Nonoscillation and Oscillation: Theory for Functional Differential Equations

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Preface

This book is devoted to a rapidly developing branch of the qualitative theory of differential equations with or without delays. It summarizes the most recent contributions of the authors and their colleagues in this area and will be a stimulus to its further development.

There are eight chapters in this book. After the preliminaries in Chapter 1, we present oscillatory and nonoscillatory properties of first order delay differential equations and first order neutral delay differential equations in Chapters 2 and 3, respectively. Classification schemes and existence of positive solutions of neutral delay differential equations with variable coefficients are also considered. In Chapter 4, oscillation and nonoscillation of second order nonlinear differential equations without delays is investigated. Chapter 5 is devoted to classification schemes and existence of positive solutions of second order delay differential equations with or without neutral terms. Nonoscillation and oscillation of higher order delay differential equations is considered in Chapter 6. Chapter 7 features oscillation and nonoscillation for two-dimensional systems of nonlinear differential equations. Finally, in Chapter 8, we give some first results on the oscillation of dynamic equations on time scales. Time scales have been introduced in order to unify continuous and discrete analysis and to extend those theories to cases "in between". Many results given in the first seven chapters of this book may be generalized within the time scales setting (hence accommodating differential equations and difference equations simultaneously), and in this final chapter we present some of those results.

This book is addressed to a wide audience of specialists such as mathematicians, physicists, engineers and biologists. It can be used as a textbook at the graduate level and as a reference book for several disciplines.

Thanks are due to Xiao-Yun Cao for her assistance in typing portions of the book and a very special thank you to Dr. Murat Adıvar, Dr. Elvan Akın-Bohner, Dr. Xiang-Li Fei, and Dr. Hai-Feng Huo for their help in proofreading. Finally, we wish to express our thanks to the staff of Marcel Dekker, Inc., in particular Maria Allegra and Elizabeth Draper, for their cooperation during the preparation of this book for publication.

Ravi Agarwal
Martin Bohner
Wan-Tong Li

Contents

CHAPTER 1

Preliminaries

1.1. Introduction

This chapter is essentially introductory in nature. Its main purpose is to present some basic concepts from the theory of delay differential equations and to sketch some preliminary results which will be used throughout the book. In this respect, this is almost a self-contained monograph. The reader may glance at the material covered in this chapter and then proceed to Chapter 2.

Section 1.2 is concerned with the statement of the basic initial value problems and classification of equations with delays. In Section 1.3 we provide definition of oscillation of solutions with or without delays. Section 1.4 states some fixed point theorems which are important tools in oscillation theory, especially, when one proves the existence of nonoscillatory solutions.

1.2. Initial Value Problems

Let us consider the ordinary differential equation (ODE)

$$x'(t) = f(t, x) \tag{1.1}$$

together with the initial condition

$$x(t_0) = x_0. \tag{1.2}$$

It is well known that under certain assumptions on f the initial value problem (1.1) and (1.2) has a unique solution and is equivalent to the integral equation

$$x(t) = x(t_0) + \int_{t_0}^{t} f\Big(s, x(s)\Big)\, ds \quad \text{for} \quad t \geq t_0.$$

Next, we consider a differential equation of the form

$$x'(t) = f\Big(t, x(t), x(t-\tau)\Big) \quad \text{with} \quad \tau > 0 \quad \text{and} \quad t \geq t_0, \tag{1.3}$$

in which the right-hand side depends not only on the instantaneous position $x(t)$, but also on $x(t-\tau)$, the position at τ units back, that is to say, the equation has past memory. Such an equation is called an *ordinary differential equation with delay* or *delay differential equation*. Whenever necessary, we shall consider the integral equation

$$x(t) = x(t_0) + \int_{t_0}^{t} f\Big(s, x(s), x(s-\tau)\Big)\, ds,$$

which is equivalent to (1.3). In order to define a solution of (1.3), we need to have a known function φ on $[t_0 - \tau, t_0]$, instead of just the initial condition $x(t_0) = x_0$.

The basic initial value problem for a delay differential equation is posed as follows: On the interval $[t_0, T]$, $T \leq \infty$, we seek a continuous function x that satisfies (1.3) and an initial condition

$$x(t) = \varphi(t) \quad \text{for all} \quad t \in E_{t_0}, \tag{1.4}$$

where t_0 is an initial point, $E_{t_0} = [t_0 - \tau, t_0]$ is the initial set; the known function φ on E_{t_0} is called the *initial function*. Usually, it is assumed that $\varphi(t_0 + 0) = \varphi(t_0)$. We always mean a one-sided derivative when we speak of the derivative at an endpoint of an interval.

Under general assumptions, the existence and uniqueness of solutions to the initial value problem (1.3) and (1.4) can be established (see, for example, Győri and Ladas [**118**]). The solution sometimes is denoted by $x(t, \varphi)$. In the case of a variable delay $\tau = \tau(t) > 0$ in (1.3), it is also required to find a solution of this equation for $t > t_0$ such that on the initial set

$$E_{t_0} = t_0 \cup \left\{ t - \tau(t) : \ t - \tau(t) < t_0, \ t \geq t_0 \right\},$$

x coincides with the given initial function φ. If it is required to determine the solution on the interval $[t_0, T]$, then the initial set is

$$E_{t_0 T} = \{t_0\} \cup \left\{ t - \tau(t) : \ t - \tau(t) < t_0, \ t_0 \leq t \leq T \right\}.$$

Example 1.2.1. For the equation

$$y'(t) = f\left(t, y(t), y(t - \cos^2 t)\right),$$

$t_0 = 0$, $E_0 = [-1, 0]$, and the initial function φ must be given on the interval $[-1, 0]$.

The initial set E_{t_0} depends on the initial point t_0, as can be seen from the following example.

Example 1.2.2. For the equation

$$y'(t) = ay(t/2),$$

we have $\tau(t) = t/2$ so that

$$E_0 = \{0\} \quad \text{and} \quad E_1 = [1/2, 1].$$

Now we consider the differential equation of nth order with l deviating arguments, of the form

$$\begin{aligned} y^{(m_0)}(t) = f\Big(&t, y(t), \ldots, y^{(m_0-1)}(t), y(t - \tau_1(t)), \ldots, y^{(m_1-1)}(t - \tau_1(t)), \ldots, \\ &y(t - \tau_l(t)), \ldots, y^{(m_l-1)}(t - \tau_l(t))\Big), \end{aligned} \tag{1.5}$$

where the deviations $\tau_i(t) > 0$, and $\max_{0 \leq i \leq l} m_i = n$.

In order to formulate the initial value problem for (1.5), we shall need the following notation. Let t_0 be the given initial point. Each deviation $\tau_i(t)$ defines the initial set $E_{t_0}^{(i)}$ given by

$$E_{t_0}^{(i)} = \{t_0\} \cup \left\{ t - \tau_i(t) : \ t - \tau_i(t) < t_0, \ t \geq t_0 \right\}.$$

We denote $E_{t_0} = \cup_{i=1}^{l} E_{t_0}^{(i)}$, and on E_{t_0} continuous functions φ_k, $k = 0, 1, \ldots, \mu$, must be given, with $\mu = \max_{0 \le i \le l} m_i$. In applications, it is most natural to consider the case where on E_{t_0},

$$\varphi_k(t) = \varphi_0^{(k)}(t) \quad \text{for} \quad k = 0, 1, \ldots, \mu,$$

but it is not generally necessary.

For the nth order differential equation, there should be given initial values $y_0^{(k)}$, $k = 0, 1, 2, \ldots, n-1$. Now let $y_0^{(k)} = \varphi_k(t_0)$, $k = 0, 1, 2, \ldots, \mu$. If $\mu < n-1$, then, in addition, the numbers $y_0^{(\mu+1)}, \ldots, y_0^{(n-1)}$ are given. If the point t_0 is an isolated point of E_{t_0}, then $y_0^{(0)}, \ldots, y_0^{(n)}$ are also given.

For (1.5), the basic initial value problem consists of the determination of an $(n-1)$ times continuously differentiable function y that satisfies (1.5) for $t > t_0$ and the conditions

$$y^{(k)}(t_0 + 0) = y_0^{(k)}$$

for $k = 0, 1, \ldots, n-1$, and

$$y^{(k)}(t - \tau_i(t)) = \varphi_k(t - \tau_i(t)) \quad \text{if} \quad t - \tau_i(t) < t_0$$

for $k = 0, 1, \ldots, \mu$ and $i = 1, 2, \ldots, l$. At the point $t_0 + (k-1)\tau$ the derivative $y^{(k)}(t)$, generally speaking, is discontinuous, but the derivatives of lower order are continuous.

Example 1.2.3. Consider

$$y''(t) = f\left(t, y(t), y'(t), y(t - \cos^2 t), y\left(\frac{t}{2}\right)\right). \tag{1.6}$$

For $t_0 = 0$, we have $n = 2$, $l = 2$, $\mu = 0$, the initial sets $E_0^{(1)} = [-1, 0]$, $E_0^{(2)} = \{0\}$, and $E_0 = [-1, 0]$, on which is given the initial function φ_0, $y_0^{(0)} = \varphi_0(0)$, and $y_0^{(1)}$ is any given number.

For (1.5) a classification method was proposed by Kamenskiĭ [**141**]. We let $\lambda = m_0 - \mu$. If $\lambda > 0$, (1.5) is called an equation with *retarded* arguments or with delay. If $\lambda = 0$, it is called an equation of *neutral* type. If $\lambda < 0$, it is called an equation of *advanced* type.

Example 1.2.4. The equations

$$y'(t) + a(t)y(t - \tau) = 0 \quad \text{with} \quad \tau > 0,$$
$$y'(t) + a(t)y(t + \tau) = 0 \quad \text{with} \quad \tau > 0,$$

and

$$y'(t) + a(t)y(t) + b(t)y'(t - \tau) = 0 \quad \text{with} \quad \tau > 0$$

are of retarded type ($\lambda = 1$), advanced type ($\lambda = -1$), and neutral type ($\lambda = 0$), respectively.

In applications, the equation with retarded arguments is most important; the theory of such equations has been developed extensively. In this book we study mainly equations with or without delays.

1.3. Definition of Oscillation

Before we define oscillation of solutions, let us consider some simple examples.

Example 1.3.1. The equation

$$y'' + y = 0$$

has periodic solutions $y_1(t) = \cos t$ and $y_2(t) = \sin t$.

Example 1.3.2. Consider the equation

$$y''(t) - \frac{1}{t}y'(t) + 4t^2 y(t) = 0,$$

whose solution is $y(t) = \sin t^2$. This solution is not periodic but has an oscillatory property.

Example 1.3.3. Consider the equation

$$y''(t) + \frac{1}{2}y'(t) - \frac{1}{2}y(t-\pi) = 0 \quad \text{for} \quad t \geq 0, \tag{1.7}$$

whose solution $y(t) = 1 - \sin t$ has an infinite sequence of multiple zeros. This solution also has an oscillatory property.

Example 1.3.4. Consider the equation

$$y''(t) - y(-t) = 0.$$

This equation has an oscillatory solution $y_1(t) = \sin t$ and a nonoscillatory solution $y_2(t) = e^t + e^{-t}$.

Let us now restrict our discussion to those solutions y of the equation

$$y''(t) + a(t)y(t - \tau(t)) = 0 \tag{1.8}$$

which exist on some ray $[T_y, \infty)$ and satisfy $\sup\{|y(t)| : t \geq T\} > 0$ for every $T \geq T_y$. In other words, $|y(t)| \not\equiv 0$ on any infinite interval $[T, \infty)$. Such a solution sometimes is said to be a *regular solution*.

We usually assume that $a(t) \geq 0$ or $a(t) \leq 0$ in (1.8), and in doing so we mean to imply that $a(t) \not\equiv 0$ on any infinite interval $[T, \infty)$.

There are various possibilities of defining oscillation of solutions of ODEs (with or without delays). In this section, we give two definitions of oscillation, which are used in the rest of the book; these are the ones most frequently used in the literature.

As we see from the above examples, the definition of oscillation of regular solutions can have two different forms.

Definition 1.3.5. A nontrivial solution y (implying a regular solution always) is said to be *oscillatory* if it has arbitrarily large zeros for $t \geq t_0$, that is, there exists a sequence of zeros $\{t_n\}$ (i.e., $y(t_n) = 0$) of y such that $\lim_{n\to\infty} t_n = \infty$. Otherwise, y is said to be *nonoscillatory*.

For nonoscillatory solutions there exists t_1 such that

$$y(t) \neq 0 \quad \text{for all} \quad t \geq t_1.$$

Definition 1.3.6. A nontrivial solution y is said to be *oscillatory* if it changes sign on (T, ∞), where T is any number.

When $\tau(t) \equiv 0$ and $a(t)$ is continuous in (1.8), the two definitions given above are equivalent. This is because of the fact that the uniqueness of the solution makes multiple zeros impossible. However, as Example 1.2.3 suggests, a differential equation with delay can have solutions with multiple zeros. Then the two definitions are different, especially for higher order ordinary differential equations which may have solutions with multiple zeros.

Definition 1.3.5 is more general than Definition 1.3.6. The solution $y(t) = 1 - \sin t$ of (1.7) is oscillatory according to Definition 1.3.5 and is nonoscillatory according to Definition 1.3.6.

In Example 1.3.3, the possibility of multiple zeros of nontrivial solutions is a consequence of the retardation, since if $\tau(t) \equiv 0$, the corresponding equation has no solutions with multiple zeros.

For the system of first order equations with deviating arguments

$$\begin{cases} x' = f_1(t, x, x \circ \tau_1, y, y \circ \tau_2), \\ y' = f_2(t, x, x \circ \tau_1, y, y \circ \tau_2), \end{cases}$$

the solution (x, y) is said to be *strongly* (*weakly*) *oscillatory* if each (at least one) of its components is oscillatory. Otherwise, it is said to be strongly (weakly) nonoscillatory if each (at least one) of its nontrivial components is nonoscillatory.

1.4. Some Fixed Point Theorems

Fixed point theorems are important tools in proving the existence of nonoscillatory solutions. In this section we state some fixed point theorems that we need later. Let us begin with the following notation.

Let S be any fixed set and C_S be the relation of strict inclusion on subsets of S:

$$C_S = \Big\{(A, B) : \ A \subseteq B \subseteq S \text{ and } A \neq B\Big\}$$

We write $A \subset_S B$ instead of the notation $(A, B) \in C_S$.

For the set of real numbers, we have the usual ordering relation $<$. For any distinct real numbers x and y, either $x < y$ or $y < x$.

Definition 1.4.1. A *partial ordering* is a relation R satisfying

(i) if xRy and yRz, then xRz (i.e., R is transitive),
(ii) if xRy and yRx, then $x = y$ (i.e., R is antisymmetric).

If $<$ is such a relation, then we can define $x \le y$ if either $x < y$ or $x = y$. It is easy to see that $x \le y < z$ implies $x < z$.

Lemma 1.4.2. *Assume that $<$ is a partial ordering. Then for any x, y, and z, at most one of the three alternatives*

$$x < y, \quad x = y, \quad y < x$$

can hold. Also, $x \le y \le x$ implies $x = y$.

Definition 1.4.3. Suppose that $<$ is a partial ordering on A, and consider a subset C of A. An upper bound of C is an element $b \in A$ such that $x \le b$ for all $x \in C$. Here b may or may not belong to C. If b is the least element of the set of all upper bounds for C, then b is called the *least upper bound* (or *supremum*) of C. We write

$b = \sup C$. Similarly we define the *greatest lower bound* or *infimum* a of C and write $a = \inf C$.

Example 1.4.4. Consider a fixed set S. The set consisting of all subsets of S is denoted by $\mathcal{P}(S)$. Let the partial ordering be $\subset_S$ on S. For A and B in $\mathcal{P}(S)$, the set $\{A, B\}$ has a least upper bound (w.r.t. $\subset_S$), namely $A \cup B$.

Theorem 1.4.5. *Let $<$ be a partial ordering relative to a field A, and suppose that every $B \subseteq A$ has a least upper bound and that* $\inf A \in A$. *Suppose that F maps A into A in such a way that for all $x, y \in A$,*

$$x \leq y \quad \text{implies} \quad Fx \leq Fy.$$

Then F has a fixed point *in A, i.e., $Fx = x$ for some $x \in A$.*

Definition 1.4.6. A subset S of a normed space X is called *bounded* if there is a number M such that $\|x\| \leq M$ for all $x \in S$.

Definition 1.4.7. A set S in a vector space X is called *convex* if, for any $x, y \in S$, $\lambda x + (1 - \lambda)y \in S$ for all $\lambda \in [0, 1]$.

Definition 1.4.8. Let M, N be normed linear spaces, and $X \subset N$. An operator $T : X \to M$ is called *continuous* at a point $x \in X$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\|Tx - Ty\| < \varepsilon$ whenever $y \in X$ with $\|x - y\| < \delta$. The operator T is called continuous on X, or simply continuous, if it is continuous at all points of X.

Theorem 1.4.9. *Every continuous mapping of a closed bounded convex set in $\mathbb{R}^n$ into itself has a fixed point.*

Definition 1.4.10. A subset S of a normed space B is called *compact* if every infinite sequence of elements of S has a subsequence which converges to an element of S.

We can prove that compact sets are closed and bounded, but vice versa this is in general not true.

Lemma 1.4.11. *Continuous mappings take compact sets into compact sets. In other words, if M, N are normed linear spaces, $X \subset M$ is compact, and $T : X \to N$ is continuous, then the image of X under T, i.e., the set $T(X) = \{Tx : x \in X\}$, is compact.*

Definition 1.4.12. A subset S of a normed linear space N is called *relatively compact* if every sequence in S has a subsequence converging to an element of N.

It is obvious that every subset of a compact or relatively compact set is relatively compact.

Lemma 1.4.13. *The closure of a relatively compact set is compact, and a closed and relatively compact set is compact.*

Definition 1.4.14. A function $f : \mathbb{R} \to \mathbb{C}$ is called *bounded* on an interval $I \subset \mathbb{R}$ if there exists $M > 0$ such that $|f(x)| \leq M$ for all $x \in I$. A family $\mathcal{F}$ of functions is called *uniformly bounded* on I if there exists $M > 0$ such that $|f(x)| \leq M$ for all $x \in I$ and all $f \in \mathcal{F}$.

Lemma 1.4.15. *Continuous mappings on compact sets are uniformly continuous.*

Definition 1.4.16. A family $\mathcal{F}$ of functions is called *equicontinuous* on an interval $I \subset \mathbb{R}$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $f \in \mathcal{F}$, $|f(x) - f(y)| < \varepsilon$ whenever $x, y \in I$ with $|x - y| < \delta$.

Theorem 1.4.17 (Arzelà–Ascoli). *A set of functions in $C[a, b]$ with*

$$\|f\| = \sup_{x \in [a,b]} |f(x)|$$

is relatively compact iff it is uniformly bounded and equicontinuous on $[a, b]$.

Theorem 1.4.18 (Schauder's First Fixed Point Theorem). *If S is a convex and compact subset of a normed linear space, then every continuous mapping of S into itself has a fixed point.*

Theorem 1.4.19 (Schauder's Second Fixed Point Theorem). *If S is a convex closed subset of a normed linear space and R a relatively compact subset of S, then every continuous mapping of S into R has a fixed point.*

Theorem 1.4.19 is the more useful form for the theory of ordinary differential equations or delay differential equations.

Remark 1.4.20. We should point out that we need to use Theorem 1.4.17 carefully, because we usually discuss problems on the infinite interval $[t_0, \infty)$ in the qualitative theory of ODEs. That is, we usually want to prove that the family of functions is uniformly bounded and equicontinuous on $[t_0, \infty)$. Levitan's result [**168**] provides a correct formulation. According to his result, the family of functions is equicontinuous on $[t_0, \infty)$ if for any given $\varepsilon > 0$, the interval $[t_0, \infty)$ can be decomposed into a finite number of subintervals in such a way that on each subinterval all functions of the family have oscillations less than ε.

Definition 1.4.21. A real-valued function ρ defined on a linear space X is called a *seminorm* on X if

(i) $\rho(x + y) \leq \rho(x) + \rho(y)$ for all $x, y \in X$,
(ii) $\rho(\alpha x) = |\alpha|\, \rho(x)$ for all $x \in X$ and all scalars α.

From this definition, we can prove that a seminorm ρ satisfies $\rho(0) = 0$,

$$\rho(x_1 - x_2) \geq |\rho(x_1) - \rho(x_2)|\,,$$

and in particular $\rho(x) \geq 0$. However, it may happen that $\rho(x) = 0$ for $x \neq 0$.

Definition 1.4.22. A family $\mathcal{P}$ of semimorms on X is said to be *separating* if to each $x \neq 0$ there corresponds at least one $\rho \in \mathcal{P}$ with $\rho(x) \neq 0$.

For a separating seminorm family $\mathcal{P}$, if $\rho(x) = 0$ for every $\rho \in \mathcal{P}$, then $x = 0$.

Definition 1.4.23. A topology $\mathcal{T}$ on a linear space E is called *locally convex* if every neighborhood of the element 0 includes a convex neighborhood of 0.

A locally convex topology $\mathcal{T}$ on a linear space E is determined by a family of seminorms $\{\rho_\alpha : \alpha \in I\}$, I being the index set.

Let E be a locally convex space, $x \in E$, $\{x_n\} \subset E$. We say that $x_n \to x$ in E if $\rho_\alpha(x_n - x) \to 0$ as $n \to \infty$ for every $\alpha \in I$.

A set $S \subset E$ is bounded if and only if the set of numbers $\{\rho_\alpha(x) : x \in S\}$ is bounded for every $\alpha \in I$.

Definition 1.4.24. A complete metrizable locally convex space is called a *Fréchet space.*

Theorem 1.4.25 (Schauder–Tychonov Theorem)**.** *Let X be a locally convex topological linear space, C a compact convex subset of X, and $f : C \to C$ a continuous mapping such that $f(C)$ is compact. Then f has a fixed point in C.*

For example, $C([t_0, \infty), \mathbb{R})$ is a locally convex space consisting of the set of all continuous functions. The topology of C is the topology of uniform convergence on every compact interval of $[t_0, \infty)$. The seminorm of the space $C([t_0, \infty), \mathbb{R})$ is defined by

$$\rho_\alpha(x) = \max_{t\in[t_0,\alpha]} |x(t)| \quad \text{for} \quad x \in C \quad \text{and} \quad \alpha \in [t_0, \infty).$$

Let X be any set. A *metric* in X is a function $d : X \times X \to \mathbb{R}$ having the following properties for all $x, y, z \in X$:

(i) $d(x, y) \geq 0$, and $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, z) \leq d(x, y) + d(y, z)$.

A *metric space* is a set X together with a given metric in X. A *complete* metric space is a metric space X in which every Cauchy sequence converges to a point in X. A *Banach space* is a normed space that is complete with respect to the metric $d(x, y) = \|x - y\|$ defined by the norm.

Let (X, d) be a metric space and let $T : X \to X$. If there exists a number $L \in [0, 1)$ such that

$$d(Tx, Ty) \leq Ld(x, y) \quad \text{for all} \quad x, y \in X,$$

then we say that T is a *contraction mapping* on X.

Theorem 1.4.26 (Banach's Contraction Mapping Principle)**.** *A contraction mapping on a complete metric space has exactly one fixed point.*

Theorem 1.4.27 (Krasnosel′skiĭ's Fixed Point Theorem)**.** *Let X be a Banach space, let Ω be a bounded closed convex subset of X, and let A, B be maps of Ω into X such that $Ax + By \in \Omega$ for every pair $x, y \in \Omega$. If A is a contraction and B is completely continuous, then the equation*

$$Ax + Bx = x$$

has a solution in Ω.

A nonempty and closed subset K of a Banach space X is called a *cone* if it possesses the following properties:

(i) If $\alpha \in \mathbb{R}^+$ and $x \in K$, then $\alpha x \in K$;
(ii) if $x, y \in K$, then $x + y \in K$;
(iii) if $x \in K \setminus \{0\}$, then $-x \notin K$.

Theorem 1.4.28 (Knaster's Fixed Point Theorem)**.** *Let X be a partially ordered Banach space with ordering $\leq$. Let M be a subset of X with the following properties: The infimum of M belongs to M and every nonempty subset of M has a supremum which belongs to M. Let $T : M \to M$ be an increasing mapping, i.e., $x \leq y$ implies $Tx \leq Ty$. Then T has a fixed point in M.*

Let X be a Banach space, let K be a cone in X, and let $\leq$ be the ordering in X induced by K, i.e., $x \leq y$ if and only if $y - x \in K$. Let D be a subset of K and $T : D \to K$ a mapping.

We denote by $\langle x, y \rangle$ the closed ordered interval between x and y, i.e.,

$$\langle x, y \rangle = \left\{ z \in X : \; x \leq z \leq y \right\}.$$

We assume that the cone K is normal in X, which implies that ordered intervals are norm bounded. The cones of nonnegative functions are normal in the space of continuous functions with supremum norm and in the space L^p.

Theorem 1.4.29. *Let X be a Banach space, K a normal cone in X, D a subset of K such that if $x, y \in D$ with $x \leq y$, then $\langle x, y \rangle \subset D$, and let $T : D \to K$ be a continuous decreasing mapping which is compact on any closed ordered interval contained in D. Suppose that there exists $x_0 \in D$ such that $T^2 x_0$ is defined (where $T^2 x_0 = T(Tx_0)$) and furthermore Tx_0, $T^2 x_0$ are (order) comparable to x_0. Then T has a fixed point in D provided that either*

(i) *$Tx_0 \leq x_0$ and $T^2 x_0 \leq x_0$, or $Tx_0 \geq x_0$ and $T^2 x_0 \geq x_0$, or*
(ii) *the complete sequence of iterates $\{T^n x_0\}_{n=0}^{\infty}$ is bounded and there exists $y_0 \in D$ such that $Ty_0 \in D$ and $y_0 \leq T^n x_0$ for all $n \in \mathbb{N}_0$.*

Theorem 1.4.30. *Let X be a Banach space and $A : X \to X$ a completely continuous mapping such that $I - A$ is one-to-one. Let Ω be a bounded set with $0 \in (I - A)(\Omega)$. Then the completely continuous mapping $S : \Omega \to X$ has a fixed point in the closure $\overline{\Omega}$ if for any $\lambda \in (0, 1)$, the equation*

$$x = \lambda S x + (1 - \lambda) A x$$

has no solution x on the boundary $\partial\Omega$ of Ω.

1.5. Notes

The material in Chapter 1 is based on Erbe, Kong, and Zhang [**92**], Ladde, Lakshmikantham, and Zhang [**166**], and Zhong, Fan, and Chen [**304**].

CHAPTER 2

First Order Delay Differential Equations

2.1. Introduction

In this chapter, we will describe some of the recent developments in oscillation theory of first order delay differential equations. This theory is interesting from the theoretical as well as the practical point of view. It is well known that homogeneous ordinary differential equations (ODEs) of first order do not possess oscillatory solutions. But the presence of deviating arguments can cause oscillation of solutions. In this chapter we will see these phenomena and we will show various techniques used in oscillation and nonoscillation theory of differential equations with delays. We will present some criteria for oscillation and for the existence of positive solutions of delay differential equations of first order.

2.2. Equations with a Single Delay: General Case

We consider linear delay differential inequalities and equations of the form

$$x'(t) + p(t)x\left(\tau(t)\right) \le 0, \tag{2.1}$$

$$x'(t) + p(t)x\left(\tau(t)\right) \ge 0, \tag{2.2}$$

and

$$x'(t) + p(t)x\left(\tau(t)\right) = 0, \tag{2.3}$$

where $p, \tau \in C([t_0, \infty), \mathbb{R}^+)$, $\tau(t) \le t$, and $\lim_{t\to\infty} \tau(t) = \infty$. Set

$$m = \liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds \quad \text{and} \quad M = \limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds. \tag{2.4}$$

The following lemmas will be used to prove the main results of this section. All inequalities in this section and in the later parts hold eventually if it is not mentioned specifically.

Lemma 2.2.1. *Suppose that $m > 0$ and set*

$$\delta(t) = \max\left\{\tau(s) : \ s \in [t_0, t]\right\}.$$

Then we have

$$\liminf_{t\to\infty} \int_{\delta(t)}^{t} p(s)ds = \liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds = m. \tag{2.5}$$

Proof. Clearly, $\delta(t) \ge \tau(t)$ and so

$$\int_{\delta(t)}^{t} p(s)ds \le \int_{\tau(t)}^{t} p(s)ds.$$

Hence

$$\liminf_{t\to\infty}\int_{\delta(t)}^{t} p(s)ds \le \liminf_{t\to\infty}\int_{\tau(t)}^{t} p(s)ds.$$

If (2.5) does not hold, then there exist $m' > 0$ and a sequence $\{t_n\}$ such that $t_n \to \infty$ as $n \to \infty$ and

$$\lim_{n\to\infty}\int_{\delta(t_n)}^{t_n} p(s)ds < m' < m.$$

By definition, $\delta(t_n) = \max\{\tau(s) : s \in [t_0, t_n]\}$, and hence there exists $t'_n \in [t_0, t_n]$ such that $\delta(t_n) = \tau(t'_n)$. Hence

$$\int_{\delta(t_n)}^{t_n} p(s)ds = \int_{\tau(t'_n)}^{t_n} p(s)ds > \int_{\tau(t'_n)}^{t'_n} p(s)ds.$$

It follows that $\left\{\int_{\tau(t'_n)}^{t'_n} p(s)ds\right\}_{n=1}^{\infty}$ is a bounded sequence having a convergent subsequence, say

$$\int_{\tau(t'_{n_k})}^{t'_{n_k}} p(s)ds \to c \le m' \quad \text{as} \quad k \to \infty,$$

which implies that

$$\liminf_{t\to\infty}\int_{\tau(t)}^{t} p(s)ds \le m',$$

contradicting the first definition in (2.4). □

Lemma 2.2.2. *Let x be an eventually positive solution of* (2.1)*.*

(i) *If $m > \frac{1}{e}$, then*

$$\lim_{t\to\infty}\frac{x(\tau(t))}{x(t)} = \infty. \tag{2.6}$$

(ii) *If $m \le \frac{1}{e}$, then*

$$\lim_{t\to\infty}\frac{x(\tau(t))}{x(t)} \ge \lambda,$$

where λ is the smallest positive root of the equation

$$\lambda = e^{m\lambda}. \tag{2.7}$$

Proof. Let t_1 be a sufficiently large number so that $x(\tau(t)) > 0$ for $t \ge t_1$. Hence x is decreasing on $[t_1, \infty)$ and

$$\frac{x'(t)}{x(t)} \le -p(t)\frac{x(\tau(t))}{x(t)} \le -p(t). \tag{2.8}$$

Integrating (2.8) from $\tau(t)$ to t we have that eventually

$$\frac{x(\tau(t))}{x(t)} \ge \exp\left(\int_{\tau(t)}^{t} p(s)ds\right).$$

Then, for any $\varepsilon > 0$, there exists T_ε such that

$$\frac{x(\tau(t))}{x(t)} \ge e^m - \varepsilon \quad \text{for all} \quad t \ge T_\varepsilon. \tag{2.9}$$

Substituting (2.9) into (2.8) we have $\frac{x'(t)}{x(t)} \leq -(e^m - \varepsilon)p(t)$ for $t \geq T_\varepsilon$, and hence

$$\liminf_{t\to\infty} \frac{x(\tau(t))}{x(t)} \geq \exp\left(me^m\right).$$

Set $\lambda_0 = 1$ and recursively $\lambda_n = \exp(m\lambda_{n-1})$ for all $n \in \mathbb{N}$. For a sequence $\{\varepsilon_n\}$ with $\varepsilon_n > 0$ and $\varepsilon_n \to 0$ as $n \to \infty$, there exists a sequence $\{t_n\}$ such that $t_n \to \infty$ as $n \to \infty$ and

$$\frac{x(\tau(t))}{x(t)} \geq \lambda_n - \varepsilon_n \quad \text{for all} \quad t \geq t_n.$$

If $m > \frac{1}{e}$, then $\lim_{n\to\infty} \lambda_n = \infty$, and (2.6) holds. If $m = \frac{1}{e}$, then $\lim_{n\to\infty} \lambda_n = e$, and if $m < \frac{1}{e}$, then λ_n tends to the smallest root of (2.7). □

Remark 2.2.3. From Theorem 2.2.6 we will see that (2.1) has no eventually positive solutions if $m > \frac{1}{e}$.

Lemma 2.2.4. *Assume τ is nondecreasing, $0 \leq m \leq \frac{1}{e}$, and x is an eventually positive solution of* (2.1). *Set*

$$r = \liminf_{t\to\infty} \frac{x(t)}{x(\tau(t))}.$$

Then

$$A(m) := \frac{1 - m - \sqrt{1 - 2m - m^2}}{2} \leq r \leq 1. \tag{2.10}$$

Proof. Assume that $x(t) > 0$ for $t > T_1 \geq t_0$ and that there exists a sequence $\{T_n\}$ such that $T_1 < T_2 < T_3 < \ldots$ and $\tau(t) > T_n$ for $t > T_{n+1}$, $n \in \mathbb{N}$. Hence $x(\tau(t)) > 0$ for $t > T_2$. In view of (2.1), $x'(t) \leq 0$ on (T_2, ∞). Clearly, (2.10) holds for $m = 0$. If $0 < m \leq \frac{1}{e}$, for any $\varepsilon \in (0, m)$, there exists N_ε such that

$$\int_{\tau(t)}^{t} p(s)ds > m - \varepsilon \quad \text{for} \quad t > N_\varepsilon. \tag{2.11}$$

Let $\varepsilon > 0$ and $t > N_\varepsilon$. Then

$$f(\lambda) := \int_t^\lambda p(s)ds \quad \text{is continuous and} \quad \lim_{\lambda\to\infty} f(\lambda) > m - \varepsilon > 0 = f(t).$$

Hence there exists $\lambda_t > t$ such that $f(\lambda_t) = m - \varepsilon$, i.e.,

$$\int_t^{\lambda_t} p(s)ds = m - \varepsilon$$

holds. From (2.11) we have

$$\int_{\tau(\lambda_t)}^{\lambda_t} p(s)ds > m - \varepsilon = \int_t^{\lambda_t} p(s)ds$$

and therefore $\tau(\lambda_t) < t$.

Integrating (2.1) from $t > \max\{T_4, N_\varepsilon\}$ to λ_t we have

$$x(t) - x(\lambda_t) \geq \int_t^{\lambda_t} p(y)x(\tau(y))dy. \tag{2.12}$$

We see that $\tau(t) \leq \tau(y) \leq \tau(\lambda_t) < t$ for $t \leq y \leq \lambda_t$.

Integrating (2.1) from $\tau(y)$ to t we have that for $t \le y \le \lambda_t$

$$\begin{aligned} x(\tau(y)) - x(t) &\ge \int_{\tau(y)}^{t} p(u)x(\tau(u))du \\ &\ge x(\tau(t)) \int_{\tau(y)}^{t} p(u)du \\ &= x(\tau(t)) \left(\int_{\tau(y)}^{y} p(u)du - \int_{t}^{y} p(u)du \right) \\ &> x(\tau(t)) \left[(m-\varepsilon) - \int_{t}^{y} p(u)du \right]. \end{aligned} \tag{2.13}$$

From (2.12) and (2.13) we have

$$\begin{aligned} x(t) &\ge x(\lambda_t) + \int_{t}^{\lambda_t} p(y)x(\tau(y))dy \\ &> x(\lambda_t) + \int_{t}^{\lambda_t} p(y) \left\{ x(t) + x(\tau(t)) \left[(m-\varepsilon) - \int_{t}^{y} p(u)\, du \right] \right\} dy \\ &= x(\lambda_t) + x(t)(m-\varepsilon) + x(\tau(t)) \left[(m-\varepsilon)^2 - \int_{t}^{\lambda_t} p(y) \int_{t}^{y} p(u)dudy \right]. \end{aligned} \tag{2.14}$$

Noting the known formula

$$\int_{t}^{\lambda_t} \int_{t}^{y} p(y)p(u)dudy = \int_{t}^{\lambda_t} \int_{u}^{\lambda_t} p(y)p(u)dydu = \int_{t}^{\lambda_t} \int_{y}^{\lambda_t} p(y)p(u)dudy,$$

we have

$$\begin{aligned} \int_{t}^{\lambda_t} \int_{t}^{y} p(y)p(u)dudy &= \frac{1}{2} \left[\int_{t}^{\lambda_t} \int_{t}^{y} p(y)p(u)dudy + \int_{t}^{\lambda_t} \int_{y}^{\lambda_t} p(y)p(u)dudy \right] \\ &= \frac{1}{2} \int_{t}^{\lambda_t} \int_{t}^{\lambda_t} p(y)p(u)dudy \\ &= \frac{1}{2} \left[\int_{t}^{\lambda_t} p(s)ds \right]^2 \\ &= \frac{1}{2}(m-\varepsilon)^2. \end{aligned}$$

Substituting this into (2.14) we have

$$x(t) > x(\lambda_t) + (m-\varepsilon)x(t) + \frac{1}{2}(m-\varepsilon)^2 x(\tau(t)). \tag{2.15}$$

Hence (note that $1 - m + \varepsilon > 0$)

$$\frac{x(t)}{x(\tau(t))} > \frac{(m-\varepsilon)^2}{2(1-m+\varepsilon)} =: d_1, \tag{2.16}$$

and then

$$x(\lambda_t) > \frac{(m-\varepsilon)^2}{2(1-m+\varepsilon)} x(\tau(\lambda_t)) = d_1 x(\tau(\lambda_t)) \ge d_1 x(t).$$

Substituting this into (2.15) we obtain

$$x(t) > (m + d_1 - \varepsilon)x(t) + \frac{1}{2}(m-\varepsilon)^2 x(\tau(t)),$$

and hence

$$\frac{x(t)}{x(\tau(t))} > \frac{(m-\varepsilon)^2}{2(1-m-d_1+\varepsilon)} =: d_2.$$

In general we have

$$\frac{x(t)}{x(\tau(t))} > \frac{(m-\varepsilon)^2}{2(1-m-d_n+\varepsilon)} =: d_{n+1} \quad \text{for} \quad n \in \mathbb{N}.$$

It is not difficult to see that if ε is small enough, then $1 \geq d_{n+1} > d_n$ for all $n \in \mathbb{N}$. Hence $\lim_{n\to\infty} d_n = d$ exists and satisfies

$$-2d^2 + 2d(1-m+\varepsilon) = (m-\varepsilon)^2,$$

i.e.,

$$d = \frac{1-m+\varepsilon \pm \sqrt{1-2(m-\varepsilon)-(m-\varepsilon)^2}}{2}.$$

Therefore, for all large t,

$$\frac{x(t)}{x(\tau(t))} \geq \frac{1-m+\varepsilon - \sqrt{1-2(m-\varepsilon)-(m-\varepsilon)^2}}{2}.$$

Letting $\varepsilon \to 0$, we obtain that

$$\frac{x(t)}{x(\tau(t))} \geq \frac{1-m-\sqrt{1-2m-m^2}}{2} = A(m).$$

This shows that (2.10) holds. □

Lemma 2.2.5. *Assume that $M \in (0,1]$ and that τ is nondecreasing. Let x be an eventually positive solution of* (2.1)*. Set*

$$\liminf_{t\to\infty} \frac{x(\tau(t))}{x(t)} = l.$$

Then

$$l \leq B(M) := \left(\frac{1+\sqrt{1-M}}{M}\right)^2. \tag{2.17}$$

Proof. For a given $\varepsilon \in (0, M)$, there exists a sequence $\{t_n\}$ such that $t_n \to \infty$ as $n \to \infty$ and

$$\int_{\tau(t_n)}^{t_n} p(s)ds > M - \varepsilon, \quad t_n > T, \quad n \in \mathbb{N}.$$

Set $\theta_\varepsilon = 1 - \sqrt{1-(M-\varepsilon)}$. It is easy to see that $0 < \theta_\varepsilon < M - \varepsilon$ for small ε. Hence there exists $\{\lambda_n\}$ such that $\tau(t_n) < \lambda_n < t_n$ and

$$\int_{\lambda_n}^{t_n} p(s)ds = \theta_\varepsilon \quad \text{for} \quad n \in \mathbb{N}.$$

Integrating (2.1) from λ_n to t_n, we obtain

$$x(\lambda_n) - x(t_n) \geq \int_{\lambda_n}^{t_n} p(s)x(\tau(s))ds \geq x(\tau(t_n)) \int_{\lambda_n}^{t_n} p(s)ds = \theta_\varepsilon x(\tau(t_n)).$$

Similarly, we have

$$\begin{aligned} x(\tau(t_n)) - x(\lambda_n) &\geq \int_{\tau(t_n)}^{\lambda_n} p(s)x(\tau(s))ds \\ &\geq x(\tau(\lambda_n)) \int_{\tau(t_n)}^{\lambda_n} p(s)ds \\ &= x(\tau(\lambda_n)) \left[\int_{\tau(t_n)}^{t_n} p(s)ds - \int_{\lambda_n}^{t_n} p(s)ds \right] \\ &> x(\tau(\lambda_n))(M - \varepsilon - \theta_\varepsilon). \end{aligned}$$

From the above inequalities we get

$$x(\lambda_n) > \theta_\varepsilon x(\tau(t_n)) > \theta_\varepsilon \Big(x(\lambda_n) + x(\tau(\lambda_n))(M - \varepsilon - \theta_\varepsilon) \Big)$$

and then

$$\frac{x(\tau(\lambda_n))}{x(\lambda_n)} < \frac{1 - \theta_\varepsilon}{\theta_\varepsilon(M - \varepsilon - \theta_\varepsilon)} \quad \text{for} \quad n \in \mathbb{N},$$

which implies that

$$l \leq \frac{1 - \theta_\varepsilon}{\theta_\varepsilon(M - \varepsilon - \theta_\varepsilon)} \quad \text{for all} \quad \varepsilon \in (0, M).$$

Now, $\theta_\varepsilon \to 1 - \sqrt{1 - M}$ as $\varepsilon \to 0$, and then we obtain

$$l \leq \frac{\sqrt{1 - M}}{(1 - \sqrt{1 - M})(M - 1 + \sqrt{1 - M})} = \left(\frac{1 + \sqrt{1 - M}}{M} \right)^2,$$

which is (2.17). □

We are now in a position to state oscillation criteria for (2.3).

Theorem 2.2.6. *Assume $m > \frac{1}{e}$. Then*

(i) (2.1) *has no eventually positive solutions;*
(ii) (2.2) *has no eventually negative solutions;*
(iii) *every solution of* (2.3) *is oscillatory.*

Proof. It is sufficient to prove (i) as (ii) and (iii) follow from (i). Suppose the contrary is true, and let x be an eventually positive solution of (2.1). In view of Lemma 2.2.1, we may assume that τ is nondecreasing. By Lemma 2.2.2,

$$\liminf_{t \to \infty} \frac{x(\tau(t))}{x(t)} = \infty.$$

On the other hand, from (2.16), $\frac{x(\tau(t))}{x(t)}$ is bounded above. This contradiction proves (i). □

Remark 2.2.7. If τ is nondecreasing and $M \in (0, 1]$, then the condition $m > \frac{1}{e}$ in Theorem 2.2.6 can be replaced by

$$m > \frac{\ln b}{b} \quad \text{with} \quad b = \min\{e, B(M)\}, \tag{2.18}$$

where $B(M)$ is defined in (2.17).

Proof. To see this, let x be a positive solution of (2.1). Set $w(t) = \frac{x(\tau(t))}{x(t)}$. By Lemma 2.2.5, $\liminf_{t\to\infty} w(t) = l \le B(M)$. From (2.1), we obtain

$$-\frac{x'(t)}{x(t)} \ge p(t)w(t) \quad \text{for all} \quad t \ge T,$$

where T is sufficiently large. Integrating from $\tau(t)$ to t we obtain

$$\ln w(t) \ge \int_{\tau(t)}^{t} p(s)w(s)ds = w(\xi_t)\int_{\tau(t)}^{t} p(s)ds$$

for some $\xi_t \in [\tau(t), t]$ and hence

$$\ln l = \liminf_{t\to\infty} \ln w(t) \ge lm$$

and

$$m \le \frac{\ln l}{l} \le \frac{\ln b}{b}$$

which contradicts (2.18). Therefore (2.1) has no eventually positive solutions. □

Theorem 2.2.8. *Assume* $0 \le m \le \frac{1}{e}$ *and* τ *is nondecreasing. Furthermore, suppose*

$$M > 1 - A(m), \tag{2.19}$$

where $A(m)$ *is defined in* (2.10), *or*

$$M > \frac{\ln\lambda + 1}{\lambda}, \tag{2.20}$$

where λ *is the smallest positive root of the equation* (2.7). *Then the conclusions of Theorem 2.2.6 are true.*

Proof. As in Theorem 2.2.6, it is sufficient to show that under our assumptions (2.1) has no eventually positive solutions. We assume that x is an eventually positive solution of (2.1). Integrating (2.1) from $\tau(t)$ to t we obtain

$$x(\tau(t)) - x(t) \ge \int_{\tau(t)}^{t} p(s)x(\tau(s))ds \ge x(\tau(t))\int_{\tau(t)}^{t} p(s)ds.$$

Then if (2.19) holds, by Lemma 2.2.4, we have

$$\begin{aligned} M &= \limsup_{t\to\infty}\int_{\tau(t)}^{t} p(s)ds \le \limsup_{t\to\infty}\left[1 - \frac{x(t)}{x(\tau(t))}\right] \\ &= 1 - \liminf_{t\to\infty}\frac{x(t)}{x(\tau(t))} = 1 - r \le 1 - A(m), \end{aligned} \tag{2.21}$$

which contradicts (2.10).

If (2.20) holds, choose $m' < m$ sufficiently close to m such that

$$M = \limsup_{t\to\infty}\int_{\tau(t)}^{t} p(s)ds > \frac{\ln\lambda' + 1}{\lambda'}, \tag{2.22}$$

where λ' is the smallest root of the equation $\lambda = e^{m'\lambda}$.

Clearly, $\lambda' < \lambda$ and hence $\frac{\ln\lambda'+1}{\lambda'} > \frac{\ln\lambda+1}{\lambda}$. By Lemma 2.2.2, we have

$$\frac{x(\tau(t))}{x(t)} > \lambda' \quad \text{for all large} \quad t. \tag{2.23}$$

From (2.22), there exists t_1 so that (2.23) holds for all $t > \tau(\tau(t_1))$, and

$$\int_{\tau(t_1)}^{t_1} p(s)ds > \frac{\ln \lambda' + 1}{\lambda'}. \tag{2.24}$$

Without loss of generality denote $t_0 = \tau(t_1)$. We shall show that $x(t) > 0$ on $[t_0, t_1]$ will lead to a contradiction. In fact, let $t_2 \in [t_0, t_1]$ be a point at which $x(t_0)/x(t_2) = \lambda'$. If such a point does not exist, take $t_2 = t_1$. Integrating (2.1) over $[t_2, t_1]$ and noting that $x(\tau(t)) \geq x(t_0)$, we have

$$\int_{t_2}^{t_1} p(s)ds \leq \frac{1}{\lambda'}. \tag{2.25}$$

On the other hand, dividing (2.1) by $x(t)$ and integrating it over $[t_0, t_2]$ we find

$$\int_{t_0}^{t_2} p(s)ds \leq -\frac{1}{\lambda'} \int_{t_0}^{t_2} \frac{x'(s)}{x(s)} ds = \frac{\ln \lambda'}{\lambda'}. \tag{2.26}$$

Combining (2.25) and (2.26) we get

$$\int_{t_0}^{t_1} p(s)ds \leq \frac{\ln \lambda' + 1}{\lambda'},$$

which contradicts (2.24). $\square$

Example 2.2.9. Consider the equation

$$x'(t) + \frac{0.85}{a\pi + \sqrt{2}} (2a + \cos t)\, x\left(t - \frac{\pi}{2}\right) = 0, \tag{2.27}$$

where $a = 1.137$. Then (2.27) is in the form (2.3) with

$$p(t) = \frac{0.85}{a\pi + \sqrt{2}} (2a + \cos t) \qquad \text{and} \qquad \tau(t) = t - \frac{\pi}{2}.$$

We have

$$\int_{\tau(t)}^{t} p(s)ds = \frac{0.85}{a\pi + \sqrt{2}} \left(a\pi + \sqrt{2}\cos\left(t - \frac{\pi}{4}\right)\right).$$

Hence

$$m = \liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds = 0.85\frac{a\pi - \sqrt{2}}{a\pi + \sqrt{2}} = 0.367837 < \frac{1}{e}$$

and

$$M = \limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds = 0.85.$$

It is easy to see that (2.19) holds. Therefore every solution of (2.27) is oscillatory.

In the following we will consider the existence of positive solutions of a linear delay differential equation of the form

$$x'(t) + x(t - \tau(t)) = 0, \tag{2.28}$$

where $\tau \in C([t_0, \infty), \mathbb{R}^+)$ and $\lim_{t\to\infty}(t - \tau(t)) = \infty$. Set $T_0 = \inf_{t\geq t_0}\{t - \tau(t)\}$.

Definition 2.2.10. A solution x is called *positive with respect to the initial point* t_0, if x is a solution of (2.28) on (t_0, ∞) and $x(t) > 0$ for all $t \in [T_0, \infty)$.

Theorem 2.2.11. *Equation* (2.28) *has a positive solution with respect to* t_0 *if and only if there exists a real continuous function* λ_0 *on* $[T_0, \infty)$ *such that* $\lambda_0(t) > 0$ *for all* $t \geq t_0$ *and*

$$\tau(t) \leq t - \Lambda_0^{-1}\left(\Lambda_0(t) - \ln \lambda_0(t)\right) \quad \text{for all} \quad t \geq t_0, \tag{2.29}$$

where $\Lambda_0(t) = \int_{T_0}^t \lambda_0(s)ds$, *and* Λ_0^{-1} *denotes the inverse function of* Λ_0.

Proof. We first prove necessity. Let x_0 be a positive solution of (2.28) with respect to t_0. Then $x_0(t) > 0$ for all $t \in [T_0, \infty)$. Set

$$\lambda_0(t) = \frac{x_0(t - \tau(t))}{x_0(t)} \quad \text{for all} \quad t \geq T_0.$$

Clearly, $\lambda_0(t) > 0$ for all $t \geq t_0$ and hence $\Lambda_0(t) = \int_{T_0}^t \lambda_0(s)ds$ defines a function Λ_0 on $[T_0, \infty)$, which is strictly increasing on $[t_0, \infty)$. We have for $t \geq t_0$

$$\begin{aligned} \ln \lambda_0(t) &= \ln\left(\frac{x_0(t-\tau(t))}{x_0(t)}\right) = -\int_{t-\tau(t)}^t \frac{x_0'(s)}{x_0(s)}ds \\ &= \int_{t-\tau(t)}^t \lambda_0(s)ds = \Lambda_0(t) - \Lambda_0(t - \tau(t)) \end{aligned}$$

and therefore

$$t - \tau(t) = \Lambda_0^{-1}\left(\Lambda_0(t) - \ln \lambda_0(t)\right).$$

Then

$$\tau(t) = t - \Lambda_0^{-1}\left(\Lambda_0(t) - \ln \lambda_0(t)\right)$$

so that (2.29) holds.

Now we prove sufficiency. If there exists a function λ_0 such that (2.29) holds, then

$$\Lambda_0\left(t - \tau(t)\right) \geq \Lambda_0(t) - \ln \lambda_0(t)$$

and

$$\lambda_0(t) \geq \exp\left(\int_{t-\tau(t)}^t \lambda_0(s)ds\right).$$

Define

$$\lambda_1(t) = \begin{cases} \exp\left(\int_{t-\tau(t)}^t \lambda_0(s)ds\right) & \text{if} \quad t \geq t_0 \\ \lambda_1(t_0) + \lambda_0(t) - \lambda_0(t_0) & \text{if} \quad t \in [T_0, t_0). \end{cases}$$

Clearly, $\lambda_1(t) \leq \lambda_0(t)$ for $t \geq T_0$ and $0 \leq \lambda_1(t) \leq \lambda_0(t)$ for $t \geq t_0$. In general, we define

$$\lambda_n(t) = \begin{cases} \exp\left(\int_{t-\tau(t)}^t \lambda_{n-1}(s)ds\right)a & \text{if} \quad t \geq t_0 \\ \lambda_n(t_0) + \lambda_0(t) - \lambda_0(t_0) & \text{if} \quad t \in [T_0, t_0). \end{cases}$$

Thus

$$\lambda_0(t) - \lambda_0(t_0) \leq \lambda_n(t) \leq \lambda_{n-1}(t) \leq \ldots \leq \lambda_0(t) \quad \text{for all} \quad t \geq T_0$$

and $\lambda_n(t) \geq 0$ for all $t \geq t_0$. Then $\lim_{n\to\infty} \lambda_n(t) = \lambda(t)$ exists for $t \geq T_0$ and

$$\lim_{n\to\infty} \int_{t-\tau(t)}^{t} \lambda_n(s)ds = \int_{t-\tau(t)}^{t} \lambda(s)ds \quad \text{for all} \quad t \geq t_0.$$

Hence

$$\lambda(t) = \exp\left(\int_{t-\tau(t)}^{t} \lambda(s)ds\right) \quad \text{for all} \quad t \geq t_0.$$

Set

$$x(t) = \exp\left(-\int_{T_0}^{t} \lambda(s)ds\right) \quad \text{for} \quad t \geq T_0.$$

Then x is a positive solution of (2.28) with respect to t_0. $\square$

Remark 2.2.12. If we take $\lambda_0(t) \equiv \lambda > 0$ in Theorem 2.2.11, then condition (2.29) becomes

$$\tau(t) \leq \frac{\ln \lambda}{\lambda} \quad \text{for all} \quad t \geq t_0. \tag{2.30}$$

In particular, if $\lambda = e$, then (2.30) becomes

$$\tau(t) \leq \frac{1}{e} \quad \text{for all} \quad t \geq t_0, \tag{2.31}$$

i.e., (2.31) is a sufficient condition for the existence of positive solutions of (2.28).

Let $t_0 = \frac{1}{2}$ and $\lambda_0(t) = 2t$. Then by Theorem 2.2.11, if

$$\tau(t) = t - \sqrt{t^2 - \ln 2t},$$

then (2.28) has a positive solution with respect to $t_0 = \frac{1}{2}$. In fact, $x(t) = e^{-t^2}$ is such a solution. We note that

$$\tau\left(\frac{e}{2}\right) = \frac{e}{2} - \sqrt{\left(\frac{e}{2}\right)^2 - 1} > \frac{1}{e}.$$

This example shows that (2.31) is not necessary for the existence of a positive solution of (2.28).

We now consider the linear equation of the form

$$x'(t) + p(t)x\left(t - \tau(t)\right) = 0, \tag{2.32}$$

where $p, \tau \in C([t_0, \infty), \mathbb{R}^+)$, $\tau(t) \leq t$, and $\lim_{t\to\infty}(t - \tau(t)) = \infty$. As before, set $T_0 = \inf_{t\geq t_0}\{t - \tau(t)\}$. Similarly as in Theorem 2.2.11 we can prove the following result.

Theorem 2.2.13. *Equation* (2.32) *has a positive solution with respect to t_0 if and only if there exists a continuous function λ_0 on $[T_0, \infty)$ such that $\lambda_0(t) > 0$ for $t \geq t_0$ and*

$$\lambda_0(t) \geq p(t) \exp\left(\int_{t-\tau(t)}^{t} \lambda_0(s)ds\right) \quad \textit{for all} \quad t \geq t_0. \tag{2.33}$$

Remark 2.2.14. If $p(t) > 0$, then (2.33) can be replaced by

$$\tau(t) \leq t - \Lambda_0^{-1}\left(\Lambda_0(t) - \ln \frac{\lambda_0(t)}{p(t)}\right) \quad \text{for all} \quad t \geq t_0.$$

Corollary 2.2.15. *If*

$$\int_{t-\tau(t)}^{t} p(s)ds \leq \frac{1}{e} \quad \textit{for all} \quad t \geq t_0,$$

then (2.32) *has a positive solution with respect to* t_0.

Proof. If we take $\lambda_0(t) = ep(t)$, then (2.33) is satisfied. Then the corollary follows from Theorem 2.2.13. □

Theorem 2.2.16. *Assume that* $\tau(t) \equiv \tau > 0$ *and* $\int_{t_0}^{\infty} p(t)dt = \infty$. *Then* (2.32) *has a positive solution with respect to* t_0 *if and only if there exists a continuous function* λ_0 *on* $[t_0 - \tau, \infty)$ *such that*

$$\int_{t-\tau}^{t} p(s)ds \leq t - \Lambda_0^{-1}\left(\Lambda_0(t) - \ln \lambda_0(t)\right) \quad \textit{for all} \quad t \geq t_0. \tag{2.34}$$

Proof. Set $u = P(t) = \int_{t_0}^{t} p(s)ds$ for $t \geq t_0$. Then

$$t - \tau = P^{-1}\left(u - \int_{P^{-1}(u)-\tau}^{P^{-1}(u)} p(s)ds\right).$$

Denote

$$z(u) = x\left(P^{-1}(u)\right).$$

Then (2.32) becomes

$$z'(u) + z\left(u - \int_{P^{-1}(u)-\tau}^{P^{-1}(u)} p(s)ds\right) = 0. \tag{2.35}$$

By Theorem 2.2.11, (2.34) is a necessary and sufficient condition for (2.35) to have a positive solution with respect to 0. From the transformation, it is equivalent to (2.32) having a positive solution with respect to t_0. □

Remark 2.2.17. If we choose $\lambda_0(t) \equiv e$ in (2.34), then we obtain

$$\int_{t-\tau}^{t} p(s)ds \leq \frac{1}{e} \quad \text{for all} \quad t \geq t_0. \tag{2.36}$$

As we have mentioned, (2.36) is a sufficient condition and is not a necessary condition for the existence of a positive solution of (2.32).

Combining Theorem 2.2.6 and (2.36), we obtain the following corollary.

Corollary 2.2.18. *Let* $p(t) \equiv p > 0$ *and* $\tau(t) \equiv \tau > 0$. *Then a necessary and sufficient condition for all solutions of* (2.32) *to be oscillatory is that* $p\tau e > 1$.

Remark 2.2.19. The above techniques can be used on the first order advanced type equations

$$x'(t) = x\left(t + \tau(t)\right) \tag{2.37}$$

and

$$x'(t) = p(t)x\left(t + \tau(t)\right), \tag{2.38}$$

where $p, \tau \in C([t_0, \infty), \mathbb{R}^+)$. For example, (2.37) has a positive solution if and only if there exists a continuous function $\lambda \in C([T_0, \infty), \mathbb{R})$ such that

$$\tau(t) \leq \Lambda^{-1}\Big(\Lambda(t) + \ln \lambda(t)\Big) - t, \tag{2.39}$$

where $\Lambda(t) = \int_{t_0}^t \lambda(s)ds$. If we let $\lambda(t) \equiv \lambda > 0$, then (2.39) becomes $\tau(t) \leq \frac{1}{e}$, which is a sufficient condition for the existence of a positive solution of (2.37).

For (2.38), assume

$$\liminf_{t\to\infty} \int_t^{t+\tau(t)} p(s)ds > \frac{1}{e}.$$

Then every solution of (2.38) oscillates.

If $\tau(t) \equiv \tau > 0$, then (2.38) has a positive solution if and only if there exists a continuous function λ such that

$$\int_t^{t+\tau} p(s)ds \leq \Lambda^{-1}\Big(\Lambda(t) + \ln \lambda(t)\Big) - t.$$

Corollary 2.2.18 is also true for (2.38).

2.3. Equations with Variable Delay: Critical Case

In this section we will consider oscillatory solutions of (2.3) in the critical case

$$\lim_{t\to\infty} \int_{\tau(t)}^t p(s)ds = \frac{1}{e}$$

under the assumption

$$\int_{\tau(t)}^t p(s)ds \geq \frac{1}{e}. \tag{2.40}$$

We suppose that the delay function τ in (2.3) is strictly increasing on $[t_0, \infty)$ with $\tau(t) < t$ and $\lim_{t\to\infty} \tau(t) = \infty$, and define recursively

$$t_{k+1} := \tau^{-1}(t_k) \quad \text{for all} \quad k \in \mathbb{N}_0.$$

Clearly $t_k \to \infty$ as $k \to \infty$. Moreover, the coefficient p is assumed to be a piecewise continuous function satisfying (2.40). We define a set $\mathcal{A}_\lambda$ for $0 < \lambda \leq 1$ as follows.

Definition 2.3.1. The piecewise continuous function $p : [t_0, \infty] \to [0, \infty]$ belongs to $\mathcal{A}_\lambda$ if (2.40) holds for all $t \geq t_1$ and

$$\int_{\tau(t)}^t p(s)ds \geq \frac{1}{e} + \lambda_k \left(\int_{t_k}^{t_{k+1}} p(s)ds - \frac{1}{e}\right), \; t \in (t_k, t_{k+1}], \; k \in \mathbb{N} \tag{2.41}$$

for some $\lambda_k \geq 0$ with

$$\liminf_{k\to\infty} \lambda_k = \lambda > 0.$$

Remark 2.3.2. If $\int_{\tau(t)}^t p(s)ds$ is a nonincreasing function and $\int_{\tau(t)}^t p(s)ds \geq \frac{1}{e}$, then $p \in \mathcal{A}_1$, since we may choose $\lambda_k = 1$ in (2.41). However, the monotonicity is not a necessary condition; e.g., in the case $\tau(t) = t - 1$, the condition that the function

$$p(s) = \frac{1}{e} + K\frac{\sin^2(\pi s)}{s^\alpha} \quad \text{with} \quad K > 0 \text{ and } 0 \leq \alpha \leq 2 \tag{2.42}$$

belongs to $\mathcal{A}_1$ becomes the condition that $\int_{t-1}^{t} \sin^2(\pi s)/s^\alpha ds$ is a nonincreasing function.

Lemma 2.3.3. *Assume that x is a positive solution of (2.3) on $[t_{k-2}, t_{k+1}]$ for some $k \geq 2$. Let N be defined by*

$$N = \min_{t_k \leq t \leq t_{k+1}} \frac{x(\tau(t))}{x(t)}.$$

Then $N < (2e)^2$.

Proof. Let L be the integral

$$L := \int_{t_k}^{t_{k+1}} p(s)ds \geq \frac{1}{e}.$$

By Lemma 2.2.5, we obtain

$$N < \left(\frac{1+\sqrt{1-L}}{L}\right)^2.$$

Since the right-hand side is a decreasing function of L, we get

$$N < \left(\frac{1+\sqrt{1-\frac{1}{e}}}{\frac{1}{e}}\right)^2 < (2e)^2.$$

The proof is complete. □

Lemma 2.3.4. *Assume that x is a positive solution of (2.3) on $[t_{k-3}, t_{k-1}]$ for some $k \geq 3$ and $p \in \mathcal{A}_\lambda$. Let M and N be defined by*

$$M = \min_{t_{k-1} \leq t \leq t_k} \frac{x(\tau(t))}{x(t)} \quad \text{and} \quad N = \min_{t_k \leq t \leq t_{k+1}} \frac{x(\tau(t))}{x(t)}.$$

Then

$$M > 1 \quad \text{and} \quad N \geq \exp\left(M\left[\frac{1}{e} + \lambda_k\left(\int_{t_k}^{t_{k+1}} p(s)ds - \frac{1}{e}\right)\right]\right) \geq M.$$

Proof. Following the lines of the proof of Elbert and Stavroulakis [**82**, Lemma 1], we have $\min\{M, N\} = M$, and by (2.41) for $t_k < t \leq t_{k+1}$

$$\frac{x(\tau(t))}{x(t)} \geq \exp\left(M \int_{\tau(t)}^{t} p(s)ds\right) \geq \exp\left(M\left[\frac{1}{e} + \lambda_k\left(\int_{t_k}^{t_{k+1}} p(s)ds - \frac{1}{e}\right)\right]\right),$$

which implies the inequality concerning N. On the other hand, x is a strictly decreasing function on $[t_{k-2}, t_{k+1}]$. Hence $x(\tau(t))/x(t) > 1$ on $[t_{k-1}, t_k]$, and therefore $M > 1$. The proof is complete. □

The next lemma deals with some properties of the sequence $\{r_i\}_{i=0}^\infty$ defined recursively by

$$r_0 = 1 \quad \text{and} \quad r_{i+1} = e^{r_i/e} \text{ for } i \in \mathbb{N}_0. \tag{2.43}$$

Lemma 2.3.5. *For the sequence $\{r_i\}_{i=0}^\infty$ defined in (2.43) we have:*

(i) $r_i < r_{i+1}$;
(ii) $r_i < e$;
(iii) $\lim_{i\to\infty} r_i = e$;

(iv) $r_i > e - 2e/(i+2)$.

Proof. The first two relations can be proved by induction. As a consequence of (i) and (ii), $\lim_{i\to\infty} r_i = r$ exists and is finite. Then by (2.43) we have

$$r = e^{r/e}.$$

It is easy to check that

$$e^{x/e} > x \quad \text{for} \quad x \neq e. \tag{2.44}$$

This inequality implies that the limit r is equal to e.

Now we give the proof of (iv). For $i = 0$ and $i = 1$ it is immediate. For $i \in \mathbb{N}$ the proof can be done induction, so we have

$$r_{i+1} = e^{r_i/e} > e^{1-2/(i+2)},$$

and it is sufficient to show

$$e^{1-2/(i+2)} > e - \frac{2e}{i+3}$$

or

$$f(i+2) > 1, \qquad \text{where} \qquad f(x) = e^{-2/x} + \frac{2}{x+1}.$$

Since

$$f'(x) = \frac{2}{x^2}\left(e^{-1/x} + \frac{x}{x+1}\right)\left(e^{-1/x} - \frac{x}{x+1}\right)$$

and

$$e^{1/x} > 1 + \frac{1}{x} = \frac{x+1}{x},$$

we have $f'(x) < 0$ and $f(i+2) > \lim_{x\to\infty} f(x) = 1$, which was to be shown. The proof is complete. $\square$

Theorem 2.3.6. *Assume $p \in \mathcal{A}_\lambda$ for some $\lambda \in (0,1]$. If*

$$\sum_{i=1}^{\infty}\left(\int_{t_{i-1}}^{t_i} p(s)ds - \frac{1}{e}\right) = \infty, \tag{2.45}$$

then every solution of (2.3) *oscillates.*

Proof. Suppose the contrary. Then we may assume that, without loss of generality, there exists a solution x such that $x(t) > 0$ for $t \geq t_{k-3}$ for some $k \geq 3$. Let the sequence $\{N_i\}_{i=0}^{\infty}$ be defined by

$$N_i = \min_{t_{k+i-1} \leq t \leq t_{k+i}} \frac{x(\tau(t))}{x(t)}. \tag{2.46}$$

By Lemma 2.3.4 we have $N_0 > 1$ and

$$N_{i+1} \geq \exp\left(\frac{N_i}{e}\right)\exp\left(N_i\lambda_{k+i}\left(\int_{t_{k+i}}^{t_{k+i+1}} p(s)ds - \frac{1}{e}\right)\right) \geq N_i, \tag{2.47}$$

therefore the sequence $\{N_i\}_{i=0}^{\infty}$ is nondecreasing. On the other hand, it is bounded by Lemma 2.3.3. Consequently the sequence converges. Let

$$\lim_{i\to\infty} N_i = N.$$

Then (2.47) implies

$$N \geq \exp\left(\frac{N}{e}\right).$$

Hence by (2.44) we have $N = e$ and

$$1 < N_0 < N_1 < \ldots < e. \tag{2.48}$$

From (2.47), in view of (2.44), we obtain

$$N_{i+1} > N_i \left(1 + N_i \lambda_{k+1} \left(\int_{t_{k+i}}^{t_{k+i+1}} p(s)ds - \frac{1}{e}\right)\right).$$

Thus

$$N_{i+1} - N_i > N_i^2 \lambda_{k+1} \left(\int_{t_{k+i}}^{t_{k+i+1}} p(s)ds - \frac{1}{e}\right). \tag{2.49}$$

From the definition of $\mathcal{A}_\lambda$ we know that $\lambda = \liminf_{k\to\infty} \lambda_k > 0$, so for any sufficiently small $\varepsilon > 0$ there exists a value c_ε such that $\lambda_{k+i} > \lambda - \varepsilon$ for $k + i > c_\varepsilon$. Thus, for such indices i, from (2.49) and (2.48) we have

$$N_{i+1} - N_i > N_i^2 (\lambda - \varepsilon) \left(\int_{t_{k+i}}^{t_{k+i+1}} p(s)ds - \frac{1}{e}\right)$$

and

$$\begin{aligned} N_{i+2} - N_{i+1} &> N_{i+1}^2 (\lambda - \varepsilon) \left(\int_{t_{k+i+1}}^{t_{k+i+2}} p(s)ds - \frac{1}{e}\right) \\ &> N_i^2 (\lambda - \varepsilon) \left(\int_{t_{k+i+1}}^{t_{k+i+2}} p(s)ds - \frac{1}{e}\right). \end{aligned}$$

Summing up the inequalities above, we obtain for $k + i \geq c_\varepsilon$

$$e - 1 > e - N_i > N_i^2 (\lambda - \varepsilon) \sum_{j=1}^{\infty} \left(\int_{t_{k+i+j-1}}^{t_{k+i+j}} p(s)ds - \frac{1}{e}\right). \tag{2.50}$$

This last inequality contradicts assumption (2.45). The proof is complete. □

In the next theorem we consider the case where the sum in (2.45) is convergent.

Theorem 2.3.7. *Assume $p \in \mathcal{A}_\lambda$ for some $0 < \lambda \leq 1$ and either*

$$\lambda \limsup_{k\to\infty} k \sum_{i=k}^{\infty} \left(\int_{t_{i-1}}^{t_i} p(s)ds - \frac{1}{e}\right) > \frac{2}{e} \tag{2.51}$$

or

$$\lambda \liminf_{k\to\infty} k \sum_{i=k}^{\infty} \left(\int_{t_{i-1}}^{t_i} p(s)ds - \frac{1}{e}\right) > \frac{1}{2e}. \tag{2.52}$$

Then every solution of (2.3) *oscillates.*

Proof. Suppose the contrary. Then, as in the proof of Theorem 2.3.6, the sequence $\{N_i\}_{i=0}^{\infty}$ defined by (2.46) satisfies the inequalities (2.47)–(2.50). In particular, from (2.47) we have

$$N_{i+1} \geq \exp\left(\frac{N_i}{e}\right).$$

Comparing the last inequality with (2.43), we obtain by induction

$$N_0 > r_0 = 1 \quad \text{and} \quad N_i > r_i \text{ for } i \in \mathbb{N}.$$

Then by Lemma 2.3.5 (iv) we have

$$e - N_i < e - r_i < \frac{2e}{i+2}. \tag{2.53}$$

Multiplying (2.50) by $k+i \geq c_\varepsilon$, we obtain from (2.53)

$$(k+i)\frac{2e}{i+2} > N_i^2(\lambda-\varepsilon)(k+i)\sum_{j=k+i}^{\infty}\left(\int_{t_j}^{t_{j+1}} p(s)ds - \frac{1}{e}\right).$$

Taking the limit as $i \to \infty$, we get

$$2e \geq e^2\lambda \limsup_{k\to\infty} k\sum_{j=k}^{\infty}\left(\int_{t_j}^{t_{j+1}} p(s)ds - \frac{1}{e}\right),$$

which contradicts (2.51).

Now let A be defined by

$$A = \liminf_{k\to\infty} k\sum_{j=k}^{\infty}\left(\int_{t_j}^{t_{j+1}} p(s)ds - \frac{1}{e}\right).$$

If $A = \infty$, then every solution oscillates by (2.51). Therefore we consider the case $0 < A < \infty$. For any sufficiently small $\varepsilon > 0$ there exists a value $\bar{c}_\varepsilon$ such that for $\bar{\lambda} = \lambda - \varepsilon > 0$ and $\bar{A} = A - \varepsilon > 0$

$$\lambda_k > \bar{\lambda} \quad \text{and} \quad \sum_{j=k}^{\infty}\left(\int_{t_j}^{t_{j+1}} p(s)ds - \frac{1}{e}\right) > \frac{\bar{A}}{k} \quad \text{for} \quad k \geq \bar{c}_\varepsilon.$$

If we use the inequality

$$\exp\left(\frac{x}{e}\right) > x + \frac{1}{2}\exp\left(\frac{\xi}{e}\right)\left(1-\frac{x}{e}\right)^2 \quad \text{for} \quad \xi < x < e$$

in (2.47), then we obtain for $N_i > \xi$ and $k+i > \bar{c}_\varepsilon$

$$\begin{aligned} N_{i+1} &\geq \exp\left(\frac{N_i}{e}\right)\exp\left(N_i\bar{\lambda}\left(\int_{t_{k+i}}^{t_{k+i+1}} p(s)ds - \frac{1}{e}\right)\right) \\ &> \left[N_i + \frac{1}{2}\exp\left(\frac{\xi}{e}\right)\left(1-\frac{N_i}{e}\right)^2\right]\left[1 + N_i\bar{\lambda}\left(\int_{t_{k+i}}^{t_{k+i+1}} p(s)ds - \frac{1}{e}\right)\right]. \end{aligned}$$

Consequently

$$N_{i+1} - N_i > \frac{1}{2}\exp\left(\frac{\xi}{e}\right)\left(1-\frac{N_i}{e}\right)^2 + \xi^2\bar{\lambda}\left(\int_{t_{k+i}}^{t_{k+i+1}} p(s)ds - \frac{1}{e}\right)$$

and summing up,

$$\begin{aligned} e - N_i &> \frac{1}{2}\exp\left(\frac{\xi}{e}\right)\sum_{j=i}^{\infty}\left(1-\frac{N_j}{e}\right)^2 + \xi^2\bar{\lambda}\sum_{j=k+i}^{\infty}\left(\int_{t_j}^{t_{j+1}} p(s)ds - \frac{1}{e}\right) \\ &> \frac{1}{2}\exp\left(\frac{\xi}{e}\right)\sum_{j=i}^{\infty}\left(1-\frac{N_j}{e}\right)^2 + \frac{\xi^2\bar{\lambda}\bar{A}}{k+i}. \end{aligned} \tag{2.54}$$

In particular,

$$e - N_i > \frac{U_0}{k+i} \quad \text{with} \quad U_0 = \xi^2\bar{\lambda}\bar{A}.$$

By iteration we can improve this inequality to

$$e - N_i > \frac{U_n}{k+i} \quad \text{for} \quad n \in \mathbb{N}_0. \tag{2.55}$$

Namely by (2.54) we have

$$\begin{aligned} e - N_i &> \frac{1}{2}\exp\left(\frac{\xi}{e}\right)\sum_{j=i}^{\infty}\left(\frac{U_n}{e\,(k+j)}\right)^2 + \frac{U_0}{k+i} \\ &> \frac{U_n^2}{2e^2}\exp\left(\frac{\xi}{e}\right)\frac{1}{k+i} + \frac{U_0}{k+i} = \frac{U_{n+1}}{k+i}, \end{aligned}$$

where

$$U_{n+1} = \frac{U_n^2}{2e^2}\exp\left(\frac{\xi}{e}\right) + U_0 \quad \text{for} \quad n \in \mathbb{N}_0.$$

From this it is clear that the sequence $\{U_n\}_{n=0}^{\infty}$ is increasing. Moreover, comparing inequalities (2.53) and (2.55), we see that $U_n \le 2e$. Therefore the sequence has a limit, say U, which satisfies the equation

$$U = \frac{U^2}{2e^2}\exp\left(\frac{\xi}{e}\right) + \xi^2\bar{\lambda}\bar{A}.$$

This is a quadratic equation with real roots and therefore the discriminant is not negative, i.e.,

$$1 - 2\exp\left(\frac{\xi}{e} - 2\right)\xi^2\bar{\lambda}\bar{A} \ge 0.$$

Let $\varepsilon \to 0$ and $\xi \to e$. Then the last inequality becomes

$$1 - 2e\lambda A \ge 0,$$

which contradicts (2.52). The proof is complete. □

Remark 2.3.8. If the function $\int_{\tau(t)}^{t} p(s)ds$ is monotone, then the value of λ in conditions (2.51) and (2.52) of Theorem 2.3.7 is equal to one.

In the following theorem we give a criterion for nonoscillation.

Theorem 2.3.9. *Let $\tau(t) = t - 1$, $p(t) = \frac{1}{e} + a(t)$, and $t_0 = 1$ in (2.3), i.e., (2.3) has the form*

$$x'(t) + \left(\frac{1}{e} + a(t)\right)x(t-1) = 0, \quad t \ge 1. \tag{2.56}$$

Assume that

$$a(t) \le \frac{1}{8et^2}.$$

Then (2.56) *has a solution x satisfying $x(t) \geq \sqrt{t}e^{-t}$.*

Proof. The proof is based on known comparison theorems (see Myškhis [**230**]). Let the functions A, B, and C on $(1, \infty)$ be defined by

$$A(t) = \frac{1}{e} + a(t), \quad B(t) = \frac{1}{e} + \frac{1}{8et^2}, \quad \text{and} \quad C(t) = \frac{1}{e}\frac{1 - \frac{1}{2t}}{\sqrt{1 - \frac{1}{t}}}.$$

By the assumption we have $A(t) \leq B(t)$. We are going to show that the inequality $B(t) < C(t)$ also holds. Namely, for $\theta = \frac{1}{2t} \in (0, 1/2)$, we have

$$C(t) - B(t) = \frac{\theta^3\left(\frac{1}{2}\theta^2 - \frac{1}{4}\theta + 2\right)}{e\sqrt{1-2\theta}\left[1 - \theta + \left(1 + \frac{1}{2}\theta^2\right)\sqrt{1-2\theta}\right]} > 0.$$

Now we will compare the differential equations

$$\begin{aligned} x'(t) + A(t)x(t-1) &= 0, \\ z'(t) + B(t)z(t-1) &= 0, \\ u'(t) + C(t)u(t-1) &= 0. \end{aligned}$$

Let us observe that the function $u(t) = \sqrt{t}e^{-t}$ is a solution of the last differential equation. Let the initial function φ be the function defined by $\varphi(t) = \sqrt{t}e^{-t}$ on $[0, 1]$, and let x and z be the solutions of the first and the second differential equations, respectively, associated with this initial function φ. Then by the comparison theorems mentioned above we have

$$x(t) \geq z(t) > u(t) = \sqrt{t}e^{-t} \quad \text{for} \quad t > 1,$$

which was to be shown. The proof is complete. □

Remark 2.3.10. For (2.56) we have $t_k = k + 1$ and

$$\limsup_{k\to\infty} k \sum_{i=k}^{\infty}\left(\int_{t_{i-1}}^{t_i} p(s)ds - \frac{1}{e}\right) = \limsup_{k\to\infty} k\int_k^\infty a(s)ds \leq \frac{1}{8e}.$$

Now the question arises naturally whether or not the bounds in conditions (2.51) and (2.52) of Theorem 2.3.7 can be replaced by smaller ones.

Remark 2.3.11. It is to be emphasized that in Theorem 2.3.9 we require

$$\text{neither} \quad p(t) \geq 0 \quad \text{nor} \quad \int_{\tau(t)}^t p(s)ds \geq \frac{1}{e}.$$

Remark 2.3.12. Applying Theorems 2.3.6 and 2.3.7, we see that, under (2.42), (2.3) oscillates for any $K > 0$ if $0 \leq \alpha < 2$ and $K > \frac{1}{e}$ if $\alpha = 2$. On the other hand, it has a nonoscillatory solution for $K < \frac{1}{8e}$ if $\alpha = 2$.

2.4. Equations with Constant Delay

Consider the delay differential equation

$$x'(t) + p(t)x(t-\tau) = 0, \quad t \geq t_0, \tag{2.57}$$

where

$$p \in C([t_0, \infty), [0, \infty)) \quad \text{and} \quad \tau \text{ is a positive constant.} \tag{2.58}$$

Theorem 2.4.1. *Assume that* (2.58) *holds and that there exists* $\bar{t}_0 > t_0 + \tau$ *such that*

$$\int_{t-\tau}^{t} p(s)ds \geq \frac{1}{e} \quad \textit{for all} \quad t \geq \bar{t}_0 \tag{2.59}$$

and

$$\int_{t_0+\tau}^{\infty} p(t)\left[\exp\left(\int_{t-\tau}^{t} p(s)ds - \frac{1}{e}\right) - 1\right] dt = \infty. \tag{2.60}$$

Then every solution of (2.57) *oscillates.*

Proof. Assume, for the sake of contradiction, that (2.57) has an eventually positive solution x. Then there exists $t_1 \geq \bar{t}_0$ such that for $t \geq t_1$

$$x(t) > 0, \quad x(t-\tau) > 0, \quad x'(t) \leq 0, \quad x(t-\tau) \geq x(t).$$

Set

$$w(t) = \frac{x(t-\tau)}{x(t)} \quad \text{for} \quad t \geq t_1. \tag{2.61}$$

Then

$$w(t) \geq 1 \quad \text{for all} \quad t \geq t_1. \tag{2.62}$$

Dividing both sides of (2.57) by $x(t)$, we obtain

$$\frac{x'(t)}{x(t)} + p(t)w(t) = 0 \quad \text{for} \quad t \geq t_1. \tag{2.63}$$

Integrating both sides of (2.63) from $t - \tau$ to t yields

$$w(t) = \exp\left(\int_{t-\tau}^{t} p(s)w(s)ds\right) \quad \text{for} \quad t \geq t_1 + \tau. \tag{2.64}$$

By (2.59), for $t \geq t_1 + \tau$, there exists $\gamma(t)$ with $0 < \gamma(t) \leq \tau$, such that

$$\int_{t-\gamma(t)}^{t} p(s)ds = \frac{1}{e} \quad \text{for} \quad t \geq t_1 + \tau. \tag{2.65}$$

It follows that, for $t \geq t_1 + \tau$,

$$\begin{aligned}
w(t) &= \exp\left(\int_{t-\gamma(t)}^{t} p(s)w(s)ds + \int_{t-\tau}^{t-\gamma(t)} p(s)w(s)ds\right) \\
&\geq \exp\left(\int_{t-\gamma(t)}^{t} p(s)w(s)ds + \int_{t-\tau}^{t-\gamma(t)} p(s)ds\right) \\
&= \exp\left(\int_{t-\gamma(t)}^{t} p(s)w(s)ds + \int_{t-\tau}^{t} p(s)ds - \frac{1}{e}\right) \\
&= \exp\left(\int_{t-\gamma(t)}^{t} p(s)w(s)ds\right) \exp\left(\int_{t-\tau}^{t} p(s)ds - \frac{1}{e}\right).
\end{aligned}$$

One can easily show that

$$e^x \geq ex \quad \text{for all} \quad x > 0,$$

and so

$$w(t) \geq e \left(\int_{t-\gamma(t)}^{t} p(s)w(s)ds \right) \exp \left(\int_{t-\tau}^{t} p(s)ds - \frac{1}{e} \right) \tag{2.66}$$

for $t \geq t_1 + \tau$ or

$$p(t)w(t) \geq ep(t) \left(\int_{t-\gamma(t)}^{t} p(s)w(s)ds \right) \exp \left(\int_{t-\tau}^{t} p(s)ds - \frac{1}{e} \right)$$

for $t \geq t_1 + \tau$ or by (2.59), (2.62), and (2.65), for $t \geq t_1 + \tau$,

$$\begin{aligned} & p(t) \left(w(t) - e \int_{t-\gamma(t)}^{t} p(s)w(s)ds \right) \\ \geq\ & ep(t) \left(\int_{t-\gamma(t)}^{t} p(s)w(s)ds \right) \left[\exp \left(\int_{t-\tau}^{t} p(s)ds - \frac{1}{e} \right) - 1 \right] \\ \geq\ & p(t) \left[\exp \left(\int_{t-\tau}^{t} p(s)ds - \frac{1}{e} \right) - 1 \right]. \end{aligned}$$

By integrating both sides from $t_2 = t_1 + 2\tau$ to $T > t_2$, we find

$$\begin{aligned} & \int_{t_2}^{T} p(t) \left(w(t) - e \int_{t-\gamma(t)}^{t} p(s)w(s)ds \right) dt \\ \geq\ & \int_{t_2}^{T} p(t) \left[\exp \left(\int_{t-\tau}^{t} p(s)ds - \frac{1}{e} \right) - 1 \right] dt. \end{aligned} \tag{2.67}$$

We select a function $N \in C^1([t_1, \infty), (0, \infty))$ such that

$$N'(t) = \max_{t_1 \leq s \leq t+\tau} p(s). \tag{2.68}$$

From (2.59) and (2.68), we have

$$N'(t) \geq \frac{1}{e} \quad \text{for} \quad t \geq t_1.$$

Thus N is increasing on $[t_1, \infty)$ and $\lim_{t\to\infty} N(t) = \infty$. One can easily show that

$$\int_{t_1}^{\infty} \exp(-N(t))\, dt < \infty \tag{2.69}$$

and

$$\int_{t_1+\tau}^{\infty} p(t) \exp \Big(-N(t-\tau) \Big)\, dt < \infty. \tag{2.70}$$

Set

$$q(t) = p(t) + \exp(-N(t)) \quad \text{for} \quad t \geq t_1.$$

In view of (2.66),

$$w(t) - e \int_{t-\gamma(t)}^{t} p(s)w(s)ds \geq 0 \quad \text{for} \quad t \geq t_1 + \tau.$$

It follows that

$$\begin{aligned}
(2.71)\qquad & \int_{t_2}^{T} p(t)\left[w(t) - e\int_{t-\gamma(t)}^{t} p(s)w(s)ds\right]dt \\
&\le \int_{t_2}^{T} q(t)\left[w(t) - e\int_{t-\gamma(t)}^{t} p(s)w(s)ds\right]dt \\
&= \int_{t_2}^{T} q(t)\left[w(t) - e\int_{t-\gamma(t)}^{t} q(s)w(s)ds\right]dt \\
&\quad + e\int_{t_2}^{T} q(t)\left[\int_{t-\gamma(t)}^{t} \exp(-N(s))w(s)ds\right]dt.
\end{aligned}$$

Now we claim that

$$\limsup_{t\to\infty} w(t) = \infty. \tag{2.72}$$

Otherwise, there exists an $M > 0$ such that

$$w(t) \le M \quad \text{for all} \quad t \ge t_1. \tag{2.73}$$

Then, by using the decreasing nature of $\exp(-N(t))$, we have

$$\int_{t_2}^{T} q(t)\left[\int_{t-\gamma(t)}^{t} \exp\left(-N(s)\right)w(s)ds\right]dt \le \tau M\int_{t_2}^{T} q(t)\exp\left(-N(t-\tau)\right)dt.$$

From this, (2.60), (2.67), (2.68), (2.69), (2.70), and (2.71), we have

$$\lim_{T\to\infty}\int_{t_2}^{T} q(t)\left[w(t) - e\int_{t-\gamma(t)}^{t} q(s)w(s)ds\right]dt = \infty. \tag{2.74}$$

Set

$$u = Q(t) := \int_{t_2-\tau}^{t} q(s)ds \quad \text{for} \quad t \ge t_2 - \tau. \tag{2.75}$$

Then $Q(t) \to \infty$ as $t \to \infty$, Q is strictly increasing and thus Q^{-1} exists. Set

$$z(u) = w\left(Q^{-1}(u)\right).$$

Then

$$\begin{aligned}
& \int_{t_2}^{T} q(t)\left[w(t) - e\int_{t-\gamma(t)}^{t} q(s)w(s)ds\right]dt \\
&= \int_{Q(t_2)}^{Q(T)}\left[w\left(Q^{-1}(u)\right) - e\int_{Q^{-1}(u)-\gamma(Q^{-1}(u))}^{Q^{-1}(u)} q(s)w(s)ds\right]du \\
&= \int_{Q(t_2)}^{Q(T)}\left[w\left(Q^{-1}(u)\right) - e\int_{Q(Q^{-1}(u)-\gamma(Q^{-1}(u)))}^{Q\left(Q^{-1}(u)\right)} w\left(Q^{-1}(\xi)\right)d\xi\right]du \\
&= \int_{Q(t_2)}^{Q(T)}\left[z(u) - e\int_{Q(Q^{-1}(u)-\gamma(Q^{-1}(u)))}^{u} z(\xi)d\xi\right]du. \\
(2.76)\qquad &\le \int_{Q(t_2)}^{Q(T)}\left[z(u) - e\int_{u-\frac{1}{e}}^{u} z(s)ds\right]du,
\end{aligned}$$

where the last inequality (2.76) is true because of

$$\begin{aligned} Q\left(Q^{-1}(u)-\gamma\left(Q^{-1}(u)\right)\right) &= Q\left(t-\gamma(t)\right) = \int_{t_2-\tau}^{t-\gamma(t)} q(s)ds \\ &\leq \int_{t_2-\tau}^{t} q(s)ds - \int_{t-\gamma(t)}^{t} p(s)ds \leq u-\frac{1}{e}. \end{aligned}$$

From (2.74), (2.75), and (2.76), we obtain

$$\lim_{A\to\infty}\int_{Q(t_2)}^{A}\left(z(u)-e\int_{u-\frac{1}{e}}^{u} z(s)ds\right)du=\infty. \tag{2.77}$$

By interchanging the order of integration, we obtain that for $A > Q(t_2)$,

$$\begin{aligned} \int_{Q(t_2)}^{A} e\left(\int_{u-\frac{1}{e}}^{u} z(s)ds\right)du &= \int_{Q(t_2)-\frac{1}{e}}^{Q(t_2)} e\left(\int_{Q(t_2)}^{s+\frac{1}{e}} z(s)du\right)ds \\ &\quad + \int_{Q(t_2)}^{A-\frac{1}{e}} e\left(\int_{s}^{s+\frac{1}{e}} z(s)du\right)ds + \int_{A-\frac{1}{e}}^{A} e\left(\int_{s}^{A} z(s)du\right)ds \\ &= \int_{Q(t_2)-\frac{1}{e}}^{Q(t_2)}\left(es+1-eQ(t_2)\right)z(s)ds \\ &\quad + \int_{Q(t_2)}^{A} z(s)ds + \int_{A-\frac{1}{e}}^{A} e(A-s)z(s)ds. \end{aligned} \tag{2.78}$$

On the right-hand side of (2.78), the first term is a constant independent of A and the last term is positive. From (2.77) and (2.78), we obtain

$$\lim_{A\to\infty}\int_{A-\frac{1}{e}}^{A} z(s)ds=\infty.$$

This shows that

$$\limsup_{u\to\infty} z(u)=\infty$$

and thus

$$\limsup_{t\to\infty} w(t)=\infty$$

which contradicts (2.73). Hence, (2.72) holds.

Because of (2.59), for any $t \geq t_1+\tau$, there exists $\xi \in (t-\tau, t)$ such that

$$\int_{\xi}^{t} p(s)ds \geq \frac{1}{2e} \quad \text{and} \quad \int_{t}^{\xi+\tau} p(s)ds \geq \frac{1}{2e}. \tag{2.79}$$

By integrating (2.57) over the intervals $[\xi, t]$ and $[t, \xi+\tau]$, we find that

$$x(t)-x(\xi)+\int_{\xi}^{t} p(s)x(s-\tau)ds=0 \tag{2.80}$$

and

$$x(\xi+\tau)-x(t)+\int_{t}^{\xi+\tau} p(s)x(s-\tau)ds=0. \tag{2.81}$$

By omitting the first terms in (2.80) and (2.81) and using the decreasing nature of x and (2.79), we find that

$$-x(\xi) + \frac{1}{2e}x(t-\tau) < 0 \quad \text{and} \quad -x(t) + \frac{1}{2e}x(\xi) < 0,$$

i.e.,

$$x(t) > \frac{1}{2e}x(\xi) > \left(\frac{1}{2e}\right)^2 x(t-\tau),$$

i.e.,

$$w(t) < (2e)^2 \quad \text{for all} \quad t \geq t_1 + \tau.$$

This contradicts (2.72) and completes the proof. □

Corollary 2.4.2. *Suppose that* (2.58) *holds. If*

$$\liminf_{t\to\infty} \int_{t-\tau}^{t} p(s)ds > \frac{1}{e}, \tag{2.82}$$

then every solution of (2.57) *oscillates.*

Proof. From (2.82) one can easily see that

$$\int_{t_0}^{\infty} p(t)dt = \infty$$

and that there exists $c > 0$ such that for sufficiently large t,

$$\int_{t-\tau}^{t} p(s)ds - \frac{1}{e} > c.$$

Hence condition (2.60) is satisfied. By Theorem 2.4.1, every solution of (2.57) oscillates. □

Corollary 2.4.3. *Suppose that* (2.58) *holds. If* (2.59) *holds and*

$$\int_{t_0+\tau}^{\infty} p(t)\left(\int_{t-\tau}^{t} p(s)ds - \frac{1}{e}\right)dt = \infty, \tag{2.83}$$

then every solution of (2.57) *oscillates.*

Proof. By using (2.59) and the fact that $e^c - 1 \geq c$ for $c \geq 0$, we obtain

$$\exp\left(\int_{t-\tau}^{t} p(s)ds - \frac{1}{e}\right) - 1 \geq \int_{t-\tau}^{t} p(s)ds - \frac{1}{e} \quad \text{for all} \quad t \geq t_1.$$

Therefore (2.83) implies (2.60). By Theorem 2.4.1, every solution of (2.57) oscillates. □

Example 2.4.4. Consider the delay differential equation

$$x'(t) + \left(\frac{1}{1+t} + \frac{1}{e}\right)x(t-1) = 0 \quad \text{for} \quad t \geq 0. \tag{2.84}$$

Clearly, for $t \geq 1$,

$$\int_{t-1}^{t} \left(\frac{1}{1+t} + \frac{1}{e}\right)dt = \ln\frac{1+t}{t} + \frac{1}{e} > \frac{1}{e}$$

and

$$\lim_{t\to\infty} \int_{t-1}^{t} \left(\frac{1}{1+t} + \frac{1}{e}\right)dt = \frac{1}{e}.$$

Hence (2.82) is not satisfied. But for any $T > 1$,

$$\int_1^T \left(\frac{1}{1+t} + \frac{1}{e}\right) \ln \frac{1+t}{t} dt \geq \frac{1}{e} \int_1^T \ln \frac{1+t}{t} dt \to \infty \quad \text{as} \quad T \to \infty.$$

Therefore, by Corollary 2.4.3, every solution of (2.84) oscillates.

Next, in order to improve conditions (2.59) and (2.60), we use a different method to obtain new sufficient conditions for oscillation of (2.57).

Assume (2.58) and define the following sequences of functions:

$$\begin{cases} p_1(t) = \displaystyle\int_{t-\tau}^t p(s)ds, \quad t \geq t_0 + \tau, \\ p_{k+1}(t) = \displaystyle\int_{t-\tau}^t p(s)p_k(s)ds, \quad t \geq t_0 + (k+1)\tau \quad \text{for} \quad k \in \mathbb{N}, \\ \bar{p}_1(t) = \displaystyle\int_t^{t+\tau} p(s)ds, \quad t \geq t_0, \\ \bar{p}_{k+1}(t) = \displaystyle\int_t^{t+\tau} p_k(s)\bar{p}_k(s)ds, \quad t \geq t_0 \quad \text{for} \quad k \in \mathbb{N}. \end{cases} \tag{2.85}$$

Theorem 2.4.5. *Suppose that* (2.58) *holds. If there exist* $t_1 > t_0 + \tau$ *and* $n \in \mathbb{N}$ *such that*

$$p_n(t) \geq \frac{1}{e^n} \quad \text{and} \quad \bar{p}_n(t) \geq \frac{1}{e^n} \quad \text{for all} \quad t \geq t_1 \tag{2.86}$$

(*where* p_n *and* $\bar{p}_n$ *are defined by* (2.85)) *and*

$$\int_{t_0+n\tau}^\infty p(t) \left[\exp\left(e^{n-1}p_n(t) - \frac{1}{e}\right) - 1\right] dt = \infty, \tag{2.87}$$

then every solution of (2.57) *oscillates.*

Proof. Assume, for the sake of contradiction, that (2.57) has an eventually positive solution x. Then there exists $t_2 \geq t_1$ such that

$$x(t-\tau) \geq x(t) > 0 \quad \text{and} \quad x'(t) \leq 0 \quad \text{for all} \quad t \geq t_2.$$

Let w be defined by (2.61). As in the proof of Theorem 2.4.1, we have that (2.64) holds. It is easy to show that $e^c \geq ec$ for all $c \geq 0$ and so

$$w(t) \geq e \int_{t-\tau}^t p(s)w(s)ds \quad \text{for all} \quad t \geq t_2 + \tau. \tag{2.88}$$

Set $w_0 = w$ and for $1 \leq k \leq n$,

$$w_k(t) = \int_{t-\tau}^t p(s)w_{k-1}(s)ds \quad \text{for} \quad t \geq t_2 + k\tau,$$

$v_0 = v := w - 1$ and for $1 \leq k \leq n$,

$$v_k(t) = \int_{t-\tau}^t p(s)v_{k-1}(s)ds \quad \text{for} \quad t \geq t_2 + k\tau. \tag{2.89}$$

By (2.62),

$$v_k(t) \geq 0, \quad t \geq t_2 + k\tau \quad \text{for} \quad 0 \leq k \leq n. \tag{2.90}$$

From (2.64) and (2.88), we easily obtain

$$w(t) \geq e^{n-1} w_{n-1}(t) \quad \text{for} \quad t \geq t_2 + (n-1)\tau$$

and

$$w(t) \geq \exp\left(e^{n-1} \int_{t-\tau}^{t} p(s) w_{n-1}(s) ds\right) \quad \text{for} \quad t \geq t_2 + n\tau. \tag{2.91}$$

In view of (2.85), (2.89), and (2.90), (2.91) can be written as

$$\begin{aligned} w(t) &\geq \exp\left(e^{n-1} \int_{t-\tau}^{t} p(s) v_{n-1}(s) ds + e^{n-1} p_n(t)\right) \\ &= \exp\left(e^{n-1} \int_{t-\tau}^{t} p(s) v_{n-1}(s) ds + \frac{1}{e}\right) \exp\left(e^{n-1} p_n(t) - \frac{1}{e}\right) \end{aligned}$$

for $t \geq t_2 + n\tau$, and so

$$w(t) \geq \left(e^{n} \int_{t-\tau}^{t} p(s) v_{n-1}(s) ds + 1\right) \exp\left(e^{n-1} p_n(t) - \frac{1}{e}\right)$$

for $t \geq t_2 + n\tau$. By (2.86) and (2.90),

$$\begin{aligned} p(t)\left(v(t) - e^{n} v_n(t)\right) &\geq p(t)\left[w(t) - e^{n}\left(\int_{t-\tau}^{t} p(s) v_{n-1}(s) ds + 1\right)\right] \\ &\geq p(t)\left[e^{n} \int_{t-\tau}^{t} p(s) v_{n-1}(s) ds + 1\right]\left[\exp\left(e^{n-1} p_n(t) - \frac{1}{e}\right) - 1\right] \\ &\geq p(t)\left[\exp\left(e^{n-1} p_n(t) - \frac{1}{e}\right) - 1\right] \end{aligned}$$

for $t \geq t_2 + n\tau$. By integrating both sides from $t_3 = t_2 + n\tau$ to $T > t_3 + n\tau$, we obtain

$$\int_{t_3}^{T} p(t)\left(v(t) - e^{n} v_n(t)\right) dt \geq \int_{t_3}^{T} p(t)\left[\exp\left(e^{n-1} p_n(t) - \frac{1}{e}\right) - 1\right] dt.$$

From this and (2.87), we have

$$\lim_{T \to \infty} \int_{t_3}^{T} p(t)\left(v(t) - e^{n} v_n(t)\right) dt = \infty. \tag{2.92}$$

Since

$$\begin{aligned} e^{n} \int_{t_3}^{T} p(t) v_n(t) dt &= e^{n} \int_{t_3}^{T} p(t) dt \int_{t-\tau}^{t} p(s) v_{n-1}(s) ds \\ &\geq e^{n} \int_{t_3}^{T-\tau} p(s) v_{n-1}(s) ds \int_{s}^{s+\tau} p(t) dt \\ &= e^{n} \int_{t_3}^{T-\tau} p(t) \bar{p}_1(t) dt \int_{t-\tau}^{t} p(s) v_{n-2}(s) ds \\ &\geq e^{n} \int_{t_3}^{T-2\tau} p(s) v_{n-2}(s) ds \int_{s}^{s+\tau} p(t) \bar{p}_1(t) dt \\ &= e^{n} \int_{t_3}^{T-2\tau} p(t) \bar{p}_2(t) v_{n-2}(s) ds, \end{aligned}$$

we have by induction

$$e^n \int_{t_3}^{T} p(t)v_n(t)dt \ge e^n \int_{t_3}^{T-n\tau} p(t)v(t)\bar{p}_n(t)dt \ge \int_{t_3}^{T-n\tau} p(t)v(t)dt.$$

Thus,

$$\begin{aligned}\int_{t_3}^{T} p(t)\Big(v(t) - e^n v_n(t)\Big)\, dt \ &\le \ \int_{t_3}^{T} p(t)v(t)dt - \int_{t_3}^{T-n\tau} p(t)v(t)dt \\ &= \ \int_{T-n\tau}^{T} p(t)v(t)dt.\end{aligned}$$

In view of (2.92), we have

$$\lim_{T\to\infty} \int_{T-n\tau}^{T} p(t)v(t)dt = \infty.$$

This shows that either

$$\lim_{T\to\infty} \int_{T-n\tau}^{T} p(t)dt = \infty \tag{2.93}$$

or

$$\limsup_{t\to\infty} v(t) = \infty. \tag{2.94}$$

If (2.93) holds, then

$$\limsup_{t\to\infty} \int_{t-\tau}^{t} p(s)ds = \infty.$$

By a known result in [**166**], every solution of (2.57) oscillates. Next, if (2.94) holds, then

$$\limsup_{t\to\infty} w(t) = \infty. \tag{2.95}$$

On the other hand, integrating both sides of (2.57) from $t-\tau$ to t, we have

$$x(t) - x(t-\tau) + \int_{t-\tau}^{t} p(s)x(s-\tau)ds = 0 \quad \text{for all} \quad t \ge t_2,$$

and so

$$x(t-\tau) > \int_{t-\tau}^{t} p(s)x(s-\tau)ds \quad \text{for all} \quad t \ge t_2.$$

From this, by successively substituting $(n-2)$ times and using the decreasing nature of x, it follows that

$$x(t-\tau) > \int_{t-\tau}^{t} p(s)p_{n-2}(s)x(s-\tau)ds > x(t-\tau)\int_{t-\tau}^{t} p(s)p_{n-2}(s)ds,$$

and so

$$x(t-\tau) > x(t-\tau)p_{n-1}(t) \quad \text{for all} \quad t \ge t_2 + (n-2)\tau. \tag{2.96}$$

By (2.86), for any $t \ge t_1 + \tau$ there exists $\xi \in (t-\tau, t)$ such that

$$\int_{\xi}^{t} p(s)p_{n-1}(s)ds \ge \frac{1}{2e^n} \quad \text{and} \quad \int_{t}^{\xi+\tau} p(s)p_{n-1}(s)ds \ge \frac{1}{2e^n}. \tag{2.97}$$

By integrating both sides of (2.57) over $[\xi, t]$ and $[t, \xi+\tau]$, we have

$$x(t) - x(\xi) + \int_{\xi}^{t} p(s)x(s-\tau)ds = 0, \quad t \geq t_2 + (n-1)\tau \tag{2.98}$$

and

$$x(\xi+\tau) - x(t) + \int_{t}^{\xi+\tau} p(s)x(s-\tau)ds = 0, \quad t \geq t_2 + (n-1)\tau. \tag{2.99}$$

Substituting (2.96) into (2.98) and (2.99), omitting the first term in (2.98) and (2.99), and using the decreasing nature of x and (2.97), we find

$$x(t) > \frac{1}{4e^{2n}} x(t-\tau),$$

i.e.,

$$w(t) < 4e^{2n} \quad \text{for all} \quad t \geq t_2 + (n-1)\tau.$$

This contradicts (2.95) and completes the proof. □

Theorem 2.4.6. *Suppose that* (2.58) *holds. If there exists* $t > t_0 + \tau$ *such that* (2.59) *and* (2.87) *hold, then every solution of* (2.57) *oscillates.*

Proof. Because (2.59) implies (2.86), Theorem 2.4.5 implies Theorem 2.4.6. □

Corollary 2.4.7. *Suppose that* (2.58) *holds and that, for some* $n \in \mathbb{N}$,

$$\liminf_{t\to\infty} p_n(t) > \frac{1}{e^n} \quad \textit{and} \quad \liminf_{t\to\infty} \bar{p}_n(t) > \frac{1}{e^n},$$

where p_n *and* $\bar{p}_n$ *are defined by* (2.85). *Then every solution of* (2.57) *oscillates.*

Corollary 2.4.8. *Suppose that* (2.58) *holds. If* (2.59) *holds, and for some* $n \in \mathbb{N}$,

$$\int_{t_0+n\tau}^{\infty} p(t) \left(e^{n-1} p_n(t) - \frac{1}{e} \right) dt = \infty, \tag{2.100}$$

where p_n *is defined by* (2.85), *then every solution of* (2.57) *oscillates.*

Corollary 2.4.9. *Suppose that* (2.58) *holds. If* (2.86) *and* (2.100) *hold, then every solution of* (2.57) *oscillates.*

Example 2.4.10. Consider the delay differential equation

$$x'(t) + \frac{1}{2e}(1+\cos t)\, x(t-\pi) = 0, \quad t \geq 0. \tag{2.101}$$

Clearly, for $t \geq \pi$,

$$p_1(t) = \int_{t-\pi}^{t} \frac{1}{2e}(1+\cos s)\, ds = \frac{1}{2e}(\pi + 2\sin t)$$

and

$$\liminf_{t\to\infty} \int_{t-\pi}^{t} \frac{1}{2e}(1+\cos s)\, ds = \frac{1}{2e}(\pi - 2) < \frac{1}{e}.$$

This shows that (2.82) and (2.59) do not hold. But

$$\begin{aligned} p_2(t) &= \int_{t-\pi}^{t} p(s)p_1(s)ds = \frac{1}{4e^2}\int_{t-\pi}^{t} (1+\cos s)(\pi + 2\sin s)\, ds \\ &= \frac{1}{4e^2}\left(\pi^2 + 2\pi \sin t - 4\cos t\right), \end{aligned}$$

$$p_3(t) = \int_{t-\pi}^{t} p(s)p_2(s)ds = \frac{1}{8e^3}\left(\pi^3 - 2\pi + (2\pi^2 - 8)\sin t - 4\pi\cos t\right),$$

$$\begin{aligned} p_4(t) &= \int_{t-\pi}^{t} p(s)p_3(s)ds \\ &= \frac{1}{16e^4}\left(\pi^4 - 4\pi^2 + 2(\pi^3 - 6\pi)\sin t - 4(\pi^2 - 4)\cos t\right), \end{aligned}$$

$$\liminf_{t\to\infty} p_4(t) = \frac{1}{16e^4}\left(\pi^4 - 4\pi^2 - 2\sqrt{(\pi^3 - 6\pi)^2 + 4(\pi^2 - 4)^2}\right) > \frac{22}{16e^4},$$

and

$$\bar{p}_1(t) = \int_{t}^{t+\pi} \frac{1}{2e}(1 + \cos s)ds = \frac{1}{2e}(\pi - 2\sin t),$$

$$\bar{p}_2(t) = \int_{t}^{t+\pi} p(s)\bar{p}_1(s)ds = \frac{1}{4e^2}\left(\pi^2 - 2\pi\sin t - 4\cos t\right),$$

$$\bar{p}_3(t) = \int_{t}^{t+\pi} p(s)\bar{p}_2(s)ds = \frac{1}{8e^3}\left(\pi^3 - 2\pi - (2\pi^2 - 8)\sin t - 4\pi\cos t\right),$$

$$\begin{aligned} \bar{p}_4(t) &= \int_{t}^{t+\pi} p(s)\bar{p}_3(s)ds \\ &= \frac{1}{16e^4}\left(\pi^4 - 4\pi^2 - 2(\pi^3 - 6\pi)\sin t - 4(\pi^2 - 4)\cos t\right), \end{aligned}$$

$$\liminf_{t\to\infty} \bar{p}_4(t) = \frac{1}{16e^4}\left(\pi^4 - 4\pi^2 - 2\sqrt{(\pi^3 - 6\pi)^2 + 4(\pi^2 - 4)^2}\right) > \frac{22}{16e^4}.$$

Then, by Corollary 2.4.7, every solution of (2.101) oscillates.

Next, we will present a criterion for oscillation of (2.57) which indicates that conditions (2.59) or (2.82) or even the condition

$$\liminf_{t\to\infty} \int_{t-\tau}^{t} p(s)ds > 0$$

is no longer necessary.

Before stating the main results, we need the following lemmas which are applicable to equations with several delays of the form

$$x'(t) + \sum_{i=1}^{n} p_i(t)x(t - \tau_i) = 0, \quad t \ge t_0. \tag{2.102}$$

Lemma 2.4.11. *If*

$$\limsup_{t\to\infty} \int_{t}^{t+\tau_i} p_i(s)ds > 0 \quad \textit{for some} \quad i \in \{1, \ldots, n\}$$

and x is an eventually positive solution of (2.102), *then*

$$\liminf_{t\to\infty} \frac{x(t-\tau_i)}{x(t)} < \infty.$$

Proof. In view of the assumption, there exist a constant $d > 0$ and a sequence $\{t_k\}$ such that $t_k \to \infty$ as $k \to \infty$ and

$$\int_{t_k}^{t_k+\tau_i} p_i(s)ds \geq d \quad \text{for} \quad k \in \mathbb{N}.$$

Then for every $k \in \mathbb{N}$ there exists $\xi_k \in (t_k, t_k + \tau_i)$ such that

$$\int_{t_k}^{\xi_k} p_i(s)ds \geq \frac{d}{2} \quad \text{and} \quad \int_{\xi_k}^{t_k+\tau_i} p_i(s)ds \geq \frac{d}{2}. \tag{2.103}$$

On the other hand, (2.102) implies

$$x'(t) + p_i(t)x(t - \tau_i) \leq 0 \tag{2.104}$$

eventually. By integrating (2.104) over the intervals $[t_k, \xi_k]$ and $[\xi_k, t_k + \tau_i]$, we find

$$x(\xi_k) - x(t_k) + \int_{t_k}^{\xi_k} p_i(s)x(s - \tau_i)ds \leq 0 \tag{2.105}$$

and

$$x(t_k + \tau_i) - x(\xi_k) + \int_{\xi_k}^{t_k+\tau_i} p_i(s)x(s - \tau_i)ds \leq 0. \tag{2.106}$$

By omitting the first terms in (2.105) and (2.106) and by using the decreasing nature of x and (2.103), we find

$$\frac{x(\xi_k - \tau_i)}{x(\xi_k)} \leq \left(\frac{2}{d}\right)^2.$$

The proof is complete. □

Lemma 2.4.12. *If* (2.102) *has an eventually positive solution, then eventually*

$$\int_t^{t+\tau_i} p(s)ds \leq 1 \quad \textit{for all} \quad i \in \{1, \ldots, n\}.$$

Proof. See the proof of [**166**, Theorem 2.1.3]. □

Theorem 2.4.13. *Suppose that* (2.58) *holds and that* $\int_t^{t+\tau} p(s)ds > 0$ *for* $t \geq t_0$ *for some* $t_0 > 0$ *and*

$$\int_{t_0}^{\infty} p(t) \ln\left(e \int_t^{t+\tau} p(s)ds\right) dt = \infty. \tag{2.107}$$

Then every solution of (2.57) *oscillates.*

Proof. Assume the contrary. Then there exists an eventually positive solution x of (2.57). Obviously x is eventually monotone decreasing. Let $\lambda = -x'/x$. Clearly, the function λ is eventually nonnegative and continuous, and

$$x(t) = x(t_1) \exp\left(-\int_{t_1}^t \lambda(s)ds\right),$$

where $x(t_1) > 0$ for some $t_1 \geq t_0$. Furthermore, λ satisfies the generalized characteristic equation

$$\lambda(t) = p(t) \exp\left(\int_{t-\tau}^t \lambda(s)ds\right).$$

One can easily show that

$$e^{rx} \geq x + \frac{\ln(er)}{r} \quad \text{for} \quad r > 0,$$

and thus

$$\begin{aligned}\lambda(t) &= p(t)\exp\left(A(t)\cdot\frac{1}{A(t)}\int_{t-\tau}^{t}\lambda(s)ds\right)\\ &\geq p(t)\left(\frac{1}{A(t)}\int_{t-\tau}^{t}\lambda(s)ds + \frac{\ln\left(eA(t)\right)}{A(t)}\right),\end{aligned}$$

where $A(t) = \int_t^{t+\tau} p(s)ds$. It follows that

$$\lambda(t)\int_t^{t+\tau} p(s)ds - p(t)\int_{t-\tau}^{t}\lambda(s)ds \geq p(t)\ln\left(e\int_t^{t+\tau} p(s)ds\right). \tag{2.108}$$

Then, for $N > T$, note that the inequality

$$\int_T^N\int_{t-\tau}^{t} p(t)\lambda(s)dsdt \geq \int_T^{N-\tau}\int_s^{s+\tau} p(t)\lambda(s)dtds$$

can be derived by interchanging the order of integration, and use this together with (2.108) to obtain

$$\begin{aligned}\int_T^N p(t)\ln\left(e\int_t^{t+\tau}p(s)ds\right)dt &\leq \int_T^N\int_t^{t+\tau}\lambda(t)p(s)dsdt - \int_T^N\int_{t-\tau}^{\tau}p(t)\lambda(s)dsdt\\ &\leq \int_T^N\int_t^{t+\tau}\lambda(t)p(s)dsdt - \int_T^{N-\tau}\int_s^{s+\tau}p(t)\lambda(t)dtds\\ &= \int_{N-\tau}^N\int_t^{t+\tau}\lambda(t)p(s)dsdt \leq \int_{N-\tau}^N\lambda(t)dt = \ln\frac{x(N-\tau)}{x(N)},\end{aligned}$$

where we have used Lemma 2.4.12 for the last inequality. In view of (2.107),

$$\lim_{t\to\infty}\frac{x(t-\tau)}{x(t)} = \infty. \tag{2.109}$$

On the other hand, (2.107) implies that there exists a sequence $\{t_n\}$ with $t_n \to \infty$ as $n \to \infty$ such that

$$\int_{t_n}^{t_n+\tau} p(s)ds \geq \frac{1}{e} \quad \text{for all} \quad n \in \mathbb{N}.$$

Hence by Lemma 2.4.11, we obtain

$$\lim_{t\to\infty}\frac{x(t-\tau)}{x(t)} < \infty.$$

This contradicts (2.109) and completes the proof. □

Remark 2.4.14. Theorem 2.4.13 substantially improves condition (2.82). In fact, if (2.82) holds, then

$$\int_{t_0}^{\infty} p(s)ds = \infty \tag{2.110}$$

and there exists $c > 0$ such that for large t,

$$\ln\left(e\int_t^{t+\tau} p(s)ds\right) \geq c. \tag{2.111}$$

Note that (2.110) and (2.111) imply (2.107). Condition (2.107) is an evaluation of $p(t)$ and $\int_t^{t+\tau} p(s)ds$ in an infinite interval. Obviously, $\int_t^{t+\tau} p(s)ds > 0$ is a necessary condition for (2.107).

Example 2.4.15. Consider the delay differential equation

$$x'(t) + \exp(k \sin t - 1)\, x(t-1) = 0, \tag{2.112}$$

where $p(t) = \exp(k \sin t - 1)$ and k is a constant. Clearly,

$$\liminf_{t\to\infty} \int_{t-1}^{t} p(s)ds < \frac{1}{e}.$$

So condition (2.82) is not satisfied. By Jensen's inequality,

$$\begin{aligned} \int_0^\infty p(t) \ln\left(e \int_t^{t+1} p(s)ds\right) dt &\geq \int_0^\infty p(t) \int_t^{t+1} k \sin s ds dt \\ &= \frac{2k \sin \frac{1}{2}}{2} \int_0^\infty \exp(k \sin t) \sin\left(t + \frac{1}{2}\right) dt. \end{aligned}$$

On the other hand, it is easy to see that $\int_0^{t_0} \exp(k \sin t) \cos t dt$ is bounded and

$$\int_0^{2\pi} \exp(k \sin t) \sin t dt > 0.$$

It follows that

$$\int_0^\infty p(t) \ln\left(e \int_t^{t+1} p(s)ds\right) dt = \infty.$$

By Theorem 2.4.13, every solution of (2.112) oscillates.

2.5. Equations with Several Delays

Consider the delay differential equation

$$x'(t) + \sum_{i=1}^{n} p_i(t) x(t - \tau_i) = 0, \tag{2.113}$$

where p_i are continuous and nonnegative functions and τ_i are positive constants, $1 \leq i \leq n$. In the following we give sufficient conditions for the oscillation of all solutions of (2.113).

Theorem 2.5.1. *Assume* $\tau_n = \max\{\tau_1, \tau_2, \ldots, \tau_n\}$. *Suppose that*

$$\sum_{i=1}^{n} \int_t^{t+\tau_i} p_i(s)ds > 0$$

for $t \geq t_0$ *for some* $t_0 > 0$ *and that*

$$\limsup_{t\to\infty} \int_t^{t+\tau_n} p_n(s)ds > 0.$$

If, in addition,

$$\int_{t_0}^\infty \left(\sum_{i=1}^{n} p_i(t)\right) \ln\left(e \sum_{i=1}^{n} \int_t^{t+\tau_i} p_i(s)ds\right) dt = \infty, \tag{2.114}$$

then every solution of (2.113) *oscillates.*

Proof. Assume the contrary. Then (2.113) has an eventually positive and decreasing solution x. Let $\lambda = -x'/x$. Then λ is nonnegative and continuous, and there exists $t_1 \geq t_0$ with $x(t_1) > 0$ such that $x(t) = x(t_1)\exp\left(-\int_{t_1}^t \lambda(s)ds\right)$. Furthermore, λ satisfies the generalized characteristic equation

$$\lambda(t) = \sum_{i=1}^n p_i(t)\exp\left(\int_{t-\tau_i}^t \lambda(s)ds\right).$$

Let

$$B(t) = \sum_{i=1}^n \int_t^{t+\tau_i} p_i(s)ds.$$

By using $e^{rx} \geq x + \frac{\ln(er)}{r}$ for $r > 0$, we find

$$\begin{aligned}\lambda(t) &= \sum_{i=1}^n p_i(t)\exp\left(B(t)\cdot\frac{1}{B(t)}\int_{t-\tau_i}^t \lambda(s)ds\right) \\ &\geq \sum_{i=1}^n p_i(t)\left(\frac{1}{B(t)}\int_{t-\tau_i}^t \lambda(s)ds + \frac{\ln(eB(t))}{B(t)}\right)\end{aligned}$$

and hence

$$\begin{aligned}&\lambda(t)\left(\sum_{i=1}^n \int_t^{t+\tau_i} p_i(s)ds\right) - \sum_{i=1}^n p_i(t)\int_{t-\tau_i}^t \lambda(s)ds \\ &\quad\geq \sum_{i=1}^n p_i(t)\ln\left(e\sum_{i=1}^n \int_t^{t+\tau_i} p_i(s)ds\right).\end{aligned}$$

Then for $N \geq T$,

$$\begin{aligned}&\int_T^N \left(\sum_{i=1}^n p_i(t)\right)\ln\left(e\sum_{i=1}^n \int_t^{t+\tau_i} p_i(s)ds\right)dt \\ &\leq \int_T^N \left(\sum_{i=1}^n \int_t^{t+\tau_i} p_i(s)ds\right)\lambda(t)dt - \sum_{i=1}^n \int_T^N p_i(t)\int_{t-\tau_i}^t \lambda(s)dsdt \\ &\leq \sum_{i=1}^n \int_T^N \int_t^{t+\tau_i} p_i(s)\lambda(t)dsdt - \sum_{i=1}^n \int_T^{N-\tau_i} \int_s^{s+\tau_i} p_i(t)\lambda(s)dtds \\ &= \sum_{i=1}^n \int_{N-\tau_i}^N \int_t^{t+\tau_i} \lambda(t)p_i(s)dsdt \\ &\leq \sum_{i=1}^n \int_{N-\tau_i}^N \lambda(t)dt = \sum_{i=1}^n \ln\frac{x(N-\tau_i)}{x(N)} = \ln\left(\prod_{i=1}^n \frac{x(N-\tau_i)}{x(N)}\right),\end{aligned}$$

where we also used Lemma 2.4.12. In view of (2.114),

$$\lim_{t\to\infty}\prod_{i=1}^n \frac{x(t-\tau_i)}{x(t)} = \infty.$$

This implies

$$\lim_{t\to\infty}\frac{x(t-\tau_n)}{x(t)} = \infty. \tag{2.115}$$

However, by Lemma 2.4.11, we have

$$\liminf_{t\to\infty} \frac{x(t-\tau_n)}{x(t)} < \infty.$$

This contradicts (2.115) and completes the proof. □

Corollary 2.5.2. *If*

$$\liminf_{t\to\infty} \sum_{i=1}^{n} \int_t^{t+\tau_i} p_i(s)ds > \frac{1}{e}, \tag{2.116}$$

then every solution of (2.113) *oscillates.*

Proof. Let $\tau_1 < \tau_2 < \ldots < \tau_n$. Then it follows from (2.116) that there is an m with $1 \le m \le n$ such that

$$\limsup_{t\to\infty} \int_t^{t+\tau_m} p_m(s)ds > 0$$

and

$$\liminf_{t\to\infty} \sum_{i=1}^{m} \int_t^{t+\tau_i} p_i(s)ds > \frac{1}{e}. \tag{2.117}$$

Now assume, for the sake of contradiction, that (2.113) has an eventually positive solution x. Then x is also an eventually positive solution of the inequality

$$x'(t) + \sum_{i=1}^{m} p_i(t)x(t-\tau_i) \le 0.$$

So, by [**118**, Corollary 3.2.2], we know that the equation

$$y'(t) + \sum_{i=1}^{m} p_i(t)y(t-\tau_i) = 0$$

has an eventually positive solution as well. On the other hand, from (2.117) we see that for some $t_0 > 0$,

$$\int_{t_0}^{\infty} \left(\sum_{i=1}^{m} p_i(t)\right) \ln\left(e\sum_{i=1}^{m}\int_t^{t+\tau_i} p_i(s)ds\right) dt = \infty.$$

Then by Theorem 2.5.1, every solution of (2.113) oscillates. □

2.6. Equations with Piecewise Constant Argument

Consider the linear delay differential equation with piecewise constant deviating argument of the form

$$x'(t) + a(t)x(t) + b(t)x\left([t-1]\right) = 0, \quad t \ge 0, \tag{2.118}$$

where a and b are continuous functions on $[-1,\infty)$, $b(t) \ge 0$ (but not identically zero) for $t \ge 0$, and $[\cdot]$ denotes the greatest integer function.

By a solution of (2.118) we mean a function x which is defined on the set $\{-1,0\} \cup (0,\infty)$ and which satisfies the conditions

(i) x is continuous on $[0,\infty)$;
(ii) the derivative $x'(t)$ exists at each point $t \in [0,\infty)$ with the possible exception of the points $t \in \mathbb{N}_0$, where one-sided derivatives exist;

(iii) (2.118) is satisfied on each interval $[n, n+1]$ for $n \in \mathbb{N}_0$.

Definition 2.6.1. For $n_0 \in \mathbb{N}_0$ we define $N(n_0) := \{n_0, n_0+1, n_0+2, \ldots\}$. A nontrivial sequence $\{A_n\}_{n=n_0}^{\infty}$ is called *oscillatory* if for every $n_1 \in N(n_0)$ there exists $n \geq n_1$ such that $A_n \cdot A_{n+1} \leq 0$. Otherwise, it is called *nonoscillatory.*

Let $A_{-1}, A_0 \in \mathbb{R}$. Then the following lemma shows that (2.118) has a unique solution x satisfying the conditions

$$x(-1) = A_{-1} \quad \text{and} \quad x(0) = A_0. \tag{2.119}$$

Lemma 2.6.2. *Let A_{-1} and A_0 be given. Then* (2.118) *and* (2.119) *has a unique solution x given on $[n, n+1]$, $n \in \mathbb{N}_0$ by*

$$x(t) = A_n \exp\left(-\int_n^t a(s)ds\right) - A_{n-1}\int_n^t b(s)\exp\left(-\int_s^t a(u)du\right)ds, \tag{2.120}$$

where the sequence $\{A_n\}$ satisfies the difference (recurrence) equation

$$A_{n-1} = A_n \exp\left(\int_{n-1}^n a(s)ds\right) + A_{n-2}\int_{n-1}^n b(t)\exp\left(\int_{n-1}^t a(s)ds\right)dt \tag{2.121}$$

for $n \in \mathbb{N}$.

Proof. Let x be a solution of (2.118) and (2.119). Then on $[n, n+1)$ for any $n \in \mathbb{N}_0$, (2.118) can be written in the form

$$x'(t) + a(t)x(t) + b(t)A_{n-1} = 0, \quad t \in [n, n+1), \tag{2.122}$$

where we used the notation

$$A_n = x(n) \quad \text{for} \quad n \in N(-1).$$

Equation (2.122) can be rewritten as

$$\left(x(t)\exp\left(\int_n^t a(s)ds\right)\right)' + b(t)\exp\left(\int_n^t a(s)ds\right)A_{n-1} = 0, \quad t \in [n, n+1).$$

Integrating from n to $t \in [n, n+1)$, we have

$$x(t)\exp\left(\int_n^t a(s)ds\right) - A_n + \left[\int_n^t b(s)\exp\left(\int_n^s a(u)du\right)ds\right]A_{n-1} = 0. \tag{2.123}$$

This implies (2.120). From (2.120) and by continuity, letting $t \to n+1$ and replacing n by $n-1$, we obtain (2.121).

Conversely, let $\{A_n\}$ be the solution of (2.121) and define x on $\{-1, 0\} \cup (0, \infty)$ by (2.119) and (2.120). Then, clearly, for every $n \in \mathbb{N}_0$ and $t \in [n, n+1)$, (2.120) implies (2.122) and, in turn, (2.122) is equivalent to (2.118) in the interval $[n, n+1)$. The proof is complete. $\square$

Lemma 2.6.3. *Equation* (2.118) *has a nonoscillatory solution if and only if the difference equation* (2.121) *has a nonoscillatory solution.*

Proof. Assume that x is a nonoscillatory solution of (2.118). Then $\{A_n = x(n)\}$ is a nonoscillatory solution of (2.121).

Conversely, assume that $\{A_n\}$ is a nonoscillatory solution of (2.121) such that eventually $A_n > 0$ (the case where eventually $A_n < 0$ is similar and is omitted). From (2.123), letting $t \to n+1$ and by continuity, we have for n sufficiently large

$$A_{n+1}\exp\left(\int_n^{n+1} a(u)du\right) = A_n - A_{n-1}\int_n^{n+1} b(s)\exp\left(\int_n^s a(u)du\right)ds > 0.$$

Then, by (2.123), we obtain for $n \le t < n+1$ with n sufficiently large

$$\begin{aligned} x(t)\exp\left(\int_n^t a(s)ds\right) &= A_n - A_{n-1}\int_n^t b(s)\exp\left(\int_n^s a(u)du\right)ds \\ &\ge A_n - A_{n-1}\int_n^{n+1} b(s)\exp\left(\int_n^s a(u)du\right)ds > 0. \end{aligned}$$

This shows that eventually $x(t) > 0$, and so x is a nonoscillatory solution of (2.118). The proof is complete. □

Theorem 2.6.4. *Equation* (2.118) *is nonoscillatory if and only if it has a nonoscillatory solution. This also implies that* (2.118) *is oscillatory if and only if it has an oscillatory solution.*

Proof. From the proof of Lemma 2.6.3, we also can see that if all solutions of (2.121) are nonoscillatory, then all solutions of (2.118) are nonoscillatory. Since (2.121) is a second order linear difference (recurrence) equation, by the known result in Fort [**97**], we see that if one solution of (2.121) is nonoscillatory, then all its solutions are nonoscillatory. On the basis of this discussion and by Lemma 2.6.3 and a simple analysis, we see that Theorem 2.6.4 is true. □

In the following, for convenience, we let for any $n \in \mathbb{N}_0$

$$P_n = \exp\left(\int_{n-1}^n a(t)dt\right) \quad \text{and} \quad Q_n = \int_{n-1}^n b(t)\exp\left(\int_{n-1}^t a(s)ds\right)dt. \tag{2.124}$$

Then

$$Q_nP_{n-1} = \int_{n-1}^n b(t)\exp\left(\int_{n-2}^t a(s)ds\right)dt.$$

Observe that, by (2.124), the difference equation (2.121) can be rewritten as

$$A_{n-1} = P_nA_n + Q_nA_{n-2}, \quad n \in \mathbb{N}, \tag{2.125}$$

and, by Lemma 2.6.2, if x is a solution of (2.118), then $A_n = x(n)$ satisfies (2.125). In the following we will assume that all occurring inequalities involving values of functions or sequences are satisfied eventually for all large t or n.

Lemma 2.6.5. *Assume that there exists $h \in [0, 1/4]$ such that*

$$Q_nP_{n-1} \ge h \quad \text{for large} \quad n. \tag{2.126}$$

Let $\{A_n\}$ be an eventually positive solution of (2.125)*. Set for $n \in \mathbb{N}$*

$$W_n^{(1)} = \frac{A_{n-1}}{A_{n-2}}P_{n-1} \quad \text{and} \quad W_n^{(2)} = \frac{A_{n-2}}{A_{n-1}}Q_n.$$

Then

$$\limsup_{n\to\infty} W_n^{(i)} \le \frac{1+\sqrt{1-4h}}{2} \quad \text{for} \quad i \in \{1, 2\}. \tag{2.127}$$

Proof. We first prove (2.127) for $i = 1$. From (2.125), we have

$$A_{n-2} \geq P_{n-1}A_{n-1}. \tag{2.128}$$

This implies $\limsup_{n\to\infty} W_n^{(1)} \leq 1$, and so (2.127) holds for $h = 0$. We now consider the case when $0 < h \leq 1/4$. From (2.128), it follows that

$$\frac{A_{n-1}}{A_{n-2}}P_{n-1} \leq 1 =: \lambda_1,$$

i.e.,

$$\frac{A_{n-2}}{A_{n-1}} \geq \frac{P_{n-1}}{\lambda_1}. \tag{2.129}$$

Dividing both sides of (2.125) by A_{n-1}, then using (2.129) and (2.126), we find

$$1 \geq P_n\frac{A_n}{A_{n-1}} + \frac{Q_nP_{n-1}}{\lambda_1} \geq P_n\frac{A_n}{A_{n-1}} + \frac{h}{\lambda_1}.$$

This yields

$$W_{n+1}^{(1)} \leq \frac{\lambda_1 - h}{\lambda_1} =: \lambda_2.$$

Following this iterative procedure, we obtain

$$W_{n+m}^{(1)} \leq \frac{\lambda_m - h}{\lambda_m} =: \lambda_{m+1} \quad \text{for all} \quad m \in \mathbb{N}.$$

It is not difficult to see that $1 = \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m \geq \lambda_{m+1} > 0$ for $m \in \mathbb{N}$. Hence the limit $\lim_{m\to\infty} \lambda_m =: \lambda$ exists and satisfies $\lambda^2 - \lambda + h = 0$. Therefore we have

$$\limsup_{n\to\infty} W_n^{(1)} \leq \frac{1 + \sqrt{1-4h}}{2}.$$

This shows (2.127) for $i = 1$. Next we prove (2.127) for $i = 2$. From (2.125), we have

$$A_{n-1} \geq Q_nA_{n-2} \tag{2.130}$$

and

$$A_{n-2} = P_{n-1}A_{n-1} + Q_{n-1}A_{n-3}. \tag{2.131}$$

Inequality (2.130) yields

$$W_n^{(2)} = \frac{A_{n-2}}{A_{n-1}}Q_n \leq 1 =: \lambda_1. \tag{2.132}$$

Thus (2.127) holds for $h = 0$. In the case when $0 < h \leq 1/4$, from (2.131), (2.132), and (2.126), we have

$$1 = P_{n-1}\frac{A_{n-1}}{A_{n-2}} + Q_{n-1}\frac{A_{n-3}}{A_{n-2}} \geq P_{n-1}\frac{Q_n}{\lambda_1} + Q_{n-1}\frac{A_{n-3}}{A_{n-2}} \geq \frac{h}{\lambda_1} + Q_{n-1}\frac{A_{n-3}}{A_{n-2}}.$$

This leads to

$$W_{n+1}^{(2)} \leq \frac{\lambda_1 - h}{\lambda_1} =: \lambda_2.$$

Following this iterative procedure, we have

$$W_{n+m}^{(2)} \leq \frac{\lambda_m - h}{\lambda_m} =: \lambda_{m+1} \quad \text{for all} \quad m \in \mathbb{N}.$$

Now the conclusion follows from the above inequalities and by the same arguments as in the case when $i = 1$. ☐

Lemma 2.6.6. *Let A_n satisfy* (2.125)*. Then the equality*

$$A_{n-2} = P_{n-1}A_{n-1} + A_{n-k-4}\prod_{j=0}^{k+1} Q_{n-j-1} + \sum_{i=0}^{k} A_{n-i-2}P_{n-i-2}\prod_{j=0}^{i} Q_{n-j-1} \tag{2.133}$$

holds for any $k \in \mathbb{N}_0$.

Proof. Clearly,

$$\begin{aligned} P_{n-1}A_{n-1} + A_{n-4}Q_{n-1}Q_{n-2} + P_{n-2}A_{n-2}Q_{n-1} &= P_{n-1}A_{n-1} + Q_{n-1}A_{n-3} \\ &= A_{n-2} \end{aligned}$$

by applying (2.125) first for $n-2$ and then for $n-1$. Hence (2.133) holds for $k = 0$. Now we assume that (2.133) holds for some $k \in \mathbb{N}_0$. Then

$$\begin{aligned} &P_{n-1}A_{n-1} + A_{n-k-5}\prod_{j=0}^{k+2} Q_{n-j-1} + \sum_{i=0}^{k+1} P_{n-i-2}A_{n-i-2}\prod_{j=0}^{i} Q_{n-j-1} \\ = {} & P_{n-1}A_{n-1} + A_{n-k-5}Q_{n-k-3}\prod_{j=0}^{k+1} Q_{n-j-1} + \sum_{i=0}^{k} P_{n-i-2}A_{n-i-2}\prod_{j=0}^{i} Q_{n-j-1} \\ & + P_{n-k-3}A_{n-k-3}\prod_{j=0}^{k+1} Q_{n-j-1} \\ = {} & P_{n-1}A_{n-1} + A_{n-k-4}\prod_{j=0}^{k+1} Q_{n-j-1} + \sum_{i=0}^{k} P_{n-i-2}A_{n-i-2}\prod_{j=0}^{i} Q_{n-j-1} \\ = {} & A_{n-2} \end{aligned}$$

(we used (2.125) for $n-k-3$), i.e., (2.133) holds for $k+1$. Hence, by induction, (2.133) holds for all $k \in \mathbb{N}_0$. ☐

Theorem 2.6.7. *Equation* (2.118) *is oscillatory if*

$$\liminf_{n\to\infty} (Q_n P_{n-1}) > \frac{1}{4} \tag{2.134}$$

and nonoscillatory if

$$Q_n P_{n-1} \le \frac{1}{4} \quad \textit{for large} \quad n \in \mathbb{N}. \tag{2.135}$$

Proof. We first prove that if (2.134) holds, then (2.118) is oscillatory. Suppose to the contrary that (2.118) has an eventually positive solution x. Then, by (2.124) and Lemma 2.6.3, $A_n = x(n)$ satisfies (2.125). Let

$$R_n = \frac{A_{n-1}}{P_n A_n}.$$

Then (2.125) reduces to

$$1 = \frac{1}{R_n} + Q_n P_{n-1} R_{n-1},$$

and hence

$$Q_{n+1}P_nR_n = \frac{Q_{n+1}P_n}{1-Q_nP_{n-1}R_{n-1}}. \tag{2.136}$$

Let

$$\alpha_n = Q_nP_{n-1}R_{n-1} \tag{2.137}$$

so that $\alpha_n = 1 - 1/R_n$ and $\alpha_{n+1}(1-\alpha_n) = Q_{n+1}P_n$. Hence $0 < \alpha_n < 1$. This implies $\alpha_n(1-\alpha_n) \le 1/4$ because of $\max_{0\le x\le 1} x(1-x) = 1/4$. From (2.136) and (2.137), we have

$$\begin{aligned} Q_{n+1}P_n &= \alpha_{n+1}(1-\alpha_n) = \frac{\alpha_{n+1}}{\alpha_n}\alpha_n(1-\alpha_n) \\ &\le \frac{\alpha_{n+1}}{4\alpha_n} = \frac{Q_{n+1}P_nR_n}{4Q_nP_{n-1}R_{n-1}} \\ &= \frac{1}{4Q_nP_{n-1}}\left(\frac{A_{n-1}}{A_n}Q_{n+1}\right)\left(\frac{A_{n-1}}{A_{n-2}}P_{n-1}\right). \end{aligned}$$

By (2.134), there exists a number c such that

$$Q_nP_{n-1} \ge c > \frac{1}{4} \quad \text{for large} \quad n \in \mathbb{N}. \tag{2.138}$$

Note that (2.126) is satisfied with $h = 1/4$. Then, by Lemma 2.6.5, for any number $\varepsilon \in (0, 1/4)$, we have for large n

$$\frac{A_{n-1}}{A_n}Q_{n+1} \le \frac{1}{2-\varepsilon} \quad \text{and} \quad \frac{A_{n-1}}{A_{n-2}}P_{n-1} \le \frac{1}{2-\varepsilon}.$$

We choose such an ε with $1/(2-\varepsilon)^2 < c$. Thus we obtain

$$Q_{n+1}P_n \le \frac{1}{4Q_nP_{n-1}} \cdot \frac{1}{(2-\varepsilon)^2} \le \frac{1}{4(2-\varepsilon)^2c} < \frac{1}{4}.$$

This contradicts (2.138). Hence (2.118) is oscillatory.

Next we prove that (2.135) implies that (2.118) is nonoscillatory. To this end, we first show that the difference equation

$$x_n = \frac{1}{1-a_nx_{n-1}}, \quad n \in \mathbb{N} \tag{2.139}$$

has an eventually positive solution $\{x_n\}$, where

$$a_n = Q_nP_{n-1}, \quad n \in \mathbb{N}.$$

By (2.135), without loss of generality, we may assume

$$0 \le a_n \le \frac{1}{4}, \quad n \in \mathbb{N}. \tag{2.140}$$

Set

$$\gamma = \begin{cases} \dfrac{1-\sqrt{1-4a_1}}{2a_1} & \text{if} \quad a_1 > 0, \\ 1 & \text{if} \quad a_1 = 0. \end{cases} \tag{2.141}$$

Then γ satisfies

$$\gamma = \frac{1}{1-a_1\gamma}.$$

We claim that

$$1 \le \gamma \le 2. \tag{2.142}$$

Indeed, let

$$f(x) = \frac{1-\sqrt{1-4x}}{2x}, \quad 0 < x \le \frac{1}{4}.$$

Then

$$f'(x) = \frac{1-2x-\sqrt{1-4x}}{2x^2\sqrt{1-4x}}, \quad 0 < x < \frac{1}{4}.$$

Set

$$F(x) = 1-2x-\sqrt{1-4x}, \quad 0 \le x \le \frac{1}{4}.$$

Then

$$F'(x) = 2\left(\frac{1}{\sqrt{1-4x}} - 1\right) > 0, \quad 0 < x < \frac{1}{4}.$$

Thus, F is strictly increasing on $(0, 1/4)$. Since $F(0) = 0$, it follows that $F(x) > 0$ for $0 < x < 1/4$. Therefore $f'(x) > 0$ for $0 < x < 1/4$. Hence f is strictly increasing on $(0, 1/4)$. Notice that $f(1/4) = 2$ and

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{1-\sqrt{1-4x}}{2x} = \lim_{x\to 0^+} \frac{1}{\sqrt{1-4x}} = 1,$$

and therefore we have $1 < f(x) < 2$ for $0 < x < 1/4$. This and (2.141) lead to (2.142). Now we define a sequence $\{x_n\}$ by

$$x_0 = \gamma \quad \text{and} \quad x_n = \frac{1}{1-a_n x_{n-1}}, \quad n \in \mathbb{N}.$$

It is clear that $1 \le x_0 \le 2$ and

$$1 \le x_1 = \frac{1}{1-a_1x_0} = \frac{1}{1-a_1\gamma} = \gamma \le 2.$$

Thus, by (2.140), we have

$$1 \le x_2 = \frac{1}{1-a_2x_1} \le \frac{1}{1-\frac{1}{4}\cdot 2} = 2.$$

By induction, we have $1 \le x_n \le 2$ for $n \in \mathbb{N}$. This shows that (2.139) has a solution $\{x_n\}$ such that $x_n > 0$, $n \in \mathbb{N}$, and $\{x_n\}$ satisfies

$$x_n = \frac{1}{1-Q_nP_{n-1}x_{n-1}}, \quad n \in \mathbb{N}. \tag{2.143}$$

Next we define

$$A_{-1} = 1 \quad \text{and} \quad A_n = (P_nx_n)^{-1}A_{n-1}, \quad n \in \mathbb{N}_0.$$

Clearly, $A_n > 0$ for $n \in N(-1)$. Substituting $x_n = A_{n-1}/(P_nA_n)$ into (2.143), we obtain

$$\frac{A_{n-1}}{P_nA_n}\left(1 - Q_nP_{n-1}\frac{A_{n-2}}{P_{n-1}A_{n-1}}\right) = 1,$$

i.e.,

$$A_{n-1} = P_nA_n + Q_nA_{n-2}, \quad n \in \mathbb{N}.$$

This proves that $\{A_n\}$ is a nonoscillatory solution of (2.125). By Lemma 2.6.3 and Theorem 2.6.4, we see that (2.118) is ocillatory. $\square$

Theorem 2.6.8. *Assume that for some $k \in \mathbb{N}_0$,*

$$\limsup_{n\to\infty} \left(Q_n P_{n-1} + \sum_{i=0}^{k} \prod_{j=0}^{i} Q_{n-j-1} P_{n-j-2} \right) > 1. \tag{2.144}$$

Then every solution of (2.118) *oscillates.*

Proof. Assume, for the sake of contradiction, that (2.118) has an eventually positive solution x. Then, by Lemma 2.6.3, $A_n = x(n)$ satisfies (2.125). So $A_{n-1} \geq P_n A_n$. By induction, the iterative formula

$$A_{n-i} \geq A_n \prod_{j=0}^{i-1} P_{n-j} \quad \text{for all} \quad i \in \mathbb{N} \tag{2.145}$$

holds. By Lemma 2.6.6, A_n satisfies (2.133). Now using $A_{n-1} \geq Q_n A_{n-2}$ and (2.145) in (2.133), we obtain

$$A_{n-2} \geq Q_n P_{n-1} A_{n-2} + \sum_{i=0}^{k} \prod_{j=0}^{i} Q_{n-j-1} P_{n-j-2} A_{n-2}.$$

Dividing both sides of the above inequality by A_{n-2}, we arrive at

$$1 \geq \limsup_{n\to\infty} \left(Q_n P_{n-1} + \sum_{i=0}^{k} \prod_{j=0}^{i} Q_{n-j-1} P_{n-j-2} \right).$$

This contradicts (2.144). The proof is complete. $\square$

In the following we establish other type (also "best possible") oscillation criteria for (2.118). The results are formulated in terms of the numbers m and M defined by

$$m = \liminf_{n\to\infty} \left(Q_n P_{n-1}\right) \quad \text{and} \quad M = \limsup_{n\to\infty} \left(Q_n P_{n-1}\right).$$

Theorem 2.6.9. *Assume that $0 \leq m \leq 1/4$ and that for some $k \in \mathbb{N}_0$,*

$$\limsup_{n\to\infty} \left(L Q_n P_{n-1} + \sum_{i=0}^{k} L^i \prod_{j=0}^{i} Q_{n-j-1} P_{n-j-2} \right) > 1, \tag{2.146}$$

where

$$L = \left(\frac{1+\sqrt{1-4m}}{2} \right)^{-1}.$$

Then (2.118) *is oscillatory.*

Proof. By Theorem 2.6.8, the conclusion holds when $m = 0$. To prove the conclusion when $0 < m \leq 1/4$, suppose to the contrary that (2.118) has an eventually positive solution x. Then, by Lemma 2.6.3, $A_n = x(n)$ satisfies (2.125). Since for any $\eta \in (0, m)$ we have $Q_n P_{n-1} \geq m - \eta$ for large n, by Lemma 2.6.5, we have

$$\limsup_{n\to\infty} \frac{A_{n-1}}{A_{n-2}} P_{n-1} \leq \frac{1+\sqrt{1-4(m-\eta)}}{2}$$

and

$$\limsup_{n\to\infty} \frac{A_{n-2}}{A_{n-1}} Q_n \leq \frac{1+\sqrt{1-4(m-\eta)}}{2}.$$

Letting $\eta \to 0$, we see that the above two inequalities hold for $\eta = 0$. Thus, for any sufficiently small $\varepsilon > 0$, the inequalities

$$A_{n-1} \geq L_\varepsilon P_n A_n, \tag{2.147}$$

and

$$A_{n-1} \geq L_\varepsilon Q_n A_{n-2} \tag{2.148}$$

hold for sufficiently large n, where

$$L_\varepsilon = \left(\frac{1 + \sqrt{1 - 4m}}{2} + \varepsilon \right)^{-1}.$$

From (2.147), by induction, we have the iterative formula

$$A_{n-1} \geq L_\varepsilon^i \prod_{j=0}^{i-1} P_{n-j} A_n \quad \text{for all} \quad i \in \mathbb{N}. \tag{2.149}$$

By Lemma 2.6.6, A_n satisfies (2.133). Now using (2.148) and (2.149) in (2.133), we have

$$\begin{aligned} A_{n-2} &\geq L_\varepsilon Q_n P_{n-1} A_{n-2} + A_{n-2} \sum_{i=0}^{k} P_{n-i-2} L_\varepsilon^i \prod_{j=0}^{i-1} P_{n-j-2} \prod_{j=0}^{i} Q_{n-j-1} \\ &= L_\varepsilon Q_n P_{n-1} A_{n-2} + A_{n-2} \sum_{i=0}^{k} L_\varepsilon^i \prod_{j=0}^{i} Q_{n-j-1} P_{n-j-2}. \end{aligned}$$

Dividing both sides of the above inequality by A_{n-2} and taking the limit, we obtain

$$1 \geq \limsup_{n \to \infty} \left(L_\varepsilon Q_n P_{n-1} + \sum_{i=0}^{k} L_\varepsilon^i \prod_{j=0}^{i} Q_{n-j-1} P_{n-j-2} \right).$$

Letting $\varepsilon \to 0$ we have $L_\varepsilon \to L$ so that (2.146) and the above inequality lead to the contradiction

$$1 \geq \limsup_{n \to \infty} \left(L Q_n P_{n-1} + \sum_{i=0}^{k} L^i \prod_{j=0}^{i} Q_{n-j-1} P_{n-j-2} \right) > 1.$$

The proof is complete. □

Corollary 2.6.10. *Assume that* $0 \leq m \leq 1/4$ *and*

$$M > \left(\frac{1 + \sqrt{1 - 4m}}{2} \right)^2. \tag{2.150}$$

Then (2.118) *is oscillatory.*

Proof. If $m = 0$, then the conclusion holds (see [**118**]). Let $0 < m \leq 1/4$. It suffices to prove that (2.150) implies (2.146). Indeed, notice

$$\frac{1 + \sqrt{1 - 4m}}{2} = 1 - \frac{m}{1 - Lm},$$

and by (2.150), there exists $\varepsilon \in (0, m)$ such that $Q_n P_{n-1} \geq m - \varepsilon$ and

$$L \limsup_{n \to \infty} (Q_n P_{n-1}) > 1 - \frac{m - \varepsilon}{1 - L(m - \varepsilon)}.$$

From this and the fact that $[L(m-\varepsilon)]^n \to 0$ as $n \to \infty$, we have

$$\begin{aligned} L \limsup_{n\to\infty} (Q_n P_{n-1}) &> 1 - \frac{(m-\varepsilon)\left[1 - L(m-\varepsilon)\right]^{k+1}}{1 - L(m-\varepsilon)} \\ &= 1 - (m-\varepsilon) \sum_{i=0}^{k} [L(m-\varepsilon)]^i, \end{aligned}$$

where $k \in \mathbb{N}$ is sufficiently large. The last inequality leads to (2.146) because

$$\sum_{i=0}^{k} L^i \prod_{j=0}^{i} Q_{n-j-1} P_{n-j-2} \geq (m-\varepsilon) \sum_{i=0}^{k} [L(m-\varepsilon)]^i.$$

The proof is complete. □

Remark 2.6.11. Observe that $0 \leq m \leq 1/4$ implies $L \geq 1$ and that $L = 1$ if and only if $m = 0$. Also note that when $m \to 0$, condition (2.146) reduces to (2.144). However, it is clear that (2.146) improves (2.144) when $0 \leq m \leq 1/4$. It is interesting to observe that when $m \to 1/4$, condition (2.150) reduces to $M > 1/4$, which cannot be improved in the sense that the lower bound $1/4$ cannot be replaced by a smaller number (cf. (2.135)).

The following is an illustrative example.

Example 2.6.12. Consider the equation

$$x'(t) + \frac{1}{2+t} x(t) + b\left(1 + \cos\frac{\pi t}{2}\right) x([t-1]) = 0, \quad t \geq 0, \tag{2.151}$$

where $b = \pi/(5(\pi - 2))$. It is not difficult to see that

$$\begin{aligned} \int_{n-1}^{n} b(t) \exp\left(\int_{n-2}^{t} a(s)ds\right) dt &= b \int_{n-1}^{n} \left(1 + \cos\frac{\pi t}{2}\right) \frac{t+2}{n} dt \\ = \frac{b}{n} &\left\{ \frac{2n+3}{2} + \frac{2}{\pi}\left[(2+n)\sin\frac{n\pi}{2} - (1+n)\sin\frac{(n-1)\pi}{2}\right]\right\} \\ &+ \frac{4b}{n\pi^2}\left(\cos\frac{n\pi}{2} - \cos\frac{(n-1)\pi}{2}\right), \end{aligned}$$

and so

$$m = \liminf_{n\to\infty} \int_{n-1}^{n} b(t) \exp\left(\int_{n-2}^{t} a(s)ds\right) dt = \frac{b(\pi-2)}{\pi} = \frac{1}{5} < \frac{1}{4}$$

and

$$\limsup_{n\to\infty} \int_{n-1}^{n} b(t) \exp\left(\int_{n-2}^{t} a(s)ds\right) dt = \frac{b(\pi+2)}{\pi} = \frac{\pi+2}{5(\pi-2)} < 1.$$

Thus, it is easy to see that condition (2.150) is satisfied. So, by Corollary 2.6.10, (2.151) is oscillatory.

2.7. Notes

The results in Section 2.2 are taken from Erbe, Kong, and Zhang [**92**], and Section 2.3 is taken from Elbert and Stavroulakis [**83**]. Theorems 2.4.1, 2.4.5, 2.4.6, and 2.4.13 are due to Li [**169**], Tang and Shen [**259**], and Li [**170**], respectively, while Theorem 2.5.1 is proved by Li [**170**]. The contents of Section 2.6 is taken from Shen and Stavroulakis [**252**].

CHAPTER 3

First Order Neutral Differential Equations

3.1. Introduction

In general, the theory of neutral delay differential equations is more complicated than the theory of delay differential equations without neutral terms. For example, Snow (see also Győri and Ladas [**118**, Chapter 6]) has shown that even though the characteristic roots of a neutral differential equation may all have negative real parts, it is still possible for some solutions to be unbounded.

In this chapter, we will present some recent results in the oscillation theory of first order neutral delay differential equations, and consequently this will be a useful source for researchers in this field.

In Section 3.2, we consider nonlinear neutral delay differential equations. We first present a comparison theorem for oscillation, and then some sufficient conditions for oscillation are established. Section 3.3 is concerned with oscillation of neutral delay differential equations with positive and negative coefficients by using generalized characteristic equations. Some oscillation criteria are given. In Section 3.4, we deal with neutral delay differential equations with positive and negative coefficients; here the comparison method plays an important rôle. In Section 3.5, we consider nonoscillation of neutral delay differential equations with positive and negative coefficients. Some criteria for existence of positive solutions are given. In Section 3.6, we give classification schemes of eventually positive solutions of neutral differential equations in terms of their asymptotic magnitude, and provide necessary and/or sufficient conditions for the existence of solutions. Finally, Section 3.7 is concerned with the existence of positive solutions of neutral perturbed differential equations.

3.2. Comparison Theorems and Oscillation

We consider a first order neutral delay nonlinear differential equations of the form

$$\text{(3.1)} \qquad \Big(x(t) - R(t)x(t-r)\Big)' + P(t)\prod_{i=1}^{m}\Big|x(t-\tau_i)\Big|^{\alpha_i}\operatorname{sgn} x(t-\tau_i) = 0$$

along with the corresponding inequality

$$\text{(3.2)} \qquad \Big(x(t) - R(t)x(t-r)\Big)' + P(t)\prod_{i=1}^{m}\Big|x(t-\tau_i)\Big|^{\alpha_i}\operatorname{sgn} x(t-\tau_i) \le 0,$$

where $P, R \in C([t_0,\infty),\mathbb{R}^+)$, $r \in (0,\infty)$ and $\tau_1, \tau_2, \dots, \tau_m$ are nonnegative numbers, $R(t) \ge 0$, and $P(t) \ge 0$ for $t \ge t_0$ such that $P(t)$ is not identically zero for all large t, and each α_i is a positive number for $1 \le i \le m$ such that $\sum_{i=1}^{m} \alpha_i = 1$. When $m = 1$, (3.1) reduces to the linear equation

$$\Big(x(t) - R(t)x(t-r)\Big)' + P(t)x(t-\tau) = 0.$$

Let $T_0 = \max\{r, \tau_1, \dots, \tau_m\}$. By a solution of the equation (3.1) we mean a function $x \in C([t_0 - T_0, \infty), \mathbb{R})$ such that $x(t) - R(t)x(t-r)$ is continuously differentiable and satisfies (3.1) on $[t_0, \infty)$. Let $\phi \in C([t_0 - T_0, t_0], \mathbb{R})$ be a given initial function. One can easily see by the method of steps that the equation (3.1) has a unique solution $x \in C([t_0 - T_0, \infty), \mathbb{R})$ such that $x(t) = \phi(t)$ for $t_0 - T_0 \le t \le t_0$.

Lemma 3.2.1. *Assume that there exists $t^* \ge t_0$ such that*

$$R(t^* + mr) \le 1 \quad \textit{for all} \quad m \in \mathbb{N}_0. \tag{3.3}$$

Then for any eventually positive solution x of (3.2), the function y defined by

$$y(t) = x(t) - R(t)x(t-r) \tag{3.4}$$

satisfies

$$y'(t) \le 0 \quad \textit{and} \quad y(t) > 0.$$

Proof. It is clear from (3.1) that $y'(t) \le 0$ and is not identically zero for all large t. Thus $y(t)$ is eventually positive or eventually negative. If $x(t) > 0$, $y'(t) \le 0$ and $y(t) < 0$ for $t \ge T$, then $y(t) \le y(T) < 0$ for $t > T$. By choosing n so large that $t^* + nr \ge T$, we claim that

$$x(t^* + nr + kr) \le ky(T) + x(t^* + nr) \tag{3.5}$$

holds for all $k \in \mathbb{N}_0$. In fact, (3.5) is clear for $k = 0$, and if (3.5) holds for some $k \in \mathbb{N}_0$, then by (3.4)

$$\begin{aligned} x\Big(t^* + nr + (k+1)r\Big) &= y\Big(t^* + nr + (k+1)r\Big) \\ &\quad + R\Big(t^* + nr + (k+1)r\Big)\, x(t^* + nr + kr) \\ &\le y(T) + x(t^* + nr + kr) \\ &\le (k+1)y(T) + x(t^* + nr). \end{aligned}$$

Hence (3.5) holds for $k+1$, and by induction it holds for all $k \in \mathbb{N}_0$. By letting k in (3.5) tend to infinity, we see that the right-hand side diverges to $-\infty$, which is contrary to our assumption that $x(t) > 0$. The proof is complete. $\square$

Theorem 3.2.2. *Assume that* (3.3) *holds and that either*

$$R(t) + P(t)r > 0 \tag{3.6}$$

or

$$r > 0 \quad \textit{and} \quad P(s) \not\equiv 0 \textit{ for } s \in [t, t+r]. \tag{3.7}$$

Then every solution of (3.1) *oscillates if and only if the corresponding differential inequality* (3.2) *has no eventually positive solution.*

Proof. The sufficiency is obvious. To prove the necessity, we assume that x is an eventually positive solution of (3.2). Define y as in (3.4). Then by Lemma 3.2.1, we find

$$y(t) \ge \int_t^{\infty} P(s) \prod_{i=1}^{m} [x(s - \tau_i)]^{\alpha_i}\, ds,$$

i.e.,

$$x(t) \geq R(t)x(t-r) + \int_t^\infty P(s) \prod_{i=1}^m [x(s-\tau_i)]^{\alpha_i} \, ds. \tag{3.8}$$

Let $T > t_0$ be fixed so that (3.8) holds for all $t \geq T$. Now we consider the set of functions

$$E = \left\{ u \in C([T-T_0,\infty),\mathbb{R}^+) : \ 0 \leq u(t) \leq 1 \text{ for } t \geq T - T_0 \right\}$$

and define a mapping F on E as

$$(Fu)(t) = \begin{cases} \dfrac{1}{x(t)} \left[R(t)u(t-r)x(t-r) + \displaystyle\int_t^\infty P(s) \prod_{i=1}^m [u(s-\tau_i)x(s-\tau_i)]^{\alpha_i} \, ds \right] & \\ & \text{if} \quad t \geq T, \\ \dfrac{t-T+T_0}{T_0}(Fu)(T) + \left(1 - \dfrac{t-T+T_0}{T_0}\right) & \text{if} \quad T-T_0 \leq t < T. \end{cases}$$

It is easy to see, using (3.8), that F maps E into itself. Moreover, for any $u \in E$ we have $(Fu)(t) > 0$ for $T - T_0 \leq t < T$. Next we define the sequence $\{u_k\} \subset E$ by

$$u_0(t) = 1 \text{ for } t \geq T - T_0, \quad u_{k+1} = Fu_k \text{ for } k \in \mathbb{N}_0.$$

Then, by using (3.8) and a simple induction, we can easily see that

$$0 \leq u_{k+1}(t) \leq u_k(t) \leq 1 \quad \text{for} \quad t \geq T - T_0 \text{ and } k \in \mathbb{N}_0.$$

Set

$$u(t) = \lim_{k\to\infty} u_k(t), \quad t \geq T - T_0.$$

Then it follows from Lebesgue's dominated convergence theorem that u satisfies

$$u(t) = \frac{1}{x(t)} \left[R(t)u(t-r)x(t-r) + \int_t^\infty P(s) \prod_{i=1}^m [u(s-\tau_i)x(s-\tau_i)]^{\alpha_i} \, ds \right]$$

for $t \geq T$ and

$$u(t) = \frac{t-T+T_0}{T_0}(Fu)(T) + \left(1 - \frac{t-T+T_0}{T_0}\right)$$

for $T - T_0 \leq t \leq T$. Now set

$$w = ux.$$

Then w satisfies $w(t) > 0$ for $T - T_0 \leq t < T$ and

$$w(t) = R(t)w(t-r) + \int_t^\infty P(s) \prod_{i=1}^m [w(s-\tau_i)]^{\alpha_i} \, ds, \quad t \geq T. \tag{3.9}$$

Hence w solves (3.1). Clearly, w is continuous on $[T - T_0, \infty)$.

It remains to show that $w(t)$ is positive for all $t \geq T - T_0$. Assume that there exists $t^* \geq T - T_0$ such that $w(t) > 0$ for $T - T_0 \leq t < t^*$ and $w(t^*) = 0$. Then $t^* \geq T$, and by (3.9) we get

$$0 = w(t^*) = R(t^*)w(t^*-r) + \int_{t^*}^\infty P(s) \prod_{i=1}^m [w(s-\tau_i)]^{\alpha_i} \, ds,$$

which implies

$$R(t^*) = 0 \quad \text{and} \quad P(t)\prod_{i=1}^{m}[w(t-\tau_i)]^{\alpha_i} \equiv 0 \text{ for } t \ge t^*.$$

This contradicts (3.6) or (3.7). Therefore $w(t)$ is positive on $[T-T_0, \infty)$. □

We now compare (3.1) with the equation

$$\Big(x(t) - R^*(t)x(t-r)\Big)' + P^*(t)\prod_{i=1}^{m}\Big|x(t-\tau_i)\Big|^{\alpha_i} \operatorname{sgn} x(t-\tau_i) = 0 \tag{3.10}$$

to obtain the following comparison theorem.

Theorem 3.2.3. *Assume that* (3.3) *holds and that* (3.6) *or* (3.7) *holds for* $P^*(t)$ *and* $Q^*(t)$. *Further assume that*

$$R(t) \ge R^*(t) \quad \text{and} \quad P(t) \ge P^*(t). \tag{3.11}$$

If every solution of (3.10) *oscillates, then every solution of* (3.1) *oscillates as well.*

Proof. Suppose the contrary and let x be an eventually positive solution of (3.1). Set y as in (3.4). Then by Lemma 3.2.1 we have eventually

$$y'(t) \le 0 \quad \text{and} \quad y(t) > 0.$$

Thus, by integrating (3.1) from t to $T > t$, we obtain

$$y(t) = y(T) + \int_t^T P(s)\prod_{i=1}^{m}[x(s-\tau_i)]^{\alpha_i}\,ds \ge \int_t^T P(s)\prod_{i=1}^{m}[x(s-\tau_i)]^{\alpha_i}\,ds.$$

Therefore $y(t) \ge \int_t^\infty P(s)\prod_{i=1}^m [x(s-\tau_i)]^{\alpha_i}ds$, and noting (3.11), we obtain

$$\begin{aligned} x(t) &\ge R(t)x(t-r) + \int_t^\infty P(s)\prod_{i=1}^{m}[x(s-\tau_i)]^{\alpha_i}\,ds \\ &\ge R^*(t)x(t-r) + \int_t^\infty P^*(s)\prod_{i=1}^{m}[x(s-\tau_i)]^{\alpha_i}\,ds \end{aligned}$$

which (note that (3.11) implies that (3.3) also holds for R^*) implies in view of Theorem 3.2.2 that (3.11) also has an eventually positive solution. This is a contradiction and the proof is complete. □

We now turn to the question as to when (3.1) is oscillatory. This question is important if we want to apply Theorem 3.2.3.

Lemma 3.2.4. *Assume that* $R(t) \ge 1$ *for* $t \ge t_0$ *and*

$$\int_{t_0}^\infty P(s)\exp\left(\frac{1}{r}\int_{t_0}^s uP(u)du\right)ds = \infty. \tag{3.12}$$

Let x *be an eventually positive solution of* (3.1) *and define* y *by* (3.4). *Then eventually*

$$y'(t) \le 0 \quad \textit{and} \quad y(t) < 0. \tag{3.13}$$

Proof. From (3.1) and (3.4), we have

$$y'(t) = -P(t)\prod_{i=1}^{m}[x(t-\tau_i)]^{\alpha_i} \le 0.$$

Therefore, if (3.13) does not hold, then eventually $y(t) > 0$, i.e.,

$$x(t) \ge R(t)x(t-r) \ge x(t-r).$$

Let $t_1 \ge t_0$ be such that $x(t-r) > 0$ for $t \ge t_1$ and such that (3.8) holds for $t \ge t_1$. Define

$$M = \min\Big\{x(t):\ t\in[t_1-r,t_1]\Big\}.$$

Then $x(t) \ge M$ for $t \ge t_1$. Set $\tau^* = \max\{r, \tau_1, \ldots, \tau_m\}$, and we have

$$x(t) \ge M \quad \text{for} \quad t \ge t_1 + \tau^* = t_2.$$

For convenience, we denote

$$N(t) = \left[\frac{t-t_2}{r}\right],$$

where $[(t-t_2)/r]$ is the greatest integer part of $(t-t_2)/r$. We claim that

$$x(t) \ge \sum_{k=0}^{n-1} y(t-kr) + x(t-nr), \quad t \ge t_2 \tag{3.14}$$

holds for all $0 \le n \le N(t)$. Clearly, (3.14) is true for $n = 0$. If (3.14) is true for some $0 \le n < N(t)$, then

$$\begin{aligned}
x(t) &\ge \sum_{k=0}^{n-1} y(t-kr) + x(t-nr)\\
&= \sum_{k=0}^{n-1} y(t-kr) + y(t-nr) + R(t-nr)x(t-nr-r)\\
&= \sum_{k=0}^{n} y(t-kr) + R(t-nr)x\Big(t-(n+1)r\Big)\\
&\ge \sum_{k=0}^{n} y(t-kr) + x\Big(t-(n+1)r\Big)
\end{aligned}$$

for $t \ge t_2$ and so (3.14) is true for $n+1$. Hence (3.14) is true for all $0 \le n \le N(t)$. We note that y is decreasing and $x(t-N(t)r) \ge M$ for $t \ge t_2$. Thus, by (3.10) we obtain from (3.14) with $n = N(t)$

$$x(t) \ge N(t)y(t) + M, \quad t \ge t_2.$$

Substituting this into (3.1), we have

$$y'(t) + P(t)\prod_{i=1}^{m}\Big(N(t-\tau_i)y(t-\tau_i) + M\Big)^{\alpha_i} \le 0, \quad t \ge t_2 + \tau_0 = t_3,$$

where $\tau_0 = \max\{\tau_1, \ldots, \tau_m\}$. By Hölder's inequality [**118**], we have

$$\prod_{i=1}^{m}\Big(N(t-\tau_i)y(t-\tau_i) + M\Big)^{\alpha_i} \ge \prod_{i=1}^{m}[N(t-\tau_i)]^{\alpha_i}\prod_{i=1}^{m}[y(t-\tau_i)]^{\alpha_i} + M$$

and so

$$y'(t) + P(t)\prod_{i=1}^{m}[N(t-\tau_i)]^{\alpha_i}\,y(t) + P(t)M \le 0, \quad t \ge t_3.$$

Therefore

$$\left[y(t)\exp\left(\int_{t_3}^{t} P(s)\prod_{i=1}^{m}[N(s-\tau_i)]^{\alpha_i}\,ds\right)\right]' + MP(t)\exp\left(\int_{t_3}^{t} P(s)\prod_{i=1}^{m}[N(s-\tau_i)]^{\alpha_i}\,ds\right) \le 0, \quad t \ge t_3.$$

Integrating this inequality from t_3 to $t \ge t_3$, we find

$$y(t)\exp\left(\int_{t_3}^{t} P(s)\prod_{i=1}^{m}[N(s-\tau_i)]^{\alpha_i}\,ds\right) - y(t_3) + M\int_{t_3}^{t} P(s)\exp\left(\int_{t_3}^{s} P(u)\prod_{i=1}^{m}[N(u-\tau_i)]^{\alpha_i}\,du\right)ds \le 0, \quad t \ge t_3. \tag{3.15}$$

If the condition

$$\int_{t_0}^{\infty} P(s)ds = \infty$$

is satisfied, then it is easy to see that every solution of (3.1) oscillates. Hence we assume

$$\int_{t_0}^{\infty} P(s)ds < \infty.$$

Noting $\prod_{i=1}^{m}[N(t-\tau_i)]^{\alpha_i}/t \to 1/r$ as $t \to \infty$, it is easy to see that

$$\int_{t_3}^{\infty} P(s)\left\{\frac{s}{r} - \prod_{i=1}^{m}[N(s-\tau_i)]^{\alpha_i}\right\}ds$$

is absolutely convergent and

$$\lim_{s\to\infty}\frac{\exp\left(\int_{t_3}^{s} P(u)\prod_{i=1}^{m}[N(u-\tau_i)]^{\alpha_i}\,du\right)}{\exp\left(\frac{1}{r}\int_{t_3}^{s} uP(u)du\right)}$$

exists. By condition (3.12), we obtain

$$\int_{t_3}^{\infty} P(s)\exp\left(\int_{t_3}^{s} P(u)\prod_{i=1}^{m}[N(u-\tau_i)]^{\alpha_i}\,du\right)ds = \infty.$$

Letting $t \to \infty$ in (3.15), we obtain a contradiction. □

In view of Lemmas 3.2.1 and 3.2.4, it is now easy to obtain the following result.

Theorem 3.2.5. *Assume that* (3.12) *holds and*

$$R(t) \equiv 1.$$

Then every solution of (3.1) *oscillates.*

Example 3.2.6. Consider the neutral delay differential equation

$$\Big(x(t) - x(t-r)\Big)' + t^{-\alpha}x(t-\tau) = 0, \tag{3.16}$$

where $r > 0$, $\tau > 0$, and $1 < \alpha < 2$. Note that (3.16) satisfies all conditions of Theorem 3.2.5. Hence all solutions of (3.16) oscillate. On the other hand, we see by [**301**, Theorem 1] that (3.16) has a bounded nonoscillatory solution if and only if $\alpha > 2$. Therefore, it remains an **open problem** to determine the oscillation of all solutions of (3.16) with $\alpha = 2$.

Theorem 3.2.7. *Assume that* (3.3) *and* (3.12) *hold, and that*

$$\prod_{i=1}^{m} [R(t-\tau_i)]^{\alpha_i}\, P(t) \geq P(t-r). \tag{3.17}$$

Then every solution of (3.1) *oscillates.*

Proof. Otherwise, (3.1) would have an eventually positive solution x. Let y be defined by (3.4). Then by Lemma 3.2.1, we have eventually

$$y'(t) \leq 0 \quad \text{ and } \quad y(t) > 0.$$

From (3.1), (3.4), and (3.17), we find

$$\begin{aligned} y'(t) &= -P(t)\prod_{i=1}^{m}[x(t-\tau_i)]^{\alpha_i} \\ &= -P(t)\prod_{i=1}^{m}\Big(y(t-\tau_i) + R(t-\tau_i)x(t-r-\tau_i)\Big)^{\alpha_i} \\ &\leq -P(t)\prod_{i=1}^{m}[y(t-\tau_i)]^{\alpha_i} + y'(t-r), \end{aligned}$$

where we have used Hölder's inequality in the last step. This implies that y is a positive solution of the inequality

$$y'(t) - y'(t-r) + P(t)\prod_{i=1}^{m}[y(t-\tau_i)]^{\alpha_i} \leq 0, \tag{3.18}$$

which satisfies all conditions of Lemma 3.2.1, hence $u(t) = y(t) - y(t-r) > 0$ eventually. On the other hand, since (3.18) satisfies all conditions of Lemma 3.2.4, $u(t) = y(t) - y(t-r) < 0$ eventually, which is a contradiction. $\square$

In case the assumption (3.17) is not satisfied, we may check to see if there is some number $c \in [0,1)$ such that $cP(t-r) \leq P(t)\prod_{i=1}^{m}[R(t-\tau_i)]^{\alpha_i}$ for all large t.

Theorem 3.2.8. *Assume that* (3.3) *and* (3.12) *hold and that there is some number* $c \in [0,1)$ *such that*

$$cP(t-r) \leq P(t)\prod_{i=1}^{m}[R(t-\tau_i)]^{\alpha_i} \qquad \textit{for all large} \quad t.$$

Then (3.1) *is oscillatory provided that the inequality*

$$z'(t) + \frac{c}{1-c}P(t)z(t-r-\tau) \leq 0 \qquad \textit{with} \qquad \tau = \min\{\tau_1, \ldots, \tau_m\} \tag{3.19}$$

does not have an eventually positive solution.

Proof. Suppose to the contrary that x is an eventually positive solution of (3.1). Then by means of Lemma 3.2.1, the function y defined by (3.4) satisfies $y(t) > 0$ and $y'(t) \le 0$ for all large t. Then, as in the proof of Theorem 3.2.7, we see that

$$y'(t) - cy'(t-r) + P(t)\prod_{i=1}^{m}[y(t-\tau_i)]^{\alpha_i} \le 0$$

for all large t. Since y is nonincreasing, we have

$$y'(t) - cy'(t-r) + P(t)y(t-\tau) \le y'(t) - cy'(t-r) + P(t)\prod_{i=1}^{m}[y(t-\tau_i)]^{\alpha_i} \le 0$$

for all large t, where $\tau = \min\{\tau_1, \dots, \tau_m\}$. Let $z(t) = y(t) - cy(t-r)$ for $t \ge t_0$. By means of Lemma 3.2.1, it is clear that $z'(t) \le 0$ and $z(t) > 0$ for t greater than or equal to some number T. Without loss of generality, we may also assume that $y(t) > 0$ for $t \ge T$. Now

$$\begin{aligned} y(t) &= z(t) + cy(t-r) \\ &\ge z(t) + cz(t-r) + \ldots + c^j z(t-jr) + c^{j+1} y\left(t-(j+1)r\right) \\ &> \left(c + c^2 + \ldots + c^{j+1}\right) z(t-r) \\ &= \frac{c(1-c^{j+1})}{1-c} z(t-r) \end{aligned}$$

for $t > (j+1)r + T + \tau$ and hence

$$\begin{aligned} 0 &\ge y'(t) - cy'(t-r) + P(t)y(t-\tau) \;=\; z'(t) + P(t)y(t-\tau) \\ &> z'(t) + \frac{c}{1-c}P(t)z(t-r-\tau) \end{aligned}$$

for all large t, which is contrary to our hypothesis. □

Remark 3.2.9. Several explicit conditions ensuring that (3.19) cannot have an eventually positive solution have already been established. A sample of these conditions can be found in the book by Győri and Ladas [**118**].

Theorem 3.2.10. *Suppose* $P(t) \ge p > 0$, $R(t) \ge 0$ *for* $t \ge t_0$ *such that* (3.3) *holds. Suppose further that* $\tau_i \in (0, \infty)$, $1 \le i \le m$ *and*

$$\inf_{t \ge t_0, \lambda > 0} \left\{ \frac{P(t)}{P(t-r)} \prod_{i=1}^{m} [R(t-\tau_i)]^{\alpha_i} e^{r\lambda} + \frac{1}{l\lambda} e^{\lambda \sum_{i=1}^{m} \alpha_i \tau_i} \int_t^{t+l} P(s)ds \right\} > 1 \tag{3.20}$$

for all $l \in \{r, \tau_1, \dots, \tau_m\}$*. Then* (3.1) *is oscillatory.*

Proof. Suppose to the contrary that x is an eventually positive solution of (3.1). Then arguments similar to those used in the proof of Theorem 3.2.7 show that the function y defined by (3.4) is positive, nonincreasing, and satisfies

$$y'(t) - \frac{P(t)}{P(t-r)} \prod_{i=1}^{m} [R(t-\tau_i)]^{\alpha_i} y'(t-r) + P(t)\prod_{i=1}^{m}[y(t-\tau_i)]^{\alpha_i} \le 0$$

for $t \ge t_0$. For the sake of convenience, let us write

$$Q(t) = \frac{P(t)}{P(t-r)} \prod_{i=1}^{m} [R(t-\tau_i)]^{\alpha_i}, \quad t \ge t_0.$$

Let $w(t) = -\frac{y'(t)}{y(t)}$ for $t \geq t_0$. Then $w(t) > 0$ for $t \geq t_0$ and

$$w(t) \geq Q(t)w(t-r)\exp\left(\int_{t-r}^{t} w(s)ds\right) + P(t)\prod_{i=1}^{m}\left[\exp\left(\int_{t-\tau_i}^{t} w(s)ds\right)\right]^{\alpha_i}$$

for $t \geq t_0 + \tau^*$, where $\tau^* = \max\{r, \tau_1, \ldots, \tau_m\}$.

The rest of the proof of Theorem 3.2.10 is similar to that of [**91**, Theorem 2.1] and hence will be omitted. □

Remark 3.2.11. When $m = 1$ and $R(t) \equiv R > 0$, $P(t) \equiv P$ for $t \geq t_0$, the condition (3.20) is sharp.

3.3. Oscillation of Equations with Variable Coefficients (I)

Consider first order neutral delay differential equations with positive and negative coefficients of the form

$$\left(x(t) - R(t)x(t-r)\right)' + P(t)x(t-\tau) - Q(t)x(t-\delta) = 0, \tag{3.21}$$

where

$$P, Q, R \in C([t_0, \infty), \mathbb{R}^+), \quad r \in (0, \infty) \quad \text{and} \quad \tau, \delta \in \mathbb{R}^+ \tag{3.22}$$

and

$$\tau > \delta, \quad \bar{P}(t) = P(t) - Q(t-\tau+\delta) \geq 0 \quad \text{and not identically zero.} \tag{3.23}$$

Let $T_0 = \max\{r, \tau, \delta\}$. By a solution of the equation (3.21) we mean a function $x \in C([t_0 - T_0, \infty), \mathbb{R})$, for some $t_1 \geq t_0$ such that $x(t) - R(t)x(t-r)$ is continuously differentiable on $[t_1, \infty)$ and such that (3.21) is satisfied for $t \geq t_1$.

In this section we give some sharp sufficient conditions for the oscillation of all solutions of (3.21). Before stating our main results, we need the following lemmas.

Lemma 3.3.1. *Assume that* (3.22) *and* (3.23) *hold and*

$$R(t) + \int_{t-\tau+\delta}^{t} Q(s)ds \leq 1 \quad \textit{for} \quad t \geq t_1 \geq t_0. \tag{3.24}$$

Let x be an eventually positive solution of (3.21) *and set*

$$y(t) = x(t) - R(t)x(t-r) - \int_{t-\tau+\delta}^{t} Q(s)x(s-\delta)ds. \tag{3.25}$$

Then eventually

$$y'(t) \leq 0 \quad \textit{and} \quad y(t) > 0. \tag{3.26}$$

Proof. From (3.21), (3.23), and (3.25), we see that

$$y'(t) = -\bar{P}(t)x(t-\tau) \leq 0, \quad t \geq t_2 \geq t_1.$$

Denote $\lim_{t\to\infty} y(t) = l$. We show that $l \geq 0$.

First we consider the case that x is unbounded, i.e., $\limsup_{t\to\infty} x(t) = \infty$. Then there exists a sequence $\{s_n\}$ with $\lim_{n\to\infty} s_n = \infty$, $\lim_{n\to\infty} x(s_n) = \infty$, and $x(s_n) = \max\{x(t) : t_2 \le t \le s_n\}$ for $n \in \mathbb{N}$. Hence from (3.23) and (3.24)

$$\begin{aligned} y(s_n) &= x(s_n) - R(s_n)x(s_n - r) - \int_{s_n-\tau+\delta}^{s_n} Q(s)x(s-\delta)ds \\ &\ge x(s_n)\left[1 - R(s_n) - \int_{s_n-\tau+\delta}^{s_n} Q(s)ds\right] \ge 0 \end{aligned}$$

for $n \in \mathbb{N}$. Therefore $l = \lim_{t\to\infty} y(t) = \lim_{n\to\infty} y(s_n) \ge 0$.

Next we consider the case that x is bounded. Set $\limsup_{t\to\infty} x(t) = \bar{l} < \infty$. Then there exists a sequence $\{\bar{s}_n\}$ such that $\lim_{n\to\infty} \bar{s}_n = \infty$, $\lim_{n\to\infty} x(\bar{s}_n) = \bar{l}$. Denote $r_1 = \min\{r, \delta\}$, $r_2 = \max\{r, \tau\}$, and

$$x(\xi_n) = \max\left\{x(s) : \ \bar{s}_n - r_2 \le s \le \bar{s}_n - r_1\right\}, \quad n \in \mathbb{N}.$$

Clearly, $\lim_{n\to\infty} \xi_n = \infty$ and $\lim_{n\to\infty} x(\xi_n) \le \bar{l}$. From (3.23) and (3.24),

$$\begin{aligned} x(\bar{s}_n) - y(\bar{s}_n) &= R(\bar{s}_n)x(\bar{s}_n - r) + \int_{\bar{s}_n-\tau+\delta}^{\bar{s}_n} Q(s)x(s-\delta)ds \\ &\le x(\xi_n)\left[R(\bar{s}_n) + \int_{\bar{s}_n-\tau+\delta}^{\bar{s}_n} Q(s)ds\right] \le x(\xi_n). \end{aligned}$$

Taking limit superior on both sides as $n \to \infty$, we obtain $\bar{l} - l \le \bar{l}$, and so $l \ge 0$. Then $y(t) > l \ge 0$ for $t \ge t_2$. □

Lemma 3.3.2. *Assume that* $\delta > 0$, $Q \in C([t_0, \infty), \mathbb{R}^+)$, $\lambda \in C([t_0 - \delta, \infty), \mathbb{R}^+)$, *and*

$$\lambda(t) \ge Q(t) \exp\left(\int_{t-\delta}^{t} \lambda(s)ds\right), \quad t \ge t_0. \tag{3.27}$$

Then the condition

$$\liminf_{t\to\infty} \int_{t-\delta}^{t} Q(s)ds > 0 \tag{3.28}$$

implies that

$$\liminf_{t\to\infty} \int_{t-\delta}^{t} \lambda(s)ds < \infty. \tag{3.29}$$

Proof. Define

$$\bar{Q}(t) = \int_{t_0}^{t} Q(s)ds, \quad t \ge t_0.$$

The condition (3.28) implies that $\lim_{t\to\infty} \bar{Q}(t) = \infty$, and $\bar{Q}(t)$ is strictly increasing. Then $\bar{Q}^{-1}(t)$ is well defined, strictly increasing, and $\lim_{t\to\infty} \bar{Q}^{-1}(t) = \infty$. The condition (3.28) implies that there exist $c > 0$ and $T_1 \ge t_0$ such that

$$\bar{Q}(t) - \bar{Q}(t-\delta) \ge \frac{c}{2} \quad \text{for} \quad t \ge T_1$$

and thus

$$\bar{Q}^{-1}\left(\bar{Q}(t) - \frac{c}{2}\right) \ge t - \delta \quad \text{for} \quad t \ge T_1.$$

Set
$$\Lambda(t) = \exp\left(-\int_{T_1}^{t} \lambda(s)ds\right).$$
Now (3.27) implies that
$$\Lambda'(t) \le -Q(t)\Lambda(t-\delta), \quad t \ge t_0.$$
By [**92**, Lemma 2.1.3], $\Lambda(t-\delta)/\Lambda(t)$ is bounded above under the condition (3.28). Then (3.29) is true. □

Theorem 3.3.3. *Assume that* (3.22), (3.23), *and* (3.24) *hold and*
$$\liminf_{t\to\infty} \int_{t-\tau}^{t} \bar{P}(s)ds > 0. \tag{3.30}$$
Assume moreover that there exists a positive continuous function H such that
$$\lim_{t\to\infty} \int_{t-\tau}^{t} H(s)ds > 0 \tag{3.31}$$
and that either
$$\begin{aligned} 1 \quad < \quad & \inf_{\lambda>0, t\ge T} \left\{ \frac{R(t-\tau)\bar{P}(t)H(t-r)}{\bar{P}(t-r)H(t)} \exp\left(\lambda \int_{t-r}^{t} H(s)ds\right) \right. \\ & + \frac{\bar{P}(t)}{H(t)\lambda} \exp\left(\lambda \int_{t-\tau}^{t} H(s)ds\right) \\ & \left. + \frac{\bar{P}(t)}{H(t)} \int_{t-\tau+\delta}^{t} \frac{Q(s-\tau)H(s-\delta)}{\bar{P}(s-\delta)} \exp\left(\lambda \int_{s-\delta}^{t} H(u)du\right) ds \right\} \end{aligned} \tag{3.32}$$
or
$$\begin{aligned} 1 \quad < \quad & \inf_{\lambda>0, t\ge T} \left\{ \frac{1}{H(t)\lambda} \exp\left(\lambda \int_{t-\tau}^{t} H(s)\bar{P}(s)ds\right) \right. \\ & + \frac{H(t-r)R(t-\tau)}{H(t)} \exp\left(\lambda \int_{t-r}^{t} H(s)\bar{P}(s)ds\right) \\ & \left. + \frac{1}{H(t)} \int_{t-\tau+\delta}^{t} Q(s-\tau)H(s-\delta) \exp\left(\lambda \int_{s-\delta}^{t} H(u)\bar{P}(u)du\right) ds \right\}. \end{aligned} \tag{3.33}$$
Then every solution of (3.21) *is oscillatory.*

Proof. Without loss of generality, assume that (3.21) has an eventually positive solution x. Let y be defined by (3.25). Then by Lemma 3.3.1 we have
$$y'(t) \le 0 \quad \text{and} \quad y(t) > 0 \quad \text{for} \quad t \ge t_1 \ge t_0.$$
From (3.21) we have
$$\begin{aligned} y'(t) \;&=\; -\bar{P}(t)x(t-\tau) \\ &= -\bar{P}(t)\left[y(t-\tau) + R(t-\tau)x(t-\tau-r) + \int_{t-\tau+\delta}^{t} Q(s-\tau)x(s-\tau-\delta)ds\right] \\ &= -\bar{P}(t)y(t-\tau) + \frac{R(t-\tau)\bar{P}(t)}{\bar{P}(t-r)}y'(t-r) + \bar{P}(t)\int_{t-\tau+\delta}^{t} \frac{Q(s-\tau)}{\bar{P}(s-\delta)}y'(s-\delta)ds. \end{aligned} \tag{3.34}$$
Set
$$\lambda(t)H(t) = -\frac{y'(t)}{y(t)}.$$

Then (3.34) reduces to

$$
\begin{aligned}
\lambda(t)H(t) \;&\geq\; \bar{P}(t)\exp\left(\int_{t-\tau}^{t}\lambda(s)H(s)ds\right) \\
&+\lambda(t-r)H(t-r)\frac{R(t-\tau)\bar{P}(t)}{\bar{P}(t-r)}\exp\left(\int_{t-r}^{t}\lambda(s)H(s)ds\right) \\
&+\bar{P}(t)\int_{t-\tau+\delta}^{t}\frac{Q(s-\tau)}{\bar{P}(s-\delta)}\lambda(s-\delta)H(s-\delta)\exp\left(\int_{s-\delta}^{t}\lambda(u)H(u)du\right)ds.
\end{aligned}
\tag{3.35}
$$

It is obvious that $\lambda(t)H(t) > 0$ for $t \geq t_0$. From (3.35) we have

$$\lambda(t)H(t) \geq \bar{P}(t)\exp\left(\int_{t-\tau}^{t}\lambda(s)H(s)ds\right).$$

In view of (3.30) and Lemma 3.3.2, we get

$$\liminf_{t\to\infty}\int_{t-\tau}^{t}\lambda(s)H(s)ds < \infty,$$

which implies, by using (3.31), that $\liminf_{t\to\infty}\lambda(t) < \infty$. Now we show that

$$\liminf_{t\to\infty}\lambda(t) > 0.$$

In fact, if $\liminf_{t\to\infty}\lambda(t) = 0$, then there exists a sequence $\{t_n\}$ such that $t_n \geq t_1$, $t_n \to \infty$ as $n \to \infty$, and $\lambda(t_n) \leq \lambda(t)$ for $t \in [t_1, t_n]$. From (3.35) we have

$$
\begin{aligned}
\lambda(t_n)H(t_n) \;&\geq\; \bar{P}(t_n)\exp\left(\lambda(t_n)\int_{t_n-\tau}^{t_n}H(s)ds\right) \\
&+\lambda(t_n)H(t_n-r)\frac{R(t_n-\tau)\bar{P}(t_n)}{\bar{P}(t_n-r)}\exp\left(\lambda(t_n)\int_{t_n-r}^{t_n}H(s)ds\right) \\
&+\bar{P}(t_n)\int_{t_n-\tau+\delta}^{t_n}\frac{Q(s-\tau)}{\bar{P}(s-\delta)}\lambda(t_n)H(s-\delta)\exp\left(\lambda(t_n)\int_{s-\delta}^{t_n}H(u)du\right)ds.
\end{aligned}
$$

Hence

$$
\begin{aligned}
1 \;&\geq\; \frac{\bar{P}(t_n)}{\lambda(t_n)H(t_n)}\exp\left(\lambda(t_n)\int_{t_n-\tau}^{t_n}H(s)ds\right) \\
&+\frac{H(t_n-r)R(t_n-\tau)\bar{P}(t_n)}{H(t_n)\bar{P}(t_n-r)}\exp\left(\lambda(t_n)\int_{t_n-r}^{t_n}H(s)ds\right) \\
&+\frac{\bar{P}(t_n)}{H(t_n)}\int_{t_n-\tau+\delta}^{t_n}\frac{Q(s-\tau)H(s-\delta)}{\bar{P}(s-\delta)}\exp\left(\lambda(t_n)\int_{s-\delta}^{t_n}H(u)du\right)ds,
\end{aligned}
$$

which contradicts (3.32). Now, let us first assume that (3.32) holds. Then

$$\liminf_{t\to\infty}\lambda(t) = h \in (0,\infty). \tag{3.36}$$

By (3.32), there exists an $\alpha \in (0,1)$ such that

$$
\begin{aligned}
(3.37)\quad 1 \quad < \quad & \alpha \inf_{\lambda>0, t\geq T} \left\{ \frac{R(t-\tau)\bar{P}(t)H(t-r)}{\bar{P}(t-r)H(t)} \exp\left(\lambda \int_{t-r}^{t} H(s)ds\right) \right. \\
& + \frac{\bar{P}(t)}{H(t)\lambda} \exp\left(\lambda \int_{t-\tau}^{t} H(s)ds\right) \\
& \left. + \frac{\bar{P}(t)}{H(t)} \int_{t-\tau+\delta}^{t} \frac{Q(s-\tau)H(s-\delta)}{\bar{P}(s-\delta)} \exp\left(\lambda \int_{s-\delta}^{t} H(u)du\right) ds \right\}.
\end{aligned}
$$

In view of (3.36), there exists $t_2 > t_1$ such that

$$\lambda(t) > \alpha h \quad \text{for all} \quad t \geq t_2.$$

Substituting (3.37) into (3.35), we obtain

$$
\begin{aligned}
\lambda(t)H(t) \quad \geq \quad & \bar{P}(t) \exp\left(h\alpha \int_{t-\tau}^{t} H(s)ds\right) \\
& + h\alpha \frac{H(t-r)R(t-\tau)\bar{P}(t)}{\bar{P}(t-r)} \exp\left(h\alpha \int_{t-r}^{t} H(s)ds\right) \\
& + \bar{P}(t)\alpha h \int_{t-\tau+\delta}^{t} \frac{Q(s-\tau)H(s-\delta)}{\bar{P}(s-\delta)} \exp\left(h\alpha \int_{s-\delta}^{t} H(u)du\right) ds
\end{aligned}
$$

for $t \geq t_2 + T_0$. Hence

$$
\begin{aligned}
h \quad \geq \quad & \liminf_{t\to\infty} \left\{ \frac{\bar{P}(t)}{H(t)} \exp\left(\alpha h \int_{t-\tau}^{t} H(s)ds\right) \right. \\
& + \alpha h \frac{H(t-r)R(t-\tau)\bar{P}(t)}{H(t)\bar{P}(t-r)} \exp\left(\alpha h \int_{t-r}^{t} H(s)ds\right) \\
& \left. + \frac{\bar{P}(t)}{H(t)} \alpha h \int_{t-\tau+\delta}^{t} \frac{Q(s-\tau)H(s-\delta)}{\bar{P}(s-\delta)} \exp\left(\alpha h \int_{s-\delta}^{t} H(u)du\right) ds \right\},
\end{aligned}
$$

which implies that there exists a sequence $\{\bar{t}_n\}$ such that $\bar{t}_n \geq \max\{T, t_2 + T_0\}$, $\bar{t}_n \to \infty$ as $n \to \infty$, and

$$
\begin{aligned}
h \quad \geq \quad & \lim_{n\to\infty} \left\{ \frac{\bar{P}(\bar{t}_n)}{H(\bar{t}_n)} \exp\left(\alpha h \int_{\bar{t}_n-\tau}^{\bar{t}_n} H(s)ds\right) \right. \\
& + \alpha h \frac{H(\bar{t}_n-r)R(\bar{t}_n-\tau)\bar{P}(\bar{t}_n)}{H(\bar{t}_n)\bar{P}(\bar{t}_n-r)} \exp\left(\alpha h \int_{\bar{t}_n-r}^{\bar{t}_n} H(s)ds\right) \\
& \left. + \frac{\bar{P}(\bar{t}_n)}{H(\bar{t}_n)} \alpha h \int_{\bar{t}_n-\tau+\delta}^{\bar{t}_n} \frac{Q(s-\tau)H(s-\delta)}{\bar{P}(s-\delta)} \exp\left(\alpha h \int_{s-\delta}^{\bar{t}_n} H(u)du\right) ds \right\}.
\end{aligned}
$$

If we set $\lambda = h\alpha$, then

$$
\begin{aligned}
1 \quad \geq \quad & \alpha \lim_{n\to\infty} \left\{ \frac{\bar{P}(\bar{t}_n)}{H(\bar{t}_n)\lambda} \exp\left(\lambda \int_{\bar{t}_n-\tau}^{\bar{t}_n} H(s)ds\right) \right. \\
& + \frac{H(\bar{t}_n-r)R(\bar{t}_n-\tau)\bar{P}(\bar{t}_n)}{H(\bar{t}_n)\bar{P}(\bar{t}_n-r)} \exp\left(\lambda \int_{\bar{t}_n-r}^{\bar{t}_n} H(s)ds\right) \\
& \left. + \frac{\bar{P}(\bar{t}_n)}{H(\bar{t}_n)} \int_{\bar{t}_n-\tau+\delta}^{\bar{t}_n} \frac{Q(s-\tau)H(s-\delta)}{\bar{P}(s-\delta)} \exp\left(\lambda \int_{s-\delta}^{\bar{t}_n} H(u)du\right) ds \right\},
\end{aligned}
$$

which contradicts (3.37) and completes the proof of this theorem under condition (3.32).

Now we assume that condition (3.33) holds. Let

$$\lambda(t)H(t)\bar{P}(t) = -\frac{y'(t)}{y(t)}.$$

Then (3.34) becomes

$$\begin{aligned}(3.38)\quad \lambda(t)H(t) \ \geq\ & \exp\left(\int_{t-\tau}^{t} \lambda(s)H(s)\bar{P}(s)ds\right)\\ &+\lambda(t-r)H(t-r)R(t-\tau)\exp\left(\int_{t-r}^{t} \lambda(s)H(s)\bar{P}(s)ds\right)\\ &+\int_{t-\tau+\delta}^{t} Q(s-\tau)\lambda(s-\delta)H(s-\delta)\exp\left(\int_{s-\delta}^{t} \lambda(u)H(u)\bar{P}(u)du\right)ds.\end{aligned}$$

As before, by Lemma 3.3.2, we can prove that

$$\liminf_{t\to\infty} \int_{t-\tau}^{t} \lambda(s)H(s)\bar{P}(s)ds < \infty. \tag{3.39}$$

From (3.30), (3.31), and (3.39), we may conclude that $\liminf_{t\to\infty}\lambda(t) < \infty$. In view of (3.38), $\lambda(t) \geq 1$, and so

$$\liminf_{t\to\infty}\lambda(t) = h \in (0,\infty).$$

From (3.33), there exists $\alpha \in (0,1)$ such that

$$\begin{aligned}1 \ <\ \alpha & \inf_{\lambda>0, t\geq T}\left\{\frac{1}{\lambda H(t)}\exp\left(\lambda\int_{t-\tau}^{t} H(s)\bar{P}(s)ds\right)\right.\\ &+\frac{H(t-r)R(t-\tau)}{H(t)}\exp\left(\lambda\int_{t-r}^{t} H(s)\bar{P}(s)ds\right)\\ &\left.+\frac{1}{H(t)}\int_{t-\tau+\delta}^{t} Q(s-\tau)H(s-\delta)\exp\left(\lambda\int_{s-\delta}^{t} H(u)\bar{P}(u)du\right)ds\right\}.\end{aligned}$$

By using a similar method as in the first part of the proof, we can derive a contradiction. The proof is complete. □

Remark 3.3.4. Conditions (3.32) and (3.33) are equivalent when $\bar{P}(t) > 0$ for $t \geq T$. In fact, if $\bar{P}(t) > 0$ for $t \geq T$, set $K(t) = H(t)\bar{P}(t)$. Then condition (3.33) becomes (3.32). Conversely, if we let $K(t) = H(t)/\bar{P}(t)$, then (3.32) reduces to (3.33).

Since $e^x \geq ex$ and $e^x \geq 1$ for $x \geq 0$, (3.32) and (3.33) lead to the following corollary.

Corollary 3.3.5. *Assume that* (3.22), (3.23), (3.24), (3.30), *and* (3.31) *hold. Further assume that either*

$$\begin{aligned}1 < \liminf_{t\to\infty}\left\{\frac{R(t-\tau)\bar{P}(t)H(t-r)}{\bar{P}(t-r)H(t)} + \frac{e\bar{P}(t)}{H(t)}\int_{t-\tau}^{t} H(s)ds\right.\\ \left.+\frac{\bar{P}(t)}{H(t)}\int_{t-\tau+\delta}^{t}\frac{Q(s-\tau)H(s-\delta)}{\bar{P}(s-\delta)}ds\right\}\end{aligned}$$

(if $\bar{P}(t) > 0$ *for* $t \geq T$*) or*

$$1 < \liminf_{t\to\infty} \left\{ \frac{e}{H(t)} \int_{t-\tau}^{t} H(s)\bar{P}(s)ds + \frac{H(t-r)R(t-\tau)}{H(t)} + \frac{1}{H(t)} \int_{t-\tau+\delta}^{t} Q(s-\tau)H(s-\delta)ds \right\}.$$

Then every solution of (3.21) *is oscillatory.*

From Corollary 3.3.5, we can obtain different sufficient conditions for oscillation of (3.21) by different choices of $H(t)$. For instance, if we choose $H(t) = \bar{P}(t) > 0$ for $t \geq T$ or $H(t) \equiv 1$, then the first condition in Corollary 3.3.5 becomes

$$\liminf_{t\to\infty} \left\{ R(t-\tau) + e\int_{t-\tau}^{t} \bar{P}(s)ds + \int_{t-\tau+\delta}^{t} Q(s-\tau)ds \right\} > 1 \tag{3.40}$$

or

$$\liminf_{t\to\infty} \left\{ \frac{\bar{P}(t)R(t-\tau)}{\bar{P}(t-r)} + e\bar{P}(t)\tau + \bar{P}(t)\int_{t-\tau+\delta}^{t} \frac{Q(s-\tau)}{\bar{P}(s-\delta)}ds \right\} > 1 \tag{3.41}$$

(if $\bar{P}(t) > 0$ for $t \geq T$). If we select $H(t) = \bar{P}(t) > 0$ for $t \geq T$, then the second condition in Corollary 3.3.5 becomes

$$1 < \liminf_{t\to\infty} \left\{ \frac{e}{\bar{P}(t)} \int_{t-\tau}^{t} \left[\bar{P}(s)\right]^2 ds + \frac{\bar{P}(t-r)R(t-\tau)}{\bar{P}(t)} + \frac{1}{\bar{P}(t)} \int_{t-\tau+\delta}^{t} Q(s-\tau)\bar{P}(s-\delta)ds \right\}. \tag{3.42}$$

Corollary 3.3.6. *Assume that* (3.22), (3.23), (3.24), *and* (3.40) *hold. Then every solution of* (3.21) *is oscillatory.*

Corollary 3.3.7. *Assume that* (3.22), (3.23), (3.24), *and* (3.41) *hold. Then every solution of* (3.21) *is oscillatory.*

Corollary 3.3.8. *Assume that* (3.22), (3.23), (3.24), *and* (3.42) *hold. Then every solution of* (3.21) *is oscillatory.*

Remark 3.3.9. Condition (3.40) is the same as in [**289**, Theorem 1] (obtained by a different technique). But we should point out that the proof in [**289**, Theorem 1] is inaccurate (see Zhang [**297**]). On the other hand, if $\bar{P}(t) > 0$ is nonincreasing, then it is easy to see that

$$\frac{R(t-\tau)\bar{P}(t)}{\bar{P}(t-r)} \leq R(t-\tau), \quad \tau\bar{P}(t) \leq \int_{t-\tau}^{t} \bar{P}(s)ds,$$

and

$$\bar{P}(t)\int_{t-\tau+\delta}^{t} \frac{Q(s-\tau)}{\bar{P}(s-\delta)}ds \leq \frac{\bar{P}(t)}{\bar{P}(t-\delta)}\int_{t-\tau+\delta}^{t} Q(s-\tau)ds \leq \int_{t-\tau+\delta}^{t} Q(s-\tau)ds.$$

It follows that

$$\frac{R(t-\tau)\bar{P}(t)}{\bar{P}(t-r)} + e\bar{P}(t)\tau + \bar{P}(t)\int_{t-\tau+\delta}^{t} \frac{Q(s-\tau)}{\bar{P}(s-\delta)}ds \leq R(t-\tau) + e\int_{t-\tau}^{t} \bar{P}(s)ds + \int_{t-\tau+\delta}^{t} Q(s-\tau)ds.$$

If $\bar{P}(t) > 0$ is nondecreasing, then all inequalities above are reversed.

Example 3.3.10. Consider the neutral differential equation

$$\left(x(t) - \frac{t-1}{2t}x(t-1)\right)' + \left(\frac{1}{8e} + \frac{1}{t}\right)x(t-2) - \frac{1}{t+1}x(t-1) = 0 \tag{3.43}$$

for $t \geq 4$. Here, $\tau = 2$, $\delta = 1$,

$$R(t) = \frac{t-1}{2t}, \quad P(t) = \frac{1}{8e} + \frac{1}{t}, \quad Q(t) = \frac{1}{t+1}$$

so that $\bar{P}(t) = 1/(8e)$. It is easy to verify that the assumptions of Corollary 3.3.7 are satisfied. Therefore, every solution of (3.43) is oscillatory.

Corollary 3.3.11. *Assume that* $R(t) = r_0 > 0$, $P(t) = p > 0, Q(t) = q \geq 0$, $p > q$, $\tau > \delta$ *and* $r_0 + q(\tau - \delta) \leq 1$. *Then every solution of* (3.21) *oscillates if and only if*

$$f_1(\lambda) = -\lambda(p-q) + qe^{\lambda p\tau} + \lambda(p-q)r_0e^{\lambda(p-q)r} - qe^{\lambda(p-q)\delta} > 0, \quad \lambda > 0.$$

Corollary 3.3.12. *Assume that the assumptions of Corollary 3.3.7 hold. Then every solution of* (3.21) *oscillates if and only if*

$$f_2(\lambda) = -\lambda + \lambda r_0 e^{\lambda\tau} + pe^{\lambda\tau} - qe^{\lambda\delta} > 0, \quad \lambda > 0. \tag{3.44}$$

Proof. Sufficiency is obvious. We will prove the necessity. Assume that the condition (3.44) is false, then there exists $\lambda_0 > 0$ such that $f_2(\lambda_0) \leq 0$, and $f_2(0) = p - q > 0$. Thus there exists $\lambda_1 \in (0, \lambda_0]$ such that $f_2(\lambda_1) = 0$. In fact, $x(t) = \exp(-\lambda_1 t)$ is nonoscillatory solution of (3.21). This is a contradiction and the proof is complete. □

Remark 3.3.13. Corollaries 3.3.11 and 3.3.12 imply that the conditions of Theorem 3.3.3 are sharp.

3.4. Oscillation of Equations with Variable Coefficients (II)

In this section we continue to study the oscillation of the first order neutral delay differential equation (3.21) by a comparison theorem.

Theorem 3.4.1. *Assume that* (3.22), (3.23), *and* (3.24) *hold and that either*

$$R(t) + \Big(P(t) - Q(t-\tau+\delta)\Big)\tau > 0 \tag{3.45}$$

or

$$\tau > 0 \quad \text{and} \quad \bar{P}(s) \not\equiv 0 \text{ for } s \in [t, t+\tau]. \tag{3.46}$$

Then every solution of (3.21) *oscillates if and only if the corresponding differential inequality*

$$\Big(x(t) - R(t)x(t-r)\Big)' + P(t)x(t-\tau) - Q(t)x(t-\delta) \leq 0 \tag{3.47}$$

has no eventually positive solution.

Proof. The sufficiency is obvious. To prove the necessity, we assume that x is an eventually positive solution of (3.47). Define y as in (3.25). Then by (3.24) and Lemma 3.3.1, we find

$$y(t) \geq \int_t^\infty \bar{P}(s)x(s-\tau)ds,$$

i.e.,

$$x(t) \geq R(t)x(t-r) + \int_{t-\tau+\delta}^{t} Q(s)x(s-\delta)ds + \int_{t}^{\infty} \bar{P}(s)x(s-\tau)ds. \tag{3.48}$$

Let $T > t_0$ be fixed so that (3.48) holds for all $t \geq T$. Now we consider the set of functions

$$E = \Big\{ u \in C([T-T_0, \infty), \mathbb{R}^+) : \ 0 \leq u(t) \leq 1 \text{ for } t \geq T - T_0 \Big\}$$

and define a mapping F on E as

$$(Fu)(t) = \begin{cases} \dfrac{1}{x(t)} \left[R(t)u(t-r)x(t-r) + \displaystyle\int_{t-\tau+\delta}^{t} Q(s)u(s-\delta)x(s-\delta)ds \right. \\ \qquad \left. + \displaystyle\int_{t}^{\infty} \bar{P}(s)u(s-\tau)x(s-\tau)ds \right] & \text{if} \quad t \geq T, \\ \dfrac{t-T+T_0}{T_0}(Fu)(T) + \left(1 - \dfrac{t-T+T_0}{T_0}\right) & \text{if} \quad T - T_0 \leq t < T. \end{cases}$$

It is easy to see, using (3.48), that F maps E into itself. Moreover, for any $u \in E$ we have $(Fu)(t) > 0$ for $T - T_0 \leq t < T$. Next we define the sequence $\{u_k\} \subset E$ by

$$u_0(t) = 1 \text{ for } t \geq T - T_0, \quad u_{k+1} = Fu_k \text{ for } k \in \mathbb{N}_0.$$

Then, by using (3.48) and a simple induction, we can easily see that

$$0 \leq u_{k+1}(t) \leq u_k(t) \leq 1 \quad \text{for} \quad t \geq T - T_0 \text{ and } k \in \mathbb{N}_0.$$

Set

$$u(t) = \lim_{k\to\infty} u_k(t), \quad t \geq T - T_0.$$

Then it follows from Lebesgue's dominated convergence theorem that u satisfies

$$u(t) = \frac{1}{x(t)} \left[R(t)u(t-r)x(t-r) + \int_{t-\tau+\delta}^{t} Q(s)u(s-\delta)x(s-\delta)ds \right. \\ \left. + \int_{t}^{\infty} \bar{P}(s)u(s-\tau)x(s-\tau)ds \right]$$

for $t \geq T$ and

$$u(t) = \frac{t-T+T_0}{T_0}(Fu)(T) + \left(1 - \frac{t-T+T_0}{T_0}\right)$$

for $T - T_0 \leq t \leq T$. Set

$$w = ux.$$

Then w satisfies $w(t) > 0$ for $T - T_0 \leq t < T$ and

$$w(t) = R(t)w(t-r) + \int_{t-\tau+\delta}^{t} Q(s)w(s-\delta)ds + \int_{t}^{\infty} \bar{P}(s)w(s-\tau)ds \tag{3.49}$$

for $t \geq T$. So w solves (3.21). Clearly, w is continuous on $[T - T_0, \infty)$.

It remains to show that $w(t)$ is positive for all $t \geq T - T_0$. Assume that there exists $t^* \geq T - T_0$ such that $w(t) > 0$ for $T - T_0 \leq t < t^*$ and $w(t^*) = 0$. Then $t^* \geq T$, and by (3.49) we get

$$0 = w(t^*) = R(t^*)w(t^*-r) + \int_{t^*-\tau+\delta}^{t^*} Q(s)w(s-\delta)ds + \int_{t^*}^{\infty} \bar{P}(s)w(s-\tau)ds,$$

which implies

$$R(t^*) = 0 \quad \text{and} \quad Q(t) \equiv 0 \text{ for } t \in [t^* - \tau + \delta, t^*]$$

and

$$\bar{P}(t)w(t-\tau) \equiv 0 \quad \text{for all} \quad t \geq t^*.$$

This contradicts (3.45) or (3.46). Therefore $w(t)$ is positive on $[T - T_0, \infty)$. ☐

We now compare (3.21) with the equation

$$\left(x(t) - R^*(t)x(t-r)\right)' + P^*(t)x(t-\tau) - Q^*(t)x(t-\delta) = 0 \tag{3.50}$$

to obtain the following comparison theorem.

Theorem 3.4.2. *Assume that* (3.22), (3.23), *and* (3.24) *hold and that* (3.23) *as well as* (3.45) *or* (3.46) *hold for* P^* *and* Q^*. *Further assume that*

$$R(t) \geq R^*(t), \quad \bar{P}(t) \geq \bar{P}^*(t), \quad \text{and} \quad Q(t) \geq Q^*(t). \tag{3.51}$$

If every solution of (3.50) *oscillates, then every solution of* (3.21) *oscillates as well.*

Proof. Suppose the contrary and let x be an eventually positive solution of (3.21). Set y as in (3.25). Then by Lemma 3.3.1 we have eventually

$$y'(t) \leq 0 \quad \text{and} \quad y(t) > 0.$$

Thus, by integrating (3.21) from t to $T > t$, we obtain

$$y(t) = y(T) + \int_t^T \bar{P}(s)x(s-\tau)ds \geq \int_t^T \bar{P}(s)x(s-\tau)ds.$$

Therefore $y(t) \geq \int_t^\infty \bar{P}(s)x(s-\tau)ds$. Using (3.51), we obtain

$$\begin{aligned} x(t) &= y(t) + R(t)x(t-r) + \int_{t-\tau+\delta}^t Q(s)x(s-\delta)ds \\ &\geq R^*(t)x(t-r) + \int_{t-\tau+\delta}^t Q^*(s)x(s-\delta)ds + \int_t^\infty \bar{P}^*(s)x(s-\tau)ds. \end{aligned}$$

Note that (3.51) implies that (3.24) also holds for R^* and Q^*. Therefore, by Theorem 3.4.1, (3.50) has also an eventually positive solution. This is a contradiction and the proof is complete. ☐

In what follows, we will derive some sufficient conditions for the oscillation of all solutions of (3.21). The following lemma plays an important rôle.

Lemma 3.4.3. *Assume that* (3.22) *and* (3.23) *hold and that*

$$R(t) + \int_{t-\tau+\delta}^t Q(s)ds \geq 1 \tag{3.52}$$

and

$$\int_{t_0}^\infty \bar{P}(s) \exp\left(\frac{1}{r^*}\int_{t_0}^s u\bar{P}(u)du\right) ds = \infty, \tag{3.53}$$

where $r^* = \min\{\delta, r\} > 0$. *Let* x *be an eventually positive solution of inequality* (3.47), *and define* y *by* (3.25). *Then eventually*

$$y'(t) \leq 0 \quad \text{and} \quad y(t) < 0. \tag{3.54}$$

Proof. From (3.22) and (3.23), we have

$$y'(t) \le -\bar{P}(t)x(t-\tau) \le 0. \tag{3.55}$$

Therefore, if (3.54) does not hold, then eventually $y(t) > 0$. That is,

$$x(t) \ge R(t)x(t-r) + \int_{t-\tau+\delta}^{t} Q(s)x(s-\delta)ds. \tag{3.56}$$

Let $t_1 \ge t_0$ be such that $x(t-r) > 0$, $x(t-\tau) > 0$ for $t \ge t_1$ and such that (3.56) holds for $t \ge t_1$. Define

$$m = \min\Big\{x(t):\ t \in [t_1 - \tau^*, t_1]\Big\}, \qquad \text{where} \qquad \tau^* = \max\{r, \tau\}.$$

Then for $t \in [t_1, t_1 + r^*]$, we have by (3.56) and (3.52)

$$x(t) > m\left[R(t) + \int_{t-\tau+\delta}^{t} Q(s)ds\right] \ge m.$$

Thus, by induction, we can show that

$$x(t) > m \qquad \text{for} \qquad t \in [t_1 + (n-1)r^*, t_1 + nr^*], \quad n \in \mathbb{N},$$

and so

$$x(t) > m \qquad \text{for} \qquad t \ge t_1 - \tau^*.$$

For convenience, we denote

$$N(t) = \left[\frac{t-t_1}{r^*}\right].$$

Since $r^* = \min\{\delta, r\} > 0$ and y is nonincreasing, from (3.25) and (3.52) we have

$$\begin{aligned}
x(t) &= y(t) + R(t)x(t-r) + \int_{t-\tau+\delta}^{t} Q(s)x(s-\delta)ds \\
&= y(t) + R(t)\left[y(t-r) + R(t-r)x(t-2r) + \int_{t-\tau+\delta-r}^{t-r} Q(s)x(s-\delta)ds\right] \\
&\quad + \int_{t-\tau+\delta}^{t} Q(s)\left[y(s-\delta) + R(s-\delta)x(s-r-\delta) + \int_{s-\tau}^{s-\delta} Q(u)x(u-\delta)du\right] ds \\
&\ge y(t) + \left[R(t) + \int_{t-\tau+\delta}^{t} Q(s)ds\right] y(t-r^*) \\
&\quad + R(t)\left[R(t-r)x(t-2r) + \int_{t-\tau+\delta-r}^{t-r} Q(s)x(s-\delta)ds\right] \\
&\quad + \int_{t-\tau+\delta}^{t} Q(s)\left[R(s-\delta)x(s-r-\delta) + \int_{s-\tau}^{s-\delta} Q(u)x(u-\delta)du\right] ds \\
&\ge y(t) + y(t-r^*) + R(t)\left[R(t-r)x(t-2r) + \int_{t-\tau+\delta-r}^{t-r} Q(s)x(s-\delta)ds\right] \\
&\quad + \int_{t-\tau+\delta}^{t} Q(s)\left[R(s-\delta)x(s-r-\delta) + \int_{s-\tau}^{s-\delta} Q(u)x(u-\delta)du\right] ds \\
&= y(t) + y(t-r^*) + R(t)\Big\{R(t-r)\Big[y(t-2r) + R(t-2r)x(t-3r) \\
&\quad + \int_{t-\tau+\delta-2r}^{t-2r} Q(s)x(s-\delta)ds\Big]
\end{aligned}$$

$$+\int_{t-\tau+\delta-r}^{t-r} Q(s)\Big[y(s-\delta)+R(s-\delta)x(s-r-\delta)$$

$$+\int_{s-\tau}^{s-\delta} Q(u)x(u-\delta)du\Big]\,ds\Big\}+\int_{t-\tau+\delta}^{t} Q(s)\Big\{R(s-\delta)\Big[y(s-r-\delta)$$

$$+R(s-r-\delta)x(s-2r-\delta)+\int_{s-r-\tau}^{s-r-\delta} Q(u)x(u-\delta)du\Big]$$

$$+\int_{s-\tau}^{s-\delta} Q(u)\Big[y(u-\delta)+R(u-\delta)x(u-r-\delta)$$

$$+\int_{u-\tau}^{u-\delta} Q(v)x(v-\delta)dv\Big]\,du\Big\}\,ds.$$

In general, by induction, we can show that

$$x(t)\geq y(t)+y(t-r^*)+\ldots+y\Big(t-(N(t)-1)r^*\Big)+x\Big(t-N(t)r^*\Big) \tag{3.57}$$

for $t\geq t_1$. Noting that $y(t)$ is nonincreasing and $x(t-N(t)r^*)\geq m$ for $t\geq t_1$, by (3.57) we obtain that

$$x(t)\geq N(t)y(t)+m,\quad t\geq t_1.$$

Substituting this into (3.55), we find

$$y'(t)+\bar{P}(t)\Big[N(t-\tau)y(t-\tau)+m\Big]\leq 0,\quad t\geq t_1+\tau=t_2,$$

and hence

$$y'(t)+\bar{P}(t)N(t-\tau)y(t)+\bar{P}(t)m\leq 0,\quad t\geq t_2.$$

Then

$$0\geq\left[y(t)\exp\left(\int_{t_2}^{t}\bar{P}(s)N(s-\tau)ds\right)\right]'+m\bar{P}(t)\exp\left(\int_{t_2}^{t}\bar{P}(s)N(s-\tau)ds\right)$$

for $t\geq t_2$. Integrating this inequality from t_2 to $t\geq t_2$, we have

$$\begin{aligned}0\geq y(t)\exp\left(\int_{t_2}^{t}\bar{P}(s)N(s-\tau)ds\right)&-y(t_2)\\&+m\int_{t_2}^{t}\bar{P}(s)\exp\left(\int_{t_2}^{s}\bar{P}(u)N(u-\tau)du\right)ds\end{aligned} \tag{3.58}$$

for $t\geq t_2$. If the condition

$$\int_{t_0}^{\infty}\bar{P}(s)ds=\infty$$

is satisfied, then it is easy to see from (3.55) and the fact that $x(t)\geq m$ for $t\geq t_1-\tau^*$, that $\lim_{t\to\infty}y(t)=-\infty$, which is a contradiction. Hence, we assume that

$$\int_{t_0}^{\infty}\bar{P}(s)ds<\infty.$$

Because $N(t-\tau)/t \to 1/r^*$ holds as $t \to \infty$, it is easy to see that $\int_{t_2}^{\infty} \bar{P}(s)\left[\frac{s}{r^*} - N(s-\tau)\right] ds$ is absolutely convergent and

$$\lim_{s\to\infty} \frac{\exp\left(\int_{t_2}^{s} \bar{P}(u)N(u-\tau)du\right)}{\exp\left(\frac{1}{r^*}\int_{t_2}^{s} u\bar{P}(u)du\right)} \quad \text{exists.}$$

By condition (3.53), we obtain

$$\int_{t_2}^{\infty} \bar{P}(s)\exp\left(\int_{t_2}^{s} \bar{P}(u)N(u-\tau)du\right) ds = \infty.$$

Letting $t \to \infty$ in (3.58), we obtain a contradiction. □

As an immediate consequence of Lemmas 3.3.1 and 3.4.3, we obtain the following result.

Theorem 3.4.4. *Assume that* (3.22), (3.23), *and* (3.53) *hold and that*

$$R(t) + \int_{t-\tau+\delta}^{t} Q(s)ds \equiv 1.$$

Then every solution of (3.21) *oscillates.*

Example 3.4.5. Consider the neutral delay differential equation

$$\Big(x(t) - (1-\alpha)x(t-1)\Big)' + \left(\alpha + t^{-\beta}\right)x(t-2) - \alpha x(t-1) = 0, \quad t > 0,$$

where $0 < \alpha < 1$ and $1 < \beta < 2$. This equation satisfies all conditions given in Theorem 3.4.4. Hence all solutions of the equation oscillate.

Theorem 3.4.6. *Assume that* (3.22), (3.23), (3.24), *and* (3.53) *hold, and that*

$$R(t-\tau)\bar{P}(t) \geq \bar{P}(t-r). \tag{3.59}$$

Then every solution of (3.21) *oscillates.*

Proof. By way of contradiction, we assume that the conclusion is false. Then (3.21) would have an eventually positive solution x. Let y be defined by (3.25). Then by Lemma 3.3.1 we have eventually

$$y'(t) \leq 0 \quad \text{and} \quad y(t) > 0.$$

From (3.21), (3.24), and (3.59), we have

$$\begin{aligned} y'(t) &= -\bar{P}(t)x(t-\tau) \\ &= -\bar{P}(t)\left[y(t-\tau) + R(t-\tau)x(t-r-\tau) + \int_{t-\tau+\delta}^{t} Q(s-\tau)x(s-\tau-\delta)ds\right] \\ &\leq -\bar{P}(t)y(t-\tau) + y'(t-r), \end{aligned}$$

i.e.,

$$y'(t) - y'(t-r) + \bar{P}(t)y(t-\tau) \leq 0,$$

which, in view of Theorem 3.4.1, implies that the equation

$$y'(t) - y'(t-r) + \bar{P}(t)y(t-\tau) = 0 \tag{3.60}$$

has an eventually positive solution. But, on the other hand, by Theorem 3.4.4 we see that (3.53) implies that (3.60) cannot have an eventually positive solution. This is a contradiction and the proof is complete. □

Theorem 3.4.7. *Assume that* (3.22), (3.23), (3.52), *and* (3.53) *hold and that*

$$R(t-\tau)\bar{P}(t) \le h_1 \bar{P}(t-r), \tag{3.61}$$

and also suppose that $1/\bar{P}$ *is nondecreasing and satisfies*

$$\bar{P}(t)Q(t-\tau) \le h_2 \bar{P}(t-\delta), \tag{3.62}$$

where h_1, h_2 *are nonnegative constants satisfying*

$$h_1 + h_2(\tau - \delta) = 1.$$

Then every solution of (3.21) *oscillates.*

Proof. If the above conclusion does not hold, then (3.21) has an eventually positive solution x. Let y be defined by (3.25). From Lemma 3.4.3 we have $y(t) < 0$. By (3.61) and (3.62), we get

$$\begin{aligned}
y'(t) &= -\bar{P}(t)x(t-\tau) \\
&= -\bar{P}(t)\left[y(t-\tau) + R(t-\tau)x(t-r-\tau) + \int_{t-\tau+\delta}^{t} Q(s-\tau)x(s-\delta-\tau)ds\right] \\
&\ge -\bar{P}(t)y(t-\tau) - h_1\bar{P}(t-r)x(t-r-\tau) \\
&\quad -\bar{P}(t)\int_{t-\tau+\delta}^{t} \frac{Q(s-\tau)}{\bar{P}(s-\delta)}\left[-y'(s-\delta)\right] ds \\
&\ge -\bar{P}(t)y(t-\tau) + h_1 y'(t-r) + h_2\int_{t-\tau+\delta}^{t} y'(s-\delta)ds \\
&= -\left[\bar{P}(t) + h_2\right] y(t-\tau) + h_2 y(t-\delta) + h_1 y'(t-r),
\end{aligned}$$

i.e.,

$$\Big(y(t) - h_1 y(t-r)\Big)' + \left[\bar{P}(t) + h_2\right] y(t-\tau) - h_2 y(t-\delta) \ge 0$$

so that $-y$ is a positive solution of the inequality

$$\Big(z(t) - h_1 z(t-r)\Big)' + \left[\bar{P}(t) + h_2\right] z(t-\tau) - h_2 z(t-\delta) \le 0.$$

This yields a contradiction by Lemma 3.4.3. □

Example 3.4.8. If we take $h_1 = 1/4$ and $h_2 = 1/2$, then the equation

$$\left(x(t) - \frac{t+2}{2(t+1)}x(t-1)\right)' + \left(\frac{1}{2} + t^{-\beta}\right)x(t-2) - \frac{1}{2}x(t-1) = 0,$$

where $1 < \beta < 2$, satisfies all the assumptions of Theorem 3.4.7. Hence all solutions of this equation oscillate.

In the next theorem, we set

$$R_0(t) = \frac{R(t-\tau)\bar{P}(t)}{\bar{P}(t-r)}$$

for $\bar{P}(t) > 0$.

Theorem 3.4.9. *Assume that* (3.22), (3.23), *and* (3.24) *hold and that*

$$\bar{P}(t) > 0 \quad \text{and} \quad 0 < r_0 \le R_0(t) < 1. \tag{3.63}$$

Then either

$$r_0 \limsup_{t\to\infty} \int_{t-\tau+\delta}^{t} \bar{P}(s)ds > 1 - r_0 \tag{3.64}$$

or

$$r_0 \liminf_{t\to\infty} \int_{t-\tau+\delta}^{t} \bar{P}(s)ds > \frac{1-r_0}{e} \tag{3.65}$$

implies that every solution of (3.21) *oscillates.*

Proof. Assume, for the sake of contradiction, that (3.21) has an eventually positive solution x. Define y by (3.25). Then by Lemma 3.3.1, (3.26) holds. Also, y satisfies the equation

$$\begin{aligned} 0 &= y'(t) - R_0(t)y'(t-r) + \bar{P}(t)y(t-\tau) + \bar{P}(t)\int_{t-\tau+\delta}^{t} Q(s-\tau)x(s-\tau-\delta)ds \\ &\geq y'(t) - r_0 y'(t-r) + \bar{P}(t)y(t-\tau), \end{aligned}$$

where we also have used (3.63). In view of Theorem 3.4.1, the corresponding differential equation

$$y'(t) - r_0 y'(t-r) + \bar{P}(t)y(t-\tau) = 0$$

has an eventually positive solution y. Set $w(t) = y(t) - r_0 y(t-r)$. Then $w'(t) \leq 0$ and $w(t) > 0$ by Lemma 3.3.1. Thus, for any $n \in \mathbb{N}$, we have

$$\begin{aligned} y(t) &= w(t) + r_0 w(t-r) + \ldots + r_0^n w(t-nr) + r_0^n y\Big(t-(n+1)r\Big) \\ &> \Big(r_0 + r_0^2 + \ldots + r_0^n\Big)\, w(t-r) \;=\; \frac{r_0(1-r_0^n)}{1-r_0} w(t-r) \end{aligned}$$

and so

$$w'(t) + \frac{r_0(1-r_0^n)}{1-r_0}\bar{P}(t)w(t-r-\tau) \leq 0. \tag{3.66}$$

Since $0 < r_0 < 1$, it follows for sufficiently large $n \in \mathbb{N}$ that (3.64) or (3.65) implies respectively

$$\frac{r_0(1-r_0^n)}{1-r_0} \limsup_{t\to\infty} \int_{t-\tau+\delta}^{t} \bar{P}(s)ds > 1 \tag{3.67}$$

or

$$\frac{r_0(1-r_0^n)}{1-r_0} \liminf_{t\to\infty} \int_{t-\tau+\delta}^{t} \bar{P}(s)ds > \frac{1}{e}. \tag{3.68}$$

It is well known that (3.67) or (3.68) implies that (3.66) has no eventually positive solution (see e.g., [**118**, Theorem 2.3.3 on page 46]). This is a contradiction and so the proof is complete. ∎

3.5. Existence of Nonoscillating Solutions

The purpose of this section is to study the existence of nonoscillatory solution of the neutral delay differential equation (3.21) under conditions (3.22) and (3.23).

Theorem 3.5.1. *Assume that* (3.22) *and* (3.23) *hold and that*

$$0 \le P(t) \le M, \qquad \text{where } M \text{ is a constant,}$$

and that there exists a positive number α *such that*

$$0 \le R(t)e^{\alpha r} \le c < 1 \tag{3.69}$$

and

$$\begin{aligned} 1 \ \ge\ & R(t)e^{\alpha r} + e^{\alpha\delta}\int_{t-\tau+\delta}^{t} Q(s)e^{-\alpha(s-t)}ds \\ & + \int_t^\infty \Big(P(s) - Q(s-\tau+\delta)\Big)\, e^{-\alpha(s-t)}e^{\alpha\tau}ds. \end{aligned} \tag{3.70}$$

Then (3.21) *has an eventually positive solution which approaches zero exponentially.*

Proof. Let (3.69) and (3.70) hold for $t \ge t_0$. Let $BC = BC[t_0 - T_0, \infty)$ denote the Banach space of all bounded and continuous real-valued functions defined on $[t_0 - T_0, \infty)$, and the norm in BC is the sup norm. Let Ω be the subset of BC defined by

$$\Omega = \Big\{ y \in BC : \ 0 \le y(t) \le 1 \text{ for } t \ge t_0 - T_0 \Big\}.$$

Now we define operators F_1 and F_2 on Ω as

$$(F_1y)(t) = \begin{cases} R(t)e^{\alpha r}y(t-r) & \text{if} \quad t \ge t_0, \\ \dfrac{t}{t_0}(F_1y)(t_0) + \left(1 - \dfrac{t}{t_0}\right) & \text{if} \quad t_0 - T_0 \le t < t_0 \end{cases}$$

and

$$(F_2y)(t) = \begin{cases} \displaystyle\int_{t-\tau+\delta}^{t} Q(s)e^{\alpha\delta}e^{-\alpha(s-t)}y(s-\delta)ds + \int_t^\infty e^{\alpha\tau}\bar{P}(s)e^{-\alpha(s-t)}y(s-\tau)ds \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if} \quad t \ge t_0, \\ \dfrac{t}{t_0}(F_2y)(t_0) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \text{if} \quad t_0 - T_0 \le t < t_0. \end{cases}$$

In view of (3.69) and (3.70) for every $y_1, y_2 \in \Omega$ we have

$$F_1y + F_2y \in \Omega,$$

and F_1 is a contraction on Ω. Since $P(t) - Q(t-\tau+\delta) \ge 0$ and $P(t)$ is bounded, $Q(t)$ is bounded. It is easy to see that

$$\left| \frac{d}{dt}(F_2y)(t) \right| \le N$$

for some positive number N. Hence F_2 is completely continuous on Ω. By the Krasnosel′skiĭ fixed point theorem (Theorem 1.4.27), there exists $y \in \Omega$ such that

$$F_1y + F_2y = y.$$

That is, for $t \ge t_0$,

$$\begin{aligned} y(t) = R(t)e^{\alpha r}y(t-r) + \int_{t-\tau+\delta}^{t} & Q(s)e^{\alpha\delta}e^{-\alpha(s-t)}y(s-\delta)ds \\ & + \int_t^\infty e^{\alpha\tau}\bar{P}(s)e^{-\alpha(s-t)}y(s-\tau)ds, \end{aligned}$$

and for $t_0 - T_0 \le t < t_0$,

$$y(t) = \frac{t}{t_0} y(t_0) + \left(1 - \frac{t}{t_0}\right) > 0.$$

It follows that $y(t) > 0$ for all $t \ge t_0 - T_0$. Set

$$x(t) = y(t) e^{-\alpha t}.$$

Then

$$x(t) = R(t)x(t-r) + \int_{t-\tau+\delta}^{t} Q(s)x(s-\delta)ds + \int_t^\infty \bar{P}(s)x(s-\tau)ds.$$

Consequently,

$$\Big(x(t) - R(t)x(t-r)\Big)' + P(t)x(t-\tau) - Q(t)x(t-\delta) = 0, \quad t \ge t_0,$$

i.e., x is a positive solution of (3.21) and $x(t)$ approaches zero exponentially. The proof is complete. □

Remark 3.5.2. When $R(t) \equiv R$, $P(t) \equiv P$, $Q(t) \equiv Q$ are nonnegative constants, $P > Q$, $\tau \ge \delta \ge 0$, $r > 0$, $R + Q(\tau - \delta) \le 1$, then condition (3.70) becomes

$$Re^{\alpha r} + \frac{P}{\alpha} e^{\alpha\tau} - \frac{Q}{\alpha} e^{\alpha\delta} \le 1$$

for some $\alpha > 0$. This condition provides not only a sufficient condition but also a necessary condition for the existence of a nonoscillatory solution of (3.21) (see Theorem 3.3.3 in Section 3.3).

Theorem 3.5.3. *Assume that*

$$R \in C^1([t_0,\infty),\mathbb{R}),\ P, Q \in C([t_0,\infty),\mathbb{R}),\ r,\tau,\delta \in [0,\infty),$$

and that there exists a positive number μ such that

$$\mu|R(t)|e^{\mu r} + |R'(t)|e^{\mu r} + |P(t)|e^{\mu\tau} + |Q(t)|e^{\mu\delta} \le \mu, \quad t \ge t_0. \tag{3.71}$$

Then (3.21) *has a positive solution on $t \ge t_0$.*

Proof. Set

$$P_1(t) = \begin{cases} R(t) & \text{if} \quad t \ge t_0, \\ \dfrac{t - t_0 + r}{r} R(t_0) & \text{if} \quad t_0 - r \le t < t_0, \\ 0 & \text{if} \quad t_0 - T_0 - r \le t < t_0 - r, \end{cases}$$

$$P_2(t) = \begin{cases} R'(t) & \text{if} \quad t \ge t_0, \\ \dfrac{t - t_0 + r}{r} R'(t_0) & \text{if} \quad t_0 - r \le t < t_0, \\ 0 & \text{if} \quad t_0 - T_0 - r \le t < t_0 - r, \end{cases}$$

$$P_3(t) = \begin{cases} P(t) & \text{if} \quad t \ge t_0, \\ \dfrac{t - t_0 + r}{r} P(t_0) & \text{if} \quad t_0 - r \le t < t_0, \\ 0 & \text{if} \quad t_0 - T_0 - r \le t < t_0 - r, \end{cases}$$

and

$$P_4(t) = \begin{cases} Q(t) & \text{if} \quad t \geq t_0, \\ \dfrac{t - t_0 + r}{r} Q(t_0) & \text{if} \quad t_0 - r \leq t < t_0, \\ 0 & \text{if} \quad t_0 - T_0 - r \leq t < t_0 - r. \end{cases}$$

Then P_i for $i \in \{1, 2, 3, 4\}$ are continuous on $[t_0 - T_0 - r, \infty)$. From (3.71) we have

$$\mu |P_1(t)| e^{\mu r} + |P_2(t)| e^{\mu r} + |P_3(t)| e^{\mu \tau} + |P_4(t)| e^{\mu \delta} \leq \mu, \quad t \geq t_0 - T_0 - r.$$

We introduce the Banach space $BC[t_0 - T_0 - r, \infty)$ of all bounded continuous functions $x : [t_0 - T_0 - r, \infty) \to \mathbb{R}$ with norm

$$\|x\| = \sup_{t \geq t_0 - T_0 - r} |x(t)| e^{-\eta t},$$

where $\eta > 0$ satisfies the inequality

$$|P_1(t)| e^{\mu r} \left(e^{-\eta r} + \frac{\mu}{\eta} \right) + \frac{e^{\mu r}}{\eta} |P_2(t)| + \frac{e^{\mu \tau}}{\eta} |P_3(t)| + \frac{e^{\mu \delta}}{\eta} |P_4(t)| \leq \frac{1}{2}$$

for $t \geq t_0 - r$. We consider the subset Ω of BC as

$$\Omega = \left\{ \lambda \in BC : \ |\lambda(t)| \leq \mu \text{ for } t \geq t_0 - T_0 - r \right\}.$$

Clearly, Ω is a bounded, closed, and convex subset of BC. Define the operator F on Ω as

$$(F\lambda)(t) = \begin{cases} \lambda(t - r) P_1(t) \exp\left(\displaystyle\int_{t-r}^{t} \lambda(s) ds \right) + P_2(t) \exp\left(\displaystyle\int_{t-r}^{t} \lambda(s) ds \right) \\ \qquad + P_3(t) \exp\left(\displaystyle\int_{t-\tau}^{t} \lambda(s) ds \right) - P_4(t) \exp\left(\displaystyle\int_{t-\delta}^{t} \lambda(s) ds \right) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if} \quad t \geq t_0 - r, \\ 0 \qquad\qquad\qquad\qquad\qquad\qquad\quad \text{if} \quad t_0 - T_0 - r \leq t < t_0 - r. \end{cases}$$

In view of (3.71), we have for $t \geq t_0 - r$

$$|F\lambda(t)| \leq \mu |P_1(t)| e^{\mu r} + |P_2(t)| e^{\mu r} + |P_3(t)| e^{\mu \tau} + |P_4(t)| e^{\mu \delta} \leq \mu,$$

which shows that F maps Ω into itself. Next, we show that F is a contraction on Ω. In fact, for any $\lambda_1, \lambda_2 \in \Omega$ and $t \geq t_0 - r$,

$$\begin{aligned}
&\left|(F\lambda_1)(t) - (F\lambda_2)(t)\right| \\
\leq\ & |P_1(t)| \left|\lambda_1(t-r)\exp\left(\int_{t-r}^{t}\lambda_1(s)ds\right) - \lambda_2(t-r)\exp\left(\int_{t-r}^{t}\lambda_2(s)ds\right)\right| \\
& + |P_2(t)| \left|\exp\left(\int_{t-r}^{t}\lambda_1(s)ds\right) - \exp\left(\int_{t-r}^{t}\lambda_2(s)ds\right)\right| \\
& + |P_3(t)| \left|\exp\left(\int_{t-\tau}^{t}\lambda_1(s)ds\right) - \exp\left(\int_{t-\tau}^{t}\lambda_2(s)ds\right)\right| \\
& + |P_4(t)| \left|\exp\left(\int_{t-\delta}^{t}\lambda_1(s)ds\right) - \exp\left(\int_{t-\delta}^{t}\lambda_2(s)ds\right)\right| \\
\leq\ & |P_1(t)| \left[e^{\mu r}\left|\lambda_1(t-r) - \lambda_2(t-r)\right| + \mu e^{\mu r}\int_{t-r}^{t}\left|\lambda_1(s) - \lambda_2(s)\right| ds\right] \\
& + e^{\mu r}\left|P_2(t)\right| \int_{t-r}^{t}\left|\lambda_1(s) - \lambda_2(s)\right| ds + e^{\mu\tau}\left|P_3(t)\right| \int_{t-\tau}^{t}\left|\lambda_1(s) - \lambda_2(s)\right| ds \\
& + e^{\mu\delta}\left|P_4(t)\right| \int_{t-\delta}^{t}\left|\lambda_1(s) - \lambda_2(s)\right| ds \\
\leq\ & \left\|\lambda_1 - \lambda_2\right\| \left\{e^{\mu r}\left|P_1(t)\right| e^{\eta t}\left[e^{-\eta r} + \frac{\mu}{\eta}(1 - e^{-\eta r})\right] + \frac{e^{\mu r}}{\eta}\left|P_2(t)\right|\left(1 - e^{-\eta r}\right)e^{\eta t}\right. \\
& \left. + \frac{1}{\eta}\left|P_3(t)\right| e^{\mu\tau}\left(1 - e^{-\eta\tau}\right)e^{\eta t} + \frac{1}{\eta}e^{\mu\delta}\left|P_4(t)\right|\left(1 - e^{-\eta\delta}\right)e^{\eta t}\right\} \\
\leq\ & \left\|\lambda_1 - \lambda_2\right\| e^{\eta t}\left\{e^{\mu r}|P_1(t)|\left(e^{-\eta r} + \frac{\mu}{\eta}\right) + \frac{1}{\eta}e^{\mu r}|P_2(t)|\right. \\
& \left. + \frac{1}{\eta}|P_3(t)|e^{\mu\tau} + \frac{1}{\eta}e^{\mu\delta}|P_4(t)|\right\} \\
\leq\ & \frac{1}{2}e^{\eta t}\left\|\lambda_1 - \lambda_2\right\|.
\end{aligned}$$

Hence,

$$\begin{aligned}
\left\|F\lambda_1 - F\lambda_2\right\| &= \sup_{t \geq t_0 - T_0 - r}\left|(F\lambda_1)(t) - (F\lambda_2)(t)\right| e^{-\eta t} \\
&= \sup_{t \geq t_0 - r}\left|(F\lambda_1)(t) - (F\lambda_2)(t)\right| e^{-\eta t} \\
&\leq \frac{1}{2}\left\|\lambda_1 - \lambda_2\right\|,
\end{aligned}$$

i.e., F is a contraction on Ω. Therefore (see Theorem 1.4.26) there exists $\lambda \in \Omega$ such that $F\lambda = \lambda$. That is,

$$\lambda(t) = \begin{cases} \lambda(t-r)P_1(t)\exp\left(\int_{t-r}^{t}\lambda(s)ds\right) + P_2(t)\exp\left(\int_{t-r}^{t}\lambda(s)ds\right) \\ \qquad +P_3(t)\exp\left(\int_{t-\tau}^{t}\lambda(s)ds\right) - P_4(t)\exp\left(\int_{t-\delta}^{t}\lambda(s)ds\right) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if} \quad t \geq t_0 - r, \\ 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\;\; \text{if} \quad t_0 - T_0 - r \leq t < t_0 - r. \end{cases}$$

According to the definition of P_i, we have

$$\lambda(t) = \lambda(t-r)R(t)\exp\left(\int_{t-r}^{t}\lambda(s)ds\right) + R'(t)\exp\left(\int_{t-r}^{t}\lambda(s)ds\right) \\ + P(t)\exp\left(\int_{t-\tau}^{t}\lambda(s)ds\right) - Q(t)\exp\left(\int_{t-\delta}^{t}\lambda(s)ds\right)$$

for $t \geq t_0$. Set

$$x(t) = \exp\left(-\int_{t_0}^{t}\lambda(s)ds\right).$$

Then it is easy to see that x is a positive solution of (3.21) on $[t_0, \infty)$. The proof is complete. ☐

3.6. Classification Schemes of Positive Solutions

This section is concerned with the first order neutral differential equation

$$\Big(x(t) - P(t)x(t-\tau)\Big)' + Q(t)x(t-\sigma) = 0, \quad t \geq t_0, \tag{3.72}$$

where $P \in C([t_0, \infty), \mathbb{R})$, $Q \in C([t_0, \infty), \mathbb{R}^+)$, $\tau > 0$, and $\sigma \geq 0$.

When $P(t) \leq 0$ or $0 \leq P(t) \leq 1$, (3.72) has been discussed in the monographs [**29, 92, 118**]. Therefore, in this section we consider (3.72) with $P(t) \geq 1$. When $P(t) \geq 1$ and is nondecreasing, it is easy to see that there exists a nondecreasing function $r \in C([t_0 - \tau, \infty), (0, \infty))$ such that

$$P(t) = \frac{r(t)}{r(t-\tau)} \quad \text{for} \quad t \geq t_0.$$

In what follows with respect to (3.72), we shall assume that

$$R(t) = \int_{t_0}^{t}\frac{ds}{r(s)} \to \infty \quad \text{as} \quad t \to \infty.$$

First, we shall provide some comparison theorems for oscillation and nonoscillation of solutions of (3.72), and show that all solutions of (3.72) oscillate if there exists an oscillatory solution.

Lemma 3.6.1. *Let x be an eventually positive solution of the differential inequality*

$$\left(x(t) - \frac{r(t)}{r(t-\tau)}x(t-\tau)\right)' + Q(t)x(t-\sigma) \leq 0, \quad t \geq t_0$$

and set

$$z(t) = x(t) - \frac{r(t)}{r(t-\tau)}x(t-\tau). \tag{3.73}$$

Then eventually

$$z'(t) \le 0 \quad \text{and} \quad z(t) \ge 0. \tag{3.74}$$

Proof. Since r is nondecreasing and $\int_{t_0}^{\infty} [r(s)]^{-1}\, ds = \infty$, we have

$$\sum_{k=1}^{\infty} \frac{1}{r(t_0 + k\tau)} \ge \frac{1}{\tau} \int_{t_0+\tau}^{\infty} \frac{ds}{r(s)} = \infty.$$

The conclusion (3.74) now follows by [**251**, Lemma 2]. □

Lemma 3.6.2 ([**92**]). *If the integral inequality*

$$y(t) \ge \frac{r(t)}{r(t-\tau)}y(t-\tau) + \int_t^{\infty} Q(s)y(s-\sigma)ds, \quad t \ge T > t_0$$

has a continuous positive solution $y : [T-\rho, \infty) \to (0,\infty)$, then the corresponding integral equation

$$x(t) = \frac{r(t)}{r(t-\tau)}x(t-\tau) + \int_t^{\infty} Q(s)x(s-\sigma)ds, \quad t \ge T \tag{3.75}$$

also has a continuous positive solution $x : [T-\rho, \infty) \to (0,\infty)$ with $0 < x(t) \le y(t)$ for $t \ge T$, where $\rho = \max\{\tau, \sigma\}$.

Lemma 3.6.3. *Assume that $p \in C([t_0,\infty), \mathbb{R}^+)$. If the differential inequality*

$$(r(t)y')' + p(t)y \le 0, \quad t \ge T^* > t_0$$

has a continuous positive solution $y : [T^*, \infty) \to (0,\infty)$, then the corresponding differential equation

$$(r(t)y')' + p(t)y = 0, \quad t \ge T^* > t_0$$

also has a continuous positive solution $x : [T^*, \infty) \to (0,\infty)$.

Proof. The proof is easy and will be omitted. □

Lemma 3.6.4. *Assume that $v \in C([T-\tau,\infty), \mathbb{R}^+)$, $u \in ([T,\infty), \mathbb{R}^+)$, and that u is nonincreasing on $[T,\infty)$. Then the following hold:*

(i) *If $v(t) - [r(t)/r(t-\tau)]\, v(t-\tau) \ge u(t)$ for $t \ge T$, then*

$$v(t) \ge \frac{r(t)}{\tau}\left(\int_{T+2\tau}^{t} \frac{u(s)}{r(s)}ds + \tau m\right) \quad \text{for} \quad t \ge T + 2\tau;$$

(ii) *if $v(t) - [r(t)/r(t-\tau)]\, v(t-\tau) \le u(t)$ for $t \ge T$, then*

$$v(t) \le \frac{r(t)}{\tau}\left(\int_{T}^{t} \frac{u(s)}{r(s)}ds + \tau M\right) \quad \text{for} \quad t \ge T,$$

where $m = \min\left\{\frac{v(t)}{r(t)} : T \le t \le T+\tau\right\}$ and $M = \max\left\{\frac{v(t)}{r(t)} : T \le t \le T+\tau\right\}$.

Proof. To show (i), set $v(t) = r(t)W(t)$ for $t \geq t - T$ and $n = \left[\frac{t-\tau}{\tau}\right]$ for $t \geq T$. Then we have

$$W(t) - W(t-\tau) \geq \frac{u(t)}{r(t)}, \quad t \geq T$$

and so

$$W(t) \geq \sum_{k=0}^{n-1} \frac{u(t-k\tau)}{r(t-k\tau)} + W(t-n\tau), \quad t \geq T.$$

In view of the nonincreasing nature of u/r and $T \leq t - n\tau \leq T + \tau$, we find

$$W(t) \geq \frac{1}{\tau} \sum_{k=0}^{n-2} \int_{t-(k+1)\tau}^{t-k\tau} \frac{u(s)}{r(s)} ds + m \geq \frac{1}{\tau} \int_{T+2\tau}^{t} \frac{u(s)}{r(s)} ds + m$$

for $t \geq T + 2\tau$. Thus it follows that

$$v(t) \geq \frac{r(t)}{\tau} \int_{T+2\tau}^{t} \frac{u(s)}{r(s)} ds + m, \quad t \geq T + 2\tau.$$

To show (ii), set $W(t)$ and n as in (i). Then we have

$$\begin{aligned} W(t) &\leq \sum_{k=0}^{n-1} \frac{u(t-k\tau)}{r(t-k\tau)} + W(t-n\tau) \\ &\leq \frac{1}{\tau} \sum_{k=0}^{n-1} \int_{t-(k+1)\tau}^{t-k\tau} \frac{u(s)}{r(s)} ds + M \\ &\leq \frac{1}{\tau} \int_{T}^{t} \frac{u(s)}{r(s)} ds + M \end{aligned}$$

for $t \geq T$. Thus it follows that

$$v(t) \leq \frac{r(t)}{\tau} \left(\int_{T}^{t} \frac{u(s)}{r(s)} ds + \tau M \right), \quad t \geq T.$$

The proof is complete. $\square$

Theorem 3.6.5. *Equation* (3.72) *has a positive solution if and only if the ordinary differential equation*

$$(r(t)y')' + \frac{1}{\tau} Q(t) r(t-\sigma) y = 0, \quad t \geq t_0 \tag{3.76}$$

has a positive solution.

Proof. Let x be a positive solution of (3.72) and define z as before in (3.73). By Lemma 3.6.1, there exists $t_1 > t_0$ such that

$$x(t-\rho) > 0, \quad z(t) \geq 0, \quad \text{and} \quad z'(t) \leq 0 \quad \text{for} \quad t \geq t_1,$$

where $\rho = \max\{\tau, \sigma\}$. This implies that z is nonincreasing on $[t_1, \infty)$. Set

$$m = \min\left\{ \frac{x(t)}{r(t)} : t_1 \leq t \leq t_1 + \tau \right\}, \quad t_2 = t_1 + 2\tau.$$

Then by Lemma 3.6.4 we have

$$x(t) \geq \frac{r(t)}{\tau} \left(\int_{t_2}^{t} \frac{z(s)}{r(s)} ds + \tau m \right), \quad t \geq t_2,$$

and so

$$x(t-\sigma) \geq \frac{r(t-\sigma)}{\tau}\left(\int_{t_2}^{t-\sigma}\frac{z(s)}{r(s)}ds+\tau m\right) \geq \frac{r(t-\sigma)}{\tau}\left(\int_{t_3}^{t-\sigma}\frac{z(s)}{r(s)}ds+\tau m\right) \tag{3.77}$$

for $t \geq t_3 = t_2 + \sigma$. Set

$$y(t) = \int_{t_3}^{t}\frac{z(s)}{r(s)}ds + \tau m, \quad t \geq t_3.$$

Then

$$y(t) > 0 \quad \text{and} \quad y'(t) = \frac{z(t)}{r(t)} \quad \text{for} \quad t \geq t_3. \tag{3.78}$$

Substituting this into (3.77), we obtain

$$x(t-\sigma) \geq \frac{r(t-\sigma)}{\tau}y(t), \quad t \geq t_3,$$

which, together with (3.72), (3.73), and (3.78), leads to

$$(ry')'(t) + \frac{1}{\tau}Q(t)r(t-\sigma)y(t) \leq 0, \quad t \geq t_3.$$

By Lemma 3.6.3, this implies that (3.76) has a positive solution.

Next, we assume that (3.76) has a positive solution y. It is easy to see that there exists $T \geq t_0$ such that

$$y(t) > 0, \quad y'(t) \geq 0, \quad \text{and} \quad (ry')'(t) \leq 0 \quad \text{for} \quad t \geq T. \tag{3.79}$$

In view of the nondecreasing nature of r and (3.79) it follows that y' is nonincreasing on $[T, \infty)$. Define a function v by

$$v(t) = \begin{cases} \dfrac{y(T) + (t-T)y'(T+\tau)}{\tau} & \text{if} \quad T \leq t \leq T+\tau, \\ y'(t) + v(t-\tau) & \text{if} \quad T + k\tau \leq t \leq T + (k+1)\tau, \ k \in \mathbb{N}. \end{cases}$$

It is easy to see that v is continuous and positive on $[0, \infty)$, and

$$v(t) \leq \frac{y(t)}{\tau}, \quad T \leq t \leq T+\tau, \tag{3.80}$$

$$v(t) = y'(t) + v(t-\tau), \quad t \geq T+\tau. \tag{3.81}$$

For $T + \tau \leq t \leq T + 2\tau$, we have by (3.80) and (3.81) that

$$v(t) = y'(t) + v(t-\tau) \leq \frac{1}{\tau}\Big(y(t) - y(t-\tau)\Big) + \frac{1}{\tau}y(t-\tau) = \frac{y(t)}{\tau}.$$

By induction, we can show in general that

$$v(t) \leq \frac{y(t)}{\tau}, \quad T + k\tau \leq t \leq T + (k+1)\tau, \quad k \in \mathbb{N},$$

and so

$$v(t) \leq \frac{y(t)}{\tau}, \quad t \geq T,$$

which, together with (3.79), yields

$$v(t-\sigma) \leq \frac{y(t-\sigma)}{\tau} \leq \frac{y(t)}{\tau}, \quad t \geq T+\sigma. \tag{3.82}$$

Substituting (3.81) and (3.82) into (3.77), we obtain

$$\left[r(t)\left(v(t)-v(t-\tau)\right)\right]' + Q(t)r(t-\sigma)v(t-\sigma) \le 0 \tag{3.83}$$

for $t \ge T+\sigma+\tau$. Set $\bar{x} = rv$. It follows from (3.83) that

$$\left[\bar{x}(t) - \frac{r(t)}{r(t-\tau)}\bar{x}(t-\tau)\right]' + Q(t)\bar{x}(t-\sigma) \le 0, \quad t \ge T+\sigma+\tau.$$

By Lemma 3.6.1, we have

$$\bar{x}(t) \ge \frac{r(t)}{r(t-\tau)}\bar{x}(t-\tau) + \int_t^\infty Q(s)\bar{x}(s-\sigma)ds, \quad t \ge T+\sigma+\tau.$$

By Lemma 3.6.2, this implies that the corresponding integral equation (3.75) has a positive solution x. Clearly, this x is a positive solution of (3.72). □

Next, we shall compare (3.72) with the equation

$$\left[x(t) - \frac{\bar{r}(t)}{\bar{r}(t-\tau)}x(t-\tau)\right]' + \bar{Q}(t)x(t-\sigma) = 0, \quad t \ge t_0, \tag{3.84}$$

where $\bar{r}$ and $\bar{Q}$ satisfy the same hypotheses as r and Q. By Hille–Wintner's comparison theorem (see [**255**]), the following result is immediate.

Theorem 3.6.6. *Suppose* $0 < \bar{r}(t) \le r(t)$ *and*

$$0 \le \int_t^\infty Q(s)r(s-\sigma)ds \le \int_t^\infty \bar{Q}(s)\bar{r}(s-\sigma)ds$$

hold for all $t \ge t_1 \ge t_0$. *If every solution of* (3.72) *oscillates, then every solution of* (3.84) *oscillates as well.*

In what follows, we shall show that all positive solutions of (3.72) can be classified into four types.

Definition 3.6.7. A positive solution of (3.72) is said to be of

(i) *A-type* if it can be expressed as

$$x(t) = r(t)\left(\alpha R(t) + \beta(t)\right),$$

where $\alpha > 0$ is a constant and $\beta : [t_x, \infty) \to \mathbb{R}$ is a bounded continuous function;

(ii) *B-type* if it can be expressed as

$$x(t) = r(t)\left(\alpha R(t) + \theta(t)\right),$$

where $\alpha > 0$ is a constant and $\theta : [t_x, \infty) \to (0,\infty)$ is an unbounded continuous function with $\lim_{t\to\infty} \frac{\theta(t)}{R(t)} = 0$;

(iii) *C-type* if x/r is unbounded and $\lim_{t\to\infty} \frac{x(t)}{r(t)R(t)} = 0$;

(iv) *D-type* if x/r is bounded.

Theorem 3.6.8. *If x is a positive solution of* (3.72), *then x is either A, B, C, or D-type.*

Proof. Suppose that x is a positive solution of (3.72). Define z as in (3.73). By Lemma 3.6.1, there exists $T > t_0$ such that

$$x(t-\rho) > 0, \quad z(t) \geq 0, \quad z'(t) \leq 0, \quad t \geq T. \tag{3.85}$$

Let $k = \lim_{t\to\infty} z(t)$. Clearly, $k \geq 0$. We claim that

$$\lim_{t\to\infty} \frac{x(t)}{r(t)R(t)} = \frac{k}{\tau}, \tag{3.86}$$

and if $\theta(t) = \frac{x(t)}{r(t)} - k\frac{R(t)}{\tau}$ is unbounded, then $\theta(t)$ is eventually positive. From (3.85) we have $z(t) \geq k$ for $t \geq T$. Let

$$m = \min_{t\in[T,T+\tau]} \frac{x(t)}{r(t)} \quad \text{and} \quad M = \max_{t\in[T,T+\tau]} \frac{x(t)}{r(t)}.$$

Then by Lemma 3.6.4, we have

$$\frac{r(t)}{\tau}\left(\int_{T+2\tau}^{t} \frac{z(s)}{r(s)}ds + \tau m\right) \leq x(t) \leq \frac{r(t)}{\tau}\left(\int_{T}^{t} \frac{z(s)}{r(s)}ds + \tau M\right) \tag{3.87}$$

for $t \geq T + 2\tau$. Since by L'Hôpital's rule

$$\lim_{t\to\infty} \frac{\frac{r(t)}{\tau}\left(\int_{T+2\tau}^{t} \frac{z(s)}{r(s)}ds + \tau m\right)}{r(t)R(t)} = \lim_{t\to\infty} \frac{\frac{z(t)}{\tau r(t)}}{\frac{1}{r(t)}} = \frac{k}{\tau}$$

and

$$\lim_{t\to\infty} \frac{\frac{r(t)}{\tau}\left(\int_{T}^{t} \frac{z(s)}{r(s)}ds + \tau M\right)}{r(t)R(t)} = \lim_{t\to\infty} \frac{\frac{z(t)}{\tau r(t)}}{\frac{1}{r(t)}} = \frac{k}{\tau},$$

it follows by (3.87) that (3.86) holds. From (3.87) we have

$$\begin{aligned}\frac{1}{\tau}\left(\int_{T+2\tau}^{t} \frac{z(s)-k}{r(s)}ds + \tau m - kR(T+2\tau)\right) &\leq \frac{x(t)}{r(t)} - \frac{kR(t)}{\tau} \\ &\leq \frac{1}{\tau}\left(\int_{T}^{t} \frac{z(s)-k}{r(s)}ds + \tau M - kR(T)\right).\end{aligned}$$

If $\theta(t) = \frac{x(t)}{r(t)} - k\frac{R(t)}{\tau}$ is unbounded, then

$$\lim_{t\to\infty} \int_{T}^{t} \frac{z(s)-k}{r(s)}ds = \infty,$$

and hence $\theta(t)$ is eventually positive. If $\lim_{t\to\infty} z(t) = 0$, then by (3.86) we have

$$\lim_{t\to\infty} \frac{x(t)}{r(t)R(t)} = 0.$$

Hence x is either of C-type or D-type. If $\lim_{t\to\infty} z(t) = k > 0$, then we have

$$\lim_{t\to\infty} \frac{x(t)}{r(t)R(t)} = \frac{k}{\tau}.$$

If $\theta(t) = \frac{x(t)}{r(t)} - \frac{kR(t)}{\tau}$ is unbounded, then x is of B-type. If $\beta(t) = \frac{x(t)}{r(t)} - \frac{kR(t)}{\tau}$ is bounded, then x is of A-type. The proof is complete. $\square$

By (3.86), the following corollary is immediate.

Corollary 3.6.9. *If x is a positive solution of* (3.72), *then the limit*

$$\lim_{t\to\infty} \frac{x(t)}{r(t)R(t)} = k \geq 0$$

must exist and

$$\lim_{t\to\infty} z(t) = \tau k.$$

In order to further justify our classification scheme, we derive several necessary and/or sufficient conditions for the existence of each type of positive solution.

Lemma 3.6.10. *Assume that there exists a constant p such that*

$$\frac{r(t)}{r(t-\tau)} \leq p \quad \textit{for all} \quad t \geq t_0. \tag{3.88}$$

If

$$\int_{t_0}^{\infty} Q(s)r(s-\sigma)R(s-\sigma)ds < \infty,$$

then the equation

$$\left[r(t)\Big(R(t)-R(t-\tau)\Big) + x(t) - \frac{r(t)}{r(t-\tau)}x(t-\tau)\right]' + Q(t)x(t-\sigma) + Q(t)r(t-\sigma)R(t-\sigma) = 0 \tag{3.89}$$

has a positive solution.

Proof. Choose $T > t_0 + 2\rho$ sufficiently large such that

$$\int_{T}^{\infty} Q(s)r(s-\sigma)R(s-\sigma)ds \leq \frac{\tau}{p+2} \quad \text{and} \quad R(T) > \frac{\tau}{r(T)}. \tag{3.90}$$

Set

$$H(t) = \begin{cases} \dfrac{(p+1)\tau}{r(t)} & \text{if} \quad t \geq T, \\ \dfrac{t+\tau-T}{\tau}H(T) & \text{if} \quad T-\tau \leq t < T, \\ 0 & \text{if} \quad t < T-\tau. \end{cases}$$

Clearly, $H \in C(\mathbb{R}, \mathbb{R}^+)$. Define

$$y(t) = \sum_{k=0}^{\infty} H(t-k\tau), \quad t \geq T. \tag{3.91}$$

It is obvious that $y \in C([T,\infty), \mathbb{R}^+)$ and $y(t) = y(t-\tau) + H(t)$. Since r is nondecreasing, H is nonincreasing on $[T,\infty)$. Then by Lemma 3.6.4, we have

$$\begin{aligned} y(t) &\leq \frac{1}{\tau}\int_T^t H(s)ds + H(T) \\ &= (p+1)\Big(R(t)-R(T)\Big) + \frac{(p+1)\tau}{r(T)} \\ &\leq (p+1)R(t) \end{aligned}$$

for $t \geq T$. Define a set X as

$$X = \Big\{x \in C[t_0,\infty) : \ 0 \leq x(t) \leq r(t)y(t), \ t \geq T\Big\}$$

and an operator S on X by

$$(Sx)(t) = \begin{cases} (p+1)\tau - r(t)\Big(R(t) - R(t-\tau)\Big) + \dfrac{r(t)}{r(t-\tau)}x(t-\tau) \\ \qquad + \displaystyle\int_t^\infty Q(s)(x+rR)(s-\sigma)ds \qquad \text{if} \quad t \ge T+\rho, \\ \dfrac{(Sx)(T+\rho)r(t)y(t)}{(T+\rho)r(T+\rho)y(T+\rho)}t + r(t)y(t)\left(1 - \dfrac{t}{T+\rho}\right) \\ \qquad\qquad\qquad\qquad\qquad\qquad\quad \text{if} \quad T \le t < T+\rho. \end{cases}$$

For any $x \in X$ and $t \ge T+\rho$, by (3.90) and (3.91) we have

$$\begin{aligned} (Sx)(t) &\le (p+1)\tau - \tau + r(t)y(t-\tau) + (p+2)\int_t^\infty Q(s)r(s-\sigma)R(s-\sigma)ds \\ &\le r(t)\left(y(t-\tau) + \frac{(p+1)\tau}{r(t)}\right) \\ &= r(t)y(t) \end{aligned}$$

and

$$(Sx)(t) \ge (p+1)\tau - p\tau > 0.$$

For $T \le t \le T+\rho$, it is easy to see that $0 \le (Sx)(t) \le r(t)y(t)$. So, $SX \subset X$.

Define a sequence of functions $\{x_k\}_{k=0}^\infty$ as

$$x_0 = ry \quad \text{and} \quad x_k = Sx_{k-1} \text{ for } k \in \mathbb{N} \quad \text{on} \quad [T,\infty).$$

By induction we can prove that

$$r(t)y(t) = x_0(t) \ge x_1(t) \ge x_2(t) \ge \ldots \ge 0, \quad t \ge T.$$

Thus for $t \ge T$, $u(t) = \lim_{k\to\infty} x_k(t)$ exists and

$$u(t) = \begin{cases} (p+1)\tau - r(t)\Big(R(t) - R(t-\tau)\Big) + \dfrac{r(t)}{r(t-\tau)}u(t-\tau) \\ \qquad + \displaystyle\int_t^\infty Q(s)(u+rR)(s-\sigma)ds \qquad \text{if} \quad t \ge T+\rho, \\ \dfrac{u(T+\rho)r(t)y(t)}{(T+\rho)r(T+\rho)y(T+\rho)}t + r(t)y(t)\left(1 - \dfrac{t}{T+\rho}\right) \\ \qquad\qquad\qquad\qquad\qquad\qquad\quad \text{if} \quad T \le t < T+\rho. \end{cases}$$

Now it is not difficult to show that $u(t) > (p+1)\tau - p\tau > 0$ on $[T+\rho,\infty)$. Further, u is continuous on $[T,\infty)$ and is a solution of (3.89). The proof is complete. $\square$

Lemma 3.6.11. *Suppose* (3.72) *has a positive solution x. Then the following hold:*

(i) *If x is an A-type solution, then*

$$\int^\infty \frac{1}{r(s)}\int_s^\infty Q(u)r(u-\sigma)R(u-\sigma)duds < \infty. \tag{3.92}$$

(ii) *If x is a B-type solution, then*

$$\begin{cases} \displaystyle\int^\infty Q(s)r(s-\sigma)R(s-\sigma)ds < \infty, \\ \displaystyle\int^\infty \frac{1}{r(s)}\int_s^\infty Q(u)r(u-\sigma)R(u-\sigma)duds = \infty. \end{cases} \tag{3.93}$$

(iii) *If x is a D-type solution, then*

$$\int^{\infty} Q(s)r(s-\sigma)R(s-\sigma)ds < \infty. \tag{3.94}$$

Proof. We first show (i). Suppose x is an A-type solution of (3.72). Then x can be expressed as

$$x(t) = r(t)\left(\alpha R(t) + \beta(t)\right),$$

where $\alpha > 0$ and $\beta(t)$ is bounded. Choose $T_1 \geq t_0$ sufficiently large such that

$$x(t) > \frac{\alpha r(t)R(t)}{2}, \quad t \geq T_1.$$

Since

$$\lim_{t\to\infty} \frac{x(t)}{r(t)R(t)} = \alpha,$$

by Corollary 3.6.9 we have

$$\lim_{t\to\infty} z(t) = \alpha\tau.$$

Obviously,

$$z'(t) = -Q(t)x(t-\sigma).$$

Integrating this on both sides from t to ∞, we get

$$\begin{aligned} z(t) &= \alpha\tau + \int_t^{\infty} Q(s)x(s-\sigma)ds \\ &\geq \alpha\tau + \frac{\alpha}{2}\int_t^{\infty} Q(s)r(s-\sigma)R(s-\sigma)ds \end{aligned}$$

for $t \geq T_1 + \sigma$, i.e.,

$$x(t) - \frac{r(t)}{r(t-\tau)}x(t-\tau) \geq \alpha\tau + \frac{\alpha}{2}\int_t^{\infty} Q(s)r(s-\sigma)R(s-\sigma)ds$$

for $t \geq T_1 + \sigma$. By Lemma 3.6.4, we have

$$x(t) \geq \frac{r(t)}{\tau}\int_{T_1+\sigma+2\tau}^{t} \frac{1}{r(s)}\left(\alpha\tau + \frac{\alpha}{2}\int_s^{\infty} Q(u)r(u-\sigma)R(u-\sigma)du\right)ds$$

for $t \geq T_1 + \sigma + 2\tau$, and hence

$$\beta(t) + \alpha R(T_1+\sigma+2\tau) \geq \frac{\alpha}{2\tau}\int_{T_1+\sigma+2\tau}^{t} \frac{1}{r(s)}\int_s^{\infty} Q(u)r(u-\sigma)R(u-\sigma)duds$$

for $t \geq T_1 + \sigma + 2\tau$. In the above inequality as $t \to \infty$, we obtain (3.92).

Next we show (ii). Suppose $x(t) = r(t)\left[\alpha R(t) + \theta(t)\right]$ is a B-type solution of (3.72) with

$$\lim_{t\to\infty} \frac{\theta(t)}{R(t)} = 0 \quad \text{and} \quad \limsup_{t\to\infty} \theta(t) = \infty.$$

Choose $T_2 > t_0$ sufficiently large such that

$$\alpha r(t)R(t) \leq x(t) \leq 2\alpha r(t)R(t), \quad t \geq T_2.$$

As in the proof of (i), we have

$$z(t) = \alpha\tau + \int_t^{\infty} Q(s)x(s-\sigma)ds \geq \alpha\tau + \alpha\int_t^{\infty} Q(s)r(s-\sigma)R(s-\sigma)ds$$

for $t \geq T_2$, and so

$$\int^{\infty} Q(s)r(s-\sigma)R(s-\sigma)ds < \infty.$$

On the other hand,

$$z(t) \leq \alpha\tau + 2\alpha \int_t^{\infty} Q(s)r(s-\sigma)R(s-\sigma)ds, \quad t \geq T_2 + \sigma. \tag{3.95}$$

Rewrite (3.95) as

$$x(t) - \frac{r(t)}{r(t-\tau)}x(t-\tau) \leq \alpha\tau + 2\alpha \int_t^{\infty} Q(s)r(s-\sigma)R(s-\sigma)ds$$

for $t \geq T_2 + \sigma$. By Lemma 3.6.4, we get

$$x(t) \leq \frac{r(t)}{\tau}\left[\int_{T_2+\sigma}^{t} \frac{1}{r(s)}\left(\alpha\tau + 2\alpha\int_s^{\infty} Q(u)r(u-\sigma)R(u-\sigma)du\right)ds + \tau\bar{M}\right]$$

for $t \geq T_2 + \sigma$, where

$$\bar{M} = 2\alpha R(T_2 + \sigma + \tau).$$

Thus it follows that

$$\theta(t) + \alpha R(T_2 + \sigma) - \bar{M} \leq \frac{2\alpha}{\tau}\int_{T_2+\sigma}^{t} \frac{1}{r(s)}\int_s^{\infty} Q(u)r(u-\sigma)R(u-\sigma)duds$$

for $t \geq T_2 + \sigma$. Letting $t \to \infty$ in the above inequality, (3.93) follows.

Finally we show (iii). Suppose that x is a D-type positive solution of (3.72). Then x/r is bounded. Let z be defined by (3.73). By Lemma 3.6.1, there exists $T_3 \geq t_0$ such that (3.85) holds. Set

$$m = \min\left\{\frac{x(t)}{r(t)} : T_3 \leq t \leq T_3 + \rho\right\}.$$

It is easy to see that $x(t) \geq mr(t)$ for $t \geq T_3$. From (3.72) and (3.85), we have

$$z(t) \geq m\int_t^{\infty} Q(s)r(s-\sigma)ds, \quad t \geq T_3 + \sigma. \tag{3.96}$$

Thus, by Lemma 3.6.4, (3.73), (3.85), and (3.96), we get

$$\begin{aligned}
\frac{x(t)}{r(t)} &\geq \frac{1}{\tau}\left(\int_{T_3+2\tau}^{t} \frac{z(s)}{r(s)}ds + \tau m\right) \\
&\geq \frac{m}{\tau}\int_{T_3+2\tau+\sigma}^{t} \frac{1}{r(s)}\int_s^{\infty} Q(u)r(u-\sigma)duds
\end{aligned}$$

for $t \geq T_3 + 2\tau + \sigma$. Letting $t \to \infty$ in the above inequality, we obtain

$$\int^{\infty} \frac{1}{r(s)}\int_s^{\infty} Q(u)r(u-\sigma)duds < \infty \tag{3.97}$$

and so (3.94) holds. The proof is complete. □

Theorem 3.6.12. *Assume that* (3.88) *holds. Then* (3.72) *has an A-type positive solution if and only if* (3.92) *holds.*

Proof. The necessity follows from Lemma 3.6.11. Now we show sufficiency. If (3.92) holds, then by Lemma 3.6.10, (3.89) has a positive solution u. Thus $x = rR + u$ is a positive solution of (3.72). By Corollary 3.6.9, the limit

$$\lim_{t\to\infty}\left[1+\frac{u(t)}{r(t)R(t)}\right] = \lim_{t\to\infty}\frac{x(t)}{r(t)R(t)} = k$$

must exist and $k \geq 1$. Hence, by Theorem 3.6.8, x is either an A-type or B-type solution of (3.72). But by (3.92) and Lemma 3.6.11, it is obvious that x is not a B-type solution. Therefore, x is an A-type solution of (3.72). The proof is complete. □

Example 3.6.13. Consider the neutral differential equation

$$\left(x(t) - \frac{t}{t-2}x(t-2)\right)' + Q_1(t)x(t-1) = 0, \quad t \geq 3, \tag{3.98}$$

where

$$Q_1(t) = \frac{2-(t-2)\left(\ln t - \ln(t-2)\right)}{(t-1)(t-2)\ln(t-1)}.$$

Set

$$r(t) = t \quad \text{so that} \quad R(t) = \int_3^t \frac{ds}{s} = \ln t - \ln 3.$$

By direct calculation, it is easy to see that

$$\int^\infty \frac{1}{r(s)}\int_s^\infty Q_1(u)r(u-1)R(u-1)du ds < \infty.$$

Thus Theorem 3.6.12 ensures that (3.98) has an A-type solution. In fact,

$$x(t) = r(t)\left(R(t) + \ln 3\right) = t\ln t$$

is such a solution of (3.98).

Theorem 3.6.14. *Assume that* (3.88) *holds. Then* (3.72) *has a B-type positive solution if and only if* (3.93) *holds.*

Proof. The necessity part follows from Lemma 3.6.11. The proof of the sufficiency part is similar to that of the proof of Theorem 3.6.12. □

Example 3.6.15. Consider the neutral differential equation

$$\left(x(t) - \frac{\sqrt{t}}{\sqrt{t-1}}x(t-1)\right)' + Q_2(t)x(t-2) = 0, \quad t \geq 2, \tag{3.99}$$

where

$$Q_2(t) = \frac{\left[\left(1-\frac{1}{t}\right)^{\frac{1}{2}} + \left(1-\frac{1}{t}\right)^{-\frac{1}{2}} - 2\right] + \frac{1}{4}t^{-\frac{1}{4}}\left[2\left(1-\frac{1}{t}\right)^{\frac{1}{4}} + \left(1-\frac{1}{t}\right)^{\frac{3}{4}} - 3\right]}{2(t-2)+(t-2)^{\frac{3}{4}}}.$$

Set

$$r(t) = t^{\frac{1}{2}} \quad \text{so that} \quad R(t) = \int_2^t \frac{ds}{\sqrt{s}} = 2t^{\frac{1}{2}} - 2^{\frac{3}{2}}.$$

By direct calculation, it is easy to see that

$$\int^\infty \frac{1}{r(s)}\int_s^\infty Q_2(u)r(u-2)R(u-2)du ds = \infty$$

and

$$\int^{\infty} Q_2(s)r(s-2)R(s-2)ds < \infty.$$

Thus Theorem 3.6.14 guarantees that (3.99) has a B-type solution. In fact,

$$x(t) = r(t)\left(R(t) + t^{\frac{1}{4}} + 2^{\frac{3}{2}}\right) = 2t + t^{\frac{3}{4}}$$

is such a solution of (3.99).

Theorem 3.6.16. *Equation* (3.72) *has a D-type solution if and only if* (3.94) *holds.*

Proof. The necessity part follows from Lemma 3.6.11. Now we show sufficiency. If (3.94) holds, then (3.97) holds. Set

$$Q^*(t) = \frac{1}{r(t)}\int_t^{\infty} Q(s)r(s-\sigma)ds, \quad t \geq t_0 + \sigma.$$

Clearly, Q^* is nonincreasing on $[t_0+\sigma, \infty)$. Choose $t_1 > t_0 + \sigma$ sufficiently large such that

$$\frac{1}{\tau}\int_{t_1}^{\infty}\frac{1}{r(s)}\int_s^{\infty} Q(u)r(u-\sigma)duds + \frac{2}{r(t_1)}\int_{t_1}^{\infty} Q(s)r(s-\sigma)ds \leq 1. \tag{3.100}$$

Set

$$u(t) = \begin{cases} Q^*(t) & \text{if} \quad t \geq t_1, \\ \dfrac{t-t_1+\tau}{\tau}Q^*(t_1) & \text{if} \quad t_1 - \tau \leq t < t_1, \\ 0 & \text{if} \quad t < t_1 - \tau. \end{cases}$$

Clearly, $u \in C(\mathbb{R}, \mathbb{R}^+)$. Define

$$v(t) = \sum_{k=0}^{\infty} u(t-k\tau), \quad t \geq t_1 - \tau.$$

It is obvious that $v \in C([t_1-\tau,\infty), \mathbb{R}^+)$ and $v(t) = v(t-\tau) + u(t)$ for $t \geq t_1$. Set $\bar{x}(t) = r(t)v(t)$ for $t \geq t_1 - \tau$. Then we have

$$\bar{x}(t) - \frac{r(t)}{r(t-\tau)}\bar{x}(t-\tau) = r(t)u(t) = \int_t^{\infty} Q(s)r(s-\sigma)ds \tag{3.101}$$

for $t \geq t_1$. By Lemma 3.6.4, (3.101) yields

$$\bar{x}(t) \leq \frac{r(t)}{\tau}\left(\int_{t_1}^{t}\frac{1}{r(s)}\int_s^{\infty} Q(u)r(u-\sigma)duds + \frac{2\tau}{r(t_1)}\int_{t_1}^{\infty} Q(s)r(s-\sigma)ds\right)$$

for $t \geq t_1$, which, in view of (3.100), gives

$$\bar{x}(t) \leq r(t), \quad t \geq t_1. \tag{3.102}$$

Substituting (3.102) into (3.101), we obtain

$$\bar{x}(t) \geq \frac{r(t)}{r(t-\tau)}\bar{x}(t-\tau) + \int_t^{\infty} Q(s)\bar{x}(s-\sigma)ds, \quad t \geq t_1 + \sigma.$$

By Lemma 3.6.2, this implies that the corresponding integral equation (3.75) has a positive solution x with $0 < x(t) \leq \bar{x}(t)$ for $t \geq t_1$. Since $x/r \leq \bar{x}/r \leq 1$, x is a D-type positive solution of (3.72). The proof is complete. □

Example 3.6.17. Consider the neutral differential equation

$$\left(x(t) - \frac{t\ln t}{(t-1)\ln(t-1)}x(t-1)\right)' + Q_3(t)x(t-1) = 0, \quad t \geq 3, \tag{3.103}$$

where

$$Q_3(t) = \frac{t\ln t - t + 1}{t(t-1)^2(t-2)\ln(t-1)}.$$

Set

$$r(t) = t\ln t \quad \text{so that} \quad R(t) = \int_3^t \frac{ds}{s\ln s} = \ln\ln t - \ln\ln 3.$$

Then we have

$$\int^\infty Q_3(s)r(s-1)R(s-1)ds < \infty.$$

Thus by Theorem 3.6.16, (3.103) has a D-type solution. In fact

$$x(t) = (t-1)\ln t$$

is such a solution.

Lemma 3.6.18 ([**282**]). *Assume that* $p \in C([t_0,\infty),\mathbb{R}^+)$ *and*

$$\int_{t_0}^t \frac{ds}{r(s)} \int_t^\infty p(s)ds < \frac{1}{4}, \quad t \geq t_1 \geq t_0.$$

Then the ordinary differential equation

$$(r(t)y')' + p(t)y = 0, \quad t \geq t_0 \tag{3.104}$$

has a positive solution.

Theorem 3.6.19. *Assume that*

$$\int^\infty Q(s)r(s-\sigma)R(s-\sigma)ds = \infty \tag{3.105}$$

and that there exists $T > t_0$ *such that*

$$R(t)\int_t^\infty Q(s)r(s-\sigma)ds \leq \frac{\tau}{4}, \quad t \geq T. \tag{3.106}$$

Then (3.72) *has a C-type positive solution.*

Proof. In view of Lemma 3.6.18, (3.106) implies that (3.76) has a positive solution. Thus by Theorem 3.6.5, (3.72) has a positive solution x. By Theorems 3.6.12, 3.6.14, and 3.6.16, (3.105) implies that x is neither one of A-type, B-type, or D-type positive solution of (3.72). Hence, by Theorem 3.6.8, x is a C-type positive solution of (3.72). The proof is complete. □

Example 3.6.20. Consider the neutral differential equation

$$\left(x(t) - \frac{t^{\frac{1}{3}}}{(t-1)^{\frac{1}{3}}}x(t-1)\right)' + Q_4(t)x(t) = 0, \quad t \geq 3, \tag{3.107}$$

where

$$Q_4(t) = \frac{3-(t-1)\left(\ln t - \ln(t-1)\right)}{3t(t-1)\ln t}.$$

Let

$$r(t) = t^{\frac{1}{3}} \quad \text{so that} \quad R(t) = \int_3^t \frac{ds}{\sqrt[3]{s}} = \frac{3}{2}\left(t^{\frac{2}{3}} - 3^{\frac{2}{3}}\right).$$

Then

$$\int^{\infty} Q_4(s)r(s)R(s)ds = \infty$$

and

$$R(t)\int_t^{\infty} Q_4(s)r(s)ds < \frac{1}{4}, \quad t > e^{10}.$$

Thus by Theorem 3.6.19, (3.107) has a C-type solution. In fact,

$$x(t) = t^{\frac{1}{3}} \ln t$$

is such a solution.

Finally, we shall obtain a sharp oscillation result for (3.72).

Lemma 3.6.21 ([**282**]). *Assume that* $p \in C([t_0, \infty), \mathbb{R}^+)$ *and*

$$\liminf_{t\to\infty} \int_{t_0}^t \frac{ds}{r(s)} \int_t^{\infty} p(u)du > \frac{1}{4}.$$

Then every solution of (3.104) *oscillates.*

Theorem 3.6.22. *Assume that*

$$\liminf_{t\to\infty} R(t) \int_t^{\infty} Q(s)r(s-\sigma)ds > \frac{\tau}{4}.$$

Then every solution of (3.72) *oscillates.*

From Theorems 3.6.12, 3.6.14, 3.6.16, 3.6.19, and 3.6.22, the following result is true.

Corollary 3.6.23. *Consider the neutral differential equation*

$$\left(x(t) - \frac{r(t)}{r(t-\tau)}x(t-\tau)\right)' + c\,[r(t)r(t-\sigma)]^{-1}\,[R(t)]^{-\alpha}x(t-\sigma) = 0 \tag{3.108}$$

for $t \geq t_0$*, where* $c > 0$ *and* $\alpha \in \mathbb{R}$*. Then every solution of* (3.108) *oscillates if and only if* $\alpha < 2$*, or* $\alpha = 2$ *and* $c > \frac{\tau}{4}$*.*

3.7. Positive Solutions of Neutral Perturbed Equations

Consider the first order neutral differential equation

$$\Big(x(t) + h(t)x\left(\tau(t)\right)\Big)' + \sigma f\Big(t, x(g(t))\Big) = 0, \tag{3.109}$$

where $\sigma \in \{1, -1\}$. It is assumed that

(H$_1$) $\tau : [t_0, \infty] \to \mathbb{R}$ is continuous and strictly increasing, $\tau(t) < t$ for $t \geq t_0$ and $\lim_{t\to\infty} \tau(t) = \infty$.
(H$_2$) $h : [\tau(t_0), \infty] \to \mathbb{R}$ is continuous.
(H$_3$) $g : [t_0, \infty) \to \mathbb{R}$ is continuous and $\lim_{t\to\infty} g(t) = \infty$.
(H$_4$) $f : [t_0, \infty) \times (0, \infty) \to [0, \infty)$ is continuous and $f(t, u)$ is nondecreasing in $u \in (0, \infty)$ for any fixed $t \in [t_0, \infty)$.

Theorem 3.7.1. *Equation* (3.109) *has a solution* x *satisfying*

$$0 < \liminf_{t\to\infty} x(t) \le \limsup_{t\to\infty} x(t) < \infty \tag{3.110}$$

if and only if

$$\int_{t_0}^{\infty} f(t,a)dt < \infty \quad \text{for some} \quad a > 0, \tag{3.111}$$

when one of the following cases holds:

(i) $|h(t)| \le \lambda < 1$ *and* $h(t)h(\tau(t)) \ge 0$*;*
(ii) $h(t) = 1$ *and* $\tau(t) = t - p$ *with* $p > 0$*;*
(iii) $1 < \mu < h(t) \le \lambda < \infty$*,*

where λ*,* μ*, and* p *are constants.*

Proof. The proof is similar to the proofs of the theorems in Section 3.6 and will be omitted here. □

In what follows, we consider the existence of a solution x of (3.109) satisfying (3.110) in the case

$$h(t) > -1 \quad \text{and} \quad h(\tau(t)) = h(t) \quad \text{for} \quad t \ge t_0. \tag{3.112}$$

Pairs of functions

$$\tau(t) = t - 2\pi \quad \text{and} \quad h(t) = 1 + \frac{3}{2}\sin t,$$

$$\tau(t) = \frac{t}{e} \quad \text{and} \quad h(t) = 1 + \frac{3}{2}\sin(2\pi \ln t), \quad t_0 > 0,$$

or

$$\tau(t) = t^{1/e} \quad \text{and} \quad h(t) = 1 + \frac{3}{2}\sin\Big(2\pi\ln(\ln t)\Big), \quad t_0 > 1$$

give typical examples satisfying (3.112). We easily see that if (3.112) holds, then for any positive constant b,

$$x(t) = \frac{b}{1+h(t)}$$

is a positive solution of the unperturbed equation

$$\Big(x(t) + h(t)x(\tau(t))\Big)' = 0,$$

and so it is natural to expect that, if f is small enough, (3.109) possesses a solution x that behaves like the function $b/[1+h(t)]$ as $t \to \infty$. In fact, the following theorem will be shown.

Theorem 3.7.2. *Suppose that* (3.112) *holds. Then* (3.109) *has a positive solution* x *satisfying*

$$x(t) = \frac{b}{1+h(t)} + \mathrm{o}(1) \quad \text{as} \quad t \to \infty \quad \text{for some} \quad b > 0 \tag{3.113}$$

if and only if (3.111) *holds.*

We remark that if (3.112) holds, then there are constants μ and λ such that $-1 < \mu \le h(t) \le \lambda < \infty$ for $t \ge t_0$. In fact, assume that (3.112) holds. Then note that

$$[t_0, \infty) = \bigcup_{p=0}^{\infty} \left[\tau^{-p}(t_0), \tau^{-(p+1)}(t_0)\right]$$

and that the range of $h(t)$ for $t \in [t_0, \tau^{-1}(t_0)]$ is identical to the range of $h(t)$ $(= h(\tau^p(t)))$ for $t \in [\tau^{-p}(t_0), \tau^{-(p+1)}(t_0)]$, $p \in \mathbb{N}_0$. Let

$$\mu = \min h\left([t_0, \tau^{-1}(t_0)]\right) \quad \text{and} \quad \lambda = \max h\left([t_0, \tau^{-1}(t_0)]\right).$$

Then we find that $-1 < \mu \le h(t) \le \lambda < \infty$ for all $t \ge t_0$. Also it is worthwhile to note that a positive solution x with the asymptotic property (3.113) satisfies (3.110).

We give an example illustrating the above theorem.

Example 3.7.3. Consider the first order neutral differential equation

$$\left(x(t) + h(t)x(t-\tau)\right)' + \sigma e^{-t} \left[p(g(t))\right]^{-\gamma} \left[x(g(t))\right]^{\gamma} = 0, \tag{3.114}$$

where $\sigma \in \{1, -1\}$, $\gamma > 0$, $\tau = \ln(4/3)$, $g \in C[t_0, \infty)$, $\lim_{t\to\infty} g(t) = \infty$, $g(t) \ge 0$ for $t \ge t_0$, $h(t) = 1 + (3/2)\sin(2\pi t/\tau)$, and

$$p(t) = \frac{11}{1 + h(t)} + \sigma \frac{3e^{-t}}{3 + 4h(t)} = \frac{22}{4 + 3\sin(2\pi t/\tau)} + \sigma \frac{3e^{-t}}{7 + 6\sin(2\pi t/\tau)}$$

for $t \ge 0$. Clearly, $h(t) > -1$, $h(t) = h(t-\tau)$ for $t \ge t_0$, and $p(t) \ge 1/7$ for $t \ge 0$. Then it is easy to check that

$$\int_{t_0}^{\infty} e^{-t} \left[p(g(t))\right]^{-\gamma} a^{\gamma} dt < \infty \quad \text{if} \quad a > 0.$$

By Theorem 3.7.2, we conclude that (3.114) has a positive solution x satisfying

$$x(t) = \frac{b}{4 + 3\sin(2\pi t/\tau)} + \mathrm{o}(1) \quad \text{as} \quad t \to \infty \quad \text{for some} \quad b > 0.$$

Indeed, note that $p(t) + h(t)p(t-\tau) = 11 + \sigma e^{-t}$, and therefore $x = p$ is such a positive solution.

In order to prove Theorem 3.7.2, we make some preparation and use the following notation:

$$\tau^0(t) = t; \quad \tau^k(t) = \tau\left(\tau^{k-1}(t)\right), \quad \tau^{-k}(t) = \tau^{-1}\left(\tau^{-(k-1)}(t)\right), \quad k \in \mathbb{N},$$

where τ^{-1} is the inverse function of τ. We note here that $\tau^{-p}(t) \to \infty$ as $p \to \infty$ for each fixed $t \ge t_0$. Otherwise, there is a constant $c \ge t_0$ such that $\lim_{p\to\infty} \tau^{-p}(t) = c$, because of $\tau^{-p}(t) < \tau^{-(p+1)}(t)$. Letting $p \to \infty$ in $\tau^{-p}(t) = \tau^{-1}(\tau^{-(p-1)}(t))$, we have $c = \tau^{-1}(c)$ which contradicts $\tau(t) < t$ for $t \ge t_0$.

Recall that

$$\max\left\{h(t) : t \in [t_0, \infty)\right\} = \max\left\{h(t) : t \in \left[\tau^{-p}(t_0), \tau^{-(p+1)}(t_0)\right]\right\}$$

for $p \in \mathbb{N}_0$ and that $\tau^{-p}(t) \to \infty$ as $p \to \infty$ for each fixed $t \ge t_0$. Thus it is possible to take a sufficiently large number $T \ge t_0$ such that

$$h(T) = \max\left\{h(t) : t \in [t_0, \infty)\right\}.$$

Define

$$T_* := \min\left\{\tau(T), \inf\{g(t) : t \geq T\}\right\} \geq t_0.$$

Let $C[T_*, \infty)$ denote the Fréchet space of all continuous functions on $[T_*, \infty)$ with the topology of uniform convergence on every compact subinterval of $[T_*, \infty)$. Let $\eta \in C[T, \infty)$ be fixed such that $\eta(t) \geq 0$ for $t \geq T$ and $\lim_{t\to\infty} \eta(t) = 0$. We consider the set Y of all functions $y \in C[T_*, \infty)$ that are nonincreasing on $[T, \infty)$ and satisfy

$$y(t) = y(T) \text{ for } t \in [T_*, T], \quad 0 \leq y(t) \leq \eta(t) \text{ for } t \geq T.$$

It is easy to see that Y is a closed and convex subset of $C[T_*, \infty)$.

Proposition 3.7.4. *Suppose that* (3.112) *holds. Let $\eta \in C[T_*, \infty]$ with $\eta(t) \geq 0$ for $t \geq T$ and $\lim_{t\to\infty} \eta(t) = 0$. For this η, define Y as above. Then there exists a mapping $\Phi : Y \to C[T_*, \infty)$, which possesses the following properties:*

(i) *For each $y \in Y$, $\Phi[y]$ satisfies*

$$\Phi[y](t) + h(t)\Phi[y]\left(\tau(t)\right) = y(t) \text{ for } t \geq T \qquad \text{and} \qquad \lim_{t\to\infty} \Phi[y](t) = 0.$$

(ii) *Φ is continuous on Y in the $C[T_*, \infty)$-topology, i.e., if $\{y_j\}_{j\in\mathbb{N}}$ is a sequence in Y converging to $y \in Y$ uniformly on every compact subinterval of $[T_*, \infty)$, then $\Phi[y_j]$ converges to $\Phi[y]$ uniformly on every compact subinterval of $[T_*, \infty)$.*

Before proving Proposition 3.7.4, we give several lemmas.

Assume that (3.112) holds. For each $y \in Y$, we define the function $\Psi[y]$ by

$$\Psi[y] = \begin{cases} \displaystyle\sum_{k=1}^{\infty} (-1)^{k+1} \left[H(t)\right]^{k} y\left(\tau^{-k}(t)\right) & \text{if} \quad t \geq \tau(T), \\ \Psi[y]\left(\tau(T)\right) & \text{if} \quad t \in [T_*, \tau(T)), \end{cases}$$

where $H(t) = \max\{1, h(t)\}$. We note that $H(\tau(t)) = H(t)$ and $H(t) \geq 1$ for $t \geq t_0$.

Lemma 3.7.5. *Assume that* (3.112) *holds.*

(i) *For each $y \in Y$, the series*

$$\sum_{k=1}^{\infty} (-1)^{k+1} \left[H(t)\right]^{-k} y\left(\tau^{-k}(t)\right)$$

converges uniformly on $[\tau(T), \infty)$; hence, $\Psi[y]$ is well defined and continuous on $[T_*, \infty)$.

(ii) *For each $y \in Y$, $\Psi[y]$ satisfies*

$$0 \leq \Psi[y] \leq \eta\left(\tau^{-1}(t)\right), \quad t \geq \tau(T) \tag{3.115}$$

and

$$\Psi[y](t) + H(t)\Psi[y]\left(\tau(t)\right) = y(t), \quad t \geq T. \tag{3.116}$$

(iii) *Ψ is continuous on Y in the $C[T_*, \infty)$-topology.*

Proof. We first show (i). Let $y \in Y$. We set

$$\Psi_m[y](t) = \sum_{k=1}^{m} (-1)^{k+1} [H(t)]^{-k} y\left(\tau^{-k}(t)\right), \quad t \geq \tau(T), \quad m \in \mathbb{N}.$$

Now we claim that

$$0 \leq \Psi_m[y](t) \leq \eta\left(\tau^{-1}(t)\right), \quad t \geq \tau(T) \tag{3.117}$$

for all $m \in \mathbb{N}$. Since y is nonincreasing on $[T, \infty)$ and $H(t) \geq 1$, we have

$$y\left(\tau^{-1}(t)\right) - [H(t)]^{-1} y\left(\tau^{-2}(t)\right) \geq 0, \quad t \geq \tau(T) \tag{3.118}$$

and

$$[H(t)]^{-1} y\left(\tau^{-1}(t)\right) \leq \eta\left(\tau^{-1}(t)\right), \quad t \geq \tau(T). \tag{3.119}$$

Hence, we easily see that (3.117) holds for the cases $m = 1$ and $m = 2$. If $m \geq 3$ is odd, then we can rewrite $\Psi_m[y](t)$ as

$$\begin{aligned} \Psi_m[y](t) &= \sum_{k=1}^{(m-1)/2} [H(t)]^{-(2k-1)} \left[y\left(\tau^{-(2k-1)}(t)\right) - [H(t)]^{-1} y\left(\tau^{-2k}(t)\right)\right] \\ &\quad + [H(t)]^{-m} y\left(\tau^{-m}(t)\right) \end{aligned}$$

and

$$\begin{aligned} \Psi_m[y](t) &= [H(t)]^{-1} y\left(\tau^{-1}(t)\right) \\ &\quad - \sum_{k=1}^{(m-1)/2} [H(t)]^{-2k} \left[y\left(\tau^{-2k}(t)\right) - [H(t)]^{-1} y\left(\tau^{-(2k+1)}(t)\right)\right]. \end{aligned}$$

If $m \geq 4$ is even, then we can rewrite $\Psi_m[y](t)$ as

$$\Psi_m[y](t) = \sum_{k=1}^{m/2} [H(t)]^{-(2k-1)} \left[y\left(\tau^{-(2k-1)}(t)\right) - [H(t)]^{-1} y\left(\tau^{-2k}(t)\right)\right]$$

and

$$\begin{aligned} \Psi_m[y](t) &= [H(t)]^{-1} y\left(\tau^{-1}(t)\right) \\ &\quad - \sum_{k=1}^{(m/2)-1} [H(t)]^{-2k} \left[y\left(\tau^{-2k}(t)\right) - [H(t)]^{-1} y\left(\tau^{-(2k+1)}(t)\right)\right] \\ &\quad - [H(t)]^{-m} y\left(\tau^{-m}(t)\right). \end{aligned}$$

From (3.118) and (3.119), we conclude that (3.117) holds for all $m \in \mathbb{N}$. Using (3.117), we find that if $m \geq p \geq 1$, then

$$\begin{aligned} &\left|\sum_{k=p}^{m} (-1)^{k+1} [H(t)]^{-k} y\left(\tau^{-k}(t)\right)\right| \\ &= \left|\sum_{k=1}^{m-p+1} (-1)^{(k+p-1)+1} [H(t)]^{-(k+p-1)} y\left(\tau^{-k}\left(\tau^{-p+1}(t)\right)\right)\right| \\ &= \left|(-1)^{p-1} [H(t)]^{-(p-1)} \Psi_{m-p+1}[y]\left(\tau^{-p+1}(t)\right)\right| \\ &\leq \eta\left(\tau^{-p}(t)\right) \end{aligned} \tag{3.120}$$

for $t \geq \tau(T)$. Here, we have used the equality $H(t) = H(\tau^{-p+1}(t))$, $p \in \mathbb{N}$. Since $\eta(\tau^{-p}(t)) \to 0$ as $p \to \infty$, the series $\sum_{k=1}^{\infty}(-1)^{k+1}[H(t)]^{-k}y(\tau^{-k}(t))$ converges for each fixed $t \in [\tau(T), \infty)$. From (3.120) it follows that

$$\begin{aligned}
\sup_{t\geq\tau(T)} \left| \sum_{k=p}^{\infty}(-1)^{k+1}\left[H(t)\right]^{-k} y\left(\tau^{-k}(t)\right) \right| &\leq \sup_{t\geq\tau(T)} \eta\left(\tau^{-p}(t)\right) \\
&= \sup_{t\geq\tau^{-p+1}(T)} \eta(t) \\
&\to 0 \quad \text{as} \quad p \to \infty,
\end{aligned}$$

and hence the series $\sum_{k=1}^{\infty}(-1)^{k+1}[H(t)]^{-k}y(\tau^{-k}(t))$ converges uniformly on $[\tau(T), \infty)$.

Now we show (ii). Letting $m \to \infty$ in (3.117), we have (3.115). To verify (3.116), we calculate

$$\begin{aligned}
H(t)\Psi[y](\tau(t)) &= H(t)\sum_{k=1}^{\infty}(-1)^{k+1}\left[H(\tau(t))\right]^{-k} y\left(\tau^{-k}(\tau(t))\right) \\
&= H(t)\sum_{k=1}^{\infty}(-1)^{k+1}\left[H(t)\right]^{-k} y\left(\tau^{-k+1}(t)\right) \\
&= \sum_{k=1}^{\infty}(-1)^{k+1}\left[H(t)\right]^{-k+1} y\left(\tau^{-k+1}(t)\right) \\
&= \sum_{k=0}^{\infty}(-1)^{k+2}\left[H(t)\right]^{-k} y\left(\tau^{-k}(t)\right) \\
&= y(t) - \Psi[y](t).
\end{aligned}$$

Finally we show (iii). Let $\varepsilon > 0$. There exists $p \in \mathbb{N}$ such that

$$\sup_{t\in[\tau(T),\infty)} \eta\left(\tau^{-(p+1)}(t)\right) = \sup_{t\in[\tau^{-p}(T),\infty)} \eta(t) < \frac{\varepsilon}{3}.$$

Let $\{y_j\}_{j\in\mathbb{N}}$ be a sequence in Y converging to $y \in Y$ uniformly on every compact subinterval of $[T_*, \infty)$. Take an arbitrary compact subinterval I of $[\tau(T), \infty)$. There exists $j_0 \in \mathbb{N}$ such that

$$\sum_{k=1}^{p}\left|y_j\left(\tau^{-k}(t)\right) - y\left(\tau^{-k}(t)\right)\right| < \frac{\varepsilon}{3}, \quad t \in I, \quad j \geq j_0.$$

It follows from (3.120) that

$$\begin{aligned}
\left|\Psi[y_j](t) - \Psi[y](t)\right| &\leq \sum_{k=1}^{p}\left[H(t)\right]^{-k}\left|y_j\left(\tau^{-k}(t)\right) - y\left(\tau^{-k}(t)\right)\right| \\
&\quad + \left|\sum_{k=p+1}^{\infty}(-1)^{k+1}\left[H(t)\right]^{-k} y_j\left(\tau^{-k}(t)\right)\right| + \left|\sum_{k=p+1}^{\infty}(-1)^{k+1}\left[H(t)\right]^{-k} y\left(\tau^{-k}(t)\right)\right| \\
&\leq \sum_{k=1}^{p}\left|y_j\left(\tau^{-k}(t)\right) - y\left(\tau^{-k}(t)\right)\right| + 2\eta\left(\tau^{-(p+1)}(t)\right) < \varepsilon
\end{aligned}$$

for $t \in I$ and $j \geq j_0$, which implies that $\Psi[y_j]$ converges to $\Psi[y]$ uniformly on I. It is easy to see that $\Psi[y_j] \to \Psi[y]$ uniformly on $[T_*, \tau(T)]$. Consequently, we conclude that Ψ is continuous on Y. □

To each $y \in Y$ we assign the function $\varphi[y]$ as follows:

$$\varphi[y](t) = \begin{cases} \dfrac{y(T)}{1+h(T)} & \text{if} \quad h(T) < 1, \\ \Psi[y](t) & \text{if} \quad h(T) \geq 1, \end{cases}$$

$t \in [T_*, T]$.

Lemma 3.7.6. (i) *For each $y \in Y$, $\varphi[y]$ satisfies*

$$\varphi[y](T) + h(T)\varphi[y]\left(\tau(T)\right) = y(T).$$

(ii) *Suppose that $\{y_j\}_{j=1}^{\infty}$ is a sequence in Y converging to $y \in Y$ uniformly on every compact subinterval of $[T_*, \infty)$. Then $\varphi[y_i]$ converges to $\varphi[y]$ uniformly on $[T_*, T]$.*

Proof. It is obvious that (i) and (ii) hold for the case $h(T) < 1$. For the case $h(T) \geq 1$, (i) and (ii) follow from (ii) and (iii) of Lemma 3.7.5. □

For each $y \in Y$, we define the function $\Phi[y]$ as follows:

$$\Phi[y](t) = \begin{cases} \displaystyle\sum_{k=0}^{m} (-1)^k \left[h(t)\right]^k y\left(\tau^k(t)\right) + (-1)^{m+1}\left[h(t)\right]^{m+1} \varphi[y]\left(\tau^{m+1}(t)\right) & \\ \qquad\qquad\qquad \text{if} \quad t \in \left[\tau^{-m}(T), \tau^{-(m+1)}(T)\right], \quad m \in \mathbb{N}_0, & \\ \varphi[y](t) \qquad\qquad \text{if} \quad t \in [T_*, T]. & \end{cases}$$

Lemma 3.7.7. *Let $y \in Y$.*

(i) *$\Phi[y]$ is continuous on $[T_*, \infty)$;*
(ii) *$\Phi[y]$ satisfies*

$$\Phi[y](t) + h(t)\Phi[y]\left(\tau(t)\right) = y(t), \quad t \geq T;$$

(iii) *for $t \in [\tau(T), \infty)$ with $h(t) \geq 1$,*

$$\Phi[y](t) = \Psi[y](t);$$

(iv) *Φ is continuous on Y in the $C[T_*, \infty)$-topology.*

Proof. To show (i), it is easy to see that $\Phi[y]$ is continuous on

$$[T_*, \infty) \setminus \{\tau^{-m}(T) : \ m \in \mathbb{N}_0\}.$$

From Lemma 3.7.6 (i), it follows that

$$\lim_{t \to T^-} \Phi[y](t) = \varphi[y](T) = y(T) - h(T)\varphi[y](\tau(T)) = \lim_{t \to T^+} \Phi[y](t)$$

and that if $m \in \mathbb{N}$, then

$$\begin{aligned}
\lim_{t\to(\tau^{-m}(T))^-} \Phi[y](t) &= \sum_{k=0}^{m-1}(-1)^k \left[h\left(\tau^{-m}(T)\right)\right]^k y\left(\tau^{k-m}(T)\right) \\
&\quad +(-1)^m \left[h\left(\tau^{-m}(T)\right)\right]^m \varphi[y](T) \\
&= \sum_{k=0}^{m-1}(-1)^k \left[h\left(\tau^{-m}(T)\right)\right]^k y\left(\tau^{k-m}(T)\right) \\
&\quad +(-1)^m \left[h\left(\tau^{-m}(T)\right)\right]^m \Big(y(T) - h(T)\varphi[y]\left(\tau(T)\right)\Big) \\
&= \sum_{k=0}^{m}(-1)^k \left[h\left(\tau^{-m}(T)\right)\right]^k y\left(\tau^{k-m}(T)\right) \\
&\quad +(-1)^{m+1} \left[h\left(\tau^{-m}(T)\right)\right]^{m+1} \varphi[y]\Big(\tau^{m+1}\left(\tau^{-m}(T)\right)\Big) \\
&= \lim_{t\to(\tau^{-m}(T))^+} \Phi[y](t).
\end{aligned}$$

Consequently, $\Phi[y]$ is continuous on $[T_*, \infty)$.

An easy computation shows that (ii) follows.

Now we show (iii). If $h(T) < 1$, then there is no number $t \in [\tau(T), \infty)$ such that $h(t) \geq 1$ (recall the choice of T). Assume that $h(T) \geq 1$. Then

$$\Phi[y](t) = \varphi[y](t) = \Psi[y](t) \quad \text{for} \quad t \in [\tau(T), T].$$

We suppose that there is $m \in \mathbb{N}_0$ such that $\Phi[y] = \Psi[y]$ on $[\tau^{-(m-1)}(T), \tau^{-m}(T)]$ with $h(t) \geq 1$. In view of Lemma 3.7.7 (ii) and (3.116), we find that if $t \in [\tau^{-m}(T), \tau^{-(m+1)}(T)]$ and if $h(t) \geq 1$, then

$$\Phi[y](t) = y(t) - h(t)\Phi[y](\tau(t)) = y(t) - H(t)\Psi[y](\tau(t)) = \Psi[y](t).$$

By induction, we conclude that $\Phi[y](t) = \Psi[y](t)$ for $t \in [\tau(T), \infty)$ with $h(t) \geq 1$.

Finally we show (iv). Let $\{y_j\}_{j=1}^{\infty}$ be a sequence in Y converging to $y \in Y$ uniformly on every compact subinterval of $[T_*, \infty)$. Lemma 3.7.6 implies that $\Phi[y_j]$ converges to $\Phi[y]$ uniformly on $[T_*, t]$. It suffices to prove that $\Phi[y_j] \to \Phi[y]$ uniformly on $I_m := [\tau^{-m}(T), \tau^{-(m+1)}(T)]$, $m \in \mathbb{N}_0$. Since $|h(t)| \leq \lambda$ on $[t_0, \infty)$ for some $\lambda \geq 1$, we observe that

$$\begin{aligned}
\sup_{t\in I_m} \Big|\Phi[y_j](t) - \Phi[y](t)\Big| &\leq \sum_{k=0}^{m} \lambda^k \sup_{t\in I_m} \Big|y_j\left(\tau^k(t)\right) - y\left(\tau^k(t)\right)\Big| \\
&\quad +\lambda^{m+1} \sup_{t\in I_m} \Big|\varphi[y_j]\left(\tau^{m+1}(t)\right) - \varphi[y]\left(\tau^{m+1}(t)\right)\Big| \\
&\leq \lambda^m \sum_{k=0}^{m} \sup_{t\in I_{m-k}} \Big|y_j(t) - y(t)\Big| + \lambda^{m+1} \sup_{t\in[T_*,T]} \Big|\varphi[y_j](t) - \varphi[y](t)\Big|.
\end{aligned}$$

Then, $\sup_{t\in I_m} |\Phi[y_j](t) - \Phi[y](t)| \to 0$ as $j \to \infty$, so that $\Phi[y_j]$ converges to $\Phi[y]$ uniformly on I_m for $m \in \mathbb{N}_0$. □

Lemma 3.7.8. *Let $\{t_j\}_{j=0}^{\infty}$ be a sequence with $\lim_{j\to\infty} t_j = \infty$ and $h(t_j) \leq \nu < 1$, $j \in \mathbb{N}$, for some $\nu > 0$. Then $\lim_{j\to\infty} \Phi[y](t_j) = 0$ for each $y \in Y$.*

Proof. Let $y \in Y$. Since $\lim_{t\to\infty} y(t) = 0$, for each $\varepsilon > 0$, there exists $p \in \mathbb{N}$ such that
$$\frac{y(\tau^{-p}(T))}{1-\nu} < \frac{\varepsilon}{3}.$$
There exists $q \in \mathbb{N}$ such that
$$\frac{y(T)\nu^{r-p+1}}{1-\nu} < \frac{\varepsilon}{3} \quad \text{and} \quad \nu^{r+1} \sup_{t\in[T_*,T]} \Big|\varphi[y](t)\Big| < \frac{\varepsilon}{3} \quad \text{for all} \quad r \geq p+q.$$
Let $m \geq p+q$. Then $\tau^{m-p}(t) \geq \tau^{-p}(T)$ for $t \in [\tau^{-m}(T), \tau^{-(m+1)}(T)]$. In view of the monotonicity of y, we see that if $t \in [\tau^{-m}(T), \tau^{-(m+1)}(T)]$ and $|h(t)| \leq \nu$, then
$$\begin{aligned}
\Big|\Phi[y](t)\Big| &\leq \sum_{k=0}^{m} \nu^k y\left(\tau^k(t)\right) + \nu^{m+1}\Big|\varphi[y]\left(\tau^{m+1}(t)\right)\Big| \\
&\leq \sum_{k=0}^{m-p} \nu^k y\left(\tau^k(t)\right) + \sum_{k=m-p+1}^{m} \nu^k y\left(\tau^k(t)\right) + \frac{\varepsilon}{3} \\
&\leq y\left(\tau^{m-p}(t)\right) \sum_{k=0}^{m-p} \nu^k + y(T)\nu^{m-p+1} \sum_{k=0}^{p-1} \nu^k + \frac{\varepsilon}{3} \\
&\leq \frac{y(\tau^{-p}(T))}{1-\nu} + \frac{y(T)\nu^{m-p+1}}{1-\nu} + \frac{\varepsilon}{3} \; < \; \varepsilon.
\end{aligned}$$
This implies that $|\Phi[y](t)| < \varepsilon$ for $t \in [\tau^{-(p+q)}(T), \infty)$ with $|h(t)| \leq \nu$, and hence the conclusion follows. □

Lemma 3.7.9. *Let $m \in \mathbb{N}_0$. If t satisfies $t \geq \tau^{-m}(T)$ and $0 \leq h(t) \leq 1$, then*
$$\left|\sum_{k=0}^{m} (-1)^k \left[h(t)\right]^k y\left(\tau^k(t)\right)\right| \leq 2y\left(\tau^m(t)\right), \quad y \in Y. \tag{3.121}$$

Proof. Let $t \geq \tau^{-m}(T)$ satisfy $0 \leq h(t) \leq 1$ and let $y \in Y$. Put
$$A(t) := \sum_{k=0}^{m} (-1)^k [h(t)]^k y(\tau^k(t)).$$
It is easy to see that (3.121) holds for $m = 0$ and $m = 1$. If $m \geq 3$ is odd, then we can rewrite $A(t)$ as
$$A(t) = y(t) - \sum_{j=1}^{(m-1)/2} \left[h(t)\right]^{2j-1} \Big(y\left(\tau^{2j-1}(t)\right) - h(t)y\left(\tau^{2j}(t)\right)\Big) - \left[h(t)\right]^m y\left(\tau^m(t)\right)$$
and
$$A(t) = \sum_{j=0}^{(m-1)/2} \left[h(t)\right]^{2j} \Big(y\left(\tau^{2j}(t)\right) - h(t)y\left(\tau^{2j+1}(t)\right)\Big).$$
If $m \geq 2$ is even, then we can rewrite $A(t)$ as
$$A(t) = y(t) - \sum_{j=1}^{m/2} \left[h(t)\right]^{2j-1} \Big(y\left(\tau^{2j-1}(t)\right) - h(t)y\left(\tau^{2j}(t)\right)\Big)$$

and

$$A(t)=\sum_{j=0}^{(m/2)-1}[h(t)]^{2j}\Big(y\left(\tau^{2j}(t)\right)-h(t)y\left(\tau^{2j+1}(t)\right)\Big)+[h(t)]^m\,y\left(\tau^m(t)\right).$$

Since y is nonincreasing on $[T,\infty)$, we see that

$$y(t)-h(t)y(\tau(t))\le[1-h(t)]y(t),\quad t\ge\tau^{-1}(T).$$

Hence, for the case when $m\ge3$ is odd, we have

$$\begin{aligned}
A(t)&\ge-\sum_{j=1}^{(m-1)/2}[h(t)]^{2j-1}[1-h(t)]\,y\left(\tau^{2j-1}(t)\right)-[h(t)]^m\,y\left(\tau^m(t)\right)\\
&\ge-\sum_{j=1}^{(m-1)/2}[h(t)]^{2j-1}[1-h(t)]\,y\left(\tau^{m}(t)\right)-[h(t)]^m\,y\left(\tau^m(t)\right)\\
&=y\left(\tau^m(t)\right)\sum_{k=1}^{m}(-1)^k\,[h(t)]^k\\
&=-y\left(\tau^m(t)\right)h(t)\frac{1-[-h(t)]^m}{1+h(t)}\\
&\ge-2y\left(\tau^m(t)\right).
\end{aligned}$$

In the same way, we can show that $A(t)\le2y(\tau^m(t))$ for the case when $m\ge3$ is odd, and that $-2y(\tau^m(t))\le A(t)\le2y(\tau^m(t))$ for the case when $m\ge2$ is even. $\square$

Lemma 3.7.10. *Let $y\in Y$. Then $\lim_{t\to\infty}\Phi[y](t)=0$.*

Proof. Assume that $\lim_{t\to\infty}\Phi[y](t)=0$ does not hold. Then we first claim that there is a sequence $\{t_j\}_{j=1}^{\infty}$ such that

$$\text{(3.122)}\qquad\begin{cases}\lim\limits_{j\to\infty}t_j=\infty,\quad \lim\limits_{j\to\infty}\Phi[y](t_j)\quad\text{exists in}\quad[-\infty,\infty]\setminus\{0\},\\ 0<h(t_j)<1\quad\text{for}\quad j\in\mathbb{N},\quad\text{and}\quad\lim\limits_{j\to\infty}h(t_j)=1.\end{cases}$$

By assumption, there exists a sequence $\{s_j\}_{j=1}^{\infty}$ for which $s_j\to\infty$ as well as $\Phi[y](s_j)\to c\in[-\infty,\infty]\setminus\{0\}$ as $j\to\infty$. Since $-1<\mu\le h(t)\le\lambda$ for $t\ge t_0$, there exists a subsequence $\{t_j\}_{j=1}^{\infty}$ of $\{s_j\}_{j=1}^{\infty}$ such that $\lim_{j\to\infty}h(t_j)=d\in[\mu,\lambda]$. Lemma 3.7.8 implies that $d\ge1$. It can be shown that $h(t_j)<1$, $j\ge j_0$ for some $j_0\in\mathbb{N}$. Otherwise, there exists a subsequence $\{\widetilde{t}_j\}_{j=1}^{\infty}$ of $\{t_j\}_{j=1}^{\infty}$ such that $h(\widetilde{t}_j)\ge1$ for all $j\in\mathbb{N}$. From Lemma 3.7.7 (iii) and Lemma 3.7.5 (ii), it follows that

$$|c|=\left|\lim_{j\to\infty}\Phi[y](\widetilde{t}_j)\right|=\left|\lim_{j\to\infty}\Psi[y](\widetilde{t}_j)\right|\le\lim_{j\to\infty}\eta\left(\tau^{-1}(\widetilde{t}_j)\right)=0,$$

which is a contradiction. Since $d\ge1$, we see that $d=1$, so that $0<h(t_j)<1$, $j\ge j_1$ for some $j_1\ge j_0$. This proves the existence of $\{t_j\}_{j=1}^{\infty}$ satisfying (3.122).

Suppose that $\{t_j\}_{j=1}^{\infty}$ is a sequence satisfying (3.122). Let $\varepsilon>0$ be arbitrary. There exists $p\in\mathbb{N}$ such that

$$\eta(t)<\varepsilon,\quad t\ge\tau^{-p-1}(T).$$

There is $\delta > 0$ such that if $s_1, s_2 \in [\tau^{-p}(T), \tau^{-(p+1)}(T)]$ with $|s_1 - s_2| < \delta$, then

$$\left|\Phi[y](s_1) - \Phi[y](s_2)\right| < \varepsilon. \tag{3.123}$$

Consider the mapping $N : [\tau^{-p}(T), \infty) \to \mathbb{N}_0$ such that

$$\tau^{N(t)}(t) \in \left[\tau^{-p}(T), \tau^{-(p+1)}(T)\right) \quad \text{for} \quad t \geq \tau^{-p}(T).$$

We note that $\lim_{t\to\infty} N(t) = \infty$. It is easily verified that $\{t_j\}_{j=1}^{\infty}$ has a subsequence $\{u_j\}_{j=1}^{\infty}$ such that

$$\lim_{j\to\infty} \tau^{N(u_j)}(u_j) \quad \text{exists in} \quad \left[\tau^{-p}(T), \tau^{-(p+1)}(T)\right].$$

Put $\bar{u} = \lim_{j\to\infty} \tau^{N(u_j)}(u_j)$. Then we find that

$$h(\bar{u}) = \lim_{j\to\infty} h\left(\tau^{N(u_j)}(u_j)\right) = \lim_{j\to\infty} h(u_j) = 1.$$

There exists $j_0 \in \mathbb{N}$ such that $u_j \geq \tau^{-p}(T)$ and $|\tau^{N(u_j)}(u_j) - \bar{u}| < \delta$ for $j \geq j_0$. From Lemma 3.7.7 (ii), we observe that

$$\begin{aligned} \Phi[y](t) &= y(t) - h(t)\Phi[y]\left(\tau(t)\right) \\ &= y(t) - h(t)y\left(\tau(t)\right) + [h(t)]^2\,\Phi[y]\left(\tau^2(t)\right) \\ &= \sum_{k=0}^{m-1}(-1)^k\,[h(t)]^k\,y\left(\tau^k(t)\right) + (-1)^m\,[h(t)]^m\,\Phi[y]\left(\tau^m(t)\right) \end{aligned}$$

for $t \geq \tau^{-m+1}(T)$. Since $h(\bar{u}) = 1$, we have

$$\begin{aligned} \left|\Phi[y](u_j) - \Phi[y]\left(\tau^{-N(u_j)}(\bar{u})\right)\right| \leq{} & \left|\sum_{k=0}^{N(u_j)-1}(-1)^k\,[h(u_j)]^k\,y\left(\tau^k(u_j)\right)\right| \\ &+ \left|\sum_{k=0}^{N(u_j)-1}(-1)^k y\left(\tau^k\left(\tau^{-N(u_j)}(\bar{u})\right)\right)\right| \\ &+ \left|[h(u_j)]^{N(u_j)}\,\Phi[y]\left(\tau^{N(u_j)}(u_j)\right) - \Phi[y]\left(\tau^{N(u_j)}\left(\tau^{-N(u_j)}(\bar{u})\right)\right)\right|. \end{aligned} \tag{3.124}$$

Lemma 3.7.9 implies that if $j \geq j_0$, then

$$\begin{aligned} \sum_{k=0}^{N(u_j)-1}(-1)^k\,[h(u_j)]^k\,y\left(\tau^k(u_j)\right) &\leq 2y\left(\tau^{N(u_j)-1}(u_j)\right) \\ &\leq 2\eta\left(\tau^{N(u_j)-1}(u_j)\right) < 2\varepsilon \end{aligned} \tag{3.125}$$

and

$$\begin{aligned} \sum_{k=0}^{N(u_j)-1}(-1)^k y\left(\tau^k\left(\tau^{-N(u_j)}(\bar{u})\right)\right) &\leq 2y\left(\tau^{N(u_j)-1}\left(\tau^{-N(u_j)}(\bar{u})\right)\right) \\ &\leq 2\eta\left(\tau^{-1}(\bar{u})\right) < 2\varepsilon. \end{aligned} \tag{3.126}$$

From Lemma 3.7.7 (iii), Lemma 3.7.5 (ii), and the fact that $h(\bar{u}) = 1$, it follows that

$$|\Phi[y](\bar{u})| = |\Psi[y](\bar{u})| \leq \eta\left(\tau^{-1}(\bar{u})\right) < \varepsilon.$$

Then we observe that for $j \geq j_0$,

$$\begin{aligned}
(3.127)\qquad & \left|[h(u_j)]^{N(u_j)}\,\Phi[y]\left(\tau^{N(u_j)}(u_j)\right) - \Phi[y]\left(\tau^{N(u_j)}\left(\tau^{-N(u_j)}(\bar{u})\right)\right)\right| \\
\leq\ & \left|[h(u_j)]^{N(u_j)}\right|\left|\Phi[y]\left(\tau^{N(u_j)}(u_j)\right) - \Phi[y](\bar{u})\right| + \left|[h(u_j)]^{N(u_j)} - 1\right|\left|\Phi[y](\bar{u})\right| \\
\leq\ & \left|\Phi[y]\left(\tau^{N(u_j)}(u_j)\right) - \Phi[y](\bar{u})\right| + 2\left|\Phi[y](\bar{u})\right| \\
<\ & 3\varepsilon
\end{aligned}$$

because of (3.123). Combining (3.124)–(3.127), we obtain

$$\left|\Phi[y](u_j) - \Phi[y]\left(\tau^{-N(u_j)}(\bar{u})\right)\right| < 7\varepsilon, \quad j \geq j_0.$$

This means that

$$\lim_{j\to\infty}\left|\Phi[y](u_j) - \Phi[y]\left(\tau^{-N(u_j)}(\bar{u})\right)\right| = 0.$$

On the other hand, in view of Lemma 3.7.7 (iii) and Lemma 3.7.5 (ii), we see that

$$\lim_{j\to\infty}\left|\Phi[y]\left(\tau^{-N(u_j)}(\bar{u})\right)\right| \leq \lim_{j\to\infty}\eta\left(\tau^{-N(u_j)-1}(\bar{u})\right) = 0.$$

From (3.122) it follows that

$$\lim_{j\to\infty}\left|\Phi[y](u_j) - \Phi[y]\left(\tau^{-N(u_j)}(\bar{u})\right)\right| \qquad \text{exists and is not equal to} \quad 0.$$

This is a contradiction. The proof is complete. □

Proof of Proposition 3.7.4. This result follows from Lemmas 3.7.7 and 3.7.10. □

Proof of Theorem 3.7.2. First we prove the "only if" part of Theorem 3.7.2. Let x be a solution of (3.109) which satisfies (3.113). Put $y(t) = x(t) + h(t)x(\tau(t))$. Then (3.112) implies that $y(t) = b + o(1)$ as $t \to \infty$. Integration of (3.109) over $[T, \infty)$ yields

$$b - y(T) + \sigma\int_T^\infty f\left(s, x\left(g(s)\right)\right)ds = 0,$$

where $T \geq t_0$. Hence we obtain

$$\int_T^\infty f\left(s, x\left(g(s)\right)\right)ds < \infty.$$

Noting that x satisfies (3.110) and using the monotonicity of f, we conclude that (3.111) holds.

Now let us show the "if" part of Theorem 3.7.2. Put

$$\eta(t) = \int_t^\infty f(s,a)ds, \quad t \geq T.$$

We use Proposition 3.7.4 for this η. We can take constants $b > 0$, $\delta > 0$, and $\varepsilon > 0$ such that

$$0 < \delta + \varepsilon \leq \frac{b}{1+h(t)} \leq a - \varepsilon, \quad t \geq T_*.$$

Define the mapping $\Theta : Y \to C[T_*, \infty)$ as follows:

$$(\Theta y)(t) = \begin{cases} \displaystyle\int_t^\infty F\left(s, \frac{b}{1+h\left(g(s)\right)} + \sigma\Phi[y]\left(g(s)\right)\right)ds & \text{if} \quad t \geq T, \\ (\Theta y)(T) & \text{if} \quad t \in [T_*, T), \end{cases}$$

where

$$F(t,u)=\begin{cases} f(t,\delta) & \text{if} \quad u\le\delta,\\ f(t,u) & \text{if} \quad \delta<u<a,\\ f(t,a) & \text{if} \quad u\ge a.\end{cases}$$

It is easy to see that Θ is well defined on Y and maps Y into itself. Since Φ is continuous on Y, the Lebesgue dominated convergence theorem shows that Θ is continuous on Y. Let I be an arbitrary compact subinterval of $[T,\infty)$. We find that

$$\left|(\Theta y)'(t)\right|\le \max\left\{f(s,a):\ s\in I\right\},\quad t\in I,$$

so that $\{(\Theta y)'(t)\}_{y\in Y}$ is uniformly bounded on I. The mean value theorem shows that $\Theta(Y)$ is equicontinuous on I. Since

$$\left|(\Theta y)(t_1)-(\Theta y)(t_2)\right|=0 \quad \text{for} \quad t_1,t_2\in[T_*,T],$$

we conclude that $\Theta(Y)$ is equicontinuous on every compact subinterval of $[T_*,\infty)$. Obviously, $\Theta(Y)$ is uniformly bounded on $[T_*,\infty)$. Hence, by the Arzelà–Ascoli theorem (Theorem 1.4.17), $\Theta(Y)$ is relatively compact. Consequently, we may apply the Schauder–Tychonov fixed point theorem (Theorem 1.4.25) to the operator Θ and we conclude that there exists $\widetilde{y}\in Y$ such that $\widetilde{y}=\Theta\widetilde{y}$. Set

$$x(t)=\frac{b}{1+h(t)}+\sigma\Phi[\widetilde{y}](t).$$

Proposition 3.7.4 implies that x satisfies (3.113) and that there exists a number $\widetilde{T}\ge T$ such that $\delta\le x(g(t))\le a$ for $t\ge\widetilde{T}$. Then $F(t,x(g(t)))=f(t,x(g(t)))$ for $t\ge\widetilde{T}$. Observe that

$$\begin{aligned} x(t)+h(t)x\left(\tau(t)\right) &= \frac{b}{1+h(t)}+\frac{bh(t)}{1+h\left(\tau(t)\right)}+\sigma\Big(\Phi[\widetilde{y}](t)+h(t)\Phi[\widetilde{y}]\left(\tau(t)\right)\Big)\\ &= b+\sigma\widetilde{y}(t)\\ &= b+\sigma\int_t^\infty f\Big(s,x\left(g(s)\right)\Big)\,ds \end{aligned}$$

for $t\ge\widetilde{T}$. By differentiating the above equation, we see that x is a solution of (3.109). The proof is complete. ☐

3.8. Notes

Lemma 3.2.1 is obtained by Yu, Wang, and Qian [**293**]. The rest of Section 3.2 is taken from Li [**180**]. Lemma 3.3.1 is based on Yu [**289**]. Lemma 3.3.2 is obtained by Győri [**117**]. The rest of Section 3.3 is adopted from Li and Yan [**206**]. The treatment in Section 3.4 is based on [**203**]. The material of Section 3.5 is taken from He and Li [**122**], a special case is obtained by Yu [**289**]. Section 3.6 is taken from Agarwal, Tang, and Wang [**10**], and related work can be found also in Yang and Zhang [**286**]. The contents of Section 3.7 is based on Tanaka [**256**].

CHAPTER 4

Second Order Ordinary Differential Equations

4.1. Introduction

The study of oscillation for second order ordinary differential equations is interesting from the theoretical as well as the practical point of view. There are many articles and books studying this topic. The survey paper by Wong [**267**] contains a complete bibliography up to 1968. Also, for a detailed account on second order nonlinear differential equations, see the paper by Kartsatos [**142**], which gives more than 300 references. In this chapter, we will present some criteria for oscillation and for the existence of positive solutions of nonlinear differential equations of second order, mainly including some contributions of the authors and their colleagues.

4.2. Oscillation of Superlinear Equations

Consider the second order nonlinear ordinary differential equation

$$y''(t) + a(t)f(y(t)) = 0, \tag{4.1}$$

where a is a continuous real-valued function on an interval $[t_0, \infty)$ without any restriction on its sign, and f is a continuous real-valued function on the real line $\mathbb{R}$, which is continuously differentiable on $\mathbb{R} \setminus \{0\}$ and satisfies $yf(y) > 0$ and $f'(y) \geq 0$ for every $y \neq 0$. The prototype of (4.1) is the so-called *Emden–Fowler equation*

$$y''(t) + a(t)|y(t)|^{\gamma} \operatorname{sgn} y(t) = 0 \quad \text{with} \quad \gamma > 0. \tag{4.2}$$

Here we are interested in the oscillation of solutions of (4.1) when $f(y)$ satisfies, in addition, the superlinearity condition

$$0 < \int_{\varepsilon}^{\infty} \frac{dy}{f(y)} < \infty \quad \text{and} \quad 0 < \int_{-\varepsilon}^{-\infty} \frac{dy}{f(y)} < \infty \quad \text{for all} \quad \varepsilon > 0. \tag{4.3}$$

Throughout the section, we shall restrict our attention only to those solutions of the differential equation (4.1) which exist on some ray $[T, \infty)$, where $T \geq t_0$ may depend on the particular solution. Note that under quite general conditions there will always exist solutions of (4.1) which are extendable to an interval $[T, \infty)$, $T \geq t_0$, even though there will also exist nonextendable solutions [**59**].

In the linear case, i.e., (4.2) with $\gamma = 1$, Kamenev [**140**] showed that the condition

$$\limsup_{t \to \infty} \frac{1}{t^{\alpha}} \int_{t_0}^{t} (t-s)^{\alpha} a(s) ds = \infty \quad \text{for some} \quad \alpha > 1 \tag{4.4}$$

alone is sufficient for oscillation of (4.2). Wong [**269**] extended this result to equation (4.2) in the sublinear case ($0 < \gamma < 1$) and proposed [**272**] the following conjecture.

Wong's Conjecture. Condition (4.4) together with the compatibility condition

$$\liminf_{t\to\infty} \frac{1}{t^\beta} \int_{t_0}^{t} (t-s)^\beta a(s)ds > -\infty \quad \text{for some} \quad \beta \geq 1 \tag{4.5}$$

is sufficient for oscillation of (4.2) in the superlinear case ($\gamma > 1$).

Remark 4.2.1. The special case when $\beta = 0$ in (4.5) has been answered affirmatively in an earlier paper [**268**]. In addition, Wong [**273**] proved the conjecture in the case when conditions (4.4) and (4.5) hold for $\alpha > 1$ and $\beta \geq 1$, where $\alpha, \beta \in \mathbb{N}$. But it remained an open problem when $\alpha > 1$ and $\beta \geq 1$ are positive real numbers.

In the following, we first prove the above conjecture and extend it to the general superlinear differential equation (4.1), subject to the nonlinearity condition [**272**]

$$f'(y)G(y) \geq d > 1 \quad \text{for all} \quad y, \quad \text{where} \quad G(y) = \int_y^\infty \frac{du}{f(u)}. \tag{4.6}$$

Theorem 4.2.2. *Assume that* (4.3) *and* (4.6) *hold. Then conditions* (4.4) *and* (4.5) *imply that* (4.1) *is oscillatory.*

For (4.2) with $\gamma > 1$, (4.6) becomes

$$f'(y)G(y) = \gamma(\gamma-1)^{-1} = d > 1.$$

Hence, our theorem leads to the following oscillation result.

Corollary 4.2.3. *Equation* (4.2) *with* $\gamma > 1$ *is oscillatory if* (4.4) *and* (4.5) *are satisfied.*

This oscillation criterion is Wong's conjecture.

Proof of Theorem 4.2.2. Assume that the differential equation (4.1) admits a nonoscillatory solution y on an interval $[T,\infty)$, $T \geq \max\{t_0, 1\}$. Without loss of generality, this solution can be supposed to be such that $y(t) \neq 0$ for $t \geq T$. Furthermore, we observe that the substitution $u = -y$ transforms (4.1) into the equation $u''(t) + a(t)\bar{f}(u(t)) = 0$, where $\bar{f}(y) = -f(-y)$, $y \in \mathbb{R}$. The function $\bar{f}$ is subject to the same conditions as f. So there is no loss of generality to restrict our discussion to the case where the solution y is positive on the interval $[T,\infty)$. Define $w(t) = G(y(t))$. Then we obtain

$$w''(t) = a(t) + [w'(t)]^2 f'(y(t)) \quad \text{for} \quad t \geq T. \tag{4.7}$$

Since $f'(y) \geq 0$, we note that the integral of $f'(y(t))[w'(t)]^2$ over $[T,\infty)$ exists, finite or infinite. Hence we consider the two cases when $\int_T^\infty [w'(t)]^2 f'(y(t))dt$ is finite or infinite.

First, if $\int_T^\infty [w'(t)]^2 f'(y(t))dt < \infty$, then the proof given in Wong [**273**] is also valid for positive reals α, β, and hence for brevity is not reproduced. Second, we consider the case when $\int_T^\infty [w'(s)]^2 f'(y(s))ds = \infty$. In view of (4.6), we can choose a real number $k \geq \max\{\beta, 3\}$ such that

$$\frac{k-1}{k-2} < d. \tag{4.8}$$

As $k \geq \beta$, it is easy to verify that condition (4.5) implies that

$$\liminf_{t\to\infty} \frac{1}{t^{k-1}} \int_T^t (t-s)^{k-1} a(s)ds > -\infty. \tag{4.9}$$

Now, for each $t \geq T$, from (4.7) we obtain

$$\begin{aligned}\int_T^t (t-s)^{k-1} a(s)ds + \int_T^t (t-s)^{k-1} [w'(s)]^2 f'(y(s))ds \\ = \int_T^t (t-s)^{k-1} w''(s)ds \\ = -(t-T)^{k-1} w'(T) + (k-1) \int_T^t (t-s)^{k-2} w'(s)ds.\end{aligned}$$

So we have

$$\begin{aligned}\frac{1}{t^{k-1}} \int_T^t (t-s)^{k-1} a(s)ds &= -\left(1 - \frac{T}{t}\right)^{k-1} w'(T) \\ &+ \frac{k-1}{t^{k-1}} \int_T^t (t-s)^{k-2} w'(s)ds - \frac{1}{t^{k-1}} \int_T^t (t-s)^{k-1} [w'(s)]^2 f'(y(s))ds\end{aligned} \tag{4.10}$$

for $t \geq T$. Next, by taking into account (4.8), we choose a constant d_1 with

$$\frac{k-1}{d(k-2)} < d_1 < 1. \tag{4.11}$$

We claim that, for every $t^* \geq T$, there exists $t \geq t^*$ such that

$$\int_T^t (t-s)^{k-1} [w'(s)]^2 f'(y(s))ds > \frac{k-1}{d_1} \int_T^t (t-s)^{k-2} w'(s)ds.$$

Otherwise, there is $t^* \geq T$ such that

$$\int_T^t (t-s)^{k-1} [w'(s)]^2 f'(y(s))ds \leq \frac{k-1}{d_1} \int_T^t (t-s)^{k-2} w'(s)ds \tag{4.12}$$

for all $t \geq t^*$. By using the Schwarz inequality and (4.12) and taking into account the definitions of w and d, for $t \geq t^*$ we find

$$\begin{aligned}0 &\leq \int_T^t (t-s)^{k-2} w'(s)ds \\ &\leq \left\{\int_T^t (t-s)^{k-1} [w'(s)]^2 f'(y(s))ds\right\}^{1/2} \left\{\int_T^t \frac{(t-s)^{k-3}}{f'(y(s))} ds\right\}^{1/2} \\ &\leq \left\{\frac{k-1}{d_1} \int_T^t (t-s)^{k-2} w'(s)ds\right\}^{1/2} \left\{\frac{1}{d} \int_T^t (t-s)^{k-3} w(s)ds\right\}^{1/2},\end{aligned}$$

and so we obtain

$$\int_T^t (t-s)^{k-2} w'(s)ds \leq \frac{k-1}{d_1 d} \int_T^t (t-s)^{k-3} w(s)ds \quad \text{for} \quad t \geq t^*. \tag{4.13}$$

But for any $t \geq t^*$ we have

$$\int_T^t (t-s)^{k-3} w(s)ds = \frac{1}{k-2} \left((t-T)^{k-2} w(T) + \int_T^t (t-s)^{k-2} w'(s)ds\right).$$

So, (4.13) gives

$$\left(d_1 d \frac{k-2}{k-1} - 1\right) \int_T^t (t-s)^{k-2} w'(s)ds \leq (t-T)^{k-2} w(T)$$

for every $t \geq t^*$. Therefore, by (4.12), we find

$$\left(d_1 d\frac{k-2}{k-1} - 1\right)\frac{1}{t^{k-2}}\int_T^t (t-s)^{k-1}[w'(s)]^2 f'(y(s))ds \leq \frac{k-1}{d_1}\left(1-\frac{T}{t}\right)^{k-2} w(T) \tag{4.14}$$

for $t \geq t^*$. But one has

$$\int_T^\infty [w'(s)]^2 f'(y(s))ds = \infty$$

and hence

$$\lim_{t\to\infty}\frac{1}{t^{k-2}}\int_T^t (t-s)^{k-1}[w'(s)]^2 f'(y(s))ds = \infty.$$

On the other hand, (4.11) implies that $d_1 d(k-2)/(k-1) - 1 > 0$. Thus, (4.14) leads to a contradiction. Hence our claim is proved and so we can consider a sequence $\{t_n\}_{n=1}^\infty$ of points in the interval $[T, \infty)$ with $\lim_{n\to\infty} t_n = \infty$ and such that

$$\int_T^{t_n} (t_n - s)^{k-1}[w'(s)]^2 f'(y(s))ds > \frac{k-1}{d_1}\int_T^{t_n}(t_n-s)^{k-2} w'(s)ds, \quad n \in \mathbb{N}.$$

Then from (4.10) it follows that

$$\begin{aligned}\frac{1}{t_n^{k-1}}\int_T^{t_n}(t_n-s)^{k-1}a(s)ds \;&=\; -\left(1-\frac{T}{t_n}\right)^{k-1} w'(T) \\ &+\frac{k-1}{t_n^{k-1}}\int_T^{t_n}(t_n-s)^{k-2}w'(s)ds - \frac{1}{t_n^{k-1}}\int_T^{t_n}(t_n-s)^{k-1}[w'(s)]^2 f'(y(s))ds \\ &< -\left(1-\frac{T}{t_n}\right)^{k-1} w'(T) + (d_1 - 1)\frac{1}{t_n^{k-1}}\int_T^{t_n}(t_n-s)^{k-1}[w'(s)]^2 f'(y(s))ds \qquad (4.15)\end{aligned}$$

for $n \in \mathbb{N}$. Since

$$\lim_{t\to\infty}\frac{1}{t^{k-1}}\int_T^t (t-s)^{k-1}[w'(s)]^2 f'(y(s))ds = \int_T^\infty [w'(s)]^2 f'(y(s))ds = \infty$$

and, by (4.11), $d_1 < 1$, inequality (4.15) ensures that

$$\lim_{n\to\infty}\frac{1}{t_n^{k-1}}\int_T^{t_n}(t_n-s)^{k-1}a(s)ds = -\infty.$$

This shows that

$$\liminf_{t\to\infty}\frac{1}{t^{k-1}}\int_T^t (t-s)^{k-1}a(s)ds = -\infty. \tag{4.16}$$

Finally, for every $t \geq T$, we obtain

$$\begin{aligned}\int_{t_0}^t (t-s)^{k-1}a(s)ds \;&\leq\; \int_{t_0}^T (t-s)^{k-1}|a(s)|ds + \int_T^t (t-s)^{k-1}a(s)ds \\ &\leq\; (t-t_0)^{k-1}\int_{t_0}^T |a(s)|ds + \int_T^t (t-s)^{k-1}a(s)ds.\end{aligned}$$

Thus, from (4.16) it follows that

$$\liminf_{t\to\infty}\frac{1}{t^{k-1}}\int_{t_0}^t (t-s)^{k-1}a(s)ds = -\infty,$$

which contradicts (4.9). The proof of the theorem is now complete. □

In Theorem 4.2.2, condition (4.4) is not necessary. In fact, condition (4.4) can be replaced by another condition. That is to say, we have the following result.

Theorem 4.2.4. *Assume that* (4.3) *and* (4.6) *hold. Then* (4.5) *and*

$$\limsup_{t\to\infty} \frac{1}{t}\int_{t_0}^{t} [B(s)]^2\, ds = \infty, \qquad \textit{where} \qquad B(t) = \int_{t_0}^{t} a(s)ds \tag{4.17}$$

imply that (4.1) *is oscillatory.*

Corollary 4.2.5. *Conditions* (4.5) *and* (4.17) *are sufficient for oscillation of* (4.2) *with* $\gamma > 1$.

Proof of Theorem 4.2.4. Similarly as in the proof of Theorem 4.2.2, assume that y is positive on the interval $[T,\infty)$. Define $w(t) = G(y(t))$ and use (4.1) to obtain (4.7). We consider the two cases when $\int_T^\infty [w'(t)]^2 f'(y(t))dt$ is finite or infinite.

If $\int_T^\infty [w'(s)]^2 f'(y(s))ds = \infty$, then the proof is similar to that of Theorem 4.2.2 and hence will be omitted. Now we assume that $\int_T^\infty [w'(t)]^2 f'(y(t))dt$ is finite. We shall show that

$$\lim_{t\to\infty} \frac{w(t)}{t} = 0. \tag{4.18}$$

Since $(w')^2 f' \in L^1[T,\infty)$, for each $\varepsilon > 0$ we can choose $T_1 \geq T$ such that

$$\int_{T_1}^{\infty} [w'(s)]^2 f'(y(s))ds \leq \frac{\varepsilon}{4}. \tag{4.19}$$

Furthermore, by using the Schwarz inequality, for $t \geq T_1$ we obtain

$$\begin{aligned} w(t) - w(T_1) &\leq \left| \int_{T_1}^{t} w'(s)ds \right| \\ &\leq \left\{ \int_{T_1}^{t} [w'(s)]^2 f'(y(s))ds \right\}^{1/2} \left\{ \int_{T_1}^{t} \frac{1}{f'(y(s))} ds \right\}^{1/2}, \end{aligned}$$

and so, in view of (4.19), we have

$$w(t) \leq w(T_1) + \frac{\sqrt{\varepsilon}}{2} \left\{ \int_{T_1}^{t} \frac{1}{f'(y(s))} ds \right\}^{1/2} \qquad \text{for} \qquad t \geq T_1. \tag{4.20}$$

If

$$\int_{T_1}^{\infty} \frac{1}{f'(y(s))} ds < \infty,$$

then from (4.20) it follows that w is bounded on $[T_1,\infty)$ and hence (4.18) is satisfied. So we assume that

$$\int_{T_1}^{\infty} \frac{1}{f'(y(s))} ds = \infty.$$

Then there exists a point $T_2 > T_1$ such that

$$w(T_1) \leq \frac{\sqrt{\varepsilon}}{2} \left\{ \int_{T_1}^{t} \frac{1}{f'(y(s))} ds \right\}^{1/2} \qquad \text{for} \qquad t \geq T_2,$$

and consequently (4.20) and then (4.6) gives

$$w(t) \leq \sqrt{\varepsilon} \left\{ \int_{T_1}^{t} \frac{1}{f'(y(s))} ds \right\}^{1/2} \leq \sqrt{\varepsilon/d} \left\{ \int_{T_1}^{t} w(s)ds \right\}^{1/2} \qquad \text{for} \qquad t \geq T_2. \tag{4.21}$$

Hence

$$\frac{d}{dt}\left\{\int_{T_1}^{t} w(s)ds\right\}^{1/2} \leq \frac{\sqrt{\varepsilon/d}}{2} \quad \text{for} \quad t \geq T_2. \tag{4.22}$$

Upon integrating (4.22), we have for $t \geq T_2$

$$\left\{\int_{T_1}^{t} w(s)ds\right\}^{1/2} - \left\{\int_{T_1}^{T_2} w(s)ds\right\}^{1/2} \leq \frac{\sqrt{\varepsilon/d}}{2}(t-T_2) < \frac{\sqrt{\varepsilon/d}}{2}t.$$

So, by letting

$$T_3 = \max\left\{T_2,\ \frac{2}{\sqrt{\varepsilon/d}}\left(\int_{T_1}^{T_2} w(s)ds\right)^{1/2}\right\},$$

we obtain for $t \geq T_3$

$$\begin{aligned}\left\{\int_{T_1}^{t} w(s)ds\right\}^{1/2} &< \frac{\sqrt{\varepsilon/d}}{2}t + \left\{\int_{T_1}^{T_2} w(s)ds\right\}^{1/2} \\ &\leq \frac{\sqrt{\varepsilon/d}}{2}t + \frac{\sqrt{\varepsilon/d}}{2}t \\ &= \sqrt{\frac{\varepsilon}{d}}t.\end{aligned}$$

Hence, (4.21) gives

$$w(t) < \frac{\varepsilon t}{d} \quad \text{for all} \quad t \geq T_3.$$

Since $\varepsilon > 0$ is arbitrary, this proves (4.18).

Integrating (4.7) from T to t, one obtains

$$B(t) = \int_{t_0}^{t} a(s)ds = -w'(T) + B(T) + w'(t) - \int_{T}^{t} [w'(s)]^2 f'(y(s))ds,$$

from which it follows that

$$[B(t)]^2 \leq 3\left(C_0^2 + k_0^2 + [w'(t)]^2\right), \tag{4.23}$$

where

$$C_0 = w'(T) + B(T) \quad \text{and} \quad k_0 = \int_{T}^{\infty} [w'(s)]^2 f'(y(s))ds.$$

Next we estimate the integral of $(w')^2$ as

$$\begin{aligned}\frac{1}{t}\int_{T}^{t} [w'(s)]^2 ds &\leq \frac{1}{t}\max_{T\leq s\leq t}\frac{1}{f'(y(s))}\int_{T}^{t} f'(y(u))[w'(u)]^2 du \\ &\leq \frac{k_0}{dt}\max_{T\leq s\leq t} w(s).\end{aligned} \tag{4.24}$$

By (4.18), we can choose $T_4 \geq T$ such that $|w(t)| \leq t$ for $t \geq T$, hence

$$\max_{T\leq s\leq t} w(s) \leq \max_{T\leq s\leq T_4} w(s) + t = K_1 + t, \tag{4.25}$$

where $K_1 = \max_{T\leq s\leq T_4} w(s)$. Combining (4.24) and (4.25) in (4.23), we find

$$\frac{1}{t}\int_{T}^{t} [B(s)]^2\, ds \leq 3\left(C_0^2 + k_0^2\right) + \frac{3k_0}{dt}(K_1 + t). \tag{4.26}$$

Letting t tend to infinity in (4.26) we obtain the desired contradiction to (4.17). $\square$

4.3. Oscillation of Sublinear Equations

In this section we continue to study oscillation of (4.1) under the condition

$$(4.27)\qquad 0 < \int_0^{\varepsilon} \frac{dy}{f(y)} < \infty \quad \text{and} \quad 0 < \int_0^{-\varepsilon} \frac{dy}{f(y)} < \infty \quad \text{for all} \quad \varepsilon > 0,$$

which corresponds to the special case $f(y) = |y|^\gamma \operatorname{sgn} y$ when $0 < \gamma < 1$. The coefficient $a(t)$ is allowed to be negative for arbitrarily large t.

Theorem 4.3.1. *Assume* (4.27) *and*

$$(4.28)\qquad f'(y)F(y) \geq \frac{1}{c} > 0 \quad \textit{for all} \quad y, \quad \textit{where} \quad F(y) = \int_0^y \frac{du}{f(u)}.$$

Then condition (4.4) *implies that* (4.1) *is oscillatory.*

Corollary 4.3.2. *Condition* (4.4) *is sufficient for* (4.2) *to be oscillatory in the sublinear case, i.e., when* $0 < \gamma < 1$.

Proof of Theorem 4.3.1. Assume that the differential equation (4.1) admits a nonoscillatory solution y on an interval $[T, \infty)$, $T \geq \max\{t_0, 1\}$. Without loss of generality, this solution can be supposed to be such that $y(t) > 0$ for $t \geq T$. Define $w(t) = F(y(t))$. Then we obtain

$$(4.29)\qquad w''(t) + a(t) + [w'(t)]^2 f'(y(t)) = 0 \quad \text{for} \quad t \geq T.$$

Since $f'(y) \geq 0$, we note that the integral of $f'(y(t))[w'(t)]^2$ over $[T, \infty)$ exists, finite or infinite. Hence we consider the two cases when $\int_T^\infty [w'(t)]^2 f'(y(t))dt$ is finite or infinite.

If $\int_T^\infty [w'(t)]^2 f'(y(t))dt < \infty$, then we can show in exactly the same way as in the proof of Theorem 4.2.4 that (4.18) holds. Now we choose a real number $k \geq \max\{\alpha, 3\}$. From (4.29) we obtain

$$\begin{aligned}\frac{1}{t^{k-1}} \int_T^t (t-s)^{k-1} a(s)ds &= -\left(1 - \frac{T}{t}\right)^{k-1} w'(T) \\ &\quad + \frac{k-1}{t^{k-1}} \int_T^t (t-s)^{k-2} w'(s)ds - \frac{1}{t^{k-1}} \int_T^t (t-s)^{k-1}[w'(s)]^2 f'(y(s))ds\end{aligned}$$

for $t \geq T$. Set

$$g(t) = \frac{1}{t^{k-1}} \int_T^t (t-s)^{k-1}[w'(s)]^2 f'(y(s))ds.$$

Since

$$g(t) = \sum_{i=0}^{k-1} \binom{k-1}{i} \int_T^t \frac{(-1)^{k-1-i}}{t^{k-1-i}} s^{k-1-i}[w'(s)]^2 f'(y(s))ds$$

and

$$0 \leq \int_T^t \frac{s^{k-1-i}}{t^{k-1-i}} [w'(s)]^2 f'(y(s))ds \leq \int_T^t [w'(s)]^2 f'(y(s))ds < \infty,$$

g is bounded on $[T, \infty)$. On the other hand,

$$\begin{aligned} g'(t) &= \frac{k-1}{t^k}\left(t\int_T^t (t-s)^{k-2}[w'(s)]^2 f'(y(s))ds \right. \\ &\qquad \left. - \int_T^t (t-s)^{k-1}[w'(s)]^2 f'(y(s))ds\right) \\ &\geq \frac{k-1}{t^{k-1}}\left(\int_T^t (t-s)^{k-1}[w'(s)]^2 f'(y(s))ds \right. \\ &\qquad \left. - \int_T^t (t-s)^{k-1}[w'(s)]^2 f'(y(s))ds\right) \\ &= 0. \end{aligned}$$

Hence $\lim_{t\to\infty} g(t)$ exists, say $\lim_{t\to\infty} g(t) = k_0$. By (4.18) we have

$$\frac{1}{t^{k-1}}\int_T^t (t-s)^{k-1}a(s)ds = -w'(T) - k_0.$$

But, for any $t \geq T$, we obtain

$$\begin{aligned} \int_T^t (t-s)^{k-1}a(s)ds &\leq \int_{t_0}^T (t-s)^{k-1}\left|a(s)\right| ds + \int_T^t (t-s)^{k-1}a(s)ds \\ &\leq (t-t_0)^{k-1}\int_{t_0}^T \left|a(s)\right| ds + \int_T^t (t-s)^{k-1}a(s)ds. \end{aligned}$$

So we derive

$$\limsup_{t\to\infty} \frac{1}{t^{k-1}}\int_T^t (t-s)^{k-1}a(s)ds \leq \int_{t_0}^T \left|a(s)\right| ds - w'(T) - k_0 < \infty,$$

which contradicts (4.4).

Second, we consider the case when $\int_T^\infty [w'(t)]^2 f'(y(t))dt = \infty$. Choose a real number $k \geq \max\{\beta, 3\}$. Because of $w(t) > 0$, it follows that

$$\begin{aligned} \text{(4.30)}\quad \frac{1}{t^{k-1}}\int_T^t (t-s)^{k-1}a(s)ds &= -\left(1-\frac{T}{t}\right)^{k-1} w'(T) \\ &\quad + \frac{k-1}{t^{k-1}}(t-T)^{k-2}w(T) - \frac{1}{t^{k-1}}\int_T^t (t-s)^{k-1}w'(s)ds \end{aligned}$$

for $t \geq T$. Since

$$\lim_{t\to\infty} \frac{1}{t^{k-1}}\int_T^t (t-s)^{k-1}[w'(s)]^2 f'(y(s))ds = \int_T^\infty [w'(s)]^2 f'(y(s))ds = \infty,$$

we obtain from (4.30)

$$\lim_{t\to\infty} \frac{1}{t^{k-1}}\int_T^t (t-s)^{k-1}a(s)ds = -\infty,$$

which contradicts (4.4). □

Theorem 4.3.3. *Assume that* (4.27) *and* (4.28) *hold. Then* (4.5) *and* (4.17) *imply that* (4.1) *is oscillatory.*

Proof. The proof is similar to the proofs of Theorems 4.3.1 and 4.2.2. □

Corollary 4.3.4. *Conditions* (4.5) *and* (4.17) *are sufficient for* (4.2) *to be oscillatory with* $0 < \gamma < 1$.

4.4. Oscillation of Nonlinear Equations

In this section we consider nonlinear differential equations of the form

$$\left(a(t)[y'(t)]^\sigma\right)' + q(t)f(y(t)) = 0, \tag{4.31}$$

where σ is a positive quotient of odd integers, a is an eventually positive function, q is continuous on an interval $[t_0, \infty)$ without any restriction on its sign, and f is a continuous real-valued function on the real line $\mathbb{R}$ and satisfies

$$uf(u) > 0 \quad \text{and} \quad f'(u) \geq 0 \quad \text{for every} \quad u \neq 0.$$

Lemma 4.4.1 ([**159**]). *Let the function* $K : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ *be such that for fixed* t *and* s, *the function* $K(t, s, \cdot)$ *is nondecreasing. Further, let* p *be a given function and* u_1 *and* u_2 *be functions satisfying, for* $t \geq t_0$

$$u_1(t) \geq p(t) + \int_{t_0}^t K\left(t, s, u_1(s)\right) ds \quad \text{and} \quad u_2(t) \leq p(t) + \int_{t_0}^t K\left(t, s, u_2(s)\right) ds.$$

If v_1 *is the minimal solution and* v_2 *is the maximal solution of*

$$v(t) = p(t) + \int_{t_0}^t K\left(t, s, v(s)\right) ds,$$

then

$$u_1(t) \geq v_1(t) \quad \text{and} \quad u_2(t) \leq v_2(t) \quad \text{for} \quad t \geq t_0.$$

Lemma 4.4.2. *Assume that* $f'(y) \geq 0$. *Let* σ *be a quotient of odd integers. Suppose that* y *is a positive solution of* (4.31) *for* $t \in [t_0, \alpha]$, *and there exist* $t_1 \in [t_0, \alpha]$ *and* $m > 0$ *such that*

$$-\frac{a(t_0)[y'(t_0)]^\sigma}{f(y(t_0))} + \int_{t_0}^t q(s)ds + \int_{t_0}^{t_1} \frac{a(s)f'(y(s))[y'(s)]^{\sigma+1}}{[f(y(s))]^2} ds \geq m \tag{4.32}$$

for all $t \in [t_1, \alpha]$. *Then*

$$a(t)[y'(t)]^\sigma \leq -mf(y(t_1)) \quad \text{for} \quad t \in [t_1, \alpha]. \tag{4.33}$$

If y *is a negative solution of* (4.31), *then the result remains true if the inequality in* (4.33) *is reversed.*

Proof. Define $w = a(y')^\sigma$. From (4.31) we have

$$\frac{w'(t)}{f(y(t))} = -q(t). \tag{4.34}$$

Then it follows from (4.34) that

$$\left[\frac{w(t)}{f(y(t))}\right]' = -q(t) - \frac{w(t)f'(y(t))y'(t)}{[f(y(t))]^2}. \tag{4.35}$$

Integrating (4.35) from t_0 to t, where $t \in [t_1, \alpha]$, and using (4.32), we find

$$\begin{aligned}(4.36)\quad -\frac{w(t)}{f(y(t))} &= -\frac{w(t_0)}{f(y(t_0))} + \int_{t_0}^{t} q(s)ds + \int_{t_0}^{t} \frac{a(s)\,[y'(s)]^{\sigma+1}\, f'(y(s))}{[f(y(s))]^2} ds \\ &\geq m + \int_{t_1}^{t} \frac{a(s)\,[y'(s)]^{\sigma+1}\, f'(y(s))}{[f(y(s))]^2} ds \;>\; 0.\end{aligned}$$

We first consider the case when $y(t)$ is positive. Then (4.36) implies $-w(t) > 0$, or eventually $y'(t) < 0$, $t \in [t_1, \alpha]$. Let $u = -w = -a(y')^\sigma$. Then (4.36) becomes

$$u(t) \geq mf(y(t)) + \int_{t_1}^{t} \frac{f(y(t))\,[-y'(s)]\, f'(y(s))}{[f(y(s))]^2} u(s)ds$$

for $t \in [t_1, \alpha]$. Define

$$K(t, s, z) = \frac{f(y(t))\,[-y'(s)]\, f'(y(s))}{[f(y(s))]^2} z, \quad s, t \in [t_1, \alpha], \quad z \in \mathbb{R}^+. \tag{4.37}$$

Since $y'(t) < 0$, $t \in [t_1, \alpha]$, we observe that for fixed t and s, $K(t, s, \cdot)$ is nondecreasing. With $p(t) = mf(y(t))$, we apply Lemma 4.4.1 to get

$$u(t) \geq v(t) \quad \text{for all} \quad t \in [t_1, \alpha], \tag{4.38}$$

where v is the minimal solution of the equation

$$v(t) = mf(y(t)) + \int_{t_1}^{t} \frac{f(y(t))\,[-y'(s)]\, f'(y(s))}{[f(y(s))]^2} v(s)ds, \tag{4.39}$$

provided $v(t) \in \mathbb{R}^+$ for all $t \in [t_1, \alpha]$. From (4.39) we find

$$\begin{aligned}(4.40)\quad \left[\frac{v(t)}{f(y(t))}\right]' &= \left[m + \int_{t_1}^{t} \frac{(-y'(s))\, f'(y(s))}{[f(y(s))]^2} v(s)ds\right]' \\ &= \frac{(-y'(t))\, f'(y(t))}{[f(y(t))]^2} v(t).\end{aligned}$$

On the other hand,

$$\left[\frac{v(t)}{f(y(t))}\right]' = \frac{v'(t)}{f(y(t))} - \frac{v(t) f'(y(t)) y'(t)}{[f(y(t))]^2}. \tag{4.41}$$

Equating (4.40) and (4.41), we obtain $v'(t) \equiv 0$ and so $v(t) \equiv v(t_1) = mf(y(t_1))$. The inequality (4.33) is now immediate from (4.38).

In the second case, we suppose that $y(t)$ is negative. Then (4.36) gives $w(t) > 0$, or equivalently $y'(t) > 0$, $t \in [t_1, \alpha]$. Let $u = w = a(y')^\sigma$. It follows from (4.36) that

$$u(t) \geq -mf(y(t)) + \int_{t_1}^{t} \frac{[-f(y(t))]\, y'(s) f'(y(s))}{[f(y(s))]^2} u(s)ds$$

for $t \in [t_1, \alpha]$. With K defined as in (4.37), we note that for fixed t and s, $K(t, s, \cdot)$ is nondecreasing. Applying Lemma 4.4.1 with $p(t) = -mf(y(t))$, we get (4.38), where v is the minimal solution of the equation

$$v(t) = -mf(y(t)) + \int_{t_1}^{t} \frac{[-f(y(t))]\, y'(s) f'(y(s))}{[f(y(s))]^2} v(s)ds.$$

As in the first case, $v'(t) \equiv 0$ and hence $v(t) \equiv v(t_1) = -mf(y(t_1))$. The inequality (4.38) immediately reduces to (4.33). $\square$

Corollary 4.4.3. *Assume that $f'(y) \geq 0$. Let y be a positive solution of* (4.31). *If*

$$\liminf_{t\to\infty} \int_{t_1}^{t} q(s)ds > -\infty$$

and

$$\int_{t_1}^{\infty} \frac{ds}{[a(s)]^{1/\sigma}} = \infty, \tag{4.42}$$

then

$$\int_{t_1}^{\infty} \frac{a(s)f'(y(s))[y'(s)]^{\sigma+1}}{[f(y(s))]^2} ds < \infty.$$

Proof. Otherwise,

$$\int_{t_1}^{\infty} \frac{a(s)f'(y(s))[y'(s)]^{\sigma+1}}{[f(y(s))]^2} ds = \infty,$$

and hence there exists $t_1^* \geq t_1$ such that (4.32) holds (with $t_1 = t_1^*$). Therefore, by Lemma 4.4.2

$$a(t)[y'(t)]^{\sigma} \leq -mf(y(t_1^*)) \quad \text{for} \quad t \geq t_1^*. \tag{4.43}$$

Since σ is a quotient of odd integers, by (4.43) we have

$$y'(t) \leq -[mf(y(t_1^*))]^{1/\sigma} \frac{1}{[a(t)]^{1/\sigma}} \quad \text{for} \quad t \geq t_1^*.$$

In view of (4.42), relation (4.43) implies that $y(t)$ is negative eventually, which is a contradiction. $\square$

Corollary 4.4.4. *Assume $f'(y) \geq 0$ and* (4.42). *If*

$$\int_{t_0}^{\infty} q(s)ds = \infty,$$

then every solution of (4.31) *is oscillatory.*

We now consider the case when $\lim_{t\to\infty} \int_{t_0}^{t} q(s)ds$ exists.

Lemma 4.4.5. *Let $\sigma \geq 1$ be a quotient of odd integers. Assume $f'(y) \geq 0$ and* (4.42). *Suppose further that*

(i) $\lim_{|y|\to\infty} f(y) = \infty$;
(ii) $\lim_{t\to\infty} \int_{t_0}^{t} q(s)ds$ *exists.*

Let y be a nonoscillatory solution of (4.31). *Then*

$$\int_{t_0}^{\infty} \frac{a(s)f'(y(s))[y'(s)]^{\sigma+1}}{[f(y(s))]^2} ds < \infty, \tag{4.44}$$

$$\lim_{t\to\infty} \frac{a(t)[y'(t)]^{\sigma}}{f(y(t))} = 0, \tag{4.45}$$

and

$$\frac{a(t)[y'(t)]^{\sigma}}{f(y(t))} = \int_{t}^{\infty} q(s)ds + \int_{t}^{\infty} \frac{a(s)f'(y(s))[y'(s)]^{\sigma+1}}{[f(y(s))]^2} ds \tag{4.46}$$

for sufficiently large t.

Proof. Let y be a nonoscillatory solution of (4.31). Without loss of generality, assume $y(t) > 0$ for $t \geq t_0$. By Corollary 4.4.3 it follows that (4.44) holds. From (4.31) we have

$$\frac{w'(t)}{f(y(t))} = -q(t), \qquad \text{where} \qquad w = a(y')^\sigma. \tag{4.47}$$

Then it follows from (4.47) that

$$\left[\frac{w(t)}{f(y(t))}\right]' = -q(t) - \frac{w(t)f'(y(t))y'(t)}{[f(y(t))]^2}.$$

Integrating this equation from t_0 to t, we find

$$\begin{aligned} \frac{w(t)}{f(y(t))} &= \frac{w(t_0)}{f(y(t_0))} - \int_{t_0}^{t} q(s)ds - \int_{t_0}^{t} \frac{a(s)f'(y(s))[y'(s)]^{\sigma+1}}{[f(y(s))]^2}ds \\ &= \frac{w(t_0)}{f(y(t_0))} - \int_{t_0}^{\infty} q(s)ds - \int_{t_0}^{\infty} \frac{a(s)f'(y(s))[y'(s)]^{\sigma+1}}{[f(y(s))]^2}ds \\ &\quad + \int_{t}^{\infty} q(s)ds + \int_{t}^{\infty} \frac{a(s)f'(y(s))[y'(s)]^{\sigma+1}}{[f(y(s))]^2}ds \\ &= \beta + \int_{t}^{\infty} q(s)ds + \int_{t}^{\infty} \frac{a(s)f'(y(s))[y'(s)]^{\sigma+1}}{[f(y(s))]^2}ds, \end{aligned} \tag{4.48}$$

where

$$\beta = \frac{w(t_0)}{f(y(t_0))} - \int_{t_0}^{\infty} q(s)ds - \int_{t_0}^{\infty} \frac{a(s)f'(y(s))[y'(s)]^{\sigma+1}}{[f(y(s))]^2}ds.$$

We claim that $\beta = 0$.

If $\beta < 0$, then we choose t_2 so large that

$$\left|\int_{t_2}^{t} q(s)ds\right| \leq -\frac{\beta}{4} \qquad \text{for all} \qquad t \in [t_2, \infty)$$

and

$$\int_{t_2}^{\infty} \frac{a(s)f'(y(s))[y'(s)]^{\sigma+1}}{[f(y(s))]^2}ds < -\frac{\beta}{4}.$$

We take $t_0 = t_1 = t_2$ in Lemma 4.4.2. Then all assumptions of Lemma 4.4.2 (with $m = -\beta/2$) hold. From Lemma 4.4.2 we obtain

$$y'(t) \leq -[mf(y(t_2))]^{1/\sigma}\frac{1}{[a(t)]^{1/\sigma}} \qquad \text{for} \qquad t \geq t_2,$$

which contradicts the positivity of $y(t)$ since (4.42) holds.

If $\beta > 0$, then from (4.48) we have

$$\lim_{t\to\infty} \frac{a(t)[y'(t)]^\sigma}{f(y(t))} = \beta > 0,$$

which implies that $y'(t) > 0$ eventually. Hence there exists $t_1 \geq t_0$ such that

$$\frac{a(t)[y'(t)]^\sigma}{f(y(t))} \geq \frac{\beta}{2} \qquad \text{for} \qquad t \geq t_1. \tag{4.49}$$

Thus

$$\infty > \int_{t_1}^{\infty} \frac{a(s)f'(y(s))[y'(s)]^{\sigma+1}}{[f(y(s))]^2}ds \geq \frac{\beta}{2}\int_{t_1}^{\infty} \frac{f'(y(s))y'(s)}{f(y(s))}ds = \frac{\beta}{2}\lim_{t\to\infty} \ln\frac{f(y(t))}{f(y(t_1))}.$$

Therefore $\lim_{t\to\infty} \ln f(y(t)) < \infty$, which implies $\lim_{t\to\infty} f(y(t)) < \infty$. Due to condition (i) and since y is eventually increasing, it follows that y must be bounded. On the other hand, from (4.49) and the monotonicity of f, we get

$$a(t)[y'(t)]^\sigma \geq \frac{\beta}{2} f(y(t)) \geq \frac{\beta}{2} f(y(t_1)),$$

and so

$$y'(t) \geq \left[\frac{\beta}{2} f(y(t_1))\right]^{1/\sigma} \frac{1}{[a(t)]^{1/\sigma}} \quad \text{for} \quad t \geq t_1.$$

By (4.42), it follows that $\lim_{t\to\infty} y(t) = \infty$, which contradicts the boundedness of y. The proof is complete. □

Example 4.4.6. Consider the nonlinear differential equation

$$\left(\frac{1}{t}(y')^\sigma\right)' + \frac{1}{t^5} y^\gamma = 0, \quad t \geq 1, \tag{4.50}$$

where $\gamma > 0$ and $\sigma \geq 1$ is a quotient of odd integers. It is easy to verify that the assumptions of Lemma 4.4.5 hold. Hence, every nonoscillatory solution y of (4.50) satisfies (4.44), (4.45), and (4.46).

Theorem 4.4.7. *Let $\sigma \geq 1$ be a quotient of odd integers. Assume $f'(y) \geq 0$ and* (4.42). *Suppose further that*

(i) $0 < \int_\varepsilon^\infty \frac{dy}{[f(y)]^{1/\sigma}} < \infty$ *and* $0 < \int_{-\varepsilon}^{-\infty} \frac{dy}{[f(y)]^{1/\sigma}} < \infty$ *for any* $\varepsilon > 0$;

(ii) $\int_{t_0}^\infty q(s)ds$ *exists and* $\lim_{t\to\infty} \int_{t_0}^t \frac{1}{[a(s)]^{1/\sigma}} \left(\int_s^\infty q(u)du\right)^{1/\sigma} ds = \infty$.

Then every solution of (4.31) *is oscillatory.*

Proof. Suppose the contrary. Without loss of generality, we assume that (4.31) has an eventually positive solution. Under our assumptions, Lemma 4.4.5 holds. Let y be an eventually positive solution of (4.31). Then (4.46) is satisfied. Since f is nondecreasing and $[y'(t)]^{\sigma+1} \geq 0$, the second integral in (4.46) is nonnegative. Hence

$$\frac{y'(t)}{[f(y(t))]^{1/\sigma}} \geq \frac{1}{[a(t)]^{1/\sigma}} \left(\int_t^\infty q(s)ds\right)^{1/\sigma}.$$

Integrating from t_0 to t provides

$$\infty > \int_{y(t_0)}^\infty \frac{du}{[f(u)]^{1/\sigma}} \geq \int_{t_0}^t \frac{y'(s)}{[f(y(s))]^{1/\sigma}} ds \geq \int_{t_0}^t \frac{1}{[a(s)]^{1/\sigma}} \left(\int_s^\infty q(u)du\right)^{1/\sigma} ds,$$

which contradicts condition (ii). Similarly, one can prove that (4.31) does not possess eventually negative solutions. □

Example 4.4.8. Consider the superlinear differential equation

$$\left((t-1)^{-\beta} y'(t)\right)' + \frac{1}{t^2(t-1)^\alpha} |y(t)|^\gamma \operatorname{sgn} y(t) = 0, \quad t \geq 2,$$

where $\gamma > 1$, $\alpha > 0$, and $\beta > 0$ are constants. If $\beta \geq \alpha$, then

$$\begin{aligned}\int_{t_0}^{\infty} \frac{1}{a(s)} \int_s^{\infty} q(u)duds &= \int_2^{\infty} (s-1)^{\beta} \int_s^{\infty} \frac{1}{u^2(u-1)^{\alpha}} duds \\ &= \int_2^{\infty} \frac{1}{u^2(u-1)^{\alpha}} \int_2^{u} (s-1)^{\beta} dsdu \\ &= \frac{1}{\beta+1} \int_2^{\infty} \frac{(u-1)^{\beta+1}-1}{u^2(u-1)^{\alpha}} du \\ &= \infty.\end{aligned}$$

By Theorem 4.4.7, this equation is oscillatory.

We note that if (ii) holds, then

$$Q_0(t) = \int_t^{\infty} q(s)ds, \quad t \geq t_0$$

is finite. Assume that $Q_0(t) \geq 0$ for sufficiently large t. Define for $n \in \mathbb{N}$ the sequence

$$Q_1(t) = \int_t^{\infty} Q_0(s) g^{-1}(Q_0(s))\, g^{-1}\left(\frac{1}{a(s)}\right) ds, \qquad \text{where} \qquad g(y) = y^{\sigma}$$

and

$$Q_{n+1}(t) = \int_t^{\infty} [Q_0(s) + \lambda Q_n(s)]\, g^{-1}\Big(Q_0(s) + \lambda Q_n(s)\Big)\, g^{-1}\left(\frac{1}{a(s)}\right) ds.$$

Condition (H). For every $\lambda > 0$, there exists $N \in \mathbb{N}$ such that $Q_n(t)$ is finite for $n \in \{1, 2, \ldots, N-1\}$ and $Q_N(t)$ is infinite.

Theorem 4.4.9. *Suppose that conditions* (i) *and* (ii) *from Lemma 4.4.5, $f'(y) \geq 0$,* (4.42), *and condition* (H) *hold and that*

$$[f(y)]^{\frac{1-\sigma}{\sigma}} f'(y) \geq \lambda > 0 \quad \textit{for all} \quad y \neq 0. \tag{4.51}$$

Then every solution of (4.31) *is oscillatory.*

Proof. Suppose to the contrary that y is a nonoscillatory, without loss of generality, positive solution of (4.31). Hence, by Lemma 4.4.5, y satisfies (4.44) and (4.46), which implies that

$$\frac{a(t)[y'(t)]^{\sigma}}{f(y(t))} \geq Q_0(t) + \int_t^{\infty} \frac{a(s)[y'(s)]^{\sigma+1} f'(y(s))}{[f(y(s))]^2} ds \geq Q_0(t) \geq 0 \tag{4.52}$$

for $t \geq t_1$, and so

$$y'(t) \geq [Q_0(t)]^{1/\sigma} [f(y(t))]^{1/\sigma} \left(\frac{1}{a(t)}\right)^{1/\sigma}. \tag{4.53}$$

From (4.51), (4.52), and (4.53), we have

$$\begin{aligned}
\int_t^\infty \frac{a(s)[y'(s)]^{\sigma+1} f'(y(s))}{[f(y(s))]^2} ds &\geq \int_t^\infty \frac{Q_0(s) y'(s) f'(y(s))}{f(y(s))} ds \\
&\geq \int_t^\infty f'(y(s))[f(y(s))]^{\frac{1-\sigma}{\sigma}} [Q_0(s)]^{\frac{1+\sigma}{\sigma}} \left(\frac{1}{a(s)}\right)^{1/\sigma} ds \\
&\geq \lambda \int_t^\infty [Q_0(s)]^{\frac{1+\sigma}{\sigma}} \left(\frac{1}{a(s)}\right)^{1/\sigma} ds \\
&= \lambda Q_1(t) \qquad (4.54)
\end{aligned}$$

for $t \geq t_1$. If $N = 1$ in Condition (H), then the right-hand side of (4.54) is infinite. This is a contradiction to (4.44).

Next, it follows from (4.52) and (4.54) that

$$\frac{a(t)[y'(t)]^\sigma}{f(y(t))} \geq Q_0(t) + \lambda Q_1(t),$$

and as before we obtain

$$\begin{aligned}
\int_t^\infty \frac{a(s)[y'(s)]^{\sigma+1} f'(y(s))}{[f(y(s))]^2} ds &\geq \lambda \int_t^\infty [Q_0(s) + \lambda Q_1(s)]^{\frac{\sigma+1}{\sigma}} \left(\frac{1}{a(s)}\right)^{1/\sigma} ds \\
&= \lambda Q_2(t)
\end{aligned}$$

for $t \geq t_1$. If $N = 2$ in Condition (H), then once again we get a contradiction to (4.44). A similar argument yields a contradiction for any integer $N > 2$. This completes the proof of the theorem. □

Example 4.4.10. Consider the nonlinear differential equation

$$\left(\frac{1}{t}(y')^3\right)' + \frac{1}{t^2} y^3 = 0, \quad t \geq 1. \qquad (4.55)$$

Here $q(t) = 1/t^2$, $a(t) = 1/t$, $f(y) = g(y) = y^3$, and $\sigma = 3$. We have

$$\frac{f'(y)}{[f(y)]^{(\sigma-1)/\sigma}} = \frac{3y^2}{y^2} = 3 > 0, \quad \int_1^\infty q(s) ds = \int_1^\infty \frac{ds}{s^2} = 1,$$

$$\int_1^\infty \frac{ds}{[a(s)]^{1/\sigma}} = \int_1^\infty s^{1/3} ds = \infty, \quad Q_0(t) = \int_t^\infty q(s) ds = \frac{1}{t}, \quad t \geq 1,$$

and

$$Q_1(t) = \int_t^\infty (s^{-1})^{4/3} s^{1/3} ds = \int_t^\infty s^{-4/3} s^{1/3} ds = \int_t^\infty \frac{ds}{s} = \infty.$$

Hence, by Theorem 4.4.9, every solution of (4.55) is oscillatory.

Remark 4.4.11. Consider the second order nonlinear differential equation

$$\Big(a(t) g\left(y'(t)\right)\Big)' + q(t) f\left(y(t)\right) = 0.$$

If g satisfies $yg(y) > 0$ and $g'(y) \geq 0$, then the above results are true, see [**262**].

The remaining results in this section are for the nonlinear equation (4.31) with $\sigma = 1$, i.e.,

$$\Big(a(t) y'(t)\Big)' + q(t) f\left(y(t)\right) = 0, \qquad (4.56)$$

where we assume that a is an eventually positive continuously differentiable function.

Theorem 4.4.12. *Let $\sigma = 1$. Assume that*

$$f'(y) \geq \mu > 0 \quad \text{for} \quad y \neq 0. \tag{4.57}$$

Let $D_0 = \{(t,s) : t > s \geq t_0\}$ and $D = \{(t,s) : t \geq s \geq t_0\}$. Let $H \in C(D,\mathbb{R})$ satisfy the following two conditions:

(i) *$H(t,t) = 0$ for $t \geq t_0$, $H(t,s) > 0$ for $t > s \geq t_0$;*
(ii) *H has a continuous and nonpositive partial derivative on D_0 with respect to the second variable.*

Suppose that $h : D_0 \to \mathbb{R}$ is a continuous function with

$$-\frac{\partial H}{\partial s}(t,s) = h(t,s)\sqrt{H(t,s)} \quad \text{for all} \quad (t,s) \in D_0.$$

If there exists a function $g \in C^1[t_0,\infty)$ such that

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)\phi(s) - \frac{1}{4\mu} r(s)a(s)[h(t,s)]^2 \right] ds = \infty, \tag{4.58}$$

where

$$r(s) = \exp\left(-2\mu \int^s g(v)dv\right), \quad \phi(s) = r(s)\left\{q(s) + \mu a(s)[g(s)]^2 - (ag)'(s)\right\},$$

then every solution of (4.56) is oscillatory.

Proof. Let y be a nonoscillatory solution of (4.56). Without loss of generality, we may assume that $y(t) > 0$ on $[T_0,\infty)$ for some $T_0 \geq t_0$. Define

$$u(t) = r(t)a(t)\left\{\frac{y'(t)}{f(y(t))} + g(t)\right\} \quad \text{for all} \quad t \geq T_0. \tag{4.59}$$

By (4.56), (4.57), and (4.59), we obtain

$$\begin{aligned}
u'(t) &= -2\mu g(t)u(t) + r(t)\left\{\frac{(ay')'(t)}{f(y(t))} - a(t)\frac{[y'(t)]^2 f'(y(t))}{[f(y(t))]^2} + (ag)'(t)\right\} \\
&\leq -2\mu g(t)r(t)a(t)\left\{\frac{y'(t)}{f(y(t))} + g(t)\right\} - r(t)q(t) \\
&\quad -r(t)a(t)\frac{[y'(t)]^2\mu}{[f(y(t))]^2} + r(t)(ag)'(t) \\
&= -2\mu g(t)r(t)a(t)\frac{y'(t)}{f(y(t))} - 2\mu r(t)a(t)[g(t)]^2 - r(t)q(t) \\
&\quad -r(t)a(t)\frac{[y'(t)]^2\mu}{[f(y(t))]^2} + r(t)(ag)'(t) \\
&= -\frac{\mu[u(t)]^2}{a(t)r(t)} - \phi(t)
\end{aligned}$$

for $t \geq T_0$. It follows that for all $t \geq T \geq T_0$,

$$\begin{aligned}
\int_T^t H(t,s)\phi(s)ds \;&\leq\; -\int_T^t H(t,s)u'(s)ds - \int_T^t H(t,s)\frac{\mu[u(s)]^2}{a(s)r(s)}ds \\
&= -H(t,s)u(s)\Big|_{s=T}^{t} - \int_T^t \left\{-\frac{\partial H}{\partial s}(t,s)u(s) + H(t,s)\frac{\mu[u(s)]^2}{a(s)r(s)}\right\}ds \\
&= H(t,T)u(T) - \int_T^t \left\{h(t,s)\sqrt{H(t,s)}u(s) + H(t,s)\frac{\mu[u(s)]^2}{a(s)r(s)}\right\}ds \\
&= H(t,T)u(T) - \int_T^t \left\{\sqrt{\frac{\mu H(t,s)}{a(s)r(s)}}u(s) + \frac{1}{2}\sqrt{\frac{a(s)r(s)}{\mu}}h(t,s)\right\}^2 ds \\
&\quad + \frac{1}{4\mu}\int_T^t a(s)r(s)[h(t,s)]^2 ds.
\end{aligned}$$

Then, for all $t \geq T \geq T_0$,

$$\begin{aligned}
\int_T^t &\left[H(t,s)\phi(s) - \frac{1}{4\mu}r(s)a(s)[h(t,s)]^2\right]ds \\
&\leq H(t,T)u(T) - \int_T^t \left\{\sqrt{\frac{\mu H(t,s)}{a(s)r(s)}}u(s) + \frac{1}{2}\sqrt{\frac{a(s)r(s)}{\mu}}h(t,s)\right\}^2 ds.
\end{aligned}\tag{4.60}$$

This implies that for every $t \geq T_0$,

$$\begin{aligned}
\int_{T_0}^t \left[H(t,s)\phi(s) - \frac{1}{4\mu}r(s)a(s)[h(t,s)]^2\right]ds &\leq H(t,T_0)u(T_0) \\
&\leq H(t,T_0)\,|u(T_0)| \leq H(t,t_0)\,|u(T_0)|\,.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_{t_0}^t \left[H(t,s)\phi(s) - \frac{1}{4\mu}r(s)a(s)[h(t,s)]^2\right]ds \\
&= \int_{t_0}^{T_0} \left[H(t,s)\phi(s) - \frac{1}{4\mu}r(s)a(s)[h(t,s)]^2\right]ds \\
&\quad + \int_{T_0}^t \left[H(t,s)\phi(s) - \frac{1}{4\mu}r(s)a(s)[h(t,s)]^2\right]ds \\
&\leq H(t,t_0)\int_{t_0}^{T_0} |\phi(s)|ds + H(t,t_0)|u(T_0)| \\
&= H(t,t_0)\left\{\int_{t_0}^{T_0} |\phi(s)|ds + |u(T_0)|\right\}
\end{aligned}$$

for all $t \geq T_0$. This gives

$$\begin{aligned}
\limsup_{t\to\infty} \frac{1}{H(t,t_0)}\int_{t_0}^t &\left[H(t,s)\phi(s) - \frac{1}{4\mu}r(s)a(s)[h(t,s)]^2\right]ds \\
&\leq \int_{t_0}^{T_0} |\phi(s)|ds + |u(T_0)|,
\end{aligned}$$

which contradicts (4.58). The proof is complete. □

From Theorem 4.4.12, we can obtain different sufficient conditions for oscillation of all solutions of (4.56) by different choices of $H(t,s)$. For example, let

$$H(t,s) = (t-s)^\lambda, \quad t \geq s \geq t_0,$$

where $\lambda > 1$ is a constant. By Theorem 4.4.12, we have the following result.

Corollary 4.4.13. *Assume that* (4.57) *holds. Let* $\lambda > 1$ *be a constant. Suppose that there is a function* $g \in C^1(t_0, \infty)$ *satisfying*

$$\limsup_{t\to\infty} \frac{1}{t^\lambda} \int_{t_0}^t \left[(t-s)^\lambda \phi(s) - \frac{\lambda^2}{4\mu}(t-s)^{\lambda-2} r(s)a(s) \right] ds = \infty,$$

where

$$r(s) = \exp\left(-2\mu \int^s g(v)dv\right), \quad \phi(s) = r(s)\left\{ q(s) + \mu a(s)[g(s)]^2 - (ag)'(s) \right\}.$$

Then every solution of (4.56) *is oscillatory.*

Define

$$A(t) = \int_{t_0}^t \frac{1}{a(s)} ds, \quad t \geq t_0 \tag{4.61}$$

and let

$$H(t,s) = [A(t) - A(s)]^\lambda, \quad t \geq t_0,$$

where $\lambda > 1$ is a constant. By Theorem 4.4.12, we have the following oscillation criterion.

Corollary 4.4.14. *Assume that* (4.57) *holds. If*

$$\limsup_{t\to\infty} \frac{1}{[A(t)]^\lambda} \int_{t_0}^t [A(t) - A(s)]^\lambda q(s) ds = \infty \quad \text{for some} \quad \lambda > 1,$$

then every solution of (4.56) *is oscillatory.*

Corollary 4.4.15. *Assume that* (4.57) *holds and that* $\lim_{t\to\infty} A(t) = \infty$. *If*

$$\liminf_{t\to\infty} A(t) \int_t^\infty q(s) ds > \frac{1}{4\mu}, \tag{4.62}$$

where $A(t)$ *is defined by* (4.61), *then every solution of* (4.56) *is oscillatory.*

Proof. By (4.62), there are two numbers $T \geq t_0$ and $k > 1/(4\mu)$ such that

$$A(t) \int_t^\infty q(s) ds > k \quad \text{for} \quad t \geq T \quad \text{and} \quad \lim_{t\to\infty} A(t) = \infty.$$

Let

$$H(t,s) = [A(t) - A(s)]^2 \quad \text{and} \quad g(t) = -\frac{1}{2\mu a(t) A(t)}.$$

Then

$$h(t,s) = 2A'(s) = \frac{2}{a(s)} \quad \text{and} \quad r(t) = A(t).$$

Thus

$$\begin{aligned} H(t,s)\phi(s) - \frac{1}{4\mu} r(s)a(s)[h(t,s)]^2 & \\ &= [A(t) - A(s)]^2 A(s) \left\{ q(s) - \frac{1}{4\mu a(s)[A(s)]^2} \right\} - \frac{A(s)}{\mu a(s)}. \end{aligned}$$

Define

$$Q(t) = \int_t^\infty q(s)ds.$$

Then, for all $t \ge T$,

$$\begin{aligned}
&\int_T^t \left(H(t,s)\phi(s) - \frac{1}{4\mu} r(s)a(s)[h(t,s)]^2 \right) ds \\
&= \int_T^t [A(t) - A(s)]^2 A(s) \frac{d}{ds} \left(-Q(s) + \frac{1}{4\mu A(s)} \right) ds - \int_T^t \frac{A(s)}{\mu a(s)} ds \\
&= [A(t) - A(T)]^2 A(T) \left(Q(T) - \frac{1}{4\mu A(T)} \right) - \frac{1}{2\mu} \left([A(t)]^2 - [A(T)]^2 \right) \\
&\quad + \int_T^t \left[A(s)Q(s) - \frac{1}{4\mu} \right] \left[-4A(t) + 3A(s) + \frac{[A(t)]^2}{A(s)} \right] A'(s) ds \\
&\ge \left(k - \frac{1}{4\mu} \right) \left[\left(-\frac{5}{2} - \ln A(T) \right) [A(t)]^2 + [A(t)]^2 \ln A(t) \right] \\
&\quad - \frac{1}{2\mu} [A(t)]^2 - \left[\left(k - \frac{1}{4\mu} \right) \frac{3}{2} - \frac{1}{2\mu} \right] [A(T)]^2.
\end{aligned}$$

This and (4.62) imply that

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^t \left[H(t,s)\phi(s) - \frac{1}{4\mu} r(s)a(s)[h(t,s)]^2 \right] ds = \infty.$$

It follows from Theorem 4.4.12 that every solution of (4.56) is oscillatory. □

Example 4.4.16. Consider the nonlinear differential equation

$$(ty')' + \frac{\gamma}{t(\ln t)^2} \left(y + y^3 \right) = 0, \quad t \ge 1. \tag{4.63}$$

Then

$$A(t) = \int_{t_0}^t \frac{ds}{a(s)} = \int_1^t \frac{ds}{s} = \ln t, \quad f'(y) \ge 1 = \mu,$$

and

$$\liminf_{t\to\infty} \ln t \int_t^\infty \frac{\gamma}{s(\ln s)^2} ds = \gamma.$$

Hence, by Corollary 4.4.15, every solution of (4.63) is oscillatory if $\gamma > 1/4$.

Theorem 4.4.17. *Suppose that* (4.57) *holds. Let $H(t,s)$ and $h(t,s)$ be as in Theorem 4.4.12, and let*

$$0 < \inf_{s \ge t_0} \left\{ \liminf_{t\to\infty} \frac{H(t,s)}{H(t,t_0)} \right\} \le \infty. \tag{4.64}$$

Suppose that there exist two functions $g \in C^1[t_0,\infty)$ and $B \in C[t_0,\infty)$ such that

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^t a(s)r(s)[h(t,s)]^2 ds < \infty \tag{4.65}$$

and

$$\int_T^\infty \frac{[B_+(s)]^2}{a(s)r(s)} ds = \infty, \tag{4.66}$$

where $B_+(t) = \max\{B(t), 0\}$, and r and ϕ are defined as in Theorem 4.4.12. If for every $T \geq t_0$,

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[H(t,s)\phi(s) - \frac{1}{4\mu} r(s)a(s)[h(t,s)]^2\right] ds \geq B(t), \tag{4.67}$$

then every solution of (4.56) *is oscillatory.*

Proof. Without loss of generality, we may assume that there exists a solution of (4.56) such that $y(t) > 0$ on $[T_0, \infty)$ for some $T_0 \geq t_0$. Define u as in (4.59). As in the proof of Theorem 4.4.12, we can obtain (4.60). Then

$$\begin{aligned}
&\frac{1}{H(t,T)} \int_T^t \left[H(t,s)\phi(s) - \frac{1}{4\mu} r(s)a(s)[h(t,s)]^2\right] ds \\
&\qquad \leq u(T) - \frac{1}{H(t,T)} \int_T^t \left\{\sqrt{\frac{\mu H(t,s)}{a(s)r(s)}} u(s) + \frac{1}{2}\sqrt{\frac{a(s)r(s)}{\mu}} h(t,s)\right\}^2 ds
\end{aligned}$$

for $t > T \geq T_0$. Consequently, by (4.67),

$$\begin{aligned}
B(t) &\leq \limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[H(t,s)\phi(s) - \frac{1}{4\mu} r(s)a(s)[h(t,s)]^2\right] ds \\
&\leq u(T) - \liminf_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left\{\sqrt{\frac{\mu H(t,s)}{a(s)r(s)}} u(s) + \frac{1}{2}\sqrt{\frac{a(s)r(s)}{\mu}} h(t,s)\right\}^2 ds \\
&\leq u(T)
\end{aligned}$$

for all $T \geq T_0$. Thus

$$u(T) \geq B(T) \quad \text{for all} \quad T \geq T_0 \tag{4.68}$$

and

$$\begin{aligned}
&\liminf_{t\to\infty} \left\{\frac{1}{H(t,T_0)} \int_{T_0}^t h(t,s)\sqrt{H(t,s)} u(s) ds + \frac{1}{H(t,T_0)} \int_{T_0}^t H(t,s) \frac{\mu [u(s)]^2}{a(s)r(s)} ds\right\} \\
&\quad \leq \liminf_{t\to\infty} \frac{1}{H(t,T)} \int_{T_0}^t \left\{\sqrt{\frac{\mu H(t,s)}{a(s)r(s)}} u(s) + \frac{1}{2}\sqrt{\frac{a(s)r(s)}{\mu}} h(t,s)\right\}^2 ds \\
&\quad \leq u(T_0) - B(T_0) < \infty.
\end{aligned} \tag{4.69}$$

Define

$$P(t) = \frac{1}{H(t,T_0)} \int_{T_0}^t H(t,s) \frac{\mu [u(s)]^2}{a(s)r(s)} ds$$

and

$$Q(t) = \frac{1}{H(t,T_0)} \int_{T_0}^t h(t,s)\sqrt{H(t,s)} u(s) ds$$

for $t > T_0$. Then (4.69) implies that

$$\liminf_{t\to\infty} [P(t) + Q(t)] < \infty. \tag{4.70}$$

Now we claim that

$$\int_{T_0}^t \frac{[u(s)]^2}{a(s)r(s)} ds < \infty. \tag{4.71}$$

Suppose to the contrary that

$$\int_{T_0}^{\infty} \frac{[u(s)]^2}{a(s)r(s)} ds = \infty. \tag{4.72}$$

By (4.64), there exists a positive constant K_1 such that

$$\inf_{s \geq t_0} \left\{ \liminf_{t\to\infty} \frac{H(t,s)}{H(t,t_0)} \right\} > K_1 > 0. \tag{4.73}$$

Let $K_2 > 0$ be arbitrary. Then it follows from (4.72) that there exists $T_1 > T_0$ such that

$$\int_{T_0}^{t} \frac{[u(s)]^2}{a(s)r(s)} ds \geq \frac{K_2}{K_1} \qquad \text{for all} \qquad t \geq T_1.$$

Therefore,

$$\begin{aligned}
P(t) &= \frac{\mu}{H(t,T_0)} \int_{T_0}^{t} H(t,s) \frac{d}{ds} \left(\int_{T_0}^{s} \frac{[u(v)]^2}{a(v)r(v)} dv \right) ds \\
&= \frac{\mu}{H(t,T_0)} \int_{T_0}^{t} \left(-\frac{\partial H}{\partial s}(t,s) \right) \left(\int_{T_0}^{s} \frac{[u(v)]^2}{a(v)r(v)} dv \right) ds \\
&\geq \frac{\mu}{H(t,T_0)} \int_{T_1}^{t} \left(-\frac{\partial H}{\partial s}(t,s) \right) \left(\int_{T_0}^{s} \frac{[u(v)]^2}{a(v)r(v)} dv \right) ds \\
&\geq \frac{K_2 \mu}{K_1 H(t,T_0)} \int_{T_1}^{t} \left(-\frac{\partial H}{\partial s}(t,s) \right) ds \\
&= \frac{K_2 \mu H(t,T_1)}{K_1 H(t,T_0)}
\end{aligned}$$

for $t \geq T_1$. By (4.73), there exists $T_2 \geq T_1$ such that

$$\frac{H(t,T_1)}{H(t,T_0)} \geq K_1 \qquad \text{for all} \qquad t \geq T_2,$$

which implies

$$P(t) \geq \mu K_2 \qquad \text{for all} \qquad t \geq T_2.$$

Since K_2 is arbitrary,

$$\lim_{t\to\infty} P(t) = \infty. \tag{4.74}$$

Next, consider a sequence $\{t_n\}_{n=1}^{\infty} \subset (t_0, \infty)$ with $\lim_{n\to\infty} t_n = \infty$ satisfying

$$\lim_{n\to\infty} \left[P(t_n) + Q(t_n) \right] = \liminf_{t\to\infty} \left[P(t) + Q(t) \right].$$

In view of (4.70), there exists a constant M such that

$$P(t_n) + Q(t_n) \leq M \qquad \text{for} \qquad n \in \mathbb{N}. \tag{4.75}$$

It follows from (4.74) that

$$\lim_{n\to\infty} P(t_n) = \infty. \tag{4.76}$$

This and (4.75) give

$$\lim_{n\to\infty} Q(t_n) = -\infty. \tag{4.77}$$

Then, by (4.75) and (4.76),

$$1 + \frac{Q(t_n)}{P(t_n)} \leq \frac{M}{P(t_n)} < \frac{1}{2} \qquad \text{for large enough} \qquad n \in \mathbb{N}.$$

Thus

$$\frac{Q(t_n)}{P(t_n)} < \frac{1}{2} \quad \text{for all large} \quad n \in \mathbb{N}.$$

This and (4.77) imply that

$$\lim_{n\to\infty} \frac{[Q(t_n)]^2}{P(t_n)} = \infty. \tag{4.78}$$

On the other hand, by the Schwarz inequality, we have

$$\begin{aligned}
[Q(t_n)]^2 &= \left\{ \frac{1}{H(t_n, T_0)} \int_{T_0}^{t_n} h(t_n, s)\sqrt{H(t_n, s)}u(s)ds \right\}^2 \\
&\leq \left\{ \frac{1}{H(t_n, T_0)} \int_{T_0}^{t_n} a(s)r(s)[h(t_n, s)]^2 ds \right\} \\
&\quad \times \left\{ \frac{1}{H(t_n, T_0)} \int_{T_0}^{t_n} \frac{H(t_n, s)}{a(s)r(s)}[u(s)]^2 ds \right\} \\
&\leq \frac{P(t_n)}{\mu} \left\{ \frac{1}{H(t_n, T_0)} \int_{T_0}^{t_n} a(s)r(s)[h(t_n, s)]^2 ds \right\}
\end{aligned}$$

for any $n \in \mathbb{N}$. But (4.73) guarantees that

$$\liminf_{t\to\infty} \frac{H(t, T_0)}{H(t, t_0)} > K_1.$$

This means that there exists $T_3 \geq T_0$ such that

$$\frac{H(t, T_0)}{H(t, t_0)} > K_1 \quad \text{for large enough} \quad n \in \mathbb{N}$$

and therefore

$$\frac{[Q(t_n)]^2}{P(t_n)} \leq \frac{1}{\mu K_1 H(t_n, t_0)} \int_{t_0}^{t_n} a(s)r(s)[h(t_n, s)]^2 ds \quad \text{for all large} \quad n \in \mathbb{N}.$$

It follows from (4.78) that

$$\lim_{n\to\infty} \frac{1}{H(t_n, t_0)} \int_{t_0}^{t_n} a(s)r(s)[h(t_n, s)]^2 ds = \infty.$$

This gives

$$\limsup_{t\to\infty} \frac{1}{H(t, t_0)} \int_{t_0}^{t} a(s)r(s)[h(t, s)]^2 ds = \infty,$$

which contradicts (4.65). Thus (4.71) holds. Then, by (4.68),

$$\int_{T_0}^{\infty} \frac{[B_+(s)]^2}{a(s)r(s)} ds \leq \int_{T_0}^{\infty} \frac{[u(s)]^2}{a(s)r(s)} ds < \infty,$$

which contradicts (4.66). This completes the proof. □

Similarly, we can prove the following result.

Theorem 4.4.18. *Suppose that* (4.57) *holds. Let* $H(t, s)$ *and* $h(t, s)$ *be as in Theorem 4.4.12, and let* (4.64) *hold. Suppose that there exist two functions* $g \in C^1[t_0, \infty)$ *and* $B \in C[t_0, \infty)$ *such that* (4.66) *and the two conditions*

$$\liminf_{t\to\infty} \frac{1}{H(t, t_0)} \int_{t_0}^{t} H(t, s)\phi(s)ds < \infty$$

and

$$\liminf_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[H(t,s)\phi(s) - \frac{1}{4\mu} r(s)a(s)[h(t,s)]^2 \right] ds \geq B(t)$$

hold for every $T \geq t_0$*, where* B_+*,* r*, and* ϕ *are as in Theorem 4.4.17. Then every solution of* (4.56) *is oscillatory.*

From Theorems 4.4.17 and 4.4.18, we can obtain different sufficient conditions for oscillation of all solutions of (4.56) by different choices of $H(t,s)$. For example, let

$$H(t,s) = (t-s)^\lambda \quad \text{for some} \quad \lambda > 1.$$

By Theorem 4.4.17, we have the following corollary.

Corollary 4.4.19. *Suppose that* (4.57) *holds. Let* $\lambda > 1$ *be a constant and suppose that there exist two functions* $g \in C^1[t_0,\infty)$ *and* $B \in C[t_0,\infty)$ *such that* (4.64)*,* (4.66)*,*

$$\limsup_{t\to\infty} \frac{1}{t^\lambda} \int_{t_0}^t (t-s)^\lambda a(s)r(s)ds < \infty,$$

and

$$\limsup_{t\to\infty} \frac{1}{t^\lambda} \int_T^t \left[(t-s)^\lambda \phi(s) - \frac{\lambda^2}{4\mu}(t-s)^{\lambda-2} r(s)a(s)[h(t,s)]^2 \right] ds \geq B(t)$$

hold for every $T \geq t_0$*, where* B_+*,* r*, and* ϕ *are as in Theorem 4.4.17. Then every solution of* (4.56) *is oscillatory.*

Remark 4.4.20. In Theorems 4.4.12, 4.4.17, and 4.4.18, we always assume that (4.57) holds. In fact, if we replace condition (4.57) with

$$\frac{f(y)}{y} \geq \mu > 0 \quad \text{for all} \quad y \neq 0,$$

then Theorems 4.4.12, 4.4.17, and 4.4.18 remain true, but the function q should be nonnegative; one can refer to [**245**] for details.

4.5. Forced Oscillation of Nonlinear Equations

In this section we consider the second order nonlinear differential equation

$$\left(a(t)|y'(t)|^{\sigma-1}y'(t)\right)' + q(t)f(y(t)) = r(t), \tag{4.79}$$

where $\sigma > 0$ is a quotient of odd integers, a is an eventually positive function, q and r are continuous on an interval $[t_0,\infty)$ without any restriction on their sign, and f is a continuous real-valued function on the real line $\mathbb{R}$ and satisfies

$$uf(u) > 0 \quad \text{and} \quad f'(u) \geq 0 \quad \text{for every} \quad u \neq 0.$$

The following lemma is a generalization of Lemma 4.4.2.

Lemma 4.5.1. *Suppose that* y *is a positive solution of* (4.79) *on* $t \in [t_0,\alpha]$*, and there exist* $t_1 \in [t_0,\alpha]$ *and* $m > 0$ *such that*

$$\begin{aligned} m \leq -\frac{a(t_0)|y'(t_0)|^{\sigma-1}y'(t_0)}{f(y(t_0))} &+ \int_{t_0}^t \left[q(s) - \frac{r(s)}{f(y(s))} \right] ds \\ &+ \int_{t_0}^{t_1} \frac{a(s)f'(y(s))|y'(s)|^{\sigma-1}[y'(t)]^2}{[f(y(s))]^2} ds \end{aligned} \tag{4.80}$$

for all $t \in [t_1, \alpha]$. *Then*

$$a(t)|y'(t)|^{\sigma-1}y'(t) \leq -mf(y(t_1)), \quad t \in [t_1, \alpha]. \tag{4.81}$$

If y *is a negative solution of* (4.79), *then the result remains true with the inequality reversed in* (4.81).

For simplicity, we list the conditions used in the main results as

$$\int^{\infty} |r(s)|ds < \infty, \tag{4.82}$$

$$-\infty < \int_{t_0}^{\infty} q(s)ds < \infty, \tag{4.83}$$

$$\int^{\infty} \frac{ds}{[a(s)]^{1/\sigma}} = \infty, \tag{4.84}$$

and

$$\lim_{|y|\to\infty} |f(y)| = \infty. \tag{4.85}$$

Theorem 4.5.2. *Let conditions* (4.82), (4.83), (4.84), *and* (4.85) *hold and let* y *be a nonoscillatory solution of* (4.79) *such that* $\liminf_{t\to\infty} |y(t)| > 0$. *Then*

$$\int_{t_0}^{\infty} \frac{a(s)f'(y(s))|y'(s)|^{\sigma-1}[y'(t)]^2}{[f(y(s))]^2} ds < \infty \tag{4.86}$$

and

$$\frac{a(t)|y'(t)|^{\sigma-1}y'(t)}{f(y(t))} = \int_t^{\infty} \left[q(s) - \frac{r(s)}{f(y(s))}\right] ds + \int_t^{\infty} \frac{a(s)f'(y(s))|y'(s)|^{\sigma-1}[y'(s)]^2}{[f(y(s))]^2} ds \tag{4.87}$$

for sufficiently large t.

Proof. Since $\liminf_{t\to\infty} |y(t)| > 0$, there exist $t_1 \geq t_0$ and $m_1, m_2 > 0$ such that $|y(t)| \geq m_1$ and $|f(y(t))| \geq m_2$ for $t \geq t_1$. Then it follows from (4.82) that

$$\left| \int_{t_1}^{t} \frac{r(s)}{f(y(s))} ds \right| \leq \int_{t_1}^{t} \left| \frac{r(s)}{f(y(s))} \right| ds \leq \frac{1}{m_2} \int_{t_1}^{t} |r(s)|\, ds \leq m_3 \tag{4.88}$$

for $t \geq t_1$, where m_3 is a finite positive constant. Suppose now that (4.86) does not hold. Then, in view of (4.83) and (4.88), we see that (4.80) is satisfied for $t \geq t_1$ if t_1 is sufficiently large. Suppose $y(t)$ is positive for $t \geq t_1$. Applying Lemma 4.5.1, we obtain

$$a(t)|y'(t)|^{\sigma-1}y'(t) \leq -mf(y(t_1)) \quad \text{and} \quad y'(t) < 0$$

for $t \geq t_1$. Since

$$|y'(t)|^{\sigma-1}y'(t) = -[-y'(t)]^{\sigma},$$

we have

$$y(t) \leq y(t_1) - [mf(y(t_1))]^{1/\sigma} \int_{t_1}^{t} \frac{ds}{[a(s)]^{1/\sigma}},$$

which in view of condition (4.84) contradicts the fact that $y(t) > 0$ for $t \geq t_1$. The case when $y(t)$ is negative for $t \geq t_1$ follows by a similar argument. Hence (4.86) is proved.

Dividing (4.79) by $f(y(t))$ and integrating it from t_0 to t, we obtain

$$\begin{aligned}(4.89)\quad \frac{a(t)|y'(t)|^{\sigma-1}y'(t)}{f(y(t))} &= \frac{a(t_0)|y'(t_0)|^{\sigma-1}y'(t_0)}{f(y(t_0))} \\ &\quad -\int_{t_0}^{t}\left[q(s)-\frac{r(s)}{f(y(s))}\right]ds-\int_{t_0}^{t}\frac{a(s)|y'(s)|^{\sigma-1}[y'(s)]^2f'(y(s))}{[f(y(s))]^2}ds \\ &= \beta+\int_{t}^{\infty}\left[q(s)-\frac{r(s)}{f(y(s))}\right]ds+\int_{t}^{\infty}\frac{a(s)|y'(s)|^{\sigma-1}[y'(s)]^2f'(y(s))}{[f(y(s))]^2}ds,\end{aligned}$$

where

$$\begin{aligned}\beta = \frac{a(t_0)|y'(t_0)|^{\sigma-1}y'(t_0)}{f(y(t_0))}&-\int_{t_0}^{\infty}\left[q(s)-\frac{r(s)}{f(y(s))}\right]ds \\ &-\int_{t_0}^{\infty}\frac{a(s)|y'(s)|^{\sigma-1}[y'(s)]^2f'(y(s))}{[f(y(s))]^2}ds.\end{aligned}$$

Hence (4.87) is proved if we can show that $\beta = 0$.

If $\beta < 0$, then in view of (4.83), (4.86), and (4.88), we choose t_1 so large that

$$(4.90)\qquad \left|\int_{t}^{\infty} q(s)ds\right| \le -\frac{\beta}{6}, \quad \left|\int_{t}^{\infty}\frac{r(s)}{f(y(s))}ds\right| \le -\frac{\beta}{6}, \quad t \ge t_1,$$

and

$$(4.91)\qquad \int_{t_1}^{\infty}\frac{a(s)|y'(s)|^{\sigma-1}[y'(s)]^2f'(y(s))}{[f(y(s))]^2}ds \le -\frac{\beta}{6}.$$

Let $t = t_0$ in (4.89) to obtain

$$\begin{aligned}(4.92)\quad \frac{a(t_0)|y'(t_0)|^{\sigma-1}y'(t_0)}{f(y(t_0))} = \beta &+ \int_{t_0}^{\infty}\left[q(s)-\frac{r(s)}{f(y(s))}\right]ds \\ &+\int_{t_0}^{\infty}\frac{a(s)|y'(s)|^{\sigma-1}[y'(s)]^2f'(y(s))}{[f(y(s))]^2}ds.\end{aligned}$$

Using (4.90), (4.91), (4.92), and the fact that $f'(y(t)) \ge 0$, we see that

$$\begin{aligned}&-\frac{a(t_0)|y'(t_0)|^{\sigma-1}y'(t_0)}{f(y(t_0))}+\int_{t_0}^{t}\left[q(s)-\frac{r(s)}{f(y(s))}\right]ds \\ &\quad +\int_{t_0}^{t_1}\frac{a(s)|y'(s)|^{\sigma-1}[y'(s)]^2f'(y(s))}{[f(y(s))]^2}ds \\ &= \beta-\int_{t}^{\infty}\left[q(s)-\frac{r(s)}{f(y(s))}\right]ds-\int_{t_1}^{\infty}\frac{a(s)|y'(s)|^{\sigma-1}[y'(s)]^2f'(y(s))}{[f(y(s))]^2}ds \\ &> -\beta+\frac{\beta}{6}+\frac{\beta}{6}+\frac{\beta}{6} = -\frac{\beta}{2} = m_0 > 0\end{aligned}$$

for $t \ge t_1$, i.e., (4.80) is satisfied. Hence we can apply Lemma 4.5.1 and obtain a contradiction as earlier.

If $\beta > 0$, then from (4.89) we have

$$\lim_{t\to\infty}\frac{a(t)|y'(t)|^{\sigma-1}y'(t)}{f(y(t))} = \beta > 0,$$

which implies that $y'(t) > 0$ eventually. Hence there exists $t_1 \geq t_0$ such that

$$\frac{a(t)|y'(t)|^{\sigma-1}y'(t)}{f(y(t))} \geq \frac{\beta}{2}, \quad t \geq t_1. \tag{4.93}$$

Thus

$$\begin{aligned}
\infty &> \int_{t_1}^{\infty} \frac{a(s)|y'(s)|^{\sigma-1}[y'(s)]^2 f'(y(s))}{[f(y(s))]^2} ds \\
&\geq \frac{\beta}{2} \int_{t_1}^{\infty} \frac{f'(y(s))y'(s)}{f(y(s))} ds = \frac{\beta}{2} \lim_{t\to\infty} \ln \frac{f(y(t))}{f(y(t_1))}.
\end{aligned}$$

Therefore, $\ln f(y(t)) < \infty$, which implies that $f(y(t)) < \infty$ as $t \to \infty$. Due to condition (4.85) and the fact that y is eventually increasing, y is bounded. On the other hand, from (4.93) and the monotonicity of f, we have

$$a(t)\left|y'(t)\right|^{\sigma-1} y'(t) \geq \frac{\beta}{2} f(y(t)) \geq \frac{\beta}{2} f\left(y(t_1)\right), \quad t \geq t_1.$$

Since $y'(t) > 0$, we further have

$$y'(t) \geq \left[\frac{\beta}{2} f(y(t_1))\right]^{1/\sigma} \frac{1}{[a(t)]^{1/\sigma}}, \quad t \geq t_1.$$

Hence condition (4.84) implies that $\lim_{t\to\infty} y(t) = \infty$, which contradicts the boundedness of y. The proof is complete. □

Next, we obtain a sufficient condition for the oscillation of (4.79) subject to the condition

$$\frac{f'(y)}{[f(y)]^{\frac{\sigma-1}{\sigma}}} \geq \lambda > 0 \quad \text{for all} \quad y \neq 0. \tag{4.94}$$

We note that if (4.82) and (4.83) hold, then

$$h_0(t) = \int_t^{\infty} \Big(q(s) - l|r(s)|\Big)\, ds, \quad t \geq t_0$$

is finite for any positive constant l. Assume that $h_0(t) > 0$ for sufficiently large t. Define, for $n \in \mathbb{N}$, the sequence

$$h_1(t) = \int_t^{\infty} \frac{[h_0(s)]^{\frac{\sigma+1}{\sigma}}}{[a(s)]^{1/\sigma}} ds$$

and

$$h_{n+1}(t) = \int_t^{\infty} \frac{[h_0(s) + \lambda h_n(s)]^{\frac{\sigma+1}{\sigma}}}{[a(s)]^{1/\sigma}} ds \quad \text{for} \quad n \in \mathbb{N}.$$

Condition (H). For every $\lambda > 0$, there exists $N \in \mathbb{N}$ such that $h_n(t)$ is finite for $n \in \{1, 2, \ldots, N-1\}$ and $h_N(t)$ is infinite.

Theorem 4.5.3. *Suppose conditions* (4.82), (4.83), (4.84), (4.85), (4.94), *and* (H) *hold. Then every solution* y *of* (4.79) *is either oscillatory or satisfies* $\liminf_{t\to\infty} |y(t)| = 0$.

Proof. Suppose to the contrary that y is a nonoscillatory solution of (4.79) such that $\liminf_{t\to\infty}|y(t)|>0$. Hence, by Theorem 4.5.2, y satisfies (4.86) and (4.87). Furthermore there exist $t_1\geq t_0$ and $m_1,m_2>0$ such that $|y(t)|\geq m_1$ and $|f(y(t))|\geq m_2$ for $t\geq t_1$. Hence, from (4.86) and (4.87) we find

$$\begin{aligned}(4.95)\qquad \frac{a(t)|y'(t)|^{\sigma-1}y'(t)}{f(y(t))} &\geq h_0(t)+\int_t^\infty \frac{a(s)|y'(s)|^{\sigma-1}[y'(s)]^2f'(y(s))}{[f(y(s))]^2}ds\\ &\geq h_0(t)\geq 0\end{aligned}$$

for $t\geq t_1$, and so

$$(4.96)\qquad y'(t)\geq [h_0(t)]^{1/\sigma}[f(y(t))]^{1/\sigma}[a(t)]^{-1/\sigma}.$$

From (4.94), (4.95), and (4.96), we have

$$\begin{aligned}\int_t^\infty \frac{a(s)|y'(s)|^{\sigma-1}[y'(s)]^2f'(y(s))}{[f(y(s))]^2}ds &\geq \int_t^\infty \frac{h_0(s)y'(s)f'(y(s))}{f(y(s))}ds\\ &\geq \int_t^\infty \frac{[h_0(s)]^{\frac{\sigma+1}{\sigma}}[f(y(s))]^{1/\sigma}f'(y(s))}{f(y(s))[a(s)]^{1/\sigma}}ds\\ &\geq \lambda\int_t^\infty \frac{[h_0(s)]^{\frac{\sigma+1}{\sigma}}}{[a(s)]^{1/\sigma}}ds = \lambda h_1(t)\end{aligned}$$

for $t\geq t_1$. If $N=1$ in Condition (H), then the right-hand side of the above inequality is infinite. This is a contradiction to (4.86).

Next, it follows from (4.95) and the above inequality that

$$\frac{a(t)[y'(t)]^\sigma}{f(y(t))}\geq h_0(t)+\lambda h_1(t),$$

and as before we obtain

$$\int_t^\infty \frac{a(s)[y'(s)]^{\sigma+1}f'(y(s))}{[f(y(s))]^2}ds\geq \lambda\int_t^\infty \frac{[h_0(s)+\lambda h_1(s)]^{\frac{\sigma+1}{\sigma}}}{[a(s)]^{1/\sigma}}ds=\lambda h_2(t)$$

for $t\geq t_1$. If $N=2$ in Condition (H), then once again we get a contradiction to (4.86). A similar argument yields a contradiction for any integer $N>2$. $\square$

Next we consider (4.79) with $\sigma=1$, namely,

$$(4.97)\qquad (a(t)y'(t))'+q(t)f(y(t))=r(t),$$

where $uf(u)>0$ and $f'(u)\geq 0$ for $u\neq 0$.

Define the set D and the function H as in Theorem 4.4.12. Also, in order to simplify notation, we define

$$W(t)=\frac{a(t)y'(t)}{f(y(t))}$$

for any nonoscillatory solution y of (4.97).

Theorem 4.5.4. *Suppose that for any $\lambda_1>0$ there exists $\lambda_2>0$ such that*

$$(4.98)\qquad f'(y)\geq\lambda_2\quad \textit{for}\quad |y|\geq\lambda_1.$$

Suppose that

$$\int_{t_0}^t a(s)[h(t,s)]^2ds<\infty\quad \textit{for}\quad t\geq t_0$$

and

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[H(t,s)\Big(q(s) - K|r(s)|\Big) - La(s)[h(t,s)]^2\right] ds = \infty \tag{4.99}$$

for every $T \geq t_0$ *and any positive constants* K *and* L. *Then any solution* y *of* (4.97) *is either oscillatory or satisfies* $\liminf_{t\to\infty} |y(t)| = 0$.

Proof. Suppose on the contrary that y is a nonoscillatory solution of (4.97) such that $\liminf_{t\to\infty} |y(t)| > 0$. It then follows from $f'(y) \geq 0$ for $y \neq 0$ that there exist $M > 0$ and $T_1 \geq t_0$ such that $|f(y(t))| \geq \frac{1}{M}$ for all $t \geq T_1$. This implies that

$$\frac{r(t)}{f(y(t))} \leq \left|\frac{r(t)}{f(y(t))}\right| \leq M|r(t)| \quad \text{for} \quad t \geq T_1. \tag{4.100}$$

Since

$$q(t) - \frac{r(t)}{f(y(t))} = -W'(t) - \frac{f'(y(t))}{a(t)}[W(t)]^2,$$

we have for $t \geq T_1$

$$\begin{aligned}
&\int_{T_1}^t H(t,s)\left(q(s) - \frac{r(s)}{f(y(s))}\right) ds \\
&= -\int_{T_1}^t H(t,s)W'(s)ds - \int_{T_1}^t H(t,s)\frac{f'(y(s))}{a(s)}[W(s)]^2 ds \\
&= -H(t,s)W(s)\Big|_{s=T_1}^t - \int_{T_1}^t \left[-\frac{\partial H}{\partial s}(t,s)W(s) + H(t,s)\frac{f'(y(s))}{a(s)}[W(s)]^2\right] ds \\
&= H(t,T_1)W(T_1) - \int_{T_1}^t \left[h(t,s)\sqrt{H(t,s)}W(s) + H(t,s)\frac{f'(y(s))}{a(s)}[W(s)]^2\right] ds.
\end{aligned}$$

Define

$$\varphi(t,s) = \sqrt{H(t,s)}\sqrt{\frac{f'(y(s))}{a(s)}}W(s) + h(t,s)\frac{\sqrt{a(s)}}{2\sqrt{f'(y(s))}}. \tag{4.101}$$

Then, for all $t \geq T_1$,

$$\int_{T_1}^t \left[H(t,s)\left(q(s) - \frac{r(s)}{f(y(s))}\right) ds - \frac{1}{4}\int_{T_1}^t [h(t,s)]^2 \frac{a(s)}{f'(y(s))}\right] ds$$

$$= H(t,T_1)W(T_1) - \int_{T_1}^t [\varphi(t,s)]^2 ds \tag{4.102}$$

$$\leq H(t,T_1)W(T_1). \tag{4.103}$$

By (4.98), there exists $L > 0$ with $f'(y(s)) \geq 1/(4L)$ for $s \geq T_1$. It follows from (4.100) that

$$H(t,T_1)W(T_1) \geq \int_{T_1}^t \left[H(t,s)\left(q(s) - \frac{r(s)}{f(y(s))}\right) - \frac{1}{4}[h(t,s)]^2 \frac{a(s)}{f'(y(s))}\right] ds$$

$$\geq \int_{T_1}^t \left[H(t,s)\Big(q(s) - M|r(s)|\Big) - La(s)[h(t,s)]^2\right] ds \tag{4.104}$$

for all $t \geq T_1$, which together with (4.103) give that

$$\limsup_{t\to\infty} \frac{1}{H(t,T_1)} \int_{T_1}^t \left[H(t,s)\Big(q(s) - M|r(s)|\Big) - La(s)[h(t,s)]^2\right] ds \leq W(T_1),$$

contradicting our assumption (4.99). □

Theorem 4.5.5. *Let*

$$0 < \inf_{s \ge t_0} \left[\liminf_{t\to\infty} \frac{H(t,s)}{H(t,t_0)} \right] \le \infty. \tag{4.105}$$

Suppose that for any $\lambda_1 > 0$ there exists $\lambda_2 > 0$ such that (4.98) holds, and that

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} a(s)[h(t,s)]^2 ds < \infty. \tag{4.106}$$

If there exists a function $A \in C([t_0,\infty))$ such that

$$\int_{t_0}^{\infty} \frac{[A_+(s)]^2}{p(s)} ds = \infty \quad \text{with} \quad A_+(s) = \max\{A(s), 0\} \tag{4.107}$$

holds, and for every $T \ge t_0$ and any positive constants K and L,

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[H(t,s)\left(q(s) - K|r(s)|\right) - La(s)[h(t,s)]^2 \right] ds \ge A(T), \tag{4.108}$$

then any solution y of (4.97) is either oscillatory or satisfies $\liminf_{t\to\infty} |y(t)| = 0$.

Proof. Suppose y is a nonoscillatory solution of (4.97) with $\liminf_{t\to\infty} |y(t)| > 0$. Then as in the proof of Theorem 4.5.4, we can obtain (4.100), (4.102), and (4.104). From (4.102) and (4.104), we deduce

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[H(t,s)\left(q(s) - M|r(s)|\right) - \frac{L}{4} a(s)[h(t,s)]^2 \right] ds$$
$$\le W(T) - \liminf_{t\to\infty} \frac{1}{H(t,T)} \int_T^t [\varphi(t,s)]^2 ds$$

for all $T \ge T_1$. Thus, by (4.108)

$$W(T) \ge A(T) + \liminf_{t\to\infty} \frac{1}{H(t,T)} \int_T^t [\varphi(t,s)]^2 ds$$

for all $T \ge T_1$. This shows that

$$W(T) \ge A(T) \quad \text{for} \quad T \ge T_1 \tag{4.109}$$

and

$$\liminf_{t\to\infty} \frac{1}{H(t,T_1)} \int_{T_1}^t [\varphi(t,s)]^2 ds \le W(T_1) - A(T_1) < \infty.$$

Let

$$P(t) = \frac{1}{H(t,T_1)} \int_{T_1}^t H(t,s) \frac{f'(y(s))}{a(s)} [W(s)]^2 ds$$

and

$$Q(t) = \frac{1}{H(t,T_1)} \int_{T_1}^t h(t,s)\sqrt{H(t,s)} W(s) ds$$

for all $t \ge T_1$. Then

$$\liminf_{t\to\infty} [P(t) + Q(t)] \le \liminf_{t\to\infty} \frac{1}{H(t,T_1)} \int_{T_1}^t [\varphi(t,s)]^2 ds < \infty. \tag{4.110}$$

Now, suppose that

$$\int_{T_1}^{\infty} \frac{f'(y(s))}{a(s)} [W(s)]^2 ds = \infty. \tag{4.111}$$

By condition (4.105), there exists a positive constant ξ such that

$$\inf_{s\geq t_0}\left[\liminf_{t\to\infty}\frac{H(t,s)}{H(t,t_0)}\right]>\xi>0. \tag{4.112}$$

Let $\mu>0$ be arbitrary. Since (4.111) holds, there exists $T_2>T_1$ such that

$$\int_{T_1}^t\frac{f'(y(s))}{a(s)}[W(s)]^2ds\geq\frac{\mu}{\xi}\qquad\text{for}\qquad t\geq T_2.$$

Therefore, for all $t\geq T_2$,

$$\begin{aligned}
P(t) &= \frac{1}{H(t,T_1)}\int_{T_1}^t H(t,s)\frac{d}{ds}\left(\int_{T_1}^s\frac{f'(y(\tau))}{a(\tau)}[W(\tau)]^2d\tau\right)ds\\
&= -\frac{1}{H(t,T_1)}\int_{T_1}^t\frac{\partial H}{\partial s}(t,s)\left(\int_{T_1}^s\frac{f'(y(\tau))}{a(\tau)}[W(\tau)]^2d\tau\right)ds\\
&\geq -\frac{1}{H(t,T_1)}\int_{T_2}^t\frac{\partial H}{\partial s}(t,s)\left(\int_{T_1}^s\frac{f'(y(\tau))}{a(\tau)}[W(\tau)]^2d\tau\right)ds\\
&\geq -\frac{\mu}{\xi H(t,T_1)}\int_{T_2}^t\frac{\partial H}{\partial s}(t,s)ds \;=\; \frac{\mu H(t,T_2)}{\xi H(t,T_1)} \;\geq\; \frac{\mu H(t,T_2)}{\xi H(t,t_0)}.
\end{aligned}$$

By (4.112), there exists $T_3\geq T_2$ such that $\frac{H(t,T_2)}{H(t,t_0)}\geq\xi$ for all $t\geq T_3$, and accordingly $P(t)\geq\mu$ for all $t\geq T_3$. Since μ is arbitrary,

$$\lim_{t\to\infty}P(t)=\infty. \tag{4.113}$$

Further, consider a sequence $\{t_n\}_{n=1}^\infty\subset(t_0,\infty)$ with $\lim_{n\to\infty}t_n=\infty$ and such that

$$\lim_{n\to\infty}[P(t_n)+Q(t_n)]=\liminf_{t\to\infty}[P(t)+Q(t)].$$

Because of (4.110), there is a constant ρ such that

$$P(t_n)+Q(t_n)\leq\rho\qquad\text{for all}\qquad n\in\mathbb{N}. \tag{4.114}$$

Furthermore, (4.113) shows that

$$\lim_{n\to\infty}P(t_n)=\infty, \tag{4.115}$$

and hence (4.114) gives

$$\lim_{n\to\infty}Q(t_n)=-\infty. \tag{4.116}$$

Then, from (4.114) and (4.115), we get for large enough $n\in\mathbb{N}$

$$1+\frac{Q(t_n)}{P(t_n)}\leq\frac{\rho}{P(t_n)}<\frac{1}{2},$$

which together with (4.116) ensures that

$$\lim_{n\to\infty}\frac{[Q(t_n)]^2}{P(t_n)}=\infty. \tag{4.117}$$

On the other hand, using the Schwarz inequality, we obtain for any $n \in \mathbb{N}$

$$\begin{aligned}
[Q(t_n)]^2 &= \left[\frac{1}{H(t_n,T_1)}\int_{T_1}^{t_n} h(t_n,s)\sqrt{H(t_n,s)}W(s)ds\right]^2 \\
&\le \left[\frac{1}{H(t_n,T_1)}\int_{T_1}^{t_n} H(t_n,s)\frac{f'(y(s))}{a(s)}[W(s)]^2 ds\right] \\
&\quad \times \left[\frac{1}{H(t_n,T_1)}\int_{T_1}^{t_n} [h(t_n,s)]^2\frac{a(s)}{f'(y(s))}ds\right] \\
&= \frac{P(t_n)}{H(t_n,T_1)}\int_{T_1}^{t_n} [h(t_n,s)]^2\frac{a(s)}{f'(y(s))}ds.
\end{aligned}$$

Further, (4.112) guarantees that there exists $T_4 \ge T_0$ such that

$$\frac{H(t,T_1)}{H(t,t_0)} \ge \xi \quad \text{for all} \quad t \ge T_4,$$

which means that for large enough $n \in \mathbb{N}$, we have

$$\frac{H(t_n,T_1)}{H(t_n,t_0)} \ge \xi$$

and accordingly,

$$\frac{[Q(t_n)]^2}{P(t_n)} \le \frac{1}{\xi H(t_n,t_0)}\int_{t_0}^{t_n} [h(t_n,s)]^2\frac{a(s)}{f'(y(s))}ds.$$

Now, using assumption (4.98), we deduce that there is a constant $L > 0$ such that

$$\frac{[Q(t_n)]^2}{P(t_n)} \le \frac{L}{H(t_n,t_0)}\int_{t_0}^{t_n} a(s)[h(t_n,s)]^2 ds \quad \text{for large enough} \quad n \in \mathbb{N}.$$

Then it follows from (4.117) that

$$\lim_{n\to\infty}\frac{1}{H(t_n,t_0)}\int_{t_0}^{t_n} a(s)[h(t_n,s)]^2 ds = \infty, \tag{4.118}$$

contradicting the assumption (4.106). We have proved that (4.111) fails to hold, i.e.,

$$\int_{T_1}^{\infty}\frac{f'(y(s))}{a(s)}[W(s)]^2 ds < \infty.$$

Then, by (4.98) and (4.109), we obtain for some $L > 0$

$$L\int_{T_1}^{\infty}\frac{[A_+(s)]^2}{p(s)}ds \le \int_{T_1}^{\infty}[A_+(s)]^2\frac{f'(y(s))}{a(s)}ds \le \int_{T_1}^{\infty}[W(s)]^2\frac{f'(y(s))}{a(s)}ds < \infty,$$

which contradicts (4.107). This completes our proof. □

Theorem 4.5.6. *Suppose that* (4.105) *holds as well as that for any* $\lambda_1 > 0$ *there exists* $\lambda_2 > 0$ *such that* (4.98) *holds. Assume*

$$\liminf_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t} H(t,s)\Big(q(s) - K|r(s)|\Big)\,ds < \infty. \tag{4.119}$$

If there exists a function $A \in C([t_0,\infty))$ *such that* (4.107) *and*

$$\liminf_{t\to\infty}\frac{1}{H(t,T)}\int_{T}^{t}\Big[H(t,s)\Big(q(s) - K|r(s)|\Big) - La(s)[h(t,s)]^2\Big]\,ds \ge A(T) \tag{4.120}$$

hold for every $T \geq t_0$ *and any positive constants* K *and* L*, then any solution* y *of* (4.97) *is either oscillatory or satisfies* $\liminf_{t\to\infty} |y(t)| = 0$.

Proof. Let y be a nonoscillatory solution of (4.97) satisfying $\liminf_{t\to\infty} |y(t)| > 0$. As in the proof of Theorem 4.5.4, (4.100) and (4.102) is fulfilled for each $t \geq T_1 \geq t_0$. Thus, for all $T \geq T_1$

$$\liminf_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[H(t,s)\left(q(s) - M|r(s)|\right) - La(s)[h(t,s)]^2\right] ds$$
$$\leq W(T) - \limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t [\varphi(t,s)]^2 ds,$$

where $\varphi(t,s)$ is defined by (4.101). Therefore, by (4.120), we have

$$W(T) \geq A(T) + \limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t [\varphi(t,s)]^2 ds$$

for all $T \geq T_1$. Hence, (4.109) holds and

$$\limsup_{t\to\infty} \frac{1}{H(t,T_1)} \int_{T_1}^t [\varphi(t,s)]^2 ds \leq W(T_1) - A(T_1) < \infty.$$

This implies that

$$\limsup_{t\to\infty} [P(t) + Q(t)] \leq \limsup_{t\to\infty} \frac{1}{H(t,T_1)} \int_{T_1}^t [\varphi(t,s)]^2 ds < \infty, \tag{4.121}$$

where P and Q are defined as in the proof of Theorem 4.5.5. By using (4.120) we have

$$\begin{aligned} A(t_0) &\leq \liminf_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^t \left[H(t,s)\left(q(s) - K|r(s)|\right) - La(s)[h(t,s)]^2\right] ds \\ &\leq \liminf_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^t H(t,s)\left(q(s) - K|r(s)|\right) ds \\ &\quad - L \liminf_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^t a(s)[h(t,s)]^2 ds, \end{aligned}$$

which together with (4.119) implies that

$$\liminf_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^t a(s)[h(t,s)]^2 ds < \infty.$$

Then there exists a sequence $\{t_n\}_{n=1}^\infty \subset (t_0, \infty)$ with $\lim_{n\to\infty} t_n = \infty$ satisfying

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{H(t_n,t_0)} \int_{t_0}^{t_n} a(s)[h(t_n,s)]^2 ds \\ &\qquad = \liminf_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^t a(s)[h(t,s)]^2 ds < \infty. \end{aligned} \tag{4.122}$$

Now, suppose that (4.111) holds. Proceeding as in the proof of Theorem 4.5.5, we conclude that (4.113) is satisfied. Because of (4.121), there exists a constant ρ such that (4.114) holds. Then, as in the proof of Theorem 4.5.5, we obtain (4.118), which contradicts (4.122). This proves that (4.111) fails. Since the remainder of the proof is similar to the proof of Theorem 4.5.5, it will be omitted. □

4.6. Positive Solutions of Nonlinear Equations

In this section we first consider quasilinear differential equations of the form

$$\left(r(t)[y'(t)]^\sigma\right)' + q(t)[y(t)]^\sigma = 0, \quad t \geq t_0, \tag{4.123}$$

where σ is a quotient of positive odd integers, $q : [t_0, \infty) \to [0, \infty)$ is a continuous function such that $q(t) \not\equiv 0$, and $r : [t_0, \infty) \to (0, \infty)$ is a continuous function.

First, we assume that

$$\int_{t_0}^{\infty} \frac{ds}{[r(s)]^{1/\sigma}} = \infty. \tag{4.124}$$

We begin by assuming that y is a positive solution of (4.123). Then we see from (4.123) that

$$\left(r(t)[y'(t)]^\sigma\right)' = -q(t)[y(t)]^\sigma \leq 0 \quad \text{for} \quad t \geq t_0.$$

Furthermore, since $q(t) \geq 0$ and $q(t) \not\equiv 0$, the nonincreasing function $r(y')^\sigma$ is either eventually positive or negative. If the latter holds, then

$$r(t)[y'(t)]^\sigma \leq c < 0$$

for t greater than or equal to, say T. But then

$$y'(t) \leq \frac{c^{1/\sigma}}{[r(t)]^{1/\sigma}}, \quad t \geq T$$

so that integrating from T to $t \geq T$ provides

$$y(t) - y(T) \leq c^{1/\sigma} \int_T^t \frac{ds}{[r(s)]^{1/\sigma}} \to -\infty,$$

which is a contradiction. We have thus shown that if y is a positive solution of (4.123), then $r(y')^\sigma$ is a positive nonincreasing function, and y' is a positive function.

Let the function w be defined by

$$w(t) = \frac{r(t)[y'(t)]^\sigma}{[y(t)]^\sigma}, \quad t \geq t_0. \tag{4.125}$$

Then by means of what we have just shown, $w(t) > 0$ and $w'(t) \leq 0$ for $t \geq t_0$. Furthermore, since

$$\frac{y'(t)}{y(t)} = \left(\frac{w(t)}{r(t)}\right)^{1/\sigma},$$

we see from (4.123) that

$$w'(t) = -q(t) - w(t)\frac{\sigma y'(t)}{y(t)} = -q(t) - w(t)\sigma\left(\frac{w(t)}{r(t)}\right)^{1/\sigma},$$

i.e.,

$$w'(t) + w(t)\sigma\left(\frac{w(t)}{r(t)}\right)^{1/\sigma} + q(t) = 0, \quad t \geq t_0. \tag{4.126}$$

For the sake of convenience, we will write

$$F(x, y, z) = z\left(\frac{x}{y}\right)^{1/z}.$$

Then we can also write (4.126) in the simpler form

$$w'(t) + w(t)F\left(w(t), r(t), \sigma\right) + q(t) = 0, \quad t \geq t_0.$$

Note that $F(x, y, z) > 0$ for $x, y, z > 0$. Furthermore,

$$F_x(x, y, z) = x^{\frac{1}{z}-1}y^{-\frac{1}{z}},$$

which is positive for $x, y, z > 0$, and

$$F_y(x, y, z) = -x^{\frac{1}{z}}y^{-\frac{1}{z}-1},$$

which is negative for $x, y, z > 0$. These properties of F will be referred later as the monotone nature of F.

Theorem 4.6.1. *Equation* (4.123) *has a positive solution y for $t \geq t_0$ if and only if there is a positive and continuous function u on $[t_0, \infty)$ which satisfies the integral inequality*

$$\int_t^\infty u(s)F\left(u(s), r(s), \sigma\right) ds + \int_t^\infty q(s)ds \leq u(t), \quad t \geq t_0. \tag{4.127}$$

Proof. If y is a positive solution of (4.123), then the function w defined by (4.125) is a positive solution of the inequality

$$\int_t^\infty w(s)F\left(w(s), r(s), \sigma\right) ds + \int_t^\infty q(s)ds \leq w(t), \quad t \geq t_0,$$

obtained by integrating (4.126) from t to ∞. Conversely, let u be a positive and continuous function which satisfies (4.127). Let us define a mapping $T : C([t_0, \infty), (0, \infty)) \to C([t_0, \infty), (0, \infty))$ as follows:

$$(Tv)(t) = \int_t^\infty v(s)F\left(v(s), r(s), \sigma\right) ds + \int_t^\infty q(s)ds, \quad t \geq t_0,$$

where $v \in C([t_0, \infty), (0, \infty))$. Note that in view of (4.127), $(Tv)(t) \leq v(t)$ for $t \geq t_0$. Consider the successive approximating sequence $\{v_n\}_{n\in\mathbb{N}_0}$, defined by

$$v_0(t) = 0 \quad \text{and} \quad v_{n+1}(t) = (Tv_n)(t), \quad n \in \mathbb{N}_0 \quad \text{for} \quad t \geq t_0. \tag{4.128}$$

By means of the monotone properties of $F(w, r, \sigma)$, it is not difficult to see that

$$v_0(t) < v_1(t) \leq v_2(t) \leq \ldots \leq v_n(t) \leq \ldots \leq u(t)$$

for $n \in \mathbb{N}_0$ and $t \geq t_0$. Thus, by letting v^* be the positive function defined by

$$v^*(t) = \lim_{n\to\infty} v_n(t), \quad t \geq t_0,$$

we may then take limits on both sides of the recursive definition in (4.128) and infer from Lebesgue's dominated convergence theorem that $v^* = Tv^*$.

In view of (4.125), the function y on $[t_0, \infty)$ defined by $y(t_0) = c_0 > 0$ and

$$y(t) = y(t_0)\exp\left(\int_{t_0}^t \left(\frac{v^*(s)}{r(s)}\right)^{1/\sigma} ds\right), \quad t \geq t_0$$

is a positive solution of (4.123). The proof is complete. $\square$

We remark that by slightly modifying the arguments used in the proof of the above theorem, we see that the following variant holds.

Theorem 4.6.2. *Equation* (4.123) *has a positive solution if and only if the sequence* $\{v_n\}$ *defined by* (4.128) *is well defined and pointwise convergent.*

The approximating sequence defined by (4.128) is not the only one that is available. Indeed, let us introduce another formal sequence of functions $\{\phi_n\}$ defined as follows: First we define a mapping $S : C([t_0, \infty), (0, \infty)) \to C([t_0, \infty), (0, \infty))$ by

$$(Su)(t) = \int_t^\infty u(s)F\left(u(s), r(s), \sigma\right) ds, \quad t \geq t_0,$$

where $u \in C([t_0, \infty), (0, \infty))$. Then we define

$$\text{(4.129)} \qquad \begin{cases} \phi_0(t) = \displaystyle\int_t^\infty q(s)ds \quad \text{and} \\ \phi_{n+1}(t) = (S\phi_n)(t), \quad n \in \mathbb{N}_0 \quad \text{for} \quad t \geq t_0. \end{cases}$$

If the sequence $\{\phi_n\}$ is well defined, then by means of the monotone properties of $F(w, r, \sigma)$, $\phi_1(t) > 0$ for $t \geq t_0$. Furthermore,

$$\phi_2(t) = \left(S(\phi_0 + \phi_1)\right)(t) \geq (S\phi_0)(t) = \phi_1(t), \quad t \geq t_0$$

and

$$\phi_3(t) = \left(S(\phi_0 + \phi_2)\right)(t) \geq \left(S(\phi_0 + \phi_1)\right)(t) = \phi_2(t), \quad t \geq t_0.$$

Inductively, we see that

$$0 < \phi_1(t) \leq \phi_2(t) \leq \ldots, \quad t \geq t_0.$$

Therefore, if we assume in addition that $\{\phi_n\}$ is pointwise convergent to ϕ, then by Lebesgue's monotone convergence theorem, we see from (4.129) that

$$\phi_0(t) + \phi(t) = \phi_0(t) + \int_t^\infty \left(\phi_0(s) + \phi(s)\right) F\left(\phi_0(s) + \phi(s), r(s), \sigma\right) ds, \quad t \geq t_0.$$

In other words, we have found a positive function $\phi_0 + \phi$ which satisfies (4.127). Conversely, if we assume that y is a positive function which satisfies (4.127), i.e.,

$$(Sy)(t) + \phi_0(t) \leq y(t), \quad t \geq t_0,$$

then

$$\phi_0(t) \leq (Sy)(t) + \phi_0(t) \leq y(t), \quad t \geq t_0,$$
$$\phi_0(t) + \phi_1(t) = \phi_0(t) + (S\phi_0)(t) \leq \phi_0(t) + (Sy)(t) \leq y(t), \quad t \geq t_0,$$

and

$$\phi_0 + \phi_{n+1}(t) = \phi_0(t) + \left(S(\phi_0 + \phi_n)\right)(t) \leq \phi_0(t) + (Sy)(t) \leq y(t), \quad t \geq t_0.$$

Thus the sequence $\{\phi_n\}$ is well defined. Therefore,

$$0 < \phi_0(t) \leq \phi_2(t) \leq \ldots \leq y(t), \quad t \geq t_0,$$

which implies that $\{\phi_n\}$ is also pointwise convergent.

Theorem 4.6.3. *Equation* (4.123) *has a positive solution if and only if the sequence* $\{\phi_n\}$ *defined by* (4.129) *is well defined and pointwise convergent.*

We now deduce two important implications from these existence criteria. First of all, if u is a positive function which satisfies (4.127), then by means of the monotone properties of the function F, it also satisfies

$$\int_t^\infty u(s)F\left(u(s), R(s), \sigma\right) ds + \int_t^\infty Q(s)ds \le u(t), \quad t \ge t_0,$$

where $Q : [t_0, \infty) \to [0, \infty)$ is a continuous function which satisfies

$$\int_t^\infty Q(s)ds \le \int_t^\infty q(s)ds, \quad t \ge t_0, \tag{4.130}$$

and $R : [t_0, \infty) \to [0, \infty)$ is a continuous function which satisfies $0 < r(t) \le R(t)$ for $t \ge t_0$. The following Hille–Wintner type comparison theorem is now clear from Theorem 4.6.1.

Theorem 4.6.4. *Assume that R is a positive and continuous function which satisfies $0 < r(t) \le R(t)$ for $t \ge t_0$ and*

$$\int_{t_0}^\infty \frac{ds}{[R(s)]^{1/\sigma}} = \infty,$$

and that Q is a nonnegative and continuous function which satisfies (4.130). *If* (4.123) *has a positive solution, then so does the equation*

$$\left(R(t)[y'(t)]^\sigma\right)' + Q(t)[y(t)]^\sigma = 0, \quad t \ge t_0.$$

Next, let us assume that $\sigma = 1$ and

$$\int_t^\infty \frac{1}{r(s)} \left(\int_s^\infty q(u)du\right)^2 ds \ge \frac{1+\delta}{4} \int_t^\infty q(s)ds, \quad t \ge t_0, \tag{4.131}$$

where $\delta > 0$ is arbitrary. In view of the function ϕ_0 defined by (4.129), (4.131) is equivalent to

$$\int_t^\infty \frac{[\phi_0(s)]^2}{r(s)} ds \ge \frac{1+\delta}{4}\phi_0(t), \quad t \ge t_0.$$

Now the functions ϕ_1 and ϕ_2 defined in (4.129) satisfy on $[t_0, \infty)$

$$\phi_1(t) \ge c_0\phi_0(t),$$

where $c_0 = (1+\delta)/4$, and with $c_1 = (1+c_0)^2 c_0$,

$$\begin{aligned}
\phi_2(t) &\ge \int_t^\infty \left(\phi_0(s) + \phi_1(s)\right) F\left(\phi_0(s) + \phi_1(s), r(s), 1\right) ds \\
&\ge \int_t^\infty (1+c_0)\phi_0(s)F\left((1+c_0)\phi_0(s), r(s), 1\right) ds \\
&\ge (1+c_0)^2 \int_t^\infty \phi_0(s)\frac{\phi_0(s)}{r(s)} ds \\
&\ge (1+c_0)^2 c_0\phi_0(t) \;=\; c_1\phi_0(t),
\end{aligned}$$

respectively. By induction we easily see that

$$\phi_{n+1}(t) \ge c_n\phi_0(t), \quad t \ge t_0, \quad n \in \mathbb{N},$$

where

$$c_n = (1+c_{n-1})^2 c_0 \quad \text{for} \quad n \in \mathbb{N}.$$

It is also easy to see that the sequence $\{c_n\}$ is increasing. We assert further that it is unbounded. Otherwise, $c_n \to c$ as $n \to \infty$ would imply $c = (1+c)^2 c_0$, i.e.,

$$c_0 c^2 + (2c_0 - 1)c + c_0 = 0.$$

However, this quadratic equation cannot have a real solution if $c_0 > 1/4$. Thus the assumption that $c_n \to c$ is impossible. Hence the sequence $\{\phi_n\}$ cannot be pointwise convergent. The following is now clear from Theorem 4.6.3.

Corollary 4.6.5. *Assume that $\sigma = 1$ and that the function q satisfies* (4.131) *for some number $\delta > 0$. Then* (4.123) *cannot have any positive solution.*

We remark that the condition (4.131) in Corollary 4.6.5 is sharp in the following sense.

Corollary 4.6.6. *Assume that $\sigma = 1$, that*

$$\int_{t_0}^{\infty} q(s)ds < \infty,$$

and that

$$\int_t^{\infty} \frac{\left(\int_s^{\infty} q(u)du\right)^2}{r(s)} ds \le \mu \int_t^{\infty} q(s)ds, \quad t \ge t_0,$$

where $\mu \le 1/4$. Then (4.123) *has a positive solution.*

Proof. The crux of our proof lies in the observation that $F(x, y, \sigma) = x/y$ when $\sigma = 1$. More specifically, consider the sequence $\{\phi_n\}$ defined by (4.129). Note that

$$\phi_0(t) = \int_t^{\infty} q(s)ds < \infty, \quad t \ge t_0$$

and

$$\phi_1(t) = (S\phi_0)(t) = \int_t^{\infty} \phi_0(s) F\left(\phi_0(s), r(s), 1\right) ds = \int_t^{\infty} \frac{[\phi_0(s)]^2}{r(s)} ds \le c_0 \phi_0(t)$$

for $t \ge t_0$, where $c_0 = \mu$. Next,

$$\begin{aligned}
\phi_2(t) &= \left(S(\phi_0 + \phi_1)\right)(t) \\
&= \int_t^{\infty} \frac{[\phi_0(s) + \phi_1(s)]^2}{r(s)} ds \\
&\le \int_t^{\infty} \frac{[\phi_0(s) + c_0\phi_0(s)]^2}{r(s)} ds \\
&\le \int_t^{\infty} \frac{(1+c_0)^2 [\phi_0(s)]^2}{r(s)} ds \\
&\le c_1 \phi_0(t)
\end{aligned}$$

for $t \ge t_0$, where $c_1 = (1+c_0)^2 c_0$. Inductively, we see that

$$\phi_{n+1}(t) \le c_n \phi_0(t), \quad t \ge t_0, \quad n \in \mathbb{N}$$

where

$$c_n = (1 + c_{n-1})c_0 \quad \text{for} \quad n \in \mathbb{N}.$$

It is also easy to see that the sequence $\{c_n\}$ is nondecreasing and convergent. We may see this as follows. Consider the fixed point problem

$$x = g(x), \quad \text{where} \quad g(x) = \mu(1+x)^2.$$

As is customary, we find fixed points by means of the iteration scheme

$$x_n = \mu(1+x_{n-1})^2, \quad n \in \mathbb{N}.$$

Note that when $\mu = 1/4$, the graph of g is a parabola which has a unique minimum at $x = -1$ and touches the line $y = x$ at $(x, y) = (1, 1)$. Therefore, if we choose $x_0 = \mu$, then we see that the approximating sequence $\{x_n\}$ is strictly increasing and converges to $x = 1$. If $c_0 < 1/4$, then clearly $c_n < x_n < 1$ for all $n \in \mathbb{N}$. This shows that $\{c_n\}$ is bounded and hence converges.

We have thus shown that the sequence $\{\phi_n\}$ is well defined and pointwise convergent. The proof is now complete in view of Theorem 4.6.3. □

In order to compare with (4.123), we will consider the following class of advanced type differential equations of the form

$$\left(r(t)[y'(t)]^\sigma\right)' + q(t)[y(t+\tau)]^\sigma = 0, \quad t \geq t_0, \tag{4.132}$$

where $\tau \geq 0$. By means of the same Riccati transformation (4.125), we may proceed in a similar manner as above and obtain the following extension of Theorem 4.6.1.

Theorem 4.6.7. *Equation* (4.132) *has a positive solution* y *if and only if there is a positive and continuous function* u *which satisfies on* $[t_0, \infty)$ *the inequality*

$$\int_t^\infty u(s)F\left(u(s), r(s), \sigma\right) ds + \int_t^\infty q(s)\left\{\exp\left(\int_s^{s+\tau}\left(\frac{u(v)}{r(v)}\right)^{\frac{1}{\sigma}} dv\right)\right\}^\sigma ds \leq u(t).$$

Since

$$\int_t^\infty q(s)ds \leq \int_t^\infty q(s)\left\{\exp\left(\int_s^{s+\tau}\left(\frac{u(v)}{r(v)}\right)^{\frac{1}{\sigma}} dv\right)\right\}^\sigma ds$$

for $u > 0$, we immediately obtain from Theorems 4.6.1 and 4.6.7 the following corollary.

Corollary 4.6.8. *If* (4.132) *has a positive solution, then so does* (4.123).

A partial converse of the above Corollary 4.6.8 can be obtained as follows. Let y be a positive solution of (4.123) such that

$$1 \leq \left[\exp\left(\int_t^{t+\tau}\left(\frac{u(s)}{r(s)}\right)^{\frac{1}{\sigma}} ds\right)\right]^\sigma \leq \Gamma, \quad t \geq t_0. \tag{4.133}$$

Then

$$\int_t^\infty u(s)F\left(u(s), r(s), \sigma\right) ds + \frac{1}{\Gamma}\int_t^\infty q(s)\left[\exp\left(\int_s^{s+\tau}\left(\frac{u(v)}{r(v)}\right)^{\frac{1}{\sigma}} dv\right)\right]^\sigma ds \leq u(t)$$

for $t \geq t_0$, so that there exists a positive solution of

$$\left(r(t)[y'(t)]^\sigma\right)' + \frac{1}{\Gamma}q(t)[y(t)]^\sigma = 0, \quad t \geq t_0.$$

At this point, it is not clear which conditions are needed for a positive solution y to exist such that the additional property (4.133) holds. However, we may twist the above arguments slightly and conclude that if y is a positive solution of

$$\left(r(t)[y'(t)]^\sigma\right)' + \beta q(t)[y(t)]^\sigma = 0, \quad t \geq t_0, \tag{4.134}$$

where $\beta \geq 1$ while the other parameters are the same as in (4.123), and if

$$1 \leq \left[\exp\left(\int_t^{t+\tau} \left(\frac{u(s)}{r(s)}\right)^{\frac{1}{\sigma}} ds\right)\right]^{\sigma} \leq \beta, \quad t \geq t_0,$$

then (4.132) has a positive solution.

Now we assert that if y is a positive solution of (4.134), then

$$\frac{r(t)[y'(t)]^{\sigma}}{[y(t)]^{\sigma}} \to 0 \quad \text{as} \quad t \to \infty.$$

Indeed, from (4.134), we see that $(r(y')^{\sigma})'(t) \leq 0$ and $r(t)[y'(t)]^{\sigma} > 0$ for $t \geq t_0$. Thus either $r(t)[y'(t)]^{\sigma}$ decreases to zero or to a constant $c > 0$. In the former case,

$$\frac{r(t)[y'(t)]^{\sigma}}{[y(t)]^{\sigma}} \leq \frac{r(t)[y'(t)]^{\sigma}}{[y(t_0)]^{\sigma}} \to 0 \quad \text{as} \quad t \to \infty,$$

as desired. In the latter case, we have

$$y'(t) \geq \left(\frac{c}{r(t)}\right)^{\frac{1}{\sigma}},$$

which implies, in view of (4.124), that

$$y(t) \geq y(t_0) + \int_{t_0}^{t} \frac{c^{1/\sigma}}{[r(s)]^{1/\sigma}} ds \to \infty \quad \text{as} \quad t \to \infty.$$

Thus,

$$\frac{r(t)[y'(t)]^{\sigma}}{[y(t)]^{\sigma}} \leq \frac{r(t_0)[y'(t_0)]^{\sigma}}{[y(t)]^{\sigma}} \to 0 \quad \text{as} \quad t \to \infty$$

also. As a consequence, if $1/r$ is bounded, then given any number $\beta > 1$, the condition will automatically hold for all large t. Therefore, we may now conclude that if $\beta > 1$ and $1/r$ is bounded, and if (4.134) has a positive solution, then (4.132) has an eventually positive solution as well.

In the above discussion we always assumed that (4.124) holds. In the following we give some results for the existence of positive nondecreasing solutions of (4.123) without requiring condition (4.124).

We have shown that if y is a positive nondecreasing solution of (4.123), then w defined by (4.125) satisfies the inequality

$$w'(t) + w(t)F\Big(w(t), r(t), \sigma\Big) + q(t) \leq 0, \quad t \geq t_0. \tag{4.135}$$

Note that $w(t)$ is nonnegative. Therefore, if we now integrate (4.135) from t to ∞, we obtain

$$w(t) \geq \int_t^{\infty} w(s)F\Big(w(s), r(s), \sigma\Big)\, ds + \int_t^{\infty} q(s)ds, \quad t \geq t_0. \tag{4.136}$$

Theorem 4.6.9. *Suppose that $\int^{\infty} q(s)ds$ exists. Then* (4.123) *has a positive nondecreasing solution y on $[t_0, \infty)$ if and only if there is a nonnegative and continuous function w on $[t_0, \infty)$ which satisfies the integral inequality* (4.136).

Proof. We need to show that if (4.136) has a nonnegative solution w, then (4.123) has a positive nondecreasing solution. Let $w \in C([t_0,\infty),[0,\infty))$ be a function such that $\int_t^\infty w(s)F(w(s),r(s),\sigma)ds$ exists. Define a mapping Tw by

$$(Tw)(t) = \int_t^\infty w(s)F\left(w(s),r(s),\sigma\right)ds + \int_t^\infty q(s)ds, \quad t \ge t_0.$$

Thus $Tw \in C([t_0,\infty),[0,\infty))$. Note that in view of (4.136), $(Tw)(t) \le w(t)$ for $t \ge t_0$. Consider the successive approximating sequence $\{w_n\}_{n\in\mathbb{N}_0}$ defined by

$$\begin{cases} w_0(t) = \displaystyle\int_t^\infty q(s)ds \quad \text{and} \\ w_{n+1}(t) = (Tw_n)(t), \quad n \in \mathbb{N}_0 \quad \text{for} \quad t \ge t_0. \end{cases} \tag{4.137}$$

By means of the monotone properties of $F(w,r,\sigma)$, it is not difficult to see that

$$w_0(t) < w_1(t) \le w_2(t) \le \ldots \le w_n(t) \le \ldots \le w(t)$$

for $n \in \mathbb{N}_0$ and $t \ge t_0$. Thus, by letting w^* be the positive function defined by

$$w^*(t) = \lim_{n\to\infty} w_n(t), \quad t \ge t_0,$$

we may then take limits on both sides of the recursive definition in (4.137) and infer from Lebesgue's dominated convergence theorem that $w^* = Tw^*$.

In view of (4.125), the function $y : [t_0,\infty) \to \mathbb{R}$ defined by $y(t_0) = c_0 > 0$ and

$$y(t) = y(t_0)\exp\left(\int_{t_0}^t \left(\frac{w^*(s)}{r(s)}\right)^{\frac{1}{\sigma}} ds\right), \quad t \ge t_0$$

is a positive nondecreasing solution of (4.123). The proof is complete. $\square$

As a direct application, we deduce comparison theorems for the existence of a positive nondecreasing solution of (4.123). Consider, together with (4.123), the equation

$$\left(R(t)[y'(t)]^\sigma\right)' + Q(t)[y(t)]^\sigma = 0, \quad t \ge t_0, \tag{4.138}$$

where R and Q satisfy the same conditions as those imposed on r and q. The following is now clear from Theorem 4.6.9.

Theorem 4.6.10. *In addition to the conditions imposed on the equations* (4.123) *and* (4.138), *suppose further that* $r(t) \ge R(t) > 0$ *for* $t \ge t_0$ *and*

$$\int_t^\infty Q(s)ds \ge \int_t^\infty q(s)ds \quad \text{for} \quad t \ge t_0.$$

If (4.138) *has a positive nondecreasing solution, then so does* (4.123).

Now we compare (4.123) with the equation

$$\left(R(t)[y'(t)]^\sigma\right)' + \lambda(t)Q(t)[y(t)]^\sigma = 0. \tag{4.139}$$

Theorem 4.6.11. *Suppose that* $\int^\infty Q(s)ds$ *exists and*

$$r(t) \ge R(t) > 0 \quad \text{and} \quad Q(t) \ge q(t) \quad \text{for} \quad t \ge t_0.$$

Suppose further that λ *is a differentiable function such that* $\lambda(t) \ge 1$ *and* $\lambda'(t) \ge 0$ *for* $t \ge t_0$. *If* (4.139) *has a positive nondecreasing solution, then so does* (4.123).

Proof. By our assumptions and arguing as in (4.126), it follows that the equation

$$u'(t) + u(t)F\Big(u(t), R(t), \sigma\Big) + \lambda(t)Q(t) = 0, \quad t \geq t_0 \tag{4.140}$$

has a nonnegative solution u. Thus, dividing (4.140) by $\lambda(t)$, we obtain, in view of the homogeneity property of F, that

$$\frac{u'(t)}{\lambda(t)} + \frac{u(t)}{\lambda(t)}F\Big(u(t), R(t), \sigma\Big) + Q(t) = \frac{u'(t)}{\lambda(t)} + \frac{u(t)}{\lambda(t)}F\left(\frac{u(t)}{\lambda(t)}, \frac{R(t)}{\lambda(t)}, \sigma\right) + Q(t) = 0$$

for $t \geq t_0$. In view of $\lambda'(t) \geq 0$, we have

$$\left(\frac{u}{\lambda}\right)'(t) = \frac{u'(t)}{\lambda(t)} - \frac{u(t)\lambda'(t)}{\lambda^2(t)} \leq \frac{u'(t)}{\lambda(t)}.$$

Additionally, since $Q(t) \geq q(t)$ and

$$F\left(\frac{u(t)}{\lambda(t)}, \frac{R(t)}{\lambda(t)}, \sigma\right) \geq F\left(\frac{u(t)}{\lambda(t)}, \frac{r(t)}{\lambda(t)}, \sigma\right) \geq F\left(\frac{u(t)}{\lambda(t)}, r(t), \sigma\right),$$

we have

$$\left(\frac{u}{\lambda}\right)'(t) + \frac{u(t)}{\lambda(t)}F\left(\frac{u(t)}{\lambda(t)}, r(t), \sigma\right) + q(t) \leq 0, \quad t \geq t_0.$$

Setting $w = u/\lambda$, the preceding inequality implies that

$$w'(t) + w(t)F\Big(w(t), r(t), \sigma\Big) + q(t) \leq 0, \quad t \geq t_0$$

has a nonnegative solution. By Theorem 4.6.9, (4.123) has a positive nondecreasing solution. $\square$

There is a dual to the above theorem as follows.

Theorem 4.6.12. *Suppose that $\int^\infty q(s)ds$ exists and*

$$0 < r(t) \leq R(t) \quad \text{and} \quad Q(t) \leq q(t) \quad \text{for} \quad t \geq t_0.$$

Suppose further that λ is a differentiable function such that $0 < \lambda(t) \leq 1$ and $\lambda'(t) \leq 0$ for $t \geq t_0$. If (4.123) *has a positive nondecreasing solution, then so does* (4.139).

Proof. By our assumptions, the equation

$$u'(t) + u(t)F\Big(u(t), r(t), \sigma\Big) + q(t) = 0, \quad t \geq t_0$$

has a nonnegative solution u. It thus follows that

$$u'(t) + u(t)F\Big(u(t), r(t), \sigma\Big) + Q(t) \leq 0, \quad t \geq t_0,$$

which, after multiplying by $\lambda(t)$, becomes

$$\lambda(t)u'(t) + \lambda(t)u(t)F\Big(\lambda(t)u(t), \lambda(t)r(t), \sigma\Big) + \lambda(t)Q(t) \leq 0, \quad t \geq t_0.$$

Since $\lambda'(t) \leq 0$, we have

$$(\lambda u)'(t) = \lambda(t)u'(t) + \lambda'(t)u(t) \leq \lambda(t)u'(t)$$

and

$$F\Big(u(t)\lambda(t), r(t)\lambda(t), \sigma\Big) \geq F\Big(u(t)\lambda(t), r(t), \sigma\Big) \geq F\Big(u(t)\lambda(t), R(t), \sigma\Big).$$

Thus

$$(u\lambda)'(t) + u(t)\lambda(t)F\Big(u(t)\lambda(t), R(t), \sigma\Big) + \lambda(t)Q(t) \le 0, \quad t \ge t_0.$$

This implies that

$$w'(t) + w(t)F\Big(w(t), R(t), \sigma\Big) + \lambda(t)Q(t) \le 0, \quad t \ge t_0$$

has a nonnegative solution. By Theorem 4.6.9, (4.139) has a positive nondecreasing solution. ☐

As another application, we derive an explicit existence criterion based on Theorem 4.6.9.

Theorem 4.6.13. *Suppose that*

$$\int_{t_0}^{\infty} q(s)ds < \infty$$

and let

$$\phi(t) = 2\int_t^{\infty} q(s)ds < \infty, \quad t \ge t_0.$$

Suppose further that

$$\int_{t_0}^{\infty} \left(\frac{\phi(s)}{r(s)}\right)^{\frac{1}{\sigma}} ds \le \frac{1}{2\sigma}. \tag{4.141}$$

Then (4.123) *has a nonnegative solution.*

Proof. It suffices to show that $w = \phi$ satisfies (4.136). Indeed, for $t \ge t_0$,

$$\begin{aligned}\int_t^{\infty} w(s)F\Big(w(s), r(s), \sigma\Big)\,ds &= \int_t^{\infty} \sigma\phi(s)\left(\frac{\phi(s)}{r(s)}\right)^{\frac{1}{\sigma}} ds \\ &\le \phi(t)\int_t^{\infty} \sigma\left(\frac{\phi(s)}{r(s)}\right)^{\frac{1}{\sigma}} ds.\end{aligned}$$

Hence, in view of (4.141), we have

$$\int_t^{\infty} w(s)F\Big(w(s), r(s), \sigma\Big)\,ds + \int_t^{\infty} q(s)ds \le \frac{\phi(t)}{2} + \frac{\phi(t)}{2} = \phi(t) = w(t).$$

The proof is complete. ☐

4.7. Oscillation of Half-Linear Equations

In this section we consider the problem of oscillation of the second order half-linear damped differential equation

$$\Big(r(t)\,|y'(t)|^{\alpha-1}\,y'(t)\Big)' + p(t)\,|y'(t)|^{\alpha-1}\,y'(t) + q(t)\,|y(t)|^{\alpha-1}\,y(t) = 0 \tag{4.142}$$

on the half-line $[t_0, \infty)$. In (4.142) we assume that

$$p, q \in C[t_0, \infty) \quad \text{and} \quad r \in C^1([t_0, \infty), (0, \infty)),$$

and $\alpha > 0$ is a constant.

We recall that a function $y : [t_0, t_1) \to (-\infty, \infty)$, $t_1 > t_0$ is called a solution of (4.142) if y satisfies (4.142) on $[t_0, t_1)$. In the sequel it will be always assumed that solutions of (4.142) exist for any $t_0 \ge 0$.

In order to prove our theorems, we use the following well-known inequality due to Hardy, Littlewood, and Pólya [**120**].

Lemma 4.7.1. *If A, B are nonnegative, then*

$$A^q + (q-1)B^q \geq qAB^{q-1} \quad \text{for} \quad q > 1,$$

where equality holds if and only if $A = B$.

We say that a function $H = H(t,s)$ belongs to a function class Y, denoted by $H \in Y$, if $H \in C(D, \mathbb{R})$, where $D = \{(t,s) : -\infty < s \leq t < \infty\}$, which satisfies

$$H(t,t) = 0 \quad \text{and} \quad H(t,s) > 0 \text{ for } t > s$$

and has a partial derivative $\partial H/\partial s$ on D such that (compare Section 4.4)

$$\frac{\partial H}{\partial s} = -h(t,s)\sqrt{H(t,s)}, \tag{4.143}$$

where h is a nonnegative and continuous function on D.

Theorem 4.7.2. *If there exists $H \in Y$ such that*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)q(s) - \frac{r(s)\left|h(t,s) + \frac{p(s)}{r(s)}\sqrt{H(t,s)}\right|^{\alpha+1}}{(\alpha+1)^{\alpha+1}[H(t,s)]^{(\alpha-1)/2}} \right] ds = \infty, \tag{4.144}$$

then every solution of (4.142) *is oscillatory.*

Proof. Let y be a nonoscillatory solution of (4.142). Assume that $y(t) \neq 0$ for $t \geq t_0$. We define

$$u(t) = \frac{r(t)\left|y'(t)\right|^{\alpha-1} y'(t)}{\left|y(t)\right|^{\alpha-1} y(t)}, \quad t \geq t_0. \tag{4.145}$$

Then for every $t \geq t_0$, we have

$$u'(t) = -q(t) - \frac{p(t)}{r(t)}u(t) - \alpha \frac{\left|u(t)\right|^{(\alpha+1)/\alpha}}{[r(t)]^{1/\alpha}},$$

and consequently

$$\begin{aligned} \int_{t_0}^{t} H(t,s)q(s)ds &= -\int_{t_0}^{t} H(t,s)u'(s)ds - \int_{t_0}^{t} H(t,s)\frac{p(s)}{r(s)}u(s)ds \\ &\quad -\alpha \int_{t_0}^{t} H(t,s)\frac{\left|u(s)\right|^{(\alpha+1)/\alpha}}{[r(s)]^{1/\alpha}} ds. \end{aligned}$$

Since

$$\int_{t_0}^{t} H(t,s)u'(s)ds = -H(t,t_0)u(t_0) - \int_{t_0}^{t} \frac{\partial H(t,s)}{\partial s}u(s)ds$$

and in view of (4.143), the previous equality becomes

$$\begin{aligned} \int_{t_0}^{t} H(t,s)q(s)ds &\leq H(t,t_0)u(t_0) \\ &+ \int_{t_0}^{t} \left|h(t,s)\sqrt{H(t,s)} + \frac{p(s)}{r(s)}H(t,s)\right| \left|u(s)\right| ds - \alpha \int_{t_0}^{t} H(t,s)\frac{\left|u(s)\right|^{(\alpha+1)/\alpha}}{[r(s)]^{1/\alpha}} ds. \end{aligned} \tag{4.146}$$

In Lemma 4.7.1, we let

$$q = \frac{\alpha+1}{\alpha}, \quad A = [\alpha H(t,s)]^{\alpha/(\alpha+1)} \frac{|u(s)|}{[r(s)]^{1/\alpha}} ds,$$

and

$$B = \frac{\alpha^{\alpha/(\alpha+1)}}{(\alpha+1)^{\alpha+1}} \frac{[r(s)]^{\alpha/(\alpha+1)} \left| h(t,s)\sqrt{H(t,s)} + \frac{p(s)}{r(s)} H(t,s) \right|^{\alpha}}{[H(t,s)]^{\alpha^2/(\alpha+1)}}.$$

From Lemma 4.7.1, we then obtain for $t > s \geq t_0$

$$\begin{aligned} &\left| h(t,s)\sqrt{H(t,s)} + \frac{p(s)}{r(s)} H(t,s) \right| |u(s)| - \alpha H(t,s) \frac{|u(s)|}{[r(s)]^{1/\alpha}} \\ &\leq \quad \frac{r(s) \left| h(t,s)\sqrt{H(t,s)} + \frac{p(s)}{r(s)} H(t,s) \right|^{\alpha+1}}{(\alpha+1)^{\alpha+1} [H(t,s)]^{\alpha}} \\ &= \quad \frac{r(s) \left| h(t,s) + \frac{p(s)}{r(s)} \sqrt{H(t,s)} \right|^{\alpha+1}}{(\alpha+1)^{\alpha+1} [H(t,s)]^{(\alpha-1)/2}}. \end{aligned}$$

Hence, (4.146) implies

$$\begin{aligned} \text{(4.147)} \quad \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s) q(s) ds &\leq u(t_0) \\ &+ \frac{1}{(\alpha+1)^{\alpha+1} H(t,t_0)} \int_{t_0}^{t} \frac{r(s) \left| h(t,s) + \frac{p(s)}{r(s)} \sqrt{H(t,s)} \right|^{\alpha+1}}{[H(t,s)]^{(\alpha-1)/2}} ds \end{aligned}$$

for $t \geq t_0$. Consequently,

$$\begin{aligned} &\frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s) q(s) ds \\ &\qquad - \frac{1}{(\alpha+1)^{\alpha+1} H(t,t_0)} \int_{t_0}^{t} \frac{r(s) \left| h(t,s) + \frac{p(s)}{r(s)} \sqrt{H(t,s)} \right|^{\alpha+1}}{[H(t,s)]^{(\alpha-1)/2}} ds \leq u(t_0) \end{aligned}$$

for $t \geq t_0$. Taking the limit superior as $t \to \infty$ in the above, we obtain a contradiction to (4.144), which completes the proof. □

As immediate consequences of Theorem 4.7.2 we obtain the following corollaries.

Corollary 4.7.3. *If there exists $H \in Y$ such that*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[\frac{r(s) \left| h(t,s) + \frac{p(s)}{r(s)} \sqrt{H(t,s)} \right|^{\alpha+1}}{(\alpha+1)^{\alpha+1} [H(t,s)]^{(\alpha-1)/2}} \right] ds < \infty$$

and

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s) q(s) ds = \infty,$$

then every solution of (4.142) *is oscillatory.*

Corollary 4.7.4. *Let $\alpha = 1$ and $p(t) \equiv 0$, and let the functions h and H be as in Theorem 4.7.2. If*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)q(s) - \frac{r(s)}{4}[h(t,s)]^2 \right] ds = \infty,$$

then every solution of (4.142) *is oscillatory.*

With an appropriate choice of the functions H and h, we can derive from Theorem 4.7.2 a number of oscillation criteria for (4.142). Let us consider, for example, the function $H(t,s)$ defined by

$$H(t,s) = (t-s)^{\lambda}, \quad (t,s) \in D,$$

where $\lambda > \alpha$ is a constant. Clearly, H belongs to the class Y. Furthermore, the function

$$h(t,s) = \lambda(t-s)^{(\lambda-2)/2}, \quad (t,s) \in D$$

is continuous on $[t_0, \infty)$ and satisfies condition (4.143). Then, by Theorem 4.7.2, we obtain the following oscillation criteria.

Corollary 4.7.5. *If $p(t) \equiv 0$ and*

$$\limsup_{t\to\infty} \frac{1}{t^{\lambda}} \int_{t_0}^{t} \left[(t-s)^{\lambda} q(s) - \frac{\lambda^{\alpha+1} r(s)}{(\alpha+1)^{\alpha+1}} (t-s)^{\lambda-\alpha-1} \right] ds = \infty,$$

then every solution of (4.142) *is oscillatory.*

Corollary 4.7.6. *Suppose $p(t) \equiv 0$ and there is a function $b \in C([t_0,\infty),(0,\infty))$ such that for some $\lambda > 1$,*

$$\limsup_{t\to\infty} \frac{1}{[B(t)]^{\lambda}} \int_{t_0}^{t} \left\{ [B(t)-B(s)]^{\lambda} q(s) - \frac{[b(s)\lambda]^{\alpha+1} r(s)[B(t)-B(s)]^{\lambda-\alpha-1}}{(\alpha+1)^{\alpha+1}} \right\} ds = \infty,$$

where $B(t) = \int_{t_0}^{t} b(s)ds$. Then every solution of (4.142) *is oscillatory.*

Proof. Let us put

$$H(t,s) = [B(t)-B(s)]^{\lambda}, \quad (t,s) \in D.$$

Then with the choice

$$h(t,s) = \lambda b(t) \left[B(t)-B(s)\right]^{(\lambda-2)/2}, \quad (t,s) \in D,$$

the conclusion follows directly from Theorem 4.7.2. □

Theorem 4.7.7. *Suppose that there exists $H \in Y$ such that*

$$0 < \inf_{s \ge t_0} \left[\liminf_{t\to\infty} \frac{H(t,s)}{H(t,t_0)} \right] \le \infty \tag{4.148}$$

and

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[\frac{r(s) \left| h(t,s) + \frac{p(s)}{r(s)} \sqrt{H(t,s)} \right|^{\alpha+1}}{(\alpha+1)^{\alpha+1} [H(t,s)]^{(\alpha-1)/2}} \right] ds < \infty. \tag{4.149}$$

If there exists a function $\phi \in C[t_0, \infty)$ such that for every $T \geq t_0$

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[H(t,s)q(s) - \frac{r(s)\left| h(t,s) + \frac{p(s)}{r(s)}\sqrt{H(t,s)} \right|^{\alpha+1}}{(\alpha+1)^{\alpha+1}[H(t,s)]^{(\alpha-1)/2}} \right] ds \geq \phi(T) \tag{4.150}$$

and

$$\int_{t_0}^{\infty} \frac{[\phi_+(s)]^{(\alpha+1)/\alpha}}{[r(s)]^{1/\alpha}} ds = \infty, \tag{4.151}$$

where $\phi_+(t) = \max\{\phi(t), 0\}$, then every solution of (4.142) is oscillatory.

Proof. Suppose that there exists a solution y of (4.142) such that $y(t) \neq 0$ for $t \geq t_0$. Define u as in (4.145). As in the proof of Theorem 4.7.2, we can obtain (4.146). Then, for $t > T \geq t_0$, we have

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[H(t,s)q(s) - \frac{r(s)\left| h(t,s) + \frac{p(s)}{r(s)}\sqrt{H(t,s)} \right|^{\alpha+1}}{(\alpha+1)^{\alpha+1}[H(t,s)]^{(\alpha-1)/2}} \right] ds \leq u(T).$$

Therefore, by (4.150), we have

$$\phi(T) \leq u(T), \quad T \geq t_0 \tag{4.152}$$

and

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^t H(t,s)q(s)ds \geq \phi(t_0). \tag{4.153}$$

Define

$$P(t) = \frac{1}{H(t,t_0)} \int_{t_0}^t \left| h(t,s)\sqrt{H(t,s)} + \frac{p(s)}{r(s)} H(t,s) \right| |u(s)|\, ds$$

and

$$Q(t) = \frac{\alpha}{H(t,t_0)} \int_{t_0}^t H(t,s) \frac{|u(s)|^{(\alpha+1)/\alpha}}{[r(s)]^{1/\alpha}} ds.$$

Then, by (4.146) and (4.153), we see that

$$\begin{aligned} \liminf_{t\to\infty} [Q(t) - P(t)] &\leq u(t_0) - \limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^t H(t,s)q(s)ds \\ &\leq u(t_0) - \phi(t_0) \ < \ \infty. \end{aligned}$$

Now we claim that

$$\int_{t_0}^{\infty} \frac{|u(s)|^{(\alpha+1)/\alpha}}{[r(s)]^{1/\alpha}} ds < \infty. \tag{4.154}$$

Suppose to the contrary that

$$\int_{t_0}^{\infty} \frac{|u(s)|^{(\alpha+1)/\alpha}}{[r(s)]^{1/\alpha}} ds = \infty. \tag{4.155}$$

By (4.148), there exists a constant $k_1 > 0$ such that

$$\inf_{s \geq t_0} \left[\liminf_{t\to\infty} \frac{H(t,s)}{H(t,t_0)} \right] > k_1. \tag{4.156}$$

Let $k_2 > 0$ be arbitrary. Then it follows from (4.155) that there exists $t_1 \geq t_0$ such that

$$\int_{t_0}^{t} \frac{|u(s)|^{(\alpha+1)/\alpha}}{[r(s)]^{1/\alpha}} ds \geq \frac{k_2}{\alpha k_1} \quad \text{for all} \quad t \geq t_1.$$

Therefore,

$$\begin{aligned} Q(t) &= \frac{\alpha}{H(t,t_0)} \int_{t_0}^{t} H(t,s) \frac{d}{ds} \left(\int_{t_0}^{s} \frac{|u(\tau)|^{(\alpha+1)/\alpha}}{[r(\tau)]^{1/\alpha}} d\tau \right) ds \\ &= \frac{\alpha}{H(t,t_0)} \int_{t_0}^{t} \left(-\frac{\partial H}{\partial s}(t,s) \right) \left(\int_{t_0}^{s} \frac{|u(\tau)|^{(\alpha+1)/\alpha}}{[r(\tau)]^{1/\alpha}} d\tau \right) ds \\ &\geq \frac{\alpha}{H(t,t_0)} \int_{t_1}^{t} \left(-\frac{\partial H}{\partial s}(t,s) \right) \left(\int_{t_0}^{s} \frac{|u(\tau)|^{(\alpha+1)/\alpha}}{[r(\tau)]^{1/\alpha}} d\tau \right) ds \\ &\geq \frac{k_2}{k_1 H(t,t_0)} \int_{t_1}^{t} \left(-\frac{\partial H}{\partial s}(t,s) \right) ds \;=\; \frac{k_2}{k_1} \frac{H(t,t_1)}{H(t,t_0)}. \end{aligned}$$

By (4.156), there exists $t_2 \geq t_1$ such that

$$\frac{H(t,t_1)}{H(t,t_0)} \geq k_1 \quad \text{for all} \quad t \geq t_2,$$

which implies that $Q(t) \geq k_2$. Since k_2 is arbitrary,

$$\lim_{t \to \infty} Q(t) = \infty. \tag{4.157}$$

Next, consider a sequence $\{T_n\}_{n=1}^{\infty} \subset (t_0, \infty)$ with $\lim_{n\to\infty} T_n = \infty$ satisfying

$$\lim_{n \to \infty} [Q(T_n) - P(T_n)] = \liminf_{t \to \infty} [Q(t) - P(t)] < \infty.$$

Then there exists a constant M such that

$$Q(T_n) - P(T_n) \leq M \tag{4.158}$$

for all sufficiently large $n \in \mathbb{N}$. Since (4.157) ensures that

$$\lim_{n \to \infty} Q(T_n) = \infty, \tag{4.159}$$

(4.158) implies that

$$\lim_{n \to \infty} P(T_n) = \infty. \tag{4.160}$$

Furthermore, (4.158) and (4.159) lead to the inequality

$$\frac{P(T_n)}{Q(T_n)} - 1 \geq -\frac{M}{Q(T_n)} > -\frac{1}{2}$$

for $n \in \mathbb{N}$ large enough. Thus

$$\frac{P(T_n)}{Q(T_n)} > \frac{1}{2}$$

for $n \in \mathbb{N}$ large enough, which together with (4.160) implies

$$\lim_{n \to \infty} \frac{[P(T_n)]^{\alpha+1}}{[Q(T_n)]^{\alpha}} = \infty. \tag{4.161}$$

On the other hand, by Hölder's inequality, we have for every $n \in \mathbb{N}$

$$\begin{aligned}
P(T_n) &= \frac{1}{H(T_n,t_0)} \int_{t_0}^{T_n} \left| h(T_n,s)\sqrt{H(T_n,s)} + \frac{p(s)}{r(s)} H(T_n,s) \right| |u(s)|\, ds \\
&= \int_{t_0}^{T_n} \left(\frac{\alpha^{\alpha/(\alpha+1)}}{[H(T_n,t_0)]^{\alpha/(\alpha+1)}} \frac{|u(s)|\, [H(T_n,t_0)]^{\alpha/(\alpha+1)}}{[r(s)]^{1/(\alpha+1)}} \right) \\
&\quad \times \left(\frac{\alpha^{-\alpha/(\alpha+1)}}{[H(T_n,t_0)]^{1/(\alpha+1)}} \frac{[r(s)]^{1/(\alpha+1)} \left| h(T_n,s)\sqrt{H(T_n,s)} + \frac{p(s)}{r(s)} H(T_n,s) \right|}{[H(T_n,t_0)]^{\alpha/(\alpha+1)}} \right) ds \\
&\le \left(\frac{\alpha}{H(T_n,t_0)} \int_{t_0}^{T_n} \frac{|u(s)|^{(\alpha+1)/\alpha} H(T_n,t_0)}{[r(s)]^{1/\alpha}} ds \right)^{\frac{\alpha}{\alpha+1}} \\
&\quad \times \left(\frac{1}{\alpha^\alpha H(T_n,t_0)} \int_{t_0}^{T_n} \frac{r(s) \left| h(T_n,s)\sqrt{H(T_n,s)} + \frac{p(s)}{r(s)} H(T_n,s) \right|^{\alpha+1}}{[H(T_n,t_0)]^\alpha} ds \right)^{\frac{1}{\alpha+1}},
\end{aligned}$$

and accordingly,

$$\begin{aligned}
\frac{[P(T_n)]^{\alpha+1}}{[Q(T_n)]^\alpha} &\le \frac{1}{\alpha^\alpha H(T_n,t_0)} \int_{t_0}^{T_n} \frac{r(s) \left| h(T_n,s)\sqrt{H(T_n,s)} + \frac{p(s)}{r(s)} H(T_n,s) \right|^{\alpha+1}}{[H(T_n,t_0)]^\alpha} ds \\
&= \frac{1}{\alpha^\alpha H(T_n,t_0)} \int_{t_0}^{T_n} \frac{r(s) \left| h(T_n,s) + \frac{p(s)}{r(s)} \sqrt{H(T_n,s)} \right|^{\alpha+1}}{[H(T_n,t_0)]^{(\alpha-1)/2}} ds.
\end{aligned}$$

So, because of (4.161), we have

$$\lim_{n\to\infty} \frac{1}{H(T_n,t_0)} \int_{t_0}^{T_n} \frac{r(s) \left| h(T_n,s) + \frac{p(s)}{r(s)} \sqrt{H(T_n,s)} \right|^{\alpha+1}}{[H(T_n,t_0)]^{(\alpha-1)/2}} ds = \infty,$$

which gives

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \frac{r(s) \left| h(t,s) + \frac{p(s)}{r(s)} \sqrt{H(t,s)} \right|^{\alpha+1}}{[H(t,t_0)]^{(\alpha-1)/2}} ds = \infty,$$

contradicting (4.149). Therefore (4.154) holds. Now, from (4.152) we obtain

$$\int_{t_0}^{\infty} \frac{[\phi_+(s)]^{(\alpha+1)/\alpha}}{[r(s)]^{1/\alpha}} ds \le \int_{t_0}^{\infty} \frac{|u(s)|^{(\alpha+1)/\alpha}}{[r(s)]^{1/\alpha}} ds < \infty,$$

which contradicts (4.151). This completes the proof. □

The following result is a direct consequence of Theorem 4.7.7 and uses the same choice of the functions H and h as in Corollary 4.7.5 above.

Corollary 4.7.8. *Suppose that there exists a function* $\phi \in C[t_0,\infty)$ *such that* (4.151) *along with*

$$\limsup_{t\to\infty} \frac{1}{t^\lambda} \int_{t_0}^{t} r(s)(t-s)^{\lambda-\alpha-1} \left| \lambda + \frac{p(s)}{r(s)}(t-s) \right|^{\alpha+1} ds < \infty$$

holds and

$$\limsup_{t\to\infty} \frac{1}{t^\lambda} \int_T^t \left[(t-s)^\lambda q(s) - \frac{r(s)(t-s)^{\lambda-\alpha-1} \left| \lambda + \frac{p(s)}{r(s)}(t-s) \right|^{\alpha+1}}{(\alpha+1)^{\alpha+1}} \right] ds \geq \phi(T)$$

for all $T \geq t_0$ *and for some* $\lambda > \alpha$. *Then every solution of* (4.142) *is oscillatory.*

Proof. The only thing to be checked is condition (4.148). With the above choice of the functions H and h, this is fulfilled automatically since

$$\lim_{t\to\infty} \frac{H(t,s)}{H(t,t_0)} = \lim_{t\to\infty} \frac{(t-s)^\lambda}{(t-t_0)^\lambda} = 1$$

for any $s \geq t_0$. □

Theorem 4.7.9. *Suppose that there exists a function* $H \in Y$ *such that* (4.148) *holds and*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^t H(t,s)q(s)ds < \infty. \tag{4.162}$$

If there exists $\phi \in C[t_0,\infty)$ *such that for every* $T \geq t_0$

$$\liminf_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[H(t,s)q(s) - \frac{r(s) \left| h(t,s) + \frac{p(s)}{r(s)}\sqrt{H(t,s)} \right|^{\alpha+1}}{(\alpha+1)^{\alpha+1}[H(t,s)]^{(\alpha-1)/2}} \right] ds \geq \phi(T) \tag{4.163}$$

and (4.151) *hold, then every solution of* (4.142) *is oscillatory.*

Proof. For a nonoscillatory solution y of (4.142), as in the proof of Theorem 4.7.2, (4.146) and (4.147) are satisfied. As in the proof of Theorem 4.7.7, (4.152) holds for $t \geq T \geq t_0$. Using (4.162), we conclude that

$$\limsup_{t\to\infty} [Q(t) - P(t)] \leq u(t_0) - \liminf_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^t H(t,s)q(s)ds < \infty.$$

It follows from (4.163) that

$$\begin{aligned} \phi(t_0) \leq \liminf_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^t H(t,s)q(s)ds \\ - \liminf_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^t \frac{r(s) \left| h(t,s) + \frac{p(s)}{r(s)}\sqrt{H(t,s)} \right|^{\alpha+1}}{(\alpha+1)^{\alpha+1}[H(t,s)]^{(\alpha-1)/2}} ds. \end{aligned}$$

Hence (4.162) implies

$$\liminf_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^t \frac{r(s) \left| h(t,s) + \frac{p(s)}{r(s)}\sqrt{H(t,s)} \right|^{\alpha+1}}{(\alpha+1)^{\alpha+1}[H(t,s)]^{(\alpha-1)/2}} ds < \infty.$$

We consider a sequence $\{T_n\}_{n=1}^\infty \subset (t_0,\infty)$ with $\lim_{n\to\infty} T_n = \infty$ such that

$$\lim_{n\to\infty} [Q(T_n) - P(T_n)] = \limsup_{t\to\infty} [Q(t) - P(t)].$$

Then, using the procedure of the proof of Theorem 4.7.7, we conclude that (4.154) holds. The remainder of the proof proceeds as in the proof of Theorem 4.7.7 and hence is omitted here. □

Example 4.7.10. Consider the nonlinear differential equation

$$\left(t^{-\beta}\left|y'(t)\right|^{\alpha-1}y'(t)\right)' - t^{-\beta}\left|y'(t)\right|^{\alpha-1}y'(t) + t^{\gamma}\left(\gamma\frac{2-\cos t}{t}+\sin t\right)\left|y(t)\right|^{\alpha-1}y(t) = 0 \tag{4.164}$$

for $t \geq 1$, where α, β, γ are arbitrary positive constants and $\alpha \neq 2$. Then, for any $t \geq 1$, we have

$$\int_1^t q(s)ds = \int_1^t \frac{d}{ds}\left(s^{\gamma}(2-\cos s)\right)ds = t^{\gamma}(2-\cos t) - (2-\cos 1) \geq t^{\gamma} - k_0,$$

where $k_0 = 2 - \cos 1$. Taking $H(t,s) = (t-s)^2$ for $t \geq s \geq 1$, we have

$$\begin{aligned}
&\frac{1}{t^2}\int_1^t \left[(t-s)^2 q(s) - \frac{1}{(\alpha+1)^{\alpha+1}}\frac{|2-(t-s)|^{\alpha+1}}{s^{\beta}(t-s)^{\alpha-1}}\right]ds \\
&\quad= \frac{1}{t^2}\int_1^t \left[2(t-s)\left(\int_1^s q(\tau)d\tau\right) - \frac{1}{(\alpha+1)^{\alpha+1}}\frac{|2-(t-s)|^{\alpha+1}}{s^{\beta}(t-s)^{\alpha-1}}\right]ds \\
&\quad\geq \frac{2}{t^2}\int_1^t (t-s)(s^{\gamma}-k_0)ds - \frac{2^{\alpha+1}}{(\alpha+1)^{\alpha+1}t^2}\int_1^t (t-s)^{1-\alpha}ds \\
&\quad= \frac{2t^{\gamma}}{(\gamma+1)(\gamma+2)} + \frac{k_1}{t^2} + \frac{k_2}{t} - k_0 - \frac{1}{t^{\alpha}}\left(1-\frac{1}{t}\right)^{2-\alpha},
\end{aligned}$$

where

$$k_1 = \frac{2}{\gamma+2} - k_0, \quad k_2 = 2k_0 - \frac{2}{\gamma+1}, \quad k_3 = \frac{2^{\alpha+1}}{(\alpha+1)^{\alpha+1}(2-\alpha)}.$$

Consequently, (4.144) holds. Hence, (4.164) is oscillatory by Theorem 4.7.2.

Example 4.7.11. Consider the differential equation

$$\left(t^{\beta}\left|y'(t)\right|^{\alpha-1}y'(t)\right)' + t^{\beta}\left|y'(t)\right|^{\alpha-1}y'(t) + t^{\gamma}\cos t\left|y(t)\right|^{\alpha-1}y(t) = 0, \tag{4.165}$$

where $t \geq 1$ and α, β, γ are constants such that $-1 < \gamma \leq 1$, $0 < \alpha \neq 2$, $\alpha > \beta$, and $\gamma(\alpha+1) \geq \beta - \alpha$. For example, $\alpha = 3$, $\beta = 1$, and $\gamma = 1$ satisfy the above assumptions. Taking $H(t,s) = (t-s)^2$ for $t \ge s \ge 1$, we find

$$\begin{aligned}
\frac{1}{t^2}\int_1^t s^\beta \frac{|2-(t-s)|^{\alpha+1}}{(t-s)^{\alpha-1}}ds &\le \frac{2^{\alpha+1}}{t^2}\int_1^t \frac{s^\beta}{(t-s)^{\alpha-1}}ds \\
&= \begin{cases} 2^{\alpha+1}t^{\beta-2}\dfrac{(t-1)^{2-\alpha}}{2-\alpha} & \text{if} \quad \beta > 0 \\ \dfrac{2^{\alpha+1}}{t^2}\dfrac{(t-1)^{2-\alpha}}{2-\alpha} & \text{if} \quad \beta < 0 \end{cases} \\
&= \begin{cases} \dfrac{2^{\alpha+1}t^{\beta-\alpha}}{2-\alpha}\left(1-\dfrac{1}{t}\right)^{2-\alpha} & \text{if} \quad \beta > 0 \\ \dfrac{2^{\alpha+1}}{(2-\alpha)t^\alpha}\left(1-\dfrac{1}{t}\right)^{2-\alpha} & \text{if} \quad \beta < 0. \end{cases}
\end{aligned}$$

Therefore, (4.149) holds, and for an arbitrary small constant $\varepsilon > 0$ there exists $t_1 \ge 1$ such that for $T \ge t_1$,

$$\limsup_{t\to\infty} \frac{1}{t^2}\int_1^t \left[(t-s)^2 s^\gamma \cos s - \frac{s^\beta\,|2-(t-s)|^{\alpha+1}}{(\alpha+1)^{\alpha+1}(t-s)^{\alpha-1}}\right] ds \ge -T^\gamma \cos T - \varepsilon.$$

Set $\phi(T) = -T^\gamma \cos T - \varepsilon$. Then there is $N \in \mathbb{N}$ such that $(2N+1)\pi - \pi/4 > t_1$ and if $n \in \mathbb{N}$,

$$(2n+1)\pi - \frac{\pi}{4} \le T \le (2n+1)\pi + \frac{\pi}{4}, \quad \phi(T) \ge \delta T^\gamma,$$

where δ is a small constant. Taking into account that $\gamma(\alpha+1) \ge \beta - \alpha$, we obtain

$$\begin{aligned}
\int_1^\infty \frac{[\phi_+(s)]^{(\alpha+1)/\alpha}}{[r(s)]^{1/\alpha}}ds &\ge \sum_{n=N}^\infty \delta^{(\alpha+1)/\alpha}\int_{(2n+1)\pi-\pi/4}^{(2n+1)\pi+\pi/4} s^{[\gamma(\alpha+1)-\beta]/\alpha}ds \\
&\ge \sum_{n=N}^\infty \delta^{(\alpha+1)/\alpha}\int_{(2n+1)\pi-\pi/4}^{(2n+1)\pi+\pi/4} \frac{ds}{s} = \infty.
\end{aligned}$$

Accordingly, all conditions of Theorem 4.7.7 are satisfied, and hence (4.165) is oscillatory.

4.8. Notes

Theorem 4.2.2 is obtained by Li and Yan [**205**]. Theorem 4.2.4 is taken from Li and Agarwal [**191**]. Theorem 4.3.1 is adopted from Li and Quan [**202**] and Theorem 4.3.3 is obtained by Li and Agarwal [**191**]. Theorem 4.4.7 is based on Li [**179**]. Theorem 4.4.9 is taken from Li [**262**], a special case is obtained by Wong and Agarwal [**278**], while Theorems 4.4.12 and 4.4.17 are adopted from Li, Zhang and Fei [**209**]. Lemma 4.5.1 is given by Wong and Agarwal [**278**]. The rest of Section 4.5 is based on Li [**200**]. The material of Section 4.6 is adopted from Li [**181**] and Li and Fan [**195**]. The results in Section 4.7 are given by Li, Zhong, and Fan [**213**].

CHAPTER 5

Second Order Delay Differential Equations

5.1. Introduction

In this chapter we investigate the existence of nonoscillatory solutions of second order delay differential equations.

In Section 5.2 we present results on the existence of nonoscillatory solutions of delay differential equations. In Section 5.3 we give a classification scheme for eventually positive solutions of a class of second order nonlinear iterative differential equations, and provide necessary and/or sufficient conditions for the existence of solutions. In Sections 5.4 and 5.5 we introduce the classification of nonoscillatory solutions for second order nonlinear neutral differential equations under the conditions $\int^{\infty} ds/r(s) < \infty$ and $\int^{\infty} ds/r(s) = \infty$. Various existence results of nonoscillatory solutions of different type are given, respectively.

5.2. Nonoscillation of Half-Linear Equations

In this section we are interested in the existence and asymptotic behavior of nonoscillatory solutions of second order half-linear functional differential equations of the form

$$\left(|x'(t)|^{\alpha-1} x'(t)\right)' = \sum_{i=1}^{n} p_i(t) \left|x\left(g_i(t)\right)\right|^{\alpha-1} x\left(g_i(t)\right), \tag{5.1}$$

where $\alpha > 0$ is a constant, $p_i : [0,\infty) \to [0,\infty)$ are continuous functions such that $\sup\{p_i(t) : t \geq T_x\} > 0$ for any $T_x \geq a$, and $g_i : [0,\infty) \to \mathbb{R}$ are continuously differentiable functions with $g_i(t) < t$, $g_i'(t) \geq 0$ for $t \geq a$, and $\lim_{t\to\infty} g_i(t) = \infty$, $1 \leq i \leq n$.

By a solution of (5.1) we mean a function $x \in C^1[T_x,\infty)$, $T_x \geq a$, which has the property $|x'|^{\alpha-1} x \in C^1[T_x,\infty)$ and satisfies (5.1) for all sufficiently large $t \geq T_x$. Our attention will be restricted to those solutions x of (5.1) which satisfy $\sup\{|x(t)| : t \geq T\} > 0$ for all $T \geq T_x$. It is assumed that (5.1) does possess such a solution.

If x is a nonoscillatory solution of (5.1), then there exists $t_0 > a$ such that either

$$x(t)x'(t) > 0, \quad t \geq t_0 \tag{5.2}$$

or

$$x(t)x'(t) < 0, \quad t \geq t_0. \tag{5.3}$$

If (5.2) holds, then x is unbounded and the limit $x'(\infty) = \lim_{t\to\infty} x'(t)$, either finite or infinite, exists. If (5.3) holds, then x is bounded and the finite limit $x(\infty) = \lim_{t\to\infty} x(t)$ exists.

In what follows we need only to consider eventually positive solutions of (5.1), since if x satisfies (5.1), then so does $-x$. Let x be an eventually positive solution of (5.1) satisfying (5.2) and having a finite limit $x'(\infty) = \lim_{t\to\infty} x'(t) > 0$. Integrating (5.1) twice yields

$$x(t) = x(t_1) + \int_{t_1}^{t} \left([x'(\infty)]^{\alpha} - \int_{s}^{\infty} \sum_{i=1}^{n} p_i(r) \left[x\left(g_i(r)\right)\right]^{\alpha} dr \right)^{\frac{1}{\alpha}} ds, \; t \geq t_1, \tag{5.4}$$

where $t_1 > t_0$ is chosen so that $\inf_{t\geq t_1} g_i(t) \geq t_0$, $1 \leq i \leq n$. Let x be an eventually positive solution of (5.1) satisfying (5.3). Then we have

$$x(t) = x(\infty) + \int_{t}^{\infty} \left(\int_{s}^{\infty} \sum_{i=1}^{n} p_i(r) \left[x\left(g_i(r)\right)\right]^{\alpha} dr \right)^{\frac{1}{\alpha}} ds, \quad t \geq t_1, \tag{5.5}$$

after integrating (5.1) twice from t to ∞.

Based on these integral representations (5.4) and (5.5), we can prove the following existence theorems.

Theorem 5.2.1. *Equation* (5.1) *has a nonoscillatory solution x such that*

$$\lim_{t\to\infty} \frac{x(t)}{t} = \textit{constant} \neq 0$$

if and only if

$$\int^{\infty} p_i(t)[g_i(t)]^{\alpha} dt < \infty \quad \textit{for all} \quad i \in \{1, \ldots, n\}. \tag{5.6}$$

Proof. (The "only if" part) Let x be a nonoscillatory solution of (5.1) satisfying $\lim_{t\to\infty} \frac{x(t)}{t} = c > 0$. Then from (5.4) we see that

$$\int^{\infty} \sum_{i=1}^{n} p_i(t) \left[x(g_i(t))\right]^{\alpha} dt < \infty.$$

This, combined with the relation $\lim_{t\to\infty} \frac{x(g_i(t))}{g_i(t)} = c$, $1 \leq i \leq n$, immediately implies that (5.6) holds. Let $k > 0$ be arbitrary and fixed and take $T > a$ so large that

$$T_* = \min_{1\leq i\leq n} \left\{ \inf_{t\geq T} g_i(t) \right\} \geq a \tag{5.7}$$

and

$$\sum_{i=1}^{n} \int_{T}^{\infty} p_i(t)[g_i(t)]^{\alpha} dt \leq \frac{2^{\alpha} - 1}{2^{\alpha}}.$$

Consider the set $X \subset C[T_*, \infty)$ and the mapping $F : X \to C[T_*, \infty)$ defined by

$$X = \left\{ x \in C[T_*, \infty) : \frac{k}{2}(t - T) \leq x(t) \leq k(t - T), \; t \geq T, \; x(t) = 0, \; T_* \leq t \leq T \right\}$$

and

$$(Fx)(t) = \begin{cases} \displaystyle\int_{T}^{t} \left(k^{\alpha} - \int_{s}^{\infty} \sum_{i=1}^{n} p_i(r) \left[x(g_i(r))\right]^{\alpha} dr \right)^{\frac{1}{\alpha}} ds & \text{if} \quad t \geq T, \\ 0 & \text{if} \quad T_* \leq t < T. \end{cases}$$

It is clear that X is a closed convex subset of the Fréchet space $C[T,\infty)$ of continuous functions on $[T_*,\infty)$ with the usual metric topology and that F is well defined and continuous on X. It can be shown without difficulty that F maps X into itself and $F(X)$ is relatively compact in $C[T_*,\infty)$. Therefore, by the Schauder–Tychonov fixed point theorem (Theorem 1.4.25), F has a fixed element $x \in X$, which satisfies

$$x(t) = \int_T^t \left(k^\alpha - \int_s^\infty \sum_{i=1}^n p_i(r)\,[x(g_i(r))]^\alpha\,dr \right)^{1/\alpha} ds, \quad t \ge T.$$

By differentiating this equation, we see that x is a solution of (5.1) on $[T,\infty)$ and $\lim_{t\to\infty} \frac{x(t)}{t} = \lim_{t\to\infty} x'(t) = k$. □

Theorem 5.2.2. *Equation* (5.1) *has a nonoscillatory solution* x *such that*

$$\lim_{t\to\infty} x(t) = \textit{constant} \ne 0$$

if and only if

$$\int^\infty \left(\int_t^\infty p_i(s)ds \right)^{1/\alpha} dt < \infty \quad \textit{for all} \quad i \in \{1,\dots,n\}. \tag{5.8}$$

Proof. Note that our assumptions imply (5.3), and then the "only if" part follows readily from (5.5). To prove the "if" part, suppose that (5.8) is satisfied. Choose $T > a$ so that (5.7) holds and

$$\int_T^\infty \left(\int_t^\infty \sum_{i=1}^n p_i(s)ds \right)^{1/\alpha} dt \le \frac{1}{2}.$$

Define $Y \subset C[T_*,\infty)$ and $G : Y \to C[T_*,\infty)$ by

$$Y = \left\{ y \in C[T_*,\infty) : \ k \le y(t) \le 2k,\ t \ge T_* \right\},$$

$k > 0$ being a fixed constant, and

$$(Gy)(t) = \begin{cases} k + \displaystyle\int_t^\infty \left(\int_s^\infty \sum_{i=1}^n p_i(r)\,[y(g_i(r))]^\alpha\,dr \right)^{\frac{1}{\alpha}} ds & \text{if} \quad t \ge T, \\ (Gy)(T) & \text{if} \quad T_* \le t < T. \end{cases}$$

As in the proof of Theorem 5.2.1 one can verify that G maps Y into a relatively compact subset of Y, so that there exists $y \in Y$ such that

$$y(t) = k + \int_t^\infty \left(\int_s^\infty \sum_{i=1}^n p_i(r)\,[y(g_i(r))]^\alpha\,dr \right)^{1/\alpha} ds, \quad t \ge T.$$

Differentiating this equation twice, one sees that y satisfies (5.1) on $[T,\infty)$. Since $y(t) \to k$ as $t \to \infty$, y is a solution of (5.1) with the desired asymptotic property. This completes the proof. □

It remains to discuss the existence of an unbounded nonoscillatory solution x of (5.1) with the property $\lim_{t\to\infty} \frac{|x(t)|}{t} = \infty$ and of a bounded solution x of (5.1) with the property $\lim_{t\to\infty} x(t) = 0$. Below we confine our attention to the case where at least one g_i is retarded and show that some sufficient conditions can be derived under which (5.1) has a nonoscillatory solution that tends to zero. Our derivation is based on the following theorem which is essentially due to Philos [**239**].

Theorem 5.2.3. *Suppose that there exists* $i_0 \in \{1,2,\dots,n\}$ *such that*

$$g_{i_0}(t) < t \quad \text{and} \quad p_{i_0}(t) \ge 0 \quad \text{for} \quad t \ge a. \tag{5.9}$$

Suppose, in addition, that there exists a positive decreasing function ϕ *on* $[t_0,\infty)$ *satisfying*

$$\phi(t) \ge \int_t^\infty \left(\int_s^\infty \sum_{i=1}^n p_i(r) \left[\phi(g_i(r))\right]^\alpha dr \right)^{1/\alpha} ds, \quad t \ge t_0, \tag{5.10}$$

where t_0 *is chosen so that* $\inf_{t \ge t_0} g_i(t) \ge a$ *for all* $1 \le i \le n$*. Then* (5.1) *has a nonoscillatory solution tending to zero as* $t \to \infty$*.*

Proof. Let Z denote the set

$$Z = \left\{ z \in C[t_0,\infty) : \ 0 \le z(t) \le \phi(t), \ t \ge t_0 \right\}.$$

With each $z \in Z$ we associate the function $\tilde{z} \in C[a,\infty)$ defined by

$$\tilde{z}(t) = \begin{cases} z(t) & \text{if} \quad t \ge t_0, \\ z(t_0) + [\phi(t) - \phi(t_0)] & \text{if} \quad a \le t < t_0. \end{cases} \tag{5.11}$$

Define the mapping $H : Z \to C[t_0,\infty)$ by

$$(Hz)(t) = \int_t^\infty \left(\int_s^\infty \sum_{i=1}^n p_i(r) \left[\tilde{z}(g_i(r))\right]^\alpha dr \right)^{1/\alpha} ds, \quad t \ge t_0.$$

Then it can be shown that H is a continuous mapping which sends Z into a relatively compact subset of Z. It follows therefore that there exists $z \in Z$ such that $z = Hz$, i.e.,

$$z(t) = \int_t^\infty \left(\int_s^\infty \sum_{i=1}^n p_i(r) \left[\tilde{z}(g_i(r))\right]^\alpha dr \right)^{1/\alpha} ds, \quad t \ge t_0.$$

Differentiating the above twice shows that

$$\left(- \left[-z'(t)\right]^\alpha \right)' = \sum_{i=1}^n p_i(t) \left[\tilde{z}(g_i(t))\right]^\alpha, \quad t \ge t_0,$$

which, in view of (5.11), implies that $z(t)$ is a solution of (5.1) for all sufficiently large t. That $z(t) > 0$ for $t \ge t_0$ can be seen exactly as in Philos [**239**, page 170], and so the details are omitted. This completes the proof. $\square$

In order to apply Theorem 5.2.3 to construct decaying nonoscillatory solutions of (5.1), we distinguish the following three cases:

$$\int^\infty \sum_{i=1}^n p_i(t) dt < \infty \quad \text{and} \quad \int^\infty \left(\int_t^\infty \sum_{i=1}^n p_i(s) ds \right)^{1/\alpha} dt < \infty, \tag{5.12}$$

$$\int^\infty \sum_{i=1}^n p_i(t) dt < \infty \quad \text{but} \quad \int^\infty \left(\int_t^\infty \sum_{i=1}^n p_i(s) ds \right)^{1/\alpha} dt = \infty, \tag{5.13}$$

and

$$\int^\infty \sum_{i=1}^n p_i(t) dt = \infty. \tag{5.14}$$

The condition (5.12), which is nothing else but (5.8), always guarantees the existence of a decaying nonoscillatory solution of (5.1).

Theorem 5.2.4. *Suppose that* (5.9) *holds for some* $i_0 \in \{1, 2, \ldots, n\}$. *If* (5.8) *is satisfied, then* (5.1) *possesses a nonoscillatory solution tending to zero as* $t \to \infty$.

Proof. Let t_0 be large enough so that $\min_{1\le i\le n}\{\inf_{t\ge t_0} g_i(t)\} \ge \max\{a, 1\}$ and

$$\int_{t_0}^{\infty}\left(\int_s^{\infty}\sum_{i=1}^{n} p_i(r)dr\right)^{1/\alpha} ds \le \frac{1}{2}. \tag{5.15}$$

Choose $\phi(t) = 1 + \frac{1}{t}$. Using (5.15), we see that ϕ satisfies (5.10):

$$\begin{aligned}
&\int_t^{\infty}\left(\int_s^{\infty}\sum_{i=1}^{n} p_i(r)\left[\phi(g_i(r))\right]^{\alpha} dr\right)^{1/\alpha} ds \\
&= \int_t^{\infty}\left(\int_s^{\infty}\sum_{i=1}^{n} p_i(r)\left(1 + \frac{1}{g_i(r)}\right)^{\alpha} dr\right)^{1/\alpha} ds \\
&\le 2\int_t^{\infty}\left(\int_s^{\infty}\sum_{i=1}^{n} p_i(r)dr\right)^{1/\alpha} ds \\
&\le 1 \le \phi(t)
\end{aligned}$$

for $t \ge t_0$. The conclusion follows from Theorem 5.2.3. □

We now state existence theorems of decaying nonoscillatory solutions which are applicable to the cases (5.13) and (5.14).

Theorem 5.2.5. *Suppose that* (5.9) *holds for some* $i_0 \in \{1, 2, \ldots, n\}$ *and that*

$$\limsup_{t\to\infty}\int_{g_*(t)}^{t}\left(\int_s^{\infty}\sum_{i=1}^{n} p_i(r)dr\right)^{1/\alpha} ds < \frac{1}{e}, \tag{5.16}$$

where $g_*(t) = \min_{1\le i\le n} g_i(t)$. *Then* (5.1) *possesses a nonoscillatory solution tending to zero as* $t \to \infty$.

Proof. We put

$$P(t) = \left(\int_t^{\infty}\sum_{i=1}^{n} p_i(s)ds\right)^{1/\alpha}$$

and choose $t_0 > 0$ so that $\inf_{t\ge t_0} g_*(t) \ge a$ and

$$P_{t_0} := \sup_{t\ge t_0}\int_{g_*(t)}^{t} P(s)ds \le \frac{1}{e}. \tag{5.17}$$

Define

$$\phi(t) = \exp\left(-\frac{1}{P_{t_0}}\int_a^t P(s)ds\right).$$

Since, for $1 \le i \le n$,

$$\begin{aligned}\phi\left(g_i(t)\right) &= \exp\left(\frac{1}{P_{t_0}}\int_{g_i(t)}^t P(s)ds\right)\exp\left(-\frac{1}{P_{t_0}}\int_a^t P(s)ds\right)\\ &\le e\exp\left(-\frac{1}{P_{t_0}}\int_a^t P(s)ds\right) = e\phi(t)\end{aligned}$$

for $t \ge t_0$, we see, in view of (5.17), that ϕ satisfies (5.10):

$$\begin{aligned}&\int_t^\infty\left(\int_s^\infty\sum_{i=1}^n p_i(r)\left[\phi(g_i(r))\right]^\alpha dr\right)^{1/\alpha} ds \le e\int_t^\infty P(s)\phi(s)ds\\ &= e\int_t^\infty P(s)\exp\left(-\frac{1}{P_{t_0}}\int_a^s P(r)dr\right)ds\\ &\le eP_{t_0}\exp\left(-\frac{1}{P_{t_0}}\int_a^t P(s)ds\right)\\ &= eP_{t_0}\phi(t) \le \phi(t)\end{aligned}$$

for $t \ge t_0$. By Theorem 5.2.3, (5.1) has a decaying nonoscillatory solution. □

Theorem 5.2.6. *Suppose that* (5.9) *holds for some* $i_0 \in \{1, 2, \ldots, n\}$. *Further, suppose that there exists* $t_0 > a$ *such that* $\inf_{t\ge t_0} g_*(t) \ge a$ *and*

$$(5.18)\quad P'_{t_0} := \inf P(t) > 0 \quad \textit{and} \quad \sup_{t\ge t_0}\int_{g_*(t)}^t\sum_{i=1}^n p_i(s)ds \le \frac{\alpha+1}{e}\left(\frac{P'_{t_0}}{\alpha}\right)^{\alpha/(\alpha+1)}.$$

Then (5.1) *possesses a nonoscillatory solution tending to zero as* $t\to\infty$.

Proof. Put

$$Q_{t_0} = \sup_{t\ge t_0}\int_{g_*(t)}^t\sum_{i=1}^n p_i(s)ds \quad \text{and} \quad \phi(t) = \exp\left(-\frac{\alpha+1}{\alpha Q_{t_0}}\int_a^t\sum_{i=1}^n p_i(s)ds\right).$$

We see that

$$\phi\left(g_i(t)\right) \le \exp\left(\frac{\alpha+1}{\alpha}\right)\phi(t), \quad t \ge t_0, \quad 1 \le i \le n,$$

and hence that

$$\begin{aligned}&\int_t^\infty\sum_{i=1}^n p_i(s)\left[\phi(g_i(s))\right]^\alpha ds \le e^{\alpha+1}\int_t^\infty\left(\sum_{i=1}^n p_i(s)\right)\left[\phi(s)\right]^\alpha ds\\ &= e^{\alpha+1}\int_t^\infty\sum_{i=1}^n p_i(s)\exp\left(-\frac{\alpha+1}{Q_{t_0}}\int_a^s\sum_{i=1}^n p_i(r)dr\right)ds\\ &\le \frac{Q_{t_0}}{\alpha+1}e^{\alpha+1}\exp\left(-\frac{\alpha+1}{Q_{t_0}}\int_a^t\sum_{i=1}^n p_i(s)ds\right)\end{aligned}$$

for $t \geq t_0$. Consequently, we obtain

$$\begin{aligned}
&\int_t^\infty \left(\int_s^\infty \sum_{i=1}^n p_i(r) \left[\phi(g_i(r))\right]^\alpha dr \right)^{1/\alpha} ds \\
&\leq \left(\frac{Q_{t_0}}{\alpha+1} \right)^{1/\alpha} e^{(\alpha+1)/\alpha} \int_t^\infty \exp\left(-\frac{\alpha+1}{\alpha Q_{t_0}} \int_a^s \sum_{i=1}^n p_i(r) dr \right) ds \\
&\leq \frac{1}{P'_{t_0}} \left(\frac{Q_{t_0}}{\alpha+1} \right)^{1/\alpha} e^{(\alpha+1)/\alpha} \int_t^\infty \sum_{i=1}^n p_i(s) \exp\left(-\frac{\alpha+1}{\alpha Q_{t_0}} \int_a^s \sum_{i=1}^n p_i(r) dr \right) ds \\
&\leq \frac{\alpha Q_{t_0}}{(\alpha+1) P'_{t_0}} \left(\frac{Q_{t_0}}{\alpha+1} \right)^{1/\alpha} e^{(\alpha+1)/\alpha} \exp\left(-\frac{\alpha+1}{\alpha Q_{t_0}} \int_a^t \sum_{i=1}^n p_i(s) ds \right) \\
&\leq \phi(t)
\end{aligned}$$

for $t \geq t_0$, where (5.18) has been used. This establishes the existence of a strictly decreasing positive function satisfying (5.10), and so the proof is complete via Theorem 5.2.3. ☐

Example 5.2.7. Consider the equation

$$\left(|x'(t)| \, x'(t) \right)' = t^{-\lambda} \left| x\left(\frac{t}{\theta}\right) \right| x\left(\frac{t}{\theta}\right), \tag{5.19}$$

where $\lambda > 1$ and $\theta > 1$ are constants. This is a special case of (5.1) in which $\alpha = 2$, $n = 1$, $p_1(t) = t^{-\lambda}$, and $g_1(t) = \frac{t}{\theta}$.

(i) Let $\lambda > 3$. Then both (5.6) and (5.7) hold for (5.19), and so by Theorem 5.2.1 and Theorem 5.2.2, (5.19) has nonoscillatory solutions x_1 and x_2 such that $\lim_{t\to\infty} \frac{x_1(t)}{t}$ = constant $\neq 0$ and $\lim_{t\to\infty} x_2(t)$ = constant $\neq 0$ regardless of the values of $\theta > 0$.

(ii) Let $\lambda = 3$. An easy computation shows that (5.16) is satisfied for (5.19) if $1 < \theta < \exp\left(\frac{\sqrt{2}}{e}\right)$, since

$$\int_{g_1(t)}^t \left(\int_s^\infty p_1(r) dr \right)^{1/\alpha} ds = \int_{t/\theta}^t \left(\int_s^\infty \frac{dr}{r^3} \right)^{1/2} ds = 2^{1/2} \ln \theta.$$

From Theorem 5.2.5 it follows that, for such a θ, (5.19) possesses a nonoscillatory solution tending to zero as $t \to \infty$.

(iii) Let $1 < \lambda < 3$. Then (5.18) is satisfied for (5.19) since $P_{t_0} = 1$ and

$$\int_{g_1(t)}^t p_1(s) ds = \int_{t/\theta}^t \frac{ds}{s^\lambda} = \frac{\theta^{\lambda-1} - 1}{\lambda - 1} t^{1-\lambda} \to 0 \quad \text{as} \quad t \to \infty.$$

Therefore there exists a decaying nonoscillatory solution of (5.19) by Theorem 5.2.6.

5.3. Classification Schemes for Iterative Equations

In this section we are concerned with the general class of second order nonlinear differential equations

$$\left(r(t) \left[x'(t)\right]^\sigma \right)' + f\left(t, x(t), x\left(\Delta\left(t, x(t)\right)\right) \right) = 0 \tag{5.20}$$

with the conditions $\int_0^\infty ds/[r(s)]^{1/\sigma} = \infty$ and $\int_0^\infty ds/[r(s)]^{1/\sigma} < \infty$, respectively. We give a classification scheme for eventually positive solutions of this equation in terms of their asymptotic magnitude, and provide necessary and/or sufficient conditions for the existence of solutions.

Let $T \in \mathbb{R}^+ = [0, \infty)$. Define $T_{-1} = \inf\{\Delta(t, x) : t \geq T, x \in \mathbb{R}\}$.

Definition 5.3.1. The function x is called a solution of the differential equation (5.20) in the interval $[T, \infty)$, if $x(t)$ is defined for $t \geq T_{-1}$, is twice differentiable, and satisfies (5.20) for $t \geq T$.

Definition 5.3.2. The solution x of (5.20) is called *regular*, if it is defined on some interval $[T_x, \infty)$ and $\sup\{|x(t)| : t \geq T\} > 0$ for $t \geq T_x$.

Throughout this section, we assume that the following conditions hold:

(H1) $r \in C(\mathbb{R}^+, \mathbb{R}^+)$ and $r(t) > 0$, $t \in \mathbb{R}^+$.
(H2) $f \in C(\mathbb{R}^+ \times \mathbb{R}^2, \mathbb{R}^+)$.
(H3) There exists $T \in \mathbb{R}^+$ such that $uf(t, u, v) > 0$ for $t \geq T$, $uv > 0$, and $f(t, u, v)$ is nondecreasing in u and v for each fixed $t \geq T$.
(H4) $\Delta \in C(\mathbb{R}^+ \times \mathbb{R}, \mathbb{R})$.
(H5) There exist a function $\Delta_* \in C(\mathbb{R}^+, \mathbb{R})$ and $T \in \mathbb{R}^+$ such that $\lim_{t\to\infty} \Delta_*(t) = \infty$ and $\Delta_*(t) \leq \Delta(t, x)$ for $t \geq T$, $x \in \mathbb{R}$.
(H6) There exist a function $\Delta^* \in C(\mathbb{R}^+, \mathbb{R})$ and $T \in \mathbb{R}^+$ such that $\Delta^*(t)$ is nondecreasing for $t \geq T$ and $\Delta(t, x) \leq \Delta^*(t) \leq t$ for $t \geq T$, $x \in \mathbb{R}$.
(H7) σ is a quotient of odd integers.

For the sake of convenience, we will employ the following notation:

$$R(t) = \int_t^\infty \frac{ds}{[r(s)]^{1/\sigma}}, \quad R(T, t) = \int_T^t \frac{ds}{[r(s)]^{1/\sigma}}, \quad R_0 = \int_0^\infty \frac{ds}{[r(s)]^{1/\sigma}}.$$

Lemma 5.3.3. *Suppose x is an eventually positive solution of* (5.20)*. Then $x'(t)$ is of constant sign eventually.*

Proof. Assume that there exists $t_0 \geq 0$ such that $x(t) > 0$ for $t \geq t_0$. It follows from (H6) that there exists $t_1 \geq t_0$ such that $x(\Delta(t, x(t))) > 0$ for $t \geq t_1$. From (H4) and (5.20) we conclude that $(r(x')^\sigma)'(t) < 0$ for $t \geq t_1$. If $x'(t)$ is not eventually positive, then there exists $t_2 \geq t_1$ such that $x'(t_2) \leq 0$. Therefore, $r(t_2)[x'(t_2)]^\sigma \leq 0$. Integrating (5.20) from t_2 to t provides

$$r(t)\,[x'(t)]^\sigma - r(t_2)\,[x'(t_2)]^\sigma + \int_{t_2}^t f\left(s, x(s), x(\Delta(s, x(s)))\right) ds = 0.$$

Thus

$$r(t)\,[x'(t)]^\sigma \leq -\int_{t_2}^t f\left(s, x(s), x(\Delta(s, x(s)))\right) ds < 0$$

for $t \geq t_2$. This shows that $x'(t) < 0$ for $t \geq t_2$. The proof is complete. □

As a consequence, an eventually positive solution x of (5.20) either satisfies $x(t) > 0$ and $x'(t) > 0$ for all large t, or $x(t) > 0$ and $x'(t) < 0$ for all large t.

Lemma 5.3.4. *Suppose that*

$$R_0 = \int_0^\infty \frac{ds}{[r(s)]^{1/\sigma}} < \infty$$

and that x is an eventually positive solution of (5.20)*. Then $\lim_{t\to\infty} x(t)$ exists.*

Proof. If not, then we have $\lim_{t\to\infty} x(t) = \infty$ by Lemma 5.3.3. On the other hand, we have noted that $r(x')^\sigma$ is monotone decreasing eventually. Therefore, there exists $t_1 \geq 0$ such that

$$r(t)\,[x'(t)]^\sigma \leq r(t_1)\,[x'(t_1)]^\sigma \quad \text{for all} \quad t \geq t_1.$$

Then

$$x'(t) \leq [r(t_1)]^{1/\sigma}\, x'(t_1) \frac{1}{[r(t)]^{1/\sigma}} \tag{5.21}$$

for $t \geq t_1$, and after integrating,

$$x(t) - x(t_1) \leq [r(t_1)]^{1/\sigma}\, x'(t_1) R(t_1, t)$$

for $t \geq t_1$. But this is contrary to the fact that $\lim_{t\to\infty} x(t) = \infty$ and the assumption that $R_0 < \infty$. The proof is complete. □

Lemma 5.3.5. *Suppose that $R_0 < \infty$. Let x be an eventually positive solution of* (5.20)*. Then there exist $a_1 > 0$, $a_2 > 0$, and $T \geq 0$ such that $a_1 R(t) \leq x(t) \leq a_2$ for $t \geq T$.*

Proof. By Lemma 5.3.4, there exists $t_0 \geq 0$ such that $x(t) \leq a_2$ for some number $a_2 > 0$. We know that $x'(t)$ is of constant sign eventually by Lemma 5.3.3. If $x'(t) > 0$ eventually, then $R(t) \leq x(t)$ eventually because $\lim_{t\to\infty} R(t) = 0$. If $x'(t) < 0$ eventually, then since $r(t)[x'(t)]^\sigma$ is also eventually decreasing, we may assume that $x'(t) < 0$ and $r(t)[x'(t)]^\sigma$ is monotone decreasing for $t \geq T$. By (5.21), we have

$$x(s) - x(t) \leq [r(T)]^{1/\sigma} x'(T) R(t, s), \quad s \geq t \geq T.$$

Taking the limit as $s \to \infty$ on both sides of the above inequality, we find

$$x(t) \geq -[r(T)]^{1/\sigma} x'(T) R(t)$$

for $t \geq T$. The proof is complete. □

Our next result is concerned with necessary conditions for the function f to hold in order that an eventually positive solution of (5.20) exists.

Lemma 5.3.6. *Suppose that $R_0 < \infty$ and that x is an eventually positive solution of* (5.20)*. Then*

$$\int_0^\infty \left(\frac{1}{r(t)} \int_0^t f\left(s, x(s), x(\Delta(s, x(s)))\right) ds \right)^{1/\sigma} dt < \infty.$$

Proof. In view of Lemma 5.3.3, we may assume without loss of generality that $x(t) > 0$, and $x'(t) > 0$ or $x'(t) < 0$ for $t \geq 0$. From (5.20), we have

$$r(t)\,[x'(t)]^\sigma - r(0)\,[x'(0)]^\sigma + \int_0^t f\left(s, x(s), x\left(\Delta(s, x(s))\right)\right) ds = 0.$$

Thus, if $x'(t) > 0$ for $t \geq 0$, then we have

$$\int_0^u \frac{1}{[r(t)]^{1/\sigma}} \left(\int_0^t f\left(s, x(s), x\left(\Delta(s, x(s))\right)\right) ds \right)^{1/\sigma} dt$$
$$\leq [r(0)]^{1/\sigma} x'(0) \int_0^u \frac{1}{[r(t)]^{1/\sigma}} dt$$

for $u \geq 0$, and

$$\int_0^u \frac{1}{[r(t)]^{1/\sigma}} \left(\int_0^t f\left(s, x(s), x(\Delta(s, x(s)))\right) ds \right)^{1/\sigma} dt \leq [r(0)]^{1/\sigma} x'(0) R_0 < \infty.$$

If $x'(t) < 0$ for $t \geq 0$, then we have

$$\int_0^u \frac{1}{[r(t)]^{1/\sigma}} \left(\int_0^t f\left(s, x(s), x(\Delta(s, x(s)))\right) ds \right)^{1/\sigma} dt \leq -\int_0^\infty x'(s) ds \leq x(0) < \infty.$$

The proof is complete. □

We now consider the case when $R_0 = \infty$.

Lemma 5.3.7. *Suppose that*

$$R_0 = \int_0^\infty \frac{ds}{[r(s)]^{1/\sigma}} = \infty. \tag{5.22}$$

Let x be an eventually positive solution of (5.20). *Then $x'(t)$ is eventually positive and there exist $c_1 > 0$, $c_2 > 0$, and $T \geq 0$ such that $c_1 \leq x(t) \leq c_2 R(t, T)$ for $t \geq T$.*

Proof. In view of Lemma 5.3.3, $x'(t)$ is of constant sign eventually. If $x(t) > 0$ and $x'(t) < 0$ for $t \geq T$, then we have

$$r(t) [x'(t)]^\sigma \leq r(T) [x'(T)]^\sigma < 0.$$

Thus

$$x'(t) \leq [r(T)]^{1/\sigma} x'(T) \frac{1}{[r(t)]^{1/\sigma}}, \quad t \geq T,$$

which after integrating yields

$$x(t) - x(T) \leq [r(T)]^{1/\sigma} x'(T) \int_T^t \frac{ds}{[r(s)]^{1/\sigma}}.$$

The left-hand side tends to $-\infty$ in view of (5.22), which is a contradiction. Thus $x'(t)$ is eventually positive, and thus $x(t) \geq c_1$ eventually for some constant $c_1 > 0$. Furthermore, the same reasoning just used also leads to

$$x(t) \leq x(T_0) + [r(T_0)]^{1/\sigma} x'(T_0) \int_{T_0}^t \frac{ds}{[r(s)]^{1/\sigma}}$$

for $t \geq T_0$, where T_0 is a number such that $x(t) > 0$ and $x'(t) > 0$ for $t \geq T_0$. Since $R_0 = \infty$, there exists $c_2 > 0$ such that $x(t) \leq c_2 R(T, t)$ for all large t. The proof is complete. □

We have shown that when x is an eventually positive solution of (5.20), then $r(x')^\sigma$ is eventually decreasing and $x'(t)$ is eventually of constant sign. We have also shown that under the assumption that $R_0 < \infty$, $x(t)$ must converge to some (nonnegative) constant. As a consequence, under the condition $R_0 < \infty$, we may now classify an eventually positive solution x of (5.20) according to the limits of the functions x and $r(x')^\sigma$. For this purpose, we first denote the set of eventually positive solutions of (5.20) by P. We then single out eventually positive solutions of (5.20) which converge to zero or to positive constants, and denote the corresponding subsets by P_0 and P_α respectively. But for any $x \in P_\alpha$, $r(t)[x'(t)]^\sigma$ either tends to a finite limit or to $-\infty$, so we can further partition P_α into P_α^β and $P_\alpha^{-\infty}$.

Theorem 5.3.8. *Suppose $R_0 < \infty$. Then any eventually positive solution of* (5.20) *must belong to one of the following classes:*

$$P_0 = \left\{ x \in P : \lim_{t\to\infty} x(t) = 0 \right\},$$

$$P_\alpha^\beta = \left\{ x \in P : \lim_{t\to\infty} x(t) = \alpha > 0, \quad \lim_{t\to\infty} r(t)\left[x'(t)\right]^\sigma = \beta \right\},$$

or

$$P_\alpha^{-\infty} = \left\{ x \in P : \lim_{t\to\infty} x(t) = \alpha > 0, \quad \lim_{t\to\infty} r(t)\left[x'(t)\right]^\sigma = -\infty \right\}.$$

To justify the above classification scheme, we now derive several existence theorems.

Theorem 5.3.9. *Suppose $R_0 < \infty$. Then a necessary and sufficient condition for* (5.20) *to have an eventually positive solution $x \in P_\alpha$ is that for some $C > 0$,*

$$\int_0^\infty \left(\frac{1}{r(t)} \int_0^t f(s, C, C)\, ds \right)^{1/\sigma} dt < \infty. \tag{5.23}$$

Proof. Let x be any eventually positive solution of (5.20) with $\lim_{t\to\infty} x(t) = c > 0$. Thus, in view of (H6), there exist $C_1 > 0$, $C_2 > 0$, and $T \geq 0$ such that

$$C_1 \leq x(t) \leq C_2 \quad \text{and} \quad C_1 \leq x(\Delta(t, x(t))) \leq C_2 \quad \text{for} \quad t \geq T.$$

On the other hand, using Lemma 5.3.6 we have

$$\int_T^\infty \left(\frac{1}{r(t)} \int_0^t f\left(s, x(s), x(\Delta(s, x(s)))\right) ds \right)^{1/\sigma} dt < \infty.$$

Since $f(t, u, v)$ is nondecreasing in u and v for each fixed t, we have

$$\int_T^\infty \left(\frac{1}{r(t)} \int_0^t f(s, C_1, C_1)\, ds \right)^{1/\sigma} dt < \infty.$$

Conversely, let $a = C/2$. In view of (5.23), we may choose $T \geq 0$ so large that

$$\int_T^\infty \left(\frac{1}{r(t)} \int_0^t f(s, C, C)\, ds \right)^{1/\sigma} dt < a.$$

Define the set

$$\Omega = \left\{ x \in C([T_{-1}, \infty), \mathbb{R}) : \ a \leq x(t) \leq 2a, \ t \geq T_{-1} \right\}.$$

Then Ω is a bounded, convex, and closed subset of $C([T_{-1}, \infty), \mathbb{R})$. Let us further define an operator $F : \Omega \to C([T_{-1}, \infty), \mathbb{R})$ by

$$Fx(t) = \begin{cases} a + \displaystyle\int_t^\infty \left(\frac{1}{r(s)} \int_0^s f\left(u, x(u), x(\Delta(u, x(u)))\right) du \right)^{\frac{1}{\sigma}} ds \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if} \quad t \geq T, \\ Fx(T) \qquad\qquad\qquad\qquad\quad\ \ \text{if} \quad T_{-1} \leq t < T. \end{cases}$$

The mapping F has the following properties. F maps Ω into Ω. Indeed, if $x \in \Omega$, then

$$\begin{aligned} a \leq Fx(t) &= a + \int_t^\infty \left(\frac{1}{r(s)} \int_0^s f\left(u, x(u), x(\Delta(u, x(u)))\right) du \right)^{1/\sigma} ds \\ &\leq a + \int_t^\infty \left(\frac{1}{r(s)} \int_0^s f(u, C, C)\, du \right)^{1/\sigma} ds \leq 2a. \end{aligned}$$

Next, we show that F is continuous. To see this, let $\varepsilon > 0$. Choose $M \geq T$ so large that

$$\int_t^\infty \left(\frac{1}{r(s)} \int_0^s f(u, C, C)\, du \right)^{1/\sigma} ds < \frac{\varepsilon}{2} \quad \text{for} \quad t \geq M. \tag{5.24}$$

Let $\{x^{(n)}\} \subset \Omega$ be a sequence such that $x^{(n)} \to x$ as $n \to \infty$. Since Ω is closed, we have $x \in \Omega$. Furthermore, for any $s \geq t \geq M$,

$$\begin{aligned} &\left| Fx^{(n)}(t) - Fx(t) \right| \\ &\leq \int_t^\infty \left(\frac{1}{r(s)} \int_0^s f(u, C, C)\, du \right)^{1/\sigma} ds + \int_t^\infty \left(\frac{1}{r(s)} \int_0^s f(u, C, C)\, du \right)^{1/\sigma} ds \\ &= 2\int_t^\infty \left(\frac{1}{r(s)} \int_0^s f(u, C, C)\, du \right)^{1/\sigma} ds < \varepsilon. \end{aligned}$$

For $T \leq t \leq s \leq M$,

$$\begin{aligned} &\left| Fx^{(n)}(t) - Fx(t) \right| \\ &\leq \int_M^\infty \left(\frac{1}{r(s)} \int_0^s f(u, C, C) du \right)^{1/\sigma} ds + \int_M^\infty \left(\frac{1}{r(s)} \int_0^s f(u, C, C) du \right)^{1/\sigma} ds \\ &\quad + \int_t^M \left(\frac{1}{r(s)} \int_0^s f(u, C, C) du \right)^{1/\sigma} ds - \int_s^M \left(\frac{1}{r(s)} \int_0^s f(u, C, C) du \right)^{1/\sigma} ds \\ &\leq \varepsilon + \int_t^s \left(\frac{1}{r(s)} \int_0^s f(u, C, C) du \right)^{1/\sigma} ds \\ &\leq \varepsilon + \max_{T \leq u \leq M} \frac{1}{r(u)} \int_0^u f(v, C, C) dv \, |s - t| \\ &\leq \varepsilon + C_0 |s - t| < 2\varepsilon \end{aligned}$$

if

$$|s - t| < \frac{\varepsilon}{C_0}, \quad \text{where} \quad C_0 = \max_{T \leq u \leq M} \frac{1}{r(u)} \int_0^u f(v, C, C) dv.$$

Also, for $T_{-1} \leq t \leq s < T$,

$$\left| Fx^{(n)}(t) - Fx(t) \right| = 0.$$

These statements show that $\left\| Fx^{(n)} - Fx \right\|$ tends to zero, i.e., F is continuous. When $s, t \geq M$, by (5.24) we have

$$\begin{aligned} |Fx(s) - Fx(t)| &\leq \int_t^\infty \left(\frac{1}{r(s)} \int_0^s f(u, C, C) du \right)^{1/\sigma} ds \\ &\quad + \int_t^\infty \left(\frac{1}{r(s)} \int_0^s f(u, C, C) du \right)^{1/\sigma} ds < \varepsilon, \end{aligned}$$

which holds for any $x \in \Omega$. Therefore, $F\Omega$ is precompact. In view of Schauder's fixed point theorem, we see that there exists $x^* \in \Omega$ such that $Fx^* = x^*$. It is easy to check that x^* is an eventually positive solution of (5.20). The proof is complete. $\square$

Theorem 5.3.10. *Suppose $R_0 < \infty$. A necessary and sufficient condition for* (5.20) *to have an eventually positive solution $x \in P_\alpha^\beta$ is that* (5.23) *holds for some $C > 0$ and that for some $D > 0$,*

$$\int_0^\infty f(t, D, D)dt < \infty. \tag{5.25}$$

Proof. If $x \in P_\alpha^\beta$ is an eventually positive solution, then, in view of Theorem 5.3.9, we see that (5.23) holds. Furthermore, as in the proof of Theorem 5.3.9, $0 < C_1 \le x(t) \le C_2$ and $C_1 \le x(\Delta(t, x(t))) \le C_2$ for $t \ge T$. In view of (5.20), we see that

$$\begin{aligned} \int_T^\infty f(s, C_1, C_1)ds &\le \int_T^\infty f\Big(s, x(s), x\left(\Delta(s, x(s))\right)\Big)\, ds \\ &= r(T)\left[x'(T)\right]^\sigma - \lim_{t\to\infty} r(t)\left[x'(t)\right]^\sigma < \infty. \end{aligned}$$

Conversely, in view of (5.25), we can choose $T \ge 0$ such that

$$\int_T^\infty f(t, D, D)dt < \left(\frac{D}{2R_0}\right)^\sigma.$$

We define the subset Ω of $C([T_{-1}, \infty), \mathbb{R})$ by

$$\Omega = \left\{x \in C([T_{-1}, \infty), \mathbb{R}) : \frac{D}{2} \le x(t) \le D,\ t \ge T_{-1}\right\}.$$

Then Ω is a bounded, convex, and closed subset of $C([T_{-1}, \infty), \mathbb{R})$. In view of R_0 and (5.25), we can further define an operator $F : \Omega \to C([T_{-1}, \infty), \mathbb{R})$ as

$$Fx(t) = \begin{cases} D - \displaystyle\int_t^\infty \left(\frac{1}{r(s)} \int_s^\infty f\Big(u, x(u), x\left(\Delta(u, x(u))\right)\Big)\, du\right)^{\frac{1}{\sigma}} ds & \\ & \text{if} \quad t \ge T, \\ Fx(T) & \text{if} \quad T_{-1} \le t < T. \end{cases}$$

Then arguments similar to those in the proof of Theorem 5.3.9 show that F has a fixed point u which satisfies

$$r(t)[u'(t)]^\sigma = \int_t^\infty f\Big(s, u(s), u\left(\Delta(s, u(s))\right)\Big)\, ds, \quad t \ge T.$$

Hence $\lim_{t\to\infty} r(t)[u'(t)]^\sigma = 0$ as required. Choose $T \ge 0$ such that

$$\int_T^\infty f(t, D, D)dt < \left(\frac{D}{4R_0}\right)^\sigma \quad \text{and} \quad R(t) < \left(\frac{D}{4R_0}\right)^\sigma$$

for $t \ge T$, and let

$$Fx(t) = \begin{cases} D - \displaystyle\int_t^\infty \left(\frac{1}{r(s)} + \frac{1}{r(s)} \int_0^s f\Big(u, x(u), x\left(\Delta(u, x(u))\right)\Big)\, du\right)^{\frac{1}{\sigma}} ds & \\ & \text{if} \quad t \ge T, \\ Fx(T) & \text{if} \quad T_{-1} \le t < T. \end{cases}$$

Then under the same conditions (5.23) and (5.25), we can show that F has a fixed point u which satisfies $\lim_{t\to\infty} u(t) = D > 0$ and

$$r(t)[u'(t)]^\sigma = 1 + \int_t^\infty f\Big(s, u(s), u\left(\Delta(s, u(s))\right)\Big)\, ds, \quad t \geq T.$$

Therefore, $\lim_{t\to\infty} r(t)[u'(t)]^\sigma = 1 > 0$, and the proof is complete. □

In view of Theorem 5.3.10, the following result is obvious.

Theorem 5.3.11. *Suppose $R_0 < \infty$. A necessary and sufficient condition for (5.20) to have an eventually positive solution $x \in P_\alpha^{-\infty}$ is that (5.23) holds for some $C > 0$ and that for any $D > 0$,*

$$\int_0^\infty f(t, D, D)dt = \infty.$$

Our next result concerns the existence of eventually positive solutions in P_0.

Theorem 5.3.12. *Suppose $R_0 < \infty$ and $\sigma = 1$. If for some $C > 0$,*

$$\int_0^\infty f\Big(t, CR(t), CR(\Delta_*(t))\Big)\, dt < \infty, \tag{5.26}$$

then (5.20) has an eventually positive solution in P_0. Conversely, if the equation (5.20) has an eventually positive solution x such that $\lim_{t\to\infty} x(t) = 0$ and $\lim_{t\to\infty} r(t)[x'(t)]^\sigma = d \neq 0$, then for some $C > 0$, (5.26) holds.

Proof. Suppose (5.26) holds. Then there exists $T \geq 0$ such that

$$\int_t^\infty f\Big(s, CR(s), CR(\Delta_*(s))\Big)\, ds < \frac{C}{2} \quad \text{for} \quad t \geq T.$$

Consider the equation

$$x(t) = \begin{cases} R(t)\left(\dfrac{C}{2} + \displaystyle\int_T^t f\Big(s, x(s), x\left(\Delta(s, x(s))\right)\Big)\, ds\right) & \\ \qquad + \displaystyle\int_t^\infty R(s) f\Big(s, x(s), x\left(\Delta(s, x(s))\right)\Big)\, ds & \text{if} \quad t \geq T, \\ Fx(T) & \text{if} \quad T_{-1} \leq t < T. \end{cases}$$

It is easy to check that a solution of the above equation must be a solution of (5.20). We shall show that the above equation has a positive solution $x \in P_0$ by means of the method of successive approximations. Consider the sequence $\{x_k\}$ of successive approximating sequences defined as

$$x_1(t) = 0 \quad \text{and} \quad x_{n+1}(t) = Fx_n(t), \quad n \in \mathbb{N} \quad \text{for} \quad t \geq T_{-1},$$

where F is defined by

$$Fx(t) = \begin{cases} R(t)\left(\dfrac{C}{2} + \displaystyle\int_T^t f\Big(s, x(s), x\left(\Delta(s, x(s))\right)\Big)\, ds\right) & \\ \qquad + \displaystyle\int_t^\infty R(s) f\Big(s, x(s), x\left(\Delta(s, x(s))\right)\Big)\, ds & \text{if} \quad t \geq T, \\ Fx(T) & \text{if} \quad T_{-1} \leq t < T. \end{cases}$$

In view of (H3), it is easy to see that $0 \leq x_n(t) \leq x_{n+1}(t)$ for $t \geq T$ and $n \in \mathbb{N}$. On the other hand,

$$x_2(t) = Fx_1(t) = \frac{C}{2}R(t) \leq CR(t), \quad t \geq T,$$

and inductively,

$$\begin{aligned} Fx_k(t) &\leq \frac{C}{2}R(t) + R(t)\int_T^t f\left(s, CR(s), CR(\Delta^*(s))\right) ds \\ &\quad + R(t)\int_t^\infty f\left(s, CR(s), CR(\Delta^*(s))\right) ds \\ &\leq \frac{C}{2}R(t) + R(t)\int_T^\infty f\left(s, CR(s), CR(\Delta^*(s))\right) ds \\ &\leq CR(t) \end{aligned}$$

for $k \geq 2$. Therefore, by means of Lebesgue's dominated convergence theorem, we see that $Tx^* = x^*$. Furthermore, it is clear that $x(t)$ converges to zero as $t \to \infty$.

Let x be an eventually positive solution of (5.20) such that $x(t) \to 0$ and $r(t)[x'(t)]^\sigma \to d < 0$ (the proof of the case $d > 0$ is similar). Then there exist $C_1 > 0$, $C_2 > 0$, and $T \geq 0$ such that $-C_1 < r(t)[x'(t)]^\sigma < -C_2$ for $t \geq T$. Hence,

$$-C_1^{1/\sigma}\frac{1}{[r(t)]^{1/\sigma}} < x'(t) < -C_2^{1/\sigma}\frac{1}{[r(t)]^{1/\sigma}},$$

and, after integrating,

$$-C_1^{1/\sigma}R(t,s) < x(s) - x(t) < -C_2^{1/\sigma}R(t,s)$$

for $s > t \geq T$. Letting $s \to \infty$ provides

$$-C_1^{1/\sigma}R(t) < -x(t) < -C_2^{1/\sigma}R(t), \quad \text{i.e.,} \quad C_2^{1/\sigma}R(t) < x(t) < C_1^{1/\sigma}R(t).$$

On the other hand, by (5.20),

$$r(t)\left[x'(t)\right]^\sigma = r(T)\left[x'(T)\right]^\sigma - \int_T^t f\left(s, x(s), x\left(\Delta(s, x(s))\right)\right) ds, \quad t \geq T.$$

Since $\lim_{t\to\infty} r(t)[x'(t)]^\sigma = d < 0$, we have

$$\int_T^\infty f\left(s, x(s), x\left(\Delta(s, x(s))\right)\right) ds = r(T)\left[x'(T)\right]^\sigma - d < \infty.$$

Thus,

$$\int_T^\infty f\left(s, C_1^{1/\sigma}R(s), C_1^{1/\sigma}R(\Delta_*(s))\right) ds \leq \int_T^\infty f\left(s, x(s), x\left(\Delta(s, x(s))\right)\right) ds < \infty.$$

The proof is complete. □

We now consider the existence of eventually positive solutions of (5.20) in the case $R_0 = \infty$.

Recall that if $x \in P$, then $r(x')^\sigma$ is eventually decreasing. Furthermore, in view of Lemma 5.3.7, we see that $x'(t)$, and hence $r(t)[x'(t)]^\sigma$, are eventually positive. Hence $x(t)$ either tends to a positive constant or to positive infinity, and $r(t)[x'(t)]^\sigma$

tends to a nonnegative constant. Note that if $x(t)$ tends to a positive constant, then $r(t)[x'(t)]^\sigma$ must tend to zero. Otherwise $r(t)[x'(t)]^\sigma \geq d > 0$ for $t \geq T$ so that

$$x'(t) \geq d^{1/\sigma} \frac{1}{[r(t)]^{1/\sigma}}$$

and

$$x(t) \geq x(T) d^{1/\sigma} \int_T^t \frac{1}{[r(s)]^{1/\sigma}} ds \to \infty \quad \text{as} \quad t \to \infty,$$

which is a contradiction.

Theorem 5.3.13. *Suppose that* $R_0 = \infty$. *Then any eventually positive solution* x *of* (5.20) *must belong to one of the following classes:*

$$P_\alpha^0 = \left\{ x \in P : \lim_{t\to\infty} x(t) \in (0,\infty), \quad \lim_{t\to\infty} r(t)\,[x'(t)]^\sigma = 0 \right\},$$
$$P_\infty^0 = \left\{ x \in P : \lim_{t\to\infty} x(t) = \infty, \quad \lim_{t\to\infty} r(t)\,[x'(t)]^\sigma = 0 \right\},$$

or

$$P_\infty^\beta = \left\{ x \in P : \lim_{t\to\infty} x(t) = \infty, \quad \lim_{t\to\infty} r(t)\,[x'(t)]^\sigma = \beta \neq 0 \right\}.$$

In order to justify our classification scheme, we present the following three results.

Theorem 5.3.14. *Suppose that* $R_0 = \infty$. *A necessary and sufficient condition for* (5.20) *to have an eventually positive solution* $x \in P_\alpha^0$ *is that for some* $C > 0$,

$$\int_0^\infty \left(\frac{1}{r(t)} \int_t^\infty f(s, C, C) ds \right)^{1/\sigma} dt < \infty. \tag{5.27}$$

Proof. Suppose $x \in P_\alpha^0$ is an eventually positive solution of (5.20). This implies $\lim_{t\to\infty} x(t) = \alpha > 0$ and $\lim_{t\to\infty} r(t)[x'(t)]^\sigma = 0$. Then there exist $C_1 > 0$, $C_2 > 0$, and $T \geq 0$ such that $C_1 \leq x(t) \leq C_2$ and $C_1 \leq x(\Delta(t, x(t))) \leq C_2$ for $t \geq T$. On the other hand, in view of (5.20) we have

$$r(t)\,[x'(t)]^\sigma = \int_t^\infty f\Big(s, x(s), x\left(\Delta(s, x(s))\right)\Big) ds$$

for $t \geq T$. After integrating, we see that

$$\begin{aligned}
&\int_T^\infty \left(\frac{1}{r(t)} \int_t^\infty f\left(s, C_1, C_1\right) ds \right)^{1/\sigma} dt \\
&\leq \int_0^\infty \left(\frac{1}{r(t)} \int_t^\infty f\Big(s, x(s), x\left(\Delta(s, x(s))\right)\Big) ds \right)^{1/\sigma} dt \\
&\leq \alpha - x(T).
\end{aligned}$$

The proof of the converse is similar to that of Theorem 5.3.9 and hence is only sketched. In view of (5.27), we may choose $T \geq 0$ so large that

$$\int_T^\infty \left(\frac{1}{r(t)} \int_t^\infty f(s, C, C) ds \right)^{1/\sigma} dt < \frac{C}{2}.$$

Define a bounded, convex, and closed subset Ω of $C([T_{-1},\infty),\mathbb{R})$ and an operator $F:\Omega\to\Omega$ by

$$\Omega = \left\{ x \in C([T_{-1},\infty),\mathbb{R}) : \ \frac{C}{2} \le x(t) \le C, \ t \ge T_{-1} \right\}$$

and

$$Fx(t) = \begin{cases} \dfrac{C}{2} + \displaystyle\int_t^\infty \left(\frac{1}{r(s)} \int_s^\infty f\Big(u, x(u), x\left(\Delta(u, x(u))\right)\Big)\, du \right)^{\frac{1}{\sigma}} ds & \\ & \text{if} \quad t \ge T, \\ Fx(T) & \text{if} \quad T_{-1} \le t < T, \end{cases}$$

respectively. As in the proof of Theorem 5.3.10, we prove that F maps Ω into Ω, that F is continuous, and that $F\Omega$ is precompact. The fixed point $x^*(t)$ of F converges to $C/2$ and satisfies (5.20). $\square$

Theorem 5.3.15. *Suppose $R_0 = \infty$. If for a constant $C > 0$,*

$$\int_0^\infty f\Big(t, CR(t,0), CR(\Delta^*(t),0)\Big)\, dt < \infty,$$

then (5.20) has a solution in P_∞^β. Conversely, if (5.20) has a solution $x \in P_\infty^\beta$, then for some positive constant C,

$$\int_0^\infty f\Big(t, CR(t,0), CR(\Delta_*(t),0)\Big)\, dt < \infty.$$

In view of Theorems 5.3.14 and 5.3.15, the following result is clear.

Theorem 5.3.16. *Suppose $R_0 = \infty$. If for any $C > 0$ and for some $D > 0$,*

$$\int_0^\infty \left(\frac{1}{r(t)} \int_t^\infty f\,(s, C, C)\, ds \right)^{1/\sigma} dt = \infty$$

and

$$\int_0^\infty f\Big(t, DR(t,0), DR(\Delta^*(t),0)\Big)\, dt < \infty,$$

then (5.20) has a solution in P_∞^0.

5.4. Nonoscillation of Nonlinear Equations with $\int^\infty ds/r(s) < \infty$

In this section we consider the classification of all nonoscillatory solutions of second order neutral nonlinear differential equations of the form

$$\Big(r(t)\,[x(t) - p(t)x(t-\tau)]'\Big)' + f\Big(t, x(t-\delta)\Big) = 0 \tag{5.28}$$

in the case

$$\int_{t_0}^\infty \frac{du}{r(u)} < \infty,$$

and give necessary and/or sufficient conditions for their existence, where we assume $\tau > 0$, $\delta \ge 0$, $p \in C([t_0,\infty),\mathbb{R}^+)$ such that $0 \le p(t) \le \rho < 1$ for $t \ge t_0$, and $r \in C([t_0,\infty),(0,\infty))$. We further assume that for any $t \ge t_0$, $f(t,\cdot)$ is continuous on $\mathbb{R}$ and $xf(t,x) > 0$ for $x \ne 0$, $t \ge t_0$.

Hereafter, the term solution of (5.28) is always used to denote a real function x satisfying (5.28) for which $\sup_{t\ge t_1} |x(t)| > 0$ for any $t_1 \ge t_0$. We assume that (5.28) always has such solutions. A solution of (5.28) is called *nonoscillatory* if it is eventually positive or eventually negative. We use the functions $R(s,t)$ and $R(s)$ defined by $R(s,t) = \int_s^t \frac{1}{r(u)} du$ and $R(s) = \int_s^\infty \frac{1}{r(u)} du$, where $s \geq t_0$. We also let $R_0 = \lim_{t\to\infty} R(t_0, t)$.

We shall say that condition (H) is met if the following conditions hold:

(H1) $xf(t,x) > 0$ for $x \neq 0$, $t \geq t_0$ and $f(t,x_1) \geq f(t,x_2)$ for $x_1 \geq x_2 > 0$ or $x_1 \leq x_2 < 0$, $t \geq t_0$;
(H2) $0 \leq p(t) < \rho < 1$ for $t \geq t_0$.

Let x be a solution of (5.28). We define an associated function y by

$$y(t) = x(t) - p(t)x(t-\tau), \quad t \geq t_0. \tag{5.29}$$

Note that if $x(t)$ is eventually positive, then the fact that

$$(ry')'(t) = -f\left(t, x(t-\tau)\right) < 0 \quad \text{for all large} \quad t$$

implies that $y'(t)$ is of constant positive or constant negative sign eventually. This fact, in turn, implies that $y(t)$ is eventually positive or eventually negative.

Lemma 5.4.1. *Suppose that* (H) *holds. Let x be an eventually positive or an eventually negative solution of* (5.28), *and let y be defined by* (5.29). *Then $y'(t)$ is eventually of one sign.*

Lemma 5.4.2. *Suppose that* (H) *holds. Let x be an eventually positive* (*negative*) *solution of* (5.28) *and let y be defined by* (5.29). *If* $\limsup_{t\to\infty} x(t) > 0$ ($\limsup_{t\to\infty}(-x(t)) > 0$), *then $y(t)$ is eventually positive* (*respectively negative*).

Proof. Let x be an eventually positive solution of (5.28) and $\limsup_{t\to\infty} x(t) > 0$. Then $y(t) > 0$. If not, then we have $y(t) < 0$ for all large t. If $x(t)$ is unbounded, then there exists a sequence $\{t_k\}$ which tends to infinity and

$$x(t_k) = \max_{t \leq t_k} x(t)$$

such that $\lim_{k\to\infty} x(t_k) = \infty$. Then, from (5.29), we have

$$y(t_k) = x(t_k) - p(t_k)x(t_k - \tau) \geq x(t_k)(1-\rho).$$

From this inequality we obtain $\lim_{k\to\infty} y(t_k) = \infty$. This is a contradiction. If $x(t)$ is bounded, then there exists a sequence $\{t_k\}$ which tends to infinity such that

$$\lim_{k\to\infty} x(t_k) = \limsup_{t\to\infty} x(t) = L > 0.$$

Since $\lim_{k\to\infty} x(t_k - \tau) \leq L$, we have

$$0 \geq \lim_{k\to\infty} y(t_k) \geq L(1-\rho) > 0.$$

This is also a contradiction and the proof is complete. $\square$

The following lemma is independent of (5.28) and is due to Lu [**214**].

Lemma 5.4.3. *Suppose that* (H2) *holds, $x(t) > 0$, and y is defined by* (5.29).

(i) *If* $\lim_{t\to\infty} p(t) = p_0 \in [0,1)$ *and* $\lim_{t\to\infty} y(t) = a \in \mathbb{R}$, *then*

$$\lim_{t\to\infty} x(t) = \frac{a}{1-p_0}.$$

(ii) *If* $\lim_{t\to\infty} y(t) = \infty$, *then* $\lim_{t\to\infty} x(t) = \infty$.

If $x(t) < 0$, then the statement remains true if ∞ in (ii) *is replaced by $-\infty$.*

Proof. To show (i), assume that $x(t) > 0$ and $\lim_{t\to\infty} y(t) = a \in \mathbb{R}$. Then y is bounded. Similar as in the proof of Lemma 5.4.2, it follows that x is bounded. Hence there exists a sequence $\{t_k\}$ which tends to infinity such that $\lim_{k\to\infty} x(t_k) = \limsup_{t\to\infty} x(t)$. Since $\{p(t_k)\}$ and $\{x(t_k - \tau)\}$ are bounded, without loss of generality, we assume that $\lim_{k\to\infty} p(t_k)$ and $\lim_{k\to\infty} x(t_k - \tau)$ exist. Then

$$\begin{aligned} a &= \lim_{k\to\infty} y(t_k) = \lim_{k\to\infty} x(t_k) - \lim_{k\to\infty} p(t_k)x(t_k - \tau) \\ &\geq \limsup_{t\to\infty} x(t)\left[1 - \lim_{k\to\infty} p(t_k)\right] = \limsup_{t\to\infty} x(t)\left[1 - p_0\right], \end{aligned}$$

and so

$$\frac{a}{1-p_0} \geq \limsup_{t\to\infty} x(t).$$

On the other hand, there exists a sequence $\{t'_k\}$ which tends to infinity such that $\lim_{k\to\infty} x(t'_k) = \liminf_{t\to\infty} x(t)$. Since $\{p(t'_k)\}$ and $\{x(t'_k - \tau)\}$ are bounded, without loss of generality, we assume that $\lim_{k\to\infty} p(t'_k)$ and $\lim_{k\to\infty} x(t'_k - \tau)$ exist. Then

$$\begin{aligned} a &= \lim_{k\to\infty} y(t'_k) = \lim_{k\to\infty} x(t'_k) - \lim_{k\to\infty} p(t'_k)x(t'_k - \tau) \\ &\leq \liminf_{t\to\infty} x(t)\left[1 - \lim_{k\to\infty} p(t'_k)\right] = \liminf_{t\to\infty} x(t)\left[1 - p_0\right], \end{aligned}$$

and so

$$\frac{a}{1-p_0} \leq \liminf_{t\to\infty} x(t).$$

Therefore, we have $\lim_{t\to\infty} x(t) = a/(1-p_0)$. For the case $x(t) < 0$, the proof is similar and is omitted here.

Finally, we show (ii). Assume that $x(t) > 0$. If $\lim_{t\to\infty} y(t) = \infty$, then, in view of $x(t) \geq y(t)$, we have $\lim_{t\to\infty} x(t) = \infty$. For the case $x(t) < 0$, the proof is similar and we omit it here. □

We have already remarked that if x is an eventually positive solution of (5.28), then y and y' are also of one sign eventually. These sign regularities provide additional asymptotic information as will be seen in the following two lemmas.

Lemma 5.4.4. *Suppose that* (H) *holds. If x is a nonoscillatory solution of* (5.28), *then y defined by* (5.29) *is eventually increasing or decreasing and* $\lim_{t\to\infty} y(t) = L$ *exists, where L is a finite constant.*

Proof. Suppose x is an eventually positive solution of (5.28). If $\lim_{t\to\infty} x(t) = 0$, then $\lim_{t\to\infty} y(t) = 0$. If $\limsup_{t\to\infty} x(t) > 0$, by Lemma 5.4.2, we have $y(t) > 0$ for all large t. Thus, there exists $t_1 \geq t_0$ such that $x(t-\delta) > 0$ and $y(t) > 0$ for $t \geq t_1$. It follows from (5.28) that $(ry')'(t) < 0$. Hence

$$y(t) < y(s) + r(s)y'(s)\int_s^t \frac{du}{r(u)} = y(s) + r(s)y'(s)R(s,t) \tag{5.30}$$

for $t \geq s \geq t_1$. If there exists $t_2 \geq t_1$ such that $y'(t_2) \leq 0$, then it follows from (5.30) that $y(t) < y(s)$ for $t > s \geq t_2$. This means that y is eventually decreasing. If y is eventually decreasing, it follows from $y(t) > 0$ that $\lim_{t\to\infty} y(t) = L$ exists and

$|L| < \infty$. If there does not exist $s \geq t_1$ such that $y'(s) \leq 0$, then $y'(s) > 0$ for all $s \geq t_1$. This means that y is eventually increasing. Since $R_0 < \infty$ and $y'(t) > 0$, we see from (5.30) that y is bounded. Therefore $\lim_{t\to\infty} y(t) = L$ exists and $|L| < \infty$.

Similarly, we may discuss the case when x is an eventually negative solution of (5.28). The proof is complete. □

Lemma 5.4.5. *Suppose that* (H) *holds. If x is a nonoscillatory solution of* (5.28) *and y is defined by* (5.29), *then there exist $a_1 > 0$, $a_2 > 0$, and $t_1 \geq t_0$ such that either*

$$a_1 R(t) \leq y(t) \leq a_2 \quad \text{or} \quad -a_2 \leq y(t) \leq -a_1 R(t)$$

for all $t \geq t_1$.

Proof. Let x be an eventually positive solution of (5.28). By Lemma 5.4.1, y is eventually of one sign. We have four cases to consider:

(i) $y(t) > 0$ and $y'(t) > 0$ eventually;
(ii) $y(t) > 0$ and $y'(t) < 0$ eventually;
(iii) $y(t) < 0$ and $y'(t) > 0$ eventually;
(iv) $y(t) < 0$ and $y'(t) < 0$ eventually.

We shall only consider cases (i) and (ii) in detail since the other two cases are similar.

First we show (i). If y is eventually increasing, then (5.30) holds. In view of Lemma 5.4.4, there exists a constant $a_2 > 0$ such that $y(t) \leq a_2$. Since we are assuming that y is positive and increasing and since $R(t) \to 0$ as $t \to \infty$, there exist a constant $a_1 > 0$ and $t_1 \geq t_0$ such that $y(t) \geq a_1 R(t)$ for all $t \geq t_1$.

Next we show (ii). If y is eventually decreasing, then, from (5.30),

$$y(s) \geq y(t) - r(s)y'(s)R(s,t).$$

By Lemma 5.4.4, $\lim_{t\to\infty} y(t) = L \geq 0$. Taking the limit as $t \to \infty$ on both sides of the last inequality, we can see that

$$y(s) \geq L - r(s)y'(s)R(s).$$

Since ry' is eventually decreasing, we can choose t_1 so large that $r(t_1)y'(t_1) < 0$. Then $r(s)y'(s) \leq r(t_1)y'(t_1) = -a_1$ for all $s \geq t_1$. Therefore

$$y(s) \geq a_1 R(s)$$

for $s \geq t_1$, where $a_1 > 0$ is independent of s. Changing s to t, we can see that

$$y(t) \geq a_1 R(t), \quad a_1 > 0$$

for $t \geq t_1$.

Similarly, we can prove the case when x is an eventually negative solution of (5.28). This completes the proof. □

The following result is one of the main classification theorems.

Theorem 5.4.6. *Suppose that* (H) *holds and that* $\lim_{t\to\infty} p(t) = p_0 \in [0,1)$. *Then any nonoscillatory solution of* (5.28) *must belong to one of the following four types:*

$$S(b,a,c): \quad x(t) \to b = \frac{a}{1-p_0} \neq 0,\ y(t) \to a \neq 0,\ r(t)y'(t) \to c \quad (t\to\infty);$$
$$S(b,a,\infty): \quad x(t) \to b = \frac{a}{1-p_0} \neq 0,\ y(t) \to a \neq 0,\ r(t)y'(t) \to \pm\infty \quad (t\to\infty);$$
$$S(0,0,c): \quad x(t) \to 0,\ y(t) \to 0,\ r(t)y'(t) \to c \neq 0 \quad (t\to\infty);$$
$$S(0,0,\infty): \quad x(t) \to 0,\ y(t) \to 0,\ r(t)y'(t) \to \pm\infty \quad (t\to\infty),$$

where a, b, c *are some finite constants.*

Proof. Suppose that x is a nonoscillatory solution of (5.28). By Lemmas 5.4.1 and 5.4.4, y is eventually of one sign and $\lim_{t\to\infty} y(t) = L$, where L is a finite constant. So there are only two possibilities: $\lim_{t\to\infty} y(t) = a \neq 0$ or $\lim_{t\to\infty} y(t) = 0$, where a is a finite constant. According to Lemma 5.4.3, $\lim_{t\to\infty} x(t) = b \neq 0$ or $\lim_{t\to\infty} x(t) = 0$.

In addition, by our assumption that $xf(t,x) > 0$ for $x \neq 0$, we see from (5.28) that ry' is eventually decreasing or increasing. Again there are only two possibilities: $\lim_{t\to\infty} r(t)y'(t) = c$ or $\lim_{t\to\infty} r(t)y'(t) = \pm\infty$, where c is a finite constant.

Based on the above discussion, we see that x must belong to one of the four types as stated, except that we have not yet shown that for the case $S(0,0,c)$, the constant $c \neq 0$. We do this next.

Suppose x is a nonoscillatory solution of (5.28) which belongs to $S(0,0,c)$, that is, $\lim_{t\to\infty} x(t) = 0$, $\lim_{t\to\infty} y(t) = 0$, and $\lim_{t\to\infty} r(t)y'(t) = c$. Then $c \neq 0$. In fact, consider the functions y and R. From our assumption, it is easy to see that

$$\lim_{t\to\infty} y(t) = 0, \quad \lim_{t\to\infty} R(t) = 0, \quad R'(t) < 0,$$

and

$$\lim_{t\to\infty} \frac{y'(t)}{R'(t)} = \lim_{t\to\infty} (-r(t)y'(t)) = -c.$$

By L'Hôpital's rule, $\lim_{t\to\infty}(y(t)/R(t))$ exists, and

$$\lim_{t\to\infty} \frac{y(t)}{R(t)} = \lim_{t\to\infty} \frac{y'(t)}{R'(t)} = -c.$$

If $y(t)$ is eventually positive, then, by Lemma 5.4.5, there exist two positive constants a_1 and a_2 such that $a_1 R(t) \leq y(t) \leq a_2$. Thus $y(t)/R(t) \geq a_1$ holds, hence $a_1 \leq -c$. It follows that $c \neq 0$. If $y(t)$ is eventually negative, then by Lemma 5.4.5, there exist two positive constants a_1 and a_2 such that $-a_2 \leq y(t) \leq -a_1 R(t)$. Thus $y(t)/R(t) \leq -a_1$, which means $-c \leq -a_1$. Once more it follows that $c \neq 0$, which completes the proof. □

Next we derive two existence theorems.

Theorem 5.4.7. *Assume that* (H) *holds and* $\lim_{t\to\infty}(R(t-\tau)/R(t)) = 1$. *A necessary and sufficient condition for* (5.28) *to have a nonoscillatory solution in* $S(0,0,c)$ *is that*

$$\int_{t_0}^{\infty} \Big| f(t, \lambda R(t-\delta)) \Big| \, dt < \infty \quad \textit{for some} \quad \lambda \neq 0. \tag{5.31}$$

Proof. Necessity. Let $x \in S(0,0,c)$ be any nonoscillatory positive solution of (5.28), i.e., $\lim_{t\to\infty} x(t) = 0$, $\lim_{t\to\infty} y(t) = 0$, and $\lim_{t\to\infty} r(t)y'(t) = c \neq 0$. Without loss of generality we may assume that $c < 0$. Since ry' is monotone, there exist $\lambda_1 > 0$, $\lambda_2 > 0$, and $t_1 \geq t_0$ such that $-\lambda_1 \leq r(t)y'(t) \leq -\lambda_2$ for $t \geq t_1$. It follows that

$$-\lambda_1 R(t,s) \leq y(s) - y(t) \leq -\lambda_2 R(t,s)$$

for $s > t$, $t \geq t_1$. If $s \to \infty$, then $-\lambda_1 R(t) \leq -y(t) \leq -\lambda_2 R(t)$. That is, $\lambda_2 R(t) \leq y(t) \leq \lambda_1 R(t)$. On the other hand, by (5.28),

$$r(t)y'(t) = r(t_1)y'(t_1) - \int_{t_1}^{t} f\left(s, x(s-\delta)\right) ds.$$

Since $\lim_{t\to\infty} r(t)y'(t) = c < 0$, we have

$$\int_{t_1}^{\infty} \left| f\left(s, x(s-\delta)\right)\right| ds < \infty.$$

Furthermore, since (H) holds, by $y(t) \leq x(t)$, we have

$$f\left(s, x(s-\delta)\right) \geq f\left(s, y(s-\delta)\right) \geq f\left(s, \lambda_2 R(s-\delta)\right).$$

This means that

$$\int_{t_1}^{\infty} \left| f\left(s, \lambda_2 R(s-\delta)\right)\right| ds < \infty.$$

Sufficiency. Suppose that (5.31) holds for $\lambda > 0$. A similar argument can be applied if $\lambda < 0$. Since $\lim_{t\to\infty}(R(t-\tau)/R(t)) = 1$, we may choose $A \in (\rho, 1)$ and $t_1 \geq t_0$ such that $p(t)\frac{R(t-\tau)}{R(t)} \leq A$ and

$$\int_{t_1}^{\infty} f\left(s, \lambda R(s-\tau)\right) ds < a(1-A), \qquad \text{where} \qquad a = \frac{\lambda}{2}.$$

Consider the equation

$$x(t) = p(t)x(t-\tau) + R(t)\left[(1-A)a + \int_{t_1}^{t} f\left(s, x(s-\delta)\right) ds\right] + \int_{t}^{\infty} R(s) f\left(s, x(s-\delta)\right) ds \tag{5.32}$$

for $t \geq t_1 + \max\{\delta, \tau\}$. It is easy to see that a solution of (5.32) must also be a solution of (5.28). By means of the method of successive approximations, we shall show that (5.32) has a nonoscillatory solution $x \in S(0,0,c)$. Consider the sequence $\{x_n\}$ of successive approximating sequences defined by

$$x_1(t) = 0 \quad \text{and} \quad x_{n+1}(t) = (Fx_n)(t), \quad n \in \mathbb{N} \quad \text{for} \quad t \geq t_1,$$

where F is defined by

$$(Fx)(t) = \begin{cases} p(t)x(t-\tau) + R(t)\left[(1-A)a + \int_{t_1}^{t} f\left(s, x(s-\delta)\right) ds\right] & \\ \qquad + \int_{t}^{\infty} R(s) f\left(s, x(s-\delta)\right) ds & \text{if} \quad t \geq t_1 + \max\{\delta, \tau\}, \\ (Fx)\left(t_1 + \max\{\delta, \tau\}\right) & \text{if} \quad t_1 \leq t < t_1 + \max\{\delta, \tau\}. \end{cases}$$

In view of (H1), it is easy to see that

$$0 \leq x_n(t) \leq x_{n+1}(t), \quad t \geq t_1, \quad n \in \mathbb{N}.$$

On the other hand,

$$x_2(t) = (Fx_1)(t) = a(1-A)R(t) \le 2aR(t), \quad t \ge t_1,$$

and inductively, we have

$$\begin{aligned}(Fx_n)(t) &\le 2p(t)aR(t-\tau) + (1-A)aR(t) + (1-A)aR(t) \\ &\le 2AaR(t) + 2(1-A)aR(t) \ = \ 2aR(t)\end{aligned}$$

for $n \ge 2$. Thus by means of Lebesgue's dominated convergence theorem, we see that $Fx = x$. Furthermore, it is clear that $x(t)$, and hence its associated function $y(t)$, converge to zero (since $R(t)$ does). Finally, in view of (5.32), we see that

$$\begin{aligned}-(1-A)a \ > \ r(t)y'(t) &= -(1-A)a - \int_{t_1}^t f\Big(s, x(s-\delta)\Big)\,ds \\ &> -(1-A)a - \int_{t_1}^t f\Big(s, \lambda R(s-\delta)\Big)\,ds \\ &> -(1-A)a - (1-A)a \ = \ -2(1-A)a,\end{aligned}$$

which implies $\lim_{t\to\infty} r(t)y'(t) = c \ne 0$ as required. The proof is complete. □

Theorem 5.4.8. *Assume that* (H) *holds and that* $\lim_{t\to\infty} p(t) = p_0 \in [0,1)$. *A necessary and sufficient condition for* (5.28) *to have a nonoscillatory solution in* $S(b,a,c) \cup S(b,a,\infty)$ *is that for* $t_2 \ge t_1 \ge t_0$,

$$\int_{t_2}^\infty \frac{1}{r(s)} \int_{t_1}^s |f(u,\lambda)|\,du ds < \infty \quad \textit{for some} \quad \lambda \ne 0. \tag{5.33}$$

Proof. Necessity. Let x be any nonoscillatory positive solution of (5.28) such that $\lim_{t\to\infty} x(t) = b > 0$. By (5.29), we have $\lim_{t\to\infty} y(t) = a = b(1-p_0) > 0$, which means that $y(t)$ is eventually positive and tends monotonically to a. Thus there exist $c_1 > 0$, $c_2 > 0$, and $t_1 \ge t_0$ such that $c_1 \le y(t) \le c_2$ for $t \ge t_1$. It follows from (5.28) that

$$y(t) = y(s) + r(t_1)y'(t_1)\int_s^t \frac{du}{r(u)} - \int_s^t \frac{1}{r(u)}\int_{t_1}^u f\Big(v, x(v-\delta)\Big)\,dvdu.$$

Taking the limit as $t \to \infty$ on both sides of the last equality, we obtain

$$a = y(s) + r(t_1)y'(t_1)\int_s^\infty \frac{du}{r(u)} - \int_s^\infty \frac{1}{r(u)}\int_{t_1}^u f\Big(v, x(v-\delta)\Big)\,dvdu.$$

This means that

$$0 \le \int_s^\infty \frac{1}{r(u)}\int_{t_1}^u f\Big(v, x(v-\delta)\Big)\,dvdu < \infty. \tag{5.34}$$

By (H), we have $f(t,c_1) \le f(t, y(t-\delta)) \le f(t, x(t-\delta))$. It follows from (5.34) that

$$\int_s^\infty \frac{1}{r(u)}\int_{t_1}^u f(v,c_1)dvdu < \infty.$$

Sufficiency. Suppose that (5.33) holds for $\lambda > 0$. A similar argument can be applied if $\lambda < 0$. Choose $t_1 \ge t_0$ so large that

$$\int_s^\infty \frac{1}{r(u)}\int_{t_1}^u f(v,\lambda)dvdu < (1-\rho)a, \quad \text{where} \quad a = \frac{\lambda}{2}. \tag{5.35}$$

Consider the equation

$$x(t) = p(t)x(t-\tau) + (1-\rho)a + \int_t^\infty \frac{1}{r(u)} \int_{t_1}^u f\Big(v, x(v-\delta)\Big)\, dvdu. \tag{5.36}$$

It is easy to verify that a solution of (5.36) must also be a solution of (5.28). Consider the Banach space Φ of all bounded functions x with norm $\sup_{t\ge t_1}|x(t)|$, endowed with the usual pointwise ordering $\le$: For $x, y \in \Phi$, $x \le y$ means $x(t) \le y(t)$ for all $t \ge t_1$. Then Φ is partially ordered. Define a subset Ω of Φ by

$$\Omega = \Big\{x \in \Phi : \ (1-\rho)a \le x(t) \le 2a, \ t \ge t_1\Big\}.$$

For any subset $B \subset \Omega$, it is obvious that $\inf B \in \Omega$ and $\sup B \in \Omega$. We also define an operator $F : \Omega \to \Phi$ as

$$(Fx)(t) = \begin{cases} p(t)x(t-\tau) + (1-\rho)a + \displaystyle\int_t^\infty \frac{1}{r(u)} \int_{t_1}^u f\Big(v, x(v-\delta)\Big)\, dvdu \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if} \quad t \ge t_1 + \delta, \\ (Fx)(t_1+\delta) \qquad\qquad\qquad\qquad\quad \text{if} \quad t_1 \le t < t_1 + \delta. \end{cases}$$

The mapping F satisfies the assumptions of Knaster's fixed point theorem. Namely, it satisfies the following: First, F maps Ω into itself. Indeed, if $x \in \Omega$, by (5.35), we have

$$(1-\rho)a \le (Fx)(t) \le 2p(t)a + (1-\rho)a + (1-\rho)a \le 2\rho a + 2(1-\rho)a = 2a.$$

Second, by (H), F is nondecreasing. That is, for any $x, y \in \Omega$, $x \le y$ implies $Fx \le Fy$. Hence, by Knaster's fixed point theorem (Theorem 1.4.28), there exists an $x \in \Omega$ such that $Fx = x$, that is, x is a nonoscillatory solution of (5.28) which belongs to $S(b, a, c)$ or $S(b, a, \infty)$. Note that $b \ne 0$. Since $x \in \Omega$ and by (5.29) and Lemma 5.4.3, $\lim_{t\to\infty} y(t) = a$ and $\lim_{t\to\infty} x(t) = a/(1-p_0) = b \ne 0$ must exist. This completes the proof. $\square$

5.5. Nonoscillation of Nonlinear Equations with $\int^\infty ds/r(s) = \infty$

In this section we consider nonoscillatory solutions of (5.28) in the case $R_0 = \infty$. Before stating the main results, we give several lemmas.

Lemma 5.5.1. *Suppose that* (H) *holds. Let x be an eventually positive solution of* (5.28) *and let y be defined by* (5.29). *If $R_0 = \infty$ and $\lim_{t\to\infty} x(t) = 0$, then y is eventually increasing and negative and $\lim_{t\to\infty} y(t) = 0$. The statement remains true if "positive" is replaced by "negative" and "increasing" is replaced by "decreasing".*

Proof. Let x be an eventually positive solution of (5.28) and define y by (5.29). If $\lim_{t\to\infty} x(t) = 0$, then $\lim_{t\to\infty} y(t) = 0$. Suppose to the contrary that $y(t) > 0$ for all large t. Then $y'(t) < 0$ for all large t. From $(ry')'(t) < 0$, we obtain

$$y(t) \le y(s) + r(s)y'(s) \int_s^t \frac{du}{r(u)} = y(s) + r(s)y'(s)R(s,t) \tag{5.37}$$

for $t \ge s$, where $s \ge t_0$ such that $x(t) > 0$ for $t \ge s$. Since ry' is eventually decreasing, we can choose t_1 so large that $r(t_1)y'(t_1) < 0$. So $r(s)y'(s) < r(t_1)y'(t_1) = -a_1$ for $s \ge t_1$. Therefore

$$y(t) \le y(s) - a_1 R(s,t), \quad a_1 > 0.$$

If $R_0 = \infty$, taking the limit as $t \to \infty$ on both sides of the last inequality and in view of $\lim_{t\to\infty} y(t) = 0$, we see that

$$y(s) \geq \infty,$$

which is a contradiction. □

Lemma 5.5.2. *Suppose that* (H) *holds. Let x be an eventually positive solution of* (5.28) *and let y be defined by* (5.29). *If $R_0 = \infty$, then y' is eventually positive. The statement remains true if "positive" is replaced by "negative".*

Proof. Suppose x is an eventually positive solution of (5.28) and let y be defined by (5.29). We assert that $y'(t) > 0$ for all large t. If $\lim_{t\to\infty} x(t) = 0$, in view of Lemma 5.5.1, the conclusion holds. If $\limsup_{t\to\infty} x(t) > 0$, by Lemma 5.4.2, we have $y(t) > 0$ for all large t. Suppose to the contrary that $y'(t)$ is eventually negative. Note that in view of (5.28), $(ry')'(t) = -f(t, x(t-\delta)) < 0$ for all large t. Thus there exists $t_1 \geq t_0$ such that $y(t) > 0$, $y'(t) < 0$, and $(ry')'(t) < 0$ for all large $t \geq t_1$. Then

$$y(t) \leq y(t_1) + r(t_1)y'(t_1) \int_{t_1}^{t} \frac{ds}{r(s)} \to -\infty \quad \text{as} \quad t \to \infty,$$

which is a contradiction. The proof is complete. □

Lemma 5.5.3. *Suppose that* (H) *holds. If x is an eventually positive* (*negative*) *solution of* (5.28) *and* $\limsup_{t\to\infty} x(t) > 0$ $(\limsup_{t\to\infty}(-x(t)) > 0)$, *then there exist $a_1 > 0$, $a_2 > 0$, and $t_1 \geq t_0$ such that the function y defined by* (5.29) *is monotone increasing* (*decreasing*) *and satisfies*

$$a_1 \leq y(t) \leq a_2 R(t_1, t) \quad (-a_2 R(t_1, t) \leq y(t) \leq -a_1) \tag{5.38}$$

for all $t \geq t_1$.

Proof. Let x be an eventually positive solution of (5.28) and $\limsup_{t\to\infty} x(t) > 0$. Then, by Lemma 5.4.2, we have $y(t) > 0$ eventually. Thus, there exists $s \geq t_0$ such that $x(t) > 0$ and $y(t) > 0$ for all $t \geq s$. It follows from (5.28) that $(ry')'(t) < 0$ and thus ry' is eventually decreasing. By Lemma 5.5.2, we have that $y'(t) > 0$ eventually. Thus, y is eventually increasing and $r(t)y'(t)$ is positive. By (5.37) we then see that (5.38) holds. This completes the proof. □

Theorem 5.5.4. *Suppose that* (H) *holds and* $\lim_{t\to\infty} p(t) = p_0 \in [0, 1)$. *Then any nonoscillatory solution of* (5.28) *must belong to one of the following five types:*

$$S(0,0,0): \quad x(t) \to 0,\ y(t) \to 0,\ r(t)y'(t) \to 0 \quad (t \to \infty);$$
$$S(0,0,c): \quad x(t) \to 0,\ y(t) \to 0,\ r(t)y'(t) \to c \neq 0 \quad (t \to \infty);$$
$$S(b,a,0): \quad x(t) \to b = \frac{a}{1-p_0} \neq 0,\ y(t) \to a \neq 0,\ r(t)y'(t) \to 0 \quad (t \to \infty);$$
$$S(\infty,\infty,c): \quad x(t) \to \infty,\ y(t) \to \infty,\ r(t)y'(t) \to c \neq 0 \quad (t \to \infty);$$
$$S(\infty,\infty,0): \quad x(t) \to \infty,\ y(t) \to \infty,\ r(t)y'(t) \to 0 \quad (t \to \infty),$$

where a, b, c are some finite constants.

Proof. For a nonoscillatory solution x of (5.28), without loss of generality, we may suppose that x is an eventually positive solution. If $\lim_{t\to\infty} x(t) = 0$, then, by Lemma 5.5.1, $\lim_{t\to\infty} y(t) = 0$ and $y'(t) > 0$ for all large t. Since $(ry')'(t) < 0$ for all large t, we have

$$\lim_{t\to\infty} r(t)y'(t) = c \geq 0.$$

If $\limsup_{t\to\infty} x(t) > 0$, in view of Lemma 5.4.2, we have $y(t) > 0$ for all large t. By Lemma 5.5.3, we know that y is eventually increasing, and ry' is positive and decreasing. Thus there exist only the possibilities

$$\lim_{t\to\infty} y(t) = a \in (0, \infty) \quad \text{or} \quad \lim_{t\to\infty} y(t) = \infty$$

and

$$\lim_{t\to\infty} r(t)y'(t) = c \geq 0.$$

Since ry' is eventually decreasing and $\lim_{t\to\infty} r(t)y'(t) = c$, we have

$$r(t)y'(t) \geq c, \quad \text{i.e.,} \quad y'(t) \geq \frac{c}{r(t)}$$

for all large t. Integrating this inequality, we obtain

$$y(t) \geq y(s) + c\int_s^t \frac{du}{r(u)}. \tag{5.39}$$

If $c > 0$ and $R_0 = \infty$, then it follows that

$$\lim_{t\to\infty} y(t) = \infty.$$

By Lemma 5.4.3, we have $\lim_{t\to\infty} x(t) = \infty$. If $c = 0$ and $\lim_{t\to\infty} y(t) = a$ (or ∞), then, by Lemma 5.4.3, it follows that $\lim_{t\to\infty} x(t) = b = a/(1-p_0)$ (or ∞). Therefore x must belong to one of the five types as stated. $\square$

Next we derive several existence criteria for nonoscillatory solutions of (5.28).

Theorem 5.5.5. *Assume that* (H) *holds and* $\lim_{t\to\infty} p(t) = p_0 \in [0, 1)$. *A necessary and sufficient condition for* (5.28) *to have a nonoscillatory solution which belongs to* $S(b, a, 0)$ *is that*

$$\int_{t_0}^{\infty} \frac{1}{r(u)} \int_u^{\infty} |f(v, \lambda)|\, dv du < \infty \quad \textit{for some} \quad \lambda \neq 0. \tag{5.40}$$

Proof. Necessity. Let $x \in S(b, a, 0)$ be a nonoscillatory solution of (5.28), i.e.,

$$\lim_{t\to\infty} x(t) = b, \quad \lim_{t\to\infty} y(t) = a, \quad \text{and} \quad \lim_{t\to\infty} r(t)y'(t) = 0.$$

Without loss of generality, we may suppose that $b > 0$. By (5.29), it follows that $\lim_{t\to\infty} y(t) = a = b(1 - p_0) > 0$. Then there exist $c_1 > 0$, $c_2 > 0$, and $t_1 \geq t_0$ such that

$$c_1 \leq y(t) \leq c_2 \quad \text{for} \quad t \geq t_1.$$

On the other hand, by (5.28),

$$r(t)y'(t) = r(s)y'(s) - \int_s^t f\Big(u, x(u-\delta)\Big)\, du \tag{5.41}$$

for $t \geq s \geq t_1$. After taking the limits as $t \to \infty$ on both sides of (5.41) and using $\lim_{t\to\infty} r(t)y'(t) = 0$, we obtain

$$r(s)y'(s) = \int_s^\infty f\left(u, x(u-\delta)\right) du \tag{5.42}$$

for $s \geq t_1$. Now it follows from (5.42) that

$$y(s) = y(t_1) + \int_{t_1}^s \frac{1}{r(u)} \int_u^\infty f\left(v, x(v-\delta)\right) dvdu$$

for $s \geq t_1$. Let $s \to \infty$. Then

$$a = y(t_1) + \int_{t_1}^\infty \frac{1}{r(u)} \int_u^\infty f\left(v, x(v-\delta)\right) dvdu.$$

Therefore

$$\int_{t_1}^\infty \frac{1}{r(u)} \int_u^\infty f\left(v, x(v-\delta)\right) dvdu < \infty.$$

By (H), $f(t, c_1) \leq f(t, y(t-\delta)) \leq f(t, x(t-\delta))$, so (5.40) holds.

Sufficiency. Without loss of generality, we may assume that (5.40) holds for $\lambda > 0$. Then there exists $t_1 \geq t_0$ such that

$$\int_{t_1}^\infty \frac{1}{r(u)} \int_u^\infty f(v, \lambda) dvdu < (1-\rho)a, \qquad \text{where} \qquad a = \frac{\lambda}{2}.$$

Now consider the equation

$$x(t) = p(t)x(t-\tau) + (1-\rho)a + \int_{t_1}^t \frac{1}{r(u)} \int_u^\infty f\left(v, x(v-\delta)\right) dvdu \tag{5.43}$$

for $t \geq t_1$. Consider the Banach space Φ of all bounded real functions x with norm $\sup_{t\geq t_1} |x(t)|$, with the usual pointwise ordering $\leq$: For $x, y \in \Phi$, $x \leq y$ means $x(t) \leq y(t)$ for all $t \geq t_1$. Then Φ is partially ordered. Define a subset Ω of Φ as follows:

$$\Omega = \left\{x \in \Phi : \ (1-\rho)a \leq x(t) \leq 2a, \ t \geq t_1\right\}.$$

If $x \in \Phi$, also let

$$(Fx)(t) = \begin{cases} p(t)x(t-\tau) + (1-\rho)a + \displaystyle\int_{t_1}^t \frac{1}{r(u)} \int_u^\infty f\left(v, x(v-\delta)\right) dvdu \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if} \quad t \geq t_1 + \delta, \\ (Fx)(t_1+\delta) \qquad\qquad\qquad\qquad\quad\ \ \text{if} \quad t_1 \leq t < t_1 + \delta. \end{cases}$$

Then, by using Knaster's fixed point theorem and Lemma 5.5.3, we can show that there exists a nonoscillatory solution of (5.43) and thus of (5.28) which belongs to $S(b, a, 0)$. We omit the details. □

Theorem 5.5.6. *Assume that* (H) *holds. A necessary and sufficient condition for* (5.28) *to have a nonoscillatory solution which belongs to* $S(\infty, \infty, c)$ *is that for* $t_1 \geq t_0$,

$$\int_{t_1}^\infty \left| f\left(u, \lambda R(t_1, u-\delta)\right)\right| du < \infty \qquad \textit{for some} \qquad \lambda \neq 0. \tag{5.44}$$

Proof. Necessity. Suppose that $x \in S(\infty, \infty, c)$ is an eventually positive solution of (5.28), i.e.,

$$\lim_{t\to\infty} x(t) = \infty, \quad \lim_{t\to\infty} y(t) = \infty, \quad \text{and} \quad \lim_{t\to\infty} r(t)y'(t) = c > 0.$$

Then it follows from (5.38) and (5.39) that there exist $c_1 > 0$, $c_2 > 0$, and $t_1 \ge t_0$ such that

$$c_1 R(t_1, t) \le y(t) \le c_2 R(t_1, t)$$

for $t \ge t_1$. On the other hand, by (5.28),

$$r(t)y'(t) = r(t_1)y'(t_1) - \int_{t_1}^{t} f\left(u, x(u-\delta)\right) du. \tag{5.45}$$

Let $t \to \infty$ on both sides of (5.45) to obtain

$$\int_{t_1}^{\infty} \left| f\left(u, x(u-\delta)\right)\right| du < \infty. \tag{5.46}$$

Since (H) holds, $f(t, c_1 R(t_1, t-\delta)) \le f(t, y(t-\delta)) \le f(t, x(t-\delta))$, and we can conclude from (5.46) that (5.44) holds for some $\lambda \ne 0$ and $t_1 \ge t_0$.

Sufficiency. Without loss of generality, we may assume that (5.44) holds for $\lambda > 0$ and $t_1 \ge t_0$. Then there exists $t_2 \ge t_1$ such that

$$\int_{t_2}^{\infty} f\left(u, \lambda R(t_1, u-\delta)\right) du < (1-\rho)a, \quad \text{where} \quad a = \frac{\lambda}{2}.$$

Now consider the equation

$$\begin{aligned} x(t) = p(t)x(t-\tau) + (1-\rho)aR(t_1, t) + \int_{t_2}^{t} R(t_2, u) f\left(u, x(u-\delta)\right) du \\ + R(t_1, t)\int_{t}^{\infty} f\left(u, x(u-\delta)\right) du \end{aligned} \tag{5.47}$$

for $t \ge t_2$. We introduce a Banach space Φ of all bounded real functions x which satisfy

$$\sup_{t\ge t_2} \frac{|x(t)|}{R(t_1, t)} < \infty \quad \text{with the norm} \quad \|x\| = \sup_{t\ge t_2} \frac{|x(t)|}{R(t_1, t)}.$$

Now Φ is considered to be endowed with the usual pointwise ordering $\le$: For $x, y \in \Phi$, $x \le y$ means $x(t) \le y(t)$ for all $t \ge t_2$. Then Φ is partially ordered. Define a subset Ω of Φ and an operator $F : \Omega \to \Phi$ by

$$\Omega = \left\{x \in \Phi : \ (1-\rho)aR(t_1, t) \le x(t) \le 2aR(t_1, t), \ t \ge t_2\right\},$$

and for $x \in \Phi$,

$$(Fx)(t) = \begin{cases} p(t)x(t-\tau) + (1-\rho)aR(t_1, t) + \int_{t_2}^{t} R(t_2, u) f\left(u, x(u-\delta)\right) du & \\ \quad +R(t_1, t)\int_{t}^{\infty} f\left(u, x(u-\delta)\right) du & \text{if} \quad t \ge t_2 + \delta, \\ (Fx)(t_2+\delta) & \text{if} \quad t_2 \le t < t_2 + \delta. \end{cases}$$

Similar to the proof of Theorem 5.4.8, we can show that there exists a nonoscillatory solution of (5.47) and also of (5.28) which belongs to $S(\infty, \infty, c)$. This completes the proof. $\square$

Theorem 5.5.7. *Assume that* (H) *holds. A sufficient condition for* (5.28) *to have a nonoscillatory solution which belongs to* $S(\infty, \infty, 0)$ *is that*

$$\int_{t_0}^{\infty} R(t_0, u) \left| f\left(u, \lambda_1 R(t_1, u-\delta)\right) \right| du = \infty \tag{5.48}$$

and

$$\int_{t_0}^{\infty} \left| f\left(u, \lambda_2 R(t_1, u-\delta)\right) \right| du < \infty$$

for some λ_1, λ_2 *such that* $\lambda_1 \lambda_2 > 0$ *and* $t_1 \ge t_0$.

Proof. Without loss of generality, we may assume $\lambda_1 > 0$ and $\lambda_2 > 0$. We take a large $t_2 \ge t_1$ such that for $t \ge t_2$,

$$\frac{\lambda_1}{R(t_1, t)\lambda_2} + p(t) + \frac{1}{\lambda_2} \int_{t_2}^{\infty} f\left(u, \lambda_2 R(t_1, u-\delta)\right) du < 1. \tag{5.49}$$

Define an operator F as

$$(Fx)(t) = \begin{cases} \dfrac{\lambda_1}{R(t_1,t)} + \dfrac{R(t_1, t-\tau)}{R(t_1,t)} p(t) x(t-\tau) & \\ \qquad + \dfrac{1}{R(t_1,t)} \displaystyle\int_{t_1}^{t} R(t_1, u) f\left(u, R(t_1, u-\delta) x(u-\delta)\right) du & \\ \qquad + \displaystyle\int_{t}^{\infty} f\left(u, R(t_1, u-\delta) x(u-\delta)\right) du & \text{if} \quad t \ge t_2, \\ Fx(t_2) & \text{if} \quad t_1 \le t < t_2, \end{cases}$$

and let

$$x_1(t) = 0 \quad \text{and} \quad x_{n+1}(t) = (Fx_n)(t), \quad n \in \mathbb{N} \quad \text{for} \quad t \ge t_1.$$

By (H) and induction, it is easy to see that $0 \le x_n(t) \le x_{n+1}(t)$ for $t \ge t_1$ and $n \in \mathbb{N}$. On the other hand, $x_2(t) \le \lambda_2$ for $t \ge t_1$. It follows from (5.49) that

$$x_{n+1}(t) \le \lambda_2 \left[\frac{\lambda_1}{R(t_1,t)\lambda_2} + p(t) + \frac{1}{\lambda_2} \int_{t_2}^{\infty} f\left(u, \lambda_2 R(t_1, u-\delta)\right) du \right] \le \lambda_2. \tag{5.50}$$

By induction, we have $x_n(t) \le \lambda_2$ for $t \ge t_1$, $n \in \mathbb{N}$. Let

$$\lim_{n\to\infty} x_n(t) = x^*(t), \quad t \ge t_2.$$

Using Lebesgue's dominated convergence theorem, we get

$$x^*(t) = (Fx^*)(t), \quad t \ge t_1.$$

By (5.49) and (5.50), it is easy to see that $\frac{\lambda_1}{R(t_1,t)} \le x^*(t) \le \lambda_2$. Set

$$z(t) = R(t_1, t) x^*(t), \quad t \ge t_1.$$

Then we have $\lambda_1 \le z(t) \le \lambda_2 R(t_1, t)$ and

$$z(t) = \lambda_1 + p(t) z(t-\tau) + \int_{t_1}^{t} R(t_1, u) f\left(u, z(u-\delta)\right) du \\ + R(t_1, t) \int_{t}^{\infty} f\left(u, z(u-\delta)\right) du$$

for $t \ge t_2$. Again set

$$w(t) = z(t) - p(t) z(t-\tau).$$

Then, by (5.48), we have $\lim_{t\to\infty} w(t) = \infty$. From Lemma 5.4.2, it follows that $\lim_{t\to\infty} z(t) = \infty$. Since

$$r(t)w'(t) = \int_t^\infty f\left(u, z(u-\delta)\right) du,$$

it follows that $\lim_{t\to\infty} r(t)w'(t) = 0$. This means that $z \in S(\infty, \infty, 0)$ is a positive solution of (5.28). □

Theorem 5.5.8. *Assume that* (H) *holds, that* $|p(t) - p(s)| \leq K|t-s|$, *and that* r *is nondecreasing. Furthermore, suppose that there exist* $K_1 > K_2 > 0$ *such that*

$$p(t)e^{K_1\tau} > 1 \geq p(t)e^{K_2\tau}, \tag{5.51}$$

and for large t_1 *and* $t \geq t_1$,

$$\left(p(t)e^{K_1\tau} - 1\right) e^{-K_1 t} \geq \int_t^\infty \frac{1}{r(u)} \int_u^\infty f\left(v, e^{-K_2(v-\delta)}\right) dv du. \tag{5.52}$$

Then (5.28) *has a nonoscillatory solution* $x \in S(0,0,0)$.

Proof. Set

$$\Omega = \left\{x \in C[0,\infty) :\, e^{-K_1 t} \leq x(t) \leq e^{-K_2 t},\ |x(t) - x(s)| \leq L|t-s|,\ t \geq s \geq t_0\right\},$$

where $L \geq \max\{K, K_2\}$. It is easy to show that Ω is nonempty, bounded, convex, and closed in the Banach space Φ defined to be all bounded real functions x with norm $\sup_{t\geq t_0} |x(t)|$. Define an operator F on Ω as

$$(Fx)(t) = \begin{cases} p(t)x(t-\tau) - \displaystyle\int_t^\infty \frac{1}{r(u)} \int_u^\infty f\left(v, x(v-\delta)\right) dv du & \text{if} \quad t \geq t_2, \\ \exp\left(\dfrac{\ln(Fx(t_2))}{t_2} t\right) & \text{if} \quad t_0 \leq t < t_2, \end{cases}$$

where $t_2 \geq t_1$ and for given $\alpha \in (\rho, 1)$ such that

$$\frac{1}{r(t)} \int_{t_2}^\infty f\left(s, e^{-K_2(s-\delta)}\right) ds \leq \left(\alpha - p(t)\right) L, \quad t \geq t_2 \tag{5.53}$$

and

$$\alpha + e^{-K_2(t-\tau)} \leq 1, \quad t \geq t_2. \tag{5.54}$$

The mapping F satisfies the assumptions of Schauder's fixed point theorem. Namely, it satisfies the following:

First, F maps Ω into Ω. For any $x \in S$, by (5.51) and (5.52) we obtain

$$(Fx)(t) \leq p(t)x(t-\tau) \leq p(t)\exp\left(-K_2(t-\tau)\right) \leq \exp\left(-K_2 t\right)$$

and

$$\begin{aligned}(Fx)(t) &\geq p(t)\exp\Big(-K_1(t-\tau)\Big)\\ &\quad -\int_t^{\infty}\frac{1}{r(u)}\int_u^{\infty} f\Big(v, \exp\left(-K_2(v-\delta)\right)\Big)\,dvdu\\ &= \exp\left(-K_1 t\right) + \Big(p(t)\exp\left(K_1\tau\right)-1\Big)\exp\left(-K_1 t\right)\\ &\quad -\int_t^{\infty}\frac{1}{r(u)}\int_u^{\infty} f\Big(v, \exp\left(-K_2(v-\delta)\right)\Big)\,dvdu\\ &\geq \exp(-K_1 t),\end{aligned}$$

and so

$$K_2 \leq \frac{-\ln(Fx(t_2))}{t_2} \leq K_1.$$

For any $s \geq t \geq t_2$, by (5.53) and (5.54) we get

$$\begin{aligned}&\Big|(Fx)(s)-(Fx)(t)\Big|\\ &\leq \left[\Big(p(s)+e^{-K_2(t-\tau)}\Big)L + \frac{1}{r(t)}\int_t^{\infty} f\Big(u, \exp\left(-K_2(u-\delta)\right)\Big)\,du\right]|s-t|\\ &\leq \Big(\exp(-K_2(t-\tau))+\alpha\Big)L|s-t| \;\leq\; L|s-t|,\end{aligned}$$

and for $t_0 \leq t \leq s \leq t_2$,

$$\Big|(Fx)(s)-(Fx)(t)\Big| = \left|\exp\left(\frac{\ln(Fx(t_2))}{t_2}s\right) - \exp\left(\frac{\ln(Fx(t_2))}{t_2}t\right)\right| \leq L\left|s-t\right|.$$

Hence, F maps Ω into Ω.

Second, F is continuous. Let $\{x_n\} \subset \Omega$ such that

$$\lim_{n\to\infty}\|x_n - x\| = 0.$$

Since Ω is closed, $x \in \Omega$. Then we get

$$\begin{aligned}&\Big|(Fx_n)(t)-(Fx)(t)\Big|\\ &\leq p(t)\left\|x_n - x\right\| + \int_{t_2}^{\infty}\frac{1}{r(u)}\int_u^{\infty}\Big|f\Big(v, x_n(v-\delta)\Big) - f\Big(v, x(v-\delta)\Big)\Big|\,dvdu\\ &\leq \rho\left\|x_n - x\right\| + \int_{t_2}^{\infty}\frac{1}{r(u)}\int_u^{\infty}\Big|f\Big(v, x_n(v-\delta)\Big) - f\Big(v, x(v-\delta)\Big)\Big|\,dvdu.\end{aligned}$$

By the continuity of f and Lebesgue's dominated convergence theorem, it follows that

$$\lim_{n\to\infty}\left(\sup_{t\geq t_2}\Big|(Fx_n)(t)-(Fx)(t)\Big|\right) = 0. \tag{5.55}$$

We can easily show that

$$\sup_{t_0\leq t\leq t_2}\Big|(Fx_n)(t)-(Fx)(t)\Big| \leq \Big|\ln((Fx_n)(t_2)) - \ln((Fx)(t_2))\Big|. \tag{5.56}$$

Using (5.55) and (5.56), it is easy to show that

$$\lim_{n\to\infty}\|Fx_n - Fx\| = 0.$$

Third, $F\Omega$ is precompact. Let $x \in \Omega$ and $s > t \geq t_2$. Then by (5.54) we have

$$\begin{aligned}
\Big|(Fx)(t) - (Fx)(s)\Big| &= \Big|p(t)x(t-\tau) - p(s)x(s-\tau) \\
&\quad + \int_s^\infty \frac{1}{r(u)} \int_u^\infty f\Big(v, x(v-\delta)\Big)\, dvdu - \int_t^\infty \frac{1}{r(u)} \int_u^\infty f\Big(v, x(v-\delta)\Big)\, dvdu\Big| \\
&\leq \Big|p(t)x(t-\tau) - p(s)x(s-\tau)\Big| + \left|\int_s^t \frac{1}{r(u)} \int_u^\infty f\Big(s, x(v-\delta)\Big)\, dvdu\right| \\
&\leq \Big|p(t)x(t-\tau) - p(s)x(s-\tau)\Big| + \left|\int_s^\infty \frac{1}{r(u)} \int_u^\infty f\Big(s, x(v-\delta)\Big)\, dvdu\right| \\
&\leq 2e^{-K_2(t-\tau)} + \Big(p(t)e^{K_1\tau} - 1\Big)\, e^{-K_1 t}.
\end{aligned}$$

Since $e^{-K_1 t} \to 0$ and $e^{-K_2(t-\tau)} \to 0$ as $t \to \infty$, we conclude from the above inequalities that, for any given $\varepsilon > 0$, there exists $t_3 \geq t_2$ such that for all $x \in \Omega$ and $s, t \geq t_3$,

$$\Big|(Fx)(t) - (Fx)(s)\Big| < \varepsilon.$$

This means that $F(\Omega)$ is relatively compact in the topology of the Fréchet space $C[t_0, \infty)$.

By using Schauder's fixed point theorem we can conclude that there is an $x \in \Omega$ such that $x = Fx$. That is, x is a positive solution of (5.28). Since $\lim_{t\to\infty} x(t) = 0$, by Lemma 5.4.3 it follows that $\lim_{t\to\infty} y(t) = 0$ and $\lim_{t\to\infty} y'(t) = 0$. The proof is complete. □

Remark 5.5.9. It would not be difficult to extend all the results in this section to an equation whose nonlinear term has the form

$$f\Big(t, x(t-\delta_1), \ldots, x(t-\delta_m)\Big).$$

5.6. Notes

The material of Section 5.2 is adopted from Kusano and Lalli [**155**]. The results in Section 5.3 are obtained by Fan, Li, and Zhong [**95**]. Sections 5.4 and 5.5 are taken from Li [**177**].

CHAPTER 6

Higher Order Delay Differential Equations

6.1. Introduction

In this chapter we describe some of the recent developments in the oscillation and nonoscillation theory of higher order delay differential equations.

In Section 6.2, we consider nonlinear neutral delay differential equations with variable coefficients of the form

$$\frac{d^n}{dt^n}\Big(x(t) - P(t)x(t-\tau)\Big) + Q(t)f\Big(x(t-\delta(t))\Big) = 0$$

under the assumption

$$P(t^* + k\tau) \leq 1 \quad \text{for} \quad k \in \mathbb{N}$$

and establish a comparison theorem for oscillation. Some necessary and/or sufficient conditions for all solutions to be oscillatory are presented. In Section 6.3, we consider linear neutral delay differential equations of the form

$$\frac{d^n}{dt^n}\Big(x(t) - px(t-\tau)\Big) + \sum_{i=1}^{m} q_i(t)x(t-\sigma_i) = 0$$

under the assumption $p \geq 1$ or $0 \leq p < 1$, respectively. Some oscillation criteria are presented. In Section 6.4, we are concerned with the asymptotic behavior of nonoscillatory solutions of nonlinear neutral differential equations of the form

$$\frac{d^n}{dt^n}\left(x(t) - \sum_{i=1}^{m} P_i(t)x(t-r_i)\right) + \delta\sum_{j=1}^{k} Q_j(t)f_j\Big(x\left(h_j(t)\right)\Big) = 0$$

under the condition $\sum_{i=1}^{m} |P_i(t)| \leq \lambda < 1$, where $p_i(t)$ is allowed to oscillate about zero. Section 6.5 deals with the nonlinear neutral delay differential equation

$$\frac{d^n}{dt^n}\Big(x(t) - p(t)x(t-\tau)\Big) + q(t)\prod_{i=1}^{m}\Big|x(t-\sigma_i)\Big|^{\alpha_i}\operatorname{sgn} x(t-\sigma_i) = 0$$

under the conditions $p(t) \geq 1$ and $0 \leq p(t) \leq 1$. We present some sufficient conditions for all solutions to be oscillatory. In Section 6.6, we consider the existence of positive solutions of the neutral delay differential equation

$$\frac{d^n}{dt^n}\Big(x(t) - px(t-\tau)\Big) + q(t)x(g(t)) = 0,$$

where $p \neq 1$ and $p = 1$, respectively. In Sections 6.7 and 6.8, we give a classification scheme of eventually positive solutions of our two type equations in terms of their asymptotic magnitude and provide necessary and/or sufficient conditions for the existence of these solutions.

6.2. Comparison Theorems and Oscillation

Our aim in this section is to obtain comparison theorems and sufficient conditions for the oscillation of all solutions of the neutral nonlinear delay differential equation

$$\frac{d^n}{dt^n}\Big(x(t) - P(t)x(t-\tau)\Big) + Q(t)f\Big(x\left(t-\delta(t)\right)\Big) = 0, \tag{6.1}$$

where

$$P, Q, \delta \in C([t_0,\infty),\mathbb{R}^+), \ n \in \mathbb{N} \text{ is odd}, \ \tau > 0, \ \lim_{t\to\infty}\Big(t-\delta(t)\Big) = \infty, \tag{6.2}$$

$$\begin{cases} f \in C(\mathbb{R},\mathbb{R}), & f \text{ is nondecreasing in } x, \\ f(-x) = -f(x), & xf(x) > 0 \text{ for } x \neq 0. \end{cases} \tag{6.3}$$

Before stating our main results we need the following lemma.

Lemma 6.2.1. *Assume that* (6.2) *and* (6.3) *hold and that there exists* $t^* \geq t_0$ *such that*

$$P(t^* + k\tau) \leq 1 \quad \textit{for} \quad k \in \mathbb{N}. \tag{6.4}$$

Suppose that $Q(t)$ *is not identically equal zero. Let* x *be an eventually positive solution of* (6.1) *and set*

$$y(t) = x(t) - P(t)x(t-\tau). \tag{6.5}$$

Then

$$y(t) > 0 \quad \textit{eventually.} \tag{6.6}$$

Proof. From (6.1), we have

$$y^{(n)}(t) = -Q(t)f\Big(x\left(t-\delta(t)\right)\Big) \leq 0,$$

which implies, in view of the hypothesis that $Q(t)$ is not identically zero, that $y^{(i)}$, $0 \leq i \leq n-1$, is eventually monotone. Hence, if (6.6) does not hold, then eventually $y(t) < 0$. Since n is odd, we have eventually $y'(t) < 0$ and so there exist $t_1 > t_0$ and $\alpha > 0$ such that

$$y(t) \leq -\alpha, \quad t \geq t_1,$$

i.e.,

$$x(t) \leq -\alpha + P(t)x(t-\tau), \quad t \geq t_1. \tag{6.7}$$

Choose $k^* \in \mathbb{N}$ so large that $t^* + k^*\tau \geq t_1$. Hence, by (6.4) and (6.7), we have

$$x\Big(t^* + (k^*+k)\tau\Big) \leq -\alpha + x\Big(t^* + (k^*+k-1)\tau\Big), \quad k \in \mathbb{N}_0,$$

and by induction

$$x\Big(t^* + (k^*+k)\tau\Big) \leq -(k+1)\alpha + x\Big(t^* + (k^*-1)\tau\Big) \to -\infty \quad \text{as} \quad k \to \infty.$$

This is a contradiction and so $y(t)$ is eventually positive. The proof is complete. □

Theorem 6.2.2. *Assume that* (6.2), (6.3), *and* (6.4) *hold and that* $Q(t)$ *is not identically equal to zero eventually. Suppose also that either*

$$P(t) + Q(t)\delta(t) > 0 \tag{6.8}$$

or

$$\delta(t) > 0 \quad \textit{and} \quad Q(s) \not\equiv 0 \quad \textit{for} \quad s \in [t, T^*], \tag{6.9}$$

where T^* *satisfies* $T^* - \delta(T^*) = t$. *Then every solution of* (6.1) *oscillates if and only if the corresponding differential inequality*

$$\frac{d^n}{dt^n}\Big(x(t) - P(t)x(t-\tau)\Big) + Q(t)f\Big(x\left(t - \delta(t)\right)\Big) \le 0 \tag{6.10}$$

has no eventually positive solutions.

Proof. The sufficiency is obvious. To prove the necessity, assume that x is an eventually positive solution of (6.10). Set y as in (6.5). From (6.8) we have that

$$y^{(n)}(t) \le -Q(t)f\Big(x(t - \delta(t))\Big) \le 0, \tag{6.11}$$

which implies that $y^{(i)}$, $0 \le i \le n-1$, is eventually monotone. As in the proof of Lemma 6.2.1, we have $y(t) > 0$. Therefore, there exists a nonnegative even integer $n^* \le n-1$ (recall $n \in \mathbb{N}$ is odd) such that

$$\begin{cases} y^{(i)}(t) > 0 & \text{for} \quad i \in \{0, 1, \dots, n^*\}, \\ (-1)^i y^{(i)}(t) > 0 & \text{for} \quad i \in \{n^*, \dots, n-1\}. \end{cases} \tag{6.12}$$

We consider the following possible cases.

Case I: $n^* = 0$. By using (6.12) and integrating (6.11) from t to ∞, we obtain

$$y(t) \ge \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} Q(s) f\Big(x\left(s - \delta(s)\right)\Big)\, ds,$$

i.e.,

$$x(t) \ge P(t)x(t-\tau) + \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} Q(s) f\Big(x\left(s - \delta(s)\right)\Big)\, ds. \tag{6.13}$$

Let $T > t_0$ be such that (6.13) holds for all $t \ge T$. Set

$$T_0 = \max\left\{\tau, \min_{t \ge T} \delta(t)\right\}.$$

Now we consider the set of functions

$$\Omega = \Big\{z \in C\left([T - T_0, \infty), \mathbb{R}^+\right) :\ 0 \le z(t) \le 1 \text{ for } t \ge T - T_0\Big\}$$

and define a mapping F on Ω as

$$(Fz)(t) = \begin{cases} \dfrac{1}{x(t)}\left[P(t)(zx)(t-\tau) + \displaystyle\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} Q(s) f\Big((zx)(s - \delta(s))\Big)\, ds\right] \\ \hfill \text{if} \quad t \ge T, \\ \dfrac{t - T + T_0}{T_0}(Fz)(T) + 1 - \dfrac{t - T + T_0}{T_0} \qquad \text{if} \quad T - T_0 \le t < T. \end{cases}$$

It is easy to see by using (6.13) that F maps Ω into itself, and for any $z \in \Omega$, we have $(Fz)(t) > 0$ for $T - T_0 \le t < T$.

Next we define the sequence $\{z_k\}_{k\in\mathbb{N}_0} \subset \Omega$ by

$$z_0(t) \equiv 1 \quad \text{and} \quad z_{k+1}(t) = (Fz_k)(t), \quad k \in \mathbb{N}_0 \quad \text{for} \quad t \geq T - T_0.$$

Then, by using (6.13) and a simple induction, we can easily see that

$$0 \leq z_{k+1}(t) \leq z_k(t) \leq 1 \quad \text{for} \quad t \geq T - T_0, \quad k \in \mathbb{N}_0.$$

Set

$$z(t) = \lim_{k\to\infty} z_k(t), \quad t \geq T - T_0.$$

Then it follows from Lebesgue's dominated convergence theorem that z satisfies

$$z(t) = \frac{1}{x(t)}\Big[P(t)z(t-\tau)x(t-\tau) \\ + \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}Q(s)f\Big(z\left(s-\delta(s)\right)x\left(s-\delta(s)\right)\Big)\,ds\Big]$$

for $t \geq T$ and

$$z(t) = \frac{t-T+T_0}{T_0}(Fz)(T) + 1 - \frac{t-T+T_0}{T_0} > 0$$

for $T - T_0 \leq t < T$. Again, set

$$w = zx.$$

Then w satisfies $w(t) > 0$ for $T - T_0 \leq t < T$ and

$$w(t) = P(t)w(t-\tau) + \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}Q(s)f\Big(w\left(s-\delta(s)\right)\Big)\,ds, \quad t \geq T. \tag{6.14}$$

Clearly, w is continuous on $[T-T_0, T)$. Then, by the method of steps, in view of (6.14) we see that w is continuous on $[T-T_0, \infty)$.

Finally, it remains to show that $w(t)$ is positive for all $t \geq T - T_0$. Assume that there exists $t^* \geq T - T_0$ such that $w(t) > 0$ for $T - T_0 \leq t < t^*$ and $w(t^*) = 0$. Then $t^* \geq T$ and by (6.14), we obtain

$$0 = w(t^*) = P(t^*)w(t^*-\tau) + \int_{t^*}^\infty \frac{(s-t^*)^{n-1}}{(n-1)!}Q(s)f\Big(w\left(s-\delta(s)\right)\Big)\,ds,$$

which implies

$$P(t^*) = 0 \quad \text{and} \quad Q(s)f\Big(w\left(s-\delta(s)\right)\Big) \equiv 0$$

for all $t \geq t^*$. This contradicts (6.8) or (6.9). Therefore $w(t) > 0$ on $[T-T_0, \infty)$. It is easy to see that w is a positive solution of (6.1), which implies that inequality (6.10) having no eventually positive solution is a necessary condition for the oscillation of all solutions of (6.1).

Case II: $2 \leq n^* \leq n-1$. By using (6.12) and integrating (6.11) from t to ∞, we obtain

$$y^{(n^*)}(t) \geq \int_t^\infty \frac{(s-t)^{n-n^*-1}}{(n-n^*-1)!}Q(s)f\Big(x\left(s-\delta(s)\right)\Big)\,ds. \tag{6.15}$$

Let $T > t_0$ be such that (6.15) holds for all $t \geq T$. Integrating (6.15) and using (6.12), we have

$$y(t) \geq \int_T^t \frac{(t-s)^{n^*-1}}{(n^*-1)!}\int_s^\infty \frac{(u-s)^{n-n^*-1}}{(n-n^*-1)!}Q(u)f\Big(x\left(u-\delta(u)\right)\Big)\,du ds$$

for $t \geq T$. That is,

$$x(t) \geq P(t)x(t-\tau)+\int_T^t \frac{(t-s)^{n^*-1}}{(n^*-1)!}\int_s^\infty \frac{(u-s)^{n-n^*-1}}{(n-n^*-1)!}Q(u)f\Big(x\left(u-\delta(u)\right)\Big)\,duds$$

for $t \geq T$, which, using a method similar to the proof of Case I, yields that (6.1) also has a positive solution. Hence the proof is complete. □

Now we give some applications of Theorem 6.2.2. We compare (6.1) with the equation

$$\frac{d^n}{dt^n}\Big(x(t)-P^*(t)x(t-\tau)\Big)+Q^*(t)f^*\Big(x\left(t-\delta(t)\right)\Big)=0 \tag{6.16}$$

and state the following comparison theorem.

Theorem 6.2.3. *Assume that* (6.2), (6.3), *and* (6.8) *or* (6.9) *hold. Furthermore assume that* f^* *satisfies* (6.3), *that*

$$\begin{cases} xf^*(x) \geq xf(x) \text{ for all } x \in \mathbb{R}, \\ P^*(t) \geq P(t) \quad \text{and} \quad Q^*(t) \geq Q(t), \end{cases} \tag{6.17}$$

and that there exists $t^* \geq t_0$ *with*

$$P^*(t^*+k\tau) \leq 1, \quad k \in \mathbb{N}_0.$$

If every solution of (6.1) *oscillates, then every solution of* (6.16) *also oscillates.*

Proof. Suppose the contrary and let x be an eventually positive solution of (6.16). Set

$$y(t)=x(t)-P^*(t)x(t-\tau).$$

Then by Lemma 6.2.1, we have

$$y^{(n)}(t) \leq 0 \quad \text{and} \quad y(t)>0.$$

Therefore there exists a nonnegative even integer $n^* \leq n-1$ such that (6.12) holds.

Suppose $n^*=0$. By integrating (6.16) from t to ∞, we obtain

$$x(t) \geq P^*(t)x(t-\tau)+\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}Q^*(s)f^*\Big(x\left(s-\delta(s)\right)\Big)\,ds.$$

Noting condition (6.17), we find

$$x(t) \geq P(t)x(t-\tau)+\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}Q(s)f\Big(x\left(s-\delta(s)\right)\Big)\,ds.$$

Using a method similar to the proof of Theorem 6.2.2, we see that (6.1) also has an eventually positive solution, and this is a contradiction.

Let $2 \leq n^* \leq n-1$. Integrating (6.16), we obtain

$$x(t) \geq P^*(t)x(t-\tau)$$
$$+\int_T^t \frac{(t-s)^{n^*-1}}{(n^*-1)!}\int_s^\infty \frac{(u-s)^{n-n^*-1}}{(n-n^*-1)!}Q^*(u)f^*\Big(x\left(u-\delta(u)\right)\Big)\,duds$$

for $t \geq T$. From condition (6.17), we find

$$x(t) \geq P(t)x(t-\tau) + \int_T^t \frac{(t-s)^{n^*-1}}{(n^*-1)!} \int_s^\infty \frac{(u-s)^{n-n^*-1}}{(n-n^*-1)!} Q(u) f\Big(x\left(u-\delta(u)\right)\Big)\, duds$$

for $t \geq T$, which again yields a contradiction. Hence the proof is complete. □

Theorem 6.2.4. *Assume that* (6.2) *and* (6.4) *hold and that there exist* $m, N \in \mathbb{N}_0$ *with* $m \leq N$ *and*

$$Q(s) \prod_{j=1}^{i_0} P\Big(s-\delta(s)-(j-1)\tau\Big) \not\equiv 0, \quad s \in [t, t+\delta(t)+i_0\tau], \quad i_0 \in \{m, \dots, N\}.$$

If every solution of the delay differential equation

$$y^{(n)}(t) + Q(t) \sum_{j=m}^{N} \left(\prod_{i=1}^{j} P\Big(t-\delta(t)-(i-1)\tau\Big)\, y\Big(t-\delta(t)-i\tau\Big) \right) = 0 \tag{6.18}$$

oscillates, then every solution of (6.1) *is oscillatory.*

Proof. Without loss of generality, assume that there exists an eventually positive solution x of (6.1). Then, from Lemma 6.2.1 we know that for all sufficiently large t we have $y(t) > 0$ and $y\left(t-\delta(t)-i\tau\right) > 0$ for $i \in \{1, \dots, N\}$. Thus we have

$$\begin{aligned} x(t) &= P(t)x(t-\tau) + y(t) \\ &= y(t) + P(t)\Big[y(t-\tau) + P(t-\tau)x(t-2\tau)\Big] \\ &\geq \sum_{j=m}^{N} \prod_{i=1}^{j} P\Big(t-(i-1)\tau\Big)\, y(t-i\tau), \end{aligned}$$

so

$$y^{(n)}(t) + Q(t) \sum_{j=m}^{N} \left(\prod_{i=1}^{j} P\Big(t-\delta(t)-(i-1)\tau\Big)\, y\Big(t-\delta(t)-i\tau\Big) \right) \leq 0.$$

Since $y(t) > 0$, it follows that (6.18) also has an eventually positive solution. This is a contradiction, and the proof is complete. □

Next, for the sake of completeness, we state the following lemma, which is very useful in the subsequent results.

Lemma 6.2.5 ([**68**]). *Suppose that* $\phi \in C^{(n)}([T,\infty), \mathbb{R}^+)$, $T \geq 0$, *such that* $\phi^{(i)}(t)$, $i < n$, *is of one sign in* $[T, \infty)$ *and* $\phi^{(n)}(t) \leq 0$, $t \geq T$. *Then* $\mu > 0$ *implies that*

$$\phi(t-\mu) \geq \frac{\mu^{n-1}}{(n-1)!} \phi^{(n-1)}(t), \quad t \geq T + 2\mu.$$

Theorem 6.2.6. *Assume that* (6.3) *holds,* $0 \leq \alpha \leq P(t) \leq \beta < 1$, $n > 1$ *is an odd integer,* $\delta(t) \equiv \delta > 0$, *and there exists* $\mu \in (0,1)$ *such that all solutions of the first order delay differential equation*

$$y'(t) + Q(t) f\left(\frac{\mu}{(1-\alpha)(n-1)!} \left(\frac{n-1}{n}\right)^{n-1} y\left(t - \frac{\delta}{n}\right) \right) = 0 \tag{6.19}$$

are oscillatory. Then all solutions of (6.1) *are oscillatory.*

Proof. Assume that x is an eventually positive solution of (6.1). Set y as in (6.5). From Lemma 6.2.1 we have $y(t) > 0$ eventually. Since all solutions of (6.19) being oscillatory implies that $\int^{\infty} Q(s)ds = \infty$ by [**284**, Lemma 2.1], we have $y'(t) < 0$ and $y^{(n-1)}(t) > 0$ eventually. Suppose that $y(t) > 0$ for $t \geq t_1$. There exists $k \in \mathbb{N}$ such that $1 - \alpha^{k+1} > \mu$. Thus we obtain by using $y'(t) < 0$,

$$\begin{aligned} x(t) &= y(t) + P(t)x(t-\tau) \;\geq\; y(t) + \alpha x(t-\tau) \\ &\geq \sum_{i=0}^{k} \alpha^i y(t - i\tau) + \alpha^{k+1} x\Big(t - (k+1)\tau\Big) \\ &\geq y(t)\sum_{i=0}^{k} \alpha^i \;\geq\; \frac{\mu}{1-\alpha} y(t) \end{aligned} \tag{6.20}$$

for $t \geq t_1 + (k+1)\tau$. From (6.1), (6.20), and the fact that $y(t) < x(t)$ for $t \geq t_1 + \tau$, we obtain

$$y^{(n)}(t) + Q(t) f\left(\frac{\mu}{1-\alpha} y(t-\delta)\right) \leq 0, \quad t \geq t_1 + (k+1)\tau + \delta.$$

By Lemma 6.2.5, we have eventually

$$y^{(n)}(t) + Q(t) f\left(\frac{\mu}{(1-\alpha)(n-1)!}\left(\frac{n-1}{n}\right)^{n-1} \delta^{n-1} y^{(n-1)}\left(t - \frac{\delta}{n}\right)\right) \leq 0. \tag{6.21}$$

From (6.21) we see that $y^{(n-1)}$ is an eventually positive solution of the first order differential inequality. By Theorem 6.2.2, (6.19) has an eventually positive solution. This contradiction completes the proof. $\square$

Corollary 6.2.7. *Assume that P and n satisfy the hypotheses of Theorem 6.2.6. If $\delta > 0$, $Q(t) \geq 0$, and either*

$$\liminf_{t\to\infty} \int_{t-\frac{\delta}{n}}^{t} \delta^{n-1} Q(s)ds > \frac{1}{e}\left(\frac{n}{n-1}\right)^{n-1} (1-\alpha)(n-1)!$$

or

$$\limsup_{t\to\infty} \int_{t-\frac{\delta}{n}}^{t} \delta^{n-1} Q(s)ds > \left(\frac{n}{n-1}\right)^{n-1} (1-\alpha)(n-1)!$$

holds, then every solution of (6.19) *is oscillatory.*

Example 6.2.8. Consider the neutral delay differential equation

$$\left(x(t) - \frac{23}{24}x(t-\tau)\right)^{(5)} + 5\left(\frac{626}{256e} + e^{-t}\right) x(t-1) = 0, \quad t \geq 1. \tag{6.22}$$

Here $\tau > 0$, $P(t) = \frac{23}{24}$, $Q(t) = 5\left(\frac{626}{256e} + e^{-t}\right)$, $\delta = 1$, $n = 5$, $\alpha = \frac{23}{24}$, $f(x) = x$, and

$$\liminf_{t\to\infty} \int_{t-\frac{\delta}{n}}^{t} (t-s)^{n-1} Q(s)ds = \frac{5}{2e} < (n-1)!(1-\alpha) = 1.$$

Then

$$\liminf_{t\to\infty} \int_{t-\frac{\delta}{n}}^{t} \delta^{n-1} Q(s)ds = \frac{626}{256e} > \frac{625}{256e} = \frac{1}{e}\left(\frac{n}{n-1}\right)^{n-1} (1-\alpha)(n-1)!$$

so that Corollary 6.2.7 implies that every solution of (6.22) is oscillatory.

Remark 6.2.9. For the general equation

$$\frac{d^n}{dt^n}\Big(x(t) - P(t)x(t-\tau)\Big) + \sum_{i=1}^{m} Q_i(t) f_i\Big(x(t-\delta)\Big) = 0,$$

the above results also hold if each f_i, $1 \le i \le m$, satisfies (6.3).

6.3. Oscillation Criteria for Neutral Equations

In this section we are concerned with the odd order neutral delay differential equation

$$\frac{d^n}{dt^n}\Big(y(t) - py(t-\tau)\Big) + \sum_{i=1}^{m} q_i(t) y(t-\sigma_i) = 0 \tag{6.23}$$

and mainly examine the oscillatory characteristics of (6.23) in the cases $p \ge 1$ and $0 \le p < 1$. For the sake of convenience, let

$$z(t) = y(t) - py(t-\tau).$$

We also define

$$\sigma_* = \min\{\sigma_1, \sigma_2, \dots, \sigma_m\}, \quad \sigma^* = \max\{\sigma_1, \sigma_2, \dots, \sigma_m\},$$

and the set

$$\mathcal{L} = \left\{\tau, \frac{\tau-\sigma_1}{n}, \frac{\tau-\sigma_2}{n}, \dots, \frac{\tau-\sigma_m}{n}\right\}.$$

Lemma 6.3.1. *Assume $p \in [1,\infty)$ and $\sigma_*, \tau, \tau-\sigma^* \in (0,\infty)$, Suppose further that $q_i \in C([t_0,\infty),\mathbb{R}^+)$, $\sum_{i=1}^m q_i(t) \not\equiv 0$ on any subinterval of $[t_0,\infty)$, and*

$$\liminf_{t\to\infty} \int_t^{t+\frac{\tau-\sigma^*}{n}} \sum_{i=1}^{m} q_i(s)ds > 0. \tag{6.24}$$

(i) *If y is eventually positive, then either $z^{(i)}$ is decreasing with*

$$z^{(i)}(t) \to -\infty \quad as \quad t \to \infty \tag{6.25}$$

for $i \in \{0,1,2,\dots,n-1\}$, or $z^{(i)}$ is monotone and

$$z^{(i)}(t) \to 0 \quad as \quad t \to \infty \quad and \quad z^{(i)}(t) z^{(i+1)}(t) < 0 \tag{6.26}$$

for $i \in \{0,1,2,\dots,n-1\}$.

(ii) *If y is eventually negative, then either $z^{(i)}$ is increasing with*

$$z^{(i)}(t) \to \infty \quad as \quad t \to \infty$$

for $i \in \{0,1,2,\dots,n-1\}$, or (6.26) holds.

(iii) *If n is odd and (6.26) holds, then $z(t) > 0$ for $y(t) > 0$, and $z(t) < 0$ for $y(t) < 0$ eventually.*

Proof. Condition (6.24) implies that

$$\int_{t_0}^{\infty} \sum_{i=1}^{m} q_i(t)dt = \infty.$$

By [**109**, Lemma 2], it is easy to see that (i), (ii), and (iii) hold. $\square$

Lemma 6.3.2 ([**166**]). *Let $\phi \in C^n([t_0,\infty),\mathbb{R})$ be of constant sign and suppose*

$$\phi(t)\phi^{(n)}(t) < 0 \quad (\phi(t)\phi^{(n)}(t) > 0).$$

Then there exist $t^* \geq t_0$ and $l \in \{0,1,\dots,n\}$ such that $l+n$ is odd (even), and for $t \geq t^*$ the following inequalities hold:

$$\begin{cases} \phi(t)\phi^{(i)}(t) > 0 & \text{for} \quad i \in \{0,1,\dots,l\}, \\ (-1)^{l+i}\phi(t)\phi^{(i)}(t) > 0 & \text{for} \quad i \in \{l+1,\dots,n\}. \end{cases}$$

Lemma 6.3.3. *Suppose that $\phi \in C^n([t_0,\infty),\mathbb{R})$, $t_0 \geq 0$, such that $\phi^{(i)}(t)$, $i < n$, is of one sign in $[t_0,\infty)$ and $\phi^{(n)}(t) \leq 0$, $t \geq t_0$. Then $\alpha > 0$ implies the following:*

(i) *If $\phi(t) > 0$, then*

$$\phi(t-\alpha) \geq \frac{\alpha^{n-1}}{(n-1)!}\phi^{(n-1)}(t), \quad t \geq t_0 + 2\alpha.$$

(ii) *If $\phi(t) < 0$, then*

$$\phi(t+\alpha) \leq \frac{\alpha^{n-1}}{(n-1)!}\phi^{(n-1)}(t), \quad t \geq t_0. \tag{6.27}$$

Proof. Part (i) is Lemma 6.2.5. Now we shall prove part (ii). Since $\phi^{(n)}(t) \leq 0$, by Lemma 6.3.2, there exists an integer $k \in \{0,1,\dots,n\}$ such that

$$\begin{cases} \phi^{(j)}(t) < 0 & \text{for} \quad j \leq k, \\ \phi^{(j)}(t)\phi^{(j+1)}(t) < 0 & \text{for} \quad k \leq j < n. \end{cases}$$

By Lemma 6.3.2, we have that $k+n$ is even. Hence we may note that k is odd (even) if and only if n is odd (even).

If $k = n-1$ or $k = n$, then, expanding $\phi(t)$ by Taylor's theorem, there exists $\xi \in (t, t+\alpha)$ such that

$$\phi(t+\alpha) = \sum_{j=0}^{n-1}\frac{\alpha^j}{j!}\phi^{(j)}(t) + \frac{\alpha^n}{n!}\phi^{(n)}(\xi) \leq \frac{\alpha^{n-1}}{(n-1)!}\phi^{(n-1)}(t).$$

This shows that (6.27) holds.

If $k < n-1$, then, by Lemma 6.3.2, we have $\phi^{(n-1)}(t) > 0$. Obviously, (6.27) holds also. The proof is complete. $\square$

Lemma 6.3.4. *Assume $p \in [0,1)$ and* (6.24)*. Then z satisfies* (6.26)*.*

Proof. Since (6.24) holds, we have

$$\int_{t_0}^{\infty}\sum_{i=1}^{m} q_i(t)dt = \infty.$$

By [**109**, Lemma 2 (d)], we have $\lim_{t\to\infty} z(t) = 0$. The proof is complete. $\square$

Theorem 6.3.5. *Assume $p \in [1,\infty)$, $n \in \mathbb{N}$ is odd, and $\sigma_*, \tau, \tau - \sigma^* \in (0,\infty)$. Suppose further that* (6.24) *holds and for $\mu > 0$ and $l \in \mathcal{L}$,*

$$\liminf_{t\to\infty}\left\{\frac{1}{p}e^{\mu\tau} + \frac{\left(\frac{n-1}{n}\right)^{n-1}}{lp\mu(n-1)!}\sum_{i=1}^{m}(\tau-\sigma_i)^{n-1}e^{\mu\left(\frac{\tau-\sigma_i}{n}\right)}\int_t^{t+l} q_i(s)ds\right\} > 1 \tag{6.28}$$

and

$$q_i(t) \le q_i(t-\tau), \quad i \in \{1, 2, \ldots, m\}. \tag{6.29}$$

Then every solution of (6.23) *oscillates.*

Proof. Suppose the contrary and let y be an eventually positive solution of (6.23). Then from (6.23), $z^{(n)}(t) \le 0$ eventually. By Lemma 6.3.1 (i), we have that (6.25) or (6.26) holds. If (6.26) holds, then, by Lemma 6.3.1 (iii), we have $z(t) > 0$ eventually. Then there exist $M > 0$ and $T \ge t_0$ such that $y(t-\sigma_i) \ge M$ for $1 \le i \le m$ and $t \ge T$. From (6.23), we have

$$z^{(n)}(t) + M \sum_{i=1}^{m} q_i(t) \le 0. \tag{6.30}$$

Condition (6.24) implies that

$$\int_T^\infty \sum_{i=1}^m q_i(t)dt = \infty.$$

Integrating (6.30) from T to $t \ge T$ provides

$$z^{(n-1)}(t) - z^{(n-1)}(T) \le -M \int_T^t \sum_{i=1}^m q_i(s)ds.$$

Letting $t \to \infty$, we obtain $\lim_{t\to\infty} z^{(n-1)}(t) = -\infty$ and hence $\lim_{t\to\infty} z(t) = -\infty$, which is a contradiction. Hence (6.25) holds.

From (6.23) and (6.29) we have

$$\begin{aligned}
z^{(n)}(t) &= -\sum_{i=1}^m q_i(t)y(t-\sigma_i) \\
&= -\sum_{i=1}^m q_i(t)z(t-\sigma_i) - p\sum_{i=1}^m q_i(t)y(t-\tau-\sigma_i) \\
&\ge -\sum_{i=1}^m q_i(t-\tau)z(t-\sigma_i) - p\sum_{i=1}^m q_i(t-\tau)y(t-\tau-\sigma_i) \\
&= -\sum_{i=1}^m q_i(t-\tau)z(t-\sigma_i) + pz^{(n)}(t-\tau).
\end{aligned}$$

Hence

$$z^{(n)}(t) \le \frac{z^{(n)}(t+\tau)}{p} + \frac{1}{p}\sum_{i=1}^m q_i(t)z(t+\tau-\sigma_i). \tag{6.31}$$

Dividing both sides of (6.31) by $z^{(n-1)}(t)$ and noting that $z^{(n-1)}(t)$ is negative, we have

$$\frac{z^{(n)}(t)}{z^{(n-1)}(t)} \ge \frac{z^{(n)}(t+\tau)}{pz^{(n-1)}(t)} + \frac{1}{pz^{(n-1)}(t)}\sum_{i=1}^m q_i(t)z(t+\tau-\sigma_i). \tag{6.32}$$

Using Lemma 6.3.3 (ii) with $\alpha = \frac{n-1}{n}(\tau-\sigma_i)$ for the term $z(t+\tau-\sigma_i)$, $1 \le i \le m$, we find

$$z(t+\tau-\sigma_i) \le \frac{(\tau-\sigma_i)^{n-1}}{(n-1)!}\left(\frac{n-1}{n}\right)^{n-1} z^{(n-1)}\left(t+\frac{\tau-\sigma_i}{n}\right). \tag{6.33}$$

Let

$$\lambda = \frac{z^{(n)}}{z^{(n-1)}}.$$

Then $\lambda(t) > 0$. By (6.32), (6.33), we obtain

$$\begin{aligned}\lambda(t) \geq{}& \frac{\lambda(t+\tau)}{p}\exp\left(\int_t^{t+\tau}\lambda(s)ds\right)\\&+\frac{1}{p(n-1)!}\left(\frac{n-1}{n}\right)^{n-1}\sum_{i=1}^m q_i(t)\,(\tau-\sigma_i)^{n-1}\exp\left(\int_t^{t+\frac{\tau-\sigma_i}{n}}\lambda(s)ds\right).\end{aligned}$$

Define $\{\lambda_k(t)\}$ for $k \in \mathbb{N}$ and $t \geq T$ and a sequence of numbers $\{\mu_k\}$ for $k \in \mathbb{N}$ as follows: $\lambda_1(t) \equiv 0$,

$$\begin{aligned}\lambda_{k+1}(t) ={}& \frac{\lambda_k(t+\tau)}{p}\exp\left(\int_t^{t+\tau}\lambda_k(s)ds\right)\\&+\frac{1}{p(n-1)!}\left(\frac{n-1}{n}\right)^{n-1}\sum_{i=1}^m q_i(t)(\tau-\sigma_i)^{n-1}\exp\left(\int_t^{t+\frac{\tau-\sigma_i}{n}}\lambda_k(s)ds\right),\end{aligned}$$

$\mu_1 = 0$,

$$\begin{aligned}\mu_{k+1} = \inf_{t\geq T}\Bigg\{\min_{l\in\mathcal{L}}\Bigg[&\frac{\mu_k}{p}e^{\mu_k\tau}+\frac{1}{pl(n-1)!}\left(\frac{n-1}{n}\right)^{n-1}\\&\times\sum_{i=1}^m(\tau-\sigma_i)^{n-1}e^{\mu_k\left(\frac{\tau-\sigma_i}{n}\right)}\int_t^{t+l}q_i(s)ds\Bigg]\Bigg\}.\end{aligned}\tag{6.34}$$

It is easy to see that

(i) $0 = \mu_1 < \mu_2 < \ldots$;
(ii) $\lambda_k(t) \leq \lambda(t)$ for $k \in \mathbb{N}$;
(iii) $\frac{1}{l}\int_t^{t+l}\lambda_k(s)ds \geq \mu_k$ for $k \in \mathbb{N}$, $t \geq T$, and $l \in \mathcal{L}$.

Since (i) and (ii) are obvious, we shall consider only (iii). It is evident for $k = 1$ that (iii) holds. If (iii) is true for $k \in \mathbb{N}$, then

$$\begin{aligned}\frac{1}{l}\int_t^{t+l}\lambda_{k+1}(s)ds ={}& \frac{1}{pl}\int_t^{t+l}\lambda_k(s+\tau)\exp\left(\int_s^{s+\tau}\lambda_k(u)du\right)ds\\&+\frac{\left(\frac{n-1}{n}\right)^{n-1}}{pl(n-1)!}\int_t^{t+l}\sum_{i=1}^m(\tau-\sigma_i)^{n-1}q_i(s)\exp\left(\int_s^{s+\frac{\tau-\sigma_i}{n}}\lambda_k(u)du\right)ds\\\geq{}& \frac{\mu_k}{p}e^{\mu_k\tau}+\frac{1}{pl(n-1)!}\left(\frac{n-1}{n}\right)^{n-1}\sum_{i=1}^m(\tau-\sigma_i)^{n-1}\,e^{\mu_k\left(\frac{\tau-\sigma_i}{n}\right)}\int_t^{t+l}q_i(s)ds\\\geq{}& \mu_{k+1}.\end{aligned}$$

Thus (iii) holds for $k \in \mathbb{N}$. From (6.28) and (6.34), it is easy to see that there exists $\alpha > 1$ such that $\mu_{k+1} \geq \alpha\mu_k$. Hence $\lim_{k\to\infty}\mu_k = \infty$. From properties (ii) and (iii), we have

$$\frac{z^{(n-1)}(t+l)}{z^{(n-1)}(t)} = \exp\left(\int_t^{t+l}\lambda(s)ds\right) \to \infty \quad \text{as} \quad t\to\infty.\tag{6.35}$$

On the other hand, since $y(t) > 0$, we have $z(t) > -py(t-\tau)$. Substituting the last inequality into (6.23), we find

$$z^{(n)}(t) = -\sum_{i=1}^{m} q_i(t)y(t-\sigma_i) < \frac{1}{p}\sum_{i=1}^{m} q_i(t)z\left(t+\tau-\sigma_i\right).$$

Furthermore, by Lemma 6.3.3 (ii) (as in (6.33)), we have

$$\begin{aligned} z^{(n)}(t) &< \frac{1}{p}\sum_{i=1}^{m} q_i(t)z(t+\tau-\sigma_i) \\ &\le \frac{1}{p(n-1)!}\left(\frac{n-1}{n}\right)^{n-1}\sum_{i=1}^{m} q_i(t)\left(\tau-\sigma_i\right)^{n-1} z^{(n-1)}\left(t+\frac{\tau-\sigma_i}{n}\right) \\ &\le \frac{(1-1/n)^{n-1}}{p(n-1)!}\sum_{i=1}^{m} q_i(t)(\tau-\sigma_i)^{n-1} z^{(n-1)}\left(t+\frac{\tau-\sigma^*}{n}\right). \end{aligned}$$

Set

$$w = z^{(n-1)}.$$

Then

$$w'(t) < \frac{(1-1/n)^{n-1}}{p(n-1)!}\sum_{i=1}^{m} q_i(t)(\tau-\sigma_i)^{n-1} w\left(t+\frac{\tau-\sigma^*}{n}\right). \tag{6.36}$$

Integrating (6.36) from t to $t+(\tau-\sigma^*)/(2n)$, we find

$$\begin{aligned} w\left(t+\frac{\tau-\sigma^*}{2n}\right) &\le w\left(t+\frac{\tau-\sigma^*}{2n}\right) - w(t) \\ &\le \frac{(1-1/n)^{n-1}}{p(n-1)!}\int_t^{t+\frac{\tau-\sigma^*}{2n}}\sum_{i=1}^{m} q_i(s)(\tau-\sigma_i)^{n-1} w\left(s+\frac{\tau-\sigma^*}{n}\right) ds \\ &\le w\left(t+\frac{\tau-\sigma^*}{n}\right)\frac{(1-1/n)^{n-1}}{p(n-1)!}\int_t^{t+\frac{\tau-\sigma^*}{2n}}\sum_{i=1}^{m} q_i(s)(\tau-\sigma_i)^{n-1} ds. \end{aligned}$$

Hence

$$\frac{w\left(t+\frac{\tau-\sigma^*}{n}\right)}{w\left(t+\frac{\tau-\sigma^*}{2n}\right)}\frac{\left(\frac{n-1}{n}\right)^{n-1}}{p(n-1)!}\int_t^{t+\frac{\tau-\sigma^*}{2n}}\sum_{i=1}^{m} q_i(s)(\tau-\sigma_i)^{n-1} ds \le 1. \tag{6.37}$$

From (6.24) and (6.37), we obtain that

$$\frac{w\left(t+\frac{\tau-\sigma^*}{n}\right)}{w\left(t+\frac{\tau-\sigma^*}{2n}\right)}$$

is bounded, and this contradicts (6.35). This completes the proof. □

Remark 6.3.6. The application of Lemma 6.3.3 with $\alpha = \frac{n-1}{n}(\tau-\sigma_i)$ in (6.32) is not totally unjustified. Suppose that Lemma 6.3.3 is applied to (6.32) with t replaced by $t-\alpha_i$ in $z(t+\tau-\sigma_i)$. Then the reduced inequality is

$$\frac{\alpha_i^{n-1}}{(n-1)!} z^{n-1}\left(t-\alpha_i+\tau-\sigma_i\right) \ge z(t+\tau-\sigma_i).$$

If we choose α_i such that $0 < \alpha_i < \tau-\sigma_i$, then the function

$$G(\alpha_i) = \alpha_i^{n-1}\left(\tau-\sigma_i-\alpha_i\right)$$

attains its maximum value at $\alpha_i = \frac{n-1}{n}(\tau - \sigma_i)$. A similar remark also holds for Theorem 6.3.11 below.

Remark 6.3.7. Condition (6.24) can be replaced by

$$\liminf_{t\to\infty} \int_t^{t+\frac{\tau-\sigma_i}{n}} q_i(s)ds > 0 \quad \text{for some} \quad i \in \{1, 2, \dots, m\}.$$

Remark 6.3.8. When $n = 1$, condition (6.28) becomes

$$\liminf_{t\to\infty} \left\{ \frac{1}{p} e^{\mu\tau} + \frac{1}{\mu p l} \sum_{i=1}^{m} e^{\mu(\tau-\sigma_i)} \int_t^{t+l} q_i(s)ds \right\} > 1.$$

Corollary 6.3.9. *Assume that for $l \in \mathcal{L}$,*

$$\liminf_{t\to\infty} \left\{ \frac{K}{(n-1)!} \sum_{k=0}^{\infty} \frac{e\left[\left(k+\frac{1}{n}\right)\tau - \frac{\sigma_i}{n}\right]}{p^{k+1}} \right\} > 1,$$

where

$$K = \left(\frac{n-1}{n}\right)^{n-1} \sum_{i=1}^{m} (\tau - \sigma_i)^{n-1} \left(\frac{1}{l} \int_t^{t+l} q_i(s)ds \right).$$

Then every solution of (6.23) *is oscillatory.*

Proof. In fact, for $\frac{1}{p}e^{\mu\tau} < 1$, we have

$$\begin{aligned}
&\liminf_{t\to\infty} \frac{\left(1-\frac{1}{n}\right)^{n-1}}{\mu p l (n-1)!} \sum_{i=1}^{m} e^{\mu\left(\frac{\tau-\sigma_i}{n}\right)} (\tau-\sigma_i)^{n-1} \int_t^{t+l} q_i(s)ds \left(1 - \frac{1}{p}e^{\mu\tau}\right)^{-1} \\
= \;& \liminf_{t\to\infty} \frac{\left(1-\frac{1}{n}\right)^{n-1}}{\mu p l (n-1)!} \sum_{i=1}^{m} e^{\mu\left(\frac{\tau-\sigma_i}{n}\right)} (\tau-\sigma_i)^{n-1} \int_t^{t+l} q_i(s)ds \sum_{k=0}^{\infty} \frac{e^{k\mu\tau}}{p^k} \\
\geq \;& \liminf_{t\to\infty} \frac{\left(1-\frac{1}{n}\right)^{n-1}}{(n-1)!} \sum_{i=1}^{m} (\tau-\sigma_i)^{n-1} \sum_{k=0}^{\infty} \left(\frac{1}{l} \int_t^{t+l} q_i(s)ds \right) \frac{e\left(\left(k+\frac{1}{n}\right)\tau - \frac{\sigma_i}{n}\right)}{p^{k+1}} \\
> \;& 1,
\end{aligned}$$

i.e., (6.28) holds. By Theorem 6.3.5, every solution of (6.23) oscillates. If $\frac{1}{p}e^{\mu\tau} \geq 1$, then (6.28) is satisfied also. Therefore the corollary holds. $\square$

Example 6.3.10. Consider the equation

$$\frac{d^3}{dt^3}\Big(y(t) - py(t-\tau)\Big) + \left(1 + \frac{1}{t}\right) y(t-\sigma) = 0, \quad t \geq 1,$$

where $p \geq 1$, $\tau > 0$, $\sigma > 0$, and $\tau - \sigma > 0$. Let $q(t) = 1 + \frac{1}{t}$. It is easy to see that for all $l > 0$,

$$\lim_{t\to\infty} \frac{1}{l} \int_t^{t+l} q(s)ds = \lim_{t\to\infty} \frac{1}{l} \int_t^{t+l} \left(1 + \frac{1}{s}\right) ds = \lim_{t\to\infty} \left(1 + \frac{1}{l} \ln\left(1 + \frac{l}{t}\right)\right) = 1.$$

According to Theorem 6.3.5, the above equation is oscillatory if for all $\mu > 0$,

$$\frac{1}{p}e^{\mu\tau} + \frac{2}{9p\mu}(\tau-\sigma)^2 e^{\mu\left(\frac{\tau-\sigma}{3}\right)} > 1.$$

Next we establish a sufficient condition for the oscillation of (6.23) in the case $p \in [0, 1)$.

Theorem 6.3.11. *Assume $p \in [0,1)$, $n \in \mathbb{N}$ is odd, and $\tau, \sigma_* \in (0,\infty)$. Assume further that*

$$\liminf_{t\to\infty} \int_t^{t+\frac{\sigma^*}{n}} \sum_{i=1}^m q_i(s)ds > 0$$

holds and for $\mu > 0$, $l \in \left\{\frac{\sigma_i}{n} : 1 \le i \le m\right\}$,

$$\liminf_{t\to\infty} \left\{ pe^{\mu\tau} + \frac{\left(\frac{n-1}{n}\right)^{n-1}}{l\mu(n-1)!} \sum_{i=1}^m \sigma_i^{n-1} e^{\mu\frac{\sigma_i}{n}} \int_t^{t+l} q_i(s)ds \right\} > 1$$

and

$$q_i(t) \ge q_i(t-\tau) \quad \textit{for} \quad i \in \{1,2,\ldots,m\}. \tag{6.38}$$

Then every solution of (6.23) *oscillates.*

Proof. Suppose the contrary and let y be an eventually positive solution of (6.23). Then from (6.23), $z^{(n)}(t) \le 0$ eventually. By Lemmas 6.3.1 and 6.3.4 we have $z(t) > 0$ eventually and $\lim_{t\to\infty} z(t) = 0$. Hence $z'(t) < 0$ eventually. In view of Lemma 6.3.2 we obtain $z^{(n-1)}(t) > 0$ eventually. From (6.23) and (6.38) we have

$$\begin{aligned} z^{(n)}(t) &= -\sum_{i=1}^m q_i(t)y(t-\sigma_i) \\ &= -\sum_{i=1}^m q_i(t)z(t-\sigma_i) - p\sum_{i=1}^m q_i(t)y(t-\tau-\sigma_i) \\ &\le -\sum_{i=1}^m q_i(t)z(t-\sigma_i) - p\sum_{i=1}^m q_i(t-\tau)y(t-\tau-\sigma_i) \\ &= -\sum_{i=1}^m q_i(t)z(t-\sigma_i) + pz^{(n)}(t-\tau). \end{aligned}$$

Hence

$$z^{(n)}(t) \le pz^{(n)}(t-\tau) - \sum_{i=1}^m q_i(t)z(t-\sigma_i). \tag{6.39}$$

Dividing both sides of (6.39) by $-z^{(n-1)}(t)$ and noting that $z^{(n-1)}(t) > 0$ eventually, we have

$$\frac{z^{(n)}(t)}{-z^{(n-1)}(t)} \ge \frac{pz^{(n)}(t-\tau)}{-z^{(n-1)}(t)} + \frac{1}{z^{(n-1)}(t)} \sum_{i=1}^m q_i(t)z(t-\sigma_i). \tag{6.40}$$

Using Lemma 6.3.3 (i) with $\alpha = \frac{n-1}{n}\sigma_i$ for the term $z(t-\sigma_i)$, $1 \le i \le m$, we have

$$z(t-\sigma_i) \ge \frac{\sigma_i^{n-1}}{(n-1)!} \left(\frac{n-1}{n}\right)^{n-1} z^{(n-1)}\left(t - \frac{\sigma_i}{n}\right). \tag{6.41}$$

Let

$$\lambda(t) = -\frac{z^{(n)}(t)}{z^{(n-1)}(t)} > 0. \tag{6.42}$$

By (6.40), (6.41), and (6.42), we obtain

$$\lambda(t) \geq p\lambda(t-\tau)\exp\left(\int_{t-\tau}^{t}\lambda(s)ds\right) + \frac{1}{(n-1)!}\left(\frac{n-1}{n}\right)^{n-1}\sum_{i=1}^{m} q_i(t)\sigma_i^{n-1}\exp\left(\int_{t-\frac{\sigma_i}{n}}^{t}\lambda(s)ds\right).$$

Repeating the proof of Theorem 6.3.5, we can obtain that the conclusion is valid. The proof is complete. □

In the following we establish a comparison theorem. We shall consider the equation

$$\frac{d^n}{dt^n}\Big(y(t) - P(t)y(t-\tau)\Big) + Q(t)y(t-\sigma) = 0 \tag{6.43}$$

and the equation

$$\frac{d^n}{dt^n}\Big(y(t) - P^*(t)y(t-\tau)\Big) + Q^*(t)y(t-\sigma) = 0. \tag{6.44}$$

Lemma 6.3.12 ([**290**]). *Assume that $P(t) \geq 1$ and $Q \in C([t_0,\infty),\mathbb{R}^+)$ such that*

$$\int_{t_0}^{\infty} s^n Q(s)\int_s^{\infty}(u-s)^{n-1}Q(u)duds = \infty. \tag{6.45}$$

If y is an eventually positive solution of (6.43) *and $z(t) = y(t) - P(t)y(t-\tau)$, then $z(t) < 0$ eventually.*

Theorem 6.3.13. *Suppose that $P(t) \geq 1$ and $Q^* \in C([t_0,\infty),\mathbb{R}^+)$ satisfies* (6.45). *Suppose further that*

$$P(t) \leq P^*(t) \quad \text{and} \quad Q(t) \geq Q^*(t). \tag{6.46}$$

Then oscillation of (6.44) *implies oscillation of* (6.43).

Proof. Suppose the conclusion is not valid and let y be an eventually positive solution of (6.43). Let $z(t) = y(t) - P(t)y(t-\tau)$. Then, by Lemma 6.3.12, we have $z(t) < 0$ and $z^{(n)}(t) \leq 0$ eventually.

Since n is odd, by [**166**, Lemma 5.2.2], we have either

$$z^{(i)}(t) < 0 \quad \text{for all} \quad i \in \{0,1,\ldots,n\} \tag{6.47}$$

or

$$\begin{cases} z^{(i)}(t) < 0 & \text{for} \quad i \in \{0,1,\ldots,l\}, \\ (-1)^i z^{(i)}(t) \leq 0 & \text{for} \quad i \in \{l+1,\ldots,n\}, \end{cases} \tag{6.48}$$

where l is odd.

Let $T_1 \geq t_0$ be such that

$$z^{(n)}(t) = -Q(t)y(t-\sigma) \tag{6.49}$$

holds for all $t \geq T_1$. If (6.47) holds, then integrating (6.49) from T_1 to t provides

$$z(t) \leq -\int_{T_1}^{t}\int_{T_1}^{s_{n-1}}\cdots\int_{T_1}^{s_1} Q(u)y(u-\sigma)duds_1\cdots ds_{n-1},$$

i.e.,

$$P(t)y(t-\tau) \geq y(t) + \int_{T_1}^{t}\int_{T_1}^{s_{n-1}}\cdots\int_{T_1}^{s_1} Q(u)y(u-\sigma)duds_1\cdots ds_{n-1}.$$

Since $P(t) \geq 1$, we find

$$y(t) \geq \frac{1}{P(t+\tau)}\left[y(t+\tau) + \int_{T_1}^{t+\tau}\int_{T_1}^{s_{n-1}}\cdots\int_{T_1}^{s_1} Q(u)y(u-\sigma)duds_1\cdots ds_{n-1}\right].$$

By condition (6.46), we have

$$\begin{aligned}(6.50)\quad y(t) \geq \frac{1}{P^*(t+\tau)}\Big[&y(t+\tau)\\ &+\int_{T_1}^{t+\tau}\int_{T_1}^{s_{n-1}}\cdots\int_{T_1}^{s_1} Q^*(u)y(u-\sigma)duds_1\cdots ds_{n-1}\Big]\end{aligned}$$

for $t \geq T_1+\tau$. Let $T > T_1+\tau$ such that (6.50) holds for all $t \geq T$. Set

$$T_0 = \max\{\tau,\sigma\}.$$

Now we consider the set of functions

$$\Omega = \Big\{w \in C([T-T_0,\infty),\mathbb{R}^+):\ 0 \leq w(t) \leq 1 \text{ for } t \geq T-T_0\Big\}$$

and define a mapping F on Ω as

$$(Fw)(t) = \begin{cases} \dfrac{1}{y(t)P^*(t+\tau)}\Big[(wy)(t+\tau) + \displaystyle\int_{T_1}^{t+\tau}\int_{T_1}^{s_{n-1}}\cdots\int_{T_1}^{s_1} Q^*(u)w(u-\sigma) & \\ \qquad\times\, y(u-\sigma)duds_1\cdots ds_{n-1}\Big] & \text{if} \quad t \geq T,\\ \dfrac{t-T+T_0}{T_0}(Fw)(T) + \left(1-\dfrac{t-T+T_0}{T_0}\right) & \text{if} \quad T-T_0 \leq t < T.\end{cases}$$

It is easy to see by using (6.50) that F maps Ω into itself, and for any $w \in \Omega$, we have $(Fw)(t) > 0$ for $T-T_0 \leq t < T$.

Next we define a sequence $\{w_k\}_{k\in\mathbb{N}_0} \subset \Omega$ by

$$w_0(t) = 1 \quad \text{and} \quad w_{k+1}(t) = (Fw_k)(t), \quad k \in \mathbb{N}_0 \quad \text{for} \quad t \geq T-T_0.$$

Then, by using (6.50) and a simple induction, we can easily see that

$$0 \leq w_{k+1}(t) \leq w_k(t) \leq 1 \quad \text{for} \quad t \geq T-T_0 \quad \text{and} \quad k \in \mathbb{N}_0.$$

Set

$$w(t) = \lim_{k\to\infty} w_k(t), \quad t \geq T-T_0.$$

Then it follows from Lebesgue's dominated convergence theorem that w satisfies

$$\begin{aligned}w(t) = \frac{1}{y(t)P^*(t+\tau)}\Big[&w(t+\tau)y(t+\tau)\\ &+\int_{T_1}^{t+\tau}\int_{T_1}^{s_{n-1}}\cdots\int_{T_1}^{s_1} Q^*(u)w(u-\sigma)y(u-\sigma)duds_1\cdots ds_{n-1}\Big]\end{aligned}$$

for $t \geq T$ and

$$w(t) = \frac{t-T+T_0}{T_0}(Fw)(T) + \left(1-\frac{t-T+T_0}{T_0}\right) > 0$$

for $T - T_0 \le t < T$. Again set

$$v = wy.$$

Then v satisfies $v(t) > 0$ for $T - T_0 \le t < T$ and

$$v(t) = \frac{1}{P^*(t+\tau)} \left[v(t+\tau) + \int_{T_1}^{t+\tau} \int_{T_1}^{s_{n-1}} \cdots \int_{T_1}^{s_1} Q^*(u) v(u-\sigma) du ds_1 \cdots ds_{n-1} \right]$$

for $t \ge T$. Clearly v is continuous on $[T - T_0, T)$. Then by the method of steps we see in view of the above that v is continuous on $[T - T_0, \infty)$.

Since $v(t) > 0$ for $T - T_0 \le t < T$, it is easy to see that $v(t) > 0$ for $t \ge T - T_0$. Hence, v is a positive solution of (6.44). This is a contradiction.

If (6.48) holds, then, integrating (6.49) from T_1 to t, we get

$$z^{(l)}(t) \ge - \int_{T_1}^{t} \int_{T_1}^{s_{n-l-1}} \cdots \int_{T_1}^{s_1} Q(s) y(s-\sigma) ds ds_1 \cdots ds_{n-l-1} \tag{6.51}$$

for $t \ge T_1$. Again integrating (6.51) from T_1 to t, we have

$$\begin{aligned} y(t) \ge \frac{1}{P(t+\tau)} \Bigg[& y(t+\tau) \\ & + \int_{T_1}^{t} \frac{(t-s)^{l-1}}{(l-1)!} \int_{T_1}^{s} \int_{T_1}^{s_{n-l-1}} \cdots \int_{T_1}^{s_1} Q(u) y(u-\sigma) du ds_1 \cdots ds_{n-l-1} \Bigg]. \end{aligned}$$

By condition (6.46), we have

$$\begin{aligned} y(t) \ge \frac{1}{P^*(t+\tau)} \Bigg[& y(t+\tau) \\ & + \int_{T_1}^{t} \frac{(t-s)^{l-1}}{(l-1)!} \int_{T_1}^{s} \int_{T_1}^{s_{n-l-1}} \cdots \int_{T_1}^{s_1} Q^*(u) y(u-\sigma) du ds_1 \cdots ds_{n-l-1} \Bigg], \end{aligned}$$

which, using a method similar to the proof of the former case, yields that (6.44) has a positive solution. This is also a contradiction. The proof is complete. □

Remark 6.3.14. When $n = 1$, (6.45) becomes

$$\int_{t_0}^{\infty} s Q(s) \int_{s}^{\infty} Q(u) du ds = \infty.$$

Clearly, the last condition is weaker than $\int_{t_0}^{\infty} Q(s) ds = \infty$.

Remark 6.3.15. It would not be difficult to extend the comparison result from Theorem 6.3.13 to nonlinear equations.

6.4. Asymptotic Behavior of Nonoscillatory Solutions

We consider the nth order neutral delay differential equation

$$\frac{d^n}{dt^n} \left(x(t) - \sum_{i=1}^{m} P_i(t) x(t - r_i) \right) + \delta \sum_{j=1}^{k} Q_j(t) f_j \Big(x(h_j(t)) \Big) = 0, \tag{6.52}$$

where $n \in \mathbb{N}$, $\delta \in \{-1, 1\}$, $r_i \ge 0$, and $h_j, P_i, Q_j : [t_0, \infty) \to \mathbb{R}$, $t_0 \ge 0$, are continuous with h_j and Q_j nonnegative, $h_j(t) \le t$ and $h_j(t) \to \infty$ as $t \to \infty$, $P_i, Q_j \not\equiv 0$ on any half-line $[t, \infty)$, and $f_j : \mathbb{R} \to \mathbb{R}$ are continuous with $u f_j(u) > 0$ for $u \ne 0$, $i \in I_m = \{1, 2, \ldots, m\}$, $j \in I_k = \{1, 2, \ldots, k\}$.

In this section, our aim is to study the asymptotic behavior of nonoscillatory solutions of (6.52), without requiring that $P_i(t)$, $i \in I_m$, has constant sign, namely, $P_i(t)$, $i \in I_m$, is allowed to oscillate about zero.

Since we are interested in asymptotic behavior of nonoscillatory solutions of (6.52), we only consider those solutions x that are extendable and nontrivial, i.e., x is defined on $[t_x, \infty)$ for some $t_x \geq t_0$ and $\sup\{|x(t)| : t \geq t_1\} > 0$ for every $t_1 > t_x$.

We first consider (6.52) when $\delta = 1$, i.e.,

$$\frac{d^n}{dt^n}\left(x(t) - \sum_{i=1}^{m} P_i(t)x(t - r_i)\right) + \sum_{j=1}^{k} Q_j(t)f_j\left(x(h_j(t))\right) = 0. \tag{6.53}$$

We will frequently require that

(6.54) $f_j(u)$ is bounded away from zero if u is bounded away from zero, $j \in I_k$,

$$\sum_{i=1}^{m} |P_i(t)| \leq \lambda < 1, \tag{6.55}$$

and

$$\sum_{j=1}^{k} \int_{t_0}^{\infty} Q_j(s)ds = \infty. \tag{6.56}$$

For notational purposes, we let

$$y(t) = x(t) - \sum_{i=1}^{m} P_i(t)x(t - r_i). \tag{6.57}$$

All proofs in this section will be done only for the case when a nonoscillatory solution of (6.53) is eventually positive, since, in each instance, the proof for an eventually negative solution is similar.

We begin with two lemmas that are useful in proving a number of our asymptotic results.

Lemma 6.4.1. *Assume that* (6.54) *and* (6.55) *hold and that x is an eventually positive* (*negative*) *solution of* (6.53). *Then*

(i) *If* $\lim_{t\to\infty} x(t) = 0$, *then* $\lim_{t\to\infty} y(t) = 0$, $y^{(i)}$ *is monotone, and*

$$\lim_{t\to\infty} y(t) = 0 \quad \textit{and} \quad y^{(i)}(t)y^{(i+1)}(t) < 0 \tag{6.58}$$

for $i \in \{0, 1, 2, \ldots, n-1\}$. *If n is even, then* $y(t) < 0$ ($y(t) > 0$). *If n is odd, then* $y(t) > 0$ ($y(t) < 0$).

(ii) *If* $x(t) \not\to 0$ *as* $t \to \infty$, *then* $y(t) > 0$ ($y(t) < 0$).

Proof. Let x be an eventually positive solution of (6.53), say $x(t - r_i) > 0$, $i \in I_m$, and $x(h_j(t)) > 0$, $j \in I_k$, for $t \geq t_1 \geq t_0$. If $x(t) \to 0$ as $t \to \infty$, it is easy to see that $y(t) \to 0$ as $t \to \infty$. By (6.53), $y^{(n-1)}$ is decreasing. If $y^{(n-1)}(t) \to L < 0$ as $t \to \infty$, then clearly there exist $L_1 < 0$ and $t_2 \geq t_1$ such that $y^{(n-1)}(t) \leq L_1$ for $t \geq t_2$. But the last inequality contradicts $y(t) \to 0$ as $t \to \infty$. On the other hand, if $y^{(n-1)}(t) \to L > 0$ as $t \to \infty$, then $y^{(n-1)}(t) \geq L$ for $t \geq t_1$, which again contradicts $y(t) \to 0$ as $t \to \infty$. Thus we conclude that $y^{(n-1)}(t) \to 0$ as $t \to \infty$. Moreover, since $y^{(n-1)}$ is decreasing and $Q_j(t)$, $j \in I_k$, is not identically zero on any half-line, we see that $y^{(n-1)}(t) > 0$ on $[t_1, \infty)$. So, if $n \geq 2$, then $y^{(n-2)}$ is increasing and hence $y^{(n-2)}(t) \to L_2 > -\infty$ as $t \to \infty$. If $L_2 < 0$, then $y^{(n-2)}(t) \leq L_2$ for $t \geq t_1$, contradicting $y(t) \to 0$ as $t \to \infty$. Now suppose $L_2 > 0$. Then there exist $L_3 > 0$ and $t_3 \geq T_1$ such that $y^{(n-2)}(t) \geq L_3$ for $t \geq t_3$, which again contradicts $y(t) \to 0$ as $t \to \infty$. Therefore $y^{(n-2)}(t) \to 0$ as $t \to \infty$ and, since $y^{(n-2)}$ is increasing, we have $y^{(n-2)}(t) < 0$ on $[t_1, \infty)$. Continuing in this manner we obtain (6.58).

If $x(t) \not\to 0$ as $t \to \infty$, then $\limsup_{t\to\infty} x(t) > 0$. We claim that $y(t)$ is eventually positive. Otherwise we have $y(t) < 0$ eventually. If x is unbounded, then there exists an increasing sequence $\{t_k\}$, $t_k \to \infty$ as $k \to \infty$ such that $x(t_k) = \max_{t \leq t_k} x(t)$ and $x(t_k) \to \infty$ as $k \to \infty$. By (6.57) we have

$$\text{(6.59)}\quad \begin{aligned} y(t_k) &= x(t_k) - \sum_{i=1}^{m} P_i(t_k)x(t_k - r_i) \;\geq\; x(t_k) - \sum_{i=1}^{m} |P_i(t_k)|x(t_k) \\ &= x(t_k)\left[1 - \sum_{i=1}^{m} |P_i(t_k)|\right] \;\geq\; x(t_k)(1-\lambda). \end{aligned}$$

From (6.55) and (6.59) we have $y(t_k) \to \infty$ as $k \to \infty$. This is a contradiction. If x is bounded, then there exists a sequence $\{t_k\}$ such that $t_k \to \infty$ as $k \to \infty$ and $\lim_{k\to\infty} x(t_k) = \limsup_{t\to\infty} x(t) > 0$. The sequence $\{x(t_k - r_i)\}$, $i \in I_m$, is bounded. Thus it has a convergent subsequence. Therefore, without loss of generality, we may suppose that $\lim_{k\to\infty} x(t_k - r_i)$, $i \in I_m$, exists. Hence

$$\begin{aligned} 0 \;\geq\; \lim_{k\to\infty} y(t_k) &= \lim_{k\to\infty}\left[x(t_k) - \sum_{i=1}^{m} P_i(t_k)x(t_k - r_i)\right] \\ &\geq \lim_{k\to\infty}\left[x(t_k) - \sum_{i=1}^{m} |P_i(t_k)|x(t_k)\right] \\ &= \limsup_{t\to\infty} x(t)(1-\lambda) \;>\; 0. \end{aligned}$$

This is also a contradiction and the proof is complete. □

The following lemma is independent of (6.52).

Lemma 6.4.2. *Suppose that* (6.55) *holds, and that* $x(t) > 0$ ($x(t) < 0$).

(i) *If* $\lim_{t\to\infty} y(t) = 0$, *then* $\lim_{t\to\infty} x(t) = 0$.
(ii) *If* $\lim_{t\to\infty} y(t) = \infty$ $(-\infty)$, *then* $\lim_{t\to\infty} x(t) = \infty$ $(-\infty)$.

Proof. First we show (i). Suppose that $x(t) > 0$. We prove that x is bounded. Since $\lim_{t\to\infty} y(t) = 0$, we have that y is bounded. If x is unbounded, then there exists an increasing sequence $\{t_k\}$ with $\lim_{k\to\infty} t_k = \infty$ such that $x(t_k) = \max_{t \leq t_k} x(t)$ and $\lim_{k\to\infty} x(t_k) = \infty$. By (6.57), we have that (6.59) holds, which, in view of (6.55), implies that $\lim_{k\to\infty} y(t_k) = \infty$. This is a contradiction.

Next we prove that $\lim_{t\to\infty} x(t) = 0$. Let $\{t_k\}$ be a sequence of points in $[t_0, \infty)$ with $t_k \to \infty$ as $k \to \infty$ such that

$$\lim_{k\to\infty} x(t_k) = \limsup_{t\to\infty} x(t) = M > 0.$$

By (6.57), we have

$$\begin{aligned} 0 &= \lim_{k\to\infty} y(t_k) = \lim_{k\to\infty}\left(x(t_k) - \sum_{i=1}^{m} P_i(t_k)x(t_k - r_i)\right) \\ &\geq \lim_{k\to\infty} x(t_k)(1-\lambda) \geq M(1-\lambda). \end{aligned}$$

This is a contradiction. Therefore $\limsup_{t\to\infty} x(t) = 0$ and $\lim_{t\to\infty} x(t) = 0$.

Now we show (ii). By (6.57), we have

$$y(t) = x(t) - \sum_{i=1}^{m} P_i(t)x(t-r_i) \leq x(t) + \sum_{i=1}^{m} |P_i(t)|x(t-r_i).$$

Since $y(t) \to \infty$ as $t \to \infty$, it follows that x is unbounded. Now we will prove that $x(t) \to \infty$ as $t \to \infty$. Since x is unbounded, $x(t - r_i)$, $i \in I_m$, is also unbounded. Without loss of generality, we may assume $r_1 > r_2 > \ldots > r_m$, and therefore $\liminf_{t\to\infty} x(t - r_m) = b \geq 0$. Then there is a sequence $\{t_k\}$ with $\lim_{k\to\infty} t_k = \infty$, $x(t_k) = \min_{t\leq t_k} x(t)$, and $\lim_{k\to\infty} x(t_k - r_m) = \liminf_{t\to\infty} x(t - r_m) = b \geq 0$. By (6.57), in view of $x(t_k) \leq x(t_k - r_i)$, $i \in I_m$, we have

$$\begin{aligned} y(t_k) &= x(t_k) - \sum_{i=1}^{m} P_i(t_k)x(t_k - r_i) \leq x(t_k) + \sum_{i=1}^{m} |P_i(t_k)|x(t_k - r_i) \\ &\leq x(t_k - r_m) + \lambda x(t_k - r_m) = (1+\lambda)x(t_k - r_m). \end{aligned}$$

Letting $k \to \infty$ we obtain

$$\infty = \lim_{k\to\infty} y(t_k) \leq (1+\lambda)\lim_{k\to\infty} x(t_k - r_m) = (1+\lambda)b.$$

This is a contradiction. Therefore $\liminf_{t\to\infty} x(t) = \infty$ and $\lim_{t\to\infty} x(t) = \infty$. The proof is complete. $\square$

The following lemma is extracted from [**109**].

Lemma 6.4.3. *Suppose that* (6.54) *and* (6.56) *hold and that x is an eventually positive* (*negative*) *solution of* (6.53). *Then*

(i) $y^{(n-1)}$ *is an eventually decreasing* (*increasing*) *function and satisfies* $y^{(n-1)}(t) \to L < \infty$ $(> -\infty)$ *as* $t \to \infty$*;*
(ii) *If* $L > -\infty$ $(< \infty)$, *then* $\liminf_{t\to\infty} |x(t)| = 0$.

Theorem 6.4.4. *Suppose that* (6.54), (6.55), *and* (6.56) *hold. Then every nonoscillatory solution of* (6.53) *satisfies* $x(t) \to 0$ *as* $t \to \infty$.

Proof. Suppose that x is an eventually positive solution of (6.53) and $x(t) \to 0$ as $t \to \infty$. By Lemma 6.4.1 we have $y(t) > 0$ eventually. From Lemma 6.4.3 (i), we have that $y^{(n-1)}$ is decreasing on $[t_1, \infty)$, where $t_1 \geq t_0$ such that $x(t - r_i) > 0$, $i \in I_m$, $x(h_j(t)) > 0$, $j \in I_k$, for $t \geq t_1$, and $y^{(n-1)}(t) \to L \geq -\infty$ as $t \to \infty$. Notice first that if $L = -\infty$, then $y^{(i)}(t) \to -\infty$ as $t \to \infty$, $0 \leq i \leq n-1$. This contradicts $y(t) > 0$. If $L > -\infty$, by Lemma 6.4.3 (ii), we have $\liminf_{t\to\infty} x(t) = 0$. If $y^{(n-1)}(t) \to L < 0$ as $t \to \infty$, then $y(t)$ is eventually negative, in contradiction to $y(t) > 0$. Also, if $y^{(n-1)}(t) \to L > 0$ as $t \to \infty$, and $n \geq 2$, then $y^{(n-1)}(t) \geq L$ for $t \geq t_1$. Thus $y^{(i)}(t) \to \infty$ as $t \to \infty$, $0 \leq i \leq n-2$. By Lemma 6.4.2, $x(t) \to \infty$ as $t \to \infty$, which contradicts $\liminf_{t\to\infty} x(t) = 0$. If $n = 1$ and $y(t) \to L > 0$ as $t \to \infty$, then by Lemma 6.4.3 we have $\liminf_{t\to\infty} x(t) = 0$. Without loss of generality, we assume that $r_1 > r_2 > \ldots > r_m$. Then there exists an increasing sequence $\{t_k\}$ with $\lim_{k\to\infty} t_k = \infty$ such that $x(t_k) = \min_{t\le t_k} x(t)$ and $\lim_{k\to\infty} x(t_k - r_1) = 0$. By (6.57), in view of $x(t_k) \le x(t_k - r_i)$, $i \in I_m$, we have

$$\begin{aligned} y(t_k) &\le x(t_k) + \sum_{i=1}^{m} |P_i(t_k)| x(t_k - r_i) \ \le\ x(t_k - r_1) + \lambda x(t_k - r_1) \\ &= (1+\lambda) x(t_k - r_1). \end{aligned}$$

Letting $k \to \infty$, we have $\lim_{k\to\infty} y(t_k) = 0$, which contradicts $\lim_{t\to\infty} y(t) = L > 0$. If $\lim_{t\to\infty} y(t) = 0$, then by Lemma 6.4.2 (i), we have $\lim_{t\to\infty} x(t) = 0$, which contradicts $x(t) \not\to 0$ as $t \to \infty$. The proof is complete. □

Remark 6.4.5. For the case $0 \le P_i(t) < p_i < 1$ or $-1 < p_i \le P_i(t) \le 0$, $i \in I_m$, of (6.53), it was proved in [**109**] that every nonoscillatory solution of (6.53) satisfies $x(t) \to 0$ as $t \to \infty$. This means that Theorem 6.4.4 generalizes [**104**, Theorem 3], [**108**, Theorem 5], [**109**, Theorems 4 and 7], [**110**, Theorem 3], [**112**, Theorem 3], parts of [**113**, Theorem 1(b)], parts of [**164**, Theorem 1(b)], [**218**, Theorem 1(b)], and [**279**, Theorem 4].

The results in the following are for (6.52) with $\delta = -1$, i.e., for

$$\frac{d^n}{dt^n}\left(x(t) - \sum_{i=1}^{m} P_i(t) x(t - r_i) \right) - \sum_{j=1}^{k} Q_j(t) f_j\left(x(h_j(t)) \right) = 0. \tag{6.60}$$

Lemma 6.4.6. *Assume that* (6.54) *and* (6.55) *hold and that x is an eventually positive* (*negative*) *solution of* (6.60). *Then we have:*

(i) *If* $\lim_{t\to\infty} x(t) = 0$, *then* $\lim_{t\to\infty} y(t) = 0$, $y^{(i)}$ *is monotone and*
$$\lim_{t\to\infty} y(t) = 0 \qquad \textit{and} \qquad y^{(i)}(t) y^{(i+1)}(t) < 0$$
for $i \in \{0, 1, 2, \ldots, n-1\}$. *If n is even, then* $y(t) > 0$ ($y(t) < 0$). *If n is odd, then* $y(t) < 0$ ($y(t) > 0$).

(ii) *If* $x(t) \to 0$ *as* $t \to \infty$, *then* $y(t) > 0$ ($y(t) < 0$).

Proof. The proof is similar to that of the corresponding results in Lemma 6.4.1 and will be omitted. □

Lemma 6.4.7 ([**109**]). *Suppose that* (6.54) *and* (6.56) *hold and that x is an eventually positive* (*negative*) *solution of* (6.60). *Then*

(i) $y^{(n-1)}$ *is an eventually increasing* (*decreasing*) *function and satisfies* $y^{(n-1)}(t) \to M > -\infty$ ($< \infty$) *as* $t \to \infty$;

(ii) *If* $M < \infty$ ($> -\infty$), *then* $\liminf_{t\to\infty} |x(t)| = 0$.

Theorem 6.4.8. *Suppose that* (6.54), (6.55), *and* (6.56) *hold. Then every nonoscillatory solution of* (6.60) *satisfies* $x(t) \to 0$ *or* $|x(t)| \to \infty$ *as* $t \to \infty$.

Proof. Suppose that x is an eventually positive solution of (6.60) and $x(t) \to 0$ as $t \to \infty$. By Lemma 6.4.6 we have $y(t) > 0$ eventually. From Lemma 6.4.7 (i), we have that $y^{(n-1)}$ is increasing on $[t_1, \infty)$, where $t_1 \ge t_0$ such that $x(t - r_i) > 0$, $i \in I_m$, and $x(h_j(t)) > 0$, $j \in I_k$, for $t \ge t_1$, and $y^{(n-1)}(t) \to M \le \infty$ as $t \to \infty$. If $M = \infty$, then $y^{(i)}(t) \to \infty$ as $t \to \infty$, $0 \le i \le n-1$. By Lemma 6.4.2, we have $\lim_{t\to\infty} x(t) = \infty$. If $M > 0$ and $n \ge 2$, then there exist $M_1 > 0$ and

$t_2 \geq t_1$ such that $y^{(n-1)}(t) \geq M_1$ for $t \geq t_1$. But the last inequality implies that $y^{(n-2)}(t) \geq y^{(n-2)}(t_2) + M_1(t - t_2) \to \infty$ as $t \to \infty$, and hence $y^{(i)}(t) \to \infty$ as $t \to \infty$, $0 \leq i \leq n-2$. By Lemma 6.4.2, we have $x(t) \to \infty$ as $t \to \infty$. If $n = 1$ and $y(t) \to M < 0$ as $t \to \infty$, then $y(t)$ is eventually negative, contradicting $y(t) > 0$. If $M > 0$, then by Lemma 6.4.7, we have $\liminf_{t\to\infty} x(t) = 0$. The rest of the proof is similar to that of Theorem 6.4.4 and hence is omitted here. □

Example 6.4.9. Consider the equation

$$\frac{d^n}{dt^n}\left(x(t) - \frac{\sin t}{2}x(t-1)\right) + \delta Q(t)x(t-2) = 0, \tag{6.61}$$

where Q is continuous on $[3, \infty)$ and $\int_3^\infty Q(t)dt = \infty$. If $\delta = 1$, by Theorem 6.4.4, we have that every nonoscillatory solution of (6.61) satisfies $\lim_{t\to\infty} x(t) = 0$. If $\delta = -1$, then, by Theorem 6.4.8, we have that $\lim_{t\to\infty} x(t) = 0$ or $\lim_{t\to\infty} x(t) = \infty$.

Remark 6.4.10. The case $\sum_{i=1}^m |P_i(t)| \geq 1$ of (6.52) remains as an *open problem.*

Remark 6.4.11. We point out that when $Q_j(t)$, $j \in I_k$, is oscillatory, the problem is far more difficult, and any such results, even for linear equations, would be of interest.

Remark 6.4.12. It would not be difficult to extend all results in this section to equations whose nonlinear term has the form

$$F\left(t, x(h_{01}(t)), \ldots, x(h_{0m}(t)), x'(h_{11}(t)), \ldots, x'(h_{1m}(t)), \ldots, x^{(n-1)}(h_{(n-1)m}(t))\right).$$

We leave the formulation and proof of such results to the reader.

6.5. Positive Solutions of Nonlinear Equations

In this section we consider the nonlinear differential equation of the form

$$\frac{d^n}{dt^n}\left(y(t) - p(t)y(t-\tau)\right) + q(t)\prod_{i=1}^m \left|y(t-\sigma_i)\right|^{\alpha_i} \operatorname{sgn} y(t-\sigma_i) = 0, \tag{6.62}$$

where $n \in \mathbb{N}$ is odd, $\tau > 0$, and $\sigma_1, \sigma_2, \ldots, \sigma_m \geq 0$, $p, q \in C([t_0, \infty), \mathbb{R})$, $q(t) \geq 0$, $q(t)$ is not identically zero for all large t, and each α_i is a positive number for $1 \leq i \leq m$ with $\alpha_1 + \alpha_2 + \ldots + \alpha_m = 1$. As in Section 6.3 we also define again

$$\sigma_* = \min\{\sigma_1, \sigma_2, \ldots, \sigma_m\}, \quad \sigma^* = \max\{\sigma_1, \sigma_2, \ldots, \sigma_m\},$$

and the set

$$\mathcal{L}' = \left\{\tau, \frac{\sigma_1}{n}, \frac{\sigma_2}{n}, \ldots, \frac{\sigma_m}{n}\right\} \setminus \{0\}.$$

When $m = 1$, (6.62) reduces to the linear equation

$$\frac{d^n}{dt^n}\left(y(t) - p(t)y(t-\tau)\right) + q(t)y(t-\sigma) = 0.$$

Let $\mu = \max\{\sigma^*, \tau\}$. Then by a solution of (6.62) we mean a function y that is defined for $t \geq t_0 - \mu$ such that (6.62) is satisfied. It is clear that if $y(t)$ is given for $t_0 - \mu \leq t \leq t_0$, then (6.62) has a unique solution satisfying these initial values. We will be concerned with the existence and nonexistence as well as the asymptotic behaviors of eventually positive solutions of (6.62). Moreover, we deal with the asymptotic behavior of eventually positive solutions of (6.62) with oscillating coefficient.

To this end, we consider an associated inequality relation

$$\frac{d^n}{dt^n}\Big(y(t) - P(t)y(t-\tau)\Big) + Q(t)\prod_{i=1}^{m}\Big|y(t-\sigma_i)\Big|^{\alpha_i}\operatorname{sgn} y(t-\sigma_i) \le 0, \tag{6.63}$$

where $\sigma_1, \dots, \sigma_m$ and $\alpha_1, \dots, \alpha_m$ are the same as before, and P and Q satisfy the same assumptions satisfied by p and q.

Theorem 6.5.1. *Assume $p(t) + q(t)\sigma_* > 0$, or $\sigma_* > 0$ and $q(s) \not\equiv 0$ on $[t, t+\sigma^*]$. Suppose furthermore that P and Q are two functions such that $P(t) \ge p(t) \ge 0$ and $Q(t) \ge q(t)$ for all large t, and there exists $t^* \ge t_0$ such that $P(t^* + k\tau) \le 1$ for $k \ge 0$. If* (6.63) *has an eventually positive solution y, then so does* (6.62).

Proof. The proof is similar to that of Theorem 6.2.2, and we omit it here. □

As an immediate consequence of Theorem 6.5.1, we have the following result.

Corollary 6.5.2. *Assume that there exists $t^* \ge t_0$ such that $0 \le p(t^* + k\tau) \le 1$ for $k \ge 0$. Suppose furthermore that either $p(t) + q(t)\sigma_* > 0$, or $\sigma_* > 0$ and $q(s) \not\equiv 0$ on $[t, t+\sigma^*]$. Then every solution of* (6.62) *is oscillatory if and only if*

$$\frac{d^n}{dt^n}\Big(y(t) - p(t)y(t-\tau)\Big) + q(t)\prod_{i=1}^{m}\Big|y(t-\sigma_i)\Big|^{\alpha_i}\operatorname{sgn} y(t-\sigma_i) \le 0$$

does not have an eventually positive solution.

As an immediate corollary of Theorem 6.5.1 and Corollary 6.5.2, we have the following comparison result.

Corollary 6.5.3. *Under the assumptions of Theorem 6.5.1, if* (6.62) *is oscillatory, then so is the equation*

$$\frac{d^n}{dt^n}\Big(y(t) - P(t)y(t-\tau)\Big) + Q(t)\prod_{i=1}^{m}\Big|y(t-\sigma_i)\Big|^{\alpha_i}\operatorname{sgn} y(t-\sigma_i) = 0.$$

As another standard application, we may consider equations of the form

$$\frac{d^n}{dt^n}\Big(y(t) - P(t)y(t-\tau)\Big) + Q(t)\prod_{i=1}^{m}\Big|y(t-\sigma_i)\Big|^{\alpha_i}\operatorname{sgn} y(t-\sigma_i) = f\Big(t, y(t)\Big),$$

where $xf(\cdot, x) \le 0$ whenever $x > 0$. If such an equation has an eventually positive solution, then so does (6.63). We may then conclude that some solution of (6.62) is not oscillatory.

We now turn to the question as to when equation (6.62) is oscillatory. This question is important if we want to apply Corollary 6.5.3. In view of Lemmas 6.2.1 and 6.3.12, it is easy to see that the following results hold.

Theorem 6.5.4. *Suppose that $p(t) \equiv 1$, $q(t) \ge 0$ for $t \ge t_0$ and*

$$\int_{t_0}^{\infty} s^n q(s) \int_s^{\infty} (u-s)^{(n-1)} q(u)\,du\,ds = \infty. \tag{6.64}$$

Then (6.62) *cannot have an eventually positive solution.*

An oscillation criterion can be derived as a consequence of Theorem 6.5.4.

Theorem 6.5.5. *Suppose that there exists $t^* \geq t_0$ such that $p(t^* + k\tau) \leq 1$ for $k \geq 0$, $p(t) \geq 0$ and $q(t) \geq 0$ for $t \geq t_0$, and that* (6.64) *holds. Suppose further that*

$$q(t) \prod_{i=1}^{m} [p(t-\sigma_i)]^{\alpha_i} \geq q(t-\tau) \tag{6.65}$$

for all large t. Then (6.62) *cannot have an eventually positive solution.*

Proof. Suppose to the contrary that y is an eventually positive solution of (6.62). Then, by Lemma 6.2.1, the function $z(t) = y(t) - p(t)y(t-\tau)$ satisfies $z(t) > 0$ for all large t. In view of (6.62), we have

$$\begin{aligned} z^{(n)}(t) &= -q(t) \prod_{i=1}^{m} [y(t-\sigma_i)]^{\alpha_i} \\ &= -q(t) \prod_{i=1}^{m} [z(t-\sigma_i) + p(t-\sigma_i) y(t-\tau-\sigma_i)]^{\alpha_i} \\ &\leq -q(t) \left\{ \prod_{i=1}^{m} [z(t-\sigma_i)]^{\alpha_i} + \prod_{i=1}^{m} [p(t-\sigma_i)]^{\alpha_i} \prod_{i=1}^{m} [y(t-\tau-\sigma_i)]^{\alpha_i} \right\} \end{aligned}$$

for all large t, where we have used Hölder's inequality to obtain our last inequality. Since (6.62) implies

$$z^{(n)}(t-\tau) + q(t-\tau) \prod_{i=1}^{m} [y(t-\tau-\sigma_i)]^{\alpha_i} = 0,$$

we have

$$\begin{aligned} & z^{(n)}(t) - z^{(n)}(t-\tau) + q(t) \prod_{i=1}^{m} [z(t-\sigma_i)]^{\alpha_i} \\ & \qquad \leq \left(q(t-\tau) - q(t) \prod_{i=1}^{m} [p(t-\sigma_i)]^{\alpha_i} \right) \prod_{i=1}^{m} [y(t-\tau-\sigma_i)]^{\alpha_i}. \end{aligned}$$

In view of our hypothesis (6.65),

$$\Big(z(t) - z(t-\tau) \Big)^{(n)} + q(t) \prod_{i=1}^{m} [z(t-\sigma_i)]^{\alpha_i} \leq 0$$

for all large t. Hence (6.63) (with $P(t) \equiv 1$ and $Q = q$) has the eventually positive solution z. By Theorem 6.5.1, (6.62) (with $p(t) \equiv 1$) has an eventually positive solution. But this is contrary to Theorem 6.5.4, and the proof is complete. □

In case assumption (6.65) is not satisfied, we may check to see if there is some number $r \in [0, 1)$ such that $rq(t-\tau) \leq q(t) \prod_{i=1}^{m} [p(t-\sigma_i)]^{\alpha_i}$ for all large t.

Theorem 6.5.6. *Suppose that there exists $t^* \geq t_0$ such that $p(t^* + k\tau) \leq 1$ for $k \geq 0$, $p(t) \geq 0$ and $q(t) \geq 0$ for $t \geq t_0$, and that* (6.64) *holds. Suppose further that*

$$q(t) \prod_{i=1}^{m} [p(t-\sigma_i)]^{\alpha_i} \geq rq(t-\tau)$$

for all large t. Then (6.62) *does not have an eventually positive solution provided that the inequalities*

$$w^{(n)}(t) + \frac{r}{1-r} w(t-\tau-\sigma_*) \le 0 \quad \textit{and} \quad w^{(n)}(t) + \frac{r}{1-r} w(t-\tau-\sigma^*) \le 0$$

for $t \ge t_0$ do not have an eventually positive solution.

Proof. Suppose to the contrary that y is an eventually positive solution of (6.62). Then by means of Lemma 6.2.1, the function $z(t) = y(t) - p(t)y(t-\tau)$ satisfies $z(t) > 0$ and $z^{(n)}(t) \le 0$ for all large t. Then, as in the proof of Theorem 6.5.5, we see that

$$\Big(z(t) - rz(t-\tau)\Big)^{(n)} + q(t)\prod_{i=1}^{m} [z(t-\sigma_i)]^{\alpha_i} \le 0$$

for all large t. If $n^* = 0$ (n^* defined by (6.12)), then $z'(t) < 0$ eventually. Thus, we have

$$\begin{aligned}\Big(z(t) - rz(t-\tau)\Big)^{(n)} &+ q(t)z(t-\sigma_*) \\ &\le \Big(z(t) - rz(t-\tau)\Big)^{(n)} + q(t)\prod_{i=1}^{m} [z(t-\sigma_i)]^{\alpha_i} \le 0\end{aligned}$$

for all large t. Let $w(t) = z(t) - rz(t-\tau)$ for $t \ge t_0$. By Lemma 6.2.1, it is clear that $w^{(n)}(t) \le 0$ and $w(t) > 0$ for t greater than or equal to some number T. Without loss of generality, we may assume that $z(t) > 0$ for $t \ge T$. Now

$$\begin{aligned} z(t) &= w(t) + rz(t-\tau) \\ &= w(t) + rw(t-\tau) + \ldots + r^j w(t-j\tau) + r^{j+1} z\Big(t-(j+1)\tau\Big) \\ &> \Big(r + r^2 + \ldots + r^{j+1}\Big) w(t-\tau) \\ &= \frac{r(1-r^{j+1})}{1-r} w(t-\tau) \end{aligned}$$

for $t > (j+1)\tau + T + \sigma_*$. Thus

$$w^{(n)}(t) + \frac{r}{1-r} w(t-\tau-\sigma_*) \le 0$$

for all large t, which is contrary to our hypothesis.

If $n^* \ge 2$, then $z'(t) > 0$. Repeating the procedure of the proof for the case $n^* = 0$, we can obtain a contradiction. The proof is complete. □

Lemma 6.5.7 ([**109**]). *Suppose that $0 \le p(t) \le 1$ and $q(t) \ge 0$ for $t \ge t_0$ with $\int_{t_0}^\infty q(s)ds = \infty$. Let y be a nonoscillatory solution of* (6.62). *Then the function $z(t) = y(t) - p(t)y(t-\tau)$ satisfies $z^{(i)}(t) \to 0$ as $t \to \infty$ and $z^{(i)}(t)z^{(i+1)}(t) < 0$ for $i \in \{0, 1, \ldots, n-1\}$.*

Theorem 6.5.8. *Suppose that $n > 1$, $p(t) \equiv p \in [0,1]$ and $q(t) \ge q(t-\tau) \ge q > 0$ for all t. Suppose further that for all $\lambda > 0$ and $l \in \mathcal{L}'$,*

$$\liminf_{t\to\infty} \left\{ pe^{\lambda\tau} + \frac{\left(\frac{n-1}{n}\right)^{n-1}}{(n-1)!} \prod_{i=1}^{m} \sigma_i^{n-1} \left(e^{\frac{\lambda\sigma_i}{n}}\right)^{\alpha_i} \frac{1}{l} \int_t^{t+l} q(s)ds \right\} > 1. \tag{6.66}$$

Then (6.62) *cannot have an eventually positive solution.*

Proof. Suppose to the contrary that y is an eventually positive solution of (6.62). By Lemma 6.2.1, we see that $z(t) = y(t) - py(t-\tau) > 0$ for all large t. Since n is odd, using $\int_{t_0}^{\infty} q(s)ds = \infty$ and Lemma 6.5.7, we have

$$z^{(i)}(t) \to 0 \quad \text{as} \quad t \to \infty \quad \text{and} \quad z^{(i)}(t)z^{(i+1)}(t) < 0 \tag{6.67}$$

for $i \in \{0, 1, \dots, n-1\}$. Since $z(t) > 0$ eventually, $z'(t) < 0$ and $z^{(n-1)}(t) > 0$ eventually. By arguments similar to those used in the proof of Theorem 6.5.5, the function z satisfies

$$z^{(n)}(t) - pz^{(n)}(t-\tau) + q(t) \prod_{i=1}^{m} [z(t-\sigma_i)]^{\alpha_i} \le 0 \tag{6.68}$$

for $t \ge t_0$ for some t_0. By Lemma 6.2.5, with $\mu = \frac{n-1}{n}\sigma_i$ for the term $z(t-\sigma_i)$, $1 \le i \le m$, we have

$$z(t-\sigma_i) \ge \frac{\sigma_i^{n-1}}{(n-1)!} \left(\frac{n-1}{n}\right)^{n-1} z^{(n-1)}\left(t - \frac{\sigma_i}{n}\right). \tag{6.69}$$

Combining (6.68) and (6.69) yields

$$z^{(n)}(t) - pz^{(n)}(t-\tau) + q(t) \frac{\left(\frac{n-1}{n}\right)^{n-1}}{(n-1)!} \prod_{i=1}^{m} \left(\sigma_i^{n-1}\right)^{\alpha_i} \left(z^{(n-1)}\left(t - \frac{\sigma_i}{n}\right)\right)^{\alpha_i} \le 0. \tag{6.70}$$

Set

$$w = z^{(n-1)} \quad \text{and} \quad u = -\frac{w'}{w}.$$

By (6.67), we have $w(t) > 0$, and (6.70) reduces to

$$w'(t) - pw'(t-\tau) + q(t) \frac{\left(\frac{n-1}{n}\right)^{n-1}}{(n-1)!} \prod_{i=1}^{m} \left(\sigma_i^{n-1}\right)^{\alpha_i} \left(w\left(t - \frac{\sigma_i}{n}\right)\right)^{\alpha_i} \le 0, \tag{6.71}$$

i.e.,

$$\begin{aligned} u(t) \ge{}& pu(t-\tau) \exp\left(\int_{t-\tau}^{t} u(s)ds\right) \\ &+ q(t) \frac{1}{(n-1)!} \left(\frac{n-1}{n}\right)^{n-1} \prod_{i=1}^{m} \left(\sigma_i^{n-1}\right)^{\alpha_i} \left(\exp\left(\int_{t-\frac{\sigma_i}{n}}^{t} u(s)ds\right)\right)^{\alpha_i} \end{aligned}$$

for $t \ge t_0 + \mu$ with $\mu = \max\{\tau, \sigma^*\}$.

We now define a sequence of functions $\{u_k(t)\}$ for $k \in \mathbb{N}$ and $t \ge t_0$ and a sequence of numbers $\{\lambda_k\}$ for $k \in \mathbb{N}_0$ as follows: $u_1(t) \equiv 0$ for $t \ge t_0$, and for $k \in \mathbb{N}$, $t \ge t_0 + k\mu$,

$$\begin{aligned} u_{k+1}(t) ={}& pu_k(t-\tau) \exp\left(\int_{t-\tau}^{t} u_k(s)ds\right) \\ &+ q(t) \frac{1}{(n-1)!} \left(\frac{n-1}{n}\right)^{n-1} \prod_{i=1}^{m} \left(\sigma_i^{n-1}\right)^{\alpha_i} \left(\exp\left(\int_{t-\frac{\sigma_i}{n}}^{t} u_k(s)ds\right)\right)^{\alpha_i}, \end{aligned} \tag{6.72}$$

$\lambda_1 = 0$, and for $k \in \mathbb{N}$,

$$\lambda_{k+1} = \inf_{t \geq t_0} \min_{l \in \mathcal{L}'} \left\{ p\lambda_k e^{\lambda_k \tau} + q(t) \frac{1}{(n-1)!} \left(\frac{n-1}{n} \right)^{n-1} \times \prod_{i=1}^{m} (\sigma_i^{n-1})^{\alpha_i} \left(e^{\frac{\lambda_k \sigma_i}{n}} \right)^{\alpha_i} \frac{1}{l} \int_t^{t+l} q(s)ds \right\}. \tag{6.73}$$

We claim that the following inequalities hold:

(i) $0 = \lambda_1 < \lambda_2 < \ldots$;
(ii) $u_k(t) \leq u(t)$ for $t \geq t_0 + (k-1)\mu$ and $k \in \mathbb{N}$;
(iii) $\frac{1}{l} \int_t^{t+l} u_k(s)ds \geq \lambda_k$ for $t \geq t_0 + (k+1)\mu$, $k \in \mathbb{N}$, and $l \in \mathcal{L}'$.

Clearly, $\lambda_2 > \lambda_1 = 0$ and $u_1(t) \leq u(t)$ for $t \geq t_0$. By induction we see that (i) and (ii) are true. We now show that (iii) also holds. Obviously (iii) is true for $k = 1$. Assume (iii) is true for some $k \in \mathbb{N}$. Then (6.72) and (6.73) imply that for $t \geq t_0 + k\mu$ and $l \in \mathcal{L}'$,

$$\begin{aligned}
\frac{1}{l} \int_t^{t+l} u_{k+1}(s)ds &= \frac{1}{l} \int_t^{t+l} \frac{q(s)}{(n-1)!} \left(\frac{n-1}{n} \right)^{n-1} \prod_{i=1}^{m} (\sigma_i^{n-1})^{\alpha_i} \\
&\quad \times \left[\exp \left(\int_{s-\frac{\sigma_i}{n}}^{s} u_k(\xi)d\xi \right) \right]^{\alpha_i} ds \\
&\quad + \frac{p}{l} \int_t^{t+l} u_k(s-\tau) \exp \left(\int_{s-\tau}^{s} u_k(\xi)d\xi \right) ds \\
&\geq p\lambda_k e^{\lambda_k \tau} + \frac{\left(\frac{n-1}{n}\right)^{n-1}}{(n-1)!} \prod_{i=1}^{m} (\sigma_i^{n-1})^{\alpha_i} \left(e^{\frac{\lambda_k \sigma_i}{n}} \right)^{\alpha_i} \frac{1}{l} \int_t^{t+l} q(s)ds \\
&\geq \lambda_{k+1}.
\end{aligned}$$

Hence (iii) holds.

Now let $\lambda^* = \lim_{k \to \infty} \lambda_k$. From (6.66) and (6.73), there exists $\beta > 1$ such that $\lambda_{k+1} \geq \beta \lambda_k$, $k \in \mathbb{N}$, and this means that $\lambda^* = \infty$. By (ii) and (iii) we have that $\lim_{t \to \infty} \int_t^{t+\sigma_i} u(s)ds = \infty$, and so

$$\limsup_{t \to \infty} \int_t^{t+\frac{\sigma_*}{2}} u(s)ds = \infty,$$

where $\sigma_* > 0$. Integrating both sides of the equation $u = -w'/w$ from t to $t + \frac{\sigma_*}{2}$ for t sufficiently large we obtain

$$\limsup_{t \to \infty} \frac{w(t)}{w(t + \frac{\sigma_*}{2})} = \limsup_{t \to \infty} \exp \left(\int_t^{t+\frac{\sigma_*}{2}} u(s)ds \right) = \infty.$$

By (6.71),

$$\begin{aligned} w'(t) &\le -q(t)\frac{1}{(n-1)!}\left(\frac{n-1}{n}\right)^{n-1}\prod_{i=1}^{m}\left(\sigma_i^{n-1}\right)^{\alpha_i}\left(w\left(t-\frac{\sigma_i}{n}\right)\right)^{\alpha_i} \\ &\le -q\frac{1}{(n-1)!}\left(\frac{n-1}{n}\right)^{n-1}\prod_{i=1}^{m}\left(\sigma_i^{n-1}\right)^{\alpha_i}\left(w\left(t-\frac{\sigma_i}{n}\right)\right)^{\alpha_i} \\ &\le -q\frac{1}{(n-1)!}\left(\frac{n-1}{n}\right)^{n-1} w\left(t-\frac{\sigma_*}{n}\right)\prod_{i=1}^{m}\left(\sigma_i^{n-1}\right)^{\alpha_i}. \end{aligned}$$

Integrating both sides from $t+\sigma_*/2$ to $t+\sigma_*$ and using the decreasing nature of w, we find for t sufficiently large

$$0 < w(t+\sigma_*) \le w\left(t+\frac{\sigma_*}{2}\right) - q\frac{\sigma_*}{2}\frac{1}{(n-1)!}\left(\frac{n-1}{n}\right)^{n-1}\prod_{i=1}^{m}\left(\sigma_i^{n-1}\right)^{\alpha_i} w(t).$$

Thus,

$$\frac{w(t)}{w\left(t+\frac{\sigma_*}{2}\right)} \le \frac{2}{q\sigma_*}\left[\frac{1}{(n-1)!}\left(\frac{n-1}{n}\right)^{n-1}\prod_{i=1}^{m}\left(\sigma_i^{n-1}\right)^{\alpha_i}\right]^{-1}.$$

This is a contradiction and the proof is complete. $\square$

Next we consider the stability of eventually positive solutions of (6.62) with oscillating coefficient.

Lemma 6.5.9. *Suppose that there exists $t^* \ge t_0$ such that*

$$0 \le |P(t^*+k\tau)| \le 1, \quad k \ge 0. \tag{6.74}$$

Then for any eventually positive solution y of the inequality (6.63), *the function $z(t) = y(t) - P(t)y(t-\tau)$ satisfies*

$$z(t) > 0 \quad \text{and} \quad z^{(n)}(t) \le 0 \quad \text{eventually.}$$

Proof. It is clear from (6.63) that $z^{(n)}(t) \le 0$ and is not identically zero for all large t. This implies that $z^{(i)}$ for $i \in \{0, 1, \dots, n-1\}$ are eventually monotone. Suppose to the contrary that $z(t) < 0$. Since n is odd, $z'(t) < 0$ eventually. Thus, there exist $t_1 \ge t_0$ and $\alpha > 0$ such that $z(t) \le -\alpha$ for $t \ge t_1$. That is,

$$y(t) \le -\alpha + P(t)y(t-\tau), \quad t \ge t_1. \tag{6.75}$$

By choosing $k^* \ge 1$ such that $t^* + k^*\tau \ge t_1$, we see from (6.74) and (6.75) that

$$\begin{aligned} y(t^*+k^*\tau+j\tau) &\le -\alpha + P\left(t^*+k^*\tau+j\tau\right)y\left(t^*+(k^*+j-1)\tau\right) \\ &\le -\alpha + \left|P\left(t^*+k^*\tau+j\tau\right)\right| y\left(t^*+(k^*+j-1)\tau\right) \\ &\le -\alpha + y\left(t^*+(k^*+j-1)\tau\right) \le \dots \\ &\le -(j+1)\alpha + y\left(t^*+(k^*-1)\tau\right) \end{aligned}$$

for $j \ge 0$. By letting $j \to \infty$, we see that the right-hand side diverges to $-\infty$, which is contrary to our assumption that $y(t) > 0$ for $t \ge t_1$. The proof is complete. $\square$

Lemma 6.5.10. *Suppose that*

$$|P(t)| \le p < 1$$

and that $y(t) > 0$ $(y(t) < 0)$. *Define* $z(t) = y(t) - P(t)y(t-\tau)$.

(i) *If* $\lim_{t\to\infty} z(t) = 0$, *then* $\lim_{t\to\infty} y(t) = 0$;
(ii) *if* $\lim_{t\to\infty} z(t) = \infty$ $(-\infty)$, *then* $\lim_{t\to\infty} y(t) = \infty$ $(-\infty)$.

Proof. To show (i), suppose that $y(t) > 0$. First, we prove that y is bounded. Since $\lim_{t\to\infty} z(t) = 0$, z is bounded. If y is unbounded, then there exists an increasing sequence $\{t_k\}$ with $\lim_{k\to\infty} t_k = \infty$ such that $y(t_k) = \max_{t\le t_k} y(t)$ and $\lim_{k\to\infty} y(t_k) = \infty$. By the definition of z, we have

$$z(t_k) = y(t_k) - P(t_k)y(t_k - \tau) \ge y(t_k) - |P(t_k)|y(t_k) \ge (1-p)y(t_k) > 0.$$

Letting $k \to \infty$, we obtain a contradiction.

Second, we prove that $\lim_{t\to\infty} y(t) = 0$. Let $\{t_k\}$ be a sequence of points in $[t_0, \infty)$ with $t_k \to \infty$ as $k \to \infty$ such that

$$\lim_{k\to\infty} y(t_k) = \limsup_{t\to\infty} y(t) = M > 0.$$

By the definition of z, we have

$$\begin{aligned} 0 &= \lim_{k\to\infty} z(t_k) = \lim_{k\to\infty} \Big(y(t_k) - P(t_k)y(t_k-\tau)\Big) \\ &\ge \lim_{k\to\infty} y(t_k)(1-p) = M(1-p) > 0. \end{aligned}$$

This is a contradiction. Therefore $\limsup_{t\to\infty} y(t) = 0$ and $\lim_{t\to\infty} y(t) = 0$.

Now we show (ii). By the definition of z, we have

$$z(t) = y(t) - P(t)y(t-\tau) \le y(t) + |P(t)|y(t-\tau) \le y(t) + py(t-\tau).$$

Since $z(t) \to \infty$ as $t \to \infty$, we have that y is unbounded. Now we show $y(t) \to \infty$ as $t \to \infty$. Since $y(t)$ is unbounded, $y(t-\tau)$ is also unbounded. We assume that $\liminf_{t\to\infty} y(t-\tau) = b \ge 0$. Then there exists a sequence $\{t_k\}$ with $\lim_{k\to\infty} t_k = \infty$ such that $y(t_k) = \min_{t\le t_k} y(t)$ and $\lim_{k\to\infty} y(t_k - \tau) = \liminf_{t\to\infty} y(t-\tau) = b \ge 0$. By the definition of z, in view of $y(t_k) \le y(t_k - \tau)$, we have

$$\begin{aligned} z(t_k) &= y(t_k) - P(t_k)y(t_k-\tau) \le y(t_k) + |P(t_k)|y(t_k-\tau) \\ &\le y(t_k-\tau) + py(t_k-\tau) = (1+p)y(t_k-\tau). \end{aligned}$$

Letting $k \to \infty$ we obtain

$$\infty = \lim_{k\to\infty} z(t_k) \le (1+p)\lim_{k\to\infty} y(t_k-\tau) = (1+p)b.$$

This is a contradiction. Therefore $\liminf_{t\to\infty} y(t) = \infty$ and $\lim_{t\to\infty} y(t) = \infty$. The proof is complete. ☐

Theorem 6.5.11. *Suppose that* $|p(t)| \le p < 1$ *and that* $q(t) \ge 0$ *for* $t \ge t_0$ *such that*

$$\int_{t_0}^{\infty} q(s)ds = \infty. \tag{6.76}$$

Then every eventually positive solution y *of* (6.62) *satisfies* $\lim_{t\to\infty} y(t) = 0$.

Proof. Suppose that y is an eventually positive solution of the equation (6.62). In view of Lemma 6.5.9, we have $z(t) > 0$, $z^{(n)}(t) \leq 0$ and is not identically zero for all large t. This implies that $z^{(i)}$ for $0 \leq i \leq n-1$ is eventually monotone. From (6.62), $z^{(n)}(t) = -q(t)\prod_{i=1}^{m}[y(t-\sigma_i)]^{\alpha_i} \leq 0$, so $z^{(n-1)}(t)$ is decreasing and converges to $L < \infty$ as $t \to \infty$.

If $L = -\infty$, then $z(t) \to -\infty$ as $t \to \infty$. This contradicts $z(t) > 0$. If $L > -\infty$, integrating (6.62) from t_1 to t and then letting $t \to \infty$, we find

$$\int_{t_1}^{\infty} q(s)\prod_{i=1}^{m}[y(s-\sigma_i)]^{\alpha_i}\,ds = z^{(n-1)}(t_1) - L < \infty,$$

where t_1 is a sufficiently large number. By (6.62) and (6.76), we have $\liminf_{t\to\infty} y(t) = 0$.

If $\lim_{t\to\infty} z(t) = \infty$, by Lemma 6.5.10, $\lim_{t\to\infty} y(t) = \infty$, which contradicts $\liminf_{t\to\infty} y(t) = 0$.

If $\lim_{t\to\infty} z(t) = 0$, by Lemma 6.5.10, we have $\lim_{t\to\infty} y(t) = 0$. Suppose that $\lim_{t\to\infty} z(t) = M > 0$. Since $\liminf_{t\to\infty} y(t) = 0$, there exists a sequence $\{t_k\}$ with $\lim_{k\to\infty} t_k = \infty$ such that $y(t_k) = \min_{t\leq t_k} y(t)$ and $\lim_{k\to\infty} y(t_k - \sigma^*) = 0$. By the definition of z, we have

$$\begin{aligned} z(t_k) &= y(t_k) - p(t_k)\prod_{i=1}^{m}[y(t_k-\sigma_i)]^{\alpha_i} \leq y(t_k) + |p(t_k)|\prod_{i=1}^{m}[y(t_k-\sigma_i)]^{\alpha_i} \\ &\leq y(t_k-\sigma^*) + py(t_k-\sigma^*) = (1+p)y(t_k-\sigma^*). \end{aligned}$$

Letting $k \to \infty$, we obtain

$$0 < M \leq (1+p)\lim_{k\to\infty} y(t_k - \sigma^*) = 0.$$

This is a contradiction, and the proof is complete. □

6.6. Classifications of Nonoscillatory Solutions

This section is concerned with a class of higher order nonlinear delay differential equations of the form

$$\left(r(t)x^{(m-1)}(t)\right)' + f\left(t, x(t-\tau)\right) = 0, \quad t \geq t_0, \tag{6.77}$$

where $\tau \geq 0$ is a constant, $m \in \mathbb{N}\setminus\{1\}$, r is a positive continuous function, and f is a real-valued function defined on $[t_0,\infty)\times\mathbb{R}$ which is continuous in the second variable x and satisfies $f(t,x) > 0$ for $x > 0$. We will give a classification scheme of eventually positive solutions of our equation in terms of their asymptotic magnitude and provide necessary and/or sufficient conditions for the existence of these solutions. In order to accomplish our goal, additional conditions will be imposed on the coefficient function r and the function f. We will need either one of the following two assumptions for the function r so as to include the case when $r(t) \equiv 1$:

(H1) $r'(t) \geq 0$ for $t \geq t_0$;
(H2) $\int_{t_0}^{\infty} \frac{dt}{r(t)} = \infty$.

As for the function f, for each fixed t, if $f(t,x)/x$ is nondecreasing in x for $x>0$, it is called *superlinear*. If for each t, $f(t,x)/x$ is nonincreasing in x for $x>0$, then f is said to be *sublinear*. Superlinear or sublinear functions f will be assumed in later results. Here we note that if $0<a\le x\le b$, then

$$f(t,a)\le f(t,x)\le f(t,b) \quad \text{if } f \text{ is superlinear}$$

and

$$\frac{a}{b}f(t,b)\le f(t,x)\le \frac{b}{a}f(t,a) \quad \text{if } f \text{ is sublinear.}$$

For the sake of convenience, we will employ the notations

$$R(s,t)=\int_s^t \frac{du}{r(u)}, \quad t\ge s\ge t_0$$

and

$$R(s)=\int_s^\infty \frac{du}{r(u)}, \quad s\ge t_0.$$

We begin by classifying all possible positive solutions of (6.77) according to their asymptotic behavior as $t\to\infty$, on the basis of the well-known Lemma 6.3.2.

Let x be an eventually positive solution of (6.77). Then

$$\left(r(t)x^{(m-1)}(t)\right)'=-f\left(t,x(t-\tau)\right)<0$$

for all large t. Hence

$$r(t)x^{(m-1)}(t)<r(s)x^{(m-1)}(s), \quad t>s\ge t_0. \tag{6.78}$$

Since $r'(t)\ge 0$, $r(s)/r(t)\le 1$ for $t\ge s$, by (6.78), we have

$$x^{(m-1)}(t)<x^{(m-1)}(s), \quad t>s.$$

This means that $x^{(m-1)}$ is eventually strictly decreasing. We may assert further that $x^{(m-1)}(t)$ is eventually positive.

Lemma 6.6.1. *Suppose the conditions* (H1) *and* (H2) *hold. Let x be an eventually positive solution of* (6.77). *Then $x^{(m-1)}(t)$ is eventually positive.*

Proof. Assume without loss of generality that $x(t)>0$ for $t\ge t_0$. Then in view of (6.77), we obtain (6.78). If it were the case that $x^{(m-1)}(t)<0$ for some $t\ge T$, then

$$r(s)x^{(m-1)}(s)<r(T)x^{(m-1)}(T) \quad \text{for all} \quad s>T$$

implies

$$\begin{aligned} x^{(m-2)}(t)-x^{(m-2)}(T) &= \int_T^t x^{(m-1)}(s)ds \\ &< \int_T^t \frac{r(T)}{r(s)}x^{(m-1)}(T)ds \\ &= R(T,t)r(T)x^{(m-1)}(T). \end{aligned}$$

Since $\lim_{t\to\infty}R(T,t)=\infty$ by (H2), we see that the right-hand side tends to negative infinity. Thus $\lim_{t\to\infty}x^{(m-2)}(t)=-\infty$, which implies that $x(t)$ is eventually negative. This is a contradiction and the proof is complete. $\square$

Lemma 6.6.2. *Suppose the conditions* (H1) *and* (H2) *hold. Let x be an eventually positive solution of* (6.77). *Then the function $x^{(m)}$ is eventually negative.*

Proof. By means of Lemma 6.6.1, $x^{(m-1)}(t)$ is eventually positive. Furthermore, in view of (6.77) and our assumption on r, we see that

$$r(t)x^{(m)}(t) = -r'(t)x^{(m-1)}(t) - f\left(t, x(t-\tau)\right) < 0,$$

as required. □

Under the conditions (H1) and (H2), it is clear that Lemma 6.3.2 provides a classification scheme for eventually positive solutions of (6.77). Such a scheme is crude, however. We will propose an auxiliary classification scheme for eventually positive solutions of (6.77). For the sake of convenience, we will make use of the following notations in this scheme:

$$E_j(\infty, *) = \left\{x : \lim_{t\to\infty} \frac{x(t)}{t^{2j-2}} = \infty, \quad \lim_{t\to\infty} \frac{x(t)}{t^{2j-1}} = a \neq 0\right\},$$

$$E_j(\infty, 0) = \left\{x : \lim_{t\to\infty} \frac{x(t)}{t^{2j-2}} = \infty, \quad \lim_{t\to\infty} \frac{x(t)}{t^{2j-1}} = 0\right\},$$

$$E_j(*, 0) = \left\{x : \lim_{t\to\infty} \frac{x(t)}{t^{2j-2}} = a \neq 0, \quad \lim_{t\to\infty} \frac{x(t)}{t^{2j-1}} = 0\right\},$$

$$O_j(\infty, *) = \left\{x : \lim_{t\to\infty} \frac{x(t)}{t^{2j-1}} = \infty, \quad \lim_{t\to\infty} \frac{x(t)}{t^{2j}} = a \neq 0\right\},$$

$$O_j(\infty, 0) = \left\{x : \lim_{t\to\infty} \frac{x(t)}{t^{2j-1}} = \infty, \quad \lim_{t\to\infty} \frac{x(t)}{t^{2j}} = 0\right\},$$

$$O_j(*, 0) = \left\{x : \lim_{t\to\infty} \frac{x(t)}{t^{2j-1}} = a \neq 0, \quad \lim_{t\to\infty} \frac{x(t)}{t^{2j}} = 0\right\},$$

where the integer $j \in \{1, 2, \ldots, [m/2]\}$ will be specified.

Theorem 6.6.3. *Suppose the conditions* (H1) *and* (H2) *hold. Under the additional condition that m is even, there is an integer $j \in \{1, 2, \ldots, m/2\}$ such that every eventually positive solution x of* (6.77) *must belong to either one of the classes $E_j(\infty, *)$, $E_j(\infty, 0)$, or $E_j(*, 0)$. Under the additional condition that m is odd, either there is an integer $j \in \{1, 2, \ldots, (m-1)/2\}$ such that any eventually positive solution of* (6.77) *belongs to one of the classes $O_j(\infty, *)$, $O_j(\infty, 0)$, $O_j(*, 0)$, or also every eventually positive solution of* (6.77) *converges.*

Proof. First of all, we infer from Lemma 6.6.2 that $x^{(m)}(t)$ is eventually negative. Suppose m is even. In view of Lemma 6.3.2, there is an integer $l = 2j - 1$, where $j \in \{1, 2, \ldots, m/2\}$, such that for each $k \in \{0, 1, \ldots, l-1\}$, $x^{(k)}(t) > 0$ for all large t, and for each $k \in \{l, l+1, \ldots, m-1\}$, $(-1)^{k+1}x^{(k)}(t) > 0$ for all large t. In particular, $x^{(2j-2)}(t) > 0$, $x^{(2j-1)}(t) > 0$, and $x^{(2j)}(t) < 0$ for all large t. Therefore the limits

$$\lim_{t\to\infty} x^{(2j-1)}(t) = \lambda_{2j-1} \quad \text{and} \quad \lim_{t\to\infty} x^{(2j-2)}(t) = \lambda_{2j-2}$$

satisfy $0 \le \lambda_{2j-1} < \infty$ and $0 < \lambda_{2j-2} \le \infty$, respectively. If $\lambda_{2j-1} > 0$, then by L'Hôpital's rule, we find

$$\lim_{t\to\infty} \frac{x(t)}{t^{2j-1}} = \lim_{t\to\infty} \frac{x'(t)}{(2j-1)t^{2j-2}} = \ldots = \lim_{t\to\infty} \frac{x^{(2j-1)}(t)}{(2j-1)!} = \frac{\lambda_{2j-1}}{(2j-1)!} \neq 0.$$

It follows that $\lim_{t\to\infty} x(t)/t^{2j-2} = \infty$, i.e., $x \in E_j(\infty, *)$.

If $\lambda_{2j-1} = 0$ and $\lambda_{2j-2} = \infty$, then by L'Hôpital's rule again, it is easy to see that

$$\lim_{t\to\infty} \frac{x(t)}{t^{2j-1}} = 0 \quad \text{and} \quad \lim_{t\to\infty} \frac{x(t)}{t^{2j-2}} = \infty.$$

Hence $x \in E_j(\infty, 0)$. Finally, in case $\lambda_{2j-1} = 0$ and $0 < \lambda_{2j-2} < \infty$, we apply L'Hôpital's rule again to find

$$\lim_{t\to\infty} \frac{x(t)}{t^{2j-2}} = \frac{\lambda_{2j-2}}{(2j-2)!} \neq 0.$$

It follows that $\lim_{t\to\infty} x(t)/t^{2j-1} = 0$, and hence $x \in E_j(*, 0)$.

When the integer m is odd, in view of Lemma 6.3.2, there is an even integer $l \in \{0, 1, \ldots, m-1\}$ such that for each $k \in \{0, 1, \ldots, l\}$, $x^{(k)}(t) > 0$ for all large t, and for each $k \in \{l+1, \ldots, m-1\}$, $(-1)^k x^{(k)}(t) > 0$ for all large t. In case $l \in \{1, 2, \ldots, m-1\}$, the proof is similar to that given above. In case $l = 0$, we have $x(t) > 0$, $x'(t) < 0$, and $x''(t) > 0$ for all large t. It follows that $x(t)$ converges to some nonnegative constant. The proof is complete. □

Under the conditions (H1) and (H2), eventually positive solutions can be classified according to Theorem 6.6.3. We remark that there is an uncertainty involved, namely the integer j, which is needed in the definitions of the various subsets E and O. We now impose conditions which are sufficient for the existence of eventually positive solutions in these subsets.

Theorem 6.6.4. *Suppose that m is even and that* (H1) *and* (H2) *hold. Suppose further that f is superlinear or sublinear. If there exist a constant $c > 0$ and $j \in \{1, 2, \ldots, (m-1)/2\}$ such that*

$$\int_{t_0}^{\infty} \frac{s^{m-2j-1}}{r(s)} \int_s^{\infty} \left| f\left(u, c(u-\tau)^{2j-1}\right) \right| du ds < \infty, \tag{6.79}$$

then (6.77) *has an eventually positive solution in $E_j(\infty, *)$. The converse is also true.*

Proof. Let $a = c/2$ if f is superlinear and $a = c$ if f is sublinear. Set

$$K(t) = t^{2j-1}.$$

In view of (6.79), we may choose T so large that

$$\frac{1}{(2j-1)!} \int_T^{\infty} \frac{(t-T)^{m-2j-1}}{r(t)(m-2j-1)!} \int_t^{\infty} f\left(s, c(s-\tau)^{2j-1}\right) ds dt < \frac{a}{2}. \tag{6.80}$$

Let us introduce the linear space X of all real functions $x \in C[t_0, \infty)$ such that

$$\sup_{t\geq t_0} \frac{|x(t)|}{K(t)} < \infty.$$

It is not difficult to verify that X endowed with the norm

$$\|x\| = \sup_{t\geq t_0} \frac{|x(t)|}{K(t)}$$

is a Banach space. Define a subset Ω of X by

$$\Omega = \left\{ x \in X : \; aK(t) \leq x(t) \leq 2aK(t), \; t \geq t_0 \right\}.$$

Then Ω is a bounded, convex, and closed subset of X. Let us further define an operator $F : \Omega \to X$ as

$$(Fx)(t) = \begin{cases} \dfrac{3a}{2}K(t) + \displaystyle\int_T^t \frac{(t-s)^{2j-2}}{(2j-2)!} \int_s^\infty \frac{(u-s)^{m-2j-1}}{r(u)(m-2j-1)!} & \\ \qquad \times \displaystyle\int_u^\infty f\left(v, x(v-\tau)\right) dvduds & \text{if} \quad t \geq T, \\ (Fx)(T) & \text{if} \quad t_0 \leq t < T. \end{cases}$$

The mapping F has the following properties. First of all, F maps Ω into Ω. Indeed, if $x \in \Omega$, then

$$(Fx)(t) \geq \frac{3a}{2}K(t) \geq aK(t), \quad t \geq t_0.$$

Furthermore, by (6.80), we also have

$$\begin{aligned} (Fx)(t) &\leq \frac{3a}{2}K(t) \\ &\quad + \frac{(t-T)^{2j-1}}{(2j-1)!} \int_T^\infty \frac{(u-T)^{m-2j-1}}{r(u)(m-2j-1)!} \int_u^\infty f\left(v, x(v-\tau)\right) dvdu \\ &\leq \frac{3a}{2}K(t) + \frac{a}{2}K(t) = 2aK(t). \end{aligned}$$

Next, we show that F is continuous. To see this, let $\varepsilon > 0$. Choose $T_1 \geq T$ so large that

$$\int_{T_1}^\infty \frac{(t-T)^{m-2j-1}}{r(t)(m-2j-1)!} \int_t^\infty f\left(s, c(s-\tau)^{2j-1}\right) dsdt < \varepsilon \tag{6.81}$$

and

$$\int_{T_1}^\infty f\left(s, c(s-\tau)^{2j-1}\right) ds < \varepsilon. \tag{6.82}$$

Let $\{x_k\}_{k\in\mathbb{N}} \subset \Omega$ be such that $x_k \to x$. Since Ω is closed, $x \in \Omega$. Furthermore, for all large k,

$$\begin{aligned} &\left| \int_T^\infty f\left(s, x_k(s-\tau)\right) ds - \int_T^\infty f\left(s, x(s-\tau)\right) ds \right| \\ &\leq \left| \int_T^{T_1} f\left(s, x_k(s-\tau)\right) ds - \int_T^{T_1} f\left(s, x(s-\tau)\right) ds \right| \\ &\quad + \left| \int_{T_1}^\infty f\left(s, x_k(s-\tau)\right) ds \right| + \left| \int_{T_1}^\infty f\left(s, x(s-\tau)\right) ds \right| \\ &\leq 3h\varepsilon, \end{aligned}$$

where $h = 1$ if f is superlinear, and $h = 1/2$ if f is sublinear. In view of the definition of F,

$$\begin{aligned}
\Big|(Fx_k)(t) - (Fx)(t)\Big| &\leq K(t) \int_T^{T_1} \frac{(u-T)^{m-2j-1}}{r(u)(m-2j-1)!} \\
&\quad \times \int_u^\infty \Big| f\Big(s, x_k(s-\tau)\Big) - f\Big(s, x(s-\tau)\Big)\Big| \, ds du \\
&\quad + K(t) \left| \int_{T_1}^\infty \frac{(u-T)^{m-2j-1}}{r(u)(m-2j-1)!} \int_u^\infty f\Big(s, x_k(s-\tau)\Big) \, ds du \right| \\
&\quad + K(t) \left| \int_{T_1}^\infty \frac{(u-T)^{m-2j-1}}{r(u)(m-2j-1)!} \int_u^\infty f\Big(s, x(s-\tau)\Big) \, ds du \right| \\
&\leq 3h\varepsilon K(t).
\end{aligned}$$

This shows that $\|Fx_k - Fx\|$ tends to zero, i.e., F is continuous. Finally, note that when $t_2 > t_1 \geq T$,

$$\begin{aligned}
&\left(\frac{Fx}{K}\right)(t_1) - \left(\frac{Fx}{K}\right)(t_2) \\
&= \int_T^{t_1} \frac{(t_1-s)^{2j-2}}{t_1^{2j-1}(2j-2)!} \int_s^\infty \frac{(u-s)^{m-2j-1}}{r(u)(m-2j-1)!} \int_u^\infty f\Big(v, x(v-\tau)\Big) \, dvduds \\
&\quad - \int_T^{t_2} \frac{(t_2-s)^{2j-2}}{t_2^{2j-1}(2j-2)!} \int_s^\infty \frac{(u-s)^{m-2j-1}}{r(u)(m-2j-1)!} \int_u^\infty f\Big(v, x(v-\tau)\Big) \, dvduds \\
&\leq \int_T^{t_1} \frac{(t_1-s)^{2j-2}}{t_1^{2j-1}(2j-2)!} \int_s^\infty \frac{(u-s)^{m-2j-1}}{r(u)(m-2j-1)!} \int_u^\infty f\Big(v, x(v-\tau)\Big) \, dvduds \\
&\quad - \int_T^{t_1} \frac{(t_1-s)^{2j-2}}{t_2^{2j-1}(2j-2)!} \int_s^\infty \frac{(u-s)^{m-2j-1}}{r(u)(m-2j-1)!} \int_u^\infty f\Big(v, x(v-\tau)\Big) \, dvduds \\
&= \frac{t_2^{2j-1} - t_1^{2j-1}}{t_1^{2j-1} t_2^{2j-1}} \int_T^{t_1} \frac{(t_1-s)^{2j-2}}{(2j-2)!} \int_s^\infty \frac{(u-s)^{m-2j-1}}{r(u)(m-2j-1)!} \\
&\quad \times \int_u^\infty f\Big(v, x(v-\tau)\Big) \, dvduds \\
&\leq \frac{t_2^{2j-1} - t_1^{2j-1}}{t_2^{2j-1}} \int_T^\infty \frac{(u-T)^{m-2j-1}}{r(u)(m-2j-1)!} \int_u^\infty f\Big(v, x(v-\tau)\Big) \, dvdu \\
&= \frac{t_2^{2j-1} - t_1^{2j-1}}{t_2^{2j-1}} \left[\int_{T_1}^\infty \frac{(u-T)^{m-2j-1}}{r(u)(m-2j-1)!} \int_u^\infty f\Big(v, x(v-\tau)\Big) \, dvdu \right. \\
&\quad \left. + \int_T^{T_1} \frac{(u-T)^{m-2j-1}}{r(u)(m-2j-1)!} \int_u^\infty f\Big(v, x(v-\tau)\Big) \, dvdu \right] \\
&\leq \varepsilon + \frac{t_2^{2j-1} - t_1^{2j-1}}{t_2^{2j-1}} \int_T^{T_1} \frac{(u-T)^{m-2j-1}}{r(u)(m-2j-1)!} \int_u^\infty f\Big(v, x(v-\tau)\Big) \, dvdu,
\end{aligned}$$

so

$$\left| \left(\frac{Fx}{K}\right)(t_1) - \left(\frac{Fx}{K}\right)(t_2) \right|$$

$$\leq \varepsilon + \frac{t_2^{2j-1} - t_1^{2j-1}}{t_2^{2j-1}} \int_T^{T_1} \frac{(u-T)^{m-2j-1}}{r(u)(m-2j-1)!} \int_u^\infty f\left(v, x(v-\tau)\right) dvdu.$$

Hence, there exists $\delta > 0$ such that for all $x \in \Omega$,

$$\left| \left(\frac{Fx}{K}\right)(t_1) - \left(\frac{Fx}{K}\right)(t_2) \right| \leq 2\varepsilon \quad \text{if} \quad |t_1 - t_2| < \delta.$$

Therefore, $F\Omega$ is relatively compact. In view of Schauder's fixed point theorem, we see that there exists $x^* \in \Omega$ such that $Fx^* = x^*$. It is easy to check that x^* is an eventually positive solution of (6.77). Furthermore, by means of L'Hôpital's rule,

$$\begin{aligned} &\lim_{t\to\infty} \frac{1}{t^{2j-1}} \int_T^t \frac{(t-s)^{2j-2}}{(2j-2)!} \int_s^\infty \frac{(u-s)^{m-2j-1}}{r(u)(m-2j-1)!} \int_u^\infty f\left(v, x(v-\tau)\right) dvduds \\ = &\ \ldots = \lim_{t\to\infty} \frac{1}{(2j-1)!} \int_t^\infty \frac{(u-s)^{m-2j-1}}{r(u)(m-2j-1)!} \int_u^\infty f\left(v, x(v-\tau)\right) dvdu \\ = &\ 0. \end{aligned}$$

Thus

$$\lim_{t\to\infty} \frac{x^*(t)}{t^{2j-1}} = \lim_{t\to\infty} \frac{(Fx^*)(t)}{t^{2j-1}} = \frac{3a}{2} \quad \text{and} \quad \lim_{t\to\infty} \frac{x^*(t)}{t^{2j-2}} = \infty.$$

Therefore $x^* \in E_j(\infty, *)$.

We now show that the converse holds. Let $x \in E_j(\infty, *)$ be an eventually positive solution of (6.77). In view of Lemmas 6.6.1 and 6.6.2, we see that $x^{(m-1)}(t) > 0$ and $x^{(m)}(t) < 0$ for t greater than or equal to some positive t_1, and $x^{(k)}$ is eventually monotone for each $k \in \{1, 2, \ldots, m-1\}$. Since $\lim_{t\to\infty} x(t)/t^{2j-1} = a > 0$, there exists $t_2 \geq t_1$ with

$$\frac{a}{2} t^{2j-1} \leq x(t) \leq \frac{3a}{2} t^{2j-1}, \quad t \geq t_2$$

so that

$$f\left(t, x(t-\tau)\right) \geq f\left(t, \frac{a}{2}(t-\tau)^{2j-1}\right), \quad t \geq t_2 + \tau = t_3$$

if f is superlinear and

$$f\left(t, x(t-\tau)\right) \geq 3f\left(t, \frac{a}{2}(t-\tau)^{2j-1}\right), \quad t \geq t_3$$

if f is sublinear. We assert that

$$\lim_{t\to\infty} x^{(2j-1)}(t) = (2j-1)!a.$$

In fact, by means of L'Hôpital's rule,

$$a = \lim_{t\to\infty} \frac{x(t)}{t^{2j-1}} = \lim_{t\to\infty} \frac{x'(t)}{(2j-1)t^{2j-2}} = \ldots = \lim_{t\to\infty} \frac{x^{(2j-1)}(t)}{(2j-1)!}.$$

In case $j < \frac{m}{2}$, we see further that

$$0 = \lim_{t\to\infty} x^{(2j)}(t) = \lim_{t\to\infty} x^{(2j+1)}(t) = \ldots = \lim_{t\to\infty} x^{(m-1)}(t).$$

Since $x^{(i)}$ is eventually monotone for $i \in \{2j, 2j+1, \ldots, m-1\}$, we find by means of (6.77),

$$r(s)x^{(m-1)}(s) + \int_t^s f\left(v, x(v-\tau)\right) dv = r(t)x^{(m-1)}(t), \quad s > t \geq t_3$$

so that

$$x^{(m-1)}(t) \geq \frac{1}{r(t)} \int_t^\infty f\left(v, x(v-\tau)\right) dv, \quad t \geq t_3.$$

Integrating the above inequality successively and invoking (6.81) if necessary, we see that

$$x^{(2j)}(t) \geq \int_t^\infty \frac{(u-t)^{m-2j-2}}{r(u)(m-2j-2)!} \int_u^\infty f\left(v, x(v-\tau)\right) dvdu, \quad t \geq t_3.$$

Integrating the above inequality one more time, we then obtain

$$\begin{aligned} & a(2j-1)! - x^{(2j-1)}(t_3) \\ \geq \; & \int_{t_3}^\infty \int_s^\infty \frac{(u-s)^{m-2j-2}}{r(u)(m-2j-2)!} \int_u^\infty f\left(v, x(v-\tau)\right) dvduds \\ \geq \; & C \int_{t_3}^\infty \frac{(u-t_3)^{m-2j-1}}{r(u)} \int_u^\infty f\left(v, c(v-\tau)^{2j-1}\right) dvdu \end{aligned}$$

for some appropriate constants C and c. The proof is complete. □

Theorem 6.6.5. *Suppose that m is even and that* (H1) *and* (H2) *hold. Suppose further that f is superlinear or sublinear. If there exist a constant $c > 0$ and $j \in \{1, 2, \ldots, m/2\}$ such that*

$$\int_{t_0}^\infty \frac{s^{m-2j}}{r(s)} \int_s^\infty \left| f\left(u, c(u-\tau)^{2j-2}\right) \right| duds < \infty,$$

then (6.77) *has an eventually positive solution in $E_j(\infty, 0)$. The converse is also true.*

Proof. The proof of the sufficiency part is similar to that of Theorem 6.6.4 and is therefore only sketched as follows. Let $a = c/2$ if f is superlinear and $a = c$ if f is sublinear. Set

$$K(t) = t^{2j-2}, \quad t \geq t_0.$$

Then as in the proof of Theorem 6.6.4, we see that there exist a number $t_1 \geq t_0$ and a function x^* such that

$$aK(t) \leq x^*(t) \leq 2aK(t), \quad t \geq t_1 + \tau = T$$

and

$$\begin{aligned} x^*(t) = \; & \frac{3a}{2} K(t) \\ & + \int_T^t \frac{(t-s)^{2j-3}}{(2j-3)!} \int_s^\infty \frac{(u-T)^{m-2j}}{r(u)(m-2j)!} \int_u^\infty f\left(v, x^*(v-\tau)\right) dvduds. \end{aligned}$$

Then by means of L'Hôpital's rule, we may show that

$$\lim_{t\to\infty} \frac{x^*(t)}{t^{2j-2}} = \frac{3a}{2} + \beta,$$

where β is a constant satisfying

$$0 < \beta \leq \int_T^\infty \frac{(t-T)^{m-2j-1}}{r(t)(m-2j-1)!} \int_t^\infty 2f\left(s, c(s-\tau)^{2j-2}\right) dsdt.$$

It follows that

$$\lim_{t\to\infty} \frac{x^*(t)}{t^{2j-1}} = 0.$$

This shows that x^* is an eventually positive solution in $E_j(*, 0)$. $\square$

Theorem 6.6.6. *Suppose that m is odd and that* (H1) *and* (H2) *hold. Suppose further that f is superlinear or sublinear. If there exist a constant $c > 0$ and $j \in \{1, 2, \ldots, (m-1)/2\}$ such that*

$$\int_{t_0}^{\infty} \frac{s^{m-2j-2}}{r(s)} \int_s^{\infty} \left| f\left(u, c(u-\tau)^{2j}\right)\right| duds < \infty,$$

then (6.77) *has an eventually positive solution in $O_j(\infty, *)$. The converse is also true.*

Proof. The proof is similar to that of Theorem 6.6.4. We only need to note that the function K there should be replaced by

$$K(t) = t^{2j}$$

and the mapping F should be modified as

$$(Fx)(t) = \begin{cases} \dfrac{3a}{2} K(t) + \displaystyle\int_T^t \frac{(t-s)^{2j-1}}{(2j-1)!} \int_s^{\infty} \frac{(u-s)^{m-2j-2}}{r(u)(m-2j-2)!} \\ \qquad \times \displaystyle\int_u^{\infty} f\left(v, x(v-\tau)\right) dvduds & \text{if} \quad t \geq T, \\ (Fx)(T) & \text{if} \quad t_0 \leq t < T. \end{cases}$$

The rest of the proof is as in the proof of Theorem 6.6.4. $\square$

Theorem 6.6.7. *Suppose that m is odd and that* (H1) *and* (H2) *hold. Suppose further that f is superlinear or sublinear. If there exist a constant $c > 0$ and $j \in \{1, 2, \ldots, (m-1)/2\}$ such that*

$$\int_{t_0}^{\infty} \frac{s^{m-2j-1}}{r(s)} \int_s^{\infty} \left| f\left(u, c(u-\tau)^{2j-1}\right)\right| duds < \infty,$$

then (6.77) *has an eventually positive solution in $O_j(\infty, 0)$. The converse is also true.*

Theorem 6.6.8. *Suppose that m is odd and that* (H1) *and* (H2) *hold. Suppose further that f is superlinear or sublinear. If there exist a constant $c > 0$ and $j \in \{1, 2, \ldots, (m-1)/2\}$ such that*

$$\int_{t_0}^{\infty} \frac{s^{m-2}}{r(s)} \int_s^{\infty} |f(u, c)| \, duds < \infty,$$

then (6.77) *has an eventually positive solution which converges to a positive constant. The converse is also true.*

Proof. The proof is similar to that of Theorem 6.6.4. We only need to note that the function K should now be replaced by

$$K(t) \equiv 1$$

and the mapping F should be modified as

$$(Fx)(t) = \begin{cases} \dfrac{3a}{2} + \displaystyle\int_t^{\infty} \frac{(s-t)^{m-2}}{r(s)(m-2)!} \int_s^{\infty} f\left(u, x(u-\tau)\right) duds \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if} \quad t \geq T, \\ (Fx)(T) \qquad\qquad\qquad\qquad\qquad \text{if} \quad t_0 \leq t < T. \end{cases}$$

The rest of the proof follows as in the proof of Theorem 6.6.4. □

Theorem 6.6.9. *Suppose that m is odd and that* (H1) *and* (H2) *hold. Suppose further that f is nondecreasing in x. Then* (6.77) *has an eventually positive solution x which converges to zero if*

$$t\int_t^\infty \frac{(s-t)^{m-2}}{r(s)(m-2)!}\int_s^\infty \left| f\left(u, \frac{1}{u-\tau}\right)\right| duds \le 1$$

holds for $t \ge T \ge m+\tau$.

Proof. Let X be the partially ordered Banach space of all real functions endowed with the usual sup-norm and pointwise ordering. Define a subset Ω of X by

$$\Omega = \left\{x \in X : \ x \text{ is nondecreasing and } 0 \le x(t) \le 1, \ t \ge t_0\right\}.$$

For any subset M of Ω, it is clear that $\inf M \in \Omega$ and $\sup M \in \Omega$. Define an operator F on Ω by

$$(Fx)(t) = \begin{cases} t\displaystyle\int_t^\infty \frac{(s-t)^{m-2}}{r(s)(m-2)!}\int_s^\infty f\left(u, \frac{x(u-\tau)}{u-\tau}\right) duds \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if} \quad t \ge T, \\ \exp\left(\dfrac{\ln((Fx)(T))t}{T}\right) \qquad\qquad \text{if} \quad t_0 \le t < T. \end{cases}$$

By means of (6.82), we see that F maps Ω into Ω. Furthermore, it is clear that F is an increasing mapping. By means of Knaster's fixed point theorem, there exists a function y^* such that $Fy^* = y^*$. If we let

$$x^*(t) = \frac{y^*(t)}{t}, \quad t \ge T,$$

then

$$x^*(t) = \int_t^\infty \frac{(s-t)^{m-2}}{r(s)(m-2)!}\int_s^\infty f\left(u, x^*(u-\tau)\right) duds.$$

By differentiating both sides of the above equation, we may easily verify that x^* is a solution of (6.77) for all large t. Since x^* is eventually positive and converges to zero, we have found the desired solution. The proof is complete. □

Remark 6.6.10. We remark that it would not be difficult to extend all the results in this section to an equation whose nonlinear term has the form

$$f\left(t, x(t-\tau_1), \ldots, x(t-\tau_n)\right).$$

6.7. Asymptotic Trichotomy for Positive Solutions

This section is concerned with higher order nonlinear functional differential equations of the form

$$x^{(n)}(t) + \sigma f\left(t, x(g(t))\right) = 0, \quad t \ge t_0, \tag{6.83}$$

where $n \in \mathbb{N}\setminus\{1\}$, $\sigma \in \{-1, 1\}$, g is a continuous and nondecreasing function such that $\lim_{t\to\infty} g(t) = \infty$, and $f : [t_0, \infty) \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that

$$f(t, x) \ge 0 \quad \text{for} \quad (t, x) \in [t_0, \infty) \times (0, \infty). \tag{6.84}$$

It is well known [**145**] that if x is a positive solution of (6.83), then there exists $k \in \mathbb{N}$ such that $0 \le k \le n$, $(-1)^{n-k-1}\sigma = 1$, and

$$\begin{cases} x^{(i)}(t) > 0 & \text{if} \quad 0 \le i \le k-1, \\ (-1)^{i-k} x^{(i)}(t) > 0 & \text{if} \quad k \le i \le n \end{cases} \tag{6.85}$$

for $t \ge T_x$, where T_x is sufficiently large. Denote by P and P_k, respectively, the set of all positive solutions of (6.83) and the set of all positive solutions x of (6.83) satisfying (6.85). Then the above observation means that P has the decomposition

$$P = \begin{cases} P_1 \cup P_3 \cup \ldots \cup P_{n-1} & \text{if} \quad \sigma = 1 \quad \text{and} \quad n \text{ is even}, \\ P_0 \cup P_2 \cup \ldots \cup P_{n-1} & \text{if} \quad \sigma = 1 \quad \text{and} \quad n \text{ is odd}, \\ P_0 \cup P_2 \cup \ldots \cup P_n & \text{if} \quad \sigma = -1 \quad \text{and} \quad n \text{ is even}, \\ P_1 \cup P_3 \cup \ldots \cup P_n & \text{if} \quad \sigma = -1 \quad \text{and} \quad n \text{ is odd}. \end{cases}$$

In what follows our attention will be restricted to the classes P_k with k such that

$$0 < k < n \quad \text{and} \quad (-1)^{n-k-1}\sigma = 1. \tag{6.86}$$

If $x \in P_k$ for k satisfying (6.86), then, in view of (6.85), there exist $c_1 > 0$, $c_2 > 0$, and $T \ge T_x$ such that

$$c_1 t^{k-1} \le x(t) \le c_2 t^k, \quad t \ge T \tag{6.87}$$

and exactly one of the following three cases occurs:

$$\lim_{t\to\infty} x^{(k)}(t) \equiv \text{constant} > 0, \tag{6.88}$$

$$\lim_{t\to\infty} x^{(k)}(t) = 0 \quad \text{and} \quad \lim_{t\to\infty} x^{(k-1)}(t) = \infty, \tag{6.89}$$

or

$$\lim_{t\to\infty} x^{(k-1)}(t) \equiv \text{constant} > 0. \tag{6.90}$$

This suggests a further decomposition of P_k as

$$P_k = P_k[\max] \cup P_k[\text{int}] \cup P_k[\min],$$

where $P_k[\max]$, $P_k[\text{int}]$ and $P_k[\min]$ denote the sets of all $x \in P_k$ satisfying (6.88), (6.89), and (6.90), respectively, and naturally raises the question of characterizing these classes of P_k.

Theorem 6.7.1. *Suppose that $f(t,x)$ satisfies* (6.84) *and is either nondecreasing or nonincreasing in $x \in (0,\infty)$ for each fixed $t \in [t_0,\infty)$. Let $k \in \mathbb{N}$ satisfy* (6.86). *Then*

(i) $P_k[\max] \ne \emptyset$ *for* (6.83) *if and only if*

$$\int^{\infty} t^{n-k-1} f\left(t, c\,[g(t)]^k\right) dt < \infty \quad \textit{for some} \quad c > 0;$$

(ii) $P_k[\min] \ne \emptyset$ *for* (6.83) *if and only if*

$$\int^{\infty} t^{n-k} f\left(t, c\,[g(t)]^{k-1}\right) dt < \infty \quad \textit{for some} \quad c > 0. \tag{6.91}$$

Proof. The proof is similar to that of Theorem 6.6.4 and will be omitted. $\square$

In the following, we shall consider the subclasses $P_k[\text{int}]$ subject to the assumption that the nonlinear term $f(t,x)$ is *nonincreasing* in x or *nondecreasing* in x. First we give a lemma which is important for proving the necessity of the following main results.

Lemma 6.7.2 ([**156**]). *Consider the equation*

$$x^{(n)}(t) + \sigma p(t)[x(t)]^{\gamma} = 0, \quad t \geq t_0, \tag{6.92}$$

where $\gamma \in (0,1)$. *Suppose* $k \in \mathbb{N}$ *satisfies* (6.86). *Then* $P_k[\text{int}] \neq \emptyset$ *for* (6.92) *if and only if*

$$\int^{\infty} t^{n-k-1+k\gamma} p(t)dt < \infty \quad \text{and} \quad \int^{\infty} t^{n-k+(k-1)\gamma} p(t)dt = \infty.$$

Theorem 6.7.3. *In addition to* (6.84) *assume that* $f(t,x)$ *is nondecreasing in* $x \in (0,\infty)$ *for each fixed* $t \geq t_0$. *Suppose* $k \in \mathbb{N}$ *satisfies* (6.86). *Then* $P_k[\text{int}] \neq \emptyset$ *if*

$$\int^{\infty} t^{n-k-1} f\left(t, a\,[g(t)]^k\right) dt < \infty \quad \text{for some} \quad a > 0 \tag{6.93}$$

and

$$\int^{\infty} t^{n-k} f\left(t, b\,[g(t)]^{k-1}\right) dt = \infty \quad \text{for every} \quad b > 0. \tag{6.94}$$

Proof. Let $T \geq t_0$ be so large such that

$$\frac{b}{g(t)} < a \quad \text{and} \quad \frac{b}{t} + \frac{(t-T)^k}{t^k k!} \int_T^{\infty} \frac{(s-T)^{n-k-1}}{(n-k-1)!} f\left(s, a\,[g(s)]^k\right) ds < a \tag{6.95}$$

for $t \geq T$. Such a number exists since $g(t) \to \infty$ as $t \to \infty$ and (6.93) holds. Define an operator F by

$$(Fx)(t) = \begin{cases} \dfrac{b}{t} + \dfrac{1}{t^k} \displaystyle\int_T^t \frac{(t-s)^{k-1}}{(k-1)!} \int_s^{\infty} \frac{(r-s)^{n-k-1}}{(n-k-1)!} f\left(r, [g(r)]^k x(g(r))\right) drds & \\ & \text{if} \quad t \geq T, \\ (Fx)(T) & \text{if} \quad t_0 \leq t < T. \end{cases}$$

Consider the sequence $\{x_i(t)\}$ of successive approximations defined by

$$x_1(t) \equiv 0 \quad \text{and} \quad x_{i+1}(t) = (Fx_i)(t), \quad i \in \mathbb{N} \quad \text{for} \quad t \geq t_0.$$

In view of the nondecreasing property of $f(t,x)$, it is easy to see that

$$0 \leq x_i(t) \leq x_{i+1}(t), \quad t \geq t_0, \quad i \in \mathbb{N}.$$

On the other hand,

$$x_2(t) = \frac{b}{t} < a \quad \text{and} \quad x_2(g(t)) = \frac{b}{g(t)} < a \quad \text{for} \quad t \geq t_0,$$

and by induction $x_i(t) \leq a$ for $t \geq t_0$ implies by (6.95)

$$\begin{aligned} x_{i+1}(t) &\leq \frac{b}{t} + \frac{1}{t^k} \int_T^t \frac{(t-s)^{k-1}}{(k-1)!} \int_s^{\infty} \frac{(r-s)^{n-k-1}}{(n-k-1)!} f\left(r, [g(r)]^k x_i(g(r))\right) drds \\ &\leq \frac{b}{t} + \frac{(t-T)^k}{t^k k!} \int_T^{\infty} \frac{(r-T)^{n-k-1}}{(n-k-1)!} f\left(r, a[g(r)]^k\right) dr \; < \; a \end{aligned}$$

for $t \geq t_0$. Thus $\{x_i\}$ is pointwise convergent to some function x^*. By means of Lebesgue's dominated convergence theorem, we obtain $Fx^* = x^*$. In view of (6.95), it is clear that $b/t \le x^*(t) \le a$, $t \ge t_0$. We assert that the function z defined by $z(t) = x^*(t)t^k$, $t \ge t_0$, is an eventually positive solution of (6.83) in $P_k[\text{int}]$. Indeed, note that

$$z(t) = bt^{k-1} + \int_T^t \frac{(t-s)^{k-1}}{(k-1)!} \int_s^\infty \frac{(r-s)^{n-k-1}}{(n-k-1)!} f\left(r, z(r)\right) drds \tag{6.96}$$

for $t \ge t_0$. Differentiating (6.96) $k-1$ times, we see that

$$z^{(k-1)}(t) = b + \int_T^t \int_s^\infty \frac{(r-s)^{n-k-1}}{(n-k-1)!} f\left(r, z(r)\right) drds, \quad t \ge t_0 \tag{6.97}$$

and hence

$$z^{(k)}(t) = \int_t^\infty \frac{(r-t)^{n-k-1}}{(n-k-1)!} f\left(r, z(r)\right) dr, \quad t \ge t_0. \tag{6.98}$$

In view of (6.86), it is clear that $z = z(t)$ satisfies (6.83) at every point of $[T, \infty)$. By (6.98) we have $\lim_{t\to\infty} z^{(k)}(t) = 0$. Since (6.97) and (6.98) imply that $z^{(k-1)}(t)$ is positive and increasing in $[T, \infty)$, $z^{(k-1)}(t)$ either converges to some positive limit or diverges to ∞ as $t \to \infty$. Assume that the first case holds. Then this means that $z \in P_k[\text{min}]$, and so (6.91) holds by Theorem 6.7.1. But this contradicts the assumption (6.94). Thus, we conclude that $\lim_{t\to\infty} z^{(k-1)}(t) = \infty$, implying that z is a solution of (6.83) belonging to $P_k[\text{int}]$. $\square$

Necessary conditions for the existence of solutions of (6.83) belonging to $P_k[\text{int}]$ are given in the following theorem.

Theorem 6.7.4. *In addition to* (6.84) *assume that $f(t, x)$ is nondecreasing in $x \in (0, \infty)$ for each fixed $t \ge t_0$. Suppose $k \in \mathbb{N}$ satisfies* (6.86). *If* (6.83) *has a positive solution in the class $P_k[\text{int}]$, then*

$$\int^\infty t^{n-k-1} f\left(t, a\left[g(t)\right]^{k-1}\right) dt < \infty \quad \textit{for every} \quad a > 0 \tag{6.99}$$

and

$$\int^\infty t^{n-k} f\left(t, b\left[g(t)\right]^{k}\right) dt = \infty \quad \textit{for every} \quad b > 0. \tag{6.100}$$

Proof. Let $x \in P_k[\text{int}]$ for (6.83). Then

$$\lim_{t\to\infty} \frac{x(t)}{t^k} = 0 \quad \text{and} \quad \lim_{t\to\infty} \frac{x(t)}{t^{k-1}} = \infty.$$

Hence, for any $c_1 > 0$, $c_2 > 0$, there exists $t_1 \ge t_0$ such that

$$x(t) \ge c_1 t^{k-1}, \quad x(g(t)) \ge c_1 [g(t)]^{k-1}$$

and

$$x(t) \le c_2 t^k, \quad x(g(t)) \le c_2 [g(t)]^k$$

hold for $t \ge t_1$, which, in view of the nondecreasing property of $f(t, x)$, implies that

$$f\left(t, x(g(t))\right) [x(t)]^{-\gamma} \ge c_2^{-\gamma} t^{-k\gamma} f\left(t, c_1 \left[g(t)\right]^{k-1}\right)$$

and

$$f\left(t, x(g(t))\right) [x(t)]^{-\gamma} \le c_1^{-\gamma} t^{-(k-1)\gamma} f\left(t, c_2 \left[g(t)\right]^{k}\right)$$

hold for $t \ge t_1$. Lemma 6.7.2 applied to the equation

$$x^{(n)}(t) + \sigma f\left(t, x(g(t))\right) [x(t)]^{-\gamma} [x(t)]^{\gamma} = 0 \quad \text{for} \quad \gamma \in (0, 1)$$

implies that

$$\int^{\infty} t^{n-k-1+k\gamma} f\left(t, x(g(t))\right) [x(t)]^{-\gamma}\, dt < \infty$$

and

$$\int^{\infty} t^{n-k+(k-1)\gamma} f\left(t, x(g(t))\right) [x(t)]^{-\gamma}\, dt = \infty.$$

This shows that (6.99) and (6.100) hold. $\square$

Now we consider the equation

$$x^{(n)}(t) + \sigma p(t)\phi\left(x(g(t))\right) = 0, \quad t \ge t_0, \tag{6.101}$$

where $p : [t_0, \infty) \to [0, \infty)$, $\phi : (0, \infty) \to (0, \infty)$ are continuous, and g is a continuous and *nondecreasing* function such that $\lim_{t\to\infty} g(t) = \infty$. Then it is easy to prove the following result.

Theorem 6.7.5. *Assume that ϕ is nonincreasing and satisfies*

$$\lim_{x\to\infty} \phi(x) > 0, \tag{6.102}$$

and let $k \in \mathbb{N}$ satisfy (6.86). *Then* (6.101) *has a positive solution in the class* $P_k[\text{int}]$ *if and only if*

$$\int^{\infty} t^{n-k-1} p(t) dt < \infty \quad \text{and} \quad \int^{\infty} t^{n-k} p(t) dt = \infty. \tag{6.103}$$

Similarly, from Theorems 6.7.3 and 6.7.4, we can obtain the following result.

Theorem 6.7.6. *Assume that ϕ is nondecreasing and satisfies*

$$0 < \phi(x) \le c, \quad \text{where } c \text{ is a constant}, \tag{6.104}$$

and let $k \in \mathbb{N}$ satisfy (6.86). *Then* (6.101) *has a solution in the class* $P_k[\text{int}]$ *if and only if* (6.103) *holds.*

Example 6.7.7. As an example, consider the equation

$$x^{(n)}(t) + \sigma p(t) \frac{[x(g(t))]^{\gamma}}{1 + [x(g(t))]^{\gamma}} = 0, \quad t \ge t_0, \tag{6.105}$$

where $\gamma > 0$ is constant and g is defined as in Theorem 6.7.3. By Theorem 6.7.6, (6.105) has a solution in the class $P_k[\text{int}]$ if and only if (6.103) holds. But Theorem 6.7.5 is not applicable to (6.105) since ϕ is increasing.

By Theorems 6.7.5 and 6.7.6, we have the following result.

Theorem 6.7.8. *Assume that ϕ is either nondecreasing or nonincreasing and satisfies*

$$a \le \phi(x) \le b, \quad x \in (0, \infty),$$

where a and b are constants. Suppose $k \in \mathbb{N}$ satisfies (6.86). *Then* $P_k[\text{int}] \ne \emptyset$ *if and only if* (6.103) *holds.*

Next we consider (6.83) in which $f(t, x)$ is nonincreasing in x and establish conditions for this equation to possess positive solutions in the subclasses $P_k[\text{int}]$ for $k \in \mathbb{N}$ satisfying (6.86).

Theorem 6.7.9. *In addition to* (6.84) *assume that* $f(t,x)$ *is nonincreasing in* $x \in (0,\infty)$ *for each fixed* $t \ge t_0$. *Suppose* $k \in \mathbb{N}$ *satisfies* (6.86). *Then* $P_k[\text{int}] \neq \emptyset$ *if* (6.93) *and* (6.94) *hold.*

Proof. If $k > 1$, then define

$$q_{k-2}(t) = \sum_{j=0}^{k-2} \frac{\alpha_j (t-t_0)^j}{j!},$$

where $\alpha_m > 0$, $0 \le m \le k-2$, are arbitrary fixed constants, and if $k = 1$, put $q_{k-2}(t) = 0$. Let $\delta > a(k-1)!$ be fixed, where a is the number in (6.93). Define by $C[t_0,\infty)$ the space of all continuous functions on $[t_0,\infty)$ with the topology of uniform convergence on every compact subinterval of $[t_0,\infty)$, and consider the subset X of $C[t_0,\infty)$ consisting of all $x \in C[t_0,\infty)$ such that

$$\begin{aligned} q_{k-2}(t) + \frac{\delta(t-t_0)^{k-1}}{(k-1)!} &\le x(t) \le q_{k-2}(t) + \frac{\delta(t-t_0)^{k-1}}{(k-1)!} \\ &+ \int_{t_0}^t \frac{(t-s)^{k-1}}{(k-1)!} \int_s^\infty \frac{(r-s)^{n-k-1}}{(n-k-1)!} f\left(r, g\left(q_{k-2}(r) + \frac{\delta(r-t_0)^{k-1}}{(k-1)!}\right)\right) drds \end{aligned}$$

for $t \ge t_0$. A solution x of (6.83) with the required properties is obtained as a fixed point of the operator $M : X \to C[t_0,\infty)$ defined by

$$\begin{aligned} Mx(t) = q_{k-2}(t) &+ \frac{\delta(t-t_0)^{k-1}}{(k-1)!} \\ &+ \int_{t_0}^t \frac{(t-s)^{k-1}}{(k-1)!} \int_s^\infty \frac{(r-s)^{n-k-1}}{(n-k-1)!} f\Big(r, x(r)\Big)\, drds \end{aligned}$$

for $t \ge t_0$. It is clear that M is well defined on X and maps X into $C[t_0,\infty)$. A routine computation shows that

(i) M maps X into X;
(ii) M is continuous on X;
(iii) MX is relatively compact.

The Schauder–Tychonov theorem (Theorem 1.4.25) then implies that M has a fixed point in X. Let $x \in X$ be a fixed point of M. Differentiating the equation $x = Mx$, we see that

$$x^{(k-1)}(t) = \delta + \int_{t_0}^t \int_s^\infty \frac{(r-s)^{n-k-1}}{(n-k-1)!} f\Big(r, x(r)\Big)\, drds, \tag{6.106}$$

and hence

$$x^{(k)}(t) = \int_t^\infty \frac{(r-t)^{n-k-1}}{(n-k-1)!} f\Big(r, x(r)\Big)\, dr \tag{6.107}$$

hold for $t \ge t_0$. In view of (6.87), it is also clear that $x = x(t)$ satisfies (6.83) at every point of $[t_0,\infty)$. By (6.107), we have $\lim_{t\to\infty} x^{(k)}(t) = 0$. Since (6.106) and (6.107) imply that $x^{(k-1)}(t)$ is positive and increasing in $[t_0,\infty)$, $x^{(k-1)}(t)$ either converges to some positive limit or diverges to ∞ as $t \to \infty$. Assume that the first case holds. Then this means that $x \in P_k[\text{min}]$, and so (6.91) holds by Theorem 6.7.1. But this contradicts the assumption (6.94). Thus we conclude that $\lim_{t\to\infty} x^{(k-1)}(t) = \infty$, implying that x is a solution of (6.83) belonging to $P_k[\text{int}]$. □

Necessary conditions for the existence of solutions of (6.83) in $P_k[\text{int}]$ are given in the following theorem.

Theorem 6.7.10. *In addition to* (6.84) *assume that* $f(t,x)$ *is nonincreasing in* $x \in (0,\infty)$ *for each fixed* $t \geq t_0$. *Let* $k \in \mathbb{N}$ *satisfy* (6.86). *If* (6.83) *has a positive solution in the class* $P_k[\text{int}]$, *then*

$$\int^{\infty} t^{n-k-1} f\left(t, a\,[g(t)]^k\right) dt < \infty \quad \textit{for every} \quad a > 0$$

and

$$\int^{\infty} t^{n-k} f\left(t, b\,[g(t)]^{k-1}\right) dt = \infty \quad \textit{for every} \quad b > 0.$$

Proof. The proof is similar to that of Theorem 6.7.4 and will be omitted here. □

As an application, we consider the equation (6.101), where g is a continuous and *nonincreasing* function such that $\lim_{t\to\infty} g(t) = \infty$. Then it is easy to prove the following results.

Theorem 6.7.11. *Assume that* ϕ *is nonincreasing and satisfies* (6.102), *and let* $k \in \mathbb{N}$ *satisfy* (6.86). *Then* (6.101) *has a positive solution in the class* $P_k[\text{int}]$ *if and only if* (6.103) *holds.*

Theorem 6.7.12. *Assume that* ϕ *is nonincreasing and satisfies* (6.104), *and let* $k \in \mathbb{N}$ *satisfy* (6.86). *Then* (6.101) *has a solution in the class* $P_k[\text{int}]$ *if and only if* (6.103) *holds.*

6.8. Existence of Nonoscillatory Solutions

In this section we give several sufficient conditions for the existence of positive solutions of higher order neutral differential equations of the form

$$\frac{d^n}{dt^n}\Big(x(t) - cx(t-\tau)\Big) + p(t)x(g(t)) = 0, \tag{6.108}$$

where $p, g \in C((t_0,\infty),\mathbb{R})$, $c \in \mathbb{R}$, $\tau \in \mathbb{R}^+$, and $g(t) \to \infty$ as $t \to \infty$.

Theorem 6.8.1. *Assume that* $n \in \mathbb{N}$ *is even and*

(i) $c > 0$, $p(t) \geq 0$, *and* $g(t+\tau) < t$;
(ii) *there exists a constant* $\alpha > 0$ *such that*

$$\frac{1}{c}e^{-\alpha\tau} + \frac{1}{c}\int_{t+\tau}^{\infty} \frac{(s-t-\tau)^{n-1}}{(n-1)!} p(s) \exp\Big(\alpha\,(t - g(s))\Big)\, ds \leq 1 \textit{ for large } t. \tag{6.109}$$

Then (6.108) *has a positive solution* x *satisfying* $x(t) \to 0$ *as* $t \to \infty$.

Proof. If the equal sign in (6.109) holds eventually, then (6.108) has the positive solution $x(t) = e^{-\alpha t}$. In the rest of the proof we may assume that there exists $T > t_0$ such that $t - \tau \geq t_0$, $g(t) \geq t_0$ for $t \geq T$,

$$\beta := \frac{1}{c}e^{-\alpha\tau} + \frac{1}{c}\int_{T+\tau}^{\infty} \frac{(s-T-\tau)^{n-1}}{(n-1)!} p(s) \exp\Big(\alpha\,(T - g(s))\Big)\, ds < 1,$$

and condition (6.109) holds for $t \geq T$.

Let X denote the Banach space of all continuous bounded functions defined on $[t_0, \infty)$ with the sup-norm and let Ω be the subset of X defined by

$$\Omega = \Big\{ y \in X : \; 0 \le y(t) \le 1, \; t \ge t_0 \Big\}.$$

Define a map $S : \Omega \to X$ as

$$(Sy)(t) = (S_1 y)(t) + (S_2 y)(t),$$

where

$$(S_1 y)(t) = \begin{cases} \dfrac{1}{c} e^{-\alpha\tau} y(t+\tau) & \text{if} \quad t \ge T, \\ (S_1 y)(T) + \exp\Big(\varepsilon\,(T-t)\Big) - 1 & \text{if} \quad t_0 \le t < T \end{cases}$$

and

$$(S_2 y)(t) = \begin{cases} \dfrac{1}{c} \displaystyle\int_{t+\tau}^{\infty} \frac{(s-t-\tau)^{n-1}}{(n-1)!} p(s) \exp\Big(\alpha\,(t - g(s))\Big)\, y(g(s)) ds \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if} \quad t \ge T, \\ (S_2 y)(T) \qquad\qquad\qquad\qquad \text{if} \quad t_0 \le t < T, \end{cases}$$

and $\varepsilon = \ln(2-\beta)/(T-t_0)$. It is easy to see that the integral in S_2 is defined whenever $y \in \Omega$. Clearly, the set Ω is closed, bounded, and convex in X. We shall show that for every pair $x, y \in \Omega$,

$$S_1 x + S_2 y \in \Omega. \tag{6.110}$$

In fact, for any $x, y \in \Omega$, we have by (6.109)

$$\begin{aligned} (S_1 x)(t) + (S_2 y)(t) &= \frac{1}{c} e^{-\alpha\tau} x(t+\tau) \\ &\quad + \frac{1}{c} \int_{t+\tau}^{\infty} \frac{(s-t-\tau)^{n-1}}{(n-1)!} p(s) \exp\Big(\alpha\,(t-g(s))\Big)\, y(g(s)) ds \\ &\le \frac{1}{c} e^{-\alpha\tau} + \frac{1}{c} \int_{t+\tau}^{\infty} \frac{(s-t-\tau)^{n-1}}{(n-1)!} p(s) \exp\Big(\alpha\,(t-g(s))\Big)\, ds \;\le\; 1 \end{aligned}$$

and

$$\begin{aligned} (S_1 x)(t) + (S_2 y)(t) &= (S_1 x)(T) + (S_2 y)(T) + \exp\Big(\varepsilon\,(T-t)\Big) - 1 \\ &\le \beta + \exp\Big(\varepsilon(T-t)\Big) - 1 \;\le\; 1 \end{aligned}$$

for $t \ge T$. Obviously, $(S_1 x)(t) + (S_2 y)(t) \ge 0$ for $t \ge t_0$. Thus (6.110) is proved. Next, since $0 < \frac{1}{c} e^{-\alpha\tau} < 1$, it follows that S_1 is a strict contraction. We now shall show that S_2 is completely continuous. In fact, from condition (6.109), there exists a constant $M > 0$ such that

$$\frac{1}{c} \int_{t+\tau}^{\infty} \frac{(s-t-\tau)^{n-1}}{(n-1)!} p(s) \exp\Big(\alpha\,(t-g(s))\Big)\, ds \le M \qquad \text{for} \qquad t \ge T.$$

Thus we obtain

$$\begin{aligned} \left| \frac{d}{dt} (S_2 y)(t) \right| = \bigg| &\frac{-1}{c} \int_{t+\tau}^{\infty} \frac{(s-t-\tau)^{n-2}}{(n-2)!} p(s) \exp\Big(\alpha\,(t-g(s))\Big)\, y\,(g(s))\, ds \\ &+ \frac{\alpha}{c} \int_{t+\tau}^{\infty} \frac{(s-t-\tau)^{n-1}}{(n-1)!} p(s) \exp\Big(\alpha\,(t-g(s))\Big)\, y\,(g(s))\, ds \bigg| \le M + \alpha \end{aligned}$$

for $t \geq T$ and

$$\frac{d}{dt}(S_2 y)(t) = 0 \quad \text{for} \quad t_0 \leq t \leq T.$$

This implies that S_2 is relatively compact. On the other hand, it is easy to see that S_2 is continuous and uniformly bounded, and so S_2 is completely continuous. Now, by Krasnosel′skiĭ's fixed point theorem (Theorem 1.4.27), S has a fixed point $y \in \Omega$, i.e.,

$$y(t) = \begin{cases} \dfrac{1}{c} e^{-\alpha\tau} y(t+\tau) + \dfrac{1}{c} \displaystyle\int_{t+\tau}^{\infty} \frac{(s-t-\tau)^{n-1}}{(n-1)!} p(s) \exp\Big(\alpha\,(t-g(s))\Big)\, y(g(s))ds \\ \hfill \text{if} \quad t \geq T, \\ y(T) + \exp\Big(\varepsilon\,(T-t)\Big) - 1 \hfill \text{if} \quad t_0 \leq t < T. \end{cases}$$

Since $y(t) \geq \exp(\varepsilon\,(T-t)) - 1$ for $t_0 \leq t \leq T$, it follows that $y(t) > 0$ for $t \geq t_0$. Set

$$x(t) = y(t)e^{-\alpha t}.$$

Then the above equation becomes

$$x(t) = \frac{1}{c}x(t+\tau) + \frac{1}{c}\int_{t+\tau}^{\infty} \frac{(s-t-\tau)^{n-1}}{(n-1)!} p(s) x\,(g(s))\, ds \quad \text{for} \quad t \geq T.$$

Furthermore, since $n \in \mathbb{N}$ is even, we have

$$x(t) = cx(t-\tau) + \int_{t}^{\infty} \frac{(t-s)^{n-1}}{(n-1)!} p(s) x\,(g(s))\, ds \quad \text{for} \quad t \geq T+\tau.$$

It follows that

$$\frac{d^n}{dt^n}\Big(x(t) - cx(t-\tau)\Big) + p(t)x\,(g(t)) = 0 \quad \text{for} \quad t \geq T+\tau,$$

and hence x is an eventually positive solution of (6.108) satisfying $x(t) \to 0$ as $t \to \infty$. $\square$

Theorem 6.8.2. *Assume that $n \in \mathbb{N}$ is odd and*

(i) $c \in (0,1)$, $\tau > 0$, $p(t) \geq 0$, *and* $g(t+\tau) < t$;
(ii) *there exists a constant* $\alpha > 0$ *such that*

$$ce^{\alpha\tau} + \int_{t}^{\infty} \frac{(t-s)^{n-1}}{(n-1)!} p(s) \exp\Big(\alpha\,(t-g(s))\Big)\, ds \leq 1 \text{ for large } t. \tag{6.111}$$

Then (6.108) *has a positive solution x satisfying $x(t) \to 0$ as $t \to \infty$.*

Proof. If the equal sign in (6.111) holds eventually, then (6.108) has the positive solution $x(t) = e^{-\alpha t}$. Now we assume that there exists $T > t_0$ such that $t - \tau \geq t_0$, $g(t) \geq t_0$ for $t \geq T$,

$$\beta = ce^{\alpha\tau} + \int_{T}^{\infty} \frac{(T-s)^{n-1}}{(n-1)!} p(s) \exp\Big(\alpha\,(T-g(s))\Big)\, ds < 1,$$

and condition (6.111) holds for $t \geq T$.

Define the Banach space X and its subset Ω as in the proof of Theorem 6.8.1. Define a map $S : \Omega \to X$ by

$$(Sy)(t) = (S_1 y)(t) + (S_2 y)(t),$$

where

$$(S_1y)(t) = \begin{cases} ce^{\alpha\tau}y(t-\tau) & \text{if} \quad t \geq T, \\ (S_1y)(T) + \exp\Big(\varepsilon(T-t)\Big) - 1 & \text{if} \quad t_0 \leq t < T \end{cases}$$

and

$$(S_2y)(t) = \begin{cases} \displaystyle\int_t^\infty \frac{(t-s)^{n-1}}{(n-1)!} p(s) \exp\Big(\alpha\left(t-g(s)\right)\Big)\, y(g(s))ds & \text{if} \quad t \geq T, \\ (S_2y)(T) & \text{if} \quad t_0 \leq t < T, \end{cases}$$

and $\varepsilon = \ln(2-\beta)/(T-t_0)$. As in the proof of Theorem 6.8.1, we can show that the map S satisfies all conditions of Krasnosel′skiĭ's fixed point theorem, and so S has a fixed point $y \in \Omega$. Clearly, $y(t) > 0$ for $t \geq t_0$. It is easy to check that $x(t) = y(t)e^{-\alpha t}$ is a solution of (6.108), and so the proof is complete. □

Theorem 6.8.3. *Assume $c \neq 1$, $g(t+\tau) < t$, $p(t) \geq 0$, and*

$$\int_{t_0}^\infty s^{n-1}p(s)ds < \infty \qquad \textit{for some odd} \qquad n \in \mathbb{N}.$$

Then (6.108) *has a bounded positive solution.*

Proof. Let X be the Banach space of all continuous bounded functions defined on $[t_0, \infty)$ with the norm

$$\|x\| = \sup_{t \geq t_0} |x(t)|.$$

We discuss the following five possibilities.

(a) $c \in (0,1)$. Let $T > t_0$ be a sufficiently large number such that $T - \tau \geq t_0$, $T - g(T) \geq t_0$ for $t \geq T$, and

$$\int_T^\infty s^{n-1}p(s)ds \leq \frac{(1-c)(n-1)!}{2}.$$

Set

$$\Omega = \Big\{x \in X: \; 1 \leq x(t) \leq 2, \; t \geq t_0\Big\}.$$

Then Ω is closed, bounded, and convex in X. Define $S : \Omega \to X$ by

$$(Sx)(t) = \begin{cases} 1 - c + cx(t-\tau) + \displaystyle\int_t^\infty \frac{(t-s)^{n-1}}{(n-1)!} p(s)x(g(s))ds & \text{if} \quad t \geq T, \\ (Sx)(T) & \text{if} \quad t_0 \leq t < T. \end{cases}$$

Clearly, S is continuous. For every $x \in \Omega$, it is easy to see that

$$(Sx)(t) \leq 1 - c + 2c + \frac{2(1-c)}{2} = 2 \quad \text{and} \quad (Sx)(t) \geq 1 - c + c + 0 = 1$$

for $t \geq t_0$. This means that $S\Omega \subseteq \Omega$. We now shall show that S is a strict contraction. In fact, for every pair $x_1, x_2 \in \Omega$ and $t \geq T$, we have

$$\begin{aligned} \Big|(Sx_1)(t) - (Sx_2)(t)\Big| &\leq c\Big|x_1(t-\tau) - x_2(t-\tau)\Big| \\ &\quad + \int_t^\infty \frac{(t-s)^{n-1}}{(n-1)!} p(s) \Big|x_1\left(g(s)\right) - x_2\left(g(s)\right)\Big| ds \\ &\leq \|x_1 - x_2\| \left(c + \frac{1-c}{2}\right) = \left(\frac{1+c}{2}\right) \|x_1 - x_2\| \end{aligned}$$

and

$$\begin{aligned}\|Sx_1 - Sx_2\| &= \sup_{t\geq t_0}\Big|(Sx_1)(t) - (Sx_2)(t)\Big| = \sup_{t\geq T}\Big|(Sx_1)(t) - (Sx_2)(t)\Big| \\ &\leq \left(\frac{1+c}{2}\right)\|x_1 - x_2\|.\end{aligned}$$

Since $(1+c)/2 < 1$, it follows that S is a strict contraction. By the Banach contraction mapping principle (Theorem 1.4.26), S has a fixed point $x \in \Omega$. It is easy to check that x is a bounded positive solution of (6.108).

(b) $c > 1$. Let $T > t_0$ be a sufficiently large number such that $T - g(T) \geq t_0$ for $t \geq T$ and

$$\int_{T+\tau}^{\infty} s^{n-1}p(s)ds \leq \frac{(c-1)(n-1)!}{2}.$$

Set

$$\Omega = \left\{x \in X : \frac{c-1}{2} \leq x(t) \leq c,\ t \geq t_0\right\}.$$

Then Ω is closed, bounded, and convex in X. Define $S : \Omega \to X$ by

$$(Sx)(t) = \begin{cases} c - 1 + \dfrac{1}{c}x(t+\tau) - \dfrac{1}{c}\displaystyle\int_{t+\tau}^{\infty}\frac{(t+\tau-s)^{n-1}}{(n-1)!}p(s)x\,(g(s))\,ds & \\ & \text{if} \quad t \geq T, \\ (Sx)(T) & \text{if} \quad t_0 \leq t < T. \end{cases}$$

As in the proof of (a), we can show that $S\Omega \subseteq \Omega$, and for every pair $x_1, x_2 \in \Omega$, we have

$$\|Sx_1 - Sx_2\| \leq \frac{1}{c}\left(\frac{2c-1}{2c}\right)\|x_1 - x_2\|.$$

Since $0 < (2c-1)/(2c) < 1$, it follows that S is a strict contraction. By the Banach contraction mapping principle, S has a fixed point $x \in \Omega$, and x is a positive solution of (6.108).

(c) $c \in (-1, 0]$. Let $T > t_0$ be a sufficiently large number such that $T - \tau \geq t_0$, $T - g(T) \geq t_0$ for $t \geq T$, and

$$\int_{T}^{\infty} s^{n-1}p(s)ds \leq \frac{(c+1)(n-1)!}{2}.$$

Set

$$\Omega = \left\{x \in X : \frac{(c+1)^2}{1-c} \leq x(t) \leq \frac{2(c+1)^2}{1-c},\ t \geq t_0\right\}.$$

Then Ω is closed, bounded, and convex in X. Define $S : \Omega \to X$ by

$$(Sx)(t) = \begin{cases} c + 1 + cx(t-\tau) + \displaystyle\int_{t}^{\infty}\frac{(t-s)^{n-1}}{(n-1)!}p(s)x(g(s))ds & \text{if} \quad t \geq T, \\ (Sx)(T) & \text{if} \quad t_0 \leq t < T. \end{cases}$$

As in the proof of (a), we can show that

$$\frac{(c+1)^2}{1-c} \leq x(t) \leq \frac{2(c+1)^2}{1-c}, \quad t \geq t_0,$$

and for every pair $x_1, x_2 \in \Omega$, we have

$$\|Sx_1 - Sx_2\| \leq \left(\frac{1-c}{2}\right)\|x_1 - x_2\|.$$

Since $0 < (1-c)/2 < 1$, it follows that S is a strict contraction. By the Banach contraction mapping principle, S has a fixed point $x \in \Omega$, and x is a positive solution of (6.108).

(d) $c < -1$. Let $T > t_0$ be a sufficiently large number such that $t+\tau-g(t) \geq t_0$ for $t \geq T$ and

$$\int_{T+\tau}^{\infty} s^{n-1} p(s) ds \leq -\frac{(c+1)(n-1)!}{2}.$$

Set

$$\Omega = \left\{ x \in X : \frac{(c+1)^2}{c(c-1)} \leq x(t) \leq \frac{-2(c+1)}{1-c},\ t \geq t_0 \right\}.$$

Then Ω is closed, bounded, and convex in X. Define $S : \Omega \to X$ by

$$(Sx)(t) = \begin{cases} 1 + \dfrac{1}{c} + \dfrac{1}{c} x(t+\tau) - \dfrac{1}{c} \displaystyle\int_{t+\tau}^{\infty} \frac{(t+\tau-s)^{n-1}}{(n-1)!} p(s) x(g(s)) ds & \\ & \text{if} \quad t \geq T, \\ (Sx)(T) & \text{if} \quad t_0 \leq t < T. \end{cases}$$

As in the proof of (a), we can show that $S\Omega \subseteq \Omega$, and for every pair $x_1, x_2 \in \Omega$, we have

$$\|Sx_1 - Sx_2\| \leq \left(\frac{1-c}{-2c} \right) \|x_1 - x_2\|.$$

Since $0 < (c-1)/(2c) < 1$, it follows that S is a strict contraction. By the Banach contraction mapping principle, S has a fixed point $x \in \Omega$, and x is a positive solution of (6.108).

(e) $c = -1$. Let $T > t_0$ be a sufficiently large number such that $t+\tau-g(t) \geq t_0$ for $t \geq T$ and

$$\int_{T+\tau}^{\infty} s^{n-1} p(s) ds \leq \frac{(n-1)!}{2}.$$

Set

$$\Omega = \left\{ x \in X : 1 \leq x(t) \leq 2,\ t \geq t_0 \right\}.$$

Then Ω is closed, bounded, and convex in X. Define $S : \Omega \to X$ by

$$(Sx)(t) = \begin{cases} 1 + \displaystyle\sum_{i=1}^{\infty} \int_{t+(2i-1)\tau}^{t+2i\tau} \frac{(t-s)^{n-1}}{(n-1)!} p(s) x(g(s)) ds & \text{if} \quad t \geq T, \\ (Sx)(T) & \text{if} \quad t_0 \leq t < T. \end{cases}$$

As in the proof of (a), we can show that $S\Omega \subseteq \Omega$, and for every pair $x_1, x_2 \in \Omega$, we have

$$\|Sx_1 - Sx_2\| \leq \frac{1}{2} \|x_1 - x_2\|.$$

It follows that S is a strict contraction. By the Banach contraction mapping principle, S has a fixed point $x \in \Omega$, i.e.,

$$x(t) = \begin{cases} 1 + \displaystyle\sum_{i=1}^{\infty} \int_{t+(2i-1)\tau}^{t+2i\tau} \frac{(t-s)^{n-1}}{(n-1)!} p(s) x(g(s)) ds & \text{if} \quad t \geq T, \\ (Sx)(T) & \text{if} \quad t_0 \leq t < T. \end{cases}$$

From this we obtain

$$x(t) + x(t-\tau) = 2 + \int_t^{\infty} \frac{(t-s)^{n-1}}{(n-1)!} p(s) x(g(s)) ds, \quad t \geq T + \tau.$$

Hence x is a positive solution of (6.108). $\square$

Theorem 6.8.4. *Assume that $n \in \mathbb{N}$ is even and*

(i) $c \in (0,1)$, $\tau > 0$, $p(t) \leq 0$;
(ii) *there exists a constant $\alpha > 0$ such that*

$$ce^{\alpha\tau} + \int_t^\infty \frac{(t-s)^{n-1}}{(n-1)!} p(s) \exp\Big(\alpha\,(t - g(s))\Big)\,ds \leq 1 \quad \textit{for all large } t. \tag{6.112}$$

Then (6.108) *has a positive solution satisfying $x(t) \to 0$ as $t \to \infty$.*

Proof. It is easy to see that if the equal sign in (6.112) holds eventually, then (6.108) has the positive solution $x(t) = e^{-\alpha t}$. Now we assume that there exists $T > t_0$ such that $t - \tau \geq t_0$, $g(t) \geq t_0$ for $t \geq T$,

$$\beta := ce^{\alpha\tau} + \int_T^\infty \frac{(T-s)^{n-1}}{(n-1)!} p(s) \exp\Big(\alpha\,(T - g(s))\Big)\,ds < 1,$$

and condition (6.112) holds for $t \geq T$.

Define the Banach space X and its subset Ω as in the proof of Theorem 6.8.1. Define a map $S : \Omega \to X$ by

$$(Sy)(t) = (S_1y)(t) + (S_2y)(t),$$

where

$$(S_1y)(t) = \begin{cases} ce^{\alpha\tau} y(t-\tau) & \text{if} \quad t \geq T, \\ (S_1y)(T) + \exp\Big(\varepsilon\,(T-t)\Big) - 1 & \text{if} \quad t_0 \leq t < T \end{cases}$$

and

$$(S_2y)(t) = \begin{cases} \displaystyle\int_t^\infty \frac{(t-s)^{n-1}}{(n-1)!} p(s) \exp\Big(\alpha\,(t-g(s))\Big)\,y(g(s))ds & \text{if} \quad t \geq T, \\ (S_2y)(T) & \text{if} \quad t_0 \leq t < T, \end{cases}$$

and $\varepsilon = \ln(2-\beta)/(T-t_0)$. As in the proof of Theorem 6.8.1, we can show that the map S satisfies all conditions of Krasnosel′skiĭ's fixed point theorem, and so S has a fixed point $y \in \Omega$. Clearly, $y(t) > 0$ for $t \geq t_0$. It is easy to check that $x(t) = y(t)e^{-\alpha t}$ is a solution of (6.108), and so the proof is complete. $\square$

By the Banach contraction mapping principle and Krasnosel′skiĭ's fixed point theorem, we can easily show that the following two theorems hold.

Theorem 6.8.5. *If $c > 1$, $\tau > 0$, $p(t) \leq 0$, and*

$$-\int_{t_0}^\infty s^{n-1} p(s)ds < \infty \qquad \textit{for some even} \qquad n \in \mathbb{N},$$

then (6.108) *has a bounded positive solution.*

Theorem 6.8.6. *Assume that $n \in \mathbb{N}$ is odd and*

(i) $c > 0$, $p(t) \leq 0$, *and* $g(t+\tau) < t$;
(ii) *there exists a constant $\alpha > 0$ such that*

$$\frac{1}{c}e^{-\alpha\tau} - \frac{1}{c}\int_{t+\tau}^\infty \frac{(s-t-\tau)^{n-1}}{(n-1)!} p(s) \exp\Big(\alpha\,(t-g(s))\Big)\,ds \leq 1 \quad \textit{for all large } t.$$

Then (6.108) *has a positive solution satisfying $x(t) \to 0$ as $t \to \infty$.*

Theorem 6.8.7. *Assume $c > 1$ and*

$$\int_{t_0}^{\infty} s^{n-1} |p(s)|\, ds < \infty \quad \text{for some even} \quad n \in \mathbb{N}. \tag{6.113}$$

Then (6.108) *has a bounded positive solution.*

Proof. Let $T > t_0$ be a sufficiently large number such that $t + \tau \geq t_0$, $g(t+\tau) \geq t_0$ for $t \geq T$, and

$$\int_{T+\tau}^{\infty} s^{n-1} p(s) ds \leq \frac{(c-1)(n-1)!}{4}.$$

Let X be the Banach space of all continuous bounded functions defined on $[t_0, \infty)$ with the norm $\|x\| = \sup_{t \geq t_0} |x(t)|$ and set

$$\Omega = \left\{ x \in X : \ \frac{1}{2c} \leq x(t) \leq 2c, \ t \geq t_0 \right\}.$$

Then Ω is closed, bounded, and convex in X. Define $S : \Omega \to X$ by

$$(Sx)(t) = \begin{cases} c - 1 + \dfrac{1}{c} x(t+\tau) + \dfrac{1}{c} \displaystyle\int_{t+\tau}^{\infty} \frac{(s-t-\tau)^{n-1}}{(n-1)!} p(s) x(g(s)) ds & \\ & \text{if} \quad t \geq T, \\ (Sx)(T) & \text{if} \quad t_0 \leq t < T. \end{cases}$$

Clearly, S is continuous. For every $x \in \Omega$, it is easy to see that

$$(Sx)(t) \leq c - 1 + \frac{1}{c} 2c + \frac{1}{c} \int_{t+\tau}^{\infty} \frac{(s-t-\tau)^{n-1}}{(n-1)!} |p(s)|\, 2c ds \leq c + \frac{1+c}{2} < 2c$$

and

$$\begin{aligned} (Sx)(t) & \geq c - 1 + \frac{1}{c}\frac{c}{2} - \frac{1}{c} \int_{t+\tau}^{\infty} \frac{(s-t-\tau)^{n-1}}{(n-1)!} |p(s)|\, 2c ds \\ & \geq c - 1 + \frac{1}{2} - \frac{c-1}{2} = \frac{c}{2} \end{aligned}$$

for $t \geq t_0$. This means that $S\Omega \subseteq \Omega$.

We now shall show that S is a strict contraction. In fact, for every pair $x_1, x_2 \in \Omega$ and $t \geq T$, we have

$$\begin{aligned} \Big|(Sx_1)(t) - (Sx_2)(t)\Big| & \leq \frac{1}{c} \Big| x_1(t+\tau) - x_2(t+\tau) \Big| \\ & \qquad + \frac{1}{c} \int_{t+\tau}^{\infty} \frac{(s-t-\tau)^{n-1}}{(n-1)!} |p(s)| \Big| x_1(g(s)) - x_2(g(s)) \Big| ds \\ & \leq \|x_1 - x_2\| \left(\frac{1}{c} + \frac{1}{c}\frac{c-1}{4} \right) \leq \frac{1}{4}\left(1 + \frac{3}{c}\right) \|x_1 - x_2\|, \end{aligned}$$

which implies that

$$\begin{aligned} \|Sx_1 - Sx_2\| & = \sup_{t \geq t_0} \Big|(Sx_1)(t) - (Sx_2)(t)\Big| = \sup_{t \geq T} \Big|(Sx_1)(t) - (Sx_2)(t)\Big| \\ & \leq \frac{1}{4}\left(1 + \frac{3}{c}\right) \|x_1 - x_2\|. \end{aligned}$$

Since $\frac{1}{4}\left(1 + \frac{3}{c}\right) < 1$, it follows that S is a strict contraction. By the Banach contraction mapping principle, S has a fixed point $x \in \Omega$, and x is a positive solution of (6.108). $\square$

By using the above method, we can show that the following three theorems are also true.

Theorem 6.8.8. *Assume that* $c \in (0, 1)$, $\tau > 0$, *and that condition* (6.113) *holds. Then* (6.108) *has a bounded positive solution.*

Theorem 6.8.9. *Assume* $c > 1$ *and*

$$\int_{t_0}^{\infty} s^{n-1} |p(s)| \, ds < \infty \quad \textit{for some odd} \quad n \in \mathbb{N}. \tag{6.114}$$

Then (6.108) *has a bounded positive solution.*

Theorem 6.8.10. *Assume that* $c \in (0, 1)$ *and that condition* (6.114) *holds. Then* (6.108) *has a bounded positive solution.*

6.9. Notes

The results in Section 6.2 are taken from Li [**178**], a special case is obtained by Zhang, Yu, and Wang [**302**]. Section 6.3 is adopted from Zhang and Li [**300**]. The material in Section 6.4 is taken from Li [**182**], related results are given by Graef, Spikes, and Grammatikopoulos [**109**]. The results in Sections 6.5 and 6.6 are obtained by Li [**183**] and Li and Fei [**196**], respectively. The contents of Sections 6.7 and 6.8 is taken from Kusano and Singh [**156**], Li and Zhong [**212**], and Li and Ye [**208**], respectively.

CHAPTER 7

Systems of Nonlinear Differential Equations

7.1. Introduction

Oscillation and nonoscillation of systems of nonlinear differential equations is an interesting problem. In this chapter we will present some recent contributions.

In Sections 7.2 and 7.3, we consider oscillation of all solutions of systems of nonlinear differential equations with or without forcing. Some oscillation criteria are presented. In Sections 7.4 and 7.5, we classify positive solutions of our systems according to their limiting behaviors and then provide necessary and sufficient conditions for their existence. Then, in Section 7.6, we provide a classification scheme for positive solutions of two-dimensional second order differential systems and give conditions for the existence of solutions with designated asymptotic properties. Section 7.7 is concerned with nonoscillation of systems of differential equations of the Emden–Fowler type. Several necessary and/or sufficient conditions for strong nonoscillation are given. Here we use the definitions for strong oscillation and strong nonoscillation given in Chapter 1, i.e., a vector solution is said to be *strongly oscillatory* (*strongly nonoscillatory*) if each of its nontrivial components has arbitrarily large zeros (is nonoscillatory).

7.2. Oscillation of Nonlinear Systems

Consider the nonlinear two-dimensional differential system

$$\begin{cases} x'(t) = a(t) f\left(y(t)\right), \\ y'(t) = -b(t) g\left(x(t)\right), \end{cases} \tag{7.1}$$

where a and b are continuous real-valued functions on an interval $[t_0, \infty)$, and f and g are continuous real-valued functions on the real line $\mathbb{R}$ satisfying the sign property

$$uf(u) > 0 \quad \text{and} \quad ug(u) > 0 \quad \text{for all} \quad u \in \mathbb{R} \setminus \{0\}.$$

It is supposed that a is nonnegative on $[t_0, \infty)$, f is increasing on $\mathbb{R}$, and g is continuously differentiable on $\mathbb{R} \setminus \{0\}$ and satisfies

$$g'(u) \geq 0 \quad \text{for all} \quad u \neq 0.$$

Note that no restriction on the sign of the coefficient b is imposed.

Throughout this section, we shall restrict our attention only to those solutions of the differential system (7.1) which exist on some ray $[T_0, \infty)$, where $T_0 \geq t_0$ may depend on the particular solution. Note that, under quite general conditions, there will always exist solutions of (7.1) which are extendable to an interval $[T_0, \infty)$, $T_0 \geq t_0$, even though there will also exist nonextendable solutions [**152**].

The special case when

$$f(u) = |u|^\lambda \operatorname{sgn} u, \quad g(u) = |u|^\mu \operatorname{sgn} u, \quad u \in \mathbb{R} \quad \text{with} \quad \lambda, \mu > 0$$

is of particular interest. In this case, the differential system (7.1) becomes

$$\begin{cases} x'(t) = a(t)|y(t)|^\lambda \operatorname{sgn} y(t), \\ y'(t) = -b(t)|x(t)|^\mu \operatorname{sgn} x(t), \end{cases}$$

where λ and μ are positive constants.

In the following, we will be concerned with conditions which are sufficient for oscillation of all solutions of (7.1). A lemma is useful for this purpose.

Lemma 7.2.1. *Suppose $a(t) \geq 0$ ($\not\equiv 0$). Then the first component x of a nonoscillatory solution (x, y) of (7.1) is also nonoscillatory.*

Proof. Assume to the contrary that x is oscillatory but y is eventually positive. Then in view of (7.1), $x'(t) = a(t)f(y(t)) \geq 0$ for t larger than some T, and that $x'(t) \geq 0$ ($\not\equiv 0$) when $t > T$. Thus $x(t) > 0$ for all large t or $x(t) < 0$ for all large t. This is a contradiction. The case when y is eventually negative is proved similarly. □

Lemma 7.2.2. *Let (x, y) be a solution of (7.1) on an interval $[\tau, \infty)$ and $\tau \geq t_0$ be such that $x(t) > 0$ for all $t \geq \tau$. Moreover, let $\tau^* \geq \tau$ and c be a real constant. If*

$$-\frac{y(\tau)}{g(x(\tau))} + \int_\tau^t b(s)ds + \int_\tau^{\tau^*} \frac{y(s)x'(s)g'(x(s))}{[g(x(s))]^2} ds \geq c \quad \textit{for all} \quad t \geq \tau^*,$$

then

$$y(t) \leq -cg(x(\tau^*)) \quad \textit{for all} \quad t \geq \tau^*.$$

Proof. From the second equation of (7.1) we obtain for $t \geq \tau^*$,

$$\begin{aligned} \int_\tau^t b(s)ds &= \int_\tau^t \frac{-y'(s)}{g(x(s))} ds \\ &= -\frac{y(t)}{g(x(t))} + \frac{y(\tau)}{g(x(\tau))} - \int_\tau^t \frac{y(s)x'(s)g'(x(s))}{[g(x(s))]^2} ds, \end{aligned}$$

and so we have by our hypothesis

$$\begin{aligned} \frac{-y(t)}{g(x(t))} &= \frac{-y(\tau)}{g(x(\tau))} + \int_\tau^t b(s)ds + \int_\tau^{\tau^*} \frac{y(s)x'(s)g'(x(s))}{[g(x(s))]^2} ds \\ &\quad + \int_{\tau^*}^t \frac{y(s)x'(s)g'(x(s))}{[g(x(s))]^2} ds \\ &\geq c + \int_{\tau^*}^t \left\{ -\frac{y(s)}{g(x(s))} \right\} \left\{ -\frac{x'(s)g'(x(s))}{g(x(s))} \right\} ds \end{aligned}$$

for all $t \geq \tau^*$. Hence, by using Lemma 4.4.1, we conclude that

$$w(t) \leq -y(t) \quad \text{for all} \quad t \geq \tau^*,$$

where w satisfies

$$\frac{w(t)}{g(x(t))} = c + \int_{\tau^*}^t \frac{w(s)}{g(x(s))} \left\{ -\frac{x'(s)g'(x(s))}{g(x(s))} \right\} ds \quad \text{for} \quad t \geq \tau^*.$$

We can easily see that $w' = 0$ on $[\tau^*, \infty)$. Moreover, we have $w(\tau^*) = cg(x(\tau^*))$. Thus $w(t) = cg(x(\tau^*))$ for all $t \geq \tau^*$, and so the proof of the lemma is complete. $\square$

Theorem 7.2.3. *Suppose that*

$$\int^{\infty} \frac{du}{f(g(u))} < \infty \quad \textit{and} \quad \int^{-\infty} \frac{du}{f(g(u))} < \infty, \tag{7.2}$$

$$\int_{t_0}^{\infty} a(t)dt = \infty, \tag{7.3}$$

$$\int_{t_0}^{\infty} b(s)ds \quad \textit{exists as a real number,} \tag{7.4}$$

and

$$f(u)f(v) \leq f(uv) \leq f(u)[-f(-v)] \;\; \textit{for all } v > 0 \textit{ and sufficiently small } u. \tag{7.5}$$

Then the differential system (7.1) *is oscillatory if*

$$\int_{t_0}^{\infty} a(t)f(B(t))dt = \infty, \tag{7.6}$$

where

$$B(t) = \int_t^{\infty} b(s)ds \quad \textit{for} \quad t \geq t_0.$$

Proof. Assume that the differential system (7.1) admits a nonoscillatory solution (x, y) on an interval $[T_0, \infty)$, where $T_0 \geq t_0$. From (7.3) it follows that the coefficient a is not identically zero on any interval of the form $[\tau_0, \infty)$, $\tau_0 \geq t_0$. Therefore, by Lemma 7.2.1, x is always nonoscillatory. Without loss of generality, we shall assume that $x(t) \neq 0$ for all $t \geq T_0$. Furthermore, we observe that the substitution $z = -x$, $w = -y$ transforms (7.1) into the system

$$\begin{cases} z'(t) = a(t)\hat{f}\,(w(t))\,, \\ w'(t) = -b(t)\hat{g}\,(z(t))\,, \end{cases}$$

where

$$\hat{f}(u) = -f(u) \quad \text{and} \quad \hat{g}(u) = -g(-u) \quad \text{for} \quad u \in \mathbb{R}.$$

The functions $\hat{f}$ and $\hat{g}$ are subject to the same conditions as the ones imposed on f and g. Thus we can restrict our discussion to the case when x is positive on $[T_0, \infty)$. It must be noted that, from the first equation of system (7.1), it follows that the function yx' is necessarily nonnegative on the interval $[T_0, \infty)$, even though y is oscillatory.

First of all, we will show that

$$\int_{T_0}^{\infty} \frac{y(t)x'(t)g'(x(t))}{[g(x(t))]^2} dt < \infty. \tag{7.7}$$

To this end, let us assume that (7.7) fails to hold. By condition (7.4), there exists a real constant K such that

$$-\frac{y(T_0)}{g(x(T_0))} + \int_{T_0}^{t} b(s)ds \geq K, \quad t \geq T_0.$$

Furthermore, we can choose a point $T_0^* \geq T_0$ so that

$$\int_{T_0}^{T_0^*} \frac{y(s)x'(s)g'(x(s))}{[g(x(s))]^2} ds \geq 1 - K.$$

So we have

$$-\frac{y(T_0)}{g(x(T_0))} + \int_{T_0}^{t} b(s)ds + \int_{T_0}^{T_0^*} \frac{y(s)x'(s)g'(x(s))}{[g(x(s))]^2} ds \geq 1, \quad t \geq T_0^*,$$

and hence, by applying Lemma 7.2.2 with $\tau = T_0$, $\tau^* = T_0^*$, and $c = 1$, we obtain

$$y(t) \leq d, \quad t \geq T_0^*,$$

where $d = -g(x(T_0^*)) < 0$. Next, from the first equation of (7.1) we derive for $t \geq T_0^*$,

$$x(t) - x(T_0^*) = \int_{T_0^*}^{t} a(s)f(y(s))ds \leq f(d) \int_{T_0^*}^{t} a(s)ds,$$

which, in view of (7.3), gives $\lim_{t\to\infty} x(t) = -\infty$, a contradiction. Hence (7.7) holds.

Now, by taking into account (7.4) and the definition of the function B as well as (7.7), from the second equation of (7.1) we get for $t \geq T_0$,

$$\begin{aligned}
B(T_0) - B(t) &= \int_{T_0}^{t} b(s)ds = \int_{T_0}^{t} \frac{-y'(s)}{g(x(s))} ds \\
&= -\frac{y(t)}{g(x(t))} + \frac{y(T_0)}{g(x(T_0))} - \int_{T_0}^{t} \frac{y(s)x'(s)g'(x(s))}{[g(x(s))]^2} ds \\
&= -\frac{y(t)}{g(x(t))} + \frac{y(T_0)}{g(x(T_0))} - \int_{T_0}^{\infty} \frac{y(s)x'(s)g'(x(s))}{[g(x(s))]^2} ds \\
&\quad + \int_{t}^{\infty} \frac{y(s)x'(s)g'(x(s))}{[g(x(s))]^2} ds,
\end{aligned}$$

namely

$$\frac{y(t)}{g(x(t))} = \theta + B(t) + \int_{t}^{\infty} \frac{y(s)x'(s)g'(x(s))}{[g(x(s))]^2} ds, \quad t \geq T_0, \tag{7.8}$$

where the real number θ is defined by

$$\theta := \frac{y(T_0)}{g(x(T_0))} - B(T_0) - \int_{T_0}^{\infty} \frac{y(s)x'(s)g'(x(s))}{[g(x(s))]^2} ds.$$

We claim that the constant θ is nonnegative. Otherwise, from (7.4) and (7.7) it follows that there exists $T_0^{**} \geq T$ such that

$$\int_{T_0^{**}}^{\infty} \frac{y(s)x'(s)g'(x(s))}{[g(x(s))]^2} ds \leq -\frac{\theta}{4} \quad \text{and} \quad \int_{t}^{\infty} b(s)ds \leq -\frac{\theta}{4}$$

for $t \geq T_0^{**}$. Thus, by using (7.8), we find for every $t \geq T_0^{**}$,

$$\begin{aligned}
-\frac{y(T_0)}{g(x(T_0))} + \int_{T_0}^{t} b(s)ds + \int_{T_0}^{T_0^{**}} \frac{y(s)x'(s)g'(x(s))}{[g(x(s))]^2}ds &= -\theta - B(T_0) \\
- \int_{T_0}^{\infty} \frac{y(s)x'(s)g'(x(s))}{[g(x(s))]^2}ds &+ \int_{T_0}^{t} b(s)ds + \int_{T_0}^{T_0^{**}} \frac{y(s)x'(s)g'(x(s))}{[g(x(s))]^2}ds \\
= -\theta - \int_{t}^{\infty} b(s)ds - \int_{T_0^{**}}^{\infty} \frac{y(s)x'(s)g'(x(s))}{[g(x(s))]^2}ds & \\
\geq -\theta + \frac{\theta}{4} + \frac{\theta}{4} = -\frac{\theta}{2}, &
\end{aligned}$$

and so Lemma 7.2.2 ensures that

$$y(t) \leq D, \quad t \geq T_0^{**}, \quad \text{where} \quad D = \frac{\theta}{2} g(x(T_0^{**})) < 0.$$

So, exactly as in the proof of (7.7), we find the contradiction $\lim_{t\to\infty} x(t) = -\infty$, which proves $\theta \geq 0$.

Therefore, (7.8) guarantees that

$$y(t) \geq B(t)g(x(t)) \quad \text{for all} \quad t \geq T_0.$$

Hence, by taking into account the fact that $\lim_{t\to\infty} B(t) = 0$ and using condition (7.5), from the first equation of (7.1) we obtain for $t \geq T_0$,

$$x'(t) = a(t)f(y(t)) \geq a(t)f\Big(B(t)g(x(t))\Big) \geq a(t)f(B(t))f(g(x(t))),$$

and consequently

$$\int_{x(T_0)}^{x(t)} \frac{du}{f(g(u))} \geq \int_{T_0}^{t} a(s)f(B(s))ds, \quad t \geq T_0.$$

So, because of condition (7.2), we have

$$\int_{T_0}^{t} a(s)f(B(s))ds \leq \int_{x(T_0)}^{\infty} \frac{du}{f(g(u))} < \infty, \quad t \geq T_0,$$

which contradicts (7.6). The proof is complete. □

Remark 7.2.4. When $\int_{t_0}^{\infty} b(s)ds = \infty$, there are many oscillation results for the differential system (7.1). The reader can refer to [**152**, **224**, **225**, **226**].

7.3. Oscillation of Nonlinear Systems with Forcing

In this section we are concerned with the oscillation of the differential system

$$\text{(7.9)} \qquad \begin{cases} x'(t) = a(t)f(y(t)), \\ y'(t) = -b(t)g(x(t)) + r(t), \end{cases}$$

where a, b, and r are nontrivial and continuous functions defined on an interval $[t_0, \infty)$, $a(t)$ is nonnegative on $[t_0, \infty)$, f and g are real, nondecreasing, and continuously differentiable functions defined on $\mathbb{R}$ such that

$$xf(x) > 0, \quad xg(x) > 0, \quad \text{and} \quad g'(x) \geq \mu > 0 \quad \text{for} \quad x \neq 0.$$

First of all, we present two lemmas which will be used in the proofs of the following main results.

Lemma 7.3.1. *Suppose $a(t) \geq 0$ ($\not\equiv 0$). Then the first component x of a nonoscillatory solution (x, y) of (7.9) is also nonoscillatory.*

Proof. The proof is similar to the proof of Lemma 7.2.1, and we omit it here. □

Lemma 7.3.2. *Suppose (x, y) is a solution of (7.9) with $x(t) > 0$ for $t \in [t_0, \alpha]$. Suppose further that there exist $t_1 \in [t_0, \alpha]$ and $m > 0$ such that*

$$-\frac{y(t_0)}{g(x(t_0))} + \int_{t_0}^{t} \left(b(s) - \frac{r(s)}{g(x(s))}\right) ds + \int_{t_0}^{t_1} \frac{y(s)x'(s)g'(x(s))}{[g(x(s))]^2} ds \geq m \tag{7.10}$$

for all $t \in [t_1, \alpha]$. Then

$$y(t) \leq -mg\left(x(t_1)\right) \quad \textit{for} \quad t \in [t_1, \alpha].$$

Proof. The proof is similar to the proof of Lemma 7.2.2, and we omit it here. □

Before stating the main results, for simplicity, we list the conditions used as

$$\int_{t_0}^{\infty} |r(s)| ds < \infty, \tag{7.11}$$

$$\int_{t_0}^{\infty} b(s) ds = \infty, \tag{7.12}$$

and

$$-\infty < \int_{t_0}^{\infty} b(s) ds < \infty. \tag{7.13}$$

Theorem 7.3.3. *Suppose* (7.3), (7.11), *and* (7.12) *hold. Then every solution of* (7.9) *either oscillates or satisfies* $\liminf_{t\to\infty} |x(t)| = 0$.

Proof. Let (x, y) be a nonoscillatory solution of (7.9) with $\liminf_{t\to\infty} |x(t)| > 0$. Since $a(t) \geq 0$ ($\not\equiv 0$), in view of (7.3), we infer from Lemma 7.3.1 that x is nonoscillatory. Assume that $x(t)$ is eventually positive such that $x(t) > 0$ for $t \geq t_0$. Since $\liminf_{t\to\infty} |x(t)| > 0$, there exist $T \geq t_0$ and $m_1, m_2 > 0$ such that $x(t) \geq m_1$ and $g(x(t)) \geq m_2$ for $t \geq T$. Then it follows from (7.11) that

$$\left|\int_T^t \frac{r(s)}{g(x(s))} ds\right| \leq \int_T^t \left|\frac{r(s)}{g(x(s))}\right| ds \leq \frac{\int_T^t |r(s)|\, ds}{m_2} \leq m_3, \quad t \geq T, \tag{7.14}$$

where m_3 is a finite positive constant. Then, in view of (7.12) and (7.14), we see that (7.10) is satisfied for $t \geq T$. If T is sufficiently large, applying Lemma 7.3.2, we obtain

$$y(t) \leq -mg(x(T)) < 0, \quad t \geq T.$$

But since f is nondecreasing, we find

$$x'(t) = a(t)f(y(t)) \leq a(t)f\Big(-mg(x(T))\Big), \quad t \geq T. \tag{7.15}$$

By means of (7.3), $x(t)$ tends to $-\infty$, which is a contradiction. The case when $x(t)$ is eventually negative is proved similarly. □

Example 7.3.4. Consider the differential system

$$\begin{cases} x'(t) = y(t), \\ y'(t) = -x(t) + \dfrac{6+t^2}{t^4} \end{cases}$$

for $t \geq 2$. Clearly, $a(t) \equiv 1$, $b(t) \equiv 1$, $f(x) = g(x) = x$, and $r(t) = (6+t^2)/t^4$, and hence the conditions of Theorem 7.3.3 are satisfied. Therefore, every solution x of the above system either is oscillatory or satisfies $\liminf_{t\to\infty} |x(t)| = 0$. In fact,

$$x(t) = \sin t + \frac{1}{t^2}, \quad y(t) = \cos t - \frac{2}{t^3}$$

is such an oscillatory solution.

Theorem 7.3.5. *Suppose* (7.2), (7.3), (7.5), (7.11), *and* (7.13) *hold. Then every solution of* (7.9) *either oscillates or satisfies* $\liminf_{t\to\infty} |x(t)| = 0$ *if*

$$\int_{t_0}^{\infty} a(t) f\left(\int_t^{\infty} (b(s) - l|r(s)|)\, ds \right) dt = \infty \quad \text{for all} \quad l > 0. \tag{7.16}$$

Proof. Let (x, y) be a nonoscillatory solution of (7.9) with $\liminf_{t\to\infty} |x(t)| > 0$. Since $a(t) \geq 0$ ($\not\equiv 0$) in view of (7.3), we infer from Lemma 7.3.1 that x is nonoscillatory. Assume that $x(t)$ is eventually positive such that $x(t) > 0$ for $t \geq t_0$. Furthermore, there exist $t_1 \geq t_0$ and $m_1, m_2 > 0$ such that $x(t) \geq m_1$ and $g(x(t)) \geq m_2$ for $t \geq t_1$. As in the proof of Lemma 7.2.2, we obtain

$$\frac{y(t)}{g(x(t))} = \frac{y(t_0)}{g(x(t_0))} - \int_{t_0}^{t} \left(b(s) - \frac{r(s)}{g(x(s))} \right) ds - \int_{t_0}^{t} \frac{y(s)x'(s)g'(x(s))}{[g(x(s))]^2} ds$$

for $t \geq t_0$. Note that

$$\int_{t_0}^{\infty} \frac{y(s)x'(s)g'(x(s))}{[g(x(s))]^2} ds < \infty. \tag{7.17}$$

For otherwise (7.10) is valid for some $m > 0$ and t_1. Then by Lemma 7.3.2, we have $y(t) \leq -mg(x(t_1)) < 0$ for $t \geq t_1$ so that (7.15) holds, and its subsequent contradiction follows as before. Now

$$\begin{aligned} \frac{y(t)}{g(x(t))} &= \frac{y(t_0)}{g(x(t_0))} - \int_{t_0}^{\infty} \left(b(s) - \frac{r(s)}{g(x(s))} \right) ds - \int_{t_0}^{\infty} \frac{y(s)x'(s)g'(x(s))}{[g(x(s))]^2} ds \\ &\quad + \int_t^{\infty} \left(b(s) - \frac{r(s)}{g(x(s))} \right) ds + \int_t^{\infty} \frac{y(s)x'(s)g'(x(s))}{[g(x(s))]^2} ds \\ &= \beta + \int_t^{\infty} \left(b(s) - \frac{r(s)}{g(x(s))} \right) ds + \int_t^{\infty} \frac{y(s)x'(s)g'(x(s))}{[g(x(s))]^2} ds, \end{aligned} \tag{7.18}$$

where

$$\beta := \frac{y(t_0)}{g(x(t_0))} - \int_{t_0}^{\infty} \left(b(s) - \frac{r(s)}{g(x(s))} \right) ds - \int_{t_0}^{\infty} \frac{y(s)x'(s)g'(x(s))}{[g(x(s))]^2} ds.$$

We now show that $\beta \geq 0$. Indeed, if $\beta < 0$, then (7.13), (7.14), and (7.17) respectively imply

$$\left| \int_t^{\infty} b(s) ds \right| \leq -\frac{\beta}{6}, \quad \left| \int_t^{\infty} \frac{r(s)}{g(x(s))} ds \right| \leq -\frac{\beta}{6},$$

and

$$\int_t^{\infty} \frac{y(s)x'(s)g'(x(s))}{[g(x(s))]^2} ds < -\frac{\beta}{6}$$

for $t \geq T$. But then

$$\begin{aligned}
& -\frac{y(t_0)}{g(x(t_0))} + \int_{t_0}^{t} \left(b(s) - \frac{r(s)}{g(x(s))} \right) ds + \int_{t_0}^{T} \frac{y(s)x'(s)g'(x(s))}{[g(x(s))]^2} ds \\
= & \quad -\beta - \int_{t}^{\infty} \left(b(s) - \frac{r(s)}{g(x(s))} \right) ds - \int_{t}^{\infty} \frac{y(s)x'(s)g'(x(s))}{[g(x(s))]^2} ds \\
\geq & \quad -\beta + \frac{\beta}{6} + \frac{\beta}{6} + \frac{\beta}{6} = -\frac{\beta}{2} \ > \ 0.
\end{aligned}$$

In view of Lemma 7.3.2, $y(t) \leq \frac{\beta}{2} g(x(T))$ for $t \geq T$. Again, (7.15) holds with $m = -\beta/2$, which is contrary to the condition (7.3) and the assumption that $x(t) > 0$ for $t \geq t_0$, so $\beta \geq 0$. In view of (7.18) and $\beta \geq 0$, we have

$$\begin{aligned}
y(t) \quad & \geq \quad g(x(t)) \int_t^\infty \left(b(s) - \frac{r(s)}{g(x(s))} \right) ds \\
& \geq \quad g(x(t)) \left(\int_t^\infty b(s) ds - l \int_t^\infty |r(s)| ds \right)
\end{aligned}$$

for all large t, where $l = 1/m_2$. For the sake of convenience, let

$$C(t) = \int_t^\infty \Big(b(s) - l|r(s)| \Big) \, ds$$

for all large t. Then $\lim_{t\to\infty} C(t) = 0$, and in view of (7.5) and the fact that f is nondecreasing,

$$x'(t) = a(t) f\left(y(t)\right) \geq a(t) f\Big(C(t) g(x(t))\Big) \geq a(t) f\left(C(t)\right) f\left(g(x(t))\right)$$

for t larger than or equal to some number T. It is easy to see that

$$\frac{x'(s)}{f\left(g(x(s))\right)} \geq a(t) f\left(C(t)\right),$$

and thus by (7.2)

$$\int_T^\infty a(s) f\left(C(s)\right) ds \leq \int_{x(T)}^\infty \frac{du}{f\left(g(u)\right)} < \infty,$$

which is contrary to (7.16). The case when $x(t)$ is eventually negative is proved similarly. □

7.4. Classification Schemes of Positive Solutions (I)

In this section we provide several nonoscillation theorems for the two-dimensional nonlinear differential system

$$\begin{cases} x'(t) = a(t) f\left(y(t)\right), \\ y'(t) = -b(t) g\left(x(t)\right), \end{cases} \tag{7.19}$$

More specifically, we will classify the nonoscillatory solutions of (7.19) according to their limiting behaviors and then provide necessary and sufficient conditions for their existence.

We will assume that

(H1) a and b are nontrivial, nonnegative, and continuous functions defined on an interval $[t_0, \infty)$, and

(H2) f and g are real, increasing, and continuously differentiable functions defined on $\mathbb{R}$ such that $xf(x) > 0$ and $xg(x) > 0$ for $x \neq 0$.

The system (7.19) is naturally classified into four classes according to whether

$$\int_{t_0}^{\infty} a(s)ds < \infty, \int_{t_0}^{\infty} a(s)ds = \infty, \int_{t_0}^{\infty} b(s)ds < \infty, \text{ or } \int_{t_0}^{\infty} b(s)ds = \infty.$$

By symmetry considerations, we will, however, restrict our attention to the cases where

$$\int_{t_0}^{\infty} a(s)ds < \infty \quad \text{or} \quad \int_{t_0}^{\infty} a(s)ds = \infty.$$

For this reason, we will employ the following notations:

$$A(s) = \int_{s}^{\infty} a(u)du, \quad t_0 \le s$$

and

$$A(s,t) = \int_{s}^{t} a(u)du, \quad t_0 \le s \le t.$$

Lemma 7.4.1. *Suppose the conditions* (H1) *and* (H2) *hold. Suppose further that the function a is not identically zero on any interval of the form $[\tau_0, \infty)$, where $\tau_0 \ge t_0$. Then the component function x of a nonoscillatory solution (x, y) of* (7.19) *is also nonoscillatory.*

Proof. Assume to the contrary that x is oscillatory but y is eventually positive. Then in view of (7.19), $x'(t) = a(t)f(y(t)) \ge 0$ for all large t and $x'(t_i) = a(t_i)f(y(t_i)) > 0$ for an increasing and divergent sequence $\{t_i\}$. Thus $x(t) > 0$ for all large t or $x(t) < 0$ for all large t. This is a contradiction. The case when y is eventually negative is proved similarly. □

Similarly, if b is not identically zero on any interval of the form $[\tau_0, \infty)$, then the component function y of a nonoscillatory solution (x, y) is also nonoscillatory. Therefore, under the additional condition

(H3) a and b are not identically zero on any interval of the form $[\tau_0, \infty)$, where $\tau_0 \ge t_0$,

each component function of a nonoscillatory solution (x, y) of (7.19) is eventually of one sign.

If we now interpret (7.19) as a (time varying) vector field in the plane and its solutions as trajectories, then we see that each nonoscillatory solution corresponds to a trajectory which ultimately lies in one of the four open quadrants of the plane. In view of the directions of the vector field in each open quadrant, it is also clear that the component functions of a nonoscillatory trajectory must be monotone.

7.4.1. The Case $A(t_0) = \infty$.

We now impose an additional condition on (7.19), namely $A(t_0) = \infty$. We assert that any nonoscillatory solution (x, y) of (7.19) must ultimately lie in the first or the third open quadrant.

Lemma 7.4.2. *Suppose the conditions* (H1)–(H3) *hold. Suppose further that $A(t_0) = \infty$. Then any nonoscillatory trajectory (x, y) of* (7.19) *must ultimately* *lie in the first or the third open quadrant and $y(t)$ must converge. Furthermore, there exist $c_1 > 0$, $c_2 > 0$, and $T \geq t_0$ such that*

$$c_1 \leq x(t) \leq c_2 A(t_0, t) \quad \text{or} \quad -c_2 A(t_0, t) \leq x(t) \leq -c_1 \quad \text{for} \quad t \geq T.$$

Proof. Assume without loss of generality that $x(t) > 0$ and $y(t) > 0$, or $x(t) > 0$ and $y(t) < 0$ for $t \geq T_0$. The latter case cannot happen. Otherwise, $x'(t) \leq 0$ and $y'(t) \leq 0$ for $t \geq T_0$. Hence

$$x'(t) = a(t)f(y(t)) \leq a(t)f(y(T_0)), \quad t \geq T_0,$$

which implies

$$0 < x(t) \leq x(T_0) + f(y(T_0)) \int_{T_0}^t a(s)ds = x(T_0) + f(y(T_0))A(T_0, t) \to -\infty$$

as $t \to \infty$, a contradiction. Thus $x(t) > 0$, $y(t) > 0$, $x'(t) \geq 0$, and $y'(t) \leq 0$ for $t \geq T_0$. It follows that $y(t)$ monotonically decreases to a nonnegative constant, and $x(t)$ monotonically increases. Furthermore,

$$0 < x(T_0) \leq x(t) \leq x(T_0) + f(y(T_0))A(T_0, t) \leq c_2 A(t_0, t), \quad t \geq T_0$$

for some $c_2 > 0$ since $A(t_0) = \infty$. The proof is complete. □

We have shown that a nonoscillatory solution (x, y) must ultimately lie in the first or the third quadrant, and that $y(t)$ must converge. Note that since $x'(t) \geq 0$ or $x'(t) \leq 0$ for all large t, we see further that $x(t)$ either converges to some nonzero constant or diverges to positive infinity or to negative infinity as $t \to \infty$. However, if $x(t)$ converges to some nonzero constant, then $y(t)$ must converge to 0. Indeed, if $\lim_{t\to\infty} y(t) = d > 0$, then since

$$x'(t) = a(t)f(y(t)) \geq a(t)f(d)$$

for all large t,

$$x(t) \geq x(M) + f(d) \int_M^t a(s)ds \to \infty \quad \text{as} \quad t \to \infty,$$

which is a contradiction.

In view of the above considerations, we may now make the following classification. Let Ω be the set of all nonoscillatory solutions of (7.19) and Ω^+ be the subset of Ω containing those which ultimately lie in the first open quadrant. Suppose (H1)–(H3) hold and $A(t_0) = \infty$. Then any nonosillatory solution in Ω^+ must belong to one of the following three classes:

$$\Omega^+(+, 0) = \left\{(x, y) \in \Omega^+ : \lim_{t\to\infty} x(t) \in (0, \infty), \quad \lim_{t\to\infty} y(t) = 0\right\},$$
$$\Omega^+(\infty, 0) = \left\{(x, y) \in \Omega^+ : \lim_{t\to\infty} x(t) = \infty, \quad \lim_{t\to\infty} y(t) = 0\right\},$$

and

$$\Omega^+(\infty, +) = \left\{(x, y) \in \Omega^+ : \lim_{t\to\infty} x(t) = \infty, \quad \lim_{t\to\infty} y(t) \in (0, \infty)\right\}.$$

A similar classification is also available for nonoscillatory solutions which lie ultimately in the third open quadrant.

In order to further justify our classification scheme, we derive several necessary and sufficient conditions for the existence of each type of nonoscillatory solutions.

Theorem 7.4.3. *Suppose* (H1)–(H3) *hold and* $A(t_0) = \infty$. *A necessary and sufficient condition for* (7.19) *to have a nonoscillatory solution* $(x, y) \in \Omega^+(+, 0)$ *is that*

$$\int_{t_0}^{\infty} a(t) \left| f\left(\int_t^{\infty} b(s)g(c)ds\right)\right| dt < \infty \quad \textit{for some} \quad c > 0. \tag{7.20}$$

Proof. Let $(x, y) \in \Omega^+(+, 0)$ be such that $\lim_{t\to\infty} x(t) = \alpha > 0$. Then there exist $c_1 > 0$ and T such that $c_1 \le x(t)$ for $t \ge T$. In view of (7.19),

$$y(t) = \int_t^{\infty} b(s)g(x(s))ds, \quad t \ge T$$

and

$$\begin{aligned}
\infty \ > \ \alpha - x(T) \ &= \ \int_T^{\infty} a(s)f\left(y(s)\right)ds \\
&= \ \int_T^{\infty} a(s)f\left(\int_s^{\infty} b(u)g\left(x(u)\right)du\right)ds \\
&\ge \ \int_T^{\infty} a(s)f\left(\int_s^{\infty} b(u)g(c_1)du\right)ds.
\end{aligned}$$

Conversely, choose a number M so large that

$$\int_M^{\infty} a(s)f\left(\int_s^{\infty} b(u)g(c)du\right)ds < \frac{c}{2}.$$

Let X be the set of all bounded, continuous, real-valued functions x defined on $[M, \infty)$ with norm $\|x\| = \sup_{t\ge M} |x(t)|$. Let Ψ be the subset of the Banach space X defined by

$$\Psi = \left\{x \in X : \frac{c}{2} \le x(t) \le c, \ t \ge T\right\}.$$

Then Ψ is a bounded, convex, and closed subset of X. Let us define an operator $F : \Psi \to X$ by

$$(Fx)(t) = c - \int_t^{\infty} a(s)f\left(\int_s^{\infty} b(u)g(x(u))du\right)ds \quad \text{for} \quad t \ge M.$$

The mapping F has the following properties. First of all, F maps Ψ into Ψ. Indeed, if $x \in \Psi$, then

$$\begin{aligned}
c \ \ge \ (Fx)(t) \ &= \ c - \int_t^{\infty} a(s)f\left(\int_s^{\infty} b(u)g\left(x(u)\right)du\right)ds \\
&\ge \ c - \int_M^{\infty} a(s)f\left(\int_s^{\infty} b(u)g(c)du\right)ds \ \ge \ \frac{c}{2}.
\end{aligned}$$

Next, we show that F is continuous. Let $\{x_i\}$ be a convergent sequence of functions in Ψ such that $\lim_{i\to\infty} \|x_i - x\| = 0$. Since Ψ is closed, $x \in \Psi$. By the definition of F, we have

$$\begin{aligned}
&\left|(Fx_i)(t) - (Fx)(t)\right| \\
&= \left|\int_t^{\infty} a(s)f\left(\int_s^{\infty} b(u)g\left(x_i(u)\right)du\right)ds - \int_t^{\infty} a(s)f\left(\int_s^{\infty} b(u)g\left(x(u)\right)du\right)ds\right| \\
&\le \int_t^{\infty} a(s)\left|f\left(\int_s^{\infty} b(u)g\left(x_i(u)\right)du\right) - f\left(\int_s^{\infty} b(u)g\left(x(u)\right)du\right)\right| ds.
\end{aligned}$$

By the continuity of f and g and Lebesgue's dominated convergence theorem, it follows that $\lim_{i\to\infty}\|Fx_i - Fx\| = 0$.

Finally, we show that $F\Psi$ is precompact. Let $x \in \Psi$ and $s, t \geq M$. Then, assuming without loss of generality that $s > t$,

$$\begin{aligned}\Big|(Fx)(s) - (Fx)(t)\Big| &= \left|\int_t^s a(u) f\left(\int_u^\infty b(v) g\left(x(v)\right) dv\right) du\right| \\ &\leq \int_t^s a(u) f\left(\int_u^\infty b(v) g(c) dv\right) du.\end{aligned}$$

In view of (7.20), for any $\varepsilon > 0$, there exists $\delta > 0$ such that $|s-t| < \delta$ implies

$$\Big|(Fx)(s) - (Fx)(t)\Big| < \varepsilon.$$

This means that $F\Psi$ is precompact.

By Schauder's fixed point theorem, we may conclude that there exists $x \in \Psi$ such that $x = Fx$. Set

$$y(t) = \int_t^\infty b(v) g(x(v)) dv, \quad t \geq M.$$

Then $\lim_{t\to\infty} y(t) = 0$ and $y'(t) = -b(t)g(x(t))$. On the other hand,

$$\begin{aligned}x(t) &= (Fx)(t) = c - \int_t^\infty a(s) f\left(\int_s^\infty b(u) g(x(u)) du\right) ds \\ &= c - \int_t^\infty a(s) f(y(s)) ds,\end{aligned}$$

and thus $\lim_{t\to\infty} x(t) = c$ and $x'(t) = a(t) f(y(t))$. Hence $(x, y) \in \Omega^+(+, 0)$. $\square$

Theorem 7.4.4. *Suppose* (H1)–(H3) *hold and* $A(t_0) = \infty$. *A necessary and sufficient condition for* (7.19) *to have a nonoscillatory solution* $(x, y) \in \Omega^+(\infty, +)$ *is that*

$$\int_{t_0}^\infty b(t) \Big| g\left(cA(t_0, t)\right) \Big| dt < \infty \quad \textit{for some} \quad c > 0.$$

Proof. Let $(x, y) \in \Omega^+(\infty, +)$ be such that $\lim_{t\to\infty} y(t) = \beta > 0$. Then there exist four positive constants c_1, c_2, d_1, d_2 and $T \geq t_0$ such that

$$c_1 \leq x(t) \leq c_2 A(t_0, t) \quad \text{and} \quad d_1 \leq y(t) \leq d_2 \quad \text{for} \quad t \geq T.$$

In view of (7.19), we have with $c = f(d_1)$

$$\begin{aligned}x(t) &= x(t_0) + \int_{t_0}^t a(s) f(y(s)) ds \ \geq\ x(t_0) + \int_{t_0}^t a(s) f(d_1) ds \\ &\geq f(d_1) A(t_0, t) \ =\ cA(t_0, t),\end{aligned}$$

and hence

$$\infty > y(T) - \beta = \int_T^\infty b(s) g(x(s)) ds \geq \int_T^\infty b(s) g\left(cA\left(t_0, s\right)\right) ds.$$

Conversely, pick a number $T \geq t_0$ so that

$$\int_T^\infty b(s) g\left(cA(T, s)\right) ds < d, \quad \text{where} \quad d = \frac{f^{-1}(c)}{2}.$$

Let X be the partially ordered Banach space of all continuous real-valued functions x with the norm

$$\|x\| = \sup_{t\geq T} \frac{|x(t)|}{A(T,t)}$$

and the usual pointwise ordering. Let Ψ be the subset of X defined by

$$\Psi = \left\{ x \in X : \ f(d)A(T,t) \leq x(t) \leq f(2d)A(T,t), \ t \geq T \right\}.$$

For any subset B of Ψ, it is obvious that $\inf B \in X$ and $\sup B \in X$. Let us further define an operator $F : \Psi \to X$ by

$$(Fx)(t) = \int_T^t a(s) f\left(d + \int_s^\infty b(u)g(x(u))du \right) ds \quad \text{for} \quad t \geq T.$$

The mapping F satisfies the assumptions of Knaster's fixed point theorem: F maps Ψ into itself and F is increasing. The latter is easy to see. As for the former statement, note that for any $x \in \Psi$,

$$(Fx)(t) \geq A(T,t)f(d)$$

and

$$\begin{aligned} (Fx)(t) &\leq \int_T^t a(s) f\left(d + \int_T^\infty b(u) g\left(f(2d)A(T,u) \right) du \right) ds \\ &= \int_T^t a(s) f\left(d + \int_T^\infty b(u) g(cA(T,u))du \right) ds \\ &\leq f(2d)A(T,t) \end{aligned}$$

for $t \geq T$, as desired. By Knaster's fixed point theorem, we may conclude that there exists $x \in \Psi$ such that $x = Fx$. Set

$$y(t) = d + \int_t^\infty b(u)g(x(u))du, \quad t \geq T.$$

Then $\lim_{t\to\infty} y(t) = d$ and $y'(t) = -b(t)g(x(t))$. On the other hand,

$$x(t) = (Fx)(t) = \int_T^t a(s)f(y(s))ds \geq f(d) \int_T^t a(s)ds = f(d)A(T,t)$$

so that $\lim_{t\to\infty} x(t) = \infty$ and $x'(t) = a(t)f(y(t))$. Hence $(x,y) \in \Omega^+(\infty,+)$. □

Finally, we provide a sufficient condition for the existence of a solution in $\Omega^+(\infty,0)$.

Theorem 7.4.5. *Suppose* (H1)–(H3) *hold and* $A(t_0) = \infty$. *A sufficient condition for* (7.19) *to have a nonoscillatory solution* $(x,y) \in \Omega^+(\infty,0)$ *is that*

$$\int_{t_0}^\infty b(t) \left| g(cA(t_0,t)) \right| dt < \infty \quad \textit{for some} \quad c > 0$$

and

$$\int_{t_0}^\infty a(t) \left| f\left(\int_t^\infty b(s)g(d)ds \right) \right| dt = \infty \quad \textit{for any} \quad d > 0. \tag{7.21}$$

Proof. Let $\delta > c$ be fixed. Denote by $C[t_0, \infty)$ the space of all continuous functions on $[t_0, \infty)$ with the topology of uniform convergence on every compact subinterval of $[t_0, \infty)$. Consider the subset Ψ of $C[t_0, \infty)$ consisting of all $x \in C[t_0, \infty)$ such that

$$\delta \leq x(t) \leq \delta + \int_T^t a(s) f\left(\int_s^\infty b(u) g(x(u)) du\right) ds, \quad t \geq T.$$

Let us define an operator $F : \Psi \to C[t_0, \infty)$ by

$$(Fx)(t) = \delta + \int_T^t a(s) f\left(\int_s^\infty b(u) g(x(u)) du\right) ds \quad \text{for} \quad t \geq T.$$

It is clear that F is well defined and maps Ψ into $C[t_0, \infty)$. By reasonings similar to those in the proof of Theorem 7.4.3, we may also show that F maps Ψ into Ψ, that F is continuous on Ψ, and that $F\Psi$ is relatively compact. Schauder's fixed point theorem then implies that F has a fixed point $x \in \Psi$. Set

$$y(t) = \int_t^\infty b(u) g(x(u)) du, \quad t \geq T.$$

Then $\lim_{t\to\infty} y(t) = 0$ and $y'(t) = -b(t)g(x(t))$. On the other hand,

$$x(t) = (Fx)(t) = \delta + \int_T^t a(s) f(y(s)) ds$$

so that $x'(t) = a(t)f(y(t))$ for $t \geq T$. We assert that $\lim_{t\to\infty} x(t) = \infty$. Indeed, $x(t)$ either converges to some positive limit or diverges to ∞. If $\lim_{t\to\infty} x(t) = d > 0$, then $x(s) \geq d/2$ for $s \geq S$. In view of (7.21),

$$\begin{aligned} x(t) &= \delta + \int_T^t a(s) f\left(\int_s^\infty b(u) g(x(u)) du\right) ds \\ &\geq \delta + \int_T^t a(s) f\left(\int_s^\infty b(u) g\left(\frac{d}{2}\right) du\right) ds = \infty, \end{aligned}$$

which is a contradiction. Hence $(x, y) \in \Omega^+(\infty, 0)$. □

7.4.2. The Case $A(t_0) < \infty$.

We now impose another condition on (7.19), namely $A(t_0) < \infty$. We assert that any nonoscillatory solution (x, y) of (7.19) approaches some vertical line as $t \to \infty$.

Lemma 7.4.6. *Suppose* (H1)–(H3) *hold and* $A(t_0) < \infty$. *Then any nonoscillatory trajectory* $(x(t), y(t))$ *of* (7.19) *tends to some vertical line* $x = \alpha$ *as* $t \to \infty$. *Furthermore, there exist* $c_1 > 0$, $c_2 > 0$, *and* $T \geq t_0$ *such that*

$$c_1 A(t) \leq x(t) \leq c_2 \quad \text{or} \quad -c_2 \leq x(t) \leq -c_1 A(t) \quad \text{for} \quad t \geq T.$$

Proof. Without loss of generality, suppose $x(t) > 0$ and $y(t) > 0$ for $t \geq T_0$, or $x(t) > 0$ and $y(t) < 0$ for $t \geq T_0$. If the former case holds, then in view of (7.19), $x'(t) \geq 0$ and $y'(t) \leq 0$ for $t \geq T_0$. Thus,

$$0 \leq x'(t) = a(t) f(y(t)) \leq a(t) f(y(T_0)), \quad t \geq T_0,$$

which implies that $x(t)$ monotonically increases to a constant $\alpha \geq 0$ since

$$x(t) \leq x(T_0) + f(y(T_0)) \int_{T_0}^t a(s) ds \leq x(T_0) + f(y(T_0)) A(T_0) < \infty.$$

Furthermore, it is clear that $A(t) \le x(t) < \alpha + 1$ for all large t since $A(t) \to 0$ as $t \to \infty$.

Suppose the latter case holds. Then in view of (7.19), we see that $x'(t) \le 0$ and $y'(t) \le 0$ for $t \ge T_0$. Hence $x(t)$ monotonically decreases to a constant $\beta \ge 0$. Since

$$x'(t) = a(t)f(y(t)) \le a(t)f(y(T_0)), \quad t \ge T_0,$$

we find

$$0 \le \beta = x(s) + f(y(T_0)) \int_s^\infty a(u)du = x(s) + f(y(T_0))A(s), \quad t \ge T_0.$$

We further see that $\beta + 1 \ge x(s) \ge -f(y(T_0))A(s)$ for all large s. The proof is complete. $\square$

We have shown that a nonoscillatory solution (x, y) must ultimately lie in one of the four open quadrants of the plane. If (x, y) is eventually in the first open quadrant, then in view of our previous lemmas, $x(t)$ monotonically increases and approaches a positive constant and $y(t)$ decreases and approaches a nonnegative constant. In the case when (x, y) is eventually in the fourth quadrant, $x(t)$ decreases and converges to a nonnegative constant and $y(t)$ decreases and either converges to a negative constant or diverges to $-\infty$. The other two cases can be analyzed similarly.

In view of the above considerations, we may now make the following classification. Let Ω be the set of all nonoscillatory solutions of (7.19) and Ω_{++}, Ω_{+-} be respectively the subsets of Ω containing those which ultimately lie in the first or the fourth open quadrant. Suppose (H1)–(H3) hold and $A(t_0) < \infty$. Then any solution in Ω_{++} must belong to one of the classes

$$\Omega_{++}(+,+) = \left\{(x,y) \in \Omega_{++} : \lim_{t\to\infty} x(t) \in (0,\infty), \quad \lim_{t\to\infty} y(t) \in (0,\infty)\right\},$$
$$\Omega_{++}(+,0) = \left\{(x,y) \in \Omega_{++} : \lim_{t\to\infty} x(t) \in (0,\infty), \quad \lim_{t\to\infty} y(t) = 0\right\},$$

while any solution in Ω_{+-} must belong to one of the following classes:

$$\Omega_{+-}(+,-) = \left\{(x,y) \in \Omega_{+-} : \lim_{t\to\infty} x(t) \in (0,\infty), \quad \lim_{t\to\infty} y(t) \in (-\infty,0)\right\},$$
$$\Omega_{+-}(+,-\infty) = \left\{(x,y) \in \Omega_{+-} : \lim_{t\to\infty} x(t) \in (0,\infty), \quad \lim_{t\to\infty} y(t) = -\infty\right\},$$
$$\Omega_{+-}(0,-) = \left\{(x,y) \in \Omega_{+-} : \lim_{t\to\infty} x(t) = 0, \quad \lim_{t\to\infty} y(t) \in (-\infty,0)\right\},$$
$$\Omega_{+-}(0,-\infty) = \left\{(x,y) \in \Omega_{+-} : \lim_{t\to\infty} x(t) = 0, \quad \lim_{t\to\infty} y(t) = -\infty\right\}.$$

Theorem 7.4.7. *Suppose* (H1)–(H3) *hold and* $A(t_0) < \infty$*. A necessary and sufficient condition for* (7.19) *to have a nonoscillatory solution in* $\Omega_{+-}(0,-)$ *is that*

$$\int_{t_0}^\infty b(t)g(cA(t))dt < \infty \quad \textit{for some} \quad c > 0. \tag{7.22}$$

Proof. Let $(x, y) \in \Omega_{+-}(0,-)$ satisfy $\lim_{t\to\infty} x(t) = 0$ and $\lim_{t\to\infty} y(t) = \beta < 0$. Since $y(t)$ is monotone, there exist two positive constants c_1, c_2 and $T \ge t_0$ such that $-c_1 \le y(t) \le -c_2$ for $t \ge T$. On the other hand, in view of the second equation of (7.19), we have

$$y(t) = y(T) - \int_T^t b(s)g(x(s))ds.$$

Since $\lim_{t\to\infty} y(t) = \beta < 0$, we have

$$\int_T^\infty b(s)g(x(s))ds < \infty.$$

Furthermore, we see from Lemma 7.4.6 that

$$g(x(t)) \ge g(cA(t)) \quad \text{for some} \quad c > 0,$$

which implies that

$$\int_{t_0}^\infty b(t)g(cA(t))dt < \infty.$$

Conversely, suppose that (7.22) holds. Then, in view of $A(t_0) < \infty$, there exists $T \ge t_0$ such that

$$\int_T^\infty b(t)g(cA(t))dt < d, \qquad \text{where} \qquad d = \frac{f^{-1}(c)}{2}.$$

Let X be the Banach space of all bounded, continuous, real-valued functions y with the norm

$$\|y\| = \sup_{t\ge T} |y(t)|\,.$$

Define a subset Ω of X by

$$\Omega = \Big\{y \in X: \ -2d \le y(t) \le -d, \ t \ge T\Big\}.$$

Then Ω is a bounded, convex, and closed subset of X. Let us further define an operator $F : \Omega \to X$ by

$$(Fy)(t) = -2d - \int_t^\infty b(s)g\left(\int_s^\infty a(u)f(y(u))du\right) ds \quad \text{for} \quad t \ge T.$$

The mapping F has the following properties. First of all, F maps Ω into itself. Indeed, if $y \in \Omega$, then

$$\begin{aligned}(Fy)(t) &= -2d + \int_t^\infty b(s)g\left(-\int_s^\infty a(u)f(y(u))du\right) ds \\ &\ge -2d + \int_t^\infty b(s)g\Big(f(d)A(s)\Big)\, ds \ \ge\ -2d\end{aligned}$$

and

$$\begin{aligned}(Fx)(t) &= -2d + \int_t^\infty b(s)g\left(-\int_s^\infty a(u)f(y(u))du\right) ds \\ &\le -2d + \int_t^\infty b(s)g\Big(f(2d)A(s)\Big)\, ds \\ &\le -2d + d \ = \ -d\end{aligned}$$

for all $t \geq T$. Next, we show that F is continuous. Let $y, y_l \in \Omega$ such that $\lim_{l\to\infty} \|y_l - y\| = 0$. Since Ω is closed, $y \in \Omega$. Then

$$\begin{aligned}
&\left|(Fy_l)(t) - (Fy)(t)\right| \\
&= \left|\int_t^\infty b(s)g\left(\int_s^\infty a(u)f(y_l(u))du\right)ds - \int_t^\infty b(s)g\left(\int_s^\infty a(u)f(y(u))du\right)ds\right| \\
&\leq \int_t^\infty b(s)\left|g\left(\int_s^\infty a(u)f(y_l(u))du\right) - g\left(\int_s^\infty a(u)f(y(u))du\right)\right|ds.
\end{aligned}$$

By the continuity of f and g and Lebesgue's dominated convergence theorem, it follows that

$$\lim_{l\to\infty}\sup_{t\geq T}\left|(Fy_l)(t) - (Fy)(t)\right| = 0.$$

This shows that

$$\lim_{l\to\infty}\|Fy_l - Fy\| = 0,$$

i.e., F is continuous.

Finally, we will show that $F\Omega$ is precompact. Let $y \in \Omega$ and $s, t \geq T$. Then we have for $s > t$

$$\begin{aligned}
\left|(Fy)(s) - (Fy)(t)\right| &\leq \int_t^s b(u)g\left(\int_u^\infty a(v)\,|f(y(v))|\,dv\right)du \\
&\leq \int_t^\infty b(u)g\Big(f(2d)A(u)\Big)\,du.
\end{aligned}$$

In view of (7.22), this means that $F\Omega$ is precompact.

By Schauder's fixed point theorem, we can conclude that there exists $y \in \Omega$ such that $y = Fy$. Set

$$x(t) = -\int_t^\infty a(s)f\left(y(s)\right)ds.$$

Since $y(t) < 0$ implies $x(t) > 0$, we have

$$x'(t) = a(t)f\left(y(t)\right) \qquad \text{and} \qquad \lim_{t\to\infty} x(t) = 0.$$

On the other hand,

$$y(t) = (Fy)(t) = -2d + \int_t^\infty b(s)g\left(x(s)\right)ds,$$

which implies that $\lim_{t\to\infty} y(t) = -2d$ and

$$y'(t) = -b(t)g\left(x(t)\right).$$

Hence $(x, y) \in \Omega_{+-}(0, -)$. □

Theorem 7.4.8. *Suppose* (H1)–(H3) *hold and* $A(t_0) < \infty$. *A necessary and sufficient condition for* (7.19) *to have a nonoscillatory solution in* $\Omega_{++}(+,+)$ *is that*

$$\int_T^\infty a(s)f\left(\beta + \int_s^\infty b(u)g(c)du\right)ds < \infty \quad \textit{for some} \quad c > 0,\ \beta > 0. \tag{7.23}$$

Proof. Let (x, y) be a solution of (7.19) such that $\lim_{t\to\infty} x(t) = \alpha > 0$ and $\lim_{t\to\infty} y(t) = \beta > 0$. Then there exist two positive constants c_1, c_2 and $T \geq t_0$ such that $c_1 \leq x(t) \leq c_2$ for $t \geq T$. In view of the first equation of (7.19), we have

$$x(t) = x(T) + \int_T^t a(s)f(y(s))ds.$$

Since $\lim_{t\to\infty} x(t) = \alpha > 0$, we have

$$\int_T^\infty a(s)f(y(s))ds < \infty. \tag{7.24}$$

Furthermore, we see from the second equation of (7.19) that

$$y(t) = \beta + \int_t^\infty b(s)g(x(s))ds$$

and

$$\beta + \int_t^\infty b(s)g(c_1)ds \leq y(t) \leq \beta + \int_t^\infty b(s)g(c_2)ds.$$

Together with (7.24), this means that

$$\int_T^\infty a(s)f\left(\beta + \int_s^\infty b(u)g(c_1)du\right)ds < \infty.$$

Conversely, suppose that (7.23) holds. Choose $T \geq t_0$ so large that

$$\int_T^\infty a(s)f\left(\beta + \int_s^\infty b(u)g(c)du\right)ds < d, \qquad \text{where} \qquad d = \frac{c}{2}.$$

Let X be the Banach space of all bounded, continuous, real-valued functions x with the norm

$$\|x\| = \sup_{t\geq T} |x(t)|$$

endowed with the usual pointwise ordering $\leq$: For $x_1, x_2 \in X$, $x_1 \leq x_2$ means $x_1(t) \leq x_2(t)$ for all $t \geq T$. Then X is partially ordered. Define a subset Ω of X by

$$\Omega = \Big\{x \in X :\ d \leq x(t) \leq 2d,\ t \geq T\Big\}.$$

For any subset $B \subset \Omega$, it is obvious that $\inf B \in \Omega$ and $\sup B \in \Omega$. Let us further define an operator $F : \Omega \to X$ by

$$(Fx)(t) = d + \int_T^t a(s)f\left(\beta + \int_s^\infty b(u)g(x(u))du\right)ds \quad \text{for} \quad t \geq T.$$

The mapping F satisfies the assumptions of Knaster's fixed point theorem. Namely, it satisfies the following:

(i) F maps Ω into itself. Indeed, if $x \in \Omega$, then for $t \geq T$,

$$\begin{aligned} d \ &\leq\ (Fx)(t) \leq d + \int_T^t a(s)f\left(\beta + \int_s^\infty b(u)g(2d)du\right)ds \\ &=\ d + \int_T^t a(s)f\left(\beta + \int_s^\infty b(u)g(c)du\right)ds \ \leq\ 2d. \end{aligned}$$

(ii) By the assumptions on f and g, F is increasing. That is, for any $x_1, x_2 \in \Omega$, $x_1 \leq x_2$ implies $Fx_1 \leq Fx_2$.

By Knaster's fixed point theorem, we can conclude that there exists $x \in \Omega$ such that $x = Fx$. Set

$$y(t) = \beta + \int_t^\infty b(u)g(x(u))du, \quad t \geq T.$$

Then

$$y'(t) = -b(t)g(x(t)), \quad \lim_{t\to\infty} y(t) = \beta > 0, \quad \text{and} \quad x'(t) = a(t)f(y(t)).$$

Hence $(x, y) \in \Omega_{++}(+,+)$. □

Similar to the proof of Theorem 7.4.8, we can prove the following two theorems.

Theorem 7.4.9. *Suppose* (H1)–(H3) *hold and* $A(t_0) < \infty$. *A necessary and sufficient condition for* (7.19) *to have a nonoscillatory solution in* $\Omega_{++}(+,0)$ *is that*

$$\int_T^\infty a(s)f\left(\int_s^\infty b(u)g(c)du\right)ds < \infty \quad \textit{for some} \quad c > 0.$$

Theorem 7.4.10. *Suppose* (H1)–(H3) *hold and* $A(t_0) < \infty$. *A necessary and sufficient condition for* (7.19) *to have a nonoscillatory solution in* $\Omega_{+-}(+,-)$ *is that*

$$\int_T^\infty a(s)\left|f\left(\beta + \int_s^\infty b(u)g(c)du\right)\right|ds < \infty \quad \textit{for some} \quad c > 0,\ \beta < 0.$$

Next, we derive two criteria for the existence of nonoscillatory solutions in $\Omega_{+-}(+,-\infty)$ and $\Omega_{+-}(0,-\infty)$.

Theorem 7.4.11. *Suppose* (H1)–(H3) *hold and* $A(t_0) < \infty$. *Suppose further that* $f(-u) = -f(u)$. *A necessary and sufficient condition for* (7.19) *to have a nonoscillatory solution in* $\Omega_{+-}(+,-\infty)$ *is that*

$$\begin{cases} \displaystyle\int_T^\infty a(s)f\left(\int_T^s b(u)g(c)du\right)ds < \infty \quad \textit{for some} \quad c > 0, \\ \displaystyle\int_{t_0}^\infty b(s)ds = \infty. \end{cases} \tag{7.25}$$

Proof. Let (x, y) be a solution of (7.19) such that $\lim_{t\to\infty} x(t) = \alpha > 0$ and $\lim_{t\to\infty} y(t) = -\infty$. Then there exist two positive constants c_1, c_2 and $T \geq t_0$ such that $c_1 \leq x(t) \leq c_2$ for $t \geq T$. In view of the first equation of (7.19), we have

$$x(t) = x(T) + \int_T^t a(s)f(y(s))ds.$$

Since $\lim_{t\to\infty} x(t) = \alpha > 0$, we have

$$\int_T^\infty a(s)\left|f(y(s))\right|ds < \infty.$$

Furthermore, we see from the second equation of (7.19) that

$$y(t) = y(T) - \int_T^t b(s)g(x(s))ds$$

and

$$y(T) - \int_T^t b(s)g(c_2)ds \leq y(t) \leq -\int_T^t b(s)g(c_1)ds.$$

Since $|f(y(t))| \geq f\left(\int_T^t b(s)g(c_1)ds\right)$, we have

$$\int_T^\infty a(s)f\left(\int_T^s b(u)g(c_1)du\right)ds < \infty.$$

On the other hand, in view of (7.19) and $y(t) \to -\infty$ as $t \to \infty$, we have

$$\int_T^\infty b(s)ds = \infty,$$

which implies that (7.25) holds.

Conversely, suppose that (7.25) holds. Choose $T \geq t_0$ so large that

$$\int_T^\infty a(s)f\left(\int_T^s b(u)g(c)du\right)ds < d, \qquad \text{where} \qquad d = \frac{c}{2}.$$

Let X be the Banach space of all bounded, continuous, real-valued functions x with the norm

$$\|x\| = \sup_{t \geq T}|x(t)|$$

endowed with the usual pointwise ordering $\leq$: For $x_1, x_2 \in X$, $x_1 \leq x_2$ means $x_1(t) \leq x_2(t)$ for all $t \geq T$. Then X is partially ordered. Define a subset Ω of X by

$$\Omega = \left\{x \in X :\ d \leq x(t) \leq 2d,\ t \geq T\right\}.$$

For any subset $B \subset \Omega$, it is obvious that $\inf B \in \Omega$ and $\sup B \in \Omega$. Let us further define an operator $F : \Omega \to X$ by

$$(Fx)(t) = d + \int_t^\infty a(s)f\left(\int_T^s b(u)g(x(u))du\right)ds \qquad \text{for} \qquad t \geq T.$$

The mapping F satisfies the assumptions of Knaster's fixed point theorem. Namely, it satisfies the following:

(i) F maps Ω into itself. Indeed, if $x \in \Omega$, then for $t \geq T$,

$$\begin{aligned} d \ \leq\ (Fx)(t) \ &\leq\ d + \int_t^\infty a(s)f\left(\int_T^s b(u)g(2d)du\right)ds \\ &=\ d + \int_t^\infty a(s)f\left(\int_T^s b(u)g(c)du\right)ds \ \leq\ 2d. \end{aligned}$$

(ii) By the assumptions on f and g, F is increasing. That is, for any $x_1, x_2 \in \Omega$, $x_1 \leq x_2$ implies $Fx_1 \leq Fx_2$.

By Knaster's fixed point theorem, we can conclude that there exists $x \in \Omega$ such that $x = Fx$. Set

$$y(t) = -\int_T^t b(u)g(x(u))du, \quad t \geq T.$$

Then

$$y'(t) = -b(t)g(x(t)), \quad -g(2d)\int_T^t b(s)ds \leq y(t) \leq -g(d)\int_T^t b(s)ds$$

and

$$x'(t) = -a(t)f(-y(t)) = a(t)f(y(t)), \quad \lim_{t\to\infty} y(t) = -\infty$$

since $\int_{t_0}^\infty b(s)ds = \infty$, where we have used the assumption $f(-u) = -f(u)$. Hence $(x, y) \in \Omega_{+-}(+, -\infty)$. $\square$

Theorem 7.4.12. *Suppose* (H1)–(H3) *hold and* $A(t_0) < \infty$. *If*

$$\int_T^\infty a(s) f\left(\int_s^\infty b(u) g(cA(u)) du\right) ds < \infty \quad \text{for some} \quad c > 0$$

and

$$\int_{t_0}^\infty b(s) g(dA(s)) ds = \infty \quad \text{for any} \quad d > 0,$$

then (7.19) *has a nonoscillatory solution in* $\Omega_{+-}(0, -\infty)$.

Proof. The proof is similar to that of Theorem 7.4.11, and we omit it here. □

Finally, we derive a necessary condition for (7.19) to have a nonoscillatory solution in $\Omega_{+-}(0, -\infty)$.

Theorem 7.4.13. *Suppose* (H1)–(H3) *hold and* $A(t_0) < \infty$. *A necessary condition for* (7.19) *to have a nonoscillatory solution in* $\Omega_{+-}(0, -\infty)$ *is that*

$$\int_T^\infty a(s) f\left(\int_s^\infty b(u) g(cA(u)) du\right) ds < \infty \quad \text{for some} \quad c > 0$$

and

(7.26) $$\int_{t_0}^\infty b(s) ds = \infty.$$

Proof. Let (x, y) be a solution of (7.19) such that $\lim_{t\to\infty} x(t) = 0$ and $\lim_{t\to\infty} y(t) = -\infty$. Then, by Lemma 7.4.6, there exist two positive constants c_1, c_2 and $T \geq t_0$ such that $c_1 A(t) \leq x(t) \leq c_2$ for $t \geq T$. In view of the first equation of (7.19), we have

$$\infty > x(t) = -\int_t^\infty a(s) f(y(s)) ds > 0,$$

and therefore

$$\int_{t_0}^\infty a(s) \left| f(y(s)) \right| ds < \infty.$$

Furthermore, we see from the second equation of (7.19) that

$$y(t) = y(T) - \int_T^t b(s) g(x(s)) ds$$

and

$$y(T) - \int_T^t b(s) g(c_2) ds \leq y(t) \leq -\int_T^t b(s) g(c_1 A(s)) ds.$$

Since $|f(y(t))| \geq f\left(\int_T^t b(s) g(c_1 A(s)) ds\right)$, we have

$$\int_T^\infty a(s) f\left(\int_T^s b(u) g(c_1 A(u)) du\right) ds < \infty.$$

On the other hand, in view of (7.19) and $y(t) \to -\infty$ as $t \to \infty$, we have

$$\int_T^\infty b(s) ds = \infty,$$

which implies that (7.26) holds. The proof is complete. □

7.5. Classification Schemes of Positive Solutions (II)

In Section 7.4, we classified the positive solutions of (7.19) under the assumption $b(t) \geq 0$ according to their limiting behavior and provided necessary and sufficient conditions for their existence. However, a remaining problem is to characterize the case $b(t) < 0$. In this section, we study this problem (see [**185**]). For the sake of convenience, we rewrite (7.19) as

$$\begin{cases} x'(t) = a(t)f\left(y(t)\right), \\ y'(t) = b(t)g\left(x(t)\right), \end{cases} \tag{7.27}$$

where

(H1) a and b are real-valued nonzero functions such that $a(t) > 0$ and $b(t) > 0$ for $t \geq t_0$, and
(H2) f and g are continuous, real-valued and increasing functions on the real line $\mathbb{R}$ and satisfy $xf(x) > 0$ and $xg(x) > 0$ for $x \neq 0$.

The system (7.27) is naturally classified into the four classes

$$\begin{aligned} &\int_{t_0}^{\infty} a(s)ds = \infty \quad \text{and} \quad \int_{t_0}^{\infty} b(s)ds = \infty, \\ &\int_{t_0}^{\infty} a(s)ds = \infty \quad \text{and} \quad \int_{t_0}^{\infty} b(s)ds < \infty, \\ &\int_{t_0}^{\infty} a(s)ds < \infty \quad \text{and} \quad \int_{t_0}^{\infty} b(s)ds = \infty, \\ &\int_{t_0}^{\infty} a(s)ds < \infty \quad \text{and} \quad \int_{t_0}^{\infty} b(s)ds < \infty. \end{aligned}$$

For this reason, we will employ the following notations:

$$A(t) = \int_{t}^{\infty} a(s)ds \quad \text{and} \quad B(t) = \int_{t}^{\infty} b(s)ds \quad \text{for} \quad t \geq t_0.$$

7.5.1. The Case $A(t_0) = \infty$ and $B(t_0) = \infty$. Assume that $A(t_0) = \infty$ and $B(t_0) = \infty$. Let (x, y) be a solution of (7.27) such that $x(t) > 0$ and $y(t) > 0$ for $t \geq t_0$. Then, from (7.27) we have $y'(t) > 0$ and $x'(t) > 0$ for $t \geq t_0$, which implies that y and x are increasing. Therefore,

$$x(t) = x(t_0) + \int_{t_0}^{t} a(s)f(y(s))ds \geq x(t_0) + f(y(t_0)) \int_{t_0}^{t} a(s)ds \to \infty \quad \text{as} \quad t \to \infty$$

and

$$y(t) = y(t_0) + \int_{t_0}^{t} b(s)g(x(s))ds \geq y(t_0) + g(x(t_0)) \int_{t_0}^{t} b(s)ds \to \infty \quad \text{as} \quad t \to \infty,$$

which imply that $x(t) \to \infty$ and $y(t) \to \infty$ as $t \to \infty$.

In view of our considerations, we may now make the following classification. Let C be the set of all positive solutions of (7.27). Then we have the following result.

Theorem 7.5.1. *Suppose that $A(t_0) = \infty$ and $B(t_0) = \infty$. Then any positive solution of* (7.27) *must belong to the set*

$$C(\infty, \infty) = \left\{ (x, y) \in C : \lim_{t \to \infty} x(t) = \infty, \quad \lim_{t \to \infty} y(t) = \infty \right\}.$$

7.5.2. The Case $A(t_0) = \infty$ and $B(t_0) < \infty$. Assume that $A(t_0) = \infty$ and $B(t_0) < \infty$. If (x, y) is a positive solution of (7.27), i.e., $x(t) > 0$ and $y(t) > 0$ for $t \geq t_0$, then, in view of (7.27), we have $x'(t) > 0$ and $y'(t) > 0$ for $t \geq t_0$, which imply that x and y are increasing. By the first equation of (7.27), we have

$$x(t) = x(t_0) + \int_{t_0}^{t} a(s) f(y(s)) ds \geq x(t_0) + f(y(t_0)) \int_{t_0}^{t} a(s) ds \to \infty \quad \text{as} \quad t \to \infty,$$

and so $\lim_{t\to\infty} x(t) = \infty$.

In view of our considerations, we may now make the following classification.

Theorem 7.5.2. *Suppose that $A(t_0) = \infty$ and $B(t_0) < \infty$. Then any positive solution of (7.27) must belong to one of the following classes:*

$$C(\infty, +) = \left\{(x, y) \in C : \lim_{t\to\infty} x(t) = \infty, \quad \lim_{t\to\infty} y(t) > 0\right\},$$
$$C(\infty, \infty) = \left\{(x, y) \in C : \lim_{t\to\infty} x(t) = \infty, \quad \lim_{t\to\infty} y(t) = \infty\right\}.$$

Theorem 7.5.3. *Suppose that $A(t_0) = \infty$ and $B(t_0) < \infty$. A necessary and sufficient condition for (7.27) to have a positive solution $(x, y) \in C(\infty, +)$ is that*

$$\int_{t_0}^{\infty} b(t) g\left(\int_{t_0}^{t} a(s) f(c)\, ds\right) dt < \infty \quad \textit{for some} \quad c > 0. \tag{7.28}$$

Proof. Let (x, y) be a positive solution of (7.27) such that $\lim_{t\to\infty} x(t) = \infty$ and $\lim_{t\to\infty} y(t) = \beta > 0$. Then there exist two positive constants c_1 and c_2 such that $c_1 \leq y(t) \leq c_2$ for $t \geq t_1 \geq t_0$. In view of the first equation of (7.27) we have

$$x(t) = x(t_1) + \int_{t_1}^{t} a(s) f(y(s)) ds.$$

After integrating the second equation of (7.27), we see that

$$y(t) = y(t_1) + \int_{t_1}^{t} b(s) g(x(s)) ds,$$

and so

$$\begin{aligned} \infty > \beta \geq y(t) &= y(t_1) + \int_{t_1}^{t} b(s) g\left(x(t_1) + \int_{t_1}^{s} a(u) f(y(u)) du\right) ds \\ &\geq y(t_1) + \int_{t_1}^{t} b(s) g\left(\int_{t_1}^{s} a(u) f(y(u)) du\right) ds, \end{aligned}$$

which implies (7.28) with $c = c_1$.

Conversely, choose T so large that

$$\int_{T}^{\infty} b(t) g\left(\int_{T}^{t} a(s) f(c) ds\right) dt < \frac{c}{2}.$$

Let X be the set of all bounded, continuous, real-valued functions y on $[T, \infty)$ equipped with the norm $\|y\| = \sup_{t\geq T} |y(t)|$. Then X is a Banach space. We define a subset Ω of X by

$$\Omega = \left\{y \in X : \frac{c}{2} \leq y(t) \leq c, \ t \geq T\right\}.$$

Then Ω is a bounded, convex, and closed subset of X. Let us further define an operator $F : \Omega \to X$ by

$$(Fy)(t) = c - \int_t^\infty b(s) g\left(\int_T^s a(u) f(y(u)) du\right) ds \quad \text{for} \quad t \geq T.$$

The mapping F has the following properties. First of all, F maps Ω into Ω. Indeed, if $y \in \Omega$, then

$$\begin{aligned} c \;\geq\; (Fy)(t) \;&=\; c - \int_t^\infty b(s) g\left(\int_T^s a(u) f(y(u)) du\right) ds \\ &\geq\; c - \int_t^\infty b(s) g\left(\int_T^s a(u) f(c) du\right) ds \;\geq\; \frac{c}{2}. \end{aligned}$$

Next, we show that F is continuous. Let $y, y_l \in \Omega$ such that $\lim_{l\to\infty} \|y_l - y\| = 0$. Since Ω is closed, $y \in \Omega$. Then by (7.27), we have

$$\begin{aligned} &\left|(Fy_l)(t) - (Fy)(t)\right| \\ &= \left|\int_t^\infty b(s) g\left(\int_T^s a(u) f(y_l(u)) du\right) ds - \int_t^\infty b(s) g\left(\int_T^s a(u) f(y(u)) du\right) ds\right| \\ &\leq \int_t^\infty b(s) \left|g\left(\int_T^s a(u) f(y_l(u)) du\right) - g\left(\int_T^s a(u) f(y(u)) du\right)\right| ds. \end{aligned}$$

By the continuity of f and g and Lebesgue's dominated convergence theorem, it follows that

$$\lim_{l\to\infty} \sup_{t\geq T} \left|(Fy_l)(t) - (Fy)(t)\right| = 0.$$

This shows that

$$\lim_{l\to\infty} \|Fy_l - Fy\| = 0,$$

i.e., F is continuous.

Finally, we show that $F\Omega$ is precompact. Let $y \in \Omega$ and $s, t \geq T$. Then we have for $s > t$

$$\begin{aligned} \left|(Fy)(s) - (Fy)(t)\right| \;&\leq\; \int_t^s b(u) g\left(\int_T^u a(v) f\left(y(v)\right) dv\right) du \\ &\leq\; \int_t^\infty b(u) g\left(\int_T^u a(v) f(c) dv\right) du. \end{aligned}$$

In view of (7.28), this means that $F\Omega$ is precompact.

By Schauder's fixed point theorem, we can conclude that there exists $y \in \Omega$ such that $y = Fy$. Set

$$x(t) = \int_T^t a(s) f\left(y(s)\right) ds.$$

Then

$$x(t) \geq \int_T^t a(s) f\left(\frac{c}{2}\right) ds \to \infty \quad \text{as} \quad t \to \infty,$$

and hence $\lim_{t\to\infty} x(t) = \infty$. On the other hand,

$$y(t) = (Fy)(t) = c - \int_t^\infty b(s) g\left(x(s)\right) ds,$$

which implies that $\lim_{t\to\infty} y(t) = c$. The proof is complete. $\square$

7.5.3. The Case $A(t_0) < \infty$ and $B(t_0) = \infty$. In this subsection we consider the classification and existence for positive solutions of (7.27) under the assumption $A(t_0) < \infty$ and $B(t_0) = \infty$.

Theorem 7.5.4. *Suppose that $A(t_0) < \infty$ and $B(t_0) = \infty$. Then any positive solution of* (7.27) *must belong to one of the following classes:*

$$C(+,\infty) = \left\{(x,y) \in C : \lim_{t\to\infty} x(t) > 0, \quad \lim_{t\to\infty} y(t) = \infty\right\},$$
$$C(\infty,\infty) = \left\{(x,y) \in C : \lim_{t\to\infty} x(t) = \infty, \quad \lim_{t\to\infty} y(t) = \infty\right\}.$$

Proof. If (x,y) is a positive solution of (7.27), i.e., $x(t) > 0$ and $y(t) > 0$ for $t \geq t_0$, then, in view of (7.27), we have $x'(t) > 0$ and $y'(t) > 0$ for $t \geq t_0$, which imply that x and y are increasing. By the second equation of (7.27), we have

$$y(t) = y(t_0) + \int_{t_0}^{t} b(s)g(x(s))ds \geq y(t_0) + \int_{t_0}^{t} b(s)g(x(t_0))ds \to \infty \quad \text{as} \quad t \to \infty,$$

and so $\lim_{t\to\infty} y(t) = \infty$. The proof is complete. □

Similar to the proof of Theorem 7.5.3, we can prove the following result.

Theorem 7.5.5. *Suppose that $A(t_0) < \infty$ and $B(t_0) = \infty$. A necessary and sufficient condition for* (7.27) *to have a positive solution $(x,y) \in C(+,\infty)$ is that*

$$\int_{t_0}^{\infty} a(t)f\left(\int_{t_0}^{t} b(s)g(c)ds\right)dt < \infty \quad \textit{for some} \quad c > 0.$$

7.5.4. The Case $A(t_0) < \infty$ and $B(t_0) < \infty$. We first give a classification scheme for positive solutions of (7.27) subject to $A(t_0) < \infty$ and $B(t_0) < \infty$.

Theorem 7.5.6. *Suppose that $A(t_0) < \infty$ and $B(t_0) < \infty$. Then any positive solution of* (7.27) *must belong to one of the following classes:*

$$C(+,+) = \left\{(x,y) \in C : \lim_{t\to\infty} x(t) > 0, \quad \lim_{t\to\infty} y(t) > 0\right\},$$
$$C(\infty,\infty) = \left\{(x,y) \in C : \lim_{t\to\infty} x(t) = \infty, \quad \lim_{t\to\infty} y(t) = \infty\right\}.$$

Proof. If (x,y) is a positive solution of (7.27), i.e., $x(t) > 0$ and $y(t) > 0$ for $t \geq t_0$, then, in view of (7.27), we have $x'(t) > 0$ and $y'(t) > 0$ for $t \geq t_0$, which imply that x and y are increasing.

If $\lim_{t\to\infty} x(t) = \alpha > 0$, then $x(t) \leq \alpha$ for $t \geq t_0$ and

$$\begin{aligned} y(t) &= y(t_0) + \int_{t_0}^{t} b(s)g(x(s))ds \leq y(t_0) + g(\alpha)\int_{t_0}^{t} b(s)ds \\ &\leq y(t_0) + g(\alpha)B(t_0) < \infty, \end{aligned}$$

which implies that y is bounded, and hence $\lim_{t\to\infty} y(t) = \beta > 0$. Similarly, if $\lim_{t\to\infty} y(t) = \beta > 0$, then x is bounded, and hence $\lim_{t\to\infty} x(t) = \alpha > 0$. The proof is complete. □

Theorem 7.5.7. *A necessary and sufficient condition for* (7.27) *to have a positive solution $(x,y) \in C(+,+)$ is that*

$$A(t_0) < \infty \quad \textit{and} \quad B(t_0) < \infty. \tag{7.29}$$

Proof. Let (x, y) be a solution of (7.27) such that $\lim_{t\to\infty} x(t) = \alpha > 0$ and $\lim_{t\to\infty} y(t) = \beta > 0$. Then there exist four positive constants c_1, c_2, c_3, c_4 and $T \geq t_0$ such that $c_1 \leq x(t) \leq c_2$ and $c_3 \leq y(t) \leq c_4$ for $t \geq T$. In view of the first equation of (7.27) and $\lim_{t\to\infty} x(t) = \alpha > 0$, we have

$$x(t) = \alpha - \int_t^\infty a(s)f(y(s))ds \geq \alpha - \int_t^\infty a(s)f(c_4)ds,$$

and so

$$\int_{t_0}^\infty a(s)f(c_4)ds < \infty.$$

Furthermore, we see from the second equation of (7.27) that

$$y(t) = \beta - \int_t^\infty b(s)g(x(s))ds \geq \beta - \int_t^\infty b(s)g(c_2)ds,$$

and so

$$\int_{t_0}^\infty b(s)g(c_2)ds < \infty.$$

Conversely, suppose that (7.29) holds. Let $c, d > 0$ be arbitrary. Then there exists $T \geq t_0$ such that

$$\int_T^\infty a(s)f(2c)ds < d \quad \text{and} \quad \int_T^\infty b(s)g(2d)ds < c.$$

Let X be the Banach space of all bounded, continuous, real-valued functions (x, y) on $[T, \infty)$ endowed with the norm

$$\|(x, y)\| = \max\left\{\sup_{t\geq T} |x(t)|, \sup_{t\geq T} |y(t)|\right\}$$

and with the usual pointwise ordering $\leq$. Define a subset Ω of X by

$$\Omega = \Big\{(x, y) \in X : \ d \leq x(t) \leq 2d, \ c \leq y(t) \leq 2c, \ t \geq T\Big\}.$$

For any subset B of Ω, it is obvious that $\inf B \in \Omega$ and $\sup B \in \Omega$. Let us further define an operator $F : \Omega \to X$ by

$$F(x, y)(t) = \begin{pmatrix} d \\ c \end{pmatrix} + \begin{pmatrix} \int_T^t a(s)f(y(s))ds \\ \int_T^t b(s)g(x(s))ds \end{pmatrix} \quad \text{for} \quad (x, y) \in \Omega.$$

The mapping F satisfies the assumptions of Knaster's fixed point theorem: F is increasing and maps Ω into itself. Indeed, if $(x, y) \in \Omega$, then

$$d \leq (Fx)(t) = d + \int_T^t a(s)f(y(s))ds \leq d + f(2c)\int_T^\infty a(s)ds \leq 2d$$

and

$$c \leq (Fy)(t) = c + \int_T^t b(s)g(x(s))ds \leq c + g(2d)\int_T^\infty b(s)ds \leq 2c.$$

By Knaster's fixed point theorem, we can conclude that there exists $(x, y) \in \Omega$ such that $(x, y) = F(x, y)$. That is,

$$x(t) = d + \int_T^t a(s)f(y(s))ds \quad \text{and} \quad y(t) = c + \int_T^t b(s)g(x(s))ds.$$

Then

$$\lim_{t\to\infty} x(t) = d + \int_T^\infty a(s)f(y(s))ds \quad \text{and} \quad \lim_{t\to\infty} y(t) = c + \int_T^\infty b(s)g(x(s))ds.$$

Hence $(x, y) \in C(+,+)$. □

In the previous subsections, we have given some classification schemes for positive solutions of (7.27) under the assumptions $A(t_0) = \infty$ and $B(t_0) = \infty$, $A(t_0) = \infty$ and $B(t_0) < \infty$, $A(t_0) < \infty$ and $B(t_0) = \infty$, and $A(t_0) < \infty$ and $B(t_0) < \infty$, respectively. However, we could not give sufficient conditions for (7.27) to have a positive solution which belongs to $C(\infty,\infty)$. Now, as an open problem we leave it for the reader.

Open Problem. Obtain sufficient conditions for (7.27) to have a positive solution which belongs to $C(\infty,\infty)$ under one of the following conditions:

(i) $A(t_0) = \infty$ and $B(t_0) = \infty$;
(ii) $A(t_0) = \infty$ and $B(t_0) < \infty$;
(iii) $A(t_0) < \infty$ and $B(t_0) = \infty$;
(iv) $A(t_0) < \infty$ and $B(t_0) < \infty$.

Remark 7.5.8. If the functions f and g are bounded, $A(t_0) < \infty$ and $B(t_0) < \infty$, then $C(\infty,\infty) = \emptyset$. In fact, if $\lim_{t\to\infty} x(t) = \infty$, then

$$\begin{aligned} x(t) &= x(t_0) + \int_{t_0}^t a(s)f(y(s))ds \le x(t_0) + f(y(t)) \int_{t_0}^t a(s)ds \\ &\le x(t_0) + f(y(t))A(t_0) \to \infty \quad \text{as} \quad t \to \infty, \end{aligned}$$

which implies that $f(y(t)) \to \infty$ as $t \to \infty$. This is a contradiction since f is bounded. Similarly, if $\lim_{t\to\infty} y(t) = \infty$, then we can also obtain a contradiction.

Remark 7.5.9. All above results can be extended to two-dimensional delay differential systems of the form

$$\begin{cases} x'(t) = a(t)f(y(t-\tau)), \\ y'(t) = b(t)g(x(t-\delta)), \end{cases}$$

where $\tau, \delta > 0$.

7.6. Positive Solutions of Second Order Systems

In Sections 7.4 and 7.5, we provided a classification scheme for positive solutions of two-dimensional **first order** nonlinear differential systems and gave conditions for the existence of solutions with designated asymptotic properties.

In this section, following [**207**], we are concerned with a class of two-dimensional **second order** nonlinear differential systems of the form

$$\begin{cases} x''(t) = a(t)f(y(t)), \\ y''(t) = -b(t)g(x(t)), \end{cases} \tag{7.30}$$

where

(H1) a and b are continuous and real-valued nonzero functions such that $a(t) \ge 0$ and $b(t) \ge 0$ for $t \ge t_0$, and

(H2) f and g are continuous, real-valued, and increasing functions on the real line $\mathbb{R}$ and satisfy $xf(x) > 0$ and $xg(x) > 0$ for $x \neq 0$.

As usual, a solution (x, y) of (7.30) is said to be *positive* if both x and y are positive. Positive solutions of (7.30) are interesting for many reasons. For example, when $a(t) \equiv 1$ and $f(u) = u$, we see from (7.30) that

$$x^{(4)}(t) = -b(t)g(x(t)).$$

Therefore, a positive solution of (7.30) yields a positive and strictly concave solution of the fourth order nonlinear differential equation

$$x^{(4)}(t) + b(t)g(x(t)) = 0.$$

Other important differential equations such as

$$x^{(4)}(t) + p(t)\,|x(t)|^{\gamma}\operatorname{sgn} x(t) = 0$$

and

$$(rx'')''(t) + p(t)f(x(t)) = 0,$$

can also be written in the form (7.30).

In this section, we will be concerned with classification schemes for positive solutions of (7.30) and give necessary as well as sufficient conditions for the existence of these solutions. The system (7.30) is naturally classified into the four classes

$$\begin{aligned}
&\int_{t_0}^{\infty} a(s)ds = \infty \quad \text{and} \quad \int_{t_0}^{\infty} b(s)ds = \infty,\\
&\int_{t_0}^{\infty} a(s)ds = \infty \quad \text{and} \quad \int_{t_0}^{\infty} b(s)ds < \infty,\\
&\int_{t_0}^{\infty} a(s)ds < \infty \quad \text{and} \quad \int_{t_0}^{\infty} b(s)ds = \infty,\\
&\int_{t_0}^{\infty} a(s)ds < \infty \quad \text{and} \quad \int_{t_0}^{\infty} b(s)ds < \infty.
\end{aligned}$$

For this reason, we will employ the following notations:

$$A(t) = \int_t^{\infty} a(s)ds \quad \text{and} \quad B(t) = \int_t^{\infty} b(s)ds \quad \text{for} \quad t \geq t_0.$$

7.6.1. The Case $A(t_0) = \infty$ and $B(t_0) = \infty$. In this subsection we always assume that $A(t_0) = \infty$ and $B(t_0) = \infty$.

Theorem 7.6.1. *Suppose that $A(t_0) = \infty$ and $B(t_0) = \infty$. Then there exists no positive solution of* (7.30).

Proof. Suppose that (x, y) is a solution of (7.30) such that $x(t) > 0$ and $y(t) > 0$ for $t \geq t_0$. Then, from (7.30) we have $y''(t) < 0$ for $t \geq t_0$, which implies that y' is decreasing. Therefore, there are two possibilities:

(i) $y'(t) > 0$ for $t \geq t_0$, and
(ii) $y'(t) < 0$ for $t \geq t_0$.

If $y'(t) > 0$ for $t \geq t_0$, then y is an increasing function. Since $y(t) > 0$ for $t \geq t_0$, we have $y(t) \geq y(t_0) > 0$ for $t \geq t_0$. From the first equation of (7.30) we find $x''(t) > 0$ for $t \geq t_0$ and hence

$$x'(t) = x'(t_0) + \int_{t_0}^{t} a(s)f(y(s))ds \geq x'(t_0) + f(y(t_0)) \int_{t_0}^{t} a(s)ds \to \infty \quad \text{as} \quad t \to \infty,$$

which implies that there exists $t_1 \geq t_0$ such that $x(t) \geq x(t_1) > 0$ for $t \geq t_1$. From the second equation of (7.30), we have

$$y'(t) = y'(t_1) - \int_{t_1}^{t} b(s)g(x(s))ds \leq y'(t_1) - g(x(t)) \int_{t_1}^{t} b(s)ds \to -\infty \quad \text{as} \quad t \to \infty,$$

which contradicts the fact that $y'(t) > 0$ for $t \geq t_0$.

If $y'(t) < 0$ for $t \geq t_0$, then from $y''(t) < 0$ for $t \geq t_0$, it follows that y' is decreasing, and hence there exists a constant $c > 0$ such that

$$y'(t) \leq -c \quad \text{for} \quad t \geq t_2 \geq t_0,$$

which means that

$$y(t) \leq y(t_2) - \int_{t_2}^{t} cds \to -\infty \quad \text{as} \quad t \to \infty,$$

and so there exists $t_3 \geq t_2$ such that $y(t) < 0$ for $t \geq t_3$. This is a contradiction and completes the proof. □

7.6.2. The Case $A(t_0) = \infty$ and $B(t_0) < \infty$. Assume that $A(t_0) = \infty$ and $B(t_0) < \infty$. If (x, y) is a positive solution of (7.30), that is to say, $x(t) > 0$ and $y(t) > 0$ for $t \geq t_0$, then, in view of (7.30), we have $x''(t) > 0$ and $y''(t) < 0$ for $t \geq t_0$, which imply that x' is increasing and y' is decreasing. Hence x and y are eventually monotone functions. By the second equation of (7.30), we have

$$y'(t) = y(t_0) - \int_{t_0}^{t} b(s)g(x(s))ds, \quad t \geq t_0.$$

If there exists $t_1 \geq t_0$ such that $y'(t) < y'(t_1) < 0$ for $t \geq t_1$, then

$$y(t) = y(t_1) + \int_{t_1}^{t} y'(s)ds \leq y(t_1) + \int_{t_1}^{t} y'(t_1)ds \to -\infty \quad \text{as} \quad t \to \infty,$$

which contradicts the assumption $y(t) > 0$ for $t \geq t_0$. Hence $y'(t) > 0$ for $t \geq t_0$ and $\lim_{t\to\infty} y'(t) = c \geq 0$, which implies that $\lim_{t\to\infty} y(t) = \infty$ or $\lim_{t\to\infty} y(t) = \beta > 0$.

By the first equation of (7.30), we have

$$x'(t) = x'(t_0) + \int_{t_0}^{t} a(s)f(y(s))ds \geq x'(t_0) + f(y(t_0)) \int_{t_0}^{t} a(s)ds \to \infty \quad \text{as} \quad t \to \infty,$$

and so $\lim_{t\to\infty} x(t) = \infty$.

In view of our considerations, we may now make the following classification. Let C be the set of all positive solutions of (7.30). Then we have the following result.

Theorem 7.6.2. *Suppose that $A(t_0) = \infty$ and $B(t_0) < \infty$. Then any positive solution of* (7.30) *must belong to one of the following classes:*

$$C(\infty, +) = \left\{ (x, y) \in C : \lim_{t\to\infty} x(t) = \infty, \quad \lim_{t\to\infty} y(t) > 0 \right\},$$

$$C(\infty, \infty) = \left\{ (x, y) \in C : \lim_{t\to\infty} x(t) = \infty, \quad \lim_{t\to\infty} y(t) = \infty \right\}.$$

In order to further justify our classification scheme, we derive several sufficient conditions for the existence of each type of positive solution.

Theorem 7.6.3. *Suppose that* $A(t_0) = \infty$ *and* $B(t_0) < \infty$. *A sufficient condition for* (7.30) *to have a positive solution* $(x, y) \in C(\infty, +)$ *is that*

$$\int_{t_0}^{\infty} \int_{s}^{\infty} b(u) g\left(\int_{t_0}^{u} \int_{t_0}^{v} a(r) f(c) dr dv\right) du ds < \infty \quad \textit{for some} \quad c > 0 \tag{7.31}$$

and

$$\int_{t_0}^{\infty} \int_{t_0}^{u} a(v) dv du = \infty.$$

Proof. Choose M so large that

$$\int_{M}^{\infty} \int_{s}^{\infty} b(u) g\left(\int_{M}^{u} \int_{M}^{r} a(t) f(c) dt dr\right) du ds < \frac{c}{2}.$$

Let X be the set of all bounded, continuous, real-valued functions y equipped with the norm $\|y\| = \sup_{t \geq M} |y(t)|$. Then X is a Banach space. We define a subset Ω of X by

$$\Omega = \left\{ y \in X : \frac{c}{2} \leq y(t) \leq c, \ t \geq M \right\}.$$

Then Ω is a bounded, convex, and closed subset of X. Let us further define an operator $F : \Omega \to X$ by

$$(Fy)(t) = c - \int_{t}^{\infty} \int_{s}^{\infty} b(u) g\left(\int_{M}^{u} \int_{M}^{r} a(w) f(y(w)) dw dr\right) du ds, \quad t \geq M. \tag{7.32}$$

The mapping F has the following properties. First of all, F maps Ω into Ω. Indeed, if $y \in \Omega$, then

$$\begin{aligned} c \ \geq \ (Fy)(t) \ &= \ c - \int_{t}^{\infty} \int_{s}^{\infty} b(u) g\left(\int_{M}^{u} \int_{M}^{r} a(w) f(y(w)) dw dr\right) du ds \\ &\geq \ c - \int_{M}^{\infty} \int_{s}^{\infty} b(u) g\left(\int_{M}^{u} \int_{M}^{r} a(w) f(c) dw dr\right) du ds \ \geq \ \frac{c}{2}. \end{aligned}$$

Next, we show that F is continuous. Let $y_l \in \Omega$ such that $\lim_{l \to \infty} \|y_l - y\| = 0$. Since Ω is closed, $y \in \Omega$. Then by (7.32), we have

$$\begin{aligned} &\left|(Fy_l)(t) - (Fy)(t)\right| \\ &= \left| \int_{t}^{\infty} \int_{s}^{\infty} b(u) g\left(\int_{M}^{u} \int_{M}^{r} a(w) f(y_l(w)) dw dr\right) du ds \right. \\ &\qquad \left. - \int_{t}^{\infty} \int_{s}^{\infty} b(u) g\left(\int_{M}^{u} \int_{M}^{r} a(w) f(y(w)) dw dr\right) du ds \right| \\ &\leq \int_{t}^{\infty} \int_{s}^{\infty} b(u) \left| g\left(\int_{M}^{u} \int_{M}^{r} a(w) f(y_l(w)) dw dr\right) \right. \\ &\qquad \left. - g\left(\int_{M}^{u} \int_{M}^{r} a(w) f(c) dw dr\right) \right| du ds. \end{aligned}$$

By the continuity of f and g and Lebesgue's dominated convergence theorem, it follows that

$$\lim_{l \to \infty} \sup_{t \geq M} \left|(Fy_l)(t) - (Fy)(t)\right| = 0.$$

This shows that

$$\lim_{l\to\infty} \|Fy_l - Fy\| = 0,$$

i.e., F is continuous.

Finally, we show that $F\Omega$ is precompact. Let $y \in \Omega$ and $s, t \geq M$. Then we have for $s > t$

$$\begin{aligned}\Big|(Fy)(s) - (Fy)(t)\Big| &\leq \int_t^s \int_u^\infty b(v) g\left(\int_M^v \int_M^r a(w) f(y(w))\, dw dr\right) dv du \\ &\leq \int_t^\infty \int_u^\infty b(v) g\left(\int_M^u \int_M^r a(w) f(y(w))\, dw dr\right) dv du.\end{aligned}$$

In view of (7.31), this means that $F\Omega$ is precompact.

By Schauder's fixed point theorem, we can conclude that there exists $y \in \Omega$ such that $y = Fy$. Set

$$x(t) = \int_M^t \int_M^r a(w) f(y(w))\, dw dr.$$

Then

$$x(t) \geq \int_M^t \int_M^r a(w) f\left(\frac{c}{2}\right) dw dr \to \infty \quad \text{as} \quad t \to \infty,$$

and hence $\lim_{t\to\infty} x(t) = \infty$. On the other hand,

$$y(t) = (Fy)(t) = c - \int_t^\infty \int_u^\infty b(v) g(x(v))\, dv du,$$

which implies that $\lim_{t\to\infty} y(t) = c$. Hence $(x, y) \in C(\infty, +)$. ☐

Theorem 7.6.4. *Suppose that $A(t_0) = \infty$ and $B(t_0) < \infty$. A sufficient condition for (7.30) to have a positive solution $(x, y) \in C(\infty, \infty)$ is that*

$$\int_{t_0}^\infty a(s) f(cs) ds < \infty \quad \text{for some} \quad c > 0 \tag{7.33}$$

and

$$\int_{t_0}^\infty b(s) g(ds) ds < \infty \quad \text{for some} \quad d > 0. \tag{7.34}$$

Proof. Suppose that (7.33) and (7.34) hold. Then there exists $M \geq t_0$ such that

$$\int_M^\infty a(s) f(cs) ds < \frac{d}{2} \quad \text{and} \quad \int_M^\infty b(s) f(ds) ds < \frac{c}{2}.$$

Let X be the Banach space of all real-valued continuous functions (x, y) endowed with the norm

$$\|(x, y)\| = \max\left\{\sup_{t\geq M}\left|\frac{x(t)}{t}\right|, \quad \sup_{t\geq M}\left|\frac{y(t)}{t}\right|\right\}$$

and with the usual pointwise ordering $\leq$. Define a subset Ω of X by

$$\Omega = \left\{(x, y) \in X : \frac{dt}{2} \leq x(t) \leq dt, \ \frac{ct}{2} \leq y(t) \leq ct, \ t \geq M\right\}.$$

For any subset B of Ω, it is obvious that $\inf B \in \Omega$ and $\sup B \in \Omega$. Let us further define an operator $F : \Omega \to X$ by

$$F(x,y)(t) = \begin{pmatrix} \frac{dt}{2} + \int_M^t \int_M^s a(u)f(y(u))duds \\ \\ \frac{ct}{2} + \int_M^t \int_s^\infty b(u)g(x(u))duds \end{pmatrix} \quad \text{for} \quad t \geq M.$$

The mapping F satisfies the assumptions of Knaster's fixed point theorem: F is increasing and maps Ω into itself. Indeed, if $(x,y) \in \Omega$, then

$$\frac{dt}{2} \leq (Fx)(t) \leq \frac{dt}{2} + t\int_M^\infty a(s)f(cs)ds \leq dt, \quad t \geq M$$

and

$$\frac{ct}{2} \leq (Fy)(t) \leq \frac{ct}{2} + t\int_M^\infty b(s)g(ds)ds \leq ct, \quad t \geq M.$$

By Knaster's fixed point theorem, we can conclude that there exists $(x,y) \in \Omega$ such that $(x,y) = F(x,y)$. That is,

$$x(t) = \frac{dt}{2} + \int_M^t \int_M^s a(u)f(y(u))duds, \quad t \geq M$$

and

$$y(t) = \frac{ct}{2} + \int_M^t \int_s^\infty b(u)g(x(u))duds, \quad t \geq M.$$

Then

$$\lim_{t\to\infty} x(t) = \infty \quad \text{and} \quad \lim_{t\to\infty} y(t) = \infty.$$

Hence $(x,y) \in C(\infty,\infty)$. □

7.6.3. The Case $A(t_0) < \infty$ and $B(t_0) < \infty$. We first classify positive solutions of (7.30) under the assumption $A(t_0) < \infty$ and $B(t_0) < \infty$.

Theorem 7.6.5. *Suppose that $A(t_0) < \infty$ and $B(t_0) < \infty$. Then any positive solution of* (7.30) *must belong to one of the following six classes:*

$$C(+,+) = \left\{(x,y) \in C : \lim_{t\to\infty} x(t) > 0, \quad \lim_{t\to\infty} y(t) > 0\right\},$$
$$C(0,+) = \left\{(x,y) \in C : \lim_{t\to\infty} x(t) = 0, \quad \lim_{t\to\infty} y(t) > 0\right\},$$
$$C(\infty,+) = \left\{(x,y) \in C : \lim_{t\to\infty} x(t) = \infty, \quad \lim_{t\to\infty} y(t) > 0\right\},$$
$$C(+,\infty) = \left\{(x,y) \in C : \lim_{t\to\infty} x(t) > 0, \quad \lim_{t\to\infty} y(t) = \infty\right\},$$
$$C(0,\infty) = \left\{(x,y) \in C : \lim_{t\to\infty} x(t) = 0, \quad \lim_{t\to\infty} y(t) = \infty\right\},$$
$$C(\infty,\infty) = \left\{(x,y) \in C : \lim_{t\to\infty} x(t) = \infty, \quad \lim_{t\to\infty} y(t) = \infty\right\}.$$

Proof. Let (x,y) be a positive solution of (7.30). Then $y''(t) = -b(t)g(x(t)) < 0$ for $t \geq t_0$. Hence y' is monotone and either $y'(t) > 0$ for $t \geq t_0$ or $y'(t) < 0$ for $t \geq t_0$. If the latter holds, then $y(t) \leq y(t_0)$ for $t \geq t_0$ and $y'(t) \leq y'(t_0) < 0$ for $t \geq t_0$, and so

$$y(t) \leq y(t_0) + \int_{t_0}^t y'(t_0)ds \to -\infty \quad \text{as} \quad t \to \infty,$$

which contradicts the assumption $y(t) > 0$ for $t \geq t_0$. This means that $\lim_{t\to\infty} y(t) = \infty$ or $\lim_{t\to\infty} y(t) = \beta > 0$. On the other hand, it follows from (7.30) that $x''(t) > 0$ for $t \geq t_0$, which implies that x' is monotone and either $x'(t) > 0$ for $t \geq t_0$ or $x'(t) < 0$ for $t \geq t_0$. If the latter holds, then $\lim_{t\to\infty} x(t) = \alpha \geq 0$. If the former holds, then $\lim_{t\to\infty} x(t) = \infty$ or $\lim_{t\to\infty} x(t) = \alpha > 0$. The proof is complete. □

Again, in order to justify our classification scheme, we derive several necessary and/or sufficient conditions for the existence of each type of positive solution.

Theorem 7.6.6. *Suppose that $A(t_0) < \infty$ and $B(t_0) < \infty$. A necessary and sufficient condition for (7.30) to have a positive solution $(x, y) \in C(+,+)$ is that*

$$\int_{t_0}^{\infty}\int_{s}^{\infty} a(u)duds < \infty \quad \textit{and} \quad \int_{t_0}^{\infty}\int_{s}^{\infty} b(u)duds < \infty. \tag{7.35}$$

Proof. Let (x, y) be a solution of (7.30) such that $\lim_{t\to\infty} x(t) = \alpha > 0$ and $\lim_{t\to\infty} y(t) = \beta > 0$. Then there exist four positive constants c_1, c_2, c_3, c_4 and $M \geq t_0$ such that $c_1 \leq x(t) \leq c_2$ and $c_3 \leq y(t) \leq c_4$ for $t \geq M$. In view of the first equation of (7.30) and $\lim_{t\to\infty} x(t) = \alpha > 0$, we have

$$x(t) = \alpha + \int_{t}^{\infty}\int_{s}^{\infty} a(u)f(y(u))duds,$$

and so

$$\int_{t}^{\infty}\int_{s}^{\infty} a(u)f(c_3)duds < \infty.$$

Furthermore, we see from the second equation of (7.30) that

$$y'(t) = \int_{t}^{\infty} b(s)g(x(s))ds$$

and

$$y(t) = \beta - \int_{t}^{\infty}\int_{s}^{\infty} b(u)g(x(u))duds > 0.$$

Thus

$$\int_{t_0}^{\infty}\int_{s}^{\infty} b(u)g(c_1)duds < \beta < \infty.$$

Conversely, suppose that (7.35) holds. Let $c, d > 0$ be arbitrary. Then there exists $M \geq t_0$ such that

$$\int_{M}^{\infty}\int_{s}^{\infty} a(u)f(2c)duds < d \quad \text{and} \quad \int_{M}^{\infty}\int_{s}^{\infty} b(u)g(2d)duds < c.$$

Let X be the Banach space of all real-valued, bounded, continuous functions (x, y) endowed with the norm

$$\|(x, y)\| = \max\left\{\sup_{t\geq M}|x(t)|,\ \sup_{t\geq M}|y(t)|\right\}$$

and with the usual pointwise ordering $\leq$. Define a subset Ω of X by

$$\Omega = \left\{(x, y) \in X :\ d \leq x(t) \leq 2d,\ c \leq y(t) \leq 2c,\ t \geq M\right\}.$$

For any subset B of Ω, it is obvious that $\inf B \in \Omega$ and $\sup B \in \Omega$. Let us further define an operator $F : \Omega \to X$ by

$$F(x,y)(t) = \begin{pmatrix} d + \int_t^\infty \int_s^\infty a(u) f(y(u)) du ds \\ \\ c + \int_M^t \int_s^\infty b(u) g(x(u)) du ds \end{pmatrix} \quad \text{for} \quad t \geq M.$$

The mapping F satisfies the assumptions of Knaster's fixed point theorem: F is increasing and maps Ω into itself. Indeed, if $x \in \Omega$, then for $t \geq M$,

$$\begin{aligned} d \ \leq \ (Fx)(t) \ &= \ d + \int_t^\infty \int_s^\infty a(u) f(y(u)) du ds \\ &\leq \ d + \int_t^\infty \int_s^\infty a(u) f(2c) du ds \ \leq \ 2d \end{aligned}$$

and

$$\begin{aligned} c \ \leq \ (Fy)(t) \ &= \ c + \int_M^t \int_s^\infty b(u) g(x(u)) du ds \\ &\leq \ c + \int_M^t \int_s^\infty b(u) g(2d) du ds \ \leq \ 2c. \end{aligned}$$

By Knaster's fixed point theorem, we can conclude that there exists $(x,y) \in \Omega$ such that $(x,y) = F(x,y)$. That is,

$$x(t) = d + \int_t^\infty \int_s^\infty a(u) f(y(u)) du ds, \quad t \geq M$$

and

$$y(t) = c + \int_M^t \int_s^\infty b(u) g(x(u)) du ds, \quad t \geq M.$$

Then

$$\lim_{t\to\infty} x(t) = d \quad \text{and} \quad \lim_{t\to\infty} y'(t) = \lim_{t\to\infty} \int_t^\infty b(u) g(x(u)) du = 0,$$

and so $\lim_{t\to\infty} y(t) = \beta \geq 0$. In view of $y'(t) = \int_t^\infty b(u) g(x(u)) du > 0$, it follows that $\beta > 0$. Hence $(x,y) \in C(+,+)$. □

Theorem 7.6.7. *Suppose that $A(t_0) < \infty$ and $B(t_0) < \infty$. A necessary and sufficient condition for (7.30) to have a positive solution (x,y) which belongs to $C(0,+)$ is that*

$$(7.36) \quad \int_{t_0}^\infty \int_t^\infty b(u) g\left(\int_u^\infty \int_v^\infty a(r) f(c) dr dv \right) du ds < \infty \quad \textit{for some} \quad c > 0.$$

Proof. Let (x,y) be a solution of (7.30) such that $\lim_{t\to\infty} x(t) = 0$ and $\lim_{t\to\infty} y(t) = \beta > 0$. Then there exist two positive constants c_1, c_2 and $T \geq t_0$ such that $c_1 \leq y(t) \leq c_2$ for $t \geq T$. In view of the first equation of (7.30) and $\lim_{t\to\infty} x(t) = 0$, we have

$$x(t) = \int_t^\infty \int_s^\infty a(u) f(y(u)) du ds.$$

Furthermore, we see from the second equation of (7.30) that

$$y'(t) = \int_t^\infty b(s) f(y(s)) ds$$

and

$$y(t) = \beta - \int_t^\infty \int_s^\infty b(u) g \left(\int_u^\infty \int_v^\infty a(r) f(y(r)) dr dv \right) du ds > 0.$$

Thus,

$$\int_T^\infty \int_t^\infty b(u) g \left(\int_u^\infty \int_v^\infty a(r) f(c_1) dr dv \right) du ds < \beta.$$

Conversely, suppose that (7.36) holds. Then there exists $T \geq t_0$ such that

$$\int_T^\infty \int_t^\infty b(u) g \left(\int_u^\infty \int_v^\infty a(r) f(2c) dr dv \right) du ds < c.$$

Let X be the Banach space of all bounded, continuous, real-valued functions y endowed with the norm

$$\|y\| = \sup_{t \geq T} |y(t)|$$

and with the usual pointwise ordering $\leq$. Define a subset Ω of X by

$$\Omega = \left\{ y \in X : \ c \leq y(t) \leq 2c, \ t \geq T \right\}.$$

For any subset B of Ω, it is obvious that $\inf B \in \Omega$ and $\sup B \in \Omega$. Let us further define an operator $F : \Omega \to X$ by

$$(Fy)(t) = c + \int_T^t \int_s^\infty b(u) g \left(\int_u^\infty \int_v^\infty a(r) f(y(r)) dr dv \right) du ds \quad \text{for} \quad t \geq T.$$

The mapping F satisfies the assumptions of Knaster's fixed point theorem: F is increasing and maps Ω into itself. Indeed, if $y \in \Omega$, then

$$c \leq (Fy)(t) \leq c + \int_T^\infty \int_s^\infty b(u) g \left(\int_u^\infty \int_v^\infty a(r) f(2c) dr dv \right) du ds \leq 2c$$

for $t \geq T$. By Knaster's fixed point theorem, we can conclude that there exists $y \in \Omega$ such that $y = Fy$. Set

$$x(t) = \int_t^\infty \int_s^\infty a(u) f(y(u)) du ds.$$

Then

$$y(t) = c + \int_T^t \int_s^\infty b(u) g\left(x(u)\right) du ds$$

and

$$\lim_{t \to \infty} x(t) = 0 \quad \text{and} \quad \lim_{t \to \infty} y(t) = c + \int_T^\infty \int_s^\infty b(u) g\left(x(u)\right) du ds.$$

Hence $(x, y) \in C(0, +)$. $\square$

By means of similar reasoning used as in the proof of Theorems 7.6.3, 7.6.6, and 7.6.7 we may prove the following three theorems.

Theorem 7.6.8. *Suppose that $A(t_0) < \infty$ and $B(t_0) < \infty$. A sufficient condition for (7.30) to have a positive solution $(x, y) \in C(\infty, +)$ is that*

$$\int_{t_0}^\infty \int_{t_0}^s a(u) du ds = \infty$$

and

$$\int_{t_0}^\infty \int_s^\infty b(u) g \left(\int_{t_0}^u \int_{t_0}^v a(r) f(c) dr dv \right) du ds < \infty \quad \textit{for some} \quad c > 0.$$

Theorem 7.6.9. *Suppose that* $A(t_0) < \infty$ *and* $B(t_0) < \infty$. *A sufficient condition for* (7.30) *to have a positive solution* $(x, y) \in C(+, \infty)$ *is that*

$$\int_{t_0}^{\infty} \int_{s}^{\infty} b(u)duds = \infty$$

and

$$\int_{t_0}^{\infty} \int_{s}^{\infty} a(u)f\left(\int_{t_0}^{u} \int_{t_0}^{v} a(r)g(c)drdv\right) duds < \infty \quad \text{for some} \quad c > 0.$$

Theorem 7.6.10. *Suppose that* $A(t_0) < \infty$ *and* $B(t_0) < \infty$. *A sufficient condition for* (7.30) *to have a positive solution* $(x, y) \in C(\infty, \infty)$ *is that*

$$\int_{t_0}^{\infty} a(s)f(cs)ds < \infty \quad \text{and} \quad \int_{t_0}^{\infty} b(s)g(c_0 s)ds < \infty \quad \text{for some} \quad c > 0,\ c_0 > 0.$$

7.6.4. The Case $A(t_0) < \infty$ and $B(t_0) = \infty$. In this subsection we consider the classification and existence for positive solutions of (7.30) under the assumption $A(t_0) < \infty$ and $B(t_0) = \infty$.

Theorem 7.6.11. *Suppose that* $A(t_0) < \infty$ *and* $B(t_0) = \infty$. *Then any positive solution of* (7.30) *must belong to one of the following classes:*

$$C(0, +) = \left\{(x, y) \in C : \lim_{t\to\infty} x(t) = 0, \quad \lim_{t\to\infty} y(t) > 0\right\},$$
$$C(0, \infty) = \left\{(x, y) \in C : \lim_{t\to\infty} x(t) = 0, \quad \lim_{t\to\infty} y(t) = \infty\right\}.$$

Proof. Let (x, y) be a positive solution of (7.30). Then $y''(t) = -b(t)g(x(t)) < 0$ for $t \geq t_0$. Hence y' is monotone and either $y'(t) > 0$ for $t \geq t_0$ or $y'(t) < 0$ for $t \geq t_0$. If the latter holds, then $y(t) \leq y(t_0)$ for $t \geq t_0$ and $y'(t) \leq y'(t_0) < 0$ for $t \geq t_0$, and so

$$y(t) \leq y(t_0) + \int_{t_0}^{t} y'(t_0)ds \to -\infty \quad \text{as} \quad t \to \infty,$$

which contradicts the assumption $y(t) > 0$ for $t \geq t_0$. This means that $\lim_{t\to\infty} y(t) = \infty$ or $\lim_{t\to\infty} y(t) = \beta > 0$. On the other hand, it follows from (7.30) that $x''(t) > 0$ for $t \geq t_0$, which implies that x' is monotone and either $x'(t) > 0$ for $t \geq t_0$ or $x'(t) < 0$ for $t \geq t_0$. If the former holds, then $x(t) \geq x(t_0)$ for $t \geq t_0$. By the second equation of 7.30 we have

$$y'(t) = y'(t_0) - \int_{t_0}^{t} b(s)g(x(s))ds \leq y'(t_0) - g(x(t)) \int_{t_0}^{t} b(s)ds \to -\infty \quad \text{as} \quad t \to \infty,$$

which implies that $\lim_{t\to\infty} y'(t) = -\infty$ and hence $\lim_{t\to\infty} y(t) = -\infty$. This contradicts the assumption $y(t) > 0$ for $t \geq t_0$. If the latter holds, then $\lim_{t\to\infty} x(t) = \alpha \geq 0$. Since $x'(t) < 0$ for $t \geq t_0$, we have $x(t) \geq \alpha \geq 0$ for $t \geq t_0$. If $\alpha > 0$, then

$$y'(t) = y'(t_0) - \int_{t_0}^{t} b(s)g(x(s))ds \leq y'(t_0) - g(\alpha) \int_{t_0}^{t} b(s)ds \to -\infty \quad \text{as} \quad t \to \infty,$$

which also contradicts the assumption $y(t) > 0$ for $t \geq t_0$. □

In the following, in order to justify our classification scheme, we derive several necessary and/or sufficient conditions for the existence of each type of positive solution.

Theorem 7.6.12. *Suppose that $A(t_0) < \infty$ and $B(t_0) = \infty$. A necessary and sufficient condition for (7.30) to have a positive solution $(x, y) \in C(0, +)$ is that*

$$\int_{t_0}^{\infty} \int_{s}^{\infty} a(u)duds < \infty \tag{7.37}$$

and

$$\int_{t_0}^{\infty} \int_{s}^{\infty} b(u)g\left(\int_{u}^{\infty} \int_{v}^{\infty} a(r)f(c_0)drdv\right) duds < \infty \quad \textit{for some} \quad c_0 > 0. \tag{7.38}$$

Proof. Let (x, y) be a solution of (7.30) such that $\lim_{t\to\infty} x(t) = 0$ and $\lim_{t\to\infty} y(t) = \beta > 0$. Then there exist two positive constants c_1, c_2 and $M \geq t_0$ such that $c_1 \leq y(t) \leq c_2$ for $t \geq M$. In view of the first equation of (7.30) and $\lim_{t\to\infty} x(t) = 0$, we find

$$x'(t) = -\int_{t}^{\infty} a(s)f(y(s))ds$$

and so

$$\infty > x(t) = \int_{t_0}^{\infty} \int_{s}^{\infty} a(u)f(y(u))duds \geq \int_{t_0}^{\infty} \int_{s}^{\infty} a(u)f(c_1)duds.$$

Furthermore, we see from the second equation of (7.30) that

$$y'(t) = \int_{t}^{\infty} b(s)g(x(s))ds$$

and

$$\begin{aligned}
\infty > y(t) &= y(M) + \int_{M}^{t} \int_{s}^{\infty} b(u)g(x(u))duds \\
&\geq \int_{M}^{t} \int_{s}^{\infty} b(u)g\left(\int_{u}^{\infty} \int_{v}^{\infty} a(r)f(y(r))drdv\right) duds \\
&\geq \int_{M}^{t} \int_{s}^{\infty} b(u)g\left(\int_{u}^{\infty} \int_{v}^{\infty} a(r)f(c_1)drdv\right) duds.
\end{aligned}$$

Conversely, suppose that (7.37) and (7.38) hold. Then there exists $M \geq t_0$ such that

$$\int_{M}^{\infty} \int_{s}^{\infty} b(u)g\left(\int_{u}^{\infty} \int_{v}^{\infty} a(r)f(2c)drdv\right) duds \leq c.$$

Let X be the Banach space of all real-valued, bounded, continuous functions y endowed with the norm

$$\|y\| = \sup_{t\geq M} |y(t)|$$

and with the usual pointwise ordering $\leq$. Define a subset Ω of X by

$$\Omega = \left\{y \in X : \ c \leq y(t) \leq 2c, \ t \geq M\right\}.$$

For any subset B of Ω, it is obvious that $\inf B \in \Omega$ and $\sup B \in \Omega$. Let us further define an operator $F : \Omega \to X$ by

$$(Fy)(t) = c + \int_{M}^{t} \int_{s}^{\infty} b(u)g\left(\int_{u}^{\infty} \int_{v}^{\infty} a(r)f(y(r))drdv\right) duds, \quad t \geq M.$$

The mapping F satisfies the assumptions of Knaster's fixed point theorem: F is increasing and maps Ω into itself. Indeed, if $y \in \Omega$, then for $t \geq M$,

$$\begin{aligned} c \;\leq\; (Fy)(t) \;&=\; c + \int_M^t \int_s^\infty b(u) g\left(\int_u^\infty \int_v^\infty a(r) f(y(r)) dr dv\right) du ds \\ &\leq\; c + \int_M^\infty \int_s^\infty b(u) g\left(\int_u^\infty \int_v^\infty a(r) f(2c) dr dv\right) du ds \;\leq\; 2c. \end{aligned}$$

By Knaster's fixed point theorem, we can conclude that there exists $y \in \Omega$ such that $y = Fy$. That is,

$$y(t) = c + \int_M^t \int_s^\infty b(u) g\left(\int_u^\infty \int_v^\infty a(r) f(y(r)) dr dv\right) du ds, \quad t \geq M.$$

Set

$$x(t) = \int_t^\infty \int_s^\infty a(r) f(y(r)) dr ds.$$

Then $\lim_{t\to\infty} x(t) = 0$ and

$$y(t) = c + \int_M^t \int_s^\infty b(u) g\left(x(u)\right) du ds, \quad t \geq M,$$

and so

$$\lim_{t\to\infty} y'(t) = \lim_{t\to\infty} \int_t^\infty b(s) g(x(s)) ds = 0.$$

Hence $\lim_{t\to\infty} y(t) = \beta \geq c > 0$ and $(x, y) \in C(0, +)$. □

Theorem 7.6.13. *Suppose that $A(t_0) < \infty$ and $B(t_0) = \infty$. A sufficient condition for* (7.30) *to have a positive solution $(x, y) \in C(0, \infty)$ is that*

$$\int_{t_0}^\infty b(s) g\left(\int_s^\infty \int_u^\infty a(v) f(cv) dv du\right) ds < \infty \quad \textit{for some} \quad c > 0. \tag{7.39}$$

Proof. Suppose that (7.39) holds. Then there exists M so large that

$$\int_M^\infty b(s) g\left(\int_s^\infty \int_u^\infty a(r) f(2cr) dr du\right) ds < c.$$

Let X be the set of all real-valued continuous functions y equipped with the norm $\|y\| = \sup_{t\geq M} |y(t)/t|$. Then X is a Banach space. We define a subset Ω of X by

$$\Omega = \left\{y \in X : \ ct \leq y(t) \leq 2ct, \ t \geq M\right\}.$$

Then Ω is a bounded, convex, and closed subset of X. Let us further define an operator $F : \Omega \to X$ by

$$(Fy)(t) = ct + \int_M^t \int_v^\infty b(s) g\left(\int_s^\infty \int_u^\infty a(r) f(y(r)) dr du\right) ds dv \quad \text{for} \quad t \geq M.$$

The mapping F satisfies the assumptions of Knaster's fixed point theorem: F is increasing and maps Ω into itself. Indeed, if $y \in \Omega$, then

$$ct \leq (Fy)(t) \leq ct + \int_M^\infty b(s) g\left(\int_s^\infty \int_u^\infty a(r) f(2cr) dr du\right) ds \leq 2ct \quad \text{for} \quad t \geq M.$$

By Knaster's fixed point theorem, we can conclude that there exists $y \in \Omega$ such that $y = Fy$. Set

$$x(t) = \int_t^\infty \int_s^\infty a(r) f(y(r)) dr ds.$$

Then

$$y(t) = ct + \int_M^t \int_s^\infty b(u) g\left(x(u)\right) du ds$$

and

$$\lim_{t\to\infty} x(t) = 0 \quad \text{and} \quad \lim_{t\to\infty} y(t) = \infty.$$

Hence $(x, y) \in C(0, \infty)$. □

Remark 7.6.14. All above results also hold for the two-dimensional delay differential system

$$\begin{cases} x''(t) = a(t) f(y(t-\tau)), \\ y''(t) = -b(t) g(x(t-\delta)), \end{cases}$$

where τ and δ are positive numbers.

7.7. Nonoscillation of Emden–Fowler Systems

The differential equation

$$u''(t) = a(t)|u(t)|^\lambda \operatorname{sgn} u(t) \quad \text{with some} \quad \lambda \neq 1$$

is known in the literature as an equation of the Emden–Fowler type. Study of the equation of this type began in connection with astrophysical investigations around the turn of the century. The oscillatory and nonoscillatory behavior of solutions of Emden–Fowler equations has been investigated by many authors. A survey on such results and a fairly extensive bibliography of the earlier work can be found in the book of Kiguradze and Chanturiya [**145**].

For the system of differential equations of the Emden–Fowler type

$$\begin{cases} u_1'(t) = a_1(t)|u_2(t)|^{\lambda_1} \operatorname{sgn} u_2(t), \\ u_2'(t) = a_2(t)|u_1(t)|^{\lambda_2} \operatorname{sgn} u_1(t), \end{cases} \tag{7.40}$$

which is a generalization of the Emden–Fowler equation, considerably less research work have been done. We refer to [**224, 225, 226, 227, 228, 229**] for oscillation theorems and to [**229, 249, 250, 254**] for nonoscillation theorems.

Throughout this section we assume that the functions a_i, $i \in \{1, 2\}$, are summable on each finite segment of the interval $[0, \infty)$, $a_i(t) > 0$ for all $t \geq t_0$ and $\lambda_i > 0$, $i \in \{1, 2\}$, $\lambda_1 \lambda_2 = 1$. We consider only those solutions of the system (7.40) which exist on some ray $[t_0, \infty)$, where $t_0 \geq 0$ may depend on the particular solution. A nontrivial solution (x, y) of the system (7.40) is said to be *nonoscillatory* if we can find $t_* > t_0$ such that u_1 and u_2 are different from zero on $[t_*, \infty)$. The system (7.40) is called *nonoscillatory* if all nontrivial solutions are nonoscillatory.

It is well known that the oscillatory nature of the system (7.40) and the existence of solutions of the generalized Riccati equation

$$v'(t) + \lambda_2 a_1(t)|v(t)|^{1+\lambda_1} = a_2(t)$$

are closely related. Namely, in 1980, Skhalyakho [**249**] has established the following sufficient and necessary condition on the nonoscillation of the system (7.40).

Theorem 7.7.1. *The system* (7.40) *is nonoscillatory if and only if there exists a function* $\theta \in C^1([t_0,\infty))$ *such that*

$$\theta'(t) + \lambda_2 a_1(t)\,|\theta(t)|^{1+\lambda_1} - a_2(t) \le 0 \quad \text{for} \quad t \ge t_0 \ge 0,$$

where $\lambda_1\lambda_2 = 1$.

The purpose of this section is to establish nonoscillation theorems for the system (7.40) by the application of Theorem 7.7.1. In order to simplify notation, we define

$$\varepsilon = \frac{\lambda_1}{(1+\lambda_1)^{1+\lambda_2}}, \qquad \alpha = (1+\lambda_1)^{\lambda_2}.$$

Theorem 7.7.2. *Let* g *and* ψ *be two continuously differentiable functions on* $[t_0,\infty)$ *such that*

$$g(t) > 0, \quad g'(t) \ge a_1(t), \quad \psi'(t) \ge -a_2(t). \tag{7.41}$$

If

$$\limsup_{t\to\infty} \Big([g(t)]^{\lambda_2}\,|\psi(t)| \Big) < \varepsilon, \tag{7.42}$$

then the system (7.40) *is nonoscillatory.*

Proof. Since (7.42) holds, there exist $T \ge t_0$ and $k \in (0,\varepsilon)$ such that

$$[g(t)]^{\lambda_2}\,|\psi(t)| < k \quad \text{for} \quad t \ge T. \tag{7.43}$$

Let

$$\theta(t) = -\psi(t) + \frac{1-k\alpha}{\alpha[g(t)]^{\lambda_2}}.$$

Then

$$\alpha k < \alpha\varepsilon = (1+\lambda_1)^{\lambda_2}\frac{\lambda_1}{(1+\lambda_1)^{1+\lambda_2}} = \frac{\lambda_1}{1+\lambda_1} < 1$$

and

$$\theta'(t) = -\psi'(t) - \lambda_2\frac{1-k\alpha}{\alpha}\frac{g'(t)}{[g(t)]^{1+\lambda_2}} \le a_2(t) - \lambda_2\frac{1-k\alpha}{\alpha}\frac{a_1(t)}{[g(t)]^{1+\lambda_2}}.$$

Accordingly,

$$\begin{aligned}
&\theta'(t) + \lambda_2 a_1(t)\,|\theta(t)|^{\lambda_1+1} - a_2(t)\\
&\le \lambda_2\frac{k\alpha-1}{\alpha}\frac{a_1(t)}{[g(t)]^{1+\lambda_2}} + \lambda_2 a_1(t)\left|-\psi(t) + \frac{1-k\alpha}{\alpha[g(t)]^{\lambda_2}}\right|^{\lambda_1+1}\\
&= \lambda_2\frac{a_1(t)}{[g(t)]^{1+\lambda_2}}\left[k - \frac{1}{\alpha} + \left|\frac{1-k\alpha}{\alpha} - \psi(t)[g(t)]^{\lambda_2}\right|^{\lambda_1+1}\right].
\end{aligned}$$

Now, (7.43) implies for all $t \ge T$

$$\begin{aligned}
&\theta'(t) + \lambda_2 a_1(t)\,|\theta(t)|^{\lambda_1+1} - a_2(t)\\
&\le \lambda_2\frac{a_1(t)}{[g(t)]^{1+\lambda_2}}\left[k - \frac{1}{\alpha} + \left|\frac{1-k\alpha}{\alpha} + k\right|^{\lambda_1+1}\right]\\
&= \lambda_2\frac{a_1(t)}{[g(t)]^{1+\lambda_2}}(k-\varepsilon) \;<\; 0.
\end{aligned}$$

The conclusion follows now from Theorem 7.7.1. ☐

Theorem 7.7.3. *Let g and ψ be two continuously differentiable functions on $[t_0, \infty)$ satisfying one of the conditions*

$$g(t) > 0, \quad g'(t) \geq a_1(t), \quad \psi'(t) \leq a_2(t), \tag{7.44}$$

$$g(t) > 0, \quad g'(t) \leq -a_1(t), \quad \psi'(t) \geq -a_2(t), \tag{7.45}$$

$$g(t) > 0, \quad g'(t) \leq -a_1(t), \quad \psi'(t) \leq a_2(t). \tag{7.46}$$

If (7.42) holds, then the system (7.40) is nonoscillatory.

Proof. The proof follows by similar arguments as in the previous theorem by taking

$$\theta(t) = \psi(t) + \frac{1 - k\alpha}{\alpha[g(t)]^{\lambda_2}}$$

if (7.44) holds,

$$\theta(t) = -\psi(t) - \frac{1 - k\alpha}{\alpha[g(t)]^{\lambda_2}}$$

if (7.45) holds, and

$$\theta(t) = \psi(t) - \frac{1 - k\alpha}{\alpha[g(t)]^{\lambda_2}}$$

if (7.46) holds. □

For the next two theorems, we refer to the fact that there exists a constant $k > 0$ such that

$$\left|[g(t)]^{\lambda_2}\psi(t) - k\right|^{1+\lambda_1} \leq k, \quad k^{\frac{1}{1+\lambda_1}} + k \leq \varepsilon \tag{7.47}$$

or such that

$$\left|[g(t)]^{\lambda_2}\psi(t) + k\right|^{1+\lambda_1} \leq k, \quad k^{\frac{1}{1+\lambda_1}} - k \leq \varepsilon. \tag{7.48}$$

Theorem 7.7.4. *Let g and ψ be two continuously differentiable functions on $[t_0, \infty)$. Then we have:*

(i) *(7.41) and (7.47) imply that the system (7.40) is nonoscillatory;*
(ii) *(7.44) and (7.48) imply that the system (7.40) is nonoscillatory.*

Proof. To prove (i) let

$$\theta(t) = -\psi(t) + \frac{k}{[g(t)]^{\lambda_2}}.$$

Then, according to (7.41),

$$\theta'(t) = -\psi'(t) - k\lambda_2\frac{g'(t)}{[g(t)]^{1+\lambda_2}} \leq a_2(t) - k\lambda_2\frac{a_1(t)}{[g(t)]^{1+\lambda_2}},$$

which implies

$$\begin{aligned}
&\theta'(t) + \lambda_2 a_1(t)\,|\theta(t)|^{\lambda_1+1} - a_2(t) \\
\leq\;& -k\lambda_2\frac{a_1(t)}{[g(t)]^{1+\lambda_2}} + \lambda_2 a_1(t)\left|\psi(t) - \frac{k}{[g(t)]^{\lambda_2}}\right|^{\lambda_1+1} \\
=\;& \lambda_2\frac{a_1(t)}{[g(t)]^{1+\lambda_2}}\left[\left|\psi(t)[g(t)]^{\lambda_2} - k\right|^{1+\lambda_1} - k\right] \;\leq\; 0.
\end{aligned}$$

This proves the conclusion (i) by an application of Theorem 7.7.1.

To show (ii), let

$$\theta(t) = \psi(t) + \frac{k}{[g(t)]^{\lambda_2}}.$$

Using (7.44), we obtain

$$\theta'(t) \le a_2(t) - k\lambda_2 \frac{a_1(t)}{[g(t)]^{1+\lambda_2}}$$

and

$$\begin{aligned} &\theta'(t) + \lambda_2 a_1(t) \left|\theta(t)\right|^{\lambda_1+1} - a_2(t) \\ \le\ & -k\lambda_2 \frac{a_1(t)}{[g(t)]^{1+\lambda_2}} + \lambda_2 a_1(t) \left|\psi(t) + \frac{k}{[g(t)]^{\lambda_2}}\right|^{\lambda_1+1} \\ =\ & \lambda_2 \frac{a_1(t)}{[g(t)]^{1+\lambda_2}} \left(\left|\psi(t)[g(t)]^{\lambda_2} + k\right|^{1+\lambda_1} - k \right) \ \le\ 0, \end{aligned}$$

which establishes the conclusion (ii). □

Theorem 7.7.5. *Let g and ψ be two continuously differentiable functions on $[t_0, \infty)$. Then we have:*

(i) (7.45) *and* (7.48) *imply that the system* (7.40) *is nonoscillatory;*
(ii) (7.46) *and* (7.47) *imply that the system* (7.40) *is nonoscillatory.*

Proof. The conclusion follows according to Theorem 7.7.1 if we let

$$\theta(t) = -\psi(t) - \frac{k}{[g(t)]^{\lambda_2}}$$

for (i) and

$$\theta(t) = \psi(t) - \frac{k}{[g(t)]^{\lambda_2}}$$

for (ii). □

Theorem 7.7.6. *Let g be a continuously differentiable function on $[t_0, \infty)$ such that $g(t) > 0$ and $g'(t) \le -a_1(t)$. If there exists a continuously differentiable function ψ on $[t_0, \infty)$ such that*

$$\lim_{t\to\infty} \psi(t) \ \textit{exists} \quad \textit{and} \quad \psi'(t) \ge -a_2(t)[g(t)]^{\lambda_2}, \tag{7.49}$$

then the system (7.40) *is nonoscillatory.*

Proof. Since $\lim_{t\to\infty} \psi(t)$ exists, there exist two numbers $T \ge t_0$ and M such that

$$0 < M + \psi(t) \le 1 \quad \text{for} \quad t \ge T.$$

Then, for the function

$$\theta(t) = -\frac{M + \psi(t)}{[g(t)]^{\lambda_2}}$$

we obtain

$$\theta'(t) = -\frac{\psi'(t)}{[g(t)]^{\lambda_2}} + \lambda_2 \frac{M + \psi(t)}{[g(t)]^{1+\lambda_2}} g'(t) \le a_2(t) - \lambda_2 \frac{M + \psi(t)}{[g(t)]^{1+\lambda_2}} a_1(t),$$

which implies

$$\begin{aligned}
&\theta'(t) + \lambda_2 a_1(t)\,|\theta(t)|^{\lambda_1+1} - a_2(t)\\
&\le\ \lambda_2 a_1(t)\frac{(M+\psi(t))^{1+\lambda_1}}{[g(t)]^{1+\lambda_2}} - \lambda_2 a_1(t)\frac{M+\psi(t)}{[g(t)]^{1+\lambda_2}}\\
&\le\ \lambda_2\frac{a_1(t)}{[g(t)]^{1+\lambda_2}}\Big(M+\psi(t)-(M+\psi(t))\Big)\ =\ 0.
\end{aligned}$$

This completes the proof by an application of Theorem 7.7.1. □

Theorem 7.7.7. *Let g be a continuously differentiable function on $[t_0,\infty)$ such that $g(t)>0$ and $g'(t)\ge a_1(t)$. If there exists a continuously differentiable function ψ on $[t_0,\infty)$ satisfying (7.49), then the system (7.40) is nonoscillatory.*

Proof. The assumption (7.49) guarantees the existence of two numbers $T\ge t_0$ and K such that

$$0 < K-\psi(t)\le 1 \quad \text{for} \quad t\ge T.$$

Taking $\theta(t)=\frac{K-\psi(t)}{[g(t)]^{\lambda_2}}$, we get the conclusion as in the previous proof. □

Theorem 7.7.8. *Let $a_2(t)\ge -[h(t)]^{-1-\lambda_2}$, where $h\in C^1([t_0,\infty),(0,\infty))$. If*

$$\lim_{t\to\infty}\Big(h'(t)-a_1(t)\Big) = L \ \textit{exists} \quad \textit{and} \quad L>\frac{1}{\lambda_2},$$

then the system (7.40) is nonoscillatory.

Proof. It follows from the assumption (7.49) that there exists $T\ge t_0$ with

$$h'(t)-a_1(t)>\frac{1}{\lambda_2} \quad \text{for} \quad t\ge T.$$

Then the function $\theta(t)=[h(t)]^{-\lambda_2}$ satisfies the condition of the Theorem 7.7.1, since we have

$$\begin{aligned}
\theta'(t)+\lambda_2 a_1(t)\,|\theta(t)|^{\lambda_1+1}-a_2(t) &= -\lambda_2\frac{h'(t)}{[h(t)]^{1+\lambda_2}}+\lambda_2\frac{a_1(t)}{[h(t)]^{1+\lambda_2}}-a_2(t)\\
&\le \frac{\lambda_2}{[h(t)]^{1+\lambda_2}}\left[-h'(t)+a_1(t)+\frac{1}{\lambda_2}\right] < 0.
\end{aligned}$$

Hence we obtain the desired result according to Theorem 7.7.1. □

Theorem 7.7.9. *Let ψ be a nonnegative continuously differentiable function on $[t_0,\infty)$ such that $\psi'(t)\le a_2(t)$. If*

$$\int_t^\infty [\psi(s)]^{1+\lambda_1}a_1(s)ds \le \frac{\psi(t)}{(1+\lambda_2)^{1+\lambda_1}},$$

then the system (7.40) is nonoscillatory.

Proof. Denote $\beta=(1+\lambda_2)^{1+\lambda_1}$ and

$$\theta(t)=\psi(t)+\lambda_2\beta\int_t^\infty[\psi(s)]^{1+\lambda_1}a_1(s)ds.$$

Then

$$\theta'(t)=\psi'(t)-\lambda_2\beta[\psi(t)]^{1+\lambda_1}a_1(t)\le a_2(t)-\lambda_2\beta[\psi(t)]^{1+\lambda_1}a_1(t).$$

Therefore,

$$\begin{aligned}
&\theta'(t) + \lambda_2 a_1(t) \left|\theta(t)\right|^{\lambda_1+1} - a_2(t) \\
&\quad\leq \lambda_2 a_1(t) \left|\psi(t) + \lambda_2\beta \int_t^\infty [\psi(s)]^{1+\lambda_1} a_1(s) ds\right|^{\lambda_1+1} - \lambda_2\beta[\psi(t)]^{1+\lambda_1} a_1(t) \\
&\quad\leq \lambda_2 a_1(t) \left([\psi(t) + \lambda_2\psi(t)]^{\lambda_1+1} - \beta[\psi(t)]^{1+\lambda_1}\right) = 0.
\end{aligned}$$

Hence, by Theorem 7.7.1, the system (7.40) is nonoscillatory. $\square$

Theorem 7.7.10. *Let ψ be a nonnegative continuously differentiable function on $[t_0, \infty)$ such that $\psi'(t) \leq a_2(t)$ and let*

$$\varphi(t) = \int_t^\infty [\psi(s)]^{1+\lambda_1} a_1(s) \exp\left(\lambda_2(1+\lambda_2)^{\lambda_1} \int_t^s [\psi(\tau)]^{\lambda_1} a_1(\tau) d\tau\right) ds.$$

If $\varphi(t) \leq \frac{\psi(t)}{(1+\lambda_2)^{\lambda_1}}$, then the system (7.40) *is nonoscillatory.*

Proof. Denote $\gamma = (1+\lambda_2)^{\lambda_1}$ and $\theta(t) = \psi(t) + \lambda_2\gamma\varphi(t)$. Then we get

$$\theta'(t) = \psi'(t) + \lambda_2\gamma\varphi'(t) \leq a_2(t) - \lambda_2\gamma\left([\psi(t)]^{1+\lambda_1} a_1(t) + \lambda_2\gamma\varphi(t)[\psi(t)]^{\lambda_1} a_1(t)\right)$$

and consequently,

$$\begin{aligned}
&\theta'(t) + \lambda_2 a_1(t) \left|\theta(t)\right|^{\lambda_1+1} - a_2(t) \\
&\quad\leq \lambda_2 a_1(t) \left|\psi(t) + \lambda_2\gamma\varphi(t)\right|^{\lambda_1+1} - \lambda_2\gamma a_1(t)[\psi(t)]^{\lambda_1}\left(\psi(t) + \lambda_2\gamma\varphi(t)\right) \\
&\quad= \lambda_2 a_1(t)\left(\psi(t) + \lambda_2\gamma\varphi(t)\right)\left([\psi(t) + \lambda_2\gamma\varphi(t)]^{\lambda_1} - \gamma[\psi(t)]^{\lambda_1}\right) \\
&\quad\leq \lambda_2 a_1(t)\left(\psi(t) + \lambda_2\gamma\varphi(t)\right)\left([\psi(t) + \lambda_2\psi(t)]^{\lambda_1} - \gamma[\psi(t)]^{\lambda_1}\right) = 0.
\end{aligned}$$

It follows from Theorem 7.7.1 that the system (7.40) is nonoscillatory. $\square$

7.8. Notes

The results in Section 7.2 are based on Kordonis and Philos [**152**]; see also Mirzov [**224, 225, 226**] and Kwong and Wong [**161**]. The treatment in Section 7.3 is from Li and Huo [**198**]. The results of Section 7.4 are taken from Li and Cheng [**194**]. While Section 7.5 follows Li [**185**], Section 7.6 summarizes results by Li and Yang [**207**]. The material in Section 7.7 is adopted from Manojlović [**223**].

CHAPTER 8

Oscillation of Dynamic Equations on Time Scales

8.1. Introduction

The study of dynamic equations on a time scale goes back to its founder Stefan Hilger (1988), and is a new area of still fairly theoretical exploration in mathematics. Motivating the subject is the notion that dynamic equations on time scales can build bridges between continuous and discrete mathematics. Further, the study of time scales has led to several important applications, e.g., in the study of insect population models, neural networks, heat transfer, and epidemic models.

The development of the theory of time scales is yet in its infancy, yet as inroads are made, interest is gathering steam. Of a great deal of interest are methods being introduced for dynamic equations on time scales which now explain some discrepancies that have been encountered when results for differential equations and their discrete counterparts have been independently considered. The explanations of these seemingly discrepancies are incidentally producing unifying results via time scale methods. While there are currently many independent results on oscillation of differential equations as well as on oscillation of difference equations, the occurring discrepancies may suggest that a unification will be too hard or impossible. However, all of the problems that have been tackled so far using the time scales calculus led to a unification and hence shed light on the nature of the underlying structures.

In addition to the **unification** aspect of the theory of time scales there is an **extension** aspect, which might even have a broader impact on the future of oscillation. Instead of differential or difference equations, any other kind of dynamic equation is also applicable to the theory (for example, so-called q-difference equations), and it might be very important to understand oscillation properties of solutions of such more general equations. The applicability of dynamic equations on time scales to modeling phenomena of seasonal plant (or insect) population dynamics is of great potential value. Results in this direction will undoubtedly attract the attention of researchers in other disciplines such as the biological sciences.

In this chapter we present some first progress in direction of generalizing the oscillation results given in the earlier chapters of this book to the time scales case. In Section 8.2 we first give a general introduction into the theory of dynamic equations on time scales. The reader may consult the books [**53, 55**] for further results. Then, in Section 8.3, we present some oscillation results of second order nonlinear dynamic equations on time scales. In Section 8.4, some oscillation criteria for perturbed nonlinear dynamic equations are given. We then follow [**197**] and classify positive solutions of nonlinear dynamic equations in Section 8.5 and discuss oscillation of Emden–Fowler dynamic equations in Section 8.6. In Section 8.7 we present some oscillation criteria for first order delay dynamic equations. Finally, in Section 8.8, we consider oscillation of symplectic dynamic systems, which contain as special cases linear Hamiltonian dynamic systems and Sturm–Liouville dynamic equations of any even order.

8.2. The Time Scales Calculus

In this section we introduce some basic concepts concerning the calculus on time scales that one needs to know to read the remainder of this chapter. Most of these results will be stated without proof. Proofs can be found in the books by Bohner and Peterson [**53, 55**]. A *time scale* is an arbitrary nonempty closed subset of the real numbers. Thus $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{N}_0$, $[0,1]\cup[2,3]$, $[0,1]\cup\mathbb{N}$, and the Cantor set are examples of time scales, while $\mathbb{Q}$, $\mathbb{R}\setminus\mathbb{Q}$, $\mathbb{C}$, and $(0,1)$ are not time scales. Throughout this chapter we will denote a time scale by the symbol $\mathbb{T}$. We assume throughout that a time scale $\mathbb{T}$ has the topology that it inherits from the real numbers with the standard topology.

The calculus of time scales was initiated by Stefan Hilger in [**124**] in order to create a theory that can unify discrete and continuous analysis. Indeed, below we will introduce the delta derivative f^Δ for a function f defined on $\mathbb{T}$, and it turns out that

(i) $f^\Delta = f'$ is the usual derivative if $\mathbb{T}=\mathbb{R}$ and
(ii) $f^\Delta = \Delta f$ is the usual forward difference operator if $\mathbb{T}=\mathbb{Z}$.

We now introduce the basic notions connected to time scales. We start by defining the forward and backward jump operators.

Definition 8.2.1. Let $\mathbb{T}$ be a time scale. For $t\in\mathbb{T}$ we define the *forward jump operator* $\sigma:\mathbb{T}\to\mathbb{T}$ by

$$\sigma(t) := \inf\left\{s\in\mathbb{T}:\ s>t\right\} \quad \text{for all} \quad t\in\mathbb{T},$$

while the *backward jump operator* $\rho:\mathbb{T}\to\mathbb{T}$ is defined by

$$\rho(t) := \sup\left\{s\in\mathbb{T}:\ s<t\right\} \quad \text{for all} \quad t\in\mathbb{T}.$$

In this definition we put $\inf\emptyset = \sup\mathbb{T}$ (i.e., $\sigma(M)=M$ if $\mathbb{T}$ has a maximum M) and $\sup\emptyset=\inf\mathbb{T}$ (i.e., $\rho(m)=m$ if $\mathbb{T}$ has a minimum m), where $\emptyset$ denotes the empty set. If $\sigma(t)>t$, then we say that t is *right-scattered*, while if $\rho(t)<t$, then we say that t is *left-scattered*. Points that are right-scattered and left-scattered at the same time are called *isolated*. Also, if $t<\sup\mathbb{T}$ and $\sigma(t)=t$, then t is called *right-dense*, and if $t>\inf\mathbb{T}$ and $\rho(t)=t$, then t is called *left-dense*. Points that are right-dense or left-dense are called *dense*. If $\mathbb{T}$ has a left-scattered maximum M, then we define $\mathbb{T}^\kappa=\mathbb{T}-\{M\}$, otherwise $\mathbb{T}^\kappa=\mathbb{T}$. Finally, the *graininess function* $\mu:\mathbb{T}\to[0,\infty)$ is defined by

$$\mu(t) := \sigma(t)-t \quad \text{for all} \quad t\in\mathbb{T}.$$

Remark 8.2.2. As in this book we are concerned with oscillatory behavior as $t\to\infty$, we will assume for the remainder of this chapter that $\mathbb{T}$ is a time scale that is unbounded above.

Now we consider a function $f:\mathbb{T}\to\mathbb{R}$ and define the so-called delta (or Hilger) derivative of f at a point $t\in\mathbb{T}^\kappa$.

Definition 8.2.3. Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^\kappa$. Then we define $f^\Delta(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t (i.e., $U = (t-\delta, t+\delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$\left| f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s) \right| \le \varepsilon \left| \sigma(t) - s \right| \quad \text{for all} \quad s \in U.$$

We call $f^\Delta(t)$ the *delta* (or *Hilger*) *derivative* of f at t.

Moreover, we say that f is *delta* (or *Hilger*) *differentiable* (or in short: *differentiable*) on $\mathbb{T}^\kappa$ provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$. The function $f^\Delta : \mathbb{T}^\kappa \to \mathbb{R}$ is then called the (delta) derivative of f on $\mathbb{T}^\kappa$.

Some easy and useful relationships concerning the delta derivative are given next.

Theorem 8.2.4. *Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^\kappa$. Then we have the following:*

(i) *If f is differentiable at t, then f is continuous at t.*

(ii) *If f is continuous at t and t is right-scattered, then f is differentiable at t with*

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

(iii) *If t is right-dense, then f is differentiable at t iff the limit*

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

exists as a finite number. In this case

$$f^\Delta(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

(iv) *If f is differentiable at t, then*

$$f^\sigma(t) = f(t) + \mu(t) f^\Delta(t), \qquad \textit{where} \qquad f^\sigma := f \circ \sigma. \tag{8.1}$$

Example 8.2.5. Again we consider the two cases $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$.

(i) If $\mathbb{T} = \mathbb{R}$, then Theorem 8.2.4 (iii) yields that $f : \mathbb{R} \to \mathbb{R}$ is delta differentiable at $t \in \mathbb{R}$ iff

$$f'(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s} \quad \text{exists,}$$

i.e., iff f is differentiable (in the ordinary sense) at t. In this case we then have

$$f^\Delta(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s} = f'(t)$$

by Theorem 8.2.4 (iii).

(ii) If $\mathbb{T} = \mathbb{Z}$, then Theorem 8.2.4 (ii) yields that $f : \mathbb{Z} \to \mathbb{R}$ is delta differentiable at $t \in \mathbb{Z}$ with

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)} = \frac{f(t+1) - f(t)}{1} = f(t+1) - f(t) =: \Delta f(t),$$

where Δ is the usual *forward difference operator* defined by the last equation above.

Next, we would like to be able to find the derivatives of sums, products, and quotients of differentiable functions. This is possible according to the following theorem.

Theorem 8.2.6. *Assume $f, g : \mathbb{T} \to \mathbb{R}$ are differentiable at $t \in \mathbb{T}^\kappa$. Then:*

(i) *The sum $f + g : \mathbb{T} \to \mathbb{R}$ is differentiable at t with*

$$(f+g)^\Delta(t) = f^\Delta(t) + g^\Delta(t).$$

(ii) *For any constant α, $\alpha f : \mathbb{T} \to \mathbb{R}$ is differentiable at t with*

$$(\alpha f)^\Delta(t) = \alpha f^\Delta(t).$$

(iii) *The product $fg : \mathbb{T} \to \mathbb{R}$ is differentiable at t with*

$$(fg)^\Delta(t) = f^\Delta(t)g(t) + f(\sigma(t))g^\Delta(t) = f(t)g^\Delta(t) + f^\Delta(t)g(\sigma(t)). \tag{8.2}$$

(iv) *If $f(t)f(\sigma(t)) \neq 0$, then $\frac{1}{f}$ is differentiable at t with*

$$\left(\frac{1}{f}\right)^\Delta(t) = -\frac{f^\Delta(t)}{f(t)f(\sigma(t))}.$$

(v) *If $g(t)g(\sigma(t)) \neq 0$, then $\frac{f}{g}$ is differentiable at t and*

$$\left(\frac{f}{g}\right)^\Delta(t) = \frac{f^\Delta(t)g(t) - f(t)g^\Delta(t)}{g(t)g(\sigma(t))}. \tag{8.3}$$

The following theorem enables us to differentiate polynomials.

Theorem 8.2.7. *Let α be constant and $m \in \mathbb{N}$.*

(i) *For f defined by $f(t) = (t-\alpha)^m$ we have*

$$f^\Delta(t) = \sum_{\nu=0}^{m-1} (\sigma(t)-\alpha)^\nu (t-\alpha)^{m-1-\nu}.$$

(ii) *For g defined by $g(t) = \frac{1}{(t-\alpha)^m}$ we have*

$$g^\Delta(t) = -\sum_{\nu=0}^{m-1} \frac{1}{(\sigma(t)-\alpha)^{m-\nu}(t-\alpha)^{\nu+1}},$$

provided $(t-\alpha)(\sigma(t)-\alpha) \neq 0$.

In order to describe classes of functions that are "delta integrable", we introduce the following concept.

Definition 8.2.8. A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by

$$\mathrm{C}_{\mathrm{rd}} = \mathrm{C}_{\mathrm{rd}}(\mathbb{T}) = \mathrm{C}_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}).$$

The set of functions $f : \mathbb{T} \to \mathbb{R}$ that are differentiable and whose derivative is rd-continuous is denoted by

$$\mathrm{C}^1_{\mathrm{rd}} = \mathrm{C}^1_{\mathrm{rd}}(\mathbb{T}) = \mathrm{C}^1_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}).$$

Definition 8.2.9. A function $F : \mathbb{T} \to \mathbb{R}$ is called an *antiderivative* of $f : \mathbb{T} \to \mathbb{R}$ provided

$$F^{\Delta}(t) = f(t) \quad \text{holds for all} \quad t \in \mathbb{T}^{\kappa}.$$

Theorem 8.2.10 (Existence of Antiderivatives)**.** *Every rd-continuous function has an antiderivative.*

Definition 8.2.11. Let $f : \mathbb{T} \to \mathbb{R}$ be rd-continuous, and let $F : \mathbb{T} \to \mathbb{R}$ be an antiderivative of f. Then we define the (Cauchy) integral of f by

$$\int_r^s f(t)\Delta t = F(s) - F(r) \quad \text{for all} \quad r, s \in \mathbb{T}.$$

The following theorem gives several elementary properties of the delta integral.

Theorem 8.2.12. *If $a, b, c \in \mathbb{T}$, $\alpha \in \mathbb{R}$, and $f, g \in \mathrm{C}_{\mathrm{rd}}$, then*

(i) $\int_a^b [f(t) + g(t)]\Delta t = \int_a^b f(t)\Delta t + \int_a^b g(t)\Delta t;$
(ii) $\int_a^b (\alpha f)(t)\Delta t = \alpha \int_a^b f(t)\Delta t;$
(iii) $\int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t;$
(iv) $\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t;$
(v) $\int_a^b f(\sigma(t))g^{\Delta}(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t)g(t)\Delta t;$
(vi) $\int_a^b f(t)g^{\Delta}(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t)g(\sigma(t))\Delta t;$
(vii) $\int_a^a f(t)\Delta t = 0;$
(viii) *if $f(t) \geq 0$ for all $a \leq t < b$, then $\int_a^b f(t)\Delta t \geq 0;$*
(ix) *if $|f(t)| \leq g(t)$ on $[a, b)$, then*

$$\left| \int_a^b f(t)\Delta t \right| \leq \int_a^b g(t)\Delta t.$$

The following simple theorem is useful.

Theorem 8.2.13. *If $f : \mathbb{T} \to \mathbb{R}$ is an arbitrary function and $t \in \mathbb{T}$, then*

$$\int_t^{\sigma(t)} f(\tau)\Delta\tau = \mu(t)f(t).$$

One can then easily prove the following theorem.

Theorem 8.2.14. *Let $a, b \in \mathbb{T}$ and $f \in \mathrm{C}_{\mathrm{rd}}$.*

(i) *If $\mathbb{T} = \mathbb{R}$, then*

$$\int_a^b f(t)\Delta t = \int_a^b f(t)dt,$$

where the integral on the right is the usual Riemann integral from calculus.

(ii) *If $[a, b]$ consists of only isolated points, then*

$$\int_a^b f(t)\Delta t = \begin{cases} \sum\limits_{t \in [a,b)} \mu(t)f(t) & \text{if} \quad a < b \\ 0 & \text{if} \quad a = b \\ -\sum\limits_{t \in [b,a)} \mu(t)f(t) & \text{if} \quad a > b. \end{cases}$$

(iii) *If* $\mathbb{T} = h\mathbb{Z} = \{hk: \ k \in \mathbb{Z}\}$, *where* $h > 0$, *then*

$$\int_a^b f(t)\Delta t = \begin{cases} \displaystyle\sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} f(kh)h & \text{if} \quad a < b \\ 0 & \text{if} \quad a = b \\ \displaystyle -\sum_{k=\frac{b}{h}}^{\frac{a}{h}-1} f(kh)h & \text{if} \quad a > b. \end{cases}$$

(iv) *If* $\mathbb{T} = \mathbb{Z}$, *then*

$$\int_a^b f(t)\Delta t = \begin{cases} \displaystyle\sum_{t=a}^{b-1} f(t) & \text{if} \quad a < b \\ 0 & \text{if} \quad a = b \\ \displaystyle -\sum_{t=b}^{a-1} f(t) & \text{if} \quad a > b. \end{cases}$$

Now we will introduce the exponential function on time scales. In order to do so, we first need to look at what is called the regressive group.

Definition 8.2.15. We say that a function $p : \mathbb{T} \to \mathbb{R}$ is *regressive* provided

$$1 + \mu(t)p(t) \neq 0 \quad \text{for all} \quad t \in \mathbb{T}^\kappa \tag{8.4}$$

holds. The set of all regressive and rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by

$$\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R}).$$

Theorem 8.2.16. *For* $p, q \in \mathcal{R}$, *define "circle plus" addition by*

$$(p \oplus q)(t) := p(t) + q(t) + \mu(t)p(t)q(t) \quad \textit{for all} \quad t \in \mathbb{T}^\kappa.$$

Then $(\mathcal{R}, \oplus)$ *is an Abelian group, the so-called* regressive group. *The inverse of* $p \in \mathcal{R}$ *with respect to* $\oplus$ *is given by*

$$(\ominus p)(t) = -\frac{p(t)}{1 + \mu(t)p(t)} \quad \textit{for all} \quad t \in \mathbb{T}^\kappa.$$

If we define the "circle minus" subtraction $\ominus$ *on* $\mathcal{R}$ *by*

$$(p \ominus q)(t) := (p \oplus (\ominus q))(t) \quad \textit{for all} \quad t \in \mathbb{T}^\kappa,$$

then we have the formula

$$p \ominus q = \frac{p - q}{1 + \mu q} \quad \textit{for all} \quad p, q \in \mathcal{R}.$$

If we define the set of positively regressive *functions* $\mathcal{R}^+$ *as the set consisting of those* $p \in \mathcal{R}$ *satisfying*

$$1 + \mu(t)p(t) > 0 \quad \textit{for all} \quad t \in \mathbb{T},$$

then $(\mathcal{R}^+, \oplus)$ *is a subgroup of the regressive group. If we define the "circle square" of* $p \in \mathcal{R}$ *by*

$$(p^{②})(t) = (-p(t))(\ominus p(t)) \quad \textit{for all} \quad t \in \mathbb{T},$$

then we have the formula [**53**, (2.8)]

$$f + (\ominus f) = \mu f^{②}. \tag{8.5}$$

The main existence theorem for initial value problems with first order linear dynamic equations now reads as follows.

Theorem 8.2.17. *Suppose $p \in \mathcal{R}$ and fix $t_0 \in \mathbb{T}$. Then the initial value problem*

$$y^\Delta = p(t)y, \quad y(t_0) = 1 \tag{8.6}$$

has a unique solution on $\mathbb{T}$.

Definition 8.2.18. If $p \in \mathcal{R}$ and $t_0 \in \mathbb{T}$, then the unique solution of the initial value problem (8.6) is called the *exponential function* and denoted by $e_p(\cdot, t_0)$.

In the following theorem we collect some important properties of the exponential function. Their proofs can be found in [**53**, Theorem 2.36 and Theorem 2.39].

Theorem 8.2.19. *If $p, q \in \mathcal{R}$, then*

(i) $e_0(t,s) \equiv 1$ *and* $e_p(t,t) \equiv 1$;
(ii) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t,s)$;
(iii) $\frac{1}{e_p(t,s)} = e_{\ominus p}(t,s)$;
(iv) $e_p(t,s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s,t)$;
(v) $e_p(t,s)e_p(s,r) = e_p(t,r)$;
(vi) $e_p(t,s)e_q(t,s) = e_{p\oplus q}(t,s)$;
(vii) $\frac{e_p(t,s)}{e_q(t,s)} = e_{p\ominus q}(t,s)$;
(viii) $\left(\frac{1}{e_p(\cdot,s)}\right)^\Delta = -\frac{p(t)}{e_p^\sigma(\cdot,s)}$;
(ix) $[e_p(c,\cdot)]^\Delta = -p[e_p(c,\cdot)]^\sigma$, *where* $c \in \mathbb{T}$.

There are two versions of variation of parameters results as follows.

Theorem 8.2.20. *Suppose $f \in \mathrm{C_{rd}}$ and $p \in \mathcal{R}$. Then the unique solution of the initial value problem*

$$y^\Delta = p(t)y + f(t), \quad y(t_0) = y_0$$

is given by

$$y(t) = y_0 e_p(t,t_0) + \int_{t_0}^t e_p(t, \sigma(\tau))f(\tau)\Delta\tau.$$

Also, the unique solution of the initial value problem

$$y^\Delta = -p(t)y^\sigma + f(t), \quad y(t_0) = y_0$$

is given by

$$y(t) = y_0 e_{\ominus p}(t,t_0) + \int_{t_0}^t e_{\ominus p}(t,\tau)f(\tau)\Delta\tau.$$

We now give some examples of exponential functions. These examples can be verified easily by checking that the given functions satisfy the corresponding initial value problems (8.6).

Example 8.2.21. (i) Let $\mathbb{T} = \mathbb{R}$ and $\alpha \in \mathbb{R}$ be a constant. Then

$$e_\alpha(t,0) = e^{\alpha t} \quad \text{for all} \quad t \in \mathbb{R};$$

(ii) Let $\mathbb{T} = \mathbb{R}$ and $p : \mathbb{T} \to \mathbb{R}$ be continuous. Then

$$e_p(t,t_0) = \exp\left\{\int_{t_0}^t p(s)ds\right\} \quad \text{for all} \quad t \in \mathbb{R};$$

(iii) Let $\mathbb{T} = \mathbb{Z}$ and $\alpha \in \mathbb{R}$ be a constant. Then

$$e_\alpha(t, 0) = (1 + \alpha)^t \quad \text{for all} \quad t \in \mathbb{Z};$$

(iv) Let $\mathbb{T} = \mathbb{Z}$ and $p : \mathbb{T} \to \mathbb{R}$ be arbitrary. Then

$$e_p(t, t_0) = \prod_{s=t_0}^{t-1} \Big(1 + p(s)\Big) \quad \text{for all} \quad t \in \mathbb{Z} \cap [t_0, \infty);$$

(v) Let $\mathbb{T} = h\mathbb{N}_0$ for $h > 0$ and $\alpha \in \mathcal{R}$ be a constant. Then

$$e_\alpha(t, 0) = (1 + \alpha h)^{\frac{t}{h}} \quad \text{for all} \quad t \in \mathbb{T}.$$

(vi) Let $\mathbb{T} = \mathbb{N}_0^2 = \{n^2 : n \in \mathbb{N}_0\}$. Then

$$e_1(t, 0) = 2^{\sqrt{t}}(\sqrt{t})! \quad \text{for all} \quad t \in \mathbb{T}.$$

(vii) Let $\mathbb{T} = \{H_n : n \in \mathbb{N}_0\}$, where $H_n = \sum_{k=1}^{n} 1/k$. If $\alpha \geq 0$ is constant, then

$$e_\alpha(H_n, 0) = \binom{n + \alpha}{n} \quad \text{for all} \quad n \in \mathbb{N}.$$

(viii) Let $\mathbb{T} = q^{\mathbb{N}_0} = \left\{q^k : k \in \mathbb{N}_0\right\}$ and $p \in \mathcal{R}$. Then

$$e_p(t, 1) = \prod_{s \in \mathbb{T} \cap (0,t)} \Big(1 + (q - 1)sp(s)\Big) \quad \text{for all} \quad t \in \mathbb{T}.$$

8.3. Oscillation of Second Order Nonlinear Dynamic Equations

In this section we follow [**58**] and consider the nonlinear second order dynamic equation

$$\Big(p(t)x^\Delta\Big)^\Delta + q(t)(f \circ x^\sigma) = 0 \quad \text{for} \quad t \in [a, b], \tag{8.7}$$

where p and q are positive, real-valued rd-continuous functions defined on the time scales interval $[a, b]$ (throughout $a, b \in \mathbb{T}$ with $a < b$). Since we are interested in oscillatory behavior, we suppose that the time scale under consideration is not bounded above, i.e., it is a time scale interval of the form $[a, \infty)$. We suppose that there exists a constant $K > 0$ such that $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$xf(x) > 0 \quad \text{and} \quad f(x) \geq Kx \quad \text{for all} \quad x \neq 0. \tag{8.8}$$

Let us first recall that a solution of (8.7) is a nontrivial real function x satisfying (8.7) for $t \geq a$. A solution x of (8.7) is said to be *oscillatory* if it is neither eventually positive nor eventually negative; otherwise it is called *nonoscillatory*. Equation (8.7) is said to be *oscillatory* if all its solutions are oscillatory. Our attention is restricted to those solutions of (8.7) which exist on some half line $[t_x, \infty)$ and satisfy $\sup\{|x(t)| : t > t_0\} > 0$ for any $t_0 \geq t_x$.

The classical *Riccati transformation* technique for differential equations consists essentially in "completing the square". For dynamic equations we will need to "complete the circle square".

Lemma 8.3.1. *For $f, g \in \mathcal{R}$ we have*

$$(f \ominus g)^{②} = f^{②} + f(\ominus g) + (\ominus f)g + g^{②}. \tag{8.9}$$

Proof. We have

$$(f \ominus g)^{②} = \frac{(f \ominus g)^2}{1 + \mu(f \ominus g)} = \frac{\frac{(f-g)^2}{(1+\mu g)^2}}{1 + \mu\frac{f-g}{1+\mu g}} = \frac{(f-g)^2}{(1+\mu f)(1+\mu g)},$$

and hence

$$\begin{aligned} (f \ominus g)^{②} - f^{②} - g^{②} &= \frac{(f-g)^2}{(1+\mu f)(1+\mu g)} - \frac{f^2}{1+\mu f} - \frac{g^2}{1+\mu g} \\ &= \frac{f^2 - 2fg + g^2 - f^2 - \mu f^2 g - g^2 - \mu g^2 f}{(1+\mu f)(1+\mu g)} \\ &= -fg\frac{1 + \mu f + 1 + \mu g}{(1+\mu f)(1+\mu g)} \\ &= f(\ominus g) + (\ominus f)g \end{aligned}$$

implies (8.9). □

In the next lemma we collect some identities that are needed in the proof of our Riccati transformation result. These identities follow easily from (8.1), and hence we omit the proof.

Lemma 8.3.2. *Suppose f is differentiable with $ff^\sigma \neq 0$ and define $g = \frac{f^\Delta}{f}$. Then*

$$1 + \mu g = \frac{f^\sigma}{f}, \quad \ominus g = -\frac{f^\Delta}{f^\sigma}, \quad \text{and} \quad g^{②} = \frac{(f^\Delta)^2}{ff^\sigma}. \tag{8.10}$$

Theorem 8.3.3. *Suppose that x solves (8.7) with $x(t) \neq 0$ for all $t \geq t_0$. Let z be a differentiable function and define w on $[t_0, \infty)$ by*

$$w = \frac{z^2 p x^\Delta}{x}. \tag{8.11}$$

Then we have on $[t_0, \infty)$

$$-w^\Delta = (z^\sigma)^2 q \frac{f \circ x^\sigma}{x^\sigma} - p(z^\Delta)^2 + pzz^\sigma (r \ominus s)^{②}, \tag{8.12}$$

where

$$r = \frac{x^\Delta}{x} \quad \text{and} \quad s = \frac{z^\Delta}{z}.$$

If additionally (8.8) holds and $x(t)x^\sigma(t) > 0$ for all $t \geq t_0$, then on $[t_0, \infty)$

$$-w^\Delta \geq qK(z^\sigma)^2 - p(z^\Delta)^2. \tag{8.13}$$

Proof. We calculate

$$
\begin{aligned}
-w^\Delta &\overset{(8.2)}{=} -\left[z^\Delta\left(\frac{zpx^\Delta}{x}\right)+z^\sigma\left(\frac{zpx^\Delta}{x}\right)^\Delta\right]\\
&\overset{(8.2)}{=} -z^\Delta zpr - z^\sigma\left[z^\sigma\left(\frac{px^\Delta}{x}\right)^\Delta + z^\Delta\frac{px^\Delta}{x}\right]\\
&\overset{(8.3)}{=} -z^\Delta zpr-(z^\sigma)^2\left[\frac{(px^\Delta)^\Delta x - x^\Delta px^\Delta}{xx^\sigma}\right]-z^\sigma z^\Delta pr\\
&\overset{(8.10)}{=} -(z^\sigma)^2\frac{(px^\Delta)^\Delta}{x^\sigma}+(z^\sigma)^2pr^{②}-z^\Delta zpr-z^\sigma z^\Delta pr\\
&\overset{(8.7)}{=} (z^\sigma)^2q\frac{f\circ x^\sigma}{x^\sigma}+pzz^\sigma\left[\frac{z^\sigma}{z}r^{②}-\frac{z^\Delta}{z^\sigma}r-\frac{z^\Delta}{z}r\right]\\
&\overset{(8.1)}{=} (z^\sigma)^2q\frac{f\circ x^\sigma}{x^\sigma}+pzz^\sigma\left[\frac{z+\mu z^\Delta}{z}r^{②}-\frac{z^\Delta}{z}r-\frac{z^\Delta}{z^\sigma}r\right]\\
&\overset{(8.10)}{=} (z^\sigma)^2q\frac{f\circ x^\sigma}{x^\sigma}+pzz^\sigma\left[r^{②}+\mu sr^{②}-sr+(\ominus s)r\right]\\
&\overset{(8.5)}{=} (z^\sigma)^2q\frac{f\circ x^\sigma}{x^\sigma}+pzz^\sigma\left[r^{②}+s(\ominus r)+(\ominus s)r\right]\\
&\overset{(8.9)}{=} (z^\sigma)^2q\frac{f\circ x^\sigma}{x^\sigma}+pzz^\sigma\left[(r\ominus s)^{②}-s^{②}\right]\\
&\overset{(8.10)}{=} (z^\sigma)^2q\frac{f\circ x^\sigma}{x^\sigma}+pzz^\sigma(r\ominus s)^{②}-p(z^\Delta)^2,
\end{aligned}
$$

where we simply "completed the square". This proves (8.12). To obtain (8.13), note that

$$zz^\sigma p(r\ominus s)^{②} = \frac{z^2x(r-s)^2}{x^\sigma}$$

holds (apply the formula in the proof of Lemma 8.3.1 and the identities (8.10) from Lemma 8.3.2). This and (8.8) imply (8.13). □

8.3.1. The Case $\int_a^\infty \frac{1}{p(t)}\Delta t = \infty$. Now we assume

$$\int_a^\infty \frac{1}{p(t)}\Delta t = \infty. \tag{8.14}$$

and present some oscillation criteria for (8.7). We start with the following auxiliary result.

Lemma 8.3.4. *Assume* (8.14). *Suppose that x is a nonoscillatory solution of* (8.7). *Then there exists $t_0 \in \mathbb{T}$ such that*

$$x(t)x^\Delta(t) > 0 \quad \textit{for all} \quad t \ge t_0. \tag{8.15}$$

Proof. Since x is nonoscillatory, it is either eventually positive or eventually negative. We only prove the lemma for the first case as the second case is similar and hence omitted. Assume there exists $t_0 \in \mathbb{T}$ such that

$$x(t) > 0 \quad \text{for all} \quad t \ge t_0.$$

Define $y = px^\Delta$. Let $t \ge t_0$. Then $x(\sigma(t)) > 0$ and hence

$$y^\Delta(t) = -q(t)f(x^\sigma(t)) < 0$$

so that y is decreasing. Assume that there exists $t_1 \ge t_0$ with $y(t_1) =: c < 0$. Then

$$p(s)x^{\Delta}(s) = y(s) \le y(t_1) = c \quad \text{for all} \quad s \ge t_1$$

and therefore

$$x^{\Delta}(s) \le \frac{c}{p(s)} \quad \text{for all} \quad s \ge t_1.$$

Let $t \ge t_1$. Then

$$\begin{aligned} x(t) &= x(t_1) + \int_{t_1}^{t} x^{\Delta}(s)\Delta s \\ &\le x(t_1) + \int_{t_1}^{t} \frac{c}{p(s)}\Delta s \\ &= x(t_1) + c\left\{\int_{t_1}^{t} \frac{\Delta s}{p(s)}\right\} \\ &\overset{(8.14)}{\to} -\infty \quad \text{as} \quad t \to \infty, \end{aligned}$$

a contradiction. Hence $y(t) > 0$ for all $t \ge t_0$ and thus $x^{\Delta}(t) > 0$ for all $t \ge t_0$, i.e., (8.15) holds. □

Now we are ready to present the main results of this section.

Theorem 8.3.5. *Assume that* (8.8) *and* (8.14) *hold. Furthermore, assume that there exists a differentiable function* z *with*

$$\limsup_{t\to\infty} \int_{a}^{t} \left[Kq(s)(z^{\sigma}(s))^2 - p(s)(z^{\Delta}(s))^2\right] \Delta s = \infty. \tag{8.16}$$

Then every solution of (8.7) *is oscillatory on* $[a, \infty)$.

Proof. Suppose to the contrary that x is a nonoscillatory solution of (8.7). Then (8.15) from Lemma 8.3.4 implies that there exists $t_0 \in \mathbb{T}$ such that

$$w(t) > 0 \quad \text{for all} \quad t \ge t_0,$$

where w is defined by (8.11). All assumptions from Theorem 8.3.3 are satisfied, and hence we may integrate (8.13) from t_0 to $t \ge t_0$ to obtain

$$\begin{aligned} w(t_0) &\ge w(t_0) - w(t) \\ &= -\int_{t_0}^{t} w^{\Delta}(s)\Delta s \\ &\ge \int_{t_0}^{t} \left(Kq(s)(z^{\sigma}(s))^2 - p(s)(z^{\Delta}(s))^2\right) \Delta s \\ &\overset{(8.16)}{\to} \infty, \end{aligned}$$

which is not possible. The proof is complete. □

Corollary 8.3.6. *Assume that* (8.8) *and* (8.14) *hold. Furthermore, assume that there exists a positive function* δ *with*

$$\limsup_{t\to\infty} \int_{a}^{t} \left[Kq(s)\delta^{\sigma}(s) - p(s)\left(\frac{\delta^{\Delta}(s)}{\sqrt{\delta(s)} + \sqrt{\delta^{\sigma}(s)}}\right)^2\right] \Delta s = \infty. \tag{8.17}$$

Then every solution of (8.7) *is oscillatory on* $[a, \infty)$.

Proof. Define $z = \sqrt{\delta}$ and note that [**53**]

$$z^{\Delta} = \frac{\delta^{\Delta}}{\sqrt{\delta} + \sqrt{\delta^{\sigma}}}.$$

Since (8.17) holds for δ, we see that (8.16) holds for $z = \sqrt{\delta}$. Hence the claim follows from Theorem 8.3.5. $\square$

From Theorem 8.3.5 and Corollary 8.3.6 we can obtain different conditions for oscillation of all solutions of (8.7) by different choices of δ. E.g., if $z(t) = \delta(t) \equiv 1$, then the following oscillation criterion appears.

Corollary 8.3.7. *Assume that* (8.8) *and* (8.14) *hold. If*

$$\limsup_{t\to\infty} \int_a^t q(s)\Delta s = \infty,$$

then every solution of (8.7) *is oscillatory on* $[a, \infty)$.

If $\delta(t) = t$, then Corollary 8.3.6 yields the following result.

Corollary 8.3.8. *Assume that* (8.8) *and* (8.14) *hold. If*

$$\limsup_{t\to\infty} \int_a^t \left[K\sigma(s)q(s) - \frac{p(s)}{(\sqrt{s} + \sqrt{\sigma(s)})^2} \right] \Delta s = \infty, \tag{8.18}$$

then every solution of (8.7) *is oscillatory on* $[a, \infty)$.

Example 8.3.9. Consider the *Euler dynamic equation*

$$x^{\Delta\Delta} + \frac{\gamma}{t\sigma(t)} x^{\sigma} = 0 \quad \text{for} \quad t \in [1, \infty). \tag{8.19}$$

Here, $p(t) \equiv 1$, $K = 1$, and $q(t) = \frac{\gamma}{t\sigma(t)}$. Then (8.18) from Corollary 8.3.8 reads

$$\limsup_{t\to\infty} \int_1^t \left[\frac{\gamma}{s} - \frac{1}{(\sqrt{s} + \sqrt{\sigma(s)})^2} \right] \Delta s = \infty.$$

Note that the estimate

$$\begin{aligned} \frac{\gamma}{s} - \frac{1}{(\sqrt{s} + \sqrt{\sigma(s)})^2} &\geq \frac{\gamma}{s} - \frac{1}{(\sqrt{s} + \sqrt{s})^2} \\ &= \frac{\gamma}{s} - \frac{1}{(2\sqrt{s})^2} \\ &= \frac{\gamma - \frac{1}{4}}{s} \end{aligned}$$

implies the following result: *If* $\mathbb{T}$ *is a time scale that satisfies*

$$\int_1^{\infty} \frac{\Delta t}{t} = \infty, \tag{8.20}$$

and if $\gamma > \frac{1}{4}$, *then* (8.19) *is oscillatory.* Note that (8.20) holds for the time scales $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$ as

$$\lim_{t\to\infty} \ln t = \infty \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{1}{k} = \infty.$$

It also holds for the time scale $\mathbb{T} = q^{\mathbb{N}_0} := \{q^n : n \in \mathbb{N}_0\}$, where $q > 1$, since for this time scale

$$\int_1^\infty \frac{\Delta t}{t} = \sum_{k=0}^\infty \int_{q^k}^{q^{k+1}} \frac{\Delta t}{t} = \sum_{k=0}^\infty \frac{\mu(q^k)}{q^k} = \sum_{k=0}^\infty \frac{(q-1)q^k}{q^k} = \sum_{k=0}^\infty (q-1) = \infty.$$

In fact, in [**47**] it is shown that (8.20) holds whenever $\mathbb{T}$ is a time scale that is unbounded above. Note that our result is compatible with the well-known oscillatory behavior of (8.19) when $\mathbb{T} = \mathbb{R}$ (see [**172**]) and when $\mathbb{T} = \mathbb{Z}$ (see [**303**]). For the case $\mathbb{T} = \mathbb{Z}$, it is also known from [**303**] that for $\gamma \le 1/4$, (8.19) has a nonoscillatory solution. Hence, Theorem 8.3.5 and Corollary 8.3.8 are sharp. Finally note that the results in [**73, 86**], i.e., Corollary 8.3.7, cannot be applied to (8.19) as

$$\int_1^\infty q(t)\Delta t = \int_1^\infty \frac{\gamma \Delta t}{t\sigma(t)} = \gamma \int_1^\infty \left(-\frac{1}{t}\right)^\Delta \Delta t = \gamma \lim_{t\to\infty}\left(1 - \frac{1}{t}\right) = \gamma.$$

Example 8.3.10. Let $0 < p(t) \le 1$ for all t (e.g., $p(t) = t/(t+1)$) and consider the nonlinear dynamic equation

$$\left(p(t)x^\Delta\right)^\Delta + \frac{\gamma}{t\sigma(t)} x^\sigma(1 + (x^\sigma)^2) = 0 \quad \text{for} \quad t \ge 1. \tag{8.21}$$

Here, $K = 1$ and $q(t) = \frac{\gamma}{t\sigma(t)}$. Then (8.18) from Corollary 8.3.8 reads

$$\limsup_{t\to\infty} \int_1^t \left[\frac{\gamma}{s} - \frac{p(s)}{(\sqrt{s} + \sqrt{\sigma(s)})^2}\right] \Delta s = \infty.$$

Note that the estimate

$$\frac{\gamma}{s} - \frac{p(s)}{(\sqrt{s} + \sqrt{\sigma(s)})^2} \ge \frac{\gamma}{s} - \frac{1}{(\sqrt{s} + \sqrt{s})^2} = \frac{\gamma - \frac{1}{4}}{s}$$

implies that every solution of (8.21) is oscillatory when $\gamma > 1/4$. Note also that the results in [**73, 86**] cannot be applied to (8.21).

Theorem 8.3.11. *Assume that* (8.8) *and* (8.14) *hold. Furthermore, assume that there exists a differentiable function z and an odd $m \in \mathbb{N}$ with*

$$\limsup_{t\to\infty} \frac{1}{t^m} \int_a^t (t-s)^m \left(Kq(s)(z^\sigma(s))^2 - p(s)(z^\Delta(s))^2\right) \Delta s = \infty. \tag{8.22}$$

Then every solution of (8.7) *is oscillatory on* $[a, \infty)$.

Proof. Suppose to the contrary that x is a nonoscillatory solution of (8.7). Then (8.15) from Lemma 8.3.4 implies that there exists $t_0 \in \mathbb{T}$ such that

$$w(t) > 0 \quad \text{for all} \quad t \ge t_0,$$

where w is defined by (8.11). All assumptions from Theorem 8.3.3 are satisfied, and hence we may multiply (8.13) by $(t-s)^m$ for $t \ge s$ and integrate the resulting inequality from t_0 to $t \ge t_0$ to obtain

$$\begin{aligned}
\int_{t_0}^t (t-s)^m \left(Kq(s)(z^\sigma(s))^2 - p(s)(z^\Delta(s))^2\right) \Delta s \;&\le\; -\int_{t_0}^t (t-s)^m w^\Delta(s)\Delta s \\
= \;& -\left\{-(t-t_0)^m w(t_0) - (-1)^m \int_{t_0}^t \sum_{\nu=0}^{m-1} (\sigma(t)-s)^\nu (t-s)^{m-\nu-1} w(\sigma(s))\Delta s\right\} \\
\le \;& (t-t_0)^m w(t_0),
\end{aligned}$$

where we have used the integration by parts formula from Theorem 8.2.12 (vi), Theorem 8.2.7, and the fact that $m \in \mathbb{N}$ is odd. Therefore

$$\frac{1}{t^m}\int_{t_0}^{t}(t-s)^m\left(Kq(s)(z^\sigma(s))^2 - p(s)(z^\Delta(s))^2\right)\Delta s \le \left(1-\frac{t_0}{t}\right)^m w(t_0),$$

which is a contradiction to (8.22). The proof is complete. □

Remark 8.3.12. Note that when $z(t) \equiv 1$, then (8.22) reduces to

$$\limsup_{t\to\infty}\frac{1}{t^m}\int_{T}^{t}(t-s)^m q(s)\Delta s = \infty, \tag{8.23}$$

which can be considered as an extension of Kamenev type oscillation criteria for second order differential equations; see [140]. When $\mathbb{T} = \mathbb{R}$, then (8.23) becomes

$$\limsup_{t\to\infty}\frac{1}{t^m}\int_{T}^{t}(t-s)^m q(s)ds = \infty,$$

and when $\mathbb{T} = \mathbb{Z}$, then (8.23) becomes

$$\limsup_{t\to\infty}\frac{1}{t^m}\sum_{s=T}^{t-1}(t-s)^m q(s) = \infty.$$

8.3.2. The Case $\int_a^\infty \frac{1}{p(t)}\Delta t < \infty$.

In this subsection we consider (8.7), where p does not satisfy (8.14), i.e.,

$$\int_a^\infty \frac{1}{p(t)}\Delta t < \infty. \tag{8.24}$$

In addition to (8.8), we impose the additional assumption

$$f \quad \text{is nondecreasing.} \tag{8.25}$$

We start with the following auxiliary result.

Lemma 8.3.13. *Assume* (8.8), (8.24), (8.25), *and*

$$\int_a^\infty \frac{1}{p(t)}\int_a^t q(s)\Delta s\Delta t = \infty. \tag{8.26}$$

Suppose that x is a nonoscillatory solution of (8.7) *such that there exists $t_1 \in \mathbb{T}$ with*

$$x(t)x^\Delta(t) < 0 \quad \textit{for all} \quad t \ge t_1. \tag{8.27}$$

Then

$$\lim_{t\to\infty} x(t) \quad \textit{exists and is zero.}$$

Proof. Since x is nonoscillatory, it is either eventually positive or eventually negative. We only prove the lemma for the first case as the second case is similar and hence omitted. Assume there exists $t_1 \in \mathbb{T}$ such that

$$x(t) > 0 \quad \text{and} \quad x^\Delta(t) < 0 \quad \text{for all} \quad t \ge t_1. \tag{8.28}$$

Hence x is positive and decreasing, and therefore $\lim_{t\to\infty} x(t) =: b$ clearly exists. We have to show $b = 0$. Let us assume the opposite, i.e., $b > 0$. By (8.8), $f(b) > 0$. Hence

$$x(\sigma(t)) \ge b > 0 \quad \text{for all} \quad t \ge t_1$$

implies by (8.25)

$$f(x(\sigma(t))) \geq f(b) > 0 \quad \text{for all} \quad t \geq t_1.$$

Define $y = px^{\Delta}$ and integrate the inequality

$$y^{\Delta}(s) \leq -q(s)f(x(\sigma(s))) \leq -q(t)f(b)$$

from t_1 to $t \geq t_1$ to find

$$y(t) = y(t_1) + \int_{t_1}^{t} y^{\Delta}(s)\Delta s \leq \int_{t_1}^{t} y^{\Delta}(s)\Delta s \leq -\int_{t_1}^{t} q(s)f(b)\Delta s$$

so that

$$x^{\Delta}(t) \leq -f(b)\frac{1}{p(t)}\int_{t_1}^{t} q(s)\Delta s.$$

Now we integrate this inequality from t_1 to $T \geq t_1$ to obtain

$$\begin{aligned} x(T) &= x(t_1) + \int_{t_1}^{T} x^{\Delta}(t)\Delta t \\ &\leq -f(b)\int_{t_1}^{T} \frac{1}{p(t)}\int_{t_1}^{t} q(s)\Delta s\Delta t \\ &\overset{(8.26)}{\longrightarrow} -\infty \quad \text{as} \quad T \to \infty. \end{aligned}$$

This is contradictory to (8.28), and the proof is complete. ☐

Using Lemma 8.3.13, we can now derive the following criteria.

Theorem 8.3.14. *Assume* (8.8), (8.24), (8.25), *and* (8.26). *If there exists a differentiable function z satisfying* (8.16), *then every solution of* (8.7) *is either oscillatory or converges to zero.*

Proof. We assume that x is a nonoscillatory solution of (8.7). Hence x is either eventually positive or eventually negative, i.e., there exists $t_0 \in \mathbb{T}$ with $x(t) > 0$ for all $t \geq t_0$ or $x(t) < 0$ for all $t \geq t_0$. Let $y = px^{\Delta}$. If there exists $t_1 \geq t_0$ with $y(t_1) < 0$, then

$$y(t) \leq y(t_1) < 0 \quad \text{for all} \quad t \geq t_1$$

since y is decreasing, and hence $x^{\Delta}(t) < 0$ for all $t \geq t_1$. If, however, $y(t) > 0$ for all $t \geq t_0$, then $x^{\Delta}(t) > 0$ for all $t \geq t_0$. Altogether, either

$$x(t)x^{\Delta}(t) > 0 \quad \text{for all} \quad t \geq t_1,$$

in which case we can use Theorem 8.3.3 to derive a contradiction as in the proof of Theorem 8.3.5, or

$$x(t)x^{\Delta}(t) < 0 \quad \text{for all} \quad t \geq t_1,$$

in which case we see from Lemma 8.3.13 that $x(t)$ converges to zero as t tends to infinity. This completes the proof. ☐

Similarly we can prove the following theorem.

Theorem 8.3.15. *Assume* (8.8), (8.24), (8.25), *and* (8.26). *If there exists a differentiable function z satisfying* (8.22), *then every solution of* (8.7) *is either oscillatory or converges to zero.*

8.4. Oscillation of Perturbed Nonlinear Dynamic Equations

In this section we follow [**57**] and provide some sufficient conditions for oscillation of second order nonlinear perturbed dynamic equations of the form

$$\left(\alpha(t)\left(x^{\Delta}\right)^{\gamma}\right)^{\Delta} + F(t, x^{\sigma}) = G(t, x^{\sigma}, x^{\Delta}) \quad \text{for} \quad t \in [a, b], \tag{8.29}$$

where γ is a positive odd integer and α is a positive, real-valued rd-continuous function defined on the time scales interval $[a, b]$. Since we are interested in oscillatory behavior, we suppose that the time scale under consideration is not bounded above, i.e., it is a time scale interval of the form $[a, \infty)$. By a solution of (8.29) we mean a nontrivial real-valued function x satisfying (8.29) for $t \geq a$. A solution x of (8.29) is said to be *oscillatory* if it is neither eventually positive nor eventually negative, otherwise it is called *nonoscillatory*. Equation (8.29) is said to be *oscillatory* if all its solutions are oscillatory. Our attention is restricted to those solutions of (8.29) which exist on some half line $[t_x, \infty)$ and satisfy $\sup\{|x(t)| : t > t_0\} > 0$ for any $t_0 \geq t_x$.

Throughout this section we shall assume that

(H$_1$) $\alpha : \mathbb{T} \to \mathbb{R}$ is a positive and rd-continuous function;
(H$_2$) $\gamma \in \mathbb{N}$ is odd;
(H$_3$) $p, q : \mathbb{T} \to \mathbb{R}$ are rd-continuous functions such that $q(t) - p(t) > 0$ for all $t \in \mathbb{T}$;
(H$_4$) $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable and nondecreasing such that

$$uf(u) > 0 \quad \text{for all} \quad u \in \mathbb{R} \setminus \{0\};$$

(H$_5$) $F : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ and $G : \mathbb{T} \times \mathbb{R}^2 \to \mathbb{R}$ are functions such that

$$uF(t, u) > 0 \quad \text{and} \quad uG(t, u, v) > 0 \quad \text{for all} \quad u \in \mathbb{R} \setminus \{0\},\ v \in \mathbb{R},\ t \in \mathbb{T};$$

(H$_6$) $\frac{F(t,u)}{f(u)} \geq q(t)$ and $\frac{G(t,u,v)}{f(u)} \leq p(t)$ for all $u, v \in \mathbb{R} \setminus \{0\}$ and all $t \in \mathbb{T}$.

For simplicity, we list the conditions used in the main results as follows ($t_0 \geq a$):

$$\int_{t_0}^{\infty} \frac{\Delta t}{[\alpha(t)]^{\frac{1}{\gamma}}} = \infty, \tag{8.30}$$

$$\int_{t_0}^{\infty} \frac{\Delta t}{[\alpha(t)]^{\frac{1}{\gamma}}} < \infty, \tag{8.31}$$

$$\int_{t_0}^{\infty} [q(t) - p(t)]\Delta t = \infty, \tag{8.32}$$

$$\lim_{t\to\infty} \int_{t_0}^{t} \left\{ \frac{1}{\alpha(s)} \int_{s}^{\infty} [q(\tau) - p(\tau)]\Delta\tau \right\}^{\frac{1}{\gamma}} \Delta s = \infty, \tag{8.33}$$

$$\int_{t_0}^{\infty} [q(t) - p(t)]\Delta t > 0, \tag{8.34}$$

$$\int_{t_0}^{\infty} \left\{ \frac{M}{\alpha(s)} - \frac{1}{\alpha(s)} \int_{t_0}^{s} [q(t) - p(t)]\Delta t \right\} \Delta s = -\infty \quad \text{for all} \quad M > 0, \tag{8.35}$$

$$\int_{t_0}^{\infty} \left\{ \frac{1}{\alpha(s)} \int_{t_0}^{s} [q(t) - p(t)]\Delta t - \frac{M}{\alpha(s)} \right\}^{\frac{1}{\gamma}} \Delta s = \infty \quad \text{for all} \quad M > 0. \tag{8.36}$$

Theorem 8.4.1. *Assume* (H_1)–(H_6). *Suppose that* (8.30) *and* (8.32) *hold. Then every solution of* (8.29) *is oscillatory on* $[a, \infty)$.

Proof. Let x be a nonoscillatory solution of (8.29), say, $x(t) > 0$ for $t \geq t_0$ for some $t_0 \geq a$. We consider only this case, because the proof for the case that x is eventually negative is similar. From (8.29), (8.2), (8.3), and the chain rule (as given in [**53**, Theorem 1.87]), we have the identity (for $t \geq t_0$)

$$\left(\frac{\alpha\left(x^\Delta\right)^\gamma}{f\circ x}\right)^\Delta(t) = \frac{G\left(t, x^\sigma(t), x^\Delta(t)\right)}{f(x^\sigma(t))} - \frac{F(t, x^\sigma(t))}{f(x^\sigma(t))} - \frac{f'(x(\xi))\alpha(t)\left[x^\Delta(t)\right]^{\gamma+1}}{f(x(t))f(x(\sigma(t)))},$$

where ξ is a number in the real interval $[t, \sigma(t)]$. In view of (H_2), (H_4), (H_5), and (H_6), we have for all $t \geq t_0$

$$\left(\frac{\alpha\left(x^\Delta\right)^\gamma}{f\circ x}\right)^\Delta(t) \leq p(t) - q(t). \tag{8.37}$$

Because of (H_6) and (H_3), from (8.29) we obtain for all $t \geq t_0$

$$\left(\alpha\left(x^\Delta\right)^\gamma\right)^\Delta(t) \leq -f(x(\sigma(t)))[q(t) - p(t)] < 0, \tag{8.38}$$

which implies that $\alpha\left(x^\Delta\right)^\gamma$ is a decreasing function on $[t_0, \infty)$. We claim that $x^\Delta(t) \geq 0$ for all $t \geq t_1 \geq t_0$. If not, then there exists $t_2 \geq t_1$ such that $\alpha(t)\left[x^\Delta(t)\right]^\gamma \leq \alpha(t_2)\left[x^\Delta(t_2)\right]^\gamma = c < 0$. Hence

$$x^\Delta(t) \leq \frac{c^{\frac{1}{\gamma}}}{[\alpha(t)]^{\frac{1}{\gamma}}}. \tag{8.39}$$

Integrating (8.39) from t_2 to t provides

$$x(t) \leq x(t_2) + c^{\frac{1}{\gamma}} \int_{t_2}^t \frac{\Delta s}{[\alpha(s)]^{\frac{1}{\gamma}}} \overset{(8.30)}{\longrightarrow} -\infty \quad \text{as} \quad t \to \infty, \tag{8.40}$$

while the left-hand side of (8.40), i.e., $x(t)$, is eventually positive. This contradiction implies that $x^\Delta(t) \geq 0$ for all $t \geq t_1$. Then, integrating (8.37) from t_1 to t gives

$$\frac{\alpha(t)\left[x^\Delta(t)\right]^\gamma}{f(x(t))} \leq \frac{\alpha(t_1)\left[x^\Delta(t_1)\right]^\gamma}{f(x(t_1))} - \int_{t_1}^t [q(s) - p(s)]\Delta s \overset{(8.32)}{\longrightarrow} -\infty \tag{8.41}$$

as $t \to \infty$, while the left-hand side of (8.41) is always nonnegative, a contradiction. Therefore every solution of (8.29) oscillates. The proof is complete. □

Example 8.4.2. If $\mathbb{T} = \mathbb{R}$, then $\sigma(t) = t$ and $\mu(t) \equiv 0$. Then (8.30) and (8.32) become (the Leighton–Wintner type criteria)

$$\int_{t_0}^\infty \frac{dt}{[\alpha(t)]^{\frac{1}{\gamma}}} = \infty \quad \text{and} \quad \int_{t_0}^\infty [q(t) - p(t)]dt = \infty.$$

If $\mathbb{T} = \mathbb{Z}$, then $\sigma(t) = t + 1$ and $\mu(t) \equiv 1$. Then (8.30) and (8.32) become (the discrete analogue of Leighton–Wintner type criteria)

$$\sum_{t=t_0}^\infty \frac{1}{[\alpha(t)]^{\frac{1}{\gamma}}} = \infty \quad \text{and} \quad \sum_{t=t_0}^\infty [q(t) - p(t)] = \infty.$$

Example 8.4.3. Let $\mathbb{T} \subset [1,\infty)$ be any time scale that is unbounded above. Some of the examples included are $\mathbb{T} = [1,\infty)$, $\mathbb{T} = \mathbb{N}$, and $\mathbb{T} = \{2^k : k \in \mathbb{N}_0\}$. On $\mathbb{T}$, we consider the perturbed nonlinear dynamic equation

$$(tx^\Delta)^\Delta + x^\sigma\left(\frac{1}{t} + \frac{1}{t^2} + t^2(x^\sigma)^2\right) = \frac{(x^\sigma)^5}{2t((x^\sigma)^4+1)((x^\Delta)^2+1)}. \tag{8.42}$$

Let

$$\alpha(t) = t, \quad \gamma = 1, \quad f(u) = u, \quad p(t) = \frac{1}{2t}, \quad q(t) = \frac{1}{t},$$

and

$$F(t,u) = u\left(\frac{1}{t} + \frac{1}{t^2} + t^2u^2\right), \quad G(t,u,v) = \frac{u^5}{2t(u^4+1)(v^2+1)}.$$

Then (8.42) is in the form (8.29) and the conditions (H_1), (H_2), (H_4), and (H_5) are clearly satisfied. In [**47**, Theorem 5.11], it was shown that for an unbounded time scale $\mathbb{T} \subset [1,\infty)$ with $a \in \mathbb{T}$ we have

$$\int_a^\infty \frac{1}{t}\Delta t = \infty. \tag{8.43}$$

Hence (8.30) is satisfied, and because of $q(t) - p(t) = 1/(2t) > 0$ and (8.43), (H_3) and (8.32) are satisfied as well. Finally, (H_6) follows from

$$\frac{F(t,u)}{f(u)} = \frac{1}{t} + \frac{1}{t^2} + t^2u^2 \geq \frac{1}{t} = q(t)$$

and

$$\frac{G(t,u,v)}{f(u)} = \frac{u^4}{2t(u^4+1)(v^2+1)} \leq \frac{1}{2t}\frac{u^4}{u^4+1} \leq \frac{1}{2t} = p(t).$$

It follows from Theorem 8.4.1 that all solutions of (8.42) are oscillatory on $[1,\infty)$. Note that the same statement is also true for the equation

$$\left(t^3(x^\Delta)^3\right)^\Delta + x^\sigma\left(\frac{1}{t} + \frac{1}{t^2} + t^2(x^\sigma)^2\right) = \frac{(x^\sigma)^5}{2t((x^\sigma)^4+1)((x^\Delta)^2+1)}.$$

Theorem 8.4.4. *Assume* (H_1)–(H_6). *Suppose that* (8.30) *and* (8.33) *hold. Then any bounded solution of* (8.29) *oscillates on* $[a,\infty)$.

Proof. Suppose that x is a bounded nonoscillatory solution of (8.29), say, $x(t) > 0$ for $t \geq t_0$ for some $t_0 \geq a$. As in the proof of Theorem 8.4.1, since (8.30) holds, we have $x^\Delta(t) \geq 0$ for all $t \geq t_1 \geq t_0$ and the inequality in (8.41) holds. Since the left-hand side of (8.41) is nonnegative, we find

$$\int_{t_1}^t [q(s) - p(s)]\Delta s \leq \frac{\alpha(t_1)\left[x^\Delta(t_1)\right]^\gamma}{f(x(t_1))},$$

and therefore for $t \geq t_1$

$$\int_t^\infty [q(s) - p(s)]\Delta s \leq \frac{\alpha(t)\left[x^\Delta(t)\right]^\gamma}{f(x(t))}. \tag{8.44}$$

Integrating (8.44) from t_1 to t, we get

$$\int_{t_1}^t \left\{\frac{1}{\alpha(s)}\int_s^\infty [q(\tau) - p(\tau)]\Delta\tau\right\}^{\frac{1}{\gamma}} \Delta s \leq \int_{t_1}^t \frac{x^\Delta(s)\Delta s}{[f(x(s))]^{\frac{1}{\gamma}}}. \tag{8.45}$$

In view of (H_4) we find that $f(x(t)) \geq f(x(t_1))$ for all $t \geq t_1$. Hence, it follows from (8.45) that

$$\begin{aligned} \int_{t_1}^{t} \left\{ \frac{1}{\alpha(s)} \int_{s}^{\infty} [q(\tau) - p(\tau)]\Delta\tau \right\}^{\frac{1}{\gamma}} \Delta s &\leq \int_{t_1}^{t} \frac{x^{\Delta}(s)\Delta s}{[f(x(s))]^{\frac{1}{\gamma}}} \\ &\leq \int_{t_1}^{t} \frac{x^{\Delta}(s)\Delta s}{[f(x(t_1))]^{\frac{1}{\gamma}}} \\ &= \frac{x(t) - x(t_1)}{[f(x(t_1))]^{\frac{1}{\gamma}}}. \end{aligned}$$

By (8.33), the left-hand side of the above inequality tends to ∞ as $t \to \infty$, while the right-hand side is bounded, a contradiction. Therefore every bounded solution of (8.29) oscillates on $[a, \infty)$. $\square$

Theorem 8.4.5. *Assume* (H_1)–(H_6). *Suppose that* (8.31), (8.34), (8.35), *and* (8.36) *hold. Then every solution of* (8.29) *is oscillatory or converges to zero on* $[a, \infty)$.

Proof. Again suppose that x is a nonoscillatory solution of (8.29) that does not converge to zero, say, $x(t) > 0$ for $t \geq t_0$ for some $t_0 \geq a$. From (8.38) we have that $\alpha\left(x^{\Delta}\right)^{\gamma}$ is a decreasing function on $[t_0, \infty)$ and x^{Δ} is monotone and of one sign.

Case 1. Suppose that $x^{\Delta}(t) \geq 0$ for all $t \geq t_1 \geq t_0$. As in the proof of Theorem 8.4.1 we get the inequality in (8.41). Let

$$M = \frac{\alpha(t_1)\left[x^{\Delta}(t_1)\right]^{\gamma}}{f(x(t_1))}.$$

Then it follows from the inequality in (8.41) that for all $t \geq t_1$

$$\frac{\left[x^{\Delta}(t)\right]^{\gamma}}{f(x(t))} \leq \frac{M}{\alpha(t)} - \frac{1}{\alpha(t)} \int_{t_1}^{t} [q(s) - p(s)]\Delta s. \tag{8.46}$$

Integrating (8.46) from t_1 to t we obtain

$$\int_{t_1}^{t} \frac{\left[x^{\Delta}(s)\right]^{\gamma}}{f(x(s))} \Delta s \leq \int_{t_1}^{t} \left\{ \frac{M}{\alpha(s)} - \frac{1}{\alpha(s)} \int_{t_1}^{t} [q(\tau) - p(\tau)]\Delta\tau \right\} \Delta s. \tag{8.47}$$

By (8.35), the right-hand side of (8.47) tends to $-\infty$ as $t \to \infty$, whereas the left-hand side is nonnegative, a contradiction.

Case 2. Suppose that $x^{\Delta}(t) < 0$ for all $t \geq t_1 \geq t_0$. Hence $x(t) \to N > 0$ as $t \to \infty$, and by (H_4), $f(x(t)) \geq f(N) > 0$ for all $t \geq t_1$. From (8.46) it follows that

$$\begin{aligned} \left[x^{\Delta}(t)\right]^{\gamma} &\leq -\left\{ \frac{1}{\alpha(t)} \int_{t_1}^{t} [q(\tau) - p(\tau)]\Delta\tau - \frac{M}{\alpha(t)} \right\} f(x(t)) \\ &\leq -f(N) \left\{ \frac{1}{\alpha(t)} \int_{t_1}^{t} [q(\tau) - p(\tau)]\Delta\tau - \frac{M}{\alpha(t)} \right\}. \end{aligned}$$

Hence

$$x^{\Delta}(t) \leq -\left[f(N)\right]^{\frac{1}{\gamma}} \left\{ \frac{1}{\alpha(t)} \int_{t_1}^{t} [q(\tau) - p(\tau)]\Delta\tau - \frac{M}{\alpha(t)} \right\}^{\frac{1}{\gamma}}. \tag{8.48}$$

Integrating (8.48) from t_1 to t, we have

$$x(t) \le x(t_1) - [f(N)]^{\frac{1}{\gamma}} \int_{t_1}^{t} \left\{ \frac{1}{\alpha(s)} \int_{t_1}^{t} [q(\tau) - p(\tau)]\Delta\tau - \frac{M}{\alpha(s)} \right\}^{\frac{1}{\gamma}} \Delta s. \tag{8.49}$$

By (8.36), the right-hand side of (8.49) tends to $-\infty$ as $t \to \infty$, but this contradicts the fact that $x(t)$ is positive. This contradiction completes the proof. □

Corollary 8.4.6. *Assume* (H_1)–(H_6). *Suppose that* (8.31), (8.34), *and* (8.35) *hold. Then any bounded solution x of* (8.29) *is oscillatory or converges to zero on* $[a, \infty)$.

Proof. The condition (8.36) is used only in Case 2 of the proof of Theorem 8.4.5. Let x be a bounded nonoscillatory solution of (8.29) that does not converge to zero. In Case 2 of the proof of Theorem 8.4.5, we have $x(t) > 0$ and $x^{\Delta}(t) < 0$ for all $t \ge t_1 \ge t_0$. Hence $x(t) \to N > 0$ as $t \to \infty$, and by (H_4), $f(x(t)) \ge f(N) > 0$ for all $t \ge t_1$. From (8.46) we find

$$\begin{aligned} \int_{t_1}^{t} \left\{ \frac{M}{\alpha(s)} - \frac{1}{\alpha(s)} \int_{t_1}^{t} [q(\tau) - p(\tau)]\Delta\tau \right\} \Delta s \quad &\ge \quad \int_{t_1}^{t} \left(\frac{x^{\Delta}(s)}{[f(x(s))]^{\frac{1}{\gamma}}} \right)^{\gamma} \Delta s \\ &\ge \quad \int_{t_1}^{t} \left(\frac{x^{\Delta}(s)}{[f(N)]^{\frac{1}{\gamma}}} \right)^{\gamma} \Delta s \\ &= \quad \left(\frac{x(t) - x(t_1)}{[f(N)]^{\frac{1}{\gamma}}} \right)^{\gamma}. \end{aligned}$$

By (8.35), the left-hand side of the above inequality tends to $-\infty$ as $t \to \infty$, hence $x(t) \to -\infty$ as $t \to \infty$, but this contradicts the assumption that x is bounded. □

Theorem 8.4.7. *Assume* (H_1)–(H_6). *Suppose that* (8.31), (8.32), *and*

$$\int_{t_0}^{\infty} \left\{ \frac{1}{\alpha(t)} \int_{t_0}^{t} [q(s) - p(s)]\Delta s \right\}^{\frac{1}{\gamma}} \Delta t = \infty \tag{8.50}$$

hold. Then every solution of (8.29) *is oscillatory or converges to zero on* $[a, \infty)$.

Proof. Let x be a nonoscillatory solution of (8.29), say, $x(t) > 0$ for $t \ge t_0$ for some $t_0 \ge a$. As in the proof of Theorem 8.4.1 we see that x^{Δ} is either eventually positive or eventually negative. If x^{Δ} is eventually positive, we can derive a contradiction as in the proof of Theorem 8.4.1, since (8.32) holds. If $x^{\Delta}(t)$ is eventually negative, then $\lim_{t\to\infty} x(t) =: N$ exists. We prove that $N = 0$. If not, then $N > 0$, from which by (H_4) we have $f(x(\sigma(t))) \ge f(N) > 0$ for all $t \ge t_1$. Hence, it follows from (8.29) and (H_6) that

$$\left(\alpha \left(x^{\Delta}\right)^{\gamma}\right)^{\Delta}(t) + [q(t) - p(t)]f(N) \le 0. \tag{8.51}$$

Define the function

$$u = \alpha \left(x^{\Delta}\right)^{\gamma}.$$

Then from (8.51) for $t \ge t_1$, we obtain

$$u^{\Delta}(t) \le -[q(t) - p(t)]f(N).$$

Hence, for $t \geq t_1$, we have

$$u(t) \leq u(t_1) - f(N) \int_{t_1}^{t} [q(s) - p(s)]\Delta s < -f(N) \int_{t_1}^{t} [q(s) - p(s)]\Delta s,$$

where $u(t_1) = \alpha(t_1)\left[x^\Delta(t_1)\right]^\gamma < 0$. Integrating the last inequality from t_1 to t, we find

$$\int_{t_1}^{t} x^\Delta(s)\Delta s \leq -\left[f(N)\right]^{\frac{1}{\gamma}} \int_{t_1}^{t} \left(\frac{1}{\alpha(s)} \int_{t_1}^{s} [q(\tau) - p(\tau)]\Delta\tau\right)^{\frac{1}{\gamma}} \Delta s \overset{(8.50)}{\longrightarrow} -\infty$$

as $t \to \infty$, and so $x(t) \to -\infty$ as $t \to \infty$, a contradiction to the fact that $x(t) > 0$ for $t \geq t_0$. Thus $N = 0$ and then $x(t) \to 0$ as $t \to \infty$. □

Theorem 8.4.8. *Assume* (H_1)–(H_6). *Suppose that* (8.31), (8.35), *and* (8.50) *hold. Then every solution of* (8.29) *is oscillatory or converges to zero on* $[a, \infty)$.

Proof. Again suppose that x is a nonoscillatory solution of (8.29), say, $x(t) > 0$ for $t \geq t_0$ for some $t_0 \geq a$. Since (8.31) holds, we see from the proof of Theorem 8.4.4 that x^Δ is either eventually positive or eventually negative. If x^Δ is eventually positive, we can derive a contradiction as in Case 1 of the proof of Theorem 8.4.5, since (8.35) holds. If $x^\Delta(t)$ is eventually negative, we can prove as in Theorem 8.4.7 that $x(t)$ converges to zero, and this completes the proof. □

In the remainder of this section, by means of Riccati transformation techniques, we establish some oscillation criteria for (8.29) in terms of the coefficients. We shall now assume besides (H_1)–(H_6) that

(H_7) there exists $K > 0$ such that $f(u) \geq Ku$ for all $u \in \mathbb{R}$.

Theorem 8.4.9. *Assume* (H_1)–(H_7). *Suppose that* (8.30) *holds. Moreover assume that there exists a differentiable function* z *such that for all constants* $M > 0$,

$$\limsup_{t\to\infty} \int_a^t \left\{K[q(s) - p(s)]\left[z^\sigma(s)\right]^2 - \frac{[\alpha(s)]^{1/\gamma}}{M^{1-1/\gamma}}\left[z^\Delta(s)\right]^2\right\}\Delta s = \infty. \tag{8.52}$$

Then every solution of (8.29) *is oscillatory on* $[a, \infty)$.

Proof. Suppose that x is a solution of (8.29) with $x(t) \neq 0$ for all t and make the *Riccati substitution*

$$w = z^2 \frac{\alpha(x^\Delta)^\gamma}{x}. \tag{8.53}$$

We use the rules (8.2) and (8.3) to find

$$\begin{aligned}
-w^\Delta &= -z^\Delta z\frac{\alpha(x^\Delta)^\gamma}{x} - z^\sigma\left\{z^\sigma\left(\frac{\alpha(x^\Delta)^\gamma}{x}\right)^\Delta + z^\Delta\frac{\alpha(x^\Delta)^\gamma}{x}\right\}\\
&= -z^\Delta z\frac{\alpha(x^\Delta)^\gamma}{x} - (z^\sigma)^2\left\{\frac{[\alpha(x^\Delta)^\gamma]^\Delta}{x^\sigma} - \frac{\alpha(x^\Delta)^{\gamma+1}}{xx^\sigma}\right\} - z^\sigma z^\Delta\frac{\alpha(x^\Delta)^\gamma}{x}\\
&= (z^\sigma)^2\left\{\frac{F(t,x^\sigma)}{x^\sigma} - \frac{G(t,x^\sigma,x^\Delta)}{x^\sigma}\right\} + (z^\sigma)^2\frac{\alpha(x^\Delta)^{\gamma+1}}{xx^\sigma}\\
&\quad -z^\Delta z\frac{\alpha(x^\Delta)^\gamma}{x} - z^\sigma z^\Delta\frac{\alpha(x^\Delta)^\gamma}{x}\\
&= (z^\sigma)^2\left\{\frac{F(t,x^\sigma)}{x^\sigma} - \frac{G(t,x^\sigma,x^\Delta)}{x^\sigma}\right\}\\
&\quad +\alpha zz^\sigma(x^\Delta)^{\gamma-1}\left\{\frac{z^\sigma}{z}\frac{(x^\Delta)^2}{xx^\sigma} - \frac{z^\Delta}{z^\sigma}\frac{x^\Delta}{x} - \frac{z^\Delta}{z}\frac{x^\Delta}{x}\right\}.
\end{aligned}$$

We put

$$r = \frac{x^\Delta}{x} \quad\text{and}\quad s = \frac{z^\Delta}{z}.$$

Then

$$\begin{aligned}
\frac{z^\sigma}{z}\frac{(x^\Delta)^2}{xx^\sigma} - \frac{z^\Delta}{z^\sigma}\frac{x^\Delta}{x} - \frac{z^\Delta}{z}\frac{x^\Delta}{x} &= \frac{z+\mu z^\Delta}{z}r^{\textcircled{2}} + (\ominus s)r - sr\\
&= r^{\textcircled{2}} + \mu s r^{\textcircled{2}} - sr + (\ominus s)r\\
&= r^{\textcircled{2}} + s(\mu r^{\textcircled{2}} - r) + (\ominus s)r\\
&= r^{\textcircled{2}} + s(\ominus r) + (\ominus s)r\\
&= (r\ominus s)^{\textcircled{2}} - s^{\textcircled{2}}\\
&= (r\ominus s)^{\textcircled{2}} - \frac{(z^\Delta)^2}{zz^\sigma}.
\end{aligned}$$

Altogether we have shown now that

$$-w^\Delta = (z^\sigma)^2\left\{\frac{F(t,x^\sigma)}{x^\sigma} - \frac{G(t,x^\sigma,x^\Delta)}{x^\sigma}\right\} + \alpha zz^\sigma(x^\Delta)^{\gamma-1}(r\ominus s)^{\textcircled{2}} - \alpha(z^\Delta)^2(x^\Delta)^{\gamma-1}.$$

Hence, if $xx^\sigma > 0$, then we can estimate (apply (H_1)–(H_7))

$$-w^\Delta \ge K(z^\sigma)^2(q-p) - \alpha(z^\Delta)^2(x^\Delta)^{\gamma-1}. \tag{8.54}$$

Using these preliminaries, we now may start the actual proof of the theorem. Assume that x is a solution of (8.29) which is positive on $[t_0,\infty)$ for some $t_0 \ge a$ (a similar proof applies to the case when x is eventually negative). Define

$$y = \alpha(x^\Delta)^\gamma. \tag{8.55}$$

Then for $t \ge t_0$, $x(\sigma(t)) > 0$, $f(x^\sigma(t)) > 0$, and

$$y^\Delta(t) = G\left(t, x^\sigma(t), x^\Delta(t)\right) - F(t, x^\sigma(t)) \le f(x^\sigma(t))[p(t) - q(t)] < 0,$$

and therefore y is strictly decreasing on $[t_0,\infty)$. Assume that there exists $t_1 \ge t_0$ with $y(t_1) =: c < 0$. Then

$$\alpha(s)\left[x^\Delta(s)\right]^\gamma = y(s) \le y(t_1) = c \quad\text{for all}\quad s \ge t_1$$

and so

$$\left[x^\Delta(s)\right]^\gamma \le \frac{c}{\alpha(s)} \quad \text{for all} \quad s \ge t_1.$$

Therefore

$$x^\Delta(s) \le \frac{c^{1/\gamma}}{[\alpha(s)]^{1/\gamma}} \quad \text{for all} \quad s \ge t_1.$$

Integrating from t_1 to $t \ge t_1$ provides

$$x(t) - x(t_1) = \int_{t_1}^t x^\Delta(s)\Delta s \le c^{1/\gamma} \int_{t_1}^t \frac{\Delta s}{[\alpha(s)]^{1/\gamma}}$$

for all $t \ge t_1$ so that

$$x(t) \le x(t_1) + c^{1/\gamma} \int_{t_1}^t \frac{\Delta s}{[\alpha(s)]^{1/\gamma}} \overset{(8.30)}{\longrightarrow} -\infty,$$

contradicting the positivity of x on $[t_0, \infty)$. Therefore $y(t) > 0$ for all $t \ge t_0$ and hence $x^\Delta(t) > 0$ for all $t \ge t_0$. Now, since y is positive and decreasing on $[t_0, \infty)$, we find $0 < y(t) \le y(t_0)$ for all $t \ge t_0$. Let $M = 1/y(t_0)$. Then

$$x^\Delta(t) \le \frac{1}{[\alpha(t)M]^{1/\gamma}} \quad \text{and hence} \quad [x^\Delta(t)]^{\gamma-1} \le \frac{1}{[\alpha(t)M]^{1-1/\gamma}}$$

for all $t \ge t_0$. Using this in (8.54), we obtain

$$-w^\Delta \ge K(z^\sigma)^2(q-p) - \frac{\alpha^{1/\gamma}}{M^{1-1/\gamma}}(z^\Delta)^2. \tag{8.56}$$

Integrating (8.56) from t_0 to $t \ge t_0$ provides (note that $w(t) > 0$ for all $t \ge t_0$ by (8.53))

$$w(t_0) \ge \int_{t_0}^t \left\{ K[z^\sigma(s)]^2[q(s) - p(s)] - \frac{[\alpha(s)]^{1/\gamma}}{M^{1-1/\gamma}}[z^\Delta(s)]^2 \right\} \Delta s \overset{(8.52)}{\longrightarrow} \infty,$$

which is impossible. The proof is therefore complete. □

We remark that in case $\gamma = 1$, $M^{1-1/\gamma} = 1$ so that (8.52) is independent of the number M. Similar remarks also hold for the results that follow.

Corollary 8.4.10. *Assume* (H$_1$)–(H$_7$). *Suppose that* (8.30) *holds. Furthermore assume that there exists a positive differentiable function δ such that for all constants $M > 0$,*

(8.57)

$$\limsup_{t\to\infty} \int_a^t \left\{ K[q(s) - p(s)]\delta^\sigma(s) - \frac{[\alpha(s)]^{1/\gamma}}{M^{1-1/\gamma}} \left(\frac{\delta^\Delta(s)}{\sqrt{\delta(s)} + \sqrt{\delta^\sigma(s)}} \right)^2 \right\} \Delta s = \infty.$$

Then every solution of (8.29) *is oscillatory on* $[a, \infty)$.

Proof. Define $z = \sqrt{\delta}$ and note that

$$z^\Delta = \frac{\delta^\Delta}{\sqrt{\delta} + \sqrt{\delta^\sigma}}.$$

If (8.57) holds for δ, then (8.52) holds for $z = \sqrt{\delta}$. Thus the claim follows from Theorem 8.4.9. □

From Theorem 8.4.9 and Corollary 8.4.10, we can obtain different conditions for oscillation of all solutions of (8.29) by different choices of $\delta(t)$. For instance, let $\delta(t) \equiv 1$ or $\delta(t) = t$. By Corollary 8.4.10 we then have the following two results.

Corollary 8.4.11. *Assume* (H_1)–(H_7). *Suppose that* (8.30) *holds. If*

$$\limsup_{t\to\infty} \int_a^t [q(s) - p(s)]\Delta s = \infty, \tag{8.58}$$

then every solution of (8.29) *is oscillatory on* $[a, \infty)$.

Corollary 8.4.12. *Assume* (H_1)–(H_7). *Suppose that* (8.30) *holds. If for all constants* $M > 0$,

$$\limsup_{t\to\infty} \int_a^t \left\{ K[q(s) - p(s)]\sigma(s) - \frac{[\alpha(s)]^{1/\gamma}}{M^{1-1/\gamma}(\sqrt{s} + \sqrt{\sigma(s)})^2} \right\} \Delta s = \infty, \tag{8.59}$$

then every solution of (8.29) *is oscillatory on* $[a, \infty)$.

Example 8.4.13. Again let $\mathbb{T} \subset [1, \infty)$ be a time scale which is unbounded above. On $\mathbb{T}$ we consider the perturbed nonlinear dynamic equation

$$x^{\Delta\Delta} + x^\sigma \left(\frac{1}{t\sigma(t)} + \frac{1}{t^2} + (x^\sigma)^2 \right) = \frac{(x^\sigma)^3}{2t\sigma(t)\left((x^\sigma)^2 + (x^\Delta)^2 + 1\right)}. \tag{8.60}$$

Let

$$\alpha(t) \equiv 1, \quad \gamma = 1, \quad f(u) = u, \quad K = 1, \quad p(t) = \frac{1}{2t\sigma(t)}, \quad q(t) = \frac{1}{t\sigma(t)},$$

and

$$F(t, u) = u\left(\frac{1}{t\sigma(t)} + \frac{1}{t^2} + u^2 \right), \quad G(t, u, v) = \frac{u^3}{2t\sigma(t)(u^2 + v^2 + 1)}.$$

Then (8.60) is in the form (8.29) and the conditions (H_1), (H_2), (H_4), (H_5), (H_7), and (8.30) are clearly satisfied. Because of $q(t) - p(t) = 1/(2t\sigma(t)) > 0$, (H_3) is satisfied as well. Next, (H_6) follows from

$$\frac{F(t,u)}{f(u)} = \frac{1}{t\sigma(t)} + \frac{1}{t^2} + 2u^2 \geq \frac{1}{t\sigma(t)} = q(t)$$

and

$$\frac{G(t,u,v)}{f(u)} = \frac{u^2}{2t\sigma(t)(u^2 + v^2 + 1)} \leq \frac{1}{2t\sigma(t)} = p(t).$$

Finally, (8.59) follows from the estimate

$$\begin{aligned}
&\int_a^t \left\{ [q(s) - p(s)]\sigma(s) - \frac{1}{(\sqrt{s} + \sqrt{\sigma(s)})^2} \right\} \Delta s \\
&= \int_a^t \left\{ \frac{1}{2s} - \frac{1}{(\sqrt{s} + \sqrt{\sigma(s)})^2} \right\} \Delta s \\
&\geq \int_a^t \left\{ \frac{1}{2s} - \frac{1}{(\sqrt{s} + \sqrt{s})^2} \right\} \Delta s \\
&= \frac{1}{4} \int_a^\infty \frac{1}{s} \Delta s \\
&\overset{(8.43)}{\longrightarrow} \infty.
\end{aligned}$$

By Corollary 8.4.12, every solution of (8.60) oscillates. We remark that the same statement is also true for the equation

$$x^{\Delta\Delta} + x^\sigma \left(\frac{c}{t\sigma(t)} + \frac{1}{t^2} + (x^\sigma)^2 \right) = \frac{d(x^\sigma)^3}{t\sigma(t)\left((x^\sigma)^2 + (x^\Delta)^2 + 1\right)},$$

provided $d > 0$ and $c > d + 1/4$.

Theorem 8.4.14. *Assume* (H_1)–(H_7). *Suppose that* (8.30) *holds. Moreover assume that there exists a differentiable function z such that for all constants $M > 0$,*
(8.61)

$$\limsup_{t\to\infty} \frac{1}{t^m} \int_a^t (t-s)^m \left\{ K[q(s) - p(s)]\,[z^\sigma(s)]^2 - \frac{[\alpha(s)]^{1/\gamma}}{M^{1-1/\gamma}} \left[z^\Delta(s)\right]^2 \right\} \Delta s = \infty,$$

where $m \in \mathbb{N}$ is odd. Then every solution of (8.29) *is oscillatory on* $[a, \infty)$.

Proof. We proceed as in the proof of Theorem 8.4.9. We may assume that (8.29) has a nonoscillatory solution x such that $x(t) > 0$, $x^\Delta(t) \geq 0$, $(\alpha(x^\Delta)^\gamma)^\Delta(t) \leq 0$ for $t \geq t_0$. Define w by (8.53) as before. Then we have $w(t) > 0$ and (8.56) holds. Then from (8.56) we have, using integration by parts given in Theorem 8.2.12 (vi) and Theorem 8.2.7

$$\begin{aligned}
&\int_{t_0}^t (t-s)^m \left\{ K[z^\sigma(s)]^2[q(s) - p(s)] - \frac{[\alpha(s)]^{1/\gamma}}{M^{1-1/\gamma}}[z^\Delta(s)]^2 \right\} \Delta s \\
&\leq -\int_{t_0}^t (t-s)^m w^\Delta(s)\Delta s \\
&= (t-t_0)^m w(t_0) - (-1)^{m+1} \int_{t_0}^t \sum_{\nu=0}^{m-1} (\sigma(t) - s)^\nu\, (t-s)^{m-\nu-1} w(\sigma(s))\Delta s \\
&< (t-t_0)^m w(t_0).
\end{aligned}$$

Hence

$$\limsup_{t\to\infty} \frac{1}{t^m} \int_{t_0}^t (t-s)^m \left\{ K[z^\sigma(s)]^2[q(s) - p(s)] - \frac{[\alpha(s)]^{1/\gamma}}{M^{1-1/\gamma}}[z^\Delta(s)]^2 \right\} \Delta s \leq w(t_0),$$

which contradicts (8.61). □

Note that when $z(t) \equiv 1$, then (8.61) reduces to

$$\limsup_{t\to\infty} \frac{1}{t^m} \int_a^t (t-s)^m [q(s) - p(s)]\Delta s = \infty, \tag{8.62}$$

which can be considered as an extension of Kamenev type oscillation criteria for second order differential equations. When $\mathbb{T} = \mathbb{R}$, then (8.62) becomes

$$\limsup_{t\to\infty} \frac{1}{t^m} \int_a^t (t-s)^m [q(s) - p(s)]ds = \infty,$$

and when $\mathbb{T} = \mathbb{Z}$, then (8.62) becomes

$$\limsup_{t\to\infty} \frac{1}{t^m} \sum_{s=a}^{t-1} (t-s)^m [q(s) - p(s)] = \infty.$$

Next, we give some sufficient conditions when (8.30) does not hold, which guarantee that every solution of (8.29) oscillates or converges to zero in $[a, \infty)$.

Theorem 8.4.15. *Assume* (H_1)–(H_7). *Suppose that* (8.31) *and* (8.50) *hold. Assume there exists a differentiable function* z *such that* (8.52) *holds for all constants* $M > 0$. *Then every solution of* (8.29) *is oscillatory or converges to zero in* $[a, \infty)$.

Proof. We proceed as in Theorem 8.4.9 and assume that (8.29) has a nonoscillatory solution such that $x(t) > 0$ for $t \geq t_0 > a$. From the proof of Theorem 8.4.9 we see that there exist two possible cases of the sign of $x^\Delta(t)$. The proof when x^Δ is eventually positive is similar to the proof of Theorem 8.4.9 and hence is omitted. Now suppose that $x^\Delta(t) < 0$ for $t \geq t_1$. Then x is decreasing and $\lim_{t\to\infty} x(t) = b \geq 0$. We assert that $b = 0$. If not, then $x(\sigma(t)) > b > 0$ for $t \geq t_2 > t_1$. Then there exists $t_3 > t_2$ such that $f(x(\sigma(t))) \geq Kb$ for $t \geq t_3$. Define the function y by (8.55). Then from (8.52) for $t \geq t_3$, we obtain

$$y^\Delta(t) \leq -[q(t) - p(t)]f(x(\sigma(t))) \leq -Kb[q(t) - p(t)].$$

Hence, for $t \geq t_3$ we have

$$y(t) \leq y(t_3) - Kb \int_{t_3}^{t} [q(s) - p(s)]\Delta s < -Kb \int_{t_3}^{t} [q(s) - p(s)]\Delta s,$$

where $y(t_3) = \alpha(t_3)\left[x^\Delta(t_3)\right]^\gamma < 0$. Integrating the last inequality from t_3 to t we have

$$\int_{t_3}^{t} x^\Delta(s)\Delta s \leq -(Kb)^{1/\gamma} \int_{t_3}^{t} \left(\frac{1}{\alpha(s)} \int_{t_3}^{s} [q(\tau) - p(\tau)]\Delta\tau\right)^{1/\gamma} \Delta s.$$

By (8.50) we get

$$x(t) \to -\infty \quad \text{as} \quad t \to \infty,$$

a contradiction to the fact that $x(t) > 0$ for $t \geq t_0$. Thus $b = 0$ and $x(t) \to 0$ as $t \to \infty$. $\square$

As in the proof of Theorem 8.4.15 we can prove the following theorem.

Theorem 8.4.16. *Assume* (H_1)–(H_7). *Suppose that* (8.31) *and* (8.50) *hold. Assume there exists a differentiable function* z *such that* (8.61) *holds for all constants* $M > 0$. *Then every solution of* (8.29) *is oscillatory or converges to zero in* $[a, \infty)$.

8.5. Positive Solutions of Nonlinear Dynamic Equations

Here we give a classification scheme for the eventually positive solutions of a class of second order nonlinear dynamic equations in terms of their asymptotic magnitudes. Necessary as well as sufficient conditions for the existence of positive solutions are provided. Our presentation follows the recent paper [**197**]. We consider the second order nonlinear dynamic equation

$$y^{\Delta\Delta}(t) + r(t)f\left(y^\sigma(t)\right) = 0, \quad t \in \mathbb{T} \tag{8.63}$$

according to limiting behavior and then provide sufficient and/or necessary conditions for their existence, where $r \in \mathrm{C_{rd}}\left([t_0, \infty), [0, \infty)\right)$, $r(t) \not\equiv 0$ for $t \in \mathbb{T}$, $t_0 > 0$, and $f(y) > 0$ is nondecreasing for any $y \in \mathbb{R} \setminus \{0\}$.

We note that if $\mathbb{T} = \mathbb{R}$, then (8.63) becomes the differential equation

$$y''(t) + r(t)f(y(t)) = 0, \quad t \in \mathbb{R}. \tag{8.64}$$

The asymptotic behavior of solutions of (8.64) has been studied by several authors under different conditions, see Naito [**231**, **233**]. If $\mathbb{T} = \mathbb{Z}$, then (8.63) becomes the difference equation

$$\Delta^2 y_n + r_n f(y_{n+1}) = 0, \quad n \in \mathbb{Z}, \tag{8.65}$$

which has been discussed in detail by many authors, one can refer to [**123**, **171**, **181**, **193**, **212**].

Let y be a positive solution of (8.63). From (8.63) we have

$$y^{\Delta\Delta}(t) = -r(t) f\left(y^{\sigma}(t)\right) \le 0,$$

which implies that y^{Δ} is nonincreasing. Thus we claim that

$$y^{\Delta}(t) \ge 0 \quad \text{ for } \quad t \ge t_0.$$

If not, then there exists a sufficiently large $t_1 \ge t_0$ such that $y^{\Delta}(t) < -c$ for $t \ge t_1$, where $c > 0$ is a constant. Hence, for $t > t_1$, we obtain

$$y(t) - y(t_1) = \int_{t_1}^{t} y^{\Delta}(s)\Delta s < \int_{t_1}^{t} (-c)\Delta s = -c(t - t_1).$$

This means that $\lim_{t\to\infty} y(t) = -\infty$, which contradicts $y(t) \ge 0$.

In view of (8.63), there are positive constants α and β such that

$$\alpha \le y(t) \le \beta t \quad \text{ for } \quad t \ge t_0.$$

From above, we can see that the set of positive solutions C of (8.63) can be partitioned in the following three classes:

$$C[\max] := \left\{ y \in C : \lim_{t\to\infty} y^{\Delta}(t) = \alpha > 0 \right\},$$
$$C[\text{int}] := \left\{ y \in C : \lim_{t\to\infty} y(t) = \infty \text{ and } \lim_{t\to\infty} y^{\Delta}(t) = 0 \right\},$$

and

$$C[\min] := \left\{ y \in C : \lim_{t\to\infty} y(t) = \beta > 0 \right\}.$$

In the following, we will give several necessary and/or sufficient conditions for the existence of positive solutions of (8.63).

Theorem 8.5.1. *Equation* (8.63) *has a positive solution in the class* $C[\max]$ *if and only if*

$$\int_{t_0}^{\infty} r(s) f\left(b\sigma(s)\right) < \infty \quad \textit{for some} \quad b > 0. \tag{8.66}$$

Proof. Let $y \in C[\max]$ be a solution of (8.63). Then

$$\lim_{t\to\infty} y^{\Delta}(t) = \alpha > 0 \quad \text{ for } \quad t \ge t_0.$$

Hence there exist a sufficiently large t_1 such that

$$\frac{1}{2}\alpha < y^{\Delta}(t) < \frac{3}{2}\alpha \quad \text{ for } \quad t \ge t_1$$

so that

$$\frac{1}{2}\alpha t < y(t) < \frac{3}{2}\alpha t \quad \text{ for } \quad t > t_1.$$

Set $b = \frac{1}{2}\alpha$. Then the nondecreasing property of f implies that

$$f(y(t)) \geq f(bt) \quad \text{and} \quad f\left(y^\sigma(t)\right) \geq f\left(b\sigma(t)\right). \tag{8.67}$$

Integrating both sides of (8.63) from t_1 to t, we see

$$y^\Delta(t_1) - y^\Delta(t) = \int_{t_1}^{t} r(s) f\left(y^\sigma(s)\right) \Delta s.$$

Taking limits on both sides of the above equality, we get

$$\lim_{t\to\infty} \int_{t_1}^{t} r(s) f\left(y^\sigma(s)\right) \Delta s = y^\Delta(t_1) - \alpha,$$

which implies that

$$\int_{t_1}^{\infty} r(s) f\left(y^\sigma(s)\right) \Delta s < \infty. \tag{8.68}$$

From (8.67) and (8.68), it follows that

$$\int_{t_1}^{\infty} r(s) f\left(b\sigma(s)\right) \Delta s < \infty.$$

Conversely, assume that (8.66) holds. Then there exists a large number T such that

$$\int_{t}^{\infty} r(s) f(b\sigma(s))\Delta s < \frac{b}{2} \quad \text{for} \quad t \geq T. \tag{8.69}$$

Consider the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_0(t) = \frac{b}{2} \quad \text{for} \quad t \geq T$$

and for $n \in \mathbb{N}_0$,

$$x_{n+1}(t) = \frac{b}{2} + \frac{1}{t} \int_{T}^{t} \int_{\tau}^{\infty} r(s) f\left(\sigma(s) x_n^\sigma(s)\right) \Delta s \Delta\tau \quad \text{for} \quad t \geq T.$$

In view of (8.69), the sequence $\{x_n(t)\}_{n=0}^{\infty}$ is well defined. In fact,

$$\begin{aligned}
x_1(t) &= \frac{b}{2} + \frac{1}{t} \int_{T}^{t} \int_{\tau}^{\infty} r(s) f\left(\frac{b}{2}\sigma(s)\right) \Delta s \Delta\tau \\
&\leq \frac{b}{2} + \frac{t-T}{t} \int_{T}^{\infty} r(s) f\left(b\sigma(s)\right) \Delta s \Delta\tau \\
&\leq \frac{b}{2} + \int_{T}^{\infty} r(s) f\left(b\sigma(s)\right) \Delta s \\
&< \frac{b}{2} + \frac{b}{2} = b
\end{aligned}$$

and

$$x_1(t) \geq x_0(t) \quad \text{for} \quad t \geq T.$$

By induction and the nondecreasing property of f, we have

$$x_{n+1}(t) \geq x_n(t) \quad \text{for} \quad t \geq T, \quad n \in \mathbb{N}_0. \tag{8.70}$$

Next, we will prove that $\{x_n(t)\}_{n=0}^{\infty}$ is bounded for $t \geq T$. First,

$$x_0(t) = \frac{b}{2} < b \quad \text{and} \quad x_1(t) < b.$$

If we assume $x_n(t) < b$ for $t \geq T$, then $\sigma(s)x_n^\sigma(s) < b\sigma(s)$, and

$$\begin{aligned} x_{n+1}(t) &= \frac{b}{2} + \frac{1}{t}\int_T^t \int_\tau^\infty r(s) f\left(\sigma(s)x_n^\sigma(s)\right)\Delta s\Delta\tau \\ &\leq \frac{b}{2} + \frac{1}{t}\int_T^t \int_\tau^\infty r(s) f\left(b\sigma(s)\right)\Delta s\Delta\tau \\ &\leq \frac{b}{2} + \int_T^\infty r(s) f\left(b\sigma(s)\right)\Delta s \; < \; b \end{aligned}$$

for $t \geq T$, which, by induction implies that $\{x_n(t)\}_{n=0}^\infty$ is bounded for $t \geq T$. In view of (8.70), we know that $\{x_n(t)\}_{n=0}^\infty$ is pointwise convergent to some function $x^*(t)$. By means of Lebesgue's dominated convergence theorem, we obtain

$$x^*(t) = \frac{b}{2} + \frac{1}{t}\int_T^t \int_\tau^\infty r(s) f\left(\sigma(s)x^\sigma(s)\right)\Delta s\Delta\tau \quad \text{for} \quad t \geq T$$

and

$$\frac{b}{2} \leq x^*(t) < b.$$

Setting $y(t) = tx^*(t)$, we find

$$y(t) = \frac{b}{2}t + \int_T^t \int_\tau^\infty r(s) f\left(y^\sigma(s)\right)\Delta s\Delta\tau \quad \text{for} \quad t \geq T.$$

Obviously, $y \in C[\max]$ is a solution of (8.63). $\square$

Theorem 8.5.2. *Equation* (8.63) *has a positive solution in the class* $C[\min]$ *if and only if*

$$\int_{t_0}^\infty \int_\tau^\infty r(s) f(d)\Delta s\Delta\tau < \infty \quad \textit{for some} \quad d > 0. \tag{8.71}$$

Proof. Let $y \in C[\min]$ be a solution of (8.63). Then

$$\lim_{t\to\infty} y(t) = \beta > 0 \quad \text{and} \quad \lim_{t\to\infty} y^\Delta(t) = 0 \quad \text{for} \quad t \geq t_0.$$

Hence there exists a sufficiently large t_1 such that

$$\frac{1}{2}\beta < y(t) < \frac{3}{2}\beta \quad \text{for} \quad t \geq t_1.$$

Set $d = \frac{1}{2}\beta$. Then the nondecreasing property implies

$$f(y(t)) > f(d) \quad \text{and} \quad f\left(y^\sigma(t)\right) > f(d) \quad \text{for} \quad t > t_1.$$

By integrating both sides of (8.63) from t to ∞ for $t > t_1$, we obtain

$$\beta - y(t_1) = \int_{t_1}^\infty \int_\tau^\infty r(s) f\left(y^\sigma(s)\right)\Delta s\Delta\tau,$$

which implies

$$\int_{t_1}^\infty \int_\tau^\infty r(s) f(d)\Delta s\Delta\tau < \infty,$$

i.e., (8.71) holds.

The rest of the proof of Theorem 8.5.2 is similar to that of Theorem 8.5.1, and therefore we omit it here. The proof is complete. $\square$

Theorem 8.5.3. *If* (8.63) *has a positive solution in* $C[\mathrm{int}]$, *then*

$$\int_{t_0}^{\infty} r(s)f(a)\Delta s < \infty \quad \textit{for some} \quad a > 0 \tag{8.72}$$

and

$$\int_{t_0}^{\infty}\int_{\tau}^{\infty} r(s)f(b\sigma(s))\Delta s\Delta\tau = \infty \quad \textit{for every} \quad b > 0.$$

Proof. Let $y \in C[\mathrm{int}]$ be a solution of (8.63). Then $\lim_{t\to\infty} y(t) = \infty$ and $\lim_{t\to\infty} y^{\Delta}(t) = 0$. Hence there exist two positive constants a and b and a sufficiently large $t_1 > t_0$ such that $a < y(t) < bt$ for $t > t_1$, which, in view of the nondecreasing property of f, implies that

$$f(y(t)) \geq f(a) \quad \text{and} \quad f\left(y^{\sigma}(t)\right) \leq f(a)$$

and

$$f(y(t)) \leq f(bt) \quad \text{and} \quad f\left(y^{\sigma}(t)\right) \leq f(b\sigma(t)) \quad \text{for} \quad t > t_1. \tag{8.73}$$

From equation (8.63) we have

$$y^{\Delta}(t) + \int_{t_1}^{t} r(s)f\left(y^{\sigma}(s)\right)\Delta s = y^{\Delta}(t_1) \quad \text{for} \quad t > t_1. \tag{8.74}$$

In view of $\lim_{t\to\infty} y^{\Delta}(t) = 0$, (8.74) yields

$$\int_{t_1}^{\infty} r(s)f\left(y^{\sigma}(s)\right)\Delta s = y^{\Delta}(t_1),$$

and so

$$\int_{t_1}^{\infty} r(s)f(a)\Delta s < \infty,$$

which implies that (8.72) holds.

Further, in view of $\lim_{t\to\infty} y^{\Delta}(t) = 0$, we obtain

$$\int_{s}^{\infty} r(s)f\left(y^{\sigma}(s)\right)\Delta s = y^{\Delta}(s) \quad \text{for} \quad s > t_1. \tag{8.75}$$

Integrating both sides of (8.75) from t_1 to t, we obtain

$$y(t) - y(t_1) = \int_{t_1}^{t}\int_{\tau}^{\infty} r(s)f\left(y^{\sigma}(s)\right)\Delta s\Delta\tau \leq \int_{t_1}^{t}\int_{\tau}^{\infty} r(s)f\left(b\sigma(s)\right)\Delta s\Delta\tau$$

for $t > t_1$. Hence, (8.73) and $\lim_{t\to\infty} y(t) = \infty$ imply

$$\int_{t_1}^{\infty}\int_{\tau}^{\infty} r(s)f\left(b\sigma(s)\right)\Delta s\Delta\tau = \infty.$$

The proof is complete. □

Theorem 8.5.4. *Equation* (8.63) *has a positive solution in* $C[\mathrm{int}]$ *provided that*

$$\int_{t_0}^{\infty} r(s)f\left(a\sigma(s)\right)\Delta s < \infty \quad \textit{for some} \quad a > 0 \tag{8.76}$$

and

$$\int_{t_0}^{\infty}\int_{\tau}^{\infty} r(s)f(b)\Delta s\Delta\tau = \infty \quad \textit{for every} \quad b > 0. \tag{8.77}$$

Proof. In view of (8.76) and (8.77), there exist two positive constants a and b and a sufficiently large t_1 such that

$$\frac{b}{t} < a \quad \text{and} \quad \frac{b}{t} + \frac{1}{t}\int_{t_1}^{t}\int_{\tau}^{\infty} r(s) f\left(a\sigma(s)\right)\Delta s \Delta\tau < a \quad \text{for} \quad t \geq t_1.$$

Consider the sequence $\{x_n(t)\}_{n=0}^{\infty}$ defined by

$$x_0(t) = 0$$

and for $t \geq t_1$, $n \in \mathbb{N}_0$,

$$x_{n+1}(t) = Px_n(t) = \frac{b}{t} + \frac{1}{t}\int_{t_1}^{t}\int_{\tau}^{\infty} r(s) f\left(\sigma(s) x_n^{\sigma}(s)\right)\Delta s \Delta\tau.$$

It is easy to see that $\{x_n(t)\}_{n=0}^{\infty}$ is well defined. In fact,

$$x_1(t) = \frac{b}{t} < a \quad \text{and} \quad x_1^{\sigma}(t) < a \quad \text{for} \quad t \geq t_1$$

and

$$\begin{aligned} x_2(t) &= \frac{b}{t} + \frac{1}{t}\int_{t_1}^{t}\int_{\tau}^{\infty} r(s) f\left(\sigma(s) x_1^{\sigma}(s)\right)\Delta s \Delta\tau \\ &\leq \frac{b}{t} + \frac{1}{t}\int_{t_1}^{t}\int_{\tau}^{\infty} r(s) f\left(a\sigma(s)\right)\Delta s \Delta\tau \ < \ a \end{aligned}$$

for $t \geq t_1$. Also, if we assume that $x_n(t) < a$ for $t \geq t_1$, then $x_n^{\sigma}(t) < a$ and

$$\begin{aligned} x_{n+1}(t) &= \frac{b}{t} + \frac{1}{t}\int_{t_1}^{t}\int_{\tau}^{\infty} r(s) f\left(\sigma(s) x_n^{\sigma}(s)\right)\Delta s \Delta\tau \\ &\leq \frac{b}{t} + \frac{1}{t}\int_{t_1}^{t}\int_{\tau}^{\infty} r(s) f\left(a\sigma(s)\right)\Delta s \Delta\tau \ < \ a \end{aligned}$$

for $t \geq t_1$, which, by induction, shows that $\{x_n(t)\}_{n=0}^{\infty}$ is bounded, i.e.,

$$0 \leq x_n(t) < a \quad \text{for} \quad t \geq t_1, \quad n \in \mathbb{N}_0.$$

In view of $x_0(t) \leq x_1(t)$ and the nondecreasing property of f, we have

$$x_{n+1}(t) \geq x_n(t) \quad \text{for} \quad t \geq t_1, \quad n \in \mathbb{N}_0.$$

Hence, Lebesgue's dominated convergence theorem implies that

$$x^*(t) = \frac{a}{t} + \frac{1}{t}\int_{t_1}^{t}\int_{\tau}^{\infty} r(s) f\left(\sigma(s) x^*(\sigma(s))\right)\Delta s \Delta\tau$$

for $t \geq t_1$. Set $y(t) = tx^*(t)$. Then

$$y(t) = a + \int_{t_1}^{t}\int_{\tau}^{\infty} r(s) f\left(y^{\sigma}(s)\right)\Delta s \Delta\tau \quad \text{for} \quad t \geq t_1.$$

It is easily verified that $y \in C[\mathrm{int}]$ is a solution of (8.63). $\square$

8.6. Oscillation of Emden–Fowler Equations

In this section we explore the solution properties of

$$u^{\Delta^2}(t) + p(t)[u(\sigma(t))]^\gamma = 0 \tag{8.78}$$

on a time scale $\mathbb{T}$ (unbounded above) which contains only isolated points (a so-called *discrete time scale*), with the eventual goal of showing that if $\int_a^\infty \sigma(t)p(t)\Delta t = \infty$, then equation (8.78) is oscillatory. The function p is defined on $\mathbb{T}$ and γ is a quotient of odd positive integers. Some of the proof techniques in this section are similar to those in the book by Agarwal [**2**] on difference equations. The results presented in this section are adopted from Akın-Bohner and Hoffacker [**21, 22**].

By a solution u of the given dynamic equation we shall mean a nontrivial solution which exists on $[a, \infty)$ for some $a \in \mathbb{T}$. We now define oscillation and nonoscillation in this setting.

Definition 8.6.1. A solution u is called *oscillatory* if for any $t_1 \in [a, \infty)$, there exists $t_2 \in [t_1, \infty)$ such that $u(t_2)u(\sigma(t_2)) \le 0$.

The dynamic equation (8.78) itself is called *oscillatory* if all its solutions are oscillatory. If the solution u is not oscillatory, then it is said to be *nonoscillatory.* Equivalently the following definition can be made.

Definition 8.6.2. The solution u is *nonoscillatory* if it is eventually positive or negative, i.e., there exists $t_1 \in [a, \infty)$ such that $u(t)u(\sigma(t)) > 0$ for all $t \in [t_1, \infty)$.

The dynamic equation (8.78) itself is called *nonoscillatory* if all of its solutions are nonoscillatory.

Example 8.6.3. A given dynamic equation can have both oscillatory and nonoscillatory solutions. Take

$$u^{\Delta^2}(t) + \frac{8}{3}u^\Delta(t) + \frac{4}{3}u(t) = 0,$$

where $t \in \mathbb{T} = \mathbb{Z}$. Solutions to this difference equation are easily found (see [**144**]). Two solutions are

$$u_1(t) = (-1)^t \quad \text{and} \quad u_2(t) = \left(\frac{1}{3}\right)^t.$$

Clearly u_1 is oscillatory and u_2 is nonoscillatory.

Example 8.6.4. Let $\mathbb{T}$ be a time scale such that $\mu(t) \ge 1$ for all $t \in \mathbb{T}$. The dynamic equation

$$u^{\Delta^2}(t) + \frac{8}{3}u^\Delta(t) + \frac{4}{3}u(t) = 0$$

is regressive. Then for $t_0 \in \mathbb{T}$,

$$e_{\frac{1}{3}}(t, t_0) \quad \text{and} \quad e_{-1}(t, t_0)$$

are two solutions of the above dynamic equation. However

$$e_{\frac{1}{3}}(t, t_0)e_{\frac{1}{3}}(\sigma(t), t_0) = \left(1 + \frac{1}{3}\mu(t)\right)[e_{\frac{1}{3}}(t, t_0)]^2 > 0$$

and

$$e_{-1}(t, t_0)e_{-1}(\sigma(t), t_0) = (1 - \mu(t))[e_{-1}(t, t_0)]^2 \le 0.$$

Thus $e_{\frac{1}{3}}(t,t_0)$ and $e_{-1}(t,t_0)$ are nonoscillatory and oscillatory solutions of the above dynamic equation, respectively.

The following are some basic properties of solutions of equation (8.78).

Lemma 8.6.5. *If u is a nontrivial solution of equation* (8.78) *with*

$$u(a)u(\sigma(a)) \le 0 \quad \textit{for some} \quad a \in \mathbb{T},$$

then either

$$u(a) \neq 0 \quad \textit{or} \quad u(\sigma(a)) \neq 0.$$

Proof. Let $t = \rho(a)$ for $a \in \mathbb{T}$ and suppose $u(a) = 0$. We desire to show that $u(\sigma(a)) \neq 0$. By equation (8.78) we have $u^{\Delta^2}(\rho(a)) = 0$, or expanding

$$\frac{u^\Delta(a) - u^\Delta(\rho(a))}{\mu(\rho(a))} = 0,$$

which implies that

$$\frac{u(\sigma(a)) - u(a)}{\mu(a)\mu(\rho(a))} - \frac{u(a) - u(\rho(a))}{\mu^2(\rho(a))} = 0.$$

However if both $u(a) = 0$ and $u(\sigma(a)) = 0$, then it must be the case that $u(\rho(a)) = 0$. This process can be continued for $t = \rho^2(a)$, etc., implying that the solution u is actually trivial. But this contradicts the assumption that our solution is nontrivial. Similarly, if we assume $u(\sigma(a)) = 0$, then it must be the case that $u(a) \neq 0$. Thus either $u(a) \neq 0$ or $u(\sigma(a)) \neq 0$. $\square$

Remark 8.6.6. If in addition $u(a) = 0$, then

$$\mu(\rho(a))u(\sigma(a)) = -\mu(a)u(\rho(a)).$$

Thus an oscillatory solution of equation (8.78) must change sign infinitely many times.

Lemma 8.6.7. *Assume $p(t) \le 0$ for all $t \in \mathbb{T}$, and for every $a \in \mathbb{T}$, $p(t) < 0$ for some $t \in [\sigma(a), \infty)$. If u is a solution of equation* (8.78) *with*

$$u(\rho(a)) \le u(a) \tag{8.79}$$

and

$$u(a) \ge 0 \tag{8.80}$$

for some $a \in \mathbb{T}$, then u and u^Δ are nondecreasing and nonnegative on $[a, \infty)$.

Proof. We will show the desired result by mathematical induction on t. Let $t = \rho(a)$ for $a \in \mathbb{T}$ in equation (8.78). Then by our assumption on p, (8.79), and (8.80),

$$u^{\Delta^2}(\rho(a)) = -p(\rho(a))[u(a)]^\gamma \ge 0 \tag{8.81}$$

and

$$u^\Delta(\rho(a)) = \frac{u(a) - u(\rho(a))}{\mu(\rho(a))} \ge 0.$$

It follows from (8.81) that

$$u^{\Delta^2}(\rho(a)) = \frac{u^\Delta(a) - u^\Delta(\rho(a))}{\mu(\rho(a))} \ge 0.$$

Therefore $u^\Delta(a) \geq u^\Delta(\rho(a)) \geq 0$. Suppose the desired result is true for $t = \sigma^{n-1}(a)$ for some $n \in \mathbb{N} \setminus \{1\}$, i.e.,

$$u^\Delta(\sigma^n(a)) \geq u^\Delta(\sigma^{n-1}(a)) \geq 0 \tag{8.82}$$

and

$$u(\sigma^n(a)) \geq u(\sigma^{n-1}(a)) \geq 0. \tag{8.83}$$

We wish to show that the desired result is true for $t = \sigma(\sigma^{n-1}(a)) = \sigma^n(a)$ for some $n \in \mathbb{N} \setminus \{1\}$. By (8.82),

$$0 \leq u^\Delta(\sigma^n(a)) = \frac{u(\sigma^{n+1}(a)) - u(\sigma^n(a))}{\mu(\sigma^n(a))}.$$

Because of this and by (8.83),

$$u(\sigma^{n+1}(a)) \geq u(\sigma^n(a)) \geq 0.$$

Therefore

$$u^{\Delta^2}(\sigma^n(a)) = -p(\sigma^n(a))[u(\sigma^{n+1}(a))]^\gamma \geq 0.$$

Using

$$u^{\Delta^2}(\sigma^n(a)) = \frac{u^\Delta(\sigma^{n+1}(a)) - u^\Delta(\sigma^n(a))}{\mu(\sigma^n(a))} \geq 0$$

and (8.82),

$$u^\Delta(\sigma^{n+1}(a)) \geq u^\Delta(\sigma^n(a)) \geq 0.$$

Hence by induction the result holds. □

Remark 8.6.8. Similarly, if p is as in Lemma 8.6.7, $u(\rho(a)) \geq u(a)$, and $u(a) \leq 0$ for some $a \in \mathbb{T}$, then u and u^Δ are nonincreasing and nonpositive on $[a, \infty)$.

The next result follows immediately from Lemma 8.6.7.

Lemma 8.6.9. *If p is as in Lemma 8.6.7, then all nontrivial solutions of equation* (8.78) *are nonoscillatory and eventually monotone.*

Lemma 8.6.10. *Assume that $p(t) \geq 0$ for all $t \in \mathbb{T}$, and for every $a \in \mathbb{T}$, $p(t) > 0$ for some $t \in [\sigma(a), \infty)$. If u is a nonoscillatory solution of equation* (8.78) *such that $u(t) > 0$ for all $t \in [a, \infty)$, then*

$$u(\sigma(t)) > u(t) \quad \text{for all} \quad t \geq a \tag{8.84}$$

and

$$0 < u^\Delta(\sigma(t)) \leq u^\Delta(t) \quad \text{for all} \quad t \geq a. \tag{8.85}$$

Proof. If u is a nonoscillatory solution of equation (8.78), then since $u(t) > 0$ for $t \in [a, \infty)$, we have $u(t)u(\sigma(t)) > 0$, which implies that $u(\sigma(t)) > 0$ on $[a, \infty)$ as well. Thus on $[a, \infty)$,

$$u^{\Delta^2}(t) = -p(t)[u(\sigma(t))]^\gamma \leq 0$$

Using

$$u^{\Delta^2}(t) = \frac{u^\Delta(\sigma(t)) - u^\Delta(t)}{\mu(t)},$$

we have

$$\frac{u^\Delta(\sigma(t)) - u^\Delta(t)}{\mu(t)} \leq 0,$$

and so for $t \in [a, \infty)$

$$u^\Delta(\sigma(t)) \leq u^\Delta(t). \tag{8.86}$$

It remains to show that (8.84) holds which will imply that $0 < u^\Delta(\sigma(t))$. Assume not. Then we have $u(\sigma(b)) \leq u(b)$ for some $b \in [\sigma(a), \infty)$. By (8.86) we have

$$0 \geq u^\Delta(b) \geq u^\Delta(\sigma(b)) \geq \cdots \geq u^\Delta(\sigma^n(b)) \geq \ldots. \tag{8.87}$$

However there exists a sequence $\{t_n\}_{n\in\mathbb{N}} \in \mathbb{T}$ such that $t_n \to \infty$ and $p(t_n) < 0$. Thus

$$u^{\Delta^2}(t_n) = -p(t_n)[u(\sigma(t_n))]^\gamma < 0.$$

But

$$u^{\Delta^2}(t_n) = \frac{u^\Delta(\sigma(t_n)) - u^\Delta(t_n)}{\mu(t_n)} < 0,$$

so infinitely many of the inequalities in (8.87) must be strict, contradicting the fact that $u(t) > 0$ for all $t \in [a, \infty)$. □

Remark 8.6.11. If instead u is a nonoscillatory solution of equation (8.78) such that $u(t) < 0$ for all $t \in [a, \infty)$, then

$$u(\sigma(t)) < u(t) \quad \text{and} \quad 0 > u^\Delta(\sigma(t)) \geq u^\Delta(t) \quad \text{for all} \quad t \geq a.$$

Remark 8.6.12. For $a, t \in \mathbb{T}$ with $t > a$ we can write $t = \sigma^n(a)$ for some $n \in \mathbb{N}$. Thus we can write

$$t - \sigma(a) = \sigma^n(a) - \sigma(a) = \sum_{i=1}^{n-1} \mu(\sigma^i(a)).$$

If instead $t < a$, then we can write $t = \rho^n(a)$ for some $n \in \mathbb{N}$, so

$$\sigma(a) - t = \sigma(a) - \rho^n(a) = \sum_{i=0}^{n} \mu(\rho^i(a)).$$

Theorem 8.6.13. *Assume $p(t) \leq 0$ for all $t \in \mathbb{T}$, and for every $a \in \mathbb{T}$, $p(t) < 0$ for some $t \in [\sigma(a), \infty)$ and for some $t \in (-\infty, \rho(a)]$. Let u and v be solutions of equation* (8.78) *satisfying*

$$u(b) \leq v(b) \quad \textit{and} \quad u(\sigma(b)) > v(\sigma(b)) \tag{8.88}$$

for some $b \in \mathbb{T}$. Then for $t \in [\sigma(b), \infty)$,

$$u(t) - v(t) \geq \frac{t - b}{\mu(b)} \Big(u(\sigma(b)) - v(\sigma(b)) \Big), \tag{8.89}$$

and for $t \in (-\infty, b]$,

$$u(t) - v(t) \leq \frac{\sigma(b) - t}{\mu(b)} \Big(u(b) - v(b) \Big). \tag{8.90}$$

In addition $u(t) > v(t)$ for all $t \in [\sigma(b), \infty)$, $u(t) < v(t)$ for all $t \in (-\infty, \rho(b)]$, and $u(t) - v(t)$ is nondecreasing for all $t \in \mathbb{T}$.

Proof. Fix $r \in \mathbb{T}$ with $r > b$ and let $w(\sigma^n(r)) = u(\sigma^n(b)) - v(\sigma^n(b))$ for $n \in \mathbb{N}_0$. From (8.88) it is clear that

$$w(r) = u(b) - v(b) \leq 0 \quad \text{and} \quad w(\sigma(r)) = u(\sigma(b)) - v(\sigma(b)) > 0.$$

By induction we shall show that

$$w(\sigma^n(r)) \geq \frac{\sum\limits_{i=0}^{n-1} \mu(\sigma^i(b))}{\sum\limits_{i=0}^{n-2} \mu(\sigma^i(b))} w(\sigma^{n-1}(r)) > 0, \tag{8.91}$$

where $n \in \mathbb{N} \setminus \{1\}$. From equation (8.78) we have

$$u^{\Delta^2}(b) = -p(b)[u(\sigma(b))]^\gamma \geq -p(b)[v(\sigma(b))]^\gamma = v^{\Delta^2}(b),$$

and so it follows that for $n = 2$, $t = \sigma^2(b)$,

$$\begin{aligned}
w(\sigma^2(r)) &= u(\sigma^2(b)) - v(\sigma^2(b)) \\
&= u(\sigma(b)) + \mu(\sigma(b))u^\Delta(\sigma(b)) - v(\sigma(b)) - \mu(\sigma(b))v^\Delta(\sigma(b)) \\
&= w(\sigma(r)) + \mu(\sigma(b)) \left(u^\Delta(\sigma(b)) - v^\Delta(\sigma(b))\right) \\
&= w(\sigma(r)) + \mu(\sigma(b)) \left(u^\Delta(b) + \mu(b)u^{\Delta^2}(b) - v^\Delta(b) - \mu(b)v^{\Delta^2}(b)\right) \\
&\geq w(\sigma(r)) + \mu(\sigma(b)) \left[\frac{w(\sigma(r)) - w(r)}{\mu(b)}\right] \\
&\geq w(\sigma(r)) + w(\sigma(r))\frac{\mu(\sigma(b))}{\mu(b)} \\
&= \frac{\mu(b) + \mu(\sigma(b))}{\mu(b)} w(\sigma(r)) \ > \ 0.
\end{aligned}$$

Hence (8.91) is true for $n = 2$. Now suppose that (8.91) is true for some $n \geq 2$. We wish to show that (8.91) holds for $n + 1$. As before we have that

$$u^{\Delta^2}(\sigma^{n-1}(b)) \geq v^{\Delta^2}(\sigma^{n-1}(b)).$$

Hence

$$\begin{aligned}
&w(\sigma^{n+1}(r)) = u(\sigma^{n+1}(b)) - v(\sigma^{n+1}(b)) \\
&= u(\sigma^n(b)) + \mu(\sigma^n(b))u^\Delta(\sigma^n(b)) - v(\sigma^n(b)) - \mu(\sigma^n(b))v^\Delta(\sigma^n(b)) \\
&= w(\sigma^n(r)) + \mu(\sigma^n(b)) \left(u^\Delta(\sigma^n(b)) - v^\Delta(\sigma^n(b))\right) \\
&= w(\sigma^n(r)) + \mu(\sigma^n(b)) \Big(u^\Delta(\sigma^{n-1}(b)) + \mu(\sigma^{n-1}(b))u^{\Delta^2}(\sigma^{n-1}(b)) \\
&\qquad -v^\Delta(\sigma^{n-1}(b)) - \mu(\sigma^{n-1}(b))v^{\Delta^2}(\sigma^{n-1}(b))\Big) \\
&\geq w(\sigma^n(r)) + \mu(\sigma^n(b)) \left[\frac{w(\sigma^n(r)) - w(\sigma^{n-1}(r))}{\mu(\sigma^{n-1}(b))}\right] \\
&\geq w(\sigma^n(r)) + w(\sigma^n(r))\frac{\mu(\sigma^n(b))}{\mu(\sigma^{n-1}(b))} - \frac{\mu(\sigma^n(b)}{\mu(\sigma^{n-1}(b))} \frac{\sum\limits_{i=0}^{n-2} \mu(\sigma^i(b))}{\sum\limits_{i=0}^{n-1} \mu(\sigma^i(b))} w(\sigma^n(r)) \\
&= \left[1 + \frac{\mu(\sigma^n(b)}{\mu(\sigma^{n-1}(b))} - \frac{\mu(\sigma^n(b)}{\mu(\sigma^{n-1}(b))} \frac{\sum\limits_{i=0}^{n-2} \mu(\sigma^i(b))}{\sum\limits_{i=0}^{n-1} \mu(\sigma^i(b))}\right] w(\sigma^n(r))
\end{aligned}$$

$$= \frac{\sum_{i=0}^{n} \mu(\sigma^i(b))}{\sum_{i=0}^{n-1} \mu(\sigma^i(b))} w(\sigma^n(r)) \ > \ 0.$$

Thus (8.91) holds for $n+1$ as well. From (8.91) and (8.88), it is clear that $u(t) > v(t)$ for all $t \in [\sigma(b), \infty)$. Further we have

$$\begin{aligned}
w(\sigma^n(r)) &\geq \frac{\sum_{i=0}^{n-1} \mu(\sigma^i(b))}{\sum_{i=0}^{n-2} \mu(\sigma^i(b))} w(\sigma^{n-1}(r)) \\
&\geq \frac{\sum_{i=0}^{n-1} \mu(\sigma^i(b))}{\sum_{i=0}^{n-2} \mu(\sigma^i(b))} \frac{\sum_{i=0}^{n-2} \mu(\sigma^i(b))}{\sum_{i=0}^{n-3} \mu(\sigma^i(b))} w(\sigma^{n-2}(r)) \\
&= \frac{\sum_{i=0}^{n-1} \mu(\sigma^i(b))}{\sum_{i=0}^{n-3} \mu(\sigma^i(b))} w(\sigma^{n-2}(r)) \\
&\geq \cdots \\
&\geq \frac{\sum_{i=0}^{n-1} \mu(\sigma^i(b))}{\mu(b)} w(\sigma(r)) \\
&= \frac{\sigma^n(b) - b}{\mu(b)} w(\sigma(r)),
\end{aligned}$$

which is the same as (8.89) for $t = \sigma^n(b)$.

For the last part of the theorem, we let $w(\rho^n(r)) = u(\rho^n(b)) - v(\rho^n(b))$ for $n \in \mathbb{N}_0$. By equation (8.78) we have

$$u^{\Delta^2}(\rho(b)) = -p(\rho(b))[u(b)]^\gamma \leq -p(\rho(b))[v(b)]^\gamma = v^{\Delta^2}(\rho(b)).$$

In addition $w(r) = u(b) - v(b) \leq 0$ and $w(\sigma(r)) = u(\sigma(b)) - v(\sigma(b)) > 0$. For $t = \rho(r)$ we have

$$\begin{aligned}
w(\rho(r)) &= u(\rho(b)) - v(\rho(b)) \\
&= u(b) - \mu(\rho(b))u^\Delta(\rho(b)) - v(b) + \mu(\rho(b))v^\Delta(\rho(b)) \\
&= w(r) - \mu(\rho(b)) \left(u^\Delta(b) - \mu(\rho(b))u^{\Delta^2}(\rho(b))\right) \\
&\quad +\mu(\rho(b)) \left(v^\Delta(b) - \mu(\rho(b))v^{\Delta^2}(\rho(b))\right) \\
&= w(r) - \mu(\rho(b)) \left(u^\Delta(b) - v^\Delta(b)\right) + \mu(\rho(b))\mu(\rho(b)) \left(u^{\Delta^2}(\rho(b)) - v^{\Delta^2}(\rho(b))\right) \\
&\leq w(r) - \mu(\rho(b)) \left(u^\Delta(b) - v^\Delta(b)\right) \\
&= w(r) - \frac{\mu(\rho(b))}{\mu(b)} w(\sigma(r)) + \frac{\mu(\rho(b))}{\mu(b)} w(r)
\end{aligned}$$

$$\begin{aligned}
&= \frac{\mu(\rho(b))+\mu(b)}{\mu(b)}w(r) - \frac{\mu(\rho(b))}{\mu(b)}w(\sigma(r)) \\
&< \frac{\mu(\rho(b))+\mu(b)}{\mu(b)}w(r) \;\le\; 0,
\end{aligned}$$

so $w(\rho(r)) < 0$ holds as well. We shall show that

$$w(\rho^n(r)) < \frac{\sum\limits_{i=0}^{n}\mu(\rho^i(b))}{\sum\limits_{i=0}^{n-1}\mu(\rho^i(b))}w(\rho^{n-1}(r)) < 0, \tag{8.92}$$

where $n \ge 2$. Using the same relationships as in the previous part of the proof we have

$$\begin{aligned}
w(\rho^2(r)) &= u(\rho^2(b)) - v(\rho^2(b)) \\
&\le w(\rho(r)) - \mu(\rho^2(b))\left(u^\Delta(\rho(b)) - v^\Delta(\rho(b))\right) \\
&= w(\rho(r)) - \frac{\mu(\rho^2(b))}{\mu(\rho(b))}w(r) + \frac{\mu(\rho^2(b))}{\mu(\rho(b))}w(\rho(r)) \\
&= \frac{\mu(\rho^2(b))+\mu(\rho(b))}{\mu(\rho(b))}w(\rho(r)) - \frac{\mu(\rho^2(b))}{\mu(\rho(b))}w(r) \\
&< \frac{\mu(\rho^2(b))+\mu(\rho(b))+\mu(b)}{\mu(\rho(b))+\mu(b)}w(\rho(r)) \;<\; 0,
\end{aligned}$$

so (8.92) is true for $n = 2$. Suppose (8.92) is true for $n \ge 2$, then we wish to show that it is true for $n+1$. As before

$$u^{\Delta^2}(\rho^{n+1}(b)) \le v^{\Delta^2}(\rho^{n+1}(b)).$$

Thus

$$\begin{aligned}
w(\rho^{n+1}(r)) &= u(\rho^{n+1}(b)) - v(\rho^{n+1}(b)) \\
&= u(\rho^n(b)) - \mu(\rho^{n+1}(b))u^\Delta(\rho^{n+1}(b)) - v(\rho^n(b)) + \mu(\rho^{n+1}(b))v^\Delta(\rho^{n+1}(b)) \\
&= w(\rho^n(r)) - \mu(\rho^{n+1}(b))\left(u^\Delta(\rho^n(b)) - v^\Delta(\rho^n(b))\right) \\
&\quad + \mu(\rho^{n+1}(b))\mu(\rho^{n+1}(b))\left(u^{\Delta^2}(\rho^{n+1}(b)) - v^{\Delta^2}(\rho^{n+1}(b))\right) \\
&\le w(\rho^n(r)) - \mu(\rho^{n+1}(b))\left(u^\Delta(\rho^n(b)) - v^\Delta(\rho^n(b))\right) \\
&= w(\rho^n(r)) - \frac{\mu(\rho^{n+1}(b))}{\mu(\rho^n(b))}w(\rho^{n-1}(r)) + \frac{\mu(\rho^{n+1}(b))}{\mu(\rho^n(b))}w(\rho^n(r)) \\
&< \frac{\mu(\rho^n(b))+\mu(\rho^{n+1}(b))}{\mu(\rho^n(b))}w(\rho^n(r)) - \frac{\mu(\rho^{n+1}(b))\sum\limits_{i=0}^{n-1}\mu(\rho^i(b))}{\mu(\rho^n(b))\sum\limits_{i=0}^{n}\mu(\rho^i(b))}w(\rho^n(r)) \\
&= \frac{\sum\limits_{i=0}^{n+1}\mu(\rho^i(b))}{\sum\limits_{i=0}^{n}\mu(\rho^i(b))}w(\rho^n(r)) \;<\; 0,
\end{aligned}$$

and (8.92) holds for $n+1$. As before we can use (8.92) to obtain

$$w(\rho^n(r)) < \frac{\sigma(b)-\rho^n(b)}{\mu(b)} w(r)$$

for $n \in \mathbb{N}$, which is equivalent to (8.90). In addition $u-v$ is nondecreasing on $\mathbb{T}$ and $u(t) < v(t)$ for all $t \in (-\infty, \rho(b)]$. ☐

Remark 8.6.14. In the case $\mathbb{T} = \mathbb{Z}$, (8.89) reduces to

$$u(t) - v(t) \ge (t-b)\Big(u(b+1) - v(b+1)\Big) \quad \text{for} \quad t \ge b+1.$$

In addition, (8.90) reduces to

$$u(t) - v(t) \le (b+1-t)\Big(u(b) - v(b)\Big) \quad \text{for} \quad t \le b,$$

which is as expected from [**2**].

Remark 8.6.15. In Lemma 8.6.7 we assumed that $u(a) \ge u(\rho(a))$, $u(a) \ge 0$, and concluded that u was nondecreasing on $[a,\infty)$. If we assume $u(a) > u(\rho(a)) \ge 0$, then u is strictly increasing on $[\rho(a),\infty)$ and $u(t) \to \infty$ as $t \to \infty$.

Proof. By assumption $u^\Delta(\rho(a)) > 0$. Using Lemma 8.6.7, u^Δ is nondecreasing, but this implies that $u^\Delta(t) > 0$ for $t \in [\rho(a),\infty)$. Thus u is strictly increasing on $[\rho(a),\infty)$. Let z be a solution of equation (8.78) defined by

$$z(a) = z(\rho(a)) = u(\rho(a)).$$

By Lemma 8.6.7, z is nonnegative on $[a,\infty)$. Now apply Theorem 8.6.13 with $b = \rho(a)$. Since $u(b) = z(b)$ and $u(\sigma(b)) > z(\sigma(b))$, we have from Theorem 8.6.13 that

$$\begin{aligned} u(t) &\ge u(t) - z(t) \ge \frac{t-\rho(a)}{\mu(\rho(a))}\Big(u(a) - z(a)\Big) \\ &= \frac{t-\rho(a)}{\mu(\rho(a))}\Big(u(a) - u(\rho(a))\Big), \end{aligned}$$

where $u(a) - u(\rho(a)) > 0$. Thus $u(t) \to \infty$ as $t \to \infty$. ☐

The following corollary is a direct result of Theorem 8.6.13.

Corollary 8.6.16. *If p is as in Lemma 8.6.7 and u, v are solutions of equation* (8.78) *satisfying $u(a) = v(a)$ and $u(b) = v(b)$ for some $a < b$ with $a, b \in \mathbb{T}$, then $u(t) = v(t)$ for all $t \in \mathbb{T}$.*

Lemma 8.6.17. *If p is as in Lemma 8.6.7, then for any $\sigma(a) > b$ with $a, b \in \mathbb{T}$, there exists a unique solution of equation* (8.78) *with $u(b) = u_0$ and $u(\sigma(a)) = 0$, where u_0 is any positive constant.*

Proof. Let z be a solution of equation (8.78) such that $z(\sigma(a)) = 0$. If $z(a) > 0$ and $z(\rho(a)) \le z(a)$, then Lemma 8.6.7 implies that $z(\sigma(a)) \ge z(a) > 0$, which is a contradiction. Thus $z(\rho(a)) > z(a) > 0$. Proceeding in this way we obtain

$$z(b) > z(\sigma(b)) > \cdots > z(a) > z(\sigma(a)) = 0. \tag{8.93}$$

Since $z(\sigma(a)) = 0$, if $z(a)$ is also specified, then $z(t)$ is uniquely determined for all $t \in [b, \sigma(a)]$ by equation (8.78). Thus in particular $z(b)$ is determined by $z(a)$. Let f be the mapping from $z(a)$ to $z(b)$. From equation (8.78) it is clear that each

$z(t)$, $t \in [b, \rho(a)]$, continuously depends on $z(a)$, and so the function $z(b) = f(z(a))$ is continuous. If we let $z(a) = u_0$, then (8.93) implies that $f(u_0) > u_0$; if we let $z(a) = 0$ so that $z(\sigma(a)) = z(a) = 0$, then $z(t) = 0$ by Lemma 8.6.7, so $f(0) = 0$. Thus since f is continuous, there exists $\beta \in (0, u_0)$, such that $f(\beta) = u_0$. Therefore there exists a solution u of equation (8.78) determined by $u(\sigma(a)) = 0$ and $u(a) = \beta$ which must satisfy $u(b) = u_0$. Finally the uniqueness of the solution follows from Corollary 8.6.16. □

Theorem 8.6.18. *If p is as in Lemma 8.6.7, then equation* (8.78) *has a positive nonincreasing solution u and a positive strictly increasing solution v such that $v(t) \to \infty$ as $t \to \infty$. In addition, the nonincreasing solution u is uniquely determined once $u(a)$ is specified.*

Proof. If we choose $a \in \mathbb{T}$, $v(a) = 1$ and $v(\sigma(a)) > 1$, then the existence of an increasing solution v satisfying the stated properties is an immediate consequence of Remark 8.6.15. We wish to show the existence of a positive nonincreasing solution u. It is clear from Lemma 8.6.17 that for each $n \in \mathbb{T}$, $n \geq \max\{1, \sigma(a)\}$, there is a unique solution $u_n(t)$, $t \in \mathbb{T}$ of equation (8.78) such that

$$u_n(a) = u_a, \quad u_n(n) = 0. \tag{8.94}$$

Further, in view of (8.93) we know that for every $n \geq \max\{1, \sigma(a)\}$,

$$u_a \geq u_n(t) > u_n(\sigma(t)) \geq 0 \quad \text{for} \quad t \in [a, \rho(n)]. \tag{8.95}$$

We claim that for every $n \geq \max\{1, \sigma(a)\}$,

$$u_{\sigma(n)}(t) > u_n(t) \quad \text{for} \quad t \in [\sigma(a), \infty). \tag{8.96}$$

For this, by Theorem 8.6.13 it suffices to show that

$$u_{\sigma(n)}(\sigma(a)) > u_n(\sigma(a)).$$

By way of contradiction assume $u_{\sigma(n)}(\sigma(a)) \leq u_n(\sigma(a))$. If $u_{\sigma(n)}(\sigma(a)) = u_n(\sigma(a))$, then since $u_n(a) = u_{\sigma(n)}(a) = u_a$, the solutions u_n and $u_{\sigma(n)}$ are identically equal. However $u_{\sigma(n)}(\sigma(n)) = u_n(n) = 0$, so both u_n and $u_{\sigma(n)}$ are identically zero, which contradicts $u_n(a) = u_a > 0$. On the other hand, if $u_{\sigma(n)}(\sigma(a)) < u_n(\sigma(a))$, then from Theorem 8.6.13 we have $u_n(t) > u_{\sigma(n)}(t)$ for all $t \in [\sigma(a), \infty)$. In particular for $t = n$ we find

$$0 = u_n(n) > u_{\sigma(n)}(n) > u_{\sigma(n)}(\sigma(n)) = 0,$$

which is also a contradiction. Hence (8.96) holds.

Combining (8.95) and (8.96), we find for each $t \in [\sigma(a), \infty)$, that the sequence $\{u_n(t)\}_{n \in \mathbb{T}}$ is increasing, bounded above by u_a, and eventually positive. For each $t \in \mathbb{T}$, let

$$u(t) = \lim_{n \to \infty} u_n(t).$$

Then $0 < u(t) \leq u_a$ for $t \in \mathbb{T}$, and from (8.95) we have $u(t) \geq u(\sigma(t))$. Now since $n \in [\sigma(a), \infty)$, u_n is a solution of equation (8.78), and we have

$$u_n^{\Delta^2}(t) = -p(t) \left[u_n(\sigma(t)) \right]^\gamma.$$

Thus as $n \to \infty$, we find that u is a nonincreasing positive solution of equation (8.78).

Finally we show that the solution u is unique once u_a is specified. For this let z be any positive nonincreasing solution of equation (8.78) such that $z(a) = u_a$.

Then either $z(\sigma(a)) < u(\sigma(a))$, $z(\sigma(a)) > u(\sigma(a))$, or $z(\sigma(a)) = u(\sigma(a))$. In the first case there exists $n \in \mathbb{T}$ and a solution u_n defined by (8.94) such that

$$z(\sigma(a)) < u_n(\sigma(a)) < u(\sigma(a)).$$

Since $u_n(a) = z(a)$ and $u_n(\sigma(a)) > z(\sigma(a))$, Theorem 8.6.13 implies $u_n(t) > z(t)$ for all $t \in [\sigma(a), \infty)$. In particular this implies $0 = u_n(n) > z(n)$, which is a contradiction. If instead $z(\sigma(a)) > u(\sigma(a))$, then Theorem 8.6.13 implies that

$$z(t) - u(t) \geq \frac{t-a}{\mu(a)}\Big(z(\sigma(a)) - u(\sigma(a))\Big) \quad \text{for} \quad t \in [\sigma(a), \infty).$$

This means that $z(t)$ becomes unbounded as $t \to \infty$ since $\frac{t-a}{\mu(a)} \to \infty$ as $t \to \infty$, which is again a contradiction. Thus $z(\sigma(a)) = u(\sigma(a))$. By Corollary 8.6.16, $z(t) = u(t)$ for all $t \in \mathbb{T}$. □

Theorem 8.6.19. *Let p be as in Lemma 8.6.10, $a \in \mathbb{T}$, $a \geq 0$, and $\gamma > 1$. If*

$$\int_a^\infty \sigma(t)p(t)\Delta t = \infty,$$

then the dynamic equation (8.78) *is oscillatory.*

Proof. Let u be a nonoscillatory solution of equation (8.78) and $u(t) > 0$ for all $t \in [a, \infty)$. Multiply both sides of equation (8.78) by $\sigma(t)[u(\sigma(t))]^{-\gamma}$ to obtain

$$\sigma(t)[u(\sigma(t))]^{-\gamma}u^{\Delta^2}(t) + \sigma(t)p(t) = 0.$$

Using the integration by parts formula for $k \in [a, \infty)$,

$$\int_a^k \sigma(t)[u(\sigma(t))]^{-\gamma}u^{\Delta^2}(t)\Delta t = k[u(k)]^{-\gamma}u^\Delta(k) - a[u(a)]^{-\gamma}u^\Delta(a) - \int_a^k \left(t[u(t)]^{-\gamma}\right)^\Delta u^\Delta(t)\Delta t$$

yields

$$k[u(k)]^{-\gamma}u^\Delta(k) - a[u(a)]^{-\gamma}u^\Delta(a) - \int_a^k \left(t[u(t)]^{-\gamma}\right)^\Delta u^\Delta(t)\Delta t + \int_a^k \sigma(t)p(t)\Delta t = 0.$$

In view of Lemma 8.6.10 and the hypothesis, it must be the case that

$$\int_a^k \left(t[u(t)]^{-\gamma}\right)^\Delta u^\Delta(t)\Delta t \to \infty \quad \text{as} \quad k \to \infty. \tag{8.97}$$

We shall show that (8.97) is impossible.

Note that $u^\Delta(t) > 0$ implies $(u^{-\gamma})^\Delta(t) < 0$. Thus

$$\begin{aligned}\int_a^k \left(t[u(t)]^{-\gamma}\right)^\Delta u^\Delta(t)\Delta t &= \int_a^k \left([u(\sigma(t))]^{-\gamma} + t(u^{-\gamma})^\Delta(t)\right) u^\Delta(t)\Delta t \\ &\leq \int_a^k [u(\sigma(t))]^{-\gamma}u^\Delta(t)\Delta t,\end{aligned}$$

and it suffices to show that

$$\int_a^k [u(\sigma(t))]^{-\gamma}u^\Delta(t)\Delta t < \infty. \tag{8.98}$$

We define $r(s)$, a continuous function on $[t, \sigma(t)]$ by

$$r(s) = u(t) + (s - t)u^{\Delta}(t).$$

Notice that $r(t) = u(t)$, $r(\sigma(t)) = u(\sigma(t))$, and $r'(s) = u^{\Delta}(t) > 0$. Hence $r(s)$ is continuous and increasing for $s \in [t, \sigma(t)]$. From this we get

$$\begin{aligned} [u(\sigma(t))]^{-\gamma} u^{\Delta}(t) &= \frac{1}{\mu(t)} \int_t^{\sigma(t)} [u(\sigma(t))]^{-\gamma} u^{\Delta}(t) ds \\ &= \frac{1}{\mu(t)} \int_t^{\sigma(t)} [r(\sigma(t))]^{-\gamma} r'(s) ds \\ &\leq \frac{1}{\mu(t)} \int_t^{\sigma(t)} [r(s)]^{-\gamma} r'(s) ds \\ &= \frac{1}{\mu(t)} \frac{1}{1-\gamma} \left([r(\sigma(t))]^{1-\gamma} - [r(t)]^{1-\gamma} \right) \\ &= \frac{1}{1-\gamma} \frac{[r(\sigma(t))]^{1-\gamma} - [r(t)]^{1-\gamma}}{\mu(t)} \\ &= \frac{1}{1-\gamma} (r^{1-\gamma})^{\Delta}(t). \end{aligned}$$

This implies that for $k \in \mathbb{T}$,

$$\int_a^k [u(\sigma(t))]^{-\gamma} u^{\Delta}(t) \Delta t \leq \frac{1}{1-\gamma} \left([r(k)]^{1-\gamma} - [r(a)]^{1-\gamma} \right).$$

However since $\gamma > 1$ and r is an increasing function, it follows that (8.98) holds, completing the proof. □

Theorem 8.6.20. *Let p be as in Lemma 8.6.10, $a \in \mathbb{T}$, $a \geq 0$, and $\gamma > 1$. Then the dynamic equation* (8.78) *is oscillatory if and only if*

$$\int_a^{\infty} \sigma(l) p(l) \Delta l = \infty.$$

Proof. If $\int_a^{\infty} \sigma(l)p(l)\Delta l = \infty$, then it was shown in Theorem 8.6.19 above that the dynamic equation (8.78) is oscillatory. The other direction follows from the proof of Theorem 8.6.22 below. □

Remark 8.6.21. Since all of the points in the time scale are isolated, one can rewrite $\int_a^{\infty} \sigma(l)p(l)\Delta l$ as $\sum_{l \in [a,\infty)} \mu(l)\sigma(l)p(l)$ (see Theorem 8.2.14 (ii)).

Theorem 8.6.22. *Let p be as in Lemma 8.6.10. Then equation* (8.78) *has a bounded nonoscillatory solution if and only if*

$$\int_a^{\infty} \sigma(l) p(l) \Delta l < \infty,$$

where $a \in \mathbb{T}$, $a \geq 0$.

Proof. First suppose that equation (8.78) has a bounded nonoscillatory solution u. Then there exists $a \in \mathbb{T}$ with $a \geq 0$ such that $u(t) > 0$ for all $t \in [a, \infty)$. In view of Lemma 8.6.10, u is increasing on $[a, \infty)$. Therefore $u(t)$ is bounded above and below by positive constants for all $t \in [a, \infty)$. Using the integration by parts formula in Theorem 8.2.12 (vi), we see that any solution u of equation (8.78) also satisfies

$$tu^{\Delta}(t) = au^{\Delta}(a) + u(t) - u(a) - \int_a^t \sigma(l)p(l)[u(\sigma(l))]^{\gamma}\Delta l \tag{8.99}$$

for all $t \in [a,\infty)$. If $\int_a^t \sigma(l)p(l)\Delta l \to \infty$ as $t \to \infty$, then the right-hand side of equation (8.99) must approach $-\infty$. This implies that the left-hand side of equation (8.99) is eventually negative. But this contradicts the fact that u is increasing.

To prove the converse, suppose $\int_a^\infty \sigma(l)p(l)\Delta l < \infty$. Using [**53**, Theorem 1.117], it is easy to verify that any solution u of

$$u(t) = 1 - \int_t^\infty \Big(\sigma(l) - t\Big)\, p(l)[u(\sigma(l))]^{\gamma}\Delta l \tag{8.100}$$

is also a solution of equation (8.78). Choose $a \in \mathbb{T}$ with $a \geq 0$ sufficiently large so that

$$\max_{t\in[a,\infty)} \left\{ \int_t^\infty \Big(\sigma(l) - t\Big)\, p(l)\Delta l, \quad 2\gamma \int_t^\infty \Big(\sigma(l) - t\Big)\, p(l)\Delta l \right\} < \frac{1}{2}.$$

Consider the Banach space L_a of all bounded real functions x on $[a,\infty)$ with norm defined by

$$\|x\| = \sup_{t\in[a,\infty)} |x(t)|.$$

We define a closed and bounded subset S of L_a as

$$S := \left\{ x \in L_a : \ \frac{1}{2} \leq x(t) \leq 1 \right\}.$$

Let $T : S \to S$ be an operator such that

$$(Tx)(t) = 1 - \int_t^\infty \Big(\sigma(l) - t\Big)\, p(l)[x(\sigma(l))]^{\gamma}\Delta l, \quad t \geq a.$$

To see that the range of T is in S, note that if $x \in S$, then

$$(Tx)(t) \geq 1 - \int_t^\infty \Big(\sigma(l) - t\Big)\, p(l)\Delta l \geq \frac{1}{2}.$$

Clearly $(Tx)(t) \leq 1$. We will show that T is a contraction mapping on S. To see this, define $r(k) = k^\gamma$. By the (ordinary) mean value theorem,

$$\Big| k^\gamma - l^\gamma \Big| \leq \left(\max_{k\leq\xi\leq l} (\xi^\gamma)' \right) |k - l|.$$

Thus for any $x, y \in S$,

$$\begin{aligned}
\Big| [x(t)]^\gamma - [y(t)]^\gamma \Big| &\leq \left(\max_{x\leq\xi\leq y} (\xi^\gamma)' \right) |x(t) - y(t)| \\
&= \left(\max_{\frac{1}{2}\leq\xi\leq 1} (\xi^\gamma)' \right) |x(t) - y(t)| \\
&\leq 2\gamma|x(t) - y(t)|.
\end{aligned}$$

Therefore

$$\begin{aligned}\Big|(Tx)(t)-(Ty)(t)\Big| &\le \int_t^\infty \Big(\sigma(l)-t\Big)\,p(l)\,\Big|[x(\sigma(l))]^\gamma-[y(\sigma(l))]^\gamma\Big|\,\Delta l\\ &\le 2\gamma\int_t^\infty \Big(\sigma(l)-t\Big)\,p(l)\,\Big|x(\sigma(l))-y(\sigma(l))\Big|\,\Delta l\\ &\le 2\gamma\,\|x-y\|\int_t^\infty \Big(\sigma(l)-t\Big)\,p(l)\Delta l\\ &\le \frac{1}{2}\,\|x-y\|\,.\end{aligned}$$

It follows that $\|Tx-Ty\|\le\frac{1}{2}\,\|x-y\|$, and hence T is a contraction mapping on S. By the Banach contraction mapping principle (Theorem 1.4.26), T has a unique fixed point in S, which is our desired bounded nonoscillatory solution of (8.100). □

Example 8.6.23. Consider

$$u^{\Delta^2}(t)=-\frac{1}{t[\sigma(t)]^2}[u(\sigma(t))]^\gamma\quad\text{for}\quad t\ge a>0.$$

Note that $p(t)=\frac{1}{t[\sigma(t)]^2}$ satisfies the conditions of Lemma 8.6.10 on $[a,\infty)$. Then

$$\int_a^\infty \sigma(s)p(s)\Delta s=\int_a^\infty\frac{1}{s\sigma(s)}\Delta s=-\int_a^\infty\left(\frac{1}{s}\right)^\Delta\Delta s=\frac{1}{a}<\infty.$$

Therefore by Theorem 8.6.22 this dynamic equation has a bounded nonoscillatory solution on $[a,\infty)$, regardless of the time scale chosen.

Example 8.6.24. Consider

$$u^{\Delta^2}(t)=-\frac{1}{t\sigma(t)}[u(\sigma(t))]^\gamma\quad\text{for}\quad t\ge a>0,$$

where $\gamma>1$. Note that $p(t)=\frac{1}{t\sigma(t)}$ satisfies the conditions of Lemma 8.6.10 on $[a,\infty)$. Then

$$\int_a^\infty\sigma(s)p(s)\Delta s=\int_a^\infty\sigma(s)\frac{1}{s\sigma(s)}\Delta s=\int_a^\infty\frac{1}{s}\Delta s\overset{(8.43)}{=}\infty.$$

So by Theorem 8.6.20 this dynamic equation is oscillatory on $[a,\infty)$, regardless of the time scale chosen.

Theorem 8.6.25. *Assume p is as in Lemma 8.6.10 and $\gamma\in(0,1)$. Then equation* (8.78) *is oscillatory if and only if*

$$\int_a^\infty[\sigma(l)]^\gamma p(l)\Delta l=\infty,$$

where $a\in\mathbb{T}$, $a\ge 0$.

Proof. Let u be a nonoscillatory solution of equation (8.78) such that $u(t)>0$ for all $t\in[a,\infty)$, where $a\in\mathbb{T}$, $a\ge0$. By Lemma 8.6.10, $u(t)$ is increasing and $u^\Delta(t)$ is positive and nonincreasing for $t\in[a,\infty)$. Fix $j\in\mathbb{T}$ such that $j>2a$. Then for all $t\in[j,\infty)$, we have

$$u(t)=u(a)+\int_a^t u^\Delta(l)\Delta l>\int_a^t u^\Delta(t)\Delta l=(t-a)u^\Delta(t)>\frac{t}{2}u^\Delta(t),$$

i.e., $\frac{u(\sigma(t))}{u^\Delta(\sigma(t))} > \frac{\sigma(t)}{2}$. Dividing equation (8.78) by $[u^\Delta(\sigma(t))]^\gamma$, using this inequality, and integrating from j to t, we obtain

$$\int_j^t \frac{u^{\Delta^2}(l)}{[u^\Delta(\sigma(l))]^\gamma}\Delta l + \frac{1}{2^\gamma}\int_j^t p(l)[\sigma(l)]^\gamma \Delta l < 0, \quad t \geq j. \tag{8.101}$$

By hypothesis, the second integral in (8.101) approaches ∞ as $t \to \infty$, so the first term approaches $-\infty$. But we will show that this is impossible. To see this, let

$$r(k) = u(l) + (k-l)u^\Delta(l),$$

$l \leq k \leq \sigma(l)$, $l \geq a$, so that r is positive, continuous, and increasing. Further, let

$$s(k) = \frac{r(k+\mu(l)) - r(k)}{\mu(l)} \quad \text{for} \quad k \geq a$$

so that s is positive and continuous. Since $l \leq k \leq \sigma(l)$, we have

$$\sigma(l) \leq k + \mu(l) \leq 2\sigma(l) - l.$$

Therefore

$$\begin{aligned} r(k+\mu(l)) &= u(\sigma(l)) + \Big(k+\mu(l)-\sigma(l)\Big)\, u^\Delta(\sigma(l)) \\ &= u(\sigma(l)) + (k-l)u^\Delta(\sigma(l)). \end{aligned}$$

This implies that

$$\begin{aligned} s(k) &= \frac{u(\sigma(l)) + (k-l)u^\Delta(\sigma(l)) - u(l) - (k-l)u^\Delta(l)}{\mu(l)} \\ &= \frac{u(\sigma(l)) - u(l)}{\mu(l)} + (k-l)\left[\frac{u^\Delta(\sigma(l)) - u^\Delta(l)}{\mu(l)}\right] \\ &= u^\Delta(l) + (k-l)u^{\Delta^2}(l). \end{aligned}$$

Therefore $s'(k) = u^{\Delta^2}(l) \leq 0$ for $l < k < \sigma(l)$, which implies that s is nonincreasing and $0 < s(k) \leq s(l) = u^\Delta(l)$. Then for $l < k < \sigma(l)$ we have

$$\begin{aligned} \frac{u^{\Delta^2}(l)}{[u^\Delta(\sigma(l))]^\gamma} &= \frac{1}{\mu(l)}\int_l^{\sigma(l)} \frac{u^{\Delta^2}(l)}{[u^\Delta(\sigma(l))]^\gamma}dk \\ &\geq \frac{1}{\mu(l)}\int_l^{\sigma(l)} \frac{s'(k)}{[s(\sigma(k))]^\gamma}dk \\ &= \frac{1}{\mu(l)}\int_l^{\sigma(l)} \frac{s'(k)}{[s(k)]^\gamma}dk \\ &= \frac{1}{\mu(l)}\frac{1}{1-\gamma}\left([s(\sigma(l))]^{1-\gamma} - [s(l)]^{1-\gamma}\right) \\ &= \frac{1}{1-\gamma}(s^{1-\gamma})^\Delta(l). \end{aligned}$$

It follows that

$$\int_j^t \frac{u^{\Delta^2}(l)}{[u^\Delta(\sigma(l))]^\gamma}\Delta l \geq \frac{1}{1-\gamma}\int_j^t (s^{1-\gamma})^\Delta(l)\Delta l = \frac{1}{1-\gamma}\left([s(t)]^{1-\gamma} - [s(j)]^{1-\gamma}\right).$$

But $[s(t)]^{1-\gamma} > 0$ and $0 < \gamma < 1$ for all $t \geq a$, so $\int_j^t \frac{u^{\Delta^2}(l)}{[u^\Delta(\sigma(l))]^\gamma}\Delta l$ is bounded below which gives a contradiction and completes the proof. The necessity part is contained in the sufficiency part of the next theorem. $\square$

Definition 8.6.26. A solution of equation (8.78) is said to have *asymptotically positively bounded differences* if there are positive constants a_1 and a_2 such that

$$a_1 \leq u^{\Delta}(t) \leq a_2$$

for all $t \in [a, \infty)$ for some $a \in \mathbb{T}$.

Theorem 8.6.27. *Let p be as in Lemma 8.6.10. Equation* (8.78) *has a solution with asymptotically positively bounded differences if and only if*

$$\int_a^{\infty} p(l)[\sigma(l)]^{\gamma} \Delta l < \infty,$$

where $a \in \mathbb{T}$, $a \geq 0$.

Proof. Assume that $\int_a^{\infty} p(l)[\sigma(l)]^{\gamma} \Delta l < \infty$, and fix $a \in \mathbb{T}$ with $a \geq 0$ sufficiently large so that $\int_a^{\infty} p(l)[\sigma(l)]^{\gamma} \Delta l < \frac{1}{2}$. Let u be the solution of equation (8.78) satisfying $u(a) = 0$ and $u(\sigma(a)) = \mu(a)$ so that $u^{\Delta}(a) = 1$. We want to show that $\frac{1}{2} \leq u^{\Delta}(t) \leq 1$ for all $t \in [a, \infty)$. For this purpose, suppose that $\frac{1}{2} \leq u^{\Delta}(t) \leq 1$ for all $t \in [a, m]$, where $m \in \mathbb{T}$ with $m \geq a$. Then $u(t) > 0$ for all $t \in (a, \sigma(m)]$. However from equation (8.78), $u^{\Delta^2}(t) = -p(t)[u(\sigma(t))]^{\gamma} \leq 0$ for all $t \in [a, m]$. Therefore for all $t \in [a, \sigma(m)]$ it follows that

$$u(t) \leq u(a) + (t-a)u^{\Delta}(a) = t - a \leq t.$$

From equation (8.78) and the above inequalities we obtain

$$u^{\Delta}(m) = u^{\Delta}(a) - \int_a^m p(l)[u(\sigma(l))]^{\gamma} \Delta l \geq 1 - \int_a^m p(l)[\sigma(l)]^{\gamma} \Delta l \geq \frac{1}{2}.$$

Also $u^{\Delta}(m) \leq u^{\Delta}(a) = 1$. Therefore $\frac{1}{2} \leq u^{\Delta}(m) \leq 1$, and now by induction

$$\frac{1}{2} \leq u^{\Delta}(t) \leq 1$$

holds for all $t \in [a, \infty)$.

Conversely let u be a solution of equation (8.78) which has asymptotically positively bounded differences. Thus there exists $a \in \mathbb{T}$ with $a \geq 0$ such that $u(t) > 0$ for all $t \in [a, \infty)$. Then as in Theorem 8.6.25, we find that

$$u(t) > \frac{t}{2} u^{\Delta}(t) \quad \text{for all} \quad t \geq j > 2a.$$

Therefore for all $t \in [j, \infty)$ it follows that

$$\begin{aligned} u^{\Delta}(j) - u^{\Delta}(t) &= \int_j^t p(l)[u(\sigma(l))]^{\gamma} \Delta l \\ &> \int_j^t p(l) \left(\frac{\sigma(l)}{2} u^{\Delta}(\sigma(l)) \right)^{\gamma} \Delta l \\ &\geq \left(\frac{a_1}{2} \right)^{\gamma} \int_j^t p(l)[\sigma(l)]^{\gamma} \Delta l \ \geq \ 0. \end{aligned}$$

If $\int_a^t p(l)[\sigma(l)]^{\gamma} \Delta l \to \infty$ as $t \to \infty$, then it must be the case that $u^{\Delta}(t) \to -\infty$ as $t \to \infty$. However by Lemma 8.6.10, $u^{\Delta}(t) > 0$ for all $t \in [a, \infty)$. This implies that $\int_a^{\infty} p(l)[\sigma(l)]^{\gamma} \Delta l < \infty$. ∎

It is interesting to note that the proof technique impacts whether or not the theorem holds for $\mathbb{T} = \mathbb{R}$ as well as for the time scales we are interested in. For example, Theorem 8.6.22 uses the contraction mapping principle, and it is well known that this result is true for $\mathbb{T} = \mathbb{R}$. This suggests that a modification of the proof techniques would generate the results on a wider selection of time scales.

8.7. Oscillation of First Order Delay Dynamic Equations

In this section we follow [**37**] and present an oscillation criterion for first order delay dynamic equations on unbounded time scales, which contains well-known criteria for delay differential equations and delay difference equations as special cases. We illustrate our results by applying them to various kinds of time scales.

As is well known (see Theorem 2.2.6 or [**118**, Theorem 2.3.1]), a first order delay differential equation of the form

$$y'(t) + p(t)y(t-\tau) = 0$$

(where $t \in \mathbb{R}$, p is continuous and positive, and $\tau > 0$) is oscillatory provided

$$\liminf_{t\to\infty} \int_{t-\tau}^{t} p(s)ds > \frac{1}{e}$$

holds. It is also well known (cf. [**118**, Theorem 7.5.1]) that a first order delay difference equation of the form

$$\Delta y_n + p_n y_{n-k} = 0$$

(where $n \in \mathbb{Z}$, $p_n > 0$, $k \in \mathbb{N}$, $\Delta y_n = y_{n+1} - y_n$) is oscillatory if

$$\liminf_{n\to\infty} \left\{ \frac{1}{k} \sum_{i=n-k}^{n-1} p_i \right\} > \frac{k^k}{(k+1)^{k+1}}$$

holds. In this section we present a **generalization** and **extension** of these two results for first order delay dynamic equations (see also [**299**]) of the form

$$y^{\Delta}(t) + p(t)y(\tau(t)) = 0, \tag{8.102}$$

where $t \in \mathbb{T}$, $\mathbb{T}$ is a time scale, p is rd-continuous and positive, the *delay* function $\tau : \mathbb{T} \to \mathbb{T}$ satisfies $\tau(t) < t$ for all $t \in \mathbb{T}$ and $\lim_{t\to\infty} \tau(t) = \infty$, and $y^{\Delta}(t)$ is the delta derivative of $y : \mathbb{T} \to \mathbb{R}$ at $t \in \mathbb{T}$.

Now let us assume that (8.102) possesses a positive solution y. Then

$$y^{\Delta}(t) = -p(t)y(\tau(t)) < 0$$

so that y is decreasing and therefore

$$\begin{aligned} 0 &= -\mu(t)\Big(y^{\Delta}(t) + p(t)y(\tau(t))\Big) = y(t) - y(\sigma(t)) - \mu(t)p(t)y(\tau(t)) \\ &< y(t) - \mu(t)p(t)y(t) = \Big(1 - \mu(t)p(t)\Big)y(t). \end{aligned}$$

Hence $1 - \mu(t)p(t) > 0$, which implies that $-p \in \mathcal{R}^+$ and that there exists $\lambda > 0$ such that $-\lambda p \in \mathcal{R}^+$. The quantity

$$\alpha := \limsup_{\substack{t\to\infty \\ t\in\mathbb{T}}} \sup_{\substack{\lambda>0 \\ -\lambda p\in\mathcal{R}^+}} \Big\{\lambda e_{-\lambda p}(t, \tau(t))\Big\} \tag{8.103}$$

is therefore well defined. Now we can formulate the main result of this section.

Theorem 8.7.1. *If* (8.102) *has an eventually positive solution, then* α *defined by* (8.103) *satisfies* $\alpha \geq 1$.

The following two easy lemmas are needed in the proof of Theorem 8.7.1.

Lemma 8.7.2. *Suppose* $-p \in \mathcal{R}^+$ *and* $s \in \mathbb{T}$. *If*

$$y^\Delta(t) + p(t)y(t) \leq 0 \quad \text{for all} \quad t \geq s,$$

then

$$y(t) \leq e_{-p}(t,s)y(s) \quad \text{for all} \quad t \geq s.$$

Proof. We put $f := y^\Delta + py$ and use Theorem 8.2.20 (see also [**53**, Theorem 2.77]) to solve

$$y^\Delta = -p(t)y + f(t), \quad y(s) \text{ given}.$$

Thus for $t \geq s$,

$$y(t) = e_{-p}(t,s)y(s) + \int_s^t e_{-p}(t,\sigma(u))f(u)\Delta u.$$

The integrand is nonpositive as $-p \in \mathcal{R}^+$ and $f \leq 0$, so our claim follows. □

Lemma 8.7.3. *For nonnegative* p *with* $-p \in \mathcal{R}^+$ *we have the inequalities*

$$1 - \int_s^t p(u)\Delta u \leq e_{-p}(t,s) \leq \exp\left\{-\int_s^t p(u)\Delta u\right\} \quad \text{for all} \quad t \geq s.$$

Proof. Fix $s \in \mathbb{T}$, denote $y(t) = -\int_s^t p(u)\Delta u$, and observe that

$$y^\Delta(t) = -p(t) \leq -p(t) - p(t)y(t).$$

We put $f := y^\Delta + py + p$ and use Theorem 8.2.20 to solve

$$y^\Delta = -p(t)y + f(t) - p(t), \quad y(s) = 0.$$

Thus for $t \geq s$,

$$\begin{aligned} y(t) &= e_{-p}(t,s)y(s) + \int_s^t e_{-p}(t,\sigma(u))\left[f(u) - p(u)\right]\Delta u \\ &\leq -\int_s^t e_{-p}(t,\sigma(u))p(u)\Delta u \\ &= e_{-p}(t,s) - 1, \end{aligned}$$

where we have used Theorem 8.2.19 (ix) (see also [**53**, Theorem 2.39]) in the last step. This establishes the left part of the asserted inequality. For the right part we use the representation [**53**, (2.15)]

$$e_{-p}(t,s) = \exp\left\{\int_s^t \xi_{\mu(u)}(-p(u))\Delta u\right\},$$

where we have for any p with $-p \in \mathcal{R}^+$

$$\xi_{\mu(u)}(-p(u)) = -p(u)$$

if $\mu(u) = 0$, and if $\mu(u) > 0$,

$$\begin{aligned} \xi_{\mu(u)}(-p(u)) &= \frac{\operatorname{Log}(1-\mu(u)p(u))}{\mu(u)} = \frac{\ln(1-\mu(u)p(u))}{\mu(u)} \\ &= -p(u) - \frac{f(-\mu(u)p(u))}{\mu(u)} \leq -p(u), \end{aligned}$$

where $f : (-1,\infty) \to \mathbb{R}$ is defined by $f(x) = x - \ln(1+x)$ and hence satisfies $f(x) \geq 0$ for all $x > -1$. □

Some remarks follow.

Remark 8.7.4. Let $s \in \mathbb{T}$. If p is rd-continuous and nonnegative, then a similar proof as in Lemma 8.7.2 can be used to show that if

$$x^{\Delta}(t) + p(t)x(\sigma(t)) \leq 0 \quad \text{for all} \quad t \geq s,$$

then

$$x(s) \geq e_p(t,s)x(t) \quad \text{for all} \quad t \geq s.$$

Remark 8.7.5. If p is rd-continuous and nonnegative, then a similar proof as in Lemma 8.7.3 can be used to show

$$1 + \int_s^t p(u)\Delta u \leq e_p(t,s) \leq \exp\left\{\int_s^t p(u)\Delta u\right\} \quad \text{for all} \quad t \geq s.$$

Remark 8.7.6. Denote $P := \int_s^t p(u)\Delta u$ for $t \geq s$, where p is nonnegative with $-p \in \mathcal{R}^+$. Then by Lemma 8.7.3, $1 - P \leq e_{-p}(t,s) \leq e^P$. For all $\lambda \in (0,1]$ we have $-\lambda p \in \mathcal{R}^+$ and hence by Lemma 8.7.3, $1 - \lambda P \leq e_{-\lambda p}(t,s) \leq e^{\lambda P}$, so that

$$\lambda - \lambda^2 P \leq \lambda e_{-\lambda p}(t,s) \leq \lambda e^{\lambda P}$$

and therefore

$$\frac{1}{4P} \leq \sup_{\substack{\lambda>0 \\ -\lambda p \in \mathcal{R}^+}} \left\{\lambda e_{-\lambda p}(t,s)\right\} \leq \frac{1}{eP}.$$

Thus we always have

$$\frac{1}{4\liminf_{t\to\infty}\int_{\tau(t)}^t p(s)\Delta s} \leq \alpha \leq \frac{1}{e\liminf_{t\to\infty}\int_{\tau(t)}^t p(s)\Delta s}$$

and

$$\frac{1}{4\alpha} \leq \liminf_{t\to\infty} \int_{\tau(t)}^t p(s)\Delta s \leq \frac{1}{e\alpha}.$$

Now we have all the tools needed to prove our main result.

Proof of Theorem 8.7.1. Throughout we assume that y solves (8.102) and is eventually positive and that $\alpha < 1$. We proceed in two parts showing

$$\liminf_{t\to\infty} \frac{y(\tau(t))}{y(t)} = \infty \tag{8.104}$$

and

$$\liminf_{t\to\infty} \frac{y(\tau(t))}{y(t)} < \infty. \tag{8.105}$$

This contradiction shows $\alpha \geq 1$ and hence finishes the proof. First we show (8.104). Let $\beta \in (1, 1/\alpha)$. Then there exists $T_0 \in \mathbb{T}$ such that

$$\frac{1}{\sup_{\substack{\lambda>0 \\ -\lambda p \in \mathcal{R}^+}} \{\lambda e_{-\lambda p}(t,\tau(t))\}} \geq \beta \quad \text{for all} \quad t \geq T_0. \tag{8.106}$$

As y is eventually positive, it is eventually decreasing and hence $y(\tau(t)) \geq y(t)$ eventually so that

$$0 = y^{\Delta}(t) + p(t)y(\tau(t)) \geq y^{\Delta}(t) + p(t)y(t)$$

implies by Lemma 8.7.2 that there exists $T_1 \geq T_0$ with

$$\frac{y(\tau(t))}{y(t)} \geq \frac{1}{e_{-p}(t,\tau(t))} \overset{(8.106)}{\geq} \beta \quad \text{for all} \quad t \geq T_1.$$

Thus

$$0 = y^\Delta(t) + p(t)y(\tau(t)) \geq y^\Delta(t) + \beta p(t)y(t)$$

implies again by Lemma 8.7.2 that there exists $T_2 \geq T_1$ with

$$\frac{y(\tau(t))}{y(t)} \geq \frac{1}{e_{-\beta p}(t,\tau(t))} = \frac{\beta}{\beta e_{-\beta p}(t,\tau(t))} \overset{(8.106)}{\geq} \beta^2 \quad \text{for all} \quad t \geq T_2.$$

Proceeding in this manner we obtain a sequence $\{T_n\} \subset \mathbb{T}$ with

$$\frac{y(\tau(t))}{y(t)} \geq \beta^n \quad \text{for all} \quad t \geq T_n.$$

This proves (8.104) as $\beta > 1$. Now we show (8.105). Let $M \in (1/4, 1/(4\alpha))$. By Lemma 8.7.3 (see Remark 8.7.6) there exists $T \in \mathbb{T}$ such that

$$\int_{\tau(t)}^t p(s)\Delta s \geq M \quad \text{for all} \quad t \geq T.$$

Now

$$\int_{\tau(t)}^{\sigma(t)} p(s)\Delta s \geq \int_{\tau(t)}^t p(s)\Delta s \geq M \quad \text{for all} \quad t \geq T.$$

Let $t \geq T$. We consider the function $f : \mathbb{T} \to \mathbb{R}$ defined by

$$f(u) = \int_{\tau(t)}^u p(s)\Delta s - \frac{M}{2}$$

and find $f(\tau(t)) < 0$ and $f(t) > 0$. By the intermediate value theorem (as given in [**53**, Theorem 1.115]) there exists $t^* \in [\tau(t), t)$ such that $f(t^*) = 0$, or $f(t^*) < 0$ and $f(\sigma(t^*)) > 0$. Hence

$$\int_{\tau(t)}^{\sigma(t^*)} p(s)\Delta s = \frac{M}{2} + f(\sigma(t^*)) \geq \frac{M}{2} \tag{8.107}$$

and

$$\int_{t^*}^{\sigma(t)} p(s)\Delta s = \int_{\tau(t)}^{\sigma(t)} p(s)\Delta s - \left[f(t^*) + \frac{M}{2}\right] \geq \frac{M}{2} - f(t^*) \geq \frac{M}{2}. \tag{8.108}$$

Now we can estimate

$$\begin{aligned}
y(t^*) &\geq y(t^*) - y(\sigma(t)) \overset{(8.102)}{=} \int_{t^*}^{\sigma(t)} p(s)y(\tau(s))\Delta s \\
&\geq y(\tau(t)) \int_{t^*}^{\sigma(t)} p(s)\Delta s \overset{(8.108)}{\geq} \frac{M}{2} y(\tau(t)) \\
&\geq \frac{M}{2}\Big(y(\tau(t)) - y(\sigma(t^*))\Big) \overset{(8.102)}{=} \frac{M}{2} \int_{\tau(t)}^{\sigma(t^*)} p(s)y(\tau(s))\Delta s \\
&\geq \frac{M}{2} y(\tau(t^*)) \int_{\tau(t)}^{\sigma(t^*)} p(s)\Delta s \overset{(8.107)}{\geq} \frac{M^2}{4} y(\tau(t^*)),
\end{aligned}$$

which proves (8.105). □

Now we proceed to give some illustrative examples and applications.

Example 8.7.7. Clearly, if $\mathbb{T} = \mathbb{R}$, then we get

$$\sup_{\substack{\lambda>0 \\ -\lambda p \in \mathcal{R}^+}} \left\{ \lambda e_{-\lambda p}(t, \tau(t)) \right\} = \sup_{\lambda>0} \left\{ \lambda e^{-\lambda \int_{\tau(t)}^t p(s)ds} \right\} = \frac{1}{e \int_{\tau(t)}^t p(s)ds},$$

and hence Theorem 8.7.1 yields the well-known result cited at the beginning of this section as

$$\alpha < 1 \quad \Longleftrightarrow \quad \liminf_{t\to\infty} \int_{\tau(t)}^t p(s)ds > \frac{1}{e}.$$

We now consider a time scale of the form

$$\mathbb{T} = \{t_n : \ n \in \mathbb{Z}\}, \tag{8.109}$$

where $\{t_n\}$ is a strictly increasing sequence of real numbers such that $\mathbb{T}$ is closed. For such time scales we present the following results.

Corollary 8.7.8. *Consider a time scale as described in* (8.109)*. If*

$$y^\Delta(t) + p(t)y(\rho(t)) = 0 \quad \text{for} \quad t \in \mathbb{T}$$

has an eventually positive solution, then

$$\liminf_{t\to\infty} \{\mu(t)p(t)\} \le \frac{1}{4}.$$

Proof. We let $\tau(t) = \rho(t)$, find

$$\lambda e_{-\lambda p}(t, \tau(t)) = \lambda - \lambda^2 \mu(\tau(t))p(\tau(t)),$$

maximize, and apply Theorem 8.7.1. $\square$

Example 8.7.9. If

$$y(4t) = y(t) - \mu(t)p(t)y(t/4) \quad \text{for} \quad t \in \{4^n : \ n \in \mathbb{N}_0\}$$

has an eventually positive solution, then

$$\liminf_{n\to\infty} \{4^n p(4^n)\} \le \frac{1}{12}.$$

Corollary 8.7.10. *Consider a time scale as described in* (8.109)*. If*

$$y^\Delta(t) + p(t)y(\rho(\rho(t))) = 0 \quad \text{for} \quad t \in \mathbb{T}$$

has an eventually positive solution, then

$$\liminf_{t\to\infty} \frac{[N(t) + M(t)]\,[N(\sigma(t)) + M(t)]}{[N(t) + N(\sigma(t)) + M(t)]^3} \ge 1,$$

where

$$N(t) = \mu(t)p(t) \quad \text{and} \quad M(t) = \sqrt{[N(t)]^2 + [N(\sigma(t))]^2 - N(t)N(\sigma(t))}.$$

Proof. We let $\tau(t) = \rho(\rho(t))$, find

$$\lambda e_{-\lambda p}(t, \tau(t)) = \lambda \Big(1 - \lambda N(\rho(\rho(t)))\Big) \Big(1 - \lambda N(\rho(t))\Big),$$

maximize, and apply Theorem 8.7.1. $\square$

Example 8.7.11. Let $h > 0$. If

$$y(t+h) = y(t) - hp(t)y(t-2h) \quad \text{for} \quad t \in \{hn : n \in \mathbb{Z}\} = h\mathbb{Z}$$

has an eventually positive solution, then

$$\liminf_{\substack{t\to\infty \\ t\in h\mathbb{Z}}} \frac{\left[p(t) + \tilde{M}(t)\right]\left[p(t+h) + \tilde{M}(t)\right]}{\left[p(t) + p(t+h) + \tilde{M}(t)\right]^3} \geq h,$$

where

$$\tilde{M}(t) = \sqrt{[p(t)]^2 + [p(t+h)]^2 - p(t)p(t+h)}.$$

Theorem 8.7.12. *Consider a time scale as described in* (8.109). *Let $k \in \mathbb{N}$ and $\tau(t_n) = t_{n-k}$ for all $n \in \mathbb{Z}$. If* (8.102) *has an eventually positive solution, then*

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)\Delta s \leq \left(\frac{k}{k+1}\right)^{k+1}. \tag{8.110}$$

Proof. We assume that (8.110) does not hold and show $\alpha < 1$, which is a contradiction with Theorem 8.7.1. Note now that

$$\begin{aligned} \lambda e_{-\lambda p}(t_n, \tau(t_n)) &= \lambda \prod_{i=n-k}^{n-1} \Big(1 - \lambda\mu(t_i)p(t_i)\Big) \\ &\leq \lambda \left\{1 - \lambda \frac{\int_{\tau(t_n)}^{t_n} p(s)\Delta s}{k}\right\}^k \\ &= \lambda(1-\lambda S)^k, \end{aligned}$$

where we used the arithmetic-geometric inequality and put

$$S = \frac{1}{k}\int_{t_{n-k}}^{t_n} p(s)\Delta s = \frac{\sum_{i=n-k}^{n-1}(t_{i+1}-t_i)p(t_i)}{k}.$$

Now $f(\lambda) = \lambda(1-\lambda S)^k$ satisfies

$$f'(\lambda) = (1-\lambda S)^k - k\lambda S(1-\lambda S)^{k-1} = (1-\lambda S)^{k-1}\Big(1-(k+1)\lambda S\Big)$$

so that

$$f(\lambda) \leq f\left(\frac{1}{(k+1)S}\right) = \frac{1}{(k+1)S}\left(1 - \frac{1}{k+1}\right)^k = \frac{k^k}{S(k+1)^{k+1}}.$$

Hence $\alpha \leq \left(\frac{k}{k+1}\right)^{k+1} \frac{1}{\liminf_{t\to\infty}\int_{\tau(t)}^t p(s)\Delta s} < 1.$ □

Example 8.7.13. If we let $\mathbb{T} = \mathbb{Z}$ in Theorem 8.7.12, then we get the following result: Let $k \in \mathbb{N}$. If

$$y(n+1) = y(n) - p(n)y(n-k) \quad \text{for} \quad n \in \mathbb{Z}$$

has an eventually positive solution, then

$$\liminf_{n\to\infty} \sum_{i=n-k}^{n-1} p(i) \leq \left(\frac{k}{k+1}\right)^{k+1}.$$

Example 8.7.14. If $\mathbb{T} = \overline{q^{\mathbb{Z}}} := \left\{q^k : k \in \mathbb{Z}\right\} \cup \{0\}$ with $q > 1$ in Theorem 8.7.12, then we get the following result: Let $k \in \mathbb{N}$. If

$$y\left(q^{n+1}\right) = y\left(q^n\right) - (q-1)q^n p(q^n) y\left(q^{n-k}\right) \quad \text{for} \quad n \in \mathbb{Z}$$

has an eventually positive solution, then

$$\liminf_{n\to\infty} \sum_{i=n-k}^{n-1} q^i p(q^i) \le \frac{\left(\frac{k}{k+1}\right)^{k+1}}{q-1}.$$

For the remainder of this section we consider the equation

$$x^{\Delta}(t) + p(t)x(\tau(\sigma(t))) = 0, \tag{8.111}$$

where p and τ satisfy the same assumptions as before. Since $\lambda p \in \mathcal{R}^+$ for all $\lambda > 0$, clearly the quantity

$$\alpha^* := \liminf_{\substack{t\to\infty \\ t\in\mathbb{T}}} \inf_{\lambda>0} \left\{\frac{e_{\lambda p}(t, \tau(\sigma(t)))}{\lambda}\right\} \tag{8.112}$$

is well defined. Our main result about equation (8.111) reads as follows.

Theorem 8.7.15. *If* (8.111) *has an eventually positive solution, then* α^* *defined by* (8.112) *satisfies* $\alpha^* \le 1$.

Proof. Throughout we assume that x solves (8.111) and is eventually positive and that $\alpha^* > 1$. We proceed as in the proof of Theorem 8.7.1 and show that x satisfies

$$\liminf_{t\to\infty} \frac{x(\tau(\sigma(t)))}{x(t)} = \infty \tag{8.113}$$

and

$$\liminf_{t\to\infty} \frac{x(\tau(\sigma(t)))}{x(t)} < \infty. \tag{8.114}$$

This contradiction shows $\alpha^* \le 1$ and finishes the proof. We first show (8.113). Let $\beta^* \in (1, \alpha^*)$. Then there exists $T_0 \in \mathbb{T}$ such that

$$\inf_{\lambda>0} \left\{\frac{e_{\lambda p}(t, \tau(t))}{\lambda}\right\} \ge \beta^* \quad \text{for all} \quad t \ge T_0. \tag{8.115}$$

As x is eventually positive, it is eventually decreasing, and hence we conclude that $x(\tau(\sigma(t))) \ge x(\sigma(t))$ eventually so that

$$0 = x^{\Delta}(t) + p(t)x(\tau(\sigma(t))) \ge x^{\Delta}(t) + p(t)x(\sigma(t))$$

implies by Remark 8.7.4 that there exists $T_1 \ge T_0$ with

$$\frac{x(\tau(\sigma(t)))}{x(\sigma(t))} \ge e_p(t, \tau(\sigma(t)))\frac{x(t)}{x(\sigma(t))} \overset{(8.115)}{\ge} \beta^* \frac{x(t)}{x(\sigma(t))} \ge \beta^* \quad \text{for all} \quad t \ge T_1.$$

Therefore

$$0 = x^{\Delta}(t) + p(t)x(\tau(\sigma(t))) \ge x^{\Delta}(t) + \beta^* p(t)x(t),$$

implies again by Remark 8.7.4 that there exists $T_2 \ge T_1$ with

$$\begin{aligned}\frac{x(\tau(\sigma(t)))}{x(\sigma(t))} \quad &\ge \quad e_{\beta^* p}(t, \tau(\sigma(t)))\frac{x(t)}{x(\sigma(t))} \;=\; \beta^* \frac{e_{\beta^* p}(t, \tau(\sigma(t)))}{\beta^*}\frac{x(t)}{x(\sigma(t))} \\ &\overset{(8.115)}{\ge} \quad (\beta^*)^2 \frac{x(t)}{x(\sigma(t))} \;\ge\; (\beta^*)^2 \quad \text{for all} \quad t \ge T_2.\end{aligned}$$

Proceeding in a way similar as in the first part of the proof of Theorem 8.7.1, we obtain (8.113). Now we show (8.114). By Remark 8.7.5 (see also Remark 8.7.6) there exists $M > 0$ and $T \in \mathbb{T}$ such that

$$\int_{\tau(\sigma(t))}^{t} p(s)\Delta s \geq M \quad \text{for all} \quad t \geq T$$

so that

$$\int_{\tau(\sigma(t))}^{\sigma(t)} \geq \int_{\tau(\sigma(t))}^{t} p(s)\Delta s \geq M \quad \text{for all} \quad t \geq T$$

and hence

$$\int_{\tau(\sigma(t))}^{\sigma(t)} \geq M \quad \text{for all} \quad t \geq T \tag{8.116}$$

holds. As in the second part of the proof of Theorem 8.7.1 we find $t^* \in [\tau(\sigma(t)), t)$ such that

$$\int_{\tau(\sigma(t))}^{\sigma(t^*)} p(s)\Delta s \geq \frac{M}{2} \quad \text{and} \quad \int_{t^*}^{\sigma(t)} p(s)\Delta s \geq \frac{M}{2}. \tag{8.117}$$

Now we can estimate

$$\begin{aligned}
x(t^*) &\geq x(t^*) - x(\sigma(t)) \overset{(8.111)}{=} \int_{t^*}^{\sigma(t)} p(s)x(\tau(\sigma(s)))\Delta s \\
&\geq x(\tau(\sigma(t))) \int_{t^*}^{\sigma(t)} p(s)\Delta s \overset{(8.117)}{\geq} \frac{M}{2} x(\tau(\sigma(t))) \\
&\geq \Big(x(\tau(\sigma(t))) - x(\sigma(t^*))\Big) \overset{(8.111)}{=} \frac{M}{2} \int_{\tau(\sigma(t))}^{\sigma(t^*)} p(s)x(\tau(\sigma(s)))\Delta s \\
&\geq \frac{M}{2} x(\tau(\sigma(t^*))) \int_{\tau(\sigma(t))}^{\sigma(t^*)} p(s)\Delta s \overset{(8.117)}{\geq} \frac{M^2}{4} x(\tau(\sigma(t^*))),
\end{aligned}$$

i.e.,

$$x(t^*) \geq \frac{M^2}{4} x(\tau(\sigma(t^*))). \tag{8.118}$$

Clearly (8.118) implies (8.114). □

We can improve the condition from Theorem 8.7.15 by imposing an additional assumption. Define now

$$\tilde{\alpha} := \liminf_{\substack{t\to\infty \\ t\in\mathbb{T}}} \inf_{\lambda>0} \left\{ \frac{e_{\lambda p}(t, \tau(t))}{\lambda} \right\} \tag{8.119}$$

Theorem 8.7.16. *Assume that there exists $K > 0$ such that*

$$\int_{\tau(\sigma(t))}^{t} p(s)\Delta s \geq K \quad \textit{for all large} \quad t \in \mathbb{T}. \tag{8.120}$$

If (8.111) has an eventually positive solution, then $\tilde{\alpha}$ defined by (8.119) satisfies $\tilde{\alpha} \leq 1$.

Proof. We assume that x solves (8.111) and is eventually positive and that $\tilde{\alpha} < 1$. We proceed as in the proofs of Theorems 8.7.1 and 8.7.15 in two parts to show

$$\liminf_{t\to\infty} \frac{x(\tau(t))}{x(t)} = \infty \tag{8.121}$$

and

$$\liminf_{t\to\infty} \frac{x(\tau(t))}{x(t)} < \infty. \tag{8.122}$$

With the same notation as in the proof of the first part of Theorem 8.7.15, we find

$$\frac{x(\tau(t))}{x(\sigma(t))} \geq e_p(t, \tau(\sigma(t))) \overset{(8.115)}{\geq} \beta^* \quad \text{for all} \quad t \geq T_1$$

and may proceed as in the proof of the first part of Theorem 8.7.15 to reach (8.121). To show (8.122), note that there exists $M > 0$ and $T \in \mathbb{T}$ such that (8.116) holds. Therefore we can proceed with the same calculation as in the second part of Theorem 8.7.15 to obtain (8.118). Observe now the estimate

$$\begin{aligned} x(\tau(\sigma(t))) &\geq x(\tau(\sigma(t))) - x(t) \overset{(8.111)}{=} \int_{\tau(\sigma(t))}^{t} p(s)x(\tau(\sigma(s)))\Delta s \\ &\geq x(\tau(t)) \int_{\tau(\sigma(t))}^{t} p(s)\Delta s \overset{(8.120)}{\geq} Kx(\tau(t)) \end{aligned}$$

for large $t \in \mathbb{T}$, which combined with (8.118) yields (8.122). □

Example 8.7.17. For $\mathbb{T} = \mathbb{Z}$ and $\tau(t) = t - 2$ for $t \in \mathbb{Z}$, we have

$$\frac{e_{\lambda p}(t, \tau(t))}{\lambda} = \frac{[1 + \lambda p(t-2)]\,[1 + \lambda p(t-1)]}{\lambda},$$

which is minimized for

$$\left(\sqrt{p(t-2)} + \sqrt{p(t-1)}\right)^2.$$

Hence by Theorem 8.7.16,

$$\liminf_{n\to\infty} \left(\sqrt{p(n)} + \sqrt{p(n+1)}\right)^2 > 1$$

and there exists $K > 0$ with $p(n) \geq K$ for all large $n \in \mathbb{N}$, then

$$x(n+1) = x(n) - p(n)x(n-1) \quad \text{for} \quad n \in \mathbb{Z} \tag{8.123}$$

is oscillatory.

Example 8.7.18. For a more specific example of the kind as discussed in Example 8.7.17, consider (8.123) with

$$p(n) = \begin{cases} 1/8 & \text{for } n \text{ even} \\ 1/2 & \text{for } n \text{ odd.} \end{cases}$$

Here,

$$\liminf_{n\to\infty} p(n) = \frac{1}{8} < \frac{1}{4},$$

so the oscillation criterion from Corollary 8.7.8, i.e., the one known in the literature for difference equations, does not apply. However,

$$\liminf_{n\to\infty} \left(\sqrt{p(n)} + \sqrt{p(n+1)}\right) = \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{8}}\right) > 1,$$

so Example 8.7.17, i.e., Theorem 8.7.16 applies and shows that the equation (8.123) is oscillatory.

A similar discussion as in Example 8.7.17, applying Theorem 8.7.16 for $\mathbb{T} = \mathbb{Z}$, yields the statements given in the following last examples of this section. Of course similar examples (e.g., of the forms as presented earlier) can also be given for other time scales.

Example 8.7.19. Consider a difference equation of the form

$$x(n+1) = x(n) - p(n)x(n-2) \quad \text{for} \quad n \in \mathbb{Z}, \tag{8.124}$$

where p is three-periodic and takes values as follows:

$$p(1) = a, \quad p(2) = a, \quad p(3) = b, \quad \ldots \quad \text{with} \quad a, b > 0.$$

Then (8.124) is oscillatory provided

$$\left(\frac{3a+M}{a+M}\right)^2 \frac{a+M+2b}{2} > 1, \qquad \text{where} \qquad M = \sqrt{a^2+8ab}.$$

Example 8.7.20. Consider equation (8.124), where p is three-periodic and takes values as follows:

$$p(1) = a, \quad p(2) = b, \quad p(3) = c, \quad \ldots \quad \text{with} \quad a, b, c > 0.$$

Let

$$m = abc\left(\frac{3}{ab+ac+bc}\right)^{3/2}.$$

By the arithmetic-geometric inequality it can be shown that $0 < m \le 1$. Now define $\varphi \in [0, \pi/2)$ such that

$$m = \cos\varphi, \qquad \text{and also put} \qquad k = 2\cos\frac{\varphi}{3}.$$

Another way to calculate k is to use the formula

$$k = \left(m + i\sqrt{1-m^2}\right)^{1/3} + \left(m + i\sqrt{1-m^2}\right)^{-1/3}.$$

Then (8.124) is oscillatory provided

$$a+b+c+\frac{\sqrt{3(ab+ac+bc)}}{2}\left[k+\frac{1}{k}\right] > 1.$$

8.8. Oscillation of Symplectic Dynamic Systems

In this section we investigate oscillatory properties of a perturbed symplectic dynamic system on a time scale that is unbounded above. The unperturbed system is supposed to be nonoscillatory, and conditions on the perturbation matrix are given, which guarantee that the perturbed system becomes oscillatory. Examples illustrating the general results are given as well. The results presented in this section follow the recent paper [**42**].

We consider the symplectic dynamic system

$$z^\Delta = \mathcal{S}(t)z, \tag{8.125}$$

i.e., $\mathcal{S}$ is a symplectic and rd-continuous $2n \times 2n$ matrix-valued function, along with its perturbation

$$z^{\Delta} = (\mathcal{S}(t) + \tilde{\mathcal{S}}(t))z, \tag{8.126}$$

which is also supposed to be symplectic. Recall from [**11, 78**] that $\mathcal{S}$ is called *symplectic* (with respect to $\mathbb{T}$) if

$$\mathcal{S}^T(t)\mathcal{J} + \mathcal{J}\mathcal{S}(t) + \mu(t)\mathcal{S}^T(t)\mathcal{J}\mathcal{S}(t) = 0 \quad \text{for all} \quad t \in \mathbb{T}, \tag{8.127}$$

where $\mathbb{T}$ is the time scale under consideration, $\mathcal{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, and the superscript ^{T} stands for the transpose of the matrix indicated. Since we are concerned with the oscillatory behavior of the systems (8.125) and (8.126), we assume that $\mathbb{T}$ is unbounded above.

The results presented in this section are quite general as systems (8.125) contain a variety of important systems as special cases, e.g., linear Hamiltonian differential systems, linear Hamiltonian difference systems, Sturm–Liouville differential equations (of any order) Sturm–Liouville difference equations (of any order), self-adjoint matrix differential systems, self-adjoint matrix difference systems, and symplectic difference systems. Our oscillation criteria presented below are new even in many of theses special cases, as will be illustrated.

Example 8.8.1. In case $\mathbb{T} = \mathbb{R}$, symplectic (differential) systems (8.125) are of the form

$$z' = \mathcal{H}(t)z, \qquad \text{where} \qquad \mathcal{J}\mathcal{H} \text{ is symmetric, i.e., } \mathcal{J}\mathcal{H} = \mathcal{H}^T\mathcal{J}^T$$

(these are so-called linear Hamiltonian differential systems). In case $\mathbb{T} = \mathbb{Z}$, symplectic difference systems (8.125) are of the form

$$z(t+1) = \mathcal{S}(t)z(t), \qquad \text{where} \qquad \mathcal{S} \text{ is symplectic, i.e., } \mathcal{S}^T\mathcal{J}\mathcal{S} = \mathcal{J}.$$

To begin with we recall some basic facts concerning symplectic dynamic systems (8.125). As mentioned above, a symplectic dynamic system is a first order linear dynamic system whose coefficient matrix satisfies (8.127). This identity implies that the matrix $I + \mu(t)\mathcal{S}(t)$ is symplectic for each $t \in \mathbb{T}$, i.e.,

$$(I + \mu\mathcal{S})^T\mathcal{J}(I + \mu\mathcal{S}) = \mathcal{J}$$

holds on $\mathbb{T}$. This last identity is equivalent to $(I + \mu\mathcal{S})\mathcal{J}(I + \mu\mathcal{S})^T = \mathcal{J}$, so a symplectic dynamic system can be also characterized as a system (8.125) whose coefficient matrix satisfies

$$\mathcal{S}(t)\mathcal{J} + \mathcal{J}\mathcal{S}^T(t) + \mu(t)\mathcal{S}(t)\mathcal{J}\mathcal{S}^T(t) = 0 \quad \text{for all} \quad t \in \mathbb{T}. \tag{8.128}$$

If we write $\mathcal{S} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $n \times n$ matrix-valued functions A, B, C, and D, then (8.127) and (8.128) read

$$\begin{cases} C - C^T + \mu(A^TC - C^TA) = 0, & C - C^T + \mu(CD^T - DC^T) = 0, \\ B^T - B + \mu(B^TD - D^TB = 0), & B^T - B + \mu(AB^T - BA^T) = 0, \\ A^T + D + \mu(A^TD - C^TB) = 0, & A + D^T + \mu(AD^T - BC^T) = 0. \end{cases} \tag{8.129}$$

Next, if Z and $\tilde{Z}$ are two $2n \times n$ matrix-valued solutions of (8.125), then $Z^T \mathcal{J} \tilde{Z}$ is a constant $n \times n$ matrix (this is a so-called *Wronskian type identity*). A solution Z is said to be a *conjoined basis* if $\operatorname{rank} Z \equiv n$ and $Z^T \mathcal{J} Z \equiv 0$. Oscillatory properties of (8.125) are defined using the concept of focal points. A $2n \times n$ matrix-valued solution Z of (8.125) has *no focal point* in the interval $\mathcal{I} = (a, b] \subset \mathbb{T}$ if $X(t)$ is invertible at all dense points $t \in \mathcal{I}$ and if

$$\operatorname{Ker} X^\sigma(t) \subset \operatorname{Ker} X(t) \quad \text{and} \quad X(t)(X^\sigma(t))^\dagger B(t) \geq 0$$

on $\mathcal{I}^\kappa$ (here, $\dagger$ denotes the Moore–Penrose generalized inverse). The system (8.125) is called *disconjugate* on $\mathcal{I}$ if the solution $Z = \binom{X}{U}$ given by the initial condition $X(a) = 0$ and $U(a) = I$ (the so-called principal solution of (8.125) at a) has no focal points in $\mathcal{I}$. System (8.125) is called *nonoscillatory* if there exists $T \in \mathbb{T}$ such that it is disconjugate on $(T, T_1]$ for every $T_1 > T$, and it is said to be *oscillatory* in the opposite case.

In our treatment we will also need the concept of the principal and nonprincipal solution of (8.125) at ∞ as introduced in [**74**] and studied in [**43**]. System (8.125) is said to be *eventually controllable* if the trivial solution $z = \binom{x}{u} \equiv \binom{0}{0}$ is the only solution for which $x \equiv 0$ eventually. If (8.125) is eventually controllable and nonoscillatory, then the first component X of any conjoined basis $Z = \binom{X}{U}$ is eventually nonsingular, and for every $T \in \mathbb{T}$ there exists $t_1 > T$ such that the matrix

$$\int_T^t (X^\sigma)^{-1}(\tau) B(\tau) (X^T)^{-1}(\tau) \Delta\tau$$

is positive definite whenever $t > t_1$. Among all conjoined bases of an eventually controllable and nonoscillatory symplectic dynamic system one can distinguish the so-called *principal solution* at ∞, which is the conjoined basis $\tilde{Z} = \binom{\tilde{X}}{\tilde{U}}$ with the property that

$$\lim_{t \to \infty} X^{-1}(t) \tilde{X}(t) = 0 \tag{8.130}$$

for any conjoined basis $Z = \binom{X}{U}$ for which the (constant) matrix $Z^T \mathcal{J} \tilde{Z}$ is nonsingular. Any conjoined basis $Z = \binom{X}{U}$ for which $Z^T \mathcal{J} \tilde{Z}$ is a nonsingular matrix is called a *nonprincipal solution* at ∞. Note that the principal solution at ∞ is uniquely determined up to a right multiplicative constant nonsingular $n \times n$ matrix factor, and that (8.130) is equivalent to

$$\lim_{t \to \infty} \left(\int^t (X^\sigma)^{-1}(\tau) B(\tau) (X^T)^{-1}(\tau) \Delta\tau \right)^{-1} = 0. \tag{8.131}$$

When investigating oscillatory properties of (8.125), a fundamental rôle is played by the so-called Reid roundabout theorem, which relates oscillatory properties of (8.125) to solvability of a certain associated Riccati type equation and to positivity of the quadratic functional

$$\mathcal{F}(z; a, b) := \int_a^b z^T(\tau) \left\{ \mathcal{S}^T(\tau) \mathcal{K} + \mathcal{K} \mathcal{S}(\tau) + \mu(\tau) \mathcal{S}^T(\tau) \mathcal{K} \mathcal{S}(\tau) \right\} z(\tau) \Delta\tau$$

with $\mathcal{K} = \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}$, over the class of pairs $z = \binom{x}{u}$ such that $\mathcal{K}z^\Delta = \mathcal{K}\mathcal{S}(t)z$ and $x(a) = x(b) = 0$. This roundabout theorem for (8.125) is established in the recent paper [**128**]. Here we use only a part of this roundabout theorem, which is formulated in the next proposition (in a slightly modified form; compare with [**78**] or [**128**]).

Proposition 8.8.2. *Suppose that for every $T \in \mathbb{T}$ there exists a pair $z = \binom{x}{u}$ such that $x \in \mathrm{C}^1_{\mathrm{rd}}[T,\infty)$, $u \in \mathrm{C}_{\mathrm{rd}}[T,\infty)$ piecewise, $x^\Delta = A(t)x + B(t)u$, $\operatorname{supp} x \subset [T,\infty)$ (i.e., $x(T) = 0$ and there exists $T_1 > T$ such that $x(t) \equiv 0$ for $t > T_1$), and*

$$\mathcal{F}(z;T,\infty) = \int_T^\infty z^T(\tau)\left\{\mathcal{S}^T(\tau)\mathcal{K} + \mathcal{K}\mathcal{S}(\tau) + \mu(\tau)\mathcal{S}^T(\tau)\mathcal{K}\mathcal{S}(\tau)\right\} z(\tau)\Delta\tau < 0.$$

Then (8.125) *is oscillatory.*

Note also that for $\mathcal{S} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and $z = \binom{x}{u}$, the functional $\mathcal{F}(z;a,b)$ takes the form

$$\mathcal{F}(x,u;a,b) = \int_a^b \left\{ \begin{pmatrix} x \\ u \end{pmatrix}^T \begin{pmatrix} C^T(I+\mu A) & \mu C^T B \\ \mu B^T C & (I+\mu D)^T B \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \right\}(\tau)\Delta\tau.$$

Now we present a result concerning a certain transformation of (8.125); see [**78**]. Let $H, K : \mathbb{T} \to \mathbb{R}^{n\times n}$ be $\mathrm{C}^1_{\mathrm{rd}}$-matrices such that H is nonsingular and $H^T K = K^T H$, i.e., the matrix $\mathcal{R} = \begin{pmatrix} H & 0 \\ K & (H^T)^{-1} \end{pmatrix}$ is symplectic. Consider the transformation

$$z = \mathcal{R}\bar{z}$$

of the symplectic dynamic system (8.125). This transformation transforms (8.125) into the system

$$\bar{z}^\Delta = \bar{\mathcal{S}}(t)\bar{z} \quad \text{with} \quad \bar{\mathcal{S}} = \begin{pmatrix} \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{pmatrix}, \tag{8.132}$$

which is again symplectic, and the matrices $\bar{A}$, $\bar{B}$, $\bar{C}$, and $\bar{D}$ are given by the formulas

$$\begin{aligned} \bar{A} &= (H^\sigma)^{-1}(AH + BK - H^\Delta) \\ \bar{B} &= (H^\sigma)^{-1}B(H^T)^{-1} \\ \bar{C} &= (K^\sigma)^T(H^\Delta - AH - BK) - (H^\sigma)^T(K^\Delta - CH - DK) \\ \bar{D} &= (H^\Delta - D^T H^\sigma - B^T K^\sigma)^T (H^T)^{-1}. \end{aligned}$$

Consequently, if $\binom{X}{U}$ is a solution of (8.125) such that X is nonsingular, setting $H = X$ and $K = U$, we have $\bar{A} = 0$ and $\bar{C} = 0$ (this is obvious) and $\bar{D} = 0$ (this follows from the fact that (8.132) is again symplectic, i.e., (8.129) hold for $\bar{A}$, $\bar{B}$, $\bar{C}$, and $\bar{D}$).

In what follows we assume that the perturbation matrix $\tilde{\mathcal{S}}$ from (8.126) is of the form

$$\tilde{\mathcal{S}} = \begin{pmatrix} 0 & 0 \\ W(I+\mu A) & \mu W B \end{pmatrix}. \tag{8.133}$$

Let us briefly explain why we choose $\tilde{\mathcal{S}}$ of the form (8.133). First we require that the admissibility equation for the quadratic functional corresponding to (8.126), $\mathcal{K} z^{\Delta} = \mathcal{K}(\mathcal{S} + \tilde{\mathcal{S}})z$, is independent of $\tilde{\mathcal{S}}$, i.e., $\mathcal{K}\tilde{\mathcal{S}} z = 0$ and hence $\tilde{\mathcal{S}} = \begin{pmatrix} 0 & 0 \\ \hat{C} & \hat{D} \end{pmatrix}$. This requirement is perhaps not strictly necessary, but it is reasonable from the application point of view as we will see in the last section. Another requirement is that the perturbed system (8.126) is again a symplectic dynamic system, i.e., (8.127) and (8.128) must hold. This means that

$$\begin{cases} \hat{C}^T(I+\mu A) = (I+\mu A)^T \hat{C}, \\ (I+\mu A)\hat{C}^T = \mu B \hat{D}^T, \\ \mu \hat{D}^T B = \mu B^T \hat{D}, \\ \hat{C}(I+\mu D^T) - (I+\mu D)\hat{C}^T = \mu(\hat{D}C^T - C\hat{D}^T), \\ \hat{D}^T(I+\mu A) = \mu B^T \hat{C}. \end{cases} \tag{8.134}$$

If $\mu = 0$, then obviously $\hat{D} = 0$ and $\hat{C} = W$ is a symmetric matrix. Now suppose $\mu \neq 0$. Then the fact that $I + \mu\mathcal{S}$ is symplectic implies

$$(I+\mu A)(I+\mu D)^T - \mu^2 B C^T = I \quad \text{and} \quad \operatorname{rank}(I+\mu A,\ \mu B) = n.$$

Hence, since $(I+\mu A)\mu B^T = \mu B(I+\mu A)^T$,

$$\operatorname{Ker}(I+\mu A,\ \mu B) = \operatorname{Im}\begin{pmatrix} \mu B^T \\ -(I+\mu A)^T \end{pmatrix}.$$

Now, the second identity in (8.134) implies that

$$\begin{pmatrix} \hat{C} \\ -\hat{D} \end{pmatrix} \in \operatorname{Im}\begin{pmatrix} \mu B^T \\ -(I+\mu A)^T \end{pmatrix},$$

i.e., there exists an $n \times n$ matrix W such that

$$\hat{D}^T = \mu B^T W \quad \text{and} \quad \hat{C}^T = (I+\mu A)^T W.$$

Substituting this into (8.134), we find that W must be symmetric, and then all identities in (8.134) are satisfied.

Now we are ready to present our oscillation criteria for systems (8.126). We first give conditions that imply, assuming nonoscillation of (8.125), that the perturbed system (8.126) is oscillatory.

Theorem 8.8.3. *Suppose that* (8.125) *is nonoscillatory and eventually controllable, and let* $\binom{X}{U}$ *be its principal solution at* ∞. *If*

$$W(t) \leq 0 \quad \textit{for large} \quad t \in \mathbb{T} \tag{8.135}$$

and if there exists a pair $\binom{\tilde{x}}{\tilde{u}} : \mathbb{T} \to \mathbb{R}^{2n}$ *such that*

$$\tilde{x} \in \mathrm{C}^1_{\mathrm{rd}}, \quad \tilde{u} \in \mathrm{C}_{\mathrm{rd}}, \quad \tilde{x}^\Delta = (X^\sigma)^{-1} B (X^T)^{-1} \tilde{u}, \tag{8.136}$$

and

$$\int^\infty \left\{ \tilde{u}^T (X^\sigma)^{-1} B (X^T)^{-1} \tilde{u} + (\tilde{x}^\sigma)^T (X^\sigma)^T W X^\sigma \tilde{x}^\sigma \right\} (\tau) \Delta\tau = -\infty, \tag{8.137}$$

then (8.126) *is oscillatory.*

Proof. Consider the transformation $z = \mathcal{R}\tilde{z}$ of (8.126) with

$$\mathcal{R} = \begin{pmatrix} X & 0 \\ U & (X^T)^{-1} \end{pmatrix}.$$

This transformation preserves the oscillatory behavior of (8.126) and transforms (8.126) into

$$\tilde{z}^\Delta = \begin{pmatrix} 0 & \bar{B} \\ \overline{W} & \mu \overline{W} \bar{B} \end{pmatrix} \tilde{z}, \tag{8.138}$$

where

$$\bar{B} = (X^\sigma)^{-1} B (X^T)^{-1} \quad \text{and} \quad \overline{W} = (X^\sigma)^T W X^\sigma.$$

To prove that (8.138) is oscillatory (and hence that (8.126) is oscillatory), according to Proposition 8.8.2 it suffices to construct for every $T \in \mathbb{T}$ a pair $z = \binom{x}{u}$ such that $x^\Delta = \bar{B}u$, $x \in \mathrm{C}^1_{\mathrm{rd}}$, $u \in \mathrm{C}_{\mathrm{rd}}$ piecewise on $[T, \infty)$, $\operatorname{supp} x \subset [T, \infty)$, and

$$\tilde{\mathcal{F}}(x, u) < 0,$$

where

$$\begin{aligned} \tilde{\mathcal{F}}(x, u) &= \int_T^\infty \left\{ \begin{pmatrix} x \\ u \end{pmatrix}^T \begin{pmatrix} \overline{W} & \mu \overline{W} \bar{B} \\ \mu \bar{B}^T \overline{W} & \bar{B} + \mu^2 \bar{B}^T \overline{W} \bar{B} \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \right\} (\tau) \Delta\tau \\ &= \int_T^\infty \left\{ u^T \bar{B} u + (x + \mu \bar{B} u)^T \overline{W} (x + \mu \bar{B} u) \right\} (\tau) \Delta\tau \\ &= \int_T^\infty \left\{ u^T \bar{B} u + (x^\sigma)^T \overline{W} x^\sigma \right\} (\tau) \Delta\tau. \end{aligned}$$

Define the pair $\binom{x}{u}$ by

$$\begin{pmatrix} x \\ u \end{pmatrix} = \begin{cases} \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \text{if} \quad t \le T \\ \begin{pmatrix} x_1 \\ u_1 \end{pmatrix} & \text{if} \quad t \in [T, t_1] \\ \begin{pmatrix} \tilde{x} \\ \tilde{u} \end{pmatrix} & \text{if} \quad t \in [t_1, t_2] \\ \begin{pmatrix} x_2 \\ u_2 \end{pmatrix} & \text{if} \quad t \in [t_2, t_3] \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \text{if} \quad t \ge t_3, \end{cases}$$

where $T \in \mathbb{T}$ is arbitrary, $t_3 > t_2 > t_1 > T$ will be specified later, and $\binom{x_1}{u_1}$ and $\binom{x_2}{u_2}$ are solutions of $x^\Delta = \bar{B}u$ satisfying

$$x_1(T) = 0, \quad x_1(t_1) = \tilde{x}(t_1), \quad x_2(t_2) = \tilde{x}(t_2), \quad \text{and} \quad x_2(t_3) = 0,$$

i.e.,

$$\begin{aligned} x_1(t) &= \left(\int_T^t \bar{B}(\tau)\Delta\tau \right) \left(\int_T^{t_1} \bar{B}(\tau)\Delta\tau \right)^{-1} \tilde{x}(t_1), \\ u_1(t) &= \left(\int_T^{t_1} \bar{B}(\tau)\Delta\tau \right)^{-1} \tilde{x}(t_1), \\ x_2(t) &= \left(\int_t^{t_3} \bar{B}(\tau)\Delta\tau \right) \left(\int_{t_2}^{t_3} \bar{B}(\tau)\Delta\tau \right)^{-1} \tilde{x}(t_2), \\ u_2(t) &= -\left(\int_{t_2}^{t_3} \bar{B}(\tau)\Delta\tau \right)^{-1} \tilde{x}(t_2). \end{aligned}$$

Note that controllability of (8.125) implies that $\int_T^t \bar{B}(\tau)\Delta\tau$ is really invertible if t is sufficiently large. Then

$$\begin{aligned} \tilde{\mathcal{F}}(x, u) &= \int_T^{t_1} \left\{ u_1^T \bar{B} u_1 + (x_1^\sigma)^T \overline{W} x_1^\sigma \right\} (\tau)\Delta\tau \\ &\quad + \int_{t_1}^{t_2} \left\{ \tilde{u}^T \bar{B} \tilde{u} + (\tilde{x}^\sigma)^T \overline{W} \tilde{x}^\sigma \right\} (\tau)\Delta\tau \\ &\quad + \int_{t_2}^{t_3} \left\{ u_2^T \bar{B} u_2 + (x_2^\sigma)^T \overline{W} x_2^\sigma \right\} (\tau)\Delta\tau \\ &\le \tilde{x}^T(t_1) \left(\int_T^{t_1} \bar{B}(\tau)\Delta\tau \right)^{-1} \tilde{x}(t_1) + \tilde{x}^T(t_2) \left(\int_{t_2}^{t_3} \bar{B}(\tau)\Delta\tau \right) \tilde{x}(t_2) \\ &\quad + \int_{t_1}^{t_2} \left\{ \tilde{u}^T \bar{B} \tilde{u} + (\tilde{x}^\sigma)^T \overline{W} \tilde{x}^\sigma \right\} (\tau)\Delta\tau. \end{aligned}$$

Here we have used (8.135). Now, let $\varepsilon > 0$ be arbitrary and $t_1 > T$ be fixed. According to (8.137), $t_2 > t_1$ can be chosen in such a way that

$$\int_{t_1}^{t_2} \left\{ \tilde{u}^T \bar{B} \tilde{u} + (\tilde{x}^\sigma)^T \overline{W} \tilde{x}^\sigma \right\} (\tau)\Delta\tau < -\tilde{x}_1^T(t_1) \left(\int_T^{t_1} \bar{B}(\tau)\Delta\tau \right)^{-1} \tilde{x}_1(t_1) - \varepsilon.$$

Finally, since $\binom{X}{U}$ is the principal solution of (8.125), we have

$$\left(\int_{t_2}^{t} \bar{B}(\tau)\Delta\tau\right)^{-1} = \left(\int_{t_2}^{t} \left\{(X^\sigma)^{-1}B(X^T)^{-1}\right\}(\tau)\Delta\tau\right)^{-1} \to 0 \quad \text{as} \quad t \to \infty,$$

and hence t_3 can be chosen such that

$$\tilde{x}_2^T(t_2)\left(\int_{t_2}^{t_3} \bar{B}(\tau)\Delta\tau\right)^{-1} \tilde{x}_2(t_2) < \varepsilon.$$

Summarizing the previous computations we see that

$$\tilde{\mathcal{F}}(x,u) < 0 \quad \text{if } T < t_1 < t_2 < t_3 \text{ are chosen as above,}$$

and hence (8.138) is oscillatory. This means that (8.126) is oscillatory as well. □

Our next result offers another oscillation criterion for (8.126).

Theorem 8.8.4. *Suppose* (8.135) *and let* $\binom{X}{U}$ *and* $\binom{\tilde{x}}{\tilde{u}}$ *be as in Theorem 8.8.3; however, instead of* (8.137) *we assume that the integral*

$$\int^\infty \left\{\tilde{u}^T\bar{B}\tilde{u} + (\tilde{x}^\sigma)^T\overline{W}\tilde{x}^\sigma\right\}(\tau)\Delta\tau$$

is convergent. Moreover, we suppose that

$$\tilde{x}^T(t)\left(\int^t \bar{B}(\tau)\Delta\tau\right)^{-1}\tilde{x}(t) \to 0 \quad \text{as} \quad t\to\infty. \tag{8.139}$$

If

$$\limsup_{t\to\infty} \frac{\int_t^\infty \left\{\tilde{u}^T\bar{B}\tilde{u} + (\tilde{x}^\sigma)^T\overline{W}\tilde{x}^\sigma\right\}(\tau)\Delta\tau}{\tilde{x}^T(t)\left(\int^t \bar{B}(\tau)\Delta\tau\right)^{-1}\tilde{x}(t)} < -1, \tag{8.140}$$

then (8.126) *is oscillatory.*

Proof. First note that the lower limit of integration in the integral in the denominator of (8.140) is not important. Indeed, since

$$\left(\int^t \bar{B}(\tau)\Delta\tau\right)^{-1} \to 0 \quad \text{as} \quad t\to\infty$$

and (8.139) holds, we find for any $a, b \in \mathbb{T}$

$$\lim_{t\to\infty} \frac{\tilde{x}^T(t)\left(\int_a^t \bar{B}(\tau)\Delta\tau\right)^{-1}\tilde{x}(t)}{\tilde{x}^T(t)\left(\int_b^t \bar{B}(\tau)\Delta\tau\right)^{-1}\tilde{x}(t)} = 1.$$

We use the same $\binom{x}{u}$ as in the proof of Theorem 8.8.3. Using the computation given there, we have

$$\begin{aligned}
\tilde{\mathcal{F}}(x,u) &\le \tilde{x}^T(t_1)\left(\int_T^{t_1} \bar{B}(\tau)\Delta\tau\right)^{-1}\tilde{x}(t_1) + \tilde{x}^T(t_2)\left(\int_{t_2}^{t_3} \bar{B}(\tau)\Delta\tau\right)^{-1}\tilde{x}(t_2) \\
&\quad + \int_{t_1}^{t_2} \left\{\tilde{u}^T\bar{B}\tilde{u} + (\tilde{x}^\sigma)^T\overline{W}\tilde{x}^\sigma\right\}(\tau)\Delta\tau \\
&= \tilde{x}^T(t_1)\left(\int_T^{t_1} \bar{B}(\tau)\Delta\tau\right)^{-1}\tilde{x}(t_1)\left(1+\Gamma_1+\Gamma_2\right),
\end{aligned}$$

where

$$\Gamma_1 := \frac{\int_{t_1}^{t_2} \left\{ \tilde{u}^T \bar{B} \tilde{u} + (\tilde{x}^\sigma)^T \overline{W} \tilde{x}^\sigma \right\} (\tau) \Delta\tau}{\tilde{x}^T(t_1) \left(\int_T^{t_1} \bar{B}(\tau)\Delta\tau \right)^{-1} \tilde{x}(t_1)}$$

and

$$\Gamma_2 := \frac{\tilde{x}^T(t_2) \left(\int_{t_2}^{t_3} \bar{B}(\tau)\Delta\tau \right)^{-1} \tilde{x}(t_2)}{\tilde{x}^T(t_1) \left(\int_T^{t_1} \bar{B}(\tau)\Delta\tau \right)^{-1} \tilde{x}(t_1)}.$$

Now, let $\varepsilon > 0$ be such that the limit superior in (8.140) is less than $-1 - 3\varepsilon$. The point $t_1 > T$ is now chosen such that

$$\frac{\int_{t_1}^{\infty} \left\{ \tilde{u}^T \bar{B} \tilde{u} + (\tilde{x}^\sigma)^T \overline{W} \tilde{x}^\sigma \right\} (\tau) \Delta\tau}{\tilde{x}^T(t_1) \left(\int_T^{t_1} \bar{B}(\tau)\Delta\tau \right)^{-1} \tilde{x}(t_1)} < -1 - 2\varepsilon$$

and $t_2 > t_1$ such that

$$\Gamma_1 = \frac{\int_{t_1}^{t_2} \left\{ \tilde{u}^T \bar{B} \tilde{u} + (\tilde{x}^\sigma)^T \overline{W} \tilde{x}^\sigma \right\} (\tau) \Delta\tau}{\tilde{x}^T(t_1) \left(\int_T^{t_1} \bar{B}(\tau)\Delta\tau \right)^{-1} \tilde{x}(t_1)} < -1 - \varepsilon.$$

Finally, $t_3 > t_2$ we take such that

$$\Gamma_2 = \frac{\tilde{x}^T(t_2) \left(\int_{t_2}^{t_3} \bar{B}(\tau)\Delta\tau \right)^{-1} \tilde{x}(t_2)}{\tilde{x}^T(t_1) \left(\int_T^{t_1} \bar{B}(\tau)\Delta\tau \right)^{-1} \tilde{x}(t_1)} < \varepsilon.$$

This is possible since $\left(\int_{t_2}^{t} \bar{B}(\tau)\Delta\tau \right)^{-1} \to 0$ as $t \to \infty$. Altogether, for these values of $t_3 > t_2 > t_1 > T$ we have $\tilde{\mathcal{F}}(x, u) < 0$, and hence (8.126) is oscillatory. $\square$

If instead of the principal solution of (8.126) at ∞ we use its nonprincipal solution at ∞, then we get the following result.

Theorem 8.8.5. *Suppose that* (8.135) *holds, let* $\binom{X}{U}$ *be a nonprincipal solution of* (8.125) *at* ∞, *and let* $\binom{\tilde{x}}{\tilde{u}}$ *be as in Theorem 8.8.4. Moreover, we suppose that*

$$\lim_{t\to\infty} \tilde{x}^T(t) \left(\int_t^{\infty} \bar{B}(\tau)\Delta\tau \right)^{-1} \tilde{x}(t) = \infty. \tag{8.141}$$

If

$$\limsup_{t\to\infty} \frac{\int^{t} \left\{ \tilde{u}^T \bar{B} \tilde{u} + (\tilde{x}^\sigma)^T \overline{W} \tilde{x}^\sigma \right\} (\tau) \Delta\tau}{\tilde{x}^T(t) \left(\int_t^{\infty} \bar{B}(\tau)\Delta\tau \right)^{-1} \tilde{x}(t)} < -1, \tag{8.142}$$

then (8.126) *is oscillatory.*

Proof. First of all note that since $\binom{X}{U}$ is the nonprincipal solution of (8.126) at ∞, the matrix integral

$$\int^{\infty} \bar{B}(\tau)\Delta\tau = \int^{\infty} \left\{ (X^\sigma)^{-1} B (X^T)^{-1} \right\} (\tau) \Delta\tau$$

is really convergent [**74**]. We use again the computations from the proof of Theorem 8.8.3. For the pair $\binom{x}{u}$ defined in the proof of that theorem we have

$$\begin{aligned}\tilde{\mathcal{F}}(x,u) &\leq \tilde{x}^T(t_1)\left(\int_T^{t_1}\bar{B}(\tau)\Delta\tau\right)^{-1}\tilde{x}(t_1)+\tilde{x}^T(t_2)\left(\int_{t_2}^{t_3}\bar{B}(\tau)\Delta\tau\right)^{-1}\tilde{x}(t_2)\\ &\quad+\int_{t_1}^{t_2}\left\{\tilde{u}^T\bar{B}\tilde{u}+(\tilde{x}^\sigma)^T\overline{W}\tilde{x}^\sigma\right\}(\tau)\Delta\tau\\ &= \tilde{x}^T(t_2)\left(\int_{t_2}^{t_3}\bar{B}(\tau)\Delta\tau\right)^{-1}\tilde{x}(t_2)\left(1+\Gamma_3+\Gamma_4\right),\end{aligned}$$

where

$$\Gamma_3 := \frac{\int_{t_1}^{t_2}\left\{\tilde{u}^T\bar{B}\tilde{u}+(\tilde{x}^\sigma)^T\overline{W}\tilde{x}^\sigma\right\}(\tau)\Delta\tau}{\tilde{x}^T(t_2)\left(\int_{t_2}^{t_3}\bar{B}(\tau)\Delta\tau\right)^{-1}\tilde{x}(t_2)}$$

and

$$\Gamma_4 := \frac{\tilde{x}^T(t_1)\left(\int_T^{t_1}\bar{B}(\tau)\Delta\tau\right)^{-1}\tilde{x}(t_1)}{\tilde{x}^T(t_2)\left(\int_{t_2}^{t_3}\bar{B}(\tau)\Delta\tau\right)^{-1}\tilde{x}(t_2)}.$$

Let $t_1 > T$ be fixed and $\varepsilon > 0$ be such that the limit superior in (8.142) is less than $-1-3\varepsilon$. By (8.141) and (8.142), $t_2 > t_1$ can be chosen in such a way that

$$\frac{\int_{t_1}^{t_2}\left\{\tilde{u}^T\bar{B}\tilde{u}+(\tilde{x}^\sigma)^T\overline{W}\tilde{x}^\sigma\right\}(\tau)\Delta\tau}{\tilde{x}^T(t_2)\left(\int_{t_2}^{\infty}\bar{B}(\tau)\Delta\tau\right)^{-1}\tilde{x}(t_2)} < -1-2\varepsilon$$

and

$$\frac{\tilde{x}^T(t_1)\left(\int_T^{t_1}\bar{B}(\tau)\Delta\tau\right)^{-1}\tilde{x}(t_1)}{\tilde{x}^T(t_2)\left(\int_{t_2}^{\infty}\bar{B}(\tau)\Delta\tau\right)^{-1}\tilde{x}(t_2)} < \varepsilon.$$

Finally, we take $t_3 > t_2$ such that

$$\Gamma_3 = \frac{\int_{t_1}^{t_2}\left\{\tilde{u}^T\bar{B}\tilde{u}+(\tilde{x}^\sigma)^T\overline{W}\tilde{x}^\sigma\right\}(\tau)\Delta\tau}{\tilde{x}^T(t_2)\left(\int_{t_2}^{t_3}\bar{B}(\tau)\Delta\tau\right)^{-1}\tilde{x}(t_2)} < -1-\varepsilon$$

and also

$$\Gamma_4 = \frac{\tilde{x}^T(t_1)\left(\int_T^{t_1}\bar{B}(\tau)\Delta\tau\right)^{-1}\tilde{x}(t_1)}{\tilde{x}^T(t_2)\left(\int_{t_2}^{t_3}\bar{B}(\tau)\Delta\tau\right)^{-1}\tilde{x}(t_2)} < \varepsilon.$$

Consequently, for these $t_3 > t_2 > t_1 > T$ we have $\tilde{\mathcal{F}}(x,u) < 0$, and hence (8.126) is oscillatory. □

In the remainder of this section we present some corollaries and examples for applications of our general oscillation criteria given above.

(i) The formulation of Theorems 8.8.3, 8.8.4, and 8.8.5 simplifies if the pair $\binom{\tilde{x}}{\tilde{u}}$ appearing in these theorems is of the form $\binom{v}{0}$, where $v \in \mathbb{R}^n$ is a constant vector. We formulate this simplification only for Theorems 8.8.3 and 8.8.4. Theorem 8.8.5 simplifies accordingly.

Corollary 8.8.6. *Suppose that* (8.135) *holds and let* $\binom{X}{U}$ *be as in Theorems 8.8.3 and 8.8.4. If there exists* $v \in \mathbb{R}^n$ *such that*

$$\int^{\infty} v^T \overline{W}(\tau) v \Delta\tau = -\infty$$

or

$$\int^{\infty} v^T \overline{W}(\tau) v \Delta\tau > -\infty \quad \text{and} \quad \limsup_{t\to\infty} \frac{\int_t^\infty v^T \overline{W}(\tau) v \Delta\tau}{v^T \left(\int^t \bar{B}(\tau)\Delta\tau\right)^{-1} v} < -1,$$

then (8.126) *is oscillatory.*

Proof. The statement follows immediately from Theorems 8.8.3 and 8.8.4 taking into account that (8.139) is satisfied for $\tilde{x}(t) = v$ due to the fact that $\binom{X}{U}$ is the principal solution of (8.125). ☐

(ii) Here we consider the case $\mathbb{T} = \mathbb{R}$, i.e., $\mu \equiv 0$. In this case (8.126) is the linear Hamiltonian system

$$\binom{x}{u}' = \begin{pmatrix} A & B \\ C + W & -A^T \end{pmatrix} \binom{x}{u} \tag{8.143}$$

with symmetric matrices B, C, and W. Oscillatory properties of (8.143) in case $A \equiv 0$ (using the variational method presented above) were investigated in [**75**]. In that paper only the possibility $\binom{\tilde{x}}{\tilde{u}} = \binom{v}{0}$ with a constant vector $v \in \mathbb{R}^n$ was considered, so the results presented are new even for linear Hamiltonian differential systems.

(iii) The higher order Sturm–Liouville differential equation

$$L(y) := \sum_{\nu=0}^{n} (-1)^\nu \left(r_\nu(t) y^{(\nu)}\right)^{(\nu)} = 0$$

with $r_n(t) > 0$ can be written (using a suitable substitution) as the linear Hamiltonian system

$$x' = A(t)x + B(t)u, \quad u' = C(t)x - A^T(t)u$$

with

$$A(t) = (a_{ij})_{1\le i,j\le n}, \quad \text{where} \quad a_{ij} = \begin{cases} 1 & \text{if} \quad j = i+1,\ 1 \le i \le n-1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$B(t) = \operatorname{diag}\left\{0, 0, \dots, 0, \frac{1}{r_n(t)}\right\} \quad \text{and} \quad C(t) = \operatorname{diag}\left\{r_0(t), \dots, r_{n-1}(t)\right\}.$$

Oscillatory properties (with applications in spectral theory of differential operators) of the equation

$$L(y) + q(t)y = 0 \tag{8.144}$$

viewed as a perturbation of the nonoscillatory equation $L(y) = 0$ were investigated in several recent papers, see e.g., [**76, 127**] and the references given therein. Writing equation (8.144) as a linear Hamiltonian system (8.143), the perturbation matrix W is of the special form

$$W = \operatorname{diag}\left\{0, 0, \ldots, 0, q\right\}.$$

Using our method, one can investigate oscillatory properties of the equation

$$L(y) + M(y) = 0, \qquad \text{where} \qquad M(y) = \sum_{\nu=0}^{m} (-1)^\nu \left(q_\nu(t) y^{(\nu)}\right)^{(\nu)}$$

with $q_m(t) > 0$ and $m < n$. In this case the perturbation matrix W is

$$W = \operatorname{diag}\left\{q_0, \ldots, q_m, 0, 0, \ldots, 0\right\}.$$

If the operator M is of higher or equal order than L, i.e., $m \geq n$, then this perturbation does not fit into our setting. However, in applications, the perturbation operator is usually of lower order than the original one, and it is also a partial justification why the perturbation matrix $\tilde{S}$ is of the form as considered here.

As an example of the application of this general idea to fourth order differential equations we give the following oscillation criterion.

Corollary 8.8.7. *Consider the fourth order differential equation*

$$y'''' - (q_1(t) y')' + q_0(t) y = 0 \tag{8.145}$$

with $q_1(t) \leq 0$ *and* $q_0(t) \leq \frac{9}{16t^4}$ *eventually. If there exist* $c_1, c_2 \in \mathbb{R}$ *such that*

$$\int^\infty \left\{q_1(\tau)(h'(\tau))^2 + \left(q_0(\tau) - \frac{9}{16\tau^4}\right) h^2(\tau)\right\} d\tau = -\infty, \tag{8.146}$$

where $h(t) = c_1 t^{(3-\sqrt{10})/2} + c_2 t^{3/2}$, *then* (8.145) *is oscillatory.*

Proof. As the "unperturbed" nonoscillatory equation we take the fourth order Euler equation

$$y'''' - \frac{9}{16t^4} y = 0. \tag{8.147}$$

Equation (8.147) has solutions

$$y_1(t) = t^{(3-\sqrt{10})/2},\ y_2(t) = t^{3/2},\ \tilde{y}_1(t) = t^{3/2} \ln t,\ \tilde{y}_2(t) = t^{(3+\sqrt{10})/2}.$$

By a direct computation one can verify that

$$X = \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}, \qquad U = \begin{pmatrix} -y_1''' & -y_2''' \\ y_1'' & y_2'' \end{pmatrix}$$

is the principal solution of the linear Hamiltonian system corresponding to (8.147), and (8.145) can be written as a system (8.143) with

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} \frac{9}{16t^4} & 0 \\ 0 & 0 \end{pmatrix},$$

and

$$W(t) = \operatorname{diag}\left\{q_1(t), q_0(t) - \frac{9}{16t^4}\right\}.$$

We take $\tilde{x}(t) = c$ and $\tilde{u}(t) = 0$ with $c = \binom{c_1}{c_2}$ and apply Theorem 8.8.3. Then

$$\tilde{x}^T W \tilde{x} = c^T X^T W X c = (c_1 y_1' + c_2 y_2')^2 q_1 + (c_1 y_1 + c_2 y_2)^2 \left(q_0 - \frac{9}{16t^4} \right),$$

and (8.137) reduces to (8.146). □

Note that for the sake of simplicity in the previous corollary we used Theorem 8.8.3 (with the special choice $\tilde{x} = c$ and $\tilde{u} = 0$). Computing explicitly the expressions

$$c^T \left(\int^t \left\{ X^{-1} B (X^T)^{-1} \right\} (\tau) \right)^{-1} c \quad \text{and} \quad c^T \left(\int_t^\infty \left\{ \tilde{X}^{-1} B (\tilde{X}^T)^{-1} \right\} (\tau) \right)^{-1} c$$

with

$$B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \tilde{X} = \begin{pmatrix} \tilde{y}_1 & \tilde{y}_2 \\ \tilde{y}_1' & \tilde{y}_2' \end{pmatrix}$$

(the functions $\tilde{y}_1$ and $\tilde{y}_2$ are given in the previous proof), one can formulate also oscillation criteria which are special cases of Theorems 8.8.4 and 8.8.5. These reformulations yield new results even for the special equation (8.145).

(iv) Now we deal with the discrete case $\mathbb{T} = \mathbb{Z}$. In this case, (8.126) reduces to the symplectic difference system

$$z_{k+1} = (I + \mathcal{S}_k) z_k. \tag{8.148}$$

Basic properties of solutions of (8.148) (e.g., the Reid roundabout theorem) have been established in [**33, 40**]. However, oscillation criteria for general symplectic difference systems (in terms of the coefficient matrices $I + A$, B, C, and $I + D$ of (8.148)) have not been established yet; so the results of Theorems 8.8.3, 8.8.4, and 8.8.5 are new also for systems (8.148). We refer here to the papers [**44, 62, 77, 90, 127, 238**] and the references contained therein, where oscillatory properties of special cases of (8.148) like discrete Hamiltonian systems or higher order Sturm–Liouville difference equations are investigated.

8.9. Notes

Most of the preliminary results given in Section 8.2 are from the books by Bohner and Peterson [**53, 55**]. The reader may also refer to Hilger's original paper [**124, 125**]. Other interesting references concerning the time scales calculus contain [**4, 5, 6, 7, 12, 13, 14, 15, 16, 17, 18, 19, 20, 26, 27, 32, 38, 39, 45, 47, 48, 49, 50, 51, 52, 54, 56, 69, 85, 86, 87, 88, 125, 126, 143**]. The results from Section 8.3 and Section 8.4 are adopted from Bohner and Saker [**58**] and [**57**], respectively. For Section 8.5, see Huang and Li [**197**]. Results from the two papers by Akın-Bohner and Hoffacker [**21, 22**] are presented in Section 8.6. Results related to Section 8.7 can be found in [**37, 299**]. Finally, the contents of Section 8.8 is taken from Bohner and Došlý [**42**].

Bibliography

[1] Ravi Agarwal, Martin Bohner, Donal O'Regan, and Allan Peterson. Dynamic equations on time scales: a survey. *J. Comput. Appl. Math.*, 141(1-2):1–26, 2002. Dynamic equations on time scales.

[2] Ravi P. Agarwal. *Difference Equations and Inequalities*. Marcel Dekker, Inc., New York, 1992.

[3] Ravi P. Agarwal, Elvan Akın-Bohner, and Said R. Grace. On the oscillation of higher order neutral difference equations of mixed type. *Dynam. Systems Appl.*, 11(4):459–469, 2002.

[4] Ravi P. Agarwal and Martin Bohner. Quadratic functionals for second order matrix equations on time scales. *Nonlinear Anal.*, 33(7):675–692, 1998.

[5] Ravi P. Agarwal and Martin Bohner. Basic calculus on time scales and some of its applications. *Results Math.*, 35(1-2):3–22, 1999.

[6] Ravi P. Agarwal, Martin Bohner, and Donal O'Regan. Time scale boundary value problems on infinite intervals. *J. Comput. Appl. Math.*, 141(1-2):27–34, 2002. Special Issue on "Dynamic Equations on Time Scales", edited by R. P. Agarwal, M. Bohner, and D. O'Regan.

[7] Ravi P. Agarwal, Martin Bohner, and Patricia J. Y. Wong. Sturm–Liouville eigenvalue problems on time scales. *Appl. Math. Comput.*, 99(2-3):153–166, 1999.

[8] Ravi P. Agarwal, Said R. Grace, and Donal O'Regan. *Oscillation Theory for Difference and Functional Differential Equations*. Kluwer Academic Publishers, Dordrecht, 2000.

[9] Ravi P. Agarwal, Maria Meehan, and Donal O'Regan. *Fixed Point Theory and Applications*, volume 141 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2001.

[10] Ravi P. Agarwal, X. H. Tang, and Z. C. Wang. The existence of positive solutions to neutral differential equations. *J. Math. Anal. Appl.*, 240(2):446–467, 1999.

[11] Calvin D. Ahlbrandt, Martin Bohner, and Jerry Ridenhour. Hamiltonian systems on time scales. *J. Math. Anal. Appl.*, 250(2):561–578, 2000.

[12] Calvin D. Ahlbrandt, Martin Bohner, and Tammy Voepel. Variable change for Sturm–Liouville differential expressions on time scales. *J. Differ. Equations Appl.*, 9(1):93–107, 2003.

[13] Elvan Akın. Boundary value problems for a differential equation on a measure chain. *Panamer. Math. J.*, 10(3):17–30, 2000.

[14] Elvan Akın. Cauchy functions for dynamic equations on a measure chain. *J. Math. Anal. Appl.*, 267(1):97–115, 2002.

[15] Elvan Akın, Martin Bohner, Lynn Erbe, and Allan Peterson. Existence of bounded solutions for second order dynamic equations. *J. Difference Equ. Appl.*, 8(4):389–401, 2002. In honor of Professor Lynn Erbe.

[16] Elvan Akın, Lynn Erbe, Allan Peterson, and Billûr Kaymakçalan. Oscillation results for a dynamic equation on a time scale. *J. Differ. Equations Appl.*, 7(6):793–810, 2001. On the occasion of the 60th birthday of Calvin Ahlbrandt.

[17] Elvan Akın-Bohner and Ferhan M. Atıcı. A quasilinearization approach for two point nonlinear boundary value problems on time scales. *Rocky Mountain J. Math.*, 2004. In press.

[18] Elvan Akın-Bohner, Ferhan M. Atıcı, and Billûr Kaymakçalan. Chapter 6: Lower and upper solutions of boundary value problems. In M. Bohner and A. Peterson, editors, *Advances in Dynamic Equations on Time Scales*, pages 165–188. Birkhäuser, Boston, 2003.

[19] Elvan Akın-Bohner and Martin Bohner. Miscellaneous dynamic equations. *Methods Appl. Anal.*, 10(1):11–30, 2003.

[20] Elvan Akın-Bohner and Martin Bohner. Exponential functions and Laplace transforms for alpha derivatives. In B. Aulbach, S. Elaydi, and G. Ladas, editors, *Conference Proceedings*

of the Sixth International Conference on Difference Equations, Augsburg, 2004. Taylor and Francis. In press.

[21] Elvan Akın-Bohner and Joan Hoffacker. Oscillation properties of an Emden–Fowler type equation on discrete time scales. *J. Differ. Equations Appl.*, 9(6):603–612, 2003.

[22] Elvan Akın-Bohner and Joan Hoffacker. Solution properties on discrete time scales. *J. Differ. Equations Appl.*, 9(1):63–75, 2003.

[23] D. C. Angelova and Drumi D. Baĭnov. On the oscillation of solutions to second order functional-differential equations. *Math. Rep. Toyama Univ.*, 5:1–13, 1982.

[24] D. C. Angelova and Drumi D. Baĭnov. On oscillation of solutions of forced functional-differential equations of second order. *Math. Nachr.*, 122:289–300, 1985.

[25] O. Arino and István Győri. Necessary and sufficient condition for oscillation of a neutral differential system with several delays. *J. Differential Equations*, 81(1):98–105, 1989.

[26] Bernd Aulbach and Stefan Hilger. Linear dynamic processes with inhomogeneous time scale. In *Nonlinear dynamics and quantum dynamical systems (Gaussig, 1990)*, volume 59 of *Math. Res.*, pages 9–20. Akademie-Verlag, Berlin, 1990.

[27] Bernd Aulbach and Stefan Hilger. A unified approach to continuous and discrete dynamics. In *Qualitative theory of differential equations (Szeged, 1988)*, volume 53 of *Colloq. Math. Soc. János Bolyai*, pages 37–56. North-Holland, Amsterdam, 1990.

[28] Drumi D. Baĭnov, N. T. Markova, and P. S. Simeonov. Asymptotic behaviour of the nonoscillatory solutions of differential equations of second order with delay depending on the unknown function. *J. Comput. Appl. Math.*, 91(1):87–96, 1998.

[29] Drumi D. Baĭnov and D. P. Mishev. *Oscillation Theory for Neutral Differential Equations with Delay.* Adam Hilger Ltd., Bristol, 1991.

[30] Edwin F. Beckenbach and Richard Bellman. *Inequalities.* Springer-Verlag, New York, Inc., 1965.

[31] S. J. Bilchev, Myron K. Grammatikopoulos, and I. P. Stavroulakis. Oscillations of higher order neutral differential equations. *J. Austral. Math. Soc. Ser. A*, 52(2):261–284, 1992.

[32] Sigrun Bodine, Martin Bohner, and Donald A. Lutz. Asymptotic behavior of solutions of dynamic equations. In *Sovremennye problemy matematiki. Fundamental′nye napravleniya*, pages 30–39. Akad. Nauk SSSR Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 2003. In Russian. Translation in J. Math. Sci. (New York).

[33] Martin Bohner. Linear Hamiltonian difference systems: disconjugacy and Jacobi-type conditions. *J. Math. Anal. Appl.*, 199(3):804–826, 1996.

[34] Martin Bohner. On disconjugacy for Sturm–Liouville difference equations. *J. Differ. Equations Appl.*, 2(2):227–237, 1996. Difference equations: theory and applications (San Francisco, CA, 1995).

[35] Martin Bohner. An oscillation theorem for a Sturm–Liouville eigenvalue problem. *Math. Nachr.*, 182:67–72, 1996.

[36] Martin Bohner. Asymptotic behavior of discretized Sturm–Liouville eigenvalue problems. *J. Differ. Equations Appl.*, 3(3-4):289–295, 1998.

[37] Martin Bohner. Oscillation criteria for delay dynamic equations. 2003. Submitted.

[38] Martin Bohner and José Castillo. Mimetic methods on measure chains. *Comput. Math. Appl.*, 42(3-5):705–710, 2001. Advances in Difference Equations, III.

[39] Martin Bohner, Steven Clark, and Jerry Ridenhour. Lyapunov inequalities for time scales. *J. Inequal. Appl.*, 7(1):61–77, 2002.

[40] Martin Bohner and Ondřej Došlý. Disconjugacy and transformations for symplectic systems. *Rocky Mountain J. Math.*, 27(3):707–743, 1997.

[41] Martin Bohner and Ondřej Došlý. Trigonometric systems in oscillation theory of difference equations. In *Dynamic systems and applications, Vol. 3 (Atlanta, GA, 1999)*, pages 99–104. Dynamic, Atlanta, GA, 2001.

[42] Martin Bohner and Ondřej Došlý. Oscillation of symplectic dynamic systems. *ANZIAM J.*, 2004. In press.

[43] Martin Bohner, Ondřej Došlý, and Roman Hilscher. Linear Hamiltonian dynamic systems on time scales: Sturmian property of the principal solution. *Nonlinear Anal.*, 47:849–860, 2001.

[44] Martin Bohner, Ondřej Došlý, and Werner Kratz. An oscillation theorem for discrete eigenvalue problems. *Rocky Mountain J. Math.*, 2004. In press.

[45] Martin Bohner and Paul W. Eloe. Higher order dynamic equations on measure chains: Wronskians, disconjugacy, and interpolating families of functions. *J. Math. Anal. Appl.*, 246(2):639–656, 2000.
[46] Martin Bohner and Gusein Sh. Guseinov. Chapter 5: Riemann and Lebesgue integration. In M. Bohner and A. Peterson, editors, *Advances in Dynamic Equations on Time Scales*, pages 117–163. Birkhäuser, Boston, 2003.
[47] Martin Bohner and Gusein Sh. Guseinov. Improper integrals on time scales. *Dynam. Systems Appl.*, 12(1-2):45–66, 2003.
[48] Martin Bohner, Gusein Sh. Guseinov, and Allan Peterson. Chapter 1: Introduction to the time scales calculus. In M. Bohner and A. Peterson, editors, *Advances in Dynamic Equations on Time Scales*, pages 1–15. Birkhäuser, Boston, 2003.
[49] Martin Bohner and Roger Hering. Perturbations of dynamic equations. *J. Difference Equ. Appl.*, 8(4):295–305, 2002. In honor of Professor Lynn Erbe.
[50] Martin Bohner and Billûr Kaymakçalan. Opial inequalities on time scales. *Ann. Polon. Math.*, 77(1):11–20, 2001.
[51] Martin Bohner and V. Lakshmikantham. Formulas of Bendixson and Alekseev for difference equations. *Bull. London Math. Soc.*, 36(1):65–71, 2004.
[52] Martin Bohner and Donald A. Lutz. Asymptotic behavior of dynamic equations on time scales. *J. Differ. Equations Appl.*, 7(1):21–50, 2001. Special issue in memory of W. A. Harris, Jr.
[53] Martin Bohner and Allan Peterson. *Dynamic Equations on Time Scales: An Introduction with Applications.* Birkhäuser Boston Inc., Boston, MA, 2001.
[54] Martin Bohner and Allan Peterson. First and second order linear dynamic equations on time scales. *J. Differ. Equations Appl.*, 7(6):767–792, 2001. On the occasion of the 60th birthday of Calvin Ahlbrandt.
[55] Martin Bohner and Allan Peterson. *Advances in Dynamic Equations on Time Scales.* Birkhäuser Boston Inc., Boston, MA, 2003.
[56] Martin Bohner and Allan C. Peterson. Laplace transform and Z-transform: unification and extension. *Methods Appl. Anal.*, 9(1):151–157, 2002.
[57] Martin Bohner and Samir H. Saker. Oscillation criteria for perturbed nonlinear dynamic equations. 2004. Submitted.
[58] Martin Bohner and Samir H. Saker. Oscillation of second order nonlinear dynamic equations on time scales. *Rocky Mountain J. Math.*, 2004. In press.
[59] G. J. Butler. The existence of continuable solutions of a second order differential equation. *Canad. J. Math.*, 29(3):472–479, 1977.
[60] Ming Po Ch'ên, Jian She Yu, and L. H. Huang. Oscillations of first order neutral differential equations with variable coefficients. *J. Math. Anal. Appl.*, 185(2):288–301, 1994.
[61] Ming Po Ch'ên, Jian She Yu, and Z. C. Wang. Nonoscillatory solutions of neutral delay differential equations. *Bull. Austral. Math. Soc.*, 48(3):475–483, 1993.
[62] S. Chen and Lynn Erbe. Oscillation results for second order scalar and matrix difference equations. *Comput. Math. Appl.*, 28:55–69, 1994.
[63] Yong Shao Chen. Existence of nonoscillatory solutions of nth order neutral delay differential equations. *Funkcial. Ekvac.*, 35(3):557–570, 1992.
[64] Sui Sun Cheng and Bing-Gen Zhang. Monotone solutions of a class of nonlinear difference equations. *Comput. Math. Appl.*, 28(1-3):71–79, 1994. Advances in difference equations.
[65] Qian Chuanxi, Mustafa R. S. Kulenović, and Gerasimos Ladas. Oscillations of neutral equations with variable coefficients. *Rad. Mat.*, 5(2):321–331, 1989.
[66] Qian Chuanxi and Gerasimos Ladas. Oscillations of neutral differential equations with variable coefficients. *Appl. Anal.*, 32(3-4):215–228, 1989.
[67] Qian Chuanxi, Gerasimos Ladas, Bing-Gen Zhang, and T. Zhao. Sufficient conditions for oscillation and existence of positive solutions. *Appl. Anal.*, 35(1-4):187–194, 1990.
[68] Pitambar Das. Oscillation criteria for odd order neutral equations. *J. Math. Anal. Appl.*, 188(1):245–257, 1994.
[69] John M. Davis, Johnny Henderson, and K. Rajendra Prasad. Upper and lower bounds for the solution of the general matrix Riccati differential equation on a time scale. *J. Comput. Appl. Math.*, 141(1-2):133–145, 2002. Special Issue on "Dynamic Equations on Time Scales", edited by R. P. Agarwal, M. Bohner, and D. O'Regan.

[70] Manuel del Pino, Manuel Elgueta, and Raúl Manásevich. Generalizing Hartman's oscillation result for $(|x'|^{p-2}x')' + c(t)|x|^{p-2}x = 0$, $p > 1$. *Houston J. Math.*, 17(1):63–70, 1991.
[71] Yuri Domshlak and I. P. Stavroulakis. Oscillations of first-order delay differential equations in a critical state. *Appl. Anal.*, 61(3-4):359–371, 1996.
[72] Yuri Domshlak and I. P. Stavroulakis. Oscillations of differential equations with deviating arguments in a critical state. *Dynam. Systems Appl.*, 7(3):405–414, 1998.
[73] Ondřej Došlý and Stefan Hilger. A necessary and sufficient condition for oscillation of the Sturm–Liouville dynamic equation on time scales. *J. Comput. Appl. Math.*, 141(1-2):147–158, 2002. Dynamic equations on time scales.
[74] Ondřej Došlý. Principal and nonprincipal solutions of symplectic dynamic systems on time scales. In *Proceedings of the 6th Colloquium on the Qualitative Theory of Differential Equations (Szeged, 1999)*, pages No. 5, 14 pp. (electronic). Electron. J. Qual. Theory Differ. Equ., Szeged, 2000.
[75] Ondřej Došlý. On the existence of conjugate points for linear differential systems. *Math. Slovaca*, 40:87–99, 1990.
[76] Ondřej Došlý. Oscillation and spectral properties of a class of singular differential operators. *Math. Nachr.*, 188:49–68, 1997.
[77] Ondřej Došlý. Oscillation criteria for higher order Sturm–Liouville difference equations. *J. Differ. Equations Appl.*, 4:425–450, 1998.
[78] Ondřej Došlý and Roman Hilscher. Disconjugacy, transformations and quadratic functionals for symplectic dynamic systems on time scales. *J. Differ. Equations Appl.*, 7:265–295, 2001.
[79] M. A. El-Sayed. An oscillation criterion for a forced second order linear differential equation. *Proc. Amer. Math. Soc.*, 118(3):813–817, 1993.
[80] Árpád Elbert. A half-linear second order differential equation. In *Qualitative theory of differential equations, Vol. I, II (Szeged, 1979)*, pages 153–180. North-Holland, Amsterdam, 1981.
[81] Árpád Elbert. Oscillation and nonoscillation theorems for some nonlinear ordinary differential equations. In *Ordinary and partial differential equations (Dundee, 1982)*, pages 187–212. Springer, Berlin, 1982.
[82] Árpád Elbert and I. P. Stavroulakis. Oscillations of first order differential equations with deviating arguments. In *Recent trends in differential equations*, pages 163–178. World Sci. Publishing, River Edge, NJ, 1992.
[83] Árpád Elbert and I. P. Stavroulakis. Oscillation and nonoscillation criteria for delay differential equations. *Proc. Amer. Math. Soc.*, 123(5):1503–1510, 1995.
[84] Lynn Erbe and Stefan Hilger. Sturmian theory on measure chains. *Differential Equations Dynam. Systems*, 1(3):223–244, 1993.
[85] Lynn Erbe and Allan Peterson. Positive solutions for a nonlinear differential equation on a measure chain. *Math. Comput. Modelling*, 32(5-6):571–585, 2000. Boundary value problems and related topics.
[86] Lynn Erbe and Allan Peterson. Riccati equations on a measure chain. In *Dynamic systems and applications, Vol. 3 (Atlanta, GA, 1999)*, pages 193–199. Dynamic, Atlanta, GA, 2001.
[87] Lynn Erbe and Allan Peterson. Oscillation criteria for second-order matrix dynamic equations on a time scale. *J. Comput. Appl. Math.*, 141(1-2):169–185, 2002. Dynamic equations on time scales.
[88] Lynn Erbe, Allan Peterson, and Pavel Řehák. Comparison theorems for linear dynamic equations on time scales. *J. Math. Anal. Appl.*, 275(1):418–438, 2002.
[89] Lynn Erbe, Allan Peterson, and Samir H. Saker. Oscillation criteria for second order nonlinear dynamic equations on time scales. *J. London Math. Soc.*, 67(3):701–714, 2003.
[90] Lynn Erbe and P. Yan. On the discrete Riccati equation and its applications to discrete Hamiltonian systems. *Rocky Mountain J. Math.*, 25(1):167–178, 1995.
[91] Lynn H. Erbe and Qingkai Kong. Oscillation and nonoscillation properties of neutral differential equations. *Canad. J. Math.*, 46(2):284–297, 1994.
[92] Lynn H. Erbe, Qingkai Kong, and Bing-Gen Zhang. *Oscillation Theory for Functional-Differential Equations*. Marcel Dekker Inc., New York, 1995.
[93] Lynn H. Erbe and Jian She Yu. Linearized oscillations for neutral equations. I. Odd order. *Hiroshima Math. J.*, 26(3):557–572, 1996.
[94] Lynn H. Erbe and Jian She Yu. Linearized oscillations for neutral equations. II. Even order. *Hiroshima Math. J.*, 26(3):573–585, 1996.

[95] Xian-Ling Fan, Wan-Tong Li, and Cheng-Kui Zhong. A classification scheme for positive solutions of second order nonlinear iterative differential equations. *Electron. J. Differential Equations*, pages No. 25, 14 pp. (electronic), 2000.

[96] K. Farrell, Edward A. Grove, and Gerasimos Ladas. Neutral delay differential equations with positive and negative coefficients. *Appl. Anal.*, 27(1-3):181–197, 1988.

[97] Tomlinson Fort. *Finite Differences and Difference Equations in the Real Domain*. Oxford, at the Clarendon Press, 1948.

[98] D. A. Georgiou and C. Qian. Oscillation criteria in neutral equations of n order with variable coefficients. *Internat. J. Math. Math. Sci.*, 14(4):689–696, 1991.

[99] H. E. Gollwitzer. Nonoscillation theorems for a nonlinear differential equation. *Proc. Amer. Math. Soc.*, 26:78–84, 1970.

[100] Kondalsamy Gopalsamy, Bikkar S. Lalli, and Bing-Gen Zhang. Oscillation of odd order neutral differential equations. *Czechoslovak Math. J.*, 42(117)(2):313–323, 1992.

[101] Kondalsamy Gopalsamy and Bing-Gen Zhang. Oscillation and nonoscillation in first order neutral differential equations. *J. Math. Anal. Appl.*, 151(1):42–57, 1990.

[102] Said R. Grace and Bikkar S. Lalli. Oscillation and convergence to zero of solutions of damped second-order nonlinear differential equations. *J. Math. Anal. Appl.*, 102(2):539–548, 1984.

[103] Said R. Grace and Bikkar S. Lalli. Asymptotic and oscillatory behavior of solutions of a class of second order differential equations with deviating arguments. *J. Math. Anal. Appl.*, 145(1):112–136, 1990.

[104] John R. Graef, Myron K. Grammatikopoulos, and Paul W. Spikes. Asymptotic and oscillatory behavior of solutions of first order nonlinear neutral delay differential equations. *J. Math. Anal. Appl.*, 155(2):562–571, 1991.

[105] John R. Graef and Paul W. Spikes. On the oscillatory behavior of solutions of second order nonlinear differential equations. *Czechoslovak Math. J.*, 36(111)(2):275–284, 1986.

[106] John R. Graef and Paul W. Spikes. Some asymptotic properties of solutions of a neutral delay equation with an oscillatory coefficient. *Canad. Math. Bull.*, 36(3):263–272, 1993.

[107] John R. Graef and Paul W. Spikes. On solutions of a neutral equation with an oscillatory coefficient. In *World Congress of Nonlinear Analysts '92, Vol. I–IV (Tampa, FL, 1992)*, pages 1475–1482. de Gruyter, Berlin, 1996.

[108] John R. Graef, Paul W. Spikes, and Myron K. Grammatikopoulos. Behavior of the nonoscillatory solutions of first order neutral delay differential equations. In *Differential equations (Xanthi, 1987)*, pages 265–272. Dekker, New York, 1989.

[109] John R. Graef, Paul W. Spikes, and Myron K. Grammatikopoulos. Asymptotic behavior of nonoscillatory solutions of neutral delay differential equations of arbitrary order. *Nonlinear Anal.*, 21(1):23–42, 1993.

[110] Myron K. Grammatikopoulos, Edward A. Grove, and Gerasimos Ladas. Oscillation and asymptotic behavior of neutral differential equations with deviating arguments. *Appl. Anal.*, 22(1):1–19, 1986.

[111] Myron K. Grammatikopoulos, Edward A. Grove, and Gerasimos Ladas. Oscillations of first-order neutral delay differential equations. *J. Math. Anal. Appl.*, 120(2):510–520, 1986.

[112] Myron K. Grammatikopoulos, Edward A. Grove, and Gerasimos Ladas. Oscillation and asymptotic behavior of second order neutral differential equations with deviating arguments. In *Oscillations, bifurcation and chaos (Toronto, Ont., 1986)*, pages 153–161. Amer. Math. Soc., Providence, RI, 1987.

[113] Myron K. Grammatikopoulos, Gerasimos Ladas, and A. Meimaridou. Oscillation and asymptotic behavior of higher order neutral equations with variable coefficients. *Chinese Ann. Math. Ser. B*, 9(3):322–338, 1988. A Chinese summary appears in Chinese Ann. Math. Ser. A **9** (1988), no. 4, 507.

[114] Edward A. Grove, Mustafa R. S. Kulenović, and Gerasimos Ladas. Sufficient conditions for oscillation and nonoscillation of neutral equations. *J. Differential Equations*, 68(3):373–382, 1987.

[115] Edward A. Grove, Gerasimos Ladas, and J. Schinas. Sufficient conditions for the oscillation of delay and neutral delay equations. *Canad. Math. Bull.*, 31(4):459–466, 1988.

[116] Gusein Sh. Guseinov and Billûr Kaymakçalan. On a disconjugacy criterion for second order dynamic equations on time scales. *J. Comput. Appl. Math.*, 141(1-2):187–196, 2002. Dynamic equations on time scales.

[117] István Győri. On the oscillatory behaviour of solutions of certain nonlinear and linear delay differential equations. *Nonlinear Anal.*, 8(5):429–439, 1984.

[118] István Győri and Gerasimos Ladas. *Oscillation Theory of Delay Differential Equations.* Oxford University Press, New York, 1991.

[119] Jack Hale. *Theory of Functional Differential Equations.* Springer-Verlag, New York, second edition, 1977. Applied Mathematical Sciences, Vol. 3.

[120] Godfrey H. Hardy, John E. Littlewood, and George Pólya. *Inequalities.* Cambridge University Press, Cambridge, 1988. Reprint of the 1952 edition.

[121] B. J. Harris and Qingkai Kong. On the oscillation of differential equations with an oscillatory coefficient. *Trans. Amer. Math. Soc.*, 347(5):1831–1839, 1995.

[122] Wan-Sheng He and Wan-Tong Li. Nonoscillation and oscillation for neutral differential equations with positive and negative coefficients. *Int. J. Appl. Math.*, 6(2):183–198, 2001.

[123] X. Z. He. Oscillatory and asymptotic behaviour of second order nonlinear difference equations. *J. Math. Anal. Appl.*, 175:482–498, 1993.

[124] Stefan Hilger. *Ein Maßkettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten.* PhD thesis, Universität Würzburg, 1988.

[125] Stefan Hilger. Analysis on measure chains—a unified approach to continuous and discrete calculus. *Results Math.*, 18(1-2):18–56, 1990.

[126] Stefan Hilger. Differential and difference calculus—unified! In *Proceedings of the Second World Congress of Nonlinear Analysts, Part 5 (Athens, 1996)*, volume 30, pages 2683–2694, 1997.

[127] Roman Hilscher. Discrete spectral criteria for a certain class of singular differential and difference operators. *Comput. Math. Appl.*, 42:465–476, 2001.

[128] Roman Hilscher. Reid roundabout theorem for symplectic dynamic systems on time scales. *Appl. Math. Optim.*, 43(2):129–146, 2001.

[129] Chunchao Huang. Oscillation and nonoscillation for second order linear differential equations. *J. Math. Anal. Appl.*, 210(2):712–723, 1997.

[130] Jaroslav Jaroš. An oscillation test for a class of linear neutral differential equations. *J. Math. Anal. Appl.*, 159(2):406–417, 1991.

[131] Jaroslav Jaroš and Takaŝi Kusano. Oscillation theory of higher order linear functional-differential equations of neutral type. *Hiroshima Math. J.*, 18(3):509–531, 1988.

[132] Jaroslav Jaroš and Takaŝi Kusano. Asymptotic behavior of nonoscillatory solutions of nonlinear functional-differential equations of neutral type. *Funkcial. Ekvac.*, 32(2):251–263, 1989.

[133] Jaroslav Jaroš and Takaŝi Kusano. On a class of first order nonlinear functional-differential equations of neutral type. *Czechoslovak Math. J.*, 40(115)(3):475–490, 1990.

[134] Jaroslav Jaroš and Takaŝi Kusano. On oscillation of linear neutral differential equations of higher order. *Hiroshima Math. J.*, 20(2):407–419, 1990.

[135] Jaroslav Jaroš and Takaŝi Kusano. Existence of oscillatory solutions for functional-differential equations of neutral type. *Acta Math. Univ. Comenian. (N.S.)*, 60(2):185–194, 1991.

[136] Jaroslav Jaroš and Takaŝi Kusano. Oscillation properties of first order nonlinear functional-differential equations of neutral type. *Differential Integral Equations*, 4(2):425–436, 1991.

[137] Jaroslav Jaroš and Takaŝi Kusano. A Picone type identity for second order half-linear differential equations. *Acta Math. Univ. Comenian. (N.S.)*, 68(1):137–151, 1999.

[138] Jaroslav Jaroš, Takaŝi Kusano, and Norio Yoshida. Forced superlinear oscillations via Picone's identity. *Acta Math. Univ. Comenian. (N.S.)*, 69(1):107–113, 2000.

[139] Jaroslav Jaroš and I. P. Stavroulakis. Oscillation tests for delay equations. *Rocky Mountain J. Math.*, 29(1):197–207, 1999.

[140] I. V. Kamenev. An integral test for conjugacy for second order linear differential equations. *Mat. Zametki*, 23(2):249–251, 1978.

[141] G. A. Kamenskiĭ. On the asymptotic behavior of solutions of linear differential equations of the second order with retarded argument. *Moskov. Gos. Univ. Uč. Zap. Mat.*, 165(7):195–204, 1954.

[142] Athanassios G. Kartsatos. Recent results on oscillation of solutions of forced and perturbed nonlinear differential equations of even order. In *Stability of dynamical systems, theory and applications (Proc. Regional Conf., Mississippi State Univ., Mississippi State, Miss., 1975)*, pages 17–72. Lecture Notes in Pure and Appl. Math., Vol. 28. Dekker, New York, 1977.

[143] Billûr Kaymakçalan, V. Lakshmikantham, and Seenith Sivasundaram. *Dynamic Systems on Measure Chains*, volume 370 of *Mathematics and its Applications*. Kluwer Academic Publishers Group, Dordrecht, 1996.

[144] Walter G. Kelley and Allan C. Peterson. *Difference Equations: An Introduction with Applications*. Academic Press, San Diego, second edition, 2001.

[145] Ivan T. Kiguradze and Teimuraz A. Chanturia. *Asymptotic Properties of Solutions of Nonautonomous Ordinary Differential Equations*. Kluwer Academic Publishers Group, Dordrecht, 1993. Translated from the 1985 Russian original.

[146] Mokhtar Kirane and Yuri V. Rogovchenko. Oscillation results for a second order damped differential equation with nonmonotonous nonlinearity. *J. Math. Anal. Appl.*, 250(1):118–138, 2000.

[147] Yuichi Kitamura and Takaŝi Kusano. Oscillation and asymptotic behavior of solutions of first-order functional-differential equations of neutral type. *Funkcial. Ekvac.*, 33(2):325–343, 1990.

[148] Yuichi Kitamura and Takaŝi Kusano. Existence theorems for a neutral functional-differential equation whose leading part contains a difference operator of higher degree. *Hiroshima Math. J.*, 25(1):53–82, 1995.

[149] Yuichi Kitamura, Takaŝi Kusano, and Bikkar S. Lalli. Existence of oscillatory and nonoscillatory solutions for a class of neutral functional-differential equations. *Math. Bohem.*, 120(1):57–69, 1995.

[150] M. Kon, Yiannis G. Sficas, and I. P. Stavroulakis. Oscillation criteria for delay equations. *Proc. Amer. Math. Soc.*, 128(10):2989–2997, 2000.

[151] Qingkai Kong. Interval criteria for oscillation of second-order linear ordinary differential equations. *J. Math. Anal. Appl.*, 229(1):258–270, 1999.

[152] I.-G. E. Kordonis and Christos G. Philos. On the oscillation of nonlinear two-dimensional differential systems. *Proc. Amer. Math. Soc.*, 126(6):1661–1667, 1998.

[153] Mustafa R. S. Kulenović and Myron K. Grammatikopoulos. Some comparison and oscillation results for first-order differential equations and inequalities with a deviating argument. *J. Math. Anal. Appl.*, 131(1):67–84, 1988.

[154] Takaŝi Kusano. On even-order functional-differential equations with advanced and retarded arguments. *J. Differential Equations*, 45(1):75–84, 1982.

[155] Takaŝi Kusano and Bikkar S. Lalli. On oscillation of half-linear functional-differential equations with deviating arguments. *Hiroshima Math. J.*, 24(3):549–563, 1994.

[156] Takaŝi Kusano and Bhagat Singh. Positive solutions of functional-differential equations with singular nonlinear terms. *Nonlinear Anal.*, 8(9):1081–1090, 1984.

[157] Takaŝi Kusano and Norio Yoshida. Nonoscillation theorems for a class of quasilinear differential equations of second order. *J. Math. Anal. Appl.*, 189(1):115–127, 1995.

[158] Man Kam Kwong and William T. Patula. Comparison theorems for first order linear delay equations. *J. Differential Equations*, 70(2):275–292, 1987.

[159] Man Kam Kwong and James S. W. Wong. An application of integral inequality to second order nonlinear oscillation. *J. Differential Equations*, 46(1):63–77, 1982.

[160] Man Kam Kwong and James S. W. Wong. Nonoscillation theorems for a second order sublinear ordinary differential equation. *Proc. Amer. Math. Soc.*, 87(3):467–474, 1983.

[161] Man Kam Kwong and James S. W. Wong. Oscillation of Emden–Fowler systems. *Differential Integral Equations*, 1(2):133–141, 1988.

[162] Gerasimos Ladas and C. Qian. Oscillation in differential equations with positive and negative coefficients. *Canad. Math. Bull.*, 33(4):442–451, 1990.

[163] Gerasimos Ladas, C. Qian, and J. Yan. A comparison result for the oscillation of delay differential equations. *Proc. Amer. Math. Soc.*, 114(4):939–947, 1992.

[164] Gerasimos Ladas and Yiannis G. Sficas. Oscillations of higher-order neutral equations. *J. Austral. Math. Soc. Ser. B*, 27(4):502–511, 1986.

[165] Gerasimos Ladas, Yiannis G. Sficas, and I. P. Stavroulakis. Necessary and sufficient conditions for oscillations of higher order delay differential equations. *Trans. Amer. Math. Soc.*, 285(1):81–90, 1984.

[166] Gangaram S. Ladde, V. Lakshmikantham, and Bing-Gen Zhang. *Oscillation Theory of Differential Equations with Deviating Arguments*. Marcel Dekker Inc., New York, 1987.

[167] Bikkar S. Lalli and Bing-Gen Zhang. Oscillation of first order neutral differential equations. *Appl. Anal.*, 39(4):265–274, 1990.

[168] B. M. Levitan. Some questions of the theory of almost periodic functions. I. *Uspehi Matem. Nauk (N.S.)*, 2(5(21)):133–192, 1947.

[169] Bing Tuan Li. Oscillations of delay differential equations with variable coefficients. *J. Math. Anal. Appl.*, 192(1):312–321, 1995.

[170] Bing Tuan Li. Oscillation of first order delay differential equations. *Proc. Amer. Math. Soc.*, 124(12):3729–3737, 1996.

[171] H. J. Li and Sui Sun Cheng. Asymptotic monotone solutions of a nonlinear difference equation. *Tamkang J. Math.*, 24:269–282, 1993.

[172] Horng Jaan Li. Oscillation criteria for second order linear differential equations. *J. Math. Anal. Appl.*, 194(1):217–234, 1995.

[173] Horng Jaan Li. Oscillation criteria for second order linear differential equations. *J. Math. Anal. Appl.*, 194(1):217–234, 1995.

[174] Horng Jaan Li and Wei Ling Liu. Oscillations of second order neutral differential equations. *Math. Comput. Modelling*, 22(1):45–53, 1995.

[175] Horng Jaan Li and Cheh Chih Yeh. Nonoscillation theorems for second order quasilinear differential equations. *Publ. Math. Debrecen*, 47(3-4):271–279, 1995.

[176] Wan-Tong Li. Comparison theorems and sufficient and necessary condition for oscillation of first order neutral nonlinear differential equations. *Ann. Differential Equations*, 12(4):446–451, 1996.

[177] Wan-Tong Li. Classifications and existence of nonoscillatory solutions of second order nonlinear neutral differential equations. *Ann. Polon. Math.*, 65(3):283–302, 1997.

[178] Wan-Tong Li. Necessary and sufficient conditions for oscillation of odd order neutral nonlinear equations with variable coefficients and their applications. *Dynam. Systems Appl.*, 7(1):93–103, 1998.

[179] Wan-Tong Li. Oscillation of certain second-order nonlinear differential equations. *J. Math. Anal. Appl.*, 217(1):1–14, 1998.

[180] Wan-Tong Li. Oscillation of first order neutral nonlinear differential equations with variable coefficients. *Indian J. Pure Appl. Math.*, 29(6):605–615, 1998.

[181] Wan-Tong Li. Positive solutions of second order nonlinear differential equations. *J. Math. Anal. Appl.*, 221(1):326–337, 1998.

[182] Wan-Tong Li. Asymptotic behavior of nonoscillatory solutions of nth-order neutral nonlinear equations with oscillating coefficients. *Nonlinear Anal.*, 36(7, Ser. A: Theory Methods):891–898, 1999.

[183] Wan-Tong Li. Positive solutions of an odd-order neutral delay nonlinear differential equations. *Nonlinear Anal.*, 36(7, Ser. A: Theory Methods):899–913, 1999.

[184] Wan-Tong Li. Monotone solutions of second order quasilinear differential equations. In *Differential equations and computational simulations (Chengdu, 1999)*, pages 234–237. World Sci. Publishing, River Edge, NJ, 2000.

[185] Wan-Tong Li. Classification schemes for positive solutions of nonlinear differential systems. *Math. Comput. Modelling*, 36(4-5):411–418, 2002.

[186] Wan-Tong Li. Interval criteria for oscillation of second-order half-linear ordinary differential equations. *Acta Math. Sinica*, 45(3):509–516, 2002.

[187] Wan-Tong Li and Ravi P. Agarwal. Interval oscillation criteria for a forced second order nonlinear ordinary differential equation. *Appl. Anal.*, 75(3-4):341–347, 2000.

[188] Wan-Tong Li and Ravi P. Agarwal. Interval oscillation criteria for second-order nonlinear differential equations with damping. *Comput. Math. Appl.*, 40(2-3):217–230, 2000.

[189] Wan-Tong Li and Ravi P. Agarwal. Interval oscillation criteria related to integral averaging technique for certain nonlinear differential equations. *J. Math. Anal. Appl.*, 245(1):171–188, 2000.

[190] Wan-Tong Li and Ravi P. Agarwal. Interval oscillation criteria for second order nonlinear differential equations. *Ukraïn. Mat. Zh.*, 53(9):1161–1173, 2001.

[191] Wan-Tong Li and Ravi P. Agarwal. Oscillation criteria for second order nonlinear differential equations involving integral averages. *Comm. Appl. Nonlinear Anal.*, 8(4):77–87, 2001.

[192] Wan-Tong Li and Ravi P. Agarwal. Interval oscillation criteria for second order nonlinear perturbed differerential equations. *Comput. Math. Appl.*, 2004. In press.

[193] Wan-Tong Li and Sui Sun Cheng. Asymptotic linear solutions of a discrete Emden–Fowler equation with singular nonlinear term. *Far East J. Math. Sci. (FJMS)*, 6(4):521–542, 1998.

[194] Wan-Tong Li and Sui Sun Cheng. Limiting behaviours of non-oscillatory solutions of a pair of coupled nonlinear differential equations. *Proc. Edinburgh Math. Soc. (2)*, 43(3):457–473, 2000.

[195] Wan-Tong Li and Xian-Ling Fan. Monotone solutions of second-order nonlinear differential equations. *Appl. Math. Lett.*, 13(4):65–70, 2000.

[196] Wan-Tong Li and Xiang-Li Fei. Classifications and existence of positive solutions of higher-order nonlinear delay differential equations. *Nonlinear Anal.*, 41(3-4, Ser. A: Theory Methods):433–445, 2000.

[197] Wan-Tong Li and Can-Yun Huang. Classification and existence of positive solutions to nonlinear dynamic equations on time scales. *Electron. J. Differential Equations*, pages No. 17, 8 pp. (electronic), 2004.

[198] Wan-Tong Li and Hai-Feng Huo. Oscillation of the Emden–Fowler differential systems. *Int. J. Appl. Math.*, 2(12):1453–1461, 2000.

[199] Wan-Tong Li and Hai-Feng Huo. Interval oscillation criteria for nonlinear second-order differential equations. *Indian J. Pure Appl. Math.*, 32(7):1003–1014, 2001.

[200] Wan-Tong Li and Xiaohu Li. Oscillation criteria for second-order nonlinear differential equations with integrable coefficient. *Appl. Math. Lett.*, 13(8):1–6, 2000.

[201] Wan-Tong Li and Hong Shun Quan. Oscillation of higher order neutral differential equations with positive and negative coefficients. *Ann. Differential Equations*, 11(1):70–76, 1995.

[202] Wan-Tong Li and Hongshun Quan. An oscillation criterion for second order sublinear differential equations. *Ann. Differential Equations*, 12(3):292–296, 1996.

[203] Wan-Tong Li, Hongshun Quan, and Jianhong Wu. Oscillation of first order neutral differential equations with variable coefficients. *Commun. Appl. Anal.*, 3(1):1–13, 1999.

[204] Wan-Tong Li and Samir H. Saker. Oscillation of nonlinear neutral delay differential equations with applications. *Ann. Polon. Math.*, 77(1):39–51, 2001.

[205] Wan-Tong Li and Jurang Yan. Oscillation criteria for second order superlinear differential equations. *Indian J. Pure Appl. Math.*, 28(6):735–740, 1997.

[206] Wan-Tong Li and Jurang Yan. Oscillation of first order neutral differential equations with positive and negative coefficients. *Collect. Math.*, 50(2):199–209, 1999.

[207] Wan-Tong Li and Xiaojing Yang. Classifications and existence criteria for positive solutions of systems of nonlinear differential equations. *J. Math. Anal. Appl.*, 2004. In press.

[208] Wan-Tong Li and Qixiao Ye. On the existence of asymptotically decaying positive solutions of nth order neutral differential equations. *Pure Appl. Math.*, 11(2):53–62, 1995.

[209] Wan-Tong Li, Ming-Yue Zhang, and Xiang-Li Fe. Oscillation criteria for second order nonlinear differential equations with damped term. *Indian J. Pure Appl. Math.*, 30(10):1017–1029, 1999.

[210] Wan-Tong Li and Shunian Zhang. Classifications and existence of positive solutions of higher order nonlinear iterative functional differential equations. *J. Comput. Appl. Math.*, 139(2):351–367, 2002.

[211] Wan-Tong Li, Zhu Zhao, and Shizhou Yang. Existence theorems for second order nonlinear differential equations. *J. Gansu Univ. Technol. (Engl. Ed.)*, 2(1):81–84, 1998.

[212] Wan-Tong Li and Cheng-Kui Zhong. Unbounded positive solutions of higher-order nonlinear functional differential equations. *Appl. Math. Lett.*, 14(7):825–830, 2001.

[213] Wan-Tong Li, Cheng-Kui Zhong, and Xian-Ling Fan. Oscillation criteria for second-order half-linear ordinary differential equations with damping. *Rocky Mountain J. Math.*, 33(1):1–26, 2003.

[214] Wu Du Lu. The asymptotic and oscillatory behavior of the solutions of higher order neutral equations. *J. Math. Anal. Appl.*, 148(2):378–389, 1990.

[215] Wu Du Lu. Existence of nonoscillatory solutions of first order nonlinear neutral equations. *J. Austral. Math. Soc. Ser. B*, 32(2):180–192, 1990.

[216] Wu Du Lu. Oscillations of high order neutral differential equations with variable coefficients. *Math. Appl.*, 3(2):36–43, 1990.

[217] Wu Du Lu. Oscillations of high order neutral differential equations with oscillatory coefficient. *Acta Math. Appl. Sinica (English Ser.)*, 7(2):135–142, 1991.

[218] Wu Du Lu. Existence and asymptotic behavior of nonoscillatory solutions to nonlinear second-order equations of neutral type. *Acta Math. Sinica*, 36(4):476–484, 1993.

[219] Wu Du Lu. Nonoscillation and oscillation for first order nonlinear neutral equations. *Funkcial. Ekvac.*, 37(2):383–394, 1994.

[220] Wu Du Lu. Nonoscillation and oscillation of first order neutral equations with variable coefficients. *J. Math. Anal. Appl.*, 181(3):803–815, 1994.

[221] Jelena Manojlović. Existence of oscillatory and nonoscillatory solutions for a nonlinear system of differential equations. *Filomat*, (11):135–148, 1997.

[222] Jelena Manojlović. Oscillation theorems for a nonlinear system of differential equations. *Acta Math. Hungar.*, 81(1-2):59–67, 1998.

[223] Jelena Manojlović. Nonoscillation theorems for Emden–Fowler system of differential equations. *Indian J. Pure Appl. Math.*, 30(7):687–694, 1999.

[224] D. D. Mirzov. The oscillatoriness of the solutions of a system of nonlinear differential equations. *Differentsial′nye Uravnenija*, 9:581–583, 592, 1973.

[225] D. D. Mirzov. On the question of the oscillation of the solutions of a system of nonlinear differential equations. *Mat. Zametki*, 16:571–576, 1974.

[226] D. D. Mirzov. The oscillatory nature of the solutions of a nonlinear differential system of Emden–Fowler type. *Differentsial′nye Uravneniya*, 16(11):1980–1984, 2108, 1980.

[227] D. D. Mirzov. Oscillatory nature of the solutions of a differential system. *Differentsial′nye Uravneniya*, 17(8):1504–1508, 1535, 1981.

[228] D. D. Mirzov. An analogue of a theorem of Hartman. *Differentsial′nye Uravneniya*, 25(2):216–222, 361, 1989.

[229] J. D. Mirzov. On some analogs of Sturm's and Kneser's theorems for nonlinear systems. *J. Math. Anal. Appl.*, 53(2):418–425, 1976.

[230] A. D. Myšhkis. *Lineinye differentsialnye uravneniya s zapazdyvayushchim argumentom.* Izdat. "Nauka", Moscow, second edition, 1972.

[231] Manabu Naito. Asymptotic behavior of solutions of second order differential equations with integrable coefficients. *Trans. Amer. Math. Soc.*, 282:577–588, 1984.

[232] Manabu Naito. Nonoscillatory solutions of second order differential equations with integrable coefficients. *Proc. Amer. Math. Soc.*, 109(3):769–774, 1990.

[233] Manabu Naito. Integral averages and the asymptotic behavior of solutions of second order ordinary differential equations. *J. Math. Anal. Appl.*, 164(2):370–380, 1992.

[234] Yūki Naito. Existence and asymptotic behavior of positive solutions of neutral differential equations. *J. Math. Anal. Appl.*, 188(1):227–244, 1994.

[235] Yūki Naito. A note on the existence of nonoscillatory solutions of neutral differential equations. *Hiroshima Math. J.*, 25(3):513–518, 1995.

[236] M. Z. Nashed and James S. W. Wong. Some varaints of a fixed point theorem of Krasnoselskii and applications to nonlinear integral equations. *J. Math. Mech.*, 18:767–777, 1969.

[237] S. B. Norkin. *Diffirintsialnye uravneniya vtorogo poryadka s zapazdyvayushchim argumenton. Nekotorye voprosy teorii kolebaniisistem s zapazdyvaniem.* Izdat. "Nauka", Moscow, 1965.

[238] Allan C. Peterson. Oscillation of second order linear matrix difference equations. *J. Differential Equations*, 89:65–88, 1991.

[239] Christos G. Philos. On the existence of nonoscillatory solutions tending to zero at ∞ for differential equations with positive delays. *Arch. Math. (Basel)*, 36(2):168–178, 1981.

[240] Christos G. Philos. On a Kamenev's integral criterion for oscillation of linear differential equations of second order. *Utilitas Math.*, 24:277–289, 1983.

[241] Christos G. Philos. Oscillation theorems for linear differential equations of second order. *Arch. Math. (Basel)*, 53(5):482–492, 1989.

[242] Christos G. Philos and I. K. Purnaras. Oscillations in superlinear differential equations of second order. *J. Math. Anal. Appl.*, 165(1):1–11, 1992.

[243] Xiang Zheng Qian and Liu Sheng Liao. On the oscillations of the equation $\dot{x}(t) + px(t - \tau) - qx(t - \sigma) = 0$. *Ann. Differential Equations*, 3(3):351–357, 1987.

[244] Oscar Quijada and Antonio Tineo. Oscillatory and nonoscillatory results for first order neutral differential equations. *J. Math. Anal. Appl.*, 180(1):37–42, 1993.

[245] Yuri V. Rogovchenko. Note on: "Oscillation criteria for second order linear differential equations" [J. Math. Anal. Appl. **194** (1995), no. 1, 217–234] by H. J. Li. *J. Math. Anal. Appl.*, 203(2):560–563, 1996.

[246] Yuri V. Rogovchenko. Oscillation criteria for certain nonlinear differential equations. *J. Math. Anal. Appl.*, 229(2):399–416, 1999.

[247] Jiong Ruan. Types and criteria of nonoscillatory solutions for second-order linear neutral differential difference equations. *Chinese Ann. Math. Ser. A*, 8(1):114–124, 1987. An English summary appears in Chinese Ann. Math. Ser. B **8** (1987), no. 2, 284.

[248] Shi Gui Ruan. Oscillations for first order neutral differential equations with variable coefficients. *Bull. Austral. Math. Soc.*, 43(1):147–152, 1991.

[249] Č. A. Shaljaho. On the oscillatory and nonoscillatory natures of the solutions for a system of nonlinear differential equations. *Differentsial′nye Uravneniya*, 16(8):1523–1526, 1535–1536, 1980.

[250] Č. A. Shaljaho. Oscillatory and nonoscillatory nature of solutions of a two-dimensional system of nonlinear differential equations. *Differentsial′nye Uravneniya*, 17(9):1702–1705, 1727, 1981.

[251] J. H. Shen. New oscillation criteria for odd order neutral equations. *J. Math. Anal. Appl.*, 201(2):387–395, 1996.

[252] J. H. Shen and I. P. Stavroulakis. Oscillatory and nonoscillatory delay equations with piecewise constant argument. *J. Math. Anal. Appl.*, 248(2):385–401, 2000.

[253] V. N. Shevelo. *Ostsillyatsiya reshenii differentsialnykh uravnenii s otklonyayushchimsya argumentom.* Izdat. "Naukova Dumka", Kiev, 1978.

[254] Cherim A. Skhalyakho. On the oscillatory nature of solutions of a system of two differential equations. *Časopis Pěst. Mat.*, 107(2):139–142, 187, 1982. With a loose English summary.

[255] C. A. Swanson. *Comparison and Oscillation Theory of Linear Differential Equations.* Academic Press, New York, 1968. Mathematics in Science and Engineering, Vol. 48.

[256] Satoshi Tanaka. Existence of positive solutions for a class of first-order neutral functional-differential equations. *J. Math. Anal. Appl.*, 229(2):501–518, 1999.

[257] Satoshi Tanaka. Oscillatory and nonoscillatory solutions of neutral differential equations. *Ann. Polon. Math.*, 73(2):169–184, 2000.

[258] X. H. Tang and J. H. Shen. Oscillation and existence of positive solutions in a class of higher order neutral equations. *J. Math. Anal. Appl.*, 213(2):662–680, 1997.

[259] Xianhua Tang and Jianhua Shen. Oscillations of delay differential equations with variable coefficients. *J. Math. Anal. Appl.*, 217(1):32–42, 1998.

[260] E. Thandapani, István Győri, and Bikkar S. Lalli. An application of discrete inequality to second order nonlinear oscillation. *J. Math. Anal. Appl.*, 186(1):200–208, 1994.

[261] E. Thandapani and S. Pandian. On the oscillatory behaviour of solutions of second order nonlinear difference equations. *Z. Anal. Anwendungen*, 13(2):347–358, 1994.

[262] Bai-Yan Wang, Wan-Tong Li, and Chao Ye. Oscillation criteria for second order nonlinear differential equations. *Far East J. Math. Sci. (FJMS)*, 1(4):649–658, 1999.

[263] Zhi Cheng Wang. Necessary and sufficient conditions for oscillations. *Ann. Differential Equations*, 5(4):475–487, 1989.

[264] Zhicheng Wang. A necessary and sufficient condition for the oscillation of higher-order neutral equations. *Tohoku Math. J. (2)*, 41(4):575–588, 1989.

[265] Jun Jie Wei. Necessary and sufficient conditions for oscillations of first-order differential equations with deviating arguments and their applications. *Acta Math. Sinica*, 32(5):632–638, 1989.

[266] Aurel Wintner. On the comparison theorem of Kneser–Hille. *Math. Scand.*, 5:255–260, 1957.

[267] James S. W. Wong. On second order nonlinear oscillation. *Funkcial. Ekvac.*, 11:207–234 (1969), 1968.

[268] James S. W. Wong. An oscillation criterion for second order nonlinear differential equations. *Proc. Amer. Math. Soc.*, 98(1):109–112, 1986.

[269] James S. W. Wong. An oscillation criterion for second order sublinear differential equations. In *Oscillations, bifurcation and chaos (Toronto, Ont., 1986)*, pages 299–302. Amer. Math. Soc., Providence, RI, 1987.

[270] James S. W. Wong. Oscillation theorems for second order nonlinear differential equations. *Proc. Amer. Math. Soc.*, 106(4):1069–1077, 1989.

[271] James S. W. Wong. Oscillation criteria for second order nonlinear differential equations with integrable coefficients. *Proc. Amer. Math. Soc.*, 115(2):389–395, 1992.

[272] James S. W. Wong. Oscillation criteria for second order nonlinear differential equations involving integral averages. *Canad. J. Math.*, 45(5):1094–1103, 1993.

[273] James S. W. Wong. An oscillation criterion for second order nonlinear differential equations with iterated integral averages. *Differential Integral Equations*, 6(1):83–91, 1993.

[274] James S. W. Wong. Oscillation criteria for a forced second-order linear differential equation. *J. Math. Anal. Appl.*, 231(1):235–240, 1999.

[275] Patricia J. Y. Wong and Ravi P. Agarwal. Oscillation theorems and existence criteria of asymptotically monotone solutions for second order differential equations. *Dynam. Systems Appl.*, 4(4):477–496, 1995.

[276] Patricia J. Y. Wong and Ravi P. Agarwal. Oscillation theorems and existence of positive monotone solutions for second order nonlinear difference equations. *Math. Comput. Modelling*, 21(3):63–84, 1995.

[277] Patricia J. Y. Wong and Ravi P. Agarwal. The oscillation and asymptotically monotone solutions of second-order quasilinear differential equations. *Appl. Math. Comput.*, 79(2-3):207–237, 1996.

[278] Patricia J. Y. Wong and Ravi P. Agarwal. Oscillatory behavior of solutions of certain second order nonlinear differential equations. *J. Math. Anal. Appl.*, 198(2):337–354, 1996.

[279] Yuan Tong Xu. Asymptotic behavior of nonoscillatory solutions of higher-order neutral equations. *Ann. Differential Equations*, 5(2):199–209, 1989.

[280] Ju Rang Yan. A note on an oscillation criterion for an equation with damped term. *Proc. Amer. Math. Soc.*, 90(2):277–280, 1984.

[281] Ju Rang Yan. Oscillation theorems for second order linear differential equations with damping. *Proc. Amer. Math. Soc.*, 98(2):276–282, 1986.

[282] Ju Rang Yan. Oscillatory properties of second-order differential equations with an “integralwise small” coefficient. *Acta Math. Sinica*, 30(2):206–215, 1987.

[283] Ju Rang Yan. Oscillatory and asymptotic behavior of nth order nonlinear time-delay differential equations. *Acta Math. Sinica*, 33(4):537–543, 1990.

[284] Ju Rang Yan. Comparison theorems and applications of oscillation of neutral differential equations. *Sci. China Ser. A*, 34(3):273–283, 1991.

[285] Ju Rang Yan. Comparison theorems for differential equations of mixed type. *Ann. Differential Equations*, 7(3):316–322, 1991.

[286] Bo Yang and Bing-Gen Zhang. Qualitative analysis of a class of neutral differential equations. *Funkcial. Ekvac.*, 39(2):347–362, 1996.

[287] Cheh Chih Yeh. Oscillation theorems for nonlinear second order differential equations with damped term. *Proc. Amer. Math. Soc.*, 84(3):397–402, 1982.

[288] Jian She Yu. First-order nonlinear differential inequalities with deviating argument. *Acta Math. Sinica*, 33(2):152–159, 1990.

[289] Jian She Yu. Neutral time-delay differential equations with positive and negative coefficients. *Acta Math. Sinica*, 34(4):517–523, 1991.

[290] Jian She Yu. Oscillations of odd-order neutral equations with an “integrally small” coefficient. *Chinese Ann. Math. Ser. A*, 16(1):33–42, 1995.

[291] Jian She Yu and Ming Po Ch’ên. Oscillation in neutral equations with an “integrally small” coefficient. *Internat. J. Math. Math. Sci.*, 17(2):361–368, 1994.

[292] Jian She Yu and Zhicheng Wang. Some further results on oscillation of neutral differential equations. *Bull. Austral. Math. Soc.*, 46(1):149–157, 1992.

[293] Jian She Yu, Zhicheng Wang, and Chuan Xi Qian. Oscillation of neutral delay differential equations. *Bull. Austral. Math. Soc.*, 45(2):195–200, 1992.

[294] Jian She Yu and Ju Rang Yan. Oscillation in first order neutral differential equations with “integrally small” coefficients. *J. Math. Anal. Appl.*, 187(2):361–370, 1994.

[295] Yuan Hong Yu. Leighton type oscillation criterion and Sturm type comparison theorem. *Math. Nachr.*, 153:137–143, 1991.

[296] Yuan Hong Yu. Oscillation of solutions of neutral delay differential equations. *Acta Math. Appl. Sinica*, 14(3):404–410, 1991.

[297] Bing-Gen Zhang. Oscillation of first order neutral functional-differential equations. *J. Math. Anal. Appl.*, 139(2):311–318, 1989.

[298] Bing-Gen Zhang. The development of the theory of oscillations of functional-differential equations. *Kexue Tongbao (Chinese)*, 43(4):345–354, 1998.

[299] Bing-Gen Zhang and Xing-Hua Deng. Oscillation of delay differential equations on time scales. *Math. Comput. Modelling*, 36(11-13):1307–1318, 2002.

[300] Bing-Gen Zhang and Wan-Tong Li. On the oscillation of odd order neutral differential equations. *Fasc. Math.*, (29):167–183, 1999.

[301] Bing-Gen Zhang and Jian She Yu. Existence of positive solutions for neutral differential equations. *Sci. China Ser. A*, 35(11):1306–1313, 1992.

[302] Bing-Gen Zhang, Jian She Yu, and Z. C. Wang. Oscillations of higher order neutral differential equations. *Rocky Mountain J. Math.*, 25(1):557–568, 1995. Second Geoffrey J. Butler Memorial Conference in Differential Equations and Mathematical Biology (Edmonton, AB, 1992).

[303] Guang Zhang and Sui Sun Cheng. A necessary and sufficient oscillation condition for the discrete Euler equation. *Panamer. Math. J.*, 9(4):29–34, 1999.

[304] Cheng-Kui Zhong, Xian-Ling Fan, and W. Y. Chen. *Introduction to Nonlinear Functional Analysis*. Lanzhou University Press, 1997. In Chinese.