Lifting in Besov spaces

Petru Mironescu, Emmanuel Russ, Yannick Sire

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1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded simply connected domain and $u : \Omega \to \mathbb{S}^1$ a continuous (resp. C^k, $k \geq 1$) function. It is a well-known fact that there exists a continuous (resp. C^k) real-valued function φ such that $u = e^{\imath\varphi}$. In other words, u has a continuous (resp. C^k) lifting.

The analogous problem when u belongs to the fractional Sobolev space $W^{s,p}$, $s > 0$, $1 \leq p < \infty$, received an complete answer in [4]. Let us briefly recall the results:

1. when $n = 1$, u has a lifting in $W^{s,p}$ for all $s > 0$ and all $p \in [1, \infty)$,

2. when $n \geq 2$ and $0 < s < 1$, u has a lifting in $W^{s,p}$ if and only if $sp < 1$ or $sp \geq n$,

3. when $n \geq 2$ and $s \geq 1$, u has a lifting in $W^{s,p}$ if and only if $sp \geq 2$.

Further developments in the Sobolev context can be found in [1, 28, 24, 26].

In the present paper, we address the corresponding question in the framework of Besov spaces. More specifically, given s, p, q in suitable ranges defined later, we ask whether a map $u \in B^s_{p,q}(\Omega; \mathbb{S}^1)$ can be lifted as $u = e^{\imath\varphi}$, with $\varphi \in B^s_{p,q}(\Omega; \mathbb{R})$. We say that $B^s_{p,q}$ has the lifting property if and only if the answer is positive.

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When dealing with circle-valued functions and their phases, it is natural to consider only maps in L^1_{loc}. This is why we assume that $s > 0$,[1] and we take the exponents p and q in the classical range $p \in [1,\infty)$, $q \in [1,\infty]$.[2]

Since Besov spaces are microscopic modifications of Sobolev (or Slobodeskii) spaces, one expects a global picture similar to the one described before for Sobolev spaces. The analysis in Besov spaces is indeed partly similar to the one in Sobolev spaces, as far as the results and the techniques are concerned. However, several difficulties occur and some cases still remain open. Thus, the analysis of the lifting problem leads us to prove several new properties for Besov spaces (in connection with restriction or absence of restriction properties, sums of integer valued functions which are constant, products of functions in Besov spaces, disintegration properties for the Jacobian), which are interesting in their own right. We also provide detailed arguments for classical properties (some embeddings, Poincaré inequalities) which could not be precisely located in the literature.

Let us now describe more precisely our results and methods. When $sp > n$, functions in $B^s_{p,q}$ are continuous, which readily implies that $B^s_{p,q}$ has the lifting property (Case 1).

In the case where $sp < 1$, we rely on a characterization of $B^s_{p,q}$ in terms of the Haar basis [3, Théorème 5], to prove that $B^s_{p,q}$ has the lifting property (Case 2).

Assume now that $0 < s \le 1$, $sp = n$ and $q < \infty$. Let $u \in B^s_{p,q}(\Omega;\mathbb{S}^1)$ and let $F(x,\varepsilon) := u * \rho_\varepsilon$, where ρ is a standard mollifier. Since $B^s_{p,q} \hookrightarrow \mathrm{VMO}$, for all ε sufficiently small and all $x \in \Omega$ we have $1/2 < |F(x,\varepsilon)| \le 1$. Writing $F(x,\varepsilon)/|F(x,\varepsilon)| = e^{\imath\psi_\varepsilon}$, where ψ_ε is C^∞, and relying on a slight modification of the trace theory for weighted Sobolev spaces developed in [27], we conclude, letting εtend to 0, that $u = e^{\imath\psi_0}$, where $\psi_0 = \lim_{\varepsilon\to 0}\psi_\varepsilon \in B^s_{p,q}$, and therefore $B^s_{p,q}$ still has the lifting property (Case 3).

Consider now the case where $s > 1$ and $sp \ge 2$. Arguing as in [4, Section 3], it is easily seen that the lifting property for $B^s_{p,q}$ will follow from the following property: given $u \in B^s_{p,q}(\Omega;\mathbb{S}^1)$, if $F := u \wedge \nabla u \in L^p(\Omega;\mathbb{R}^n)$, then $(*)$ $\operatorname{curl} F = 0$. The proof of $(*)$ is much more involved than the corresponding one for $W^{s,p}$ spaces [4, Section 3]. It relies on a disintegration argument for the Jacobians, more generally applicable in $W^{1/p,p}$. This argument, in turn, relies on the fact that $\operatorname{curl} F = 0$ when $u \in \mathrm{VMO}$ and $n = 2$, and a slicing argument. In particular, we need a *restriction property for Besov spaces*, namely the fact that, for $s > 0$, $1 \le p < \infty$ and $1 \le q \le p$, for all $f \in B^s_{p,q}$, the partial maps of f still belong to $B^s_{p,p}$ (see Lemma 6.7 below). Thus, we obtain that, when $s > 1$ and $1 \le p < \infty$,

[1] However, we will discuss an appropriate version of the lifting problem when $s \le 0$; see Remark 3.1 and Case 10 below.

[2] We discard the uninteresting case where $p = \infty$. In that case, maps in $B^s_{\infty,q}$ are continuous. Arguing as in Case 1 below, we obtain the existence of a $B^s_{\infty,q}$ phase for every $u \in B^s_{\infty,q}(\Omega;\mathbb{S}^1)$.

$B^s_{p,q}$ does have the lifting property when $[1 \le q < \infty$, $n = 2$, and $sp = 2]$, or $[1 \le q \le p$, $n \ge 3$, and $sp = 2]$, or $[1 \le q \le \infty$, $n \ge 2$, and $sp > 2]$.

One can improve the conclusion of Lemma 6.7 as follows. For $s > 0$, $1 \le p < \infty$ and $1 \le q \le p$, for all $f \in B^s_{p,q}$, the partial maps of f belong to $B^s_{p,q}$ (Proposition 6.10). We emphasize the fact that this type of property holds only under the crucial assumption $q \le p$. More precisely, if $q > p$ and $s > 0$, then we exhibit a compactly supported function $f \in B^s_{p,q}(\mathbb{R}^2)$ such that, for almost every $x \in (0,1)$, $f(x,\cdot) \notin B^s_{p,\infty}(\mathbb{R})$ (Proposition 6.11). This phenomenon, which has not been noticed before, shows a picture strikingly different from the one for $W^{s,p}$, and even more generally for Triebel-Lizorkin spaces [35, Section 2.5.13].

Let us return to the case when $0 < s < 1$, $1 \le p < \infty$ and $n \ge 2$. Assume now that $[1 \le q < \infty$ and $1 \le sp < n]$, or $[q = \infty$ and $1 < sp < n]$. In this case, we show that $B^s_{p,q}$ does not have the lifting property. The argument uses embedding theorems and the following fact, for which we provide a proof: let $s_i > 0$, $1 \le p_i < \infty$, and $[s_j p_j = 1$ and $1 \le q_j < \infty]$, or $[s_j p_j > 1$ and $1 \le q_j \le \infty]$, $i = 1, 2$. Then, if $f_i \in B^{s_i}_{p_i,q_i}$ and $f_1 + f_2$ only takes integer values, then the function $f_1 + f_2$ is constant.

Assume finally that $0 < s < \infty$, $1 \le p < \infty$, $n \ge 2$ and $[1 \le q < \infty$ and $1 \le sp < 2]$ or $[q = \infty$ and $1 \le sp \le 2]$. In this case, $B^s_{p,q}$ does not have the lifting property either. We provide a counterexample of topological nature, inspired by [4, Section 4]: namely, the function $u(x) = \dfrac{(x_1, x_2)}{\left(x_1^2 + x_2^2\right)^{1/2}}$ belongs to $B^s_{p,q}$ but has no lifting in $B^s_{p,q}$.

Contrary to the case of Sobolev spaces, some cases remain open. A first case occurs when $s > 1$, $1 \le p < \infty$, $p < q < \infty$, $n \ge 3$, and $sp = 2$. In this situation, since the restriction property for $B^s_{p,q}$ does not hold, the argument sketched before does not work any longer and we do not know if $B^s_{p,q}$ has the lifting property.

The case where $s = 1$, $1 \le p < \infty$, $n \ge 3$, and $[1 \le q < \infty$ and $2 \le p < n]$ or $[q = \infty$ and $2 < p \le n]$ is also open (except when $s = 1$ and $p = q = 2$, since in this case, $B^1_{2,2} = W^{1,2}$ has the lifting property). This is related to the fact that it is not known whether the map $\varphi \mapsto e^{\imath\varphi}$ maps $B^1_{p,q}$ into itself.

When $1 \le p < \infty$, $s = 1/p$ and $q = \infty$, we do not know if $B^{1/p}_{p,\infty}$ has the lifting property. In particular, it is unclear whether the Haar system provides a basis of $B^{1/p}_{p,\infty}$. The case where $q = \infty$, $n \le p < \infty$, $n \ge 3$ and $s = n/p$ is also open. Indeed, $B^s_{p,q}$ is not embedded into VMO in this case, and the argument briefly described above is not applicable any more.

Let us summarize the main results of this paper concerning the lifting problem. We start with positive cases.

1.1 Theorem. Let $s > 0$, $1 \le p < \infty$, $1 \le q \le \infty$. The lifting problem has a positive answer in the following cases:

1. $s > 0$, $1 \le q \le \infty$, and $sp > n$,

2. $0 < s < 1$, $1 \le q \le \infty$, and $sp < 1$,

3. $0 < s \le 1$, $1 \le q < \infty$, and $sp = n$,

4. (a) $s > 1$, $1 \le q < \infty$, $n = 2$, and $sp = 2$,

   (b) $s > 1$, $1 \le q \le p$, $n \ge 3$, and $sp = 2$,

   (c) $s > 1$, $1 \le q \le \infty$, $n \ge 2$, and $sp > 2$.

The negative cases are as follows:

negative **1.2 Theorem.** Let $s > 0$, $1 \le p < \infty$, $1 \le q \le \infty$. The lifting problem has a negative answer in the following cases:

1. (a) $0 < s < 1$, $1 \le q < \infty$, $n \ge 2$, and $1 \le sp < n$,

   (b) $0 < s < 1$, $q = \infty$, $n \ge 2$, and $1 < sp < n$,

2. (a) $0 < s < \infty$, $1 \le q < \infty$, $n \ge 2$, and $1 \le sp < 2$,

   (b) $0 < s < \infty$, $1 \le p < \infty$, $q = \infty$, $n \ge 2$, and $1 < sp \le 2$.

The paper is organized as follows. In Section 2, we briefly recall the standard definition of Besov spaces and some classical characterizations of these spaces (by Littlewood-Paley theory and wavelets). In Section 3 we establish Theorem 1.1, namely the cases where $B^s_{p,q}$ does have the lifting property, while Section 4 is devoted to negative cases (Theorem 1.2). In Section 5, we discuss the remaining cases, which are widely open. The final section gathers statements and proofs of various results on Besov spaces needed in the proofs of Theorems 1.1 and 1.2.

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Notation, framework

1. Most of our results are stated in a smooth bounded domain $\Omega \subset \mathbb{R}^n$.

2. In few cases, proofs are simpler if we consider $\mathbb{Z}^n$-periodic maps $u : (0,1)^n \to \mathbb{S}^1$. In this case, we denote the corresponding function spaces $B^s_{p,q}(\mathbb{T}^n;\mathbb{S}^1)$, and the question is whether a map $u \in B^s_{p,q}(\mathbb{T}^n;\mathbb{S}^1)$ has

a lifting $\varphi \in B^s_{p,q}((0,1)^n;\mathbb{R})$. [Of course, φ need not be, in general, $\mathbb{Z}^n$-periodic.] If such a φ exists for every $u \in B^s_{p,q}(\mathbb{T}^n;\mathbb{S}^1)$, then $B^s_{p,q}(\mathbb{T}^n;\mathbb{S}^1)$ has the lifting property.

However, in these results it is not crucial to work in $\mathbb{T}^n$. An inspection of the proofs shows that, with some extra work, we could take any smooth bounded domain.

3. In the same vein, it is sometimes easier to work in $\Omega = (0,1)^n$ (with no periodicity assumption).

4. Partial derivatives are denoted ∂_j, $\partial_j\partial_k$, and so on, or ∂^α.

5. $\wedge$ denotes vector product of complex numbers: $a \wedge b := a_1b_2 - a_2b_1$. Similarly, $u \wedge \nabla v := u_1\nabla v_2 - u_2\nabla v_1$.

6. If $u : \Omega \to \mathbb{C}$ and if ϖ is a k-form on Ω (with $k \in [\![0, n-1]\!]$), then $\varpi \wedge (u \wedge \nabla u)$ denotes the $(k+1)$-form $\varpi \wedge (u_1du_2 - u_2du_1)$.

7. We let $\mathbb{R}^n_+$ denote the open set $\mathbb{R}^{n-1} \times (0,\infty)$.

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2 Crash course on Besov spaces

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We briefly recall here the basic properties of the Besov spaces in $\mathbb{R}^n$, with special focus on the properties which will be instrumental for our purposes. For a complete treatment of these spaces, see triebel2,fjw,triebel3,runstsickel [35, 18, 36, 30].

2.1 Preliminaries

In the sequel, $\mathscr{S}(\mathbb{R}^n)$ is the usual Schwartz space of rapidly decreasing C^∞ functions. Let $\mathscr{Z}(\mathbb{R}^n)$ denote the subspace of $\mathscr{S}(\mathbb{R}^n)$ consisting of functions $\varphi \in \mathscr{S}(\mathbb{R}^n)$ such that $\partial^\alpha \varphi(0) = 0$ for every multi-index $\alpha \in \mathbb{N}^n$. Let $\mathscr{Z}'(\mathbb{R}^n)$ stand for the topological dual of $\mathscr{Z}(\mathbb{R}^n)$. It is well-known triebel2 [35, Section 5.1.2] that $\mathscr{Z}'(\mathbb{R}^n)$ can be identified with the quotient space $\mathscr{S}'(\mathbb{R}^n)/\mathscr{P}(\mathbb{R}^n)$, where $\mathscr{P}(\mathbb{R}^n)$ denotes the space of all polynomials in $\mathbb{R}^n$.

We denote by $\mathscr{F}$ the Fourier transform.

For all sequence $(f_j)_{j\geq 0}$ of measurable functions on $\mathbb{R}^n$, we set

$$\|(f_j)\|_{l^q(L^p)} := \left(\sum_{j\geq 0}\left(\int_{\mathbb{R}^n}|f_j(x)|^p\,dx\right)^{q/p}\right)^{1/q},$$

with the usual modification when $p=\infty$ and/or $q=\infty$. If (f_j) is labelled by $\mathbb{Z}$, then $\|(f_j)\|_{l^q(L^p)}$ is defined analogously with $\sum_{j\geq 0}$ replaced by $\sum_{j\in\mathbb{Z}}$.

Finally, we fix some notation for finite order differences. Let $\Omega \subset \mathbb{R}^n$ be a domain and let $f : \Omega \to \mathbb{R}$. For all integers $M \geq 0$, all $t > 0$ and all $x, h \in \mathbb{R}^n$, set

$$\Delta_h^M f(x) = \begin{cases} \displaystyle\sum_{l=0}^{M}\binom{M}{l}(-1)^{M-l} f(x+lh), & \text{if } x, x+h, \dots, x+Mh \in \Omega \\ 0, & \text{otherwise} \end{cases}. \tag{2.1}$$

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2.2 Definitions of Besov spaces

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We first focus on inhomogeneous Besov spaces. Fix a sequence of functions $(\varphi_j)_{j\geq 0} \in \mathscr{S}(\mathbb{R}^n)$ such that:

1. $\operatorname{supp} \varphi_0 \subset B(0,2)$ and $\operatorname{supp} \varphi_j \subset B(0,2^{j+1}) \setminus B(0,2^{j-1})$ for all $j \geq 1$.
2. For all multi-index $\alpha \in \mathbb{N}^n$, there exists $c_\alpha > 0$ such that $\left|D^\alpha \varphi_j(x)\right| \leq c_\alpha 2^{-j|\alpha|}$, for all $x \in \mathbb{R}^n$ and all $j \geq 0$.
3. For all $x \in \mathbb{R}^n$, it holds $\sum_{j\geq 0} \varphi_j(x) = 1$.

2.1 Definition (Definition of inhomogeneous Besov spaces)**.** Let $s \in \mathbb{R}$, $1 \leq p < \infty$ and $1 \leq q \leq \infty$. Define $B^s_{p,q}(\mathbb{R}^n)$ as the space of tempered distributions $f \in \mathscr{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{B^s_{p,q}(\mathbb{R}^n)} := \left\| \left(2^{sj} \mathscr{F}^{-1} \left(\varphi_j \mathscr{F} f(\cdot) \right) \right) \right\|_{l^q(L^p)} < \infty.$$

Recall triebel2 [35, Section 2.3.2, Proposition 1, p. 46] that $B^s_{p,q}(\mathbb{R}^n)$ is a Banach space which does not depend on the choice of the sequence $(\varphi_j)_{j\geq 0}$, in the sense that two different choices for the sequence $(\varphi_j)_{j\geq 0}$ give rise to equivalent norms. Once the φ_j's are fixed, we refer to the equality $f = \sum_j f_j$ in $\mathscr{S}'$ as the Littlewood-Paley decomposition of f.

Let us now turn to the definition of homogeneous Besov spaces. Let $(\varphi_j)_{j\in\mathbb{Z}}$ be a sequence of functions satisfying:

1. $\operatorname{supp} \varphi_j \subset B(0,2^{j+1}) \setminus B(0,2^{j-1})$ for all $j \in \mathbb{Z}$.
2. For all multi-index $\alpha \in \mathbb{N}^n$, there exists $c_\alpha > 0$ such that $\left|D^\alpha \varphi_j(x)\right| \leq c_\alpha 2^{-j|\alpha|}$, for all $x \in \mathbb{R}^n$ and all $j \in \mathbb{Z}$.
3. For all $x \in \mathbb{R}^n \setminus \{0\}$, it holds $\sum_{j\in\mathbb{Z}} \varphi_j(x) = 1$.

2.2 Definition (Definition of homogeneous Besov spaces)**.** Let $s \in \mathbb{R}$, $1 \leq p < \infty$ and $1 \leq q \leq \infty$. Define $\dot{B}^s_{p,q}(\mathbb{R}^n)$ as the space of $f \in \mathscr{Z}'(\mathbb{R}^n)$ such that

$$|f|_{B^s_{p,q}(\mathbb{R}^n)} := \left\| \left(2^{sj} \mathscr{F}^{-1} \left(\varphi_j \mathscr{F} f(\cdot) \right) \right) \right\|_{l^q(L^p)} < \infty.$$

Note that this definition makes sense since, for all polynomial P and all $f \in \mathscr{S}'(\mathbb{R}^n)$, we have $|f|_{B^s_{p,q}(\mathbb{R}^n)} = |f+P|_{B^s_{p,q}(\mathbb{R}^n)}$.

Again, $\dot{B}^s_{p,q}(\mathbb{R}^n)$ is a Banach space which does not depend on the choice of the sequence $(\varphi_j)_{j\in\mathbb{Z}}$ triebel2 [35, Section 5.1.5, Theorem, p. 240].
For all $s > 0$ and all $1 \leq p < \infty$, $1 \leq q \leq \infty$, we have triebel3 [36, Section 2.3.3, Theorem], runstsickel [30, Section 2.6.2, Proposition 3]

$$B^s_{p,q}(\mathbb{R}^n) = L^p(\mathbb{R}^n) \cap \dot{B}^s_{p,q}(\mathbb{R}^n) \text{ and } \|f\|_{B^s_{p,q}(\mathbb{R}^n)} \sim \|f\|_{L^p(\mathbb{R}^n)} + |f|_{B^s_{p,q}(\mathbb{R}^n)}. \tag{2.2}$$

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Besov spaces on domains of $\mathbb{R}^n$ are defined as follows.

2.3 Definition (Besov spaces on domains)**.** Let $\Omega \subset \mathbb{R}^n$ be an open set. Then

1. $B^s_{p,q}(\Omega) := \{f \in \mathscr{D}'(\Omega);$ there exists $g \in B^s_{p,q}(\mathbb{R}^n)$ such that $f = g|_\Omega\}$, equipped with the norm
$$\|f\|_{B^s_{p,q}(\Omega)} := \inf\left\{\|g\|_{B^s_{p,q}(\mathbb{R}^n)}; g|_\Omega = f\right\}.$$

2. $\dot{B}^s_{p,q}(\Omega) := \{f \in \mathscr{D}'(\Omega);$ there exists $g \in \dot{B}^s_{p,q}(\mathbb{R}^n)$ such that $f = g|_\Omega\}$, equipped with the semi-norm
$$\|f\|_{\dot{B}^s_{p,q}(\Omega)} := \inf\left\{\|g\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)}; g|_\Omega = f\right\}.$$

Local Besov spaces are defined in the usual way: $f \in B^s_{p,q}$ near a point x if for some cutoff φ which equals 1 near x we have $\varphi f \in B^s_{p,q}$. If f belongs to $B^s_{p,q}$ near each point, then we write $f \in (B^s_{p,q})_{loc}$.

The following is straightforward.

ka3 **2.4 Lemma.** Let $f : \Omega \to \mathbb{R}$. If, for each $x \in \overline{\Omega}$, $f \in B^s_{p,q}(B(x,r) \cap \Omega)$ for some $r = r(x) > 0$, then $f \in B^s_{p,q}$.

2.3 Besov spaces on $\mathbb{T}^n$

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Let $\varphi_0 \in \mathscr{D}(\mathbb{R}^n)$ be such that
$$\varphi_0(x) = 1 \text{ for all } |x| < 1 \text{ and } \varphi_0(x) = 0 \text{ for all } |x| \ge \frac{3}{2}.$$
For all $k \ge 1$ and all $x \in \mathbb{R}^n$, define
$$\varphi_k(x) := \varphi_0(2^{-k}x) - \varphi_0(2^{-k+1}x).$$

periodicbesov **2.5 Definition.** Let $s \in \mathbb{R}$, $1 \le p < \infty$ and $1 \le q \le \infty$. Define $B^s_{p,q}(\mathbb{T}^n)$ as the space of distributions $f \in \mathscr{D}'(\mathbb{T}^n)$ whose Fourier coefficients $(a_m)_{m\in\mathbb{Z}^n}$ satisfy
$$\|f\|_{B^s_{p,q}(\mathbb{T}^n)} := \left(\sum_{j=0}^{\infty} 2^{jsq} \left\| x \mapsto \sum_{m\in\mathbb{Z}^n} a_m \varphi_j(2\pi m) e^{2\imath\pi m\cdot x} \right\|^q_{L^p(\mathbb{T}^n)}\right)^{1/q} < \infty$$
(with the usual modification when $q = \infty$). Again, the choice of the system $(\varphi_j)_{j\ge 0}$ is irrelevant, and the equality $f = \sum f_j$, with $f_j := \sum_m a_m \varphi_j(2\pi m) e^{2\imath\pi m\cdot x}$, is the Littlewood-Paley decomposition of f.

Alternatively, we have $f \in B^s_{p,q}(\mathbb{T}^n)$ if and only if f can be identified with a $\mathbb{Z}^n$-periodic distribution in $\mathbb{R}^n$, still denoted f, which belongs to $(B^s_{p,q})_{loc}(\mathbb{R}^n)$ schmeisser [31, Section 3.5.4, pp. 167-169].

2.4 Characterization by differences

Among the various characterizations of Besov spaces, we recall here the ones involving differences [35, Section 5.2.3], [30, Theorem, p. 41], [37, Section 1.11.9, Theorem 1.118, p. 74].

Proposition 2.6. Let $s>0$, $1\le p<\infty$ and $1\le q\le\infty$. Let $M>s$ be an integer. Then, with the usual modification when $q=\infty$:

1. In the space $\dot B^s_{p,q}(\mathbb{R}^n)$ we have the equivalence of semi-norms

$$|f|_{B^s_{p,q}(\mathbb{R}^n)}\sim\left(\int_{\mathbb{R}^n}|h|^{-sq}\left\|\Delta^M_h f\right\|^q_{L^p(\mathbb{R}^n)}\frac{dh}{|h|^n}\right)^{1/q}$$
$$\sim\sum_{j=1}^n\left(\int_{\mathbb{R}}|h|^{-sq}\left\|\Delta^M_{he_j}f\right\|^q_{L^p(\mathbb{R}^n)}\frac{dh}{|h|}\right)^{1/q}. \tag{2.3}$$

2. The full $B^s_{p,q}$ norm satisfies, for all $\delta>0$,

$$\|f\|_{B^s_{p,q}(\mathbb{R}^n)}\sim\|f\|_{L^p(\mathbb{R}^n)}+\left(\int_{|h|\le\delta}|h|^{-sq}\left\|\Delta^M_h f\right\|^q_{L^p(\mathbb{R}^n)}\frac{dh}{|h|^n}\right)^{1/q}.$$

2.5 Characterization by harmonic extensions

In Section 3, it will be convenient to work with extensions of maps in $B^s_{p,q}$. The connection between regularity of maps and the properties of their suitable extensions is a classical topic in the theory of function spaces. Here is a typical result in this direction. It characterizes $B^s_{p,q}$ by means of the harmonic extension [34], [35, Section 2.12.2, Theorem, p. 184]. More specifically, if f is measurable in $\mathbb{R}^n$ and $s\in(0,1)$, then we have

$$\|f\|_{B^s_{p,q}(\mathbb{R}^n)}\sim\|f\|_{L^p(\mathbb{R}^n)}+\left(\int_0^\infty t^{(1-s)q}\left\|\frac{\partial P_t f}{\partial t}(\cdot)\right\|^q_{L^p(\mathbb{R}^n)}\frac{dt}{t}\right)^{1/q}, \tag{2.4}$$

where P_t stands for the Poisson semigroup generated by $-\Delta$, so that $(x,t)\mapsto P_tf(x)$, $t>0$, $x\in\mathbb{R}^n$, is the harmonic extension of f to the upper-half space. Since when $p>1$ we have

$$\left\|\frac{\partial P_t f}{\partial t}\right\|_{L^p(\mathbb{R}^n)}=\left\|(-\Delta_x)^{1/2}P_tf\right\|_{L^p(\mathbb{R}^n)}\sim\|\nabla_x P_t f\|_{L^p(\mathbb{R}^n)},$$

one also has, for $1<p<\infty$ and $1\le q\le\infty$,

$$\|f\|_{B^s_{p,q}(\mathbb{R}^n)}\sim\|f\|_{L^p(\mathbb{R}^n)}+\left(\int_0^\infty t^{(1-s)q}\|\nabla P_tf(\cdot)\|^q_{L^p(\mathbb{R}^n)}\frac{dt}{t}\right)^{1/q} \tag{2.5}$$

(with the usual modification when $q=\infty$).

The results in the literature are not suited to our context. We will need some variants of (2.5), which will be stated and proved in Section 6.5 below.

2.6 Lizorkin type characterizations

mm1

Such characterizations involve restrictions of the Fourier transform on cubes or corridors; see e.g. [35, Section 2.5.4, pp. 85-86] or [31, Section 3.5.3, pp. 166-167]. The following special case [31, Section 3.5.3, Theorem, p. 167] will be useful later.

mm2 **Proposition 2.7.** Let $s \in \mathbb{R}$, $1 < p < \infty$ and $1 \le q \le \infty$. Set $K_0 := \{0\} \subset \mathbb{Z}^n$ and, for $j \ge 1$, let $K_j := \{m \in \mathbb{Z}^n ; 2^{j-1} \le |m| < 2^j\}$.[3] Let $f \in \mathscr{D}'(\mathbb{T}^n)$ have the Fourier series expansion $f = \sum_{m\in\mathbb{Z}^n} a_m e^{2\imath\pi m\cdot x}$. We set $f_j := \sum_{m\in K_j} a_m e^{2\imath\pi m\cdot x}$. Then we have the norm equivalence

$$\|f\|_{B^s_{p,q}(\mathbb{T}^n)} \sim \left(\sum_{j=0}^{\infty} 2^{jsq}\, \|f_j\|^q_{L^p(\mathbb{T}^n)}\right)^{1/q}$$

(with the usual modification when $q = \infty$).

2.7 Characterization by the Haar system

at7

Besov spaces can also be described via the size of their wavelet coefficients. To illustrate this, we start with low smoothness Besov spaces, which can be described using the Haar basis. (The next section is devoted to smoother spaces and bases.) For the results of this section, see e.g. [17, Corollary 5.3], [3, Section 7], [37, Theorem 1.58], [38, Theorem 2.21].
Let

$$\psi_M(x) := \begin{cases} 1, & \text{if } 0 \le x < 1/2 \\ -1, & \text{if } 1/2 \le x \le 1 \\ 0, & \text{if } x \notin [0,1] \end{cases}, \text{ and } \psi_F(x) := \left|\psi_M(x)\right|. \tag{2.6}$$ qa8

When $j \in \mathbb{N}$, we let

$$G^j := \begin{cases} \{F,M\}^n, & \text{if } j = 0 \\ \{F,M\}^n \setminus \{(F,F,\dots,F)\}, & \text{if } j > 0 \end{cases}. \tag{2.7}$$ qa1

For all $m \in \mathbb{Z}^n$, all $x \in \mathbb{R}^n$ and all $G \in \{F,M\}^n$, define

$$\Psi^G_m(x) := \prod_{r=1}^{n} \psi_{G_r}(x_r - m_r). \tag{2.8}$$ qa2

Finally, for all $m \in \mathbb{Z}^n$, all $j \in \mathbb{N}$, all $G \in G^j$ and all $x \in \mathbb{R}^n$, let

$$\Psi^{j,G}_m(x) := \begin{cases} \Psi^G_m(x), & \text{if } j = 0 \\ 2^{nj/2}\Psi^G_m(2^j x), & \text{if } j \ge 1 \end{cases}. \tag{2.9}$$ qa3

[3] Here, $|m| := \max_{l=1}^n |m_l|$.

Recall that the family $(\Psi_m^{j,G})$, called the Haar system, is an orthonormal basis of $L^2(\mathbb{R}^n)$ [37, Proposition 1.53]. Moreover, we have the following result [38, Theorem 2.21].

at11 **Proposition 2.8.** Let $s>0$, $1\le p<\infty$, and $1\le q\le\infty$ be such that $sp<1$. Let $f\in\mathscr{S}'(\mathbb{R}^n)$. Then $f\in B^s_{p,q}(\mathbb{R}^n)$ if and only if there exists a sequence $\left(\mu_m^{j,G}\right)_{j\ge0,\ G\in G^j,\ m\in\mathbb{Z}^n}$ such that

$$\sum_{j=0}^{\infty}\sum_{G\in G^j}\left(\sum_{m\in\mathbb{Z}^n}\left|\mu_m^{j,G}\right|^p\right)^{q/p}<\infty \tag{2.10}$$ qa4

(obvious modification when $q=\infty$) and

$$f=\sum_{j=0}^{\infty}\sum_{G\in G^j}\sum_{m\in\mathbb{Z}^n}\mu_m^{j,G}2^{-j(s-n/p)}2^{-nj/2}\Psi_m^{j,G}. \tag{2.11}$$ decompof

Here, the series in (2.11) converges unconditionally in $B^s_{p,q}(\mathbb{R}^n)$ when $q<\infty$. Moreover,

$$\|f\|_{B^s_{p,q}(\mathbb{R}^n)}\sim\left(\sum_{j=0}^{\infty}\sum_{G\in G^j}\left(\sum_{m\in\mathbb{Z}^n}\left|\mu_m^{j,G}\right|^p\right)^{q/p}\right)^{1/q} \tag{2.12}$$ qa5

(obvious modification when $q=\infty$).

Equivalently, Proposition 2.8 can be reformulated as follows. Consider the partition of $\mathbb{R}^n$ into standard dyadic cubes Q of side 2^{-j}. [4] For all $x\in\mathbb{R}^n$, denote by $Q_j(x)$ the unique dyadic cube of side 2^{-j} containing x. If $f\in L^1_{loc}(\mathbb{R}^n)$, define $E_j(f)(x):=\fint_{Q_j(x)}f$ for all $j\ge0$. We also set $E_{-1}(f):=0$. We have the following results (see [3, Theorem 5 with $m=0$] in $\mathbb{R}^n$; see also [4, Appendix A] in the framework of Sobolev spaces on $\mathbb{T}^n$).

caracBesov **Proposition 2.9.** Let $s>0$, $1\le p<\infty$, and $1\le q\le\infty$ be such that $sp<1$. Let $f\in L^1_{loc}(\mathbb{R}^n)$. Then

$$\|f\|^q_{B^s_{p,q}(\mathbb{R}^n)}\sim\sum_{j\ge0}2^{sjq}\|E_j(f)-E_{j-1}(f)\|^q_{L^p}$$

(obvious modification when $q=\infty$).

Similar results hold when $\mathbb{R}^n$ is replaced by $(0,1)^n$ or $\mathbb{T}^n$; it suffices to consider only dyadic cubes contained in $[0,1)^n$.

mq2 **Corollary 2.10.** Let $s>0$, $1\le p<\infty$, and $1\le q\le\infty$ be such that $sp<1$. Let $f\in L^1_{loc}(\mathbb{R}^n)$. Then

$$\|f\|^q_{B^s_{p,q}(\mathbb{R}^n)}\sim\sum_{j\ge0}2^{sjq}\|f-E_j(f)\|^q_{L^p}$$

(obvious modification when $q=\infty$).

Similar results hold when $\mathbb{R}^n$ is replaced by $(0,1)^n$ or $\mathbb{T}^n$.

[4] Thus the Q's are of the form $Q=2^{-j}\prod_{k=1}^n[m_k,m_k+1)$, with $m_k\in\mathbb{Z}$.

mp1 **Corollary 2.11.** Let $s > 0$, $1 \le p < \infty$, and $1 \le q \le \infty$ be such that $sp < 1$. Let $(\varphi_j)_{j\ge 0}$ be a sequence of functions on $(0,1)^n$ such that: for any j, φ_j is constant on each dyadic cube Q of size 2^{-j}. Assume that $\sum_{j\ge 1} 2^{sjq}\|\varphi_j - \varphi_{j-1}\|_{L^p}^q < \infty$. Then (φ_j) converges in L^p to some $\varphi \in B^s_{p,q}$, and we have

$$\|\varphi\|_{B^s_{p,q}((0,1)^n)} \lesssim \left(\sum_{j\ge 0} 2^{sjq}\|\varphi_j - \varphi_{j-1}\|_{L^p}^q\right)^{1/q}$$

(with the convention $\varphi_{-1} := 0$ and with the usual modification when $q = \infty$).

In the framework of Sobolev spaces, Corollaries 2.10 and 2.11 are easy consequences of Proposition 2.9; see [4, Appendix A, Theorem A.1] and [4, Appendix A, Corollary A.1]. The arguments in [4] apply with no changes to Besov spaces. Details are left to the reader.

2.8 Characterization via smooth wavelets

qa6

Proposition 2.8 has a counterpart when $sp \ge 1$; this requires smoother "mother wavelet" ψ_M and "father wavelet" ψ_F. Given ψ_F and ψ_M two real functions, define $\psi^{j,G}_m$ as in (2.7)–(2.9). Then [22, Chapter 6], [37, Section 1.7.3] for every integer $k > 0$ we may find some $\psi_F \in C^k_c(\mathbb{R})$ and $\psi_M \in C^k_c(\mathbb{R})$ such that the following result holds.

qb1 **Proposition 2.12.** Let $s > 0$, $1 \le p < \infty$, and $1 \le q \le \infty$ be such that $s < k$. Let $f \in \mathscr{S}'(\mathbb{R}^n)$. Then $f \in B^s_{p,q}(\mathbb{R}^n)$ if and only if there exists a sequence $\left(\mu^{j,G}_m\right)_{j\ge 0,\, G\in G^j,\, m\in\mathbb{Z}^n}$ such that

$$\sum_{j=0}^{\infty}\sum_{G\in G^j}\left(\sum_{m\in\mathbb{Z}^n}\left|\mu^{j,G}_m\right|^p\right)^{q/p} < \infty \tag{2.13}$$ qb2

(obvious modification when $q = \infty$) and

$$f = \sum_{j=0}^{\infty}\sum_{G\in G^j}\sum_{m\in\mathbb{Z}^n}\mu^{j,G}_m 2^{-j(s-n/p)}2^{-nj/2}\Psi^{j,G}_m. \tag{2.14}$$ qb3

Here, the series in (2.11) converges unconditionally in $B^s_{p,q}(\mathbb{R}^n)$ when $q < \infty$. Moreover,

$$\|f\|_{B^s_{p,q}(\mathbb{R}^n)} \sim \left(\sum_{j=0}^{\infty}\sum_{G\in G^j}\left(\sum_{m\in\mathbb{Z}^n}\left|\mu^{j,G}_m\right|^p\right)^{q/p}\right)^{1/q} \tag{2.15}$$ qb4

(obvious modification when $q = \infty$).

For further use, let us note that, if $f \in B^s_{p,q}(\mathbb{R}^n)$ for some $s > 0$, $1 \le p < \infty$ and $1 \le q \le \infty$, then we have

$$\mu^{j,G}_m = \mu^{j,G}_m(f) = 2^{j(s-n/p+n/2)} \int_{\mathbb{R}^n} f(x)\,\Psi^{j,G}_m(x)\,dx. \tag{2.16}$$ qb40

This immediately leads to the following consequence of Proposition 2.12, the proof of which is left to the reader.

qb400 **Corollary 2.13.** Let $s > 0$, $1 \le p < \infty$ and $1 \le q \le \infty$ be such that $s < k$. Assume that $f \in L^p(\mathbb{R}^n)$ is such that the coefficients $\mu^{j,G}_m$ given by (2.16) satisfy

$$\sum_{j=0}^{\infty} \sum_{G \in G^j} \left(\sum_{m \in \mathbb{Z}^n} \left| \mu^{j,G}_m \right|^p \right)^{q/p} = \infty \tag{2.17}$$ qb50

(obvious modification when $q = \infty$). Then $f \notin B^s_{p,q}(\mathbb{R}^n)$.

2.9 Nikolskiĭ type decompositions

mm8

In practice, we often do not know the Littlewood-Paley decomposition of some given f, but only a Nikolskiĭ representation (or decomposition) of f. More specifically, set $\mathscr{C}_j := B(0, 2^{j+2})$, with $j \in \mathbb{N}$. Let $f^j \in \mathscr{S}'$ satisfy

$$\operatorname{supp} \mathscr{F} f^j \subset \mathscr{C}_j,\ \forall\, j \in \mathbb{N}, \text{ and } f = \sum_j f^j \text{ in } \mathscr{S}'; \tag{2.18}$$ e230501

the decomposition $f = \sum_j f^j$ is a Nikolskiĭ decomposition of f. Note that the Littlewood-Paley decomposition is a special Nikolskiĭ decomposition.

We have the following result.

mm9 **Proposition 2.14.** Let $s > 0$, $1 \le p < \infty$, $1 \le q \le \infty$. Assume that (2.18) holds. Then we have

$$\left\| \sum_j f^j \right\|_{B^s_{p,q}} \lesssim \left(\sum_j 2^{sqj} \|f^j\|^q_{L^p} \right)^{1/q}, \tag{2.19}$$ 28024

with the usual modification when $q = \infty$.

The above was proved in [13, Lemma 1] (see also [40]) in the framework of Triebel-Lizorkin spaces $F^s_{p,q}$; the proof applies with no change to Besov spaces and will be omitted here. For related results in the framework of Besov spaces, see [35, Section 2.5.2, pp. 79-80] and [31, Section 2.3.2, Theorem, p. 105].

3 Positive cases

pos

We start with the trivial case.

tri **Case 1.** *Range.* $s>0$, $1\le p<\infty$, $1\le q\le\infty$, and $sp>n$.
Conclusion. $B^s_{p,q}(\Omega;\mathbb{S}^1)$ does have the lifting property.

Proof. Since $B^s_{p,q}(\Omega)\hookrightarrow C^0(\overline{\Omega})$ (Lemma 6.2), we may write $u=e^{\imath\varphi}$, with φ continuous. Locally, we have $\varphi=-\imath\ln u$, for some smooth determination ln of the complex logarithm. Then φ belongs to $B^s_{p,q}$ locally in $\overline{\Omega}$ (Lemma 6.24), and thus globally (Lemma 2.4). □

A **Case 2.** *Range.* $0<s<1$, $1\le p<\infty$, $1\le q\le\infty$, and $sp<1$.
Conclusion. $B^s_{p,q}(\Omega;\mathbb{S}^1)$ does have the lifting property.

Proof. The argument being essentially the one in [4, Section 1], we will be sketchy. Assume for simplicity that $\Omega=(0,1)^n$. Let $u\in B^s_{p,q}(\Omega;\mathbb{S}^1)$. For all $j\in\mathbb{N}$, consider the function U_j defined by

$$U_j(x):=\begin{cases}E_j(u)(x)/|E_j(u)(x)|, & \text{if } E_j(u)(x)\neq 0\\ 1, & \text{if } E_j(u)(x)=0\end{cases}.$$

Since $E_j(u)\to u$ a.e., we find that $U_j\to u$ a.e. on Ω. By induction on j, for all $j\in\mathbb{N}$ we construct a phase φ_j of U_j, constant on each dyadic cube of size 2^{-j}, and satisfying the inequality

$$|\varphi_j-\varphi_{j-1}|\le\pi|U_j-U_{j-1}|\quad\text{on }\Omega,\ \forall\, j\ge 1.^{5}\tag{3.1}$$ mq1

As in [4], (3.1) implies

$$|\varphi_j-\varphi_{j-1}|\lesssim|u-E_j(u)|+|u-E_{j-1}(u)|,$$

and thus, e.g. when $q<\infty$, we have

$$\sum_{j\ge1}2^{sjq}\|\varphi_j-\varphi_{j-1}\|^q_{L^p}\lesssim\sum_{j\ge0}2^{sjq}\|u-E_j(u)\|^q_{L^p}.$$

Applying Corollaries 2.10 and 2.11, we obtain that $\varphi_j\to\varphi$ in L^p to some $\varphi\in B^s_{p,q}(\Omega;\mathbb{R})$. Since φ_j is a phase of U_j and $U_j\to u$ a.e., we find that φ is a phase of u. In addition, we have the control $\|\varphi\|_{B^s_{p,q}}\lesssim\|u\|_{B^s_{p,q}}$. □

X **Case 3.** *Range.* $0<s<1$, $1\le p<\infty$, $1\le q<\infty$, and $sp=n$.
Conclusion. $B^s_{p,q}(\Omega;\mathbb{S}^1)$ does have the lifting property.

[5] Thus φ_j is the phase of U_j closest to φ_{j-1}.

Proof. Here, it will be convenient to work with $\Omega = \mathbb{T}^n$. Let $|\ |$ denote the sup norm in $\mathbb{R}^n$. Let $\rho \in C^\infty$ be a mollifier supported in $\{|x| \le 1\}$ and set $F(x,\varepsilon) := u * \rho_\varepsilon(x)$, $x \in \mathbb{T}^n$, $\varepsilon > 0$. Since $sp = n$, we have $u \in \mathrm{VMO}(\mathbb{T}^n)$, by Lemma 6.5. Let us recall that, if $u \in \mathrm{VMO}(\mathbb{T}^n;\mathbb{S}^1)$ then, for some $\delta > 0$ (depending on u) we have [14, Remark 3, p. 207]

$$\frac{1}{2} < |F(x,\varepsilon)| \le 1 \text{ for all } x \in \mathbb{T}^n \text{ and all } \varepsilon \in (0,\delta).^{6} \tag{3.2}$$

Define

$$w(x,\varepsilon) := \frac{F(x,\varepsilon)}{|F(x,\varepsilon)|} \text{ for all } x \in \mathbb{T}^n \text{ and all } \varepsilon \in (0,\delta).$$

Pick up a function $\psi \in C^\infty(\mathbb{T}^n \times (0,\delta);\mathbb{R})$ such that $w = e^{\imath\psi}$. We note that for all $j \in [\![1,n]\!]$ we have $\nabla\psi = -\imath\overline{w}\nabla w$, and $\partial_j|F| = |F|^{-1}(F\partial_j\overline{F} + \overline{F}\partial_j F)/2$. Therefore, (3.2) yields

$$\left|\nabla\psi\right| = |\nabla w| \lesssim |\nabla F|. \tag{3.3}$$

In view of (3.3) and estimate (6.41) in Lemma 6.18, we find that

$$|u|^q_{B^{s,p}_q(\mathbb{T}^n)} \gtrsim \int_0^\delta \varepsilon^{q-sq}\|(\nabla F)(\cdot,\varepsilon)\|^q_{L^p}\frac{d\varepsilon}{\varepsilon} \gtrsim \int_0^\delta \varepsilon^{q-sq}\|(\nabla\psi)(\cdot,\varepsilon)\|^q_{L^p}\frac{d\varepsilon}{\varepsilon}. \tag{3.4}$$

Combining (3.4) with the conclusion of Lemma 6.18, we obtain that the phase ψ has, on $\mathbb{T}^n$, a trace $\varphi \in B^s_{p,q}$, in the sense that the limit $\varphi := \lim_{\varepsilon\to 0}\psi(\cdot,\varepsilon)$ exists in $B^s_{p,q}$. In particular (using Lemma 6.4), we have that $\psi(\cdot,\varepsilon_j) \to \varphi$ a.e. along some sequence $\varepsilon_j \to 0$; this leads to $w(\cdot,\varepsilon_j) = e^{\imath\psi(\cdot,\varepsilon_j)} \to e^{\imath\varphi}$ a.e. Since, on the other hand, we have $\lim_{\varepsilon\to 0} w(\cdot,\varepsilon) = u$ a.e., we find that φ is a $B^s_{p,q}$ phase of u. □

The next case is somewhat similar to Case 3, so that our argument is less detailed.

Case 4. *Range.* $s = 1$, $p = n$, $1 \le q < \infty$.
Conclusion. $B^1_{n,q}(\Omega;\mathbb{S}^1)$ does have the lifting property.

Proof. We consider δ, w and ψ as in Case 3. The analog of (3.3) is the estimate

$$|\partial_j\partial_k\psi| + |\nabla\psi|^2 \lesssim |\partial_j\partial_k F| + |\nabla F|^2, \tag{3.5}$$

which is a straightforward consequence of the identities

$$\nabla\psi = -\imath\overline{w}\nabla w \text{ and } \partial_j\partial_k\psi = -\imath\overline{w}\partial_j\partial_k w + \imath w^2\partial_j w\partial_k w.$$

[6] For an explicit calculation leading to (3.2), see e.g. [23, p. 415].

Combining (3.5) with the second part of Lemma 6.19, we obtain

$$|u|^q_{B^1_{n,q}} \gtrsim \int_0^\delta \varepsilon^q \left(\sum_{j,k=1}^n \left\| \partial_j \partial_k \psi(\cdot,\varepsilon) \right\|^q_{L^n} + \left\| \partial_\varepsilon \partial_\varepsilon \psi(\cdot,\varepsilon) \right\|^q_{L^n} + \|\nabla \psi(\cdot,\varepsilon)\|^{2q}_{L^{2n}} \right) \frac{d\varepsilon}{\varepsilon}. \tag{3.6}$$ kg1a

By (3.6) and the first part of Lemma 6.19, we find that ψ has a trace $\varphi := \operatorname{tr}\psi \in B^1_{n,q}(\mathbb{T}^n)$. Clearly, φ is a $B^1_{n,q}$ phase of u. ☐

Y **Case 5.** *Range.* $s > 1$, $1 \le p < \infty$, $1 \le q < \infty$, $n = 2$, and $sp = 2$.
Or $s > 1$, $1 \le p < \infty$, $1 \le q \le p$, $n \ge 3$, and $sp = 2$.
Or: $s > 1$, $1 \le p < \infty$, $1 \le q \le \infty$, $n \ge 2$, and $sp > 2$.

Conclusion. $B^s_{p,q}(\Omega;\mathbb{S}^1)$ does have the lifting property.

Note that, in the critical case where $sp = 2$, our result is weaker in dimension $n \ge 3$ (when we ask $1 \le q \le p$) than in dimension 2 (when we merely ask $1 \le q < \infty$).

Proof. The general strategy is the same as in [4, Section 3, Proof of Theorem 3],[7] but the key argument (validity of (3.9) below) is much more involved in our case.

It will be convenient to work in $\Omega = \mathbb{T}^n$. Let $u \in B^s_{p,q}(\mathbb{T}^n;\mathbb{S}^1)$. Assume first that we do may write $u = e^{\imath\varphi}$, with $\varphi \in B^s_{p,q}((0,1)^n;\mathbb{R})$. Then $u, \varphi \in W^{1,p}$ (Lemma 6.4). We are thus in position to apply chain's rule and infer that $\nabla u = \imath u \nabla \varphi$, and therefore

$$\nabla \varphi = \frac{1}{\imath u} \nabla u = F, \text{ with } F := u \wedge \nabla u \in L^p(\mathbb{T}^n;\mathbb{R}^n). \tag{3.7}$$ at2

The assumptions on s, p, q imply that $F \in B^{s-1}_{p,q}$ (Lemma 6.22). We may now argue as follows. If φ solves (3.7), then $\nabla\varphi \in B^{s-1}_{p,q}$, and thus $\varphi \in B^s_{p,q}$ (Lemma 6.16). Next, since $u, \varphi \in W^{1,p} \cap L^\infty$, we find that

$$\nabla(u\, e^{-\imath\varphi}) = \nabla u\, e^{-\imath\varphi} - \imath u\, e^{-\imath\varphi} \nabla\varphi = \imath u\, e^{-\imath\varphi}(u \wedge \nabla u - \nabla \varphi) = 0.$$

Thus $u\, e^{-\imath\varphi}$ is constant, and therefore φ is, up to an appropriate additive constant, a $B^s_{p,q}$ phase of u.

There is a flaw in the above. Indeed, (3.7) need not have a solution. In $\mathbb{T}^n$, the necessary and sufficient conditions for the solvability of (3.7) are[8]

$$\int_{\mathbb{T}^n} F = \widehat{F}(0) = 0 \tag{3.8}$$ at5

and

$$\operatorname{curl} F = 0. \tag{3.9}$$ at1

Clearly, (3.8) holds.[9] We complete Case 5 by noting that (3.9) holds in the relevant range of s, p, q and n (Lemma 6.27). ☐

[7] See also [15].

[8] This is easily seen by an inspection of the Fourier coefficients.

[9] Expand $u \wedge \nabla u$ in Fourier series.

aa1 **3.1 Remark.** We briefly discuss the lifting problem when $s \le 0$. For such s, distributions in $B^s_{p,q}$ need not be integrable functions, and thus the meaning of the equality $u = e^{\imath\varphi}$ is unclear. We therefore address the following reasonable version of the lifting problem: let $u : \Omega \to \mathbb{S}^1$ be a measurable function such that $u \in B^s_{p,q}(\Omega)$. Is there any $\varphi \in L^1_{loc} \cap B^s_{p,q}(\Omega;\mathbb{R})$ such that $u = e^{\imath\varphi}$?

Let us note that the answer is trivially positive when $s < 0$, $1 \le p < \infty$, $1 \le q \le \infty$.

Indeed, let φ be any bounded measurable lifting of u. Then $\varphi \in B^s_{p,q}$, since $L^\infty \hookrightarrow B^s_{p,q}$ when $s < 0$ (see Lemma 6.3).

4 Negative cases

neg B **Case 6.** *Range.* $0 < s < 1$, $1 \le p < \infty$, $1 \le q < \infty$, $n \ge 2$, and $1 \le sp < n$.

Or $0 < s < 1$, $1 \le p < \infty$, $q = \infty$, $n \ge 2$, and $1 < sp < n$.

Conclusion. $B^s_{p,q}(\Omega;\mathbb{S}^1)$ does not have the lifting property.

Proof. We want to show that there exists a function $u \in B^s_{p,q}$ such that $u \ne e^{\imath\varphi}$ for any $\varphi \in B^s_{p,q}$.

For sufficiently small $\varepsilon > 0$, set $s_1 := s/(1-\varepsilon)$ and $p_1 := (1-\varepsilon)p$. By Lemma 6.1, we have $B^{s_1}_{p_1,q_1} \not\hookrightarrow B^s_{p,q}$ (for any q_1). We will use later this fact for $q_1 := (1-\varepsilon)q$.

Let $\psi \in B^{s_1}_{p_1,q_1} \setminus B^s_{p,q}$ and set $u := e^{\imath\psi}$. Then $u \in B^{s_1}_{p_1,q_1} \cap L^\infty$ (Lemma 6.23) and thus $u \in B^s_{p,q}$ (Lemma 6.6).

We claim that there is no $\varphi \in B^s_{p,q}$ such that $u = e^{\imath\varphi}$. Argue by contradiction. Since $u = e^{\imath\varphi} = e^{\imath\psi}$, the function $(\varphi-\psi)/2\pi$ belongs to $(B^s_{p,q} + B^{s_1}_{p_1,q_1})(\Omega;\mathbb{Z})$. By Lemma 6.25, this implies that $\varphi - \psi$ is constant, and thus $\psi \in B^s_{p,q}$, which is a contradiction. $\square$

xa2 **Case 7.** *Range.* $0 < s < \infty$, $1 \le p < \infty$, $1 \le q < \infty$, $n \ge 2$, and $1 \le sp < 2$.

Or $0 < s < \infty$, $1 \le p < \infty$, $q = \infty$, $n \ge 2$, and $1 < sp \le 2$.

Conclusion. $B^s_{p,q}(\Omega;\mathbb{S}^1)$ does not have the lifting property.

Proof. The proof is based on the example of a topological obstruction considering the case $n = 2$. Consider the map $u(x) = \dfrac{x}{|x|}$, $\forall\, x \in \mathbb{R}^2$.

We first prove that $u \in B^s_{p,q}(\Omega)$ for any smooth bounded domain $\Omega \subset \mathbb{R}^2$. We distinguish two cases: firstly, $q \le \infty$ and $sp < 2$ and secondly, $q = \infty$ and $sp = 2$.

In the first case, let $s_1 > s$ such that s_1 is not an integer and $1 < s_1 p < 2$, which implies $W^{s_1,p} = B^{s_1}_{p,p} \hookrightarrow B^s_{p,q}$. Since $u \in W^{s_1,p}$ [4, Section 4], we find that $u \in B^s_{p,q}$.

The second case is slightly more involved. By the Gagliardo-Nirenberg inequality (Lemma 6.6 below), it suffices to prove that $u \in B^2_{1,\infty}(\Omega)$. Using Proposition 2.6, a sufficient condition for this to hold is

$$\left\|\Delta_h^3 u\right\|_{L^1(\mathbb{R}^2)} \lesssim |h|^2, \ \forall\, h \in \mathbb{R}^2. \tag{4.1}$$

Since u is radially symmetric and 0-homogeneous, this amounts to checking that

$$\|\Delta_{e_1}^3 u\|_{L^1(\mathbb{R}^2)} < \infty. \tag{4.2}$$

However, by the mean-value theorem, for all $|x| \ge 1$ we have

$$|\Delta_{e_1}^3 u(x)| \lesssim 1/|x|^3, \tag{4.3}$$

while $\Delta_{e_1}^3 u$ is bounded in $B(0,1)$ since u is $\mathbb{S}^1$-valued. Using this fact and estimate (4.3), we obtain (4.2).

We next claim that u has no $B^s_{p,q}$ lifting in Ω provided $\Omega \subset \mathbb{R}^2$ is a smooth bounded domain containing the origin. Argue by contradiction, and assume that $u = e^{\imath\varphi}$ for some $\varphi \in B^s_{p,q}(\Omega)$. Let, as in [4, p. 50], $\theta \in C^\infty(\mathbb{R}^2 \setminus ([0,\infty) \times \{0\}))$ be such that $e^{\imath\theta} = u$.

Note that $\theta \in B^s_{p,q}(\omega)$ for every smooth bounded open set ω such that $\overline{\omega} \subset \mathbb{R}^2 \setminus ([0,\infty) \times \{0\})$. Since $(\varphi - \theta)/(2\pi)$ is $\mathbb{Z}$-valued, Lemma 6.25 yields that $\varphi - \theta$ is constant a.e. in $\Omega \setminus ([0,\infty) \times \{0\})$. Thus, $\theta \in B^s_{p,q}(\Omega)$. Similarly, $\widetilde{\theta} \in B^s_{p,q}(\Omega)$, where $\widetilde{\theta} \in C^\infty(\mathbb{R}^2 \setminus ((-\infty,0] \times \{0\}))$ is such that $e^{\imath\widetilde{\theta}} = u$. We find that $(\theta - \widetilde{\theta})/(2\pi) \in B^s_{p,q}(\Omega)$. However, this is a non constant integer-valued function. This contradicts Lemma 6.25 and proves non existence of lifting in $B^s_{p,q}$.

When $n \ge 3$, the above arguments lead to the following. Let $u(x) = \dfrac{(x_1,x_2)}{|(x_1,x_2)|}$, and let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain. Then $u \in B^s_{p,q}(\Omega;\mathbb{S}^1)$ and, if $0 \in \Omega$, then u has no $B^s_{p,q}$ lifting. □

5 Open cases

Case 8. *Range.* $s > 1$, $1 \le p < \infty$, $p < q < \infty$, $n \ge 3$, and $sp = 2$.

Discussion. This case is complementary to Case 5. In the above range, we conjecture that the conclusion of Case 5 still holds, i.e., that the space $B^s_{p,q}(\Omega;\mathbb{S}^1)$ does not have the lifting property. The non restriction property (Proposition 6.11) prevents us from extending the argument used in Case 5 to Case 8.

Case 9. *Range.* $s = 1$, $1 \le p < \infty$, $1 \le q < \infty$, $n \ge 3$, and $2 \le p < n$.

Or: $s = 1$, $1 \le p < \infty$, $q = \infty$, $n \ge 3$, and $2 < p \le n$.

Discussion. When $p = q = 2$, $B^1_{2,2}(\Omega;\mathbb{S}^1) = H^1(\Omega;\mathbb{S}^1)$ does have the lifting property [2, Lemma 1]. The remaining cases are open. The major difficulty arises from the extension of Lemma 6.22 to the range considered in Case 9.

T **Case 10.** *Range.* $s=0$, $1\le p<\infty$, $1\le q<\infty$ (and arbitrary n).

Discussion. As explained in Remark 3.1, we consider only measurable functions $u:\Omega\to\mathbb{S}^1$. We let $B^0_{p,q}(\Omega;\mathbb{S}^1):=\{u:\Omega\to\mathbb{S}^1;\, u$ measurable and $u\in B^0_{p,q}\}$, and for u in this space we are looking for a phase $\varphi\in L^1_{loc}\cap B^0_{p,q}$.

Note that $B^0_{p,\infty}(\Omega;\mathbb{S}^1)$ does have the lifting property. Indeed, in this case we have $L^\infty\subset B^0_{p,\infty}$ (Lemma 6.3) and then it suffices to argue as in the proof of Case 3.1. More generally, $B^0_{p,q}(\Omega;\mathbb{S}^1)$ has the lifting property when $L^\infty\hookrightarrow B^0_{p,q}$.[10] The remaining cases are open.

xa3 **Case 11.** *Range.* $0<s\le 1$, $p=1/s$, $q=\infty$ (and arbitrary n).

Discussion. We do not know whether $B^s_{p,q}(\Omega;\mathbb{S}^1)$ does have the lifting property.

xa4 **Case 12.** *Range.* $0<s\le 1$, $1<p<\infty$, $q=\infty$, $n\ge 3$, and $sp=n$.

Discussion. We do not know whether $B^s_{p,q}(\Omega;\mathbb{S}^1)$ does have the lifting property. The difficulty common to Cases 11 and 12 is that in these ranges $B^s_{p,\infty}\not\subset \mathrm{VMO}$, and thus we are unable to rely on the strategy used in Cases 3 and 4.

6 Analysis in Besov spaces

The results we state here are valid when Ω is a smooth bounded domain in $\mathbb{R}^n$, or $(0,1)^n$ or $\mathbb{T}^n$. However, in the proofs we will consider only one of these sets, the most convenient for the proof.

6.1 Embeddings

ape

Besovemb **6.1 Lemma.** Let $0<s_1<s_0<\infty$, $1\le p_0<\infty$, $1\le p_1<\infty$, $1\le q_0\le\infty$ and $1\le q_1\le\infty$. Then the following hold.

1. If $q_0<q_1$, then $B^s_{p,q_0}\hookrightarrow B^s_{p,q_1}$.
2. If $s_0-n/p_0=s_1-n/p_1$, then $B^{s_0}_{p_0,q_0}\hookrightarrow B^{s_1}_{p_1,q_0}$.
3. If $s_0-n/p_0>s_1-n/p_1$, then $B^{s_0}_{p_0,q_0}\hookrightarrow B^{s_1}_{p_1,q_1}$.
4. If $B^{s_0}_{p_0,q_0}\hookrightarrow B^{s_1}_{p_1,q_1}$, then $s_0-n/p_0\ge s_1-n/p_1$.

Consequently, when $q_0\le q_1$,

$$B^{s_0}_{p_0,q_0}\hookrightarrow B^{s_1}_{p_1,q_1}\iff s_0-\frac{n}{p_0}\ge s_1-\frac{n}{p_1}. \tag{6.1}$$ equiv

[10] A special case of this is $p=q=2$, since $B^0_{2,2}=L^2$. Another special case is $1<p\le 2\le q$. Indeed, in that case we have $L^\infty\hookrightarrow L^p=F^0_{p,2}\hookrightarrow B^0_{p,q}$ [35, Section 2.3.5, p. 51], [35, Section 2.3.2, Proposition 2, p. 47].

Proof. For item 1, see [35, Section 3.2.4]. For items 2 and 3, see [35, Section 3.3.1] or [30, Theorem 1, p. 82]. Item 4 follows from a scaling argument. And (6.1) is an immediate consequence of items 1–4. $\square$

For the next result, see e.g. [35, Section 2.7.1, Remark 2, pp. 130-131].

ka1 **6.2 Lemma.** Let $s>0$, $1\le p<\infty$, $1\le q\le\infty$ be such that $sp>n$. Then $B^s_{p,q}(\Omega)\hookrightarrow C^0(\overline{\Omega})$.

ia2 **6.3 Lemma.** Let $s<0$, $1\le p<\infty$ and $1\le q\le\infty$. Then $L^\infty\hookrightarrow B^s_{p,q}$.
Similarly, if $1\le p\le\infty$, then $L^\infty\hookrightarrow B^0_{p,\infty}$.

Proof. We present the argument when $\Omega=\mathbb{T}^n$. Let $f\in L^\infty$, with Fourier coefficients $(a_m)_{m\in\mathbb{Z}^n}$. Consider, as in Definition 2.5, the functions

$$f_j(x):=\sum_{m\in\mathbb{Z}^n} a_m\varphi_j(2\pi m)e^{2\imath\pi m\cdot x},\ \forall\, j\in\mathbb{N}.$$

By the (periodic version of) the multiplier theorem [35, Section 9.2.2, Theorem, p. 267] we have

$$\|f_j\|_{L^p}\lesssim\|f\|_{L^p},\ \forall\, 1\le p\le\infty,\ \forall\, j\in\mathbb{N}. \tag{6.2}$$ kb1

We find that $\|f_j\|_{L^p}\lesssim\|f\|_{L^p}\le\|f\|_{L^\infty}$, and thus (by Definition 2.5, and with the usual modification when $q=\infty$)

$$\|f\|_{B^s_{p,q}}\lesssim\left(\sum_j 2^{sjq}\right)^{1/q}<\infty.$$

The second part of the lemma follows from a similar argument. The proof is left to the reader. $\square$

An analogous proof leads to the following result. Details are left to the reader.

kc2 **6.4 Lemma.** Let $s>0$, $1\le p<\infty$ and $1\le q\le\infty$. Then $B^s_{p,q}\hookrightarrow L^p$.
More generally, if $k\in\mathbb{N}$, $s>k$, $1\le p<\infty$, and $1\le q\le\infty$, then $B^s_{p,q}\hookrightarrow W^{k,p}$.

B-VMO **6.5 Lemma.** Let $0<s<\infty$, $1\le p<\infty$ and $1\le q<\infty$ be such that $sp=n$. Then $B^s_{p,q}\hookrightarrow\mathrm{VMO}$.
Same conclusion if $0<s<\infty$, $1\le p<\infty$ and $q=\infty$ are such that $sp>n$.

Proof. Assume first that $q<\infty$. Let $p_1>\max\{n,p,q\}$ and set $s_1:=n/p_1$. By Lemma 6.1 and the fact that s_1 is not an integer, we have

$$B^s_{p,q}\hookrightarrow B^{s_1}_{p_1,q}\hookrightarrow B^{s_1}_{p_1,p_1}=W^{s_1,p_1}.$$

It then suffices to invoke the embedding

$$W^{s_1,p_1}\hookrightarrow\mathrm{VMO}\text{ when } s_1p_1=n\ [14,\text{ Example 2, p. 210}].$$

The case where $q=\infty$ is obtained via the first part of the proof. Indeed, it suffices to choose $0<s_1<\infty$, $1\le p_1<\infty$ and $0<q_1<\infty$ such that $s_1p_1=n$ and $B^s_{p,q}\hookrightarrow B^{s_1}_{p_1,q_1}$. Such s_1, p_1 and q_1 do exist, by Lemma 6.1. $\square$

For the following special case of the Gagliardo-Nirenberg embeddings, see e.g. [30, Remark 1, pp. 39-40].

gn **6.6 Lemma.** Let $0 < s < \infty$, $1 \le p < \infty$, $1 \le q \le \infty$, and $0 < \theta < 1$. Then $B^s_{p,q} \cap L^\infty \hookrightarrow B^{\theta s}_{p/\theta,q/\theta}$.

6.2 Restrictions

mo6

Captatio benevolentiæ. Let $f \in L^1(\mathbb{R}^2)$. Then, for a.e., $y \in \mathbb{R}$, the restriction $f(\cdot, y)$ of f to the line $\mathbb{R} \times \{y\}$ belongs to L^1. In this section and the next one, we examine some analogues of this property in the framework of Besov spaces.

For this purpose, we first introduce some notation for partial functions. Let $\alpha \subset \{1,\ldots,n\}$ and set $\overline{\alpha} := \{1,\ldots,n\} \setminus \alpha$. If $x = (x_1,\ldots,x_n) \in \mathbb{R}^n$, then we identify x with the couple $(x_\alpha, x_{\overline{\alpha}})$, where $x_\alpha := (x_j)_{j\in\alpha}$ and $x_{\overline{\alpha}} := (x_j)_{j\in\overline{\alpha}}$. Given a function $f = f(x_1,\ldots,x_n)$, we let $f_\alpha = f_\alpha(x_\alpha)$ denote the partial function $x_{\overline{\alpha}} \mapsto f(x)$. Another useful notation: given an integer m such that $1 \le m \le n$, set

$$I(n-m,n) := \{\alpha \subset \{1,\ldots,n\};\, \#\alpha = n-m\}.$$

Thus, when $\alpha \in I(n-m,n)$, $f_\alpha(x_\alpha)$ is a function of m variables.

When $q = p$, we have the following result.

oa1 **6.7 Lemma.** Let $1 \le m < n$. Let $s > 0$ and $1 \le p < \infty$. Let $f \in B^s_{p,p}(\mathbb{R}^n)$.

1. Let $\alpha \in I(n-m,n)$. Then, for a.e. $x_\alpha \in \mathbb{R}^{n-m}$, we have $f_\alpha(x_\alpha) \in B^s_{p,p}(\mathbb{R}^m)$.

2. We have

$$\|f\|^p_{B^s_{p,p}(\mathbb{R}^n)} \sim \sum_{\alpha\in I(n-m,n)} \int_{\mathbb{R}^{n-m}} \|f_\alpha(x_\alpha)\|^p_{B^s_{p,p}(\mathbb{R}^m)}\, dx_\alpha.$$

Proof. For the case where $m = 1$, see [35, Section 2.5.13, Theorem, (i), p. 115]. The general case is obtained by a straightforward induction on m. □

mo7 **6.8 Lemma.** Let $s > 0$, $1 \le p < \infty$ and $1 \le q \le p$. Let $1 \le m < n$ be an integer. Assume that $sp \ge m$ and let $f \in B^s_{p,q}(\mathbb{T}^n)$. Then, for every $\alpha \in I(n-m,n)$ and for a.e. $x_\alpha \in \mathbb{T}^{n-m}$, the partial map $f_\alpha(x_\alpha)$ belongs to $\mathrm{VMO}(\mathbb{T}^m)$.

Same conclusion if $s > 0$, $1 \le p < \infty$ and $1 \le q \le \infty$, and we have $sp > m$.

Similar conclusions when $\Omega = \mathbb{R}^n$ or $(0,1)^n$.

Proof. In view of the Sobolev embeddings (Lemma 6.1), we may assume that $sp = m$ and $q = p$. By Lemma 6.7 and Lemma 6.5, for a.e. x_α we have $f_\alpha(x_\alpha) \in B^s_{p,p}(\mathbb{T}^m) \hookrightarrow \mathrm{VMO}(\mathbb{T}^m)$. □

ad1 **6.9 Lemma.** Let $s>0$, $1\le p<\infty$ and $1\le q<\infty$. Let $M>s$ be an integer. Let $f\in B^s_{p,q}$. For $x'\in\mathbb{T}^{n-1}$, consider the partial map $v(x_n)=v_{x'}(x_n):=f(x',x_n)$, with $x_n\in\mathbb{T}$. Then there exists a sequence $(t_l)\subset(0,\infty)$ such that $t_l\to 0$ and for a.e. $x'\in\mathbb{T}^{n-1}$, we have

$$\lim_{l\to\infty}\frac{\left\|\Delta^M_{t_l}v\right\|_{L^p(\mathbb{T})}}{t_l^s}=0. \tag{6.3}$$ ce1

More generally, given a finite number of functions $f_j\in B^{s_j}_{p_j,q_j}$, with $s_j>0$, $1\le p_j<\infty$ and $1\le q_j<\infty$, and given an integer $M>\max_j s_j$, we may choose a common set A of full measure in $\mathbb{T}^{n-1}$ and a sequence (t_l) such that the analog of (ce1 6.3), i.e.,

$$\lim_{l\to\infty}\frac{\left\|\Delta^M_{t_l}f_j(x',\cdot)\right\|_{L^{p_j}(\mathbb{T})}}{t_l^{s_j}}=0, \tag{6.4}$$ cf1

holds simultaneously for all j and all $x'\in A$.

Proof. We treat the case of a single function; the general case is similar.

Set $g_t:=\left\|\Delta^M_{te_n}f\right\|_{L^p}$. By (equivnormhomogrn 2.3), we have $\int_0^1 t^{-sq-1}g_t^q\,dt<\infty$, which is equivalent to $\int_{1/2}^1\sum_{m\ge 0}2^{msq}g^q_{2^{-m}\sigma}\,d\sigma<\infty$. Therefore, there exists some $\sigma\in(1/2,1)$ such that

$$\sum_{m\ge 0}2^{msq}g^q_{2^{-m}\sigma}<\infty. \tag{6.5}$$ ce2

By (ce2 6.5) , we find that

$$\lim_{m\to\infty}\frac{g_{2^{-m}\sigma}}{(2^{-m}\sigma)^s}=0. \tag{6.6}$$ ce3

Using (ce3 6.6) we find that, along a subsequence (m_l), we have

$$\lim_{m\to\infty}\frac{\|\Delta_{2^{-m_l}\sigma}v\|_{L^p}}{(2^{-m_l}\sigma)^s}=0\quad\text{for a.e. } x'\in\mathbb{T}^{n-1}.$$

This implies (ce1 6.3) with $t_l:=2^{-m_l}\sigma$. □

6.3 (Non) restrictions

We now address the question whether, given $f\in B^s_{p,q}(\mathbb{R}^2)$, we have $f(x,\cdot)\in B^s_{p,q}(\mathbb{R})$ for a.e. $x\in\mathbb{R}$. This kind of questions can also be asked in higher dimensions. The answer crucially depends on the sign of $q-p$.

We start with a simple result.

qh1 **Proposition 6.10.** Let $s>0$ and $1\le q\le p<\infty$. Let $f\in B^s_{p,q}(\mathbb{R}^2)$. Then for a.e. $x\in\mathbb{R}$ we have $f(x,\cdot)\in B^s_{p,q}(\mathbb{R})$.

Proof. Let $f\in B^s_{p,q}(\mathbb{R}^2)$. Using (equivnormhomogrn 2.3) (part 2) and Hölder's inequality, we find that for every finite interval $[a,b]\subset\mathbb{R}$ and $M>s$ we have

$$\begin{aligned}\int_a^b |f(x,\cdot)|^q_{B^s_{p,q}(\mathbb{R})}\,dx &\sim \int_a^b\int_{\mathbb{R}}\frac{1}{|h|^{sq+1}}\left(\int_{\mathbb{R}}|\Delta^M_{he_2}f(x,y)|^p\,dy\right)^{q/p}dh dx\\ &\le (b-a)^{(p-q)/p}\int_{\mathbb{R}}\frac{1}{|h|^{sq+1}}\left(\int_{[a,b]\times\mathbb{R}}|\Delta^M_{he_2}f(x,y)|^p\,dxdy\right)^{q/p}dh\\ &\lesssim |f|^q_{B^s_{p,q}(\mathbb{R}^2)}<\infty\end{aligned}$$

whence the conclusion. □

When $q>p$, a striking phenomenon occurs.

17.26 **Proposition 6.11.** Let $s>0$ and $1\le p<q\le\infty$. Then there exists some compactly supported $f\in B^s_{p,q}(\mathbb{R}^2)$ such that for a.e. $x\in(0,1)$ we have $f(x,\cdot)\not\in B^s_{p,\infty}(\mathbb{R})$.

In particular, for any $1\le r<\infty$ and a.e. $x\in(0,1)$ we have $f(x,\cdot)\not\in B^s_{p,r}(\mathbb{R})$.

Before proceeding to the proof, let us note that if $f\in B^s_{p,q}(\mathbb{R}^2)$ then $f\in L^p(\mathbb{R}^2)$, and thus the partial function $f(x,\cdot)$ is a well-defined element of $L^p(\mathbb{R})$ for a.e. x.

Proof. Since $B^s_{p,q}(\mathbb{R}^2)\subset B^s_{p,\infty}(\mathbb{R}^2)$, $\forall\, q$, we may assume that $q<\infty$. We rely on the characterization of Besov spaces in terms of smooth wavelets, as in Section ga6 2.8.

We start by explaining the construction of f. Let ψ_F and ψ_M be as in Section ga6 2.8. With no loss of generality, we may assume that $\operatorname{supp}\psi_M\subset[0,a]$ with $a\in\mathbb{N}$. Consider $(\alpha,\beta)\subset(0,a)$ and $\gamma>0$ such that $\psi_M\ge\gamma$ in $[\alpha,\beta]$.

Set $\delta:=\beta-\alpha>0$ and consider some integer N such that $[0,1]\subset[\alpha-N\delta,\beta+N\delta]$. We look for an f of the form

$$f=\sum_{\ell=-N}^{N}\sum_{j\ge j_0}g_j^\ell, \tag{6.7}$$ qb5

with

$$\begin{aligned}g_j^\ell(x,y)=\mu_j 2^{-j(s-2/p)}\sum_{m_1\in I_j}&\psi_M(2^jx-m_1-\ell\delta)\\ &\times\psi_M(2^jy-m_1-2^{j+1}\ell a-\ell\delta).\end{aligned} \tag{6.8}$$ qb6

Here, the set I_j satisfying

$$I_j\subset\{0,1,\ldots,2^j\}, \tag{6.9}$$ qb7

the integer j_0 and the coefficients $\mu_j > 0$ will be defined later.

We consider the partial sums $f_J^\ell := \sum_{j=j_0}^J g_j^\ell$. Clearly, we have $f_J^\ell \in C^k$ and, provided j_0 is sufficiently large,

$$\sup f_J^\ell \subset K_l := [-N\delta, 5/4] \times [2\ell a - 1/4, (2\ell+1)a + 1/4].$$

We next note that the compacts K_ℓ are mutually disjoint. Using Proposition 2.6 item 2, we easily find that

$$\left\| \sum_{\ell=-N}^{N} f_J^\ell \right\|_{B^s_{p,q}(\mathbb{R}^2)}^q \sim \sum_{\ell=-N}^{N} \left\| f_J^\ell \right\|_{B^s_{p,q}(\mathbb{R}^2)}^q . \tag{6.10}$$

On the other hand, if ψ_M and ψ_F are wavelets such that Proposition 2.12 holds, then so are $\psi_F(\cdot - \lambda)$ and $\psi_M(\cdot - \lambda)$, $\forall\, \lambda \in \mathbb{R}$ [37, Theorem 1.61 *(ii)*, Theorem 1.64]. Combining this fact with (6.10), we find that

$$\left\| \sum_{\ell=-N}^{N} f_J^\ell \right\|_{B^s_{p,q}(\mathbb{R}^2)}^q \sim \sum_{j=j_0}^{J} \left(\# I_j (\mu_j)^p \right)^{q/p} . \tag{6.11}$$

We now make the size assumption

$$\sum_{j=j_0}^{\infty} \left(\# I_j (\mu_j)^p \right)^{q/p} < \infty. \tag{6.12}$$

By (6.11) and (6.12), we see that the formal series in (6.7) defines a compactly supported $f \in B^s_{p,q}(\mathbb{R}^2)$, with $\sum_{\ell=-N}^N f_J^\ell \to f$ in $B^s_{p,q}(\mathbb{R}^2)$ (and therefore in $L^p(\mathbb{R}^2)$) as $J \to \infty$.

We next investigate the $B^s_{p,\infty}$ norm of the restrictions $f_J^\ell(x, \cdot)$. As in (6.10), we have

$$\left\| \sum_{\ell=-N}^{N} f_J^\ell(x,\cdot) \right\|_{B^s_{p,\infty}(\mathbb{R})} \sim \sum_{\ell=-N}^{N} \| f_J^\ell(x,\cdot) \|_{B^s_{p,\infty}(\mathbb{R})} . \tag{6.13}$$

Rewriting (6.8) as

$$\begin{aligned} g_j^\ell(x,y) = \mu_j 2^{-j(s-1/p)} 2^{j/p} &\sum_{m_1 \in I_j} \psi_M(2^j x - m_1 - \ell\delta) \\ &\times \psi_M(2^j y - m_1 - 2^{j+1}\ell a - \ell\delta), \end{aligned} \tag{6.14}$$

we obtain

$$\| f_J^\ell(x,\cdot) \|_{B^s_{p,\infty}(\mathbb{R})}^p \sim \sup_{j_0 \le j \le J} 2^j (\mu_j)^p \sum_{m_1 \in I_j} |\psi_M(2^j x - m_1 - \ell\delta)|^p . \tag{6.15}$$

We now make the size assumption

$$\sup_{j \ge j_0} 2^j (\mu_j)^p \sum_{\ell=-N}^{N} \sum_{m_1 \in I_j} |\psi_M(2^j x - m_1 - \ell\delta)|^p = \infty, \ \forall\, x \in [0,1]. \tag{6.16}$$

Then we claim that for a.e. $x \in (0,1)$ we have

$$f(x,\cdot) \notin B^s_{p,\infty}(\mathbb{R}). \tag{6.17}$$

Indeed, since $\sum_{\ell=-N}^{N} f_J^\ell \to f$ in $L^p(\mathbb{R}^2)$, for a.e. $x \in \mathbb{R}$ we have

$$\sum_{\ell=-N}^{\ell} f_J^\ell(x,\cdot) \to f(x,\cdot) \text{ in } L^p(\mathbb{R}). \tag{6.18}$$

We claim that for every $x \in [0,1]$ such that (6.18) holds, we have $f(x,\cdot) \notin B^s_{p,\infty}(\mathbb{R})$. Indeed, on the one hand (6.16) implies that for some ℓ we have $\lim_{J\to\infty} \|f_J^\ell(x,\cdot)\|_{B^s_{p,\infty}(\mathbb{R})} = \infty$. We assume e.g. that this holds when $\ell = 0$. Thus

$$\sup_{j \ge j_0} 2^j (\mu_j)^p \sum_{m_1 \in I_j} |\psi_M(2^j x - m_1)|^p = \infty. \tag{6.19}$$

On the other hand, assume by contradiction that $f(x,\cdot) \in B^s_{p,\infty}(\mathbb{R})$. Then we may write $f(x,\cdot)$ as in (2.14), with coefficients as in (2.16). In particular, taking into account the explicit formula of g_j^ℓ and the fact that $\sum_{\ell=-N}^{N} f_J^\ell(x,\cdot) \to f(x,\cdot)$ in $L^p(\mathbb{R})$, we find that for $k \ge j_0$ and $m_1 \in I_j$ we have

$$\begin{aligned} \mu_{m_1}^{k,\{M\}}(f(x,\cdot)) &= \mu_{m_1}^{k,\{M\}}\left(\sum_{j=j_0}^{J} g_j^0(x,\cdot)\right) = \mu_{m_1}^{k,\{M\}}(g_k^0(x,\cdot)) \\ &= 2^{k/p}\, \mu_k\, \psi_M(2^k x - m_1),\ \forall J \ge k. \end{aligned} \tag{6.20}$$

We obtain a contradiction combining (6.19), (6.20) and Corollary 2.13.

It remains to construct I_j and μ_j satisfying (6.9), (6.12) and (6.16). We will let $I_j = [\![s_j, t_j]\!]$, with $0 \le s_j \le t_j \le 2^j$ integers to be determined later. Set $t := q/p \in (1,\infty)$ and

$$\mu_j := \left(\frac{1}{(t_j - s_j + 1)\, j^{1/t} \ln j}\right)^{1/p}.$$

Clearly, (6.9) and (6.12) hold. It remains to define I_j in order to have (6.16). Consider the dyadic segment $L_j := [s_j/2^j, t_j/2^j]$. We claim that

$$\sum_{\ell=-N}^{N} \sum_{m_1 \in I_j} |\psi_M(2^j x - m_1 - \ell\,\delta)|^p \ge \gamma^p,\ \forall x \in L_j. \tag{6.21}$$

Indeed, let $m_1 \in [s_j, t_j]$ be the integer part of $2^j x$. By the definition of δ and by choice of N, there exists some $\ell \in [\![-N, N]\!]$ such that $\alpha \le 2^j x - m_1 - \ell\,\delta \le \beta$, whence the conclusion.

By the above, (6.16) holds provided we have

$$\sup_{j \ge j_0} 2^j (\mu_j)^p\, \mathbb{1}_{L_j(x)} = \infty,\ \forall x \in [0,1]. \tag{6.22}$$

We next note that

$$2^j(\mu_j)^p \sim \frac{1}{|L_j|\, j^{1/t} \ln j} = \frac{u_j}{|L_j|}, \tag{6.23}$$ qc600

where $u_j := 1/(j^{1/t} \ln j)$ satisfies

$$\sum_{j \geq j_0} u_j = \infty. \tag{6.24}$$ qc6000

In view of (6.23) and (6.24), existence of I_j satisfying (6.22) is a consequence of Lemma 6.12 below. The proof of Proposition 6.11 is complete. □

tempSeq **6.12 Lemma.** Consider a sequence (u_j) of positive numbers such that $\sum_{j\geq j_0} u_j = \infty$. Then there exists a sequence (L_j) of dyadic intervals $L_j = [s_j/2^j, t_j/2^j]$, such that:

1. $s_j, t_j \in \mathbb{N}$, $0 \leq s_j < 2^j$.
2. $|L_j| = o(u_j)$ as $j \to \infty$.
3. Every $x \in [0,1]$ belongs to infinitely many L_j's.

Proof. Consider a sequence (v_j) of positive numbers such that $\sum_{j\geq j_0} v_j u_j = \infty$ and $v_j \to 0$. Let L_{j_0} be the largest dyadic interval of the form $[0, t_{j_0}/2^{j_0}]$ of length $\leq v_{j_0} u_{j_0}$. This defines $s_{j_0} = 0$ and t_{j_0}.

Assuming $L_j = [s_j/2^j, t_j/2^j] = [a_j, b_j]$ constructed for some $j \geq j_0$, one of the following two occurs. Either $b_j < 1$ and then we let L_{j+1} be the largest dyadic interval of the form $[2t_j/2^{j+1}, t_{j+1}/2^{j+1}]$ such that $|L_{j+1}| \leq v_{j+1} u_{j+1}$. Or $b_j \geq 1$, and then we let L_{j+1} be the largest dyadic interval of the form $[0, t_{j+1}/2^{j+1}]$ such that $|L_{j+1}| \leq v_{j+1} u_{j+1}$.

Using the assumption $\sum_{j\geq j_0} v_j u_j = \infty$ and the fact that $|L_j| \geq v_j u_j - 2^{-j}$, we easily find that for every $j \geq j_0$ there exists some $k > j$ such that $L_k = [a_k, b_k]$ satisfies $b_k \geq 1$, and thus the intervals L_j cover each point $x \in [0,1]$ infinitely many times. □

r10 **6.13 Remark.** Following a suggestion of the first author, Brasseur investigated the non restriction property established in Proposition 6.11. In [10] (which is independent of the present work), Brasseur extends Proposition 6.11 to the full range $0 < p < q \leq \infty$; the construction is somewhat similar to ours (based on the size of the coefficients μ_j in the decomposition (6.8)), but relying on a different decomposition (subatomic instead of wavelets). [10] also contains an interesting positive result: it exhibits function spaces X intermediate between $B^s_{p,q}(\mathbb{R})$ and $\bigcup_{\varepsilon>0} B^{s-\varepsilon}_{p,q}(\mathbb{R})$ such that, if $f \in B^s_{p,q}(\mathbb{R}^2)$, then for a.e. $x \in \mathbb{R}$ we have $f(x,\cdot) \in X$.

6.4 Poincaré type inequalities

The next Poincaré type inequality for Besov spaces is certainly well-known, but we were unable to find a reference in the literature.

ad2 **6.14 Lemma.** Let $0 < s < 1$, $1 \le p < \infty$, and $1 \le q \le \infty$. Then we have

$$\left\| f - \fint f \right\|_{L^p} \lesssim |f|_{B^s_{p,q}}, \quad \forall f : \Omega \to \mathbb{R} \text{ measurable function.} \tag{6.25}$$ PBesov

Recall (Proposition 2.6) that the semi-norm in (6.25) is given by

$$|f|_{B^s_{p,q}} = |f|_{B^s_{p,q}(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} |h|^{-sq} \|\Delta_h f\|^q_{L^p} \frac{dh}{|h|^n} \right)^{1/q} \tag{6.26}$$ aa4

when $q < \infty$, with the obvious modifications when $q = \infty$ or $\mathbb{R}^n$ is replaced by Ω.

Proof. By (2.2), we have $\|f\|_{B^s_{p,q}} \sim \|f\|_{L^p} + |f|_{B^s_{p,q}}$. Recall that the embedding $B^s_{p,q} \hookrightarrow L^p$ is compact [33, Theorem 3.8.3, p. 296]. From this we infer that (6.25) holds for every function $f \in B^s_{p,q}$. Indeed, assume by contradiction that this is not the case. Then there exists a sequence of functions $(f_j)_{j \ge 1} \subset B^s_{p,q}$ such that, for every j,

$$1 = \left\| f_j - \fint f_j \right\|_{L^p} \ge j \left| f_j \right|_{B^s_{p,q}}.$$

Set $g_j := f_j - \fint f_j$. Then, up to a subsequence, we have $g_j \to g$ in L^p, where $\|g\|_{L^p} = 1$ and $\fint g = 0$. We claim that g is constant in Ω (and thus $g = 0$). Indeed, by the Fatou lemma, for every $h \in \mathbb{R}^n$ we have

$$\|\Delta_h g\|_{L^p} \le \liminf \|\Delta_h g_j\|_{L^p} = \liminf \|\Delta_h f_j\|_{L^p}. \tag{6.27}$$ aa3

By (6.26), (6.27) and the Fatou lemma, we have

$$|g|_{B^s_{p,q}} \le \liminf |g_j|_{B^s_{p,q}} = \liminf |f_j|_{B^s_{p,q}} = 0;$$

thus $g = 0$, as claimed. This contradicts the fact that $\|g\|_{L^p} = 1$.

Let us now establish (6.25) only assuming that $|f|_{B^s_{p,q}} < \infty$. We start by reducing the case where $q = \infty$ to the case where $q < \infty$. This reduction relies on the straightforward estimate

$$|f|_{B^\sigma_{p,r}} \lesssim |f|_{B^s_{p,\infty}}, \quad \forall\, 0 < \sigma < s,\ \forall\, 0 < r < \infty.$$

So let us assume that $q < \infty$. For every integer $k \ge 1$, let $\Phi_k : \mathbb{R} \to \mathbb{R}$ be given by

$$\Phi_k(t) := \begin{cases} t, & \text{if } |t| \le k \\ -k, & \text{if } t \le -k \\ k, & \text{if } t \ge k \end{cases}.$$

Clearly, Φ_k is 1-Lipschitz, so that (6.26) easily yields

$$|\Phi_k(f)|_{B^s_{p,q}} \le |f|_{B^s_{p,q}} \tag{6.28}$$

controlphik

and (by dominated convergence, using $q<\infty$ and (6.26))

$$\lim_{k\to\infty} |\Phi_k(f)-f|_{B^s_{p,q}} = 0. \tag{6.29}$$

convphikf

Since $\Phi_k(f)\in L^\infty(\Omega)\subset L^p(\Omega)$, one has $\Phi_k(f)\in B^s_{p,q}$ for every k. Therefore, (6.25) and (6.28) imply

$$\|\Phi_k(f)-c_k\|_{L^p} \lesssim |\Phi_k(f)|_{B^s_{p,q}} \le |f|_{B^s_{p,q}} \tag{6.30}$$

phikck

with $c_k := \fint \Phi_k(f)$. Thanks to (6.29), we may pick up an increasing sequence of integers $(\lambda_k)_{k\ge 1}$ such that, for every k, $\left|\Phi_{\lambda_{k+1}}(f)-\Phi_{\lambda_k}(f)\right|_{B^s_{p,q}} \le 2^{-k}$. Applying (6.25) to $\Phi_{\lambda_{k+1}}(f)-\Phi_{\lambda_k}(f)$, one therefore has

$$\left\|\left(\Phi_{\lambda_{k+1}}(f)-c_{\lambda_{k+1}}\right)-\left(\Phi_{\lambda_k}(f)-c_{\lambda_k}\right)\right\|_{L^p} \lesssim \left|\Phi_{\lambda_{k+1}}(f)-\Phi_{\lambda_k}(f)\right|_{B^s_{p,q}} \le 2^{-k},$$

which entails that $\Phi_{\lambda_k}(f)-c_{\lambda_k}\to g$ in L^p as $k\to\infty$. Up to a subsequence, one can also assume that $\Phi_{\lambda_k}(f)(x)-c_{\lambda_k}\to g(x)$ for a.e. $x\in\Omega$. Take any $x\in\Omega$ such that $\Phi_{\lambda_k}(f)(x)-c_{\lambda_k}\to g(x)$. Since $\Phi_{\lambda_k}(f)(x)\to f(x)$ as $k\to\infty$, one obtains

$$\lim_{k\to\infty} c_{\lambda_k} = c\in\mathbb{C}. \tag{6.31}$$

ckc

Finally, (6.30), (6.31) and the Fatou lemma yield $\|f-c\|_{L^p}\lesssim |f|_{B^s_{p,q}}$, from which (6.25) easily follows. □

We next state and prove a generalization of Lemma 6.14.

ad3 **6.15 Lemma.** Let $0<s<1$, $1\le p<\infty$, $1\le q\le\infty$, and $\delta\in(0,1]$. Define

$$|f|_{B^s_{p,q,\delta}} := \left(\int_{|h|\le\delta} |h|^{-sq}\|\Delta_h f\|^q_{L^p}\frac{dh}{|h|^n}\right)^{1/q} \tag{6.32}$$

ad4

when $q<\infty$, with the obvious modifications when $q=\infty$ or $\mathbb{R}^n$ is replaced by Ω. Then we have

$$\left\|f-\fint f\right\|_{L^p} \lesssim |f|_{B^s_{p,q,\delta}}, \quad \forall f:\Omega\to\mathbb{R} \text{ measurable function.} \tag{6.33}$$

ad5

Proof. Recall that $\|f\|_{B^s_{p,q}} \sim \|f\|_{L^p}+|f|_{B^s_{p,q,\delta}}$ (Proposition 2.6). We continue as in the proof of Lemma 6.14. □

We end with an estimate involving derivatives.

at4 **6.16 Lemma.** Let $s > 0$, $1 < p < \infty$ and $1 \le q \le \infty$. Let $f \in \mathscr{D}'(\Omega)$ be such that $\nabla f \in B^{s-1}_{p,q}(\Omega)$. Then $f \in B^s_{p,q}(\Omega)$ and

$$\left\| f - \fint f \right\|_{B^s_{p,q}} \lesssim \|\nabla f\|_{B^{s-1}_{p,q}}. \tag{6.34}$$ at9

The above result is well-known, but we were unable to find it in the literature; for the convenience of the reader, we present the short argument when $\Omega = \mathbb{T}^n$.

Proof. We use the notation in Proposition 2.7 and the following result [16, Lemma 2.1.1, p. 16]: we have

$$\|f_j\|_{L^p} \sim 2^{-j}\|\nabla f_j\|_{L^p}, \quad \forall\, 1 \le p \le \infty,\ \forall\, j \ge 1. \tag{6.35}$$ mm3

By combining (6.35) with Proposition 2.7, we obtain, e.g. when $q < \infty$:

$$\begin{aligned} \|f - a_0\|^q_{B^s_{p,q}} &= \left\| \sum_{j\ge 1} f_j \right\|^q_{B^s_{p,q}} \sim \sum_{j\ge 1} 2^{sjq}\|f_j\|^q_{L^p} \\ &\lesssim \sum_{j \ge 1} 2^{sjq} 2^{-jq} \|\nabla f_j\|^q_{L^p} \sim \|\nabla f\|^q_{B^{s-1}_{p,q}}. \end{aligned} \tag{6.36}$$ mm4

In particular, $f \in L^1$ (Lemma 6.4), and thus $a_0 = \fint f$. Therefore, (6.36) is equivalent to (6.34). □

mn41 **6.17 Remark.** With more work, Lemma 6.16 can be extended to the case where $p = 1$. Although this will not be needed here, we sketch below the argument. With the notation in Section 2.3, consider the Littlewood-Paley decomposition $f = \sum f_j$, with $f_j := \sum a_m \varphi_j(2\pi m) e^{2\imath\pi m\cdot x}$. Note that the Littlewood-Paley decomposition of ∇f is simply given by

$$\nabla f = \sum \nabla f_j. \tag{6.37}$$ mn7

In the spirit of [16, Lemma 2.1.1, p. 16] (see also [5, Proof of Lemma 1]), one may prove that we have the following analog of (6.35):

$$\|f_j\|_{L^p} \sim 2^{-j}\|\nabla f_j\|_{L^p}, \quad \forall\, 1 \le p \le \infty,\ \forall\, j \ge 1. \tag{6.38}$$ mn6

Using Definition 2.5, (6.37) and (6.38), we obtain (6.36). We conclude as in the proof of Lemma 6.16.

6.5 Characterization of $B^s_{p,q}$ via extensions

The type of results we present in this section are classical for functions defined on the whole $\mathbb{R}^n$ and for the harmonic extension. Such results were obtained by Uspenskiĭ in the early sixties [39]. For further developments, see [35, Section 2.12.2, Theorem, p. 184]; see also Section 2.5. When the harmonic extension is replaced by other extensions by regularization, the kind of results we present below were known to experts at least for maps defined on $\mathbb{R}^n$; see [21, Section 10.1.1, Theorem 1, p. 512] and also [27] for a systematic treatment of extensions by smoothing. The local variants (involving extensions by averages in domains) we present below could be obtained by adapting the arguments we developed in a more general setting in [27], and which are quite involved. However, we present here a more elementary approach, inspired by [21], sufficient to our purpose. In what follows, we let $|\ |$ denote the $\|\ \|_\infty$ norm in $\mathbb{R}^n$.

For simplicity, we state our results when $\Omega = \mathbb{T}^n$, but they can be easily adapted to arbitrary Ω.

6.18 Lemma. Let $0 < s < 1$, $1 \le p < \infty$, $1 \le q \le \infty$, and $\delta \in (0,1]$. Set $V_\delta := \mathbb{T}^n \times (0,\delta)$.

1. Let $F \in C^\infty(V_\delta)$. If

$$\left(\int_0^{\delta/2} \varepsilon^{q-sq} \|(\nabla F)(\cdot,\varepsilon)\|^q_{L^p} \frac{d\varepsilon}{\varepsilon}\right)^{1/q} < \infty \tag{6.39}$$

(with the obvious modification when $q = \infty$), then F has a trace $f \in B^s_{p,q}(\mathbb{T}^n)$, satisfying

$$|f|_{B^s_{p,q,\delta}} \lesssim \left(\int_0^{\delta/2} \varepsilon^{q-sq} \|(\nabla F)(\cdot,\varepsilon)\|^q_{L^p} \frac{d\varepsilon}{\varepsilon}\right)^{1/q}. \tag{6.40}$$

2. Conversely, let $f \in B^s_{p,q}(\mathbb{T}^n)$. Let $\rho \in C^\infty$ be a mollifier supported in $\{|x| \le 1\}$ and set $F(x,\varepsilon) := f * \rho_\varepsilon(x)$, $x \in \mathbb{T}^n$, $0 < \varepsilon < \delta$. Then

$$\left(\int_0^{\delta} \varepsilon^{q-sq} \|(\nabla F)(\cdot,\varepsilon)\|^q_{L^p} \frac{d\varepsilon}{\varepsilon}\right)^{1/q} \lesssim |f|_{B^s_{p,q,\delta}}. \tag{6.41}$$

A word about the existence of the trace in item 1 above. We will prove below that for every $0 < \lambda < \delta/4$ we have

$$\left|F_{|\mathbb{T}^n \times \{\lambda\}}\right|_{B^s_{p,q}} \lesssim \left(\int_0^{\delta/2} \varepsilon^{q-sq} \|(\nabla F)(\cdot,\varepsilon)\|^q_{L^p} \frac{d\varepsilon}{\varepsilon}\right)^{1/q}. \tag{6.42}$$

By Lemma 6.14 and a standard argument, this leads to the existence, in $B^s_{p,q}$, of the limit $\lim_{\varepsilon\to 0} F(\cdot,\varepsilon)$. This limit is the trace of F on $\mathbb{T}^n$ and clearly satisfies (6.40).

Proof. For simplicity, we treat only the case where $q < \infty$; the case where $q = \infty$ is somewhat simpler and is left to the reader.

We claim that in item 1 we may assume that $F \in C^\infty(\overline{V_\delta})$. Indeed, assume that (6.40) holds (with $\operatorname{tr} F = F(\cdot, 0)$) for such F. By Lemma 6.14, we have the stronger inequality $\left\| \operatorname{tr} F - \fint \operatorname{tr} F \right\|_{B^s_{p,q}} \lesssim I(F)$, where $I(F)$ is the integral in (6.39). Then, by a standard approximation argument, we find that (6.40) holds for every F.

So let $F \in C^\infty(\overline{V_\delta})$, and set $f(x) := F(x, 0)$, $\forall\, x \in \mathbb{T}^n$. Denote by $I(F)$ the quantity in (6.39). We have to prove that f satisfies

$$|f|_{B^s_{p,q}} \lesssim I(F). \tag{6.43}$$

ab210

If $|h| \le \delta$, then

$$|\Delta_h f(x)| \le |f(x+h) - F(x+h/2, |h|/2)| + |f(x) - F(x+h/2, |h|/2)|. \tag{6.44}$$

cg4

By symmetry and (6.44), the estimate (6.43) will follow from

$$\left(\int_{|h| \le \delta} |h|^{-sq} \| f - F(\cdot + h/2, |h|/2) \|^q_{L^p} \frac{dh}{|h|^n} \right)^{1/q} \lesssim I(F). \tag{6.45}$$

cg5

In order to prove (6.45), we start from

$$\begin{aligned} |F(x+h/2, |h|/2) - f(x)| &= \left| \int_0^1 (\nabla F)(x + th/2, t|h|/2) \cdot (h/2, |h|/2)\, dt \right| \\ &\le |h| \int_0^1 |\nabla F(x + th/2, t|h|/2)|\, dt. \end{aligned} \tag{6.46}$$

cg8

Let $J(F)$ denote the left-hand side of (6.45). Using (6.46) and setting $r := |h|/2$, we obtain

$$\begin{aligned} [J(F)]^q &\le \int_{|h| \le \delta} |h|^{q - sq} \left(\int_0^1 \|\nabla F(\cdot + th/2, t|h|/2)\|_{L^p}\, dt \right)^q \frac{dh}{|h|^n} \\ &= \int_{|h| \le \delta} |h|^{q - sq} \left(\int_0^1 \|\nabla F(\cdot, t|h|/2)\|_{L^p}\, dt \right)^q \frac{dh}{|h|^n} \\ &\sim \int_0^{\delta/2} r^{q - sq - 1} \left(\int_0^1 \|\nabla F(\cdot, tr)\|_{L^p}\, dt \right)^q dr \\ &\sim \int_0^{\delta/2} r^{-sq - 1} \left(\int_0^r \|\nabla F(\cdot, \sigma)\|_{L^p}\, d\sigma \right)^q dr \lesssim [I(F)]^q. \end{aligned} \tag{6.47}$$

ch1

The last inequality is a special case of Hardy's inequality [32, Chapter 5, Lemma 3.14], that we recall here when $\delta = \infty$.[11] Let $1 \le q < \infty$ and $1 < \rho < \infty$. If $G \in W^{1,1}_{loc}([0, \infty))$, then

$$\int_0^\infty \frac{|G(r) - G(0)|^q}{r^\rho}\, dr \le \left(\frac{q}{\rho - 1} \right)^q \int_0^\infty \frac{|G'(r)|^q}{r^{\rho - q}}\, dr. \tag{6.48}$$

e04269

[11] But the argument adapts to a finite δ; see e.g. [9, Proof of Corollary 7.2].

We obtain (6.47) by applying (6.48) with $G'(r) := \|\nabla F(\cdot,r)\|_{L^p}$ and $\rho := sq+1$. The proof of item 1 is complete.

We next turn to item 2. We have

$$\nabla F(x,\varepsilon) = \frac{1}{\varepsilon} f * \eta_\varepsilon(x), \tag{6.49}$$

kh2

where ∇ stands for $(\partial_1,\dots,\partial_n,\partial_\varepsilon)$. Here, $\eta = (\eta^1,\dots,\eta^{n+1}) \in C^\infty(\mathbb{T}^n;\mathbb{R}^{n+1})$ is supported in $\{|x| \le 1\}$ and is given in coordinates by

$$\eta^j = \partial_j \rho,\ \forall\, j \in [\![1,n]\!],\ \eta^{n+1} = -\operatorname{div}(x\rho). \tag{6.50}$$

kh3

Noting that $\int \eta = 0$, we find that

$$\begin{aligned} |\nabla F(x,\varepsilon)| &= \frac{1}{\varepsilon}\left|\int_{|y|\le\varepsilon} (f(x-y)-f(x))\eta_\varepsilon(y)\,dy\right| \\ &\lesssim \frac{1}{\varepsilon^{n+1}}\int_{|h|\le\varepsilon} |f(x+h)-f(x)|\,dh. \end{aligned} \tag{6.51}$$

ch2

Integrating (6.51) and using Minkowski's inequality, we obtain

$$\|\nabla F(\cdot,\varepsilon)\|_{L^p} \lesssim \frac{1}{\varepsilon^{n+1}}\int_{|h|\le\varepsilon} \|\Delta_h f\|_{L^p}\,dh. \tag{6.52}$$

ci1

Let $L(F)$ be the quantity in the left-hand side of (6.41). Combining (6.52) with Hölder's inequality, we find that

$$\begin{aligned} [L(F)]^q &\lesssim \int_0^\delta \frac{1}{\varepsilon^{nq+sq+1}}\left(\int_{|h|\le\varepsilon}\|\Delta_h f\|_{L^p}\,dh\right)^q d\varepsilon \\ &\lesssim \int_0^\delta \frac{1}{\varepsilon^{nq+sq+1}}\varepsilon^{n(q-1)}\int_{|h|\le\varepsilon}\|\Delta_h f\|_{L^p}^q\,dh\,d\varepsilon \\ &\lesssim \int_{|h|\le\delta} |h|^{-sq}\|\Delta_h f\|_{L^p}^q \frac{dh}{|h|^n} = |f|^q_{B^s_{p,q,\delta}}, \end{aligned} \tag{6.53}$$

mj1

i.e, (6.41) holds. □

In the same vein, we have the following result, involving the semi-norm appearing in Proposition 2.6, more specifically the quantity

$$|f|_{B^1_{p,q,\delta}} := \left(\int_{|h|\le\delta} |h|^{-q}\|\Delta_h^2 f\|_{L^p}^q \frac{dh}{|h|^n}\right)^{1/q} \tag{6.54}$$

kd4

when $q<\infty$, with the obvious modification when $q=\infty$. We first introduce a notation. Given $F \in C^2(V_\delta)$, we let $D^2_\# F$ denote the collection of the second order derivatives of F which are either completely horizontal (that is of the form $\partial_j\partial_k F$, with $j,k \in [\![1,n]\!]$), or completely vertical (that is $\partial_{n+1}\partial_{n+1}F$).

kb2 **6.19 Lemma.** Let $1 \le p < \infty$ and $1 \le q \le \infty$. Let $F \in C^\infty(V_\delta)$ and set

$$M(F) := \left(\int_0^\delta \varepsilon^q \|(\nabla F)(\cdot,\varepsilon)\|_{L^{2p}}^{2q} \frac{d\varepsilon}{\varepsilon} \right)^{1/q}$$

and

$$N(F) := \left(\int_0^\delta \varepsilon^q \left\|(D^2_\# F)(\cdot,\varepsilon)\right\|_{L^p}^q \frac{d\varepsilon}{\varepsilon} \right)^{1/q}$$

(with the obvious modification when $q = \infty$).

1. If $M(F) < \infty$ and $N(F) < \infty$, then F has a trace $f \in B^1_{p,q}(\mathbb{T}^n)$, satisfying

$$\left\| f - \fint f \right\|_{L^p} \lesssim M(F)^{\frac{1}{2}} \tag{6.55}$$ kf1

and

$$|f|_{B^1_{p,q,\delta}} \lesssim N(F). \tag{6.56}$$ kb3

2. Conversely, let $f \in B^1_{p,q}(\mathbb{T}^n;\mathbb{S}^1)$. Let $\rho \in C^\infty$ be an even mollifier supported in $\{|x| \le 1\}$ and set $F(x,\varepsilon) := f * \rho_\varepsilon(x)$, $x \in \mathbb{T}^n$, $0 < \varepsilon < \delta$. Then

$$M(F) + N(F) \lesssim |f|_{B^1_{p,q,\delta}}. \tag{6.57}$$ kb4

The above result is inspired by the proof of mazyanew [21, Section 10.1.1, Theorem 1, p. 512]. The arguments we present also lead to a (slightly different) proof of Lemma ab1 6.18.

We start by establishing some preliminary estimates. We call $H \in \mathbb{R}^n \times \mathbb{R}$ "pure" if H is either horizontal, or vertical, i.e., either $H \in \mathbb{R}^n \times \{0\}$ or $H \in \{0\} \times \mathbb{R}$. For further use, let us note the following fact, valid for $X \in V_\delta$ and $H \in \mathbb{R}^{n+1}$.

$$H \text{ pure} \implies |D^2F(X)\cdot(H,H)| \lesssim |D^2_\# F(X)||H|^2. \tag{6.58}$$ ja2

jc1 **6.20 Lemma.** Let X,H be such that $[X, X+2H] \subset \overline{V_\delta}$. Let $F \in C^2(\overline{V_\delta})$. Then

$$|\Delta^2_H F(X)| \le \int_0^2 \tau |D^2F(X+\tau H)\cdot(H,H)|\, d\tau. \tag{6.59}$$ jc2

In particular, if H is pure and we write $H = |H|K$, then

$$|\Delta^2_H F(X)| \lesssim \int_0^{2|H|} t |D^2_\# F(X+tK)|\, dt. \tag{6.60}$$ jc3

Proof. Set

$$G(s) := F(X + (1-s)H) + F(X + (1+s)H), \ s \in [0,1],$$

so that $G \in C^2$ and in addition we have

$$G'(0) = 0, \ G''(s) = [D^2F(X + (1-s)H) + D^2F(X + (1+s)H)] \cdot (H,H), \tag{6.61}$$

and

$$\int_0^1 (1-s)G''(s)\,ds = G(1) - G(0) - G'(0) = \Delta_H^2 F(X). \tag{6.62}$$

Estimate (6.59) is a consequence of (6.61) and (6.62) (using the changes of variable $\tau := 1 \pm s$). In the special case where H is pure, we rely on (6.58) and (6.59) and obtain (6.60) via the change of variable $t := \tau|H|$. □

If we combine (6.60) (applied first with $H = (h,0)$, $h \in \mathbb{R}^n$, next with $H = (0,t)$, $t \in [0, \delta/2]$) with Minkowski's inequality, we obtain the two following consequences[12]

$$[h \in \mathbb{R}^n, \ 0 \le \varepsilon \le \delta] \Longrightarrow \|\Delta_h^2 F(\cdot,\varepsilon)\|_{L^p} \lesssim |h|^2 \|D_\#^2 F(\cdot,\varepsilon)\|_{L^p}, \tag{6.63}$$

and[13]

$$[t, \varepsilon \ge 0, \ \varepsilon + 2t \le \delta] \Longrightarrow \|\Delta_{te_{n+1}}^2 F(\cdot,\varepsilon)\|_{L^p} \lesssim \int_0^{2t} r \|D_\#^2 F(\cdot, \varepsilon + r)\|_{L^p}\,dr. \tag{6.64}$$

Proof of Lemma 6.19. We start by proving (6.55). By Lemma 6.18 (applied with $s = 1/2$ and with $2p$ (respectively $2q$) instead of p (respectively q)), F has, on $\mathbb{T}^n$, a trace $\operatorname{tr} F \in B^{1/2}_{2p,2q}$. By Lemma 6.18, item 1, and Lemma 6.15, we have

$$\left\| \operatorname{tr} F - \fint \operatorname{tr} F \right\|_{L^p} \lesssim \left\| \operatorname{tr} F - \fint \operatorname{tr} F \right\|_{L^{2p}} \lesssim M(F)^{1/2}$$

i.e., (6.55) holds.

We next establish (6.56). Arguing as at the beginning of the proof of Lemma 6.18, one concludes that it suffices to prove (6.56) when $F \in C^\infty(\overline{V_\delta})$.

So let us consider some $F \in C^\infty(\overline{V_\delta})$. We set $f(x) = F(x,0)$, $\forall x \in \mathbb{T}^n$. Then (6.56) is equivalent to

$$|f|_{B^1_{p,q,\delta}} \lesssim N(F). \tag{6.65}$$

We treat only the case where $q < \infty$; the case where $q = \infty$ is slightly simpler and is left to the reader.

[12] In (6.63), we let $\Delta_h^2 F(\cdot,\varepsilon) := F(\cdot + 2h, \varepsilon) - 2F(\cdot + h, \varepsilon) + F(\cdot, \varepsilon)$.

[13] With the slight abuse of notation $\Delta_{te_{n+1}}^2 F(\cdot,\varepsilon) := F(\cdot, \varepsilon + 2t) - 2F(\cdot, \varepsilon + t) + F(\cdot, \varepsilon)$.

The starting point is the following identity, valid when $|h| \le \delta$ and with $t := |h|$

$$\begin{aligned}\Delta_h^2 f = &\Delta_{te_{n+1}/2}^2 F(\cdot + 2h, 0) - 2\Delta_{te_{n+1}/2}^2 F(\cdot + h, 0) + \Delta_{te_{n+1}/2}^2 F(\cdot, 0)\\ &+ 2\Delta_h^2 F(\cdot, t/2) - \Delta_h^2 F(\cdot, t).\end{aligned} \tag{6.66}$$

$\boxed{\texttt{jd1}}$

By (6.63), (6.64) and (6.66), we find that

$$\begin{aligned}\|\Delta_h^2 f\|_{L^p} \lesssim &\int_0^{|h|} r \|D_\#^2 F(\cdot, r)\|_{L^p}\, dr + |h|^2 \|D_\#^2 F(\cdot, |h|/2)\|_{L^p}\\ &+ |h|^2 \|D_\#^2 F(\cdot, |h|)\|_{L^p}.\end{aligned} \tag{6.67}$$

$\boxed{\texttt{jd2}}$

Finally, (6.67) combined with Hardy's inequality (6.48) (applied to the integral $\int_0^\delta$ and with $G'(r) := r\|D_\#^2 F(\cdot, r)\|_{L^p}$ and $\rho := q+1$) yields

$$\begin{aligned}|f|_{B^1_{p,q,\delta}}^q &\lesssim \int_{|h|\le\delta} \frac{1}{|h|^q} \left(\int_0^{|h|} r \left\| D_\#^2 F(\cdot, r) \right\|_{L^p} dr \right)^q \frac{dh}{|h|^n} + [N(F)]^q\\ &\lesssim [N(F)]^q.\end{aligned} \tag{6.68}$$

$\boxed{\texttt{kf6}}$

This implies (6.65) and completes the proof of item 1.

We now turn to item 2. We claim that

$$|f|_{B^{1/2}_{2p,2q,\delta}} \lesssim |f|_{B^1_{p,q,\delta}}^{1/2}. \tag{6.69}$$

$\boxed{\texttt{kg1}}$

Indeed, it suffices to note the fact that $|\Delta_h^2 f|^{2p} \lesssim |\Delta_h^2 f|^p$ (since $|f| = 1$). By combining (6.69) with Lemma 6.18, we find that

$$M(F) = \left(\int_0^\delta \varepsilon^q \|(\nabla F)(\cdot, \varepsilon)\|_{L^{2p}}^{2q} \frac{d\varepsilon}{\varepsilon} \right)^{1/q} \lesssim |f|_{B^1_{p,q,\delta}}. \tag{6.70}$$

$\boxed{\texttt{kg2}}$

Thus, in order to complete the proof of (6.57), it suffices to combine (6.70) with the following estimate

$$N(F) \lesssim |f|_{B^1_{p,q,\delta}}, \tag{6.71}$$

$\boxed{\texttt{kg3}}$

that we now establish. The key argument for proving (6.71) is the following second order analog of (6.51):

$$|D_\#^2 F(x, \varepsilon)| \lesssim \frac{1}{\varepsilon^{n+2}} \int_{|h|\le\varepsilon} |\Delta_h^2 f(x - h)|\, dh. \tag{6.72}$$

$\boxed{\texttt{ki1}}$

The proof of (6.72) appears in [21, p. 514]. For the sake of completeness, we reproduce below the argument. First, differentiating the expression defining F, we have

$$\partial_j \partial_k F(x, \varepsilon) = \frac{1}{\varepsilon^2} f * (\partial_j \partial_k \rho)_\varepsilon, \ \forall j, k \in [\![1, n]\!]. \tag{6.73}$$

$\boxed{\texttt{ki2}}$

Using (6.73) and the fact that $\partial_j \partial_k \rho$ is even and has zero average, we obtain the identity

$$\partial_j \partial_k F(x,\varepsilon) = \frac{1}{2\varepsilon^{n+2}} \int_{|h|\le\varepsilon} \partial_j \partial_k \rho(h/\varepsilon) \Delta_h^2 f(x-h)\,dh,$$

and thus (6.72) holds for the derivatives $\partial_j \partial_k F$, with $j,k \in [\![1,n]\!]$.

We next note the identity

$$F(x,\varepsilon) = \frac{1}{2\varepsilon^n} \int \rho(h/\varepsilon) \Delta_h^2 f(x-h)\,dh + f(x), \tag{6.74}$$ ki4

which follows from the fact that ρ is even.

By differentiating twice (6.74) with respect to ε, we obtain that (6.72) holds when $j=k=n+1$. The proof of (6.72) is complete.

Using (6.72) and Minkowski's inequality, we obtain

$$\|D_\#^2 F(\cdot,\varepsilon)\|_{L^p} \lesssim \frac{1}{\varepsilon^{n+2}} \int_{|h|\le\varepsilon} \|\Delta_h^2 f\|_{L^p}\,dh, \tag{6.75}$$ mi1

which is a second order analog of (6.52). Once (6.52) is obtained, we repeat the calculation leading to (6.53) and obtain (6.71). The details are left to the reader.

The proof of Lemma 6.19 is complete. □

av1 **6.21 Remark.** One may put Lemmas 6.18 and 6.19 in the perspective of the theory of weighted Sobolev spaces. Let us start by recalling one of the striking achievements of this theory. As it is well-known, we have $\operatorname{tr} W^{1,1}(\mathbb{R}^n_+) = L^1(\mathbb{R}^{n-1})$, and, when $n \ge 2$, the trace operator has no linear continuous right-inverse $T : L^1(\mathbb{R}^{n-1}) \to W^{1,1}(\mathbb{R}^n)$ [19], [29]. The expected analogs of these facts for $W^{2,1}(\mathbb{R}^n_+)$ are both wrong. More specifically, we have $\operatorname{tr} W^{2,1}(\mathbb{R}^n_+) = B^1_{1,1}(\mathbb{R}^{n-1})$ (which is a strict subspace of $W^{1,1}(\mathbb{R}^{n-1})$), and the trace operator has a linear continuous right inverse from $B^1_{1,1}(\mathbb{R}^{n-1})$ into $W^{2,1}(\mathbb{R}^n_+)$. These results are special cases of the trace theory for weighted Sobolev spaces developed by Uspenskiĭ [39]. For a modern treatment of this theory, see e.g. [27].

6.6 Product estimates

Lemma 6.22 below is a variant of [4, Lemma D.2]. Here, Ω is either smooth bounded, or $(0,1)^n$, or $\mathbb{T}^n$.

at3 **6.22 Lemma.** Let $s > 1$, $1 \le p < \infty$ and $1 \le q \le \infty$. If $u, v \in B^s_{p,q} \cap L^\infty(\Omega)$, then $u \nabla v \in B^{s-1}_{p,q}$.

Proof. After extension to $\mathbb{R}^n$ and cutoff, we may assume that $u, v \in B^s_{p,q} \cap L^\infty$. It thus suffices to prove that $u, v \in B^s_{p,q} \cap L^\infty(\mathbb{R}^n) \Longrightarrow u\nabla v \in B^{s-1}_{p,q}(\mathbb{R}^n)$.

In order to prove the above, we argue as follows. Let $u = \sum u_j$ and $v = \sum v_j$ be the Littlewood-Paley decompositions of u and v. Set

$$f^j := \sum_{k\le j} u_k \nabla v_j + \sum_{k<j} u_j \nabla v_k.$$

Since $\operatorname{supp} \mathscr{F}(u_k \nabla v_j) \subset B(0, 2^{\max\{k,j\}+2})$, we find that $u\nabla v = \sum f^j$ is a Nikolskiĭ decomposition of $u\nabla v$; see Section 2.9. Assume e.g. that $q < \infty$. In view of Proposition 2.14, the conclusion of Lemma 6.22 follows if we prove that

$$\sum 2^{(s-1)jq} \|f^j\|^q_{L^p} < \infty. \tag{6.76}$$

In order to prove (6.76), we rely on the elementary estimates [16, Lemma 2.1.1, p. 16], [4, formulas (D.8), (D.9), p. 71]

$$\left\| \sum_{k\le j} u_k \right\|_{L^\infty} \lesssim \|u\|_{L^\infty}, \quad \forall j \ge 0, \tag{6.77}$$

$$\left\| \sum_{k<j} \nabla v_k \right\|_{L^\infty} \lesssim 2^j \|v\|_{L^\infty}, \quad \forall j \ge 0, \tag{6.78}$$

and

$$\|\nabla v_j\|_{L^p} \lesssim 2^j \|v_j\|_{L^p}, \quad \forall j \ge 0. \tag{6.79}$$

By combining (6.77)-(6.79), we obtain

$$\begin{aligned}
\sum 2^{(s-1)jq} \|f^j\|^q_{L^p} &\lesssim \sum 2^{(s-1)jq} \left(\left\| \sum_{k\le j} u_k \right\|^q_{L^\infty} \|\nabla v_j\|^q_{L^p} + \left\| \sum_{k<j} \nabla v_k \right\|^q_{L^\infty} \|u_j\|^q_{L^p} \right) \\
&\lesssim \|u\|^q_{L^\infty} \sum 2^{sjq} \|v_j\|^q_{L^p} + \|v\|^q_{L^\infty} \sum 2^{sjq} \|u_j\|^q_{L^p} \\
&\lesssim \|u\|^q_{L^\infty} \|v\|^q_{B^s_{p,q}} + \|v\|^q_{L^\infty} \|u\|^q_{B^s_{p,q}},
\end{aligned}$$

and thus (6.76) holds. □

6.7 Superposition operators

In this section, we examine the mapping properties of the operator

$$T_\Phi, \ \psi \xrightarrow{T_\Phi} \Phi \circ \psi.$$

We work in Ω smooth bounded, or $(0,1)^n$, or $\mathbb{T}^n$.

The next result is classical and straightforward; see e.g. [30, Section 5.3.6, Theorem 1].

eipsi **6.23 Lemma.** Let $0 < s < 1$, $1 \le p < \infty$, and $1 \le q < \infty$. Let $\Phi : \mathbb{R}^k \to \mathbb{R}^l$ be a Lipschitz function . Then T_Φ maps $B^s_{p,q}(\Omega;\mathbb{R}^k)$ into $B^s_{p,q}(\Omega;\mathbb{R}^l)$.

Special case: $\psi \mapsto e^{\imath\psi}$ maps $B^s_{p,q}(\Omega;\mathbb{R})$ into $B^s_{p,q}(\Omega;\mathbb{S}^1)$.

In addition, when $q < \infty$, T_Φ is continuous.

For the next result, see [30, Section 5.3.4, Theorem 2, p. 325].

ka2 **6.24 Lemma.** Let $s > 0$, $1 \le p < \infty$ and $1 \le q \le \infty$. Let $\Phi \in C^\infty(\mathbb{R}^k;\mathbb{R}^l)$. Then T_Φ maps $(B^s_{p,q} \cap L^\infty)(\Omega;\mathbb{R}^k)$ into $(B^s_{p,q} \cap L^\infty)(\Omega;\mathbb{R}^l)$.

Special case: $\psi \mapsto e^{\imath\psi}$ maps $(B^s_{p,q} \cap L^\infty)(\Omega;\mathbb{R})$ into $(B^s_{p,q} \cap L^\infty)(\Omega;\mathbb{S}^1)$.

6.8 Integer valued functions

The next result is a cousin of [4, Appendix B],[14] but the argument in [4] does not seem to apply in our situation. Lemma 6.25 can be obtained from the results in [8], but we present below a simpler direct argument.

Eunicite **6.25 Lemma.** Let $s > 0$, $1 \le p < \infty$ and $1 \le q < \infty$ be such that $sp \ge 1$. Then the functions in $B^s_{p,q}(\Omega;\mathbb{Z})$ are constant.

Same result when $s > 0$, $1 \le p < \infty$, $q = \infty$ and $sp > 1$.

The same conclusion holds for functions in $\sum_{j=1}^k B^{s_j}_{p_j,q_j}(\Omega;\mathbb{Z})$, provided we have for all $j \in [\![1,k]\!]$: either $s_j p_j = 1$ and $1 \le q_j < \infty$, or $s_j p_j > 1$ and $1 \le q_j \le \infty$.

Proof. The case where $n = 1$ is simple. Indeed, by Lemma 6.5 we have $B^s_{p,q} \hookrightarrow \mathrm{VMO}$ (and similarly $\sum_{j=1}^k B^{s_j}_{p_j,q_j} \hookrightarrow \mathrm{VMO}$). The conclusion follows from the fact that $\mathrm{VMO}((0,1);\mathbb{Z})$ functions are constant [14, Step 5, p. 229].

We next turn to the general case. Let $f = \sum_{j=1}^k f_j$, with $f_j \in B^{s_j}_{p_j,q_j}(\Omega;\mathbb{Z})$, $\forall\, j \in [\![1,k]\!]$. In view of the conclusion, we may assume that $\Omega = (0,1)^n$. By the Sobolev embeddings, we may assume that for all j we have $s_j p_j = 1$ (and thus either $1 < p_j < \infty$ and $s_j = 1/p_j$, or $p_j = 1$ and $s_j = 1$) and $1 \le q_j < \infty$. Let, as in Lemma 6.9, $A \subset (0,1)^{n-1}$ be a set of full measure such that (6.4) holds with $M = 2$. The proof of the lemma relies on the following key implication:

$$[x_1+\cdots+x_k \in \mathbb{Z},\ 1 \le p_1,\ldots,p_k < \infty] \implies |x_1+\cdots+x_k| \lesssim |x_1|^{p_1}+\cdots+|x_k|^{p_k}. \tag{6.80}$$ cf2

This leads to the following consequence: if $g := g_1 + \cdots + g_k$ is integer-valued, then

$$\|\Delta_h^2 g\|_{L^1} \lesssim \|\Delta_h^2 g_1\|^{p_1}_{L^{p_1}} + \cdots + \|\Delta_h^2 g_k\|^{p_k}_{L^{p_k}}. \tag{6.81}$$ aaa1

[14] The context there is the one of the Sobolev spaces.

By combining (6.4) with (6.81), we find that

$$\lim_{l\to\infty} \frac{\left\|\Delta^2_{t_l e_n} f(x',\cdot)\right\|_{L^1((0,1))}}{t_l} = 0, \quad \forall\, x' \in A, \text{ for some sequence } t_l \to 0. \tag{6.82}$$ cf3

By Lemma 6.26 below, we find that $f(x',\cdot)$ is constant, for every $x' \in A$. By a permutation of the coordinates, we find that for every $i \in [\![1,n]\!]$, the function

$$t \mapsto f(x_1,\ldots,x_{i-1},t,x_{i+1},\ldots,x_n) \text{ is constant, } \forall\, i \in [\![1,n]\!], \text{ a.e. } \widehat{x}_i \in (0,1)^{n-1}; \tag{6.83}$$ aa2

here, $\widehat{x}_i := (x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n) \in (0,1)^{n-1}$.

We next invoke the fact that every measurable function satisfying (6.83) is constant [12, Lemma 2]. □

cf4 **6.26 Lemma.** Let $g \in L^1((0,1);\mathbb{Z})$ be such that, for some sequence $t_l \to 0$, we have

$$\lim_{l\to\infty} \frac{\left\|\Delta^2_{t_l} g\right\|_{L^1((0,1))}}{t_l} = 0. \tag{6.84}$$ cf5

Then g is constant.

Proof. In order to explain the main idea, let us first assume that $g = \mathbb{1}_B$ for some $B \subset (0,1)$. Let $h \in (0,1)$. If $x \in B$ and $x+2h \notin B$, then $\Delta^2_h g(x)$ is odd, and thus $|\Delta^2_h g(x)| \geq 1$. The same holds if $x \notin B$ and $x+2h \in B$. On the other hand, we have $|\Delta_{2h} g(x)| \leq 1$, with equality only when either $x \in B$ and $x+2h \notin B$, or $x \notin B$ and $x+2h \in B$. By the preceding, we obtain the inequality

$$|\Delta^2_h g(x)| \geq |\Delta_{2h} g(x)|, \quad \forall\, x, \forall\, h. \tag{6.85}$$ cf6

Using (6.84) and (6.85), we obtain

$$g' = \lim_{l\to\infty} \frac{\Delta_{2t_l} g}{2t_l} = 0.^{15} \tag{6.86}$$ cf7

Thus either $g = 0$, or $g = 1$.

We next turn to the general case. Consider some $k \in \mathbb{Z}$ such that the measure of the set $g^{-1}(\{k\})$ is positive. We may assume that $k = 0$, and we will prove that $g = 0$. For this purpose, we set $B := g^{-1}(2\mathbb{Z})$, and we let $\overline{g} := \mathbb{1}_B$. Arguing as above, we have $|\Delta^2_h g(x)| \geq |\Delta_{2h}\overline{g}(x)|$, $\forall\, x$, $\forall\, h$, and thus $\overline{g} = 0$. We find that g takes only even values. We next consider the integer-valued map $g/2$. By the above, $g/2$ takes only even values, and so on. We find that $g = 0$. □

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[15] In (6.86), the first limit is in $\mathscr{D}'$, the second one in L^1.

6.9 Disintegration of the Jacobians

The purpose of this section is to prove and generalize the following result, used in the analysis of Case 5.

6.27 Lemma. Let $s>1$, $1\le p<\infty$, $1\le q\le p$ and $n\ge 3$, and assume that $sp\ge 2$. Let $u\in B^s_{p,q}(\Omega;\mathbb{S}^1)$ and set $F:=u\wedge\nabla u$. Then $\operatorname{curl} F=0$.

Same conclusion if $s>1$, $1\le p<\infty$, $1\le q\le\infty$ and $n\ge 2$, and we have $sp>2$.

Same conclusion if $s>1$, $1\le p<\infty$, $1\le q<\infty$ and $n=2$, and we have $sp=2$.

In view of the conclusion, we may assume that $\Omega=(0,1)^n$.

Note that in the above we have $n\ge 2$; for $n=1$ there is nothing to prove.

Since the results we present in this section are of independent interest, we go beyond what is actually needed in Case 5.

The conclusion of (the generalization of) Lemma 6.27 relies on three ingredients. The first one is that it is possible to define, as a distribution, the product $F:=u\wedge\nabla u$ for u in a low regularity Besov space; this goes back to [7] when $n=2$, and the case where $n\ge 3$ is treated in [9]. The second one is a Fubini (disintegration) type result for the distribution $\operatorname{curl} F$. Again, this result holds even in Besov spaces with lower regularity than the ones in Lemma 6.27; see Lemma 6.28 below. The final ingredient is the fact that when $u\in \mathrm{VMO}((0,1)^2;\mathbb{S}^1)$ we have $\operatorname{curl} F=0$; see Lemma 6.29. Lemma 6.27 is obtained by combining Lemmas 6.28 and 6.29 via a dimensional reduction (slicing) based on Lemma 6.8; a more general result is presented in Lemma 6.30.

Now let us proceed. First, following [7] and [9], we explain how to define the Jacobian $Ju:=1/2\operatorname{curl} F$ of low regularity unimodular maps $u\in W^{1/p,p}((0,1)^n;\mathbb{S}^1)$, with $1\le p<\infty$.[16] Assume first that $n=2$ and that u is smooth. Then, in the distributions sense, we have

$$\begin{aligned}
\langle Ju,\zeta\rangle &= \frac12\int_{(0,1)^2}\operatorname{curl} F\,\zeta = -\frac12\int_{(0,1)^2}\nabla\zeta\wedge(u\wedge\nabla u)\\
&= \frac12\int_{(0,1)^2}[(u\wedge\partial_1 u)\partial_2\zeta-(u\wedge\partial_2 u)\partial_1\zeta]\\
&= \frac12\int_{(0,1)^2}(u_1\nabla u_2\wedge\nabla\zeta - u_2\nabla u_1\wedge\nabla\zeta),\quad \forall\,\zeta\in C^\infty_c((0,1)^2).
\end{aligned}\tag{6.87}$$

In higher dimensions, it is better to identify Ju with the 2-form (or rather a 2-current) $Ju\equiv 1/2\,d(u\wedge du)$.[17] With this identification and modulo the action

[16] In [7] and [9], maps are from $\mathbb{S}^n$ (instead of $(0,1)^n$) into $\mathbb{S}^1$, but this is not relevant for the validity of the results we present here.

[17] We recover the two-dimensional formula (6.87) via the usual identification of 2-forms on $(0,1)^2$ with scalar functions (with the help of the Hodge $*$-operator).

of the Hodge $*$-operator, Ju acts either or $(n-2)$-forms, or on 2-forms. The former point of view is usually adopted, and is expressed by the formula

$$\begin{aligned}\langle Ju,\zeta\rangle &= \frac{(-1)^{n-1}}{2}\int_{(0,1)^n} d\zeta\wedge(u\wedge\nabla u)\\ &= \frac{(-1)^{n-1}}{2}\int_{(0,1)^n} d\zeta\wedge(u_1\,du_2-u_2\,du_1),\quad \forall\,\zeta\in C_c^\infty(\Lambda^{n-2}(0,1)^n).^{18}\end{aligned}\tag{6.88}$$

The starting point in extending the above formula to lower regularity maps u is provided by the identity (6.89) below; when u is smooth, (6.89) is obtained by a simple integration by parts. More specifically, consider any smooth extension $U:(0,1)^n\times[0,\infty)\to\mathbb{C}$, respectively $\varsigma\in C_c^\infty(\Lambda^{n-2}((0,1)^n\times[0,\infty)))$ of u, respectively of ζ.[19] Then we have the identity [9, Lemma 5.5]

$$\langle Ju,\zeta\rangle=(-1)^{n-1}\int_{(0,1)^n\times(0,\infty)} d\varsigma\wedge\, dU_1\wedge dU_2.\tag{6.89}$$

For a low regularity u and for a well-chosen U, we take the right-hand side of (6.89) as the definition of Ju. More specifically, let $\Phi\in C^\infty(\mathbb{R}^2;\mathbb{R}^2)$ be such that $\Phi(z)=z/|z|$ when $|z|\ge 1/2$, and let v be a standard extension of u by averages, i.e., $v(x,\varepsilon)=u*\rho_\varepsilon(x)$, $x\in(0,1)^n$, $\varepsilon>0$, with ρ a standard mollifier. Set $U:=\Phi(v)$. With this choice of U, the right-hand side of (6.89) does not depend on ς (once ζ is fixed) [9, Lemma 5.4] and the map $u\mapsto Ju$ is continuous from $W^{1/p,p}((0,1)^n;\mathbb{S}^1)$ into the set of 2- (or $(n-2)$-)currents. When $p=1$, continuity is straightforward. For the continuity when $p>1$, see [9, Theorem 1.1 item 2]. In addition, when u is sufficiently smooth (for example when $u\in W^{1,1}((0,1)^n;\mathbb{S}^1)$), Ju coincides[20] with $\operatorname{curl} F$ [9, Theorem 1.1 item 1]. Finally, we have the estimate [9, Theorem 1.1 item 3]

$$|\langle Ju,\zeta\rangle|\lesssim |u|^p_{W^{1/p,p}}\|d\zeta\|_{L^\infty},\quad \forall\,\zeta\in C_c^\infty(\Lambda^{n-2}(0,1)^n).\tag{6.90}$$

We are now in position to explain disintegration along two-planes. We use the notation in Section 6.2. Let $u\in W^{1/p,p}((0,1)^n;\mathbb{S}^1)$, with $n\ge 3$. Let $\alpha\in I(n-2,n)$. Then for a.e. $x_\alpha\in(0,1)^{n-2}$, the partial map $u_\alpha(x_\alpha)$ belongs to $W^{1/p,p}((0,1)^2;\mathbb{S}^1)$ (Lemma 6.7), and therefore $Ju_\alpha(x_\alpha)$ makes sense and acts on functions.[21] Let now $\zeta\in C_c^\infty(\Lambda^{n-2}(0,1)^n)$. Then we may write

$$\zeta=\sum_{\alpha\in I(n-2,n)}\zeta^\alpha\,dx^\alpha=\sum_{\alpha\in I(n-2,n)}\left(\zeta^\alpha\right)_\alpha(x_{\overline{\alpha}})\,dx^\alpha.$$

Here, dx^α is the canonical $(n-2)$-form induced by the coordinates x_j, $j\in\alpha$, and $(\zeta^\alpha)_\alpha(x_{\overline{\alpha}})=\zeta^\alpha(x_\alpha,x_{\overline{\alpha}})$ belongs to $C_c^\infty((0,1)^2)$ (for fixed x_α).

[18] Here, $C_c^\infty(\Lambda^{n-2}(0,1)^n)$ denotes the space of smooth compactly supported $(n-2)$-forms on $(0,1)^n$.

[19] We do not claim that U is $\mathbb{S}^1$-valued. When u is not smooth, existence of $\mathbb{S}^1$-valued extensions is a delicate matter [25].

[20] Up to the action of the $*$ operator.

[21] Or rather on 2-forms, in order to be consistent with our construction in dimension ≥ 3.

We next note the following formal calculation. Fix $\alpha \in I(n-2,n)$, and let $\overline{\alpha} = \{j,k\}$, with $j<k$. Then

$$\begin{aligned}
2(-1)^{n-1}\langle Ju, \zeta^\alpha\, dx^\alpha\rangle &= \int_{(0,1)^n} d(\zeta^\alpha\, dx^\alpha)\wedge(u\wedge\nabla u)\\
&= \int_{(0,1)^n} (\partial_j\zeta^\alpha\, dx_j + \partial_k\zeta^\alpha\, dx_k)\wedge dx^\alpha\wedge u\wedge(\partial_j u\, dx_j + \partial_k u\, dx_k)\\
&= \int_{(0,1)^n} (\partial_j\zeta^\alpha\, u\wedge\partial_k u - \partial_k\zeta^\alpha\, u\wedge\partial_j u)\, dx_j\wedge dx^\alpha\wedge dx_k,
\end{aligned}$$

that is,

$$\langle Ju, \zeta\rangle = \frac{1}{2}\sum_{\alpha\in I(n-2,n)} \varepsilon(\alpha)\int_{(0,1)^{n-2}} \langle Ju_\alpha, (\zeta^\alpha)_\alpha(x_\alpha)\rangle\, dx_\alpha, \tag{6.91}$$ oc2

where $\varepsilon(\alpha)\in\{-1,1\}$ depends on α.

When $u \in W^{1,1}((0,1)^n;\mathbb{S}^1)$, it is easy to see that (6.91) is true (by Fubini's theorem). The validity of (6.91) under weaker regularity assumptions is the content of our next result.

mo2 **6.28 Lemma.** Let $1\le p<\infty$ and $n\ge 3$. Let $u \in W^{1/p,p}((0,1)^n;\mathbb{S}^1)$. Then (6.91) holds.

Proof. The case $p=1$ being clear, we may assume that $1<p<\infty$. We may also assume that $\zeta = \zeta^\alpha\, dx^\alpha$ for some fixed $\alpha\in I(n-2,n)$. A first ingredient of the proof of (6.91) is the density of $W^{1,1}((0,1)^n;\mathbb{S}^1)\cap W^{1/p,p}((0,1)^n;\mathbb{S}^1)$ into $W^{1/p,p}((0,1)^n;\mathbb{S}^1)$ [6, Lemma 23], [7, Lemma A.1]. Next, we note that the left-hand side of (6.91) is continuous with respect to the $W^{1/p,p}$ convergence of unimodular maps [9, Theorem 1.1 item 2]. In addition, as we noted, (6.91) holds when $u\in W^{1,1}((0,1)^n;\mathbb{S}^1)$. Therefore, it suffices to prove that the right-hand side of (6.91) is continuous with respect to $W^{1/p,p}$ convergence of $\mathbb{S}^1$-valued maps. This is proved as follows. Let $u_j, u \in W^{1/p,p}((0,1)^n;\mathbb{S}^1)$ be such $u_j\to u$ in $W^{1/p,p}$. By a standard argument, since the right-hand side of (6.91) is uniformly bounded with respect to j by (6.90), it suffices to prove that the right-hand side of (6.91) corresponding to u_j tends to the one corresponding to u possibly along a subsequence.

In turn, convergence up to a subsequence is proved as follows. Recall the following vector-valued version of the "converse" to the dominated convergence theorem [11, Theorem 4.9, p. 94]. If X is a Banach space, ω a measured space and $f_j\to f$ in $L^p(\omega,X)$, then (possibly along a subsequence) for a.e. $\varpi\in\omega$ we have $f_j(\varpi,\cdot)\to f(\varpi,\cdot)$ in X, and in addition there exists some $g\in L^p(\omega)$ such that $\|f_j(\varpi,\cdot)\|_X\le g(\varpi)$ for a.e. $\varpi\in\omega$.

Using the above and Lemma 6.7 item 2 (applied with $s=1/p$), we find that, up to a subsequence, we have

$$(u_j)_\alpha(x_\alpha)\to u_\alpha(x_\alpha) \text{ in } W^{1/p,p}((0,1)^2;\mathbb{S}^1) \text{ for a.e. } x_\alpha\in(0,1)^{n-2}, \tag{6.92}$$ oc3

and in addition we have, for some $g \in L^p((0,1)^{n-2})$,

$$|(u_j)_\alpha(x_\alpha)|_{W^{1/p,p}((0,1)^2)} \le g(x_\alpha) \text{ for a.e. } x_\alpha \in (0,1)^{n-2}. \tag{6.93}$$ oc4

The continuity of the right-hand side of (6.91) (along some subsequence) is obtained by combining (6.92) and (6.93) with (6.90) (applied with $n = 2$).[22] □

mo3 **6.29 Lemma.** Let $1 \le p < \infty$. Let $u \in W^{1/p,p} \cap \mathrm{VMO}((0,1)^2;\mathbb{S}^1)$. Then $Ju = 0$.

Proof. Assume first that in addition we have $u \in C^\infty$. Then $u = e^{\imath\varphi}$ for some $\varphi \in C^\infty$, and thus $Ju = 1/2\,\mathrm{curl}(u \wedge \nabla u) = 1/2\,\mathrm{curl}\,\nabla\varphi = 0$.

We now turn to the general case. Let $F(x,\varepsilon) := u * \rho_\varepsilon(x)$, with ρ a standard mollifier. Since $u \in \mathrm{VMO}((0,1)^2;\mathbb{S}^1)$, there exists some $\delta > 0$ such that $1/2 < |F(x,\varepsilon)| \le 1$ when $0 < \varepsilon < \delta$ (see (3.2) and the discussion in Case 3). Let $\Phi \in C^\infty(\mathbb{R}^2;\mathbb{R}^2)$ be such that $\Phi(z) := z/|z|$ when $|z| \ge 1/2$, and define $F_\varepsilon(x) := F(x,\varepsilon)$ and $u_\varepsilon := \Phi \circ F_\varepsilon$, $\forall\, 0 < \varepsilon < \delta$. Then $F_\varepsilon \to u$ in $W^{1/p,p}$ and (by Lemma 6.23 when $p > 1$, respectively by a straightforward argument when $p = 1$) we have $u_\varepsilon = \Phi(F_\varepsilon) \to \Phi(u) = u$ in $W^{1/p,p}((0,1)^2;\mathbb{S}^1)$ as $\varepsilon \to 0$. Since (by the beginning of the proof) we have $Ju_\varepsilon = 0$, we conclude via the continuity of J in $W^{1/p,p}((0,1)^2;\mathbb{S}^1)$ [9, Theorem 1.1 item 2]. □

We may now state and prove the following generalization of Lemma 6.27.

mo4 **6.30 Lemma.** Let $s > 0$, $1 \le p < \infty$, $1 \le q \le p$, $n \ge 3$, and assume that $sp \ge 2$. Let $u \in B^s_{p,q}(\Omega;\mathbb{S}^1)$. Then $Ju = 0$.

Same conclusion if $s > 0$, $1 \le p < \infty$, $1 \le q \le \infty$, $n \ge 2$, and we have $sp > 2$.

Same conclusion if $s > 0$, $1 \le p < \infty$, $1 \le q < \infty$, $n = 2$, and we have $sp = 2$.

Proof. We may assume that $\Omega = (0,1)^n$. By the Sobolev embeddings (Lemma 6.1), it suffices to consider the limiting case where:

1. $s > 0$, $1 \le p < \infty$, $1 \le q < \infty$, $n = 2$, and $sp = 2$.

   Or

2. $s > 0$, $1 \le p < \infty$, $q = p$, $n \ge 3$, and $sp = 2$.

In view of Lemmas 6.1 and 6.5, the case where $n = 2$ is covered by Lemma 6.29. Assume that $n \ge 3$. Then the desired conclusion is obtained by combining Lemmas 6.7, 6.8, 6.28 and 6.29. □

oc1 **6.31 Remark.** Arguments similar to the one developed in this section lead to the conclusion that the Jacobians of maps $u \in W^{s,p}((0,1)^n;\mathbb{S}^k)$, defined when $sp \ge k$ [7], [9], disintegrate over $(k+1)$-planes. When $s = 1$ and $p \ge k$, this assertion is implicit in [20, Proof of Proposition 2.2, pp. 701-704].

[22] In order to be complete, we should also check that the right-hand side of (6.91) is measurable with respect to x_α. This is clear when $u \in W^{1,1}((0,1)^n;\mathbb{S}^1)$. The general case follows by density and (6.92).

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