Section 3.2

PRE-ACTIVITY PREPARATION

Finding Prime Factors

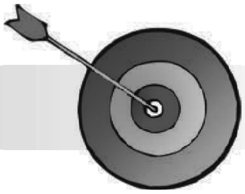
Note: *While this section on finding prime factors does not include fraction notation, it does address an intermediate and necessary concept to learn before you further explore fractions.*

Numbers are categorized in various ways. You are already familiar with whole numbers, decimal numbers, and fractions; but there are other special characteristics of numbers, which may be useful to consider as well. Numbers might be classified as even or odd, as perfect squares, as repeating or non-repeating decimals, for example.

2 3 5 7 11
13 17 19 23
29 31 37 41
43 47 53 59

Your ability to identify the whole number numerators and denominators as falling into one of two categories—primes or composites—will make the mathematical processes of working with fractions easier.

Writing a number as the product of its prime factors will be useful for several of the processes you will encounter in this course, but the important critical thinking skill of breaking something down into its basic components is a life skill with applications far beyond mathematics.



LEARNING OBJECTIVES

- Identify a whole number as a prime number or a composite number.
- Determine the prime factorization of a whole number.

TERMINOLOGY

PREVIOUSLY USED

factor

NEW TERMS TO LEARN

base
composite
divisibility tests
exponent
exponent form
exponential notation
multiple
power
prime
prime factor
prime factoring
prime factorization
whole number factor

BUILDING MATHEMATICAL LANGUAGE

- Recall that numbers to be multiplied together are called **factors**. These same numbers can also be referred to as *factors of the product* because they will divide evenly into the product.

- A **whole number factor** is a whole number that divides evenly (no remainder) into a given number.

- A **prime** number is a whole number *greater than one (1)* which has *exactly* two whole number factors—the number itself and one (1).

  For example, **29** is a prime number because 1 and 29 are its two whole number factors.
  That is to say, 1 and 29 are the only whole numbers that will divide into 29 evenly. 29 = 1 × 29.

- On the other hand, a **composite** number is a whole number that can be constructed (*composed*) by multiplying two smaller whole numbers, each greater than the number one (1).

  For example, **35** is a composite number.
  It has four whole number factors: 1, 5, 7, and 35. That is, each of these numbers will divide into 35 evenly.
  Thirty-five can be written as the product of 1 and 35 (1 × 35).
  However, it can also be written as the product of 5 and 7 (5 × 7).
  The ability to construct 35 this second way is what makes it a composite number.

- A **prime factor** is a whole number factor that is also a prime number.

  For example, the prime factors of 35 are **5** and **7**.

- The **multiples** of a number are the numbers that result from multiplying the number by 1, 2, 3, 4, and so on.

  For example, multiples of 5 are: 1 × 5 or **5**, 2 × 5 or **10**, 3 × 5 or **15**, 4 × 5 or **20**, and so on.

Finding Prime Numbers

- Finding prime numbers has always been of interest to mathematicians. Modern day mathematicians are still pursuing methods for finding out if the list of prime numbers is infinite or if there is a greatest prime number. Computer programs have been written to generate prime numbers and to see if any given number is prime. One of the simplest ways to get a good picture of prime numbers was devised by Eratosthenes in the year 250 B.C. Here is his method.

  Refer to the table of numbers from 1 to 100 on the next page.

Follow the directions listed below the table to discover the prime numbers between 1 and 100.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

- Put a box around the number one (1). ***The number one is neither composite nor prime, as it does not have two different whole number factors.***
- Circle the number two (2) and cross out all the even numbers (the multiples of two).
- Circle the number three (3) and cross off all the multiples of three (2 × 3, 3 × 3, 4 × 3, and so on).
- Circle the next number after 3 that has not been crossed off. What number is it?______
- Cross off all the multiples of this number.
- Continue in this manner until you have circled or crossed off all the numbers.

All numbers from two through one hundred should be "accounted for"—that is, either circled or crossed off. The numbers you have circled are those prime numbers between 1 and 100. Each of the crossed off numbers is a composite number.

The Fundamental Theorem of Arithmetic

This important principle, The Fundamental Theorem of Arithmetic, states that every composite number can be written as the product of a unique set of *prime* factors. This way of presenting the number is called its **prime factorization**.

For example, 42 can be written as 1 × 42, as 2 × 21, as 3 × 14, or as 6 × 7; but notice that there is a composite number factor in each representation (42 in the first, 21 in the second, 14 in the third, and 6 in the last). There is only one set (a unique set) consisting only of *prime numbers* which, when multiplied together, will equal 42. We say that the prime factorization of 42 is 2 × 3 × 7.

In a prime factorization, the factors are customarily written in sequence from smallest to largest factor.

The process of determining the prime factorization of a number is sometimes simply referred to as **prime factoring** the number.

Shortcut Notation for Repeated Factors

When prime factors occur more than once in a prime factorization, there is a mathematical notation you may use to present the factors more compactly.

Example: The prime factorization of $343 = 7 \times 7 \times 7$

$$343 = 7 \times 7 \times 7 = \underset{\textbf{base}}{7}^{\overset{\textbf{exponent}}{3}}$$

Read, "seven to the third power."

The **power** refers to how many times a single factor repeats itself.

The **exponent** 3 denotes three factors of the **base** number 7.

$7 \times 7 \times 7$ written as 7^3 is said to be written in **exponent form** (or **exponential notation**).

Example: $1350 = 2 \times 3 \times 3 \times 3 \times 5 \times 5$, or $2 \times 3^3 \times 5^2$ in exponent form.
Read, "1350 equals two times three to the third power times five to the second power."

It is helpful, when determining the prime factors of a number, to use a set of **divisibility tests** to determine if specific prime numbers will divide evenly into the given number. Three of the most useful divisibility tests to use are those for the prime numbers 2, 3, and 5:

Divisibility Tests for the Numbers 2, 3, and 5

- If a number is even (ending in 0, 2, 4, 6, or 8), then it is divisible by 2.
- If the sum of the digits of a number is divisible by 3, then the original number is divisible by 3.

  Example: 228 is divisible by 3 because 2 + 2 + 8 = 12 which is divisible by 3.

$$\begin{array}{r} 76 \\ 3\overline{)228} \\ \underline{-21} \\ 18 \\ \underline{-18} \\ 0 \end{array}$$

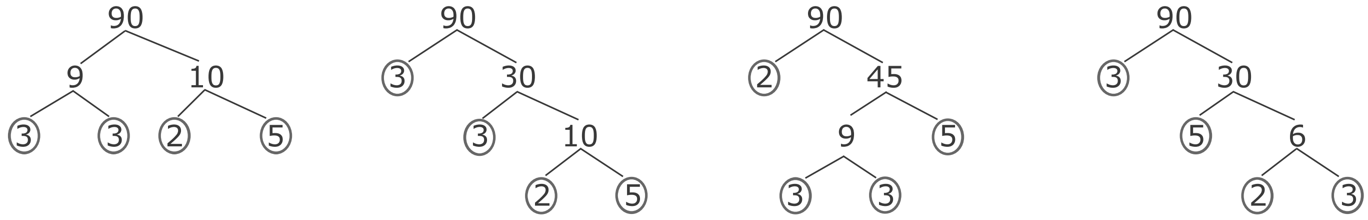
- If the number ends in 5 or 0, it is divisible by 5.

Determining the Prime Factorization of a Composite Number

If the number is small, you can factor it by simply using mental math. Example: $15 = 3 \times 5$

For a larger number, one way is to produce a "factor tree," whereby you break down the number into two factors. If either factor is composite, you break *it* down to two factors, continuing this process until all the "tree branches" end in a prime number. The written product of these end numbers is the prime factorization. For many numbers, there may be several "trees" that can be produced, all with the same correct result.

Examples of ways you might prime factor the number 90:



90 = 2 × 3 × 3 × 5

While this method works, you must be careful so as not to overlook a branch-ending prime number by losing it in your diagram.

The following methodology, which produces what might be called a "factor ladder," offers you a process that is both orderly (the prime factors appear arranged from smallest to largest) and compact (one vertical arrangement) to determine the prime factorization of a number. For the remainder of this book, it will be the preferred methodology used for prime factoring numbers.

METHODOLOGY

Determining the Prime Factorization of a Number *(using a "factor ladder")*

▶ **Example 1**: Determine the prime factorization of 504.

▶ **Example 2**: Determine the prime factorization of 90 using this methodology. → *Try It!*

Steps in the Methodology		Example 1	Example 2
Step 1 **Write the number.**	Set up the number with enough work space under it for divisions.	$\underline{\quad|504}$	
Step 2 **Divide by its smallest prime number factor.**	Divide the number by the smallest prime number that will divide into it evenly, and write your result below it. Continue to divide by that same prime number until it is no longer a factor of your result. ***Special Case***: The given number is a prime number (*see page 249*, Model 2)	Recall the divisibility test for 2: 504 is even, so it is divisible by 2. *THINK* $\underline{2\,|\,504}$ *504 ÷ 2 = 252* $\underline{2\,|\,252}$ *252 ÷ 2 = 126* $\underline{2\,|\,126}$ *126 ÷ 2 = 63* 63	

Steps in the Methodology		Example 1	Example 2
Step 3 **Divide by the next prime number factor.**	Repeat Step 2, dividing by the next larger prime number that is a factor of your result. Continue to perform divisions until that prime number no longer divides evenly.	Recall the divisibility test for 3: If 3 divides the sum of the digits, it divides the number. $2 \mid 504$ $2 \mid 252$ $2 \mid 126$ $3 \mid 63$ $3 \mid 21$ 7 *THINK* *63 ÷ 3 = 21* *21 ÷ 3 = 7*	
Step 4 **Divide by prime numbers until the quotient is one (1).**	Continue dividing by the next larger prime that is a factor of the result until the final division produces one (1) as a quotient.	$2 \mid 504$ $2 \mid 252$ $2 \mid 126$ $3 \mid 63$ $3 \mid 21$ $7 \mid 7$ 1 ← 7 ÷ 7 = 1 *THINK* *7 is prime* The prime factoring is complete.	
Step 5 **Collect prime divisors.**	Collect all of the prime factors on the left side, from smallest to largest, and **use each as many times as it appears**.	$2\times2\times2\times3\times3\times7$	
Step 6 **Present the answer.**	Present your answer.	$504 = 2\times2\times2\times3\times3\times7$ or, written in exponent form, $504 = 2^3 \times 3^2 \times 7$	
Step 7 **Validate your answer.**	Verify that all factors are prime. Validate that the prime factors are correct by finding their product.	2, 3, and 7 are all prime $2\times2\times2\times3\times3\times7 =$ $8 \times 9 \times 7 =$ $72 \times 7 =$ 504 ✓	

MODELS

Model 1

Determine the prime factorization of the following numbers. Use the methodology.

A Prime factorization of 630.

Steps 1-4

		<u>Divisibilty test results</u>:
2	630	*even*
3	315	*divisible by 3*
3	105	*divisible by 3*
5	35	*divisible by 5*
7	7	*7 is prime*
	1	

Steps 5 & 6

$630 = 2 \times 3 \times 3 \times 5 \times 7$

$= \mathbf{2 \times 3^2 \times 5 \times 7}$

Step 7 **Validate:**

2, 3, 5, and 7 are all prime.

$2 \times 3 \times 3 \times 5 \times 7 = 2 \times 9 \times 5 \times 7$

$= 18 \times 5 \times 7$

$= 90 \times 7$

$= 630$ ✓

B Prime factorization of 3465.

Steps 1-4

		<u>Divisibilty test results</u>:
3	3465	*not even, but divisible by 3*
3	1155	*divisible by 3*
5	385	*not divisible by 3, but divisible by 5*
7	77	*not divisible by 5, but divisible by 7*
11	11	*11 is prime*
	1	

Steps 5 & 6

$3465 = 3 \times 3 \times 5 \times 7 \times 11$

$= \mathbf{3^2 \times 5 \times 7 \times 11}$

Step 7 **Validate:**

3, 5, 7, and 11 are all prime.

$3 \times 3 \times 5 \times 7 \times 11 = 9 \times 5 \times 7 \times 11$

$= 45 \times 7 \times 11$

$= 315 \times 11$

$= 3465$ ✓

Model 2

Special Case: The Given Number is a Prime Number

Determine the prime factorization of 167.

		<u>Divisibilty test results</u>:
	167	not even
		not divisible by 3
		not divisible by 5

If you are uncertain as to whether a number is prime, try in succession the next larger prime numbers. You can stop when the prime number you are testing times itself is greater than the number.

Try the next prime number,

Try 7

$$\begin{array}{r} 23 \\ 7\overline{)167} \\ \underline{-14} \\ 27 \\ \underline{-21} \\ 6 \end{array}$$

7 does not divide evenly into 167

7×7=49 and 49<167

Try 11

$$\begin{array}{r} 15 \\ 11\overline{)167} \\ \underline{-11} \\ 57 \\ \underline{-55} \\ 2 \end{array}$$

11 does not divide evenly into 167

11×11=121 and 121<167

Try 13

$$\begin{array}{r} 12 \\ 13\overline{)167} \\ \underline{-13} \\ 37 \\ \underline{-26} \\ 11 \end{array}$$

13 does not divide evenly into 167

Because 13 × 13 =169, which is greater than 167, you can stop trying larger primes.

Answer: 167 is a prime number.

ADDRESSING COMMON ERRORS

Issue	Incorrect Process	Resolution	Correct Process	Validation
Not testing enough prime numbers	Determine the prime factorization of 1938. $\begin{array}{r\|l} 2 & 1938 \\ \hline 3 & 969 \\ \hline & 323 \end{array}$ 323 is not divisible by 3 nor 5. ***Try 7*** ***Try 11*** $\begin{array}{r} 46 \\ 7\overline{)323} \\ \underline{-28} \\ 43 \\ \underline{-42} \\ 1 \end{array}$ $\begin{array}{r} 29 \\ 11\overline{)323} \\ \underline{-22} \\ 103 \\ \underline{-99} \\ 4 \end{array}$ not by 7 not by 11 1938 = 2×3×323	Continue to test prime numbers for divisibility until the prime number, times itself, is **greater** than the number.	Try larger prime factors because 11 × 11 = 121 and 121 < 323 ***Try 13*** ***Try 17*** $\begin{array}{r} 24 \\ 13\overline{)323} \\ \underline{-26} \\ 63 \\ \underline{-52} \\ 11 \end{array}$ $\begin{array}{r} 19 \\ 17\overline{)323} \\ \underline{-17} \\ 153 \\ \underline{-153} \\ 0 \end{array}$ not by 13 17 **is** a factor 13×13=169 and 169<323 $\begin{array}{r\|l} 2 & 1938 \\ \hline 3 & 969 \\ \hline 17 & 323 \\ \hline 19 & 19 \\ \hline & 1 \end{array}$ 19 is prime 1938 = 2×3×17×19	2, 3, 17, and 19 are all prime factors. 2×3×17×19 = 6 × 17 × 19 = 102 × 19 = 1938 ✓

Issue	Incorrect Process	Resolution	Correct Process	Validation
Making mathematical errors when dividing by primes	Determine the prime factorization of 360. 2 | 360 2 | 170 5 | 85 17 | 17 1 360=2×2×5×17	Validate the final answer.	2×2×5×17=340, not 360 Re-work: 2 | 360 2 | 1**8**0 2 | 90 3 | 45 3 | 15 5 | 5 1 360 = 2×2×2×3×3×5	2, 3, and 5 are all prime factors. 2×2×2×3×3×5 = 8 × 9 × 5 = 72 × 5 = 360 ✓
Having non-prime (composite) factors in the prime factorization	Find the prime factorization of 990. 2 | 990 5 | 495 9 | 99 11 | 11 1 990=2×5×9×11	Beginning with the prime number 2, test the divisibility of prime factors **in sequence**. Continue to divide the resulting number by the **same** prime number **until it is no longer a factor** before moving to the next prime divisor. Always verify that the factors in your answer are all prime numbers.	Find the prime factorization of 990. 2 | 990 3 | 495 3 | 165 5 | 55 11 | 11 1 990 = 2×3×3×5×11 $= 2 \times 3^2 \times 5 \times 11$	2, 3, 5, and 11 are all prime factors. $2\times3^2\times5\times11$ = 2×9×5×11 = 18 ×5×11 = 90 ×11 = 990 ✓

Preparation Inventory

Before proceeding, you should have an understanding of each of the following:

☐ the terminology and notation associated with prime factorization

☐ the characteristics of prime numbers and composite numbers

☐ how to determine the prime factorization of a number

☐ how to assure that the prime factorization of a number is accurate

ACTIVITY

Finding Prime Factors

PERFORMANCE CRITERIA

- Correctly identifying a number as prime or composite
- Writing any composite number as a product of its prime factors
  - following a methodology for determining the prime factorization
  - validation of the prime factorization

CRITICAL THINKING QUESTIONS

1. What is the first prime number and why is it prime?

2. In the table of 100 numbers (in the Pre-Activity Preparation, page 245), what makes the crossed out numbers composite numbers?

3. What are some methodologies for determining a prime factorization?

4. How do you make sure the prime factors of a number are truly correct?

5. What are divisibility tests and how do you apply them when finding prime number factors?

6. At what point can you stop applying divisibility tests and conclude that a number is prime?

7. Skim through the methodologies in the remaining sections of Chapter 3. In which ones do you see prime factorization used?

TIPS FOR SUCCESS

- Know, at least, the first ten prime numbers: ___, ___, ___, ___, ___, ___, ___, ___, ___, ___,
- Know and use the Divisibility Tests to begin a prime factorization.
- Test larger primes when necessary.
- Always validate the prime factorization.

DEMONSTRATE YOUR UNDERSTANDING

Determine the prime factorization of:	Validation
1) 48	
2) 135	
3) 187	

Determine the prime factorization of:	Validation
4) 127	
5) 650	

IDENTIFY AND CORRECT THE ERRORS

In the second column, identify the error(s) you find in each of the following worked solutions. If the answer appears to be correct, validate it in the second column and label it "*Correct*." If the worked solution is incorrect, solve the problem correctly in the third column and validate your answer in the last column.

The directions are to determine the prime factorization of each of the following numbers.

Worked Solution *What is Wrong Here?*	Identify Errors or Validate	Correct Process	Validation
1) 36 2 | 36 2 | 18 9 | 9 1 $36 = 2 \times 2 \times 9$	A prime number has only two factors, 1 and itself. 9 is not prime. It is divisible by 3.	2 | 36 2 | 18 3 | 9 3 | 3 1 Answer: $36 = 2 \times 2 \times 3 \times 3$ or $2^2 \times 3^2$	$2 \times 2 \times 3 \times 3$ $= 4 \times 9$ $= \boxed{36}$ ✓

Worked Solution ***What is Wrong Here?***	**Identify Errors or Validate**	**Correct Process**	**Validation**
2) 484 2 | 484 2 | 242 11 | 121 11 | 11 1 $484 = 2^2 \times 11^2$			
3) 105 5 | 105 23 | 23 1 $105 = 5 \times 23$			
4) 51 51 is prime			
5) 99 3 | 99 3 | 33 11 | 11 1 $99 = 3 \times 3 \times 11 \times 1$			

ADDITIONAL EXERCISES

Determine the prime factorization of each of the following numbers. Validate each answer.

1) 735 2) 143 3) 144 4) 101 5) 201