT we mean the composition $G \circ T$ of single-valued mappings and λ is a scalar. Recently, several authors [37, 38, 135] have taken advantage of the representation $F = GT$ and established fixed point theorems for F-self mapping on closed convex subset of Banach spaces. In applications to construct a set Ω of a space E such that F takes Ω back into Ω is very difficult and sometimes impossible. As a result, it makes sense to discuss maps $F : \Omega \longrightarrow X$. To do this, one of the most important tools in nonlinear analysis is the Leray–Schauder principle. Due to the lack of compactness for many problems posed in L^1-spaces, we also give alternatives of Leray–Schauder type for some nonlinear weakly compact composite operators $F = GT$ in Dunford–Pettis spaces, where G and T verify some sequential conditions (($\mathcal{H}_1$) and ($\mathcal{H}_2$)). Let S be a nonlinear operator from a Banach space X into itself. We will use the following two conditions.

$$(\mathcal{H}_1) \qquad \begin{cases} \textit{If } \{x_n\}_{n\in\mathbb{N}} \text{ is a weakly convergent sequence in } E, \textit{ then} \\ \{Sx_n\}_{n\in\mathbb{N}} \text{ has a strongly convergent subsequence in } E. \end{cases}$$

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$$(\mathcal{H}_2) \qquad \begin{cases} \textit{If } \{x_n\}_{n\in\mathbb{N}} \text{ is a weakly convergent sequence in } E, \textit{ then} \\ \{Sx_n\}_{n\in\mathbb{N}} \text{ has a weakly convergent subsequence in } E. \end{cases}$$

In the literature a continuous map satisfying the condition $(\mathcal{H}_1)$ is called (ws)-compact [100].

The following fixed point result will be used throughout this section. The proof follows from the Schauder fixed point theorem.

Theorem 2.1. *Let Ω be a nonempty closed convex subset of a Banach space X. Assume that $F : \Omega \longrightarrow \Omega$ is a continuous map which verifies $(\mathcal{H}_1)$. If $F(\Omega)$ is weakly relatively compact, then there exists $x \in \Omega$ such that $Fx = x$.*

2.2 Nonlinear Eigenvalue Problems

We use Theorem 2.1 to obtain a nonlinear alternative of Leray–Schauder type for decomposable nonlinear weakly compact operators in Dunford–Pettis spaces.

Theorem 2.2. *Let X be a Dunford–Pettis space, Ω a nonempty closed convex subset of X, U a relatively open subset of Ω and $z \in U$. If $G : X \to X$ and $T : \overline{U} \to X$ are operators satisfying:*

1. *G is a bounded linear weakly compact operator.*
2. *T is a nonlinear continuous operator satisfying $(\mathcal{H}_2)$.*
3. *$T(\overline{U})$ is bounded and $G(T(\overline{U})) \subset \Omega$.*

Then, either

$(\mathcal{A}_1)$ *GT has a fixed point in $\overline{U}$, or*
$(\mathcal{A}_2)$ *there is a point $x \in \partial_\Omega U$ (the boundary of U in Ω) and $\lambda \in (0,1)$ with $x = (1-\lambda)z + \lambda GTx$.*

Remark 2.1. (a) Since Ω is closed, the closure in Ω of U and closure are the same, for $U \subset \Omega$.
(b) For $U \subset \Omega$, we have $\partial_\Omega U = \overline{U} \cap \overline{\Omega \backslash U}$.

Proof. Consider $GT : \overline{U} \to \Omega$. Suppose $(\mathcal{A}_2)$ does not hold. Also without loss of generality, assume that the operator GT has no fixed point in $\partial_\Omega U$ (otherwise we are finished, i.e., $(\mathcal{A}_1)$ occurs). Let D be the set defined by

$$D = \left\{ x \in \overline{U} \ : \ x = (1-\lambda)z + \lambda GTx, \text{ for some } \lambda \in [0,1] \right\}.$$

Now $D \neq \emptyset$ since $z \in D$. Also D is closed. To see this, let (x_n) be a sequence in D such that $x_n \to x \in \overline{U}$. For all $n \in \mathbb{N}$, there exists a $\lambda_n \in [0,1]$ such that $x_n = (1-\lambda)z + \lambda_n GTx_n$. Now $\lambda_n \in [0,1]$, so we can extract a subsequence $\{\lambda_{n_j}\}_j$ such that $\lambda_{n_j} \longrightarrow \lambda \in [0,1]$. So, by the continuity of the operators G and T we obtain that $(1-\lambda_{n_j})z+\lambda_{n_j}GTx_{n_j} \longrightarrow (1-\lambda)z+\lambda GTx$. Hence $x=(1-\lambda)z+\lambda GTx$ and $x \in D$. Next, we shall prove that the set D is sequentially compact. To see this, let $\{x_n\}_n$ be any sequence in D. Since $GT(D)$ is weakly relatively compact, we obtain by the Eberlein–Šmulian theorem (Theorem 1.8) that there exists a subsequence $\{x_{n_j}\}_j$ of $\{x_n\}_n$ with $GTx_{nj} \rightharpoonup y$ for some $y \in \Omega$. We have $x_{n_j}=(1-\lambda_{n_j})z+\lambda_{n_j}GTx_{n_j}$ for some $\lambda_{n_j} \in [0,1]$. Passing eventually to a new subsequence, we may assume that $\lambda_{n_j} \to \lambda$ for $\lambda \in [0,1]$. So, $\{x_{n_j}\}_j \rightharpoonup (1-\lambda)z+\lambda y$. Next, since T verifies $(\mathcal{H}_2)$, then $\{Tx_{n_j}\}_j$ has a weakly convergent subsequence, say $\{Tx_{n_{j_k}}\}_k$. Using the fact that the linear operator G is weakly compact together with Proposition 1.18, we infer that the sequence $\{GTx_{n_{j_k}}\}_k$ is strongly convergent. Hence $\{x_{n_{j_k}}\}_k$ is strongly convergent as well. Hence D is compact. Because E is a Hausdorff locally convex space, we have that E is completely regular (see Remark 1.11). Since $D \cap (\Omega \setminus U)=\emptyset$, then by Proposition 1.1, there is a continuous function $\varphi : \Omega \longrightarrow [0,1]$, such that $\varphi(x)=1$ for $x \in D$ and $\varphi(x)=0$ for $x \in \Omega \setminus U$. Since Ω is convex, $z \in \Omega$, we can define the operator $\widehat{GT}: \Omega \to \Omega$ by

$$\widehat{GT}x = \begin{cases} (1-\varphi(x))z+\varphi(x)GTx, & \text{if } x \in U, \\ z, & \text{if } x \in \Omega \setminus U. \end{cases}$$

We first check that $\widehat{GT}$ is continuous and satisfies $(\mathcal{H}_1)$. Indeed, we have $\partial_\Omega U = \partial_\Omega \overline{U}$ and the operators φ and GT are continuous, so $\widehat{GT}$ is continuous. By an argument similar to the one used above, it is easy to show that GT satisfies $(\mathcal{H}_1)$. Since $[0,1]$ is compact, it follows that $\widehat{GT}$ satisfies $(\mathcal{H}_1)$. Now, the set $GT(\overline{U})$ is weakly relatively compact. Applying the Krein -Šmulian theorem (Theorem 1.10), we have that the set $D_* = \overline{\text{conv}}(GT(\overline{U}) \cup \{z\})$ is convex and weakly compact. Moreover, $\widehat{GT}(D_*) \subset D_*$. Thus, all the assumptions of Theorem 2.1 are satisfied for the operator $\widehat{GT}$. Therefore, there exists $x_0 \in \Omega$ with $\widehat{GT}x_0 = x_0$. From the definition of $\widehat{GT}$, x_0 must be an element of U. Then $x_0=(1-\varphi(x_0))z+\varphi(x_0)GTx_0$, which implies that $x_0 \in D$ and so $\varphi(x_0)=1$. Accordingly, $GTx_0=x_0$ and the proof is complete. ■

Corollary 2.1. *Let E be a Dunford–Pettis space, Ω a nonempty closed convex subset of E, U a relatively open subset of Ω and $z \in U$. Suppose $G: E \to E$ is a bounded linear weakly compact operator and $T: \overline{U} \to E$ is a nonlinear continuous operator satisfying $(\mathcal{H}_2)$, $T(\overline{U})$ is bounded and $G(T(\overline{U})) \subset \Omega$. Also, assume that GT satisfies the Leray–Schauder boundary condition*

$$x \neq (1-\lambda)z+\lambda GTx, \ \ \lambda \in (0,1), \ \ x \in \partial_\Omega U,$$

then the set of fixed point of GT in $\overline{U}$ is nonempty and compact.

Proof. By Theorem 2.2, the operator GT has a fixed point in $\overline{U}$. Let $\mathcal{S}=\Big\{x \in \overline{U} : GTx=x,\Big\}$ be the fixed point set of GT. Since the operators G and T

are continuous, $\mathcal{S}$ is obviously a closed subset of $\overline{U}$ such that $GT(\mathcal{S}) = \mathcal{S}$. Following an argument similar to that in Theorem 2.2, we obtain that $\mathcal{S}$ is sequentially compact and, hence, it is compact. ■

As a special case, we obtain a fixed point theorem of Rothe type [175] for decomposable nonlinear weakly compact operators.

Corollary 2.2. *Let E be a Dunford–Pettis space, Ω a nonempty closed convex subset of E, U a relatively open subset of Ω and $z \in U$. Suppose $G : E \to E$ is a bounded linear weakly compact operator and $T : \overline{U} \to E$ is a nonlinear continuous operator satisfying $(\mathcal{H}_2)$ and $T(\overline{U})$ is bounded. In addition, assume that $\overline{U}$ is starshaped with respect to z and $G(T(\partial_\Omega U)) \subseteq \overline{U}$. Then the set of fixed points of F in $\overline{U}$ is nonempty and compact.*

Proof. Because $\overline{U}$ is starshaped with respect to θ and $GT(\partial_\Omega U) \subseteq \overline{U}$, then $x \neq (1-\lambda)z + \lambda GTx, \lambda \in (0,1), x \in \partial_\Omega U$. Applying Corollary 2.1, the set of fixed points of GT in $\overline{U}$ is nonempty and compact. ■

Corollary 2.3. *Let E be a Dunford–Pettis space, Ω a nonempty closed convex subset of E, U a relatively open subset of Ω and $\theta \in U$. Suppose $G : E \to E$ is a bounded linear weakly compact operator and $T : \overline{U} \to E$ is a nonlinear continuous operator satisfying $(\mathcal{H}_2)$, $T(\overline{U})$ is bounded and $G(T(\overline{U})) \subset \Omega$. In addition, suppose GT has no fixed point in $\overline{U}$. Then, there exist an $x \in \partial_\Omega U$ and $\lambda \in (0,1)$ such that $x = \lambda GTx$.*

Corollary 2.4. *Let E be a Dunford–Pettis space, Ω a nonempty closed convex subset of E, U a relatively open subset of Ω with $\theta \in U$ and $\alpha \geq 1$. Suppose $G : E \to E$ is a bounded linear weakly compact operator and $T : \overline{U} \to E$ is a nonlinear continuous operator satisfying $(\mathcal{H}_2)$, $T(\overline{U})$ is bounded and $G(T(\overline{U})) \subset \Omega$. In addition, assume that there is a real number $k > \alpha$ such that*

$$G(T(\overline{U})) \cap (k.U) = \emptyset. \tag{2.2}$$

Then there exist an $x \in \partial_\Omega U$ and $\lambda \geq k$ such that $GTx = \lambda x$.

Remark 2.2. θ is the zero vector of E.

Proof. Consider $F = GT : \overline{U} \to \Omega$. We suppose that for all $x \in \partial_\Omega U$ and $\lambda \geq k, F(x) \neq \lambda x$. Let $F_1 = \frac{1}{k}F$ and

$$D = \left\{ x \in \overline{U} \ : \ x = \lambda F_1 x, \text{ for some } \lambda \in [0,1] \right\}.$$

The set D is nonempty because $\theta \in D$. Following an argument similar to that in Theorem 2.2, we obtain that D is compact. Now we show that $D \cap (\Omega \setminus U) = \emptyset$. If this is not the case, there exists an $x \in \Omega \setminus U$ and $\beta \in [0,1]$ such that $\beta F_1 x = x$.

If $\beta = 0$, then $x = \theta$, which contradicts $\theta \in U$. If $\beta \neq 0$, then $Fx = \frac{k}{\beta}x$ ($\frac{k}{\beta} \geq k$), which contradicts (3.15). Thus, $D \cap (\Omega \setminus U) = \emptyset$. Let $F_1^* : \Omega \to \Omega$ the operator defined by

$$F_1^* x = \begin{cases} \varphi(x) F_1 x, & \text{if } x \in U, \\ z, & \text{if } x \in \Omega \setminus U. \end{cases}$$

Following arguments similar to those used in the proof of Theorem 2.2, we prove that F_1^* has a fixed point $y \in \Omega$. If $y \notin U$, $\varphi(y) = 0$ and $y = \theta$, which contradicts the hypothesis $\theta \in U$. Then $y \in U$ and $y = \varphi(y)Fy$, which implies that $y \in D$, and so $\varphi(y) = 1$ and $Fy = ky$. Hence, $F(\overline{U}) \cap (k.U) \neq \emptyset$, another contradiction. Accordingly, there exist an $x \in \partial_\Omega U$ and $\lambda \geq k$ such that $Fx = GTx = \lambda x$. ■

In the rest of this section we shall discuss nonlinear Leray–Schauder alternatives for decomposable nonlinear positive operators. Let E_1 and E_2 be two Banach lattice spaces, with positive cones E_1^+ and E_2^+, respectively. An operator F from E_1 into E_2 is said to be positive if it carries the positive cône E_1^+ into E_2^+ (i.e., $F(E_1^+) \subset E_2^+$).

Theorem 2.3. *Let Ω be a nonempty closed convex subset of a Banach lattice E such that $\Omega^+ := \Omega \cap E^+ \neq \emptyset$. Assume $F : \Omega \longrightarrow \Omega$ is a positive continuous operator satisfying $(\mathcal{H}_1)$. If $F(\Omega)$ is weakly relatively compact, then F has at least a positive fixed point in Ω.*

Proof. Clearly, the set Ω^+ is a closed convex subset of E^+ and $F(\Omega^+) \subseteq \Omega^+$. Also, $F(\Omega^+) \subseteq F(\Omega)$, so $F(\Omega^+)$ is weakly relatively compact. Now, it suffices to apply Theorem 2.1 to prove that F has a fixed point in $\Omega^+ \subseteq \Omega$. ■

Theorem 2.4. *Let E be a Dunford–Pettis lattice space, Ω a nonempty closed convex subset of E, U a relatively open subset of Ω and $z \in U \cap E^+$. Suppose $G : E \to E$ is a positive bounded linear weakly compact operator and $T : \overline{U} \to E$ is a positive nonlinear continuous operator satisfying $(\mathcal{H}_2)$, $T(\overline{U})$ is bounded and $G(T(\overline{U})) \subset \Omega$. Then, either*

- $(\mathcal{A}_1)$ *GT has a positive fixed point in $\overline{U}$, or*
- $(\mathcal{A}_2)$ *there is a point $x \in \partial_\Omega U \cap E^+$ (the positive boundary of U in Ω) and $\lambda \in (0, 1)$ with $x = (1 - \lambda)z + \lambda GTx$.*

Proof. Consider $GT : \overline{U} \to \Omega$. Suppose $(\mathcal{A}_2)$ does not hold. Also without loss of generality, assume that the operator GT has no positive fixed point in $\partial_\Omega U$ (otherwise we are finished, i.e., $(\mathcal{A}_1)$ occurs). Let D be the set defined by

$$D = \left\{ x \in \overline{U} \cap E^+ : x = (1 - \lambda)z + \lambda GTx, \text{ for some ł} \in [0, 1] \right\}.$$

Now $D \neq \emptyset$ since $z \in D$. Because E is a normed lattice, E^+ is closed, and so, $\overline{U} \cap E^+$ is a closed subset of Ω. Following arguments similar to those used in the proof of Theorem 2.2, we prove that D is compact. Because E is a Hausdorff locally convex space, we have that E is completely regular (see Remark 1.11). Since $D \cap (\Omega \setminus U) = \emptyset$, then by Proposition 1.1, there is a continuous function $\varphi : \Omega \longrightarrow [0, 1]$, such that $\varphi(x) = 1$ for $x \in D$ and $\varphi(x) = 0$ for $x \in \Omega \setminus U$. Since Ω is convex, $z \in \Omega$, we can define the operator $\widehat{GT} : \Omega \to \Omega$ by

$$\widehat{GT}x = \begin{cases} (1 - \varphi(x))z + \varphi(x)GTx, & \text{if } x \in U, \\ z, & \text{if } x \in \Omega \setminus U. \end{cases}$$

Clearly, the operator $\widehat{GT}$ is positive. By an argument similar to that in Theorem 2.2 and using Theorem 2.3, we prove that there exists a positive element $x_0 \in \Omega$ with $\widehat{GT}x_0 = x_0$. If $x_0 \notin U$, $\varphi(x_0) = 0$ and $x_0 = z$, which contradicts the hypothesis $z \in U$. Then, $x_0 \in U$ and $x_0 = (1-\varphi(x_0))z + \varphi(x_0)GTx_0$, which implies that $x_0 \in D$ and so $\varphi(x_0) = 1$. Accordingly, $GTx_0 = x_0$ and x_0 is a positive fixed point of GT which completes the proof. ∎

Corollary 2.5. *Let E be a Dunford–Pettis lattice space, Ω a nonempty closed convex subset of E, U a relatively open subset of Ω and $z \in U \cap E^+$. Suppose $G : E \to E$ is a positive bounded linear weakly compact operator and $T : \overline{U} \to E$ is a positive nonlinear continuous operator satisfying $(\mathcal{H}_2)$, $T(\overline{U})$ is bounded and $G(T(\overline{U})) \subset \Omega$. Also, assume that GT satisfies the Leray–Schauder boundary condition*

$$x \neq (1 - \lambda)z + \lambda GTx, \ \lambda \in (0, 1), \ x \in \partial_\Omega U \cap E^+,$$

then the set of positive fixed points of GT in $\overline{U}$ is nonempty and compact.

Chapter 3
Fixed Point Theory in Locally Convex Spaces

3.1 Leray–Schauder Alternatives

In this section we discuss the existence of fixed points for weakly sequentially continuous mappings on domains of Banach spaces. We first present some applicable Leray–Schauder type theorems for weakly condensing and 1-set weakly contractive operators. The main condition is formulated in terms of De Blasi's measure of weak noncompactness β (see Sect. 1.12).

Definition 3.1. Let D be a nonempty subset of Banach space E. If F maps D into E, we say that

(a) F is condensing (with respect to β) if F is bounded and $\beta(F(V)) < \beta(V)$ for all bounded subsets V of D with $\beta(V) > 0$,
(b) F is 1-set contractive (with respect to β) if F is bounded and $\beta(F(V)) \leq \beta(V)$ for all bounded subsets V of D.

Theorem 3.1. *Let X be a Banach space, $\Omega \subset E$ a closed convex subset, $U \subset \Omega$ a weakly open set (with respect to the weak topology of Ω) and such that $\theta \in U$. Assume that $\overline{U^w}$ is a weakly compact subset of Ω and $F : \overline{U^w} \longrightarrow \Omega$ is a weakly sequentially continuous mapping. Then, either*

(A_1) *F has a fixed point, or*
(A_2) *there is a point $u \in \partial_\Omega U$ (the weak boundary of U in Ω) and $\lambda \in (0, 1)$ with $u = \lambda F u$.*

Proof. Suppose (A_2) does not hold. We observe that supposition is satisfied also for $\lambda = 0$ (since $\theta \in U$). If (A_2) is satisfied for $\lambda = 1$, then in this case we have a fixed point in $u \in \partial_\Omega U$ and there is nothing to prove. In conclusion, we can consider that the supposition is satisfied for any $x \in \partial_\Omega U$ and any $\lambda \in [0, 1]$. Let D be the set defined by

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$$D = \left\{ x \in \overline{U^w} \ : \ x = \lambda F(x), \ \text{for some } \lambda \in [0,1] \right\}.$$

The set D is nonempty because $\theta \in U$. We will show that D is weakly compact. The weak sequential continuity of F implies that D is weakly sequentially closed. For that, let $\{x_n\}_n$ a sequence of D such that $x_n \rightharpoonup x$, $x \in \overline{U^w}$. For all $n \in \mathbb{N}$, there exists a $\lambda_n \in [0,1]$ such that $x_n = \lambda_n F(x_n)$. Now $\lambda_n \in [0,1]$, so we can extract a subsequence $\{\lambda_{n_j}\}_j$ such that $\lambda_{n_j} \longrightarrow \lambda \in [0,1]$. So, $\lambda_{n_j} F x_{n_j} \rightharpoonup \lambda F x$. Hence $x = \lambda F(x)$ and $x \in D$. Let $x \in \overline{U^w}$, weakly adherent to D. Since $\overline{D^w}$ is weakly compact, by the Eberlein–Šmulian theorem, there exists a sequence $\{x_n\}_n \subset D$ such that $x_n \rightharpoonup x$, so $x \in D$. Hence $\overline{D^w} = D$ and D is a weakly closed subset of the weakly compact set $\overline{U^w}$. Therefore D is weakly compact. Because E endowed with its weak topology is a Hausdorff locally convex space, we have that X is completely regular. Since $D \cap (\Omega \setminus U) = \emptyset$, it follows by Proposition 1.1 that there is a weakly continuous function $\varphi : \Omega \longrightarrow [0,1]$, such that $\varphi(x) = 1$ for $x \in D$ and $\varphi(x) = 0$ for $x \in \Omega \setminus U$. Let $F^* : \Omega \longrightarrow \Omega$ be the mapping defined by:

$$F^* x = \begin{cases} \varphi(x) F x, & \text{if } x \in U, \\ \theta, & \text{if } x \in \Omega \setminus U. \end{cases}$$

Because $\partial_\Omega U = \partial_\Omega \overline{U^w}$, φ is weakly continuous and F is weakly sequentially continuous, we have that F^* is weakly sequentially continuous. Also, $F^*(\Omega) \subset \overline{\text{co}}(F(\overline{U^w}) \cup \{\theta\})$. Let $D_* = \overline{\text{conv}}(F(\overline{U^w}) \cup \{\theta\})$. It follows, using the Krein–Šmulian theorem and the weakly sequential continuity of F that D_* is a weakly compact convex set. Moreover $F^*(D_*) \subset D_*$. Since F^* is weakly sequentially continuous, it follows using Theorem 1.31 that F^* has a fixed point $x_0 \in \Omega$. If $x_0 \notin U$, $\varphi(x_0) = 0$ and $x_0 = \theta$, which contradicts the hypothesis $\theta \in U$. Then $x_0 \in U$ and $x_0 = \varphi(x_0) F(x_0)$, which implies that $x_0 \in D$, and so $\varphi(x_0) = 1$ and the proof is complete. ■

Remark 3.1. The condition $\overline{U^w}$ is weakly compact in the statement of Theorem 3.1 can be removed if we assume that $F(\overline{U^w})$ is relatively weakly compact.

Theorem 3.2. *Let Ω a nonempty, convex closed set in a Banach space E. Assume $F : \Omega \longrightarrow \Omega$ is a weakly sequentially continuous map and condensing with respect to β. In addition, suppose that $F(\Omega)$ is bounded. Then F has a fixed point.*

Proof. Let $x_0 \in \Omega$. We consider the family $\mathcal{F}$ of all closed bounded convex subsets D of Ω such that $x_0 \in D$ and $F(D) \subset D$. Obviously $\mathcal{F}$ is nonempty, since $\overline{\text{co}}(F(\Omega) \cup \{x_0\}) \in \mathcal{F}$. We let $K = \bigcap_{D \in \mathcal{F}} D$. We have that K is closed convex and $x_0 \in K$. If $x \in K$, then $Fx \in D$ for all $D \in \mathcal{F}$ and hence $F(K) \subset K$. Therefore we have that $K \in \mathcal{F}$. We will prove that K is weakly compact. Denoting by $K_* = \overline{\text{co}}(F(K) \cup \{x_0\})$, we have $K_* \subset K$, which implies that $F(K_*) \subset F(K) \subset K_*$. Therefore $K_* \in \mathcal{F}$, $K \subset K_*$.

Hence $K = K_*$. Since K is weakly closed, it suffices to show that K is relatively weakly compact. If $\beta(K) > 0$, we obtain

$$\beta(K) = \beta(\overline{\text{co}}(F(K) \cup \{x_0\})) \leq \beta(F(K)) < \beta(K),$$

which is a contradiction. Hence, $\beta(K) = 0$ and so K is relatively weakly compact. Now, F is a weakly sequentially continuous map of K into itself. From Theorem 1.31, F has a fixed point in $K \subset \Omega$. ■

Theorem 3.3. *Let Ω be a closed convex subset of a Banach space E. In addition, let U be a weakly open subset of Ω with $\theta \in U$, and $F : \overline{U^w} \longrightarrow \Omega$ a weakly sequentially continuous map, condensing with respect to β and $F(\overline{U^w})$ is bounded. Then, either*

(A_1) *F has a fixed point, or*
(A_2) *there is a point $u \in \partial_\Omega U$ (the weak boundary of U in Ω) and $\lambda \in (0, 1)$ with $u = \lambda F u$.*

Proof. Suppose (A_2) does not hold and F does not have a fixed point in $\partial_\Omega U$ (otherwise, we are finished, i.e., (A_1) occurs). Let D be the set defined by

$$D = \left\{ x \in \overline{U^w} \ : \ x = \lambda F(x), \text{ for some } \lambda \in [0, 1] \right\}.$$

Now D is nonempty and bounded, because $\theta \in D$ and $F(\overline{U^w})$ is bounded. We have $D \subset \text{conv}(\{\theta\} \cup F(D))$. So, $\beta(D) \neq 0$ implies

$$\beta(D) \leq \beta(\text{co}(\{0\} \cup F(D)) \leq \beta(F(D)) < \beta(D),$$

which is a contradiction. Hence, $\beta(D) = 0$ and D is relatively weakly compact. Now, we prove that D is weakly closed. Arguing as in the proof of Theorem 3.1, we prove that D is weakly sequentially closed. Let $x \in \overline{U^w}$, and weakly adherent to D. Since $\overline{D^w}$ is weakly compact, by the Eberlein–Šmulian theorem, there exists a sequence $\{x_n\}_n \subset D$ such that $x_n \rightharpoonup x$, so $x \in D$. Hence $\overline{D^w} = D$ and D is a weakly closed. Therefore D is weakly compact. Because E endowed with its weak topology is a Hausdorff locally convex space, we have that E is completely regular. Since $D \cap (\Omega \setminus U) = \emptyset$, by Proposition 1.1 there is a weakly continuous function $\varphi : \Omega \longrightarrow [0, 1]$, such that $\varphi(x) = 1$ for $x \in D$ and $\varphi(x) = 0$ for $x \in \Omega \setminus U$. Let $F^* : \Omega \longrightarrow \Omega$ be the mapping defined by:

$$F^* x = \begin{cases} \varphi(x) F x, & \text{if } x \in U, \\ \theta, & \text{if } x \in \Omega \setminus U. \end{cases}$$

Clearly, $F^*(\Omega)$ is bounded. Because φ is weakly continuous and F is weakly sequentially continuous, we have that F^* is weakly sequentially continuous.

We claim that the mapping F^* is condensing. Indeed, let $X \subset \Omega$, bounded, with $\beta(X) \neq 0$. Then, since

$$F^*(X) \subset \mathrm{co}(\{\theta\} \cup F(X \cap U),$$

we have

$$\beta(F^*(X)) \leq \beta(F(X \cap U)).$$

If $X \cap U$ is relatively weakly compact, then $F(X \cap U)$ is relatively weakly compact and $\beta(F(X \cap U)) = 0 < \beta(X)$. If $\beta(X \cap U) \neq 0$, then $\beta(F(X \cap U)) < \beta(X \cap U) \leq \beta(X)$ and $\beta(F^*(X)) < \beta(X)$. So, F^* is condensing with respect to β. Therefore Theorem 3.2, implies that F^* has a fixed point $x_0 \in \Omega$. If $x_0 \notin U$, $\varphi(x_0) = 0$ and $x_0 = \theta$, which contradicts the hypothesis $\theta \in U$. Then $x_0 \in U$ and $x_0 = \varphi(x_0)F(x_0)$, which implies that $x_0 \in D$, and so $\varphi(x_0) = 1$ and the proof is complete. ■

Theorem 3.4. *Let Ω be a closed convex subset of a Banach space E. In addition, let U be a weakly open subset of Ω with $\theta \in U$, and $F : \overline{U^w} \longrightarrow \Omega$ a weakly sequentially continuous, 1-set contraction map with respect to β and $F(\overline{U^w})$ is bounded. Assume that*

(a) $u \neq \lambda F u, \lambda \in (0, 1), u \in \partial_\Omega U$.
(b) $(I - F)(\overline{U^w})$ is closed.

Then, F has a fixed point in $\overline{U^w}$.

Proof. Let $F_n = t_n F$ $n = 1, 2, \ldots$, where $\{t_n\}_n$ is a sequence of $(0, 1)$ such that $t_n \longrightarrow 1$. Since $\theta \in \Omega$ and Ω is convex, it follows that F_n maps $\overline{U^w}$ into Ω. Suppose that $\lambda_n F_n y_n = y_n$ for some $y_n \in \partial_\Omega U$ and for some $\lambda_n \in (0, 1)$. Then we have $\lambda_n t_n F_n y_n = y_n$ which contradicts hypothesis (a) since $\lambda_n t_n \in (0, 1)$. Let X an arbitrary bounded subset of $\overline{U^w}$. Then we have

$$\beta(F_n(X)) = \beta(t_n F(X)) \leq t_n \beta(F(X)) \leq t_n \beta(X).$$

So, if $\beta(X) \neq 0$ we have

$$\beta(F_n(X)) < \beta(X).$$

Therefore F_n is condensing with respect to β on $\overline{U^w}$. From Theorem 3.3, F_n has a fixed point, say, x_n in $\overline{U^w}$. Therefore, $x_n - Fx_n = (1 - t_n)Fx_n \longrightarrow \theta$ as $n \longrightarrow \infty$, since $t_n \longrightarrow 1$ as $n \longrightarrow \infty$ and $F(\overline{U^w})$ is bounded. From, condition (b), we obtain $\theta \in (I - F)(\overline{U^w})$ and hence there is a point x_0 in $\overline{U^w}$ such that $\theta = (I - F)x_0$. Thus x_0 is a fixed point of F in $\overline{U^w}$. ■

3.2 Fixed Point Theory for 1-Set Weakly Contractive Operators

In this section we consider an axiomatic definition of the measure of weak noncompactness (see Appell's survey on measures of noncompactness [10]). Let E be a Banach space. In what follows we make use of the following notation $\mathcal{P}_{bd}(E) = \{D \subseteq E : D \text{ bounded}\}$.

A function $\mu : \mathcal{P}_{bd}(E) \longrightarrow \mathbb{R}^+$ is said to be a measure of weak noncompactness if, for every $\Omega \in \mathcal{P}_{bd}(E)$, the following properties are satisfied:

(1) $\mu(\Omega) = 0 \Longleftrightarrow \Omega \in \mathcal{W}(E)$.
(2) $\mu(\overline{\text{co}}(\Omega)) = \mu(\Omega)$,
(3) $\Omega_1 \subseteq \Omega_2 \Longrightarrow \mu(\Omega_1) \leq \mu(\Omega_2)$,
(4) $\mu(\Omega_1 \cup \Omega_2) = \max\{\mu(\Omega_1), \mu(\Omega_2)\}$,
(5) $\mu(\Omega_1 + \Omega_2) \leq \mu(\Omega_1) + \mu(\Omega_2)$,
(6) $\mu(\lambda\Omega) = |\lambda|\,\mu(\Omega), \lambda \in \mathbb{R}$.

In [26] a measure of weak noncompactness in the above sense is called to be regular. We note that De Blasi's measure β is a regular measure of weak noncompactness. For more examples and properties of measures of weak noncompactness we refer the reader to [6, 23, 26, 123, 124].

Throughout we let μ be a measure of weak noncompactness on E.

Definition 3.2. Let D be a nonempty subset of Banach space E. If F maps D into E, we say that

(a) F is μ-condensing if F is bounded and $\mu(F(V)) < \mu(V)$ for all bounded subsets V of D with $\mu(V) > 0$,
(b) F is μ-nonexpansive if F is bounded and $\mu(F(V)) \leq \mu(V)$ for all bounded subsets V of D.

Lemma 3.1. *Let C be a nonempty weakly closed set of a Banach space E and $F : C \longrightarrow E$ a weakly sequentially continuous and μ-condensing operator with $F(C)$ is bounded. Then*

(a) for all weakly compact subset K of E, $(I - F)^{-1}(K)$ is weakly compact.
(b) $I - F$ maps weakly closed subset of C onto weakly sequentially closed sets in E.

Proof. (a) Let $K \subset E$ be a nonempty weakly compact set and let $D = (I - F)^{-1}(K)$. Since $I - F$ is weakly sequentially continuous, D is weakly sequentially closed. Moreover, we have

$$\mu(D) \leq \mu(K) + \mu(F(D)) = \mu(F(D)).$$

Since F is μ-condensing, it follows that $\mu(D) = 0$. Let $x \in C$, be weakly adherent to D. Since $\overline{D^w}$ is weakly compact, by the Eberlein–Šmulian theorem, there exists a sequence $\{x_n\}_n \subset D$ such that $x_n \rightharpoonup x$, so $x \in D$. Hence $\overline{D^w} = D$ and D is a weakly closed subset of C. Therefore D is weakly compact.

(b) Let $D \subset C$ be a weakly closed set and consider $x_n \in (I-F)(D)$ such that $x_n \rightharpoonup x$ in E. We have $x_n = (I-F)(u_n), \forall n \geq 1$ with $u_n \in D$. The set $K = \overline{\{x_n\}^w}$ is weakly compact and so $(I-F)^{-1}(K)$ is weakly compact. Therefore, we may assume that $u_n \rightharpoonup u$ in D, for some $u \in D$. Due to the weak sequential continuity of $I-F$, we have $x = (I-F)(u)$ and so $x \in (I-F)(D)$. Accordingly $(I-F)(D)$ is weakly sequentially closed.

■

Definition 3.3. A subset D of a Banach space is called weakly sequentially closed if, whenever $x_n \in D$ for all $n \in \mathbb{N}$ and $x_n \rightharpoonup x$, then $x \in D$.

Definition 3.4. Let D be a nonempty weakly closed set of a Banach space E and $F : D \longrightarrow E$ a weakly sequentially continuous operator. F is said to be weakly semi-closed operator at θ if the conditions $x_n \in D$, $x_n - F(x_n) \longrightarrow \theta$ imply that there exists $x \in D$ such that $F(x) = x$ (here θ means the zero vector of the space E).

It should be noted that this class of operators, as special cases, includes the weakly sequentially continuous operators which are weakly compact, weakly contractive, μ-condensing, $(I-F)(D)$ is weakly sequentially closed and others (see Lemma 3.1).

The following fixed point result, stated in [35] is an analogue of Sadovskii's fixed point result [3], will be used throughout this section. The proof follows from Theorem 1.31

Theorem 3.5. *Let Ω be a nonempty, convex closed set in a Banach space E. Assume $F : \Omega \longrightarrow \Omega$ is a weakly sequentially continuous and μ-condensing mapping. In addition, suppose that $F(\Omega)$ is bounded. Then F has a fixed point.*

Proposition 3.1. *Let Ω be a nonempty unbounded closed convex subset of a Banach space E. Suppose that $F : \Omega \longrightarrow \Omega$ is a weakly sequentially continuous μ-nonexpansive operator and $F(\Omega)$ is bounded. In addition, assume that F is weakly semi-closed at θ. Then F has a fixed point in Ω.*

Proof. Let x_0 be a fixed element of Ω. Define $F_n = t_n F + (1-t_n)x_0 \ n = 1, 2, \ldots$, where $(t_n)_n$ is a sequence of $(0, 1)$ such that $t_n \longrightarrow 1$. Since $x_0 \in \Omega$ and Ω is convex, it follows that F_n maps Ω into itself. Clearly F_n is weakly sequentially continuous and $F_n(\Omega)$ is bounded. Let X an arbitrary bounded subset of Ω. Then we have

$$\mu(F_n(X)) = \mu(t_n F(X) + \{(1-t_n)x_0\}) \leq t_n \mu(F(X)) + \mu(\{(1-t_n)x_0\}) \leq t_n \mu(X).$$

So, if $\mu(X) \neq 0$ we have

$$\mu(F_n(X)) < \mu(X).$$

Therefore F_n is μ-condensing on Ω. From Theorem 3.5, F_n has a fixed point, say, x_n in Ω. Consequently, $\|x_n - F(x_n)\| = \|(t_n - 1)(F(x_n) - x_0)\| \longrightarrow 0$ as $n \longrightarrow \infty$, since $t_n \longrightarrow 1$ as $n \longrightarrow \infty$ and $F(\Omega)$ is bounded. Since F is weakly semi-closed at θ, there exists $x \in \Omega$ such that $F(x) = x$. Accordingly, F has a fixed point in Ω. ∎

Theorem 3.6. *Let Ω be a closed wedge of a Banach space E, $x_0 \in \Omega$, U a weakly open subset of Ω such that $x_0 \in U$ and $\alpha \geq 1$. Suppose that $F : \overline{U^w} \longrightarrow \Omega$ is weakly sequentially continuous, μ-condensing operator and $F(\overline{U^w})$ is bounded. Then, either*

(A_1) *the operator equation $F(x) = \alpha x$ has a solution in $\overline{U^w}$, or*
(A_2) *there is a point $x \in \partial_\Omega U$ and $k > \alpha$ with $Fx - \alpha x_0 = k(x - x_0)$.*

Proof. Suppose (A_2) does not hold. If (A_2) is satisfied for $k = \alpha$, then the result follows. In conclusion, we can consider that the supposition is not satisfied for any $x \in \partial_\Omega U$ and any $k \geq \alpha$. Let D be the set defined by

$$D = \left\{ x \in \overline{U^w} \ : \ x = \frac{\lambda}{\alpha} F(x) + (1-\lambda)x_0, \ \text{for some } \lambda \in [0,1] \right\}.$$

Now D is nonempty and bounded, because $x_0 \in D$ and $F(\overline{U^w})$ is bounded. We have $D \subset \text{co}(\{x_0\} \cup (\frac{1}{\alpha}F(D)))$. Because the set $\{x_0\}$ is weakly compact and $\alpha \geq 1$, then $\mu(D) \neq 0$ implies

$$\mu(D) \leq \mu(\overline{\text{conv}}(\{x_0\} \cup (\frac{1}{\alpha}F(D))) \leq \frac{1}{\alpha}(F(D)) < \mu(D),$$

which is a contradiction. Hence, $\mu(D) = 0$ and D is relatively weakly compact. Now, we prove that D is weakly closed. The weak sequentially continuity of F implies that D is weakly sequentially closed. For that, let $\{x_n\}_n$ a sequence of D such that $x_n \rightharpoonup x$, $x \in \overline{U^w}$. For all $n \in \mathbb{N}$, there exists a $\lambda_n \in [0,1]$ such that $x_n = \frac{\lambda_n}{\alpha}F(x_n) + (1-\lambda_n)x_0$. Now $\lambda_n \in [0,1]$, so we can extract a subsequence $\{\lambda_{n_j}\}_j$ such that $\lambda_{n_j} \longrightarrow \lambda \in [0,1]$. So, $\frac{\lambda_{n_j}}{\alpha}F(x_{n_j}) + (1-\lambda_{n_j})x_0 \rightharpoonup \frac{\lambda}{\alpha}F(x) + (1-\lambda)x_0$. Hence $x = \frac{\lambda}{\alpha}F(x) + (1-\lambda)x_0$ and $x \in D$. Let $x \in \overline{U^w}$, be weakly adherent to D. Since $\overline{D^w}$ is weakly compact, by the Eberlein–Šmulian theorem, there exists a sequence $\{x_n\}_n \subset D$ such that $x_n \rightharpoonup x$, so $x \in D$. Hence $\overline{D^w} = D$ and D is a weakly closed subset of $\overline{U^w}$. Therefore D is weakly compact. We prove that $D \cap (\Omega \setminus U) = \emptyset$. In fact, let $x \in D$, then there exists $\lambda \in (0,1]$ such that $x = \frac{\lambda}{\alpha}F(x) + (1-\lambda)x_0$ (if $\lambda = 0$ then $x = x_0 \notin \Omega \setminus U$). So, $F(x) - \alpha x_0 = \frac{\alpha}{\lambda}(x - x_0)$ and thus $x \notin \Omega \setminus U$ ($\frac{\alpha}{\lambda} \geq \alpha$). Because E endowed with its weak topology is a Hausdorff locally convex space, we have that E is completely regular. Since $D \cap (\Omega \setminus U) = \emptyset$, by Proposition 1.1 there is a weakly continuous function $\varphi : \Omega \longrightarrow [0,1]$, such that $\varphi(x) = 1$ for $x \in D$ and $\varphi(x) = 0$ for $x \in \Omega \setminus U$. Since Ω is a wedge, $x_0 \in \Omega$, we can define the operator $F^* : \Omega \longrightarrow \Omega$ by:

$$F^*(x) = \begin{cases} \frac{\varphi(x)}{\alpha} F(x) + (1 - \varphi(x))x_0, \text{ if } x \in \overline{U^w}, \\ \theta, \text{ if } x \in \Omega \backslash \overline{U^w} \end{cases}$$

Clearly, $F^*(\Omega)$ is bounded. Because $\partial_\Omega U = \partial_\Omega \overline{U^w}$, φ is weakly continuous and F is weakly sequentially continuous, we have that F^* is weakly sequentially continuous. Let $X \subset \Omega$, bounded. Then, since

$$F^*(X) \subset \text{co}(\{x_0\} \cup F(X \cap U)),$$

we have

$$\mu(F^*(X)) \leq \mu(\overline{\text{co}}(\{x_0\} \cup (\frac{1}{\alpha} F(X \cap U))) \leq \mu(F(X \cap U)), (\alpha \geq 1).$$

If $X \cap U$ is relatively weakly compact, then $F(X \cap U)$ is relatively weakly compact and $\mu(F(X \cap U)) = 0 < \mu(X)$. If $\mu(X \cap U) \neq 0$, then $\mu(F(X \cap U)) < \mu(X \cap U) \leq \mu(X)$. So, $\mu(F^*(X)) < \mu(X)$ if $\mu(X) \neq 0$ and hence F^* is μ-condensing. Therefore by Theorem 3.5 F^* has a fixed point $x_1 \in \Omega$. If $x_1 \notin U$, $\varphi(x_1) = 0$ and $x_1 = x_0$, which contradicts the hypothesis $x_0 \in U$. Then $x_1 \in U$ and $x_1 = \frac{\varphi(x_1)}{\alpha} F(x_1) + (1 - \varphi(x_1))x_0$, which implies that $x_1 \in D$. Accordingly, $\varphi(x_1) = 1$ and so $F(x_1) = \alpha x_1$ and the proof is complete. ■

Remark 3.2. If either $\alpha = 1$ or $x_0 = \theta$, then we obtain the same conclusion by only assuming that Ω is a nonempty unbounded closed convex subset of E.

Corollary 3.1. *Let Ω be a closed wedge of a Banach space E, $x_0 \in \Omega$, U a weakly open subset of Ω such that $x_0 \in U$ and $\alpha \geq 1$. Suppose that $F : \overline{U^w} \longrightarrow \Omega$ is a weakly sequentially continuous, μ-condensing operator and $F(\overline{U^w})$ is bounded. Assume that*

$$Fx - \alpha x_0 \neq k(x - x_0), \qquad \textit{for all } x \in \partial_\Omega U, k > \alpha.$$

Then the equation $F(x) = \alpha x$ has at least a solution in $\overline{U^w}$.

Corollary 3.2. *Let Ω be a closed wedge of a Banach space E, $x_0 \in \Omega$, U a weakly open subset of Ω such that $x_0 \in U$ and $\alpha \geq 1$. Suppose that $F : \overline{U^w} \longrightarrow \Omega$ is a weakly sequentially continuous, weakly compact operator and $F(\overline{U^w})$ is bounded. Assume that*

$$F(x) - \alpha x_0 \neq k(x - x_0), \qquad \textit{for all } x \in \partial_\Omega U, k > \alpha.$$

Then the equation $F(x) = \alpha x$ has at least a solution in $\overline{U^w}$.

Remark 3.3. The conditions F is a weakly compact operator and $F(\overline{U^w})$ is bounded in the statement of Corollary 3.2 can be removed if we assume that $\overline{U^w}$ is weakly compact.

Corollary 3.3. *Let Ω be a nonempty unbounded closed convex set of a Banach space E, $x_0 \in \Omega$, U a weakly open subset of Ω such that $x_0 \in U$. Suppose that $F : \overline{U^w} \longrightarrow \Omega$ is a weakly sequentially continuous, μ-condensing operator and $F(\overline{U^w})$ is bounded. Assume that*

$$F(x) - x_0 \neq k(x - x_0), \qquad \text{for all } x \in \partial_\Omega U, k > 1.$$

Then F has a fixed point in $\overline{U^w}$.

Theorem 3.7. *Let Ω be a closed wedge of a Banach space E, $x_0 \in \Omega$, U a weakly open subset of Ω such that $x_0 \in U$ and $\alpha \geq 1$. Suppose that $F : \overline{U^w} \longrightarrow \Omega$ is a weakly sequentially continuous μ-nonexpansive operator and $F(\overline{U^w})$ is bounded. In addition, assume that F is weakly semi-closed at θ. Then, either*

(A_1) *the operator equation $F(x) = \alpha x$ has a solution in $\overline{U^w}$, or*
(A_2) *there is a point $x \in \partial_\Omega U$ and $k > \alpha$ with $F(x) - \alpha x_0 = k(x - x_0)$.*

Proof. Suppose (A_2) does not hold. Let $F_n = \frac{t_n}{\alpha}F + (1 - t_n)x_0$ $n = 1, 2, \ldots$, where $\{t_n\}_n$ is a sequence of $(0, 1)$ such that $t_n \longrightarrow 1$. Since $x_0 \in \Omega$ and Ω is a wedge, it follows that F_n maps $\overline{U^w}$ into Ω. Clearly $F_n(\overline{U^w})$ is bounded. Suppose that $\lambda_n F_n(y_n) + (1 - \lambda_n)x_0 = y_n$ for some $y_n \in \partial_\Omega U$ and for some $\lambda_n \in (0, 1)$. So,

$$\begin{aligned} y_n &= \lambda_n F_n(y_n) + (1 - \lambda_n)x_0, \\ &= \frac{\lambda_n t_n}{\alpha} F(y_n) + \lambda_n(1 - t_n)x_0 + (1 - \lambda_n)x_0, \\ &= \frac{\lambda_n t_n}{\alpha} F(y_n) + (1 - \lambda_n t_n x_0). \end{aligned}$$

Hence $F(y_n) - \alpha x_0 = \frac{\alpha}{\lambda_n t_n}(y_n - x_0)$, a contradiction with the fact that $\frac{\alpha}{\lambda_n t_n} > \alpha$. Let X an arbitrary bounded subset of $\overline{U^w}$. Then we have

$$\mu(F_n(X)) = \mu(\frac{t_n}{\alpha}F(X) + \{(1 - t_n)x_0\}) \leq \frac{t_n}{\alpha}\mu(F(X)) + \mu(\{(1 - t_n)x_0\}) \leq t_n \mu(X).$$

So, if $\mu(X) \neq 0$ we have

$$\mu(F_n(X)) < \mu(X).$$

Therefore F_n is μ-condensing on $\overline{U^w}$ (note that $\alpha \geq 1$). From Corollary 3.3, F_n has a fixed point, say, x_n in $\overline{U^w}$. Therefore, $\|x_n - \frac{1}{\alpha}F(x_n)\| = (1 - t_n)\|\frac{1}{\alpha}F(x_n) - x_0\| \longrightarrow 0$ as $n \longrightarrow \infty$, since $t_n \longrightarrow 1$ as $n \longrightarrow \infty$ and $F(\overline{U^w})$ is bounded. Since $\frac{1}{\alpha}F$ is either μ-condensing (if $\alpha > 1$) or μ-nonexpansive (if $\alpha = 1$), by Lemma 3.1 and the condition that F is weakly semi-closed at θ, we obtain that there exists a point x_1 in $\overline{U^w}$ such that $\theta = (I - \frac{1}{\alpha}F)(x_1)$. Thus $F(x_1) = \alpha x_1$. ■

Corollary 3.4. *Let Ω be a closed wedge of a Banach space E, $x_0 \in \Omega$, U a weakly open subset of Ω such that $x_0 \in U$ and $\alpha \geq 1$. Suppose that $F : \overline{U^w} \longrightarrow \Omega$ is a weakly sequentially continuous μ-nonexpansive operator weakly semi-closed at θ and $F(\overline{U^w})$ is bounded. In addition we suppose that F satisfies the following condition*

$$F(x) - \alpha x_0 \neq k(x - x_0), \qquad \text{for all } x \in \partial_\Omega U, k > \alpha. \tag{3.1}$$

Then the equation $F(x) = \alpha x$ has at least one solution in $\overline{U^w}$.

Corollary 3.5. *Let Ω be a nonempty unbounded closed convex subset of a Banach space E and U a weakly open subset of Ω. Suppose $F : \overline{U^w} \longrightarrow \Omega$ is a weakly sequentially continuous μ-nonexpansive operator weakly semi-closed at θ and $F(\overline{U^w})$ is bounded. In addition, assume that there exists $x_0 \in U$ such that*

$$x \neq \lambda F(x) + (1 - \lambda)x_0 \qquad \text{for all } x \in \partial_\Omega U, \lambda \in (0, 1).$$

Then F has a fixed point in $\overline{U^w}$.

Corollary 3.6. *Let Ω be a nonempty unbounded closed convex of a Banach space E, U a weakly open subset of Ω such that $\theta \in U$. Suppose $F : \overline{U^w} \longrightarrow \Omega$ is a weakly sequentially continuous μ-nonexpansive operator weakly semi-closed at θ and $F(\overline{U^w})$ is bounded. In addition, assume that F satisfies the Leray–Schauder boundary condition*

$$\lambda F(x) \neq x \qquad \text{for all } x \in \partial_\Omega U, \lambda \in (0, 1). \tag{L-S}$$

Then F has a fixed point in $\overline{U^w}$.

Remark 3.4. Corollary 3.6 generalizes the Leray–Schauder's theorem to the case of weakly sequentially continuous, μ-nonexpansive and semi-weakly closed operator at θ.

Theorem 3.8. *Let E, Ω, U and F be as the same as in Corollary 3.6. In addition, assume that*

$$\|F(x)\| \leq \|x\|, \qquad \text{for all } x \in \partial_\Omega U. \tag{3.2}$$

Then F has a fixed point in $\overline{U^w}$.

Proof. It suffices to prove that (3.2) satisfies condition $(L{-}S)$. Suppose the contrary. Then there exists $x_0 \in \partial_\Omega U, \lambda_0 \in (0, 1)$ such that $\lambda_0 F(x_0) = x_0$. So, $\|F(x)\| = \frac{1}{\lambda_0}\|x_0\| > \|x_0\|$, contradicting (3.2). ■

Theorem 3.9. *Let Ω be a closed wedge of a Banach space E, U a weakly open subset of Ω and $\alpha \geq 1$. Suppose that $F : \overline{U^w} \longrightarrow \Omega$ is a weakly sequentially continuous μ-nonexpansive operator weakly semi-closed at θ and $F(\overline{U^w})$ is bounded. In addition, assume that F satisfies one of the following conditions*

(a) $\theta \in U$, $F(x) \neq \lambda x$, *for* $x \in \partial_\Omega U, \lambda > \alpha$,
(b) $x_0 \in U$, $\|F(x) - \alpha x_0\| \leq \alpha \|x - x_0\|$ *for all* $x \in \partial_\Omega U$.

Then the operator equation $F(x) = \alpha x$ *has a solution in* $\overline{U^w}$.

Proof. Suppose that (a) is satisfied. We only need to let $x_0 = \theta$ in Corollary 3.4. If (b) is satisfied, we suppose that the operator equation $F(x) = \alpha x$ has no solution in $\partial_\Omega U$ (otherwise we are finished). In order to apply Corollary 3.4, we prove that (3.1) is satisfied. Suppose that (3.1) is not true, that is, there exist $k_0 > \alpha$ and $x_1 \in \partial_\Omega U$ such that $F(x_1) - \alpha x_0 = k_0(x_1 - x_0)$. From (b), we obtain $k_0\|x_1 - x_0\| \leq \alpha \|x_1 - x_0\|$. Since $x_1 \in \partial_\Omega U$ and U is a weakly open subset of Ω, thus $x_1 - x_0 \neq \theta$. Therefore, $\|x_1 - x_0\| \neq 0$ and we obtain $k_0 \leq \alpha$ and this contradicts $k_0 > \alpha$. So (3.1) holds. Accordingly, from Corollary 3.4 the operator equation $F(x) = \alpha x$ has a solution in $\overline{U^w}$. ∎

Remark 3.5. In Theorem 3.9, if the operator F satisfies the condition (a), then it suffices to take Ω a nonempty unbounded closed convex subset of E.

As a consequence we have the following fixed point result.

Corollary 3.7. *Let* Ω *be a closed wedge of a Banach space* E, $x_0 \in \Omega$, U *a weakly open subset of* Ω *and* $\alpha \geq 1$. *Suppose that* $F : \overline{U^w} \longrightarrow \Omega$ *is a weakly sequentially continuous* μ*-nonexpansive operator, weakly semi-closed at* θ *and* $F(\overline{U^w})$ *is bounded. In addition, assume that F satisfies one of the following condition*

$$\|F(x) - x_0\| \leq \|x - x_0\|,$$

for every $x \in \partial_\Omega, U$. *Then the operator F has a fixed point in* $\overline{U^w}$.

The next lemma holds easily

Lemma 3.2. *When* $y > 1$ *and* $\beta > 0$, *the following inequality holds:*

$$(y-1)^{\beta+1} < y^{\beta+1} - 1.$$

Theorem 3.10. *Let E,* Ω, *U, F be the same as in Corollary 3.6. In addition, assume that there exists* $\beta > 0$ *such that*

$$\|F(x) - x\|^{\beta+1} \geq \|F(x)\|^{\beta+1} - \|x\|^{\beta+1} \tag{3.3}$$

for every $x \in \partial_\Omega U$. *Then the operator F has a fixed point in* $\overline{U^w}$.

Proof. We suppose that the operator F has no fixed point in $\partial_\Omega U$ (otherwise we are finished). In order to apply Corollary 3.6, we prove that

$$x \neq \lambda F(x), \lambda \in (0,1), \ x \in \partial_\Omega U. \tag{3.4}$$

Suppose that (3.4) is not true, that is, there exist $\lambda_0 \in (0, 1)$ and $x_0 \in \overline{U^w}$, such that $\lambda_0 F(x_0) = x_0$. That is $F(x_0) = \frac{1}{\lambda_0} x_0$. Inserting $F(x_0) = \frac{1}{\lambda_0} x_0$ into (3.3), we obtain

$$\|\frac{1}{\lambda_0} x_0 - x_0\|^{\beta+1} \geq \|\frac{1}{\lambda_0} x_0\|^{\beta+1} - \|x_0\|^{\beta+1}.$$

This implies

$$(\frac{1}{\lambda_0} - 1)^{\beta+1} \|x_0\|^{\beta+1} \geq (\frac{1}{\lambda_0^{\beta+1}} - 1) \|x_0\|^{\beta+1}. \tag{3.5}$$

Since $x_0 \in \partial_\Omega U$, we see $x_0 \neq \theta$. Therefore, $\|x_0\|^{\beta+1} \neq 0$ and by (3.5), we obtain

$$(\frac{1}{\lambda_0} - 1)^{\beta+1} \geq \frac{1}{\lambda_0^{\beta+1}} - 1,$$

and this contradicts Lemma 3.2, since $\frac{1}{\lambda_0} \in (1, \infty)$. Hence

$$x \neq \lambda F(x), \lambda \in (0, 1), x \in \partial_\Omega U.$$

Accordingly, by Corollary 3.6, F has a fixed point in $\overline{U^w}$. ■

Remark 3.6. Theorem 3.10 is a generalization of the Altman fixed point theorem in the case of weakly sequentially, μ-nonexpansive and weakly semi-closed operator at θ.

Corollary 3.8. *Let E, Ω, U, F be the same as in Corollary 3.6. In addition, assume that there exists $\beta > 0$ and $\alpha \geq 1$ such that*

$$\|F(x) - \alpha x\|^{\beta+1} \geq \|F(x)\|^{\beta+1} - \|\alpha x\|^{\beta+1} \tag{3.6}$$

for every $x \in \partial_\Omega U$. Then the operator equation $F(x) = \alpha x$ has a solution in $\overline{U^w}$.

Proof. Using (3.6), we obtain

$$\frac{1}{\alpha^{\beta+1}} \|F(x) - \alpha x\|^{\beta+1} \geq \frac{1}{\alpha^{\beta+1}} \|F(x)\|^{\beta+1} - \frac{1}{\alpha^{\beta+1}} \|\alpha x\|^{\beta+1} \qquad \text{for } x \in \partial_\Omega U.$$

So,

$$\|\frac{1}{\alpha} F(x) - x\|^{\beta+1} \geq \|\frac{1}{\alpha} F(x)\|^{\beta+1} - \|x\|^{\beta+1}.$$

Consequently, the operator $\frac{1}{\alpha} F$, which is weakly sequentially continuous μ-nonexpansive, weakly semi-closed at θ, satisfies the conditions of Theorem 3.10. It follows from Theorem 3.10 that the conclusion of Corollary 3.8 holds true. ■

Theorem 3.11. *Let E, Ω, U, F be the same as in Corollary 3.6. In addition, assume that*

$$\|F(x)\| \leq \|F(x) - x\| \tag{3.7}$$

for every $x \in \partial_\Omega U$. Then the operator F has a fixed point in $\overline{U^w}$.

Proof. It suffices to prove that (3.7) implies (3.6). ■

Remark 3.7. Theorem 3.11 is a generalization of the Petryshyn fixed point theorem in the case of weakly sequentially, μ-nonexpansive and weakly semi-closed operator at θ.

Corollary 3.9. *Let E, Ω, U, F be the same as in Corollary 3.6. In addition, assume that there exists $\alpha \geq 1$ such that*

$$\|F(x)\| \leq \|F(x) - \alpha x\| \tag{3.8}$$

for every $x \in \partial_\Omega U$. Then the operator equation $F(x) = \alpha x$ has a solution in $\overline{U^w}$.

Similarly, we can obtain the following results by using the above-mentioned methods. We omit their proofs.

Lemma 3.3. *When $y > 1$ and $\beta \in (-\infty, 0) \cup (0, 1)$, the following inequality holds:*

$$(y-1)^\beta > y^\beta - 1.$$

Theorem 3.12. *Let E, Ω, U, F be the same as in Corollary 3.6. In addition, assume that there exists $\beta \in]-\infty, 0) \cup (0, 1)$ such that*

$$\|F(x) - x\|^\beta \leq \|F(x)\|^\beta - \|x\|^\beta \tag{3.9}$$

for every $x \in \partial_\Omega U$. Then the operator F has a fixed point in $\overline{U^w}$.

Corollary 3.10. *Let E, Ω, U, F be the same as in Corollary 3.6. In addition, assume that there exists $\beta \in (-\infty, 0) \cup (0, 1)$ and $\alpha \geq 1$ such that*

$$\|F(x) - \alpha x\|^\beta \leq \|F(x)\|^\beta - \|\alpha x\|^\beta \tag{3.10}$$

for every $x \in \partial_\Omega U$. Then the operator equation $F(x) = \alpha x$ has a solution in $\overline{U^w}$.

Lemma 3.4. *When $y > 1$ and $\beta > 0$, the following inequality holds:*

$$(y+1)^{\beta+1} > y^{\beta+1} + 1.$$

Theorem 3.13. *Let E, Ω, U, F be the same as in Corollary 3.6. In addition, assume that there exists $\beta > 1$ such that*

$$\|F(x) + x\|^{\beta} \leq \|F(x)\|^{\beta} + \|x\|^{\beta} \tag{3.11}$$

for every $x \in \partial_{\Omega} U$. Then the operator F has a fixed point in $\overline{U^{w}}$.

Corollary 3.11. *Let E, Ω, U, F be the same as in Corollary 3.6. In addition, assume that there exists $\beta > 1$ and $\alpha \geq 1$ such that*

$$\|F(x) + \alpha x\|^{\beta} \leq \|F(x)\|^{\beta} + \|\alpha x\|^{\beta} \tag{3.12}$$

for every $x \in \partial_{\Omega} U$. Then the operator equation $F(x) = \alpha x$ has a solution in $\overline{U^{w}}$.

Lemma 3.5. *When $y > 1$ and $\beta \in (-\infty, 0) \cup (0, 1)$, the following inequality holds:*

$$(y + 1)^{\beta} < y^{\beta} + 1.$$

Theorem 3.14. *Let E, Ω, U, F be the same as in Corollary 3.6. In addition, assume that there exists $\beta \in (-\infty, 0) \cup (0, 1)$ such that*

$$\|F(x) + x\|^{\beta} \geq \|F(x)\|^{\beta} + \|x\|^{\beta} \tag{3.13}$$

for every $x \in \partial_{\Omega} U$. Then the operator F has a fixed point in $\overline{U^{w}}$.

Corollary 3.12. *Let E, Ω, U, F be the same as in Corollary 3.6. In addition, assume that there exists $\beta \in (-\infty, 0) \cup (0, 1)$ and $\alpha \geq 1$ such that*

$$\|F(x) + \alpha x\|^{\beta} \geq \|F(x)\|^{\beta} + \|\alpha x\|^{\beta} \tag{3.14}$$

for every $x \in \partial_{\Omega} U$. Then the operator equation $F(x) = \alpha x$ has a solution in $\overline{U^{w}}$.

3.3 Fixed Point Theorems for Function Spaces

We discuss some Sadovskii fixed point type results for function spaces which guarantee existence results for the general operator equation

$$x(t) = Fx(t), \;\; t \in [0, T], T > 0$$

relative to the weak uniform convergence topology which is not metrizable.

Let $I = [0, T]$ an interval of the real line equipped with the usual topology. Let E be a Banach space with norm $\|.\|$. E^{*} will denote the dual of E and E_{w} will denote the space E when endowed with its weak topology. On the space $C(I, E_{w})$ of continuous functions from I to E_{w} we define a topology as follows. Let $\mathrm{Fin}(E^{*})$ be the class of all nonempty and finite subsets in E^*, Let $\mathcal{O} \in \text{Fin}(E^*)$ and let us define $\|.\|_{\mathcal{O}} : C(I, E_w) \longrightarrow \mathbb{R}_+$ by

$$\|f\|_{\mathcal{O}} := \sup_{t\in I} \sup_{x^*\in\mathcal{O}} |x^*(f(t))|$$

for each $f \in C(I, E_w)$. One may see that $\{\|.\|_{\mathcal{O}}; \mathcal{O} \in \text{Fin}(E^*)\}$ is a family of seminorms on $C(I, E_w)$ which defines a topology of a locally convex, separated space, called the uniform weak convergence topology. We emphasize that this topology (except for the case in which E is finite dimensional) is not metrizable. We will denote by $C_w(I, E)$ the space of weakly continuous functions on I with the topology of weak uniform convergence. For more details see [176].

Definition 3.5. A function $f : I \times E \longrightarrow E$ is said to be weakly–weakly continuous at (t_0, x_0) if given $\varepsilon > 0$ and $x^* \in E^*$, there exists $\delta > 0$ and a weakly open set U containing x_0 such that $|x^*(f(t, x) - f(t_0, x_0))| < \varepsilon$ whenever $|t - t_0| < \delta$ and $x \in U$.

Definition 3.6. A family $\mathcal{F} = \{f_i, i \in \mathcal{I}\}$ (where $\mathcal{I}$ is some index set) of E^I is said to be weakly equicontinuous if given $\varepsilon > 0, x^* \in E^*$ there exists $\delta > 0$ such that, for $t, s \in I$, if $|t - s| < \delta$, then $|x^*(f_i(t) - f_i(s))| < \varepsilon$ for all $i \in \mathcal{I}$.

Lemma 3.6. *(a) Let V be a bounded subset of $C(I, E)$. Then*

$$\sup_{t\in I} \beta(V(t)) \leq \beta(V)$$

where $V(t) = \{x(t) : x \in V\}$.
(b) Let $V \subseteq C(I, E)$ be a family of strongly equicontinuous functions. Then

$$\beta(V) = \sup_{t\in I} \beta(V(t)) = \beta(V(I))$$

where $V(I) = \bigcup_{t\in I} \{x(t) : x \in V\}$, and the function $t \longmapsto \beta(V(t))$ is continuous.

Theorem 3.15. *Let E be a Banach space with Q a nonempty subset of $C(I, E)$. Also assume that Q is a closed convex subset of $C_w(I, E)$, $F : Q \longrightarrow Q$ is continuous with respect to the weak uniform convergence topology, $F(Q)$ is bounded and F is β-condensing (i.e., $\beta(F(X)) < \beta(X)$ for all bounded subsets $X \subset Q$ such that $\beta(X) \neq 0$). In addition, suppose the family $F(Q)$ is weakly equicontinuous. Then the set of fixed points of F is nonempty and compact in $C_w(I, E)$.*

Proof. Let $\mathcal{F}$ the fixed points set of F in Q. We claim that $\mathcal{F}$ is nonempty. Indeed, let $x_0 \in F(Q)$ and $\mathcal{G}$ be the family of all closed bounded convex subsets D of $C(I, E)$ such that $x_0 \in D$ and $F(D) \subset D$. Obviously $\mathcal{G}$ is nonempty, since $\overline{\text{co}}(F(Q)) \in \mathcal{G}$ (the closed convex hull of $F(Q)$ in $C(I, E)$). We let $K = \bigcap_{D\in\mathcal{G}} D$. We have that K is closed convex and $x_0 \in K$. If $x \in K$, then $F(x) \in D$ for all $D \in \mathcal{G}$ and hence $F(K) \subset K$. Therefore we have that $K \in \mathcal{G}$. We claim that K is a compact subset of $C_w(I, E)$. Denoting by $K_* = \overline{\text{co}}(F(K) \cup \{x_0\})$(the closed convex hull of $F(K)$ in $C(I, E)$), we have $K_* \subset K$, which implies that $F(K_*) \subset F(K) \subset K_*$. Therefore $K_* \in \mathcal{G}$, $K \subset K_*$. Hence $K = K_*$. Clearly K is bounded and if $\beta(K) \neq 0$, we obtain

$$\beta(K) = \beta(\overline{\text{co}}(F(K) \cup \{x_0\}) \leq \beta(\text{co}(F(K) \cup \{x_0\})) \leq \beta(F(K)) < \beta(K),$$

which is a contradiction, so $\beta(K) = 0$. Since K is a weakly closed subset of $C(I,E)$ (notice a convex subset of a Banach space is closed iff it is weakly closed), then K is a weakly compact subset of $C(I,E)$. We claim that K is closed in $C_w(I,E)$. To see this, let $\mathcal{S} = E^I$ be endowed with the product topology. We consider $C(I,E)$ as a vector subspace of $\mathcal{S}$. Hence its weak topology is the topology induced by the weak topology of $\mathcal{S}$. Suppose (x_α) is a net in K with $x_\alpha \longrightarrow z$ in $C_w(I,E)$. Then $x_\alpha(t)$ tends weakly to $z(t)$ for each $t \in I$. For each $t \in I$, let $H_t = \{x_\alpha(t)\}$. Clearly the weak closure of H_t is a weakly compact subset of E. But the weak topology of E^I is the product topology of the weak topology of E. Hence the subset $H = \prod_{t\in I} \overline{H_t^w}$ is a weakly compact subset of $\mathcal{S}$ by the Tychonoff theorem. Obviously the subset $\{x_\alpha, z\} \subset H$. The set $H \cap K$ is weakly compact in K, hence in $C(I,E)$. Using the fact that for each $x^* \in E^*$ and $t \in I$ the point evaluation mapping $y \longmapsto x^*y(t)$ is a continuous linear functional on $C(I,E)$, we get $z \in K$. Now we apply the Arzelà–Ascoli theorem. Because the family $F(Q)$ is weakly equicontinuous, we have that the family $\overline{\text{co}}(F(Q))$ (the closure is taken in $C_w(I,E)$) is weakly equicontinuous and therefore, K is weakly equicontinuous. Thus, it remains to show that for each $t \in I$, the set $K(t) = \{x(t), x \in K\}$ is weakly relatively compact in E. By Lemma 3.6 (a), $\beta(K(t)) \leq \beta(K)$. Then $\beta(K(t)) = 0$ for each $t \in I$. Thus for each $t \in I$, $K(t)$ is weakly relatively compact in E. Now we apply Tychonoff theorem (Theorem 1.30) with the locally convex Hausdorff space $C_w(I,E)$ to obtain that $\mathcal{F} \neq \emptyset$. It remains to show that $\mathcal{F}$ is compact in $C_w(I,E)$. To do this, we let $\mathcal{H}$ be the family of all closed bounded convex subsets D of $C(I,E)$ such that $\mathcal{F} \subset D$ and $F(D) \subset D$. Obviously $\mathcal{H}$ is nonempty, since $\overline{\text{co}}(F(Q)) \in \mathcal{H}$ (the closed convex hull of $F(Q)$ in $C(I,E)$). We let $R = \bigcap_{D\in\mathcal{H}} D$. Arguing as above, we prove that R is compact in $C_w(I,E)$, $F(R) \subset R$ and $\mathcal{F} \subset R$. Finally, applying the Schauder fixed point theorem, we deduce that $\mathcal{F}$ is compact. ■

Corollary 3.13. *Let E be a Banach space and Q be a nonempty subset of $C(I,E)$. Also assume that Q is a closed convex subset of $C_w(I,E)$, $F : Q \longrightarrow Q$ is continuous with respect to the weak uniform convergence topology, $F(Q)$ is bounded, and F is β-condensing. In addition, suppose the family $F(Q)$ is strongly equicontinuous. Then the set of fixed points of F is nonempty and compact in $C_w(I,E)$.*

Proof. Thanks to Theorem 3.15, it suffices to prove that the family $F(Q)$ is weakly equicontinuous which is the case. ■

Corollary 3.14. *Let E be a Banach space and Q be a nonempty subset of $C(I,E)$. Also assume that Q is a closed convex subset of $C_w(I,E)$, $F : Q \longrightarrow Q$ is continuous* *with respect to the weak uniform convergence topology, and the family $F(Q)$ is bounded and strongly equicontinuous. In addition, suppose*

$$\textit{For each } t \in I, F(Q)(t) \textit{ is relatively weakly compact in } E.$$

Then the set of fixed points of F is nonempty and compact in $C_w(I,E)$

Proof. We claim that the set $F(Q)$ is relatively weakly compact in $C(I,E)$. Indeed, the family $F(Q)$ of $C(I,E)$ is bounded and strongly equicontinuous, so by Lemma 3.6, we have $\beta(F(Q)) = \sup_{t\in I}\beta(F(Q)(t)) = 0$. Therefore $F(Q)$ is a relatively weakly compact subset of $C(I,E)$. Accordingly, F is β-condensing. The result now follows from Corollary 3.13. ■

Remark 3.8. If $F(Q)$ is bounded and E is reflexive, then for each $t \in I$, $F(Q)(t)$ is relatively weakly compact in E since a subset of a reflexive Banach space is weakly compact iff it is closed in the weak topology and bounded in the norm topology.

We close this section by stating a fixed point theorem for weakly sequentially continuous mappings.

Theorem 3.16. *Let E a Banach space and Q be a nonempty, convex closed set in E. Assume $F : Q \longrightarrow Q$ is a weakly sequentially continuous map and the family $F(Q)$ is bounded and strongly equicontinuous. In addition, suppose*

$$\textit{For each } t \in I, F(Q)(t) \textit{ is relatively weakly compact in } E.$$

Then F has a fixed point.

Proof. Arguing as in the proof of Corollary 3.14, we obtain that $F(Q)$ is a relatively weakly compact subset of $C(I,E)$. Hence, F is β-condensing. It suffices now to apply Theorem 3.2 to prove the result. ■

Remark 3.9. It can be proved that the set of fixed points of F is weakly compact in $C(I,E)$.

3.4 Fixed Point Theory for the Sum of Two Operators

In this section we present fixed point theorems with two topologies. We introduce the notion of τ-measure of noncompactness in a Hausdorff topological vector space (E,Γ) where τ is a weaker Hausdorff locally convex vector topology on E ($\tau \leq \Gamma$). We also introduce the concept of demi-τ-compact and τ-semi-closed operator at the origin.

3.4.1 Preliminaries

Throughout this section we assume that (E, Γ) is a Hausdorff topological vector space (HTVS, in short) with zero element θ and τ is a weaker Hausdorff locally convex vector topology on E $(\tau \leq \Gamma)$. If E is a normed space, the symbol $B_r(z)$ will denote the closed ball centered at z and with radius r. To denote the convergence in (E, τ) we write $\xrightarrow{\tau}$ while the symbol $\rightarrow$ denotes the convergence in (E, Γ).

We now introduce the following definition of the concept of a τ-measure of noncompactness (τ-MNC, in short).

Definition 3.7. Let C be a lattice with a least element denoted by 0. A function Φ_τ defined on the family $\mathcal{M}_E$ of all nonempty and bounded subsets of (E, Γ) with values in C will be called a τ-MNC in E if it satisfies the following conditions:

(i) $\Phi_\tau(\overline{co}^\tau(\Omega)) \leq \Phi_\tau(\Omega)$ for each $\Omega \in \mathcal{M}_E$, where the symbol $\overline{co}^\tau((\Omega))$ denotes the closed convex hull of Ω in (E, τ).
(ii) $\Omega_1 \subset \Omega_2 \Rightarrow \Phi_\tau(\Omega_1) \leq \Phi_\tau(\Omega_2)$.
(iii) $\Phi_\tau(\{a\} \cup \Omega) = \Phi_\tau(\Omega)$ for any $a \in E$ and $\Omega \in \mathcal{M}_E$.
(iv) $\Phi_\tau(\Omega) = 0$ if and only if Ω is relatively τ-compact in E.

Observe that (i) still holds true if we had $\Phi_\tau(\overline{co}^\Gamma(\Omega)) \leq \Phi_\tau(\Omega)$.

In the case when C has additionally the structure of a cone in a linear space over the field of real numbers, we will say that a τ-MNC Φ_τ is *positively homogeneous* provided $\Phi_\tau(\lambda\Omega) = \lambda\Phi_\tau(\Omega)$ for all $\lambda > 0$ and for $\Omega \in \mathcal{M}_E$. Moreover, Φ_τ is referred to as *subadditive* if $\Phi_\tau(\Omega_1 + \Omega_2) \leq \Phi_\tau(\Omega_1) + \Phi_\tau(\Omega_2)$ for all $\Omega_1, \Omega_2 \in \mathcal{M}_E$.

As an example of τ-MNC we have the important and well-known De Blasi measure of weak noncompactness β.

Definition 3.8. Let Ω be a nonempty subset of E and let Φ_τ be a τ-MNC in E with values in a lattice C with a least element 0 and being a cone. If T maps Ω into E, we say that:

(a) T is Φ_τ-Lipschitzian if $T(D) \in \mathcal{M}_E$ for any bounded subset D of Ω and there exists a constant $k \geq 0$ such that $\Phi_\tau(T(D)) \leq k\Phi_\tau(D)$ for $D \in \mathcal{M}_E$, $D \subset \Omega$.
(b) T is Φ_τ-contraction if T is Φ_τ-Lipschitzian with $k < 1$.
(c) T is Φ_τ-condensing if T is Φ_τ-Lipschitzian with $k = 1$ and $\Phi_\tau(T(D)) < \Phi_\tau(D)$ for $D \in \mathcal{M}_E$ such that $D \subset \Omega$ and $\Phi_\tau(D) > 0$.
(d) T is Φ_τ-nonexpansive if T is Φ_τ-Lipschitzian with $k = 1$.

Observe that in the formulation of points (c) and (d) of the above definition we can remove the assumption that C has a cone structure.

Starting from now on we will always assume that a lattice C has a cone structure (i.e., C is a lattice with a least element 0 which is a cone in a real linear space) provided we require that Φ_τ is a positively homogeneous or subadditive τ-MNC in E.

Now we formulate other definitions needed in our considerations.

Definition 3.9. Let Ω be a nonempty subset of E. An operator $T : \Omega \to E$ is said to be τ-compact if for any nonempty and bounded subset D of Ω the set $T(D)$ is relatively τ-compact.

Definition 3.10. An operator $T : \Omega \to E$ (Ω is a nonempty subset of E) is said to be τ-sequentially continuous on Ω if for each sequence $\{x_n\} \subset \Omega$ with $x_n \xrightarrow{\tau} x$, $x \in \Omega$, we have that $Tx_n \xrightarrow{\tau} Tx$.

Remark 3.10. It is worthwhile mentioning that in several situations it is rather easy to show that a mapping between Banach spaces is weakly sequentially continuous, while the proof of weak continuity of that mapping can be difficult. In many applications involving integral equation-problems, one of the reasons for this difficulty is the fact that the Lebesgue dominated convergence theorem fails to work for nets.

Remark 3.11. If X is angelic, then any sequentially continuous map on a compact set is continuous.

Remark 3.12. Hereafter, by bounded sets in E, we will mean Γ-bounded sets.

3.4.2 Fixed Point Results

We start with the following fixed point result.

Theorem 3.17. *Let Ω be a nonempty, convex subset of a Hausdorff topological vector space (E, Γ) and let Φ_τ be a τ-MNC in E. Assume that compact sets in (E, τ) are angelic. Then the following assertions hold, for every Φ_τ-condensing and τ-sequentially continuous map $T : \Omega \to \Omega$ with bounded range:*

(i) T has a τ-approximate fixed point sequence, i.e., a sequence $\{x_n\} \subset \Omega$ so that the sequence $\{x_n - Tx_n\}$ converges to θ in (E, τ).
(ii) if Ω is τ-closed, then the set $F(T)$ of fixed points of T is nonempty and τ-compact.

Proof. (i) Fix arbitrarily $x_0 \in \Omega$ and consider the family

$$\mathcal{F} = \{Q \subset \Omega : Q \text{ is } \tau - \text{bounded, convex, } x_0 \in Q \text{ and } T(Q) \subset Q\}.$$

We have $\mathcal{F} \neq \emptyset$. To see this, first note that $T(\Omega)$ is τ-bounded since $\tau \leq \Gamma$. Thus, since τ is locally convex, we get $co(T(\Omega) \cup \{x_0\})$ is also τ-bounded (see [64]). Now it is easy to see that $co(T(\Omega) \cup \{x_0\}) \in \mathcal{F}$. Let now $G = \bigcap_{Q \in \mathcal{F}} Q$ and let $H = co(T(G) \cup \{x_0\})$. We claim that $G = H$. Indeed, since $x_0 \in G$ and $T(G) \subset G$ one sees that $H \subset G$. In particular, we get $T(H) \subset T(G) \subset H$. On the other hand, since $H \subset \Omega$ and H is τ-bounded (notice $H \subset co(T(\Omega) \cup \{x_0\})$), convex and $x_0 \in H$, we have that $H \in \mathcal{F}$ and $G \subset H$. Therefore $G = H$ as claimed.

Now we claim that $\Phi_\tau(H) = \Phi_\tau(T(H))$. Clearly from (ii)-Definition 3.7, we have $\Phi_\tau(T(H)) \leq \Phi_\tau(H)$ (since $T(H) \subset H$). Now using $(i) - (iii)$ of Definition 3.7 and the fact that $G = H$, we get

$$\Phi_\tau(H) \leq \Phi_\tau(\overline{co}^\tau(T(G) \cup \{x_0\})) \leq \Phi_\tau((T(G) \cup \{x_0\})) = \Phi_\tau(T(G)) = \Phi_\tau(T(H)).$$

Keeping in mind that T is Φ_τ-condensing, we conclude (via (iv)-Definition 3.7) that $\Phi_\tau(H) = 0$ and so $\overline{H}^\tau$ is τ-compact. Since $T(H) \subset H$ we get that $T|_H : H \longrightarrow \overline{H}^\tau$ is a τ-sequentially continuous mapping. By Theorem 2.1 in [32], we get

$$\theta \in \overline{\{x - Tx, x \in H\}}.$$

Thus, there is a net $\{x_\sigma\} \subset H$ so that $x_\sigma - Tx_\sigma \xrightarrow{\tau} \theta$.

Claim. There exists a sequence $\{x_n\} \subset \{x - Tx : x \in H\}$ so that $x_n - Tx_n \xrightarrow{\tau} \theta$. Indeed, since $\overline{H}^\tau$ is compact, so is $\overline{H}^\tau - \overline{H}^\tau$. By assumption, this set is angelic. In particular, since $\theta \in \overline{\{x_\sigma - Tx_\sigma : \sigma\}}^\tau \subset \overline{H}^\tau - \overline{H}^\tau$, there is a sequence in $\{x_n\}$ so that $x_n - Tx_n \xrightarrow{\tau} \theta$.

(ii) Let $C = \overline{H}^\tau$. Redefine the set $\mathcal{F}$ to be

$$\mathcal{F}_* = \{Q \subset \Omega : Q \text{ is } \tau-\text{bounded}, \tau-\text{closed, convex}, x_0 \in Q \text{ and } T(Q) \subset Q\}.$$

We can prove (using the same argument as in the proof of (i)) and the angelicity of C) that $T(C) \subset C$. Hence $T|_C : C \longrightarrow C$ is τ-sequentially continuous map on C. Again by Theorem 2.1 in [32], we get

$$\theta \in \overline{\{x - Tx, x \in C\}}^\tau.$$

Using this and once more the fact that C is angelic, we can find a point $x \in C$ so that $Tx = x$. So $F(T)$ is nonempty. In addition, we have $T(F(T)) = F(T)$ and $F(T)$ is τ-bounded. Hence $\Phi_\tau(F(T)) = 0$ which means that $F(T)$ is relatively τ-compact. Moreover in view of the τ-sequential continuity of T, we deduce that $F(T)$ is τ-sequentially closed. Now we show that $F(T)$ is τ-closed. To this end let $x \in \Omega$ be in $\overline{F(T)}^\tau$. Since $\overline{F(T)}^\tau$ is τ-compact, by the angelicity of $\overline{F(T)}^\tau$, there exists a sequence $\{x_n\} \subset F(T)$ such that $x_n \xrightarrow{\tau} x$. Hence $x \in F(T)$. Thus $\overline{F(T)}^\tau = F(T)$ which means that $F(T)$ is τ-compact. The proof is complete.

■

The next example shows that the angelicity assumption in Theorem 3.17 cannot be dropped.

Example 3.1. There is a Hausdorff locally convex space X equipped with its weak topology, a nonempty compact convex subset $\Omega \subset X$, and a sequentially continuous map $T : \Omega \longrightarrow \Omega$ with no approximate fixed point sequence.

Let $X = (l_\infty^*, w^*)$ and $\Omega = \{\mu \in X \,:\, \mu \geq 0 \text{ and } \|\mu\| \leq 1\}$. Then X is a locally convex space, the topology is its weak one, and Ω is a nonempty convex subset of X. It remains to construct the map T.

The space l_∞^* can be canonically identified with the space $M(\beta\mathbb{N})$ of signed radon measures on the compact space $\beta\mathbb{N}$ (Čech–Stone compactification of natural numbers). Let $P \,:\, M(\beta\mathbb{N}) \longrightarrow M(\beta\mathbb{N})$ be defined by

$$P(\mu) = \sum_{n=1}^{\infty} \mu(n)\delta_n, \qquad \mu \in M(\beta\mathbb{N}),$$

where δ_x denotes the Dirac measure supported by x. Then P is a bounded linear operator. We set $\Omega_0 = P(\Omega)$. Then $\Omega_0 \subset \Omega$ and Ω_0 is a convex subset of l_∞^* which is not totally bounded in norm. Hence, by Theorem 1 in [142], there is a Lipschitz map $F : \Omega_0 \longrightarrow \Omega_0$ without an approximate fixed point sequence (with respect to the norm).

Set $T = F \circ P \,|_\Omega$. We claim that T is weak*-to-weak* sequentially continuous and has no approximate fixed point sequence in the weak* topology.

To show the first assertion, let (μ_n) be a sequence in Ω weak* converging to some $\mu \in \Omega$. Since l_∞ is a Grothendieck space, $\mu_n \longrightarrow \mu$ weakly in l_∞^*. Since P is a bounded linear operator, it is also weak-to-weak continuous, hence $P\mu_n \longrightarrow P\mu$ weakly in l_∞^*. Since $P(l_\infty^*)$ is isometric to the space l^1, by the Schur property we have $P\mu_n \longrightarrow P\mu$ in the norm, so $F(P\mu_n) \longrightarrow F(P\mu)$ in the norm. We conclude that $T(\mu_n) \longrightarrow T(\mu)$ in the norm, and hence is also in the weak* topology. This completes the proof that T is sequentially continuous.

Next, suppose that (μ_n) is an approximate fixed point sequence in Ω. Then $\mu_n - T(\mu_n) \longrightarrow \theta$ in the weak* topology. By the Grothendieck property of l_∞ we get that $\mu_n - T(\mu_n) \longrightarrow \theta$ weakly in l_∞. Since P is a bounded linear operator, we get $P\mu_n - PT(\mu_n) \longrightarrow \theta$ weakly, so $P\mu_n - PT(\mu_n) \longrightarrow \theta$ in the norm by the Schur theorem. Further,

$$P\mu_n - PT(\mu_n) = P\mu_n - T(\mu_n) = P\mu_n - F(P\mu_n),$$

so $(P\mu_n)$ is an approximate fixed point sequence for F with respect to the norm. This is a contradiction, completing the proof.

Corollary 3.15. *Let Ω be a nonempty, convex subset of a Hausdorff topological vector space (E, Γ) and let Φ_τ be a τ-MNC in E. Assume that compact sets in (E, τ) are angelic, Ω is τ-closed and $T : \Omega \to \Omega$ is τ-sequentially continuous and τ-compact mapping with bounded range. Then the set $F(T)$ of fixed points of T is nonempty and τ-compact.*

Indeed, the above assertion is an immediate consequence of Theorem 3.17 since T is obviously Φ_τ-condensing, where Φ_τ is an arbitrary τ-MNC in E.

Corollary 3.16. *Let Ω be a nonempty, convex subset of a Hausdorff topological vector space (E,Γ) and let Φ_τ be a subadditive τ-MNC in E. Assume that compact sets in (E,τ) are angelic and Ω is τ-closed. Let $T:\Omega\to E$ and $S:\Omega\to E$ be two mappings satisfying the following conditions:*

(i) T is τ-sequentially continuous and τ-compact.
(ii) S is Φ_τ-condensing and τ-sequentially continuous.
(iii) $(T+S)(\Omega)$ is a bounded subset of Ω.

Then there exists $x\in\Omega$ such that $x=Tx+Sx$

Proof. Obviously $T+S$ is τ-sequentially continuous. Suppose D is a bounded subset of Ω. Then we have

$$\Phi_\tau((T+S)(D))\le\Phi_\tau(T(D)+S(D))\le\Phi_\tau(T(D))+\Phi_\tau(S(D))\le\Phi_\tau(S(D)),$$

since $T(D)$ is relatively τ-compact. Thus, if $\Phi_\tau(D)>0$, we get

$$\Phi_\tau((T+S)(D))<\Phi_\tau(D),$$

which yields that $T+S$ is Φ_τ-condensing and we can apply Theorem 3.17 to conclude that there exists $x\in\Omega$ such that $x=Tx+Sx$. The proof is complete.

■

Corollary 3.17. *Let Ω be a nonempty, closed and convex subset of a Banach space E and let $T:\Omega\to E$, $S:E\to E$ be weakly sequentially continuous mappings satisfying the following conditions:*

(i) T is weakly compact.
(ii) S is a nonlinear contraction, i.e., there exists a continuous nondecreasing function $\Psi:\mathbb{R}_+\to\mathbb{R}+=[0,\infty)$ with $\Psi(z)<z$ for $z>0$ and such that $||Sx-Sy||\le\Psi(||x-y||)$ for $x,y\in\Omega$.
(iii) $(T+S)(\Omega)$ is a bounded subset of Ω.

Then there exists $x\in\Omega$ such that $x=Tx+Sx$.

Proof. Keeping in mind Corollary 3.16 it is sufficient to show that S is β-condensing. To this end take a bounded subset D of Ω. Suppose that $\beta(D)=d>0$. Let $\varepsilon>0$, and then there exists a weakly compact set K of E with $D\subseteq K+B_{d+\varepsilon}(\theta)$. So for $x\in D$ there exist $y\in K$ and $z\in B_{d+\varepsilon}(\theta)$ such that $x=y+z$ and so

$$||Sx-Sy||\le\Psi(||x-y||)\le\Psi(d+\varepsilon).$$

It follows immediately that

$$S(D)\subseteq S(K)+B_{\Psi(d+\varepsilon)}(\theta).$$

Moreover, since S is a weakly sequentially continuous mapping and K is weakly compact (see Remark 3.11) then $\overline{S(K)}^w$ is weakly compact. Therefore, $\beta(S(D)) \leq \Psi(d+\varepsilon)$. Since $\varepsilon > 0$ is arbitrary, it follows that $\beta(S(D)) \leq \Psi(d) < d = \beta(D)$. Accordingly, S is β-condensing and the proof is complete. ■

Theorem 3.18. *Let Ω be a nonempty, convex subset of a Hausdorff topological vector space (E, Γ) and let Φ_τ be a subadditive τ-MNC in E. Assume that compact sets in (E, τ) are angelic and Ω is τ-closed. If $T : \Omega \to E$ and $S : \Omega \to E$ are τ-sequentially continuous mappings satisfying the following conditions:*

(i) T is τ-compact,
(ii) S is Φ_τ-condensing,
(iii) $I - S$ is invertible on $T(\Omega)$,
(iv) $[y = Tx + Sy,\ x \in \Omega] \Rightarrow y \in \Omega$,
(v) $(I - S)^{-1}T(\Omega)$ is bounded,

then there exists $x \in \Omega$ such that $x = Tx + Sx$.

Proof. First we show that the mapping $(I - S)^{-1}T$ transforms Ω into itself. In fact, by assumption (iii) for each $y \in \Omega$ the equation $z = Ty + Sz$ has a unique solution z. On the other hand, assumption (iv) implies that $z = (I - S)^{-1}Ty$ is in Ω.

Further, define the mapping $F : \Omega \to \Omega$ by putting

$$Fx = (I - S)^{-1}Tx\,.$$

Let $D = \overline{co}^\tau F(\Omega)$. Observe that the set D is τ-closed, convex, τ-bounded and $F(D) \subset D \subset \Omega$. Next, denote $D_1 = \overline{co}F(D)$. Obviously, D_1 is also τ-closed, convex, τ-bounded and $F(D_1) \subset D_1 \subset D \subset \Omega$.

We claim that D_1 is τ-compact. If this is not the case, then $\Phi_\tau(D_1) > 0$. Since $F(D) \subset T(D) + SF(D)$, we obtain

$$\begin{aligned}\Phi_\tau(D_1) &\leq \Phi_\tau(F(D)) \leq \Phi_\tau(T(D) + SF(D))\\ &\leq \Phi_\tau(T(D)) + \Phi_\tau(SF(D))\,.\end{aligned}$$

Since T is τ-compact, we have $\Phi_\tau(T(D)) = 0$. Thus, taking into account that S is Φ_τ-condensing, we get

$$\Phi_\tau(D_1) \leq \Phi_\tau(F(D)) \leq \Phi_\tau(S(F(D))) < \Phi_\tau(F(D))\,,$$

which is absurd. Hence we obtain that D_1 is τ-compact.

In view of Corollary 3.15 it remains to show that $F : D_1 \to D_1$ is τ-sequentially continuous. To do this take a sequence $\{x_n\} \subset D_1$ such that $x_n \xrightarrow{\tau} x$ and $x \in D_1$. Because the set $\{Fx_n\}$ is relatively τ-compact then applying the angelicity of (E, τ) and passing to a subsequence $\{x_{n_j}\}$ of the sequence (x_n), we get that $Fx_{n_j} \xrightarrow{\tau} y$, $y \in D_1$. Hence we have that

$$-Tx_{n_j} + Fx_{n_j} \xrightarrow{\tau} -Tx + y\,.$$

On the other hand, by virtue of the τ-sequential continuity of S we deduce that $SFx_{n_j} \xrightarrow{\tau} Sy$. Combining the above established facts with the equality

$$SF = -T + F,$$

we derive that $y = Fx$.

Now we claim that $Fx_n \xrightarrow{\tau} Fx$. Suppose that this is not the case. Then there exists a subsequence $\{x_{n_k}\}$ and a neighborhood V of Fx in (E, τ) such that $Fx_{n_k} \notin V$ for all k. On the other hand we have that $x_{n_k} \xrightarrow{\tau} x$, so arguing as before we can find a subsequence $\{x_{n_{k_s}}\}$ such that $Fx_{n_{k_s}} \xrightarrow{\tau} Fx$. Thus we obtain a contradiction. Hence it follows that F is τ-sequentially continuous.

Finally, applying Corollary 3.15 we conclude that F has a fixed point $x \in D_1$, which means that $Tx + Sx = x$. This completes the proof. ■

Theorem 3.19. *Let Ω be a nonempty, closed and convex subset of a Banach space E and let $T : \Omega \to E$, $S : E \to E$ be weakly sequentially continuous mappings satisfying the following conditions:*

(i) T is weakly compact.
(ii) S is a nonlinear contraction.
(iii) There exists a bounded subset D of E such that $T(\Omega) \subset (I - S)(D)$.
(iv) $[y = Tx + Sy,\ x \in \Omega] \Rightarrow y \in \Omega$.

Then there exists $x \in \Omega$ such that $x = Tx + Sx$.

Proof. First we show that $I - S$ is invertible on $T(\Omega)$. To this end fix arbitrarily $y \in \Omega$ and consider the mapping $S_y : E \to E$, defined in the following way:

$$S_y z = Ty + Sz.$$

Obviously S_y is a nonlinear contraction. Thus, by a result from [12] we infer that the operator S_y has a unique fixed point $z \in E$. Joining this statement with assumption (iv) we derive that $z \in \Omega$. This means that $I - S$ is invertible on $T(\Omega)$.

Further observe that in view of the above facts and assumption (iv) we have that

$$(I - S)^{-1} T(\Omega) \subset D.$$

Therefore, the conclusion of our theorem follows from Theorem 3.18. The proof is complete. ■

Definition 3.11. Let Ω be a subset of a Hausdorff topological vector space E. A mapping $T : \Omega \to E$ is said to be demi-τ-compact whenever for any sequence $\{x_n\} \subset \Omega$ such that the sequence $x_n - Tx_n \xrightarrow{\tau} y \in E$, there exists a τ-convergent subsequence of the sequence $\{x_n\}$. In the case when $y = \theta$, we say that T is demi-τ-compact at θ.

Definition 3.12. Let Ω be a subset of a Banach space E. A mapping $T : \Omega \to E$ is said to be demi-weakly compact whenever for any sequence $\{x_n\} \subset \Omega$ such that the sequence $x_n - Tx_n \rightharpoonup y \in E$, there exists a weakly convergent subsequence of the sequence $\{x_n\}$. In the case when $y = \theta$, we say that T is demi-weakly compact at θ.

Theorem 3.20. *Suppose Ω is a nonempty, closed, and convex subset of a Banach space E. Next assume that the operators $T : \Omega \to E$, $S : E \to E$ are weakly sequentially continuous and satisfy the following conditions:*

(i) *T is weakly compact.*
(ii) *S is nonexpansive (i.e., $||Sx - Sy|| \leq ||x - y||$ for all $x, y \in E$) and demi-weakly compact.*
(iii) *There exists a bounded subset D of E and a sequence $(\lambda_n) \subset (0,1)$ such that $\lambda_n \to 1$, $T(\Omega) \subset (I - \lambda_n S)(D)$ and $[y = \lambda_n Sy + Tx,\ x \in \Omega] \Rightarrow y \in \Omega$ for all $n = 1, 2, \ldots$.*

Then there exists $x \in \Omega$ such that $x = Tx + Sx$.

Proof. Observe that from the imposed assumptions it follows easily that for any natural number n the mapping $\lambda_n S$ is weakly sequentially continuous and is a nonlinear contraction. Thus, applying Theorem 3.19 to the mapping $T + \lambda_n S$ we conclude that there exists a fixed point of this mapping belonging to Ω, i.e., there exists $x_n \in \Omega$ such that

$$x_n = Tx_n + \lambda_n Sx_n$$

for $n = 1, 2, \ldots$.

Next, notice that $\{x_n\}$ is a bounded sequence in the set D mentioned in assumption (iii). Indeed, this statement is a consequence of the fact that $I - \lambda_n S$ is invertible on $T(\Omega)$ (to show this it is sufficient to adopt a suitable part of Theorem 3.19), assumption (iii) and the equalities:

$$(I - \lambda_n S)x_n = Tx_n \in (I - \lambda_n S)(D) ,$$

$$x_n = (I - \lambda_n S)^{-1} Tx_n \in D$$

for $n = 1, 2, \ldots$.

Now, in view of assumption (i), without loss of generality we can assume that $Tx_n \rightharpoonup y$, $y \in E$. Since the sequence $\{Sx_n\}$ is bounded and $\lambda_n \to 1$, in light of the above we deduce that

$$x_n - Sx_n = Tx_n + (\lambda_n - 1)Sx_n \rightharpoonup y ,$$

where $y \in E$.

Further, taking into account the demi-weak compactness of the operator S, we derive that there exists a weakly convergent subsequence $\{x_{n_k}\}$ of the sequence $\{x_n\}$, i.e., $x_{n_k} \rightharpoonup x$, $x \in \Omega$. Obviously, we have

$$x_{n_k} = Tx_{n_k} + \lambda_{n_k} Sx_{n_k}$$

Hence, using the weak sequential continuity of T and S, we conclude that $x = Tx + Sx$. Thus x is a fixed point of the operator $T + S$ belonging to Ω. The proof is complete. ■

Theorem 3.21. *Let Ω be a nonempty, convex subset of a Hausdorff topological vector space (E, Γ) and let Φ_τ be a positively homogeneous and subadditive τ-MNC in E. Assume that compact sets in (E, τ) are angelic and Ω is τ-closed. In addition, assume that $T : \Omega \to E$, $S : \Omega \to E$ are τ-sequentially continuous operators satisfying the following conditions:*

(i) T is τ-compact.
(ii) S is Φ_τ-nonexpansive and demi-τ-compact.
(iii) There exists a bounded subset Ω_0 of Ω and a sequence $(\lambda_n) \subset (0, 1)$ such that $\lambda_n \to 1$ and $(T + \lambda_n S)(\Omega) \subset \Omega_0$.

Then there exists $x \in \Omega$ such that $x = Tx + Sx$.

Proof. Define the sequence of operators by putting $G_n = T + \lambda_n S$ for $n = 1, 2, \ldots$. Assumption (iii) implies that $G_n(\Omega)$ is bounded for $n = 1, 2, \ldots$.

Further, take an arbitrary bounded subset D of Ω. Then we obtain

$$\Phi_\tau(G_n(D)) \le \Phi_\tau(T(D) + \lambda_n S(D))$$
$$\le \Phi_\tau(T(D)) + \lambda_n \Phi_\tau(S(D)) = \lambda_n \Phi_\tau(S(D)) .$$

Hence, if $\Phi_\tau(D) > 0$, we get

$$\Phi_\tau(G_n(D)) < \Phi_\tau(D) .$$

Thus G_n is Φ_τ-condensing on Ω. Obviously G_n is τ- sequentially continuous, so by Theorem 3.17 we infer that G_n has a fixed point x_n in Ω, for any $n = 1, 2, \ldots$.

Now, repeating a suitable part of the proof of the preceding theorem we get the desired conclusion. This completes the proof. ■

Corollary 3.18. *Let Ω be a nonempty, closed and convex subset of E and let $T : \Omega \to E$, $S : E \to E$ be weakly sequentially continuous mappings satisfying the below listed conditions:*

(i) T is weakly compact.
(ii) S is nonexpansive and demi-weakly compact.
(iii) There exists a bounded subset Ω_0 of Ω and a sequence $(\lambda_n) \subset (0, 1)$ such that $\lambda_n \to 1$ and $(T + \lambda_n S)(\Omega) \subset \Omega_0$ for $n = 1, 2, \ldots$.

Then there exists $x \in \Omega$ such that $x = Tx + Sx$.

Proof. The proof follows immediately from Theorem 3.21, provided we show that S is β-nonexpansive, where β is the De Blasi measure of weak noncompactness in E.

To do this take D a bounded subset of Ω and put $d = \beta(D)$. Fix $\varepsilon > 0$. Then there exists a weakly compact set K with $D \subset K + B_{d+\varepsilon}(\theta)$. This yields that for each $x \in D$ there exist $y \in K$ and $z \in B_{d+\varepsilon}(\theta)$ such that $x = y + z$. Moreover, we have

$$||Sx - Sy|| \le ||x - y|| \le d + \varepsilon .$$

Hence we obtain

$$S(D) \subset S(K) + B_{d+\varepsilon}(\theta) .$$

Since S is weakly sequentially continuous and K is weakly compact, then $S(K)$ is weakly compact (see Remark 3.11). This implies that $\beta(S(D)) \leq d + \varepsilon$. In view of the arbitrariness of ε we get that $\beta(S(D)) \leq d = \beta(D)$. Thus, S is β-nonexpansive which completes the proof. ■

Definition 3.13. Let Ω be a nonempty, τ-closed subset of a Hausdorff topological vector space E and let $T : \Omega \to E$ be a τ-sequentially continuous operator. T will be called a τ -semi-closed operator at θ (τ-sc, in short) if the conditions $x_n \in \Omega$, $x_n - Tx_n \to \theta$ imply that there exists $x \in \Omega$ such that $Tx = x$.

Lemma 3.7. *Let Ω be a τ-closed subset of a Hausdorff topological vector space E and let $T : \Omega \to E$ be a τ-sequentially continuous mapping being demi-τ-compact at θ. Then T is a τ-semi-closed mapping at θ.*

Proof. Suppose $\{x_n\}$ is a sequence in Ω such that $x_n - Tx_n \to \theta$. Since T is demi-τ-compact we infer that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and an element $x \in E$ such that $x_{n_k} \xrightarrow{\tau} x$.

We claim that $x \in \Omega$ and $Tx = x$. Indeed, since Ω is τ-closed, so $x \in \Omega$. Moreover, the τ-sequential continuity of T implies that $Tx_{n_k} \xrightarrow{\tau} Tx$. On the other hand, we have

$$x_{n_k} - Tx = (x_{n_k} - Tx_{n_k}) + (Tx_{n_k} - Tx) \xrightarrow{\tau} \theta .$$

This yields that $x_{n_k} \xrightarrow{\tau} Tx$. Hence we infer that $Tx = x$ and the proof is complete. ■

Theorem 3.22. *Let Ω be a nonempty, convex subset of a Hausdorff topological vector space (E, Γ) and let Φ_τ be a positively homogeneous and subadditive τ-MNC in E. Assume that compact sets in (E, τ) are angelic and Ω is τ-closed. Further, assume that $T : \Omega \to E$, $S : \Omega \to E$ are τ-sequentially mappings satisfying the following conditions:*

(i) T is τ-compact.
(ii) S is Φ_τ-nonexpansive.
(iii) T is τ-semi-closed at θ
(iv) $(T + S)(\Omega)$ is a bounded subset of Ω.

Then there exists $x \in \Omega$ such that $x = Tx + Sx$.

Proof. Fix $z \in \Omega$ and define $G_n = \lambda_n(T+S)+(1-\lambda_n)z$ $(n = 1, 2, \dots)$, where (λ_n) is a sequence in $(0, 1)$ such that $\lambda_n \to 1$. Since Ω is convex and $z \in \Omega$, in view of assumption (iv) we deduce that G_n maps Ω into itself. Moreover, $G_n(\Omega)$ is bounded for any $n = 1, 2, \dots$. Obviously G_n is τ-sequentially continuous.

Now, assume that D is an arbitrary bounded subset of Ω. Then we have

$$\begin{aligned}\Phi_\tau(G_n(D)) &= \Phi_\tau(\{\lambda_n(T+S)(D))\} + \{(1-\lambda_n)z\}) \\ &\leq \lambda_n \Phi_\tau((T+S)(D)) \\ &\leq \lambda_n \Phi_\tau(T(D)) + \lambda_n \Phi_\tau(S(D)) \\ &= \lambda_n \Phi_\tau(S(D)) \\ &\leq \lambda_n \Phi_\tau(D).\end{aligned}$$

Thus, if $\Phi_\tau(D) > 0$, we get

$$\Phi_\tau(G_n(D)) < \Phi_\tau(D)\ .$$

Therefore, G_n is Φ_τ-condensing on Ω and we can apply Theorem 3.17 to obtain a sequence (x_n) such that $(x_n) \subset \Omega$ and $G_n x_n = x_n$ for $n = 1, 2, \ldots$. Consequently, we obtain

$$x_n - (T+S)x_n = (\lambda_n - 1)[(T+S)x_n - z] \to 0\ ,$$

since $\lambda_n \to 1$ as $n \to \infty$ and $(T+S)(\Omega)$ is bounded. Finally, keeping in mind assumption (iii) we conclude that there exists $x \in \Omega$ such that $Tx + Sx = x$. The proof is complete. ■

Corollary 3.19. *Let Ω be a nonempty, closed and convex subset of a Banach space E. Let $T : \Omega \to E$ and $S : E \to E$ be weakly sequentially continuous mappings satisfying the following conditions:*

(i) *T is weakly compact.*
(ii) *S is nonexpansive.*
(iii) *$T + S$ is wsc.*
(iv) *$(T+S)(\Omega)$ is a bounded subset of Ω.*

Then there exists $x \in \Omega$ such that $x = Tx + Sx$.

3.5 Applications

3.5.1 A Volterra Integral Equation Under Henstock–Kurzweil–Pettis Integrability

We consider the existence of a weak solution to the Volterra integral equation

$$x(t) = h(t) + \int_0^t K(t,s) f(s, x(s)) ds \ \text{ on } I,$$

here "$\int$" denotes the HKP-integral.

First, we introduce the concept of Henstock–Kurzweil–Pettis integrability and give some related facts which are useful in the sequel. Throughout the next sections, E will be a real Banach space.

Definition 3.14. A function $f : I \longrightarrow E$ is said to be Henstock–Kurzweil-integrable, or simply HK-integrable on I, if there exists $w \in E$ with the following property : for $\varepsilon > 0$ there exists a gauge δ on I such that $\|\sigma(g, \mathcal{P}) - w\| < \varepsilon$ for each δ-fine Perron partition $\mathcal{P}$ of I. We set $w = (HK) \int_0^T f(s)ds$.

Remark 3.13. This definition includes the generalized Riemann integral. In a special case, when δ is a constant function, we get the Riemann integral.

The following result states that the HK-integrability for real functions is preserved under multiplication by functions of bounded variation.

Lemma 3.8. *Let $f : I \longrightarrow \mathbb{R}$ be an HK-integrable function and let $g : I \longrightarrow \mathbb{R}$ be of bounded variation. Then fg is HK-integrable.*

Let us recall the following integration by parts result inspired from the previous lemma.

Lemma 3.9. *Let $f : [a, b] \longrightarrow \mathbb{R}$ be HK-integrable function and let $g : I \longrightarrow \mathbb{R}$ be of bounded variation. Then, for every $t \in [a, b]$*

$$(HK) \int_a^t f(s)g(s)ds = g(t)(HK) \int_a^t f(s)ds - \int_a^t \left((HK) \int_a^s f(\tau)d\tau \right) dg(s),$$

the last integral being of Riemann–Stieltjes type.

The generalization of the Pettis integral obtained by replacing the Lebesgue integrability of the functions by the Henstock–Kurzweil integrability produces the Henstock–Kurzweil–Pettis integral.

Definition 3.15 ([60]). A function $f : I \longrightarrow E$ is said to be Henstock–Kurzweil–Pettis integrable, or simply HKP-integrable, on I if there exists a function $g : I \longrightarrow E$ with the following properties :

(i) $\forall x^* \in E^*, x^*f$ is Henstock–Kurzweil integrable on I.
(ii) $\forall t \in I, \forall x^* \in E^*, x^*g(t) = (HK) \int_0^t x^*f(s)ds$.

This function g will be called a primitive of f and by $g(T) = \int_0^T f(t)dt$ we will denote the Henstock–Kurzweil–Pettis integral of f on the interval I.

Remark 3.14. (i) Any HK-integrable function is HKP-integrable. The converse is not true. Then the family of all Henstock–Kurzweil–Pettis integrable functions is larger than the family of all Henstock–Kurzweil integrable ones.

(ii) Since each Lebesgue integrable function is HK-integrable, we find that any Pettis integrable function is HKP-integrable. The converse is not true.

Theorem 3.23 (Mean Value Theorem [60]). *If the function $f : [a,b] \longrightarrow E$ is HKP-integrable, then*

$$\int_J f(t)dt \in |J|\,\overline{conv}(f(J)),$$

where J is an arbitrary subinterval of $[a,b]$ and $|J|$ is the length of J.

Theorem 3.24. *Let $f : I \times E \longrightarrow E$, $h : I \longrightarrow E$ and $K : I \times I \longrightarrow \mathbb{R}$ satisfy the following conditions:*

(1) h is weakly continuous on I.
(2) For each $t \in I$, $K(t,.)$ continuous, $K(t,.) \in BV(I,\mathbb{R})$ and the application $t \longmapsto K(t,.)$ is $\|.\|_{BV}$-continuous. (Here $BV(I,\mathbb{R})$ represents the space of real bounded variation functions with its classical norm $\|.\|_{BV}$.)
(3) $f : I \times E \longrightarrow E$ is a weakly–weakly continuous function such that for all $x \in C_w(I,E)$, for all $t \in I$, $f(.,x(.))$ and $K(t,.)f(.,x(.))$ are HKP-integrable on I.
(4) For all $r > 0$ and $\varepsilon > 0$, there exists $\delta_{\varepsilon,r} > 0$ such that

$$\|\int_\tau^t f(s,x(s))ds\| < \varepsilon, \quad \forall\ |t-\tau| < \delta_{\varepsilon,r}, \forall\, x \in C_w(I,E), \|x\| \le r. \tag{3.15}$$

(5) There exists a nonnegative function $L(.,.)$ such that:

(a) For each closed subinterval J of I and bounded subset X of E,

$$\beta(f[J \times X]) \le \sup\{L(t,\beta(X)), t \in J\}, \tag{3.16}$$

(b) The function $s \longmapsto L(s,r)$ is continuous for each $r \in [0,+\infty[$, and

$$\sup_{t\in I}\left\{(HK)\int_0^t |K(t,s)|\,L(s,r)ds\right\} < r \tag{3.17}$$

for all $r > 0$.

Then there exist an interval $J = [0,a]$ such that a set of weakly continuous solutions of the Volterra-type integral equation

$$x(t) = h(t) + \int_0^t K(t,s)f(s,x(s))ds, \tag{3.18}$$

defined on J is nonempty and compact in the space $C_w(J,E)$.

Remark 3.15. (a) If $f(.,x(.))$ is *HKP*-integrable on I and for all $\tau \in I$ the application $T_{t,\tau} : E^* \longrightarrow \mathbb{R}$, defined by $y^* \longmapsto (\text{HK})\int_0^\tau K(t,s)y^*f(s,x(s))ds$, is weak*-continuous, then $K(t,.)f(.,x(.))$ is HKP-integrable on I. Indeed, for $\tau \in I$, because $T_{t,\tau}$ is a linear functional on E^* that is weak*-continuous, then by [176, Theorem 3.10] there exists $w_{t,\tau}$ in E such that $T_{t,\tau}(y^*) = y^*w_{t,\tau}$ for all $y^* \in E^*$. So, $(HK)\ \int_0^\tau K(t,s)y^*f(s,x(s))ds = (\text{HK})\int_0^\tau y^*K(t,s)f(s,x(s))ds = y^*w_{t,\tau}$ for all $y^* \in E^*$. Therefore $K(t,.)f(.,x(.))$ is HKP-integrable on I.

(b) For $\tau \in I$, if we suppose the HK-equi-integrability of the family $\{y^*K(t,.)f(.,x(.)), y^* \in E^*, \|y^*\| \leq 1\}$ on $[0,\tau]$, then we guarantee the continuity of $T_{t,\tau}$ with respect to the weak*-topology (see [80]).

Remark 3.16. The condition (3.15) is satisfied if we suppose that $f(.,x(.))$ is HKP-integrable on I and for all $r > 0$, there exists a HK-integrable function $M_r : I \longrightarrow \mathbb{R}_+$ such that

$$\|f(t,y)\| \leq M_r(t) \text{ for all } t \in I \text{ and } y \in E, \|y\| \leq r.$$

To see this, let $r > 0$ and $x^* \in E^*$ such that $\|x^*\| \leq 1$. For $0 \leq t_1 < t_2 \leq 1$, we have $\left|x^* \int_{t_1}^{t_2} f(s,x(s),Tx(s))ds\right| \leq \left|(HK)\int_{t_1}^{t_2} x^*f(s,x(s),Tx(s))ds\right|$. Because $s \longmapsto M_{b_0}(s)$ is Henstock–Kurzweil integrable and $|x^*f(s,x(s),Tx(s))| \leq \|x^*\|\|f(s,x(s),Tx(s))\| \leq M_{b_0}(s)$ for all $s \in [0,1]$, then by [128, Corollary 4.62]), $s \longmapsto x^*f(s,x(s),Tx(s))$ is absolutely Henstock–Kurzweil integrable on $[t_1,t_2]$ and

$$\left|(HK)\int_{t_1}^{t_2} x^*f(s,x(s),Tx(s))ds\right| \leq (HK)\int_{t_1}^{t_2} M_{b_0}(s)ds.$$

Thus

$$\|\int_{t_1}^{t_2} f(s,x(s),Tx(s))ds\| = \sup_{\|x^*\|\leq 1}\left|x^*\int_{t_1}^{t_2} f(s,x(s),Tx(s))ds\right| \leq (HK)\int_{t_1}^{t_2} M_{b_0}(s)ds,$$

which thanks to the continuity of the primitive in Henstock–Kurzweil integral, becomes less then ε for t_2 sufficiently close to t_1, and this proves the claim.

Remark 3.17. The inequality condition in Remark 3.16 is fulfilled if we suppose that E is reflexive and the function M_r is independent of $t \in I$ (see [161]). ◇

Proof. Let $c = \sup_{t\in I}\|h(t)\|$, $d = \sup_{t\in I}\|K(t,.)\|_{BV}$ and $\mu > 0$. There exists $b > 0$ such that $\mu < \frac{b-c}{d}$. From (3.15), there exists $a \leq T$ such that

$$\sup_{t\in[0,a]}\|\int_0^t f(s,x(s))ds\| < \mu,$$

for any $x \in C_w(I,E)$ satisfying $\|x\| \leq b$. Put $J = [0,a]$, and denote by $C_w(J,E)$ the space of weakly continuous functions $J \longrightarrow E$, endowed with the topology of weak uniform convergence, and by $\tilde{B}$ the set of all weakly continuous functions $J \longrightarrow B_b$, where $B_b = \{y \in E : \|y\| \leq b\}$. We shall consider $\tilde{B}$ as a topological subspace of $C_w(J,E)$. It is clear that the set $\tilde{B}$ is convex and closed. Put

$$F_x(t) = h(t) + \int_0^t K(t,s)f(s,x(s))ds \text{ on } J.$$

We require that $F : \tilde{B} \longrightarrow \tilde{B}$ is continuous.

1. Let $t \in [0,a]$. For any $x^* \in E^*$ such that $\|x^*\| \le 1$, and for any $x \in \tilde{B}$, $x^*F_x(t) = x^*h(t) + \int_0^t K(t,s)x^*f(s,x(s))ds$. Using Lemma 3.9 and the definition of the Riemann–Stieltjes integral, we obtain
   $\left|\int_0^t K(t,s)x^*f(s,x(s))ds\right|$

$$= \left|K(t,t)(HK)\int_0^t x^*f(s,x(s)) - \int_0^t \left((HK)\int_0^s x^*f(\tau,x(\tau)d\tau\right)dK_t\right|,$$

$$\le |K(t,t)| \sup_{\upsilon\in[0,t]} \|\int_0^\upsilon f(s,x(s))ds\| + (V[K_t;0,t]) \sup_{s\in[0,t]} \|\int_0^s f(\tau,x(\tau)d\tau\|,$$

$$\le |K(t,t)| \sup_{\upsilon\in J} \|\int_0^\upsilon f(s,x(s))ds\| + (V[K_t;0,t]) \sup_{s\in J} \|\int_0^s f(\tau,x(\tau)d\tau\|,$$

$$\le \|K(t,.)\|_{BV} \sup_{s\in J} \|\int_0^s f(\tau,x(\tau)d\tau\|.$$

   Here $K_t(.)$ denotes $K(t,.)$ and $V[K_t;0,t]$ denotes the total variation of K_t on the interval $[0,t]$. Hence,

$$|x^*F_x(t)| \le c + d\mu \le b.$$

   Then

$$\sup\{|x^*F_x(t)|, x^* \in E^*, \|x^*\| \le 1\} \le b.$$

   So, $F_x(t) \in B_b$.
2. Now, we will show that $F(\tilde{B})$ is a strongly equicontinuous subset.
   Let $t,\tau \in J$. We suppose without loss of generality that $\tau < t$ and that $F_x(t) \ne F_x(\tau)$. By the Hahn–Banach theorem, there exists $x^* \in E^*$, such that $\|x^*\| = 1$ and $\|F_x(t) - F_x(\tau)\| = x^*(F_x(t) - F_x(\tau))$

$$\le |x^*(h(t)) - x^*(h(\tau))| + \left|(HK)\int_0^\tau (K(t,s) - K(\tau,s))x^*f(s,x(s))ds\right| + \left|(HK)\int_\tau^t K(t,s)x^*f(s,x(s))ds\right|$$

$$\le |x^*(h(t)) - x^*(h(\tau))| + \|K(t,.) - K(\tau,.)\|_{BV} \sup_{\upsilon\in J} \|\int_0^\upsilon f(s,x(s))ds\| + d \sup_{\zeta\in[\tau,t]} \|\int_\tau^\zeta f(s,x(s))ds\|.$$

   So, the result follows from hypotheses (1), (2) and (3.15).

3. Now we will prove the continuity of F.

Since f is weakly continuous, we have by a Krasnoselskii type Lemma (see [191]) that for any $x^* \in E^*$, $\varepsilon > 0$ and $x \in \tilde{B}$ there exists a weak neighborhood U of 0 in E such that $|x^*(f(t,x(t)) - f(t,y(t)))| \leq \frac{\varepsilon}{ad}$ for $t \in J$ and $y \in \tilde{B}$ such that $x(s) - y(s) \in U$ for all $s \in J$. Because the function $s \longmapsto x^*(f(s,x(s)) - f(s,y(s)))$ is HK-integrable on J and the function $s \longmapsto \frac{\varepsilon}{ad}$ is Riemann integrable on J, then by [128, Corollary 4.62]), $s \longmapsto x^*(f(s,x(s)) - f(s,y(s)))$ is absolutely Henstock–Kurzweil-integrable on J and for all $t \in J$ we have:

$$\left|(HK)\int_0^t K(t,s)x^*(f(s,x(s)) - f(s,y(s)))\right| ds$$

$$\leq \sup_{\zeta \in I} \|K(\zeta,.)\|_{BV} \sup_{\tau \in [0,t]} \left(\left|(HK)\int_0^\tau x^*(f(s,x(s)) - f(s,y(s)))ds\right|\right)$$

$$\leq d \sup_{\tau \in J}\left((HK)\int_0^\tau |x^*(f(s,x(s)) - f(s,y(s)))|\, ds\right) \leq \varepsilon.$$

Thus F is continuous.

We have already shown that $F(\tilde{B})$ is bounded and strongly equicontinuous, then by Lemma 2.1 in [143], $Q = \overline{\text{conv}} F(\tilde{B})$ (the closed convex hull of $F(\tilde{B})$ in $C(J,E)$) is also bounded and strongly equicontinuous. Clearly $F(Q) \subset Q \subset \tilde{B}$. We claim that F is β-condensing on Q. Indeed, let V be a subset of Q such that $\beta(V) \neq 0$, $V(t) = \{x(t), x \in V\}$ and $F(V)(t) = \{F_x(t), x \in V\}$. Because V is bounded and strongly equicontinuous, we have by Lemma 3.6 (b) that $\sup_{t \in J} \beta(V(t)) = \beta(V) = \beta(V(J))$. Fix $t \in J$ and $\varepsilon > 0$. From the continuity of the functions $s \longmapsto K(t,s)$ and $s \longmapsto L(s, \beta(V))$ on I, it follows that there exists $\delta > 0$ such that

$$|K(t,\tau)L(q,\beta(V)) - K(t,s)L(s,\beta(V))| < \varepsilon, \tag{3.19}$$

if $|\tau - s| < \delta$, $|q - s| < \delta$, $q, s, \tau \in I$. Divide the interval $[0,t]$ into n subintervals $0 = t_0 < t_1 \ldots < t_n = t$ so that $t_i - t_{i-1} < \delta$ $(i = 1, \ldots, n)$ and put $T_i = [t_{i-1}, t_i]$. For each i, there exists $s_i \in T_i$ such that $L(s_i, \beta(V)) = \sup_{s \in T_i} L(s, \beta(V))$. By the mean value theorem for Henstock–Kurzweil–Pettis integral (see Theorem 3.23), we obtain

$$F_x(t) = h(t) + \sum_{i=1}^n \int_{t_{i-1}}^{t_i} K(t,s)f(s,x(s))ds$$

$$\in h(t) + \sum_{i=1}^n (t_i - t_{i-1})\overline{\text{conv}}\{K(t,s)f(s,x(s)), s \in T_i, x \in V\}.$$

Using (3.16), (3.17) and the properties of the measure of weak non-compactness, we have

$$\begin{aligned}\beta(F(V)(t)) &\leq \sum_{i=1}^{n}(t_i - t_{i-1})\beta(\overline{\text{conv}}\{K(t,s)f(s,x(s)), s \in T_i, x \in V\}) \\ &\leq \sum_{i=1}^{n}(t_i - t_{i-1})\beta(\{K(t,s)f(s,x(s)), s \in T_i, x \in V\}) \\ &\leq \sum_{i=1}^{n}(t_i - t_{i-1}) \sup_{s\in T_i} |K(t,s)| \, \beta(f(T_i \times V(T_i))) \\ &\leq \sum_{i=1}^{n}(t_i - t_{i-1}) \, |K(t,\tau_i)| \, L(s_i, \beta(V)),\end{aligned}$$

here for each i, $\tau_i \in T_i$ is a number such that $|K(t,\tau_i)| = \sup_{s\in T_i} |K(t,s)|$. Hence, using (3.19), we have

$$\begin{aligned}\beta(F(V)(t)) &\leq \sum_{i=1}^{n}\left((HK)\int_{t_{i-1}}^{t_i} |K(t,\tau_i)L(s_i,\beta(V)) - K(t,s)L(s,\beta(V))| \, ds\right) \\ &+ \sum_{i=1}^{n}\left((HK)\int_{t_{i-1}}^{t_i} |K(t,s)| \, L(s,\beta(V))ds\right) \\ &\leq \varepsilon t + (HK)\int_0^t |K(t,s)| \, L(s,\beta(V))ds \\ &\leq \varepsilon t + \sup\left\{(HK)\int_0^{t'} |K(t,s)| \, L(s,\beta(V)), t' \in J\right\}.\end{aligned}$$

As the last inequality is satisfied for every $\varepsilon > 0$, we get

$$\beta(F(V)(t)) \leq \sup\left\{(HK)\int_0^{t'} |K(t,s)| \, L(s,\beta(V))ds, t' \in J\right\}.$$

Applying Lemma 3.6(b) again for the bounded strongly equicontinuous subset F(V), we obtain $\beta(F(V)) = \sup_{t\in J}\{F(V)(t)\}$. Accordingly

$$\beta(F(V)) \leq \sup\left\{(HK)\int_0^{t'} |K(t,s)| \, L(s,\beta(V))ds, t' \in J\right\} < \beta(V),$$

so, F is β-condensing on Q. Since Q is a closed convex subset of $C(J,E)$, the set Q is weakly closed, and using similar arguments as in the proof of

Theorem 3.15, we can suppose that Q is a closed convex subset of $C_w(J,E)$ and so by Corollary 3.13 the set of the fixed points of F in $\tilde{B}$ is nonempty and compact. This means that there exists a set of weakly continuous solutions of the problem (3.18) on J which is nonempty and compact in $C_w(J,E)$.

■

3.5.2 Theory of Integral Equations in the Lebesgue Space

We show that our main result contained in Corollary 3.16 can be applied to the theory of nonlinear integral equations in Lebesgue space.

Suppose that I is a bounded interval in $\mathbb{R}$. For simplicity, we will assume that $I = [0,1]$. Denote by $L^1 = L^1(I)$ the space of Lebesgue integrable real functions on the interval I with the standard norm

$$||x|| = \int_0^1 |x(t)|dt\ .$$

The space L^1 is also called the Lebesgue space.

It is well known [9] that the superposition operator $\mathcal{N}_f$ generated by a function f satisfying Carathéodory conditions transforms the metric space $S(I)$ (the set of measurable (in Lebesgue sense) functions on I) into itself.

Now, let us fix $r > 0$ and denote by Q_r the subset of the ball $B_r(\theta)$ in L^1 consisting of functions being a.e. nondecreasing (or a.e. nonincreasing) on the interval I in the sense that there exists a subset P of I with $m(P) = 0$ and such that each function $x \in Q_r$ is nondecreasing on the set $I \setminus P$ (or nonincreasing on $I \setminus P$). Keeping in mind Theorem 1.28 it is easily seen that the set Q_r is compact in measure.

In what follows we will consider the nonlinear integral equation of the form

$$x(t) = a(t) + \int_0^1 k(t,s)f(s,x(s))ds + \int_0^1 u(t,s,x(s))ds\ , \tag{3.20}$$

for $t \in I$.

If we define on the space

$$(Hx)(t) = \int_0^1 k(t,s)f(s,x(s))ds\ , \tag{3.21}$$

$$(Ux)(t) = \int_0^1 u(t,s,x(s))ds\ , \tag{3.22}$$

for $x \in L^1$ and for $t \in I$, then H is the Hammerstein integral operator while U represents the Urysohn one.

Henceforth we will assume that the functions involved in Eq. (3.20) satisfy the following conditions:

(i) $a \in L^1$ is nonnegative and nondecreasing on the interval I.
(ii) $f : I \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory conditions and there exists a function $p \in L^1$ such that

$$|f(t,x)| \leq p(t)$$

for $t \in I$ and $x \in \mathbb{R}$. Moreover, $f : I \times \mathbb{R}_+ \to \mathbb{R}_+$.
(iii) $k : I \times I \to \mathbb{R}_+$ is measurable with respect to both variables and such that the integral operator K defined on the space L^1 by the formula

$$(Kx)(t) = \int_0^1 k(t,s)x(s)ds$$

maps the space L^1 into itself.

For further purposes let us recall the above assumption implies [122] that the operator K maps continuously the space L^1 into itself. In what follows we will denote by $||K||$ the norm of the linear operator K.

Further, we formulate our remaining assumptions.

(iv) The function $t \to k(t,s)$ is a.e. nondecreasing on the interval I for almost all $s \in I$.
(v) $u(t,s,x) = u : I \times I \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory conditions, that is, u is measurable with respect to (t,s) for any $x \in \mathbb{R}$ and is continuous in x for almost all $(t,s) \in I^2$.
(vi) $u(t,s,x) \geq 0$ for $(t,s) \in I^2$ and for $x \geq 0$.
(vii) The function $t \to u(t,s,x)$ is a.e. nondecreasing on the interval I for almost all $s \in I$ and for each $x \in \mathbb{R}$.
(viii) $|u(t,s,x)| \leq k_1(t,s)(q(t) + b|x|)$ for $(t,s) \in I^2$ and for $x \in \mathbb{R}$, where q is a nonnegative member of L^1, $0 \leq b =$ const. and a function $k_1 : I^2 \to \mathbb{R}_+$ is measurable and such that the linear operator K_1 generated by k_1 maps L^1 into itself.
(ix) $b||K_1|| < 1$.

Then we can formulate our existence result concerning Eq. (3.20).

Theorem 3.25. *Under assumptions (i)–(ix), Eq.* (3.20) *has at least one solution $x \in L^1$ such that x is a.e. nondecreasing on the interval I.*

Proof. Observe that in view of (3.21) and (3.22) we can write Eq. (3.20) in the form

$$x = a + Hx + Ux\,.$$

In order to show that Corollary 3.16 can be applied in our situation let us denote by T the operator defined on L^1 by the formula

$$Tx = a + Hx\,. \tag{3.23}$$

Further, observe that the Hammerstein operator H defined by (3.21) can be written as the product $H = K\mathcal{N}_f$ of the superposition operator

$$(\mathcal{N}_f x)(t) = f(t, x(t))$$

and the linear operator

$$(Kx)(t) = \int_0^1 k(t,s)x(s)ds\ .$$

Next, take an arbitrary function $x \in L^1$. Then, in view of assumptions (i)–(iii) and Theorem 1.23 we infer that $Tx \in L^1$, where T is defined by (3.23). On the other hand, keeping in mind assumptions (v) and $(viii)$ and the majorant principle (cf.[202]) we deduce that the Urysohn operator U transforms the space L^1 into itself and is continuous.

Consider the subset Ω of the space L^1 consisting of all functions $x = x(t)$ being a.e. nonnegative and nondecreasing on the interval I. It is easily seen that the operators T and U transform the set Ω into itself. In fact, this statement is an easy consequence of assumptions (i), (ii), (iv), (vi) and (vii). This allows us to infer that the sum $T + U$ of these operators transforms the set Ω into Ω.

Next, for arbitrarily fixed $x \in \Omega$, in view of the imposed assumptions we obtain

$$|((T+U)(x))(t)| \le a(t) + (Hx)(t) + (Ux)(t), \quad \forall t.$$

Thus

$$\begin{aligned}
||(T+U)(x)|| &\le ||a|| + ||Hx|| + \int_0^1 u(t,s,x(s))ds \\
&\le ||a|| + ||K\mathcal{N}_f x|| + \int_0^1 k_1(t,s)(q(s) + bx(s))ds \\
&\le ||a|| + ||K||||\mathcal{N}_f x|| + \int_0^1 k(t,s)q(s)ds + b\int_0^1 k_1(t,s)x(s)ds \\
&\le ||a|| + ||K|| \int_0^1 f(s,x(s))ds + ||K_1||||q|| + b||K_1||||x|| \\
&\le ||a|| + ||K|| \int_0^1 p(s)ds + ||K_1||||q|| + b||K_1||||x|| \\
&\le ||a|| + ||K||||p|| + ||K_1||||q|| + b||K_1||||x|| \\
&= A + b||K_1||||x||
\end{aligned}$$

where we let

$$A = ||a|| + ||K||||p|| + ||K_1||||q|| \,.$$

This implies

$$||(T + U)(x)|| \leq A + b||K_1||||x|| \,. \tag{3.24}$$

Further, denote by Ω_r the set consisting of all functions x belonging to Ω and such that $||x|| \leq r$, when $r = A/(1 - b||K_1||)$. Obviously the set Ω_r is nonempty, convex, closed and bounded. Moreover, (3.24) with the fact that $T + U$ is a self-mapping of the set Ω and taking into account assumption (ix) we deduce that the operator $T + U$ transforms the set Ω_r into itself. Notice also that both the operators T and U transform the set Ω_r into itself.

Now, assumptions (*ii*) and (*iii*) (cf. also the fact pointed out after assumption (*iii*) which asserts that the operator K is a continuous self-mapping of the space L^1) and taking into account Theorem 1.23 we infer that T transforms continuously the set Ω_r into Ω_r. We deduce that the Urysohn operator U transforms continuously the set Ω_r into itself.

Thus, in view of the fact that Ω_r is compact in measure (cf. Theorem 1.28 and the remarks made before that theorem) we infer that the operators T and U transform weakly continuously the set Ω_r into itself.

Now, we show that the operator T is weakly compact on the set Ω_r. What is more, the operator T is also weakly compact on the set Ω.

To prove this assertion take an arbitrary function $x \in \Omega$. Then, for a fixed $t \in I$ we get:

$$|(Tx)(t)| \leq a(t) + \left| \int_0^1 k(t,s) f(s, x(s)) ds \right|$$

$$\leq a(t) + \int_0^1 k(t,s)|f(s, x(s))| ds \leq a(t) + \int_0^1 k(t,s) p(s) ds \,. \tag{3.25}$$

Hence, taking into account that the function $t \to \int_0^1 k(t,s)p(s)ds$ is an element of the space L^1, from estimate (3.25) and Corollary 1.7 we infer that the set $T(\Omega)$ is weakly compact. Thus the operator T is weakly compact on the set Ω.

In what follows take a nonempty set $X \subset \Omega_r$ and fix $\varepsilon > 0$. Further, let D be a measurable subset of the interval I such that $m(D) \leq \varepsilon$. Then, for an arbitrary $x \in X$, in view of assumption (viii) we obtain

$$\int_D |(Ux)(t)|dt \le \int_D \left(\int_0^1 k_1(t,s)q(s)ds\right) dt + b \int_D \left(\int_0^1 k_1(t,s)x(s)ds\right) dt$$
$$= ||K_1 q||_{L^1(D)} + b||K_1 x||_{L^1(D)} \ ,$$

where by $L^1(D)$ we denote the Lebesgue space of real functions defined on the set D.

Now, taking into account that the operator K_1 maps the space $L^1(D)$ into itself and is continuous, we get

$$\int_D |(Ux)(x)|dt \le ||K_1||_D ||q||_{L^1(D)} + b||K_1||_D ||x||_{L^1(D)}$$
$$= ||K_1||_D \int_D q(t)dt + b||K_1||_D \int_D x(t)dt$$
$$\le ||K_1|| \int_D q(t)dt + b||K_1|| \int_D x(t)dt \ ,$$

where the symbol $||K_1||_D$ stands for the norm of the linear operator K_1 acting from the space $L^1(D)$ into itself.

Further, keeping in mind the fact that any singleton is weakly compact in the space L^1, in view of the Dunford–Pettis theorem and Formula (1.8) we derive the following inequality

$$\beta(UX) \le b||K_1||\beta(X) \ ,$$

where β denotes the De Blasi measure of weak noncompactness. Particularly, in view of assumption (ix) this statement means that the operator U is condensing with respect to β.

Finally, combining all the above established facts and applying Corollary 3.16 we complete the proof. ■

Chapter 4
Fixed Points for Maps with Weakly Sequentially Closed Graph

In this chapter, we discuss Sadovskii, Krasnoselskii, Leray–Schauder, and Furi–Pera type fixed point theorems for a class of multivalued mappings with weakly sequentially closed graph. We first discuss a Sadovskii type result for weakly condensing and 1-set weakly contractive multivalued maps with weakly sequentially closed graph. Next we discuss multivalued analogues of Krasnoselskii fixed point theorems for the sum $S + T$ on nonempty closed convex of a Banach space where T is weakly completely continuous and S is weakly condensing (resp. 1-set weakly contractive). In particular we consider Krasnoselskii type fixed point theorems and Leray–Schauder alternatives for the sum of two weakly sequentially continuous mappings, S and T by looking at the multivalued mapping $(I - S)^{-1}T$, where $I - S$ may not be injective. We note that the domains of all of the multivalued maps discussed here are not assumed to be bounded.

Now we introduce notation and give preliminary results which will be needed in this section. Let X be a Hausdorff linear topological space, and let

$$\mathcal{P}(X) = \Big\{ A \subset X : A \text{ is nonempty} \Big\},$$

$$\mathcal{P}_{bd}(X) = \Big\{ A \subset X : A \text{ is nonempty, and bounded} \Big\}$$

$$\mathcal{P}_{cv}(X) = \Big\{ A \subset X : A \text{ is nonempty, and convex} \Big\}$$

$$\mathcal{P}_{cl,cv}(X) = \Big\{ A \subset X : A \text{ is nonempty, closed, and convex} \Big\}.$$

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Let Z be a nonempty subset of a Banach space Y and $F : Z \longrightarrow \mathcal{P}(X)$ be a multivalued mapping. We let

$$R(F) = \bigcup_{y \in Z} F(y) \quad \text{and } GrF = \{(z, x) \in Z \times X : x \in F(z)\}$$

the range and the graph of F, respectively. Moreover, for every subset A of X, we put $F^{-1}(A) = \{z \in Z : F(z) \cap A \neq \emptyset\}$ and $F^{+}(A) = \{z \in Z : F(z) \subset A\}$.

- F is called upper semicontinuous on Z if $F^{-1}(A)$ is closed, for every closed subset A of X (or, equivalently, if $F^{+}(A)$ is open, for every open subset A of X).
- F is called weakly upper semicontinuous if F is upper semicontinuous with respect to the weak topologies of Z and X.

Now we suppose that X is a Banach space and Z is weakly closed in Y.

F is said to be weakly compact if the set $R(F)$ is relatively weakly compact in X. Moreover, F is said to have weakly sequentially closed graph if for every sequence $\{x_n\}_n \subset Z$ with $x_n \rightharpoonup x$ in Z and for every sequence $\{y_n\}_n$ with $y_n \in F(x_n), \forall n \in \mathbb{N}, y_n \rightharpoonup y$ in X implies $y \in F(x)$, where $\rightharpoonup$ denotes weak convergence. F is called weakly completely continuous if F has a weakly sequentially closed graph and, if A is bounded subset of Z, then $F(A)$ is a relatively weakly compact subset of X.

We now present the measure of weak noncompactness in a general setting (i.e., the values are in a lattice).

Definition 4.1. Let X be a Banach space and C a lattice with a least element, which is denoted by 0. By a measure of weak noncompactness MWNC on X, we mean a function Φ defined on the set of all bounded subsets of X with values in C, such that for any $\Omega_1, \Omega_2 \in \mathcal{P}_{\text{bd}}(X)$:

(1) $\Phi(\overline{\text{co}}(\Omega_1)) = \Phi(\Omega_1)$, where $\overline{\text{co}}$ denotes the closed convex hull of Ω.
(2) $\Omega_1 \subseteq \Omega_2 \Longrightarrow \Phi(\Omega_1) \leq \Phi(\Omega_2)$,
(3) $\Phi(\Omega_1 \cup \{a\}) = \Phi(\Omega_1)$ for all $a \in E$.
(4) $\Phi(\Omega_1) = 0$ if and only if Ω_1 is relatively weakly compact in E.

If the lattice C is a cone then the MWNC Φ is said to be positive homogenous if $\Phi(\lambda\Omega) = \lambda\Phi(\Omega)$ for all $\lambda > 0$ and $\Omega \in \mathcal{P}_{\text{bd}}(X)$ and it is called semi-additive if $\Phi(\Omega_1 + \Omega_2) \leq \Phi(\Omega_1) + \Phi(\Omega_2)$ for all $\Omega_1, \Omega_2 \in \mathcal{P}_{\text{bd}}(X)$.

Definition 4.2. Let Ω be a nonempty subset of Banach space X and Φ a MWNC on X. If $F : \Omega \longrightarrow \mathcal{P}(X)$, we say that

(a) F is Φ-condensing if $\Phi(F(D)) < \Phi(D)$ for all bounded sets $D \subseteq \Omega$ with $\Phi(D) \neq 0$.
(b) F is Φ-nonexpansive map if $\Phi(F(D)) \leq \Phi(D)$ for all bounded sets $D \subseteq \Omega$.

4.1 Sadovskii Type Fixed Point Theorems

We begin with the following interesting property for multivalued maps with weakly sequentially closed graph.

Theorem 4.1. *Let Ω be a nonempty, weakly compact subset of a Banach space X. Suppose $F : \Omega \longrightarrow \mathcal{P}(X)$ has weakly sequentially closed graph and $F(\Omega)$ is relatively weakly compact. Then F has weakly closed graph.*

Proof. Since $(X \times X)_w = X_w \times X_w$ (X_w the space X endowed its weak topology), it follows that $\Omega \times \overline{F(\Omega)}^w$ is a weakly compact subset of $X \times X$. Also, $GrF \subset \Omega \times \overline{F(\Omega)}^w$. So, GrF is relatively weakly compact. Let $(x, y) \in \Omega \times \overline{F(\Omega)}^w$ be weakly adherent to GrF, then from the Eberlein–Šmulian theorem we can find $\{(\{x_n\}, \{y_n\})\}_n \subseteq GrF$ such that $y_n \in F(x_n), x_n \rightharpoonup x$ and $y_n \rightharpoonup y$ in X. Because F has weakly sequentially closed graph, $y \in F(x)$ and so $(x, y) \in GrF$. Therefore, GrF is weakly closed. ■

Remark 4.1. With the conditions of Theorem 4.1, we prove that GrF is weakly compact.

Theorem 4.2. *Let Ω be a nonempty, closed, convex subset of a Banach space X. Suppose $F : \Omega \longrightarrow \mathcal{P}_{cv}(\Omega)$ has weakly sequentially closed graph and $F(\Omega)$ is weakly relatively compact. Then F has a fixed point.*

Proof. Set $K = \overline{\text{co}}(F(\Omega))$. It follows from the Krein–Šmulian theorem that K is a weakly compact convex set. We have $F(\Omega) \subseteq K \subseteq \Omega$. Notice also that $F : K \longrightarrow \mathcal{P}_{cv}(K)$. From Theorem 4.1 F has weakly closed graph, and so $F(x)$ is weakly closed for every $x \in K$. Thus by Theorem 1.36, F is weakly upper semicontinuous. Because X endowed with its weak topology is a Hausdorff locally convex space, we apply Theorem 1.39 to guarantee that F has a fixed point $x \in K \subseteq \Omega$. ■

Theorem 4.3. *Let Ω be a nonempty, closed, convex subset of a Banach space X. Assume Φ a MWNC on X and $F : \Omega \longrightarrow \mathcal{P}_{cv}(\Omega)$ has weakly sequentially closed graph. In addition, suppose that F is Φ-condensing and $F(\Omega)$ is bounded. Then F has a fixed point.*

Proof. Let $x_0 \in \Omega$. We consider the family $\mathcal{F}$ of all closed bounded convex subsets D of Ω such that $x_0 \in D$ and $F(x) \subseteq D$ for all $x \in D$. Obviously $\mathcal{F}$ is nonempty, since $\overline{\text{co}}(F(\Omega) \cup \{x_0\}) \in \mathcal{F}$. We let $K = \bigcap_{D \in \mathcal{F}} D$. We have that K is closed convex and $x_0 \in K$. If $x \in K$, then $F(x) \subseteq D$ for all $D \in \mathcal{F}$ and hence $F(x) \subseteq K$. Consequently, $K \in \mathcal{F}$. We now prove that K is weakly compact. Denoting by $K_* = \overline{\text{co}}(F(K) \cup \{x_0\})$, we have $K_* \subseteq K$, which implies that $F(x) \subseteq F(K) \subseteq K_*$ for all $x \in K_*$. Therefore $K_* \in \mathcal{F}$ and $K \subseteq K_*$. Consequently, $K = K_*$. Since K is weakly closed, it suffices to show that K is relatively weakly compact. If $\Phi(K) > 0$, we obtain

$$\Phi(K) = \Phi(\overline{\text{co}}(F(K) \cup \{x_0\})) \leq \Phi(F(K)) < \Phi(K),$$

which is a contradiction. Hence, $\Phi(K) = 0$ and so K is relatively weakly compact. Now, $F : K \longrightarrow \mathcal{P}_{\text{cv}}(K)$ has weakly sequentially closed graph. From Theorem 4.2, F has a fixed point in $K \subseteq \Omega$. ■

Corollary 4.1. *Let Ω be a nonempty, closed, convex subset of a Banach space X. Assume $F : \Omega \longrightarrow \mathcal{P}_{\text{cv}}(\Omega)$ is weakly completely continuous map with $F(\Omega)$ is bounded. Then F has a fixed point.*

Theorem 4.4. *Let X be a Banach space, Ω be a nonempty, closed, convex subset of X and Φ a positive homogenous MWNC on X. Assume $F : \Omega \longrightarrow \mathcal{P}_{\text{cv}}(\Omega)$ has weakly sequentially closed graph, and also suppose F is Φ-nonexpansive and $F(\Omega)$ is bounded. In addition, suppose that the implication*

$$\textit{if } \{x_n\} \subset \Omega \textit{ with } y_n \in F(x_n) \textit{ for all } n \textit{ and } x_n - y_n \longrightarrow \theta \textit{ as } n \longrightarrow \infty, \tag{4.1}$$

$$\textit{then there exists } \ x \in \Omega \textit{ with } \ x \in F(x),$$

holds. Then F has a fixed point.

Remark 4.2. θ denotes the zero of the space X.

Proof. Let $F_n = t_n F$ for $n = 1, 2, \ldots$, where $\{t_n\}_n$ is a sequence of $(0, 1)$ such that $t_n \longrightarrow 1$. Since $\theta \in \Omega$ and Ω is convex, it follows that $F_n : \Omega \longrightarrow \mathcal{P}_{\text{cv}}(\Omega)$. Also F_n has a weakly sequentially closed graph. Let $D \in \mathcal{P}_{\text{bd}}(\Omega)$. Then, we have

$$\Phi(F_n(D)) = \Phi(t_n F(D)) \leq t_n \Phi(D).$$

So, if $\Phi(D) \neq 0$ we have

$$\Phi(F_n(D)) < \Phi(D).$$

Therefore, F_n is Φ-condensing on Ω. From Theorem 4.3, F_n has a fixed point, in Ω. For all n, let $y_n \in F(x_n)$ with $x_n = t_n y_n$. Clearly the sequence $\{y_n\}_n$ is bounded and $x_n - y_n = (t_n - 1)y_n \longrightarrow \theta$ as $n \longrightarrow \infty$, since $t_n \longrightarrow 1$ as $n \longrightarrow \infty$. Thus (4.1) implies that there exists $x \in \Omega$ with $x \in F(x)$. ■

Corollary 4.2. *Let Ω be a nonempty, closed, convex subset of a Banach space X and $\theta \in \Omega$. In addition assume $F : \Omega \longrightarrow \mathcal{P}_{\text{cv}}(\Omega)$ has weakly sequentially closed graph, is β-nonexpansive, and $F(\Omega)$ is bounded. Also, suppose (4.1) holds. Then F has a fixed point.*

4.2 Leray–Schauder and Furi–Pera Type Theorems

In applications, the construction of a set Ω such that $F(\Omega) \subseteq \Omega$ is difficult and sometimes impossible. As a result we investigate maps $F : \Omega \longrightarrow \mathcal{P}(X)$ with weakly sequentially closed graph.

Lemma 4.1. *Let Ω be a weakly closed subset of a Banach space X with $\theta \in \Omega$. Assume $F : \Omega \longrightarrow \mathcal{P}(X)$ has weakly sequentially closed graph with $F(\Omega)$ is bounded. Let $\{x_n\}_n \subseteq \Omega$ and (λ_n) be a real sequence. If $x_n \rightharpoonup x$ and $\lambda_n \longrightarrow \lambda \in \mathbb{R}$, then the condition $x_n \in \lambda_n F(x_n)$ for all n implies $x \in \lambda F(x)$.*

Proof. For all n, there exists $y_n \in F(x_n)$ such that $x_n = \lambda_n y_n$. If $\lambda = 0$, then $x_n \rightharpoonup \theta$ ($F(\Omega)$ is bounded) and $x \in \{\theta\} \subseteq \Omega$. If $\lambda \neq 0$, then without loss of generality, we can suppose that $\lambda_n \neq 0$ for all n. So, $\lambda_n^{-1} x_n = y_n$ for all n implies $y_n \rightharpoonup \lambda^{-1} x$. Since F has weakly sequentially closed graph, we have $y \in F(x)$, which means that $x \in \lambda F(x)$. ■

Lemma 4.2. *Let Ω be a nonempty closed convex subset of a Banach space X, S a nonempty subset of Ω, $z \in U$ and Φ a MWNC on X such that $F(S)$ is bounded. If $F : S \longrightarrow \mathcal{P}(\Omega)$ is Φ-condensing, then there exists a nonempty closed and convex subset K of Ω such that $z \in K \cap S$, $K \cap S$ is relatively weakly compact and $F(K \cap S)$ is a subset of K.*

Proof. We consider the family $\mathcal{G} = \big\{D \subseteq \Omega : D \text{ bounded}, D = \overline{\text{co}} D, z \in D, F(D \cap S) \subseteq D\big\}$. Obviously $\mathcal{G}$ is nonempty, since $\overline{\text{co}}(F(S) \cup \{z\}) \in \mathcal{G}$. We let $K = \bigcap\limits_{D \in \mathcal{G}} D$. We have that K is bounded closed convex and $z \in K$. If $x \in K \cap S$, then $F(x) \in D$ for all $D \in \mathcal{G}$ and hence $F(K \cap S) \subseteq K$. Therefore, we have that $K \in \mathcal{G}$. We will prove that K is weakly compact. Denoting by $K_* = \overline{\text{co}}(F(K \cap S) \cup \{z\})$, we have $K_* \subseteq K$, which implies that $F(K_* \cap S) \subseteq F(K \cap S) \subseteq K_*$. Therefore $K_* \in \mathcal{G}$ and $K \subseteq K_*$. Hence $K = K_*$. If $\Phi(K \cap S) \neq 0$, we obtain

$$\Phi(K \cap S) \leq \Phi(K) \leq \Phi(\overline{\text{co}} F(K \cap S) \cup \{z\}) \leq \Phi(F(K \cap S) \cup \{z\}) \leq \Phi(F(K \cap S)) < \Phi(K \cap S),$$

which is a contradiction, so $\Phi(K \cap S) = 0$ and $K \cap S$ is relatively weakly compact. ■

Theorem 4.5. *Let X be a Banach space, Ω be a nonempty, closed, convex subset of X and U be a weakly open subset of Ω with $\theta \in U$. Assume Φ a MWNC on X and $F : \overline{U^w} \longrightarrow \mathcal{P}_{cv}(\Omega)$ has weakly sequentially closed graph. In addition, suppose F is Φ-condensing and $F(\overline{U^w})$ is bounded. Then, either*

(A_1) *F has a fixed point, or*

(A_2) *there is a point $x \in \partial_\Omega U$ (the weak boundary of U in Ω) and $\lambda \in (0, 1)$ with $x \in \lambda F(x)$.*

Remark 4.3. (a) Because Ω is convex, its closure and weak closure are the same, so the weak closure of U in Ω and weak closure are the same, for $U \subset \Omega$.

(b) For $U \subset \Omega$, we have $\partial_\Omega U = \overline{U^w} \cap \overline{\Omega \setminus U^w}$.

Proof. Suppose (A_2) does not hold and F does not have a fixed point in $\partial_\Omega U$ (otherwise, we are finished, i.e., (A_1) occurs). By Lemma 4.2, there exists a nonempty closed and convex subset K of Ω with $\theta \in K$, $K \cap U$ is relatively weakly compact and $F(K \cap U) \subset K$.

Let D be the set defined by

$$D = \left\{ x \in \overline{K \cap U^w} \ : \ x \in \lambda F(x), \ \text{for some } \lambda \in [0,1] \right\}.$$

Now D is nonempty and bounded, because $\theta \in D$ and $F(\overline{U^w})$ is bounded. Also D is weakly relatively compact. Now, we prove that D is weakly sequentially closed. For that, let $\{x_n\}_n$ a sequence of D such that $x_n \rightharpoonup x$, $x \in \overline{K \cap U^w}$. For all $n \in \mathbb{N}$, there exists a $\lambda_n \in [0,1]$ such that $x_n = \lambda_n F(x_n)$. Now $\lambda_n \in [0,1]$, so we can extract a subsequence $\{\lambda_{n_j}\}_j$ such that $\lambda_{n_j} \longrightarrow \lambda \in [0,1]$. We put $x_{n_j} = \lambda_{n_j} y_{n_j}$, where $y_{n_j} \in F(x_{n_j})$. Applying Lemma 4.1, we deduce that $x \in D$. Let $x \in \overline{U^w}$, be weakly adherent to D. Since $\overline{D^w}$ is weakly compact, by the Eberlein–Šmulian theorem, there exists a sequence $\{x_n\}_n \subset D$ such that $x_n \rightharpoonup x$, so $x \in D$. Hence $\overline{D^w} = D$ and D is a weakly closed subset of the weakly compact set $\overline{U^w}$. Therefore D is weakly compact. Because X endowed with its weak topology is a Hausdorff locally convex space, we have that X is completely regular. Since $D \cap (K \cap \partial_\Omega U) = \emptyset$, then, by Proposition 1.1 there is a weakly continuous function $\varphi \ : \ \Omega \longrightarrow [0,1]$, such that $\varphi(x) = 1$ for $x \in K \cap \partial_\Omega U$ and $\varphi(x) = 0$ for $x \in D$. Since K is convex, $\theta \in K$, and F with nonempty convex values, we can define the multivalued map $F^* \ : \ K \longrightarrow \mathcal{P}_{cv}(K)$ by:

$$F^*(x) = \begin{cases} (1-\varphi(x))F(x), \ \text{if } x \in \overline{K \cap U^w}, \\ \{\theta\}, \ \text{if } x \in K \setminus U. \end{cases}$$

Clearly, $F^*(K)$ is bounded. Because $\partial_K(K \cap U) = \partial_K(\overline{K \cap U^w})$, $[0,1]$ is compact, φ is weakly continuous and F has a weakly sequentially closed graph, by Lemma 4.1, we have F^* has weakly sequentially closed graph. Also, $F^*(K) \subseteq \overline{\text{co}}(F(\overline{K \cap U^w}) \cup \{\theta\})$. Let $H = \overline{\text{co}}(F(\overline{K \cap U^w}) \cup \{\theta\})$. Since $F(\overline{K \cap U^w})$ is relatively weakly compact, it follows from the Krein–Šmulian theorem that H is a weakly compact convex set of X. Moreover, $F^*(H) \subseteq H$. Then, $F^*(H)$ is relatively weakly compact. Theorem 4.2 shows that F^* has a fixed point $x_0 \in K$. From $\theta \in U \cap K \subseteq \text{int}^w_K(\overline{K \cap U^w})$, it follows that $x_0 \in F^*(x_0) = (1-\varphi(x_0))F(x_0)$, which implies $x_0 \in D$ and so $\varphi(x_0) = 0$. Thus, x_0 is a fixed point of F. ■

Corollary 4.3. *Let X be a Banach space, Ω be a nonempty, closed, convex subset of X and U be a weakly open subset of Ω with $\theta \in U$. Assume Φ a MWNC on X and $F \ : \ \overline{U^w} \longrightarrow \mathcal{P}_{cv}(\Omega)$ has weakly sequentially closed graph, Φ-condensing with $F(\overline{U^w})$ is bounded. In addition, suppose F satisfying the Leray–Schauder boundary condition*

$$x \notin \lambda F(x) \quad \text{for every } x \in \partial_\Omega U \text{ and } \lambda \in (0,1).$$

Then F has a fixed point in $\overline{U^w}$.

Corollary 4.4. *Let E be a Banach space, Ω be a nonempty, closed, convex subset of X and U be a weakly open subset of Ω with $\theta \in U$. Assume that $F : \overline{U^w} \longrightarrow \mathcal{P}_{cv}(\Omega)$ is a weakly completely continuous map with $F(\overline{U^w})$ is bounded. In addition, suppose F satisfying the Leray–Schauder boundary condition*

$$x \notin \lambda F(x) \quad \textit{for every } x \in \partial_\Omega U \textit{ and } \lambda \in (0, 1).$$

Then F has a fixed point in $\overline{U^w}$.

We now use Theorem 4.4 to obtain a nonlinear alternative of Leray–Schauder type for multivalued 1-set weakly contractive maps.

Theorem 4.6. *Let Ω be a nonempty, closed, convex subset of a Banach space X, let U be a weakly open subset of Ω with $\theta \in U$. Assume Φ a positive homogenous MWNC on X, $F : \overline{U^w} \longrightarrow \mathcal{P}_{cv}(\Omega)$ has weakly sequentially closed graph, is Φ-nonexpansive, $F(\overline{U^w})$ is bounded and (4.1) holds on $\overline{U^w}$. In addition, suppose F satisfies the following Leray–Schauder condition*

$$x \notin \lambda F(x) \quad \textit{for every } x \in \partial_\Omega U \textit{ and } \lambda \in (0, 1). \tag{4.2}$$

Then F has a fixed point in $\overline{U^w}$.

Proof. Let $F_n = t_n F$, for $n = 1, 2, \ldots$, where $\{t_n\}_n$ is a sequence of $(0, 1)$ such that $t_n \longrightarrow 1$. Since $\theta \in \Omega$ and Ω is convex, it follows that $F_n : \overline{U^w} \longrightarrow \mathcal{P}_{cv}(\Omega)$. Also F_n is Φ-condensing and has a weakly sequentially closed graph. Suppose that $y_n \in \lambda_n F_n(y_n)$ for some $y_n \in \partial_\Omega U$ and for some $\lambda_n \in (0, 1)$. Then we have $y_n \in \lambda_n t_n F_n(y_n)$ which contradicts hypothesis (4.2) since $\lambda_n t_n \in (0, 1)$. Now, applying Corollary 4.3, the remainder of the proof is similar to that in Theorem 4.4. ■

In applications, it is extremely difficult to construct a weakly open set U as in Theorem 4.5, so we are motivated to construct a Furi–Pera type fixed point theorems [90] for a multivalued mapping $F : \Omega \longrightarrow \mathcal{P}(X)$ with weakly sequentially closed graph. Here Ω is a closed convex subset of X with (possible) an empty weak interior.

Theorem 4.7. *Let X be a reflexive separable Banach space, C a closed bounded convex subset of X, and Q a closed convex subset of C with $\theta \in Q$. Also, assume $F : Q \longrightarrow \mathcal{P}_{cv}(C)$ has weakly sequentially closed graph. In addition, assume that the following condition is satisfied*

$$\textit{if } \{(x_j, \lambda_j)\}_{j=1}^{\infty} \textit{ is a sequence in } Q \times [0, 1] \textit{ with } x_j \rightharpoonup x \in \partial Q, \lambda_j \longrightarrow \lambda \textit{ and}$$
$$x \in \lambda F(x) \;\; 0 \le \lambda < 1, \textit{ then } \{\lambda_j F(x_j)\} \subseteq Q \textit{ for } j \textit{ sufficiently large; here } \partial Q$$
$$\textit{denotes the weak boundary of } Q \textit{ relative to } C.$$

Then F has a fixed point in Q.

Proof. We know from [115] that there exists a weakly continuous retraction $r : X \longrightarrow Q$. Consider

$$B = \{x \in X, x \in Fr(x)\}.$$

Now rF has a weakly sequentially closed graph, since F has weakly sequentially closed graph, and r is weakly continuous. Applying Theorem 4.2 we infer that there exists $y \in Q$ with $y \in rF(y)$. Let $z \in F(y)$ such that $y = r(z)$. Then $z \in B$ and $B \neq \emptyset$. In addition B is weakly sequentially closed, since Fr has a weakly sequentially closed graph. Moreover, since $B \subseteq Fr(B) \subseteq F(Q)$ it follows that B is relatively weakly compact. Now let $x \in \overline{B^w}$. Since $\overline{B^w}$ is weakly compact, there is a sequence $\{x_n\}_n$ of elements of B which converges weakly to some x. Since B is weakly sequentially closed, we deduce that $x \in B$. Thus $\overline{B^w} = B$. This implies that B is weakly compact. We now show that $B \cap Q \neq \emptyset$. Suppose $B \cap Q = \emptyset$. Now since X is separable and C is weakly compact we know from Theorem 1.14 that the weak topology on C is metrizable, let d^* denote the metric. With respect to (C, d^*) note Q is closed, B is compact, $B \cap Q = \emptyset$ so we have from [89, p. 65] that

$$d^*(B, Q) = \inf\{d^*(x, y) : x \in B, y \in Q\} > 0,$$

so there exists $\varepsilon > 0$ with $d^*(B, Q) > \varepsilon$. For $i \in \{1, 2, \ldots\}$, let

$$U_i = \left\{x \in C, d^*(x, Q) < \frac{\varepsilon}{i}\right\}.$$

For each $i \in \{1, 2, \ldots\}$ fixed U_i is open in with respect to the topology generated by d^*, and so U_i is weakly open in C. Also we have

$$\overline{U_i^w} = \overline{U_i^{d^*}} = \left\{x \in C, d^*(x, Q) \leq \frac{\varepsilon}{i}\right\}$$

and

$$\partial U_i = \left\{x \in C, d^*(x, Q) = \frac{\varepsilon}{i}\right\}.$$

Keeping in mind that $\overline{U_i^w} \cap B = \emptyset$, applying Corollary 4.4, we get that there exists $\lambda \in (0, 1)$ and $y_i \in \partial U_i$ such that $y_i \in \lambda_i Fr(y_i)$. In particular, since $y_i \in \partial U_i$, then

$$\{\lambda_i Fr(y_i)\} \nsubseteq Q \quad \text{for each } i \in \{1, 2, \ldots\}. \tag{4.3}$$

Now look at

$$D = \{x \in X : x \in \lambda Fr(x), \ \text{for some } \lambda \in [0, 1]\}.$$

Now D is nonempty, because $\theta \in D$. Also, $D \subseteq \overline{\text{conv}}(F(Q) \cup \{\theta\})$, so by the Krein - Šmulian theorem, D is relatively weakly compact. Since Fr has weakly sequentially closed graph and $[0,1]$ is compact, we deduce from Lemma 4.1 that D is weakly closed. So D is weakly compact. Then, up to a subsequence, we may assume that $\lambda_j \longrightarrow \lambda^* \in [0,1]$ and $y_j \rightharpoonup y^* \in \partial Q$. Since F has weakly sequentially closed graph then $y^* \in \lambda^* Fr(y^*)$. Note $\lambda^* \neq 1$ since $B \cap Q = \emptyset$. From the assumption in the statement of Theorem 4.7 it follows that $\{\lambda_j Fr(x_j)\} \subseteq Q$ for j sufficiently large, which is a contradiction. Thus $B \cap Q \neq \emptyset$, so there exists $x \in Q$ with $x \in Fr(x)$, i.e., $x \in Fx$. ■

Now, we state some new variants of Leray–Schauder type fixed point results for the sum of two weakly sequentially continuous mappings T and S. We look at the case when $I - S$ may not be invertible by looking at the multivalued mapping $(I-S)^{-1}T$.

Theorem 4.8. *Let Ω be a nonempty closed and convex subset of a Banach space X. In addition, let U be a weakly open subset of Ω with $\theta \in U$, $T : \overline{U^w} \longrightarrow X$ and $S : \Omega \longrightarrow X$ are two weakly sequentially continuous mappings satisfying:*

- (i) *$T(\overline{U^w})$ is relatively weakly compact,*
- (ii) *$T(\overline{U^w}) \subset (I-S)(\Omega)$.*
- (iii) *If $(I-S)(x_n) \rightharpoonup y$, then there exists a weakly convergent subsequence of $\{x_n\}_n$.*
- (iv) *For every y in the range of $I-S$, $D_y = \{x \in \Omega$ such that $(I-S)(x) = y\}$ is a convex set.*

Then, either $T+S$ has a fixed point or there is a point $x \in \partial_\Omega U$ (the weak boundary of U in Ω) and a $\lambda \in (0,1)$ with $x = \lambda T(x) + \lambda S\left(\frac{x}{\lambda}\right)$.

Proof. First, we assume that $I-S$ is invertible. For any given $y \in \overline{U^w}$, define $F : \overline{U^w} \longrightarrow \Omega$ by $F(y) := (I-S)^{-1}T(y)$. F is well defined by assumption (ii).

Step 1: $F(\overline{U^w})$ is relatively weakly compact. For any $\{x_n\}_n \subset F(\overline{U^w})$, we choose $\{x_n\}_n \subset \overline{U^w}$ such that $y_n = F(x_n)$. Taking into account assumption (i), together with the Eberlein–Šmulian's theorem we get a subsequence $\{y_{\varphi_1(n)}\}_n$ of $\{y_n\}_n$ such that $(I-S)(y_{\varphi_1(n)}) \rightharpoonup z$, for some $z \in \Omega$. Thus, by assumption (iii), there exists a subsequence $y_{\varphi_1(\varphi_2(n))}$ converging weakly to $y_0 \in \Omega$. Hence, $F(\overline{U^w})$ is relatively weakly compact.

Step 2: F is weakly sequentially continuous. Let $\{x_n\}_n \subset \overline{U^w}$ such that $x_n \rightharpoonup x$. Because $F(\overline{U^w})$ is relatively weakly compact, it follows from the Eberlein–Šmulian's theorem that there exists a subsequence $\{x_{n_k}\}_k$ of $\{x_n\}_n$ such that $F(x_{n_k}) \rightharpoonup y$, for some $y \in \Omega$. The weak sequential continuity of S leads to $SF(x_{n_k}) \rightharpoonup Sy$. Also, from the equality $SF = -T + F$, we have that

$$-T(x_{n_k}) + F(x_{n_k}) \rightharpoonup -T(x) + y.$$

So, $y = F(x)$. We claim that $F(x_n) \rightharpoonup F(x)$. Suppose that this is not the case, then there exists a subsequence $\{x_{\varphi_1(n)}\}_n$ and a weak neighborhood V^w of $(I-S)^{-1}T(x)$ such that $(I-S)^{-1}T\left(x_{\varphi_1(n)}\right) \notin V^w$, for all $n \in \mathbb{N}$. On the other hand, we have

$x_{\varphi_1(n)} \rightharpoonup x$, then arguing as before, we find a subsequence $\{x_{\varphi_1(\varphi_2(n))}\}_n$ such that $(I-S)^{-1}T(x_{\varphi_1(\varphi_2(n))})$ converges weakly to $((I-S)^{-1}T(x)$, which is a contradiction and hence F is weakly sequentially continuous.

Consequently, using Theorem 3.1, we get either F has a fixed point or there exist a $x \in \partial_\Omega U$ and a $\lambda \in (0,1)$ such that $x = \lambda F(x)$. This yields, either $T + S$ has a fixed point or there is a point $x \in \partial_\Omega U$ and a $\lambda \in (0,1)$ such that

$$\frac{x}{\lambda} = (I - S)^{-1}T(x). \tag{4.4}$$

Now (4.4) implies $(I - S)\left(\frac{x}{\lambda}\right) = T(x)$. So, $x = \lambda T(x) + \lambda S\left(\frac{x}{\lambda}\right)$.

Second, if $I-S$ is not invertible, $(I-S)^{-1}$ could be seen as a multivalued mapping. For any given $y \in \overline{U^w}$, define $H : \overline{U^w} \longrightarrow \mathcal{P}(\Omega)$ by $H(y) := (I - S)^{-1}T(y)$. H is well defined by assumption (ii). We should prove that H fulfills the hypotheses of Theorem 4.5.

Step 1: $H(x)$ is a convex set for each $x \in \overline{U^w}$. This is an immediate consequence of assumption (iv).
Step 2: H has a weakly sequentially closed graph. Let $\{x_n\}_n \subset \overline{U^w}$ such that $x_n \rightharpoonup x$ and $y_n \in H(x_n)$ such that $y_n \rightharpoonup y$. By the definition of H, we have $(I - S)(y_n) = T(x_n)$. Since T and $I - S$ are weakly sequentially continuous, we obtain $(I - S)(y) = T(x)$. Thus $y \in (I - S)^{-1}T(x)$.
Step 3: $H(\overline{U^w})$ is relatively weakly compact. This assertion is proved by using the same reasoning as the one in Step 1 of the first part of the proof. Hence, H is β-condensing.

In view of Theorem 4.5, either H has a fixed point; or there is a point $x \in \partial_\Omega U$ and a $\lambda \in (0,1)$ with $x \in \lambda H(x)$. By the definition of H, the last assertion implies that either there is a point $x \in \partial_\Omega U$ such that $(I - S)(x) = T(x)$ or there is a point $x \in \partial_\Omega U$ and a $\lambda \in (0,1)$ such that $(I-S)\left(\frac{x}{\lambda}\right) = T(x)$. This leads to either $T+S$ has a fixed point or there is a point $x \in \partial_\Omega U$ and a $\lambda \in (0,1)$ with $x = \lambda T(x) + \lambda S\left(\frac{x}{\lambda}\right)$.

■

Remark 4.4. We note that every β-condensing mapping $F : \Omega \subset X \longrightarrow X$ satisfies assumption (iii) of Theorem 4.8; here Ω is a subset of a Banach space X such that $F(\Omega)$ is bounded. In fact, suppose that $(I - F)x_n \rightharpoonup y$, for some $\{x_n\}_n \subset \Omega$ and $y \in X$. Writing x_n as $x_n = (I - F)(x_n) + F(x_n)$ and using the subadditivity of the De Blasi measure of weak noncompactness, we get

$$\beta(\{x_n\}) \leq \beta(\{(I - F)(x_n)\}) + \beta(\{F(x_n)\}).$$

Since $\overline{\{(I - F)(x_n)\}^w}$ is weakly compact, we obtain $\beta(\{x_n\}) \leq \beta(\{F(x_n)\})$. Now, we show that $\beta(\{x_n\}) = 0$. If we suppose the contrary, then since F is β-condensing, we obtain

$$\beta(\{x_n\}) \leq \beta(\{F(x_n)\}) < \beta(\{x_n\}),$$

which is absurd. So, $\beta(\{x_n\}) = 0$. Consequently, $\overline{\{x_n\}}^w$ is weakly compact and then by the Eberlein–Šmulian's theorem, there exists a weakly convergent subsequence of $\{x_n\}_n$. Hence, condition (iii) is satisfied.

Corollary 4.5. *Let Ω be a nonempty closed and convex subset of a Banach space X. In addition, let U be a weakly open subset of Ω with $\theta \in U$, $T : \overline{U^w} \longrightarrow X$ and $S : \Omega \longrightarrow X$ are two weakly sequentially continuous mappings satisfying:*

(i) *$T(\overline{U^w})$ is relatively weakly compact,*
(ii) *S is a contraction mapping such that $S(\Omega)$ is bounded.*
(iii) *$T(\overline{U^w}) + S(\Omega) \subset \Omega$.*

Then, either $T + S$ has a fixed point or there is a point $x \in \partial_\Omega U$ (the weak boundary of U in Ω) and a $\lambda \in (0, 1)$ with $x = \lambda T(x) + \lambda S\left(\frac{x}{\lambda}\right)$.

Proof. The result follows immediately from Theorem 4.8. Indeed, since S is a nonlinear contraction, we see that S satisfies assumption (iii) of Theorem 4.8. Moreover, we have $I - S$ is a homeomorphism, so for every y in the range of $I - S$, the set $D_y = \{x \in \Omega$ such that $(I - S)(x) = y\}$ is reduced to $\{(I - S)^{-1}(y)\}$ which is convex. ■

4.3 Krasnoselskii Type Fixed Point Theorems

In this subsection, by using an analogue of Sadovskii's fixed point theorem [178] for multivalued mappings with weakly sequentially closed graph (see Theorem 4.3), we present some multivalued analogues of Krasnoselskii fixed point theorem for mappings of the form $T + S$ on a nonempty closed convex set of a Banach space, where T is weakly completely continuous and S is weakly condensing (resp. 1-set weakly contractive) mapping with weakly sequentially closed graph measures of weak noncompactness.

The first fixed point result of this subsection is for weakly completely continuous multivalued mappings.

Proposition 4.1. *Let Ω be a nonempty, closed, convex subset of a Banach space X. Assume that $F : \Omega \longrightarrow \mathcal{P}_{cv}(\Omega)$ is weakly completely continuous with $F(\Omega)$ bounded. Then F has a fixed point.*

Proof. This is an immediate consequence of Theorem 4.3, since F is clearly Φ-condensing where Φ is any MWNC on X. ■

Lemma 4.3. *Let Ω be a nonempty weakly closed set of a Banach space X, Φ a semi-additive MWNC on X and $F : \Omega \longrightarrow \mathcal{P}(X)$ has a weakly sequentially closed graph, Φ-condensing with $F(\Omega)$ bounded. Then*

(a) *for all weakly compact subset K of X, $(I - F)^{-1}(K)$ is weakly compact.*
(b) *$I - F$ maps weakly closed subset of Ω onto weakly sequentially closed sets in X.*

Proof. (*a*) Let $K \subset X$ be a nonempty weakly compact set and let $D = (I - F)^{-1}(K)$. If $x \in D$, then $(x - F(x)) \cap K$ is nonempty, so $x \in F(x) + K$ and hence $D \subset F(D) + K$. Consequently

$$\Phi(D) \leq \Phi(F(D) + K) \leq \Phi(F(D)) + \Phi(K) \leq \Phi(F(D)).$$

Since F is Φ-condensing, it follows that $\Phi(D) = 0$. Thus D is relatively weakly compact. Next we will show that D is weakly closed. Let $x \in \Omega$, be weakly adherent to D. Since $\overline{D^w}$ is weakly compact, from the Eberlein–Šmulian theorem, there exists a sequence $\{x_n\} \subset D$ such that $x_n \rightharpoonup x$. We have $(I - F)(x_n) \cap K \neq \emptyset$ for all n, so there exists $y_n \in K$ such that $y_n \in (I - F)(x_n)$. Hence $y_n = x_n - z_n$, where $z_n \in F(x_n)$. Since $\{y_n\} \subset K$ and K is weakly compact, from the Eberlein–Šmulian theorem, there exists a subsequence $\{y_{n_k}\}$ with $y_{n_k} \rightharpoonup y \in K$. Since $z_{n_k} = x_{n_k} - y_{n_k}$ and F has weakly sequentially closed graph, it follows that $z_{n_k} \rightharpoonup x - y$ and $x - y \in F(x)$. Consequently, $y \in x - F(x)$ and $(x - F(x)) \cap K \neq \emptyset$. Accordingly, $x \in (I - F)^{-1}(K) = D$. Hence $\overline{D^w} = D$ and D is a weakly closed subset of Ω. Therefore D is weakly compact.

(*b*) Let $D \subset \Omega$ be a weakly closed set and consider $x_n \in (I - F)(D)$ such that $x_n \rightharpoonup x$ in X. We have $x_n \in (I - F)(y_n)$, $\forall n \geq 1$ with $y_n \in D$. The set $K = \overline{\{x_n\}^w}$ is weakly compact and so $(I - F)^{-1}(K)$ is weakly compact. Therefore, we may assume that $y_n \rightharpoonup y$ in D, for some $y \in D$. Choose $z_n \in F(y_n)$ such that $x_n = y_n - z_n$, and we obtain $z_n \rightharpoonup y - x$. Since F has weakly sequentially closed graph, it follows that $y - x \in F(y)$. Hence $x \in y - F(y) \subseteq (I - F)(D)$. Accordingly, $(I - F)(D)$ is weakly sequentially closed.

■

Lemma 4.4. *Let Ω be a weakly closed subset of a Banach space X. Assume that*

(a) $T : \Omega \longrightarrow \mathcal{P}(X)$ is weakly completely continuous,
(b) $S : \Omega \longrightarrow \mathcal{P}(X)$ has weakly sequentially closed graph.

Then, $T + S : \Omega \longrightarrow \mathcal{P}(X)$ has weakly sequentially closed graph.

Proof. Let $\{x_n\}$ be a sequence of Ω such that $x_n \rightharpoonup x \in \Omega$ and $y_n \in (T + S)(x_n)$ such that $y_n \rightharpoonup y \in X$. Then there exist $z_n \in T(x_n)$ and $w_n \in S(x_n)$ such that

$$y_n = z_n + w_n. \tag{4.5}$$

Since T is weakly completely continuous and $\{x_n\}$ is bounded, there is a subsequence $\{z_{n_k}\}$ which weakly converges to some $z \in T(x)$. Also from (4.5)

$$w_{n_k} = y_{n_k} - z_{n_k} \rightharpoonup y - z.$$

On the other hand, since S has weakly sequentially closed graph, we have $y - z \in S(x)$ and thus $y \in z + S(x)$. Since $z \in T(x)$, we have $y \in T(x) + S(x) = (T + S)(x)$. Consequently, $T + S$ has weakly sequentially closed graph. ■

Theorem 4.9. *Let Ω be a nonempty closed convex subset of a Banach space X and Φ a semi-additive MWNC on X. Assume $T : \Omega \longrightarrow \mathcal{P}(X)$ and $S : \Omega \longrightarrow \mathcal{P}(X)$, are two multivalued mappings satisfying the following conditions:*

(a) T is weakly completely continuous.
(b) S is Φ-condensing, with weakly sequentially closed graph.
(c) For all $x \in \Omega$, $(T + S)(x) \in \mathcal{P}_{cv}(\Omega)$.
(d) $(T + S)(\Omega)$ is a bounded set of Ω.

Then, there exists $x \in \Omega$ such that $x \in (T + S)(x)$.

Proof. Let $T + S : \Omega \longrightarrow \mathcal{P}_{cv}(\Omega)$. By Lemma 4.4, $T + S$ has weakly sequentially closed graph. We claim that $T+S$ is Φ-condensing. To see this, let $D \in \mathcal{P}_{bd}(\Omega)$ with $\Phi(D) \neq 0$. Now, since T is weakly completely continuous and S is Φ-condensing, we have

$$\Phi((T + S)(D)) \leq \Phi(T(D) + S(D)) \leq \Phi(T(D)) + \Phi(S(D)) < \Phi(S(D)).$$

We apply Theorem 4.3 to deduce that the mapping $T + S$ has a fixed point in Ω. ■

Corollary 4.6. *Let Ω be a nonempty closed convex subset of a Banach space X and Φ a semi-additive MWNC on X. Assume $T : \Omega \longrightarrow \mathcal{P}(X)$ and $S : \Omega \longrightarrow X$ satisfy the following conditions:*

(a) T is weakly completely continuous.
(b) S is Φ-condensing and weakly sequentially continuous.
(c) For all $x \in \Omega$, $(T + S)(x) \in \mathcal{P}_{cv}(\Omega)$.
(d) $(T + S)(\Omega)$ is a bounded set of Ω.

Then, there exists $x \in \Omega$ such that $x \in (T + S)(x)$.

Corollary 4.7. *Let Ω be a nonempty closed convex subset of a Banach space X. Assume $T : \Omega \longrightarrow \mathcal{P}(X)$ and $S : X \longrightarrow X$ satisfy the following conditions:*

(a) T is weakly completely continuous.
(b) S is a nonlinear contraction (i.e., there exists a continuous nondecreasing function $\psi : [0,\infty) \longrightarrow [0,\infty)$ satisfying $\psi(z) < z$ for $z > 0$), such that $\|S(x) - S(y)\| \leq \psi(\|x - y\|)$ for all $x, y \in X$) and weakly sequentially continuous.
(c) For all $x \in \Omega$, $(T + S)(x) \in \mathcal{P}_{cv}(\Omega)$.
(d) $(T + S)(\Omega)$ is a bounded set of Ω.

Then, there exists $x \in \Omega$ such that $x \in (T + S)(x)$.

Proof. In view of Corollary 4.6 it suffices to show that S is β-condensing. To see this, let $D \in \mathcal{P}_{bd}(\Omega)$. Suppose that $\beta(D) = d > 0$. Let $\varepsilon > 0$, and then there exists a weakly compact set K of X with $D \subseteq K + B_{d+\varepsilon}(0)$. So for $x \in D$ there exist $y \in K$ and $z \in B_{d+\varepsilon}(0)$ such that $x = y + z$ and so

$$\|S(x) - S(y)\| \leq \psi(\|x - y\|) \leq \psi(d + \varepsilon).$$

It follows immediately that

$$S(D) \subseteq S(K) + B_{d+\varepsilon}(0).$$

Moreover, since S is a weakly sequentially continuous mapping and K is weakly compact then $\overline{S(K)}^w$ is weakly compact. Therefore, $\beta(S(D)) \leq \psi(d + \varepsilon)$. Since $\varepsilon > 0$ is arbitrary, it follows that $\beta(S(D)) \leq \psi(d) < d = \beta(D)$. Accordingly, S is β-condensing. ■

Corollary 4.8. *Let Ω be a nonempty closed convex subset of a Banach space X and Φ a semi-additive MWNC on X. Assume $T : \Omega \longrightarrow \mathcal{P}(X)$ and $S : \Omega \longrightarrow X$ satisfy the following conditions:*

(a) T is weakly completely continuous.
(b) S is Φ-condensing and weakly sequentially continuous.
(c) For all $x \in \Omega$, $(T + S)(x) \in \mathcal{P}_{cv}(\Omega)$.
(d) $(T + S)(\Omega)$ is a bounded set of Ω.

Then, there exists $x \in \Omega$ such that $x \in (T + S)(x)$.

Definition 4.3. Let X be a Banach space. For $\Omega \in \mathcal{P}(X)$ and $x \in X$ we define $d(x, \Omega) = \inf\{\|x - y\| : y \in \Omega\}$.

Definition 4.4. Let Ω be a nonempty subset of Banach space X. If $F : \Omega \longrightarrow \mathcal{P}(X)$, we say that F is hemi-weakly compact if for each sequence $\{x_n\}$ has a weakly convergent subsequence whenever there exist $y_n \in F(x_n)$ such that the sequence $\{x_n - y_n\}$ is weakly convergent.

Definition 4.5. Let Ω be a nonempty subset of Banach space X. If $F : \Omega \longrightarrow \mathcal{P}(X)$, we say that F is hemi-weakly semiclosed at θ, if there exists $\{x_n\} \subset \Omega$ such that $d(x_n, F(x_n)) \longrightarrow 0, n \longrightarrow \infty$, then there is an $x \in \Omega$ such that $x \in F(x)$.

Theorem 4.10. *Let Ω be a nonempty closed convex subset of a Banach space X and Φ a positive homogenous semi-additive MWNC on X. Assume $T : \Omega \longrightarrow \mathcal{P}(X)$ and $S : \Omega \longrightarrow \mathcal{P}(X)$, are two multi-valued mappings satisfying the following conditions:*

(a) T is weakly completely continuous.
(b) S is Φ-nonexpansive and hemi-weakly compact with weakly sequentially closed graph.
(c) There exists a bounded set Ω_0 of X and a sequence $\{\lambda_n\} \subseteq (0, 1)$ such that $\lambda_n \longrightarrow 1$, for all $x \in \Omega, (T + \lambda_n S)(x) \in \mathcal{P}_{cv}(\Omega)$ and $(T + \lambda_n S)(\Omega) \subset \Omega_0$ for all n.

Then, there exists $x \in \Omega$ such that $x \in (T + S)(x)$.

Proof. Define $G_n = T + \lambda_n S$, for $n \in \mathbb{N}$. By assumption (c), it follows that G_n maps Ω into $\mathcal{P}_{cv}(\Omega)$ and $G_n(\Omega)$ is bounded. Let D be an arbitrary bounded subset of Ω. Then, we have

$$\Phi(G_n(D)) \leq \Phi(T(D) + \lambda_n S(D)) \leq \Phi(T(D)) + \lambda_n \Phi(S(D)) \leq \lambda_n \Phi(S(D)).$$

So, if $\Phi(D) \neq 0$, we have

$$\Phi(G_n(D)) < \Phi(D).$$

Therefore, G_n is Φ-condensing on Ω. From Lemma 4.4, G_n has weakly sequentially closed graph. From Theorem 4.3 we infer that G_n has a fixed point $x_n \in \Omega$, i.e.,

$$x_n \in T(x_n) + \lambda_n S(x_n).$$

Then, there exist $z_n \in T(x_n)$ and $w_n \in S(x_n)$ such that

$$x_n = z_n + \lambda_n w_n. \tag{4.6}$$

Obviously $\{x_n\}$ is bounded, so up to subsequence we can suppose that $z_n \rightharpoonup z \in \Omega$. Since the sequence $\{w_n\}$ is bounded and $\lambda_n \longrightarrow 1$, from (4.6) we obtain

$$x_n - w_n = z_n + (\lambda_n - 1)w_n \rightharpoonup z. \tag{4.7}$$

Now S is hemi-weakly compact implies that $\{x_n\}_n$ has a weakly convergent subsequence, say $\{x_{n_k}\}$. That is $x_{n_k} \rightharpoonup x \in \Omega$. Since T has weakly sequentially closed graph, it follows that $z \in T(x)$. Also from (4.7), $w_{n_k} \rightharpoonup x - z$. Keeping in mind that S has weakly sequentially closed graph, we obtain that $x - z \in S(x)$ and so $x \in z + S(x)$. Since $z \in T(x)$, we have $x \in T(x) + S(x) = (T + S)(x)$. ∎

Corollary 4.9. *Let Ω be a nonempty closed convex subset of a Banach space X. Assume $T : \Omega \longrightarrow \mathcal{P}(X)$ and $S : X \longrightarrow X$ satisfy the following conditions*

(a) T is weakly completely continuous.
(b) S is nonexpansive (i.e., $\|S(x) - S(y)\| \leq \|x - y\|$ for all $x, y \in X$), weakly sequentially continuous and hemi-weakly compact.
(c) There exists a bounded set Ω_0 of X and a sequence $\{\lambda_n\} \subseteq (0, 1)$ such that $\lambda_n \longrightarrow 1$, for all $x \in \Omega, (T + \lambda_n S)(x) \in \mathcal{P}_{cv}(\Omega)$ and $(T + \lambda_n S)(\Omega) \subset \Omega_0$ for all n.

Then, there exists $x \in \Omega$ such that $x \in (T + S)(x)$.

Proof. Since S is weakly sequentially continuous, we have that S has weakly sequentially closed graph. In view of Theorem 4.10 it suffices to show that S is

β-nonexpansive. To see this, let $D \in \mathcal{P}_{\text{bd}}(\Omega)$ and $d = \beta(D)$. Let $\varepsilon > 0$, and then there exists a weakly compact set K of X with $D \subseteq K + B_{d+\varepsilon}(0)$. So for $x \in D$ there exist $y \in K$ and $z \in B_{d+\varepsilon}(0)$ such that $x = y + z$ and so

$$\|S(x) - S(y)\| \leq \|x - y\| \leq d + \varepsilon.$$

It follows immediately that

$$\begin{aligned} S(D) &\subseteq S(K) + B_{d+\varepsilon}(0) \\ &\subseteq \overline{S(K)}^{w} + B_{d+\varepsilon}(0). \end{aligned}$$

Since S is weakly sequentially continuous and K is weakly compact then $\overline{S(K)}^{w}$ is weakly compact. Thus, $\beta(S(D)) \leq d + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, it follows that $\beta(S(D)) \leq \beta(D)$. Accordingly, S is β-nonexpansive. ■

Theorem 4.11. *Let Ω be a nonempty closed convex subset of a Banach space X and Φ a positive homogenous semi-additive MWNC on X. Assume $T : \Omega \longrightarrow \mathcal{P}(X)$ and $S : \Omega \longrightarrow \mathcal{P}(X)$, are two multivalued mappings satisfying the following conditions:*

(a) T is weakly completely continuous.
(b) S is Φ-nonexpansive, with weakly sequentially closed graph.
(c) For all $x \in \Omega$, $(T + S)(x) \in \mathcal{P}_{\text{cv}}(\Omega)$.
(d) $(T + S)(\Omega)$ is a bounded set of Ω.
(e) $T + S$ is hemi-weakly semiclosed at θ.

Then, there exists $x \in \Omega$ such that $x \in (T + S)(x)$.

Proof. Let z be a fixed element of Ω. Define $G_n = \lambda_n(T+S)+(1-\lambda_n)z, n = 1, 2, ..$ where $\{\lambda_n\}$ is a sequence of $(0, 1)$ such that $\lambda_n \longrightarrow 1$. Since $z \in \Omega$ and Ω is convex, by assumption (c) it follows that $G_n : \Omega \longrightarrow \mathcal{P}_{\text{cv}}(\Omega)$. By Lemma 4.4, $T + S$ has weakly sequentially closed graph, and by assumption (d), $G_n(\Omega)$ is bounded. Let D be an arbitrary bounded subset of Ω. Then, we have

$$\begin{aligned} \Phi(G_n(D)) = \Phi(\{\lambda_n(T + S)(D)\} + \{(1 - \lambda_n)z\}) &\leq \lambda_n \Phi((T + S)(D)) \\ &\leq \lambda_n \Phi(T(D)) + \lambda_n \Phi(S(D)) \leq \lambda_n \Phi(D). \end{aligned}$$

So, if $\Phi(D) \neq 0$ we have

$$\Phi(G_n(D)) < \Phi(D).$$

Therefore, G_n is Φ-condensing on Ω and we can apply Theorem 4.3 and obtain $\{x_n\} \subset \Omega$ such that $x_n \in G_n(x_n), n \geq 1$. Consequently, $x_n \in \lambda_n(T+S)(x_n) + (1-\lambda_n)z, n \geq 1$. Choose sequences $\{z_n\}$ and $\{w_n\}$ such that $z_n \in T(x_n)$, $w_n \in S(x_n)$ and $x_n = \lambda_n(z_n + w_n) + (1-\lambda_n)z$. So,

$$x_n - (y_n + z_n) = (\lambda_n - 1)(y_n + z_n) + (1 - \lambda_n)z \longrightarrow 0,$$

since $\lambda_n \longrightarrow 1$ as $n \longrightarrow \infty$ and $(T+S)(\Omega)$ is bounded. Furthermore,

$$d(x_n, (T+S)(x_n)) \leq \|x_n - (y_n + z_n)\| \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Now $T+S$ is hemi-weakly semiclosed at θ implies that there exists $x \in \Omega$ with $x \in (T+S)(x)$. ■

Corollary 4.10. *Let Ω be a nonempty closed convex subset of a Banach space X. Assume $T : \Omega \longrightarrow \mathcal{P}(X)$ and $S : X \longrightarrow X$ satisfy the following conditions:*

(a) T is weakly completely continuous.
(b) S is nonexpansive, and weakly sequentially continuous.
(c) For all $x \in \Omega$, $(T+S)(x) \in \mathcal{P}_{cv}(\Omega)$.
(d) $(T+S)(\Omega)$ is a bounded set of Ω.
(e) $T+S$ is hemi-weakly semiclosed at θ.

Then, there exists $x \in \Omega$ such that $x \in (T+S)(x)$.

Chapter 5
Fixed Point Theory in Banach Algebras

In this chapter we discuss

$$x = AxBx + Cx \tag{5.1}$$

in suitable Banach algebras. We present some fixed point theory in Banach spaces under a weak topology setting. One difficulty that arises is that in a Banach algebra equipped with its weak topology the product of two weakly convergent sequences is not necessarily weakly convergent.

5.1 Fixed Point Theorems

Definition 5.1. Let X be a Banach space. A mapping $\mathcal{G} : X \longrightarrow X$ is called $\mathcal{D}$-Lipschitzian if there exists a continuous and nondecreasing function $\phi_{\mathcal{G}} : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that

$$\|\mathcal{G}x - \mathcal{G}y\| \leq \phi_{\mathcal{G}}(\|x - y\|).$$

for all $x, y \in X$, with $\phi_{\mathcal{G}}(0) = 0$. Sometimes we call the function $\phi_{\mathcal{G}}$ a $\mathcal{D}$-function of $\mathcal{G}$ on X. If $\phi_{\mathcal{G}}(r) = kr$ for some $k > 0$, then $\mathcal{G}$ is called a Lipschitzian function on X with the Lipschitz constant k. Further if $k < 1$, then $\mathcal{G}$ is called a contraction on X with the contraction k.

Remark 5.1. Every Lipschitzian mapping is $\mathcal{D}$-Lipschitzian, but the converse may not be true. If $\phi_{\mathcal{G}}$ is not necessarily nondecreasing and satisfies $\phi_{\mathcal{G}}(r) < r$, for $r > 0$, the mapping $\mathcal{G}$ is called a nonlinear contraction with a contraction function $\phi_{\mathcal{G}}$.

An important fixed point theorem that is used in the theory of nonlinear differential and integral equations is the following result of Boyd and Wong [47].

A.B. Amar, D. O'Regan, *Topological Fixed Point Theory for Singlevalued and Multivalued Mappings and Applications*, DOI 10.1007/978-3-319-31948-3_5

Theorem 5.1. *Let $A : X \to X$ be a nonlinear contraction. Then A has a unique fixed point x^* and the sequence $\{A^n x\}_n$ of successive iterations of A converges to x^* for each $x \in X$.*

Theorem 5.2. *Let $\mathcal{E}$ be a Banach algebra and S be a nonempty closed convex subset of $\mathcal{E}$. Let $A, C : \mathcal{E} \longrightarrow \mathcal{E}$ and $B : S \longrightarrow \mathcal{E}$ be three operators such that*

(i) $\left(\frac{I-C}{A}\right)^{-1}$ *exists on $B(S)$.*
(ii) $\left(\frac{I-C}{A}\right)^{-1}$ *B is weakly sequentially continuous.*
(iii) $\left(\frac{I-C}{A}\right)^{-1}$ *$B(S)$ is relatively weakly compact.*
(iv) $x = AxBy + Cx \Rightarrow x \in S$*, for all $y \in S$.*

Then Eq. (5.1) has at least one solution in S.

Proof. From assumption (i), it follows that for each y in S, there is a unique $x_y \in \mathcal{E}$ such that

$$\left(\frac{I-C}{A}\right)x_y = By. \tag{5.2}$$

or, equivalently

$$Ax_yBy + Cx_y = x_y. \tag{5.3}$$

Since hypothesis (iv) holds, then $x_y \in S$. Therefore, we can define

$$\begin{cases} \mathcal{N} : S \longrightarrow S \\ \qquad y \longrightarrow \mathcal{N}y = \left(\frac{I-C}{A}\right)^{-1} By. \end{cases}$$

Using hypotheses (ii), (iii), Theorem 1.31 and the Krein–Šmulian theorem, we conclude that $\mathcal{N}$ has a fixed point y in S. Hence, y satisfies (5.1) i.e.,

$$AyBy + Cy = y.$$

■

Proposition 5.1. *Let $\mathcal{E}$ be a Banach algebra and S be a nonempty closed convex subset of $\mathcal{E}$. Let $A, C : \mathcal{E} \longrightarrow \mathcal{E}$ and $B : S \longrightarrow \mathcal{E}$ be three operators such that*

(i) *A and C are $\mathcal{D}$-Lipschitzians with the $\mathcal{D}$-functions ϕ_A and ϕ_C respectively,*
(ii) *A is regular on $\mathcal{E}$, i.e., A maps $\mathcal{E}$ into the set of all invertible elements of $\mathcal{E}$,*
(iii) *B is a bounded function with bound M.*

Then $\left(\frac{I-C}{A}\right)^{-1}$ *exists on $B(S)$ if $M\phi_A(r) + \phi_C(r) < r$, for $r > 0$.*

Proof. Let y be fixed in S and define the mapping

$$\begin{cases} \varphi_y : \mathcal{E} \longrightarrow \mathcal{E} \\ \qquad x \longrightarrow \varphi_y(x) = AxBy + Cx. \end{cases}$$

Let $x_1, x_2 \in \mathcal{E}$, the use of assumption (i) leads to

$$\begin{aligned} \|\varphi_y(x_1) - \varphi_y(x_2)\| &\leq \|Ax_1By - Ax_2By\| + \|Cx_1 - Cx_2\| \\ &\leq \|Ax_1 - Ax_2\|\|By\| + \|Cx_1 - Cx_2\| \\ &\leq M\phi_A(\|x_1 - x_2\|) + \phi_C(\|x_1 - x_2\|). \end{aligned}$$

Now, an application of Theorem 5.1 yields that there is a unique element $x_y \in \mathcal{E}$ such that

$$\varphi_y(x_y) = x_y$$

Hence, x_y satisfies (5.3) and so, by virtue of hypothesis (ii), x_y satisfies (5.2). Therefore, the mapping $\left(\frac{I-C}{A}\right)^{-1}$ is well defined on $B(S)$ and $\left(\frac{I-C}{A}\right)^{-1} By = x_y$ and the desired result is deduced. ■

In what follows, we will combine Theorem 5.2 and Proposition 5.1 to obtain the following fixed point theorems in Banach algebras.

Theorem 5.3. *Let $\mathcal{E}$ be a Banach algebra and S be a nonempty closed convex subset of $\mathcal{E}$. Let $A, C : \mathcal{E} \longrightarrow \mathcal{E}$ and $B : S \longrightarrow \mathcal{E}$ be three operators such that*

(i) *A and C are $\mathcal{D}$-Lipschitzians with the $\mathcal{D}$-functions ϕ_A and ϕ_C respectively,*
(ii) *A is regular on $\mathcal{E}$,*
(iii) *B is strongly continuous,*
(iv) *$B(S)$ is bounded with bound M,*
(v) $\left(\frac{I-C}{A}\right)^{-1}$ *is weakly compact on $B(S)$,*
(vi) $x = AxBy + Cx \Rightarrow x \in S$, *for all $y \in S$.*

Then Eq. (5.1) has at least one solution in S if $M\phi_A(r) + \phi_C(r) < r$, for all $r > 0$.

Proof. From Proposition 5.1, it follows that $\left(\frac{I-C}{A}\right)^{-1}$ exists on $B(S)$. By virtue of assumption (vi), we obtain

$$\left(\frac{I-C}{A}\right)^{-1} B(S) \subset S.$$

Moreover, the use of hypotheses (iv) and (v) leads that $\left(\frac{I-C}{A}\right)^{-1} B(S)$ is relatively weakly compact. Now, we show that $\left(\frac{I-C}{A}\right)^{-1} B$ is weakly sequentially continuous. To see this, let $\{u_n\}$ be any sequence in S such that $u_n \rightharpoonup u$ in S. From assumption (iii), we have

$$Bu_n \rightarrow Bu.$$

Since $\left(\frac{I-C}{A}\right)^{-1}$ is a continuous mapping on $B(S)$, we deduce that

$$\left(\frac{I-C}{A}\right)^{-1} Bu_n \rightarrow \left(\frac{I-C}{A}\right)^{-1} Bu.$$

This shows that $\left(\frac{I-C}{A}\right)^{-1} B$ is weakly sequentially continuous. Finally, an application of Theorem 5.2 yields that Eq. (5.1) has a solution in S. ■

Theorem 5.4. *Let S be a nonempty closed convex subset of a Banach algebra $\mathcal{E}$. Let $A, C : \mathcal{E} \longrightarrow \mathcal{E}$ and $B : S \longrightarrow \mathcal{E}$ be three operators such that*

(i) *A and C are $\mathcal{D}$-Lipschitzians with the $\mathcal{D}$-functions ϕ_A and ϕ_C respectively,*
(ii) *B is weakly sequentially continuous and $B(S)$ is relatively weakly compact,*
(iii) *A is regular on $\mathcal{E}$,*
(iv) $\left(\frac{I-C}{A}\right)^{-1}$ *is weakly sequentially continuous on $B(S)$,*
(v) $x = AxBy + Cx \Rightarrow x \in S$, *for all $y \in S$.*

Then Eq. (5.1) has at least one solution in S if $M\phi_A(r) + \phi_C(r) < r$, for all $r > 0$.

Proof. Similar reasoning as in the proof of Theorem 5.3 guarantees that $\left(\frac{I-C}{A}\right)^{-1}$ exists on $B(S)$ and

$$\left(\frac{I-C}{A}\right)^{-1} B(S) \subset S.$$

Since $\left(\frac{I-C}{A}\right)^{-1}$ and B are weakly sequentially continuous, so, by composition we have that $\left(\frac{I-C}{A}\right)^{-1} B$ is weakly sequentially continuous. Finally, we claim that $\left(\frac{I-C}{A}\right)^{-1} B(S)$ is relatively weakly compact. To see this, let $\{u_n\}$ be any sequence in S and let

$$v_n = \left(\frac{I-C}{A}\right)^{-1} Bu_n.$$

Since $B(S)$ is relatively weakly compact, there is a renamed subsequence $\{Bu_n\}$ weakly converging to an element w. This fact, together with hypothesis (iv), gives that

$$v_n = \left(\frac{I-C}{A}\right)^{-1} Bu_n \rightharpoonup \left(\frac{I-C}{A}\right)^{-1} w.$$

We infer that $\left(\frac{I-C}{A}\right)^{-1} B(S)$ is sequentially relatively weakly compact. An application of the Eberlein–Šmulian theorem yields that $\left(\frac{I-C}{A}\right)^{-1} B(S)$ is relatively weakly compact, which establishes our claim. The result is concluded immediately from Theorem 5.2. ■

Because the product of two weakly sequentially continuous functions is not necessarily weakly sequentially continuous, we will introduce:

Definition 5.2. We will say that the Banach algebra $\mathcal{E}$ satisfies condition $(\mathcal{P})$ if

$$(\mathcal{P}) \begin{cases} \text{For any sequences } \{x_n\} \text{ and } \{y_n\} \text{ in } \mathcal{E} \text{ such that } x_n \rightharpoonup x \text{ and } y_n \rightharpoonup y, \\ \text{then } x_n y_n \rightharpoonup xy; \text{ here } \rightharpoonup \text{ denotes weak convergence} \end{cases}$$

Note that, every finite dimensional Banach algebra satisfies condition $(\mathcal{P})$. Even, if X satisfies condition $(\mathcal{P})$ then $\mathcal{C}(K,X)$ is also a Banach algebra satisfying condition $(\mathcal{P})$, where K is a compact Hausdorff space. The proof is based on Dobrakov's theorem:

Theorem 5.5 ([81, Dobrakov, p. 36]). *Let K be a compact Hausdorff space and X be a Banach space. Let $(f_n)_n$ be a bounded sequence in $\mathcal{C}(K,X)$, and $f \in \mathcal{C}(K,X)$.*

Then $\{f_n\}_n$ is weakly convergent to f if and only if $\{f_n(t)\}_n$ is weakly convergent to $f(t)$ for each $t \in K$.

Theorem 5.6. *Let $\mathcal{E}$ be a Banach algebra satisfying condition $(\mathcal{P})$. Let S be a nonempty closed convex subset of $\mathcal{E}$. Let $A, C : \mathcal{E} \longrightarrow \mathcal{E}$ and $B : S \longrightarrow \mathcal{E}$ be three operators such that*

(i) *A and C are $\mathcal{D}$-Lipschitzians with the $\mathcal{D}$-functions ϕ_A and ϕ_C respectively,*
(ii) *A is regular on $\mathcal{E}$,*
(iii) *A, B and C are weakly sequentially continuous on S,*
(iv) *$B(S)$ is bounded with bound M,*
(v) $\left(\frac{I-C}{A}\right)^{-1}$ *is weakly compact on $B(S)$,*
(vi) *$x = AxBy + Cx \Rightarrow x \in S$, for all $y \in S$.*

Then Eq. (5.1) has at least one solution in S if $M\phi_A(r) + \phi_C(r) < r$, for all $r > 0$.

Proof. Similar reasoning as in the proof of Theorem 5.3 guarantees that $\left(\frac{I-C}{A}\right)^{-1}$ exists on $B(S)$,

$$\left(\frac{I-C}{A}\right)^{-1} B(S) \subset S$$

and $\left(\frac{I-C}{A}\right)^{-1} B(S)$ is relatively weakly compact. In view of Theorem 5.2, it suffices to establish that $\left(\frac{I-C}{A}\right)^{-1} B$ is weakly sequentially continuous. To see this, let $\{u_n\}$ be a weakly convergent sequence of S to a point u in S. Now, define the sequence $\{v_n\}$ of the subset S by

$$v_n = \left(\frac{I-C}{A}\right)^{-1} Bu_n.$$

Since $\left(\frac{I-C}{A}\right)^{-1} B(S)$ is relatively weakly compact, so, there is a renamed subsequence such that

$$v_n = \left(\frac{I-C}{A}\right)^{-1} Bu_n \rightharpoonup v.$$

But, on the other hand, the subsequence $\{v_n\}$ verifies

$$v_n - Cv_n = Av_nBu_n.$$

Therefore, from assumption (iii) and in view of condition $(\mathcal{P})$, we deduce that v verifies the following equation

$$v - Cv = AvBu,$$

or, equivalently

$$v = \left(\frac{I-C}{A}\right)^{-1} Bu.$$

Next we claim that the whole sequence $\{u_n\}$ verifies

$$\left(\frac{I-C}{A}\right)^{-1} Bu_n = v_n \rightharpoonup v.$$

Indeed, suppose that this is not the case, so, there is V^w a weakly neighborhood of v satisfying for all $n \in \mathbb{N}$, there exists an $N \geq n$ such that $v_N \notin V^w$. Hence, there is a renamed subsequence $\{v_n\}$ verifying the property

$$\text{for all } n \in \mathbb{N},\ v_n \notin V^w. \tag{5.4}$$

However

$$\text{for all } n \in \mathbb{N},\ v_n \in \left(\frac{I-C}{A}\right)^{-1} B(S).$$

Again, there is a renamed subsequence such that

$$v_n \rightharpoonup v'.$$

Thus we have

$$v' = \left(\frac{I-C}{A}\right)^{-1} Bu,$$

and, consequently

$$v = v',$$

which is a contradiction with (5.4). This yields that $\left(\frac{I-C}{A}\right)^{-1} B$ is weakly sequentially continuous. ■

Corollary 5.1. *Let $\mathcal{E}$ be a Banach algebra satisfying condition $(\mathcal{P})$ and let S be a nonempty closed convex subset of $\mathcal{E}$. Let $A, C : \mathcal{E} \longrightarrow \mathcal{E}$ and $B : S \longrightarrow \mathcal{E}$ be three operators such that*

- (i) *A and C are $\mathcal{D}$-Lipschitzians with the $\mathcal{D}$-functions ϕ_A and ϕ_C respectively,*
- (ii) *A is regular on $\mathcal{E}$,*
- (iii) *A, B and C are weakly sequentially continuous on S,*
- (vi) *$A(S), B(S)$ and $C(S)$ are relatively weakly compacts,*
- (v) *$x = AxBy + Cx \Rightarrow x \in S$, for all $y \in S$.*

Then Eq. (5.1) has at least one solution in S if $M\phi_A(r) + \phi_C(r) < r$, for all $r > 0$.

Proof. In view of Theorem 5.6, it is enough to prove that $\left(\frac{I-C}{A}\right)^{-1} B(S)$ is relatively weakly compact. To do this, let $\{u_n\}$ be any sequence in S and let

$$v_n = \left(\frac{I-C}{A}\right)^{-1} Bu_n. \tag{5.5}$$

Since $B(S)$ is relatively weakly compact, there is a renamed subsequence $\{Bu_n\}$ weakly converging to an element w. On the other hand, from (5.5), we obtain

$$v_n = Av_nBu_n + Cv_n. \tag{5.6}$$

Since $\{v_n\}$ is a sequence in S, so, by assumption (iv), there is a renamed subsequence such that $Av_n \rightharpoonup x$ and $Cv_n \rightharpoonup y$. Hence, in view of condition $(\mathcal{P})$ and (5.6), we obtain

$$v_n \rightharpoonup xw + y.$$

This shows that $\left(\frac{I-C}{A}\right)^{-1} B(S)$ is sequentially relatively weakly compact. An application of the Eberlein–Šmulian theorem yields that $\left(\frac{I-C}{A}\right)^{-1} B(S)$ is relatively weakly compact. ■

Now, we shall discuss briefly the existence of positive solutions. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two Banach algebras, with positive closed cones $\mathcal{E}_1^+$ and $\mathcal{E}_2^+$, respectively. An operator $\mathcal{G}$ from $\mathcal{E}_1$ into $\mathcal{E}_2$ is said to be positive if it carries the positive cone $\mathcal{E}_1^+$ into $\mathcal{E}_2^+$ (i.e., $\mathcal{G}(\mathcal{E}_1^+) \subset \mathcal{E}_2^+$).

Theorem 5.7. *Let $\mathcal{E}$ be a Banach algebra satisfying condition $(\mathcal{P})$ and S be a nonempty closed convex subset of $\mathcal{E}$ such that $S^+ = S \cap \mathcal{E}^+ \neq \emptyset$. Let $A, C : \mathcal{E} \longrightarrow \mathcal{E}$ and $B : S \longrightarrow \mathcal{E}$ be three operators such that*

- (i) *A and C are $\mathcal{D}$-Lipschitzians with the $\mathcal{D}$-functions ϕ_A and ϕ_C respectively,*
- (ii) *A is regular on $\mathcal{E}$,*
- (iii) *A, B and C are weakly sequentially continuous on S^+,*
- (iv) *$A(S^+)$, $B(S^+)$ and $C(S^+)$ are relatively weakly compacts,*
- (v) *$x = AxBy + Cx \Rightarrow x \in S^+$, for all $y \in S^+$.*

Then Eq. (5.1) has at least one solution in S^+ if $M^+\phi_A(r) + \phi_C(r) < r$, for all $r > 0$, where $M^+ = \|B(S^+)\|$.

Proof. Obviously $S^+ = S \cap \mathcal{E}^+$ is a closed convex subset of $\mathcal{E}$. From Proposition 5.1, it follows that $\left(\frac{I-C}{A}\right)^{-1}$ exists on $B(S^+)$. By virtue of assumption (v), we have

$$\left(\frac{I-C}{A}\right)^{-1} B(S^+) \subset S^+.$$

Then we can define the mapping:

$$\begin{cases} \mathcal{N} : S^+ \longrightarrow S^+ \\ \qquad y \longrightarrow \mathcal{N}y = \left(\frac{I-C}{A}\right)^{-1} By. \end{cases}$$

Now, an application of the Corollary 5.1 yields that $\mathcal{N}$ has a fixed point in S^+. As a result, by the definition of $\mathcal{N}$, Eq. (5.1) has a solution in S^+. ■

Now, we are concerned with existence of solutions of the following equation:

$$x = Ax + Lx\,Ux \tag{5.7}$$

in suitable Banach algebras which involves three nonlinear operators A, L and U (with conditions different from those above). We mention that this equation arises frequently in biology, engineering, physics, mechanics, and economics.

In the remainder of this section, we assume that the Banach space E has the structure of a Banach algebra satisfying condition $(\mathcal{P})$.

Lemma 5.1. *If K, K' are weakly compact subsets of E, then $K.K' := \{x.x' : x \in K, x' \in K'\}$ is weakly compact.*

Proof. We will show that $K.K'$ is sequentially weakly compact. For that, let $\{x_n\}_n$ be any sequence of K and let $\{x'_n\}_n$ be any sequence of K'. By hypothesis, there is a renamed subsequence $\{x_n\}_n$ such that $x_n \rightharpoonup x \in K$. Again, there is a renamed subsequence $\{x'_n\}_n$ of K' such that $x'_n \rightharpoonup x' \in K'$. This, together with condition $(\mathcal{P})$, yields that

$$x_n.x'_n \rightharpoonup x.x'.$$

This is in turn shows that $K.K'$ is sequentially weakly compact. Hence, an application of the Eberlein–Šmulian theorem yields that $K.K'$ is weakly compact. ■

In [25], Banas introduced a class of Banach algebras satisfying a certain condition (m):

$$\mu(X.Y) \leq \|X\|\mu(Y) + \|Y\|\mu(X),$$

where μ is a measure of noncompactness, X, Y are bounded subsets and $\|X\| := \sup\{\|x\| : x \in X\}$. Here we show that such Banach algebras satisfying condition $(\mathcal{P})$ verify, in a special and important case, condition (m) with the De Blasi measure of weak noncompactness β:

Lemma 5.2. *For any bounded subset V of E and for any weakly compact subset K of E, we have*

$$\beta(V.K) \leq \|K\|\beta(V).$$

Proof. Firstly, we note that K is bounded. Then, $\|K\|$ exists. Next, we may assume that $\|K\| > 0$ (otherwise we are finished). Let $\epsilon > 0$ be given. It follows, from the definition of β, that there exists a weakly compact subset K' of E such that

$$V \subset K' + \left(\beta(V) + \frac{\epsilon}{\|K\|}\right) B_1(\theta).$$

Then

$$V.K \subset K'.K + \left(\beta(V) + \frac{\epsilon}{\|K\|}\right) B_1(\theta).K,$$

from which, we infer that

$$V.K \subset K'.K + \left(\beta(V) + \frac{\epsilon}{\|K\|}\right) \|K\| B_1(\theta).$$

Now, by Lemma 5.1, one has

$$\beta(V.K) \le \|K\|\beta(V) + \epsilon,$$

which yields, since ϵ is arbitrary, that

$$\beta(V.K) \le \|K\|\beta(V).$$

■

Now, we are ready to investigate the existence of solutions of Eq. (5.7).

Theorem 5.8. *Let S be a nonempty subset of E and suppose that the operator $F : S \longrightarrow E$ is of the form $Fx = Ax + Lx\,Ux$, where:*

- (i) *$L : S \longrightarrow E$ is a λ-set-contraction with respect to the measure of weak noncompactness β, and*
- (ii) *$A, U : S \longrightarrow E$ are weakly compact.*

Suppose that $\gamma := \|U(S)\| < \infty$. Then, F is a strict set-contraction with respect to β if $\lambda\gamma < 1$.

Proof. Let us take arbitrary a bounded subset V of S. Then

$$F(V) \subset A(V) + L(V)U(V).$$

The use of property (vii) of β leads to

$$\begin{aligned}\beta(F(V)) &\le \beta(A(V)) + \beta(L(V)U(V)) \\ &\le \beta\left(\overline{A(V)}^w\right) + \beta\left(L(V)\overline{U(V)}^w\right).\end{aligned}$$

Now, by hypothesis (ii) and in view of Lemma 5.2, we obtain

$$\begin{aligned}\beta(F(V)) &\leq \gamma\beta(L(V))\\ &\leq \lambda\gamma\beta(V).\end{aligned}$$

Since $0 \leq \lambda\gamma < 1$, we infer that F is a strict set-contraction with respect to β. ∎

Combining Theorem 3.2 and Theorem 5.8, we obtain the following fixed-point result:

Theorem 5.9. *Let S be a nonempty closed convex subset of E. Let A, L and U be three operators such that*

(i) *$L : S \longrightarrow E$ is weakly sequentially continuous on S and a λ-set-contraction with respect to the measure of weak non-compactness β,*
(ii) *$A, U : S \longrightarrow E$ are weakly sequentially continuous on S and weakly compact, and*
(iii) *$Ax + LxUx \in S$ if $x \in S$.*

If $(A + LU)(S)$ and $U(S)$ are bounded subsets of E, then Eq. (5.7) has at least one solution in S if $0 \leq \lambda\gamma < 1$, where $\gamma := \|U(S)\|$.

When λ in Theorem 5.9 vanishes, we obtain the following result:

Theorem 5.10. *Let S be a nonempty closed convex subset of E. Let A, L and U be three operators such that*

(i) *$A, L, U : S \longrightarrow E$ are weakly sequentially continuous on S,*
(ii) *$A(S), L(S)$ and $U(S)$ are relatively weakly compact, and*
(iii) *$Ax + LxUx \in S$ if $x \in S$.*

Then, then Eq. (5.7) has at least one solution in S.

Taking $L \equiv \theta$, we obtain the Schauder–Tikhonov fixed-point theorem:

Corollary 5.2. *Let S be a nonempty closed convex subset of E. Assume that $A : S \longrightarrow S$ is weakly sequentially continuous on S such that $A(S)$ is relatively weakly compact. Then, the equation $x = Ax$ has at least one solution in S.*

Theorem 5.11. *Let S be a nonempty subset of E and suppose that the operator $F : S \longrightarrow E$ is of the form $Fx = Ax + LxUx$, where:*

(i) *$L : S \longrightarrow E$ is a condensing map with respect to the measure of weak noncompactness β, and*
(ii) *$A, U : S \longrightarrow E$ are weakly compact.*

If $0 \leq \gamma \leq 1$, where $\gamma := \|U(S)\|$, then F is a condensing map with respect to β.

Proof. Let us take an arbitrary bounded subset V of S. As in Theorem 5.8, one has

$$\begin{aligned}\beta(F(V)) &\leq \beta\left(L(V)\overline{U(V)}^{w}\right)\\ &\leq \gamma\beta(L(V)).\end{aligned}$$

Thus

$$\mu(F(V)) \leq \beta(L(V)).$$

Seeing that L is a condensing map with respect to β, it follows that F is a condensing map with respect to β. ∎

Combining Theorem 3.2 and Theorem 5.11, we obtain the following result:

Theorem 5.12. *Let S be a nonempty closed convex subset of E. Let A, L and U be three operators such that*

(*i*) *$L : S \longrightarrow E$ is weakly sequentially continuous on S and condensing map with respect to the measure of weak noncompactness β,*
(*ii*) *$A, U : S \longrightarrow E$ are weakly sequentially continuous on S and weakly compact, and*
(*iii*) *$Ax + LxUx \in S$ if $x \in S$.*

If $(A + LU)(S)$ and $U(S)$ are bounded subsets of E, then Eq. (5.7) has at least one solution in S if $0 \leq \gamma \leq 1$, where $\gamma := \|U(S)\|$.

Proposition 5.2. *If $L : E \longrightarrow E$ is Lipschitzian with Lipschitz constant α and weakly sequentially continuous on E, then L is α-set-contraction with respect to β.*

Proof. Let V be a bounded subset of E. We may assume that $\alpha > 0$ (otherwise we are finished). Let $\epsilon > 0$ be given, and it follows from the definition of β there exists a weakly compact subset K of E such that

$$V \subset K + \left(\beta(V) + \alpha^{-1}\epsilon\right) B_1(\theta).$$

We infer that

$$L(V) \subset L(K) + (\alpha\beta(V) + \epsilon)B_1(\theta).$$

To see this, let $x \in V$. Then, there exist $k \in K$ and $y \in B_1(\theta)$ such that

$$x = k + \left(\beta(V) + \alpha^{-1}\epsilon\right) y.$$

Now, since L is α-Lipschitzian, it follows that

$$\begin{aligned} \|L(x) - L(k)\| &\leq \alpha\|x - k\| \\ &\leq \alpha\left(\beta(V) + \alpha^{-1}\epsilon\right) = \alpha\beta(V) + \epsilon. \end{aligned}$$

Therefore, the element $\dfrac{1}{\alpha\beta(V) + \epsilon}(L(x) - L(k))$ belongs to $B_1(\theta)$. This means that

$$L(x) \in L(K) + (\alpha\beta(V) + \epsilon)B_1(\theta).$$

Thus

$$L(V) \subset L(K) + (\alpha\beta(V) + \epsilon)B_1(\theta).$$

Since K is weakly compact and L is weakly sequentially continuous on E, then $L : K \longrightarrow E$ is weakly continuous. Hence, $L(K)$ is weakly compact. Consequently

$$\beta(L(V)) \leq \alpha\beta(V) + \epsilon,$$

which yields, since ϵ is arbitrary, that

$$\beta(L(V)) \leq \alpha\beta(V).$$

■

Now, combining Theorem 5.8 and Proposition 5.2, we obtain the following result:

Theorem 5.13. *Let S be a nonempty subset of E and suppose that the operator $F : S \longrightarrow E$ is of the form $Fx = Ax + LxUx$, where:*

(i) $L : E \longrightarrow E$ is Lipschitzian with Lipschitz constant α and weakly sequentially continuous on E, and
(ii) $A, U : S \longrightarrow E$ are weakly compact.

Suppose that $\gamma := \|U(S)\| < \infty$. Then, F is a strict set-contraction with respect to β if $\alpha\gamma < 1$.

Proof. In view of Proposition 5.2, L is α-set-contraction with respect to β. Now, our desired result follows immediately from Theorem 5.8. ■

Theorem 5.14. *Let S be a nonempty closed convex bounded subset of E. Let A, L and U be three operators such that*

(i) $L : E \longrightarrow E$ is Lipschitzian with Lipschitz constant α and weakly sequentially continuous on E,
(ii) $A, U : S \longrightarrow E$ are weakly sequentially continuous on S and weakly compact, and
(iii) $Ax + LxUx \in S$ if $x \in S$.

Then, Eq. (5.7) has at least one solution in S if $\alpha\gamma < 1$, where $\gamma := \|U(S)\|$.

Proof. We will show that the operator F satisfies all the conditions of Theorem 3.2, where F is defined by:

$$\begin{cases} F : S \longrightarrow E \\ \quad x \longrightarrow Fx = Ax + LxUx. \end{cases}$$

First, since A, L and U are weakly sequentially continuous on S and together with condition $(\mathcal{P})$, we infer that F is weakly sequentially continuous. Next, by Theorem 5.13, F is a strict set-contraction with respect to β, since $\alpha\gamma < 1$. It follows that F is a condensing map with respect to β. Finally, the use of hypothesis (iii) leads to $F(S) \subseteq S$ and consequently $F(S)$ is bounded. Thus, Theorem 3.2 establishes the desired result. ∎

Taking $U \equiv 1_E$, where 1_E is the unit element of the Banach algebra E, we obtain the Krasnoselskii's fixed-point theorem

Corollary 5.3. *Let S be a nonempty closed convex bounded subset of E. Let A and L be two operators such that*

(i) *$L : E \longrightarrow E$ is a contraction and weakly sequentially continuous on E,*
(ii) *$A : S \longrightarrow E$ is weakly sequentially continuous on S and weakly compact, and*
(iii) *$Ax + Lx \in S$ if $x \in S$.*

Then, the equation $x = Ax + Lx$ has at least one solution in S.

Remark 5.2. It turns out that Corollary 5.3 remains valid in any Banach space, so, we don't require the sequential condition $(\mathcal{P})$.

Corollary 5.4. *Suppose that U is weakly sequentially continuous and weakly compact operator on E and let $x_0 \in E$. If there exists a nonempty closed convex bounded subset S of E such that $\gamma := \|U(S)\| < 1$ and $x_0 + xUx \in S$ for each $x \in S$, then the equation*

$$x = x_0 + xUx \tag{5.8}$$

has at least one solution in S.

Proof. It suffices to take L the identity map on E, A the constant map x_0, and then the desired result is deduced immediately from Theorem 5.8. ∎

5.2 Positivity

In this section, we survey some important parts of this theory by discussing the existence of positive solutions of Eq. (5.7) in an ordered Banach algebra $(E, \|.\|, \leq)$ satisfying condition $(\mathcal{P})$, with positive closed cone E^+ and $\leq$ the partial ordering defined by E^+. We recall that E^+ verifies (i) $E^+ + E^+ \subseteq E^+$, (ii) $\lambda E^+ \subseteq E^+$ for all $\lambda \in \mathbb{R}^+$, (iii) $\{-E^+\} \cap E^+ = \{0\}$, where 0 is the zero element of E and (iv) $E^+ \cdot E^+ \subseteq E^+$, where "$\cdot$" is a multiplicative composition in E. The details on cones and positive cones and their properties appear in Guo and Lakshmikantham [101] and Heikkilä and Lakshmikantham [107].

Lemma 5.3 ([74, Dhage]). *Let K be a positive cone in the ordered Banach algebra E. If $u_1, u_2, v_1, v_2 \in K$ are such that $u_1 \leq v_1$ and $u_2 \leq v_2$, then $u_1u_2 \leq v_1v_2$.*

Theorem 5.15. *Let S be a nonempty closed convex subset of E such that $S^+ = S \cap E^+ \neq \emptyset$. Let $A, L, U : S^+ \longrightarrow E$ be three operators such that*

(i) *A, L and U are weakly sequentially continuous on S^+,*
(ii) *$A(S^+), L(S^+)$ and $U(S^+)$ are relatively weakly compact, and*
(iii) *$Ax + LxUx \in S^+$ if $x \in S^+$.*

Then, Eq. (5.7) has at least one solution in S^+.

Proof. Obviously S^+ is a nonempty closed convex subset of E. The use of assumption (ii) leads to $(A + LU)(S^+)$ and $U(S^+)$ are bounded subsets of E. Now, we will apply Theorem 5.10 to infer that Eq. (5.7) has at least one solution in S^+. ■

Using Theorem 5.12, we obtain the following result:

Theorem 5.16. *Let S be a nonempty closed convex subset of E such that $S^+ = S \cap E^+ \neq \emptyset$. Let A, L and U be three operators such that*

(i) *$L : S^+ \longrightarrow E$ is weakly sequentially continuous on S^+ and condensing map with respect to the measure of weak noncompactness β,*
(ii) *$A, U : S^+ \longrightarrow E$ are weakly sequentially continuous on S^+ and weakly compact, and*
(iii) *$Ax + LxUx \in S^+$ if $x \in S^+$.*

If $(A + LU)(S^+)$ and $U(S^+)$ are bounded subsets of E, then Eq. (5.7) has at least one solution in S^+ if $0 \leq \gamma^+ \leq 1$, where $\gamma^+ := \sup_{x \in S^+} \|Ux\| = \|U(S^+)\|$.

Using Theorem 5.14, we obtain the following result:

Theorem 5.17. *Let S be a nonempty closed convex bounded subset of E such that $S^+ = S \cap E^+ \neq \emptyset$. Let A, L and U be three operators such that*

(i) *$L : E \longrightarrow E$ is Lipschitzian with Lipschitz constant α and weakly sequentially continuous on E,*
(ii) *$A, U : S^+ \longrightarrow E$ are weakly sequentially continuous on S^+ and weakly compact, and*
(iii) *$Ax + LxUx \in S^+$ if $x \in S^+$.*

Then, Eq. (5.7) has a solution in S^+ if $\alpha\gamma^+ < 1$, where $\gamma^+ := \|U(S^+)\|$.

To close this section, we will prove the existence of positive solutions of Eq. (5.8) in the Banach algebra $\mathcal{C}(K, E)$, the space of continuous functions from K into E, endowed with the sup-norm $\|.\|_\infty$ defined by $\|f\|_\infty := \sup\{\|f(t)\| : t \in K\}$, where K is a compact Hausdorff space. Moreover, we suppose that E^+ verifies the following condition $(\mathcal{H})$:

$$(\mathcal{H}) \begin{cases} \text{Let } x, y \in E^+. \text{If } x \leq y \text{ (i.e., } y - x \in E^+) \text{ then } \|x\| \leq \|y\| \\ \text{(i.e., } \|.\| \text{ is called monotone increasing or nondecreasing and } E^+ \text{ is called normal).} \end{cases}$$

It is known that if the cone E^+ is normal, then every order-bounded subset is bounded in norm.

We denote by $\mathcal{C}_+(K)$ the cone of nonnegative functions in $\mathcal{C}(K,E)$ (i.e., $\mathcal{C}_+(K) = \mathcal{C}(K,E^+)$). For all $f_1,f_2 \in \mathcal{C}(K,E)$, we will say that $f_1 \leq f_2$ or ($f_2 \geq f_1$) provided $f_2 - f_1 \in \mathcal{C}_+(K)$.

A map $\mathcal{F} : \mathcal{C}(K,E) \longrightarrow \mathcal{C}(K,E)$ will be called isotone if $f_1 \leq f_2$, then $\mathcal{F}(f_1) \leq \mathcal{F}(f_2)$.

Theorem 5.18. *Suppose that U is weakly sequentially continuous, weakly compact and isotone map of $\mathcal{C}_+(K)$ into itself. For an arbitrary x_0 in $\mathcal{C}_+(K)$, define a sequence $\{x_n\}_{n\geq 0}$ by:*

$$x_{n+1} = x_0 + x_n U x_n, \ n = 0, 1, 2, \ldots.$$

If $\gamma = \sup_{n \in \mathbb{N}} \|Ux_n\|_\infty < 1$, then the increasing sequence $\{x_n\}_{n\geq 0}$ of $\mathcal{C}_+(K)$ converges strongly to a point x in $\mathcal{C}_+(K)$ a solution of Eq. (5.8), satisfying:

$$\|x\|_\infty \leq (1-\gamma)^{-1} \|x_0\|_\infty,$$

and $x_n \leq x$, for all $n \in \mathbb{N}$.

Before we prove this theorem we begin with the following lemma.

Lemma 5.4. *Let K be a normal positive cone in the ordered Banach space E. Let $\{x_n\}_{n\geq 0}$ be an increasing sequence of E. If $\{x_n\}_{n\geq 0}$ is norm bounded and has a weakly convergent subsequence $\{x_{\varphi(n)}\}_{n\geq 0}$ to a point x in E, then the whole sequence $\{x_n\}_{n\geq 0}$ converges strongly to x.*

Proof. First, we claim that $x_{\varphi(n)} \leq x$, $n \in \mathbb{N}$. Indeed, suppose that is not the case, so, there exists $n_0 \in \mathbb{N}$ such that $x - x_{\varphi(n_0)} \notin K$. Hence, the Hahn–Banach separation theorem for convex closed subsets assures that there exist $x^* \in E^* \backslash \{\theta\}$ and $\epsilon > 0$ such that

$$x^*\left(x - x_{\varphi(n_0)}\right) + \epsilon \leq x^*\left(x_{\varphi(n)} - x_{\varphi(n_0)}\right), \ n \geq n_0$$

which is a contradiction if we pass to the limit as $n \to \infty$. Next, we will show that $\{x_{\varphi(n)}\}_{n\geq 0}$ has a strongly convergent subsequence to the point x. Otherwise, there exist $\epsilon > 0$ and $n_0 \in \mathbb{N}$ verifying the property:

$$\text{for all } n \geq n_0, \ \ \|x - x_{\varphi(n)}\| \geq \epsilon. \tag{5.9}$$

Set

$$Q_n := \{x \in E : x \leq x_{\varphi(n)}\}, \ n \geq n_0 \text{ and } Q = \bigcup_{n=n_0}^{\infty} Q_n.$$

Obviously, Q_n is nonempty, $Q_n \subset Q_{n+1}$ and Q_n is convex. Therefore, Q and $\overline{Q}$ are convex. If $y \in Q$, then $y \in Q_n$ for some $n \geq n_0$ and consequently $y \leq x_{\varphi(n)}$. Hence

$$0 \leq x - x_{\varphi(n)} \leq x - y.$$

As a result

$$\|x - x_{\varphi(n)}\| \leq \|x - y\|.$$

The use of (5.9) leads to $\epsilon \leq \|x - y\|$. Thus, $x \notin \overline{Q}$. Again, the Hahn–Banach separation theorem for convex closed subsets assures that there exist $f \in E^* \backslash \{0\}$ and $\alpha > 0$ such that

$$f(x) + \alpha \leq f(y), \ y \in \overline{Q}.$$

Consequently

$$f(x) + \alpha \leq f\left(x_{\varphi(n)}\right), \ n \geq n_0$$

which is a contradiction if we pass to the limit as $n \to \infty$. Then, there exists a subsequence $\{x_{(\varphi\circ\psi)(n)}\}_{n\geq 0}$ of $\{x_{\varphi(n)}\}_{n\geq 0}$ such that $x_{(\varphi\circ\psi)(n)} \to x$ as $n \to \infty$. Since for all $p \geq n$, $\varphi \circ \psi(p) \geq \varphi \circ \psi(n) \geq n$, then $x_n \leq x_{(\varphi\circ\psi)(p)}$. As $p \to \infty$, we infer that $x_n \leq x$. But, for all $\epsilon > 0$, there exists $p_0 \in \mathbb{N}$ such that $\|x - x_{(\varphi\circ\psi)(p_0)}\| \leq \epsilon$. Now, for all $n \geq \varphi \circ \psi(p_0)$, we get

$$0 \leq x - x_n \leq x - x_{(\varphi\circ\psi)(p_0)},$$

which implies

$$\|x - x_n\| \leq \epsilon, \ n \geq \varphi \circ \psi(p_0).$$

This means that the whole sequence $\{x_n\}_{n\geq 0}$ converges strongly to x. ■

Proof (Theorem 5.18). Define

$$\begin{cases} F : \mathcal{C}_+(K) \longrightarrow \mathcal{C}_+(K) \\ \qquad\qquad x \longrightarrow Fx = x_0 + xUx. \end{cases}$$

Then

$$x_{n+1} = Fx_n, \ n = 0, 1, 2, \ldots.$$

So, we have that F maps the subset $Q = \{x_0, x_1, x_2, \ldots.\}$ into itself. Proceeding by induction and using the fact that U is isotone together with Lemma 5.3, we get for each $t \in K$

$$x_n(t) \leq x_{n+1}(t), \ n = 0, 1, 2, \ldots.$$

Then, $\{x_n\}_{n\geq 0}$ is a nondecreasing sequence. Therefore, since E^+ is normal, we get

$$\begin{aligned}\|x_n(t)\| &\leq \|x_{n+1}(t)\| \\ &\leq \|x_0(t)\| + \|x_n(t)\| \, \|Ux_n(t)\| \\ &\leq \|x_0\|_\infty + \gamma \|x_n\|_\infty .\end{aligned}$$

Taking the supremum over t, we obtain

$$\|x_n\|_\infty \leq \|x_0\|_\infty + \gamma \|x_n\|_\infty ,$$

which, since $0 \leq \gamma < 1$, implies that

$$\|x_n\|_\infty \leq (1-\gamma)^{-1} \|x_0\|_\infty , \tag{5.10}$$

and consequently Q is bounded. Furthermore, notice $Q = \{x_0\} \cup F(Q)$, so that $\beta(F(Q))$, the measure of weak noncompactness of $F(Q)$, is just $\beta(Q)$. Now, apply Theorem 5.13 to infer that F is a strict set-contraction with respect to β, since $\alpha = 1$ and $0 \leq \gamma < 1$. Thus, $\beta(Q) = 0$, so Q is relatively weakly compact. It follows from the Eberlein–Šmulian theorem that Q is sequentially relatively weakly compact. Consequently, there is a renamed subsequence $\{x_n\}_{n\geq 0}$ which converges weakly to a point x in $\mathcal{C}_+(K)$ (since E^+ and consequently $\mathcal{C}_+(K)$ are weakly closed convex). This fact, together with (5.10), gives

$$\|x\|_\infty \leq (1-\gamma)^{-1} \|x_0\|_\infty .$$

Since U is weakly sequentially continuous and the fact that the Banach algebra $\mathcal{C}(K, E)$ satisfies condition $(\mathcal{P})$, we infer that F is weakly sequentially continuous. Keeping in mind that the positive cone E^+ is normal, then $\mathcal{C}_+(K)$ is normal. We now apply Lemma 5.4 to conclude that the whole nondecreasing sequence $\{x_n\}_{n\geq 0}$ converges strongly to x, then $Fx_n (= x_{n+1})$ converges weakly to both x and Fx, so that $Fx = x$. Thus, x fulfills the conclusion of Theorem 5.18, which ends the proof. ■

Theorem 5.19. *Let x_0 be in $\mathcal{C}_+(K)$ and $S := \{y \in \mathcal{C}_+(K) : y \leq x_0\}$. Let $U : S \longrightarrow \mathcal{C}_+(K)$ be a weakly sequentially continuous and weakly compact operator. If $U(S)$ is bounded, then the equation*

$$x_0 = x + xUx. \tag{5.11}$$

has at least one fixed-point in S if $\gamma := \sup_{x \in S} \|Ux\|_\infty < 1$.

Proof. Clearly S is nonempty closed convex bounded (with bound $\|x_0\|$) subset of the Banach algebra $\mathcal{C}(K, E)$. Note that Eq. (5.11) is equivalent to the equation

$$x = x_0 + x(-Ux).$$

Fix $x \in S$, so we have by definition $x \in \mathcal{C}_+(K)$ and $Ux \in \mathcal{C}_+(K)$. Then

$$xUx \in \mathcal{C}_+(K).$$

As result

$$x_0 + x(-Ux) \in S.$$

Now, the use of Corollary 5.4 completes the proof. ■

Corollary 5.5. *Let x_0 be in $\mathcal{C}_+(K)$ and $S := \{y \in \mathcal{C}_+(K) : y \leq x_0\}$. Assume that $U : S \longrightarrow \mathcal{C}_+(K)$ is such that:*

(*i*) *U is weakly sequentially continuous on S,*
(*ii*) *U is Lipschitzian with Lipschitz constant α, and*
(*iii*) *U is weakly compact.*

Then, there is an unique x in S solution of Eq. (5.11) if $\alpha\|x_0\| + \gamma < 1$.

Proof. The existence is proved in Theorem 5.19. Now, we will show the uniqueness. Assume that x_1 and x_2 are two solutions of Eq. (5.11). Hence, it follows that

$$x_1 + x_1Ux_1 = x_2 + x_2Ux_2.$$

Thus

$$x_1 - x_2 = x_2(Ux_2 - Ux_1) + (x_2 - x_1)Ux_1.$$

Then

$$\begin{aligned}\|x_1 - x_2\| &\leq \|x_2\|\ \|Ux_2 - Ux_1\| + \gamma\|x_2 - x_1\| \\ &\leq \|x_0\|\ \|Ux_2 - Ux_1\| + \gamma\|x_2 - x_1\| \\ &\leq (\alpha\|x_0\| + \gamma)\|x_2 - x_1\|.\end{aligned}$$

Since $\alpha\|x_0\| + \gamma < 1$, we must have $x_1 = x_2$ which ends the proof. ■

Remark 5.3. The element x_0 is invertible if and only if the solution x of Eq. (5.11) is invertible. Indeed, Eq. (5.11) is equivalent to the equation $x_0 = x(I + Ux)$ where I is the identity operator defined by $Ix = x$, $x \in \mathcal{C}_+(K)$. Since $\gamma := \sup_{y \in \Omega} \|Uy\|_\infty < 1$, then $\|Ux\|_\infty < 1$ and consequently $(I + Ux)$ is invertible. (Recall that $(I + Ux)^{-1} = \sum_{n=0}^{\infty}(-1)^n(Ux)^n$).

5.3 Leray–Schauder Alternatives

We present some nonlinear alternatives of Leray–Schauder type in Banach algebra satisfying certain sequential condition $(\mathcal{P})$ for the sum and the product of nonlinear weakly sequentially continuous operators.

Theorem 5.20. *Let Ω be a closed convex subset in a Banach algebra $\mathcal{E}$, $U \subset \Omega$ a weakly open set (with respect to the weak topology of Ω) such that $0 \in U$ and $\overline{U^w}$ is a weakly compact subset of Ω. Let $A, C : \mathcal{E} \to \mathcal{E}$ $B : \overline{U^w} \to \mathcal{E}$ be three operators satisfying*

- (i) *A and C are $\mathcal{D}$-Lipschitz with $\mathcal{D}$-functions Φ_A and Φ_C respectively,*
- (ii) *A is regular on $\mathcal{E}$, i, e., A maps $\mathcal{E}$ into the set of all invertible elements of $\mathcal{E}$,*
- (iii) *B is weakly sequentially continuous on $\overline{U^w}$,*
- (iv) *$M\Phi_A(r) + \Phi_C(r) < r$ for $r > 0$, with $M = \|B(\overline{U^w})\|$,*
- (v) *$x = AxBy + Cx \Rightarrow x \in \Omega$ for all $y \in \overline{U^w}$,*
- (vi) *$\left(\frac{I-C}{A}\right)^{-1}$ is weakly sequentially continuous on $B(\overline{U^w})$.*

Then either

- $(A1)$ *the equation $\lambda A\left(\frac{x}{\lambda}\right)Bx + \lambda C\left(\frac{x}{\lambda}\right) = x$ has a solution for $\lambda = 1$, or*
- $(A2)$ *there is an element $u \in \partial_\Omega U$ such that $\lambda A\left(\frac{u}{\lambda}\right)Bu + \lambda C\left(\frac{u}{\lambda}\right) = u$ for some $0 < \lambda < 1$.*

Proof. Let $y \in \overline{U^w}$ be fixed and define the mapping $\phi_y : \mathcal{E} \to \mathcal{E}$ by

$$\phi_y(x) = AxBy + Cx \tag{5.12}$$

for $x \in \mathcal{E}$. Then for any $x_1, x_2 \in \mathcal{E}$, we have

$$\begin{aligned}
\|\phi_y(x_1) - \phi_y(x_2)\| &\leq \|Ax_1By - Ax_2By\| + \|Cx_1 - Cx_2\| \\
&\leq \|Ax_1 - Ax_2\|\|By\| + \|Cx_1 - Cx_2\| \\
&\leq M\Phi_A(\|x_1 - x_2\|) + \Phi_C(\|x_1 - x_2\|).
\end{aligned}$$

In view of the hypothesis we deduce that ϕ_y is a nonlinear contraction on $\mathcal{E}$. Therefore an application of Theorem 5.1 yields that ϕ_y has a unique fixed point, say x in $\mathcal{E}$, i.e., there exists a unique $x \in \mathcal{E}$ with $AxBy + Cx = x$. By hypothesis, it is clear that $x \in \Omega$, so there exists a unique $x \in \Omega$ with $AxBy + Cx = x$. By hypothesis there exists a unique $x \in \Omega$ with

$$\left(\frac{I-C}{A}\right)x = By \text{ and so } x = \left(\frac{I-C}{A}\right)^{-1} By$$

Hence $\left(\frac{I-C}{A}\right)^{-1} B : \overline{U^w} :\to \Omega$ is well defined.

Since B is weakly sequentially continuous on $\overline{U^w}$ and $\left(\frac{I-C}{A}\right)^{-1}$ is weakly sequentially continuous on $B(\overline{U^w})$, so by composition we have $\left(\frac{I-C}{A}\right)^{-1} B$ is weakly sequentially continuous on $\overline{U^w}$.

Now an application of Theorem 3.1 implies that either

(A1) $\left(\frac{I-C}{A}\right)^{-1} B$ has a fixed point, or

(A2) there is a point $u \in \partial_\Omega U$ and $\lambda \in]0,1[$ with $u = \lambda \left(\frac{I-C}{A}\right)^{-1} Bu$.

Assume first that $x \in U$ is a fixed point of the operator $\left(\frac{I-C}{A}\right)^{-1} B$. Then $x = \left(\frac{I-C}{A}\right)^{-1} Bx$ which implies that

$$AxBx + Cx = x.$$

Suppose next that there is an element $u \in \partial_\Omega U$ and a real number $\lambda \in]0,1[$ such that $u = \lambda \left(\frac{I-C}{A}\right)^{-1} Bu$. Then

$$\left(\frac{I-C}{A}\right)^{-1} Bu = \frac{u}{\lambda},$$

so that

$$\lambda A\left(\frac{u}{\lambda}\right) Bu + \lambda C\left(\frac{u}{\lambda}\right) = u.$$

This completes the proof. ■

Corollary 5.6. *Let Ω be a closed convex subset in a Banach algebra $\mathcal{E}$, $U \subset \Omega$ a weakly open set (with respect to the weak topology of Ω) such that $0 \in U$ and $\overline{U^w}$ is a weakly compact subset of Ω. Let $A : \mathcal{E} \to \mathcal{E}$ $B : \overline{U^w} \to \mathcal{E}$ be two operators satisfying*

(*i*) *A is $\mathcal{D}$-Lipschitz with $\mathcal{D}$-function Φ_A ,*
(*ii*) *A is regular on $\mathcal{E}$,*
(*iii*) *B is weakly sequentially continuous on $\overline{U^w}$,*
(*iv*) *$M\Phi_A(r) < r$ for $r > 0$, with $M = \|B(\overline{U^w})\|$,*
(*v*) *$x = AxBy \Rightarrow x \in \Omega$ for all $y \in \overline{U^w}$,*
(*vi*) *$\left(\frac{I}{A}\right)^{-1}$ is weakly sequentially continuous on $B(\overline{U^w})$.*

Then either

(A1) *the equation $\lambda A\left(\frac{x}{\lambda}\right) Bx = x$ has a solution for $\lambda = 1$, or*

(A2) *there is an element $u \in \partial_\Omega U$ such that $\lambda A\left(\frac{u}{\lambda}\right) Bu = u$ for some $0 < \lambda < 1$.*

Now, we will state our results in a Banach algebra satisfying condition ($\mathcal{P}$).

Theorem 5.21. *Let Ω be a closed convex subset in a Banach algebra $\mathcal{E}$ satisfying condition ($\mathcal{P}$) and $U \subset \Omega$ a weakly open set (with respect to the weak topology of Ω) such that $0 \in U$. Let $A, C : \mathcal{E} \to \mathcal{E}$ $B : \overline{U^w} \to \mathcal{E}$ be three operators satisfying*

(*i*) *A and C are $\mathcal{D}$-Lipschitz with $\mathcal{D}$-functions Φ_A and Φ_C respectively,*
(*ii*) *A is regular on $\mathcal{E}$,*
(*iii*) *$B(\overline{U^w})$ is bounded with bound M,*
(*iv*) *$M\Phi_A(r) + \Phi_C(r) < r$ for $r > 0$,*
(*v*) *$x = AxBy + Cx \Rightarrow x \in \Omega$ for all $y \in \overline{U^w}$,*
(*vi*) *A, C are weakly sequentially continuous on Ω and B is weakly sequentially continuous $\overline{U^w}$,*
(*vii*) *$\left(\frac{I-C}{A}\right)^{-1} B(\overline{U^w})$ is relatively weakly compact.*

Then either

(*A*1) *the equation $\lambda A\left(\frac{x}{\lambda}\right) Bx + \lambda C\left(\frac{x}{\lambda}\right) = x$ has a solution for $\lambda = 1$, or*
(*A*2) *there is an element $u \in \partial_\Omega U$ such that $\lambda A\left(\frac{u}{\lambda}\right) Bu + \lambda C\left(\frac{u}{\lambda}\right) = u$ for some $0 < \lambda < 1$.*

Proof. Similarly reasoning to that in the proof of Theorem 5.20, guarantees that $\left(\frac{I-C}{A}\right)^{-1} B$ is well defined from $\overline{U^w}$ to Ω, and it suffices to establish that $\left(\frac{I-C}{A}\right)^{-1} B$ is weakly sequentially continuous on $\overline{U^w}$. To see this, let $\{u_n\}$ be a weakly convergent sequence of $\overline{U^w}$ to a point u in $\overline{U^w}$. Now, define the sequence $\{v_n\}$ of the subset Ω by

$$v_n = \left(\frac{I-C}{A}\right)^{-1} Bu_n.$$

Since $\left(\frac{I-C}{A}\right)^{-1} B(\overline{U^w})$ is relatively weakly compact, so, there is a renamed subsequence such that

$$v_n = \left(\frac{I-C}{A}\right)^{-1} Bu_n \rightharpoonup v.$$

But on the other hand, the subsequence $\{v_n\}$ verifies $v_n - Cv_n = Av_nBu_n$. Therefore, from assumption (*f*) and in view of condition ($\mathcal{P}$), we deduce that v satisfies

$$v - Cv = AvBu,$$

or equivalently

$$v = \left(\frac{I-C}{A}\right)^{-1} Bu.$$

Next we claim that the whole sequence $\{u_n\}$ verifies

$$\left(\frac{I-C}{A}\right)^{-1} Bu_n = v_n \rightharpoonup v.$$

Indeed, suppose that this is not the case, so, there is V^w a weakly neighborhood of v satisfying for all $n \in \mathbb{N}$, there exists an $N \geq n$ such that $v_N \notin V^w$. Hence, there is a renamed subsequence $\{v_n\}$ verifying the property

$$\text{for all } n \in \mathbb{N}, \{ v_n\} \notin V^w. \tag{5.13}$$

However for all $n \in \mathbb{N}$, $v_n \in \left(\frac{I-C}{A}\right)^{-1} B(\overline{U^w})$.

Again, there is a renamed subsequence such that $v_n \rightharpoonup v'$. According to the above, we have $v' = \left(\frac{I-C}{A}\right)^{-1} Bu$, and consequently $v' = v$, which contradicts (5.13). This yields that $\left(\frac{I-C}{A}\right)^{-1} B$ is weakly sequentially continuous.

In view of Remark 3.1, an application of Theorem 3.1 implies that either

$(A1)$ $\left(\frac{I-C}{A}\right)^{-1} B$ has a fixed point, or

$(A2)$ there is a point $u \in \partial_\Omega U$ and $\lambda \in]0,1[$ with $u = \lambda \left(\frac{I-C}{A}\right)^{-1} Bu$.

Assume first that $x \in \overline{U^w}$ is a fixed point of the operator $\left(\frac{I-C}{A}\right)^{-1} B$. Then $x = \left(\frac{I-C}{A}\right)^{-1} Bx$ which implies that

$$AxBx + Cx = x.$$

Suppose next that there is an element $u \in \partial_\Omega U$ and a real number $\lambda \in]0,1[$ such that $u = \lambda \left(\frac{I-C}{A}\right)^{-1} Bu$. Then

$$\left(\frac{I-C}{A}\right)^{-1} Bu = \frac{u}{\lambda},$$

so that

$$\lambda A\left(\frac{u}{\lambda}\right) Bu + \lambda C\left(\frac{u}{\lambda}\right) = u.$$

This completes the proof. ∎

Corollary 5.7. *Let Ω be a closed convex subset in a Banach algebra $\mathcal{E}$ satisfying condition $(\mathcal{P})$ and $U \subset \Omega$ a weakly open set (with respect to the weak topology of Ω) such that $0 \in U$. Let $A : \mathcal{E} \to \mathcal{E}$ $B : \overline{U^w} \to \mathcal{E}$ be two operators satisfying*

(i) *A is $\mathcal{D}$-Lipschitz with $\mathcal{D}$-functions Φ_A,*
(ii) *A is regular on $\mathcal{E}$,*
(iii) *$B(\overline{U^w})$ is bounded with bound M,*
(iv) *$M\Phi_A(r) < r$ for $r > 0$,*
(v) *$x = AxBy \Rightarrow x \in \Omega$ for all $y \in \overline{U^w}$,*
(vi) *A is weakly sequentially continuous on Ω and B is weakly sequentially continuous $\overline{U^w}$,*
(vii) *$\left(\frac{I}{A}\right)^{-1} B(\overline{U^w})$ is relatively weakly compact.*

Then either

$(A1)$ *the equation* $\lambda A\left(\frac{x}{\lambda}\right)Bx = x$ *has a solution for* $\lambda = 1$, *or*
$(A2)$ *there is an element* $u \in \partial_\Omega U$ *such that* $\lambda A\left(\frac{u}{\lambda}\right)Bu = u$ *for some* $0 < \lambda < 1$.

Theorem 5.22. *Let* Ω *be a closed convex subset in a Banach algebra* $\mathcal{E}$ *satisfying condition* $(\mathcal{P})$ *and* $U \subset \Omega$ *a weakly open set (with respect to the weak topology of* Ω*) such that* $0 \in U$. *Let* $A, C : \mathcal{E} \to \mathcal{E}$ $B : \overline{U^w} \to \mathcal{E}$ *be three operators satisfying*

(i) *A and C are* $\mathcal{D}$*-Lipschitz with* $\mathcal{D}$*-functions* Φ_A *and* Φ_C *respectively,*
(ii) *A is regular on* $\mathcal{E}$*,*
(iii) $M\Phi_A(r) + \Phi_C(r) < r$ *for* $r > 0$*,*
(iv) $x = AxBy + Cx \Rightarrow x \in \Omega$ *for all* $y \in \overline{U^w}$*,*
(v) *A, C are weakly sequentially continuous on* Ω *and B is weakly sequentially continuous* $\overline{U^w}$*,*
(vi) $A(\Omega)$*,* $B(\overline{U^w})$ *and* $C(\Omega)$ *are relatively weakly compacts.*

Then either

$(A1)$ *the equation* $\lambda A\left(\frac{x}{\lambda}\right)Bx + \lambda C\left(\frac{x}{\lambda}\right) = x$ *has a solution for* $\lambda = 1$, *or*
$(A2)$ *there is an element* $u \in \partial_\Omega U$ *such that* $\lambda A\left(\frac{u}{\lambda}\right)Bu + \lambda C\left(\frac{u}{\lambda}\right) = u$ *for some* $0 < \lambda < 1$.

Proof. It is enough to prove that $\left(\frac{I-C}{A}\right)^{-1}B(\overline{U^w})$ is relatively weakly compact. To do this, let $\{u_n\}$ be any sequence in $(\overline{U^w})$ and let

$$v_n = \left(\frac{I-C}{A}\right)^{-1} Bu_n. \tag{5.14}$$

Since $B(\overline{U^w})$ is relatively weakly compact, there is a renamed subsequence $\{Bu_n\}$ weakly converging to an element w. On the other hand, from (5.14), we obtain

$$v_n = Av_nBu_n + Cv_n. \tag{5.15}$$

Since $\{v_n\}$ is a sequence in $\left(\frac{I-C}{A}\right)^{-1}B(\overline{U^w})$, so by assumption (f), there is a renamed subsequence such that $Av_n \rightharpoonup x$ and $Cv_n \rightharpoonup y$. Hence, in view of condition $(\mathcal{P})$ and the last equation, we obtain

$$v_n \rightharpoonup xw + y.$$

This shows that $\left(\frac{I-C}{A}\right)^{-1}B(\overline{U^w})$ is relatively weakly sequentially compact. An application of the Eberlein–Šmulian theorem yields that $\left(\frac{I-C}{A}\right)^{-1}B(\overline{U^w})$ is relatively weakly compact. ■

Corollary 5.8. *Let* Ω *be a closed convex subset in a Banach algebra* $\mathcal{E}$ *satisfying condition* $(\mathcal{P})$ *and* $U \subset \Omega$ *a weakly open set (with respect to the weak topology of* Ω*) such that* $0 \in U$. *Let* $A : \mathcal{E} \to \mathcal{E}$ $B : \overline{U^w} \to \mathcal{E}$ *be two operators satisfying*

(i) *A is $\mathcal{D}$-Lipschitz with $\mathcal{D}$-functions Φ_A ,*
(ii) *A is regular on $\mathcal{E}$,*
(iii) $M\Phi_A(r) < r$ *for* $r > 0$,
(iv) $x = AxBy \Rightarrow x \in \Omega$ *for all* $y \in \overline{U^w}$,
(v) *A is weakly sequentially continuous on Ω and B is weakly sequentially continuous $\overline{U^w}$,*
(vi) *$A(\Omega)$ and $B(\overline{U^w})$ are relatively weakly compacts.*

Then either

$(A1)$ *the equation $\lambda A\left(\frac{x}{\lambda}\right)Bx = x$ has a solution for $\lambda = 1$, or*
$(A2)$ *there is an element $u \in \partial_\Omega U$ such that $\lambda A\left(\frac{u}{\lambda}\right)Bu = u$ for some* $0 < \lambda < 1$.

5.4 Applications

In this section, first we illustrate the applicability of Corollary 5.1 and Theorem 5.4 by considering nonlinear functional integral equations.

Let $(X, ||.||)$ be a Banach algebra satisfying condition $(\mathcal{P})$. Let $J = [0, 1]$ the closed and bounded interval in $\mathbb{R}$, the set of all real numbers. Let $\mathcal{E} = \mathcal{C}(J, X)$ the Banach algebra of all continuous functions from $[0, 1]$ to X, endowed with the sup-norm $|| \ ||_\infty$, defined by $||f||_\infty = \sup\{||f(t)|| \ ; \ t \in [0, 1]\}$, for each $f \in \mathcal{C}(J, X)$. We consider the nonlinear functional integral equation (in short, FIE):

$$x(t) = a(t) + (T_1x)(t)\left[\left(q(t) + \int_0^{\sigma(t)} p(t, s, x(s), x(\lambda s))ds\right).u\right], \ 0 < \lambda < 1, \tag{5.16}$$

for all $t \in J$, where $u \neq 0$ is a fixed vector of X and the functions a, q, σ, p, T_1 are given, while $x = x(t)$ is an unknown function.

We shall obtain the solution of FIE (5.16) under some suitable conditions. Suppose that the functions involved in Eq. (5.16) verify the following conditions:

(H_1) $a : J \longrightarrow X$ is a continuous function.
(H_2) $\sigma : J \longrightarrow J$ is a continuous and nondecreasing function.
(H_3) $q : J \longrightarrow \mathbb{R}$ is a continuous function.
(H_4) The operator $T_1 : \mathcal{C}(J, X) \longrightarrow \mathcal{C}(J, X)$ is such that

- (a) T_1 is Lipschitzian with a Lipschitzian constant α,
- (b) T_1 is regular on $\mathcal{C}(J, X)$,
- (c) T_1 is weakly sequentially continuous on $\mathcal{C}(J, X)$,
- (d) T_1 is weakly compact.

(H_5) The function $p : J \times J \times X \times X \longrightarrow \mathbb{R}$ is continuous such that for arbitrary fixed $s \in J$ and $x, y \in X$, the partial function $t \longrightarrow p(t, s, x, y)$ is uniformly continuous for $(s, x, y) \in J \times X \times X$.

(H_6) There exists $r_0 > 0$ such that

(a) $|p(t,s,x,y)| \leq r_0 - \|q\|_\infty$ for each $t,s \in J$; $x,y \in X$ such that $||x|| \leq r_0$ and $||y|| \leq r_0$,
(b) $\|T_1 x\|_\infty \leq \left(1 - \frac{\|a\|_\infty}{r_0}\right) \frac{1}{\|u\|}$ for each $x \in \mathcal{C}(J,X)$,
(c) $\alpha r_0 \|u\| < 1$.

Theorem 5.23. *Under assumptions* $(H_1)-(H_6)$*, Eq. (5.16) has at least one solution* $x = x(t)$ *which belongs to the space* $\mathcal{C}(J,X)$.

Proof. First, we begin by showing that $\mathcal{C}(J,X)$ verifies condition $(\mathcal{P})$. To see this, let $\{x_n\}$, $\{y_n\}$ be any sequences in $\mathcal{C}(J,X)$ such that $x_n \rightharpoonup x$ and $y_n \rightharpoonup y$. So, for each $t \in J$, we have $x_n(t) \rightharpoonup x(t)$ and $y_n(t) \rightharpoonup y(t)$ (cf. Theorem 5.5). Since X verifies condition $(\mathcal{P})$, then

$$x_n(t)y_n(t) \rightharpoonup x(t)y(t),$$

because $\{x_n y_n\}_n$ is a bounded sequence, and this, further, implies that

$$x_n y_n \rightharpoonup xy \quad \text{(cf. Theorem 5.5)},$$

which shows that the space $\mathcal{C}(J,X)$ verifies condition $(\mathcal{P})$.

Let us define the subset S of $\mathcal{C}(J,X)$ by

$$S := \{y \in \mathcal{C}(J,X),\ \|y\|_\infty \leq r_0\} = B_{r_0}.$$

Obviously S is nonempty, convex and closed. Let us consider three operators A,B and C defined on $\mathcal{C}(J,X)$ by

$$\begin{aligned}
(Ax)(t) &= (T_1 x)(t) \\
(Bx)(t) &= \left[q(t) + \int_0^{\sigma(t)} p(t,s,x(s),x(\lambda s))ds \right].u, \quad 0 < \lambda < 1 \\
(Cx)(t) &= a(t).
\end{aligned}$$

We shall prove that the operators A,B and C satisfy all the conditions of Corollary 5.1.

(i) From assumption $(H_4)(a)$, it follows that A is Lipschitzian with a Lipschitzian constant α. Clearly C is Lipschitzian with a Lipschitzian constant 0.
(ii) From assumption $(H_4)(b)$, it follows that A is regular on $\mathcal{C}(J,X)$.
(iii) Since C is constant, so, C is weakly sequentially continuous on S. From assumption $(H_4)(c)$, A is weakly sequentially continuous on S. Now, we show that B is weakly sequentially continuous on S. Firstly, we verify that if $x \in S$, then $Bx \in \mathcal{C}(J,X)$. Let $\{t_n\}$ be any sequence in J converging to a point t in J. Then

$$\begin{aligned}\|(Bx)(t_n)-(Bx)(t)\| &= \left\|\left[\int_0^{\sigma(t_n)} p(t_n,s,x(s),x(\lambda s))ds - \int_0^{\sigma(t)} p(t,s,x(s),x(\lambda s))ds\right].u\right\| \\ &\leq \left[\int_0^{\sigma(t_n)} |p(t_n,s,x(s),x(\lambda s)) - p(t,s,x(s),x(\lambda s))|ds\right] \|u\| \\ &+ \left|\int_{\sigma(t)}^{\sigma(t_n)} |p(t,s,x(s),x(\lambda s))|ds\right| \|u\| \\ &\leq \left[\int_0^{1} |p(t_n,s,x(s),x(\lambda s)) - p(t,s,x(s),x(\lambda s)|ds\right] \|u\| \\ &+ (r_0 - \|q\|_\infty)|\sigma(t_n)-\sigma(t)|\ \|u\|.\end{aligned}$$

Since $t_n \to t$, so, $(t_n, s, x(s), x(\lambda s)) \to (t, s, x(s), x(\lambda s))$, for all $s \in J$. Taking into account hypothesis (H_5), we obtain

$$p(t_n, s, x(s), x(\lambda s)) \to p(t, s, x(s), x(\lambda s)) \text{ in } \mathbb{R}.$$

Moreover, the use of assumption (H_6) leads to

$$|p(t_n, s, x(s), x(\lambda s)) - p(t, s, x(s), x(\lambda s))| \leq 2(r_0 - \|q\|_\infty)$$

for all $t, s \in J$, $\lambda \in (0, 1)$. Consider

$$\begin{cases} \varphi : J \longrightarrow \mathbb{R} \\ \quad s \longrightarrow \varphi(s) = 2(r_0 - \|q\|_\infty). \end{cases}$$

Clearly $\varphi \in L^1(J)$. Therefore, from the dominated convergence theorem and assumption (H_2), we obtain

$$(Bx)(t_n) \to (Bx)(t) \text{ in } X.$$

It follows that

$$Bx \in \mathcal{C}(J, X).$$

Next, we prove B is weakly sequentially continuous on S. Let $\{x_n\}$ be any sequence in S weakly converging to a point x in S. So, from assumptions (H_5) and (H_6) and the dominated convergence theorem, we get

$$\lim_{n\to\infty} \int_0^1 p(t, s, x_n(s), x_n(\lambda s))ds = \int_0^1 p(t, s, x(s), x(\lambda s)),$$

which implies

$$\lim_{n\to\infty}\left(q(t)+\int_0^1 p(t,s,x_n(s),x_n(\lambda s))ds\right).u=\left(q(t)+\int_0^1 p(t,s,x(s),x(\lambda s))ds\right).u.$$

Hence,

$$(Bx_n)(t)\to(Bx)(t) \text{ in X}.$$

Since $(Bx_n)_n$ is bounded by $r_0\|u\|$, then

$$Bx_n \rightharpoonup Bx \text{ (cf. Theorem 5.5)}.$$

We conclude that B is weakly sequentially continuous on S.

(iv) We will prove that $A(S), B(S)$, and $C(S)$ are relatively weakly compact. Since S is bounded by r_0 and taking into account hypothesis $(H_4)(d)$, it follows that $A(S)$ is relatively weakly compact. Now, we show $B(S)$ is relatively weakly compact.

(STEP I) By definition,

$$B(S) := \{B(x),\ \|x\|_\infty \le r_0\}.$$

For all $t\in J$, we have

$$B(S)(t) = \{(Bx)(t),\ \|x\|_\infty \le r_0\}.$$

We claim that $B(S)(t)$ is sequentially weakly relatively compact in X. To see this, let $\{x_n\}$ be any sequence in S, we have $(Bx_n)(t) = r_n(t).u$, where $r_n(t) = q(t) + \int_0^1 p(t,s,x_n(s),x_n(\lambda s))ds$. Since $|r_n(t)| \le r_0$ and $(r_n(t))$ is a real sequence, so, there is a renamed subsequence such that

$$r_n(t)\to r(t) \text{ in } \mathbb{R},$$

which implies

$$r_n(t).u \to r(t).u \text{ in } X,$$

and, consequently

$$(Bx_n)(t)\to (q(t)+r(t)).u \text{ in } X.$$

We conclude that $B(S)(t)$ is sequentially relatively compact in X, and then $B(S)(t)$ is sequentially relatively weakly compact in X.

(STEP II) We prove that $B(S)$ is weakly equicontinuous on J. If we take $\epsilon > 0, x \in S, x^* \in X^*, t, t' \in J$ such that $t \leq t'$ and $t' - t \leq \epsilon$, then

$$\begin{aligned}
|x^*((Bx)(t) - (Bx)(t'))| &= \left| \int_0^{\sigma(t)} p(t, s, x(s), x(\lambda s))ds - \int_0^{\sigma(t')} p(t', s, x(s), x(\lambda s))ds \right| \|x^*(u)\| \\
&\leq \left[\int_0^{\sigma(t)} |p(t, s, x(s), x(\lambda s)) - p(t', s, x(s), x(\lambda s))|ds \right] \|x^*(u)\| \\
&+ \left[\int_{\sigma(t)}^{\sigma(t')} |p(t', s, x(s), x(\lambda s))|ds \right] \|x^*(u)\| \\
&\leq [w(p, \epsilon) + (r_0 - \|q\|_\infty) w(\sigma, \epsilon)] \|x^*(u)\|,
\end{aligned}$$

where

$$\begin{aligned}
w(p, \epsilon) &= \sup\{|p(t, s, x, y) - p(t', s, x, y)| : t, t', s \in J; |t - t'| \leq \epsilon;\ x, y \in B_{r_0}\} \\
w(\sigma, \epsilon) &= \sup\{|\sigma(t) - \sigma(t')| : t, t' \in J; |t - t'| \leq \epsilon\}.
\end{aligned}$$

Taking into account hypothesis (H_5) and in view of the uniform continuity of the function σ on the set J, it follows that $w(p, \epsilon) \to 0$ and $w(\sigma, \epsilon) \to 0$ as $\epsilon \to 0$. From the Arzelà–Ascoli theorem, we conclude that $B(S)$ is sequentially weakly relatively compact in X. The Eberlein–Šmulian theorem yields that $B(S)$ is relatively weakly compact. As $C(S) = \{a\}$, hence $C(S)$ is relatively weakly compact.

(v) Finally, it remains to prove hypothesis (v) of Corollary 5.1. To see this, let $x \in \mathcal{C}(J, X)$ and $y \in S$ such that

$$x = AxBy + Cx,$$

or, equivalently for all $t \in J$,

$$x(t) = a(t) + (T_1 x)(t)(By)(t).$$

But, for all $t \in J$, we have

$$\|x(t)\| \leq \|x(t) - a(t)\| + \|a(t)\|.$$

Then

$$\begin{aligned}
\|x(t)\| &\leq \|(T_1 x)(t)\| r_0 \|u\| + \|a\|_\infty \\
&\leq \left(1 - \frac{\|a\|_\infty}{r_0}\right) r_0 + \|a\|_\infty \\
&= r_0.
\end{aligned}$$

From the last inequality and taking the supremum over t, we obtain

$$\|x\|_\infty \le r_0,$$

and, consequently $x \in S$.

We conclude that the operators A, B and C satisfy all the requirements of Corollary 5.1. Thus, an application of it yields that the FIE (5.16) has a solution in the space $\mathcal{C}(J, X)$. ∎

To illustrate Theorem 5.4, we consider the nonlinear functional integral equation (in short, FIE) in $\mathcal{C}(J, X)$.

$$x(t) = a(t)x(t) + (T_2x)(t)\left[\left(q(t) + \int_0^{\sigma(t)} p(t, s, x(s), x(\lambda s))ds\right).u\right],\ 0 < \lambda < 1, \tag{5.17}$$

for all $t \in J$, where $u \neq 0$ is a fixed vector of X and the functions a, q, σ, p, T_2 are given, while x in $\mathcal{C}(J, X)$ is an unknown function.

We obtain the solution of FIE (5.17) under some suitable conditions on the functions involved in (5.17). Suppose that the functions a, q, σ, p and the operator T_2 verify the following conditions:

(H_1) $a : J \longrightarrow X$ is a continuous function with $\|a\|_\infty < 1$.
(H_2) $\sigma : J \longrightarrow J$ is a continuous and nondecreasing function.
(H_3) $q : J \longrightarrow \mathbb{R}$ is a continuous function.
(H_4) The operator $T_2 : \mathcal{C}(J, X) \longrightarrow \mathcal{C}(J, X)$ is such that

- (a) T_2 is Lipschitzian with a Lipschitzian constant α,
- (b) T_2 is regular on $\mathcal{C}(J, X)$,
- (c) $\left(\dfrac{I}{T_2}\right)^{-1}$ is well defined on $\mathcal{C}(J, X)$,
- (d) $\left(\dfrac{I}{T_2}\right)^{-1}$ is weakly sequentially continuous on $\mathcal{C}(J, X)$.

(H_5) The function $p : J \times J \times X \times X \longrightarrow \mathbb{R}$ is continuous such that for arbitrary fixed $s \in J$ and $x, y \in X$, the partial function $t \longrightarrow p(t, s, x, y)$ is uniformly continuous for $(s, x, y) \in J \times X \times X$.
(H_6) There exists $r_0 > 0$ such that

- (a) $|p(t, s, x, y)| \le r_0 - \|q\|_\infty$ for each $t, s \in J$; $x, y \in X$ such that $||x|| \le r_0$ and $||y|| \le r_0$,
- (b) $\|T_2x\|_\infty \le \left(1 - \dfrac{\|a\|_\infty}{r_0}\right)\dfrac{1}{\|u\|}$ for each $x \in \mathcal{C}(J, X)$,
- (c) $\alpha r_0\|u\| < 1$.

Theorem 5.24. *Under assumptions* $(H_1)-(H_6)$*, Eq. (5.17) has at least one solution* $x = x(t)$ *which belongs to the space* $\mathcal{C}(J, X)$.

Proof. Let us consider three operators A, B and C defined on $\mathcal{C}(J, X)$ by

$$(Ax)(t) = (T_2x)(t)$$

$$(Bx)(t) = \left[q(t) + \int_0^{\sigma(t)} p(t, s, x(s), x(\lambda s))ds\right].u, \quad 0 < \lambda < 1$$

$$(Cx)(t) = a(t)x(t).$$

We prove that the operators A, B, and C satisfy all the conditions of Theorem 5.4.

(i) From assumption $(H_4)(a)$, A is Lipschitzian with a Lipschitzian constant α. Next, we show that C is Lipschitzian on $\mathcal{C}(J, X)$. To see this, fix arbitrarily $x, y \in \mathcal{C}(J, X)$. Then, if we take an arbitrary $t \in J$, we get

$$\begin{aligned}\|(Cx)(t) - (Cy)(t)\| &= \|a(t)x(t) - a(t)y(t)\| \\ &\leq \|a\|_\infty \|x(t) - y(t)\|.\end{aligned}$$

From the last inequality and taking the supremum over t, we obtain

$$\|Cx - Cy\|_\infty \leq \|a\|_\infty \|x - y\|_\infty.$$

This proves that C is Lipschitzian with a Lipschitzian constant $\|a\|_\infty$.

(ii) Arguing as in the proof of Theorem 5.23, we obtain B is weakly sequentially continuous on S and $B(S)$ is relatively weakly compact.

(iii) From assumption $(H_4)(b)$, A is regular on $\mathcal{C}(J, X)$.

(iv) We show that $\left(\frac{I - C}{A}\right)^{-1}$ is weakly sequentially continuous on $B(S)$. To see this, let $x, y \in \mathcal{C}(J, X)$ such that

$$\left(\frac{I - C}{A}\right)(x) = y,$$

or, equivalently

$$\frac{(1 - a)x}{T_2x} = y.$$

Since $\|a\|_\infty < 1$, so, $(1 - a)^{-1}$ exists on $\mathcal{C}(J, X)$, and then

$$\left(\frac{I}{T_2}\right)(x) = (1 - a)^{-1}y.$$

This implies, from assumption $(H_3)(c)$, that

$$x = \left(\frac{I}{T_2}\right)^{-1}((1-a)^{-1}y).$$

Thus

$$\left(\frac{I-C}{A}\right)^{-1}(x) = \left(\frac{I}{T_2}\right)^{-1}((1-a)^{-1}x)$$

for all $x \in \mathcal{C}(J, X)$. Now, let $\{x_n\}$ be a weakly convergent sequence of $B(S)$ to a point x in $B(S)$, then

$$(1-a)^{-1}x_n \rightharpoonup (1-a)^{-1}x,$$

and so, it follows from assumption $(H_4)(d)$ that

$$\left(\frac{I}{T_2}\right)^{-1}((1-a)^{-1}x_n) \rightharpoonup \left(\frac{I}{T_2}\right)^{-1}((1-a)^{-1}x),$$

we conclude that

$$\left(\frac{I-C}{A}\right)^{-1}(x_n) \rightharpoonup \left(\frac{I-C}{A}\right)^{-1}(x).$$

(v) Finally, similar reasoning as in the last part of Theorem 5.23 proves that condition (v) of Theorem 5.4 is fulfilled.

We conclude that the operators A, B and C satisfy all the requirements of Theorem 5.4. ■

Remark 5.4. Note that the operator C in Eq. (5.17) does not satisfy condition (iv) of Corollary 5.1. In fact, if we take $X = \mathbb{R}$ and $a \equiv \frac{1}{2}$, then $(Cx)(t) = \frac{1}{2}x(t)$. Thus

$$\begin{aligned} C(S) &= \{\frac{1}{2}x : \|x\|_\infty \leq r_0\}, \\ &= B_{\frac{r_0}{2}}. \end{aligned}$$

Because $\mathcal{C}(J, \mathbb{R})$ is infinite dimensional, $C(S)$ is not relatively compact. Furthermore, $\mathbb{R}$ is finite dimensional, so, $C(S)$ is not relatively weakly compact

Corollary 5.9. *Let $(X, \|\ \|)$ be a Banach algebra satisfying condition $(\mathcal{P})$, with positive closed cone X^+. Suppose that the assumption (H_1)–(H_6) hold. Also, assume that u belongs to X^+, $a(J) \subset X^+$, $q(J) \subset \mathbb{R}_+$, $p(J \times J \times X^+ \times X^+) \subset \mathbb{R}_+$ and $\left(\frac{I}{T_2}\right)^{-1}$ is a positive operator from the cone positive $\mathcal{C}(J, X^+)$ of $\mathcal{C}(J, X)$ into itself. Then Eq. (5.17) has at least one positive solution x in the cone $\mathcal{C}(J, X^+)$.*

Proof. Let

$$S^{+} := \{x \in S,\ x(t) \in X^{+} \text{ for all } t \in J\}.$$

Obviously S^{+} is nonempty, closed, and convex. Similarly reasoning as in the proof of Theorem 5.24 shows that

(i) A and C are Lipschitzians with a Lipschitzian constant α and $\|a\|_{\infty}$ respectively.
(ii) A is regular on $\mathcal{C}(J,X)$.
(iii) A, B and C are weakly sequentially continuous on S^{+}.
(iv) Because S^{+} is a subset of S, so, we have $A(S^{+}), B(S^{+})$ and $C(S^{+})$ are relatively weakly compact.
(v) Finally, we shall show that the hypothesis (v) of Theorem 5.7 is satisfied. In fact, fix an arbitrarily $x \in \mathcal{C}(J,X)$ and $y \in S^{+}$ such that

$$x = AxBy + Cx.$$

Arguing as in the proof of Theorem 5.24, we get $x \in S$. Moreover, the last equation leads to

$$\text{for all } t \in J,\ x(t) = a(t)x(t) + (T_2x)(t)(By)(t),$$

and thus,

$$\text{for all } t \in J,\ \frac{x(t)(1-a(t))}{(T_2x)(t)} = (By)(t).$$

Since for all $t \in J$, $\|a(t)\| < 1$, so, $(1-a(t))^{-1}$ exists in X, and

$$(1-a(t))^{-1} = \sum_{n=0}^{+\infty} a^{n}(t).$$

Since $a(t)$ belongs to the closed positive cone X^{+}, then $(1-a(t))^{-1}$ is positive. Also, we verify that for all $t \in J$, $(By)(t)$ is positive. Therefore, the map ψ defined on J by

$$\psi(t) = (1-a(t))^{-1}\left[\left(q(t) + \int_{0}^{\sigma(t)} p(t,s,x(s),x(\lambda s))ds\right).u\right]$$

belongs to the positive cone $\mathcal{C}(J,X^{+})$ of $\mathcal{C}(J,X)$. Then B maps $\mathcal{C}(J,X^{+})$ into itself. Seeing that

$$\left(\left(\frac{I}{T_2}\right)x\right)(t) = (1-a(t))^{-1}\left[\left(q(t) + \int_0^{\sigma(t)} p(t,s,x(s),x(\lambda s))ds\right).u\right] = \psi(t),$$

then

$$x = \left(\frac{I}{T_2}\right)^{-1}(\psi).$$

Thus, $x \in \mathcal{C}(J, X^+)$ and, consequently $x \in S^+$. ■

Next, we provide an example of the operator T_2 presented in Theorem 5.24.

Example of the Operator T_2 in $\mathcal{C}(J, \mathbb{R})$. Let $\mathcal{E} = \mathcal{C}(J, \mathbb{R}) = \mathcal{C}(J)$ denote the Banach algebra of all continuous real-valued functions on J with norm $\|x\|_\infty = \sup_{t\in J} |x(t)|$. Clearly $\mathcal{C}(J)$ satisfies condition $(\mathcal{P})$. Let $b : J \longrightarrow \mathbb{R}$ is continuous and nonnegative, and define

$$\begin{cases} T_2 : \mathcal{C}(J) \longrightarrow \mathcal{C}(J) \\ \qquad x \longrightarrow T_2x = \dfrac{1}{1+b|x|}. \end{cases}$$

We obtain the following functional integral equation:

$$x(t) = a(t)x(t) + \frac{1}{1+b(t)|x(t)|}\left[q(t) + \int_0^{\sigma(t)} p(t,s,x(s),x(\lambda s))ds\right],\ 0<\lambda<1. \tag{5.18}$$

We will prove all the conditions $(a)-(d)$ of (H_4) in Theorem 5.24 for Eq. (5.18):

(a) Fix $x, y \in \mathcal{C}(J)$. Then, for all $t \in J$, we have

$$\begin{aligned} |(T_2x)(t) - (T_2y)(t)| &= \left|\frac{1}{1+b(t)|x(t)|} - \frac{1}{1+b(t)|y(t)|}\right| \\ &= \frac{b(t)||y(t)| - |x(t)||}{(1+b(t)|x(t)|)\,(1+b(t)|y(t)|)} \\ &\le \|b\|_\infty\, |x(t) - y(t)|. \end{aligned}$$

Taking the supremum over t, we obtain

$$\|T_2x - T_2y\|_\infty \le \|b\|_\infty\, \|x-y\|_\infty,$$

which shows that T_2 is Lipschitzian with a Lipschitzian constant $\|b\|_\infty$.

(b) Clearly T_2 is regular on $\mathcal{C}(J)$.

(c) We show that $\left(\frac{I}{T_2}\right)^{-1}$ exists on $\mathcal{C}(J)$. To see this, let $x, y \in \mathcal{C}(J)$ such that

$$\left(\frac{I}{T_2}\right)x = y,$$

or, equivalently

$$x(1 + b|x|) = y,$$

which implies

$$|x|(1 + b|x|) = |y|,$$

hence

$$(\sqrt{b}|x|)^2 + |x| = |y|.$$

For each $t_0 \in J$ such that $b(t_0) = 0$, we have $x = y$. Then for each $t \in J$ such that $b(t) > 0$, we obtain

$$\left(\sqrt{b(t)}|x(t)| + \frac{1}{2\sqrt{b(t)}}\right)^2 = \frac{1}{4b(t)} + |y(t)|,$$

which further implies

$$\sqrt{b(t)}|x(t)| = -\frac{1}{2\sqrt{b(t)}} + \sqrt{\frac{1}{4b(t)} + |y(t)|},$$

hence

$$b(t)|x(t)| = -\frac{1}{2} + \sqrt{\frac{1}{4} + |y(t)|b(t)},$$

and, consequently

$$x(t) = \frac{y(t)}{1 + b(t)|x(t)|} = \frac{y(t)}{\frac{1}{2} + \sqrt{\frac{1}{4} + b(t)|y(t)|}}.$$

We remark that the equality is also verified for each t such that $b(t) = 0$.

Consider F the function defined by the expression

$$\begin{cases} F : \mathcal{C}(J) \longrightarrow \mathcal{C}(J) \\ \qquad x \longrightarrow F(x) = \frac{x}{\frac{1}{2}+\sqrt{\frac{1}{4}+b|x|}}. \end{cases}$$

It is easy to verify that for all $x \in \mathcal{C}(J)$

$$\left(\left(\frac{I}{T_2}\right) \circ F\right)(x) = \left(F \circ \left(\frac{I}{T_2}\right)\right)(x) = x.$$

We conclude that

$$\left(\frac{I}{T_2}\right)^{-1}(x) = \frac{x}{\frac{1}{2} + \sqrt{\frac{1}{4} + b|x|}}.$$

(d) It is easy to show that T_2 and $\left(\frac{I}{T_2}\right)^{-1}$ is weakly sequentially continuous on $B(S)$.

Remark 5.5. One can check easily that $\left(\frac{I}{T_2}\right)^{-1}$ is a positive operator from the positive cone $\mathcal{C}(J, \mathbb{R}_+)$ of $\mathcal{C}(J, \mathbb{R})$ into itself.

Next we consider the following nonlinear functional differential equation (in short, FDE) in $\mathcal{C}(J)$

$$\left(\left(\frac{x}{T_2 x}\right) - q_1\right)'(t) = \int_0^t \frac{\partial p}{\partial t}(t, s, x(s), x(\lambda s))ds + p(t, t, x(t), x(\lambda t)), \; t \in J, \; 0 < \lambda < 1 \tag{5.19}$$

satisfying the initial condition

$$x(0) = \zeta \in \mathbb{R} \tag{5.20}$$

where the functions q_1, p and the operator T_2 are given with $q_1(0) = 0$, while $x = x(t)$ is an unknown function. By a solution of the FDE (5.19)–(5.20), we mean an absolutely continuous function $x : J \longrightarrow \mathbb{R}$ that satisfies the two equations (5.19)–(5.20) on J.

Theorem 5.25. *We consider the following assumptions:*

(H_1) $q_1 : J \longrightarrow \mathbb{R}$ *is a continuous function.*
(H_2) *The operator* $T_2 : \mathcal{C}(J) \longrightarrow \mathcal{C}(J)$ *is such that*

- (a) T_2 *is Lipschitzian with a Lipschitzian constant* α*,*
- (b) T_2 *is regular on* $\mathcal{C}(J)$*,*

(c) $\left(\dfrac{I}{T_2}\right)^{-1}$ *is well defined on* $\mathcal{C}(J)$,

(d) $\left(\dfrac{I}{T_2}\right)^{-1}$ *is weakly sequentially continuous on* $\mathcal{C}(J)$,

(e) *For all* $x \in \mathcal{C}(J)$, *we have* $\|(T_2x)\|_\infty \leq 1$.

(H_3) *The function* $p : J \times J \times X \times X \longrightarrow \mathbb{R}$ *is continuous such that for arbitrary fixed* $s \in J$ *and* $x, y \in \mathbb{R}$ *the partial function* $t \longrightarrow p(t, s, x, y)$ *is* $\mathcal{C}^1$ *on* J.

(H_4) *there exists* $r_0 > 0$ *such that*

(a) *For all* $t, s \in J$; $y, z \in [-r_0, r_0]$ *and* $x \in \mathcal{C}(J)$, *we have*

$$|p(t, s, y, z)| \leq r_0 - \|q_1\|_\infty - \frac{|\zeta|}{|(T_2x)(0)|}.$$

(b) $\alpha r_0 < 1$.

Then the FDE (5.19)–(5.20) has at least one solution in $\mathcal{C}(J)$.

Proof. Note that the FDE (5.19)–(5.20) is equivalent to the integral functional equation:

$$x(t) = (T_2x)(t)\left[q_1(t) + \frac{\zeta}{(T_2x)(0)} + \int_0^t p(t, s, x(s), x(\lambda s))ds\right],\ t \in J,\ 0 < \lambda < 1. \tag{5.21}$$

Equation (5.21) represents a particular case of Eq. (5.17) with for all $t \in J$, $\sigma(t) = t, a(t) = 0, u = 1$ and $q(t) = q_1(t) + \frac{\zeta}{(T_2x)(0)}$. Therefore, we have for all $t \in J$, $(Ax)(t) = (T_2x)(t)$, $(Bx)(t) = q(t) + \int_0^t p(t, s, x(s), x(\lambda s))ds$ and $C(x)(t) = 0$.

Now, we prove that the operators A, B and C satisfy all the conditions of Theorem 5.4. Similar reasoning guarantees that

(i) A and C are Lipschitzians with a Lipschitzian constant α and 0 respectively.
(ii) B is weakly sequentially continuous on S and $B(S)$ is relatively weakly compact where $S = B_{r_0} := \{x \in \mathcal{C}(J),\ \|x\|_\infty \leq r_0\}$.
(iii) A is regular on $\mathcal{C}(J)$.
(iv) $\left(\dfrac{I - C}{A}\right)^{-1} = \left(\dfrac{I}{T_2}\right)^{-1}$ is weakly sequentially continuous on $B(S)$.

It, thus, remains to prove (v) of Theorem 5.4. First, we show that $M = \|B(S)\| \leq r_0$. To see this, fix an arbitrarily $x \in S$. Then, for $t \in J$, we get

$$\begin{aligned}
|(Bx)(t)| &\leq |q_1(t)| + \frac{|\zeta|}{|(T_2x)(0)|} + \int_0^t |p(t, s, x(s), x(\lambda s))|ds \\
&\leq |q_1(t)| + \frac{|\zeta|}{|(T_2x)(0)|} + \int_0^1 |p(t, s, x(s), x(\lambda s))|ds \\
&\leq \|q_1\|_\infty + \frac{|\zeta|}{|(T_2x)(0)|} + r_0 - \|q_1\|_\infty - \frac{|\zeta|}{|(T_2x)(0)|} \\
&= r_0.
\end{aligned}$$

Taking the supremum over t, we obtain

$$\|Bx\|_\infty \leq r_0.$$

Thus

$$M \leq r_0,$$

and, consequently

$$\alpha M + \beta = \alpha M \leq \alpha r_0 < 1.$$

Next, fix an arbitrarily $x \in \mathcal{C}(J)$ and $y \in S$ such that

$$x = AxBy + Cx,$$

or, equivalently

$$\text{for all } t \in J,\ x(t) = (T_2 x)(t)(By)(t),$$

then

$$|x(t)| \leq \|T_2 x\|_\infty \|By\|_\infty,$$

and thus, in view of assumption $(H_2)(e)$, we have that

$$|x(t)| \leq \|By\|_\infty.$$

Since $y \in S$, this further implies

$$|x(t)| \leq r_0,$$

and taking the supremum over t, we obtain

$$\|x\|_\infty \leq r_0.$$

As a result, x is in S. This proves (v). Now, applying Theorem 5.4, we see that Eq. (5.21) has at least one solution in $\mathcal{C}(J)$. ■

Next, we will illustrate the applicability of our Theorem 5.10 to establish the existence of solutions of FIE

$$x(t) = a(t) + (Tx)(t)\left[\left(q(t) + \int_0^{\sigma(t)} p(t, s, x(\zeta(s)), x(\eta(s)))ds\right) \cdot u\right] \tag{5.22}$$

Let $(X, \|.\|)$ be a Banach algebra satisfying the condition $(\mathcal{P})$. Let $J = [0, 1]$ be the closed and bounded interval in $\mathbb{R}$. Let $E = \mathcal{C}(J, X)$ be the Banach algebra of all continuous functions from $[0, 1]$ to X. Now, we shall obtain the solution of FIE (5.22) under some suitable conditions. Assume that the functions involved in Eq. (5.22) satisfy the following conditions:

(A_1) $a : J \longrightarrow X$ is a continuous function.
(A_2) $\sigma, \zeta, \eta : J \longrightarrow J$ are continuous.
(A_3) $q : J \longrightarrow \mathbb{R}$ is a continuous function.
(A_4) The operator $T : \mathcal{C}(J, X) \longrightarrow \mathcal{C}(J, X)$ is weakly sequentially continuous and weakly compact.
(A_5) The function $p : J^2 \times X^2 \longrightarrow \mathbb{R}$ is weakly sequentially continuous and the partial function $t \longrightarrow p(t, s, x, y)$ is uniformly continuous for $(s, x, y) \in J \times X^2$.
(A_6) There exists $r_0 > 0$ such that:

 (a) $|p(t, s, x, y)| \leq M$ for each $t, s \in J$; $x, y \in X$ such that $\|x\| \leq r_0$ and $\|y\| \leq r_0$,
 (b) $\|u\| \|Tx\|_\infty \leq 1$ for each $x \in \mathcal{C}(J, X)$ such that $\|x\|_\infty \leq r_0$,
 (c) $\|a\|_\infty + \|q\|_\infty + M \leq r_0$.

Theorem 5.26. *Under assumptions* $(A_1) - (A_6)$*, Eq. (5.22) has at least one solution* $x = x(t)$ *which belongs to the space* $\mathcal{C}(J, X)$.

Proof. Note that $\mathcal{C}(J, X)$ verifies condition $(\mathcal{P})$. Let us define the subset S of $\mathcal{C}(J, X)$ by:

$$S := \{y \in \mathcal{C}(J, X) : \|y\|_\infty \leq r_0\} = B_{r_0}.$$

Obviously S is nonempty closed convex bounded subset of E. Let us consider three operators A, L and U defined on S by:

$$\begin{aligned}
(Ax)(t) &= a(t)\\
(Lx)(t) &= (Tx)(t)\\
(Ux)(t) &= \left[q(t) + \int_0^{\sigma(t)} p(t, s, x(\zeta(s)), x(\eta(s)))ds \right] \cdot u.
\end{aligned}$$

We prove that operators A, L and U satisfy all the conditions of Theorem 5.10.

(i) Since A is constant, it is weakly sequentially continuous on S and weakly compact.
(ii) In view of hypothesis (A_4), L is weakly sequentially continuous on S and $L(S)$ is relatively weakly compact.
(iii) To prove that U satisfies all the conditions of Theorem 5.10, we will first prove that U maps S into $\mathcal{C}(J, X)$. For this purpose, let $\{t_n\}_{n\geq 0}$ be any sequence in J converging to a point t in J. Then

$$\|(Ux)(t_n)-(Ux)(t)\| \leq |q(t_n)-q(t)|\,\|u\|+$$
$$\left\|\left[\int_0^{\sigma(t_n)} p(t_n,s,x(\zeta(s)),x(\eta(s)))ds-\int_0^{\sigma(t)} p(t,s,x(\zeta(s)),x(\eta(s)))ds\right].u\right\| \leq$$
$$|q(t_n)-q(t)|\,\|u\|+\left[\int_0^{\sigma(t_n)} |p(t_n,s,x(\zeta(s)),x(\eta(s)))-p(t,s,x(\zeta(s)),x(\eta(s)))|\,ds\right]\|u\|+$$
$$\left|\int_{\sigma(t)}^{\sigma(t_n)} |p(t,s,x(\zeta(s)),x(\eta(s)))|\,ds\right|\|u\| \leq$$
$$|q(t_n)-q(t)|\,\|u\|+\left[\int_0^{1} |p(t_n,s,x(\zeta(s)),x(\eta(s)))-p(t,s,x(\zeta(s)),x(\eta(s)))|\,ds\right]\|u\|+$$
$$M|\sigma(t_n)-\sigma(t)|\,\|u\|.$$

Since $t_n \to t$, then $(t_n,s,x(\zeta(s)),x(\eta(s))) \to (t,s,x(\zeta(s)),x(\eta(s)))$, for all $s \in J$. Taking into account hypothesis (A_5), we obtain

$$p(t_n,s,x(\zeta(s)),x(\eta(s))) \to p(t,s,x(\zeta(s)),x(\eta(s))) \text{ in } \mathbb{R}.$$

Moreover, the use of assumption (A_6) leads to

$$|p(t_n,s,x(\zeta(s)),x(\eta(s)))-p(t,s,x(\zeta(s)),x(\eta(s)))| \leq 2M$$

for all $t,s \in J$. Now, we can apply the Dominated Convergence Theorem and since assumption (A_3) holds to get

$$(Ux)(t_n) \to (Ux)(t) \text{ in } X.$$

It follows that

$$Ux \in \mathcal{C}(J,X).$$

Next, we will prove U is weakly sequentially continuous on S. To do so, let $\{x_n\}_n$ be any sequence in S weakly converging to a point x in S. Then, $\{x_n\}_n$ is bounded. We can apply the Dobrakov's theorem to get

$$\forall t \in J,\ x_n(t) \rightharpoonup x(t).$$

Hence, by assumptions (A_5)–(A_6) and the Dominated Convergence Theorem, we obtain

$$\lim_{n\to\infty}\int_0^{\sigma(t)} p(t,s,x_n(\zeta(s)),x_n(\eta(s)))ds=\int_0^{\sigma(t)} p(t,s,x(\zeta(s)),x(\eta(s)))ds,$$

which implies

$$\lim_{n\to\infty}\left(q(t)+\int_0^{\sigma(t)} p(t,s,x_n(\zeta(s)),x_n(\eta(s)))ds\right)\cdot u =$$
$$\left(q(t)+\int_0^{\sigma(t)} p(t,s,x(\zeta(s)),x(\eta(s)))ds\right)\cdot u.$$

Hence

$$(Ux_n)(t) \to (Ux)(t) \text{ in X.}$$

Thus

$$(Ux_n)(t) \rightharpoonup (Ux)(t) \text{ in X.}$$

Since $\{Ux_n\}_n$ is bounded by $\|u\|(\|q\|_\infty + M)$, then we can again apply the Dobrakov's theorem to obtain

$$Ux_n \rightharpoonup Ux.$$

We conclude that U is weakly sequentially continuous on S. It remains to prove that U is weakly compact. Since S is bounded by r_0, it is enough to prove $U(S)$ is relatively weakly compact.

(STEP I) By definition,

$$U(S) := \{Ux : \ \|x\|_\infty \leq r_0\}.$$

For all $t \in J$, we have

$$U(S)(t) := \{(Ux)(t) : \ \|x\|_\infty \leq r_0\}.$$

We claim that $U(S)(t)$ is sequentially relatively weakly compact in X. To see this, let $\{x_n\}_{n\geq 0}$ be any sequence in S, and we have $(Ux_n)(t) = r_n(t)\cdot u$, where

$$r_n(t) = q(t) + \int_0^{\sigma(t)} p(t,s,x_n(\zeta(s)),x_n(\eta(s)))ds.$$

Since $|r_n(t)| \leq (\|q\|_\infty + M)$ and $(r_n(t))_{n\geq 0}$ is a equibounded real sequence, so, there is a renamed subsequence such that

$$r_n(t) \to r(t) \text{ in } \mathbb{R},$$

which implies

$$r_n(t) \cdot u \to r(t) \cdot u \text{ in } X,$$

and consequently

$$(Ux_n)(t) \to (q(t) + r(t)) \cdot u \text{ in } X.$$

We conclude that $U(S)(t)$ is sequentially relatively compact in X, and then $U(S)(t)$ is relatively compact in X.
(STEP II) We prove that $U(S)$ is weakly equicontinuous on J. If we take $\epsilon > 0$, $x \in S$, $x^* \in X^*$, $t, t' \in J$ such that $t \leq t'$ and $t' - t \leq \epsilon$, then

$$|x^*((Ux)(t) - (Ux)(t'))| \leq |q(t) - q(t')||x^*(u)| +$$

$$\left| \int_0^{\sigma(t)} p(t, s, x_(\zeta(s)), x(\eta(s)))ds - \int_0^{\sigma(t')} p(t', s, x(\zeta(s)), x(\eta(s)))ds \right| |x^*(u)| \leq$$

$$|q(t) - q(t')||x^*(u)| +$$

$$\left[\int_0^{\sigma(t)} |p(t, s, x(\zeta(s)), x(\eta(s))) - p(t', s, x(\zeta(s)), x(\eta(s)))| ds \right] \left| x^*(u) \right| +$$

$$\left| \left[\int_{\sigma(t)}^{\sigma(t')} |p(t', s, x(\zeta(s)), x(\eta(s)))| ds \right] \right| |x^*(u)| \leq$$

$$[w(q, \epsilon) + w(p, \epsilon) + M w(\sigma, \epsilon)]|x^*(u)|,$$

where:

$$w(q, \epsilon) := \sup\{|q(t) - q(t')| : t, t' \in J; |t - t'| \leq \epsilon\},$$
$$w(p, \epsilon) := \sup\{|p(t, s, x, y) - p(t', s, x, y)| : t, t', s \in J; |t - t'| \leq \epsilon;\ x, y \in B_{r_0}\},$$
$$w(\sigma, \epsilon) := \sup\{|\sigma(t) - \sigma(t')| : t, t' \in J; |t - t'| \leq \epsilon\}.$$

Now, observe that from the above obtained estimate, taking into account hypothesis (A_5) and in view of the uniform continuity of the functions q, σ on the set J, it follows that $w(q, \epsilon) \to 0$, $w(p, \epsilon) \to 0$ and $w(\sigma, \epsilon) \to 0$ as $\epsilon \to 0$. Now, apply the Arzelà–Ascoli Theorem to obtain that $U(S)$ is sequentially relatively weakly compact in E. Now the Eberlein–Šmulian theorem yields that $U(S)$ is relatively weakly compact.

(iv) Finally, it thus remains to prove that $Ax + LxUx \in S$ for all $x \in S$. To see this, let $x \in S$. Then, by (A_6), for all $t \in J$, one has

$$\begin{aligned}\|(Ax)(t) + (Lx)(t)(Ux)(t)\| &= \|a(t) + (Tx)(t)(Ux)(t)\| \\ &\leq \|a\|_\infty + \|(Ux)(t)\|\|Tx\|_\infty \\ &\leq \|a\|_\infty + (M + \|q\|_\infty)\|u\|\|Tx\|_\infty \\ &\leq r_0.\end{aligned}$$

From the last inequality and taking the supremum over t, we obtain

$$\|Ax + LxUx\|_\infty \leq r_0,$$

and consequently $Ax + LxUx \in S$.

Thus, all operators A, L and U fulfill the requirements of Theorem 5.10. Hence, an application of it yields that the FIE (5.22) has a solution in the space $\mathcal{C}(J, X)$. ■

Remark 5.6. When X is finite dimensional, the subset $U(S) \subset \mathcal{C}(J, X)$ is relatively compact if and only if it is weakly equicontinuous on J and $U(S)(J)$ is relatively compact in X, if and only if $U(S)$ is relatively weakly compact.

Remark 5.7. Observe that Eq. (5.22) contains many special types of functional integral equations in $\mathcal{C}(J, \mathbb{R})$:

(1) If we take:

$$\zeta(s) = s;\ \eta(s) = \lambda s,\ 0 < \lambda < 1 \text{ and } u = 1,$$

then Theorem 5.26 reduces to existence results proved in [54] for the nonlinear integral equation:

$$x(t) = a(t) + (Tx)(t)\left(q(t) + \int_0^{\sigma(t)} p(t, s, x(s), x(\lambda s))ds\right),\ 0 < \lambda < 1.$$

(2) If we take:

$$q(t) = 0;\ \sigma(t) = 1;\ \zeta(s) = s;\ \eta(s) = \lambda s, 0 < \lambda < 1 \text{ and } u = 1,$$

then Theorem 5.26 reduces to existence results of functional integral equation of Urysohn type:

$$x(t) = a(t) + (Tx)(t)\int_0^1 p(t, s, x(s), x(\lambda s))ds,\ 0 < \lambda < 1.$$

(3) If we take:

$$\sigma(t) = \zeta(t) = t;\ q(t) = 0;\ T = 1;\ p(t,s,x,y) = k(t,s)x \text{ and } u = 1,$$

then Theorem 5.26 reduces to existence results for the classical linear Volterra integral equation on bounded interval in [50]:

$$x(t) = a(t) + \int_0^t k(t,s)x(s)ds.$$

(4) If we take:

$$\sigma(t) = \zeta(t) = t;\ q(t) = 0;\ T = 1;\ p(t,s,x,y) = k(t,s)f(s,x) \text{ and } u = 1,$$

then Theorem 5.26 reduces to existence results for the nonlinear integral equation of Volterra–Hammerstein type:

$$x(t) = a(t) + \int_0^t k(t,s)f(s,x(s))ds.$$

(5) If we take:

$$a(t) = 0;\ (Tx)(t) = f(t,x(\nu(t)));\ p(t,s,x,y)) = g(s,y) \text{ and } u = 1,$$

then Theorem 5.26 reduces to existence results proved in [77] for the nonlinear integral equation:

$$x(t) = f(t,x(\nu(t)))\left(q(t) + \int_0^{\sigma(t)} g(s,x(\eta(s)))ds\right).$$

(6) If we take:

$$\eta(s) = \lambda s,\ 0 < \lambda < 1,$$

then the Theorem 5.26 reduces to existence results proved in [39] for the nonlinear integral equation:

$$x(t) = a(t) + (Tx)(t)\left(q(t) + \int_0^{\sigma(t)} p(t,s,x(s),x(\lambda s))ds\right) \cdot u,\ 0 < \lambda < 1.$$

Chapter 6
Fixed Point Theory for (*ws*)-Compact Operators

In this chapter we present fixed point theory and study eigenvalues and eigenvectors of nonlinear (*ws*)-compact operators.

6.1 (*ws*)-Compact Operators

In [100] Gowda and Isac introduced the class of (*ws*)-compact operators.

Definition 6.1. Let D be a subset of a Banach space $(E, \| \cdot \|)$. An operator (not necessarily linear) $F : D \longrightarrow E$ is said to be (*ws*)-compact if F is $\| \cdot \|$-continuous and, for every weakly convergent sequence $\{x_n\}_n$ elements of D, the sequence $\{F(x_n)\}_n$ admits a strongly convergent subsequence.

This class of operators generalizes the well-known class of strongly continuous operators extensively investigated in the literature.

Definition 6.2. Let E be a Banach space. An operator (not necessary linear) $F : E \longrightarrow E$ is said to be strongly continuous on E if for every sequence $\{x_n\}_n$ with $x_n \rightharpoonup x$, we have $F(x_n) \longrightarrow F(x)$.

Remark 6.1. Clearly, a strongly continuous operator is (*ws*)-compact. The converse of the preceding assertion is not in general true (even if E is reflexive) as the following example illustrates. Let $E = L^2(0,1)$ and let $F : E \to E$ be defined by $F(x)(t) = \int_0^1 x^2(s)\, ds = \|x\|_2^2$. Clearly F is $\|.\|$-continuous and in fact compact since the range of F is $\mathbb{R}$ and hence F is (*ws*)-compact. On the other hand, if $x_n(s) = \sin(n\pi s)$, then $x_n \rightharpoonup \theta$ in $L^2(0,1)$ but $F(x_n) \not\longrightarrow \theta$ in $L^2(0,1)$ since $\|F(x_n)\|_2 = \frac{1}{2}$ for all $n \geq 1$.

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Remark 6.2. (*i*) As examples of (*ws*)-compact operators we have compact operators and strongly continuous operators.

(*ii*) A map F is (*ws*)-compact if and only if it maps relatively weakly compact sets into relatively compact ones.

(*iii*) F is (*ws*)-compact does not imply that F is weakly sequentially continuous. In the following example, we give a broad class of (*ws*)-compact mappings which are not weakly sequentially continuous.

Let $f : (0,1) \times \mathbb{R} \longrightarrow \mathbb{R}$ be a function satisfying the Carathéodory conditions, that is, f is Lebesgue measurable in x for each $y \in \mathbb{R}$ and continuous in y for each $x \in (0,1)$. Additionally, we assume that

$$|f(x,y)| \leq a(x) + b\,|y|\,, \tag{6.1}$$

for all $(x,y) \in (0,1) \times \mathbb{R}$, where $a(x)$ is a nonnegative function Lebesgue integrable on the interval $(0,1)$ and $b \geq 0$. Consider the so-called superposition operator $\mathcal{N}_f$, generated by the function f, which to every function u defined on the interval $(0,1)$ assigns the function $\mathcal{N}_f u$ defined by the formula

$$(\mathcal{N}_f u)(x) = f(x, u(x)), \quad x \in (0,1).$$

Let $L^1 = L^1(0,1)$ denote the space of functions $u : (0,1) \longrightarrow \mathbb{R}$ which are Lebesgue integrable, equipped with the standard norm. Under the above-quoted assumptions the superposition operator $\mathcal{N}_f$ maps continuously the space L^1 into itself. Define the functional

$$L(u) = \int_0^1 \mathcal{N}_f u(x)dx = \int_0^1 f(x,u(x))dx$$

for $u \in L^1$. Notice that $L = K\mathcal{N}_f$, where K is the linear functional defined on L^1 by

$$K(u) = \int_0^1 u(x)dx, \quad u \in L^1.$$

Clearly, K is continuous with norm $\|K\| \leq 1$. Thus, L is continuous. Now, we show that L is (*ws*)-compact. To see this, let $\{y_n\}_n$ be a weakly convergent sequence of L^1, and then $\{y_n\}_n$ is uniformly bounded and by (6.1) we obtain

$$|f(x, y_n(x))| \leq a(x) + b\,|y_n(x)|\,. \tag{6.2}$$

Since $\{y_n\}_n$ is weakly compact in L^1, by the Dunford–Pettis criterion it turns out to be uniformly integrable on $(0,1)$, that is

$$\forall \varepsilon > 0, \exists\, \delta > 0, |D_0| < \delta \Longrightarrow \int_{D_0} |y_n(x)|\; dx < \varepsilon \;\; \forall n \in \mathbb{N}.$$

Therefore by (6.2), also $\{\mathcal{N}_f(y_n)\}_n$ is uniformly integrable on D, which implies the weak compactness in L^1 of $\{\mathcal{N}_f(y_n)\}_n$, and hence by the Eberlein–Šmulian theorem, $\mathcal{N}_f(y_n)_n$ has a weakly convergent subsequence, say $\{\mathcal{N}_f(y_{n_j})\}_j$.

On the other hand, the continuity of the linear operator K, implies its weak continuity on L^1. Consequently, we obtain that $\{K\mathcal{N}_f(y_{n_j})\}_j$ and so $\{L(y_{n_j})\}_j$ is pointwise converging, for almost all $x \in (0, 1)$. Using again the weak continuity of the linear operator K, we infer that $\{L(y_{n_j})\}_j$ is uniformly integrable on D. Hence, by Vitali's convergence theorem, $\{L(y_{n_j})\}_j$ is strongly convergent in L^1. Accordingly, the operator L is (ws)-compact. However, L is not weakly sequentially continuous unless L is linear with respect to the second variable.

Remark 6.3. In reflexive Banach spaces, a (ws)-compact mapping T is compact. This follows from the fact that bounded sets in reflexive Banach spaces are relatively weakly compact. However, in reflexive Banach spaces, the (ws)-compactness of an operator T does not imply its compactness even if T is a linear operator. For example, let T be the identity map injecting l_1 into l_2. T is clearly not compact. However, if $\{x_n\}_n$ is a sequence in l_1 which converges weakly to x, then $\{x_n\}_n$ converges to x in norm in l_1. It follows from the continuity of T that $\{Tx_n\}_n$ converges to Tx in l_2. Thus T is strongly continuous and so (ws)-compact.

6.2 Asymptotic Derivatives

Definition 6.3. Let $(E, \|.\|)$ be a Banach space and $L(E, E)$ the Banach space of linear continuous mappings from E into E. If F is a mapping from E to E, the asymptotic derivative of F (if it exists) is an element $L \in L(E, E)$ such that

$$\lim_{\|x\| \longrightarrow \infty} \frac{\|F(x) - L(x)\|}{\|x\|} = 0.$$

Remark 6.4. 1. It is known that if F has an asymptotic derivative, then it is unique and we denote it by $F^\infty = L$

2. If F is completely continuous then F^∞ is completely continuous too.

Example 6.1. Let $\Omega \subset \mathbb{R}^n$ be the closure of a bounded set, which has a piecewise smooth boundary, $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ and $K : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ two mappings. The Hammerstein operator defined by K, f and the Lebesgue measure is

$$A(\varphi)(t) = \int_\Omega K(t, s)f[s, \varphi(s)]ds,$$

where φ is for example in $L^2(\Omega, \mu)$, or in $L^p(\Omega, \mu)$, $(1 < p < \infty)$. If the following conditions are satisfied:

1. $\int_\Omega \int_\Omega K^2(t,s)dtds < +\infty$,
2. The substitution operator $f_0(\varphi)(s) = f[s,\varphi(s)]$, where $\varphi \in L^2$ is such that $f_0 : L^2 \longrightarrow L^2$,
3. $|f(t,u) - u| \leq \sum S_j(t)\,|u|^{1-p_j} + D(t)$, where $t \in \Omega, -\infty < u < +\infty, S_j(t) \in L^{\frac{2}{p_j}}, 0 < p_j < 1, j = 1,2,\ldots,m$ and $D \in L^2$,

then $A : L^2 \longrightarrow L^2$ and its asymptotic derivative is

$$B(\varphi)(t) = \int_\Omega K(t,s)\varphi(s)ds.$$

Proposition 6.1. *Consider the space $L^2 = L^2(\Omega,\mu)$ and suppose that the operators:*

$$A(u)(x) = \int_\Omega \mathcal{K}(x,y)u(y)dy$$

and

$$f_*(u)(x) = f[x,u(x)]$$

are well defined and are continuous from L^2 into L^2. The abstract form of the Hammerstein operator defined by A and f_* is $F(u) = (A \circ f_*)(u)$. If the operator f_* has an asymptotic derivative, denoted by f_*^∞, then the operator F is asymptotically derivable and its asymptotic derivative F^∞ is the linear operator $A \circ f_*^\infty$.

Proof. We have

$$\lim_{\|x\|\longrightarrow\infty} \frac{\|F(u) - (A \circ f_*^\infty)(u)\|}{\|u\|} = \lim_{\|x\|\longrightarrow\infty} \frac{\|(A \circ f_*)(u) - (A \circ f_*^\infty)(u)\|}{\|u\|} = 0.$$

■

6.3 Quasi-Bounded Operators

Granas introduced the notion of quasi-bounded operators [96].

Definition 6.4. Let $(E, \|.\|)$ be a Banach space or a Hilbert space and $F : E \longrightarrow E$ a mapping. We say that F is quasi-bounded if and only if

$$\limsup_{\|x\|\longrightarrow\infty} \frac{\|F(x)\|}{\|x\|} < +\infty.$$

If F is quasi-bounded we let:

$$[F]_{qb} = \limsup_{\|x\| \longrightarrow \infty} \frac{\|F(x)\|}{\|x\|} = \inf_{\rho > 0} \sup_{\|x\| \geq \rho} \frac{\|F(x)\|}{\|x\|},$$

and we say that $[F]_{qb}$ is the quasi-norm of F.

We let $[F]_b = \sup\limits_{x \neq 0} \frac{\|F(x)\|}{\|x\|}$ and if $[F]_b < \infty$ we say that F is linearly bounded.

Definition 6.5. We say that a mapping $F : E \longrightarrow E$ satisfies condition (BN) (Brezis–Nirenberg) if

$$\lim_{\|x\| \longrightarrow \infty} \frac{\|F(x)\|}{\|x\|} = 0.$$

In 1984, Weber studied the spectrum of a class of nonlinear operators which are called $\varphi-$asymptotically bounded operators.

Definition 6.6. Let φ be a function from $\mathbb{R}_+$ into $\mathbb{R}_+$ with the property that for a particular $\rho > 0, \varphi(t) > 0$ for any $t \geq \rho$. We say that a mapping $F : E \longrightarrow E$ is $\varphi-$asymptotically bounded if there exist $b, c \in \mathbb{R}_+$ such that for any x with $b \leq \|x\|$ we have $\|f(x)\| \leq c\varphi(\|x\|)$.

Proposition 6.2. *A mapping $F : E \longrightarrow E$ is quasi-bounded if and only if there exists $\rho > 0$ and two positive constants α and β such that $\|F(x)\| \leq \alpha\|x\| + \beta$ for any $x \in E$, with $\|x\| > \rho$.*

Examples 6.1. 1. If F is linearly bounded it is known that $[F]_{qb} \leq [F]_b$ and hence, any linearly bounded mapping is quasi-bounded. In particular, any linear continuous mapping from E into E is quasi-bounded and the converse is not true. Indeed, if $F : E \longrightarrow E$ is a linear continuous mapping then in this case we have

$$\limsup_{\|x\| \longrightarrow \infty} \frac{\|F(x)\|}{\|x\|} = \inf_{\rho > 0} \sup_{\|x\| \geq \rho} \frac{\|F(x)\|}{\|x\|} = \|F\| < +\infty,$$

that is, in this case we have $[F]_{qb} = \|f\|$.

2. Any operator, which satisfies condition (BN), is a quasi-bounded operator.
3. If $F : E \longrightarrow E$ has a decomposition of the form $F = G + H$ with H quasi-bounded and G linearly bounded (or satisfying condition (BN)), then F is quasi-bounded.
4. If $F : E \longrightarrow E$ is bounded, in the sense that there exists $\alpha > 0$ such that $\|F(x)\| \leq \alpha\|x\|$ for any $x \in E$ then F is quasi-bounded.
5. If there exists a mapping $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}$ such that $\lim\limits_{t \longrightarrow \infty} \frac{\varphi(t)}{t} = 0$, then any mapping $F : E \longrightarrow E$ such that for any $x \in E$, $\|F(x)\| \leq \alpha\|x\| + \varphi(\|x\|)$, with $\alpha > 0$ is a quasi-bounded mapping.
6. If F is $\varphi-$asymptotically bounded and φ satisfies condition (BN) then F is quasi-bounded.

Proposition 6.3. *If $F : E \longrightarrow E$ has an asymptotic derivative $T \in L(E, E)$ then F is quasi-bounded and $[F]_{qb} = \|T\|$.*

Proof. We have

$$[F]_{qb} = \limsup_{\|x\| \longrightarrow \infty} \frac{\|F(x)\|}{\|x\|} \leq \limsup_{\|x\| \longrightarrow \infty} \frac{\|F(x) - T(x)\|}{\|x\|} + \limsup_{\|x\| \longrightarrow \infty} \frac{\|T(x)\|}{\|x\|} = \limsup_{\|x\| \longrightarrow \infty} \frac{\|T(x)\|}{\|x\|} = \|T\|$$

and

$$\|T\| = \limsup_{\|x\| \longrightarrow \infty} \frac{\|T(x)\|}{\|x\|} \leq \limsup_{\|x\| \longrightarrow \infty} \frac{\|T(x) - F(x)\|}{\|x\|} + \limsup_{\|x\| \longrightarrow \infty} \frac{\|T(x)\|}{\|x\|} = [F]_{qb}.$$

∎

Theorem 6.1. *The quasi-norm of a quasi-bounded mapping has the following properties.*

1. *If $F : E \longrightarrow E$ is quasi-bounded and $\lambda \in P$ then $[\lambda F]_{qb} = |\lambda| \, [F]_{qb}$.*
2. *If $F_1, F_2 : E \longrightarrow E$ are quasi-bounded, then $F_1 + F_2$ is quasi-bounded and we have $[F_1 + F_2]_{qb} \leq [F_1]_{qb} + [F_2]_{qb}$.*

Proof. 1. We have $[\lambda F]_{qb} = \limsup_{\|x\| \longrightarrow \infty} \frac{\|\lambda F(x)\|}{\|x\|} = |\lambda| \limsup_{\|x\| \longrightarrow \infty} \frac{\|F(x)\|}{\|x\|} = |\lambda| \, [F]_{qb}$.
2. It suffices to use the properties of lim sup.

∎

6.4 Fixed Point Results

The following fixed point result will be used in this section. The proof follows from Schauder's fixed point theorem.

Theorem 6.2. *Let Ω be a nonempty closed convex subset of a Banach space E. Assume that $F : \Omega \longrightarrow \Omega$ is (ws)-compact. If $F(\Omega)$ is relatively weakly compact, then there exists $x \in \Omega$ such that $F(x) = x$*

First, we state and prove an analogue of Sadovskii's fixed point theorem for (ws)-compact, weakly condensing mapping defined on unbounded closed convex set.

Theorem 6.3. *Let Ω be a nonempty unbounded closed convex set in a Banach space E. Assume Φ is a MWNC on E and $F : \Omega \longrightarrow \Omega$ is a Φ-condensing (ws)-compact mapping. In addition, suppose that $F(\Omega)$ is bounded. Then the set of fixed points of F in Ω is nonempty and compact.*

Proof. Let $x_0 \in \Omega$ and $D = \{F^n(x_0), n \in \mathbb{N}\}$ where $F^0(x_0) = x_0$. Then $D = F(D) \cup \{x_0\}$ and so $\Phi(F(D)) = \Phi(D)$ which means that $\Phi(D) = 0$ and D is relatively weakly compact. By Remark 6.2 (*ii*) $F(D)$ is relatively compact. Also,

$F(F(D)) \subseteq F(D)$ so by [109, Lemma 1] one may choose a compact set $D_0 \subseteq F(D)$ with $D_0 \subseteq \overline{\text{conv}}(F(D_0))$. Let $\mathcal{T} = \{Q : D_0 \subseteq Q, Q = \overline{\text{co}}Q, F(Q) \subseteq Q\}$. It is obvious that $\mathcal{T} \neq \emptyset$, since $\Omega \in \mathcal{T}$. If ξ is a chain in the ordered set $(E, \subseteq)$ then $\bigcap_{Q \in \xi} Q$ is a lower bound of ξ, which can be easily verified. Hence by Zorn's lemma $\mathcal{T}$ has a minimal element K. From $F(K) \subseteq K$, since K is closed and convex, it follows that the set $\overline{\text{co}}(F(K))$ is a subset of K. So we have

$$F(\overline{\text{co}}(F(K))) \subseteq F(K) \subseteq \overline{\text{co}}(F(K)).$$

From $D_0 \subseteq \overline{\text{co}}(F(K))$, it follows that the set $\overline{\text{co}}(F(K))$ is in $\mathcal{T}$. Since K is a minimal element of $\mathcal{T}$ it follows that $K = \overline{\text{co}}(F(K))$. Hence, $\Phi(K) = \Phi(\overline{\text{co}}(F(K))) = \Phi(F(K))$. Since F is Φ-condensing, we obtain $\Phi(K) = \Phi(F(K)) = 0$, and $F(K)$ is relatively weakly compact. Now, F is a (ws)-compact map from the closed convex set K into itself. From Theorem 6.2, F has a fixed point in $K \subseteq \Omega$. Let $\mathcal{S} = \Bigg\{x \in \Omega\ :\ F(x) = x\Bigg\}$, be the fixed point set of F. Since F is continuous, $\mathcal{S}$ is obviously a closed subset of Ω such that $F(\mathcal{S}) = \mathcal{S}$. Since $\mathcal{S} \subseteq F(\Omega)$, $F(\mathcal{S}) = \mathcal{S}$ and F is Φ-condensing, we have $\Phi(\mathcal{S}) = 0$ and so $\mathcal{S}$ is a relatively weakly compact subset of Ω. Now, F is (ws)-compact so $F(\mathcal{S}) = \mathcal{S}$ is relatively compact. Since $\mathcal{S}$ is closed, we obtain that $\mathcal{S}$ is compact. This proof is complete. ■

Corollary 6.1. *Let Ω be a nonempty unbounded closed convex set in a Banach space E. Assume that $F : \Omega \longrightarrow \Omega$ is a (ws)-compact mapping which satisfies that $F(\Omega)$ is bounded and $F(D)$ is relatively weakly compact whenever D is a bounded set of Ω. Then the set of fixed points of F in Ω is nonempty and compact.*

Definition 6.7. A mapping $F : \Omega \to E$ is said to be demi-weakly compact at θ ((dwc) for short) if for every bounded a.f.p. sequence $\{x_n\}_n$ in Ω (i.e., $x_n - F(x_n) \to \theta$) then $\{x_n\}_n$ has a weakly convergent subsequence.

Now, let us recall the following well-known concept due to Petryshyn [169]:

Definition 6.8. A mapping $F : \Omega \to E$ is said to be *demicompact* at $\theta \in E$((dc) for short) if, for every bounded a.f.p. sequence $\{x_n\}_n$ in Ω, there exists a strongly convergent subsequence of $\{x_n\}_n$.

Remark 6.5. If Ω is a closed subset of E and $F : \Omega \longrightarrow E$ is a continuous mapping, demicompact at θ and it admits a bounded a.f.p. sequence $\{x_n\}_n$, then it has a fixed point. Indeed, suppose that $\{x_n\}_n$ is a bounded sequence in Ω such that $x_n - F(x_n) \to \theta$. It follows from the demicompactness of F that there exists a subsequence of $\{x_n\}_n$ which converges strongly to some $x \in \Omega$. Without loss of generality, we may assume that $\{x_n\}_n$ converges strongly to $x \in \Omega$. Hence, taking into account that $x_n - F(x_n) \to \theta$ and the continuity of F, we derive, $F(x) = x$.

Clearly if $F : \Omega \to E$ is demicompact at θ then it is demi-weakly compact at θ.

Lemma 6.1. *Let E be a Banach space and let Ω be a nonempty closed subset of E. Assume $F : \Omega \to E$ is a (ws)-compact and (dwc) mapping. Then, F is a continuous (dc)-mapping.*

Proof. Suppose that $\{x_n\}_n$ is a bounded sequence in Ω such that $x_n - F(x_n) \to \theta$. Since F is (dwc) we know that there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}_n$ and an element $x \in E$ such that $x_{n_k} \rightharpoonup x$.

We claim that there exists a subsequence $\{x_{n_{k_s}}\}$ of (x_n) such that $x_{n_{k_s}} \to x$. Indeed, by the definition of a (ws)-mapping, we know that there exist a subsequence $\{x_{n_{k_s}}\}$ of (x_{n_k}) and an element $y \in E$ such that $F(x_{n_{k_s}}) \to y$. Hence,

$$\|x_{n_{k_s}} - y\| \leq \|x_{n_{k_s}} - F(x_{n_{k_s}})\| + \|F(x_{n_{k_s}}) - y\| \to 0,$$

and this means that $x_{n_{k_s}} \to y$ and since Ω is closed, $x = y \in \Omega$. This completes the proof. ∎

Theorem 6.4. *Let Ω be a nonempty unbounded closed convex subset of a Banach space E. Assume Φ is a positive homogenous semi-additive MWNC on E and $F : \Omega \longrightarrow \Omega$ is a (ws)-compact (dwc) and Φ-nonexpansive mapping with $F(\Omega)$ is bounded. Then F has a fixed point in Ω.*

Proof. Let z be a fixed element of Ω. Define $F_n = t_n F + (1 - t_n)z$, $n = 1, 2, \ldots$, where $\{t_n\}_n$ is a sequence of $(0, 1)$ such that $t_n \longrightarrow 1$. Since $z \in \Omega$ and Ω is convex, it follows that F_n maps Ω into itself. Let D be an arbitrary bounded subset of Ω. Then we have

$$\Phi(F_n(D)) \leq \Phi(\{t_n F(D)\} + \{(1 - t_n)z\}) \leq t_n \Phi(F(D)) \leq t_n \Phi(D).$$

So, if $\Phi(D) \neq 0$ we have

$$\Phi(F_n(D)) < \Phi(D).$$

Therefore F_n is Φ-condensing on Ω. Clearly F_n is (ws)-compact mapping, so from Theorem 6.3, F_n has a fixed point, say, x_n in Ω. Consequently, $\|x_n - F(x_n)\| = \|(t_n - 1)(F(x_n) - z)\| \to 0$ as $n \longrightarrow \infty$, since $t_n \longrightarrow 1$ as $n \longrightarrow \infty$ and $F(\Omega)$ is bounded.

Finally, by Lemma 6.1 we have that F is a continuous (dc)- mapping. Since F admits a bounded sequence $\{x_n\}_n$ satisfying $x_n - F(x_n) \to \theta$, by Remark 6.5 we conclude that F has a fixed point. ∎

Remark 6.6. If in Theorem 6.4 we add the hypothesis $\theta \in \Omega$, then we obtain the same conclusion without assuming that Φ is semi-additive.

The next example shows that in Theorem 6.4 we cannot remove the condition F is a (ws)-compact mapping.

Example 6.2. Let E be the Banach space $(L^1[0,1], \|\cdot\|)$ and consider the Alspach mapping. First let

$$\Omega := \{f \in L^1[0,1] : 0 \le f \le 1, \ \|f\|_1 = \frac{1}{2}\}.$$

It is well known that Ω is a weakly compact convex subset of E. Now consider $F : \Omega \to \Omega$ such that for each $f \in \Omega$, $F(f)$ is defined by

$$F(f(t)) = \begin{cases} \min\{2f(2t), 1\}, & t \in [0, \frac{1}{2}] \\ \max\{2f(2t-1) - 1, 0\}, & t \in (\frac{1}{2}, 1]. \end{cases}$$

Since Ω is a weakly compact set, then it is clear that F is a β-condensing (dwc)-mapping with $F(\Omega)$ bounded. Nevertheless, in [2] it is proved that F is a fixed point free nonexpansive mapping.

Theorem 6.5. *Let E be a Banach space, Ω a nonempty unbounded closed convex subset of E and $U \subseteq \Omega$ an open set (with respect to the topology of Ω) and let z be an element of U. Assume Φ is a MWNC on E and $F : \overline{U} \longrightarrow \Omega$ a Φ-condensing (ws)-compact mapping with $F(\overline{U})$ is bounded. Then, either*

$(\mathcal{A}_1)$ *F has a fixed point, or*

$(\mathcal{A}_2)$ *there is a point $u \in \partial_\Omega U$ (the boundary of U in Ω) and $ł \in (0,1)$ with $u = \lambda F(u) + (1-ł)z$.*

Remark 6.7. $\overline{U}$ and $\partial_\Omega U$ denote the closure and boundary of U in Ω, respectively.

Proof. Suppose $(\mathcal{A}_2)$ does not hold and F does not have a fixed point in $\partial_\Omega U$ (otherwise, we are finished, i.e., $(\mathcal{A}_1)$ occurs). Let D be the set defined by

$$D = \left\{ x \in \overline{U} \ : \ x = \lambda F(x) + (1-\lambda)z, \ \text{for some } \lambda \in [0,1] \right\}.$$

Now D is nonempty and bounded, because $z \in D$ and $F(\overline{U})$ is bounded. We have $D \subseteq \text{co}(\{z\} \cup F(D))$. So, $\Phi(D) \neq 0$ implies

$$\Phi(D) \le \Phi(\text{co}(\{z\} \cup F(D)) \le \Phi(F(D)) < \Phi(D),$$

which is a contradiction. Hence, $\Phi(D) = 0$ and D is relatively weakly compact. We will show that D is compact. The continuity of F implies that D is closed. For that, let $\{x_n\}_n$ a sequence of D such that $x_n \longrightarrow x$, $x \in \overline{U}$. For all $n \in \mathbb{N}$, there exists a $\lambda_n \in [0,1]$ such that $x_n = \lambda_n F(x_n) + (1-ł_n)z)$. Now $\lambda_n \in [0,1]$, so we can extract a subsequence $\{\lambda_{n_j}\}_j$ such that $\lambda_{n_j} \longrightarrow \lambda \in [0,1]$. So, $\lambda_{n_j} F(x_{n_j}) \longrightarrow \lambda F(x)$. Hence $x = łF(x) + (1-ł)z$ and $x \in D$. Now, we prove that D is sequentially compact. To see this, let $\{x_n\}_n$ be a sequence of D. For all $n \in \mathbb{N}$, there exists a $\lambda_n \in [0,1]$ such that $x_n = \lambda_n F(x_n) + (1-\lambda_n)z$. Now $\lambda_n \in [0,1]$, so we can extract a subsequence $\{\lambda_{n_j}\}_j$ such that $\lambda_{n_j} \longrightarrow \lambda \in [0,1]$. We have that the set $\{x_n, n \in \mathbb{N}\}$ is contained in D, so it is relatively weakly compact and consequently by the Eberlein–Šmulian theorem it is weakly sequentially compact. Hence, without loss of generality, the sequence $\{x_n\}_n$ has a weakly convergent subsequence $\{x_{n_j}\}_j$. Since F is (ws)-compact, then the sequence $\{F(x_{n_j})\}_j$ has a strongly convergent subsequence, say $\{F(x_{n_{j_k}})\}_k$. Hence, the sequence $\{\lambda_{n_{j_k}} F(x_{n_{j_k}})\}_k$ is strongly convergent which means that the sequence $\{x_{n_{j_k}}\}_k$ is also strongly convergent. Accordingly, D is compact. Because E is a Hausdorff locally convex space, we have that E is completely regular. Since $D \cap (\Omega \setminus U) = \emptyset$, by Proposition 1.1, there is a continuous function $\varphi : \Omega \longrightarrow [0, 1]$, such that $\varphi(x) = 1$ for $x \in D$ and $\varphi(x) = 0$ for $x \in \Omega \setminus U$. Let $F^* : \Omega \longrightarrow \Omega$ be the mapping defined by:

$$F^*(x) = \varphi(x)F(x) + (1 - \varphi(x))z.$$

Clearly, $F^*(\Omega)$ is bounded. Because $\partial_\Omega U = \partial_\Omega \overline{U}$, φ is continuous, $[0, 1]$ is compact and F is (ws)-compact, we have that F^* is (ws)-compact. Let $X \subset \Omega$, bounded, with $\Phi(X) \neq 0$. Then, since

$$F^*(X) \subset \text{co}(\{\theta\} \cup F(X \cap U)),$$

we have

$$\Phi(F^*(X)) \leq \Phi(F(X \cap U)).$$

If $X \cap U$ is relatively weakly compact, then $F(X \cap U)$ is relatively weakly compact and $\Phi(F(X \cap U)) = 0 < \Phi(X)$. If $\Phi(X \cap U) \neq 0$, then $\Phi(F(X \cap U)) < \Phi(X \cap U) \leq \Phi(X)$ and $\Phi(F^*(X)) < \Phi(X)$. So, F^* is Φ-condensing. Therefore Theorem 6.3 implies that F^* has a fixed point $x_0 \in \Omega$. If $x_0 \notin U$, $\varphi(x_0) = 0$ and $x_0 = z$, which contradicts the hypothesis $z \in U$. Then $x_0 \in U$ and $x_0 = \varphi(x_0)F(x_0) + (1 - \varphi(x_0)z$ which implies that $x_0 \in D$, and so $\varphi(x_0) = 1$ and the proof is complete. ■

Corollary 6.2. *Let E be a Banach space, Ω a nonempty unbounded closed convex subset of E and $U \subseteq \Omega$ an open set (with respect to the topology of Ω) and let z be an element of U. Assume Φ is a MWNC on E and $F : \overline{U} \longrightarrow \Omega$ a Φ-condensing (ws)-compact mapping with $F(\overline{U})$ is bounded. Suppose that F satisfies the Leray–Schauder boundary condition*

$$u - z \neq \lambda(F(u) - z), \lambda \in (0, 1), u \in \partial_\Omega U,$$

then the set of fixed points of F in $\overline{U}$ is nonempty and compact.

Proof. By Theorem 6.5, F has a fixed point. Let $\mathcal{S} = \left\{x \in \overline{U} \ : \ F(x) = x,\right\}$ be the fixed point set of F. Since F is continuous, $\mathcal{S}$ is obviously a closed subset of $\overline{U}$ such that $F(\mathcal{S}) = \mathcal{S}$. Now, arguing as in the proof of Theorem 6.5 we have that $\mathcal{S}$ is sequentially compact and hence it is compact. ■

Corollary 6.3. *Let E be a Banach space, Ω a closed convex subset of E and $U \subseteq \Omega$ an open set (with respect to the topology of Ω) such that $\theta \in U$. Assume Φ is a MWNC on E and $F : \overline{U} \longrightarrow \Omega$ a Φ-condensing (ws)-compact mapping with $F(\overline{U})$ is bounded. In addition, assume that $\overline{U}$ is starshaped with respect to θ and $F(\partial_\Omega U) \subseteq \overline{U}$. Then the set of fixed points of F in $\overline{U}$ is nonempty and compact.*

Proof. Since $\overline{U}$ is starshaped with respect to θ and $F(\partial_\Omega U) \subseteq \overline{U}$, then $x \neq \lambda F(x)$ for every $x \in \partial_\Omega U$ and $\lambda \in (0,1)$. Applying Corollary 6.2, then the set of fixed points of F in $\overline{U}$ is nonempty and compact. ■

Corollary 6.4. *Let E be a Banach space, $\Omega \subset E$ a nonempty unbounded closed convex subset, $U \subset \Omega$ an open set (with respect to the topology of Ω) and let z be an element of U. Assume that $F : \overline{U} \longrightarrow \Omega$ is a (ws)-compact mapping which satisfies $F(\overline{U})$ is bounded and $F(D)$ is relatively weakly compact whenever D is a bounded set of $\overline{U}$. Then, either*

($\mathcal{A}_1$) *F has a fixed point, or*
($\mathcal{A}_2$) *there is a point $u \in \partial_\Omega U$ (the boundary of U in Ω) and $\lambda \in (0,1)$ with $u = \lambda F(u) + (1-\lambda)z$.*

Corollary 6.5. *Let E be a Banach space, $\Omega \subset E$ a closed convex subset, $U \subset \Omega$ an open set (with respect to the topology of Ω) and such that $\theta \in U$. Assume $F : \overline{U} \longrightarrow \Omega$ is (ws)-compact. If $\overline{U}$ is weakly compact then, either*

($\mathcal{A}_1$) *F has a fixed point, or*
($\mathcal{A}_2$) *there is a point $u \in \partial_\Omega U$ (the boundary of U in Ω) and $\lambda \in (0,1)$ with $u = \lambda Fu$.*

Proof. Suppose that ($\mathcal{A}_2$) does not hold. Also without loss of generality, assume that the operator F has no fixed point in $\partial_\Omega U$ (otherwise we are finished, i.e., ($\mathcal{A}_1$) occurs). Let D be the set defined by

$$D = \left\{ x \in \overline{U} \;:\; x = \lambda Fx, \text{ for some } \lambda \in [0,1] \right\}.$$

The set D is nonempty because $\theta \in U$. Let $\{x_n\}_n$ be a sequence of points in D. For every $n \in \mathbb{N}$, there exists $\lambda_n \in [0,1]$ such that $x_n = \lambda_n F x_n$. Now $\lambda_n \in [0,1]$, so we can extract a subsequence $\{\lambda_{n_j}\}_j$ such that $\lambda_{n_j} \longrightarrow \lambda \in [0,1]$. We have that the set $\{x_n, n \in \mathbb{N}\} \subset \overline{U}$ so it is relatively weakly compact and hence by the Eberlein–Šmulian theorem it is weakly sequentially compact. Consequently, without loss of generality, the sequence $\{x_n\}$ has a weakly convergent subsequence $\{x_{n_j}\}_j$. We can now proceed analogously as in the proof of Theorem 6.5. ■

Theorem 6.6. *Let E be a Banach space, Ω be a nonempty unbounded closed convex of E and $U \subseteq \Omega$ an open set (with respect to the topology of Ω). In addition, let Φ be a positive homogenous semi-additive MWNC on E and $F : \overline{U} \longrightarrow \Omega$ a Φ-nonexpansive (ws)-compact mapping, with $F(\overline{U})$ is bounded. Assume that*

(a) There exists $z \in U$ *such that* $u - z \neq \lambda(F(u) - z), \lambda \in (0, 1), u \in \partial_\Omega U$.
(b) F *is* (dwc).

Then, F *has a fixed point in* $\overline{U}$.

Proof. Let $F_n = t_n F + (1 - t_n)z$, $n = 1, 2, \ldots$, where $\{t_n\}_n$ is a sequence of $(0, 1)$ such that $t_n \longrightarrow 1$. Since $z \in \Omega$ and Ω is convex, it follows that F_n maps $\overline{U}$ into Ω. Suppose that $= \lambda_n(F_n(y_n) - z) = y_n - z$ for some $y_n \in \partial_\Omega U$ and for some $\lambda_n \in (0, 1)$. Then we have

$$y_n - z = \lambda_n(F_n(y_n) - z) = \lambda_n t_n F(y_n) + \text{ł}_n(1 - t_n)z - \lambda_n z = \lambda_n t_n(F(y_n) - z),$$

which contradicts hypothesis (a) since $\lambda_n t_n \in (0, 1)$. Let X an arbitrary bounded subset of $\overline{U}$. Then we have

$$\Phi(F_n(X)) = \Phi(\{t_n F(X)\} + \{(1 - t_n)z\}) \leq t_n \Phi(F(X)) \leq t_n \Phi(X).$$

So, if $\Phi(X) \neq 0$ we have

$$\Phi(F_n(X)) < \Phi(X).$$

Therefore, F_n is Φ-condensing on $\overline{U}$. From Theorem 6.5, F_n has a fixed point, say, x_n in $\overline{U}$. Now arguing as in the proof of Theorem 6.4, we can prove that F has a fixed point in $\overline{U}$. ■

Remark 6.8. If in Theorem 6.6 we add the hypothesis $\theta \in U$ and replace condition (a) by

(a') $u \neq \lambda F(u), \lambda \in (0, 1), u \in \partial_\Omega U$,

then we obtain the same conclusion without assuming that Φ is semi-additive.

Corollary 6.6. *Let* $(E, \|.\|)$ *be a Banach space,* $\Omega \subset E$ *a nonempty unbounded closed convex subset,* $U \subset \Omega$ *an open set (with respect to the topology of* Ω*) and let* z *be an element of* U*. Assume* $F : E \to E$ *is nonexpansive and* $F : \overline{U} \longrightarrow \Omega$ *is a* (ws)*-compact such that* $F(\overline{U})$ *is bounded. In addition suppose that*

(a) $u - z \neq \lambda(F(u) - z), \lambda \in (0, 1), u \in \partial_\Omega U$.
(b) F *is* (dwc).

Then, F *has a fixed point in* $\overline{U}$.

Proof. The proof follows immediately from Theorem 6.6, once we show that F is β-nonexpansive. To see this, let D be a bounded set of Ω and $d = \beta(D)$. Let $\varepsilon > 0$, and then there exists a weakly compact set K of E with $D \subseteq K + B_{d+\varepsilon}(\theta)$. So for $x \in D$ there exist $y \in K$ and $z \in B_{d+\varepsilon}(\theta)$ such that $x = y + z$ and so

$$\|F(x) - F(y)\| \leq \|x - y\| \leq d + \varepsilon.$$

It follows immediately that

$$\begin{aligned} F(D) &\subseteq F(K) + B_{d+\varepsilon}(\theta) \\ &\subseteq \overline{F(K)} + B_{d+\varepsilon}(\theta). \end{aligned}$$

Since F is a (ws)-compact mapping and K is weakly compact then $\overline{F(K)}$ is compact and hence weakly compact. Thus, $\beta(F(D)) \leq (d+\varepsilon)$. Since $\varepsilon > 0$ is arbitrary, then $\beta(F(D)) \leq \beta(D)$. Accordingly, F is β-nonexpansive. ■

Theorem 6.7. *Let Ω be a nonempty closed convex subset of a Banach space E and consider Φ a MWNC on E. Assume that $F : \Omega \to \Omega$ is a mapping with the following properties:*

(i) F is Φ-condensing,
(ii) F is a (ws)-compact mapping,
(iii) There exists $x_0 \in \Omega$ and $R > 0$ such that $F(x) - x_0 \neq \lambda(x - x_0)$ for every $\lambda > 1$ and for every $x \in \Omega \cap S_R(x_0)$.

Then F has a fixed point.

Proof. Consider $\Omega_R = \Omega \cap B_R(x_0)$. Then Ω_R is a nonempty bounded closed and convex subset of Ω. Define for each $x \in \Omega$

$$r(x) = \begin{cases} (1 - \frac{R}{\|x-x_0\|})x_0 + \frac{R}{\|x-x_0\|}x, & \text{if } \|x - x_0\| > R, \\ x, & \text{if } \|x - x_0\| \leq R. \end{cases}$$

Clearly r is a continuous retraction of Ω on Ω_R. Thus we can define the mapping $rF : \Omega_R \to \Omega_R$ by $rF(x) = r(F(x))$. Since r is continuous and F is (ws)-compact, obviously rF is also (ws)-compact. On the other hand, by hypothesis, if $D \subset \Omega_R$ is such that $\overline{D^w}$ is not weakly compact, we have that $\Phi(F(D)) < \Phi(D)$. We claim that $\Phi(rF(D)) < \Phi(D)$. If $x \in F(D) \subset F(\Omega_R)$ there are two possibilities

1. $\|x - x_0\| \leq R$, in this case $r(x) = x \in F(D) \subset co(F(D) \cup \{x_0\})$.
2. $\|x - x_0\| > R$, in this case $r(x) = (1 - \frac{R}{\|x-x_0\|})x_0 + \frac{R}{\|x-x_0\|}x \in co(F(D) \cup \{x_0\})$.

The above argument yields $rF(D) \subset co(F(D) \cup \{x_0\})$. Now, using the properties of the MWNC Φ and the properties of F, we have that

$$\Phi(rF(D)) \leq \Phi(F(D)) < \Phi(D),$$

as claimed.

The above argument shows that $rF : \Omega_R \to \Omega_R$ so Theorem 6.3 guarantees that there exists $y_0 \in \Omega_R$ such that $rF(y_0) = y_0$. We now show $F(y_0) = y_0$. Indeed, we have:

if $F(y_0) \in \Omega_R$ then $y_0 = rF(y_0) = r(F(y_0)) = F(y_0)$, otherwise if $F(y_0) \notin \Omega_R$ then $y_0 = rF(y_0) = r(F(y_0)) = (1 - \frac{R}{\|F(y_0)-x_0\|})x_0 + \frac{R}{\|F(y_0)-x_0\|}F(y_0)$.

Consequently $\|y_0 - x_0\| = R$ and, if we take $\lambda = \frac{\|F(y_0)-x_0\|}{R} > 1$, then $F(y_0) - x_0 = \lambda(y_0 - x_0)$, which is a contradiction. The proof is now complete. ∎

Corollary 6.7. *Let Ω be a nonempty closed convex subset of a Banach space E. Assume that $F : \Omega \to \Omega$ is a mapping with the following properties:*

(i) *F is weakly compact,*
(ii) *F is a (ws)-compact mapping,*
(iii) *There exists $x_0 \in \Omega$ and $R > 0$ such that $F(x) - x_0 \neq \lambda(x - x_0)$ for every $\lambda > 1$ and for every $x \in \Omega \cap S_R(x_0)$.*

Then F has a fixed point.

Theorem 6.8. *Let Ω be a nonempty closed convex subset of a Banach space E and consider Φ a positive homogenous semi-additive MWNC on E. Assume that $F : \Omega \to \Omega$ is a mapping with the following properties:*

(i) *F is Φ-nonexpansive,*
(ii) *F is a (ws)-compact mapping,*
(iii) *F is (dwc),*
(iv) *There exists $x_0 \in \Omega$ and $R > 0$ such that $F(x) - x_0 \neq \lambda(x - x_0)$ for every $\lambda > 1$ and for every $x \in \Omega \cap S_R(x_0)$.*

Then F has a fixed point.

Proof. Define $F_n = t_n F + (1 - t_n)x_0$, $n = 1, 2, \ldots$, where $\{t_n\}_n$ is a sequence of $(0, 1)$ such that $t_n \longrightarrow 1$. Since $x_0 \in \Omega$ and Ω is convex, it follows that F_n maps Ω into itself. Moreover, F_n is Φ-condensing and (ws)-compact mapping, for example see the proof of Theorem 6.4.

By assumption (iv), we have that $F_n(x) - x_0 \neq \lambda(x - x_0)$ for all $\lambda > 1$ and for every $x \in \Omega \cap S_R(x_0)$. Otherwise, we can find $z \in \Omega \cap S_R(x_0)$ and $\lambda > 1$ such that $F_n(x) - x_0 = \lambda(x - x_0)$, but if this holds, then

$$\lambda(z - x_0) = F_n(z) - x_0 = t_n(F(z) - x_0),$$

consequently $F(z) - x_0 = \frac{\lambda}{t_n}(z - x_0)$ which is a contradiction.

These properties allow us to invoke Theorem 6.7 and hence we have that there exists a bounded sequence $\{x_n\}_n$ ($\{x_n\}_n \subset \Omega \cap B_R(x_0)$) such that $x_n = F_n(x_n)$. Now, following the proof of Theorem 6.4 gives the result. ∎

In the rest of this section we shall discuss a nonlinear Leray–Schauder alternative for positive operators. Let E_1 and E_2 be two Banach lattices, with positive cones E_1^+ and E_2^+, respectively. An operator T from E_1 into E_2 is said to be positive, if it carries the positive cône E_1^+ into E_2^+ (i.e., $T(E_1^+) \subseteq E_2^+$).

Theorem 6.9. *Let Ω be a nonempty unbounded closed convex of a Banach lattice E such that $\Omega^+ := \Omega \cap E^+ \neq \emptyset$. Assume $F : \Omega \longrightarrow \Omega$ is a positive (ws)-compact operator. If $F(\Omega)$ is relatively weakly compact then, F has at least a positive fixed point in Ω.*

Proof. Now Ω^+ is a closed convex subset of E^+ and $F(\Omega^+) \subseteq \Omega^+$. Also, $F(\Omega^+) \subseteq F(\Omega)$, so $F(\Omega^+)$ is relatively weakly compact. Now, it suffices to apply Theorem 6.2 to prove that F has fixed point in $\Omega^+ \subseteq \Omega$. ■

Theorem 6.10. *Let Ω be a nonempty unbounded closed convex of a Banach lattice E such that $\Omega^+ \neq \emptyset$. Assume Φ is a MWNC on E and $F : \Omega \longrightarrow \Omega$ a positive Φ-condensing (ws)-compact mapping with $F(\Omega)$ is bounded. Then, the set of positive fixed points of F in Ω is nonempty and compact.*

Proof. Let $x_0 \in \Omega^+$ and $D = \{F^n(x_0), n \in \mathbb{N}\}$ where $F^0(x_0) = x_0$. Then $D = F(D) \cup \{x_0\}$ and $D \subseteq \Omega^+$. Arguing as in the proof of Theorem 6.3, there exists a closed convex subset K such that $K \cap E^+ \neq \emptyset$, $F(K) \subseteq K$ and $F(K)$ is relatively weakly compact. So, by Theorem 6.9, F has a positive fixed point in Ω.■

Theorem 6.11. *Let Ω be a nonempty unbounded closed convex subset of a Banach lattice space E. In addition, let $U \subseteq \Omega$ an open set (with respect to the topology of Ω) and let z be an element of $U \cap E^+$. Assume Φ is a MWNC on E and $F : \overline{U} \longrightarrow \Omega$ is a positive Φ-condensing (ws)-compact mapping with $F(\overline{U})$ is bounded. Then, either*

- $(\mathcal{A}_1)$ *F has a positive fixed point, or*
- $(\mathcal{A}_2)$ *there is a point $u \in \partial_\Omega U \cap E^+$ (the positive boundary of U in Ω) and $\lambda \in (0, 1)$ with $u = \lambda F(u) + (1 - \lambda)z$.*

Proof. Suppose $(\mathcal{A}_2)$ does not hold and F does not have a positive fixed point in $\partial_\Omega U$ (otherwise, we are finished, i.e., $\mathcal{A}_1$ occurs. Let D be the set defined by

$$D = \left\{ x \in \overline{U} \cap E^+ \ : \ x = \lambda F(x) + (1 - \lambda)z, \text{ for some } \lambda \in [0, 1] \right\}.$$

Since E is a normed lattice, E^+ is closed, and so, $\overline{U} \cap E^+$ is a closed subset of Ω. Arguing as in the proof of Theorem 6.5, we prove that D is compact and that there is a continuous function $\varphi : \Omega \longrightarrow [0, 1]$, such that $\varphi(x) = 1$ for $x \in D$ and $\varphi(x) = 0$ for $x \in \Omega \setminus U$. Let $F^* : \Omega \longrightarrow \Omega$ be the mapping defined by:

$$F^*(x) = \varphi(x)F(x) + (1 - \varphi(x))z.$$

Clearly $F^*(\Omega)$ is bounded. Because $\partial_\Omega U = \partial_\Omega \overline{U}$, φ is continuous and F is a positive (ws)-compact and Φ-condensing operator, and we have that F^* is a positive (ws)-compact and Φ-condensing operator. Therefore, following again the proof of Theorem 6.5 we obtain our result. ■

Corollary 6.8. *Let Ω be a nonempty unbounded closed convex subset of a Banach space E. In addition, let $U \subseteq \Omega$ an open set (with respect to the topology of Ω) such that $\theta \in U$, and $F : \overline{U} \longrightarrow \Omega$ a positive (ws)-compact map, Φ-condensing and $F(\overline{U})$ is bounded. We suppose that*

$$\text{for all } y \in \partial_\Omega U \cap E^+,\ y \notin \big\{\lambda F(y), \lambda \in (0,1)\big\},$$

then the set of positive fixed points of F in $\overline{U}$ is nonempty and compact.

Lemma 6.2. *Let Ω be a nonempty closed convex subset of a Banach space E, U a nonempty bounded subset of Ω, $z \in U$ and Φ a MWNC on E. If $F : U \longrightarrow \Omega$ is Φ-condensing, then there exists a nonempty closed and convex subset K of Ω such that $z \in K \cap U$, $K \cap U$ is relatively weakly compact and $F(K \cap U)$ is a subset of K.*

Proof. We consider the family $\mathcal{G} = \big\{D \subseteq \Omega : D \text{ bounded}, D = \overline{\text{co}}D, z \in D, F(D \cap U) \subseteq D\big\}$. Obviously $\mathcal{G}$ is nonempty, since $\overline{\text{co}}(F(U) \cup \{z\}) \in \mathcal{G}$. We let $K = \bigcap_{D \in \mathcal{G}} D$. We have that K is bounded closed convex and $z \in K$. If $x \in K \cap U$, then $F(x) \in D$ for all $D \in \mathcal{G}$ and hence $F(K \cap U) \subseteq K$. Therefore, we have that $K \in \mathcal{G}$. We will prove that $K \cap U$ is relatively weakly compact. Denoting by $K_* = \overline{\text{co}}(F(K \cap U) \cup \{z\})$, we have $K_* \subseteq K$, which implies that $F(K_* \cap U) \subseteq F(K \cap U) \subseteq K_*$. Therefore $K_* \in \mathcal{G}$ and $K \subseteq K_*$. Hence $K = K_*$. If $\Phi(K \cap U) \neq 0$, we obtain

$$\Phi(K \cap U) \leq \Phi(K) \leq \Phi(\overline{\text{co}}F(K \cap U) \cup \{z\}) \leq \Phi(F(K \cap U) \cup \{z\}) \leq \Phi(F(K \cap U)) < \Phi(K \cap U),$$

which is a contradiction, so $\Phi(K \cap U) = 0$ and $K \cap U$ is relatively weakly compact. ∎

Theorem 6.12. *Let Ω be an unbounded closed convex subset of a Banach space $(E, \|.\|)$ with $\theta \in \Omega$ and Φ a MWNC on E. Suppose that $F : \Omega \longrightarrow \Omega$ is a (ws)-compact and Φ-condensing operator such that it is strictly quasibounded, that is,*

$$l = \limsup_{\substack{\|x\| \longrightarrow \infty \\ x \in \Omega}} \frac{\|F(x)\|}{\|x\|} < 1.$$

Then F has a fixed point in Ω.

Proof. Consider the set

$$D = \Big\{ x \in \Omega \ : \ x = \lambda F(x), \text{ for some } \lambda \in [0,1] \Big\}.$$

The set D is nonempty because $\theta \in D$. We show that there is a $R > 0$ sufficiently large such that $S_R(\theta) \cap D = \emptyset$. Indeed, if we suppose the contrary, then for every positive integer n there exists $x_n \in \Omega$ and $\lambda_n \in [0,1]$ such that $\|x_n\| = n$ and $\lambda_n F(x_n) = x_n$. It follows that $\lambda_n \neq 0$ and consequently $F(x_n) = \lambda_n^{-1} x_n$, where $\lambda_n^{-1} \geq 1$. Thus we have

$$\frac{\|F(x_n)\|}{\|x_n\|} = \lambda_n^{-1}$$

for all $n \in \mathbb{N}$. Since $\{\lambda_n\}_n \subset [0, 1]$, there exists a convergent subsequence $\{\lambda_{n_k}\}_{k\in\mathbb{N}}$ of $\{\lambda_n\}_n$ such that $\lim_{k\longrightarrow\infty} \lambda_{n_k} = \lambda^*$. The limit can be 0 or not. Hence in both of the situations we have that

$$\lim_{k\longrightarrow\infty} \frac{\|F(x_{n_k})\|}{\|x_{n_k}\|}$$

exists and it belongs to $[1, +\infty]$. Since $\lim_{k\longrightarrow\infty} \|x_{n_k}\| = +\infty$, we have that

$$\limsup_{\substack{\|x\|\longrightarrow\infty \\ x\in\Omega}} \frac{\|F(x)\|}{\|x\|} \geq 1,$$

which is a contradiction. Hence there exists $R > 0$ with the property indicated above. Consider the set $\Omega \cap B_R(\theta)$. Obviously, $\Omega \cap B_R(\theta) \neq \emptyset$, since $\theta \in \Omega \cap B_R(\theta)$. By Lemma 6.2, there exists a nonempty closed and convex subset K of Ω with $\theta \in K$, $K \cap B_R(\theta)$ is relatively weakly compact and $F(K \cap B_R(\theta)) \subset K$. Now, by Remark 6.2 (*ii*), the set $F(K \cap B_R(\theta)) \cup \{\theta\}$ is relatively compact. Without loss of generality we may suppose that F has no fixed point in $K \cap S_R(\theta)$. Set

$$X_1 = \left\{ x \in K \cap B_R(\theta) : x = \lambda F(x), \text{ for some } \lambda \in [0, 1] \right\}.$$

We claim that X_1 is a compact set of E. Indeed, since $\theta \in X_1$, we conclude that X_1 is nonempty. Let $\{x_n\}$ be an arbitrary sequence in X_1, converging to $x^* \in K \cap B_R(\theta)$. For every $n \in \mathbb{N}$, there exists $\lambda_n \in [0, 1]$ such that $x_n = \lambda_n F(x_n)$. Since $\{\lambda_n\} \subseteq [0, 1]$, there exists a convergent subsequence $\{\lambda_{n_k}\}_k$ of $\{\lambda_n\}$ such that $\lim_{k\longrightarrow\infty} \lambda_{n_k} = \lambda^* \in [0, 1]$. So, by the continuity of the operator F, we obtain that $\lambda_{n_k} F(x_{n_k}) \longrightarrow \lambda^* F(x^*)$. Hence $x^* = \lambda^* F(x^*)$ and $x^* \in X_1$. In conclusion we have that X_1 is closed. Next, we shall prove that the set X_1 is sequentially compact. To see this, let $\{x_n\}$ be any sequence in X_1. Since $F(K \cap B_R(\theta))$ is relatively compact, there exist subsequences $\{\lambda_{n_k}\}$ and $\{F(x_{n_k})\}$ such that $\lambda_{n_k} \longrightarrow \lambda \in [0, 1]$ and $F(x_{n_k}) \longrightarrow y \in K$ and hence $x_{n_k} = \lambda_{n_k} F(x_{n_k}) \longrightarrow \lambda y$. Accordingly, X_1 is compact. Because E is a Hausdorff locally convex space, we have that E is completely regular. Since $X_1 \cap (K \cap S_R(\theta)) = \emptyset$ and $S_R(\theta) \cap K$ is closed in E, by Proposition 1.1, there exists a continuous function $\varphi : E \longrightarrow [0, 1]$ such that $\varphi(x) = 1$ for all $x \in K \cap S_R(\theta)$ and $\varphi(x) = 0$ for all $x \in X_1$. Let an operator $G : K \longrightarrow K$ be defined by

$$G(x) = \begin{cases} (1 - \varphi(x))F(x), & \text{if } x \in K \cap B_R(\theta), \\ \theta, & \text{if } x \in K \setminus \text{int}B_R(\theta). \end{cases}$$

Since $\partial_K(K \cap \text{int}B_R(\theta)) = \partial_K(K \cap B_R(\theta))$, φ is continuous and F is continuous, it is immediate that G is continuous. Now, we prove that G is (ws)-compact. To this end, let $\{x_n\}$ be a weakly convergent sequence in K. We consider two cases:

a) there exists some $n_0 \in \mathbb{N}$ such that for all n $(n \geq n_0 \Longrightarrow x_n \in K \cap B_R(\theta))$. In this case, the sequence $\{x_n\}_{n \geq n_0}$ lies in $K \cap B_R(\theta)$ and converges weakly in $K \cap B_R(\theta)$ ($K \cap B_R(\theta)$ is a closed convex subset of E and hence weakly closed). Since F is (ws)-compact, the sequence $\{F(x_n)\}_{n \geq n_0}$ has a strongly convergent subsequence $\{F(x_{n_j})\}$ whose limit y belongs to K. Using the compactness of $[0, 1]$, we can extract from $\{\varphi(x_{n_j})\}$ a convergent subsequence $\{\varphi(x_{n_{j_k}})\}$, such that $\lim\limits_{k \longrightarrow \infty} \varphi(x_{n_{j_k}}) = \lambda^{**} \in [0, 1]$. As a result, the sequence $\{G(x_{n_{j_k}})\}$ verifies $G(x_{n_{j_k}}) = (1 - \varphi(x_{n_{j_k}}))F(x_{n_{j_k}})$ and $\lim\limits_{k \longrightarrow \infty} G(x_{n_{j_k}}) = (1 - \lambda^{**})y \in K$.

b) If $\{x_n\}$ is such that for all n, there exists $n_s \in \mathbb{N}$ such that $x_{n_s} \notin K \cap B_R(\theta)$, then we may consider a subsequence $\{x_{n_s}\} \subset K \setminus (K \cap B_R(\theta))$ such that $G(x_{n_s}) = \theta \longrightarrow \theta \in K$.

From *a*) and *b*), G is (ws)-compact. Also, $G(K) \subseteq \overline{\text{co}}(F(K \cap B_R(\theta)) \cup \{\theta\})$. Let $H = \overline{\text{co}}(F(K \cap B_R(\theta)) \cup \{\theta\})$. Obviously, H is a compact convex subset of E and hence $G(H) \subseteq H$ is relatively compact. Theorem 6.3 shows that G has a fixed point $x_0 \in K$. From $x_0 \in H \subseteq K \cap B_R(\theta)$, it follows that $x_0 = G(x_0) = (1 - \varphi(x_0))F(x_0)$, which implies $x_0 \in X_1$ and so $\varphi(x_0) = 0$. Thus, x_0 is a fixed point of F. ■

Corollary 6.9. *Let Ω be an unbounded closed convex subset of a Banach space $(E, \|.\|)$ with $\theta \in \Omega$. Suppose that $F : \Omega \longrightarrow \Omega$ is a (ws)-compact and weakly compact operator such that it is strictly quasi-bounded. Then F has a fixed point in Ω.*

6.5 Positive Eigenvalues and Surjectivity for Nonlinear Operators

Given a Banach space $(E, \|.\|)$ and a general mapping $F : E \longrightarrow E$, the following question is fundamental: Does the mapping F have an eigenvalue?

We recall that a real number λ_0 is said to be an eigenvalue for F if there exists an element x_0A̧ $\in E \backslash \{\theta\}$ such that $F(x_0) = \lambda_0 x_0$. In this section we show the existence of positive eigenvalues and eigenvectors of (ws)-compact, weakly compact, and weakly condensing mappings defined on unbounded domains.

Proposition 6.4. *Let E be a Banach space, $\Omega \subset E$ a nonempty unbounded closed convex subset, $U \subset \Omega$ an open set (with respect to the topology of Ω) and such that $\theta \in U$. Assume that $F : \overline{U} \longrightarrow \Omega$ is a (ws)-compact mapping which satisfies $F(\overline{U})$ is bounded and $F(D)$ is relatively weakly compact whenever D is a bounded set of $\overline{U}$. In addition suppose F has no fixed point in $\overline{U}$. Then, there exist an $x \in \partial_\Omega U$ and $\lambda \in (0, 1)$ such that $x = \lambda F(x)$.*

Proof. This is an immediate consequence of Theorem 6.9. ■

Theorem 6.13. *Let E be a Banach space, $\Omega \subset E$ a nonempty unbounded closed convex subset, $U \subset \Omega$ an open set (with respect to the topology of Ω) and such that $\theta \in U$. In addition let Φ be a positive homogenous MWNC on E, $k \geq 1$ and $F : \overline{U} \longrightarrow \Omega$ a Φ-nonexpansive (ws)-compact mapping, with $F(\overline{U})$ is bounded. Suppose that there is a real number $c > k$ such that*

$$F(\overline{U}) \cap (c.U) = \emptyset.$$

Then there exists an $x \in \partial_\Omega U$ and $\lambda \geq c$ such that $F(x) = \lambda x$.

Proof. We suppose that for all $x \in \partial_\Omega U$ and $\lambda \geq c$, $F(x) \neq \lambda x$. Let $F_1 = \frac{1}{c}F$ and

$$D = \left\{ x \in \overline{U} \ : \ x = \lambda F_1(x), \ \text{for some } \lambda \in [0,1] \right\}.$$

Now D is nonempty and bounded, because $\theta \in D$ and $F(\overline{U})$ is bounded. We have $D \subseteq \text{co}(\{\theta\} \cup F_1(D))$. So, since $\Phi(D) \neq 0$, F is Φ-nonexpansive and $c > 1$ we have

$$\Phi(D) \leq \Phi(\text{co}(\{\theta\} \cup F_1(D)) \leq \frac{1}{c}\Phi(F(D)) < \Phi(D),$$

which is a contradiction. Hence, $\Phi(D) = 0$ and D is relatively weakly compact. Clearly F_1 is a (ws)-compact mapping and so D is compact. We claim that $D \cap (\Omega \setminus U) = \emptyset$. We suppose to the contrary that $D \cap (\Omega \setminus U) \neq \emptyset$, and then there exists an $x \in \Omega \setminus U$ and $\alpha \in [0,1]$ such that $\alpha F_1(x) = x$. If $\alpha = 0$, then $x = \theta$, which contradicts $\theta \in U$. If $\alpha \neq 0$, then $F(x) = \frac{c}{\alpha}x$ ($\frac{c}{\alpha} \geq c$), which contradicts the hypothesis. Thus, $D \cap (\Omega \setminus U) = \emptyset$. Let $F_1^* : \Omega \longrightarrow \Omega$ be the mapping defined by:

$$F_1^*(x) = \varphi(x)F_1(x),$$

where $\varphi : \Omega \longrightarrow [0,1]$, is a continuous function such that $\varphi(x) = 1$ for $x \in D$ and $\varphi(x) = 0$ for $x \in \Omega \setminus U$. Arguing as in the proof of Theorem 6.5, we prove that F_1 is a Φ-condensing (ws)-compact mapping with $F_1(\Omega)$ is bounded. Therefore, Theorem 6.3 implies that F_1^* has a fixed point $x_1 \in \Omega$. If $x_1 \notin U$, $\varphi(x_1) = 0$ and $x_1 = \theta$, which contradicts the hypothesis $\theta \in U$. Then $x_1 \in U$ and $x_1 = \varphi(x_1)F_1(x_1)$, which implies that $x_1 \in D$, and so $\varphi(x_1) = 1$ and $F(x_1) = cx_1$. Hence, $F(\overline{U}) \cap (c.U) \neq \emptyset$, another contradiction. Accordingly, there exist an $x \in \partial_\Omega U$ and $\lambda \geq c$ such that $F(x) = \lambda x$. ■

Proposition 6.5. *Let E be a Banach space, $k \geq 1$ and $F : B_1(\theta) \longrightarrow E$ a β-nonexpansive (ws)-compact mapping. Suppose that there is a real number $c > k$ such that $\|F(x)\| \geq c$ for all $x \in B_1(\theta)$. Then there exist an $x \in S_1(\theta)$ and $\lambda \geq c$ such that $F(x) = \lambda x$.*

Proof. It suffices to note that De Blasi's measure β of weak noncompactness is positive and homogenous. ■

Theorem 6.14. *Let Ω be an unbounded closed convex subset of a Banach space $(E, \|.\|)$ with $\theta \in \Omega$ and Φ a positive homogeneous MWNC on E. Suppose that $F : \Omega \longrightarrow \Omega$ is a (ws)-compact and Φ-condensing operator such that $F(\theta) \neq \theta$ and*

$$l = \limsup_{\substack{\|x\| \longrightarrow \infty \\ x \in \Omega}} \frac{\|F(x)\|}{\|x\|} < \infty.$$

If $\lambda > l$ and $\lambda \geq 1$, then λ is an eigenvalue of F.

Proof. Let $\lambda > l$ and $\lambda \geq 1$. Consider an operator $G : \Omega \longrightarrow \Omega$, defined by

$$G(x) := \frac{1}{\lambda} F(x) \qquad \text{for } x \in \Omega.$$

Let D be an arbitrary bounded subset of Ω. Then we have

$$\Phi(G(D)) = \frac{1}{\lambda} \Phi(F(D)) \leq \Phi(F(D)).$$

So, if $\Phi(D) \neq 0$ we have

$$\Phi(G(D)) < \Phi(D).$$

Therefore, G is Φ-condensing. From $\lambda > l$ it follows that

$$\limsup_{\substack{\|x\| \longrightarrow \infty \\ x \in \Omega}} \frac{\|G(x)\|}{\|x\|} = \frac{1}{\lambda} \limsup_{\substack{\|x\| \longrightarrow \infty \\ x \in \Omega}} \frac{\|F(x)\|}{\|x\|} = \frac{l}{\lambda} < 1.$$

Hence Theorem 6.12 implies that G has a fixed point $x_0 \in \Omega$. Because $F(\theta) \neq \theta$, we have that $F(x_0) = \lambda x_0$ and $x_0 \neq \theta$. Thus, λ is an eigenvalue of F. ■

Corollary 6.10. *Let Ω be an unbounded closed convex subset of a Banach space $(E, \|.\|)$ with $\theta \in \Omega$. Suppose that $F : \Omega \longrightarrow \Omega$ is a (ws)-compact and weakly compact operator such that $F(\theta) \neq \theta$ and*

$$l = \limsup_{\substack{\|x\| \longrightarrow \infty \\ x \in \Omega}} \frac{\|F(x)\|}{\|x\|} < \infty.$$

If $\lambda > l$ and $\lambda \geq 1$, then λ is an eigenvalue of F.

Theorem 6.15. *Let Ω be a closed wedge in a Banach space $(E, \|.\|)$ and Φ a positive homogeneous MWNC on E. Let $F : \Omega \longrightarrow \Omega$ be a (ws)-compact and* *k-Φ-condensing operator with* $0 < k < 1$ *such that* $F(\theta) \neq \theta$ *and*

$$l = \limsup_{\substack{\|x\| \longrightarrow \infty \\ x \in \Omega}} \frac{\|F(x)\|}{\|x\|} < \infty.$$

If $\lambda > l$ *and* $\lambda \geq k$*, then* λ *is an eigenvalue of* F.

Proof. Fix $\lambda > 1$ such that $\lambda \geq k$ and define an operator $G : \Omega \longrightarrow \Omega$ by

$$G(x) := \frac{1}{k} F(x) \qquad \text{for } x \in \Omega.$$

We show that G is Φ-condensing. Let $A \subset \Omega$ be bounded such that $\Phi(A) > 0$. Then

$$\Phi(G(A)) = \frac{1}{k} \Phi(F(A)) < \Phi(A)$$

because F is k-Φ-condensing. It follows from $F(\theta) \neq \theta$ that $G(\theta) \neq \theta$ and

$$\limsup_{\substack{\|x\| \longrightarrow \infty \\ x \in \Omega}} \frac{\|G(x)\|}{\|x\|} = \frac{l}{k} < \infty.$$

Hence by Theorem 6.14, $\lambda^{'}$ is an eigenvalue of G for every $\lambda^{'} > \frac{l}{k}$ with $\lambda^{'} > 1$. Taking $\lambda^{'} = \frac{\lambda}{k}$, there exists an $x \in \Omega$ with $x \neq \theta$ such that $G(x) = \frac{\lambda}{k} x$ or $F(x) = \lambda x$. This completes the proof. ■

Corollary 6.11. *Let* Ω *be a closed wedge in a Banach space* $(E, \|.\|)$*. Suppose that* $F : \Omega \longrightarrow \Omega$ *is a* (ws)*-compact and weakly compact operator such that* $F(\theta) \neq \theta$ *and*

$$l = \limsup_{\substack{\|x\| \longrightarrow \infty \\ x \in \Omega}} \frac{\|F(x)\|}{\|x\|} < \infty.$$

If $\lambda > l$*, then* λ *is an eigenvalue of* F.

Theorem 6.16. *Let* Ω *be a closed wedge in a Banach space* $(E, \|.\|)$ *and* Φ *a positive homogeneous and subadditive MWNC on E. Suppose that* $F : \Omega \longrightarrow \Omega$ *is a* Φ*-condensing and* (ws)*-compact operator such that* $(I - F)(\Omega) \subseteq \Omega$ *and*

$$l = \limsup_{\substack{\|x\| \longrightarrow \infty \\ x \in \Omega}} \frac{\|F(x)\|}{\|x\|} < 1.$$

Then $(I - F)|_{\Omega} : \Omega \longrightarrow \Omega$ *is surjective.*

Proof. We shall use some ideas of [119]. Fix $y \in \Omega$. From the definition of l, there is a real number $R_0 > 0$ such that

$$\|F(x)\| \leq l\|x\| \qquad \text{for all } x \in \Omega \text{ with } \|x\| \geq R_0.$$

Choose a real number $R \geq R_0$ such that $(1 - l)R > \|y\|$. Consider the set

$$D = \Big\{ x \in \Omega \cap B_R(\theta) : x = \lambda F(x) + \lambda y, \text{ for some } \lambda \in [0, 1] \Big\}.$$

Arguing as in the proof of Theorem 6.12, we prove that D is a compact set of E. We now claim that $D \cap S_R(\theta) = \emptyset$. Indeed, if $\lambda F(x) + \lambda y = x$ for some $(\lambda, x) \in [0, 1] \times S_R(\theta)$, then

$$R = \|x\| = \|\lambda F(x) + \lambda y\| \leq \|F(x)\| + \|y\| \leq l\|x\| + \|y\|$$

and hence $(1-l)R \leq \|y\|$, which contradicts our choice of R_0. Since $D \cap S_R(\theta)) = \emptyset$ and $S_R(\theta) \cap \Omega$ is closed in E, then by Proposition 1.1, there exists a continuous function $\varphi : E \longrightarrow [0, 1]$ such that $\varphi(x) = 1$ for all $x \in \Omega \cap S_R(\theta)$ and $\varphi(x) = 0$ for all $x \in D$. Let an operator $G : \Omega \longrightarrow \Omega$ be defined by

$$G(x) = \begin{cases} (1 - \varphi(x))F(x) + (1 - \varphi(x))y, & \text{if } x \in \Omega \cap B_R(\theta), \\ \theta, & \text{if } x \in \Omega \setminus \operatorname{int} B_R(\theta). \end{cases}$$

Since $G(\Omega) \subseteq \operatorname{conv}(F(\Omega \cap B_R(\theta)) + \{y\}) \cup \{\theta\})$ and the set $\Omega \cap B_R(\theta) + \{y\}$ is bounded, we deduce that G is strictly quasi-bounded. Following an argument similar to that in Theorem 6.12, we obtain that the operator G is (ws)-compact. We show that G is Φ-condensing. Let A be a bounded subset of Ω such that $\Phi(A) > 0$. Then $G(A) \subseteq \operatorname{conv}(F(A \cap B_R(\theta)) + \{y\}) \cup \{\theta\})$ and so

$$\begin{aligned} \Phi(G(A)) &\leq \Phi(\operatorname{conv}(F(A \cap B_R(\theta)) + \{y\}) \cup \{\theta\})) \leq \Phi(F(A \cap B_R(\theta)) + \{y\}) \cup \{\theta\})) \\ &\leq \Phi(F(A \cap B_R(\theta))) + \Phi(\{y\}) < \Phi(A), \end{aligned}$$

since F is Φ-condensing. This shows that G is Φ-condensing and so, by Theorem 6.12, G has a fixed point $x_0 \in \Omega$. From $\theta \in \operatorname{int}(B_R(\theta)) \cap \Omega \subseteq \operatorname{int}_\Omega(\Omega \cap B_R(\theta))$, it follows that $x_0 = G(x_0) = (1-\varphi(x_0))F(x_0) + (1-\varphi(x_0))y$, which implies $x_0 \in D$ and so $\varphi(x_0) = 0$. Thus, $(I - F)(x_0) = y$ and so $(I - F)|_\Omega$ is surjective. ■

Corollary 6.12. *Let Ω be a closed wedge in a Banach space $(E, \|.\|)$. Suppose that $F : \Omega \longrightarrow \Omega$ is a (ws)-compact and weakly compact operator such that $(I - F)(\Omega) \subseteq \Omega$ and*

$$l = \limsup_{\substack{\|x\| \longrightarrow \infty \\ x \in \Omega}} \frac{\|F(x)\|}{\|x\|} < 1.$$

Then $(I - F)|_\Omega : \Omega \longrightarrow \Omega$ *is surjective.*

Theorem 6.17. *Let* Ω *be a closed wedge in a Banach space* $(E, \|.\|)$ *and* Φ *a positive homogeneous and subadditive MWNC on E. Let* $F : \Omega \longrightarrow \Omega$ *be a* (ws)*-compact and* k*-*Φ*-condensing operator with* $0 < k < 1$ *such that* $F(\theta) \neq \theta$ *and*

$$l = \limsup_{\substack{\|x\| \longrightarrow \infty \\ x \in \Omega}} \frac{\|F(x)\|}{\|x\|} < \infty.$$

If $\lambda > l$ *and* $\lambda \geq k$*, then* $(\lambda I - F)|_\Omega : \Omega \longrightarrow \Omega$ *is surjective whenever* $(\lambda I - F)(\Omega) \subseteq (\Omega)$.

Proof. Fix $\lambda > l$ such that $\lambda \geq k$. We show that $(I - \frac{1}{\lambda}F)|_\Omega : \Omega \longrightarrow \Omega$ is surjective. Let $G : \Omega \longrightarrow \Omega$ be an operator defined by

$$G(x) = \frac{1}{\lambda}F(x) \qquad \text{for } x \in \Omega.$$

We claim that G is Φ-condensing. Indeed, let $D \subset \Omega$ be bounded such that $\Phi(D) > 0$. Then

$$\Phi(G(D)) = \frac{1}{\lambda}\Phi(F(D)) < \Phi(D)$$

because F is k-Φ-condensing. It follows from $F(\theta) \neq \theta$ that $G(\theta) \neq \theta$ and

$$\limsup_{\substack{\|x\| \longrightarrow \infty \\ x \in \Omega}} \frac{\|G(x)\|}{\|x\|} = \frac{l}{\lambda} < 1.$$

Hence by Theorem 6.16, the operator $(I - G)|_\Omega$ is surjective and so, because Ω is a wedge, we deduce that $(\lambda I - F)|_\Omega$ is surjective. ■

6.6 Applications

We discuss a nonlinear eigenvalue value problem in a general setting. In particular we discuss generalized Hammerstein type integral equations.

Theorem 6.18. *Let X, Y be finite dimensional Banach spaces, D a compact subset of* $\mathbb{R}^n$, $\lambda \in \mathbb{R}$ *and* $E = L^1(D, X)$. *Assume that*

(a) $G : B_1(\theta) \to E$ *is a ws-compact, weakly compact operator,*
(b) $f : D \times X \to Y$ *satisfies the Carathéodory conditions i.e., f is strongly measurable with respect to* $t \in D$, *for all* $x \in X$, *and continuous with respect to* $x \in X$, *for almost all* $t \in D$,
(c) There are $a \in L^1(D)$ *and* $b \geq 0$ *such that*

$$\|f(t,x)\| \leq a(t) + b\|x\|, \quad t \in D, x \in X,$$

(d) $k : D \times D \to L(Y, X)$ *(the space of bounded linear operators from Y into X) is strongly measurable and the linear operator K defined by*

$$(K(z))(t) = \int_D k(t,s)z(s)\,ds,$$

maps $L^1(D, Y)$ *into* $L^1(D, X)$ *continuously,*
(e) The functions $s \to k(t,s)$ *are in* $L^\infty(D, L(Y, X))$ *for almost all* $t \in D$,
(f) $|\lambda|\, b\|K\| \leq 1$ *(*$\|K\|$ *denotes the operator norm of K).*

Consider the nonlinear operator $F : B_1(\theta) \to E$ *given by*

$$F(y) = G(y) + L(y) = G(y) + \lambda \int_D k(t,s)f(s, y(s))\,ds.$$

Set

$$\alpha = |\lambda|\,(\|a\| + b)\|K\|, \gamma = \inf_{y \in B_1(\theta)} \|G(y)\|.$$

If $\gamma > \alpha + 1$, *then F has a positive eigenvalue whose corresponding vector lies in* $S_1(\theta)$.

Proof. First, we prove that L is a (ws)-compact, β-nonexpansive operator. From assumption (b), we obtain that the Nemytskii operator generated by f and defined by

$$\mathcal{N}_f(y)(t) := f(t, y(t)), y \in L^1(D, X),$$

maps continuously $L^1(D, X)$ into $L^1(D, Y)$. From assumption (d) the operator $L = \lambda K \mathcal{N}_f$ is continuous. Using assumptions (b), (c) and (f) we prove that the operator L is β-nonexpansive. First, we observe that for all $D_0 \subset D$ we have

$$\int_{D_0} \|\mathcal{N}_f(y)(t)\|dt \leq \int_{D_0} a(t)dt + b \int_{D_0} \|(y)(t)\|dt.$$

For any bounded subset Z of $B_1(\theta)$ we have $\beta(K(Z)) \leq \|K\|\beta(Z)$ and we see that

$$\beta(L(H)) \leq |\lambda|\, b\|K\|\beta(H) \leq \beta(H)$$

for any bounded subset H of $B_1(\theta)$. Now, let $\{y_n\}_n$ be a weakly convergent sequence of $L^1(D, X)$, and then $\{y_n\}_n$ is uniformly bounded and by assumption (b) we obtain

$$\|f(t, y_n)\| \leq a(t) + b\|y_n\|. \tag{6.3}$$

Since $\{y_n\}_n$ is weakly compact in $L^1(D, X)$, by the Dunford–Pettis criterion it is uniformly integrable on D, that is

$$\forall \varepsilon > 0, \exists\, \delta > 0, |D_0| < \delta \Longrightarrow \int_{D_0} \|y_n(t)\|\, dt < \varepsilon \;\; \forall n \in \mathbb{N}.$$

Therefore from (6.3), $\{\mathcal{N}_f(y_n)\}_n$ is uniformly integrable on D, which implies the weak compactness in $L^1(D, Y)$ of $\{\mathcal{N}_f(y_n)\}_n$, and hence by the Eberlein–Šmulian theorem, $\{\mathcal{N}_f(y_n)\}_n$ has a weakly convergent subsequence, say $\{\mathcal{N}_f(y_{n_j})\}_j$.

The continuity of the linear operator K implies its weak continuity on $L^1(D, Y)$ for almost $t \in D$. Consequently, we obtain that $\{K\mathcal{N}_f(y_{n_j})\}_j$ and so $\{L(y_{n_j})\}_j$ is pointwise converging, for almost all $t \in D$. Using again the weak continuity of the linear operator K, we infer that $\{L(y_{n_j})\}_j$ is uniformly integrable on D. Hence, by Vitali's convergence theorem, $\{L(y_{n_j})\}_j$ is strongly convergent in $L^1(D, X)$. Accordingly, the operator L is (ws)-compact. We will now apply Proposition 6.5. Note for all $y \in B_1(\theta)$, we have

$$\|F(y)\| \geq \|G(y)\| - \|L(y)\| \geq \gamma - \alpha > 1.$$

Now G is a weakly compact, (ws)-compact operator so F is a (ws)-compact and β-nonexpansive operator. Proposition 6.5, implies that F has an eigenvalue $\eta > 1$ with corresponding eigenvector $y \in S_1(\theta)$. ■

Chapter 7
Approximate Fixed Point Theorems in Banach Spaces

Let Ω be a nonempty convex subset of a topological vector space X. An approximate fixed point sequence for a map $F : \Omega \longrightarrow \overline{\Omega}$ is a sequence $\{x_n\}_n \in \Omega$ so that $x_n - F(x_n) \longrightarrow \theta$. Similarly, we can define approximate fixed point nets for F. Let us mention that F has an approximate fixed point net if and only if

$$\theta \in \overline{\{x - F(x) : x \in \Omega\}}.$$

In this chapter some approximate fixed point theorems for multivalued mappings defined on Banach spaces are presented. Weak and strong topologies play a role here and both bounded and unbounded regions are considered. Also an outline of how to use approximate fixed point theorems to guarantee that noncooperative game have approximate Nash equilibria is given.

7.1 Approximate Fixed Point Theorems

Let X be a normed space. For every $Y \subseteq X$ we denote the convex hull of Y by co$Y$. We say that the set Y is totally bounded if for every $\varepsilon > 0$, there exists $x_1, \ldots, x_{p_\varepsilon} \in X$ such that $Y \subset \bigcup_{i \in \{1,\ldots,p_\varepsilon\}} B_\varepsilon(x_i)$. For convenience let

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$$\mathcal{P}(X) = \left\{ Y \subset X : Y \text{ is nonempty} \right\},$$

$$\mathcal{P}_{\text{cv}}(X) = \left\{ Y \subset X : Y \text{ is nonempty and convex} \right\},$$

$$\mathcal{P}_{\text{cl}}(X) = \left\{ Y \subset X : Y \text{ is nonempty and closed} \right\},$$

$$\mathcal{P}_{\text{cl,cv}}(X) = \left\{ Y \subset X : Y \text{ is nonempty closed and convex} \right\}.$$

Definition 7.1 ([94]). We say that a multivalued map $F : Y \longrightarrow \mathcal{P}(X)$ is partially closed if the following property holds: if $\{x_\delta\}_{\delta\in\Delta}, x_\delta \in X, x_\delta \longrightarrow x, x \in X$, and $\{y_\delta\}_{\delta\in\Delta}, y_\delta \in F(x_\delta), y_\delta \longrightarrow y$, then $F(x) \cap L(x, y) \neq \emptyset$, where $L(x, y) = \{x + \lambda(y - x) : \lambda \geq 0\}$.

Definition 7.2. For fixed $\beta \in]0, 1[$, we say that a multivalued map $F : Y \longrightarrow \mathcal{P}(X)$ is β-partially closed if for every net $\{x_\delta\}_{\delta\in\Delta}, x_\delta \in X, x_\delta \longrightarrow x, x \in X$, and $\{y_\delta\}_{\delta\in\Delta}, y_\delta \in F(x_\delta), y_\delta \longrightarrow y$, then $F(x) \cap L\left(\frac{x}{1-\beta}, y\right) \neq \emptyset$. By considering X with the weak topology we say that F is respectively w-partially closed, β-w-partially closed.

Definition 7.3 ([57]). Suppose $(X, \|.\|)$ is a normed space and $Y \subset X$, and let $d(x, Y) = \inf y \in Y\|x - y\|$. For a fixed multivalued map $F : Y \longrightarrow \mathcal{P}(X)$, let us denote by $\mathcal{WF}(X)$ the set of all points $\bar{x} \in X$ such that there exists at least one sequence $\{x_n\}_n$ in X which weakly converges to $\bar{x}$ and such that $\lim_{n\longrightarrow\infty} d(x_n, F(x_n)) = 0$. The multivalued map $I - F$ is to be weakly demiclosed in $\bar{x} \in \mathcal{WF}(X)$ if for every sequence $\{x_n\}_n$ in X which weakly converges to $\bar{x}$ and such that $\lim_{n\longrightarrow\infty} d(x_n, F(x_n)) = 0$, we have $\bar{x} \in F(\bar{x})$.

Remark 7.1. One can easily check that the map $I - F$ is weakly demiclosed on $\mathcal{WF}(X)$ if it is weakly demiclosed at every point of the set $\mathcal{WF}(X)$ and this definition is well posed if the set $\mathcal{WF}(X)$ is nonempty.

For a normed $(X, \|.\|)$ space and for $F : Y \longrightarrow \mathcal{P}(Y)$ with $Y \subseteq X$, the set $\{x \in Y : d(x, F(x)) = \inf_{y\in F(x)} \{\|y - x\| \leq \varepsilon\}$ of ε-fixed points of the multivalued mapping F on Y is denoted by $FIX^\varepsilon(F)$ and the set of all fixed point of F is denoted by $FIX(F)$.

First, we give a result on the existence of approximate fixed points for multivalued mappings in reflexive Banach spaces.

Theorem 7.1. *Let X be reflexive real Banach space and let Ω be a bounded and convex subset of X with nonempty interior. Assume that $F : \Omega \longrightarrow \mathcal{P}_{\text{cv}}(\Omega)$ is a weakly closed multivalued map (that is, a multivalued map closed with respect to the weak topology). Then $FIX^\varepsilon(F) \neq \emptyset$ for each $\varepsilon > 0$.*

Proof. Suppose without loss of generality that $\theta \in \operatorname{int} \Omega$. Let $\alpha = \sup\{\|x\|, x \in \Omega\}$. Take $\varepsilon > 0$ and $0 < \delta < 1$ such that $\delta\alpha \leq \varepsilon$. Let C be the weakly compact and convex subset of X defined by $C = (1-\delta)\overline{\Omega}$, where $\overline{\Omega}$ is the closure of Ω. We prove the following inclusion

$$C \subset \Omega. \tag{7.1}$$

Fix $z^* \in C = (1-\delta)\overline{\Omega}$, and there exists $z \in \overline{\Omega}$ such that $z^* = (1-\delta)z$. Obviously, if $z \in \Omega$ we have $z^* \in \Omega$. Now, consider the case $\overline{\Omega} \setminus \Omega$. If we show that the segment $[\theta, z[= \{tz : t \in [0,1]\} \subset \Omega$ then $z^* \in \Omega$ again. Suppose, contrary to our claim, that there exists $w \in [\theta, z[$ such that $w \notin \Omega$. By the convexity of $\overline{Y}$ we have that $[\theta, z[\subset \overline{\Omega}$ and so $w \in \overline{\Omega}$. Then we note that, for every $n \in \mathbb{N}$, there exists a point $x_n \in B_{\frac{1}{n}}(w)$ such that

$$x_n \notin \overline{\Omega}. \tag{7.2}$$

Put $\rho = \|w\| > 0$ and $r > 0$ such that $\overline{B}_r(\theta) \subset \Omega$, and there exists $\rho^* > 0$ with $\rho = r + \rho^*$. Next, for every $n \in \mathbb{N}$, consider

$$i_n = \inf_{i \geq 0} \|z + t(x_n - z)\|. \tag{7.3}$$

Clearly, we have

$$i_n \leq t\|x_n - w\| + \|z + t(w - z)\| \quad \text{for all} \quad t \geq 0.$$

Therefore, for fixed $\bar{t} \geq 0$ such that $\theta = z + \bar{t}(w - z)$, we can deduce that the sequence $\{i_n\}_n$ converges to θ by the inequality:

$$0 \leq i_n \leq \bar{t}\|x_n - w\| \leq \frac{\bar{t}}{n} \quad \text{for all} \quad n \in \mathbb{N}.$$

Hence there exists $\overline{n} \in \mathbb{N}$ such that $\frac{1}{\overline{n}} < \rho^*$ and $i_{\overline{n}} < r$. Therefore, by (7.3) we can find $t_{\overline{n}} \geq 0$ such that the point $y_{\overline{n}} = z + t_{\overline{n}}(x_{\overline{n}} - z) \in B_r(\theta)$. If $t_{\overline{n}} \leq 1$ we deduce

$$\|z - w\| + \rho = \|z\| \leq \|y_{\overline{n}}\| + \|z - y_{\overline{n}}\| < r + t_{\overline{n}}\|x_{\overline{n}} - z\| \leq r + \|x_{\overline{n}} - w\| + \|w - z\|,$$

and so we get the contradiction $\rho < r + \|x_{\overline{n}} - w\| < r + \rho^* = \rho$.

Conversely, suppose $t_{\overline{n}} > 1$, and the convexity of $\overline{\Omega}$ implies

$$x_{\overline{n}} = \frac{y_{\overline{n}}}{t_{\overline{n}}} + (1 - \frac{1}{t_{\overline{n}}})z \in \overline{\Omega},$$

which is contrary to (7.2). Therefore, we can conclude that the segment $[\theta, z[$ is included in Y and then $z^* \in Y$ again. Consequently, (7.1) is true.

Next, define the multivalued map $G : C \longrightarrow \mathcal{P}(C)$ by $G(x) = (1-\delta)F(x)$ for all $x \in C$. Then G is a weakly closed multivalued with nonempty, convex, and weakly compact values. But, with respect to the weak topology, X is a Hausdorff locally convex topological vector space, so from the Fan-Glicksberg theorem (see Theorem 1.38) G has at least one fixed point on C. So there is an $x^* \in G(x^*) = (1-\delta) \times F(x^*)$. Then there is a $z \in F(x^*)$ such that $x^* = (1-\delta)z$, so $\|z - x^*\| = \delta\|z\| \leq \delta\alpha \leq \varepsilon$. Hence x^* is an ε-fixed point of F. ■

Theorem 7.2. *Let X be a reflexive and separable real Banach space and let Ω be a bounded and convex subset of X with nonempty interior. Assume that $F : \Omega \longrightarrow \mathcal{P}_{cv}(\Omega)$ is a weakly upper semicontinuous multivalued map (that is, a multivalued upper semicontinuous with respect to the weak topology). Then $FIX^{\varepsilon}(F) \neq \emptyset$ for each $\varepsilon > 0$.*

Proof. As in the proof of Theorem 7.1, we assume that $\theta \in \text{int}\ \Omega$ and $\alpha = \sup\{\|x\|, x \in \Omega\}$. Take $\varepsilon > 0$ and $0 < \delta < 1$ such that $\delta\alpha \leq \frac{\varepsilon}{2}$ and $C = (1-\delta)\overline{\Omega}$. Define the multivalued map $G : C \longrightarrow \mathcal{P}(C)$ by $G(x) = (1-\delta)F(x)$ for all $x \in C$. Now G is weakly upper semicontinuous. In fact, since X is a separable real Banach space and Ω is bounded, there exists a metric d_w on X such that the weak topology on Ω is induced by the metric d_w (see Theorem 1.14). Let $x \in C$ and assume that U is a weak neighborhood of $G(x)$. For $\sigma > 0$, we denote with U_σ the open set $\{y \in C : d_w(y, G(x)) < \sigma\}$. Since $G(x)$ is weakly compact, we have that $d_w(C \setminus U, G(x)) = \inf\{d_w(y,z) : y \in C \setminus U, z \in G(x)\} > 0$, where $C \setminus U = \{y \in C : y \notin U\}$. So, if $0 < \sigma' < \sigma < d_w(C \setminus U, G(x))$ we have $G(x) \subset U_{\sigma'} \subset \{y \in C : d_w(y, G(x)) \leq \sigma'\} \subset U_\sigma \subset U$. In view of the weak upper semicontinuity of the multivalued map $(1-\delta)F$, there exists an open neighborhood V of x such that $(1-\delta)F(z) \subset U_{\sigma'}$ for all $z \in V$. Therefore $G(z) = (1-\delta)\overline{F(z)} \subseteq \{y \in C : d_w(y, G(x)) \leq \sigma'\} \subset U$ for all $z \in V$. So G is a weakly upper semicontinuous multivalued map at x. In light of Theorem 1.36, G is also a weakly closed multivalued map at x. Therefore, from the Fan-Glicksberg theorem (see Theorem 1.38), there exists a point $x^* \in C$ such that $x^* \in G(x^*)$. Hence, there exists $z \in \overline{F(x^*)}$ such that $x^* = (1-\delta)z$, so $\|z - x^*\| = \delta\|z\| \leq \delta\alpha \leq \frac{\varepsilon}{2}$. Moreover, there is $z' \in F(x^*)$ such that $\|z' - z\| < \frac{\varepsilon}{2}$. Hence $\|z' - x^*\| < \varepsilon$, that is, $x^* \in FIX^{\varepsilon}(F)$. ■

In the next theorem the strong topology is involved

Theorem 7.3. *Let X be a real Banach space and let Ω be a convex and totally bounded subset of X with nonempty interior. Assume that $F : \Omega \longrightarrow \mathcal{P}_{cv}(\Omega)$ is a closed or upper semicontinuous multivalued map. Then $FIX^{\varepsilon}(F) \neq \emptyset$ for each $\varepsilon > 0$.*

Proof. Assume without loss of generality that $\theta \in \text{int}\ \Omega$. Take $\varepsilon > 0$ and $\eta > 0$. Since Ω is totally bounded there exists $m \in \mathbb{N}$ and $x_1, \ldots, x_m \in X$ such that $\Omega \subseteq \bigcup_{i=1}^{m} B_\eta(x_i)$, where $B_\eta(x_i) = \{y \in X : \|y - x_i\| < \eta\}$. Moreover, let $h = \max\{\|x_i\| : i \in \{1,\ldots,m\}\}$. If $0 < \delta < 1$ the set $C = (1-\delta)\overline{\Omega}$ is a nonempty, convex, and totally bounded subset of X. Since C is also closed, C is complete and therefore compact.

First, we assume that F is a closed multivalued map and we take $0 < \delta < 1$ such that $\delta(\eta + h) \leq \varepsilon$. Then the multivalued map $G : C \longrightarrow \mathcal{P}(C)$ by $G(x) = (1-\delta)F(x)$ for all $x \in C$, is closed. This implies from the Fan-Glicksberg theorem (see Theorem 1.38) that G possesses a fixed point x^*. Then there is a point $z \in F(x^*)$ such that $x^* = (1-\delta)z$. Since $\Omega \subseteq \bigcup_{i=1}^{m} B_\eta(x_i)$, there exists an $r \in \{1,\ldots,m\}$ such that $z \in B_\eta(x_r)$. So, $\|z - x^*\| = \delta\|z\| \leq \delta(\|z - x_r\| + \|x_r\|) < \delta(\eta + h) \leq \varepsilon$. Hence $x^* \in FIX^\varepsilon(F)$.

Assume now that F is an upper semicontinuous multivalued map. We take $0 < \delta < 1$ such that $\delta(\eta + h) \leq \frac{\varepsilon}{2}$. Let $G : C \longrightarrow \mathcal{P}(C)$, be defined by $G(x) = (1-\delta)\overline{F(x)}$ for all $z \in C$. We claim that G is upper semicontinuous. Let $x \in C$ and assume that U is an open neighborhood of $G(x)$. For each $\sigma > 0$, we denote with U_σ the open set $\{y \in C : \inf_{z \in G(x)} \|z - y\| < \sigma\}$. As in the proof of Theorem 7.2, we obtain that G is an upper semicontinuous multivalued map at x and is also a closed multivalued map at x. From the Fan-Glicksberg theorem (see Theorem 1.38), there exists a point $x^* \in C$ such that $x^* \in G(x^*)$ and $z \in \overline{F(x^*)}$ such that $x^* = (1-\delta)z$. Since $\Omega \subseteq \bigcup_{i=1}^{m} B_\eta(x_i)$, there exists $s \in \{1,\ldots,m\}$ such that $z \in B_\eta(x_s)$, so $\|z - x^*\| = \delta\|z\| \leq \delta(\|z - x_s\| + \|x_s\|) < \delta(\eta + h) \leq \frac{\varepsilon}{2}$. Moreover, there exists a point $z' \in F(x^*)$ such that $\|z' - z\| < \frac{\varepsilon}{2}$, so $\|z' - x^*\| < \varepsilon$, that is, $x^* \in FIX^\varepsilon(F)$. ■

Definition 7.4. Let X be a normed space and $\Omega \subseteq X$ with $\theta \in \Omega$. A multivalued map $F : \Omega \longrightarrow \mathcal{P}(\Omega)$ is called a tame multivalued map if, for each $\varepsilon > 0$, there is an $R > 0$ such that for each $x \in B_R(\theta) \cap \Omega$ the set $F(x) \cap B_{R+\varepsilon}(\theta)$ is nonempty, where $B_R(\theta) = \{x \in X : \|x\| \leq R\}$.

Example 7.1. The map $F : [0,\infty[\longrightarrow \mathcal{P}([0,\infty[)$ defined by

$$F(x) = [x + (x+1)^{-1}, \infty[\text{, for all } x \in [0,\infty[,$$

is a tame multivalued map on the unbounded set $[0,\infty[$.

Example 7.2. Let X be a normed space. Let $F : X \longrightarrow \mathcal{P}(X)$ be a multivalued mapping such that the image $F(X) = \{x \in X : x \in F(x) \text{ for some } x \in X\}$ of F is a bounded set. Then F is a tame multivalued map (for each $\varepsilon > 0$, take $R = 1 + \sup\{\|y\|,\ y \in F(X)\}$).

Remark 7.2. It follows from Example 7.2 that each $F : \Omega \longrightarrow \mathcal{P}(\Omega)$, where Ω is a bounded subset of a normed space X and $F(x)$ is nonempty for all $x \in \Omega$, is a tame multivalued map.

Example 7.3. Let X be a normed space. The translation $T : X \longrightarrow \mathcal{P}(X)$ given by $T(x) = x + a$, where $a \in X \setminus \{\theta\}$, is not tame and for all small $\varepsilon > 0$, T has no ε-fixed points.

Theorem 7.4. *Let Ω be a convex subset with nonempty interior, containing θ, of a reflexive real Banach space. Assume that $F : \Omega \longrightarrow \mathcal{P}_{cv}(\Omega)$ is a tame and weakly closed multivalued map. Then $FIX^{\varepsilon}(F) \neq \emptyset$ for each $\varepsilon > 0$.*

Proof. Let $\varepsilon > 0$ and $R > 0$ such that $F(x) \cap B_{R+\frac{\varepsilon}{2}}(\theta) \neq \emptyset$ for each $x \in B_R(\theta) \cap \Omega$, and let $C = B_R(\theta) \cap \Omega$. Now Ω is a nonempty, bounded, and convex set. Then $G : C \longrightarrow \mathcal{P}(C)$ defined by

$$G(x) = R(R + \frac{\varepsilon}{2})^{-1} F(x) \cap B_{R+\frac{\varepsilon}{2}}(\theta) \quad \text{for all } x \in C,$$

satisfies the conditions of Theorem 7.1. Hence there is $x^* \in FIX^{\frac{\varepsilon}{4}}(G)$ such that $d(x^*, G(x^*)) \leq \frac{\varepsilon}{4} < \frac{\varepsilon}{2}$ and there exists $x' \in G(x^*)$ such that $\|x' - x^*\| < \frac{\varepsilon}{2}$. Moreover, there exists an element $z \in F(x^*)$ such that $z = R^{-1}(R + \frac{\varepsilon}{2})x'$. This implies that

$$\|z - x^*\| \leq \|R^{-1}(R + \frac{\varepsilon}{2})x' - x'\| + \|x' - x^*\| < \frac{\varepsilon}{2} R^{-1} \|x'\| + \frac{\varepsilon}{2} \leq \varepsilon.$$

So $x^* \in FIX^{\varepsilon}(F)$. ∎

Theorem 7.5. *Let Ω be a convex subset with nonempty interior, containing θ, of a reflexive and separable real Banach space. Assume that $F : \Omega \longrightarrow \mathcal{P}_{cv}(\Omega)$ is a tame and weakly upper semicontinuous multivalued map. Then $FIX^{\varepsilon}(F) \neq \emptyset$ for each $\varepsilon > 0$.*

Proof. Using the same argument as in the proof of Theorem 7.4, we can show that the multivalued map G, defined on $B_R(\theta) \cap \Omega$ by

$$G(x) = R(R + \frac{\varepsilon}{2})^{-1} F(x) \cap B_{R+\frac{\varepsilon}{2}}(\theta),$$

satisfies the conditions of Theorem 7.2 and the conclusion follows as in Theorem 7.4. ∎

Remark 7.3. By Theorem 7.4, the multivalued map $F : [0, \infty[\longrightarrow \mathcal{P}([0, \infty[)$ defined by

$$F(x) = [x + (x + 1)^{-1}, \infty[, \quad \text{for all } \quad x \in [0, \infty[,$$

has ε-fixed points.

Theorem 7.6. *Let X be a reflexive real Banach space, let Ω be a nonempty convex subset of X and $\varepsilon > 0$. Assume $F : \Omega \longrightarrow \mathcal{P}_{cv}(\Omega)$ is a multivalued map satisfying the following properties:*

(*i*) *coF*(Ω) *has nonempty interior,*
(*ii*) $F(\Omega)$ *is bounded in X,*
(*iii*) *there exists* $\beta_\varepsilon \in]0,1[, \alpha\beta_\varepsilon \leq \varepsilon$, *such that F is* β_ε*-w-partially closed, where* $\alpha = \sup_{x \in F(\Omega)} \|x\|$.

Then, there exists $x_\varepsilon \in coF(\Omega) \cap FIX^\varepsilon(F)$.

Proof. From (*i*) and (*ii*) we have that the convex set $C = \mathrm{co}F(\Omega)$ is also bounded and has nonempty interior. Suppose without loss of generality that $\theta \in \mathrm{int}\, C$. Put

$$C_\varepsilon = (1-\beta_\varepsilon)\overline{C}, \tag{7.4}$$

where β_ε is the positive number introduced in (*iii*) and $\overline{C}$ is the closure of C in the Banach space X. Using the same argument in the proof of Theorem 7.4 we prove the following inclusion

$$C_\varepsilon \subset C. \tag{7.5}$$

Next, consider the map $G_\varepsilon : C_\varepsilon \longrightarrow \mathcal{P}_{\mathrm{cv}}(X)$ defined by

$$G_\varepsilon(x) = (1-\beta_\varepsilon)F(x), \quad x \in C_\varepsilon. \tag{7.6}$$

From (7.4) and taking into account that the values of F are convex, this multivalued map assumes values in the family $\mathcal{P}_{\mathrm{cv}}(C_\varepsilon)$. The closed and convex set C_ε in the Banach space X is also closed in the Hausdorff locally convex topological linear space X endowed with its weak topology. The reflexivity of the Banach space X guarantees that the set C_ε is weakly compact, since C_ε is normed bounded and weakly closed (see Theorem 1.3). Moreover, the multivalued map G_ε is w-partially closed. Indeed, for a fixed net $\{x_\delta\}_{\delta\in\Delta}, x_\delta \in C_\varepsilon, x_\delta \rightharpoonup x, x \in C_\varepsilon$, and fixed $\{y_\delta\}_{\delta\in\Delta}, y_\delta \in G_\varepsilon(x_\delta), y_\delta \rightharpoonup y$, we can prove that $G_\varepsilon(x) \cap L(x,y) \neq \emptyset$. Notice for every $\delta \in \Delta$, there exists $z_\delta \in F(x_\delta)$ such that $y_\delta = (1-\beta_\varepsilon)z_\delta$. The weak convergence of the net $(y_\delta)_{\delta\in\Delta}$ implies that, for every $W \in W(\theta)$ (where $W(\theta)$ is the family of all neighborhoods of θ in the weak topology) we have that there exists $\delta^* \in \Delta$ such that $y_\delta \in y + (1-\beta_\varepsilon)W$ for all $\delta \in \Delta, \delta^* \preceq \delta$, and so the net $\{z_\delta\}_{\delta\in\Delta}$,

$$z_\delta = \frac{y_\delta}{1-\beta_\varepsilon} \in \frac{y}{1-\beta_\varepsilon} + W \quad \text{for all} \quad \delta \in \Delta, \delta^* \preceq \delta,$$

converges weakly to the point $z = \frac{y}{1-\beta_\varepsilon}$. Thus, by (*iii*) we conclude that

$$F(x) \cap L\left(\frac{x}{1-\beta_\varepsilon}, z\right) \neq \emptyset,$$

so there exists a point $v \in F(x)$ and a number $\overline{\lambda} \geq 0$ characterized by

$$v = \frac{x}{1-\beta_\varepsilon} + \overline{\lambda}\left(z - \frac{x}{1-\beta_\varepsilon}\right).$$

By (7.6) we can write

$$(1-\beta_\varepsilon)v = x + \overline{\lambda}(y-x) \in G_\varepsilon(x) \cap L(x,y).$$

Since the map G_ε on the topological linear space X endowed with its weak topology satisfies all the conditions of the Glebov theorem [94], there exists a point $x^\varepsilon \in C_\varepsilon$ such that $x^\varepsilon \in G_\varepsilon(x^\varepsilon) = (1-\beta_\varepsilon)F(x^\varepsilon)$, and hence there exists a point $y^\varepsilon \in F(x^\varepsilon), x^\varepsilon = (1-\beta_\varepsilon)y^\varepsilon$. From (iii) we have

$$d(x^\varepsilon, F(x^\varepsilon)) \leq \|x^\varepsilon - y^\varepsilon\| = \|(1-\beta_\varepsilon)y^\varepsilon - y^\varepsilon\| = \beta_\varepsilon\|y^\varepsilon\| \leq \alpha\beta_\varepsilon \leq \varepsilon,$$

i.e., $x^\varepsilon \in \Omega$ is a ε-fixed point for F. Moreover, by (7.5) we also have $x^\varepsilon \in \mathrm{co}F(\Omega)$. Therefore, the desired result is established. ■

Corollary 7.1. *Let X be a reflexive real Banach space, let Ω be a nonempty convex subset of X and $\varepsilon > 0$. Assume $F : \Omega \longrightarrow \mathcal{P}_{\mathrm{cv}}(\Omega)$ is a multivalued map satisfying the following properties:*

(i) *$coF(\Omega)$ has nonempty interior,*
(ii) *$F(\Omega)$ is bounded in X,*
(iii) *for every $\varepsilon > 0$, there exists $\beta_\varepsilon \in]0,1[, \alpha\beta_\varepsilon \leq \varepsilon$, such that F is β_ε-w-partially closed, where $\alpha = \sup_{x\in F(\Omega)} \|x\|$.*

Then, for every $\varepsilon > 0$, there exists $x_\varepsilon \in coF(\Omega) \cap FIX^\varepsilon(F)$.

Example 7.4. Put $X = \mathbb{R}$ and $\Omega =]-2,2[$. Let $F : \Omega \longrightarrow \mathcal{P}(X)$ be the map defined by

$$F(x) = \begin{cases} \left[-\frac{x}{2}, 0\right], & \text{if } x \in [0,2[, \\ \left]0, -\frac{x}{2}\right], & \text{if } x \in]-2,0[, \\ \{0\}, & \text{if } \quad x = 2. \end{cases}$$

It is easy to see that F satisfies the hypotheses (i) and (ii) of Corollary 7.1. Moreover, for fixed $\varepsilon > 0$ assuming that $\beta_\varepsilon \in]0, \min\{1,\varepsilon\}[$ we see that F verifies (iii) of Corollary 7.1. On the other hand, F is not weakly closed, required in Theorem 7.1. To this end it is sufficient to note that the sequence $\{x_n\}_n, x_n = 2 - \frac{1}{n}, x_n \rightharpoonup x = 2$ and the sequence $\{z_n\}_n, z_n = -1 + \frac{1}{n} \in F(x_n), z_n \rightharpoonup z = -1$, but we have $z = -1 \notin F(2) = \{0\}$.

Next by using Corollary 7.1 and by assuming the values of F are closed, we can remove the separability assumption on the reflexive Banach space X, required in Theorem 7.2.

Proposition 7.1. *Let X be a reflexive real Banach space, let Ω be a nonempty convex subset of Ω and $\varepsilon > 0$. Assume $F : \Omega \longrightarrow \mathcal{P}_{\text{cl,cv}}(\Omega)$ is a multivalued map satisfying the following properties:*

(i) *co$F(\Omega)$ has nonempty interior,*
(ii) *$F(\Omega)$ is bounded in X,*
(iii) *F is weakly upper semicontinuous.*

Then, for every $\varepsilon > 0$, there exists $x_\varepsilon \in$ co$F(\Omega) \cap FIX^\varepsilon(F)$.

Proof. Consider the Hausdorff locally convex topological linear space X endowed with its weak topology. Since F has closed and convex values in X, we can say that its values are weakly closed. By (ii) they are also weakly compact. By Theorem 1.36 and (iii), we conclude that the graph of F is weakly closed. Next we show that F satisfies hypotheses (iii) of Corollary 7.1. Put $\alpha = \sup\limits_{x\in F(\Omega)} \|x\|$ and fix $\varepsilon > 0$, and let $\beta_\varepsilon \in]0,1[$ be such that $\alpha\beta_\varepsilon \leq \varepsilon$. Fix a net $\{x_\delta\}_{\delta\in\Delta}, x_\delta \in \Omega, x_\delta \rightharpoonup x, x \in \Omega$. For every net $\{y_\delta\}_{\delta\in\Delta}, y_\delta \in F(x_\delta), y_\delta \rightharpoonup y$, by the weak closedness of the graph of F, we deduce $y \in F(x)$ and so $F(x) \cap L\left(\frac{x}{1-\beta_\varepsilon}, x\right) \neq \emptyset$ holds. Finally we are in a position to apply Corollary 7.1 to the map F. Hence, for every $\varepsilon > 0$, there exists in co$F(\Omega)$ an ε-fixed point for F. ■

Theorem 7.7. *Let X be a real Banach space, let Ω be a nonempty convex subset of X and $\varepsilon > 0$. Assume $F : \Omega \longrightarrow \mathcal{P}_{\text{cv}}(\Omega)$ is a multivalued map satisfying the following properties:*

(i) *co$F(\Omega)$ has nonempty interior,*
(ii) *$F(\Omega)$ is totally bounded in X,*
(iii) *there exists $\beta_\varepsilon \in]0,1[, \beta_\varepsilon(\varepsilon + h_\varepsilon) \leq \varepsilon$, such that F is β_ε-w-partially closed, where $h_\varepsilon = \max\{\|x_i\| : i \in \{1,\ldots,p_\varepsilon\}\}$ for $x_1,\ldots,x_{p_\varepsilon} \in X$ such that $F(\Omega) \subset \bigcup\limits_{i\in\{1,\ldots,p_\varepsilon\}} B_\varepsilon(x_i)$.*

Then, there exists $x_\varepsilon \in$ co$F(\Omega) \cap FIX^\varepsilon(F)$.

Proof. Repeating the argument in the proof of Theorem 7.6 we can see that the convex set $C =$co$F(\Omega)$ satisfies the following properties

$$\theta \in \text{int}C, \quad C_\varepsilon = (1-\beta_\varepsilon)\overline{C} \subset C, \tag{7.7}$$

where β_ε is the positive number introduced in (iii). Clearly, the set C_ε is nonempty and convex. By (ii) we deduce that co$F(\Omega)$ is a totally bounded subset of X, so C_ε is totally bounded too. Since C_ε is also closed, we have that the metric space C_ε is complete with the metric induced by the norm in X. Therefore, it is also compact in the Banach space X.

As in the proof of Theorem 7.6, we consider the multivalued map $G_\varepsilon : C_\varepsilon \longrightarrow \mathcal{P}_{\text{cv}}(X)$ defined in (7.6). The multivalued map G_ε is partially closed. Hence, it satisfies in the Banach space X all the conditions of the Glebov Theorem [94]. Therefore, there exists a point $x^\varepsilon \in C_\varepsilon$ such that $x^\varepsilon \in G_\varepsilon(x^\varepsilon) = (1-\beta_\varepsilon)F(x^\varepsilon)$. It follows that the point $y^\varepsilon \in F(x^\varepsilon) \subset C$ such that $x^\varepsilon = (1-\beta_\varepsilon)y^\varepsilon$ satisfies

$$d(x^\varepsilon, F(x^\varepsilon)) \leq \|x^\varepsilon - y^\varepsilon\| \leq \beta_\varepsilon\|y^\varepsilon\| \leq \beta_\varepsilon(\|y^\varepsilon - x_k\| + \|x_k\|) \leq \beta_\varepsilon(\varepsilon + h_\varepsilon) \leq \varepsilon,$$

where $k \in \{1, \ldots, p_\varepsilon\}$ is such that $y^\varepsilon \in B_\varepsilon(x_k)$ (see (iii)). Moreover, (7.7) implies $x^\varepsilon \in \text{co}F(\Omega)$. Therefore the set $\text{co}F(\Omega) \cap FIX^\varepsilon(F)$ is nonempty. ■

By using the previous theorem we deduce the following proposition.

Proposition 7.2. *Let X be a real Banach space and let Ω be a nonempty convex subset of X. Assume $F : \Omega \longrightarrow \mathcal{P}_{\text{cv}}(\Omega)$ is a multivalued map satisfying the following properties:*

(i) *$coF(\Omega)$ has nonempty interior,*
(ii) *$F(\Omega)$ is totally bounded in X,*
(iii) *for every $\varepsilon > 0$, there exists $\beta_\varepsilon \in]0,1[, \beta_\varepsilon(\varepsilon + h_\varepsilon) \leq \varepsilon$, such that F is β_ε-w-partially closed, where $h_\varepsilon = \max\{\|x_i\| : i \in \{1, \ldots, p_\varepsilon\}\}$ for $x_1, \ldots, x_{p_\varepsilon} \in X$ such that $F(\Omega) \subset \bigcup_{i\in\{1,\ldots,p_\varepsilon\}} B_\varepsilon(x_i)$.*

Then for every $\varepsilon > 0$, there exists $x_\varepsilon \in coF(\Omega) \cap FIX^\varepsilon(F)$.

Theorem 7.8. *Let X be a reflexive real Banach space and let Ω be a nonempty closed and convex subset of X. Assume $F : \Omega \longrightarrow \mathcal{P}_{\text{cv}}(\Omega)$ is a multivalued map satisfying the following properties:*

(i) *$coF(\Omega)$ has nonempty interior,*
(ii) *$F(\Omega)$ is bounded in X,*
(iii) *there exists $\beta_\varepsilon \in]0,1[, \alpha\beta_\varepsilon \leq \varepsilon$, such that F is β_ε-w-partially closed, where $\alpha = \sup_{x\in F(\Omega)} \|x\|$,*
(iv) *the multivalued map $I - F$ is weakly demiclosed on $\mathcal{WF}(\Omega)$.*

Then, $FIX(F)$ is nonempty.

Proof. Set $\varepsilon = \frac{1}{n}, n \in \mathbb{N}$, and by Corollary 7.1, we can choose $x_n \in \text{co}F(\Omega) \subset \Omega$ so that $d(x_n, F(x_n)) < \frac{1}{n}$. In view of (ii) we have that the approximate fixed points sequence $\{x_n\}_n$ is bounded in the reflexive Banach space X. Therefore, there exists a subsequence $\{x_{n_k}\}_{n_k}$ of $\{x_n\}_n$ weakly converging to a point $\bar{x} \in \overline{co}F(\Omega)$. Since Ω is closed and convex, $\bar{x} \in \Omega$. The subsequence $\{x_{n_k}\}_{n_k}$ converges weakly to $\bar{x}$ and satisfies $\lim_{k\longrightarrow\infty} d(x_{n_k}, F(x_{n_k})) = 0$, and therefore, $\bar{x} \in \mathcal{WF}(\Omega)$. From (iv), $\bar{x} \in F(\bar{x})$. ■

The following example shows that the closedness of Ω in the above fixed point theorem cannot be removed.

Example 7.5. Let $\Omega =]0,1[\subset X = \mathbb{R}$ and let $F : \Omega \longrightarrow \mathcal{P}_{cv}(\Omega)$ be the multivalued map defined by

$$F(x) = \left\{ \frac{x}{x+1} \right\}, \quad x \in \Omega.$$

Note that the set Ω is not closed. Moreover, since F satisfies hypotheses (i), (ii), and (iii) of Corollary 7.1, there exists an approximate fixed points sequence $\{u_n\}_n$ having the property $\lim_{n\longrightarrow\infty} d(u_n, F(u_n)) = 0$. On the other hand, it easy to see that for every $x \in]0,1[$, any sequence $\{x_n\}_n$ converging to x does not satisfy $\lim_{n\longrightarrow\infty} d(x_n, F(x_n)) = 0$, so the set $\mathcal{WF}(\Omega)$ is empty.

Theorem 7.9. *Let X be a reflexive real Banach space and let Ω be a nonempty closed and convex subset of X. Assume $F : \Omega \longrightarrow \mathcal{P}_{cv}(\Omega)$ is a multivalued map satisfying the following properties:*

- (i) *co$F(\Omega)$ has nonempty interior,*
- (ii) *$F(\Omega)$ is bounded in X,*
- (iii) *for every $\varepsilon > 0$, there exists $\beta_\varepsilon \in]0,1[$, $\beta_\varepsilon(\varepsilon + h_\varepsilon) \leq \varepsilon$, such that F is β_ε-w-partially closed, where $h_\varepsilon = \max\{\|x_i\| : i \in \{1,\ldots,p_\varepsilon\}\}$ for $x_1,\ldots,x_{p_\varepsilon} \in X$ such that $F(\Omega) \subset \bigcup_{i\in\{1,\ldots,p_\varepsilon\}} B_\varepsilon(x_i)$,*
- (iv) *the multivalued map $I - F$ is weakly demiclosed on $\mathcal{WF}(\Omega)$.*

Then, FIX(F) is nonempty.

Proof. Set $\varepsilon = \frac{1}{n}$, $n \in \mathbb{N}$, and by Proposition 7.2, we can choose $x_n \in \text{co}F(\Omega) \subset \Omega$ so that $d(x_n, F(x_n)) < \frac{1}{n}$. Hence $\lim_{n\longrightarrow\infty} d(x_n, F(x_n)) = 0$. Since by (ii) we have that the set $\overline{F(\Omega)}$ is compact in X, so the Krein–Šmulian theorem (see Theorem 1.10) implies that the set $\overline{\text{co}F(\Omega)}$ is weakly compact. Now from the Eberlein–Šmulian theorem there exists a subsequence $\{x_{n_k}\}_{n_k}$ of $\{x_n\}_n$ weakly converging to a point $\bar{x} \in \overline{\text{co}F(\Omega)} \subset \Omega$. Therefore, $\bar{x} \in \mathcal{WF}(\Omega)$ and the weak demiclosedness of $I - F$ implies that $\bar{x} \in F(\bar{x})$. ■

7.2 Approximate Nash Equilibria for Strategic Games

In [155], Nash equilibria for n-person noncooperative games was introduced and using Kakutani's fixed point theorem (see Theorem 1.37) he showed that mixed extensions of finite n-person noncooperative games possess at least one Nash equilibrium. The aggregate best response multivalued mapping on the Cartesian product of the strategy spaces constructed with the aid of the best response multivalued mappings for each player possesses fixed points which coincide with the Nash equilibria of the game (see [67, 95, 155]). The existence of such equilibria usually requires a compactness condition on the strategy set of each player.

Of course, for many noncooperative games Nash equilibria do not exist. However some games are of interest when ε-Nash equilibria exist for each $\varepsilon > 0$. Here a strategy profile is called an ε-Nash equilibrium if the unilateral deviation of one of the players does not increase his/her payoff with more than ε. One can derive the existence of approximate equilibrium points using the following:

(*i*) develop ε-fixed point theorems and find conditions on strategy spaces and payoff functions of the game such that the aggregate ε-best response multivalued mappings satisfies conditions in an ε-fixed point theorem,
(*ii*) add extra conditions on the payoff-functions, guaranteeing that points in the Cartesian product of the strategy spaces nearby each other have payoffs sufficiently nearby.

In [40] the notion of ε-Nash equilibrium is introduced:

Definition 7.5. An n-person strategic game is a tuple $\Gamma = \langle \Omega_1, \ldots, \Omega_n, u_1, \ldots, u_n \rangle$ where for each player $i \in N = \{1, \ldots, n\}$ the Ω_i is the set of strategies and $u_i : \prod_{i \in N} \Omega_i \longrightarrow \mathbb{R}$ is the payoff function. If players $1, \ldots, n$ choose strategies $x_1, \ldots, x_n$, so that the functions $u(x_1, \ldots, x_n), \ldots, u_n(x_1, \ldots, x_n)$ are the resulting payoffs for the players $1, \ldots, n$ respectively. Let $\varepsilon > 0$. Then we say that $(x_i^*)_{i \in N} \in \prod_{i \in N} \Omega_i$ is an ε-Nash equilibrium if

$$u_i(x_i, x_{-i}^*) \leq u_i(x^*) + \varepsilon \ \text{ for all } \ x_i \in \Omega_i \ \text{ and for all } \quad i \in N.$$

Here x_{-i}^* is a shorthand for $(x_j^*)_{j \in N \setminus \{i\}}$ and we will denote by $NE^{\varepsilon}(\Gamma)$ the set of ε-Nash equilibria for the game Γ.

Note that for an $x^* \in NE^{\varepsilon}(\Gamma)$, a unilateral deviation by a player does not improve the payoff with more than ε. For each $i \in N$ we consider the ε-best response multivalued mapping $B_i^{\varepsilon} : \prod_{j \in N \setminus \{i\}} \Omega_j \longrightarrow \mathcal{P}(\Omega_i)$ defined by

$$B_i^{\varepsilon}(x_{-i}) = \left\{ x_i \in \Omega_i \,|\, u_i(x_i, x_{-i}) \geq \sup_{t_i \in \Omega_i} u_i(t_i, x_{-i}) - \varepsilon \right\}$$

and the aggregate ε-best response multivalued map $B_i^{\varepsilon} : \Omega \longrightarrow \Omega$ defined by

$$B^{\varepsilon}(x) = \prod_{i \in N} B_i^{\varepsilon}(x_{-i}).$$

Obviously, if $x^* \in B^{\varepsilon}(x^*)$, then $x^* \in NE^{\varepsilon}(\Gamma)$, and conversely. So if B^{ε} has a fixed point, then we have an ε-Nash equilibrium. If we do not know whether B^{ε} has a fixed point but we know that B^ε has δ-fixed points for each $\delta > 0$, then this leads under extra continuity conditions to the existence of approximate Nash equilibria for the game as we will see.

Proposition 7.3 (The Key Proposition). *Let* $\Gamma = \langle \Omega_1, \ldots, \Omega_n, u_1, \ldots, u_n \rangle$ *be an n-person strategic game with the following three properties:*

(i) *for each* $i \in N = \{1, \ldots, n\}$, Ω_i *is endowed with a metric* d_i,

(ii) *the payoff functions* $u_1, \ldots, u_n$ *are uniform continuous functions on* $\Omega = \prod_{i=1}^{n} \Omega_i$, *where* Ω *is endowed with the metric d defined by*

$$d(x, y) = \sum_{i=1}^{n} d_i(x_i, y_i) \quad \textit{for all } x, y \in \Omega,$$

(iii) *for each* $\varepsilon > 0$ *and* $\delta > 0$, *the aggregate* ε*-best response multivalued map* B^ε *possesses at least one* δ*-fixed point, i.e.,* $FIX^\delta(B^\varepsilon) \neq \emptyset$. *Then,* $NE^\varepsilon(\Gamma) \neq \emptyset$ *for each* $\varepsilon > 0$.

Proof. Take $\varepsilon > 0$. By (ii) we can find $\eta > 0$ such that for all $x, x' \in \Omega$ with $d(x, x') < \eta$ we have $|u_i(x) - u_i(x')| < \frac{\varepsilon}{2}$ for each $i \in N$. We will prove that

$$x^* \in FIX^{\frac{\eta}{2}}(B^{\frac{\varepsilon}{2}}) \Longrightarrow x^* \in NE^\varepsilon(\Gamma).$$

Take $x^* \in FIX^{\frac{\eta}{2}}(B^{\frac{\varepsilon}{2}})$, which is possible by ($iii$). Then there exists $\hat{x} \in B^{\frac{\varepsilon}{2}}(x^*)$ such that $d(x^*, \hat{x}) < \eta$, and, consequently, for each $i \in N$, $d((x_i^*, x_{-i}^*), (\hat{x}_i, x_{-i}^*)) < \eta$. This implies that

$$u_i(x_i^*, x_{-i}^*) \geq u_i(\hat{x}_i, x_{-i}^*) - \frac{1}{2}\varepsilon \quad \text{for all} \quad i \in N. \tag{7.8}$$

Furthermore $\hat{x} \in B^{\frac{\varepsilon}{2}}(x^*)$ implies

$$u_i(\hat{x}_i, x_{-i}^*) \geq \sup_{t_i \in \Omega_i} u_i(t_i, x_{-i}^*) - \frac{1}{2}\varepsilon \quad \text{for all} \quad i \in N \tag{7.9}$$

Combining (7.8) and (7.9) we obtain

$$u_i(x_i^*, x_{-i}^*) \geq \sup_{t_i \in \Omega_i} u_i(t_i, x_{-i}^*) - \varepsilon \quad \text{for all} \quad i \in N, \tag{7.10}$$

that is, $x^* \in NE^\varepsilon(\Gamma)$. ∎

Next we give some examples to show that with the use of the key proposition, we can obtain approximate Nash equilibrium theorems.

Example 7.6 (Games on the Open Unit Square). Let $\langle]0,1[,]0,1[, u_1, u_2\rangle$ be a game with uniform continuous payoff functions u_1 and u_2. Suppose that u_1 is concave in the first coordinate and u_2 is concave in the second coordinate. Then for each $\varepsilon > 0$, the game has an ε-Nash equilibrium point. In fact, apply the key proposition to the above game and note that (i) and (ii) are satisfied by taking the standard metric on $]0,1[$. Further, (iii) follows from Theorem 7.1 applied to the multivalued map B^ε.

Example 7.7 (Completely Mixed Approximate Nash Equilibria for Finite Games). Let A and B be $(m \times n)$-matrices of real numbers. Consider the two-person game $\langle \Delta_m, \Delta_n, u_1, u_2\rangle$, where

$$\Delta_m = \left\{ p \in \mathbb{R}^m \mid p_i > 0 \ \text{ for each } \ i \in \{1,\dots,m\}, \sum_{i=1}^m p_i = 1 \right\},$$

$$\Delta_n = \left\{ q \in \mathbb{R}^n \mid q_j > 0 \ \text{ for each } j \in \{1,\dots,n\}, \sum_{j=1}^n q_j = 1 \right\},$$

$$u_1(p,q) = p^T A q, \ \ u_2(p,q) = p^T B q \quad \text{for all } p \in \Delta_m, q \in \Delta_n.$$

Then for each $\varepsilon > 0$ this game has an ε-Nash equilibrium. Such an ε-Nash equilibrium is called completely mixed, because both players use each of their pure strategies with positive probability. The proof follows from the key proposition and Theorem 7.1 taking the standard Euclidean metric.

Example 7.8. Let X be a normed linear space such that there exists $a \in X \setminus \{\theta\}$. Let $\Gamma = \langle X, X, u_1, u_2\rangle$ be the two-person game with $u_1(x_1,x_2) = -\|x_1 - x_2\|, u_2(x_1,x_2) = -\|x_1 - x_2 - \frac{a}{1+\|x_1\|}\|$ for all $(x_1,x_2) \in X \times X$. Then $B_1(x_2) = \{x_2\}$ and $B_2(x_1) = \{x_1 - \frac{a}{1+\|x_1\|}\}$. So $B(x_1,x_2) = \{(x_2, x_1 - \frac{a}{1+\|x_1\|})\}$ for each $(x_1,x_2) \in X \times X$. Hence, $FIX(B) = \emptyset$. However, for each $\delta > 0$, $FIX^\delta(B) \neq \emptyset$ since one can take $x \in X$ with $\|x\| \geq \delta^{-1}\|a\|$ and, then, $(x,x) \in FIX^\delta(B)$ because

$$\|(x,x) - \left(x, x - \frac{a}{1+\|x\|}\right)\| = \frac{\|a\|}{1+\|x\|} \leq \frac{\|a\|}{\|x\|} \leq \delta.$$

Moreover, u_1 and u_2 are uniform continuous functions on $X \times X$. In fact,

$$\begin{aligned} |u_2(x_1,x_2) - u_2(y_1,y_2)| &\leq \|(x_1 - y_1) - (x_2 - y_2) + \frac{\|x_1\| - \|y_1\|}{(1+\|x_1\|)(1+\|y_1\|)} a\| \\ &\leq (\|x_1 - y_1\| + \|x_2 - y_2\|)(1 + \|a\|). \end{aligned}$$

Therefore, in light of the key proposition we can conclude that $NE^\varepsilon(\Gamma) \neq \emptyset$ for each $\varepsilon > 0$. In fact, for $\|x\|$ sufficiently large, $(x,x) \in NE^\varepsilon(\Gamma)$, since $u_2(x,x_2) - u_2(x,x) \leq \frac{\|a\|}{1+\|x\|}$.

Bibliography

1. R.P. Agarwal, D. O'Regan, X. Liu, A Leray-Schauder alternatives for weakly strongly sequentially continuous weakly compact maps. Fixed Point Theory Appl. **1**, 1–10 (2005)
2. D.E. Alspach, A fixed point free nonexpansive mapping. Proc. Am. Math. Soc. **82**, 423–424 (1981)
3. R.R. Akhmerov, M.I. Kamenskii, A.S. Potapov, A.E. Rodkina, B.N. Sadovskii, *Measures of Noncompactness and Condensing Operators* (Birkhäuser, Basel, 1992)
4. C.D. Aliprantis, O. Burkinshaw, *Positive Operators* (Academic, Orlando, 1985)
5. M. Altman, A fixed point theorem in Hilbert space. Bull. Acad. Pol. Sci. **5**, 19–22 (1957)
6. C. Angosto, B. Cascales, Measures of weak noncompactness in Banach spaces. Top. Appl. **156**, 1412–1421 (2009)
7. J. Appell, Implicit functions, nonlinear integral equations, and the measure of noncompactness of the superposition operator. J. Math. Anal. Appl. **83**, 251–263 (1981)
8. J. Appell, E. De Pascale, Su alcuni parametri connessi con la misura di non compattezza di Hausdorff in spazi funzioni misurabili. Boll. Un. Mat. Ital. B **6**(3), 497–515 (1984)
9. J. Appell, P.P. Zabrejko, *Nonlinear Superposition Operators*. Cambridge Tracts in Mathematics, vol. 95 (Cambridge University Press, Cambridge, 1990)
10. J. Appell, Measures of noncompactness, condensing operators and fixed points: an application-oriented survey. Fixed Point Theory **6**(2), 157–229 (2005)
11. O. Arino, S. Gautier, J.P. Penot, A fixed point theorem for sequentially continuous mapping with application to ordinary differential equations. Funct. Ekvac. **27**(3), 273–279 (1984)
12. D. Averna, S.A. Marano, Existence of solutions for operator inclusions: a unified approach. Rend. Semin. Mat. Univ. Padova **102** (1999)
13. C. Avramescu, A fixed point theorem for multivalued mappings. Electron. J. Qual. Theory Differ. Equ. **17**, 1–10 (2004)
14. C. Avramescu, C. Vladimirescu, Fixed point theorems of Krasnoselskii type in a space of continuous functions. Fixed Point Theory **5**, 1–11 (2004)
15. J.M. Ayerbe Toledano, T. Dominguez Benavides, G. López Acedo, *Measures of Noncompactness in Metric Fixed Point Theory* (Birkhäuser, Basel, 1997)
16. C. Baiocchi, A.C. Capello, *Variational and Quasi-Variational Inequalities. Applications to Free Boundary Problems* (Wiley, Chichester, 1984)
17. J.M. Ball, Weak continuity properties of mapping and semi-groups. Proc. R. Soc. Edinb. A **72**, 275–280 (1979)
18. G. Ball, Diffusion approximation of radiative transfer equations in a channel. Transp. Theory Stat. Phys. **30**(2&3), 269–293 (2001)

A.B. Amar, D. O'Regan, *Topological Fixed Point Theory for Singlevalued and Multivalued Mappings and Applications*, DOI 10.1007/978-3-319-31948-3

19. J. Banaś, On the superposition operator and integrable solutions for some functional equations. Nonlinear Anal. **12**, 777–784 (1988)
20. J. Banaś, Integrable solutions of Hammerstein and Urysohn integral equations. J. Aust. Math. Soc. **46**, 61–68 (1989)
21. J. Banaś, Applications of measures of weak noncompcatness and some classes of operators in the theory of functional equations in the Lebesgue space. Nonlinear Anal. **30** (6), 3283–3293 (1997)
22. J. Banaś, Demicontinuity and weak sequential continuity of operators in the Lebesgue space, in *Proceeedings of the 1st Polish Symposuim on Nonlinear Analysis*, Lódź (1997), pp. 124–129
23. J. Banaś, A. Chlebowicz, On existence of integrable solutions of a functional integral equation under Carathéodory conditions. Nonlinear Anal. **70**, 3172–3179 (2009)
24. J. Banaś, L. Lecko, Fixed points of the product of operators in Banach algebras. Panamer. Math. J. **12**(2), 101–109 (2002)
25. J. Banaś, L. Olszowy, On a class of measures of non-compactness in Banach algebras and their application to nonlinear integral equations. Z. Anal. Anwend. **28**(4), 475–498 (2009)
26. J. Banaś, J. Rivero, On measures of weak noncompactness. Ann. Mat. Pura Appl. **151**, 213–224 (1988)
27. J. Banaś, B. Rzepka, Monotonic solutions of a quadratic integral equation of fractional order. J. Math. Anal. Appl. **332**(2), 1371–1379 (2007)
28. J. Banaś, K. Sadarangani, Remarks on a measure of weak noncompactness in the Lebesgue space. J. Aust. Math. Soc. **52**, 279–286 (1995)
29. J. Banaś, K. Sadarangani, Solutions of some functional-integral equations in Banach algebras. Math. Comput. Model. **38**(3&4), 245–250 (2003)
30. C.S. Barroso, Krasnoselskii's fixed point theorem for weakly continuous maps. Nonlinear Anal. **55**, 25–31 (2003)
31. C.S. Barroso, E.V. Teixeira, A topological and geometric approach to fixed point results for sum of operators and applications. Nonlinear Anal. **60**(4), 625–660 (2005)
32. C.S. Barroso, O.F.K. Kalenda, M.P. Rebouças, Optimal approximate fixed point results in locally convex spaces. J. Math. Anal. Appl. **401**(1), 1–8 (2013)
33. L.P. Belluce, W.A. Kirk, Fixed-point theorems for certain classes of nonexpansive mappings. Proc. Am. Math. Soc. **20**, 141–146 (1969)
34. A. Ben Amar, J. Garcia-Falset, Fixed point theorems for 1-set weakly contractive and pseudocontractive operators on an unbounded domain. Port. Math. (N.S.) **68**(2), 125–147 (2011)
35. A. Ben Amar, M. Mnif, Leray-Schauder alternatives for weakly sequentially continuous mappings and application to transport equation. Math. Methods Appl. Sci. **33**(1), 80–90 (2010)
36. A. Ben Amar, S. Xu, Measures of weak noncompactness and fixed point theory for 1-set weakly contractive operators on unbounded domains. Anal. Theory Appl. **27**(3), 224–238 (2011)
37. A. Ben Amar, A. Jeribi, M. Mnif, On generalization of the Schauder and Krasnosel'skii fixed point theorems on Dunford-Pettis space and applications. Math. Methods Appl. Sci. **28**, 1737–1756 (2005)
38. A. Ben Amar, A. Jeribi, M. Mnif, Some fixed point theorems and application to biological model. Numer. Funct. Anal. Optim. **29**(1), 1–23 (2008)
39. A. Ben Amar, S. Chouayekh, A. Jeribi, New fixed point theorems in Banach algebras under weak topology features and applications to nonlinear integral equations. J. Funct. Anal. **259**(9), 2215–2237 (2010)
40. R. Brânzei, J. Morgan, V. Scalzo, S. Tijs, Approximate fixed point theorems in Banach spaces with applications in game theory. J. Math. Anal. Appl. **285**(2), 619–628 (2003)
41. F.E. Browder, Nonlinear nonexpansive operators in Banach spaces. Proc. Natl. Acad. Sci. USA **54**, 1041–1044 (1965)

42. F.E. Browder, *Problèmes Non-Linéaires*. Sém. Math. Supér. (Presses Univ. Montréal, Montreal, 1966)
43. F.E. Browder, Nonlinear mapping of nonexpansive and accretive type in Banach spaces. Bull. Am. Math. Soc. **73**, 875–882 (1967)
44. G. Bonanno, S.A. Marano, Positive solutions of elliptic equations with discontinuous nonlinearities. Topol. Methods Nonlinear Anal. **8**, 263–273 (1996)
45. V.A. Bondarenko, P.P. Zabrejko, The superposition operator in Hölder function spaces. Dokl. Akad. Nauk. SSSR **222**(6), 739–743 (1975, in Russian)
46. M. Boulanouar, A mathematical study in the theory of dynamic population. J. Math. Anal. Appl. **255**, 230–259 (2001)
47. D.W. Boyd, J.S.W. Wong, On nonlinear contractions. Proc. Am. Math. Soc. **20**, 458–464 (1969)
48. D. Bugajewski, On the existence of weak solutions of integral equations in Banach spaces. Comment. Math. Univ. Carol. **35**(1), 35–41 (1994)
49. D. Bugajewski, S. Szufla, Kneser's theorem for weak solutions of the Darboux problem in Banach spaces. Nonlinear Anal. **20**, 169–173 (1993)
50. T.A. Burton, *Volterra Integral and Differential Equations* (Academic, New York, 1983)
51. T.A. Burton, Integral equations, implicit functions and fixed points. Proc. Am. Math. Soc. **124**, 2383–2390 (1996)
52. T.A. Burton, A fixed-point theorem of Krasnoselskii. Appl. Math. Lett. **11**, 85–88 (1998)
53. T.A. Burton, C. Kirk, A fixed point theorem of Krasnoselskii-Schaefer type. Math. Nachr. **189**, 23–31 (1998)
54. J. Caballero, B. Lopez, K. Sadarangani, Existence of non decreasing and continuous solutions of an integral equations with linear modification of the argument. Acta Math. Sin. (Engl. Ser.), **23**(9), 1719–1728 (2007)
55. G.L. Cain Jr., M.Z. Nashed, Fixed points and stability for a sum of two operators in locally convex spaces. Pac. J. Math. **39**, 581–592 (1971)
56. C. Carathéodory, *Vorlesungen ü reele Funkionen* (De Gruyter, Leipzig, 1918)
57. T. Cardinali, On the existence of ε-fixed points. Cent. Eur. J. Math. **12**(9), 1320–1329 (2014)
58. M. Cichoń, Weak solutions of differential equations in Banach spaces. Discuss. Math. Differ. Incl. **15**, 5–14 (1995)
59. M. Cichoń, I. Kubiaczyk, On the set of solutions of the Cauchy problem in Banach spaces. Arch. Math. **63**, 251–257 (1994)
60. M. Cichoń, I. Kubiaczyk, A. Sikorska, The Henstock-Kurzweil-Pettis integrals and existence theorems for the Cauchy problem. Czech. Math. J. **54**(129), 279–289 (2004) .
61. J.B. Conway, *A Course in Functional Analysis.* (Springer, Berlin, 1990)
62. F. Cramer, V. Lakshmikantham, A.R. Mitchell, On the existence of weak solution of differential equations in nonreflexive Banach spaces. Nonlinear Anal. Theory. Methods Appl. **2**, 169–177 (1978)
63. R. Dautray, J.L. Lions, Analyse mathématiques et calcul numérique, Tome 9 (Masson, Paris, 1988)
64. M.M. Day, *Normed Linear Spaces* (Academic, New York, 1962)
65. F.S. De Blasi, On a property of the unit sphere in Banach space. Bull. Math. Soc. Sci. Math. R.S. Roumanie. **21**, 259–262 (1977)
66. D.G. Defigueiredo, L.A. Karlovitz, On the radial projection in normed spaces. Bull. Am. Math. Soc. **73**, 364–368 (1967)
67. G. Debreu, A social equilibrium existence theorem. Proc. Natl. Acad. Sci. USA **38**, 886–893 (1952)
68. B.C. Dhage, On some variants of Schauder's fixed point principle and applications to nonlinear integral equations. J. Math. Phys. Sci. **22**(5), 603–611 (1988)
69. B.C. Dhage, On a fixed point theorem of Krasnoselskii-Schaefer type. Electron. J. Qual. Theory Differ. Equ. **6**, 1–9 (2002)
70. B.C. Dhage, Local fixed point theory for sum of two operators. Fixed Point Theory. **4**, 49–60 (2003)

71. B.C. Dhage, Remarks on two fixed point theorems involving the sum and the product of two operators. Comput. Math. Appl. **46**, 1779–1785 (2003)
72. B.C. Dhage, A fixed point theorem in Banach algebras involving three operators with applications. Kyungpook Math. J. **44**(1), 145–155 (2004)
73. B.C. Dhage, On a fixed point theorem in Banach algebras with applications. Appl. Math. Lett. **18**(3), 273–280 (2005)
74. B.C. Dhage, Fixed-point theorems for discontinuous multivalued operators on ordered spaces with applications. Comput. Math. Appl. **51**(3&4), 589–604 (2006)
75. B.C. Dhage, On some nonlinear alternatives of Leray-Schauder type and functional integral equations. Arch. Math. (Brno). **42**(1), 11–23 (2006)
76. B.C. Dhage, S.K. Ntouyas, An existence theorem for nonlinear functional integral equations via a fixed point theorem of Krasnoselskii-Schaefer type. Nonlinear Digest **9**, 307–317 (2003)
77. B.C. Dhage, D. O'Regan, A fixed point theorem in Banach algebras with applications to functional integral equations. Funct. Differ. Equ. **7**(3&4) 259–267 (2000)
78. J. Diestel, A survey of results related to Dunford-Pettis property, in *Conference on Integration, Topology and Geometry in Linear Spaces. Contemporary Mathematics*, vol. 2 (American Mathematical Society, Providence, 1980), pp. 15–60
79. J. Dieudonné, Sur les espaces de Köthe. J. Anal. Math. **1**, 81–115 (1951)
80. L. Di Piazza, Kurzweil-Henstock type integration on Banach spaces. Real Anal. Exchange **29**(2), 543–555 (2003–2004)
81. I. Dobrakov, On representation of linear operators on $C_0(T,X)$. Czechoslov. Math. J. **21**(96), 13–30 (1971)
82. P. Dodds, J. Fremlin, Compact operator in Banach lattices. Israel J. Math. **34**, 287–320 (1979)
83. J. Dugundji, *Topology* (Allyn and Bacon, Inc., Boston, 1966)
84. N. Dunford, J.T. Schwartz, *Linear Operators: Part I* (Intersciences, New York, 1958)
85. R.E. Edwards, *Functional Analysis, Theory and Applications* (Holt, Reinhart and Winston, New York, 1965)
86. G. Emmanuele, Measures of weak noncompactness and fixed point theorems. Bull. Math. Soc. Sci. Math. R. S. Roumaine. **25**, 353–358 (1981)
87. G. Emmanuele, An existence theorem for Hammerstein integral equations. Port. Math. **51**, 607–611 (1994)
88. K.-J. Engel, Positivity and stability for one-sided coupled operator matrices. Positivity. **1**, 103–124 (1997)
89. K. Floret, *Weakly Compact Sets. Lecture Notes in Mathematics*, vol. 801. Springer (1980)
90. M. Furi, P. Pera, A continuation method on locally convex spaces and applications to ordinary differential equations on noncompact intervals. Ann. Polon. Math. **47**, 331–346 (1987)
91. J. Garcia-Falset, Existence of fixed points and measures of weak noncompactness. Nonlinear Anal. **71**, 2625–2633 (2009)
92. J. Garcia-Falset, Existence of fixed points for the sum of two operators. Math. Nachr. **283**(12), 1736–1757 (2010)
93. J. Garcia-Falset, E. Llorens-Fuster, Fixed points for pseudo-contractive mappings on unbounded domains. Fixed Point Theory Appl. **2010**, 17 pp. (2009). doi: 10.1155/2010/769858
94. N.I. Glebov, On a generalization of the Kakutani fixed point theorem. Soviet Math. Dokl. **10**(2), 446–448 (1969)
95. I.L. Glicksberg, A further generalisation of the Kakutani fixed point theorem, with application to Nash equilibrium points. Proc. Am. Math. Soc. **3**(1), 170–174 (1952)
96. A. Granas, On a class of nonlinear mappings in Banach spaces. Bull. Acad. Pol. Sci. **5**, 867–870 (1957)
97. G. Greiner, Spectral properties and asymptotic behavior of the linear transport equation. Math. Z. **185**, 167–177 (1984)
98. L. Górniewicz, *Topological Fixed Point Theory of Multivalued Mappings*, 2nd edn. (Springer, Berlin, 2006)

99. A. Haščàk, Fixed Point Theorems for Multivalued Mappings. Czech. Math. J. **35**(110), 533–542 (1985)
100. M.S. Gowda, G. Isac, Operators of class $(S)^1_+$, Altman's condition and the complementarity problem. J. Fac. Sci. Univ. Tokyo, Sect. IA, Math. **40**, 1–16 (1993)
101. D. Guo, V. Lakshmikantham, *Nonlinear Problems in Abstract Cones* (Academic, Boston, 1988)
102. A. Grothendieck, Sur les applications linéaires faiblement compactes d'espaces du type $C(K)$. Can. J. Math. **5**, 129–173 (1953)
103. O. Hadžić, Some fixed point and almost fixed point theorems for multivalued mappings in topological vector spaces. Nonlinear Anal. **5**, 1009–1019 (1981)
104. O. Hadžić, Almost fixed point and best approximations theorems in H-spaces. Bull. Aust. Math. Soc. **53**(3), 447–454 (1996)
105. M. Hazewinkel, M. van de Vel, On almost-fixed point theory. Can. J. Math. **30**, 673–699 (1978)
106. H.J.A.M. Heijmans, Structured populations, linear semigroups and positivity. Math. Z. **191**, 599–617 (1986)
107. S. Heikkilä, V. Lakshmikantham, *Monotone Iterative Techniques for Discontinuous Nonlinear Differential Equations* (Marcel Dekker, Inc., New York, 1994)
108. J. Himmelberg, Fixed Points of multifunctions. J. Math. Anal. Appl. **38**, 205–207 (1972)
109. C.J. Himmelberg, J.R. Porter, F.S. Van Vleck, Fixed point theorems for condensing multifunctions. Proc. Am. Math. Soc. **23**, 635–641 (1969)
110. A. Idzik, On γ-almost fixed point theorems. The single-valued case. Bull. Pol. Acad. Sci. Math. **35**(7&8), 461–464 (1987)
111. A. Idzik, Almost fixed point theorems. Proc. Am. Math. Soc. **104**, 779–784 (1988)
112. G. Isac, S.Z. Németh, Scalar derivatives and scalar asymptotic derivatives. An Altman type fixed point theorem on convex cones and some applications. J. Math. Anal. Appl. **290**(2), 452–468 (2004)
113. G. Isac, S.Z. Németh, Fixed points and positive eigenvalues for nonlinear operators. J. Math. Anal. Appl. **314**(2), 500–512 (2006)
114. I.M. James, *Topological and Uniform Spaces* (Springer, New York, 1987)
115. G.J.O. Jameson, An elementary proof of the Arens and Borusk extension theorems. J. Lond. Math. Soc. **14**, 364–368 (1976)
116. A. Jeribi, A nonlinear problem arising in the theory of growing cell populations. Nonlinear Anal. Real World Appl. **3**, 85–105 (2002)
117. M.A. Krasnoselskii, On the continuity of the operator $Fu(x) = f(x, u(x))$. Dokl. Akad. Nauk SSSR **77**, 185–188 (1951, in Russian)
118. J. Kelley, *General Topology* (D. Van Nostrand Company, Inc., Toronto 1955)
119. In.-S. Kim, Fixed Points, Eigenvalues and Surjectivity. J. Korean Math. Soc. **45(1)**, 151–161 (2008)
120. W.A. Kirk, Fixed point theorems for nonexpansive mapping satisfying certain boundary conditions. Proc. Am. Math. Soc. **50**, 143–149 (1975)
121. W.A. Kirk, R. Schöneberg, Some results on pseudo-contractive mapping. Pac. J. Math. **71** (1), 89–100 (1977)
122. M.A. Krasnosel'skii, P.P. Zabrejko, J.I. Pustyl'nik, P.J. Sobolevskii, *Integral Opertors in Spaces of Summable Functions* (Noordhoff, Leyden, 1976)
123. A. Kryczka, S. Prus, Measure of weak noncompactness under complex interpolation. Studia Math. **147**, 89–102 (2001)
124. A. Kryczka, S. Prus, M. Szczepanik, Measure of weak noncompactness and real interpolation of operators. Bull. Aust. Math Soc. **62**, 389–401 (2000)
125. I. Kubiaczyk, On the existence of solutions of differential equations in Banach spaces. Bull. Pol. Acad. Sci. Math. **33**, 607–614 (1985)
126. I. Kubiaczyk, On a fixed point theorem for weakly sequentially continuous mapping. Discuss. Math. Differ. Incl. **15**, 15–20 (1995)

127. I. Kubiaczyk, S. Szufla, Kenser's theorem for weak solutions of ordinary differential equations in Banach spaces. Publ. Inst. Math. (Beograd) (N.S.). **32**(46), 99–103 (1982)
128. D.S. Kurtz, C.W. Swartz, *Theories of Integration: The Integrals of Riemann, Lebesgue, Henstock-Kurzweil, and Mcshane* (World Scientific, Singapore, 2004)
129. V. Lakshmikantham, S. Leela, *Nonlinear Differential Equations in Abstract Spaces* (Pergamon, Oxford, 1981)
130. K. Latrach, On a nonlinear stationary problem arising in transport theory. J. Math. Phys. **37**, 1336–1348 (1996)
131. K. Latrach, Compactness results for transport equations and applications. Math. Models Methods Appl. Sci. **11**, 1182–1202 (2001)
132. K. Latrach, M. Aziz Taoudi, Existence results for generalized nonlinear Hammerstein equation on L_1 spaces. Nonlinear Anal. **66**, 2325–2333 (2007)
133. K. Latrach, A. Jeribi, A nonlinear boundary value problem arising in growing cell populations. Nonlinear Anal. Theory Methods Appl. **36**, 843–862 (1999)
134. K. Latrach, M. Mokhtar-Kharroubi, On an unbounded linear operator arising in growing cell populations. J. Math. Anal. Appl. **211**, 273–294 (1997)
135. K. Latrach, M.A. Taoudi, A. Zeghal, Some fixed point theorems of the Schauder and Krasnosel'skii type and application to nonlinear transport equations. J. Differ. Equ. **221**, 256–271 (2006)
136. J.L. Lebowitz, S.I. Rubinow, A theory for the age and generation time distribution of microbial population. J. Math. Biol. **1**, 17–36 (1974)
137. R.W. Legget, On certain nonlinear integral equations. J. Math. Anal. Appl. **57**(2), 462–468 (1977)
138. J. Leray et J. Schauder, Topologie et équations fonctionnelles. Ann. Ecole Norm. Sup. **51**, 45–78 (1934)
139. G.Z. Li, The fixed point index and the fixed point theorems for 1-set-contraction mapping. Proc. Am. Math. Soc. **104**, 1163–1170 (1988)
140. G.Z. Li, S.Y. Xu, H.G. Duan, Fixed point theorems of 1-set-contractive operators in Banach spaces. Appl. Math. Lett. **19**, 403–412 (2006)
141. M.B. Lignola, J. Morgan, Convergence for variational inequalities and generalized variational inequalities. Atti Sem. Mat. Fis. Univ. Modena. **45**, 377–388 (1997)
142. P.-K. Lin, Y. Sternfeld, Convex sets with the Lipschitz fixed point property are compact. Proc. Am. Math. Soc. **93**, 633–639 (1985)
143. L. Liu, F. Guo, C. Wu, Y. Wu, Existence theorems of global solutions for nonlinear Volterra type integral equations in Banach spaces. J. Math. Anal. Appl. **309**, 638–649 (2005)
144. Y. Liu, Z. Li, Schaefer type theorem and periodic solutions of evolution equations. J. Math. Anal. Appl. **316**, 237–255 (2006)
145. Y. Liu, Z. Li, Krasnoselskii type fixed point theorems and applications. Proc. Am. Math. Soc. **316**, 1213–1220 (2008)
146. B. Lods, On linear kinetic equations involving unbounded cross-sections. Math. Models Methods Appl. Sci. **27**, 1049–1075 (2004)
147. A. Majorana, S.A. Marano, Continuous solutions of a nonlinear integral equation on an unbounded domain. J. Integr. Equ. Appl. **6**(1), 119–128 (1994)
148. R.H. Martin, *Nonlinear Operators and Differential Equations in Banach Spaces* (Wiley, New York 1976)
149. R.E. Megginson, *An Introduction to Banach Space Theory.* Graduate Texts in Mathematics (Springer, New York, 1988)
150. A.R. Mitchell, C. Smith, An existence theorem for weak solutions of differential equations in Banach spaces, in *Nonlinear Equations in Abstract Spaces* (*Proc. Internat. Sympos.*, Univ. Texas, Arlington, TX, 1977), ed. by V. Lakshmikantham (Academic, New York, 1978), pp. 387–403
151. M. Mokhtar-Kharroubi, Quelques applications de la positivité en théorie du transport. Ann. Fac. Sci. Toulouse. **11**, 75–99 (1990)

152. C. Morales, Pseudo-contractive mapping and the Leray-Schauder boundary conditions. Comment. Math. Univ. Carol. **20**, 745–756 (1979)
153. C. Morales, The Leray-Shauder condition for continuous pseudo-contractive mappings. Proc. Am. Math. Soc. **137**, 1003–1020 (2009)
154. U. Mosco, Convergence of convex sets and solutions of variational inequalities. Lect. Notes Math. **3**, 510–585 (1969)
155. J.F. Nash, Equilibrium points in n-person games. Proc. Natl. Acad. Sci. USA **36**, 48–49 (1950)
156. V.I. Nazarov, The superposition operator in spaces of infinitely differentiable Roumieu functions. Vestsi Akad. Nauk BSSR Ser. Flz. Mat. **5**, 22–28 (1984, in Russian)
157. R. D. Nussbaum, Degree theory for local condensing maps. J. Math. Anal. Appl. **37** (1972), 741–766.
158. D. O'Regan, A continuation method for weakly condensing operators. Z. Anal. Anwend. **15** (1996), 565–578.
159. D. O'Regan, Fixed-point theory for weakly sequentially continuous mapping. Math. Comput. Modelling. **27**(5), 1–14 (1998)
160. D. O'Regan, New fixed point results for 1-set contractive set valued maps. Comput. Math. Appl. **35**(4), 27–34 (1998)
161. D. O'Regan, Operator equations in Banach spaces relative to the weak topology. Arch. Math. **71**, 123–136 (1998)
162. D. O'Regan, Fixed point theorems for weakly sequentially closed maps. Arch. Math. (Brno) **36**, 61–70 (2000)
163. D. O'Regan, M.-A. Taoudi, Fixed point theorems for the sum of two weakly sequentially continuous mappings. Nonlinear Anal. **73**(2), 283–289 (2010)
164. M. Palmucci, F. Papalini, Periodic and boundary value problems for second order differential inclusions. J. Appl. Math. Stoch. Anal. **14**(2), 161–182 (2001)
165. S. Park, Almost fixed points of multimaps having totally bounded ranges. Nonlinear Anal. **51**(1), 1–9 (2009)
166. C.V. Pao, Asymptotic behavior of the solution for the time-dependent neutron transport problem. J. Integr. Equ. **1**, 31–152 (1979)
167. J-P. Penot, A fixed-point theorem for asymptotically contractive mappings. Proc. Am. Math. Soc. **131**(8), 2371–2377 (2003)
168. A. Petruşel, Multivalued operators and fixed points. PUMA **9**, 165–170 (1998)
169. W.V. Petryshyn, Construction of fixed points of demicompact mappings in Hilbert spaces. J. Math. Anal. Appl. **14**, 276–284 (1966)
170. W.V. Petryshyn, Structure of the fixed points sets of k-set-contractions. Arch. Ration. Mech. Anal. **40**, 312–328 (1971)
171. W.V. Petryshyn, Fixed point theorems for various classes of 1-set-contractive and 1-ball-contractive mappings in Banach spaces. Trans. Am. Math. Soc. **182**, 323–352 (1973)
172. A.J.B. Potter, An elementary version of the Leray-Schauder theorem. J. Lond. Math. Soc. **5**(2), 414–416 (1972)
173. R. Pluccienik, On some properties of the superposition operator in generalized Orlicz spaces of vector-valued functions. Comment. Math. Prace Mat. **25**, 321–337 (1985)
174. M. Rotenberg, Transport theory for growing cell populations. J. Theor. Biol. **103**, 181–199 (1983)
175. E.H. Rothe, Zur Theorie der topologischen Ordunung und der Vektorfelder in Banachschen Räumen. Comp. Math. **5**, 177–197 (1938)
176. W. Rudin, *Functional Analysis,* 2nd edn. (McGraw Hill, New York, 1991)
177. J.B. Rutickij, On a nonlinear operator in Orlicz spaces. Dopo. Akad. Nauk URSR **3**, 161–166 (1952) (In Ukrainian).
178. B.N. Sadovskii, On a fixed point theorem. Funktsional. Anal. i Prilozhen. **1**, 74–76 (1967)
179. H.H. Schaefer, Neue Existenzsätze in der theorie nichtlinearer Integragleichungen. Ber. Verh. Sächs. Akad. Wiss. Leipzig Math.-Natur. Kl. **101**, 7–40 (1955)
180. H. Schaefer, Über die Methode der a priori-Schranken. Math. Ann. **129**, 415–416 (1955)

181. H.H. Schaefer, *Banach Lattices and Positive Operators*. Grundlehren Math. Wiss. Bd., vol. 215 (Springer, New York, 1974)
182. M. Schechter, On the essential spectrum of an arbitrary operator. I. J. Math. Anal. Appl. **13**, 205–215 (1966)
183. M. Schechter, *Principles of Functional Analysis* (Academic, New York, 1971)
184. V.M. Sehgal, S.P. Singh, A variant of a fixed point theorem of Ky fan. Indian J. Math. **25**, 171–174 (1983)
185. A. Sikorska, Existence theory for nonlinear Volterra integral and differential equations. J. Inequal. Appl. **6**(3), 325–338 (2001)
186. V.I. Shragin, On the weak continuity of the Nemytskii operator. Uchen. Zap. Mosk. Obl. Ped. Inst. **57**, 73–79 (1957)
187. R.E. Showalter, *Monotone Operators in Banach Space and Nonlinear Partial Differential Equations* (American Mathematical Society, Providence, 1997)
188. D.R. Smart, *Fixed Point Theorems* (Cambridge University Press, Cambridge, 1980)
189. M. Švec, Fixpunktsatz und monotone Lösungen der Differentialgleichung $y^n + +B(x, y, y', \ldots, y^{n-1}y = 0$. Arch. Math. (Brno), **2**, 43–55 (1966)
190. S. Szufla, Kneser's theorem for weak solutions of ordinary differential equations in reflexive Banach spaces. Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys. **26**, 407–413 (1978)
191. S. Szufla, Sets of fixed points of nonlinear mappings in functions spaces. Funkc. Ekvacioj. **22**, 121–126 (1979)
192. S. Szufla, On the application of measure of noncompactness to existence theorems. Rend. Semin. Mat. Univ. Padova. **75**, 1–14 (1986)
193. M.A. Taoudi, Krasnosel'skii type fixed point theorems under weak topology features. Nonlinear Anal. **72**(1), 478–482 (2010)
194. C. Van der Mee, P. Zweifel, A Fokker-Plank equation for growing cell population. J. Math. Biol. **25**, 61–72 (1987)
195. M. Väth, Fixed point theorems and fixed point index for countably condensing maps. Topol. Methods Nonlinear Anal. **13**(2), 341–363 (1999)
196. J. Voigt, On resolvent positive operators and positive C_0-semigroup on AL-spaces. Semigroup Forum. **38**, 263–266 (1989)
197. G.F. Webb, *Theory of Nonlinear Age-Dependent Population Dynamics* (Marcel Dekker, New York, 1985)
198. G. Webb, A model of proliferating cell populations with inherited cycle length. J. Math. Biol. **23**, 269–282 (1986)
199. G. Webb, Dynamics of structured populations with inherited properties. Comput. Math. Appl. **13**, 749–757 (1987)
200. T. Xiang, R. Yuan, Critical type of Krasnosel'skii fixed point theorem. Proc. Am. Math. Soc. **139**(3), 1033–1044 (2011)
201. S. Xu, New fixed point theorems for 1-set contractive operators in Banach spaces. Nonlinear Anal. **67**, 938–944 (2007)
202. P.P. Zabrejko, A.I. Koshelev, M.A. Krasnosel'skii, S.G. Mikhlin, L.S. Rakovshchik, V.J. Stecenko, *Integral Equations* (Noordhoff, Leyden, 1975)
203. E. Zeidler, *Nonlinear Functional Analysis and Its Applications*, vol. I (Springer, New York, 1986)
204. C. Zhu, Z. Xu, Inequalities and solution of an operator equation. Appl. Math. Lett. **21**, 607–611 (2008)